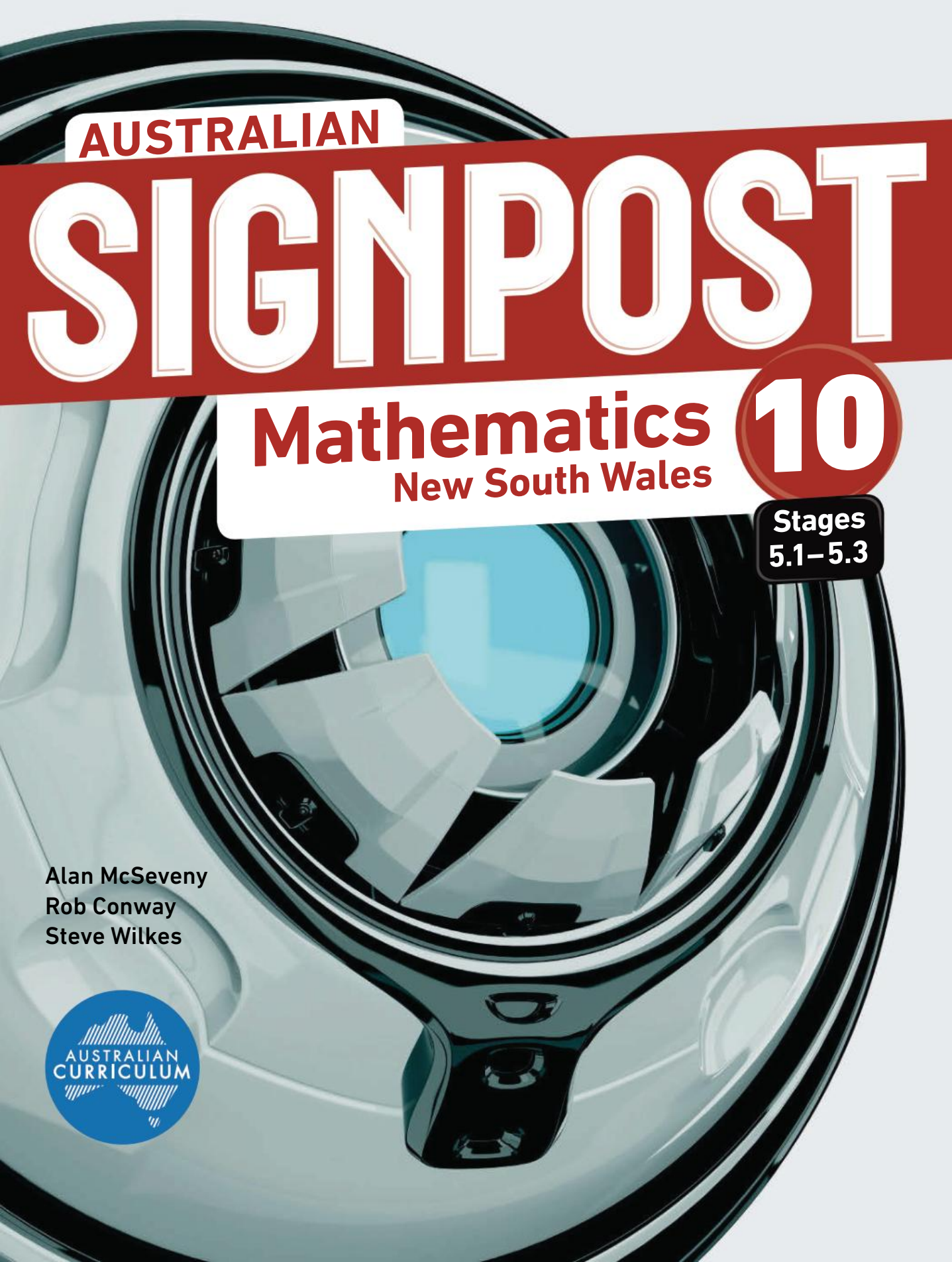


# AUSTRALIAN SIGNPOST

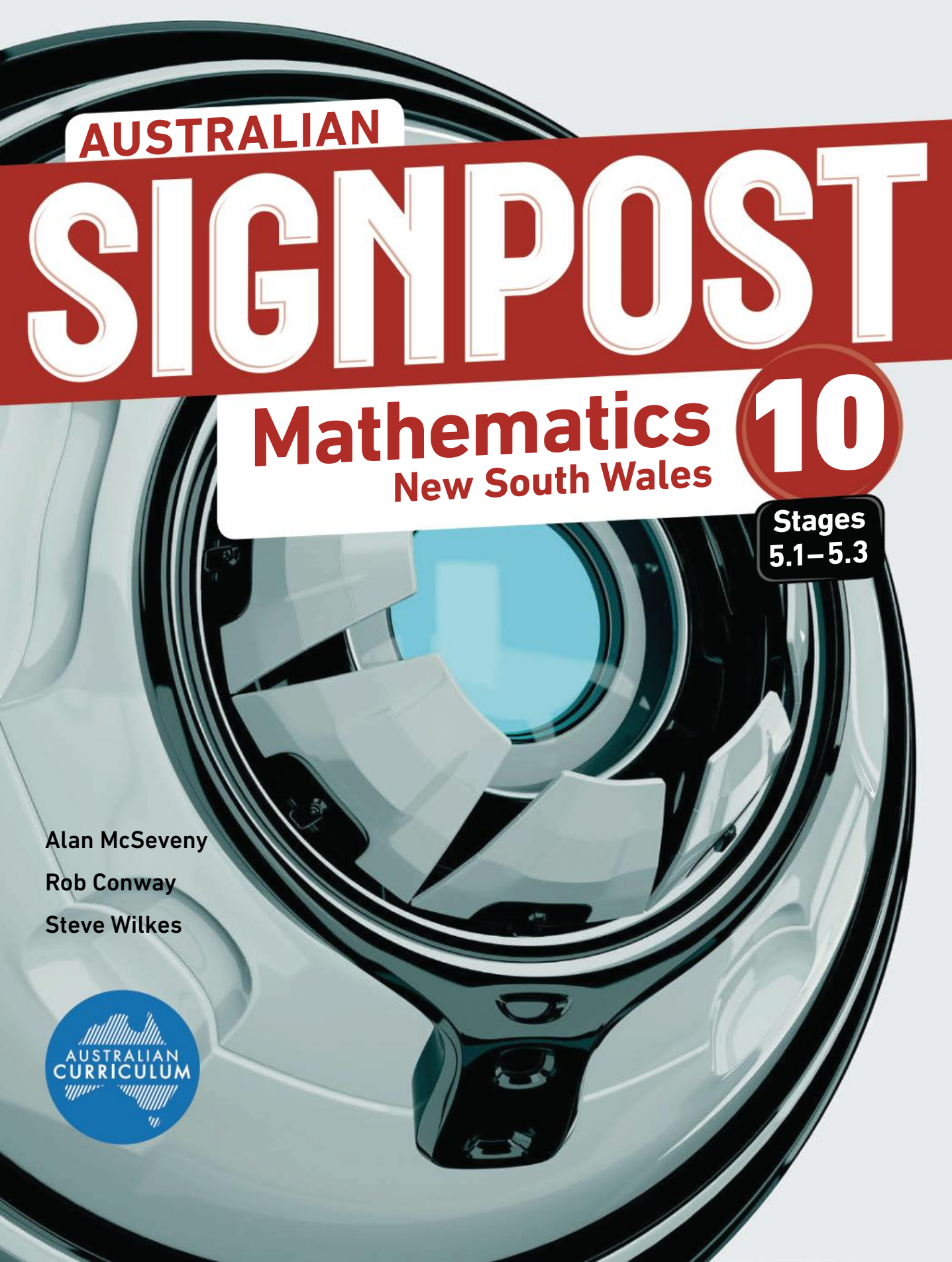
## Mathematics 10

### New South Wales

Stages 5.1–5.3

Alan McSeveny
Rob Conway
Steve Wilkes

AUSTRALIAN CURRICULUM



# AUSTRALIAN SIGNPOST

## Mathematics 10

### New South Wales

Stages 5.1–5.3

Alan McSeveny
Rob Conway
Steve Wilkes

AUSTRALIAN CURRICULUM

*Understanding is a fountain of life*
*to those who have it.*

Proverbs 16:22

**Pearson Australia**
(a division of Pearson Australia Group Pty Ltd)
707 Collins Street, Melbourne, Victoria 3008
PO Box 23360, Melbourne, Victoria 8012
www.pearson.com.au

Copyright © Pearson Australia 2014
(a division of Pearson Australia Group Pty Ltd)
First published 2014 by Pearson Australia
2022 2021 2020 2019
10 9 8 7 6 5 4 3

**Reproduction and communication for educational purposes**
The *Australian Copyright Act 1968* (the Act) allows a maximum of one chapter or 10% of the pages of this work, whichever is the greater, to be reproduced and/or communicated by any educational institution for its educational purposes provided that that educational institution (or the body that administers it) has given a remuneration notice to Copyright Agency Limited (CAL) under the Act. For details of the CAL licence for educational institutions contact Copyright Agency Limited (www.copyright.com.au).

**Reproduction and communication for other purposes**
Except as permitted under the Act (for example any fair dealing for the purposes of study, research, criticism or review), no part of this book may be reproduced, stored in a retrieval system, communicated or transmitted in any form or by any means without prior written permission. All enquiries should be made to the publisher at the address above.

This book is not to be treated as a blackline master; that is, any photocopying beyond fair dealing requires prior written permission.

Publisher: Sophie Matta and Tanya Smith
Project Manager: Jennifer Boyce
Editor: Liz Waud
Designer: Glen McClay
Typesetter: Nikki M Group Pty Ltd
Copyright & Pictures Editor: Julia Weaver
Mac Operator: Rob Curulli
Cover art: Glen McClay, David Doyle and Jenny Grigg
Illustrator: Nikki M Group Pty Ltd
Printed in Malaysia

National Library of Australia Cataloguing-in-Publication entry
Author: McSeveny, A. (Alan), author.
Title: Australian signpost mathematics New South Wales 10 (5.1-5.3) Student book / Alan McSeveny, Rob Conway, Steve Wilkes.
ISBN: 9781488656682 (paperback)
Target Audience: For secondary school age.
Subjects: Mathematics--Study and teaching (Secondary)--New South Wales. Mathematics--New South Wales-Textbooks.
Other Authors/Contributors: Conway, R. (Robert), author.
Wilkes, S. (Stephen), author.
Dewey Number: 510

Pearson Australia Group Pty Ltd ABN 40 004 245 943

**Acknowledgements**
We thank the following for their contributions to our text book:

**Alan McSeveney:** pp. 288, 300b.
**Board of Studies, NSW:** NSW Mathematics K-10 Syllabus, © Board of Studies NSW for and on behalf of the Crown in right of the State of New South Wales, 2012.
**Dreamstime:** 49, 80, 86, 94, 143, 168, 176, 201, 203, 208, 270, 271, 272, 273, 276, 277, 291, 295, 300t, 331, 333, 354, 401, 409, 427, 435, 454, 480, 483, 500, 503l, 519.
**Flickr:** p. 358.
**Fotolia:** pp. 50, 151, 400, 410, 503r.
**GeoGebra:** Created with GeoGebra (www.geogebra.org), pp. 190, 198, 207, 212, 218, 229, 238, 241, 418, 422, 431, 435, 445, 473, 480.
**Microsoft Pty Limited:** Used with permission from Microsoft., p. 315.
**Pearson Asset Library:** p. 158.
**Pearson Australia:** Pearson Australia/Alice McBroom, p. 150.
**Shutterstock:** Cover; 11, 43, 79, 81, 91, 93, 118, 136, 152b, 152c, 152t, 172, 303, 316, 319, 325, 326b, 326t, 327, 336, 407, 412, 451, 495, 497.

Every effort has been made to trace and acknowledge copyright. However, should any infringement have occurred, the publishers tender their apologies and invite copyright owners to contact them.

**Disclaimers**
The selection of internet addresses (URLs) provided for *Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.3 Student Book* was valid at the time of publication and was chosen as being appropriate for use as a secondary education research tool. However, due to the dynamic nature of the internet, some addresses may have changed, may have ceased to exist since publication, or may inadvertently link to sites with content that could be considered offensive or inappropriate. While the authors and publisher regret any inconvenience this may cause readers, no responsibility for any such changes or unforeseeable errors can be accepted by either the authors or the publisher.

Some of the images used in *Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.3 Student Book* might have associations with deceased Indigenous Australians. Please be aware that these images might cause sadness or distress in Aboriginal or Torres Strait Islander communities.

# CONTENTS


Features of Australian Signpost Mathematics vi
Treatment of syllabus content x
P Digital resources xvi
Metric equivalents xviii
The language of mathematics xix
ID Card 1 (Metric units) xix
ID Card 2 (Symbols) xix
ID Card 3 (Language) xx
ID Card 4 (Language) xxi
ID Card 5 (Language) xxii
ID Card 6 (Language) xxiii
Algebra card xxiv

## Chapter 1 Review of Year 9 1

1:01 Number and measurement 2
1:01A Order of operations 2
1:01B Fractions 2
1:01C Decimals 3
1:01D Percentages 4
1:01E Ratios 4
1:01F Rates 5
1:01G Significant figures 6
1:01H Approximations 6
1:01I Estimations 7
1:01J Using units of measurement 7
1:01K Accuracy of measurements 7
1:02 Algebra 8
1:03 Probability 10
Investigation 1:03 Blaise Pascal 11
1:04 Measurement 12
1:05 Indices 13
1:06 Equations, inequalities and formulas 14
1:07 Financial mathematics 16
1:08 Coordinate geometry 18
1:09 Simultaneous equations 19
1:10 Similarity and scale drawing 20
1:11 Trigonometry 21
1:12 Statistics 22
1:13 Proportion 24
1:14 Graphs of physical phenomena 26
Assignment 27

## Chapter 2 Quadratic equations 28

2:01 Solution using factors 29
2:02 Solution by completing the square 33
2:03 The quadratic formula 35
Investigation 2:03 How many solutions? 38
2:04 Choosing the best method 39
Fun spot 2:04 What is an Italian referee? 41
2:05 Problems involving quadratic equations 41
Investigation 2:05 Temperature and altitude 45
2:06 Equations reducible to quadratics 46
Fun spot 2:06 Did you know that 2 = 1? 48
Maths terms, Diagnostic test, Assignments 48

## Chapter 3 Properties of geometrical figures 53

3:01 Deductive reasoning in numerical exercises 54
3:01A Exercises using parallel lines 56
3:01B Exercises using triangles 58
3:01C Exercises using quadrilaterals 60
3:02 Congruent triangle proofs 62
3:03 Deducing properties of the special triangles and quadrilaterals 66
GeoGebra 3:03 The parallelogram family 68
3:04 Proving general results 69
3:05 Definitions and proof in deductive geometry 73
Investigation 3:05 Theorems and their converses 80
3:06 Polygons 81
GeoGebra activity 3:06A Convex and concave polygons 81
Investigation 3:06A The angle sum of a polygon 82
GeoGebra activity 3:06B The angle sum of a polygon 82
Investigation 3:06B The exterior angle sum of a convex polygon 83
GeoGebra activity 3:06C The exterior angle sum of a convex polygon 84
Investigation 3:06C Regular polygons and tessellations 86
Maths terms, Diagnostic test, Assignments 87

## Chapter 4 Probability 92

4:01 Probability and language 93
Investigation 4:01 Probabilities given as odds 98
4:02 Two-step chance experiments 99
Investigation 4:02 Experimental probability without replacement 102
4:03 Three-step chance experiments 103
4:04 The probability of two- and three-step events 107
Investigation 4:04 What is the chance of a boy and a girl? 112
4:05 The multiplication rule for multi-step events 113
Fun spot 4:05 The Monty Hall problem 118
4:06 The multiplication rule for dependent events 119
Maths terms, Diagnostic test, Assignments 126

## Chapter 5 Surds 132

5:01 The real number system 133
Challenge 5:01 Proof that $\sqrt{2}$ is irrational 136
Investigation 5:01 f-stops and $\sqrt{2}$ 136
5:02 Surds 137
5:03 Addition and subtraction of surds 139
5:04 Multiplication and division of surds 141
Fun spot 5:04 What do Inuits sing at birthday parties? 143
5:05 Binomial products 144
Investigation 5:05 Iteration to find square roots 146
5:06 Rationalising the denominator 147
Maths terms, Diagnostic test, Assignments 149

## Chapter 6 Financial mathematics 154

6:01 Saving money 155
Investigation 6:01 Who can you bank on? 158
6:02 Simple interest 158
6:03 Solving simple interest problems 161
Investigation 6:03 Financial spreadsheets 164
6:04 Compound interest 165
Fun spot 6:04 What is the difference between a book and a bore? 169
6:05 Depreciation 169
6:06 Compound interest and depreciation formulas 172
Investigation 6:06 Compound interest tables 177
Maths terms, Diagnostic test, Assignments 178

## Chapter 7 Linear and non-linear relationships 183

7:01 Review of coordinate geometry 184
7:02 Parallel and perpendicular lines 186
GeoGebra activity 7:02A Parallel lines 190
GeoGebra activity 7:02B Perpendicular lines 190
7:03 Using coordinate geometry 191
7:04 Special forms of the parabola 193
GeoGebra activity 7:04A Parabolas of the form $y = ax^2$ 198
GeoGebra activity 7:04B Parabolas of the form $y = ax^2 + k$ 198
GeoGebra activity 7:04C Parabolas of the form $y = (x + a)^2$ and $y = (x + a)^2 + k$ 198
Investigation 7:04 The graphs of parabolas 199
7:05 Parabolas of the form $y = ax^2 + bx + c$ 199
Fun spot 7:05 Why didn't the bald man need his keys? 206
Investigation 7:05 Describing number patterns 207
GeoGebra activity 7:05 Parabolas of the form $y = ax^2 + bx + c$ 207
7:06 Exponential graphs 208
GeoGebra activity 7:06 Exponential graphs 212
7:07 The hyperbola 212
GeoGebra activity 7:07 Hyperbolas of the form $y = \frac{k}{x}$, $y = \frac{k}{x} + c$ and $y = \frac{k}{x - b}$ 218
7:08 Circles and their equations 218
7:09 Curves of the form $y = ax^3$ and $y = ax^3 + d$ 221
7:10 Curves of the form $y = ax^n$ and $y = ax^n + d$ 224
7:11 Curves of the form $y = ax^n$ and $y = a(x - r)^n$ 228
GeoGebra activity 7:11 Curves involving powers 229
7:12 Curves of the form $y = (x - r)(x - s)(x - t)$ 230
7:13 The intersection of graphs 234
Investigation 7:13 A parabola and a circle 237
GeoGebra activity 7:13 The intersection of graphs 238
7:14 Miscellaneous equations and graphs 238
GeoGebra activity 7:14 Curves and their equations 241
Maths terms, Diagnostic test, Assignments 242

## Chapter 8 Surface area and volume 248

8:01 Review of surface area and volume 249
8:02 Surface area of a pyramid 251
8:03 Surface area of a cone 256
Investigation 8:03 The surface area of a cone 256
8:04 Surface area of a sphere 259
Investigation 8:04 The surface area of a sphere 259
Fun spot 8:04 How did the raisins win the war against the nuts? 262
8:05 Volume of a pyramid 263
Investigation 8:05 The volume of a pyramid 263
8:06 Volume of a cone 267
8:07 Volume of a sphere 269
Investigation 8:07 Estimating your surface area and volume 269
8:08 Practical applications of surface area and volume 271
Maths terms, Diagnostic test, Assignments 274

## Chapter 9 Statistics 279

9:01 Statistics: A review 280
Fun spot 9:01 Why did the teacher wear sunglasses? 288
9:02 Measures of spread: Standard deviation 289
9:03 Bivariate data 297
9:04 Scatter plots 304
Challenge 9:04 Further scatter plots 314
Investigation 9:04 Scatter plots using a spreadsheet 315
9:05 Statistics all around us 316
9:06 Statistics: Getting it together 323
Investigation 9:06 Opinion polls: How accurate are they? 328
Challenge 9:06 Use of Australian census data 329
9:07 Statistics: Can they be trusted? 331
Maths terms, Diagnostic test, Assignments 337

## Chapter 10 Similarity 346

10:01 Similar triangle proofs 347
10:02 Sides and areas of similar figures 351

10:03 Sides and volumes of similar solids 354
Investigation 10:03 King Kong: Could he have lived? 359
Maths terms, Diagnostic test, Assignments 360

## Chapter 11 Trigonometry 365

11:01 Trigonometric ratios of other angles 366
Investigation 11:01 Trigonometric graphs 370
GeoGebra activity 11:01A Using the unit circle 371
GeoGebra activity 11:01B Trigonometric graphs and the unit circle 371
GeoGebra activity 11:01C Features of trigonometric graphs 371
11:02 Trigonometric relationships between acute and obtuse angles 372
Fun spot 11:02 Why are camels terrible dancers? 376
11:03 The sine rule 377
GeoGebra activity 11:03 The sine rule 380
11:04 The sine rule: the ambiguous case 381
11:05 The cosine rule 383
Fun spot 11:05 Why did Tom's mother feed him Peter's ice-cream? 387
11:06 Area of a triangle 388
11:07 Miscellaneous problems 390
Maths terms, Diagnostic test, Assignments 392

## Chapter 12 Further equations 396

12:01 Simultaneous equations involving a non-linear equation 397
Investigation 12:01 Number patterns and algebra 401
12:02 Simple cubic equations: $ax^3 = k$ 402
12:03 Literal equations: Restrictions on variables 404
Fun spot 12:03 What small rivers flow into the Nile? 408
Challenge 12:03 Fibonacci formula 408
Maths terms, Diagnostic test, Assignments 409

## Chapter 13 Circle geometry 414

13:01 Circles 415
Investigation 13:01 Circles in space 418
GeoGebra activity 13:01 Terms used in circle geometry 418
13:02 Chord properties of circles (1) 419
GeoGebra activity 13:02 Chord properties 422
Investigation 13:02 Locating the epicentre of earthquakes 423
13:03 Chord properties of circles (2) 424
13:04 Angle properties of circles (1) 428
GeoGebra activity 13:04 Angle properties (1) 431
13:05 Angle properties of circles (2) 432
GeoGebra activity 13:05 Angle properties (2) 435
Investigation 13:05 Diameter of a circumcircle 436
13:06 Tangent properties of circles 436
13:07 Further circle properties 442
GeoGebra activity 13:07 Intersecting chords and secants 445
Fun spot 13:07 How do you make a bus stop? 446
13:08 Deductive exercises involving the circle 447
Fun spot 13:08 How many sections? 450
Maths terms, Diagnostic test, Assignments 451

## Chapter 14 Polynomials 457

14:01 Polynomials 458
14:02 Sum and difference of polynomials 460
14:03 Multiplying and dividing polynomials by linear expressions 462
14:04 Remainder and factor theorems 465
14:05 Solving polynomial equations 467
14:06 Sketching polynomials 469
GeoGebra activity 14:06 Sketching polynomials 473
Fun spot 14:06 How do you find a missing hairdresser? 474
14:07 Sketching curves related to $y = P(x)$ 474
GeoGebra activity 14:07 Sketching curves related to $y = P(x)$ 480
Maths terms, Diagnostic test, Assignments 481

## Chapter 15 Logarithms 486

15:01 Logarithms 487
Investigation 15:01 Logarithmic scales 489
15:02 Logarithmic and exponential graphs 490
15:03 Laws of logarithms 492
Investigation 15:03 Logarithmic scales and the history of calculating 495
15:04 Simple exponential equations 496
Fun spot 15:04 What do you call lumberjack music? 497
Investigation 15:04 Solving harder exponential equations by 'guess and check' 498
Challenge 15:04 Further exponential equations 498
Maths terms, Diagnostic test, Assignments 500

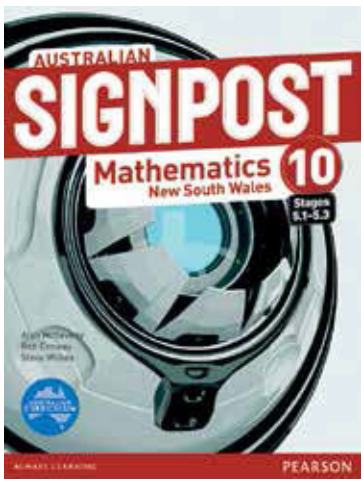
## Chapter 16 Functions and other graphs 504

16:01 Functions 505
16:02 Inverse functions 509
Investigation 16:02 Quadratic functions and inverses 513
16:03 The graphs of $y = f(x)$, $y = f(x) + k$ and $y = f(x - a)$ 514
Fun spot 16:03 Where would you get a job playing a rubber trumpet? 517
Maths terms, Diagnostic test, Assignments 518

## Answers 523

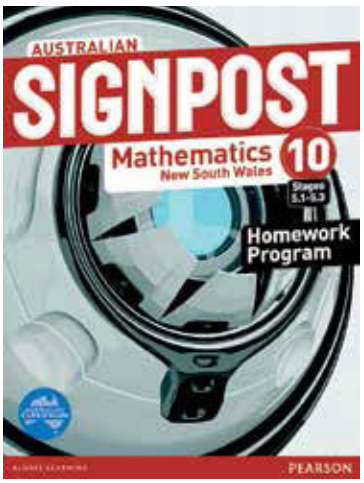
# Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.3

The **Australian Signpost Mathematics New South Wales** series has been completely updated to comprehensively cover the NSW Syllabus for the Australian Curriculum. Students and teachers are supported with a Student Book, an eBook, a Homework Program and a Teacher Companion for each stage of development. Features within each component, along with additional resources on the eBook, allow the creation of a personalised learning package for the individual. Improvements in technology, teaching resources and design allow students and teachers to approach the NSW Syllabus with confidence.



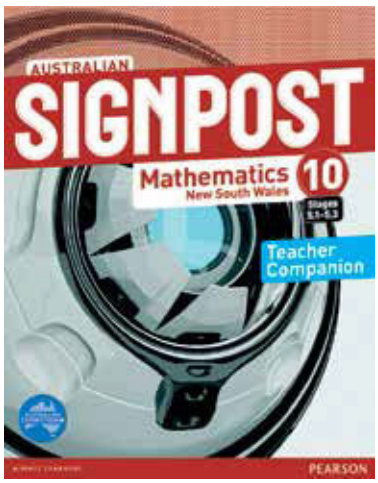

## Student Book

The *Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.2* and *Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.3* learning package addresses Stage 5 of the NSW Syllabus for the Australian Curriculum. This new series continues the Signpost tradition of expanding students' ability in a broad range of mathematical skills while emphasising problem-solving and working mathematically.




## Homework Program

The Homework Program consists of tear-out worksheets that mirror and supplement the content of the student book, providing opportunity for further practice and application of key skills.

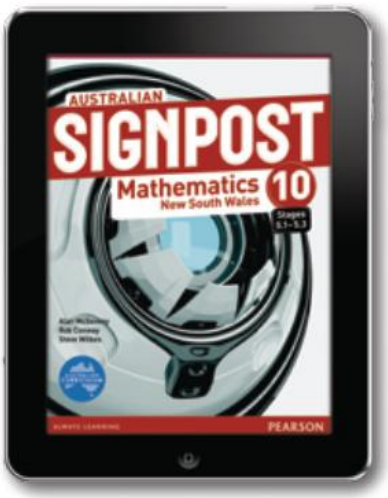



## Teacher Companion

The Teacher Companion is an invaluable resource that provides guidance and support to teachers using the **Australian Signpost Mathematics New South Wales** package. It is a practical resource featuring teaching strategies, activities, reference to Stage 5 outcomes and content statements, and links to digital learning materials.

AUSTRALIAN CURRICULUM

ALWAYS LEARNING




## Pearson eBook 3.0

### Any device, every school

*Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.3 eBook 3.0* lets you use the Student Book online or offline on any device. It allows students to independently work through each exercise, linking to a range of technological applications designed specifically for Signpost users. The eBook will engage students while consolidating learning and providing opportunities for increased depth of understanding.

PearsonDigital

Browse and buy at pearson.com.au.
Access your content at pearsonplaces.com.au.

We believe in learning.
All kinds of learning for all kinds of people, delivered in a personal style.
Because wherever learning flourishes, so do people.

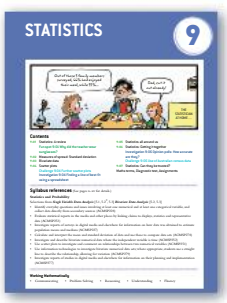
PEARSON

# HOW TO USE THE STUDENT BOOK

The **Australian Signpost Mathematics New South Wales 10 Stages 5.1–5.3** learning package covers Stage 5 outcomes of the NSW Syllabus for the Australian Curriculum.

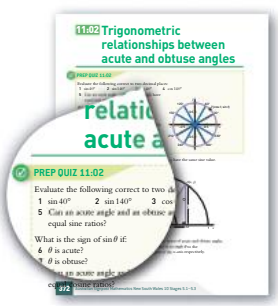
## Features of the Student Book

- a complete year's work with full coverage of the NSW Syllabus for the Australian Curriculum
- a flexible structure with carefully graded exercises, colour-coded to indicate level of difficulty
- technological applications with GeoGebra activities and Pearson eBook 3.0 resources
- integration of a broad range of mathematical skills, expanding students' ability to solve problems, work mathematically and investigate
- diagnostic tests, chapter reviews and cumulative revision components

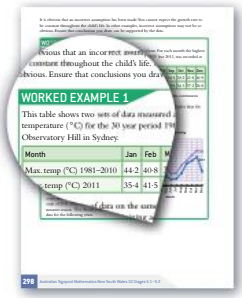



### Chapter opener pages

summarise the key content and present the Syllabus content statements addressed in each chapter. Features within each exercise such as GeoGebra activities, Fun spots and Investigations are listed here.

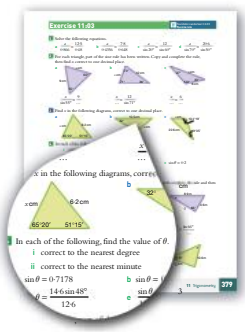



**Prep quizzes** review important skills and knowledge that students will need to complete an exercise.




### Worked examples

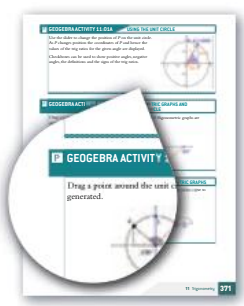
take students through key processes with step-by-step demonstrations of questions similar to those they will encounter.




### Well-graded exercises

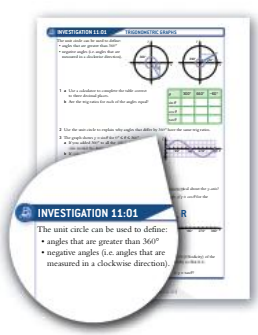
indicate each question's level of difficulty.

- 2 Foundation
- 7 Stages 5.1–5.3 level
- 12 Extension



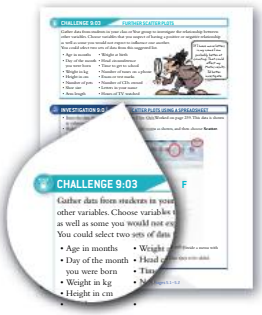

### GeoGebra activities

provide a deeper understanding of concepts covered within the exercise while providing the opportunity to integrate ICT into the classroom.



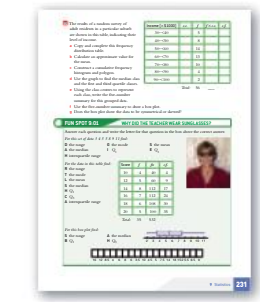

### Investigations

encourage students to seek knowledge and develop research skills. The investigative tasks provide opportunities for students to interpret, question, model and communicate mathematically.



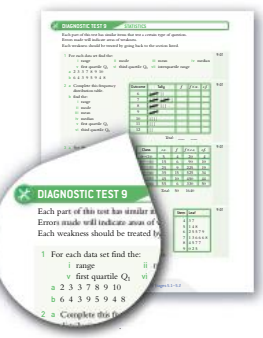

## Challenge activities

provide more difficult questions and investigations. They can be used to help students with a stronger understanding extend their knowledge.



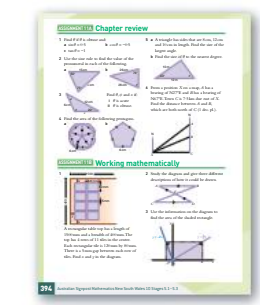
## Fun spots

provide amusement and interest while reinforcing concepts learnt within the chapter. They encourage creativity and divergent thinking, and show that mathematics is enjoyable.



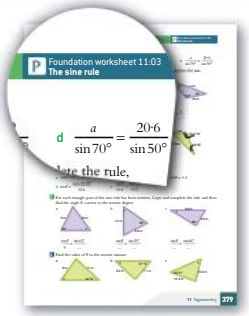

## Diagnostic tests

at the end of each chapter test students' achievement of outcomes. Importantly, they also indicate student weaknesses that need to be addressed and provide links to the relevant sections in the text or digital resources.




## Assignments

are provided at the end of each chapter. They concentrate on revising the content of previous chapters and developing students' ability to work mathematically.

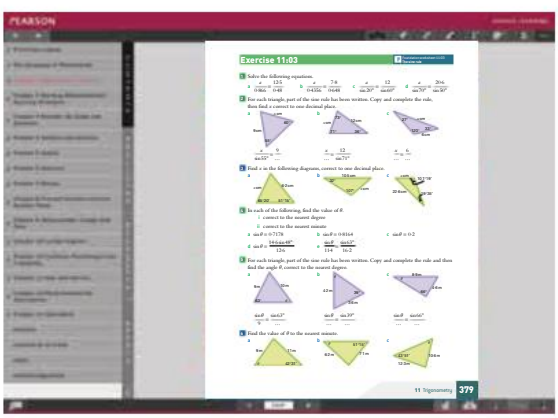



## Foundation worksheets

are indicated in the student book wherever a worksheet is available on the eBook. Foundation worksheets provide alternative exercises for consolidation of the fundamental content of the work being introduced.

## Pearson digital resources

A vast range of technology applications are available with the eBook and ProductLink to complement the content presented in the student book. Digital resources include interactive activities, Foundation worksheets and appendices. From the eBook, students can also link directly to the GeoGebra activities referred to in the student book exercises.




The eBook and ProductLink aims to create a classroom environment that embraces mathematical technologies in a way that consolidates and extends student learning

# TREATMENT OF SYLLABUS CONTENT

The syllabus outcomes and statements relevant to Stages 5.1–5.3 of the NSW Syllabus for the Australian Curriculum are listed below. Chapters where the content is treated are shown.

**Working Mathematically** (encompassing five components: Communicating, Problem Solving, Reasoning, Understanding and Fluency) is an integral part of the structure of each chapter. Special attention is given to Working Mathematically at the end of each chapter of all Year 9 and 10 Student Books.

| **Working Mathematically** | |
|---|---|
| A student: | |
| > uses appropriate terminology, diagrams and symbols in mathematical contexts | **MA5.1-1WM** |
| > selects appropriate notations and conventions to communicate mathematical ideas and solutions | **MA5.2-1WM** |
| > uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures | **MA5.3-1WM** |
| > selects and uses appropriate strategies to solve problems | **MA5.1-2WM** |
| > interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems | **MA5.2-2WM** |
| > generalises mathematical ideas and techniques to analyse and solve problems efficiently | **MA5.3-2WM** |
| > provides reasoning to support conclusions that are appropriate to the context | **MA5.1-3WM** |
| > constructs arguments to prove and justify results | **MA5.2-3WM** |
| > uses deductive reasoning in presenting arguments and formal proofs | **MA5.3-3WM** |
| **Number and Algebra** | |
| A student: | |
| > solves financial problems involving earning, spending and investing money | **MA5.1-4NA** |
| > solves financial problems involving compound interest | **MA5.2-4NA◆** |
| > recognises direct and indirect proportion, and solves problems involving direct proportion | **MA5.2-5NA** |
| > draws, interprets and analyses graphs of physical phenomena | **MA5.3-4NA** |
| > simplifies algebraic fractions, and expands and factorises quadratic expressions | **MA5.2-6NA** |
| > selects and applies appropriate algebraic techniques to operate with algebraic expressions | **MA5.3-5NA§** |
| > operates with algebraic expressions involving positive-integer and zero indices, and establishes the meaning of negative indices for numerical bases | **MA5.1-5NA** |
| > applies index laws to operate with algebraic expressions involving integer indices | **MA5.2-7NA** |
| > performs operations with surds and indices | **MA5.3-6NA§** |
| > solves linear and simple quadratic equations, linear inequalities and linear simultaneous equations, using analytical and graphical techniques | **MA5.2-8NA** |
| > solves complex linear, quadratic, simple cubic and simultaneous equations, and rearranges literal equations | **MA5.3-7NA§** |
| > determines the midpoint, gradient and length of an interval, and graphs linear relationships | **MA5.1-6NA** |
| > uses the gradient-intercept form to interpret and graph linear relationships | **MA5.2-9NA** |
| > uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line | **MA5.3-8NA§** |
| > graphs simple non-linear relationships | **MA5.1-7NA** |
| > connects algebraic and graphical representations of simple non-linear relationships | **MA5.2-10NA◆** |
| > sketches and interprets a variety of non-linear relationships | **MA5.3-9NA§** |
| > recognises, describes and sketches polynomials, and applies the factor and remainder theorems to solve problems | **MA5.3-10NA#** |
| > uses the definition of a logarithm to establish and apply the laws of logarithms | **MA5.3-11NA#** |
| > uses function notation to describe and sketch functions | **MA5.3-12NA#** |

| Measurement and Geometry | |
|---|---|
| A student:<br>> calculates the areas of composite shapes, and the surface areas of rectangular and triangular prisms<br>> calculates the surface areas of right prisms, cylinders and related composite solids<br>> applies formulas to find the surface areas of right pyramids, right cones, spheres and related composite solids | <br>**MA5.1-8MG**<br>**MA5.2-11MG**<br>**MA5.3-13MG** |
| > applies formulas to calculate the volumes of composite solids composed of right prisms and cylinders<br>> applies formulas to find the volumes of right pyramids, right cones, spheres and related composite solids | **MA5.2-12MG**<br>**MA5.3-14MG** |
| > interprets very small and very large units of measurement, uses scientific notation, and rounds to significant figures | **MA5.1-9MG** |
| > applies trigonometry, given diagrams, to solve problems, including problems involving angles of elevation and depression<br>> applies trigonometry to solve problems, including problems involving bearings<br>> applies Pythagoras' theorem, trigonometric relationships, the sine rule, the cosine rule and the area rule, to solve problems, including problems involving three dimensions | **MA5.1-10MG**<br>**MA5.2-13MG◊**<br>**MA5.3-15MG§** |
| > describes and applies the properties of similar figures and scale drawings<br>> calculates the angle sum of any polygon and uses minimum conditions to prove triangles are congruent or similar<br>> proves triangles are similar, and uses formal geometric reasoning to establish properties of triangles and quadrilaterals | **MA5.1-11MG**<br>**MA5.2-14MG**<br>**MA5.3-16MG§** |
| > applies deductive reasoning to prove circle theorems and to solve related problems | **MA5.3-17MG#** |
| **Statistics and Probability** | |
| A student:<br>> uses statistical displays to compare sets of data, and evaluates statistical claims made in the media<br>> uses quartiles and box plots to compare sets of data, and evaluates sources of data<br>> uses standard deviation to analyse data | <br>**MA5.1-12SP**<br>**MA5.2-15SP◊**<br>**MA5.3-18SP** |
| > investigates relationships between two statistical variables, including their relationship over time<br>> investigates the relationship between numerical variables using lines of best fit, and explores how data is used to inform decision-making processes | **MA5.2-16SP**<br>**MA5.3-19SP** |
| > calculates relative frequencies to estimate probabilities of simple and compound events<br>> describes and calculates probabilities in multi-step chance experiments | **MA5.1-13SP**<br>**MA5.2-17SP** |

**The NSW Syllabus for the Australian Curriculum has indicated the minimum Stage 5 content recommendations in relation to current Stage 6 Mathematics Board Developed Courses.**

◊ refers to minimum Stage 5 substrands to be covered by students planning to study Preliminary Mathematics General/HSC Mathematics General 2. All subtrands of Stage 5.1 are also part of the recommended minimum.

§ refers to minimum Stage 5 substrands to be covered by students planning to study Mathematics ('2 Unit'). *Note:* 'At least some of the content' of the Stage 5.3 subtrands, Non-Linear Relationships, Trigonometry and Pythagoras' Theorem and Properties of Geometric Figures need to be covered. All subtrands of Stage 5.1 and Stage 5.2 are also part of the recommended minimum.

# refers to the Optional Stage 5.3 substrands that are part of the recommended minimum to be covered by students planning to study Mathematics Extension 1. All subtrands of Stage 5.1, Stage 5.2 and Stage 5.3 are also part of the recommended minimum.

# Stages 5.1–5.3 content

The Stages 5.1–5.3 content is divided into three strands:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

Content statements are listed below with their Australian Curriculum code or NSW syllabus reference. The chapters that address each content statement are shown in the right-hand column.

| Number and Algebra (5.1) | References | Chapter |
|---|---|---|
| **Financial Mathematics** | | |
| • Solve problems involving earning money. | NSW (Stage 5.1) | Ch 8 (Y9) |
| • Solve problems involving simple interest. | ACMNA211 (Stage 5.1) | Ch 8 (Y9) |
| • Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies. | ACMNA229 (Stages 5.1, 5.2) | Ch 6 (Y10) |
| **Indices** | | |
| • Extend and apply the index laws to variables, using positive-integer indices and the zero index. | ACMNA212 (Stage 5.1) | Ch 6 (Y9) |
| • Simplify algebraic products and quotients using index laws. | ACMNA231 (Stage 5.1) | Ch 6 (Y9) |
| • Apply index laws to numerical expressions with integer indices. | ACMNA209 (Stage 5.1) | Ch 6 (Y9) |
| **Linear Relationships** | | |
| • Find the midpoint and gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software. | ACMNA294 (Stage 5.1) | Ch 10 (Y9) |
| • Find the distance between two points located on the Cartesian plane using a range of strategies, including graphing software. | ACMNA214 (Stage 5.1) | Ch 10 (Y9) |
| • Sketch linear graphs using the coordinates of two points. | ACMNA215 (Stage 5.1) | Ch 7 (Y9), Ch 10 (Y9) |
| • Solve problems involving parallel and perpendicular lines. | ACMNA238 (Stage 5.1) | Ch 7 (Y10) |
| **Non-linear Relationships** | | |
| • Graph simple non-linear relations, with and without the use of digital technologies. | ACMNA296 (Stage 5.1) | Ch 10 (Y9) |
| • Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technologies as appropriate. | ACMNA239 (Stage 5.1) | Ch 7 (Y10) |

| Measurement and Geometry (5.1) | References | Chapters |
|---|---|---|
| **Area and Surface Area** | | |
| • Calculate the areas of composite shapes. | ACMMG216 (Stage 5.1) | Ch 5 (Y9) |
| • Solve problems involving the surface areas of right prisms. | ACMMG218 (Stage 5.1) | Ch 5 (Y9) |
| **Numbers of Any Magnitude** | | |
| • Investigate very small and very large time scales and intervals. | ACMMG219 (Stage 5.1) | Ch 1 (Y9) |
| • Express numbers in scientific notation. | ACMMG210 (Stage 5.1) | Ch 1 (Y9), Ch 6 (Y9) |
| **Right-angled Triangles (Trigonometry)** | | |
| • Use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles. | ACMMG223 (Stage 5.1) | Ch 13 (Y9) |
| • Apply trigonometry to solve right-angled triangle problems. | ACMMG224 (Stage 5.1) | Ch 13 (Y9) |
| • Solve right-angled triangle problems, including those involving angles of elevation and depression. | ACMMG245 (Stage 5.1) | Ch 13 (Y9) |
| **Properties of Geometrical Figures** | | |
| • Use the enlargement transformation to explain similarity. | ACMMG220 (Stage 5.1) | Ch 11 (Y9) |
| • Solve problems using ratio and scale factors in similar figures. | ACMMG221 (Stage 5.1) | Ch 11 (Y9) |

| Statistics and Probability (5.1) | References | Chapters |
|---|---|---|
| **Single Variable Data Analysis** | | |
| • Identify everyday questions and issues involving at least one numerical and at least one categorical variable, and collect data directly from secondary sources. | ACMSP228 (Stage 5.1) | Ch 9 (Y10) |
| • Construct back-to-back stem-and-leaf plots and histograms and describe data, using terms including 'skewed', 'symmetric' and 'bi-modal'. | ACMSP282 (Stage 5.1) | Ch 14 (Y9) |
| • Compare data displays using mean, median and range to describe and interpret numerical data sets in terms of location (centre) and spread. | ACMSP283 (Stage 5.1) | Ch 14 (Y9) |
| • Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data. | ACMSP253 (Stage 5.1) | Ch 9 (Y10) |
| **Probability** | | |
| • Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or'. | ACMSP226 (Stage 5.1) | Ch 4 (Y9) |

| Number and Algebra (5.2) | References | Chapters |
|---|---|---|
| **Financial Mathematics**$^{\diamond}$ | | |
| • Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies. | ACMNA229 (Stages 5.1, 5.2) | Ch 6 (Y10) |
| **Ratio and Rates** | | |
| • Solve problems involving direct proportion; explore the relationship between graphs and equations corresponding to simple rate problems. | ACMNA208 (Stage 5.2) | Ch 15 (Y9) |
| **Algebraic Techniques** | | |
| • Apply the four operations to simple algebraic fractions with numerical denominators. | ACMNA232 (Stage 5.2) | Ch 3 (Y9) |
| • Apply the four operations to algebraic fractions with pronumerals in the denominator. | NSW (Stage 5.2) | Ch 3 (Y9) |
| • Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate. | ACMNA213 (Stage 5.2) | Ch 3 (Y9) |
| • Factorise algebraic expressions by taking out a common algebraic factor. | ACMNA230 (Stage 5.2) | Ch 3 (Y9), Ch 12 (Y9) |
| • Expand binomial products and factorise monic quadratic expressions using a variety of strategies. | ACMNA233 (Stage 5.2) | Ch 12 (Y9) |
| **Indices** | | |
| • Apply index laws to algebraic expressions involving integer indices. | NSW (Stage 5.2) | Ch 6 (Y9) |
| **Equations** | | |
| • Solve linear equations. | ACMNA215 (Stage 5.2) | Ch 7 (Y9) |
| • Solve linear equations involving simple algebraic fractions. | ACMNA240 (Stage 5.2) | Ch 7 (Y9) |
| • Solve simple quadratic equations using a range of strategies. | ACMNA241 (Stage 5.2) | Ch 2 (Y10) |
| • Substitute values into formulas to determine an unknown. | ACMNA234 (Stage 5.2) | Ch 7 (Y9) |
| • Solve problems involving linear equations, including those derived from formulas. | ACMNA235 (Stage 5.2) | Ch 7 (Y9) |
| • Solve linear inequalities and graph their solutions on a number line. | ACMNA236 (Stage 5.2) | Ch 7 (Y9) |
| • Solve linear simultaneous equations, using algebraic and graphical techniques, including with the use of digital technologies. | ACMNA237 (Stage 5.2) | Ch 9 (Y9) |
| **Linear Relationships** | | |
| • Interpret and graph linear relationships using the gradient-intercept form of the equation of a straight line. | NSW (Stage 5.2) | Ch 10 (Y9) |
| • Solve problems involving parallel and perpendicular lines. | ACMNA238 (Stage 5.2) | Ch 10 (Y9) |
| **Non-linear Relationships**$^{\diamond}$ | | |
| • Graph simple non-linear relationships with and without the use of digital technologies and solve simple related equations. | ACMNA296 (Stage 5.2) | Ch 10 (Y9), Ch 7 (Y10) |
| • Explore the connection between algebraic and graphical representations of relationships such as simple quadratics, circles and exponentials using digital technologies as appropriate. | ACMNA239 (Stage 5.2) | Ch 7 (Y10) |

| Measurement and Geometry (5.2) | References | Chapters |
|---|---|---|
| **Area and Surface Area** | | |
| • Calculate the surface areas of cylinders and solve related problems. | ACMMG217 (Stage 5.2) | Ch 5 (Y9) |
| • Solve problems involving surface area for a range of prisms, cylinders and composite solids. | ACMMG242 (Stage 5.2) | Ch 5 (Y9), Ch 8 (Y10) |
| **Volume** | | |
| • Solve problems involving the volumes of right prisms. | ACMMG218 (Stage 5.2) | Ch 5 (Y9) |
| • Solve problems involving volume for a range of prisms, cylinders and composite solids. | ACMMG242 (Stage 5.2) | Ch 5 (Y9) |
| **Right-angled triangles (Trigonometry)[◊]** | | |
| • Apply trigonometry to solve right-angled triangle problems. | ACMMG224 (Stage 5.2) | Ch 13 (Y9) |
| • Solve right-angled triangle problems, including those involving direction and angles of elevation and depression. | ACMMG245 (Stage 5.2) | Ch 13 (Y9) |
| **Properties of Geometrical Figures** | | |
| • Formulate proofs involving congruent triangles and angle properties. | ACMMG243 (Stage 5.2) | Ch 3 (Y10) |
| • Use the enlargement transformations to explain similarity and to develop the conditions for triangles to be similar. | ACMMG220 (Stage 5.2) | Ch 11 (Y9) |
| • Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes. | ACMMG244 (Stage 5.2) | Ch 3 (Y10), Ch 10 (Y10) |

| Statistics and Probability (5.2) | References | Chapters |
|---|---|---|
| **Single Variable Data Analysis[◊]** | | |
| • Determine quartiles and interquartile range. | ACMSP248 (Stage 5.2) | Ch 14 (Y9) |
| • Construct and interpret box plots and use them to compare data sets. | ACMSP249 (Stage 5.2) | Ch 14 (Y9) |
| • Compare shapes of box plots to corresponding histograms and dot plots. | ACMSP250 (Stage 5.2) | Ch 14 (Y9) |
| • Investigate reports of surveys in digital media and elsewhere for information on how data was obtained to estimate population means and medians. | ACMSP227 (Stage 5.2) | Ch 9 (Y10) |
| **Bivariate Data Analysis** | | |
| • Investigate and describe bivariate numerical data where the independent variable is time. | ACMSP252 (Stage 5.2) | Ch 9 (Y10) |
| • Use scatter plots to investigate and comment on relationships between two numerical variables. | ACMSP251 (Stage 5.2) | Ch 9 (Y10) |
| **Probability** | | |
| • List all outcomes for two-step chance experiments, with and without replacement, using tree diagrams or arrays; assign probabilities to outcomes and determine probabilities for events. | ACMSP225 (Stage 5.2) | Ch 4 (Y9) |
| • Describe the results of two- and three-step chance experiments, with and without replacement, assign probabilities to outcomes, and determine probabilities of events; investigate the concept of independence. | ACMSP246 (Stage 5.2) | Ch 4 (Y10) |
| • Use the language of 'if ... then', 'given', 'of', 'knowing that' to investigate conditional statements and to identify common mistakes in interpreting such language. | ACMSP247 (Stage 5.2) | Ch 4 (Y10) |

| Number and Algebra (5.3) | References | Chapters |
|---|---|---|
| **Rates and Ratio** | | |
| • Solve problems involving direct proportion; explore the relationship between graphs and equations corresponding to simple rate problems. | ACMNA208 (Stage 5.3) | Ch 15 (Y9) |
| **Algebraic Techniques[§]** | | |
| • Add and subtract algebraic fractions with numerical denominators, including those with binomial numerators. | NSW (Stage 5.3) | Ch 3 (Y9) |
| • Expand binomial products using a variety of strategies. | ACMNA233 (Stage 5.3) | Ch 3 (Y9), Ch 12 (Y9) |
| • Factorise monic and non-monic quadratic expressions. | ACMNA269 (Stage 5.3) | Ch 12 (Y9) |
| **Surds and Indices[§]** | | |
| • Define rational and irrational numbers and perform operations with surds and fractional indices. | ACMNA264 (Stage 5.3) | Ch 6 (Y9), Ch 5 (Y10) |
| **Equations[§]** | | |
| • Solve complex linear equations involving algebraic fractions. | NSW (Stage 5.3) | Ch 7 (Y9) |
| • Solve a wide range of quadratic equations derived from a variety of contexts. | ACMNA269 (Stage 5.3) | Ch 2 (Y10) |
| • Solve simple cubic equations. | NSW (Stage 5.3) | Ch 12 (Y10) |
| • Rearrange literal equations. | NSW (Stage 5.3) | Ch 7 (Y9) |
| • Solve simultaneous equations, where one equation is non-linear, using algebraic and graphical techniques, including the use of digital technologies. | NSW (Stage 5.3) | Ch 12 (Y10) |

| | | |
|---|---|---|
| **Linear Relationships[§]** | | |
| • Find the midpoint and gradient of a line segment (interval) on the Cartesian plane. | ACMNA294 (Stage 5.3) | Ch 10 (Y9) |
| • Find the distance between two points located on a Cartesian plane. | ACMNA214 (Stage 5.3) | Ch 10 (Y9) |
| • Sketch linear graphs using the coordinates of two points. | ACMNA215 (Stage 5.3) | Ch 10 (Y9) |
| • Solve problems using various standard forms of the equation of a straight line. | NSW (Stage 5.3) | Ch 7 (Y10) |
| • Solve problems involving parallel and perpendicular lines. | ACMNA238 (Stage 5.3) | Ch 9 (Y9), Ch 7 (Y10) |
| **Non-linear Relationships[§]** | | |
| • Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations. | ACMNA267 (Stage 5.3) | Ch 7 (Y10) |
| • Describe, interpret and sketch cubics, other curves and their transformations. | NSW (Stage 5.3) | Ch 7 (Y10) |
| **Polynomials[#]** | | |
| • Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems. | ACMNA266 (Stage 5.3) | Ch 14 (Y10) |
| • Apply an understanding of polynomials to sketch a range of curves and describe the features of these curves from their equations. | ACMNA268 (Stage 5.3) | Ch 14 (Y10) |
| **Logarithms[#]** | | |
| • Use the definition of a logarithm to establish and apply the laws of logarithms. | ACMNA265 (Stage 5.3) | Ch 15 (Y10) |
| • Solve simple exponential equations. | ACMNA270 (Stage 5.3) | Ch 15 (Y10) |
| **Functions and Other Graphs[#]** | | |
| • Describe, interpret and sketch functions. | NSW (Stage 5.3) | Ch 16 (Y10) |

| Measurement and Geometry (5.3) | References | Chapters |
|---|---|---|
| **Area and Surface Area** | | |
| • Solve problems involving the surface areas of right pyramids, right cones, spheres and related composite solids. | ACMMG271 (Stage 5.3) | Ch 8 (Y10) |
| **Volume** | | |
| • Solve problems involving the volumes of right pyramids, right cones, spheres and related composite solids. | ACMMG271 (Stage 5.3) | Ch 8 (Y10) |
| **Trigonometry and Pythagoras' Theorem[§]** | | |
| • Apply Pythagoras' theorem and trigonometry to solve three-dimensional problems in right-angled triangles. | ACMMG276 (Stage 5.3) | Ch 13 (Y9) |
| • Use the unit circle to define trigonometric functions, and graph them, with and without the use of digital technologies. | ACMMG274 (Stage 5.3) | Ch 11 (Y10) |
| • Solve simple trigonometric equations. | ACMMG275 (Stage 5.3) | Ch 11 (Y10) |
| • Establish the sine, cosine and area rules for any triangle and solve related problems. | ACMMG273 (Stage 5.3) | Ch 11 (Y10) |
| **Properties of Geometrical Figures[§]** | | |
| • Formulate proofs involving congruent triangles and angle properties. | ACMMG243 (Stage 5.3) | Ch 3 (Y10) |
| • Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes. | ACMMG244 (Stage 5.3) | Ch 10 (Y10) |
| **Circle Geometry[#]** | | |
| • Prove and apply angle and chord properties of circles. | ACMMG272 (Stage 5.3) | Ch 13 (Y10) |
| • Prove and apply tangent and secant properties of circles. | NSW (Stage 5.3) | Ch 13 (Y10) |

| Statistics and Probability (5.3) | References | Chapters |
|---|---|---|
| **Single Variable Data Analysis** | | |
| • Calculate and interpret the mean and standard deviation of data and use these to compare data sets. | ACMSP278 (Stage 5.3) | Ch 9 (Y10) |
| **Bivariate Data Analysis** | | |
| • Use information technologies to investigate bivariate numerical data sets; where appropriate, students use a straight line to describe the relationship, allowing for variation. | ACMSP279 (Stage 5.3) | Ch 9 (Y10) |
| • Investigate reports of studies in digital media and elsewhere for information on their planning and implementation. | ACMSP277 (Stage 5.3) | Ch 9 (Y10) |

# P DIGITAL RESOURCES

| Chapter | Drag-and-drop activities (ProductLink only) | Technology activities Excel (E)—ProductLink only GeoGebra (G)—eBook and ProductLink | Worksheets Foundation worksheets (F) eBook and ProductLink |
|---|---|---|---|
| 1 Review of Year 9 | | | Appendix summary of Year 9 |
| 2 Quadratic equations | Maths terms 2<br>Quadratic equations 1<br>Quadratic equations 2<br>Completing the square | Completing the square (E) | 2:01 Quadratic equations (F)<br>2:03 The quadratic formula (F) |
| 3 Properties of geometrical figures | Angle sum of polygons<br>Triangles<br>Quadrilaterals<br>Angles and parallel lines<br>Pythagoras' theorem<br>Maths terms 3 | The parallelogram family (G)<br>Convex and concave polygons (G)<br>The angle sum of a polygon (G)<br>The exterior angle sum of a convex polygon (G) | 3:02 Congruent triangle proofs (F)<br>3:03 Using congruent triangles (F)<br>3:04 Non-numerical proofs (F)<br>3:06 Formulas (F) |
| 4 Probability | Maths terms 4<br>Theoretical probability | | 4:01 Probability review (F) |
| 5 Surds | Simplifying surds<br>Operations with Surds<br>Maths terms 5 | | 5:02 Surds (F)<br>5:03 Addition and subtraction of surds (F)<br>5:04 Multiplication and division of surds (F)<br>5:05 Binomial products—Surds (F) |
| 6 Financial mathematics | Depreciation<br>Compound interest<br>Maths terms 6 | Who wants to be a millionaire? (E)<br>Compound interest (E) | 6:02 Simple interest (F)<br>6:04 Compound interest (F)<br>6:06 Compound interest formula (F) |
| 7 Linear and non-linear relationships | Transforming curves<br>Parabolas puzzle<br>Maths terms 7<br>Identifying graphs from equations | Parallel lines (G)<br>Perpendicular lines (G)<br>Curve sketching (E)<br>Parabolas of the form $y = ax^2$ (G)<br>Parabolas of the form $y = ax^2 + k$ (G)<br>Parabolas of the form $y = (x + a)^2$ and $(x + a)^2 + k$ (G)<br>Parabolas of the form $y = ax^2 + bx + c$ (G)<br>Exponential graphs (G)<br>Hyperbolas of the form $y = \frac{k}{x}$, $y = \frac{k}{x} + c$ and $y = \frac{k}{x - b}$ (G)<br>Parabolas in real life (E)<br>Curves involving powers (G)<br>The intersection of graphs (G)<br>Graphs and equations (G) | 7:02 Parallel and perpendicular lines (F)<br>7:03 Coordinate geometry (F)<br>7:05 The parabola $y = ax^2 + bx + c$ (F)<br>7:10 Curves of the form $y = ax^n$ and $y = ax^n + d$ (F)<br>7:11 Curves of the form $y = ax^n$ and $y = a(x - r)^n$ (F)<br>7:12 Curves of the form $y = a(x - r)(x - s)(x - t)$ (F) |
| 8 Surface area and volume | Volumes of pyramids<br>Volumes of cylinders, cones and spheres<br>Maths terms 8 | | 8:01 Surface area review (F)<br>8:02 Surface area of a pyramid (F)<br>8:03 Surface area of a cone (F)<br>8:05 Volume of a pyramid (F) |
| 9 Statistics | Maths terms 9<br>Box-and-whisker plots<br>Interquartile range | Mean and standard deviation (E) | 9:02 Standard deviation (F) |
| 10 Similarity | Maths terms 10<br>Using the scale factor | | 10:01 Similar triangle proofs (F) |

| Chapter | Drag-and-drop activities (ProductLink only) | Technology activities Excel (E)—ProductLink only GeoGebra (G)—eBook and ProductLink | Worksheets Foundation worksheets (F) eBook and ProductLink |
|---|---|---|---|
| **11 Trigonometry** | Sine rule<br>Cosine rule<br>Maths terms 11 | Using the unit circle (G)<br>Trigonometric graphs and the unit circle (G)<br>Features of trigonometric graphs [G]<br>The sine rule (G)<br>Investigating sine curves (E) | 11:02 Trig ratios of obtuse angles (F)<br>11:03 The sine rule (F)<br>11:04 The sine rule: The ambiguous case (F)<br>11:05 The cosine rule (F) |
| **12 Further equations** | Literal equations<br>Further simultaneous equations<br>Maths terms 12 | Literal equations (E) | 12:03 Literal equations (F) |
| **13 Circle geometry** | Parts of a circle<br>Circle geometry puzzle<br>Maths terms 13 | Terms used in circle geometry (G)<br>Chord properties (G)<br>Angle properties 1 (G)<br>Angle properties 2 (G)<br>Intersecting chords and secants (G)<br>Circle investigations (E) | |
| **14 Polynomials** | Polynomials<br>Maths terms 14 | Sketching polynomials (G)<br>Sketching curves related to $y = P(x)$ (G) | |
| **15 Logarithms** | Logarithms | | |
| **16 Functions and their graphs** | Function notation | | |

Teachers also have access to other digital resources on the eBook and ProductLink, including:

- syllabus overview document
- editable teaching programs
- answers to all worksheets
- answers to Prep quizzes and Investigations
- answers to the Homework Program.

# Metric equivalents

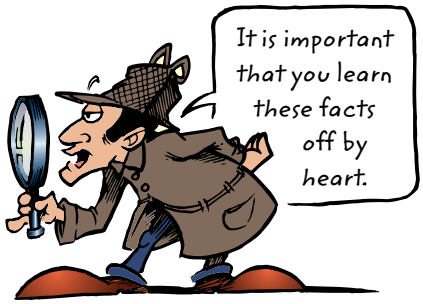
| Length |
|---|
| $1\text{ m} = 1000\text{ mm}$ |
| $= 100\text{ cm}$ |
| $= 10\text{ dm}$ |
| $1\text{ cm} = 10\text{ mm}$ |
| $1\text{ km} = 1000\text{ m}$ |
| **Area** |
| $1\text{ m}^2 = 10\,000\text{ cm}^2$ |
| $1\text{ ha} = 10\,000\text{ m}^2$ |
| $1\text{ km}^2 = 100\text{ ha}$ |
| **Mass** |
| $1\text{ kg} = 1000\text{ g}$ |
| $1\text{ t} = 1000\text{ kg}$ |
| $1\text{ g} = 1000\text{ mg}$ |
| **Volume** |
| $1\text{ m}^3 = 1\,000\,000\text{ cm}^3$ |
| $= 1000\text{ dm}^3$ |
| $1\text{ L} = 1000\text{ mL}$ |
| $1\text{ kL} = 1000\text{ L}$ |
| $1\text{ m}^3 = 1\text{ kL}$ |
| $1\text{ cm}^3 = 1\text{ mL}$ |
| $1000\text{ cm}^3 = 1\text{ L}$ |
| **Time** |
| $1\text{ min} = 60\text{ s}$ |
| $1\text{ h} = 60\text{ min}$ |
| $1\text{ day} = 24\text{ h}$ |
| $1\text{ year} = 365\text{ days}$ |
| $1\text{ leap year} = 366\text{ days}$ |

**Months of the year**

30 days each has September,
April, June and November.
All the rest have 31, except February alone,
Which has 28 days clear and 29 each leap year.

**Seasons**

*Summer:* December, January, February
*Autumn:* March, April, May
*Winter:* June, July, August
*Spring:* September, October, November



# The language of mathematics

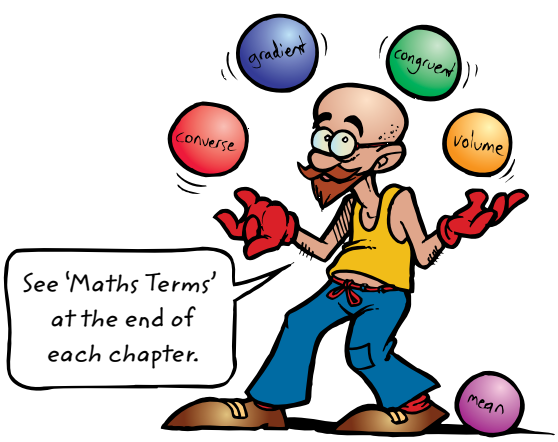
You should regularly test your knowledge by identifying the items on each card.

| ID Card 1 (Metric units) | | | |
|---|---|---|---|
| 1 m | 2 dm | 3 cm | 4 mm |
| 5 km | 6 $m^2$ | 7 $cm^2$ | 8 $km^2$ |
| 9 ha | 10 $m^3$ | 11 $cm^3$ | 12 s |
| 13 min | 14 h | 15 m/s | 16 km/h |
| 17 g | 18 mg | 19 kg | 20 t |
| 21 L | 22 mL | 23 kL | 24 °C |

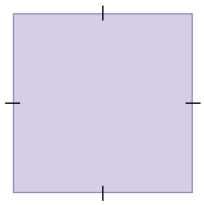
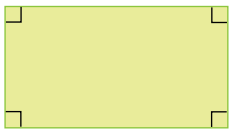
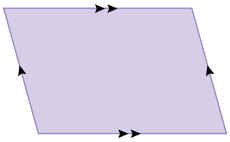
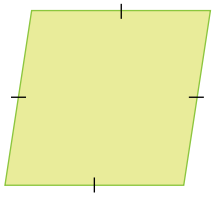
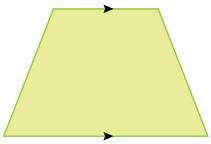
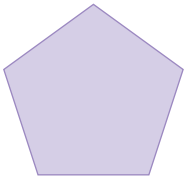
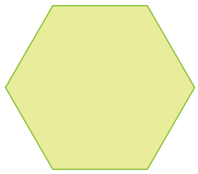
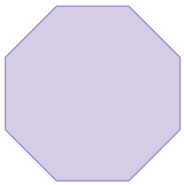
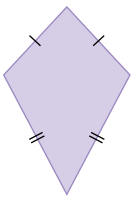
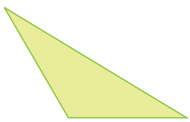
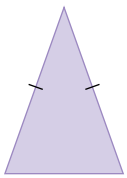
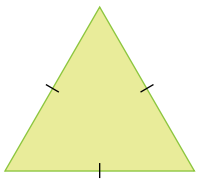
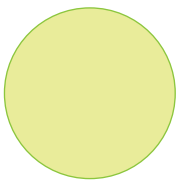
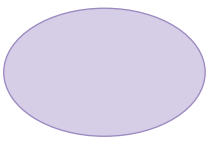
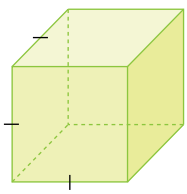
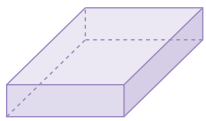
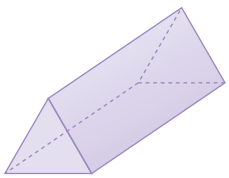
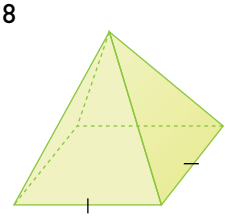
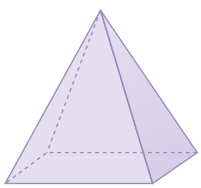
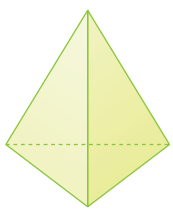
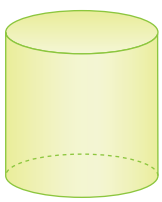
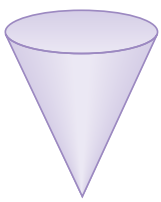
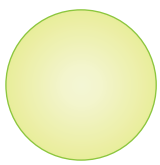
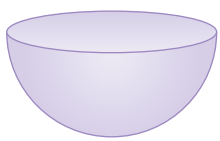
For answers see page 523.

| ID Card 2 (Symbols) | | | |
|---|---|---|---|
| 1 $=$ | 2 $\div$ or $\approx$ | 3 $\neq$ | 4 $<$ |
| 5 $\leq$ | 6 $\not<$ | 7 $>$ | 8 $\geq$ |
| 9 $4^2$ | 10 $4^3$ | 11 $\sqrt{2}$ | 12 $\sqrt[3]{2}$ |
| 13 | 14 $\parallel$ | 15 $\equiv$ | 16 $\parallel\!\parallel$ |
| 17 % | 18 $\therefore$ | 19 e.g. | 20 i.e. |
| 21 $\pi$ | 22 $\Sigma$ | 23 $\overline{x}$ | 24 $P(E)$ |

For answers see page 523.



## ID Card 3 (Language)

| | | | |
|---|---|---|---|
| 1  | 2  | 3  | 4  |
| 5  | 6  | 7  | 8  |
| 9  | 10  All sides different | 11  | 12  |
| 13  | 14  | 15  | 16  |
| 17  | 18  | 19  | 20  |
| 21  | 22  | 23  | 24  |

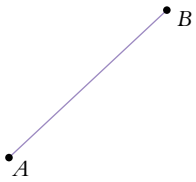
For answers see page 523.

## ID Card 4 (Language)

**1**

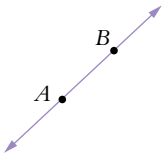
.....

**2**



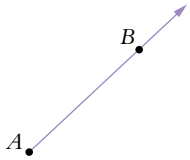

.....

**3**



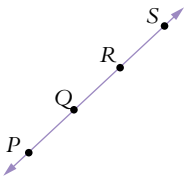

.....

**4**



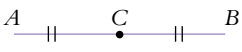

.....

**5**




..... points

**6**



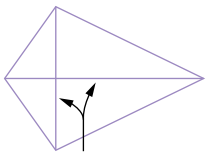

*C* is the .....

**7**

.....

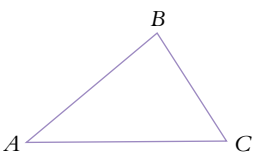
.....

**8**



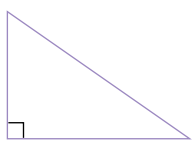
.....

**9**



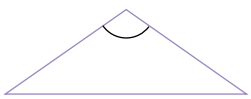

all angles less than 90°

**10**



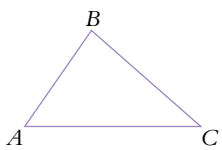
one angle 90°

**11**



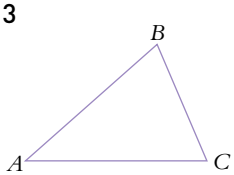
one angle greater than 90°

**12**



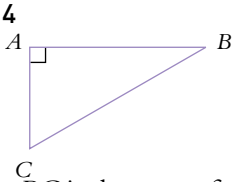

*A*, *B* and *C* are ..... of the triangle.

**13**



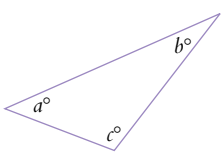

Use the vertices to name the Δ.

**14**



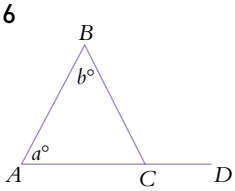

*BC* is the ..... of the right-angled Δ.

**15**



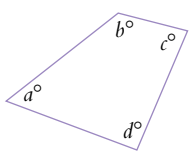

$a° + b° + c°$ = .....

**16**




$\angle BCD$ = .....

**17**

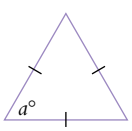



$a° + b° + c° + d°$ = ....

**18**

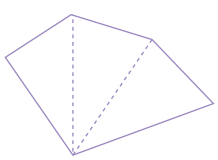
Which is true?
(a) $a° < b°$
(b) $a° = b°$
(c) $a° > b°$

**19**



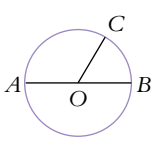

$a°$ = .....

**20**



Angle sum = .....

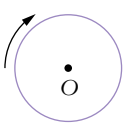
**21**




*AB* is a .....

*OC* is a .....

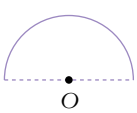
**22**




Name of distance around the circle.

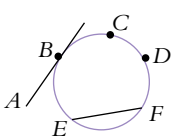
.....

**23**




s.....

**24**




*AB* is a .....

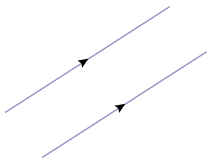
*CD* is an .....

*EF* is a .....

For answers see page 523.

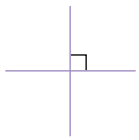
## ID Card 5 (Language)

**1**



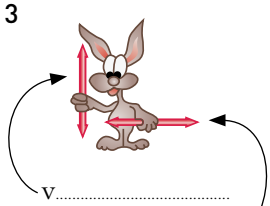
..... lines

**2**



..... lines

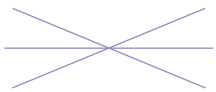
**3**



v.....

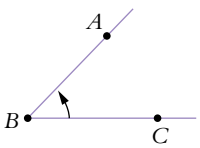
h.....

**4**



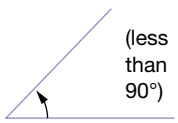
..... lines

**5**




angle .....

**6**

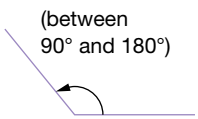



..... angle

**7**

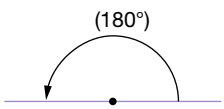
..... angle

**8**



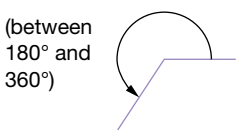

..... angle

**9**



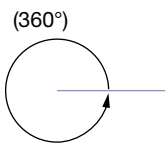

..... angle

**10**



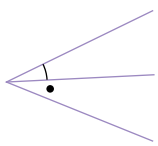

..... angle

**11**



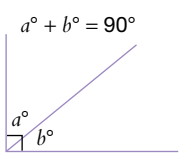

.....

**12**



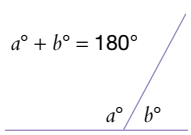
..... angles

**13**



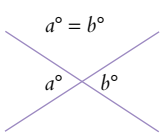

..... angles

**14**



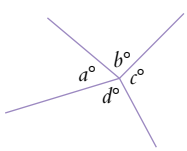

..... angles

**15**



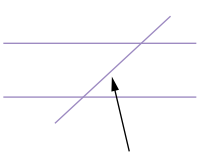

..... angles

**16**



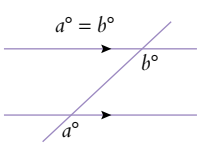

$a° + b° + c° + d° =$ ....

**17**



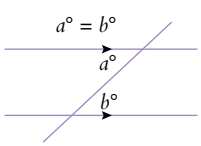
.....

**18**



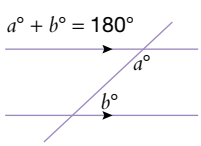

..... angles

**19**



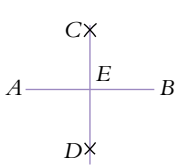

..... angles

**20**



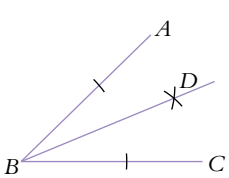

..... angles

**21**



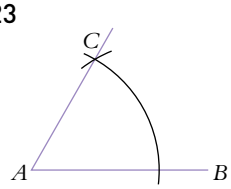

b..... an interval

**22**



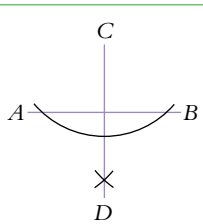

b..... an angle

**23**




$\angle CAB =$ .....

**24**




$CD$ is p..... to $AB$.

For answers see page 523.

## ID Card 6 (Language)

**1**

CE

C..... E.....

AD

A..... D.....

**2**

BCE

B..... C..... E.....

BC

B..... C.....

**3**

am

a..... m.....

means

b..... m.....

**4**

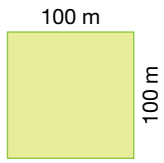
pm

p..... m.....

means

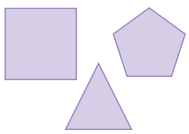
a..... m.....

**5**



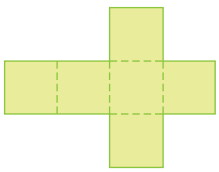

area is 1 .....

**6**



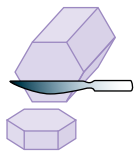
r..... shapes

**7**



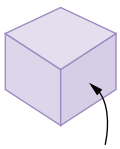
..... of a cube

**8**



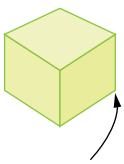
c.....-s.....

**9**



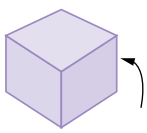
f.....

**10**



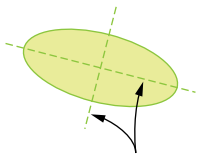
v.....

**11**



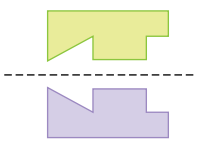
e.....

**12**



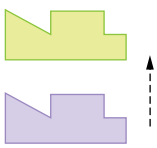
axes of .....

**13**



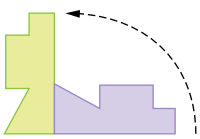
r.....

**14**



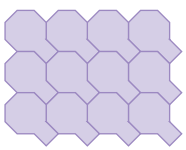
t.....

**15**



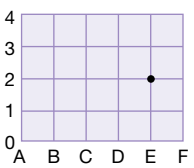
r.....

**16**



t.....

**17**




The c.....

of the dot are (E, 2).

**18**

Cars sold

| | |
|---|---|
| Mon | 𝍸 I |
| Tues | 𝍸 𝍸 𝍸 I |
| Wed | 𝍸 𝍸 II |
| Thurs | 𝍸 𝍸 𝍸 |
| Fri | 𝍸 𝍸 II |

t.....

**19**

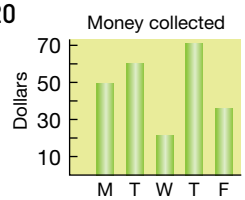
Money collected

| | |
|---|---|
| Mon | ●●●● |
| Tues | ●●●◔ |
| Wed | ●◖ |
| Thurs | ●●●● |
| Fri | ●●◢ |

● stands for $10

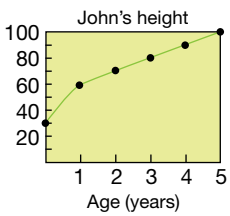
p..... graph

**20**



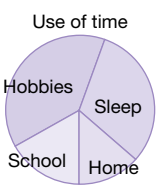

c..... graph

**21**



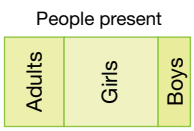

l..... graph

**22**



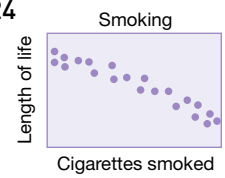

s..... graph

**23**




b..... graph

**24**




s..... d.....

For answers see page 523.

# Algebra card

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 2·1 | $\frac{1}{4}$ | $m$ | $\frac{2m}{3}$ | $-3m$ | $5m^2$ | $-5x$ | $\frac{x}{6}$ | $-3x$ | $-\frac{x}{2}$ | $x + 2$ | $x - 3$ | $2x + 1$ | $3x - 8$ |
| 2 | $-1$ | $-0{\cdot}4$ | $\frac{1}{8}$ | $-4m$ | $\frac{m}{4}$ | $2m$ | $-2m^3$ | $3x$ | $-\frac{x}{3}$ | $5x^2$ | $\frac{x}{4}$ | $x + 7$ | $x - 6$ | $4x + 2$ | $x - 1$ |
| 3 | 5 | 0·8 | $\frac{1}{3}$ | $10m$ | $-\frac{m}{4}$ | $-5m$ | $8m^5$ | $10x$ | $-\frac{2x}{7}$ | $-8x$ | $\frac{2x}{5}$ | $x + 5$ | $x + 5$ | $6x + 2$ | $x - 5$ |
| 4 | $-2$ | 1·5 | $\frac{1}{20}$ | $-8m$ | $-\frac{3m}{2}$ | $7m$ | $6m^2$ | $-15x$ | $\frac{x}{10}$ | $-4x^4$ | $-\frac{x}{5}$ | $x + 1$ | $x - 9$ | $3x + 3$ | $2x + 4$ |
| 5 | $-8$ | $-2{\cdot}5$ | $\frac{3}{5}$ | $2m$ | $-\frac{m}{5}$ | $10m$ | $m^2$ | $7x$ | $\frac{2x}{3}$ | $2x^3$ | $\frac{x}{3}$ | $x + 8$ | $x + 2$ | $3x + 8$ | $3x + 1$ |
| 6 | 10 | $-0{\cdot}7$ | $\frac{2}{7}$ | $-5m$ | $-\frac{3m}{7}$ | $-6m$ | $-9m^3$ | $9x$ | $-\frac{2x}{5}$ | $x^2$ | $\frac{3x}{5}$ | $x + 4$ | $x - 7$ | $3x + 1$ | $x + 7$ |
| 7 | $-6$ | $-1{\cdot}2$ | $\frac{3}{8}$ | $8m$ | $-\frac{m}{6}$ | $9m$ | $2m^6$ | $-6x$ | $\frac{5x}{6}$ | $5x^2$ | $\frac{2x}{3}$ | $x + 6$ | $x - 1$ | $x + 8$ | $2x - 5$ |
| 8 | 12 | 0·5 | $\frac{9}{20}$ | $20m$ | $\frac{2m}{5}$ | $-4m$ | $-3m^3$ | $-12x$ | $\frac{3x}{4}$ | $4x^3$ | $-\frac{x}{7}$ | $x + 10$ | $x - 8$ | $5x + 2$ | $x - 10$ |
| 9 | 7 | 0·1 | $\frac{3}{4}$ | $5m$ | $\frac{3m}{5}$ | $-10m$ | $m^7$ | $5x$ | $-\frac{3x}{7}$ | $-3x^5$ | $-\frac{3x}{7}$ | $x + 2$ | $x + 5$ | $2x + 4$ | $2x - 4$ |
| 10 | $-5$ | $-0{\cdot}6$ | $\frac{7}{10}$ | $-9m$ | $-\frac{4m}{5}$ | $-7m$ | $-8m^4$ | $-3x$ | $-\frac{x}{6}$ | $-7x^5$ | $\frac{2x}{9}$ | $x + 1$ | $x - 7$ | $5x + 4$ | $x + 7$ |
| 11 | $-11$ | $-1{\cdot}8$ | $\frac{1}{10}$ | $-7m$ | $\frac{m}{5}$ | $-8m$ | $-4m$ | $-4x$ | $\frac{x}{5}$ | $-x^3$ | $\frac{x}{3}$ | $x + 9$ | $x + 6$ | $2x + 7$ | $x - 6$ |
| 12 | 4 | $-1{\cdot}4$ | $\frac{2}{5}$ | $3m$ | $\frac{m}{3}$ | $12m$ | $7m^2$ | $-7x$ | $-\frac{3x}{4}$ | $x^{10}$ | $\frac{x}{6}$ | $x + 3$ | $x - 10$ | $2x + 3$ | $2x + 3$ |

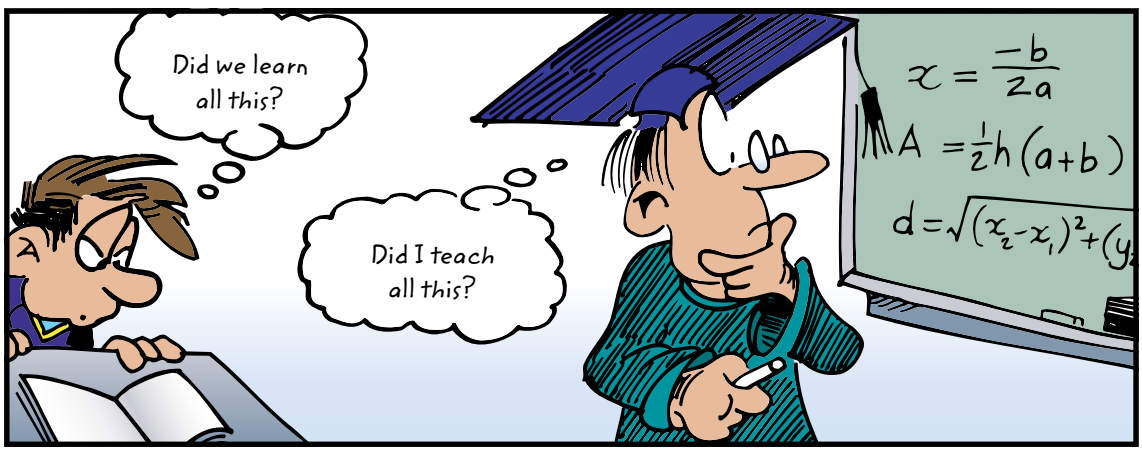
## How to use this card

As an example, if the instruction given for the Algebra card is 'column D + column F' then you would write answers for the following problems.

**1** $m + (-3m)$ **2** $(-4m) + 2m$ **3** $10m + (-5m)$ **4** $(-8m) + 7m$
**5** $2m + 10m$ **6** $(-5m) + (-6m)$ **7** $8m + 9m$ **8** $20m + (-4m)$
**9** $5m + (-10m)$ **10** $(-9m) + (-7m)$ **11** $(-7m) + (-8m)$ **12** $3m + 12m$

# REVIEW OF YEAR 9

# 1




## Contents


1:01 Number and measurement
- 1:01A Order of operations
- 1:01B Fractions
- 1:01C Decimals
- 1:01D Percentages
- 1:01E Ratios
- 1:01F Rates
- 1:01G Significant figures
- 1:01H Approximations
- 1:01I Estimations
- 1:01J Using units of measurement
- 1:01K Accuracy of measurements

1:02 Algebra

1:03 Probability

Investigation 1:03 Blaise Pascal

1:04 Measurement

1:05 Indices

1:06 Equations, inequalities and formulas

1:07 Financial mathematics

1:08 Coordinate geometry

1:09 Simultaneous equations

1:10 Similarity and scale drawing

1:11 Trigonometry

1:12 Statistics

1:13 Proportion

1:14 Graphs of physical phenomena

Assignment


## Syllabus references (See pages x–xv for details.)

**Number and Algebra, Measurement and Geometry, Statistics and Probability**

- The content covered in Year 9 is reviewed.

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

This chapter is a summary of the work covered in *Australian Signpost Mathematics 9 New South Wales Stages 5.1–5.3*. For an explanation of the work, refer to the relevant Appendix on the eBook.

# 1:01 Number and measurement

Rational numbers include integers, fractions, decimals and percentages (both positive and negative). All rational numbers can be written as terminating or recurring decimals. The following exercises revise the basic skills when working with rational numbers.

## 1:01A Order of operations

### Exercise 1:01A

P Appendix A A:01A

Simplify these expressions without using a calculator.

**1** **a** $4 - (5 - 3)$ **b** $6 - (9 - 4)$ **c** $-4 + (3 + 1)$
**d** $6 + 4 \times 2$ **e** $9 - 3 \times 4$ **f** $16 + 4 \div 4$
**g** $10 \times 4 - 4 \times 7$ **h** $30 \div 3 + 40 \div 2$ **i** $5 \times 8 + 6 \times 5$
**j** $5 \times 2^2$ **k** $3 \times 10^2$ **l** $3^2 + 4^2$
**m** $6 + 3 \times 4 + 1$ **n** $8 + 4 \div 2 + 1$ **o** $6 - (-6 - 6)$

**2** **a** $6 \times (5 - 4) + 3$ **b** $27 \div (3 + 6) - 3$ **c** $16 - [10 - (6 - 2)]$
**d** $\dfrac{30 + 10}{30 - 10}$ **e** $\dfrac{15 + 45}{45 + 5}$ **f** $\dfrac{14}{14 - 7}$
**g** $(6 + 3)^2$ **h** $(10 + 4)^2$ **i** $(19 - 9)^2$

## 1:01B Fractions

### Exercise 1:01B

P Appendix A A:01B1–A:01B9

**1** Change to mixed numerals.
**a** $\frac{7}{4}$ **b** $\frac{49}{6}$ **c** $\frac{15}{4}$ **d** $\frac{11}{8}$

**2** Change to improper fractions.
**a** $5\frac{1}{2}$ **b** $3\frac{1}{7}$ **c** $8\frac{3}{4}$ **d** $66\frac{2}{3}$

**3** Simplify the following fractions.
**a** $\frac{48}{80}$ **b** $\frac{70}{150}$ **c** $\frac{200}{300}$ **d** $\frac{250}{450}$

**4** Complete the following equivalent fractions.
**a** $\frac{3}{4} = \frac{\square}{24}$ **b** $\frac{2}{5} = \frac{\square}{50}$ **c** $\frac{2}{7} = \frac{\square}{28}$ **d** $\frac{1}{3} = \frac{\square}{120}$

Find the simplest possible answers without using a calculator.

**5** **a** $\frac{7}{15} + \frac{1}{15}$ **b** $\frac{13}{20} - \frac{2}{5}$ **c** $\frac{5}{8} + \frac{3}{10}$ **d** $\frac{6}{7} - \frac{3}{5}$

**6** **a** $6\frac{1}{2} + 2\frac{3}{5}$ **b** $4\frac{3}{4} - 2\frac{3}{10}$ **c** $4\frac{3}{4} + 6\frac{1}{10}$ **d** $5\frac{3}{8} - 1\frac{9}{10}$

**7** **a** $\frac{3}{5} \times \frac{4}{7}$ **b** $\frac{18}{25} \times \frac{15}{16}$ **c** $\frac{4}{9} \times \frac{3}{10}$ **d** $\frac{7}{10}$ of $\frac{2}{3}$

**8** **a** $6 \times \frac{3}{4}$ **b** $2\frac{1}{2} \times 1\frac{4}{5}$ **c** $1\frac{1}{3} \times 15$ **d** $10\frac{1}{2} \times 1\frac{3}{7}$

**9** **a** $\frac{9}{10} \surd \frac{2}{3}$ **b** $\frac{3}{8} \surd \frac{3}{5}$ **c** $\frac{4}{5} \surd 6$ **d** $2\frac{3}{4} \surd 1\frac{1}{2}$

# 1:01C Decimals

## Exercise 1:01C

P Appendix A A:01C1–A:01C11

**1** Place in order from smallest to largest.

**a** 0·606, 0·6, 0·66, 0·066 **b** 1·53, 0·153, 1·053

**c** 0·7, 0·017, 7, 0·77 **d** 3·5, 3·45, 3·05, 3·4

Complete the following questions without using a calculator.

**2** **a** 7·301 + 2 **b** 3·05 + 0·4 **c** 0·004 + 3·1 **d** 6 + 0·3 + 0·02

**e** 8·67 − 6·7 **f** 9·12 − 1·015 **g** 8 − 3·112 **h** 162·3 − 3

**3** **a** 0·012 × 3 **b** 0·03 × 0·2 **c** 0·45 × 1·3 **d** $(0{\cdot}05)^2$

**4** **a** 3·14 × 10 **b** 0·5 × 1000 **c** 0·0003 × 100 **d** $3{\cdot}8 \times 10^4$

**5** **a** 0·15 ÷ 5 **b** 1·06 ÷ 4 **c** 15·35 ÷ 5 **d** 0·01 ÷ 4

**6** **a** 1·3 ÷ 3 **b** 9·1 ÷ 11 **c** 14 ÷ 9 **d** 6 ÷ 7

**7** **a** 48·04 ÷ 10 **b** 1·6 ÷ 100 **c** 0·9 ÷ 1000 **d** $6{\cdot}5 \div 10^4$

**8** **a** 8·4 ÷ 0·4 **b** 0·836 ÷ 0·08 **c** 7·5 ÷ 0·005 **d** 1·4 ÷ 0·5

**9** Express as a simplified fraction or mixed numeral.

**a** 3·017 **b** 0·04 **c** 0·86 **d** 16·005

**10** Express as a decimal.

**a** $\frac{4}{5}$ **b** $\frac{7}{200}$ **c** $\frac{5}{8}$ **d** $\frac{8}{11}$

**11** Express these recurring decimals as fractions.

**a** 0·5555… **b** 0·257 257 2…

**c** $0{\cdot}\dot{7}\dot{2}$ **d** $0{\cdot}\dot{6}4\dot{2}$

**12** Express these recurring decimals as fractions.

**a** 0·8333… **b** 0·915 151 5… **c** $0{\cdot}43\dot{5}$ **d** $0{\cdot}8\dot{9}4\dot{2}$

# 1:01D Percentages

## Exercise 1:01D

Appendix A A:01D1–A:01D7

**1** Express as a fraction.

**a** 54% **b** 203% **c** $12\frac{1}{4}\%$ **d** 9·1%

**2** Express as a percentage.

**a** $\frac{11}{20}$ **b** $\frac{4}{9}$ **c** $1\frac{1}{4}$ **d** $\frac{2}{3}$

**3** Express as a decimal.

**a** 16% **b** 8·6% **c** 3% **d** $18\frac{1}{4}\%$

**4** Express as a percentage.

**a** 0·47 **b** 0·06 **c** 0·375 **d** 1·3

**5** **a** Find 36% of 400 m. **b** Find 9% of 84 g. **c** Find $8\frac{1}{2}\%$ of \$32.

**d** At a church, the offertories for 2005 amounted to \$127 000. If 68% of this money was used to pay the salary of the two full-time ministers, how much was paid to the ministers?

**6** **a** If 9% of Luke's money was spent on a \$5.40 fare, how much money did Luke have?

**b** 70% of Alana's weight is 17·5 kg. How much does Alana weigh?

**c** Lyn bought a book for a reduced price of 70 cents. This was 14% of the book's recommended retail price. What was the recommended retail price?

**d** A total of 54 minutes of mathematics lesson time was lost in one week because of other activities. If this represents 30% of the allocated weekly time for mathematics, what is this allocated time?

**7** **a** Express 85 cents as a percentage of \$2.

**b** 4 kg of sugar, 9 kg of flour and 7 kg of dried fruit were mixed. What is the percentage (by mass) of flour in the mixture?

**c** Of 32 birds in Rachel's aviary, 6 are canaries. What percentage of her birds are canaries?

**d** When Steve Waugh retired from test cricket in 2003, he had scored 32 centuries from 260 innings. In what percentage of his innings did he score centuries?

# 1:01E Ratios

## Exercise 1:01E

Appendix A A:01E1–A:01E3

**1** **a** Simplify each ratio.

**i** \$15 : \$25 **ii** 9 kg : 90 kg **iii** 75 m : 35 m **iv** $120\,m^2 : 40\,m^2$

**b** Find the ratio of 5·6 m to 40 cm in simplest terms.

**c** Naomi spent \$8 of \$20 she was given by her grandparents and saved the rest. What is the ratio of money spent to money saved?

**d** In a class, $\frac{3}{4}$ of the students walk to school and $\frac{1}{5}$ ride bicycles. Find the ratio of students who walk to students who ride bicycles.

**e** At the end of their test cricket careers, Steve Waugh had scored 50 fifties and 32 hundreds from 260 innings, whereas Mark Waugh had scored 47 fifties and 20 hundreds from 209 innings.

**i** Find the ratio of the number of hundreds scored by Steve to the number scored by Mark.

**ii** Find the ratio of the number of times Steve scored 50 or more to the number of innings.

**f** Express each ratio in the form $X:1$.

**i** $3:5$ **ii** $2:7$ **iii** $10:3$ **iv** $25:4$

**g** Express each ratio in **f** in the form $1:Y$.

**2** **a** If $x:15 = 10:3$, find the value of $x$.

**b** If the ratio of the populations of Africa and Europe is $3:2$, find the population of Africa if Europe's population is 732 million.

**c** The ratio of the average population density per $km^2$ of Asia to that of Australia is $60:1$. If the average in Asia is 152 people per $km^2$, what is the average in Australia?

**d** The ratio of the population of Sydney to the population of Melbourne is $8:7$. If 4·8 million people live in Sydney, how many people live in Melbourne?

**3** **a** If 84 jelly beans are divided between Naomi and Luke in the ratio $4:3$, how many jelly beans does each receive?

**b** The sizes of the angles of a triangle are in the ratio $2:3:4$. Find the size of each angle.

**c** A total of 22 million people live in the cities of Tokyo and Moscow. If the ratio of the populations of Tokyo and Moscow is $6:5$, what is the population of each city?

**d** In 1914, 60 baptisms were recorded in Cobargo. The ratio of males to females who were baptised was $3:2$. How many of each were baptised?

# 1:01F Rates

## Exercise 1:01F

P Appendix A A:01F

**1** Complete these equivalent rates.

**a** 5 km/min = … km/h **b** 8 km/L = … m/mL

**c** 600 kg/h = … t/day **d** 2·075 $cm^3/g$ = … $cm^3/kg$

**2** **a** Find the average speed (to the nearest mile per hour) if it took 564 hours to travel 3396 miles by horse and trap.

**b** Over a period of 30 working days, Aaron earned $1386. Find his average daily rate of pay.

**c** Shareen marked 90 books in 7 hours. What rate is this in minutes per book?

**d** On a hot day, our family used an average of 36 L of water per hour. Change this rate to $cm^3$ per second ($cm^3/s$).

# 1:01G Significant figures

## Exercise 1:01G

Appendix A A:01G

**1** State the number of significant figures in each of the following.

| | | | |
|---|---|---|---|
| **a** 21 | **b** 4·6 | **c** 2·52 | **d** 0·616 |
| **e** 16·32 | **f** 106 | **g** 3004 | **h** 2·03 |
| **i** 1·06 | **j** 50·04 | **k** 0·5 | **l** 0·003 |
| **m** 0·000 32 | **n** 0·06 | **o** 0·006 | **p** 3·0 |
| **q** 25·0 | **r** 2·60 | **s** 13·000 | **t** 6·40 |
| **u** 41 235 | **v** 600 (to nearest hundred) | | **w** 482 000 (to nearest thousand) |
| **x** 700 (to nearest ten) | | **y** 1600 | **z** 16 000 |

**2** State the number of significant figures in each of the following.

| | | | |
|---|---|---|---|
| **a** 3·0 | **b** 3·00 | **c** 0·3 | **d** 0·03 |
| **e** 0·030 | **f** 0·0030 | **g** 0·0300 | **h** 3·0300 |

# 1:01H Approximations

## Exercise 1:01H

Appendix A A:01H

**1** Round each of the following correct to one decimal place.

| | | | |
|---|---|---|---|
| **a** 4·63 | **b** 0·81 | **c** 3·17 | **d** 0·062 |
| **e** 15·176 | **f** 8·099 | **g** 0·99 | **h** 121·62 |
| **i** 0·119 | **j** 47·417 | **k** 0·35 | **l** 2·75 |

**2** Round each of the following correct to two decimal places.

| | | | |
|---|---|---|---|
| **a** 0·537 | **b** 2·613 | **c** 7·134 | **d** 1·169 |
| **e** 12·0163 | **f** 8·399 | **g** 412·678 | **h** 0·0756 |
| **i** 0·4367 | **j** 100·333 | **k** 0·015 | **l** 0·005 |

**3** Round each number correct to: **i** 1 sig. fig. **ii** 2 sig. fig.

| | | | |
|---|---|---|---|
| **a** 7·31 | **b** 84·9 | **c** 0·63 | **d** 2·58 |
| **e** 4·16 | **f** 0·0073 | **g** 0·0828 | **h** 3·05 |
| **i** 0·009 34 | **j** 0·0098 | **k** 7·52 | **l** 0·0359 |

**4** Round each of the following to the number of significant figures indicated.

| | | |
|---|---|---|
| **a** 2·3 (1 sig. fig.) | **b** 14·63 (3 sig. fig.) | **c** 2·15 (2 sig. fig.) |
| **d** 0·93 (1 sig. fig.) | **e** 4·07 (2 sig. fig.) | **f** 7·368 94 (3 sig. fig.) |
| **g** 0·724 138 (3 sig. fig.) | **h** 5·716 (1 sig. fig.) | **i** 31·685 (4 sig. fig.) |
| **j** 0·007 16 (1 sig. fig.) | **k** 0·78 (1 sig. fig.) | **l** 0·007 16 (2 sig. fig.) |

**5** Round each of the following to the number of decimal places indicated.

| | | |
|---|---|---|
| **a** 5·61 (1 dec. pl.) | **b** 0·16 (1 dec. pl.) | **c** 0·437 (2 dec. pl.) |
| **d** 15·37 (1 dec. pl.) | **e** 8·333 (2 dec. pl.) | **f** 413·789 (1 dec. pl.) |
| **g** 71·98 (1 dec. pl.) | **h** 3·0672 (3 dec. pl.) | **i** 9·99 (1 dec. pl.) |
| **j** 4·7998 (3 dec. pl.) | **k** 0·075 (2 dec. pl.) | **l** 0·0035 (3 dec. pl.) |

# 1:01I Estimations

## Exercise 1:01I

Appendix A A:01I

**1** Give an estimate for each of the following.

**a** $12{\cdot}7 \times 5{\cdot}8$ **b** $0{\cdot}55 \times 210$ **c** $17{\cdot}8 \times 5{\cdot}1 \times 0{\cdot}336$

**d** $15{\cdot}6 \div 2{\cdot}165$ **e** $(4{\cdot}62 + 21{\cdot}7) \times 4{\cdot}21$ **f** $7{\cdot}8 \times 5{\cdot}2 + 21{\cdot}7 \times 0{\cdot}89$

**g** $(0{\cdot}93 + 1{\cdot}72)(8{\cdot}5 - 1{\cdot}7)$ **h** $\dfrac{43{\cdot}7 + 18{\cdot}2}{7{\cdot}8 + 2{\cdot}9}$ **i** $\dfrac{101{\cdot}6 - 51{\cdot}7}{21{\cdot}3 - 14{\cdot}8}$

**j** $\dfrac{0{\cdot}68 \times 51}{0{\cdot}25 \times 78}$ **k** $\dfrac{11{\cdot}6 - 3{\cdot}92}{12{\cdot}7 + 6{\cdot}58}$ **l** $3{\cdot}52^2 \times \sqrt{17{\cdot}9}$

**m** $\sqrt{41{\cdot}7 \times 5{\cdot}6}$ **n** $\sqrt{4{\cdot}26} \times \sqrt{105{\cdot}6}$ **o** $3{\cdot}1^3 \times 1{\cdot}8^4$

**p** $\dfrac{4{\cdot}1 \times \sqrt{48{\cdot}12}}{26{\cdot}23}$ **q** $\dfrac{15{\cdot}7^2}{11{\cdot}3 \times 3{\cdot}1}$ **r** $\dfrac{16{\cdot}7}{2{\cdot}15} + \dfrac{41{\cdot}6}{4{\cdot}7}$

**s** $\dfrac{0{\cdot}65}{0{\cdot}01} - \dfrac{0{\cdot}75 \times 3{\cdot}6}{0{\cdot}478}$

# 1:01J Using units of measurement

## Exercise 1:01J

Appendix A A:01J

**1** Convert the following.

**a** 12 kL to L **b** 5 t to kg **c** 3·85 g to mg

**d** 8 Mm to m **e** 7 m to μm **f** 2 m to nm

**g** 6 GB to B **h** 4 TB to GB **i** 3 TB to GB

**j** 25 mm to μm **k** 5·4 MB to kB **l** 500 000 ns to μs

**2** Our computer has a capacity of 1·2 terabytes. We have 800 gigabytes of available space left. How much space have we used?

# 1:01K Accuracy of measurements

## Exercise 1:01K

Appendix A A:01K

**1** Find the absolute error (greatest possible error) for each of the following measurements.

**a** 8 g **b** 17·4 m **c** 27·625 L

**d** 32 cm **e** 1·8 t **f** 2·76 s

**g** 8000 L (measured to the nearest 1000 L)

**2** **a** Alana's bathroom scales are marked in tenths of a kilogram. She measured her mass as 63·7 kg. Find the absolute error of the measurement and her smallest and largest possible mass.

**b** A metal prism is measured as 11·3 cm thick. In what range must the actual thickness of the prism lie?

# 1:02 Algebra

## Exercise 1:02

P Appendix A A:02A–A:02I

**1** Write an expression for:

- **a** the sum of $3a$ and $4b$
- **b** the product of $3a$ and $4b$
- **c** the difference between $k$ and $m$, if $k > m$
- **d** the difference between $k$ and $m$, if $k < m$
- **e** the average of $x$, $y$ and $z$
- **f** twice the sum of $m$ and 5
- **g** the square of the difference between $a$ and $b$
- **h** the square root of the sum of $5m$ and $4n$
- **i** the next even number after $n$, if $n$ is even
- **j** the sum of three consecutive integers, if the first is $m$.

**2** If $a = 3$, $b = 5$ and $c = -6$, find the value of:

| | | |
|---|---|---|
| **a** $2a + 3b$ | **b** $a + b + c$ | **c** $2b - c$ |
| **d** $ab + bc$ | **e** $ac - b^2$ | **f** $a^2 + c^2$ |
| **g** $\frac{4a}{c}$ | **h** $\frac{3ab}{3-c}$ | **i** $\frac{3b-c}{2a}$ |
| **j** $\sqrt{ab+c}$ | **k** $\sqrt{\frac{a+b+c}{2}}$ | **l** $\sqrt{\frac{-3c}{b-a}}$ |

**3** Simplify these expressions.

| | | |
|---|---|---|
| **a** $5a + 3b - a + b$ | **b** $5ab - 2ba$ | **c** $3x^2 + x - x^2 + x$ |
| **d** $5x \times 3y$ | **e** $6ab \times 3a$ | **f** $-2m \times 5mn$ |
| **g** $15a \div 5$ | **h** $24m \div 12m$ | **i** $10a^2b \div 5ab$ |
| **j** $n \div 3n$ | **k** $15m \div 10n$ | **l** $12xy^2 \div 8x^2y$ |
| **m** $6a \times 7 \div 2a$ | **n** $20y - 2 \times 5y$ | **o** $7x + 2 \times 4x - 10x$ |

**4** Simplify these expressions involving fractions.

| | | |
|---|---|---|
| **a** $\frac{2a}{5} + \frac{4a}{5}$ | **b** $\frac{6x}{7} - \frac{4x}{7}$ | **c** $\frac{3}{y} + \frac{4}{y}$ |
| **d** $\frac{a}{3} + \frac{a}{4}$ | **e** $\frac{2m}{3} - \frac{m}{5}$ | **f** $\frac{3}{2n} - \frac{4}{3n}$ |
| **g** $\frac{a}{3} \times \frac{b}{4}$ | **h** $\frac{m}{5} \times \frac{2m}{3}$ | **i** $\frac{9a^2}{x} \times \frac{xy}{3a}$ |
| **j** $\frac{5}{m} \div \frac{2}{m}$ | **k** $\frac{ab}{5} \div \frac{a}{10}$ | **l** $\frac{xy}{z} \div \frac{y}{z^2}$ |

**5** Expand and simplify these expressions involving products.

| | | |
|---|---|---|
| **a** $3(2a + 1) - 5a$ | **b** $10m - 2(m + 5)$ | **c** $6a - (a - 5) + 10$ |
| **d** $3(2n - 1) + 2(n + 5)$ | **e** $4(2a - 1) - 3(a + 5)$ | **f** $6(1 - 2x) - (3 - 10x)$ |
| **g** $(x + 3)(x + 7)$ | **h** $(y - 4)(y - 1)$ | **i** $(k - 7)(k + 9)$ |

**j** $(2p + 3)(p - 5)$ **k** $(6x + 1)(3x - 2)$ **l** $(3m - 1)(2m - 5)$

**m** $(m - 7)(m + 7)$ **n** $(3a - 4)(3a + 4)$ **o** $(10 - 3q)(10 + 3q)$

**p** $(a + 8)^2$ **q** $(2m - 1)^2$ **r** $(4a + 5)^2$

**s** $(x + y)(x - 2y)$ **t** $(a + 2b)(a - 2b)$ **u** $(m - 3n)^2$

**6** Factorise:

**a** $15a - 10$ **b** $3m^2 - 6m$ **c** $4n + 6mn$

**d** $6mn - 4m$ **e** $10y^2 + 5y$ **f** $6a^2 - 2a + 4ab$

**g** $x^2 - 49$ **h** $100 - a^2$ **i** $16a^2 - 9b^2$

**j** $x^2 + 8x + 12$ **k** $x^2 - x - 12$ **l** $x^2 - 6x + 8$

**m** $a^2 + 6a + 9$ **n** $y^2 - 10y + 25$ **o** $1 - 4m + 4m^2$

**p** $2x^2 + 7x + 3$ **q** $3m^2 + 7m - 6$ **r** $6a^2 - 11a + 4$

**s** $4n^2 + 12n + 9$ **t** $25x^2 - 10x + 1$ **u** $9 - 24m + 16m^2$

**v** $ab - 4a + xb - 4x$ **w** $x^2 + ax - 2x - 2a$ **x** $2m^2 + 6mn - m - 3n$

**7** Factorise these expressions completely.

**a** $2x^2 - 18$ **b** $4x^2 + 4x - 24$ **c** $3a^2 - 6a - 3ab + 6b$

**d** $8n^2 - 8n + 2$ **e** $9 - 9q^2$ **f** $m^4 - m^2$

**g** $k^4 - 16$ **h** $y^3 + y^2 + y + 1$ **i** $x^3 - x^2 - x + 1$

**8** Factorise and simplify:

**a** $\dfrac{3a+12}{3}$ **b** $\dfrac{5x-15}{x-3}$ **c** $\dfrac{a+5}{a^2+7a+10}$

**d** $\dfrac{m^2-m}{m^2-1}$ **e** $\dfrac{n^2-n-6}{n^2+5n+6}$ **f** $\dfrac{2x^2-x-3}{4x^2-9}$

**9** Simplify:

**a** $\dfrac{3x+15}{2} \times \dfrac{4x}{x+5}$ **b** $\dfrac{a^2-9}{a-3} \times \dfrac{a+1}{a+3}$

**c** $\dfrac{3x+6}{10x} \div \dfrac{x+2}{5x}$ **d** $\dfrac{m^2-25}{m^2-5m} \div \dfrac{m+5}{5m}$

**e** $\dfrac{a^2+7a+12}{a^2+5a+4} \times \dfrac{a^2+6a+5}{a^2+12a+35}$ **f** $\dfrac{n^2-3n-4}{3n^2-48} \div \dfrac{n^3-n}{n^2+4n}$

**10** Simplify each of the following.

**a** $\dfrac{1}{a+4} + \dfrac{1}{a+3}$ **b** $\dfrac{3}{2x-1} - \dfrac{5}{4x+3}$

**c** $\dfrac{3}{(x+1)(x+2)} + \dfrac{2}{x(x+2)}$ **d** $\dfrac{5}{(x+3)(x+5)} - \dfrac{3}{(x+3)(x+4)}$

**11** Factorise each denominator where possible and then simplify.

**a** $\dfrac{1}{a^2-1} + \dfrac{1}{a+1}$ **b** $\dfrac{2}{3x+6} - \dfrac{1}{x^2-4}$

**c** $\dfrac{2}{x^2+x-6} + \dfrac{3}{x^2+4x+3}$ **d** $\dfrac{6}{x^2-x-2} - \dfrac{3}{x^2-2x-3}$

**e** $\dfrac{x+1}{x^2-9} + \dfrac{x-1}{x^2-5x+6}$ **f** $\dfrac{n+5}{2n^2+n-1} - \dfrac{n-3}{2n^2+5n-3}$

# 1:03 Probability

## Exercise 1:03

P Appendix A A:03A–A:03E

**1** Using the figures shown in the table, find the probability of selecting at random a matchbox containing:

**a** 50 matches
**b** 48 matches
**c** more than 50 matches
**d** at least 50 matches.

| Number of matches | 48 | 49 | 50 | 51 | 52 |
|---|---|---|---|---|---|
| Number of boxes | 3 | 6 | 10 | 7 | 4 |

**2** A single dice is rolled. What is the probability of rolling:

**a** a five
**b** less than 3
**c** an even number
**d** less than 7?

**3** A bag contains 3 red, 4 white and 5 blue marbles. If one is selected from the bag at random, find the probability that it is:

**a** white
**b** red or white
**c** not red
**d** pink

**4** A standard pack of cards has four suits, hearts and diamonds (both red), and spades and clubs (both black). In each suit there are 13 cards: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The Jack, Queen and King are called picture (court) cards.

A card is drawn from a standard pack. What is the probability that the card is:

**a** red
**b** not red
**c** a 6
**d** not a 6
**e** a picture card
**f** a red Ace
**g** a spade
**h** a red 13
**i** either a red 5 or a 10
**j** either a heart or a black Ace
**k** either a blue 5 or a 7
**l** either a heart or a black card?

(In each of these cases, the events may not be mutually exclusive.)

**m** either a picture card or a diamond
**n** either a number larger than 2 or a club
**o** either a heart or a 5
**p** either a Queen or a black picture card
**q** either a number between 2 and 8 or an even-numbered heart?

There are 4 suits, with 13 cards in each suit, so the number of cards in a standard pack is 52. (In some games a Joker is also used.)

**5** The Venn diagram shows two sports played by 90 Year 10 students. What is the probability that a randomly chosen student plays:

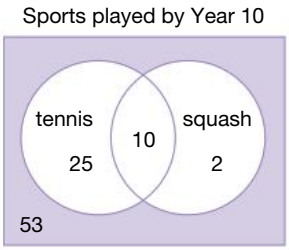
**a** both tennis and squash

**b** neither tennis nor squash

**c** tennis

**d** tennis or squash but not both?

(Give answers as a fraction and as a percentage correct to the nearest per cent.)




**6** The table shows how many females and males among 85 Year 10 students are right-handed or left-handed. What is the probability that a randomly chosen student is:

**a** both female and left-handed

**b** right-handed

**c** male

**d** a right-handed male?

(Give answers as a fraction and as a percentage correct to the nearest per cent.)

| Year 10 | left-handed | right-handed | |
|---|---|---|---|
| female | 4 | 41 | 45 |
| male | 7 | 33 | 40 |
| | 11 | 74 | 85 |

**7** **a** There are one red, two green and three yellow counters in a container. Show all possible outcomes, as an array (a table), when two counters are taken from the container, one after the other, without replacing the first counter. (Call the green counters G1 and G2, and the yellow counters Y1, Y2 and Y3.)

**b** Three counters, one red and two yellow, are in a container. Show all possible outcomes, both as an array (a table) and as a tree diagram, when two counters are taken from the container, one after the other, returning the first choice to the container before making the second choice (i.e. with replacement).

## INVESTIGATION 1:03 BLAISE PASCAL

Blaise Pascal (1623–1662) was indeed a genius. Mathematician, physicist, philosopher and theologian, he achieved fame in all these areas in his relatively short lifetime.

After discovering geometry at the age of 12, his ability became legendary. He invented the first calculating machine when he was 18, but is remembered most for establishing the foundation of probability theory (along with Fermat). He often related his philosophy to mathematics, as with the famous example called 'Pascal's wager'. Use the internet to discover more about Blaise Pascal.

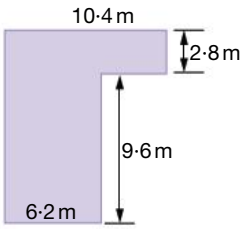
# 1:04 Measurement

## Exercise 1:04

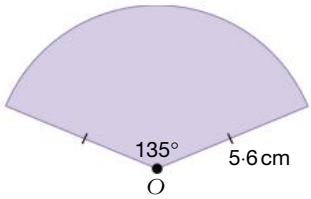
P Appendix A A:04A–A:04E

**1** Find the perimeters of the following figures correct to one decimal place.

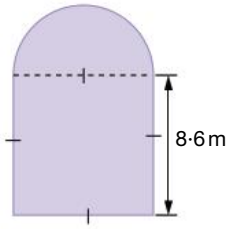
**a**




**b**



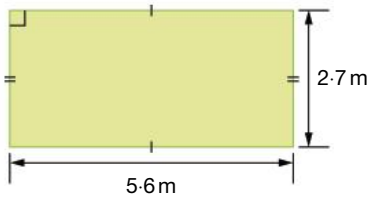

**c**



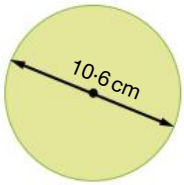

(Use $\pi = 3{\cdot}142$)

**2** Find the area of each plane shape correct to two decimal places.

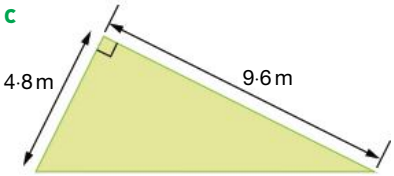
**a**



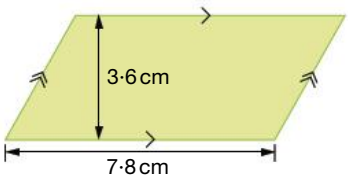

**b**



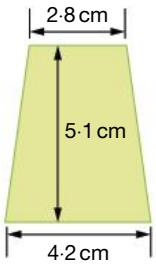

**c**



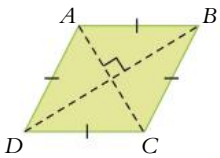

**d**




**e**



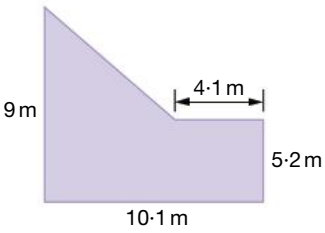

**f**



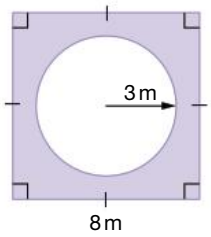

$AC = 3{\cdot}6$ cm
$BD = 6{\cdot}4$ cm

**3** Find the areas of the following shaded figures correct to three significant figures.

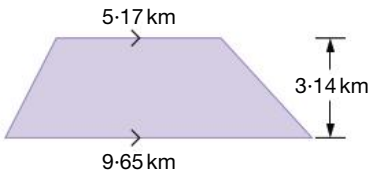
**a**




**b**

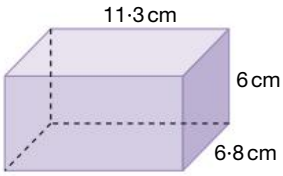



**c**



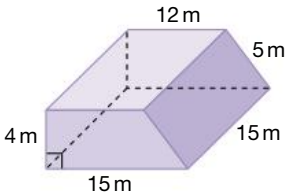

**4** Find the surface areas of the following solids.

**a**



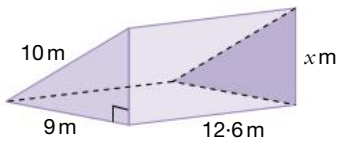

Rectangular prism

**b**




Trapezoidal prism

**c**



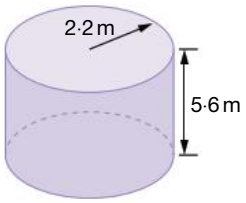

Triangular prism

(*Note:* Use Pythagoras' theorem to find $x$.)

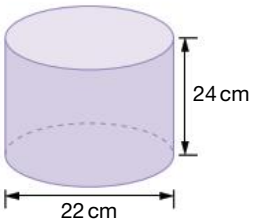
**5** For each of the following cylinders, find, correct to two decimal places:

- **i** the curved surface area
- **ii** the area of the circular ends
- **iii** the total surface area.

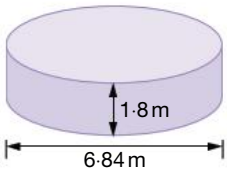
**a**




**b**




**c**




**6** Find the volume of each prism in Question **4**.

**7** Find the volume of each cylinder in Question **5**.

# 1:05 Indices

## Exercise 1:05

P Appendix A A:05A–A:05F

**1** Write in index form.

**a** $a \times a \times a$ **b** $2 \times 2 \times 2 \times 2$ **c** $n \times n \times n \times n \times n$ **d** $10 \times 10 \times 10$

**2** Simplify, giving your answers in index form.

**a** $2^4 \times 2^5$ **b** $a^3 \times a^2$ **c** $m \times m^4$ **d** $10^6 \times 10^2$

**e** $a^{10} \div a^2$ **f** $y^4 \div y^3$ **g** $b^3 \div b$ **h** $10^5 \div 10^2$

**i** $(m^3)^4$ **j** $(a^2)^3$ **k** $(x^4)^2$ **l** $(10^5)^2$

**m** $a^0 \times 3$ **n** $b^0 + c^0$ **o** $6y^0$ **p** $e^6 \div e^6$

**q** $6a^2 \times 5$ **r** $6m^3 \div 3$ **s** $6a \times 5a$ **t** $(4x^4)^2$

**3** Simplify.

**a** $6a^4 \times 5ab^3$ **b** $7a^2b^2 \times 8a^3b$ **c** $4a^2b^3 \times 6a^2b^4$ **d** $10a^7 \times a^3b^3$

**e** $(7x^3)^2$ **f** $(2m^2)^4$ **g** $(x^2y^3)^3$ **h** $(5xy^2)^4$

**i** $30a^5 \div 5a^3$ **j** $100x^4 \div 20x$ **k** $36a^3b^4 \div 12a^2b^4$ **l** $8y^7z^2 \div y^7z^2$

**4** Rewrite without a negative index.

**a** $4^{-1}$ **b** $10^{-1}$ **c** $x^{-1}$ **d** $2a^{-1}$

**e** $5^{-2}$ **f** $2^{-3}$ **g** $m^{-3}$ **h** $5x^{-2}$

**5** Rewrite each of the following with a negative index.

**a** $\frac{1}{3}$ **b** $\frac{1}{8}$ **c** $\frac{1}{a}$ **d** $\frac{3}{x}$

**e** $\frac{1}{2^4}$ **f** $\frac{1}{10^6}$ **g** $\frac{1}{y^4}$ **h** $\frac{5}{n^3}$

**6** Find the value of each of the following.

**a** $9^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}$ **c** $8^{\frac{1}{3}}$ **d** $27^{\frac{1}{3}}$

**7** Rewrite using fractional indices.

**a** $\sqrt{a}$ **b** $\sqrt[3]{y}$ **c** $5\sqrt{m}$ **d** $\sqrt{16x}$

**8** Simplify these expressions.

**a** $x^4 \times x^{-2}$ **b** $5m^3 \div m^{-2}$ **c** $4n^{-2} \times 3n^3$

**d** $6y^4 \times 3y^{\frac{1}{2}}$ **e** $12x^{\frac{3}{2}} \sqrt{6x^{\frac{1}{2}}}$ **f** $(27x^6)^{\frac{1}{3}}$

**g** $\dfrac{5a^4 \times 4a^5}{10a^8}$ **h** $\dfrac{6m^4 \times (2m)^3}{3m^2 \times 8m^5}$ **i** $\left(\dfrac{9x^3 \times (2x^3)^3}{6x^6 \div 3x^{-2}}\right)^{\frac{1}{2}}$

**9** Write these numbers in scientific (or standard) notation.

**a** 148 000 000 **b** 68 000 **c** 0·000 15 **d** 0·000 001 65

**10** Write these as basic numerals.

**a** $6{\cdot}2 \times 10^4$ **b** $1{\cdot}15 \times 10^6$ **c** $7{\cdot}4 \times 10^{-3}$ **d** $6{\cdot}91 \times 10^{-5}$

**11** Write each as bytes (B), using scientific notation.

**a** 480 000 bytes (B) **b** 1·4 kilobytes (kB) **c** 3·4 terabytes (TB)

**d** 8 GB **e** 8·50 MB **f** 4·15 GB

**12** Write each as seconds (s), using scientific notation.

**a** 5 ms **b** 4·2 µs **c** 7·5 ns **d** 125 µs **e** 65 ns **f** 80 ms

**13** Use scientific notation to write:

**a** 10·5 megalitres as litres **b** 102 gigalitres as litres

**c** 850 milligrams as grams **d** 67 nanometres as metres.

# 1:06 Equations, inequalities and formulas

## Exercise 1:06

P Appendix A A:06A–A:06G

**1** Solve the following.

**a** $a + 7 = 25$ **b** $m - 6 = -1$ **c** $5x = 75$

**d** $10 - y = 12$ **e** $3p = 7$ **f** $\dfrac{n}{4} = 3$

**g** $2x + 3 = 7$ **h** $8m + 5 = 21$ **i** $5y + 2 = 3$

**j** $9k - 1 = 5$ **k** $5 + 3x = 11$ **l** $15 - 2q = 8$

**2** Solve the following.

**a** $5m + 2 = 4m + 7$ **b** $3x - 7 = 2x - 3$ **c** $5x + 2 = 6x - 5$

**d** $2a + 3 = 3a - 5$ **e** $3m - 2 = 5m - 10$ **f** $q + 7 = 8q + 14$

**g** $10 - 2x = x + 4$ **h** $3z + 7 = z + 10$ **i** $13 - 2m = 9 - 5m$

**3** Solve these equations involving grouping symbols.

**a** $5(a + 1) = 15$ **b** $4(x - 3) = 16$ **c** $3(2x + 5) = 33$

**d** $3(5 - 2a) = 27$ **e** $4(3 - 2x) = 36$ **f** $3(2m - 5) = 11$

**g** $3(a + 2) + 2(a + 5) = 36$ **h** $2(p + 3) + p + 1 = 31$ **i** $4(2b + 7) = 2(3b - 4)$

**j** $4(2y + 3) + 3(y - 1) = 2y$ **k** $3(m - 4) - (m + 2) = 0$ **l** $2m - 3(1 - m) = 22$

**m** $5(y - 3) - 3(1 - 2y) = 4$ **n** $4(2x - 1) - 2(x + 3) = 5$

**4** Solve these equations.

**a** $\frac{5x}{2} = 10$ **b** $\frac{2a}{3} = 6$ **c** $\frac{3m}{5} = 4$

**d** $\frac{n+1}{5} = 2$ **e** $\frac{x-4}{2} = 1$ **f** $\frac{2p+5}{3} = 1$

**5** Solve these equations involving fractions.

**a** $\frac{a}{3} + \frac{a}{3} = 4$ **b** $\frac{2x}{5} - \frac{x}{5} = 3$ **c** $\frac{5p}{3} - \frac{p}{3} = 8$

**d** $\frac{q}{2} - \frac{q}{3} = 6$ **e** $\frac{2k}{3} - \frac{k}{4} = 10$ **f** $\frac{3x}{4} - \frac{x}{2} = 15$

**g** $\frac{m+6}{3} = \frac{2m+4}{4}$ **h** $\frac{n-3}{2} = \frac{n-7}{4}$ **i** $\frac{5x-1}{3} = \frac{3-x}{2}$

**j** $\frac{x+3}{2} + \frac{x+5}{5} = 8$ **k** $\frac{m+2}{5} - \frac{m+3}{6} = 1$ **l** $\frac{3a+4}{2} - \frac{a-1}{3} = \frac{2a+3}{4}$

**6** **a** Translate each sentence into an equation, using $n$ as the unknown number.

**i** A certain number is multiplied by 8, then 11 is added and the result is 39.

**ii** I think of a number, double it, add 7 and the result is 5.

**iii** I think of a number, add 4 and then divide the result by 10. The answer is 7.

**b** Solve each of the following problems by first forming an equation.

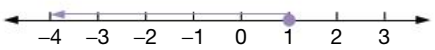
**i** If 5 is added to 3 times a certain number, the result is 38. What is the number?

**ii** If one quarter of a certain number is added to half the same number, the result is 6. What is the number?

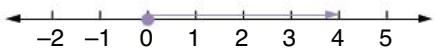
**iii** A rectangle is four times as long as it is wide. If it has a perimeter of 340 m, what are its dimensions?

**7** Write the inequality for $x$ shown on each number line below.

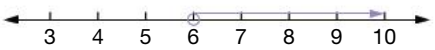
**a**



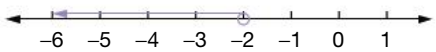

**b**




**c**




**d**




**8** Solve these inequalities and show each solution on a number line.

**a** $x + 7 > 11$ **b** $a - 5 < 3$ **c** $10 - y \geq 8$

**d** $3m \leq 21$ **e** $15 < 4x$ **f** $\frac{m}{4} \neq 1$

**g** $2x + 1 > 5$ **h** $7 - 3n > 4$ **i** $5x + 6 > x + 18$

**j** $3x - 5 < x + 6$ **k** $3 - a < 5 - 2a$ **l** $3(m + 4) < 2(m + 6)$

**m** $\frac{x}{2} + 1 < 6$ **n** $\frac{3x}{4} - 5 \geq 1$ **o** $5 - \frac{2y}{3} \leq 6$

**p** $\frac{p-1}{4} < 2$ **q** $\frac{2p+3}{2} > 7$ **r** $\frac{4+x}{5} \leq 1$

**s** $\frac{x}{2} + \frac{x}{3} > 5$ **t** $\frac{a}{4} + \frac{a}{2} \leq 6$ **u** $\frac{x}{2} - \frac{2x}{3} \leq 3$

**9** **a** If $s = ut + \frac{1}{2}at^2$, find $s$ if $u = 9$, $t = 4$ and $a = 7$.

**b** Given $F = p + qr$, find $F$ if $p = 2{\cdot}3$, $q = 3{\cdot}9$ and $r = 0{\cdot}9$.

**c** For the formula $T = a + (n - 1)d$, find $T$ if $a = 9{\cdot}2$, $n = 6$ and $d = 1{\cdot}3$.

**10** **a** Given that $V = lbh$, evaluate $b$ when $V = 4{\cdot}32$, $l = 1{\cdot}2$ and $h = 0{\cdot}9$.

**b** It is known that $S = \frac{a}{1-r}$. Find $a$ when $S = 25$ and $r = 0{\cdot}6$.

**c** $F = 32 + \frac{9C}{5}$. Find $C$ if $F = 77$.

**d** If $v^2 = u^2 + 2as$, find $a$ if $v = 2{\cdot}1$, $u = 1{\cdot}6$ and $s = 0{\cdot}3$.

**e** Given that $T = a + (n - 1)d$, find $d$ if $T = 24{\cdot}6$, $a = 8{\cdot}8$ and $n = 4$.

**11** Change the subject of each formula to $y$.

**a** $\frac{x}{a} + \frac{y}{b} = 1$ **b** $ay^2 = x$

**c** $T = \sqrt{\frac{B}{y}}$ **d** $ay = by - 1$

# 1:07 Financial mathematics

## Exercise 1:07

P Appendix A A:07A–A:07F

**1** **a** Michelle is paid $18.40 per hour and time-and-a-half for overtime. If a normal day's work is 7 hours, how much would she be paid for 10 hours work in one day?

**b** Jake receives a holiday loading of $17\frac{1}{2}\%$ on 4 weeks normal pay. If he works 37 hours in a normal week and is paid $9.20 per hour, how much money does he receive as his holiday loading?

**c** In a week, a saleswoman sells $6000 worth of equipment. If she is paid $150 plus 10% commission on sales in excess of $4000, how much does she earn?

**d** A waiter works from 5:00 pm until 1:30 am on four days in one week. His hourly rate of pay is $14.65 plus an average of $9.20 in tips per working night. Find his income for the week.

**2** **a** Find the net pay for the week if Saransh earns $1115.40, is taxed $226.20, pays $111.50 in superannuation and his miscellaneous deductions total $76.34. What percentage of his gross pay did he pay in tax? (Answer correct to one decimal place.)

**b** Find the tax payable on a taxable income of $40 180 if the tax is $4650 plus 30 cents for each $1 in excess of $37 000.

**c** James received a salary of $68 300 and an income of $496 from investments. His total tax deductions were $3050. What is his taxable income?

**d** Tom's taxable income for the year was $13 860. Find the tax that must be paid if the rate is 15 cents for each $1 in excess of $6000.

**3** **a** An item has a marked price of $87.60 in two shops. One offers a 15% discount, and the other a discount of $10.65. Which is the better buy, and by how much?

**b** Emma bought a new tyre for $100, Jade bought one for $85 and Diane bought a retread for $58. If Emma's tyre lasted 32 000 km, Jade's 27 500 km, and Diane's 16 000 km, which was the best buy? (Assume that safety and performance for the tyres are the same.)

**c** Alice wants to get the best value when buying tea. Which will she buy if Pet tea costs $1.23 for 250 g, Jet tea costs $5.50 for 1 kg and Yet tea costs $3.82 for 800 g?

**4** **a** Find the GST (10%) that needs to be added to a base price of:

**i** $75 **ii** $6.80 **iii** $18.75

**b** For each of the prices in **a**, what would the retail price be? (Retail price includes the GST.)

**c** Find the GST contained in the retail price of:

**i** $220 **ii** $8.25 **iii** $19.80

**5** **a** What is meant by the expression 'buying on terms'?

**b** Find the amount Jason will pay for a fishing line worth $87 if he pays $7 deposit and $5.69 per month for 24 months. How much extra does he pay in interest charges?

**c** Nicholas was given a discount of 10% on the marked price of a kitchen table. If the discount was $22, how much was the marked price?

**d** A factory's machinery depreciates at a rate of 15% per annum. If it is worth $642 000, what will be its value after one year?

**e** The price of a book was discounted by 20%. A regular customer was given a further discount of 15%. If the original price was $45, what was the final price of the book?

**6** **a** The cost price of a DVD player was $180 and it was sold for $240. What was:

**i** the profit as a percentage of the cost price

**ii** the profit as a percentage of the selling price?

**b** A new car worth $32 000 was sold after two years for $24 000. What was:

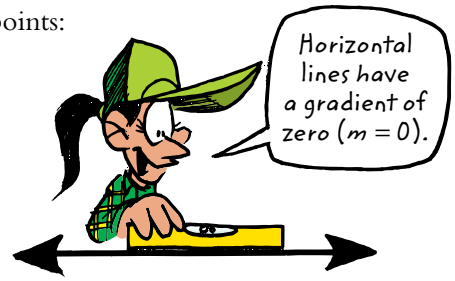
**i** the loss as a percentage of the original cost price

**ii** the loss as a percentage of the final selling price?

# 1:08 Coordinate geometry

## Exercise 1:08

P Appendix A A:08A–A:08H




**1** Find the gradient of the line that passes through the points:
- **a** (1, 2) and (2, 3)
- **b** (1, 7) and (0, 0)
- **c** (−3, −2) and (5, −2)

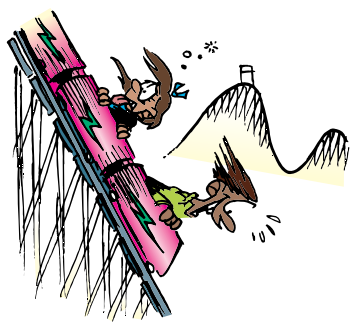
**2** Find the midpoint of the interval joining:
- **a** (2, 6) and (8, 10)
- **b** (−3, 5) and (4, −2)
- **c** (0, 0) and (7, 0)

**3**
- **a** Find the distance between (1, 4) and (5, 2).
- **b** $A$ is the point (−5, 2). $B$ is the point (−2, −6). Find the distance $AB$.
- **c** Find the distance between the origin and (6, 8).
- **d** Find the distance $AB$ between $A(-2, 1)$ and $B(5, 3)$.

**4** Sketch each of these lines on a number plane.
- **a** $y = 3x + 4$
- **b** $2x + 3y = 12$
- **c** $y = -3x$
- **d** $x = 2$

**5** Find the gradient and $y$-intercept of the lines:
- **a** $y = 3x + 5$
- **b** $y = -x - 2$
- **c** $y = -2x + 5$

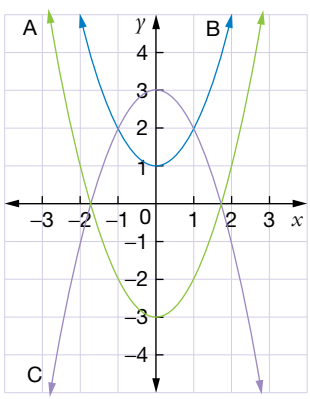
**6** Write each equation in Question **4** in general form.



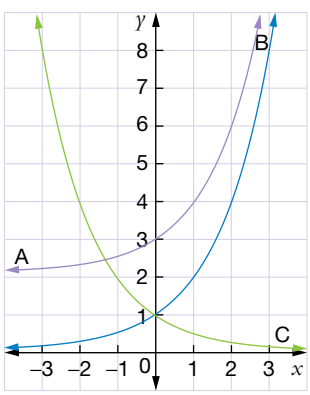
**7** Write the equation of the line that has:
- **a** a gradient of 5 and a $y$-intercept of −2
- **b** a gradient of 0 and a $y$-intercept of 4
- **c** a gradient of 2 and passes through (0, 5)
- **d** a gradient of −1 and passes through (−2, 3).

**8** Write the equation of the line that passes through:
- **a** (2, 1) and (4, 2)
- **b** (−1, 5) and (3, 1)

**9** Match each equation with the corresponding parabola in the diagram.
- **a** $y = -x^2 + 3$
- **b** $y = x^2 + 1$
- **c** $y = x^2 - 3$




**10** Match each equation with the corresponding exponential curve in the diagram.
- **a** $y = 2^x$
- **b** $y = 2^{-x}$
- **c** $y = 2 + 2^x$



# 1:09 Simultaneous equations

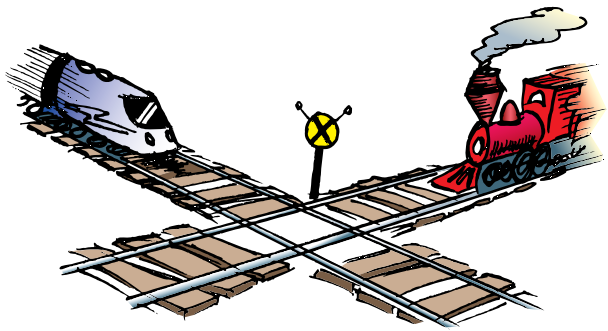
## Exercise 1:09

P Appendix A A:09A–A:09D

**1** Find the value of $y$ if:

**a** $x + y = 12$ and $x = -3$

**b** $2x - 4y = 1$ and $x = 4$



**2** Find the value of $x$ if:

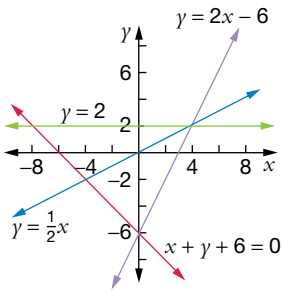
**a** $y = 5x - 4$ and $y = 21$

**b** $3x + y = 12$ and $y = -6$

**3** **a** Does the line $2x - 4y = 12$ pass through the point $(14, 4)$?

**b** Does the point $(4, 8)$ lie on the line $6x - 2y = 7$?

**4** Use the graph to solve each pair of simultaneous equations.




**a** $y = 2$, $y = 2x - 6$

**b** $y = 2$, $x + y + 6 = 0$

**c** $y = \frac{1}{2}x$, $x + y + 6 = 0$

**d** $y = \frac{1}{2}x$, $y = 2x - 6$

**e** $y = 2x - 6$, $x + y + 6 = 0$

**f** $y = 2$, $y = \frac{1}{2}x$

**5** Solve these simultaneous equations using the substitution method.

**a** $2x + y = 12$, $3x + 2y = 22$

**b** $4x - 3y = 13$, $2x = y + 9$

**c** $y = x - 2$, $2x + y = 7$

**d** $4a - b = 3$, $2a + 3b = 11$

**6** Solve these simultaneous equations using the elimination method.

**a** $5x - 3y = 20$, $2x + 3y = 15$

**b** $4a - 3b = 11$, $4a + 2b = 10$

**c** $3c + 4d = 16$, $7c - 2d = 60$

**d** $2x + 7y = 29$, $3x + 5y = 16$

**7** A theatre has 2100 seats. The rows of seats in the theatre have either 45 seats or 40 seats. If there are three times as many rows with 45 seats as there are with 40 seats, how many rows are there?

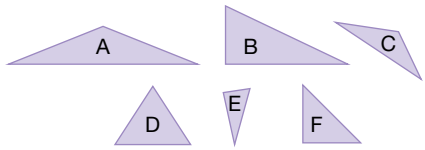
**8** Fiona has three times as much money as Jessica. If I give Jessica $100, she will have twice as much money as Fiona. How much did Jessica have originally?

# 1:10 Similarity and scale drawing

## Exercise 1:10

P Appendix A A:10A–A:10B

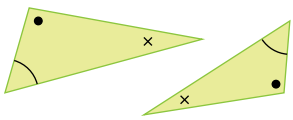
**1** **a** Explain the meaning of the term 'similar figures'.
**b** Which triangle is similar to A?
**c** Which triangle is similar to B?



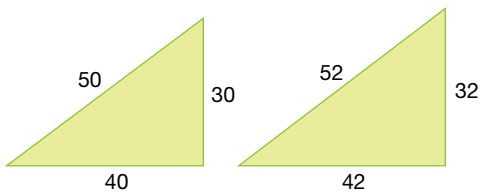

**2** True or false?
**a** When two figures are similar the matching sides are proportional.
**b** When two figures are similar the matching angles are equal.

**3** Are the triangles in each pair similar?

**a**

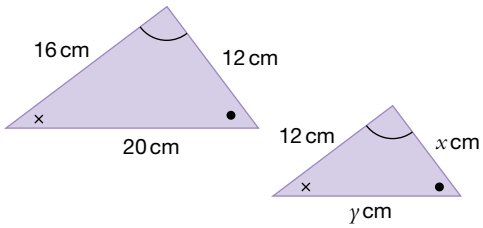


**b**

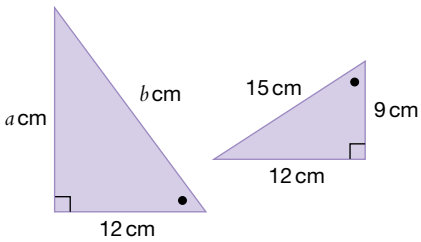



**4** Find the value of the pronumerals in each pair of similar triangles.

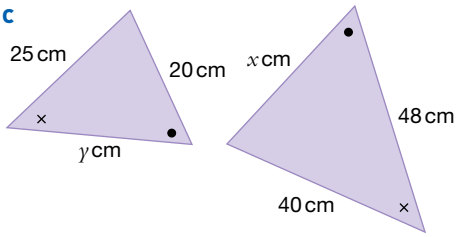
**a**



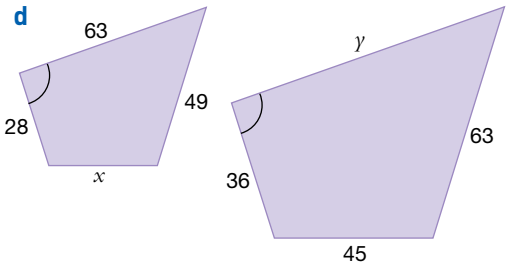

**b**




**c**




**d**




**5** A basketball court is rectangular with length 28 m and breadth 15 m. A scale drawing is to be made of the court.
**a** Calculate the dimensions of the scale drawing if the scale used is 1 : 200.
**b** What scale should be used if the length of the scale drawing must be 7 cm?

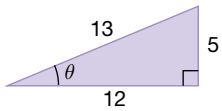
# 1:11 Trigonometry

## Exercise 1:11

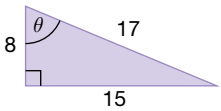
P Appendix A A:11A–A:11F

**1** State the value of $\tan\theta$ as a fraction.

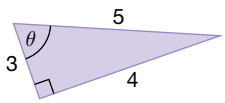
**a**




**b**

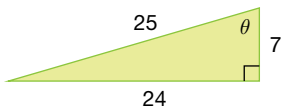



**c**



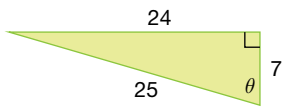

**2** State the value of each ratio as a fraction.

**a**



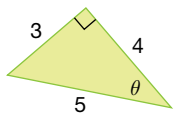

$\sin\theta = \ldots$

**b**




$\cos\theta = \ldots$

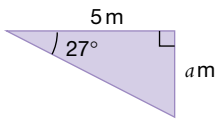
**c**



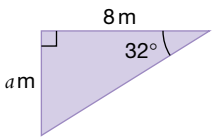

$\sin\theta = \ldots$

**3** Find the value of $a$ correct to one decimal place.

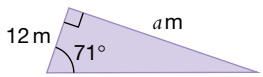
**a**




**b**

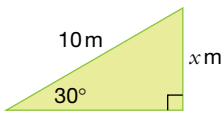



**c**

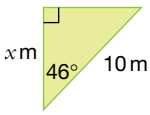



**4** Find the value of $x$ correct to one decimal place.

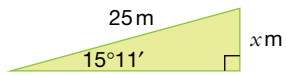
**a**




**b**

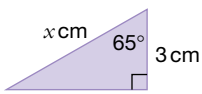



**c**

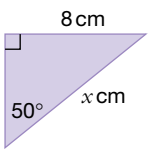



**5** Find the value of $x$ correct to one decimal place.

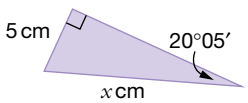
**a**




**b**

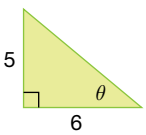



**c**

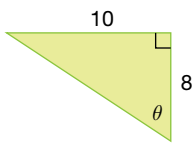



**6** Find $\theta$ to the nearest degree.

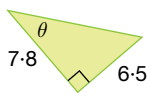
**a**




**b**

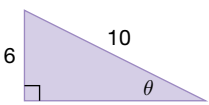



**c**

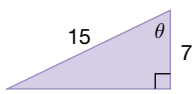



**7** Find $\theta$ correct to the nearest minute.

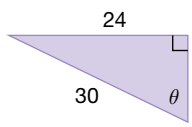
**a**




**b**




**c**




**8** The angle of depression of an object on a level plane is observed to be 19° from the top of a 21 m tower. How far from the foot of the tower is the object? Answer correct to the nearest centimetre.

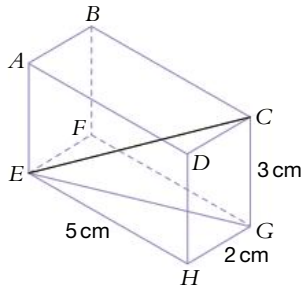
**9** The angle of elevation of the top of a vertical cliff is observed to be 23° from a boat 180 m from the base of the cliff. What is the height of the cliff? Answer correct to one decimal place.

**10** A ship sails south for 50 km, then N43°E until it is due east of its starting point. How far is the ship from its starting point to the nearest metre?

**11** For the rectangular prism shown, find:

**a** the length of $EC$ (to 2 dec. pl.)

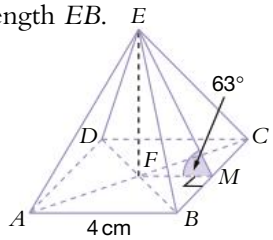
**b** $\angle CEG$ (to the nearest minute).




**12** The square pyramid shown has base edges of 4 cm in length. If $\angle EMF = 63°$, find correct to two decimal places:

**a** the length $EF$

**b** the length $EB$.




# 1:12 Statistics

## Exercise 1:12

P Appendix A A:12A–A:12H

**1** A dice was rolled 50 times and the results are shown below. Organise these results into a frequency distribution table and answer the questions.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 4 | 1 | 3 | 2 | 6 | 2 | 1 | 4 | 5 |
| 5 | 1 | 3 | 2 | 6 | 3 | 2 | 4 | 4 | 1 |
| 6 | 2 | 5 | 1 | 6 | 6 | 6 | 5 | 3 | 2 |
| 6 | 3 | 4 | 2 | 4 | 1 | 4 | 2 | 4 | 4 |
| 2 | 3 | 1 | 5 | 4 | 2 | 2 | 3 | 2 | 1 |

**a** Which number on the dice was rolled most often?

**b** Which number had the lowest frequency?

**c** How often did a 3 appear?

**d** For how many throws was the result an odd number?

**e** On how many occasions was the result greater than 3?

**2** Use the information in Question **1** to draw, on separate diagrams:

**a** a frequency histogram **b** a frequency polygon.

**3** **a** For the scores 5, 1, 8, 4, 3, 5, 5, 2, 4, find:

**i** the range **ii** the mode **iii** the mean **iv** the median.

**b** Use your frequency distribution table in Question **1** to find:

**i** the range **ii** the mode **iii** the mean **iv** the median.

**c** Add a cumulative frequency column to your frequency distribution table in Question **1**.

**i** What is the cumulative frequency of the score 4?

**ii** How many students scored 3 or less?

**iii** Does the last figure in your cumulative frequency column equal the total of the frequency column?

**4** Use your cumulative frequency table in Question **3** to draw:

**a** a frequency histogram **b** a frequency polygon.

**5** The number of cans of drink sold by a shop each day was as follows.

| | | | | | | |
|---|---|---|---|---|---|---|
| 30 | 28 | 42 | 21 | 54 | 47 | 36 |
| 37 | 22 | 18 | 25 | 26 | 43 | 50 |
| 23 | 29 | 30 | 19 | 28 | 20 | 40 |
| 33 | 35 | 31 | 27 | 42 | 26 | 44 |
| 53 | 50 | 29 | 20 | 32 | 41 | 36 |
| 51 | 46 | 37 | 42 | 27 | 28 | 31 |
| 29 | 32 | 41 | 36 | 32 | 41 | 35 |
| 41 | 29 | 39 | 46 | 36 | 36 | 33 |
| 29 | 37 | 38 | 25 | 27 | 19 | 28 |
| 47 | 51 | 28 | 47 | 36 | 35 | 40 |

The highest and lowest scores have been circled.

**a** Tabulate these results using classes of 16–22, 23–29, 30–36, 37–43, 44–50, 51–57. Make up a table using these column headings: Class, Class centre, Tally, Frequency, Cumulative frequency.

**b** What was the mean number of cans sold?

**c** Construct a cumulative frequency histogram and cumulative frequency polygon (or ogive) and find the median class.

**d** What is the modal class?

**e** Over how many days was the survey held?

**6** **a** Use the data from the first four rows of Question **5** to construct a stem-and-leaf plot.

**b** Use the answer to **a** to make this an ordered stem-and-leaf plot.

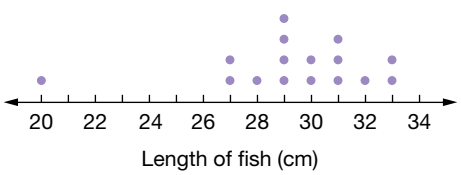
**7** **a** Find the interquartile range of the scores:

1, 2, 2, 5, 7, 9, 10, 10, 11, 11, 11, 11

**b** Draw a cumulative frequency polygon using the frequency distribution table below and use it to find the interquartile range of the scores.

| $x$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 2 | 0 | 5 | 4 | 5 | 6 | 5 | 6 | 3 | 4 |

**c** The lengths of 16 fish were measured.
The results are shown on this dot plot.
What is the interquartile range?

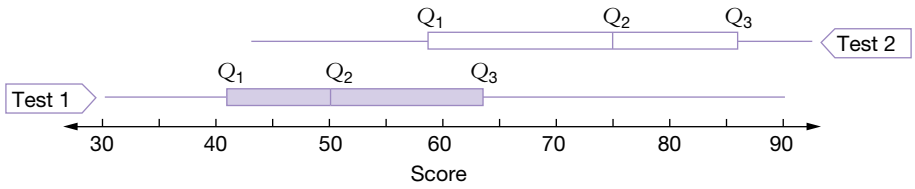



**d** What is the interquartile range of the times shown in the stem-and-leaf plot?

| Time (in days) | |
|---|---|
| Stem | Leaf |
| 3 | 8 8 |
| 4 | 0 3 4 5 8 |
| 5 | 1 3 4 4 |
| 6 | 0 |

**8** Draw a box plot for the data in Question **7a**.

**9** These box plots were drawn to compare the results of Year 10 on two tests.




**a** By how much was the median for Test 2 higher than the median of Test 1?
**b** What was the range and interquartile range of Test 1?

**10**

| Test 1 scores | | | | | |
|---|---|---|---|---|---|
| 12 | 17 | 19 | 12 | 15 | 10 |
| 9 | 22 | 24 | 11 | 18 | 8 |
| 25 | 15 | 18 | 20 | 18 | 18 |

| Test 2 scores | | | | | |
|---|---|---|---|---|---|
| 21 | 15 | 18 | 7 | 11 | 16 |
| 20 | 12 | 23 | 12 | 10 | 13 |
| 12 | 19 | 12 | 14 | 20 | 9 |

**a** Draw a dot plot for the scores in Test 1.
**b** Draw a back-to-back stem-and-leaf display to compare the scores in Tests 1 and 2.
**c** Draw a double box plot to compare the scores in Tests 1 and 2.

# 1:13 Proportion

## Exercise 1:13

P Appendix A A:13A–A:13D

**1** Write each rate in its simplest terms.
**a** 200 km in 4 hours = … km/h
**b** \$56 for 7 kg = \$…/kg
**c** 60 litres in $2\frac{1}{2}$ hours = … L/h
**d** 250 g per 10 $cm^3$ = … $g/cm^3$

**2** Convert these rates.
**a** 6 km/h = … m/min
**b** 120 000 mL/min = … L/s
**c** 15 000 kg/day = … t/day
**d** \$100 per hectare = … $c/m^2$

**3** For each example below, state whether it is an example of direct or inverse (indirect) proportion.
**a** The amount of ingredients available and the number of pizzas that can be made.
**b** The number of people eating 10 pizzas and the time taken to eat them.
**c** The speed at which a car travels and the distance covered in a given time.
**d** The speed at which a car travels and the time taken to cover a certain distance.

**4** $M$ is directly proportional to $t$, and when $t = 6$, $M = 30$.
**a** Given that $M = kt$, find the value of $k$.
**b** Find the value of $M$ when $t = 17$.
**c** Find the value of $t$ when $M = 174$.

**5** I need 8 eggs to make cakes for 3 people.

**a** How many eggs do I need to make cakes for 15 people?

**b** If I have 3 dozen eggs, for how many people can I make cakes?

**6** Travelling at a constant average speed, I can travel a distance $D$ in a certain time $T$. If I travel 120 km in 5 hours:

**a** Determine an equation for this relationship in the form $D = kT$.

**b** What is the constant of proportionality?

**c** How far can I travel in 8 hours?

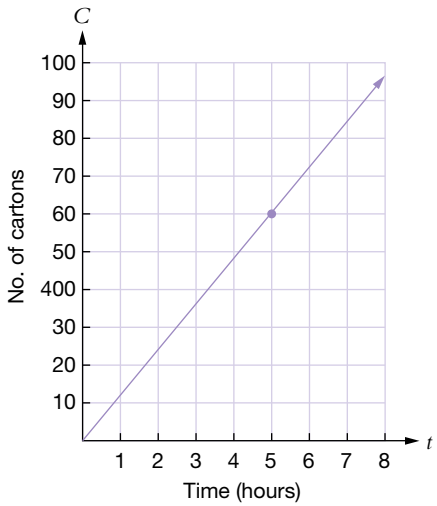
**d** How long would it take me to travel 228 km?

**7** This graph shows the number of cartons a factory can produce in a given time.

**a** By determining the gradient of the line, find the constant of proportionality for this example of direct proportion.

**b** How many cartons could be produced in 22 hours?

**c** How much time would be needed to produce 200 cartons? Give your answer in hours and minutes.




**8** The time taken ($t$) to clean the windows of a building and the number of window cleaners employed ($n$) are inversely proportional. A team of 6 window cleaners can clean the windows in 8 hours.

**a** Find an equation in the form $t = \frac{k}{n}$ for this relationship.

**b** What is the constant of proportionality?

**c** How long would it take 12 cleaners to do the same job?

**d** How many cleaners are needed to do the job in 3 hours?

**9** The time taken ($T$) for Henry to drive home is inversely proportional to the average speed ($S$) at which he drives. If his speed is 40 km/h he gets home in 3 hours.

**a** What is the constant of proportionality for this relationship?

**b** How long does he take if he increases his average speed to 50 km/h? (Give your answer in hours and minutes.)

**c** At what speed must he travel to get home in 2 hours?

**10** The frequency of a note produced on a string is inversely proportional to the length of the string. If a string of length 50 cm produces a frequency of 600 Hz:

**a** what frequency is produced by a string of length 40 cm

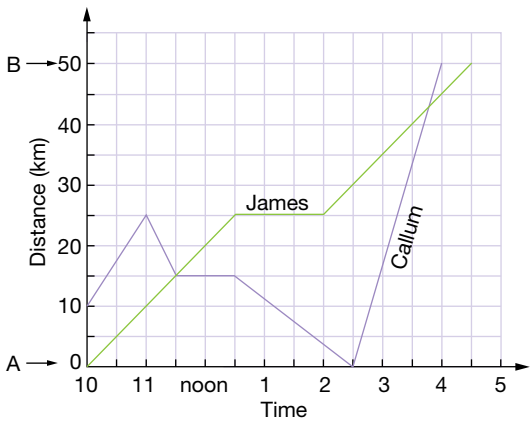
**b** what length of string would produce a frequency of 800 Hz?

# 1:14 Graphs of physical phenomena

## Exercise 1:14

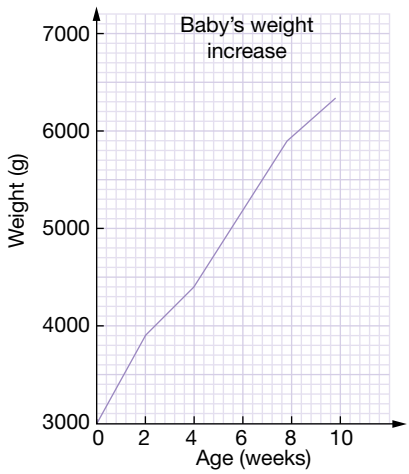
P Appendix A A:14A–A:14C

**1** The travel graph shows the journeys of James and Callum between town A and town B. (They travel on the same road.)



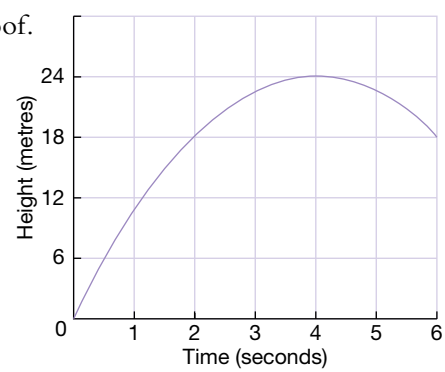

- **a** How far from A is Callum when he commences his journey?
- **b** How far is James from B at 2:30 pm?
- **c** When do James and Callum first meet?
- **d** Who reaches town B first?
- **e** At what time does Callum stop to rest?
- **f** How far does James travel?
- **g** How far apart are James and Callum when Callum is at town A?
- **h** How far does Callum travel?

**2**




- **a** What did the baby weigh at birth?
- **b** What was the baby's weight at 4 weeks of age?
- **c** By how much did the baby's weight increase in the first 2 weeks of age?
- **d** By how much did the baby's weight increase from 2 weeks of age to 4 weeks?
- **e** Considering your answer to parts **c** and **d**, in which period (0–2 weeks or 2–4 weeks) was the baby's rate of growth the greater?

**3** A ball was thrown from ground level and landed on a roof.



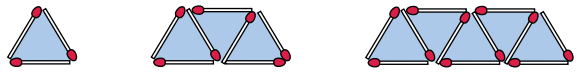

- **a** To what height did the ball rise, and how high was the roof?
- **b** After how many seconds did the ball reach its greatest height?
- **c** What was the height of the ball after 2 seconds?
- **d** When did the ball reach a height of 18 m?
- **e** What was the height of the ball after 1 second, correct to the nearest metre?

# ASSIGNMENT 1A Working mathematically

P Appendix B

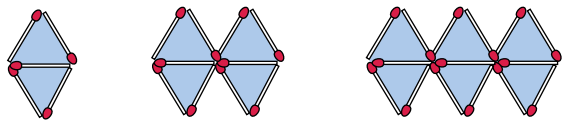
**1** Complete a table of values for each matchstick pattern below, and hence find the rule for each, linking the number of coloured triangles ($t$) to the number of matches ($m$).

**a**



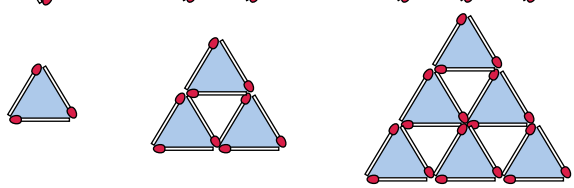
| $t$ | 1 | 3 | 5 |
|---|---|---|---|
| $m$ | | | |

**b**



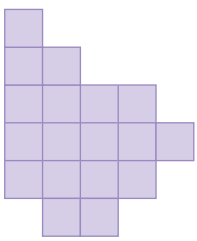
| $t$ | 2 | 4 | 6 |
|---|---|---|---|
| $m$ | | | |

**c**



| $t$ | 1 | 3 | 6 |
|---|---|---|---|
| $m$ | | | |

**2** Divide this shape into three pieces that have the same shape.



**3** Ryan answered all 50 questions in a maths competition in which he received 4 marks for each correct answer but lost one mark for each incorrect answer.

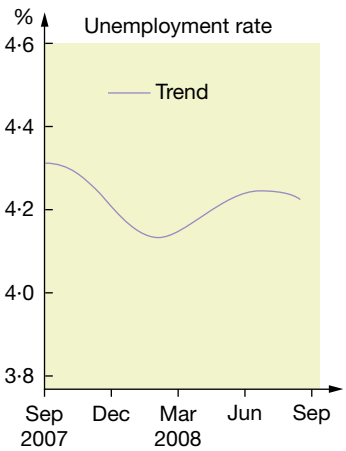
**a** What is Ryan's score if he answered 47 questions correctly?

**b** How many answers did he get right if his score was 135?

**4** It takes 3 minutes and 15 seconds to join two pieces of pipe. How long would it take to join 6 pieces of pipe into one length?

**5** A number of cards can be shared between 4 people with no remainder. When shared between 5 or 6 people, there are two cards left over. If there are fewer than 53 cards, how many are there?

**6** From September 2007 to September 2008, the unemployment rate fell from 4·3% to 4·2%.




*Source:* Australian Bureau of Statistics, Labour Force, Australia, Sep. 2008

**a** If the number of unemployed in September 2008 was 474 100, what was the size of the total workforce? Answer correct to four significant figures.

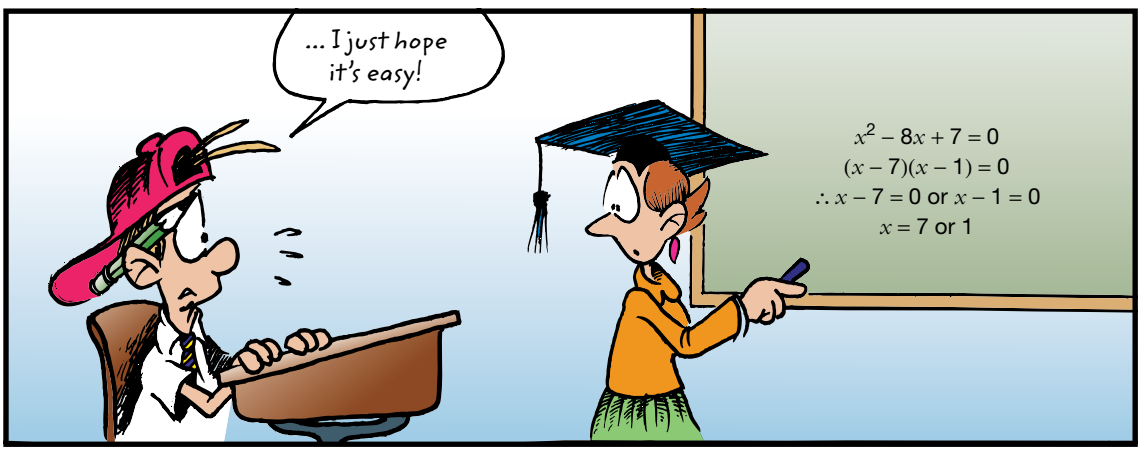
**b** How many people were unemployed in September 2007, assuming the workforce was the same size as in 2008? Answer correct to four significant figures.

**c** If the rate of 4·2% is only correct to one decimal place, the rate could really be from 4·15% to 4·25%. How many people does this approximate range of 0·1% represent?

**d** Find the current unemployment rate and compare with the data shown here.

# QUADRATIC EQUATIONS

# 2




## Contents


**2:01** Solution using factors
**2:02** Solution by completing the square
**2:03** The quadratic formula
Investigation 2:03 How many solutions?
**2:04** Choosing the best method
Fun spot 2:04 What is an Italian referee?
**2:05** Problems involving quadratic equations
Investigation 2:05 Temperature and altitude
**2:06** Equations reducible to quadratics
Fun spot 2:06 Did you know that 2 = 1?
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

Selections from ***Equations*** [Stages 5.2, 5.3§]

- Solve simple quadratic equations using a range of strategies (ACMNA241)
- Solve a wide range of quadratic equations derived from a variety of contexts (ACMNA269)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 2:01 Solution using factors

## PREP QUIZ 2:01

Factorise:
**1** $x^2 - 3x$ **2** $x^2 + 7x$ **3** $x^2 + 3x + 2$ **4** $x^2 - 4x - 12$
**5** $x^2 + x - 20$ **6** $x^2 - 8x + 7$

Solve for $x$:
**7** $3x = 12$ **8** $7x = 0$ **9** $x - 4 = 0$ **10** $x + 6 = 0$

In a quadratic equation the highest power of the pronumeral is 2.

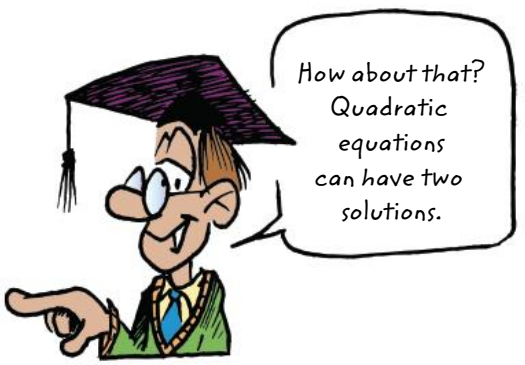
e.g. $x^2 = 9$ $5x^2 - 8 = 0$ $x^2 - 6x = 0$ $x^2 - 4x + 3 = 0$

Equations like the first two above can be solved directly but the second two require the expression to be factorised.

The term quadratic comes from *quadraticus*, which is the Latin word for square.

### Equations of the form $ax^2 = c$

To solve $x^2 = 9$ we find the square root of both sides of the equation.
The square root of 9 is 3 or −3.




So if $x^2 = 9$, then $x = \pm 3$.
The equation has two solutions: $x = 3$ and $x = -3$

## WORKED EXAMPLES

Solve these equations.
**1** $x^2 - 16 = 0$ **2** $a^2 - 5 = 0$ **3** $3m^2 = 10$ **4** $k^2 + 4 = 0$

### Solutions

**1** $x^2 - 16 = 0$
$x^2 = 16$
$x = \pm\sqrt{16}$
$\therefore x = \pm 4$

**2** $a^2 - 5 = 0$
$a^2 = 5$
$\therefore a = \pm\sqrt{5}$
(as decimal approximations
$a = 2{\cdot}236$ or $-2{\cdot}236$)

**3** $3m^2 = 10$
$m^2 = \frac{10}{3}$
$\therefore m = \pm\sqrt{\frac{10}{3}}$
(as decimal approximations
$m = 1{\cdot}826$ or $-1{\cdot}826$)

**4** $k^2 + 4 = 0$
$k^2 = -4$
The square of a real number is positive.
So this equation has no real solutions!

## Equations of the form $ax^2 + bx + c = 0$

A quadratic equation is an equation of the 'second degree'.

To solve $x^2 - 4x + 3 = 0$ we need to factorise the algebraic expression and then use the Null Factor Law.

> If $p \times q = 0$, then at least one of $p$ and $q$ must be zero.

$x^2 - 4x + 3 = 0$
$(x - 3)(x - 1) = 0$

So either $x - 3 = 0$ or $x - 1 = 0$
$\therefore x = 3$ or $x = 1$

Substituting these values into the original equation will show that they are both solutions.

## WORKED EXAMPLE 1

Solve the quadratic equations.

**a** If $(x - 1)(x + 7) = 0$
then either
$x - 1 = 0$ or $x + 7 = 0$
$\therefore x = 1$ or $x = -7$

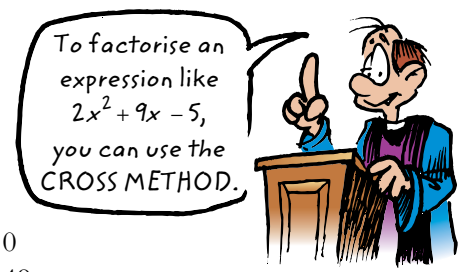
**b** If $2x(x + 3) = 0$
then either
$2x = 0$ or $x + 3 = 0$
$\therefore x = 0$ or $x = -3$

**c** If $(2x - 1)(3x + 5) = 0$
then either
$2x - 1 = 0$ or $3x + 5 = 0$
$2x = 1$ or $3x = -5$
$\therefore x = \frac{1}{2}$ or $x = -\frac{5}{3}$

> A quadratic equation can have two solutions.

## WORKED EXAMPLE 2

To solve these equations, they are factorised first.




**a** $x^2 + 4x + 3 = 0$
$(x + 3)(x + 1) = 0$
$x + 3 = 0$ or $x + 1 = 0$
$\therefore x = -3$ or $x = -1$

**b** $x^2 - 49 = 0$
$(x - 7)(x + 7) = 0$
$x - 7 = 0$ or $x + 7 = 0$
$\therefore x = 7$ or $x = -7$

or $x^2 - 49 = 0$
$x^2 = 49$
$\therefore x = 7$ or $-7$
i.e. $x = \pm 7$

**c** $2x^2 + 9x - 5 = 0$
$(2x - 1)(x + 5) = 0$
$2x - 1 = 0$ or $x + 5 = 0$
$\therefore x = \frac{1}{2}$ or $x = -5$

## WORKED EXAMPLE 3

Before these equations are solved, all the terms are gathered to one side of the equation and equated to zero.

**a**
$$x^2 + x = 12$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x + 4 = 0 \text{ or } x - 3 = 0$$
$$\therefore x = -4 \text{ or } x = 3$$

**b**
$$5x^2 = 2x$$
$$5x^2 - 2x = 0$$
$$x(5x - 2) = 0$$
$$x = 0 \text{ or } 5x - 2 = 0$$
$$\therefore x = 0 \text{ or } x = \frac{2}{5}$$

**c**
$$6x^2 = 5x + 6$$
$$6x^2 - 5x - 6 = 0$$
$$(3x + 2)(2x - 3) = 0$$
$$3x + 2 = 0 \text{ or } 2x - 3 = 0$$
$$\therefore x = -\frac{2}{3} \text{ or } x = \frac{3}{2}$$

To solve a quadratic equation:
- gather all the terms to one side of the equation
- factorise
- solve the two resulting simple equations.

You can always check your solutions by substitution. For Worked Example 3a:

Substituting $x = -4$
$x^2 + x = 12$
LHS $= (-4)^2 + (-4)$
$= 16 - 4$
$= 12$
$= \text{RHS}$

Substituting $x = 3$
$x^2 + x = 12$
LHS $= (3)^2 + (3)$
$= 9 + 3$
$= 12$
$= \text{RHS}$

LHS is left-hand side.
RHS is right-hand side.

$\therefore$ Both $x = -4$ and $x = 3$ are solutions.

## Exercise 2:01

P Foundation worksheet 2:01 Quadratic equations

**1** Solve these equations.

**a** $x^2 = 25$ **b** $x^2 = 81$ **c** $x^2 = 1$ **d** $x^2 = 100$
**e** $x^2 - 36 = 0$ **f** $x^2 - 64 = 0$ **g** $x^2 - 121 = 0$ **h** $x^2 - 400 = 0$

**2** Solve these equations, leaving your answers in square root form.

**a** $x^2 = 6$ **b** $n^2 = 3$ **c** $y^2 - 10 = 0$ **d** $k^2 - 15 = 0$
**e** $4x^2 = 8$ **f** $5p^2 = 3$ **g** $8q^2 - 10 = 0$ **h** $7z^2 - 5 = 0$

**3** Solve these equations, evaluating your answers correct to three decimal places.

**a** $x^2 = 2$ **b** $2a^2 = 7$ **c** $c^2 - 12 = 0$ **d** $8g^2 - 3 = 0$

**4** **a** Solve the equation $x^2 + 9 = 0$. **b** Give a reason for your answer in **a**.

**5** Find the two solutions for each equation. Check your answers by substitution.

**a** $x(x - 5) = 0$ **b** $x(x + 7) = 0$ **c** $2x(x + 1) = 0$
**d** $5a(a - 2) = 0$ **e** $4q(q + 5) = 0$ **f** $6p(p - 7) = 0$
**g** $(x - 2)(x - 1) = 0$ **h** $(x - 7)(x - 3) = 0$ **i** $(a - 5)(a - 2) = 0$
**j** $(y + 3)(y + 4) = 0$ **k** $(t + 3)(t + 2) = 0$ **l** $(x + 9)(x + 5) = 0$
**m** $(a - 6)(a + 6) = 0$ **n** $(y + 8)(y - 7) = 0$ **o** $(n + 1)(n - 1) = 0$
**p** $(a + 1)(2a - 1) = 0$ **q** $(3x + 2)(x - 5) = 0$ **r** $2x(3x - 1) = 0$
**s** $(4x - 1)(2x + 1) = 0$ **t** $(3a - 4)(2a - 1) = 0$ **u** $(6y - 5)(4y + 3) = 0$
**v** $6x(5x - 3) = 0$ **w** $(9y + 1)(7y + 2) = 0$ **x** $(5x - 1)(5x + 1) = 0$

**6** After factorising the left-hand side of each equation, solve the following.

**a** $x^2 + 3x = 0$ **b** $m^2 - 5m = 0$ **c** $y^2 + 2y = 0$
**d** $6x^2 + 12x = 0$ **e** $9n^2 - 3n = 0$ **f** $4x^2 + 8x = 0$
**g** $x^2 - 4 = 0$ **h** $a^2 - 49 = 0$ **i** $y^2 - 36 = 0$
**j** $a^2 - 1 = 0$ **k** $n^2 - 100 = 0$ **l** $m^2 - 64 = 0$
**m** $x^2 + 3x + 2 = 0$ **n** $a^2 - 5a + 6 = 0$ **o** $y^2 + 12y + 35 = 0$
**p** $a^2 - 10a + 21 = 0$ **q** $x^2 - 10x + 16 = 0$ **r** $m^2 - 11m + 24 = 0$
**s** $h^2 + h - 20 = 0$ **t** $x^2 + 2x - 35 = 0$ **u** $a^2 - 4a - 45 = 0$
**v** $x^2 + x - 56 = 0$ **w** $y^2 - 8y + 7 = 0$ **x** $a^2 + 9a - 10 = 0$

**7** **a** Solve the equation $x^2 - 4x + 4 = 0$.
**b** How many solutions are there for this quadratic equation?
**c** When will a quadratic equation have only one solution?

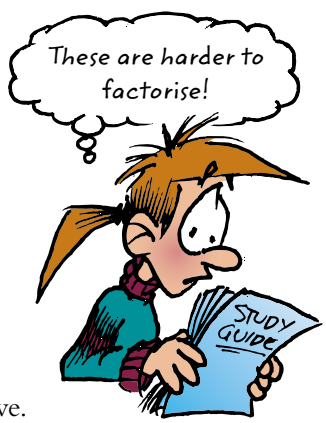
Check answers by substitution.

**8** Solve:

**a** $x^2 + 4x + 4 = 0$ **b** $m^2 - 6m + 9 = 0$ **c** $z^2 + 2z + 1 = 0$
**d** $k^2 - 8k + 16 = 0$ **e** $a^2 - 10a + 25 = 0$ **f** $y^2 + 20y + 100 = 0$

**9** Factorise and solve the following.

**a** $2x^2 + x - 1 = 0$ **b** $3x^2 + 7x + 2 = 0$
**c** $3x^2 + 17x + 10 = 0$ **d** $2x^2 - 11x + 12 = 0$
**e** $2x^2 - x - 10 = 0$ **f** $2x^2 - 11x - 21 = 0$
**g** $4x^2 + 21x + 5 = 0$ **h** $4x^2 - 19x - 5 = 0$
**i** $4x^2 - 21x + 5 = 0$ **j** $5x^2 + 16x + 3 = 0$
**k** $2x^2 + 13x - 24 = 0$ **l** $7x^2 + 48x - 7 = 0$
**m** $4x^2 - 4x - 3 = 0$ **n** $6x^2 - x - 1 = 0$
**o** $9x^2 + 9x + 2 = 0$ **p** $10x^2 + 9x + 2 = 0$
**q** $12x^2 - 7x + 1 = 0$ **r** $10x^2 - 13x + 4 = 0$




**10** Gather all the terms to one side of the equation and then solve.

**a** $x^2 = 3x$ **b** $m^2 = 8m$ **c** $t^2 = -5t$
**d** $x^2 = 5x - 4$ **e** $a^2 = 2a + 15$ **f** $y^2 = 3y - 2$
**g** $m^2 = 9m - 18$ **h** $n^2 = 7n + 18$ **i** $h^2 = 4h + 32$
**j** $x^2 + x = 2$ **k** $y^2 + 2y = 3$ **l** $x^2 - 7x = -10$
**m** $y^2 + 3y = 18$ **n** $t^2 + 3t = 28$ **o** $y^2 + 2y = 15$
**p** $2x^2 + x = 1$ **q** $2x^2 - x = 15$ **r** $4m^2 - 5m = 6$
**s** $3x^2 = 13x - 14$ **t** $5p^2 = 17p - 6$ **u** $2x^2 = 11x - 5$

# 2:02 Solution by completing the square

This method depends upon completing an algebraic expression to form a perfect square, that is, an expression of the form $(x + a)^2$ or $(x - a)^2$.

*Reminder:*
$(x + a)^2 = x^2 + 2ax + a^2$
- The middle term in the expansion is $2ax$.
- The coefficient of $x$ in the middle term is $2a$.

So, the value of $a$ can be found by halving the coefficient of the $x$ term.
This is also true for: $(x - a)^2 = x^2 - 2ax + a^2$

## WORKED EXAMPLE 1

What must be added to the following to make perfect squares?

**a** $x^2 + 8x$

**b** $x^2 - 5x$

### Solutions

**a** $x^2 + 8x + \ldots$

Half of 8 is 4, so the perfect square is:

$x^2 + 8x + 4^2 = (x + 4)^2$

$= x^2 + 8x + 16$

So 16 must be added to make a perfect square.

**b** $x^2 - 5x + \ldots$

Half of $-5$ is $-\frac{5}{2}$, so the perfect square is:

$x^2 - 5x + (-\frac{5}{2})^2 = (x - \frac{5}{2})^2$

$= x^2 - 5x + \frac{25}{4}$

So $\frac{25}{4}$ must be added to make a perfect square.

Now, to solve a quadratic equation using this technique follow these steps.

$x^2 + 4x - 21 = 0$

$x^2 + 4x = 21$ ← Move the constant to the RHS.

$x^2 + 4x + 2^2 = 21 + 2^2$ ← Add (half of $x$ coefficient)$^2$ to both sides.

$\therefore (x + 2)^2 = 25$

$x + 2 = \pm\sqrt{25}$

$x = -2 \pm 5$

$\therefore x = 3$ or $-7$

Note that $x^2 + 4x - 21 = 0$ could have been solved using the Null Factor Law as in the previous exercise.

$x^2 + 4x - 21 = 0$

$(x - 3)(x + 7) = 0$

$\therefore x = 3$ or $x = -7$

In this case, this is an easier and quicker way to find the solution. The method of completing the square, however, can determine the solution of quadratic equations that cannot be factorised. This can be seen in the following examples.

## WORKED EXAMPLE 2

Solve:

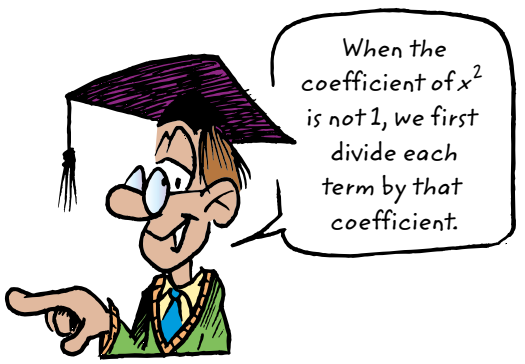
**a** $x^2 + 6x + 1 = 0$ **b** $x^2 - 3x - 5 = 0$ **c** $3x^2 - 4x - 1 = 0$

### Solutions

**a**
$$x^2 + 6x + 1 = 0$$
$$x^2 + 6x = -1$$
$$x^2 + 6x + 3^2 = -1 + 3^2$$
$$(x + 3)^2 = 8$$
$$x + 3 = \pm\sqrt{8}$$
$$\therefore x = -3 \pm \sqrt{8}$$
$$x = -3 + \sqrt{8} \text{ or } -3 - \sqrt{8}$$
$$(x \doteqdot -0{\cdot}17 \text{ or } -5{\cdot}83)$$

Note that the solution involves a square root, i.e. the solution is irrational. Using your calculator, approximations may be found.

**b**
$$x^2 - 3x - 5 = 0$$
$$x^2 - 3x = 5$$
$$x^2 - 3x + (-\tfrac{3}{2})^2 = 5 + (-\tfrac{3}{2})^2$$
$$(x - \tfrac{3}{2})^2 = 7\tfrac{1}{4}$$
$$x - \tfrac{3}{2} = \pm\sqrt{7\tfrac{1}{4}}$$
$$\therefore x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$
$$x = \frac{3 + \sqrt{29}}{2} \text{ or } \frac{3 - \sqrt{29}}{2}$$
$$(x \doteqdot 4{\cdot}19 \text{ or } -1{\cdot}19)$$




**c**
$$3x^2 - 4x - 1 = 0$$
$$x^2 - \tfrac{4}{3}x - \tfrac{1}{3} = 0$$
$$x^2 - \tfrac{4}{3}x = \tfrac{1}{3}$$
$$x^2 - \tfrac{4}{3}x + (-\tfrac{2}{3})^2 = \tfrac{1}{3} + (-\tfrac{2}{3})^2$$
$$(x - \tfrac{2}{3})^2 = \tfrac{7}{9}$$
$$x - \tfrac{2}{3} = \pm\frac{\sqrt{7}}{3}$$
$$\therefore x = \frac{2}{3} \pm \frac{\sqrt{7}}{3}$$
$$x = \frac{2 + \sqrt{7}}{3} \text{ or } \frac{2 - \sqrt{7}}{3}$$
$$(x \doteqdot 1{\cdot}55 \text{ or } -0{\cdot}22)$$

You can use the following fact to check your answers.
For the equation $ax^2 + bx + c = 0$
the two solutions must add to $-\frac{b}{a}$.
In Worked Example 2a, $(-0{\cdot}17) + (-5{\cdot}83) = -6$ or $\frac{-6}{1}$
In Worked Example 2c, $1{\cdot}55 + (-0{\cdot}22) = 1{\cdot}33 \doteqdot \frac{4}{3}$

## Exercise 2:02

**1** What number must be inserted into each expression to complete the square?

**a** $x^2 + 6x + \ldots^2 = (x + \ldots)^2$
**b** $x^2 + 8x + \ldots^2 = (x + \ldots)^2$
**c** $x^2 - 2x + \ldots^2 = (x - \ldots)^2$
**d** $x^2 - 4x + \ldots^2 = (x - \ldots)^2$
**e** $x^2 + 3x + \ldots^2 = (x + \ldots)^2$
**f** $x^2 - 7x + \ldots^2 = (x - \ldots)^2$
**g** $x^2 + 11x + \ldots^2 = (x + \ldots)^2$
**h** $x^2 - x + \ldots^2 = (x - \ldots)^2$
**i** $x^2 + \frac{5x}{2} + \ldots^2 = (x + \ldots)^2$
**j** $x^2 - \frac{2x}{3} + \ldots^2 = (x - \ldots)^2$

**2** Solve the following equations, leaving your answers in surd form.

**a** $(x - 2)^2 = 3$
**b** $(x + 1)^2 = 2$
**c** $(x + 5)^2 = 5$
**d** $(x - 1)^2 = 10$
**e** $(x - 3)^2 = 7$
**f** $(x + 2)^2 = 11$
**g** $(x + 3)^2 = 8$
**h** $(x + 10)^2 = 12$
**i** $(x - 3)^2 = 18$
**j** $(x + \frac{1}{2})^2 = 5$
**k** $(x - \frac{2}{3})^2 = 3$
**l** $(x + 1\frac{1}{2})^2 = 12$
**m** $(x - 1)^2 = 2\frac{1}{2}$
**n** $(x + 3)^2 = 4\frac{1}{2}$
**o** $(x - \frac{1}{3})^2 = \frac{5}{9}$

**3** Solve the following equations by completing the square. Also find approximations for your answers, correct to two decimal places.

**a** $x^2 + 2x - 1 = 0$
**b** $x^2 - 2x - 5 = 0$
**c** $x^2 - 4x - 8 = 0$
**d** $x^2 + 6x - 8 = 0$
**e** $x^2 - 6x + 2 = 0$
**f** $x^2 + 4x + 1 = 0$
**g** $x^2 + 10x = 5$
**h** $x^2 + 2x = 4$
**i** $x^2 - 12x = 1$
**j** $x^2 + 5x + 2 = 0$
**k** $x^2 + 7x - 3 = 0$
**l** $x^2 + x - 3 = 0$
**m** $x^2 + 9x + 3 = 0$
**n** $x^2 + 3x - 5 = 0$
**o** $x^2 - 11x + 5 = 0$
**p** $x^2 - x = 3$
**q** $x^2 + 3x = 2$
**r** $x^2 - 5x = 1$
**s** $2x^2 - 4x - 1 = 0$
**t** $2x^2 + 3x - 4 = 0$
**u** $2x^2 - 8x + 1 = 0$
**v** $3x^2 + 2x - 3 = 0$
**w** $5x^2 - 4x - 3 = 0$
**x** $4x^2 - x - 2 = 0$

# 2:03 The quadratic formula

As we have seen in the previous section, a quadratic equation is one involving a squared term. In fact, any quadratic equation can be represented by the **general form** of a quadratic equation:

$$ax^2 + bx + c = 0$$

where $a$, $b$, $c$ are all integers, and $a \neq 0$.

If any quadratic equation is arranged in this form, a formula using the values of $a$, $b$ and $c$ can be used to find the solutions.

The quadratic formula for $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof of the quadratic formula:

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

*Note:*
This proof uses the method of completing the square.

## WORKED EXAMPLE 1

Solve the following by using the quadratic formula.

**a** $2x^2 + 9x + 4 = 0$ **b** $x^2 + 5x + 1 = 0$

### Solutions

**a** For $2x^2 + 9x + 4 = 0$,
$a = 2, b = 9, c = 4$.
Substituting into the formula gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4 \times 2 \times 4}}{2 \times 2}$$

$$= \frac{-9 \pm \sqrt{81 - 32}}{4}$$

$$= \frac{-9 \pm \sqrt{49}}{4}$$

$$= \frac{-9 \pm 7}{4}$$

$$= -\frac{2}{4} \text{ or } -\frac{16}{4}$$

$\therefore x = -\frac{1}{2}$ or $-4$

**b** For $x^2 + 5x + 1 = 0$,
$a = 1, b = 5, c = 1$.
Substituting into the formula gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-5 \pm \sqrt{25 - 4}}{2}$$

$$= \frac{-5 \pm \sqrt{21}}{2}$$

There is no rational equivalent to $\sqrt{21}$ so the answer may be left as:

$$x = \frac{-5 + \sqrt{21}}{2} \text{ or } \frac{-5 - \sqrt{21}}{2}$$

Approximations for these answers may be found using a calculator.
$x \doteqdot -0{\cdot}21$ or $-4{\cdot}79$ (2 dec. pl.)

## WORKED EXAMPLE 2

Solve the following by using the quadratic formula.

**a** $3x^2 = 2x + 2$ **b** $2x^2 + 2x + 7 = 0$

### Solutions

**a** The equation $3x^2 = 2x + 2$ must first be written in the form $ax^2 + bx + c = 0$:

$3x^2 - 2x - 2 = 0$

So $a = 3, b = -2, c = -2$.

Substituting these values gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-2)}}{2 \times 3}$$

$$= \frac{2 \pm \sqrt{4 + 24}}{6}$$

$$= \frac{2 \pm \sqrt{28}}{6}$$

$$\therefore x = \frac{2 + \sqrt{28}}{6} \text{ or } \frac{2 - \sqrt{28}}{6}$$

$x \doteqdot 1{\cdot}22$ or $-0{\cdot}55$ (2 dec. pl.)

**b** For $2x^2 + 2x + 7 = 0$,

$a = 2, b = 2, c = 7$.

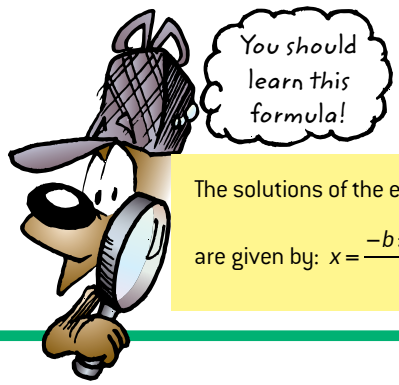
Substituting these values gives:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 2 \times 7}}{2 \times 2}$$

$$= \frac{-2 \pm \sqrt{-52}}{4}$$

But $\sqrt{-52}$ is not real!

So $2x^2 + 2x + 7 = 0$ has no real solutions.




The solutions of the equation $ax^2 + bx + c = 0$ are given by: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## Exercise 2:03

P Foundation worksheet 2:03 The quadratic formula

**1** Use the quadratic formula to solve the following equations. All have rational answers.

**a** $x^2 + 5x + 6 = 0$ **b** $x^2 + 6x + 8 = 0$ **c** $x^2 + 10x + 9 = 0$

**d** $x^2 - 3x - 10 = 0$ **e** $x^2 - 2x - 15 = 0$ **f** $x^2 + 4x - 12 = 0$

**g** $x^2 - 9x + 14 = 0$ **h** $x^2 - 8x + 12 = 0$ **i** $x^2 - 6x + 5 = 0$

**j** $3x^2 + 7x + 2 = 0$ **k** $2x^2 + 11x + 5 = 0$ **l** $4x^2 + 11x + 6 = 0$

**m** $2x^2 - 5x - 3 = 0$ **n** $5x^2 - 9x - 2 = 0$ **o** $3x^2 - 5x + 2 = 0$

**p** $6x^2 + 7x + 2 = 0$ **q** $6x^2 + 7x - 3 = 0$ **r** $8x^2 - 14x + 3 = 0$

**2** Solve the following, leaving your answers in surd form. (*Remember:* A surd is an expression involving a square root.)

**a** $x^2 + 4x + 2 = 0$ **b** $x^2 + 3x + 1 = 0$ **c** $x^2 + 5x + 3 = 0$

**d** $x^2 + x - 1 = 0$ **e** $x^2 + 2x - 2 = 0$ **f** $x^2 + 4x - 1 = 0$

**g** $x^2 - 2x - 1 = 0$ **h** $x^2 - 7x + 2 = 0$ **i** $x^2 - 6x + 3 = 0$

**j** $x^2 - 10x - 9 = 0$ **k** $x^2 - 8x + 3 = 0$ **l** $x^2 - 5x + 7 = 0$

**m** $2x^2 + 6x + 1 = 0$ **n** $2x^2 + 3x - 1 = 0$ **o** $2x^2 - 7x + 4 = 0$

**p** $3x^2 + 10x + 2 = 0$ **q** $3x^2 - 9x + 2 = 0$ **r** $5x^2 + 4x - 2 = 0$

**s** $4x^2 - x + 1 = 0$ **t** $3x^2 - 3x - 1 = 0$ **u** $4x^2 - 3x - 2 = 0$

**v** $2x^2 + 11x - 5 = 0$ **w** $2x^2 - 9x + 8 = 0$ **x** $5x^2 + 2x - 1 = 0$

**3** Use the formula to solve the following. Give answers as decimal approximations correct to two decimal places.

**a** $x^2 - 4x + 1 = 0$ **b** $x^2 - 6x + 3 = 0$ **c** $x^2 + 8x - 5 = 0$

**d** $x^2 + 9x + 1 = 0$ **e** $x^2 + 2x - 5 = 0$ **f** $x^2 + 3x - 1 = 0$

**g** $x^2 + 2 = 0$ **h** $x^2 - 7x = 2$ **i** $x^2 = 6x - 11$

**j** $2x^2 + x - 2 = 0$ **k** $2x^2 - 5x - 2 = 0$ **l** $3x^2 + 9x + 5 = 0$

**m** $2x^2 = 7x - 2$ **n** $5x^2 - 3x = 4$ **o** $6x^2 = x + 3$

## INVESTIGATION 2:03 HOW MANY SOLUTIONS?

Consider these three quadratic equations:

**a** $x^2 + 6x + 5 = 0$ **b** $x^2 + 6x + 9 = 0$ **c** $x^2 + 6x + 12 = 0$

Use the quadratic formula to solve each equation.

### Solutions

**a**
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 5}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{16}}{2}$$
$$= \frac{-6 \pm 4}{2}$$
$\therefore x = -1$ or $-5$

**b**
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{0}}{2}$$
$$= \frac{-6}{2}$$
$\therefore x = -3$

**c**
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 12}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{-12}}{2}$$
$\sqrt{-12}$ has no real solution

$\therefore x$ has no real solution

It appears that a quadratic equation may have two, one or no real solutions. The 'key' is the part of the formula under the square root sign.

The number of real solutions is determined by $b^2 - 4ac$.

If $b^2 - 4ac$ is:

- *positive* then the equation will have *two* real solutions
- *zero* then the equation will have *one* real solution
- *negative* then the equation will have *no* real solution.

### Exercises

By evaluating $b^2 - 4ac$ for each equation, determine how many real solutions it will have.

$b^2 - 4ac$ is called the *discriminant*.

**1** $x^2 + 4x + 3 = 0$ **2** $x^2 + 4x + 4 = 0$ **3** $x^2 + 4x + 5 = 0$

**4** $x^2 - x - 2 = 0$ **5** $x^2 - x = 0$ **6** $x^2 - x + 2 = 0$

**7** $4x^2 - 12x + 9 = 0$ **8** $4x^2 - 12x + 7 = 0$ **9** $4x^2 - 12x + 11 = 0$

**10** $5x^2 - x + 7 = 0$ **11** $5x^2 - x - 7 = 0$ **12** $9x^2 + 6x + 1 = 0$

# 2:04 Choosing the best method

## PREP QUIZ 2:04

Factorise:

**1** $5x^2 - 10x$ **2** $x^2 - 5x - 14$ **3** $x^2 - 81$ **4** $x^2 + 5x + 6$

Solve:

**5** $(x - 2)(x + 7) = 0$ **6** $(2x - 3)(3x + 1) = 0$ **7** $x^2 - 16 = 0$

**8** $3x^2 - 12x = 0$ **9** $x^2 - 3x + 2 = 0$

**10** Write the formula for the solutions of the equation $ax^2 + bx + c = 0$.

Some quadratic equations may appear in a different form from those seen so far, but they can always be simplified to the general form $ax^2 + bx + c = 0$. They may then be solved by factorising or by using the quadratic formula.

## WORKED EXAMPLES

Solve the following equations.

**1** $x^2 - 2x + 1 = 3x + 6$ **2** $x(x - 5) = 6$ **3** $x = \dfrac{5x - 6}{x}$

### Solutions

**1** In this example, all the terms must be gathered to the LHS.

$x^2 - 2x + 1 = 3x + 6$

$-3x - 6 = -3x - 6$

$x^2 - 5x - 5 = 0$

This cannot be factorised, so the quadratic formula is used.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25 + 20}}{2}$$

$$= \frac{5 \pm \sqrt{45}}{2}$$

$$\therefore x = \frac{5 + \sqrt{45}}{2} \text{ or } \frac{5 - \sqrt{45}}{2}$$

**2** Expand and gather the terms to the LHS.

$x(x - 5) = 6$

$x^2 - 5x = 6$

$x^2 - 5x - 6 = 0$

Factorising gives:

$(x - 6)(x + 1) = 0$

$\therefore x = 6$ or $-1$

**3** $x = \dfrac{5x - 6}{x}$

Multiplying both sides by $x$ gives:

$x^2 = 5x - 6$

$x^2 - 5x + 6 = 0$

Factorising gives:

$(x - 2)(x - 3) = 0$

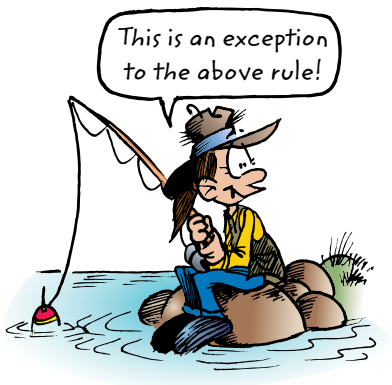
$\therefore x = 2$ or $3$

When solving a quadratic equation:

Step 1: Express the equation in general form:
$ax^2 + bx + c = 0$

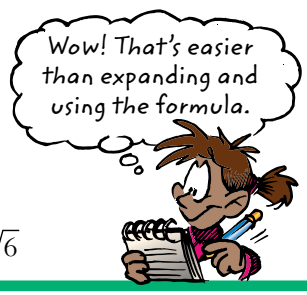
Step 2: Factorise, if you can, and solve it.
or
use the formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## WORKED EXAMPLE




$(a + 7)^2 = 6$

For equations like this, where one side is a perfect square, it is easier to follow the final steps in the method of completing the square.




### Solution

$(a + 7)^2 = 6$
$\therefore\ a + 7 = \pm\sqrt{6}$
$a = -7 \pm \sqrt{6}$
$a = -7 + \sqrt{6}$ or $-7 - \sqrt{6}$

## Exercise 2:04

**1** Solve the following quadratic equations. Give answers to two decimal places if necessary.

**a** $x^2 + 7x + 6 = 0$
**b** $x^2 - 8x + 12 = 0$
**c** $x^2 + 5x - 24 = 0$
**d** $x^2 - 3x + 1 = 0$
**e** $x^2 + 3x - 3 = 0$
**f** $x^2 + 4x + 2 = 0$
**g** $x^2 + 8x = 0$
**h** $x^2 - 10x = 0$
**i** $5x^2 - 10x = 0$
**j** $x^2 - 81 = 0$
**k** $x^2 - 121 = 0$
**l** $4x^2 - 9 = 0$
**m** $2x^2 + 4x + 1 = 0$
**n** $3x^2 - x - 1 = 0$
**o** $2x^2 - 5x + 1 = 0$
**p** $2x^2 + 6x + 4 = 0$
**q** $3x^2 + 15x + 18 = 0$
**r** $2x^2 - 6x - 8 = 0$
**s** $x^2 = 6x + 27$
**t** $x^2 = 13x - 36$
**u** $2x^2 - 5x = 12$
**v** $25 = 10x - x^2$
**w** $36 = 13x - x^2$
**x** $2 = 9x - 5x^2$

**2** Rearrange each equation below into the form $ax^2 + bx + c = 0$, and solve.

**a** $x^2 + 9x = 2x - 12$
**b** $x^2 + 20 = 8x + 5$
**c** $x^2 - 4x + 10 = 2x + 2$
**d** $3x^2 + 5x = 2x^2 - 6$
**e** $4x^2 + 5x = 3x^2 - 2x$
**f** $x^2 + 3x - 10 = 3x - 1$
**g** $x^2 + 5x = 3x + 1$
**h** $x^2 + 7 = 5 - 4x$
**i** $2x + 1 = x^2 + x$
**j** $x(x + 5) = 6$
**k** $x(x - 7) = 18$
**l** $x^2 = 4(x + 8)$
**m** $(m - 1)^2 = 4$
**n** $(x + 3)^2 = 9$
**o** $(x + 5)^2 = 11$
**p** $(2a + 1)^2 = 16$
**q** $(5y - 3)^2 = 7$
**r** $(6n - 7)^2 = 3$
**s** $x = \dfrac{2x + 15}{x}$
**t** $x = \dfrac{3x + 28}{x}$
**u** $1 = \dfrac{2 - x^2}{x}$
**v** $x = \dfrac{5x - 3}{x}$
**w** $\dfrac{3(x + 1)}{x} = x$
**x** $2(x + 2) = \dfrac{1}{x}$

## FUN SPOT 2:04 WHAT IS AN ITALIAN REFEREE?

Work out the answer to each question and write the letter for that part in the box that is above the correct answer.

*Solve:*

**A** $(x + 3)(x - 5) = 0$
**R** $(2x - 1)(x - 7) = 0$
**M** $(x + 1)(x - 1) = 0$
**R** $(3x + 5)(2x - 3) = 0$

*Solve by factorising:*

**M** $x^2 - 5x + 6 = 0$
**E** $x^2 - 5x - 6 = 0$
**A** $x^2 - x - 6 = 0$
**I** $x^2 + x - 6 = 0$

*Solve:*

**N** $(x - 2)^2 = 5$
**O** $(x + 7)^2 = 9$
**P** $(x + 1)^2 = 2$
**U** $(x + 3)^2 = 3$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | |
| $-2, 3$ | $\frac{1}{2}, 7$ | $-4, -10$ | $2, 3$ | $-3, 5$ | $2 \pm \sqrt{5}$ | $-3 \pm \sqrt{3}$ | $-1, 1$ | $-1 \pm \sqrt{2}$ | $-3, 2$ | $-\frac{5}{3}, \frac{3}{2}$ | $-1, 6$ |

# 2:05 Problems involving quadratic equations

## PREP QUIZ 2:05

From the list of numbers, $1, 2, 3\frac{1}{2}, 5{\cdot}2, 9, 10$, write the numbers that are:

**1** integers
**2** odd
**3** square

Write the next two consecutive integers after:

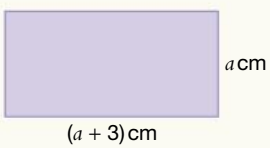
**4** 8
**5** $n$

Write the next two consecutive even numbers after:

**6** 10
**7** $x$ ($x$ is even)

Write expressions for:

**8** a number that is 3 less than $x$
**9** the area of this rectangle
**10** the perimeter of this rectangle.



When solving a problem or applying a given formula, a quadratic equation may be involved. Consider the following examples.

## WORKED EXAMPLES

1 The product of two consecutive positive even numbers is 48. Find the numbers.
(Hint: If the first number is $x$, then the next even number is $x + 2$).

2 The length of a rectangle is 5 cm longer than its breadth. If the area of the rectangle is 84 $cm^2$, find the length of the rectangle.

3 A projectile is fired vertically upwards and its height $h$, in metres, after $t$ seconds is given by the formula:
$h = 40t - 8t^2$
Find the time taken by the projectile to first reach a height of 48 m.

### Solutions

1 The problem gives the equation:
$x(x + 2) = 48$

Solving this gives:
$x^2 + 2x = 48$
$x^2 + 2x - 48 = 0$
$(x + 8)(x - 6) = 0$
$\therefore x = -8$ or $6$

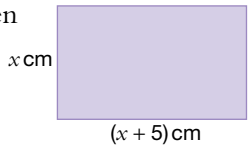
The numbers are positive, so $x = 6$.
$\therefore$ The two consecutive numbers are 6 and 8.

3 $h = 40t - 8t^2$
To find the time $t$ when the height $h = 48$ metres, substitute into the formula:
$48 = 40t - 8t^2$

Gathering all the terms on the LHS:
$8t^2 - 40t + 48 = 0$
$8(t^2 - 5t + 6) = 0$
$8(t - 2)(t - 3) = 0$
$\therefore t = 2$ or $3$

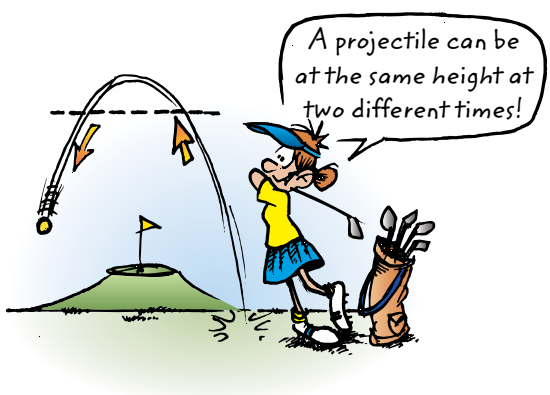
$\therefore$ The projectile was at a height of 48 m at 2 seconds and 3 seconds. So it took 2 seconds to first reach a height of 48 m.

2 If the breadth is $x$, then the length is $x + 5$. Since the area of a rectangle is equal to length times breadth, then:




$x(x + 5) = 84$
$x^2 + 5x = 84$
$x^2 + 5x - 84 = 0$
$(x + 12)(x - 7) = 0$
$\therefore x = -12$ or $7$

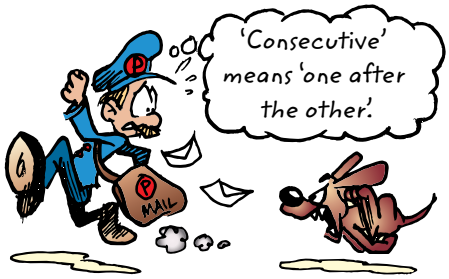
Now, since the dimensions must be positive, the breadth must be 7 cm.
$\therefore$ The length is 12 cm.



## Exercise 2:05

**1** Find the two consecutive positive integers:

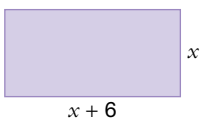
- **a** if their product is 20
- **b** if their product is 90
- **c** if they are even and their product is 120
- **d** if they are odd and their product is 63.



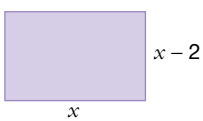

**2** **a** The sum of a positive integer and its square is 90. Find the number.

- **b** The sum of a positive integer and its square is 132. Find the number.
- **c** The difference between a positive integer and its square is 56. Find the number.
- **d** The square of a number is equal to 5 times the number. What are the two possible numbers?
- **e** When a number is subtracted from its square, the result is 42. Find the two possible numbers.

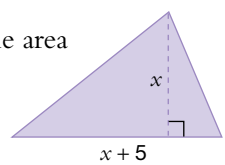
**3** **a** Find the dimensions of this rectangle if the length is 6 cm longer than the breadth and its area is 40 cm$^2$.



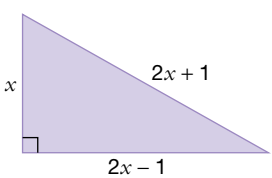

- **b** The breadth of a rectangular room is 2 m shorter than its length. If the area of the room is 255 m$^2$, find the dimensions of the room.




- **c** The base of a triangle is 5 cm longer than its height. If the area of the triangle is 7 cm$^2$, find the length of the base.




- **d** A right-angled triangle is drawn so that the hypotenuse is twice the shortest side plus 1 cm, and the other side is twice the shortest side less 1 cm. Find the length of the hypotenuse.




**4** **a** Michelle threw a ball vertically upwards. Its height $h$ metres after time $t$ seconds is given by the formula:

$h = 8t - t^2$

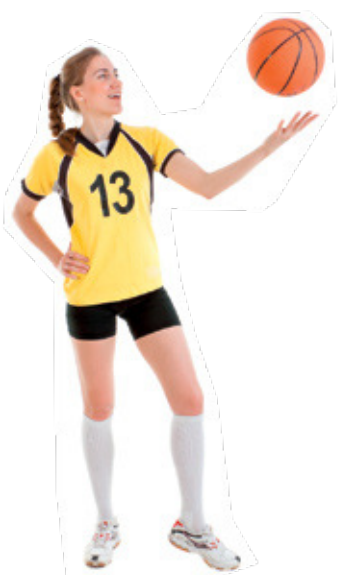
Find after what time the ball was first at a height of 12 m.

- **b** The sum, $S$, of the first $n$ positive integers is given by the formula:

$S = \frac{n}{2}(n + 1)$

Find the number of positive integers needed to give a total of 78.

- **c** For the formula $s = ut + \frac{1}{2}at^2$, find the values of $t$ if:
  - **i** $s = 18$, $u = 7$, $a = 2$
  - **ii** $s = 6$, $u = 11$, $a = 4$
  - **iii** $s = 7$, $u = 1$, $a = 6$



**5** An $n$-sided polygon has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a figure if it has 90 diagonals?

**6** Jenny is $y^2$ years old and her daughter Allyson is $y$ years old. If Jenny lives to the age of $13y$, Allyson will be $y^2$ years old. How old is Allyson now? (*Note:* the difference in ages must remain constant.)

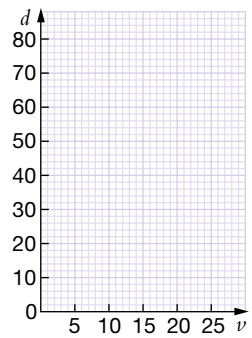
**7** Kylie bought an item for $\$x$ and sold it for \$10.56. If Kylie incurred a loss of $x$ per cent, find $x$.

**8** A relationship that is used to approximate car stopping distances ($d$) in ideal road and weather conditions is: $d = t_r v + kv^2$ where $t_r$ is the driver's reaction time, $v$ is the velocity and $k$ is a constant.

**a** Stirling's reaction time was measured to be 0·8 seconds. The distance it took him to stop while travelling at 20 m/s (72 km/h) was 51 m. Substitute this information into the formula to find the value of $k$.

**b** If, for these particular conditions, Stirling's breaking distance is given by $d = 0{\cdot}8v + 0{\cdot}0875v^2$ complete the table below, finding $d$ correct to the nearest metre in each case.

| $v$ (m/s) | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $d$ (m) | | | | | | |



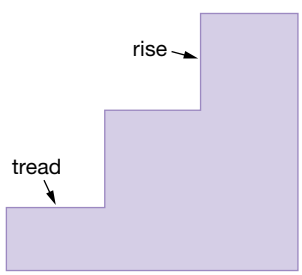

**c** Graph $d$ against $v$ using the number plane shown on the right. What kind of curve is produced?

**d** Use your graph to find the velocity (in m/s) that would produce a stopping distance of 40 m. Check your accuracy by solving the equation $40 = 0{\cdot}8v + 0{\cdot}0875v^2$ using the formula

$$v = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

**e** What factors would determine the safe car separation distance in traffic?

**9** The rise and tread of a staircase have been connected using the formula $r = \frac{1}{2}(24 - t)$, or $r = \frac{66}{t}$ where $r$ and $t$ are measured in inches (1 inch is about 2·54 cm).




**a** If the tread should not be less than 9 inches, what can be said about the rise?

**b** Graph both functions on the same set of axes and compare the information they provide.

**c** What are the points of intersection of the two graphs? Check the accuracy of your graphs by solving the two simultaneous equations:

$$r = \frac{1}{2}(24 - t) \qquad 1$$

$$r = \frac{66}{t} \qquad 2$$

(Hint: Substitute 2 into 1 and solve the resulting quadratic equation.)

**d** Convert the formulas 1 and 2 to centimetres rather than inches.

**e** Do the measurements of staircases you have experienced fit these formulas? Investigate other methods used by builders to determine $r$ and $t$.

## INVESTIGATION 2:05 TEMPERATURE AND ALTITUDE

The following formula has been used to give the boiling point of water at various heights above or below sea-level:

$h = 520(212 - T) + (212 - T)^2$

where height, $h$, is in feet and the temperature, $T$, is in degrees Fahrenheit (°F).

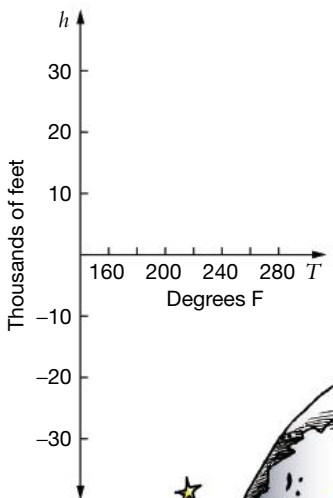
**1** Show that $h = (212 - T)(732 - T)$.

**2** At what height above sea-level does water boil at:
- **a** 200°F
- **b** 250°F?

**3** Plot a graph of $h$ against $T$. (Use values of $T$ from 160°F to 280°F. Use values of $h$ from −30 000 feet to 30 000 feet.)

| $T$ (°F) | 160 | 180 | 200 | 220 | 240 | 260 | 280 |
|---|---|---|---|---|---|---|---|
| $h$ (feet) | 29 744 | | | | | | −30 736 |

**4** Use your graph to find the temperature at which water boils, to the nearest degree:
- **a** at Flinders Peak (1155 feet above sea-level)
- **b** atop Mt Everest (29 000 feet above sea-level)
- **c** on the shore of the Dead Sea (1300 feet below sea level).




**5** Check your answers to Question **4** by substituting each height into the formula and solving the resulting quadratic equation.

**6** Change the units on the axes of your graph so that they are in degrees Celsius (°C) and thousands of metres. To do this, use the formula $C = \frac{5}{9}(F - 32)$ and the approximation, 1 foot = 0·305 metres.

**7** Discuss:
- Over what temperature range is this formula useful or valid?
- Find the height at which the International Space Station orbits. Can the formula be used there? Is the station pressurised to some equivalent height above sea-level?

# 2:06 Equations reducible to quadratics

## PREP QUIZ 2:06

Solve these equations:

**1** $2a(a - 3) = 0$ **2** $(2a + 1)(a - 5) = 0$ **3** $(3a - 4)(5a + 2) = 0$

**4** $a^2 - 5a + 6 = 0$ **5** $a^2 + 14a + 49 = 0$ **6** $a^2 - a - 12 = 0$

**7** $3a^2 + 7a + 2 = 0$ **8** $16a^2 - 24a + 9 = 0$

Solve for $x$:

**9** $x^2 - 16 = 0$ **10** $x^4 - 16 = 0$

Some equations that appear more difficult to factorise may be changed into a quadratic equation by substituting an appropriate pronumeral.

For example: $x^4 - 13x^2 + 36 = 0$
If we let $u = x^2$ and then substitute, the equation becomes: $u^2 - 13u + 36 = 0$

This equation can now be solved by factorising:
$(u - 9)(u - 4) = 0$
$\therefore u = 9$ or $u = 4$

But $u = x^2$ so $x^2 = 9$ or $x^2 = 4$
$\therefore x = \pm 3$ or $x = \pm 2$

So the equation $x^4 - 13x^2 + 36 = 0$ has four solutions, $x = -3, -2, 2$ and $3$.

## WORKED EXAMPLE 1

By making suitable substitutions, solve these equations.

**a** $3a^4 - 14a^2 + 8 = 0$
Let $u = a^2$
So equation becomes:
$3u^2 - 14u + 8 = 0$
$(3u - 2)(u - 4) = 0$
$\therefore u = \frac{2}{3}$ or $4$
But $u = x^2$, so
$x^2 = \frac{2}{3}$ or $4$
$\therefore x = \pm\sqrt{\frac{2}{3}}$ or $\pm 2$

**b** $9^x - 28(3^x) + 27 = 0$
Let $u = 3^x$
So equation becomes:
$u^2 - 28u + 27 = 0$
$(u - 27)(u - 1) = 0$
$\therefore u = 27$ or $1$
But $u = 3^x$, so
$3^x = 27$ or $1$
$3^3 = 27$ or $3^0 = 1$
$\therefore x = 3$ or $0$

*Note:* $9^x = (3^x)^2$

## WORKED EXAMPLE 2

By making suitable substitutions, solve these equations.

**a** $m^6 - 7m^3 - 8 = 0$

Let $X = m^3$

So the equation becomes:

$X^2 - 7X - 8 = 0$

$(X - 8)(X + 1) = 0$

$\therefore X = 8$ or $-1$

But $X = m^3$, so

$m^3 = 8$ or $-1$

$\therefore m = 2$ or $-1$

**b** $y^4 - 7y^2 - 18 = 0$

Let $A = y^2$

So the equation becomes:

$A^2 - 7A - 18 = 0$

$(A - 9)(A + 2) = 0$

$\therefore A = 9$ or $-2$

But $A = y^2$, so

$y^2 = 9$ or $-2$ ($y^2 \not\uparrow$ negative number)

$\therefore y = \pm 3$ are the only solutions

## Exercise 2:06

**1** Use the substitution given to reduce each equation to a quadratic equation and then solve.

**a** $x^4 - 5x^2 + 4 = 0$, $u = x^2$

**b** $a^4 - 10a^2 + 9 = 0$, $u = a^2$

**c** $n^4 - 29n^2 + 100 = 0$, $u = n^2$

**d** $k^4 - 8k^2 + 16 = 0$, $X = k^2$

**e** $m^4 - 6m^2 + 8 = 0$, $A = m^2$

**f** $z^4 - 14z^2 + 45 = 0$, $Y = z^2$

**2** By using the substitution given, solve these equations.

**a** $4x^4 - 5x^2 + 1 = 0$, $u = x^2$

**b** $9a^4 - 37a^2 + 4 = 0$, $u = a^2$

**c** $3n^4 - 28n^2 + 9 = 0$, $u = n^2$

**d** $8k^4 - 22k^2 + 9 = 0$, $X = k^2$

**3** Reduce each equation to a quadratic equation, using the substitution given, and then solve.

**a** $y^6 - 9y^3 + 8 = 0$, $u = y^3$

**b** $m^8 - 17m^4 + 16 = 0$, $u = m^4$

**c** $x^6 - 1008x^3 + 8000 = 0$, $X = x^3$

**d** $n^8 - 82n^4 + 81 = 0$, $A = n^4$

**4** By using the substitution given, solve these *exponential* equations.

**a** $4^x - 5(2^x) + 4 = 0$, $u = 2^x$ [*Note:* $4^x = (2^x)^2$]

**b** $9^x - 4(3^x) + 3 = 0$, $u = 3^x$

**c** $4^x - 12(2^x) + 32 = 0$, $X = 2^x$

**d** $9^x - 12(3^x) + 27 = 0$, $A = 3^x$

**5** By using the substitution given, solve these equations.

**a** $x^4 + 5x^2 - 6 = 0$, $u = x^2$

**b** $x^4 - 5x^2 + 6 = 0$, $u = x^2$

**c** $x^4 - 5x^2 - 6 = 0$, $u = x^2$

**d** $x^4 + 5x^2 + 6 = 0$, $u = x^2$

**e** $x^4 - 7x^2 + 10 = 0$, $u = x^2$

**f** $a^4 - 2a^2 - 15 = 0$, $u = a^2$

**6** Substitute $u$ for $(x - 1)^2$ to solve the equation: $(x - 1)^4 - 29(x - 1)^2 + 100 = 0$

**7** Solve the equation: $(a + 2)^4 - 20(a + 2)^2 + 64 = 0$

**8** Solve these equations using the substitution $X = x^2 - 2x$.

**a** $(x^2 - 2x)^2 - 23(x^2 - 2x) + 120 = 0$

**b** $(x^2 - 2x)^2 + 3(x^2 - 2x) - 40 = 0$

## FUN SPOT 2:06 DID YOU KNOW THAT 2 = 1?

Now that your algebra skills are more developed, you can be let into the secret that 2 really is equal to 1.

*Proof:* Assume that $x = y$.

| | |
|---|---|
| Multiply both sides by $x$: | $x^2 = xy$ |
| Subtract $y^2$ from both sides: | $x^2 - y^2 = xy - y^2$ |
| Factorise both sides: | $(x - y)(x + y) = y(x - y)$ |
| Divide by $(x - y)$: | $x + y = y$ |

If we let $x = y = 1$, then $2 = 1$. QED

## MATHS TERMS 2

**coefficient**

- the number that multiplies a pronumeral in an equation or algebraic expression,
  e.g. $3x^2 - x + 5 = 0$
  coefficient of $x^2$ is 3
  coefficient of $x$ is $-1$

**completing the square**

- completing an algebraic expression to form a perfect square, i.e. $(x + a)^2$ or $(x - a)^2$,
  e.g. to complete the square for $x^2 + 6x$, the number 9 is added:
  $x^2 + 6x + 9 = (x + 3)^2$

**factorise**

- to write an expression as the product of its factors
- the reverse of expand

**quadratic equation**

- an equation in which the highest power of the unknown pronumeral is 2,
  e.g. $x^2 - 16 = 0$, $x^2 + 5x + 6 = 0$
- a quadratic equation may have two, one or no solutions

**quadratic formula**

- a formula that gives the solutions to equations of the form $ax^2 + bx + c = 0$
- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## DIAGNOSTIC TEST 2 QUADRATIC EQUATIONS

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** Solve these equations, giving the answers correct to three decimal places if necessary. — 2:01

a $x^2 = 49$ b $a^2 = 7$ c $m^2 - 16 = 0$ d $3y^2 - 5 = 0$

**2** Solve these equations: — 2:01

a $(x + 7)(x - 3) = 0$ b $x(x - 5) = 0$
c $(2x - 1)(x + 1) = 0$ d $(3x + 2)(4x - 5) = 0$

**3** Factorise and solve: — 2:01

a $x^2 + 5x = 0$ b $x^2 + 9x + 14 = 0$
c $x^2 - 49 = 0$ d $2x^2 + 5x - 3 = 0$

**4** What number must be inserted to complete the square? — 2:02

a $x^2 + 6x + \ldots$ b $x^2 - 4x + \ldots$
c $x^2 + 3x + \ldots$ d $x^2 - x + \ldots$

**5** Solve the following by completing the square. — 2:05

a $x^2 + 2x - 2 = 0$ b $x^2 - 6x + 1 = 0$
c $x^2 - 3x - 5 = 0$ d $2x^2 - 10x = 1$

**6** Solve using the quadratic formula. (Leave answers in surd form.) — 2:03

a $x^2 + x - 3 = 0$ b $x^2 - 5x + 2 = 0$
c $2x^2 + 4x + 1 = 0$ d $3x^2 + 2x - 2 = 0$

**7** Solve the following: — 2:04

a $x^2 - x + 1 = 4x + 7$ b $x(x - 5) = x - 9$
c $(x + 4)^2 = 6$ d $x = \dfrac{2x + 8}{x}$

**8** Solve these equations using the substitutions given. — 2:06

a $x^4 - 20x^2 + 64 = 0, u = x^2$ b $a^4 - 11a^2 + 18 = 0, u = a^2$

One of these gear wheels has 28 teeth and the other has 29 teeth. How many revolutions of each wheel must be completed for the same two teeth to be in the same position next to each other?

## ASSIGNMENT 2A Chapter review

**1** Solve the following quadratic equations using the method you feel is most appropriate.

**a** $x^2 + x - 30 = 0$

**b** $x(x - 7) = 0$

**c** $(x + 1)^2 = 9$

**d** $2x^2 + 7x - 15 = 0$

**e** $x^2 + 2x = 24$

**f** $(x + 2)(3x - 1) = 0$

**g** $x^2 - 14x + 49 = 0$

**h** $5x(2x - 3) = 0$

**i** $x^2 - 100 = 0$

**j** $x^2 - 5x - 14 = 0$

**k** $x^2 = 28 + 3x$

**l** $x^2 + 5x + 1 = 0$

**m** $(x - 3)^2 = 2$

**n** $10x^2 - 3x - 1 = 0$

**o** $x^2 - 20 = 0$

**p** $5x^2 + 3x = 0$

**q** $x^2 + 10x + 25 = 0$

**r** $x^2 + 2x - 4 = 0$

**s** $3x^2 - x - 2 = 0$

**t** $2x^2 + 5x + 1 = 0$

**u** $x(x + 5) = 24$

**v** $(x + 2)^2 = x + 2$

**w** $x = \dfrac{5x - 2}{x}$

**x** $(2x + 1)^2 = (x + 3)^2$

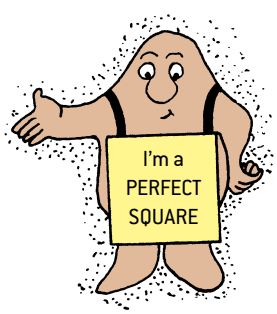
**2** Solve by completing the square.

**a** $x^2 + 4x - 32 = 0$

**b** $x^2 - 3x - 40 = 0$

**c** $x^2 - 10x + 4 = 0$

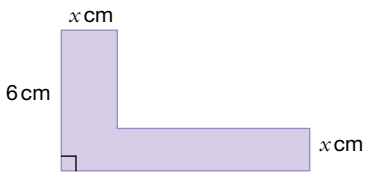
**d** $2x^2 + 6x - 3 = 0$




**3** Find three consecutive positive integers if the sum of their squares is 50.

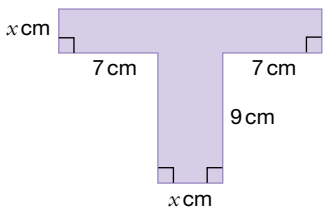
**4** Find $x$ in the following figures.

**a**




Area = 50 cm$^2$

**b**




Area = 210 cm$^2$

**5** If a rectangular field has an area of 0·28 ha and its length is 30 m more than its width, find the width of the field.

**6** Solve each equation using the substitution given.

**a** $x^4 - 26x^2 + 25 = 0$, $u = x^2$

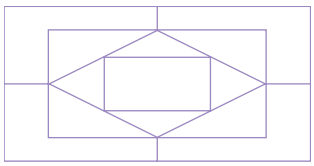
**b** $n^4 - 12n^2 + 27 = 0$, $u = n^2$

**c** $z^4 - 5z^2 - 12 = 0$, $u = z^2$

## ASSIGNMENT 2B Working mathematically

1 An odd number between 301 and 370 has three different digits. If the sum of its digits is five times the hundreds digit, what is the digit?

2 What is the minimum number of colours needed to shade this diagram if no two adjacent regions may have the same colour?



3 Emma's passbook savings account allowed her to deposit or withdraw at any time. Her interest, which was 2·5% pa, was calculated on the minimum monthly balance and was paid twice yearly into her account. She could withdraw up to $500 in cash per day or any amount in the form of a cheque. Cheques for the payment of bills (third party cheques) were provided free of charge. She was able to start her account with as little as $1.

a What is the minimum balance required?

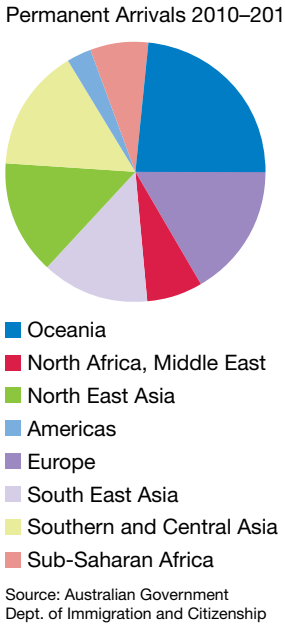
b On what amount is the interest calculated?

c Does the interest earned in one month automatically begin to earn interest during the next month?

4 Decrease $360 by 20% and then increase the result by 20%. What is the difference between $360 and your final answer?

5 50% more than what number is 25% less than 60% more than 10?

6 The immigration to Australia for 2010–2011 is shown by this pie chart.




a Which region provided:

i the most immigrants

ii the fewest immigrants?

b Measure the angle of each sector to determine the percentage of immigrants from:

i South East Asia

ii Europe.

c If the total number of immigrants was 126 536, how many (to the nearest hundred) came from:

i the Americas

ii Oceania?

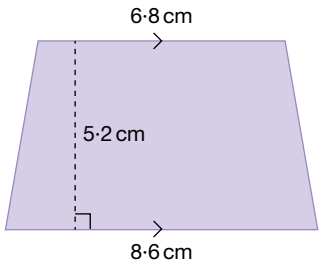
## ASSIGNMENT 2C Cumulative revision

**1** **a** Find 15% of \$125. 1:01D
**b** What percentage is \$25 of \$125?
**c** Decrease a price of \$125 by 30%.
**d** 15% discount of a price is equal to \$24. What is the full price?

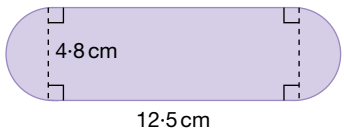
**2** Simplify the following expressions. 1:02
**a** $5x - 2y - x + y$ **b** $10ax \div 5a$ **c** $\frac{2a}{3} + \frac{3a}{5}$ **d** $\frac{2a}{3x} \surd \frac{4a}{9y}$

**3** A card is drawn from a standard pack of 52 playing cards. 1:03
What is the probability the card is:
**a** red **b** a club **c** a Jack **d** the 7 of spades?

**4** Find the area of each shape. 1:04
**a**




**b**

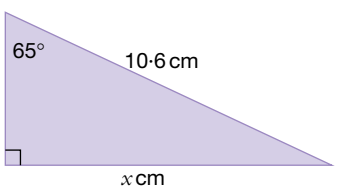


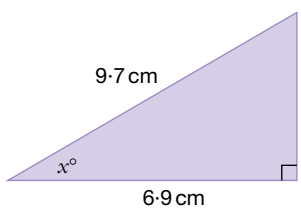

**5** Evaluate: 1:05
**a** $5^2 \times 2^5$ **b** $4^5 \div 4^4$ **c** $4^{-2}$ **d** $(2^3)^{-1}$ **e** $8^{\frac{2}{3}}$

**6** Simplify each expression: 1:06
**a** $3\sqrt{18}$ **b** $4\sqrt{8} - \sqrt{32}$ **c** $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$

**7** Yvonne is paid an hourly rate of \$28.40 for a 36 hour week. The first 6 hours overtime are paid at time-and-a-half; after that extra hours worked are paid double-time. Find Yvonne's wage for a week in which she works 45 hours. 1:08

**8** For these triangles, find: 1:12
**a** the value of $x$ to one decimal place **b** the value of $x$ to the nearest degree.

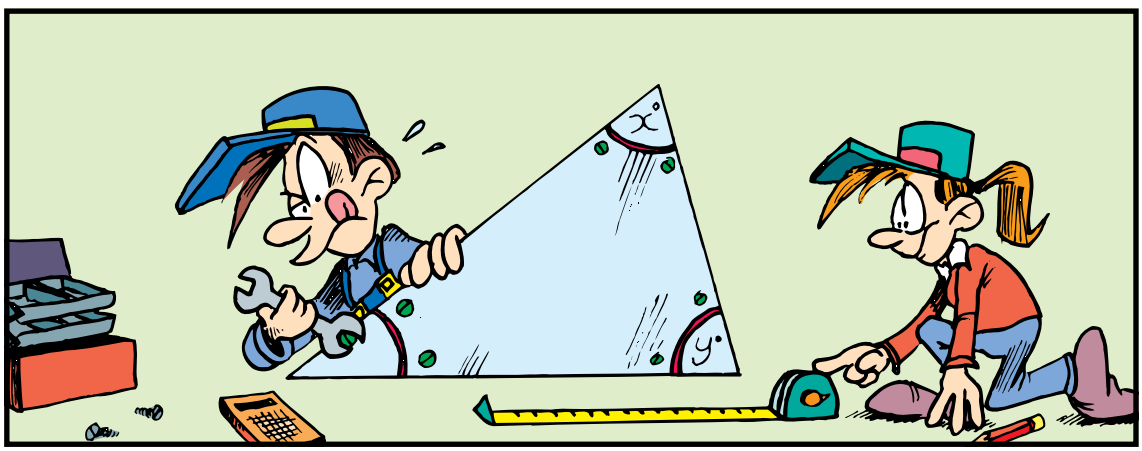



**9** For the set of scores 3, 5, 4, 7, 5, 4, 8, 3, 4, find the: 1:13
**a** range **b** mode **c** median **d** mean **e** $Q_1$ **f** $Q_3$
**g** interquartile range.

# PROPERTIES OF GEOMETRICAL FIGURES

3




## Contents


**3:01** Deductive reasoning in numerical exercises
- **3:01A** Exercises using parallel lines
- **3:01B** Exercises using triangles
- **3:01C** Exercises using quadrilaterals

**3:02** Congruent triangle proofs

**3:03** Deducing properties of the special triangles and quadrilaterals
- GeoGebra activity 3:03 The parallelogram family

**3:04** Proving general results

**3:05** Definitions and proof in deductive geometry
- Investigation 3:05 Theorems and their converses

**3:06** Polygons
- GeoGebra activity 3:06A Convex and concave quadrilaterals
- Investigation 3:06A The angle sum of a polygon
- GeoGebra activity 3:06B The angle sum of a polygon
- Investigation 3:06B The exterior angle sum of a convex polygon
- GeoGebra activity 3:06C The exterior angle sum of a convex polygon
- Investigation 3:06C Regular polygons and tessellations

Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Measurement and Geometry**

*Selections from* ***Properties of Geometrical Figures*** *[Stage 5.2]*

- Formulate proofs involving congruent triangles and angle properties (ACMMG243)
- Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes (ACMMG244)

### Working Mathematically

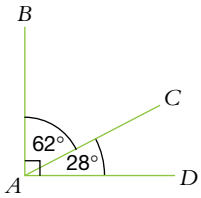
- Problem Solving
- Reasoning
- Understanding
- Fluency
- Communicating

# 3:01 Deductive reasoning in numerical exercises

Throughout Stage 4 many geometrical facts concerning angles, triangles and quadrilaterals were discovered. A summary of these facts follows.

**Adjacent angles**

- have a common vertex (or point), $A$
- have a common arm, $AC$
- lie on opposite sides of this common arm



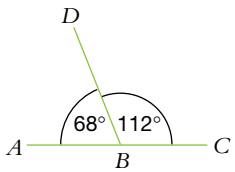

If two adjacent angles add to 90°, then together they form a right angle.

e.g. 28° + 62° = 90°

The complementary angles are adjacent, so $\angle BAD$ is a right angle.

If two adjacent angles add to 180°, then together they form a straight angle.

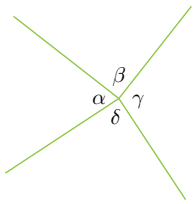
e.g. 68° + 112° = 180°




The supplementary angles are adjacent, so $ABC$ is a straight line.

If a series of adjacent angles completes a revolution, the angles must add to 360°.

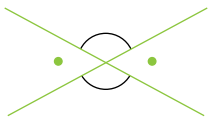
We say that the angle sum at the point is 360°.




Angle sum = 360°

**Vertically opposite angles**

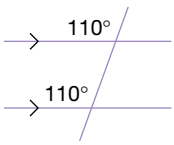
When two straight lines cross, two pairs of vertically opposite angles are formed. Vertically opposite angles are equal.

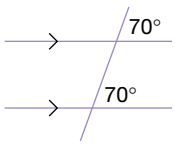


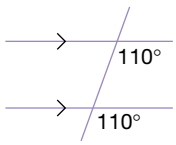
Vertically opposite angles are equal.

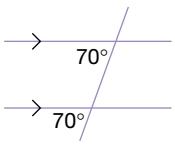
**Corresponding angles**

If a transversal cuts two parallel lines, then corresponding angles are formed. Corresponding angles are equal.



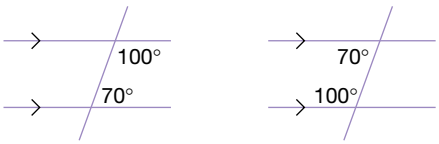



### Alternate angles

If a transversal cuts two parallel lines, then alternate angles are formed. Alternate angles are equal.

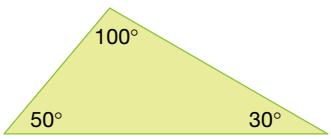
### Co-interior angles

If a transversal cuts two parallel lines, then co-interior angles are formed. Co-interior angles are supplementary.



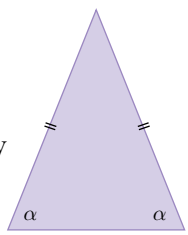

### Angle sum of a triangle

In any triangle, the angles add to 180°.



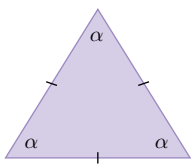

### Isosceles triangles

- have two equal sides
- have two equal angles
- have one axis of symmetry



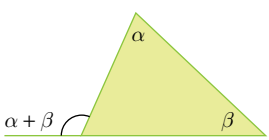

### Equilateral triangles

- have three equal sides
- have three equal angles
- have three axes of symmetry



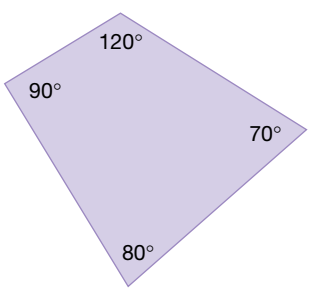

### Exterior angle of a triangle

The exterior angle of a triangle is equal to the sum of the two interior opposite angles.




### Angle sum of a quadrilateral

In any quadrilateral, the angles add to 360°.




In the following exercises you will be asked to find the sizes of unknown sides and angles, and to give a reason for each statement that you make, using the correct terminology. Reasons are usually abbreviated, but take care not to abbreviate them to such an extent that the meaning is lost.

Remember that problems can often be solved in more than one way and that usually one method is more efficient than another.

Some commonly used abbreviations and symbols are shown in the following tables.

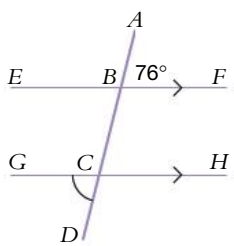
| Term | |
|---|---|
| adjacent | adj. |
| alternate | alt. |
| angles | ∠s |
| co-interior | co-int. |
| complementary | comp. |
| congruent | cong't |
| corresponding | corr. |
| is congruent to | ≡ |
| is simliar to | ||| |
| is parallel to | ‖ |

| Term | |
|---|---|
| is perpendicular to | ⊥ |
| isosceles | isos. |
| equilateral | equi. |
| triangle | Δ |
| parallelogram | par'm |
| point | pt |
| quadrilateral | quad. |
| straight | str. |
| supplementary | supp. |
| vertically opposite | vert. opp. |

# 3:01A Exercises using parallel lines

## WORKED EXAMPLES

**1**

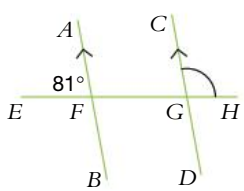



$EF \parallel GH$
Find the size of $\angle GCD$.
Give reasons.

**Solutions**

**1** $\angle EBC = 76°$ (vert. opp. ∠s)
$\angle GCD = 76°$ (corr. ∠s, $EF \parallel GH$)

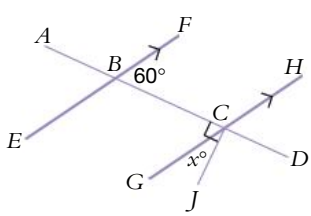
**2**




$AB \parallel CD$
Find the size of $\angle CGH$.
Give reasons.

**2** $\angle FGC = 81°$ (corr. ∠s, $AB \parallel CD$)
$\angle CGH = 99°$ (adj. supp. ∠s)

**3**



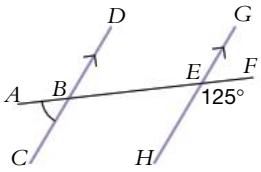

$EF \parallel GH$
Find the size of $x$.
Give reasons.

**3** $\angle BCG = 60°$ (alt. ∠s, $EF \parallel GH$)
$x + 60 = 90$ (comp. ∠s)
$x = 30$

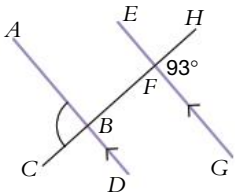
# Exercise 3:01A

**1** Find the size of $\angle ABC$ in each diagram. Give reasons.

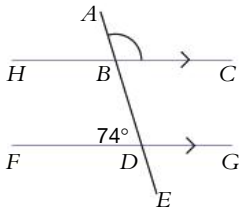
**a**




**b**

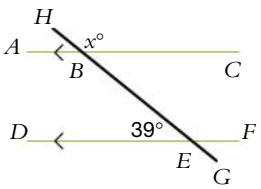



**c**

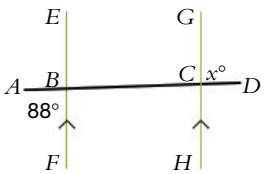



**2** Find the value of $x$ in each diagram. Give reasons.

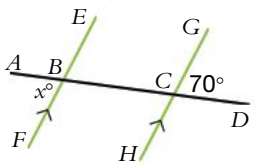
**a**



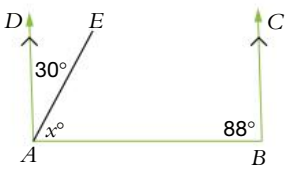

**b**



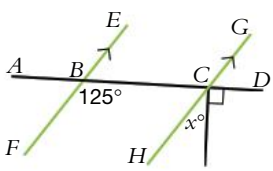

**c**



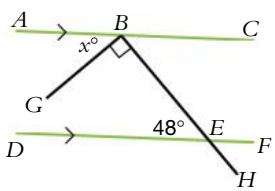

**d**




**e**

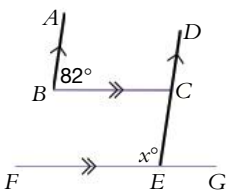



**f**

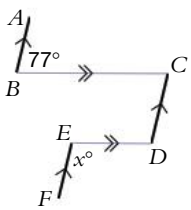



**3** Find the value of $x$ in each diagram. Give reasons.

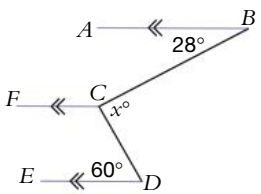
**a**



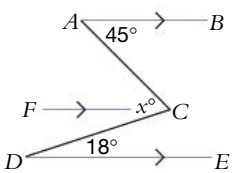

**b**



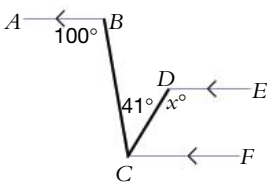

**c**



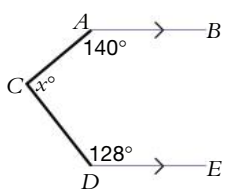

**d**



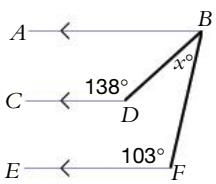

**e**



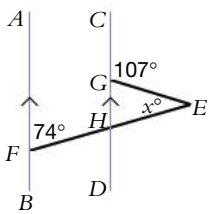

**f**



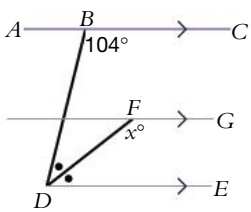

**g**




**h**




**i**

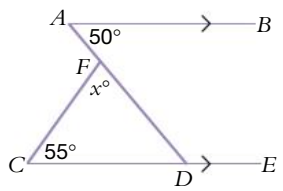



$\angle BDF = \angle FDE$

# 3:01B Exercises using triangles

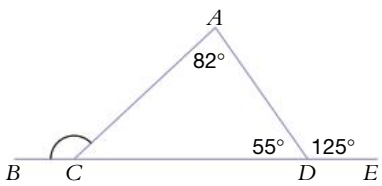
## WORKED EXAMPLES

**1**



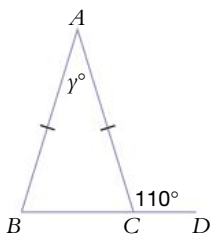

$AB \parallel CE$
Find the size of $x$.
Give reasons.

**2**




Find the size of $\angle BCA$.
Give reasons.

**3**




Find the value of $y$.
Give reasons.

### Solutions

**1**

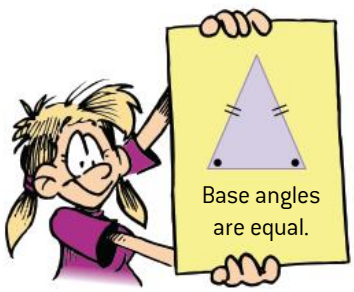
| | | |
|---|---|---|
| | $\angle FDC = 50°$ | (alt. $\angle$s, $AB \parallel CE$) |
| | $x + 55 + 50 = 180$ | ($\angle$ sum of $\Delta$) |
| | $\therefore x = 75$ | |

**2**

| | | |
|---|---|---|
| | $\angle ADC = 55°$ | (adj. supp. $\angle$s) |
| | $\angle BCA = 82° + 55°$ | (ext. $\angle$ of $\Delta$) |
| | $\therefore \angle BCA = 137°$ | |

**3**

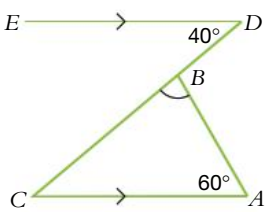
| | | |
|---|---|---|
| | $\angle ACB = 70°$ | (adj. supp. $\angle$s) |
| | $\therefore \angle ABC = 70°$ | (base $\angle$s of isos. $\Delta$) |
| | $y + 70 + 70 = 180$ | (angle sum of $\Delta$) |
| | $\therefore y = 40$ | |



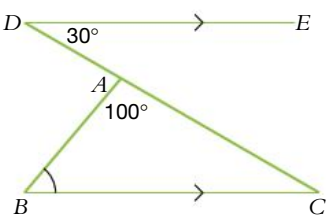

## Exercise 3:01B

**1** Find the size of $\angle ABC$ in each diagram. Give reasons.

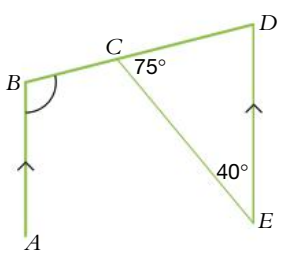
**a**




**b**

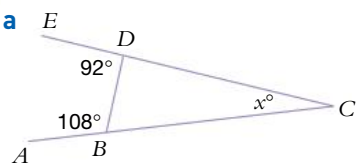



**c**

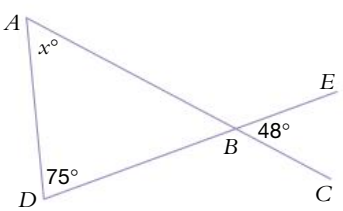



**2** Find the value of $x$ in each diagram. Give reasons.

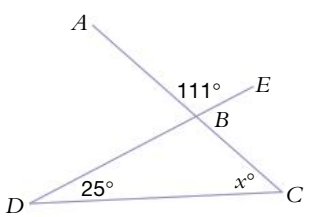
**a**




**b**

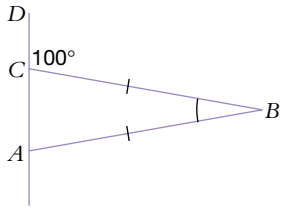



**c**

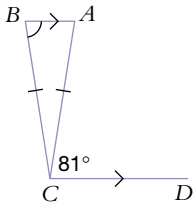


**3** Find the size of $\angle ABC$ in each diagram. Give reasons.

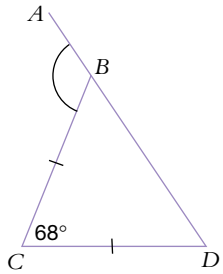
**a**




**b**

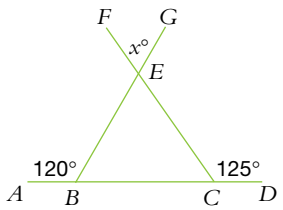



**c**

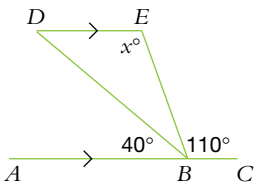



**4** Find the value of $x$ in each diagram. Give reasons.

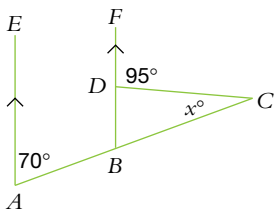
**a**



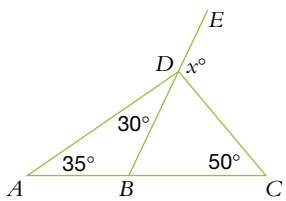

**b**



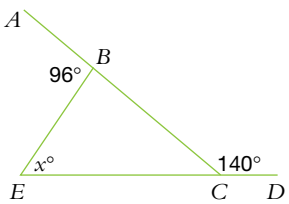

**c**



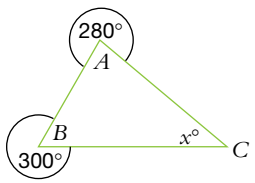

**d**




**e**



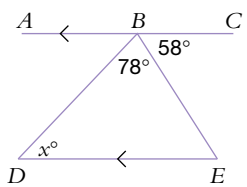

**f**



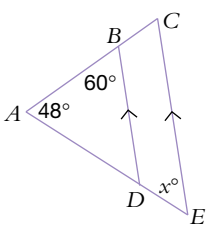

**5** Find the value of $x$ in each diagram. Give reasons.

- Copy the diagram.
- Write the sizes of as many angles as you can.
- Look for a connection.

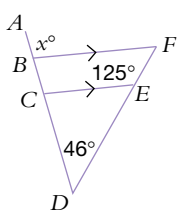
**a**



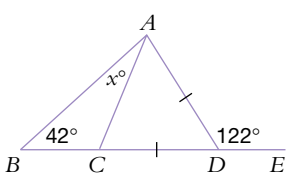

**b**



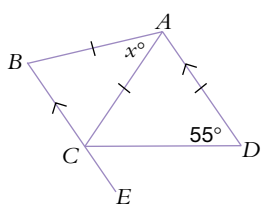

**c**




**d**

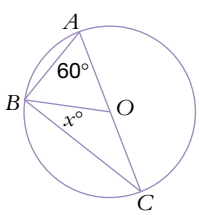



**e**




$AB = AC = AD$

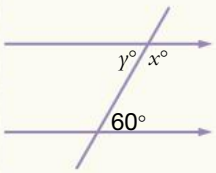
**f**




$O$ is the centre.

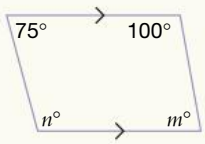
# 3:01C Exercises using quadrilaterals

## PREP QUIZ 3:01C



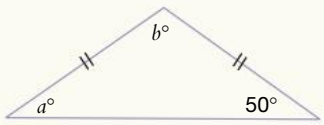

Find the value of:

**1** $x$ **2** $y$




Find the value of:

**3** $m$ **4** $n$

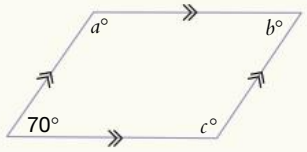



Find the value of:

**5** $a$ **6** $b$

For the parallelogram shown, find the value of:

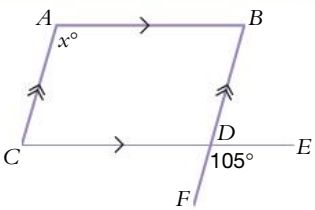
**7** $a$ **8** $b$ **9** $c$




**10** Complete: The angle sum of a quadrilateral is …

## WORKED EXAMPLES

**1**

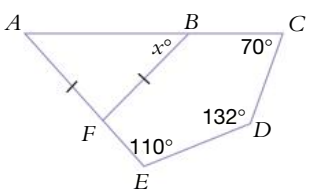



$ABDC$ is a parallelogram.
Find the size of $x$. Give reasons.

### Solution

$\angle BDC = x°$ (opp. $\angle$s of a par'm)
$\angle BDC = 105°$ (vert. opp. $\angle$s)
$\therefore\ x = 105$

**2**

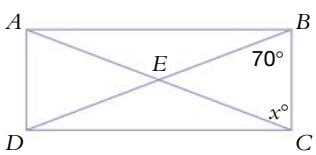



$AF = BF$
Find the size of $x$. Give reasons.

### Solution

$\angle EAC = 360° - (70° + 132° + 110°)$
($\angle$ sum of quad.)
$= 48°$
$\therefore\ x = 48$ (base $\angle$s of isos. $\Delta ABF$)

**3**




$ABCD$ is a rectangle.
Find the value of $x$. Give reasons.

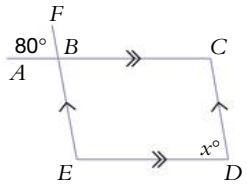
### Solution

$EB = EC$ (diagonals of a rectangle bisect each other and are equal)
$\therefore\ x = 70$ (base $\angle$s of isos. $\Delta BEC$)

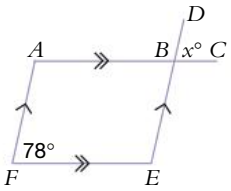
# Exercise 3:01C

**1** Find the value of $x$ in each diagram. Give reasons.

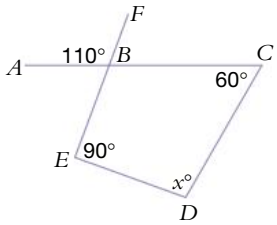
**a**



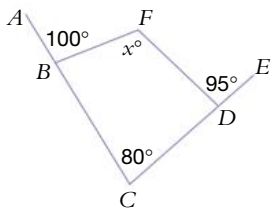

**b**



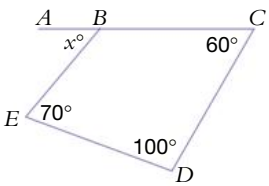

**c**



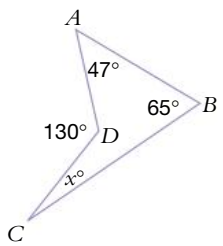

**d**




**e**

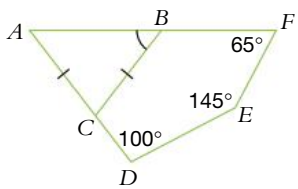



**f**

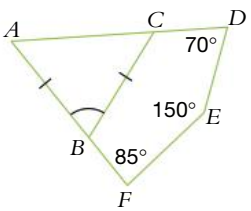



**2** Find the value of $\angle ABC$ in each diagram. Give reasons.

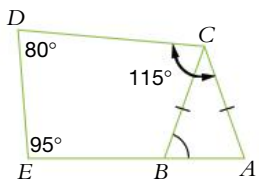
**a**



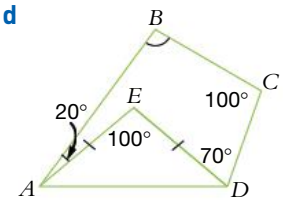

**b**



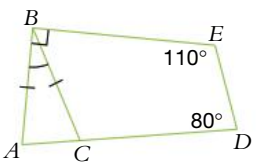

**c**



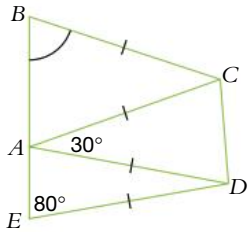

**d**




**e**

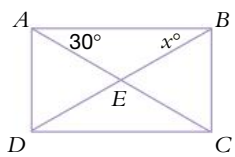



**f**

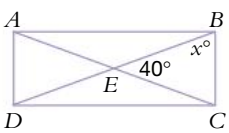



**3** $ABCD$ is a rectangle. Find the value of $x$. Give reasons.

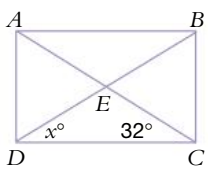
**a**




**b**

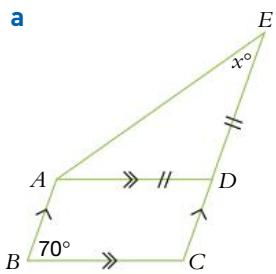



**c**

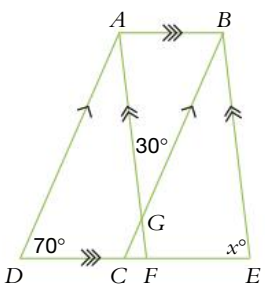



**4** Find the value of $x$. Give reasons.

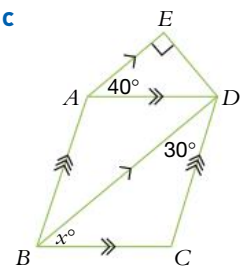
**a**




**b**



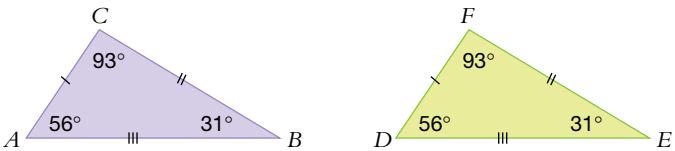

**c**



# 3:02 Congruent triangle proofs

Congruent triangles can be superimposed on one another using one or more translations, reflections or rotations.

When the angles and sides of congruent triangles are matched according to size, three equal pairs of matching sides and angles result.

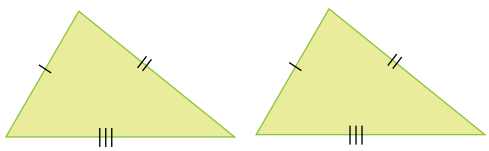



Matching angles: $\angle A$ and $\angle D$, $\angle B$ and $\angle E$, $\angle C$ and $\angle F$

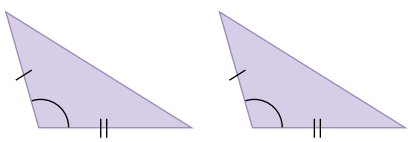
Matching sides: $AB$ and $DE$, $AC$ and $DF$, $BC$ and $EF$

To prove that two triangles are congruent, one of four congruence tests is used. Each test requires that three pieces of information about the two triangles be compared.

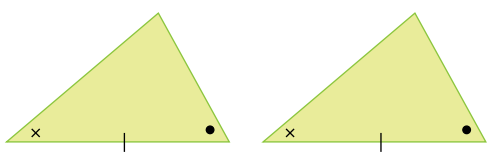
**1** SSS (side, side, side)
Two triangles are congruent if three sides of one triangle are equal to three sides of the other.

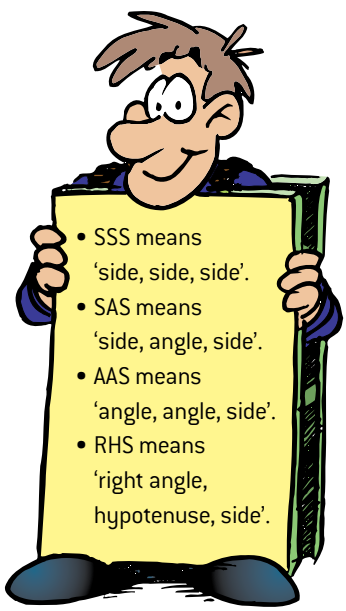


**2** SAS (two sides and an included angle)
Two triangles are congruent if two sides and the included angle of one triangle are equal to two sides and the included angle of the other.



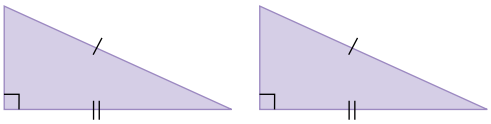
**3** AAS (two angles and a matching side)
Two triangles are congruent if two angles and a side of one triangle are equal to two angles and the matching side of the other triangle.





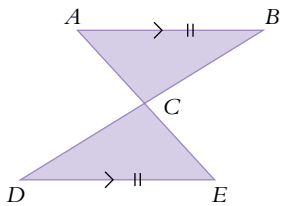
**4** RHS (right-angle, hypotenuse and side)

Two right-angled triangles are congruent if the hypotenuse and one side of one right-angled triangle are equal to the hypotenuse and one side of the other.



**Two sides (or angles) can be shown to be equal if they are the matching sides (or angles) of a pair of congruent triangles.**

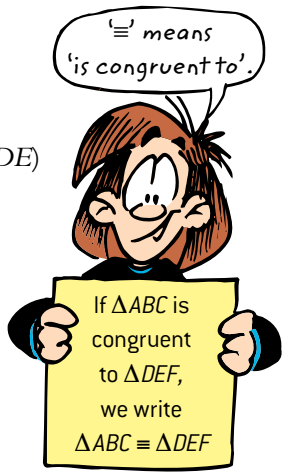
## WORKED EXAMPLE 1



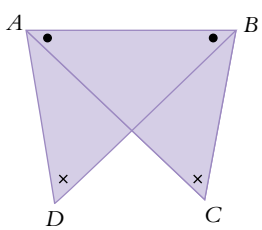

Prove that $\Delta ABC \equiv \Delta EDC$.

In $\Delta$s $ABC$ and $EDC$:

| | | |
|---|---|---|
| 1 | $\angle ACB = \angle ECD$ | (vert. opp. $\angle$s) |
| 2 | $\angle CAB = \angle CED$ | (alt. $\angle$s, $AB \parallel DE$) |
| 3 | $AB = DE$ | (given) |
| | $\therefore \Delta ABC \equiv \Delta EDC$ | (AAS) |




## WORKED EXAMPLE 2




Prove that $\Delta ABC \equiv \Delta BAD$ and hence that $AD = BC$.

In $\Delta$s $ABC$ and $BAD$:

| | | |
|---|---|---|
| 1 | $AB$ is common. | |
| 2 | $\angle BAC = \angle ABD$ | (given) |
| 3 | $\angle BCA = \angle ADB$ | (given) |
| | $\therefore \Delta ABC \equiv \Delta BAD$ | (AAS) |
| | $\therefore AD = BC$ | (matching sides of cong't $\Delta$s) |

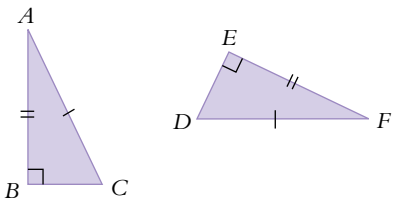
When writing congruent triangle proofs, write the vertices in matching order as shown in the examples.

# Exercise 3:02

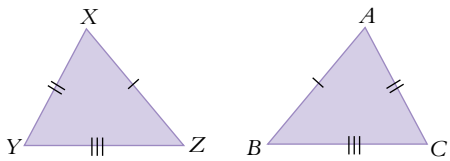
P Foundation worksheet 3:02
Congruent triangle proofs

**1** Prove that the two triangles in each pair are congruent.

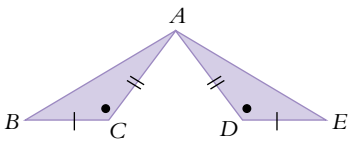
**a**



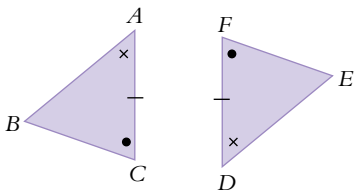

**b**



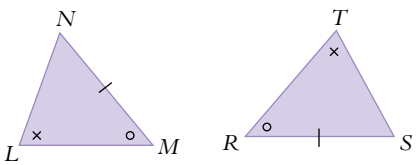

**c**




**d**

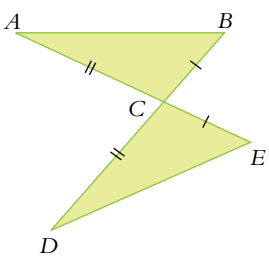



**e**

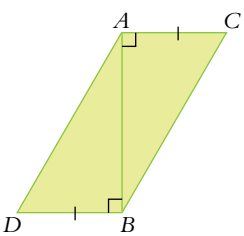


*Setting out proofs*
In Δs *ABC* and *DEF*:
**1** *AB* = *DE* (given)
**2** … = … (…)
**3** … = … (…)
∴ Δ*ABC* ≡ Δ*DEF* (…)

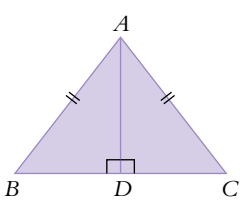
**2** **a** Prove that $\Delta ABC \equiv \Delta DEC$.



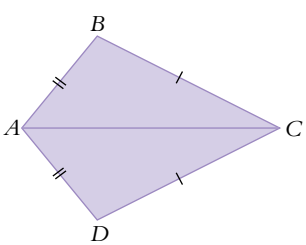

**b** Prove that $\Delta ABD \equiv \Delta BAC$.



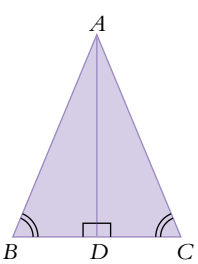

**c** Prove that $\Delta ABD \equiv \Delta ACD$.



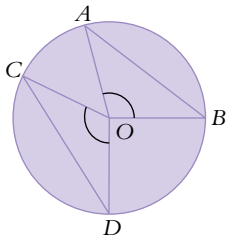

**d** Prove that $\Delta ABC \equiv \Delta ADC$.




**e** Prove that $\Delta ABD \equiv \Delta ACD$.

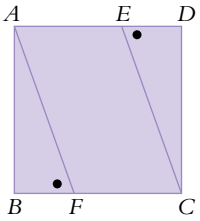


**f**



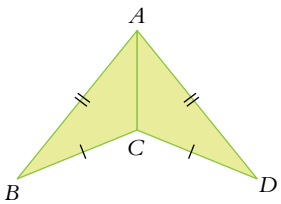

$O$ is the centre of the circle.
Prove that $\Delta AOB \equiv \Delta COD$.

**g**



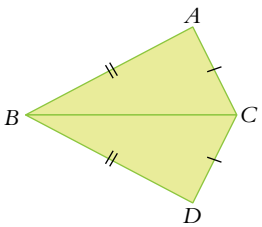

$ABCD$ is a square.
$\angle AFB = \angle CED$
Prove that $\Delta ABF \equiv \Delta CDE$.

**3 a**



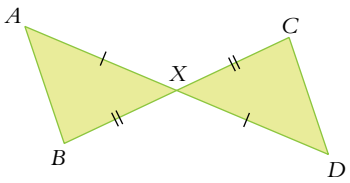

Prove that $\angle BAC = \angle DAC$.

**b**



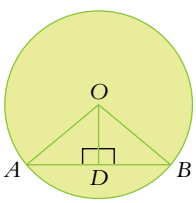

Prove that $\angle ABC = \angle CBD$.

**c**



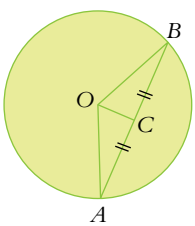

Prove that $\angle BAX = \angle CDX$, and hence that $AB \parallel CD$.

**d**



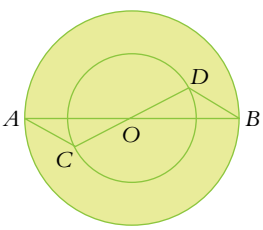

Prove that $AD = DB$.

**e**



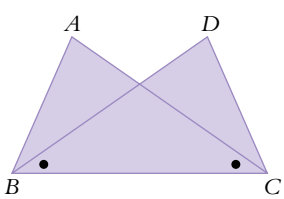

Prove that $\angle OCA = \angle OCB = 90°$.

**f**



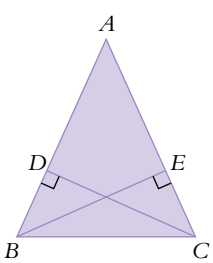

Prove that $AC = DB$ and $AC \parallel DB$.

**4 a**




$\angle DBC = \angle ACB$ and $BD = CA$.
Prove that $\Delta ABC \equiv \Delta DCB$.

**b**




$\angle ABC = \angle ACB$
Prove that $\Delta DBC \equiv \Delta ECB$.

*Hints:*

1. Write the three sides of each triangle.
2. Match up the ones that are equal.
3. Repeat the above for the angles.

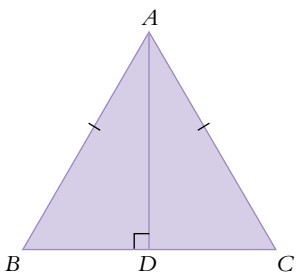
# 3:03 Deducing properties of the special triangles and quadrilaterals

In the following examples congruent triangles are used to prove the properties of isosceles and equilateral triangles as well as some properties of the special quadrilaterals.

## WORKED EXAMPLE 1

Use congruent triangles to prove that the angles opposite equal sides in an isosceles triangle are equal.

### Solution




Data: $\Delta ABC$ is isosceles with $AB = AC$.
Aim: To prove that $\angle ABC = \angle ACB$.
Construction: Draw $AD$ perpendicular to $BC$, meeting $BC$ at $D$.
Proof: In $\Delta$s $ABD$ and $ACD$:

1 $AB = AC$ (given)

2 $AD$ is common.

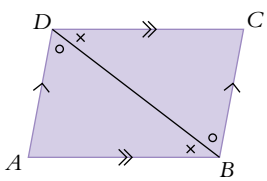
3 $\angle ADB = \angle ADC$ ($AD \perp BC$)
$\therefore \Delta ABD \equiv \Delta ACD$ (RHS)
$\therefore \angle ABD = \angle ACD$ (matching $\angle$s of cong't $\Delta$s)
$\therefore \angle ABC = \angle ACB$

## WORKED EXAMPLE 2

Prove that the opposite sides of a parallelogram are equal, given that a parallelogram is a quadrilateral with opposite sides parallel.

### Solution

Data: $ABCD$ is a parallelogram with $AB \parallel DC$ and $AD \parallel BC$.
Aim: To prove $AB = CD$ and $AD = CB$.
Construction: Join $BD$.
Proof: In $\Delta$s $ABD$ and $CDB$




1 $\angle ABD = \angle CDB$ (alt. $\angle$s, $AB \parallel DC$)

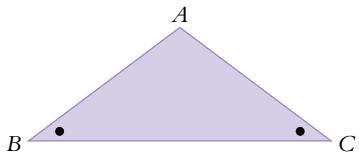
2 $\angle ADB = \angle CBD$ (alt. $\angle$s, $AD \parallel BC$)

3 $BD$ is common.
$\therefore \Delta ABD \equiv \Delta CDB$ (AAS)
$\therefore AB = CD$ and $AD = CB$ (matching sides of cong't $\Delta$s)

# Exercise 3:03

P Foundation worksheet 3:03
Using congruent triangles

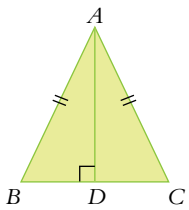
**1** $\Delta ABC$ has two equal angles. Draw $AD$ perpendicular to $BC$ (where $D$ is on $BC$) and use congruent triangles to prove that $AB = AC$.

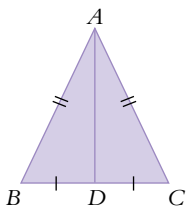



**2** Use congruent triangles to prove the following properties of isosceles triangles.

**a** A line drawn at right angles to the base of an isosceles triangle through the third vertex bisects the base.

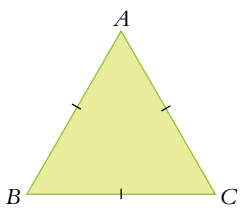
**b** A line drawn from the midpoint of the base of an isosceles triangle to the third vertex is perpendicular to the base.



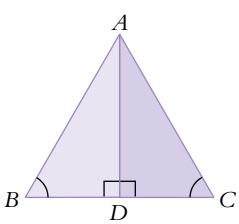



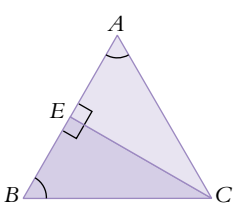
**3** **a** $\Delta ABC$ has all of its sides equal. Use the result that if a triangle has two equal sides then the angles opposite the equal sides are equal to prove the following:
$\angle A = \angle B = \angle C$

**b** How can the result in **a** be used to show that each angle of an equilateral triangle is 60°?




**4**




$\Delta ABC$ has three equal angles.
In the first diagram, a perpendicular has been drawn from $A$ to $BC$.
In the second diagram, the perpendicular has been drawn from $C$ to $AB$.
Explain why:

**a** $\Delta ABD \equiv \Delta ACD$

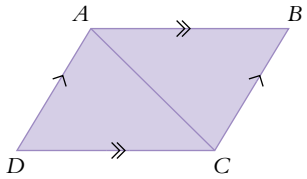
**b** $AB = AC$

**c** $\Delta CBE \equiv \Delta CAE$

**d** $BC = AC$

You have proved that if a triangle has three equal angles, then it has three equal sides.

**5** $ABCD$ is a parallelogram.

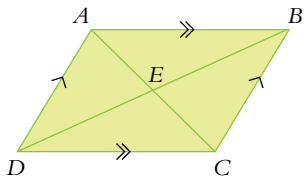



**a** Why does $\angle BAC$ equal $\angle DCA$?
**b** Why does $\angle BCA$ equal $\angle DAC$?
**c** Does $\angle BAD$ equal $\angle BCD$?
**d** Prove that $\Delta ABC \equiv \Delta CDA$. Hence prove that:
**i** $\angle ABC = \angle CDA$
**ii** $AB = DC$ and $BC = AD$

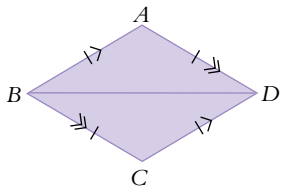
You have proved that the opposite sides and opposite angles of a parallelogram are equal.

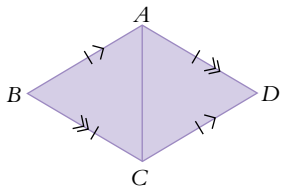
**6** Using the fact that the opposite sides of a parallelogram are equal, prove that $\Delta ABE \equiv \Delta CDE$ and hence that $AE = EC$ and $BE = ED$.

You have proved that the diagonals of a parallelogram bisect each other.




**7** A rhombus is a parallelogram with all its sides equal. Explain why:

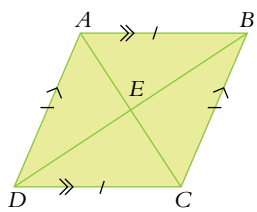



**a** $\Delta ABD \equiv \Delta CBD$
**b** $\angle ABD = \angle CBD$
**c** $\angle ADB = \angle CDB$
**d** $\Delta ABC \equiv \Delta ADC$
**e** $\angle BAC = \angle DAC$
**f** $\angle BCA = \angle DCA$

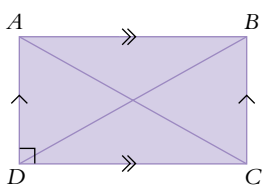
**8** **a** In the rhombus shown on the right, why does $AE = EC$?
**b** Prove that $\Delta ABE \equiv \Delta CBE$ and hence that $\angle AEB = \angle CEB = 90°$.

In the last two questions you have proved that the diagonals of a rhombus bisect each other at right angles, and that they bisect the angles through which they pass.

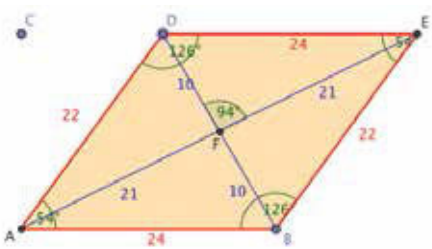



**9** A rectangle is a parallelogram with a right angle.
**a** Prove that all the angles must be right angles.
**b** Assuming the answer to **a** and the properties of a parallelogram, prove that $\Delta ABD \equiv \Delta DCA$ and hence $AC = DB$.

You have proved that all the angles of a rectangle are right angles and that its diagonals are equal in length.




## P GEOGEBRA ACTIVITY 3:03 THE PARALLELOGRAM FAMILY



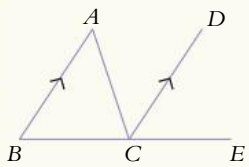

Click and drag the points to change a parallelogram into a rectangle, square or rhombus.

Reinforce your knowledge of the definitions and properties of the parallelogram family by using checkboxes to observe the changes to angles, sides and diagonals as the shapes change.

# 3:04 Proving general results

## PREP QUIZ 3:04

Copy the diagram.




**1** Mark $\angle ABC$ with '×'.

**2** Mark $\angle BAC$ with '•'.

In the diagram, which angle is equal to:

**3** $\angle ABC$?

**4** $\angle BAC$?

**5** Which angle is adjacent to $\angle DCE$?

**6** Which two adjacent angles make $\angle ACE$?

**7** If $a + b = 180$, then $180 - a = \ldots$

**8** If $a + c = 180$, then $c = \ldots$

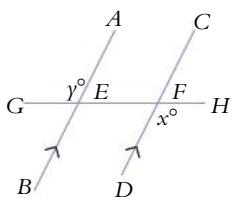
**9** If $b = 180 - a$ and $c = 180 - a$, then $b = \ldots$

**10** If $a = b$ and $b = c$, what can we say about $a$ and $c$?

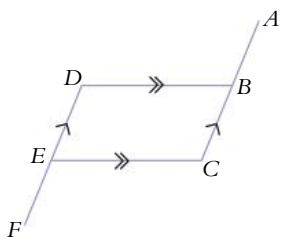
Many problems in geometry are non-numerical. In these problems, the reasoning process becomes more involved. Instead of using given values for side lengths or angles, pronumerals are used to represent unknown quantities. With the use of pronumerals, the reasoning will involve algebraic skills covered in other chapters. Questions in this exercise do not involve specific values, so the results obtained will be true for all values. The process used to answer each question is called a **proof**, and the result obtained is called a **generalisation**.

## WORKED EXAMPLE 1

**a** In the diagram, prove that $x = y$.




**b** In the diagram, prove that $\angle ABD = \angle CEF$.




### Solutions

**a** $\angle CFE = x°$ (vert. opp. $\angle$s)
$\angle CFE = y°$ (corr. $\angle$s, $AB \parallel CD$)
$\therefore\ x = y$ (both equal to $\angle CFE$)

There is usually more than one solution.

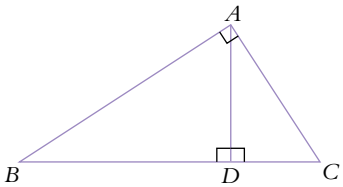
**b** $\angle ABD = \angle BDE$ (alt. $\angle$s, $AC \parallel DF$)
$\angle CEF = \angle BDE$ (corr. $\angle$s $DB \parallel EC$)
$\therefore\ \angle ABD = \angle CEF$ (both equal to $\angle BDE$)

## WORKED EXAMPLE 2

**a** In the diagram, prove that:

i $\angle ABD = \angle DAC$

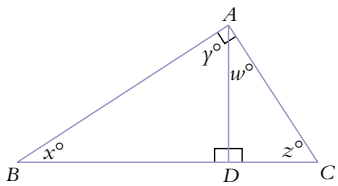
ii $\angle BAD = \angle ACD$




**b** $\Delta ABC$ is an isosceles triangle with $AB = BC$. $AB$ is extended (i.e. produced) to $D$ and $BE$ is drawn parallel to the base $AC$. Prove that $BE$ bisects $\angle CBD$.

### Solutions

**a**




Let $\angle ABD = x°$, $\angle BAD = y°$, $\angle DAC = w°$ and $\angle ACD = z°$.

i $x + y = 90$ (comp. $\angle$s in $\Delta ABD$)

$w + y = 90$ (comp. $\angle$s)

$\therefore x = w$

$\therefore \angle ABD = \angle DAC$

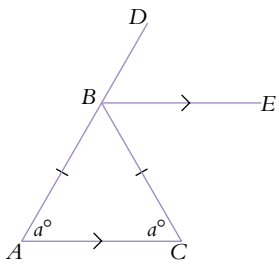
ii $x + y = 90$ (comp. $\angle$s in $\Delta ABD$)

$x + z = 90$ (comp. $\angle$s in $\Delta ABC$)

$\therefore y = z$

$\therefore \angle BAD = \angle ACD$

**b**




Let $\angle BAC = a°$

$\therefore \angle BCA = a°$ (base $\angle$s of isos. $\Delta$)

Now $\angle EBC = a°$ (alt. $\angle$s, $BE \parallel AC$)

$\angle DBE = a°$ (corr. $\angle$s, $BE \parallel AC$)

$\therefore \angle DBE = \angle EBC$

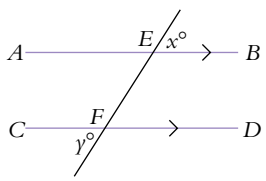
$\therefore$ $BE$ bisects $\angle CBD$.

## Exercise 3:04

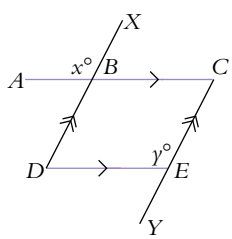
P Foundation worksheet 3:04
Non-numerical proofs

**1** Prove that $x = y$ in each of the following.

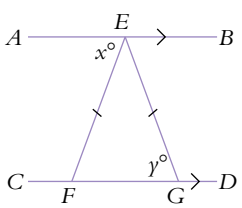
**a**



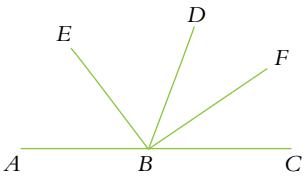

**b**




**c**

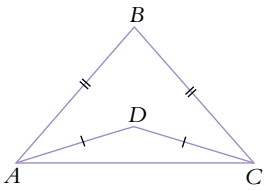


**2**

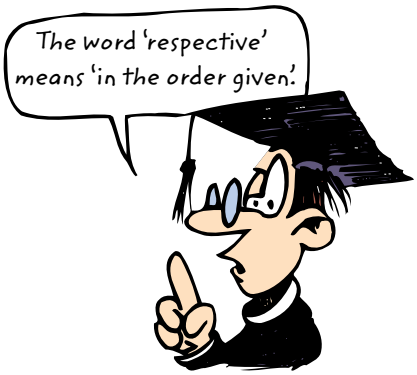



$\angle ABC$ is a straight angle. $BE$ and $BF$ bisect $\angle ABD$ and $\angle DBC$ respectively. Prove that $EB$ is perpendicular to $BF$.

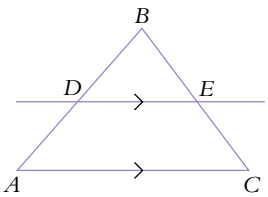
**3**




$\Delta ABC$ and $\Delta ADC$ are isosceles. $AC$ is the base of both triangles. Prove that $\angle BAD = \angle BCD$.

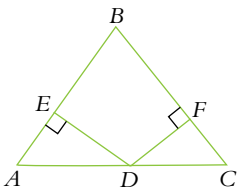



**4**



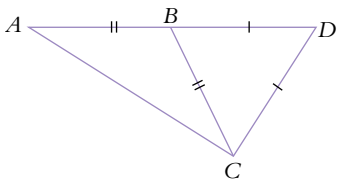

A line drawn parallel to the base $AC$ of an isosceles $\Delta ABC$ cuts the equal sides at $D$ and $E$. Prove that $\Delta DBE$ is isosceles.

**5**



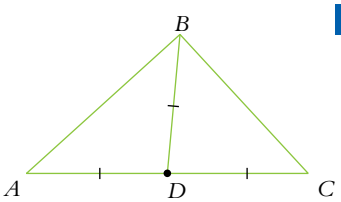

$\Delta ABC$ has $AB$ and $BC$ equal. $D$ is any point on $AC$, and $DE$ and $DF$ are perpendicular to $AB$ and $BC$ respectively. Prove that $\angle EDA = \angle FDC$.

**6**



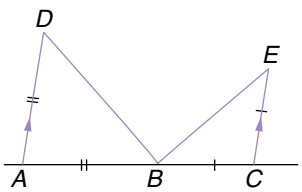

$\Delta ABC$ and $\Delta BDC$ are isosceles. $AB = BC$ and $BD = DC$. $BC$ is a common side. $A$ and $D$ lie on opposite sides of $BC$. Prove that $\angle BCD = 2 \times \angle BCA$.

**7**



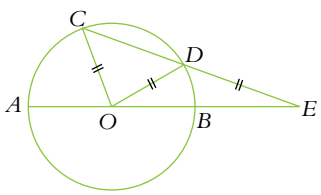

$D$ is a point on the side $AC$ of $\Delta ABC$. $D$ is equidistant from the three vertices of the triangle. Prove that $\angle ABC$ is a right angle.

**8**



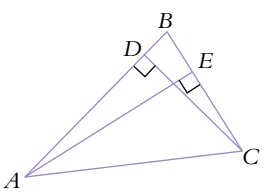

$A$, $B$ and $C$ are collinear. $AD = AB$, $BC = EC$ and $AD \parallel EC$. Prove that $\angle DBE$ is a right angle.

**9**



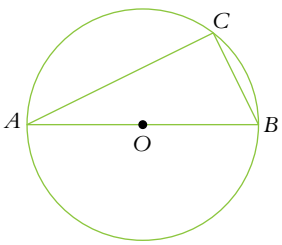

$AB$ is a diameter and $CD$ is a chord of a circle with centre $O$. $CD$ produced meets $AB$ produced at $E$ and $DE = OD$. Prove that $\angle AOC = 3 \times \angle DOB$.

**10**




In $\Delta ABC$, $AE$ and $CD$ are perpendicular to $BC$ and $AB$ respectively. Prove that $\angle BAE = \angle BCD$.

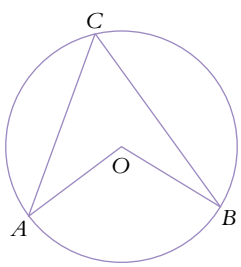
**11**




$AB$ is the diameter of a circle centred at $O$. Prove that $\angle ACB = 90°$.

*Hint!* Join $CO$.

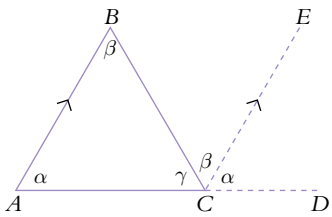
**12**




$O$ is the centre of the circle. Prove that $\angle AOB = 2 \times \angle ACB$.

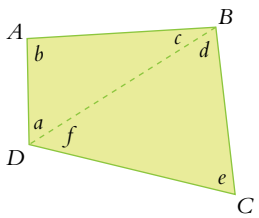
*When a result is proved, this result can then be used to prove another result. This is shown in the following set of questions.*

**13** Show that the interior angles of a triangle add up to 180° by completing the proof below.

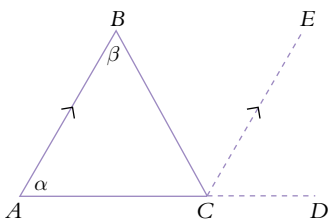



Data: $\Delta ABC$ is any triangle with angles $\alpha$, $\beta$ and $\gamma$.
Aim: To prove that $\alpha + \beta + \gamma = 180°$
(i.e. the angle sum is 180°).
Construction: Extend $AC$ to $D$. Draw $CE$ parallel to $AB$.
Proof: $\angle BCE = \beta$ (…)
$\angle ECD = \alpha$ (…)
$\angle BCA = \gamma$ (given)
$\therefore \gamma + \beta + \alpha = 180°$ (…)
$\therefore$ The angle sum of a triangle is 180°.

**14** Use the theorem proved in Question **13** to prove that the interior angles of a quadrilateral add up to 360°.



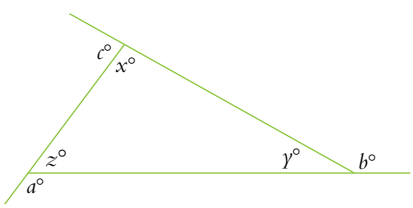

**15** It is a well-known result that 'an exterior angle of a triangle is equal to the sum of the interior opposite angles'. Complete the proof started below.



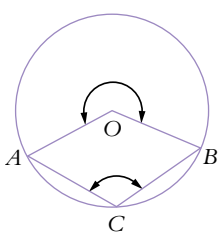

Aim: To prove that the exterior angle $\angle BCD$ is equal to the sum of the interior opposite angles (i.e. $\angle BCD = \alpha + \beta$).

Construction: Draw $CE$ parallel to $AB$.
Proof: $\angle BCE = \beta$ (…)
$\angle ECD = \alpha$ (…)
$\therefore \angle BCD =$ …

**16** Use the result above to prove that the sum of the exterior angles of a triangle is 360°.




**17** Use isosceles triangles and the exterior angle theorem to prove that the reflex angle $AOB = 2 \times \angle ACB$.




$O$ is the centre of the circle.

# 3:05 Definitions and proofs in deductive geometry

If you were asked to define an equilateral triangle, you could say:

- it is a triangle with all its sides equal, or
- it is a triangle with all its angles equal, or
- it is a triangle with all its sides equal and all its angles equal.

Geometrical figures have many properties and it is not practical to mention them all when defining the figure.

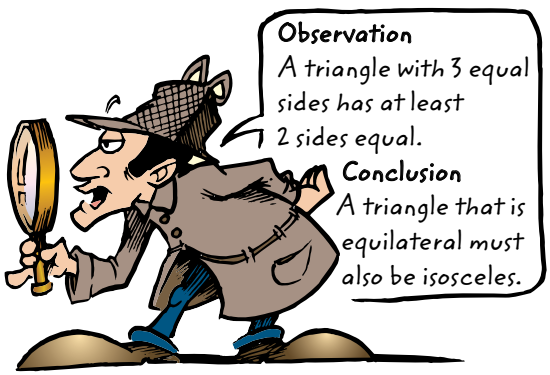
A definition is the minimum amount of information needed to identify a particular figure.

In deductive geometry, the definitions serve as starting points. The properties of the figures can then be proved using basic geometrical facts.

The proved result is known as a theorem, and this can then be used to produce other theorems.

Definitions:

- A scalene triangle is a triangle with no two sides equal in length.
- An isosceles triangle is a triangle with two sides equal in length.
- An equilateral triangle is a triangle with all sides equal in length.




The definitions imply that an equilateral triangle must also be an isosceles triangle. Hence, any property of an isosceles triangle must also be a property of an equilateral triangle.

Definitions:

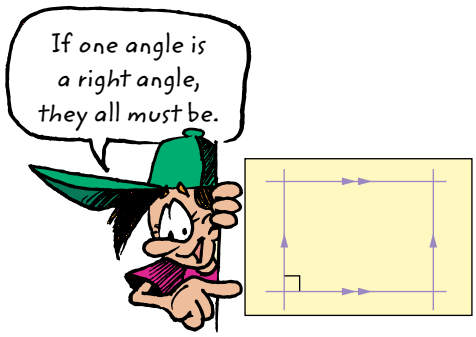
- A trapezium is a quadrilateral with at least one pair of opposite sides parallel.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
- A rhombus is a parallelogram with two adjacent sides equal in length.
- A rectangle is a parallelogram with one angle a right angle.
- A square is a rectangle with two adjacent sides equal.

or

- A square is a rhombus that contains a right angle.

Many people find the definitions above a little unusual at first.

- They start from the simplest shape and, by adding more and more constraints, end up at the most complex shape.
- The definitions are hierarchical. Each new shape is defined in terms of a simpler shape that has already been defined.
  e.g. *A rhombus is a parallelogram …*
  This saves repetition and states that a rhombus is in fact a special type of parallelogram. It has all the properties of a parallelogram and some extra properties as well.
- The definitions do not list every property of the shape. These can be deduced from the definition using other geometrical techniques.
  e.g. *A rectangle is a parallelogram with a right angle.*
  There is no need to say that it has four right angles as this can be derived using the fact that it is a parallelogram and our knowledge of co-interior angles and parallel lines.




In Exercise 3:03 many of the properties of the special triangles and quadrilaterals were proved using the definitions and congruent triangles.

- As well as the above definitions, there are other tests that can be used to show that a given quadrilateral is a parallelogram, rhombus or rectangle. These are listed below.

Tests for parallelograms:

A quadrilateral is a parallelogram if any one of the following is true.

**1** Both pairs of opposite sides are equal.

**2** Both pairs of opposite angles are equal.

**3** One pair of sides is both equal and parallel.

**4** The diagonals bisect each other.

Tests for a rhombus:

**1** All sides are equal.

or

**2** Diagonals bisect each other at right angles.

Tests for a rectangle:

**1** All angles are equal.

or

**2** Diagonals are equal and bisect each other.

In the following exercise, many of these tests, as well as some of the properties of the quadrilaterals, are proved.

## WORKED EXAMPLE

Prove that a quadrilateral is a parallelogram if its opposite angles are equal.

### Solution

Data: $ABCD$ is a quadrilateral in which $\angle A = \angle C$ and $\angle B = \angle D$.

Aim: To prove that $AB \parallel DC$ and $AD \parallel BC$.

Proof: Let $\angle A = \angle C = b°$ and $\angle B = \angle D = a°$.

$2(a + b) = 360$ ($\angle$ sum of quad.)

$\therefore a + b = 180$

$\therefore \angle ADC + \angle DAB = 180°$

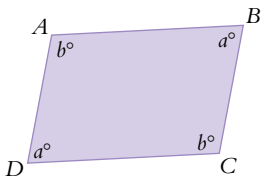
$\therefore AB \parallel DC$ (co-int. $\angle$s are supp.)

Also, $\angle ADC + \angle DCB = 180°$

$\therefore AD \parallel BC$ (co-int. $\angle$s are supp.)

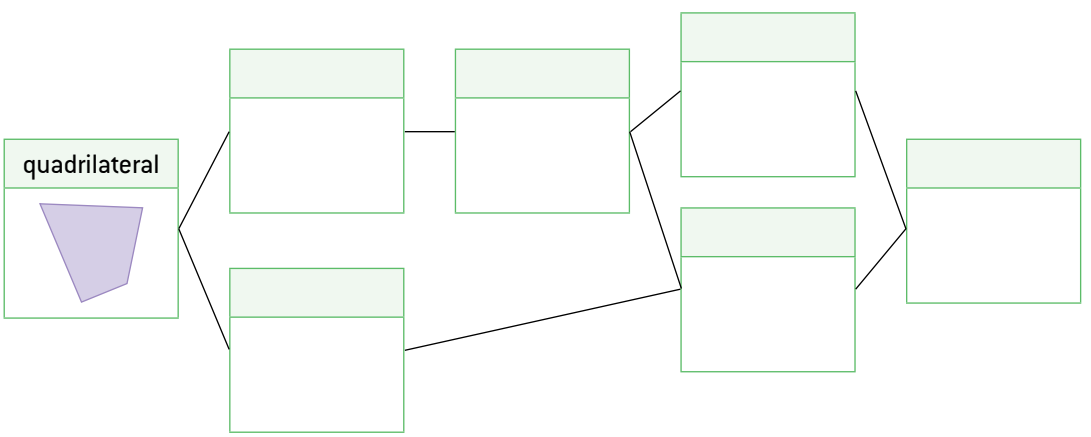
$\therefore ABCD$ has opposite sides parallel.

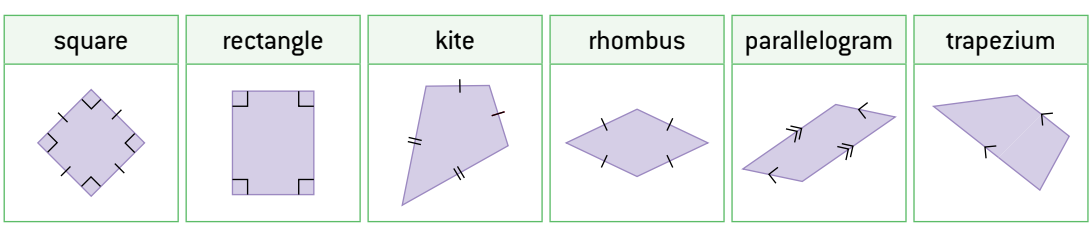
$\therefore ABCD$ is a parallelogram.




## Exercise 3:05

**1** Redraw this diagram, placing one of the shapes given (and its name) in each box, so that all shapes possess the attributes of the shapes on their left.



**2** Match each definition to the diagram below that shows the information in the definition.

**a** A scalene triangle is a triangle with no two sides equal in length.

**b** An isosceles triangle is a triangle with two sides equal in length.

**c** An equilateral triangle is a triangle with all sides equal in length.

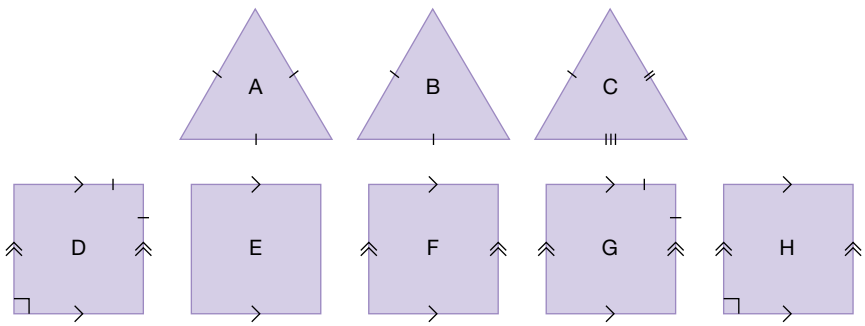
**d** A trapezium is a quadrilateral with at least one pair of opposite sides parallel.

**e** A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

**f** A rhombus is a parallelogram with two adjacent sides equal in length.

**g** A rectangle is a parallelogram with one angle a right angle.

**h** A square is a rectangle with two adjacent sides equal.




**3** Use your answer to Question **1** and the definitions in Question **2** to write true or false for each statement.

**a** Any result proven for a parallelogram would also hold for a rectangle.

**b** Any result proven for a parallelogram would also hold for a trapezium.

**c** Any result proven for a rectangle would also hold for a square.

**d** Any result proven for an isosceles triangle would also hold for an equilateral triangle.

**e** Any result proven for an equilateral triangle would also hold for an isosceles triangle.

**f** Any result proven for a rhombus would also hold for a square.

**g** Any result proven for a kite would also hold for a square.

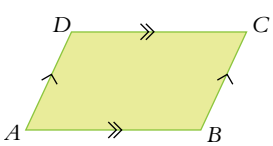
**h** Any result proven for a rectangle would also hold for a trapezium.

**4** Give reasons why:

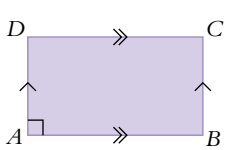
**a** a square is a rhombus but a rhombus is not necessarily a square

**b** an equilateral triangle is an isosceles triangle but an isosceles triangle is not necessarily an equilateral triangle.

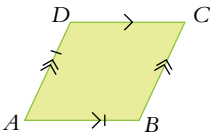
**5** Use the definition of a parallelogram (on page 73) and co-interior angles to prove that the opposite angles of a parallelogram are equal.



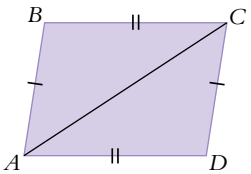

**6** Use the definition of a rectangle (on page 73) and co-interior angles to prove that a rectangle has four right angles.



**7** **a** What can you say about the opposite sides of a parallelogram?
**b** Given that a rhombus is a parallelogram with two adjacent sides equal, prove that all the sides of a rhombus are equal.




**8** $ABCD$ is a quadrilateral that has opposite sides equal. Prove that it is a parallelogram (i.e. that its opposite sides are parallel).

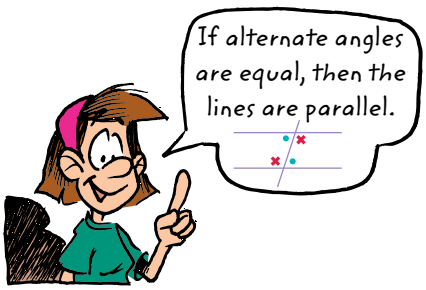


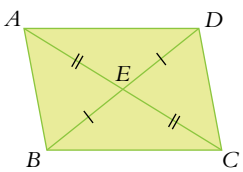

**9** The 'Data' and 'Aim' for the congruence proof of Test 4 for parallelograms (on page 74) are given below. Answer the questions to give an outline of the proof.

Data: $ABCD$ is any quadrilateral where diagonals $AC$ and $BD$ bisect each other at $E$.

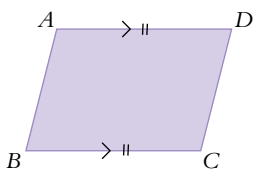
Aim: To show that $ABCD$ is a parallelogram (i.e. $AD \parallel BC$ and $AB \parallel CD$).

**a** Why does $\angle AED$ equal $\angle CEB$?
**b** Why is $\Delta AED$ congruent to $\Delta CEB$?
**c** Which angle in $\Delta CBE$ is equal to $\angle ADE$? Why?
**d** How does your answer to **c** prove that $AD \parallel BC$?
**e** Why is $\Delta AEB$ congruent to $\Delta CED$?
**f** Why is $\angle BAE$ equal to $\angle DCE$?
**g** Why is $AB$ parallel to $CD$?



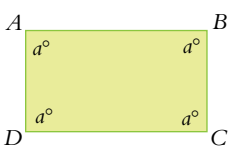




**10** Use congruent triangles to prove that if one pair of sides in a quadrilateral are both equal and parallel, then the quadrilateral is a parallelogram.




Use the tests for parallelograms on page 74 and the properties of parallelograms to do Questions **11** and **12**.

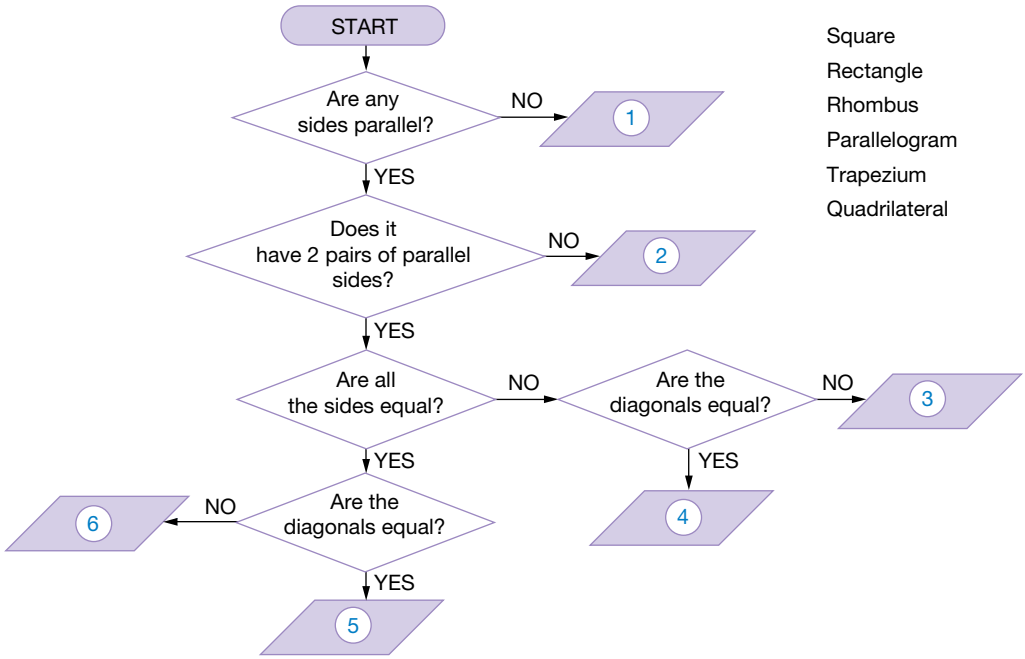
**11** Prove that a quadrilateral is a rectangle if all its angles are equal.




**12** Show how Test 1 for a rhombus (on page 74) could be proved using Test 1 for a parallelogram.

The rhombus, rectangle and square are all parallelograms. Hence they possess all the properties of a parallelogram.

**13** Follow the flowchart below and choose the correct names from the list, to be inserted into boxes 1 to 6.




**14** Use the definitions and the properties of the quadrilaterals in Exercise 3:03 to copy and complete this table by writing *yes* or *no* in each space.

| Properties | Parallelogram | Rhombus | Rectangle | Square |
|---|---|---|---|---|
| (1) Opposite sides parallel | | | | |
| (2) Opposite sides equal | | | | |
| (3) Opposite angles equal | | | | |
| (4) Diagonals bisect one another | | | | |
| (5) All sides equal | | | | |
| (6) All angles right angles | | | | |
| (7) Diagonals perpendicular | | | | |
| (8) Diagonals bisect angles through which they pass | | | | |
| (9) Diagonals are equal | | | | |

**15** Use the table above to name the properties that the following have in common.

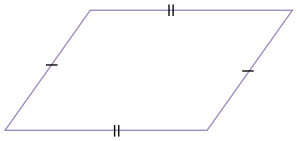
**a** square and rhombus
**b** square and rectangle
**c** rectangle and parallelogram
**d** parallelogram and rhombus

**16** **a** Use the table above to name the properties of a square that are not properties of a rectangle.
**b** Use the table above to name the properties of a square that are not properties of a rhombus.
**c** Use the table above to name the properties of a rectangle that are not properties of a parallelogram.

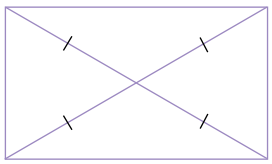
**17** **a** Does a square have all the properties of a rectangle?
**b** Is a square a special rectangle?
**c** Is a rectangle a square? Give a reason for your answer.
**d** Does a square have all the properties of a rhombus?
**e** Is a square a special rhombus?
**f** Is a rhombus a square? Give a reason for your answer.

**18** Use the tests for quadrilaterals to identify each of the following as a rectangle, rhombus or parallelogram, and then state the test used in each case.

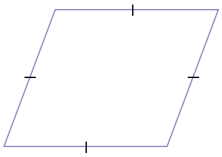
**a**



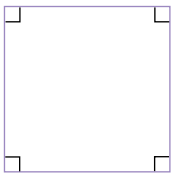
**b**



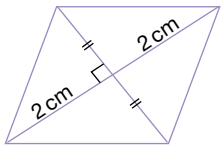
**c**



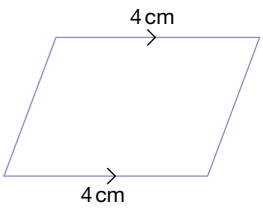
**d**



**e**

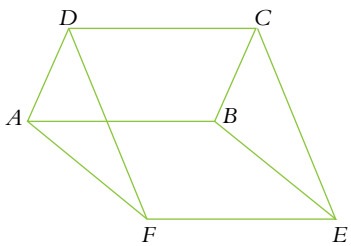



**f**

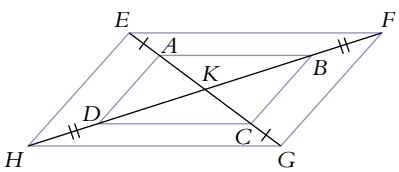



Use the tests for parallelograms and the properties of parallelograms in the following questions.

**19** In the diagram, *ABCD* and *ABEF* are parallelograms. Prove that *DCEF* is a parallelogram.




**20** *ABCD* is a parallelogram with diagonals produced so that $EA = CG$ and $DH = BF$. Prove that *EFGH* is a parallelogram.



Isosceles triangles are often found in architecture.

## INVESTIGATION 3:05 THEOREMS AND THEIR CONVERSES

A theorem usually connects two pieces of information and can be written in the form 'If $A$ then $B$'.

$A$ is usually called the *supposition* or *assumption*. $B$ is called the *conclusion*.

If $A$ and $B$ are interchanged, then we have the statement 'If $B$ then $A$'. This is called the converse of the theorem.

Even if a theorem is true, its converse may not be, as shown by the following example.

**Theorem:** If (two angles are vertically opposite), then (the angles are equal). This is true.

**Converse:** If (two angles are equal), then (they are vertically opposite). This is false.

For the following theorems, state their converse and whether the converse is true.

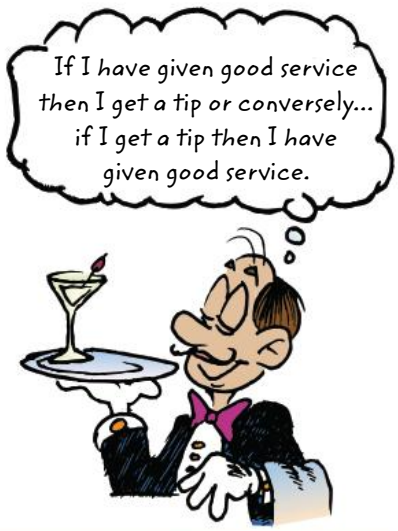
1 If a triangle has all its sides equal, then it has all angles equal.

2 If a quadrilateral is a square, then its diagonals are equal.

3 If a quadrilateral is a parallelogram, then its opposite angles are equal.

4 If a quadrilateral is a square, then it has all its sides equal.

5 If a triangle is right-angled, then the square on the longest side is equal to the sum of the squares on the two smaller sides.

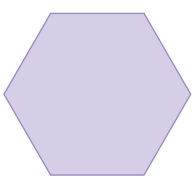


Builders often use the properties of quadrilaterals in the construction of a building.

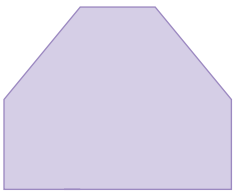
# 3:06 Polygons

The Pentagon is a famous polygonal building.

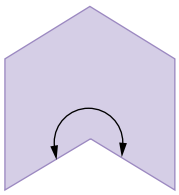
- A polygon is a plane figure with straight sides.
- A polygon is said to be regular if all of its sides and angles are equal. (If they are not, it is said to be irregular.)
- Some polygons are named according to the number of sides.



A regular hexagon



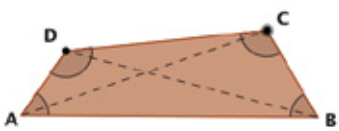
An irregular hexagon



A concave hexagon

- A polygon can be *concave* or *convex*.
- In a *convex* polygon all the angles are acute or obtuse.
- In a *concave* polygon at least one angle is reflex.

## P GEOGEBRA ACTIVITY 3:06A — CONVEX AND CONCAVE POLYGONS




**ABCD is a convex quadrilateral.**
- **All its angles are acute or obtuse.**
- **The diagonals lie inside the quadrilateral.**

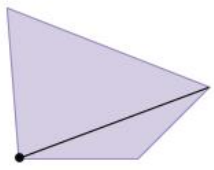
Click and drag the vertices of a quadrilateral or a hexagon and change it from convex to concave and vice versa. Then test your understanding by identifying a randomly generated polygon as either convex or concave.

## Angle sum of a polygon

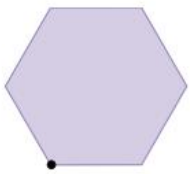
We can use the result for the angle sum of a triangle to work out the angle sum of a polygon, because all polygons can be divided into triangles.

### INVESTIGATION 3:06A THE ANGLE SUM OF A POLYGON

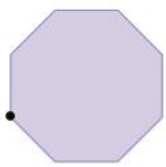
1 Divide each of these polygons into triangles by drawing diagonals from one vertex. The quadrilateral and pentagon have already been done.


Quadrilateral

Pentagon


Hexagon

Heptagon


Octagon

Nonagon

2 Sketch a decagon (10 sides) and a dodecagon (12 sides) and divide them into triangles.

3 Use the results of your work to complete this table.

| Number of sides in polygon | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| Number of triangles | 1 | 2 | 3 | | | | | | |

4 What do you think is the relationship between the 'number of sides' and the 'number of triangles'?

5 If a polygon had $n$ sides how many triangles would be formed?

6 If the angle sum of one triangle is 180°, what is the angle sum of an $n$-sided polygon?

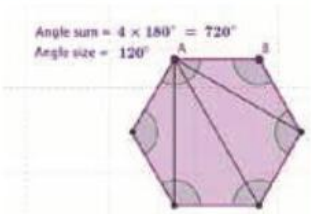
7 If the polygon is regular (all angles and sides equal), what would be the size of each angle in an $n$-sided polygon?

### GEOGEBRA ACTIVITY 3:06B THE ANGLE SUM OF A POLYGON

Use checkboxes to choose either a regular or irregular polygon and use a slider to select the number of sides (from 3 to 12).

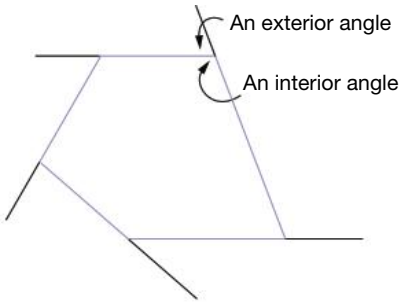
Divide the polygons up into triangles and explore the relationship between the number of sides and the angle sum of a polygon.

Work out the sizes of the internal angles in regular polygons.

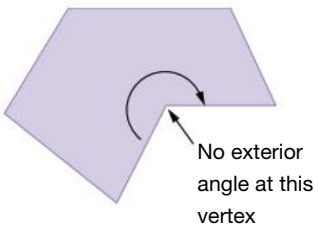


## The exterior angle sum of a convex polygon

- An exterior angle of a polygon is an angle that forms a straight line with one of the interior angles.




- If the interior angle is a reflex angle then no exterior angle exists.

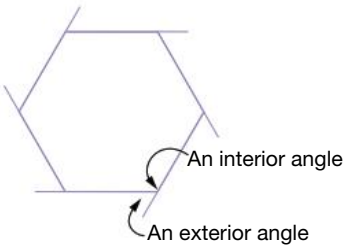



- The angles in a convex polygon are acute or obtuse so a pair of interior and exterior angles will exist at each vertex.

The sum of the exterior angles of a convex polygon is explored in the following investigation.

## INVESTIGATION 3:06B THE EXTERIOR ANGLE SUM OF A CONVEX POLYGON

**1** At each vertex of the hexagon, a side has been produced to form an exterior angle. One interior angle/exterior angle pair is formed at each vertex. How many interior angle/exterior angle pairs does a hexagon have?

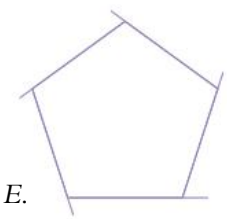



**2** At each vertex the exterior angle and interior angle make a straight angle and so sum to 180°. What is the sum of all the interior and exterior angles in the hexagon?

**3** What is the sum of all of the interior angles? (Use the result of Investigation 3:06A.)

**4** Use the answers to Questions **2** and **3** to find the sum of all of the exterior angles.

**5** Repeat Steps 1 to 4 above for the pentagon shown.



**6** What do the results above suggest about the sum of the exterior angles of any polygon?

**7** Read through the following.
Let the sum of the interior angles be $I$. Let the sum of the exterior angles be $E$.
For an $n$-sided polygon:

$$I = (n - 2) \times 180$$
$$I + E = 180n$$
$$E = 180n - (n - 2) \times 180$$
$$= 180n - 180(n - 2)$$
$$= 180n - 180n + 360$$
$$= 360$$

- The sum of the interior angles of a polygon of $n$ sides is $(n - 2) \times 180°$.
- The sum of the exterior angles of any convex polygon is 360°.

## WORKED EXAMPLES

1 Find the sum of the interior angles of an octagon and the size of an interior angle, if the octagon is regular.

2 A regular polygon has an exterior angle of 20°. How many sides does the polygon have?

### Solutions

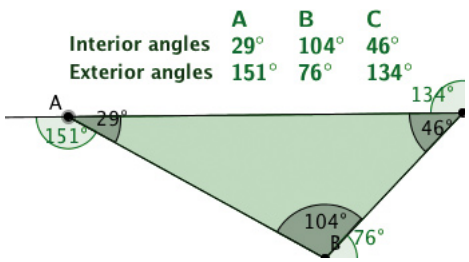
1 Sum of interior angles = $(n - 2) \times 180°$
For an octagon, $n$ is equal to 8.
Sum of interior angles = $(8 - 2) \times 180°$
$= 1080°$
If the octagon is regular, all angles are equal.
$\therefore$ Size of an interior angle = $1080° \div 8$
$= 135°$

2 Sum of exterior angles = 360°
For a polygon, the number of sides is equal to the number of exterior angles.
Number of angles = $\frac{360}{20}$
$= 18$
$\therefore$ Number of sides = 18

## GEOGEBRA ACTIVITY 3:06C

### THE EXTERIOR ANGLE SUM OF A CONVEX POLYGON

Click and drag the vertices of a polygon. Observe the relationship between the interior and exterior angle pairs. Find how the sum of the exterior angles is related to the sum of the interior angles.



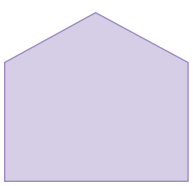

## Exercise 3:06

Foundation worksheet 3:06 Formulas

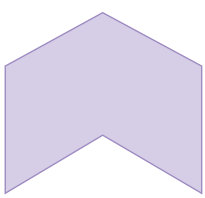
**1** Evaluate $(n - 2) \times 180$ if:
**a** $n = 6$ **b** $n = 10$ **c** $n = 15$

**2** For each of the following polygons find:
**i** the number of sides **ii** the interior angle sum.

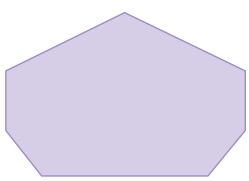
**a**



**b**



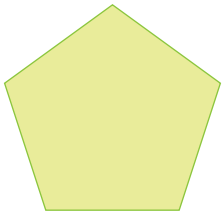
**c**



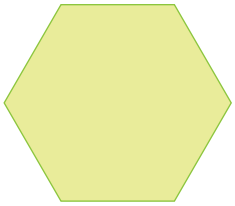
**3** Find the interior angle sum of:
**a** an octagon **b** a decagon **c** a heptagon.

**4** For each of the following regular polygons, find the size of the interior angle.

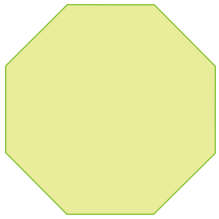
**a**



**b**



**c**



**d** a decagon (10 sides)

**e** a nonagon (9 sides)

**f** a regular polygon with 17 sides

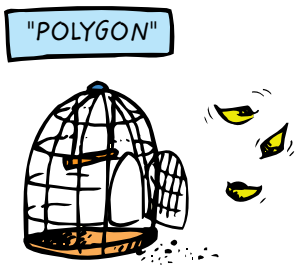
**5** A polygon has 20 sides. Find:

**a** the sum of the interior angles

**b** the sum of the exterior angles

**c** the size of an interior angle if the polygon is regular

**d** the size of an exterior angle if the polygon is regular.

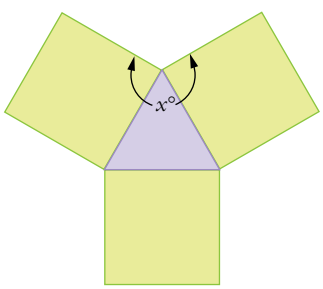



**6** Find the number of sides in a regular polygon that has an exterior angle of:

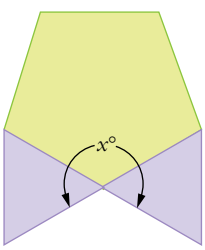
**a** 60° **b** 40° **c** 10° **d** 15°

**7** Tiles in the shapes of regular polygons have been used to make the following shapes. Find the value of the pronumerals.

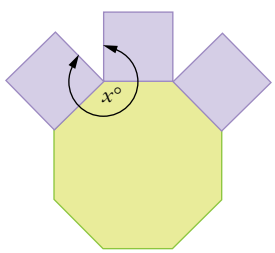
**a**




**b**

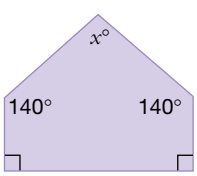



**c**

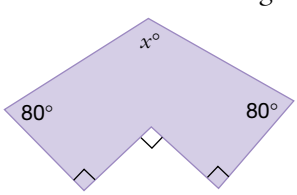



**8** Find the value of the pronumeral in each of the following.

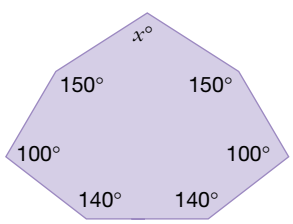
**a**



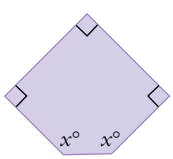

**b**



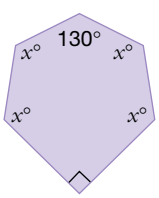

**c**



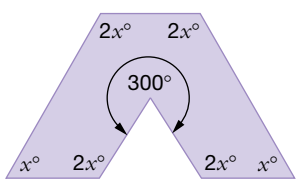

**d**




**e**

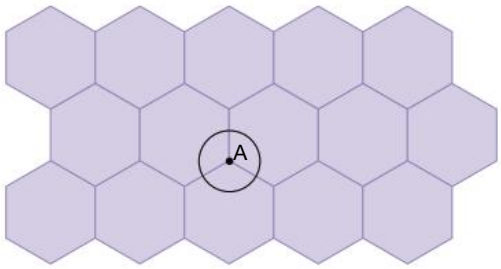



**f**



## INVESTIGATION 3:06C REGULAR POLYGONS AND TESSELLATIONS

A tessellation is a tiling pattern. If a shape can form a tile pattern on its own, it is said to tessellate. The diagram below is part of a tessellation based on a regular hexagon.



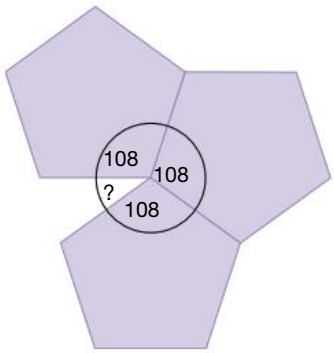

1 What is the size of an interior angle in a regular hexagon?

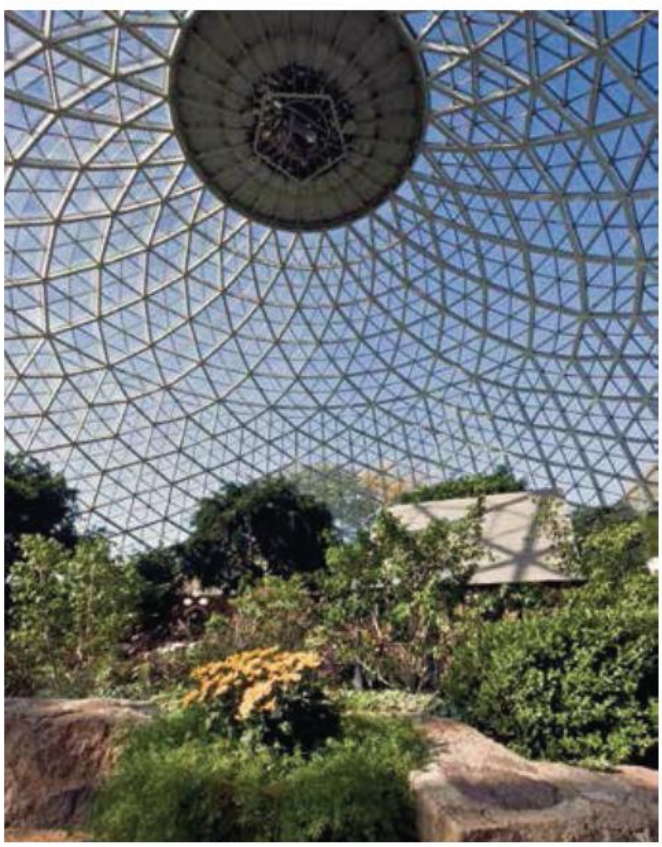
2 How many hexagons meet at the vertex, $A$?

3 If a shape tessellates, what must be the angle sum at each vertex?

4 Use the diagram on the right to explain why a regular pentagon will not tessellate.

5 What are the other two regular polygons that will tessellate?



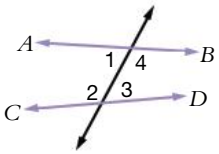



Polygons feature in the design of geodesic domes.

## MATHS TERMS 3

**alternate angles**

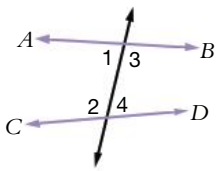
- a pair of angles on opposite sides of the transversal between the other two lines




- in the diagram, the alternate angles are 1 and 3, 2 and 4
- are equal when $AB \parallel CD$

**co-interior angles**

- a pair of angles on the same side of the transversal and between the other two lines




- in the diagram the co-interior angles are 1 and 2, 3 and 4
- are supplementary when $AB \parallel CD$

**congruent triangles**

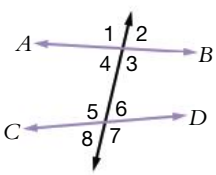
- triangles that are identical in shape and size

**congruency tests**

- a set of tests used to prove that two triangles are congruent
- the tests are usually referred to by the abbreviations SSS, SAS, AAS and RHS

**corresponding angles**

- angles that are in corresponding positions at each intersection




- in the diagram, the corresponding angles are 1 and 5, 2 and 6, 3 and 7, 4 and 8
- are equal when $AB \parallel CD$

**deductive geometry**

- a system in which results called theorems are produced from a set of basic facts that are accepted to be true

**exterior angle**

- an angle formed when the side of a convex polygon is produced

**matching angles (or sides)**

- angles (or sides) that are in the same (or corresponding) positions in congruent or similar figures

**polygon**

- a plane figure with straight sides
- regular polygons have all sides and angles equal
- convex polygons have all angles either acute or obtuse
- some polygons have special names

**proof**

- a series of steps that establishes the truth of a result

**quadrilateral**

- a polygon with four sides
- there are six special quadrilaterals

**triangle**

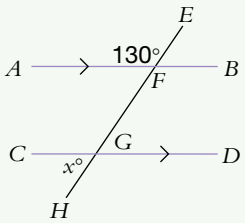
- a polygon with three sides
- equilateral, isosceles and scalene triangles have three sides, two sides and no sides equal in length respectively
- acute-angled triangles have three acute angles
- right-angled triangles have one right angle
- obtuse-angled triangles have one obtuse angle

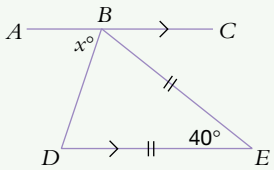
## DIAGNOSTIC TEST 3 PROPERTIES OF GEOMETRICAL FIGURES

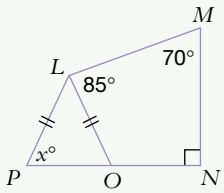
Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

3:01

1 For each figure find the value of $x$, giving reasons.

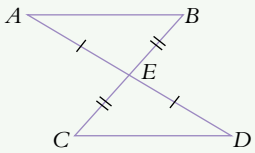
a 

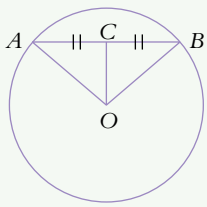

b 

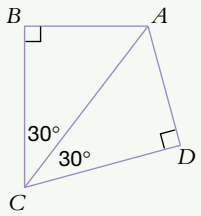

c 


3:02

2 Prove that the two triangles in each pair are congruent.

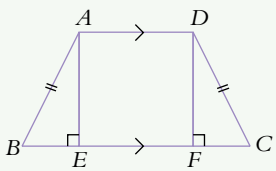
a 


b 

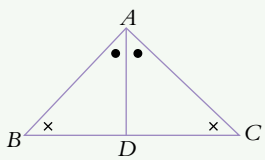

c 


3:03

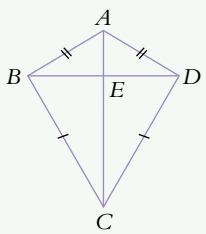
3 a Prove that $\angle ABE = \angle DCF$.



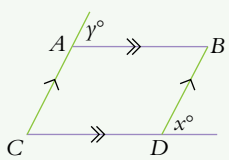

b Use congruent triangles to prove that $AD \perp BC$.



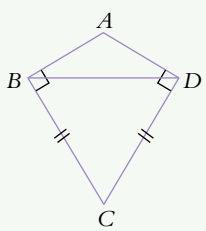

c Use congruent triangles to prove that $\angle CAB = \angle CAD$.



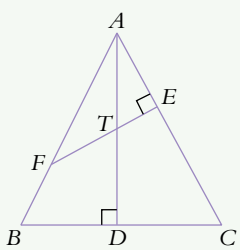

3:04

4 a 


Prove that $x = y$.

b 


Prove that $AB = AD$.

c 


Given $AB = AC$, prove that $\angle ABC = \angle FTD$.

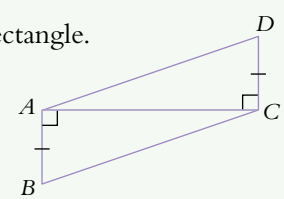
3:05

5 Prove the following.

a If all the angles of a quadrilateral are equal, then it is a rectangle.

b If the diagonals of a quadrilateral are equal in length and bisect each other at right angles, then it is a square.

c What test for parallelograms can be used to prove that $ABCD$ is a parallelogram?




3:06

6 a Find the angle sum of a polygon with 15 sides.

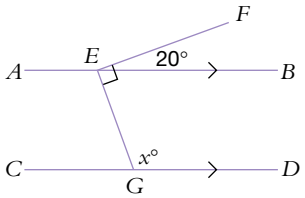
b What is the size of the interior angle in a regular octagon?

c A regular polygon has an exterior angle of 20°. How many sides does it have?

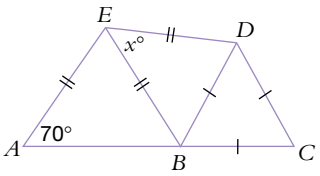
# ASSIGNMENT 3A Chapter review

**1** In each of the following, find the value of $x$. Give reasons for your answer.

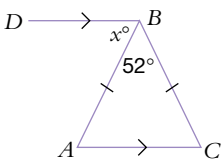
**a**



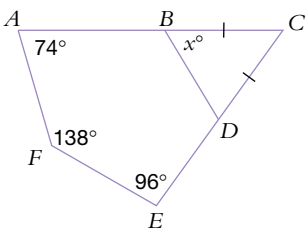

**b**



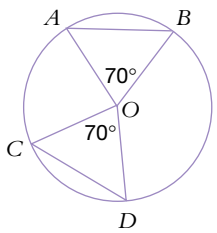

**c**



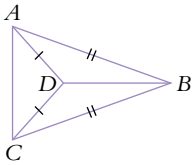

**d**



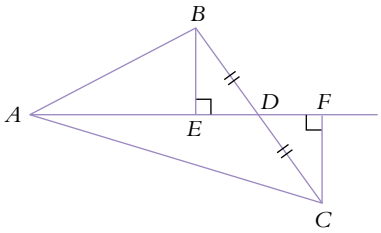

**2** **a** Prove that $\Delta AOB \equiv \Delta COD$.




**b** In the figure $AD = CD$ and $AB = CB$. Use congruent triangles to prove that $\angle ADB = \angle CDB$.




**3**




$\Delta ABC$ is any triangle. $D$ is the midpoint of $BC$, and $BE$ and $CF$ are perpendiculars drawn to $AD$, produced if necessary. Prove that $BE = CF$.

**4** A convex polygon has 20 sides.

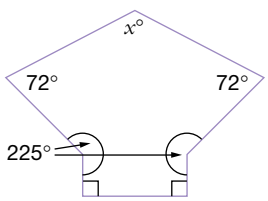
**a** Find the sum of the interior angles.

**b** If the polygon is regular (all its sides and angles are equal) find the size of:

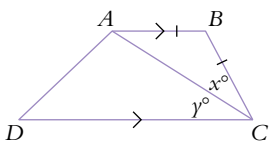
**i** an interior angle

**ii** an exterior angle.

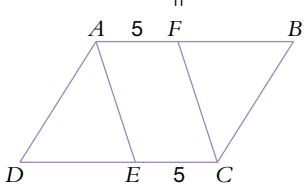
**5** Find the value of $x$.




**6** Prove that $x = y$.

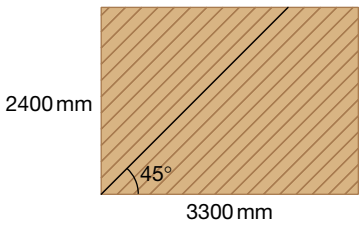



**7** $ABCD$ is a parallelogram. $AF = EC$. Prove that $AE \| FC$.



# ASSIGNMENT 3B Working mathematically

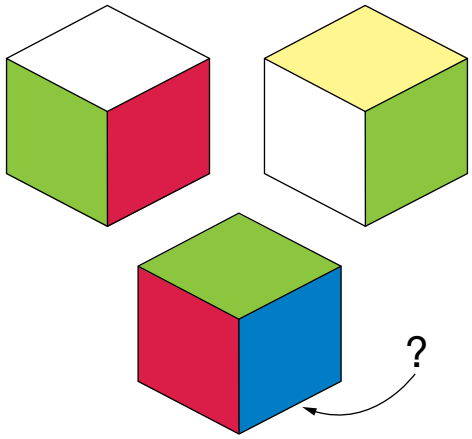
**1** Giri and Todd decide to cover their bathroom wall with cedar boards as shown in the diagram. Each board is inclined at 45° to the horizontal. What is the longest length of board that needs to be bought so that there would be no joins?



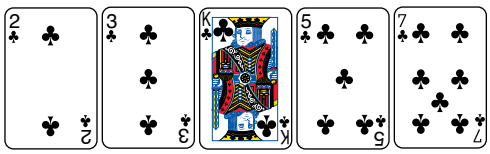

**2** The holes on a golf course have to be rated in order of degree of difficulty. The scores of all the players who played in a recent tournament are recorded in the table. Note that 51 players scored a 3 on hole number 2. Explain what calculations you would use to rate the four holes and list the holes, in order of degree of difficulty, with 1 being the hardest and 4 the easiest. Remember that par is the number of strokes that are allocated to complete a hole and that in golf a lower score is better than a higher score.

| Hole | Par | Golfers' scores | | | | | |
|---|---|---|---|---|---|---|---|
| | | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 4 | 1 | 51 | 310 | 85 | 9 | — |
| 7 | 5 | — | 20 | 211 | 198 | 23 | 4 |
| 11 | 3 | 25 | 269 | 156 | 4 | — | — |
| 16 | 3 | 50 | 292 | 99 | 13 | — | — |

**3** Three views of the same cube are given. Determine the colour of the face on the bottom of the third cube.




**4** In a card game, two points are scored when any combination of cards total 15. (*Note:* An Ace counts as 1 and the Jack, Queen and King count as 10.) For example, for the hand below, the score is 6.



$$2 + 3 + K = 15$$
$$5 + K = 15$$
$$3 + 5 + 7 = 15$$

What would the score be for these hands?

**a** 10, 10, 10, 5, 5

**b** 4, 5, 6, 5, 6

## ASSIGNMENT 3C Cumulative revision

**1** Simplify: 1:02

**a** $\frac{a}{3} \times \frac{a}{2}$ **b** $\frac{a}{3} + \frac{a}{2}$ **c** $\frac{a}{3} \surd \frac{a}{2}$ **d** $\frac{a}{3} - \frac{a}{2}$

**2** Simplify: 1:02

**a** $\frac{3ab}{2} \times \frac{4}{9a}$ **b** $\frac{3ab}{4} \surd \frac{9a}{2}$ **c** $\frac{m^2n^3}{6} \times \frac{15n}{2m^3}$ **d** $\frac{9a^2}{4y^4} \surd \frac{18a^3}{2y^2}$

**3** Convert: 1:01J

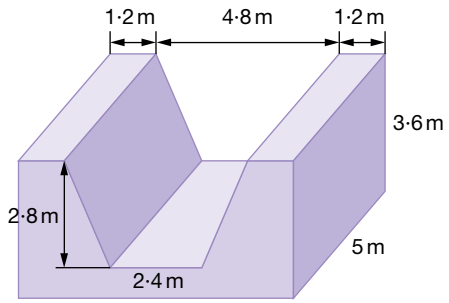
**a** 7·5 MB to B **b** 3·1 ms to μs **c** 5 TB to GB

**4** A metal bar was measured to be 6·3 cm thick. In what range must the actual thickness lie? 1:01K

**5** For the prism shown calculate: 1:04

**a** the area of the cross-section

**b** the volume.




**6** Solve, giving answers to two decimal places where necessary. 2:02, 2:04

**a** $3x^2 + 5x = 0$ **b** $x^2 - 2x - 63 = 0$

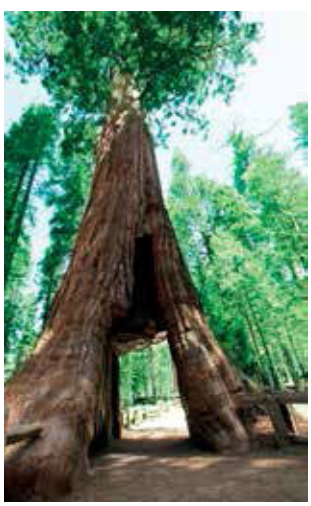
**c** $2x^2 + 7x + 1 = 0$ **d** Solve $x(x - 6) = 2(6 - x)$

**e** $(x - 7)^2 = 5$

**7** **a** Solve $x^2 + 2x - 12 = 0$ leaving your answer in surd form. 2:03, 2:06

**b** Solve this equation $x^4 - 15x^2 + 56 = 0$ using the substitution, $u = x^2$.

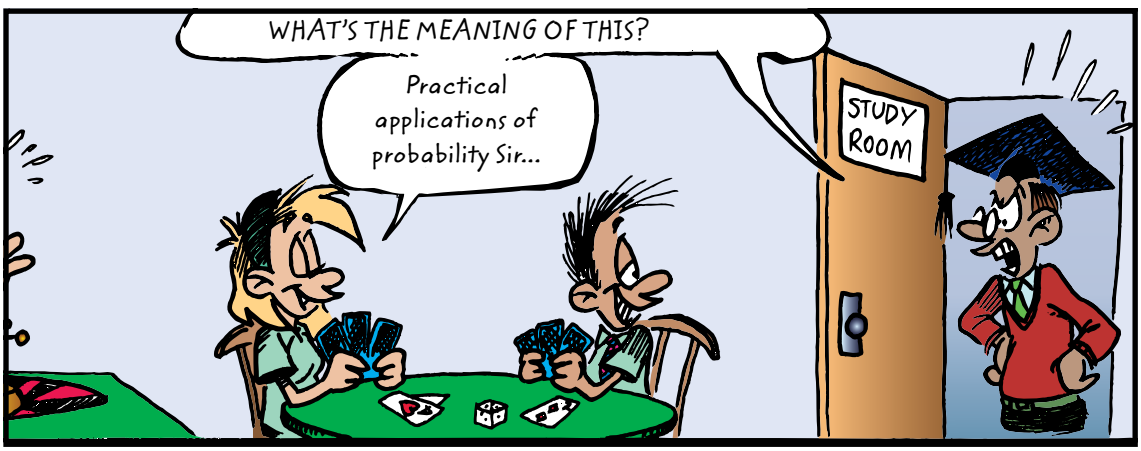
**8** The base of a triangle is 4 cm longer than its height. If the area of the triangle is 48 $cm^2$, what is the length of its base? 2:05



Estimate the total number of people that could stand inside this tree.

# PROBABILITY

# 4




## Contents


**4:01** Probability and language
Investigation 4:01 Probabilities given as odds
**4:02** Two-step chance experiments
Investigation 4:02 Experimental probability without replacement
**4:03** Three-step chance experiments
**4:04** The probability of two- and three-step events
Investigation 4:04 What is the chance of a boy and a girl?
**4:05** The multiplication rule for multi-step events
Fun spot 4:05 The Monty Hall problem
**4:06** The multiplication rule for dependent events
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Statistics and Probability**

Selections from ***Probability*** [Stage 5.2]

- List all outcomes for two-step chance experiments, with and without replacement, using tree diagrams or arrays; assign probabilities to outcomes and determine probabilities for events (ACMSP225)
- Describe the results of two- and three-step chance experiments, with and without replacement, assign probabilities to outcomes, and determine probabilities of events; investigate the concept of independence (ACMSP246)
- Use the language of 'if … then', 'given', 'of', 'knowing that' to investigate conditional statements and to identify common mistakes in interpreting such language (ACMSP247)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 4:01 Probability and language

- **Probability, $P(E)$**, is the chance of an event happening. This will be a number from 0 to 1 and can be expressed as a fraction, a decimal or a percentage.
  **Theoretical probability** is the likelihood of an event occurring under ideal circumstances.
  e.g. The probability of rolling a 6 on a normal dice is $\frac{1}{6}$.
  **Experimental probability** is the predicted chance of occurrence using the relative frequencies taken from a chance experiment.
  **Subjective probabilities** are probabilities based on opinions or judgements.
- The **sum of the probabilities** of all possible outcomes is always 1, or 100%.

The theoretical probability of an event, $E$, is the number of times the event can occur, divided by the total number of possible outcomes (as long as each outcome is equally likely to occur).

$$P(E) = \frac{n(E)}{n(S)}$$

$S$ represents the sample space, which is the set of possible outcomes.

## Complementary events

- The **complement** of 'an event happens' is 'the event does not happen'.
  The complement of $P(E)$ is written $P(E')$ or $P(\overline{E})$.
  Two events are complementary if the sum of their probabilities is 1.
  $P(E) + P(E') = 1$ or $P(E') = 1 - P(E)$

## Other terms

- The **sample space** is the set of all possible outcomes, e.g. 1, 2, 3, 4, 5, 6 on a normal dice.
- Choosing at **random** is choosing in a way that does not affect the likelihood of any outcome.
- **A trial** is one case of carrying out a chance experiment, e.g. spinning a spinner.
- **Equally likely outcomes** are present where each of the outcomes has the same chance of occurring.

- **Mutually exclusive events** have no elements in common, e.g. tossing an odd number and tossing an even number on a dice.
- **Non-mutually exclusive events** have some elements in common, e.g. rolling an odd number and rolling a number less than 4.
- The **relative frequency** of an event is the fraction of times that the event occurs. (This can also be written as a percentage.) This is the same as the experimental probability of the event occurring.

## Conditional statements

- **Conditional statements** restrict the sample space by giving extra information regarding the event.

**Example 1:**

Given that an even number has been rolled on a normal six-sided dice, calculate the probability that a 4 has been rolled.

The sample space on a normal dice is $\{1, 2, 3, 4, 5, 6\}$.

The probability of rolling a 4 on a normal dice is $\frac{1}{6}$.

However, the sample space has been reduced to $\{2, 4, 6\}$.

This knowledge reduces the question to: 'What is the chance of choosing a 4 from $\{2, 4, 6\}$?'

The probability that a 4 has been rolled is clearly $\frac{1}{3}$.

**Example 2:**

What is the chance that six heads have been tossed in a row on a fair coin, given that the first five tosses were heads?

The chance of tossing six heads in a row is $\frac{1}{64}$.

We know that the first five tosses are heads, so the only toss in question is the sixth toss. The chance of this being a head is $\frac{1}{2}$. The chance that six heads have been tossed in a row (given the extra information) is $\frac{1}{2}$.

## WORKED EXAMPLES

A card is chosen at random from a standard pack of 52 playing cards (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King in each of four suits: hearts, diamonds, clubs and spades).

**a** What is the probability of choosing the 8 of hearts?

The 8 of hearts is one card out of the 52 in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

$$P(8\text{H}) = \frac{1}{52}$$

**b** What is the probability of not choosing the 8 of hearts?

This is the complement of choosing the 8 of hearts.

$$P(E') = 1 - P(E)$$

$$P(\text{not } 8\text{H}) = 1 - \frac{1}{52}$$

$$= \frac{51}{52}$$

## Exercise 4:01

P Foundation worksheet 4:01 Probability review

**1** Which of these experiments have equally likely outcomes?

**A** spinning a tennis racquet to determine who serves

**B** selecting the winner of a raffle

**C** choosing a job applicant for an employment position from six people who have been interviewed

**2** **a** Describe the chance of each of these events happening, using the terms *certain*, *likely*, *unlikely*, *impossible* and *even chance*.

**A** It will snow in Sydney some time during the next week.
**B** We will have rain tomorrow.
**C** There will be a holiday on 25 December.
**D** I can walk from home to the Sydney Harbour Bridge without resting.
**E** I will travel overseas next month.
**F** I will live to the age of 94.
**G** I will sit for a mathematics test during the next three months.
**H** My favourite sporting team will win at least one of its next three matches.

**b** Place the events **A** to **H** in order from least likely to most likely.

**3** **a** Describe two events that have a probability of about $\frac{1}{2}$.
**b** Describe two events that have a probability of less than $\frac{1}{2}$.
**c** Describe two events that have a probability of more than $\frac{1}{2}$.

**4**

| certain | probable | even chance | improbable | impossible |
|---|---|---|---|---|

Which of the words above best describes an event that has a probability of:

**a** 0·5 **b** 2% **c** $\frac{17}{20}$ **d** 0 **e** 1 **f** 50%?

Write the answers to probability questions as fractions unless told otherwise.

**5** A glass contains eight counters. One is white, three are yellow, two are green and two are red. A counter is drawn at random. What is the probability that the counter is:

**a** white **b** red **c** white or yellow
**d** not white **e** not red **f** black or blue?

**6** A dice is rolled once. What is the probability of throwing:

**a** a 2 **b** a 6 **c** anything but a 6
**d** 2 or less **e** a 0 **f** a number less than 10?

**7** A standard pack of cards has four suits: hearts and diamonds (both red), and spades and clubs (both black). In each suit there are 13 cards: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The Jack, Queen and King are called picture (court) cards. From a standard pack of cards, a card is chosen at random. What is the probability that the card is:

**a** black **b** not black **c** yellow
**d** a 5 **e** not a 5 **f** a picture card
**g** a club **h** not a club **i** an Ace or a King?

From the results of an experiment or sample, we can calculate the experimental probability. This is used as an estimate of the theoretical probability.

**8** A factory tested a sample of 500 CDs and found four to be faulty. Use these results to estimate the probability that a CD produced by this factory is:

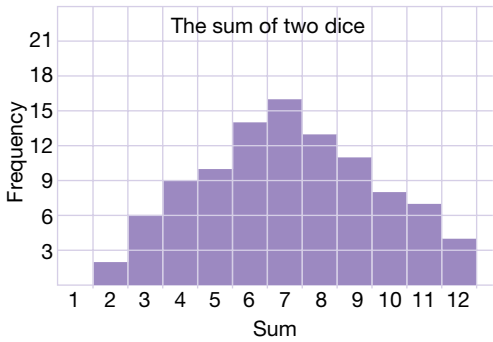
**a** faulty **b** not faulty.

**9** Two dice were thrown 100 times and the sum of the upper faces was recorded. The results are shown on this graph. Use these results to find the experimental probability of throwing a total:

**a** of 7 **b** of 6

**c** greater than 7 **d** other than 6

**e** of 6 or 7 **f** that is even.




**10** High school students were asked to vote on the predominant colour to be used on our Olympic Games uniform for the opening ceremony. The results are shown in the table on the right. If one of these people is chosen at random, what is the probability (as a percentage) that the person:

**a** is female

**b** chose gold

**c** is a male who chose orange

**d** is a female who chose green

**e** is not a female who chose green

**f** chose either green or gold

**g** chose neither green nor gold?

| | Male | Female |
|---|---|---|
| Green | 38 | 40 |
| Gold | 44 | 32 |
| Blue | 7 | 8 |
| Orange | 2 | 14 |
| Fawn | 7 | 3 |
| Other | 2 | 3 |
| Total | 100 | 100 |

Number of students = 200

**11** A barrel contains 100 blue tickets numbered 1 to 100, 50 red tickets numbered 1 to 50, and 50 green tickets numbered 1 to 50. If a ticket is drawn at random from the barrel, what is the probability that it is:

**a** green **b** green or red

**c** a 36 **d** a 72

**e** less than 51 **f** less than 60

**g** not a 50 **h** not less than 60

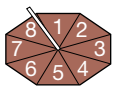
**i** either a 36 or a 72?

**12** **a** If I have already tossed three 6s in a row on a normal dice, what is the chance that I will make it four 6s in a row with my next toss?

**b** If only heads have been tossed in the first three tosses of a coin, what is the chance that the first four tosses will all be heads?

**c** Two numbers are to be chosen at random from the numbers 1 to 100. The number 9 is the first chosen. From those remaining a second number must be chosen. What is the chance that the two numbers will be 9 and 10?

**A**

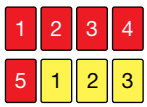



The spinner is spun.

**B**

The spinner is spun.

**C**




A card is chosen at random, then returned.

**D**

The spinner is spun.

**13** **a** Using spinner A, I spin a 5 on my first spin. What is the chance that I will spin a 5 on the second spin?

**b** Each time a card is chosen at random from C it is replaced. If the last three choices have been red, what is the chance that the next choice will be red?

**c** Given that the result of spinning A was an even number, what is the probability that the spin was a 6?

**d** Given that spinning D produced a consonant, what is the probability that U was spun?

**e** Given that spinning D produced a consonant, what is the probability that an R or an S was spun?

**f** Given that a 2 has been chosen from C, what is the chance that the card is red?

**g** Of the possible outcomes in spinning B, we know that 'orange', 'yellow' and 'white' were not spun. What is the chance that 'pink' was spun?

**h** Knowing that a number spun on A is greater than 5, what is the probability that the number spun was 7?

**i** Knowing that a card chosen from C is red, what is the probability that it is a 3?

**j** Spinner A was spun. Knowing that the result was even, what is the probability that the number is 8?

**14** **a** The weather prediction on Tuesday was 'for the weekend, there is a 50% chance that it will rain on Saturday, and a 60% chance that it will rain on Sunday'. It did not rain on Saturday. Discuss what effect this might have on the chance of it raining on Sunday.

**b** I have tossed a coin 20 times and it has landed 'heads' every time. Discuss this situation, and the chance that my next toss will be a head.

**15** Three students, A, B and C, are placed in a line in random order.
The possible orders are ABC, ACB, BAC, BCA, CAB and CBA.
Use this list to find the probability that:

**a** A is placed before B in the order

**b** A is placed before both B and C

**c** A is placed between B and C

**d** A is placed before either B or C

**e** A is not placed before either B or C.

**16** **a** The probability that two cards drawn without replacement from a standard pack of cards will both be hearts is $\frac{3}{51}$. What is the probability that the two cards drawn are *not* both hearts?

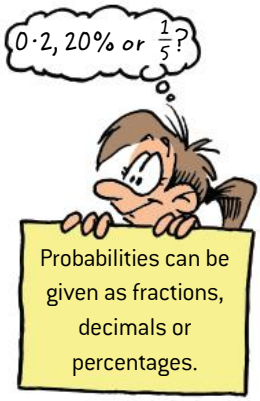
**b** The probability of throwing a sum of either 5 or 6 using two dice is 0·25. What is the probability that the sum is *neither* 5 *nor* 6?

**c** There is a 37% chance that I will be sent overseas next year. What is the chance that I will not be sent overseas?

**d** If 3 coins are tossed, the probability of getting 3 heads is 0·125.
What is the probability of getting fewer than 3 heads?

**e** **i** The probability of my dying this year is 0·5%. What is the probability that I will not die this year?

**ii** To insure my life for $10 000 during this year would cost me $135. Does this represent good value? Explain your answer.



## INVESTIGATION 4:01 PROBABILITIES GIVEN AS ODDS

### The use of odds in gambling

- Odds such as 5/1, 11/4, even money, 2/1 and 2/1 on (or 1/2) are used to give the ratio of what could be gained to what could be lost.
- The fact that for odds of 5/1, $5 might be won while only $1 might be lost, suggests that the probability of success would be 1 chance in 6. This is rarely the case, however, as odds are usually set so that the gambler is at a disadvantage. Odds of 5/1 may be offered when the probability is really 1 chance in 10. This is how the bookmaker, betting agency or insurance company makes a profit.
  This means that the ordinary gambler is sure to lose over time.
- Odds offered on a horse race or a football match are subjective, as each of these events occurs only once. This is an application of subjective probability.

### Discussion

1 What do odds of 10/1 mean?

2 Explain why the ordinary gambler must lose over time.

3 Is it healthy for a government to encourage people to gamble? Why, or why not?

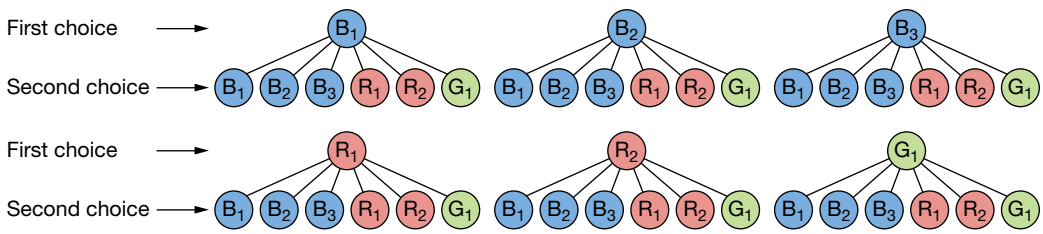
# 4:02 Two-step chance experiments

- For a chance experiment, listing all possibilities allows us to find **theoretical probabilities**.
- Carrying out the experiment allows us to find the **experimental probability** of each event.
- The experiments below involve **sampling**, the name given to the repeated drawing of items from a collection.

## Experiment 1: Choosing two counters with replacement

A bag contains one green, two red and three blue counters.
A counter is chosen at random, *replaced*, and then a second counter is chosen.

**Finding the theoretical probabilities (with replacement)**
The tree diagram shows the possible outcomes for this two-step chance experiment.

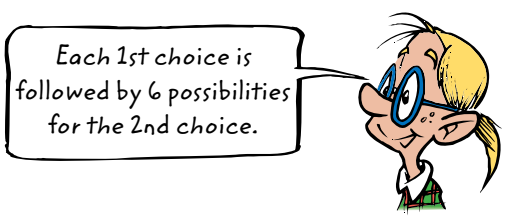



This table shows the same possible outcomes that are shown in the tree diagram above.

The outcomes for the two following cases are indicated by the coloured dots.

Second choice

| First choice | $B_1$ | $B_2$ | $B_3$ | $R_1$ | $R_2$ | $G_1$ |
|---|---|---|---|---|---|---|
| $B_1$ | BB | BB | BB | BR | BR | BG |
| $B_2$ | BB | BB | BB | BR | BR | BG |
| $B_3$ | BB | BB | BB | BR | BR | BG |
| $R_1$ | RB | RB | RB | RR | RR | RG |
| $R_2$ | RB | RB | RB | RR | RR | RG |
| $G_1$ | GB | GB | GB | GR | GR | GG |




**Case 1**

**a** There are 9 ways of choosing two blues out of the 36 possible outcomes.
$P(\text{two blues}) = \frac{9}{36}$ or 0·25 or 25%

**b** What is the effect of knowing the result of the first step of this two-step experiment?
If we know the first choice was blue, then only the 18 outcomes starting with 'blue' are possible. Of these, 9 give us two blues.
$P(\text{two blues, knowing the first choice is blue}) = \frac{9}{18}$ or 0·5 or 50%

**Case 2**

**a** There are 4 ways of choosing a red and a green out of 36 possible outcomes.
$P(\text{red and green}) = \frac{4}{36}$ or $0\dot{1}$ or $11\frac{1}{9}\%$

**b** What is the effect of knowing the result of the first step of this two-step experiment?
If we know that the first counter was red, then only the 12 outcomes starting with 'red' can occur. Of these, 2 produce the outcome 'red and green'.
$P(\text{red and green, knowing the first choice is red}) = \frac{2}{12}$ or $0\dot{1}\dot{6}$ or $16\frac{2}{3}\%$

Knowing the result of the first step of a two-step experiment changes the probabilities.
If we know that the result of the first step was 'green', then 'two blues' can't occur.

**Finding experimental probabilities (with replacement)**

- If we carry out Experiment 1 a large number of times, we can find an experimental probability for each event.
- If we carry out Experiment 1 many times, we will get a good approximation of the theoretical probabilities.

Here 100 trials of Experiment 1 have been used, so we can express the experimental probabilities as percentages.

We can use the table to find the experimental probabilities of different events. For example:

$P(\text{two blues}) = \frac{26}{100}$ or 26%

$P(\text{red and green}) = \frac{11}{100}$ or 11%

$P(\text{at least one blue}) = \frac{26 + 32 + 16}{100}$ or 74%

$P(\text{exactly one blue}) = \frac{32 + 16}{100}$ or 48%

Here are the results of 100 trials of Experiment 1.

With replacement

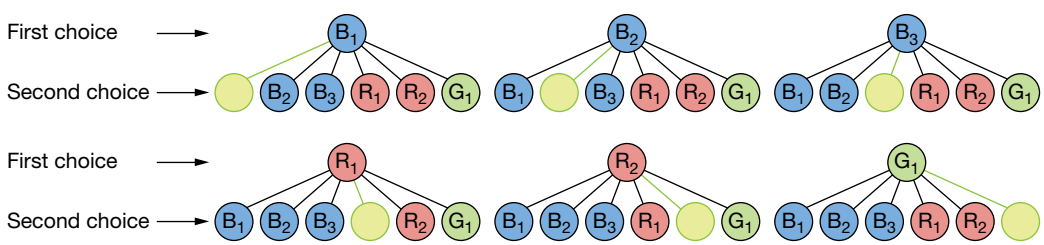
| Outcome | Frequency |
|---|---|
| two blue | 26 |
| two red | 12 |
| two green | 3 |
| blue and red | 32 |
| blue and green | 16 |
| red and green | 11 |
| *Total:* | 100 |

## Experiment 2: Choosing two counters without replacement

A bag contains one green, two red and three blue counters.
A counter is chosen at random and *not replaced*. A second counter is then chosen.
(This is the same as taking two counters at the same time.)

**Finding the theoretical probabilities (without replacement)**

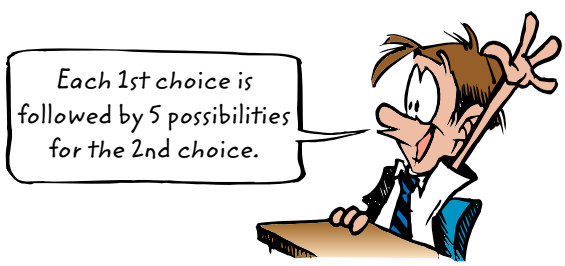
The tree diagram shows the outcomes of this two-step chance experiment.
The counter drawn in the first choice cannot be chosen as the second choice.




This table shows the same possible outcomes that are shown in the tree diagram above.

The outcomes for the two following cases are indicated by the coloured dots.

| First choice \ Second choice | $B_1$ | $B_2$ | $B_3$ | $R_1$ | $R_2$ | $G_1$ |
|---|---|---|---|---|---|---|
| $B_1$ | | BB | BB | BR | BR | BG |
| $B_2$ | BB | | BB | BR | BR | BG |
| $B_3$ | BB | BB | | BR | BR | BG |
| $R_1$ | RB | RB | RB | | RR | RG |
| $R_2$ | RB | RB | RB | RR | | RG |
| $G_1$ | GB | GB | GB | GR | GR | |



**Case 1**

**a** There are now 6 ways of getting two blues out of the 30 possible outcomes.
$P(\text{two blues}) = \frac{6}{30}$ or $\frac{1}{5}$ (This is a 20% chance.)

**b** What is the effect of knowing the result of the first step of this two-step experiment?
If we know the first choice was blue, then now, only the 15 outcomes starting with 'blue' are possible. Of these, 6 are two blues.
$P(\text{two blues, knowing the first choice is blue}) = \frac{6}{15}$ or $\frac{2}{5}$ (This is a 40% chance.)

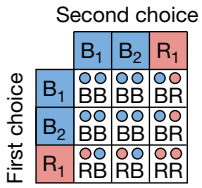
**Case 2**

**a** We can see that there are 4 ways of getting a red and a green out of 30 possible outcomes.
$P(\text{red and green}) = \frac{4}{30}$ or $\frac{2}{15}$ (This is a $13\frac{1}{3}$% chance.)

**b** What is the effect of knowing the result of the first step of this two-step experiment?
If we know that the first counter was red, then now, only the 10 outcomes starting with 'red' can occur. Of these, 2 are 'red and green'.
$P(\text{red and green, knowing the first choice is red}) = \frac{2}{10}$ or $\frac{1}{5}$ (This is a 20% chance.)

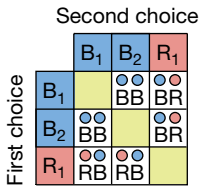
## Exercise 4:02

**1** Two counters are chosen at random, **with replacement**, from two blue counters and one red counter. The table shows all possible outcomes.
Find the probability of choosing:



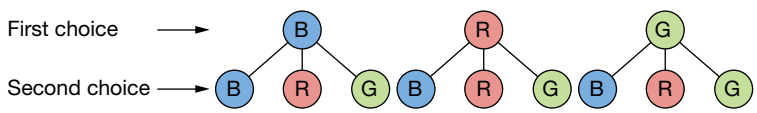

**a** two blue counters
**b** two red counters
**c** a blue counter and a red counter
**d** a blue then a red
**e** two counters that are not the same colour.

**2** Two counters are chosen at random, **without replacement**, from two blue counters and one red counter. The table shows all possible outcomes.
Find the probability of choosing:




**a** two blue counters
**b** two red counters
**c** a blue counter and a red counter
**d** a blue then a red
**e** two counters that are not the same colour.

**3** Two counters are chosen at random, **with replacement**, from one blue counter, one red counter and one green counter. This tree diagram shows all possible outcomes.



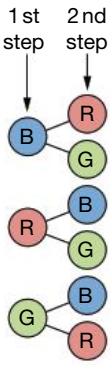

Find the probability of choosing:

**a** two blue counters
**b** two counters the same colour
**c** a blue and a green
**d** a blue then a red
**e** two counters that are not the same colour.

If we know that the first choice was blue, what is now the chance of choosing:

**f** two blue
**g** two red
**h** a blue and a green
**i** two counters of different colour?

**4** Two counters are chosen at random, **without replacement**, from one blue, one red and one green. This tree diagram shows all possible outcomes.




Find the probability of choosing:

**a** two blue counters
**b** two counters the same colour
**c** a blue and a green
**d** a blue then a green
**e** a blue as either choice
**f** a blue as the second choice
**g** no green counter
**h** two that are not the same colour.

If we know that the first choice was blue, what is now the chance of choosing:

**i** two blue
**j** two red
**k** a blue and a green
**l** two counters of different colour?

**5** 100 green raffle tickets and 100 orange raffle tickets have been placed in a barrel. Each ticket displays one of the numbers from 00 to 99. The only ticket I bought was Green 23.

**a** Before the ticket was chosen, what was my chance of winning?
**b** It was announced that the winning ticket was green. What was then my chance of winning?
**c** It was announced that the winning ticket was green and had a first digit of 2. What was then my chance of winning?
**d** It was announced that the winning ticket was number 23 but no colour was given. What was then my chance of winning?
**e** If it was announced that the winning ticket was orange, what would then be my chance of winning?

**6** Explain the effect on the probabilities of knowing the result of the first step in a two-step (or three-step) chance experiment. Use your answers to Question **5** to help with the explanation.

## INVESTIGATION 4:02 EXPERIMENTAL PROBABILITY WITHOUT REPLACEMENT

Here we will find experimental probabilities for Experiment 2 on page 100.
A bag contains one green, two red and three blue counters. One counter is taken at random. This is not replaced before the second counter is taken.

**1** Use 100 trials so you can express the experimental probabilities as percentages. Complete the table on the right (after keeping a tally). Use your results to give the experimental probability of:

**a** two blues
**b** a red and a green
**c** the two counters having at least one blue
**d** the two counters having exactly one blue.

**2** Compare your results for this experiment with the results in the table given for Experiment 1 on page 100.

**a** Explain why the probabilities are different for 'two greens'.
**b** List other outcomes that have significantly different experimental probabilities to those in Experiment 1.

Results of 100 trials of Experiment 2.

With replacement

| Outcome | Frequency |
|---|---|
| two blue | |
| two red | |
| two green | |
| blue and red | |
| blue and green | |
| red and green | |
| *Total:* | 100 |

# 4:03 Three-step chance experiments

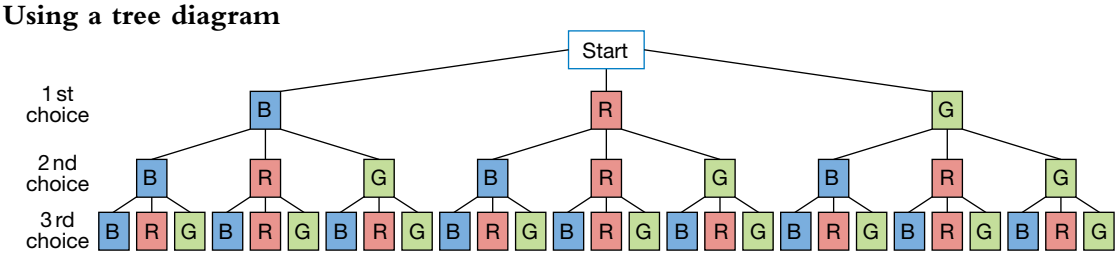
The methods we use here are an extension of the methods used in Exercise 4:02.

## Choosing three tickets with replacement

**Case 1**

A bag contains one blue, one red and one green ticket.
Three tickets are to be chosen at random, with replacement after each choice is made. One is taken at random, replaced, and then a second is taken. This is replaced and then a third is taken.
The possible outcomes can be shown in a tree diagram or a list (or even a table).

**Using a tree diagram**




- There are 27 possible ordered outcomes. One outcome would be BRB (i.e. blue red blue).
- This form of tree diagram is of limited use, as often the number of possible outcomes is large.

**Making a list**

We could list the outcomes methodically (i.e. start with the first counter mentioned and list all possibilities for that counter, then do the same for each other first choice).

The list would be:

| | | |
|---|---|---|
| BBB, BBR, BBG, | BRB, BRR, BRG, | BGB, BGR, BGG, |
| RBB, RBR, RBG, | RRB, RRR, RRG, | RGB, RGR, RGG, |
| GBB, GBR, GBG, | GRB, GRR, GRG, | GGB, GGR, GGG |

**Using a table**

Where three choices are made, a table is hard to use. In this example we have combined the second and third choices so that a two-way table could be used.

| | BB | BR | BG | RB | RR | RG | GB | GR | GG |
|---|---|---|---|---|---|---|---|---|---|
| B | BBB | BBR | BBG | BRB | BRR | BRG | BGB | BGR | BGG |
| R | RBB | RBR | RBG | RB | RRR | RRG | RGB | RGR | RGG |
| G | GBB | GBR | GBG | GRB | GRR | GRG | GGB | GGR | GGG |

2nd and 3rd choice in order

We can use the tree diagram, the list or the table to write the probabilities of events for Case 1.

$P(\text{three blue tickets}) = \frac{1}{27}$ (BBB)

$P(\text{three tickets of the same colour}) = \frac{3}{27}$ (BBB, RRR, GGG)

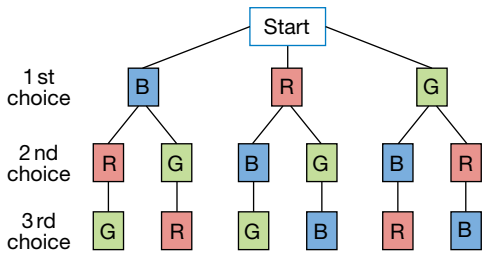
$P(\text{not all the same colour}) = 1 - \frac{3}{27} = \frac{24}{27}$ not (BBB, RRR, GGG)

## Choosing three tickets without replacement

**Case 2**

A bag contains one blue, one red and one green ticket.
Three tickets are to be chosen at random without replacement after each choice is made.
One is taken at random, not replaced, and a second is taken. This is not replaced and a third is taken.

**Using a tree diagram**



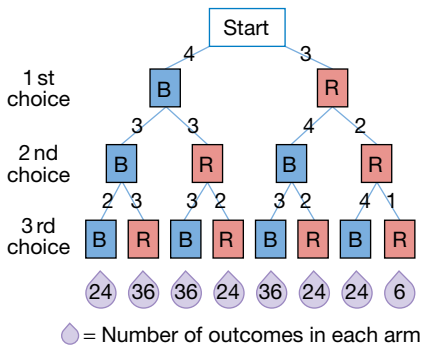

- The 6 possible ordered outcomes are BRG, BGR, RBG, RGB, GBR and GRB.

We can use the tree diagram above to write the probabilities of events for Case 2.
$P$(three blue tickets) = $\frac{0}{6}$ = 0 (This event is impossible.)
$P$(three different colours) = $\frac{6}{6}$ = 1 (This event will always happen.)
$P$(the last choice is green) = $\frac{2}{6}$ = $\frac{1}{3}$ (We need only consider the third choice.)

**Case 3**

A bag contains four blue and three red tickets.
A ticket is taken at random, not replaced, and then a second is taken. This is not replaced and then a third ticket is taken.

- As there are many identical choices possible, we can use numbers on the joining lines to show how many of that category could be chosen. By multiplying along the lines we can find the number of ways each event can occur.
- There are 7 ways of choosing the first ticket. This ticket is not returned, so there are only 6 ways of choosing the second. This ticket is also not returned, so there are only 5 ways of choosing the third.




The number of ways of choosing three tickets = 7 × 6 × 5
= 210

- We can use the tree diagram above to find the number of ways of choosing particular colours. There are 4 ways of choosing the first blue ticket, then 3 ways of choosing the second blue ticket and then 2 ways of choosing the third (once two blue tickets have been removed).

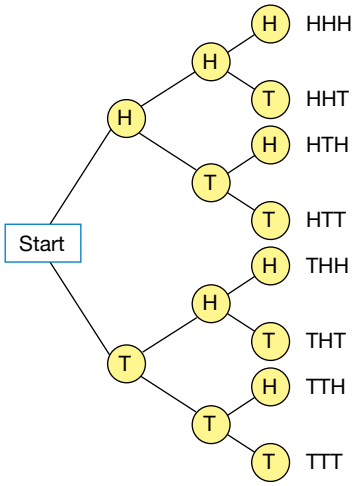
The number of ways of choosing three blue tickets = 4 × 3 × 2
= 24

- A blue and two red tickets can be chosen as BRR, RBR or RRB.

The number of ways of choosing a blue and two reds = (4 × 3 × 2) + (3 × 4 × 2) + (3 × 2 × 4)
= 24 + 24 + 24
= 72

## Exercise 4:03

**1** A coin is tossed three times. All possible outcomes are shown on this tree diagram.

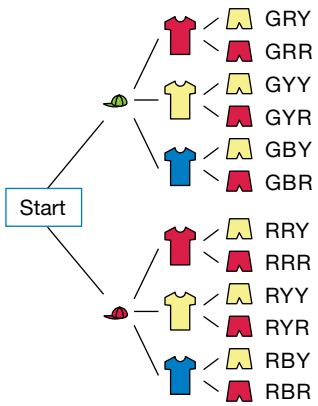



**a** How many possible outcomes are there?
**b** How many possibilities exist for each of the three tosses?
**c** Is the number of possible outcomes 2 × 2 × 2?
**d** How many outcomes contain a head and two tails?
**e** How many outcomes contain a tail and two heads?
**f** How many outcomes contain three tails?

What is the probability that there will be:

**g** three tails
**h** three heads
**i** a head and two tails
**j** a tail and two heads
**k** no heads
**l** at least two heads?

**2** Rohan chooses an outfit at random that consists of a hat, a shirt and a pair of pants. He chooses from two different hats, three different shirts and two different pairs of pants. This tree diagram shows all of the possible choices he can make.




In how many ways can he choose:

**a** a hat
**b** a shirt
**c** a pair of pants
**d** an entire outfit?
**e** Is the number of possible outcomes 2 × 3 × 2?

How many outfits have:

**f** three red items
**g** a green item
**h** two yellow items
**i** a red or green item
**j** a blue item
**k** exactly two red items?

What is the probability that the outfit chosen will have:

**l** three red items
**m** a green item
**n** two yellow items
**o** a red or green item
**p** a blue item
**q** exactly two red items?

**3** Case 1 on page 103 displays a diagram, a list and a table that represent the choosing of three tickets, drawn at random (with replacement) from a bag containing one blue, one red and one green ticket. Use those displays to answer these questions.

After each choice the ticket is replaced.

**a** How many possible outcomes are there?
**b** How many outcomes have three green tickets?
**c** How many outcomes have fewer than three green tickets?
**d** How many outcomes have exactly one green ticket?

What is the probability that the tickets chosen will be:

**e** three greens
**f** three blues
**g** a green and two reds
**h** a red and two greens
**i** no greens
**j** at least two greens?

**4** Case 2 on page 104 displays a diagram and a list that represent the choosing of three tickets, drawn at random (without replacement) from a bag. The bag contains one blue, one red and one green ticket. Use the displays to answer the questions.

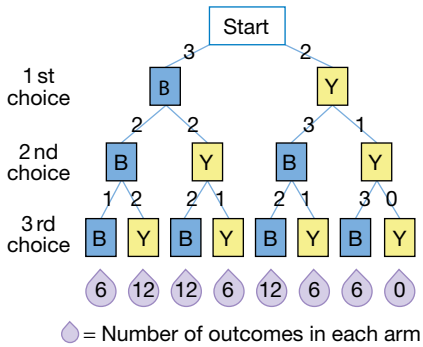
**a** How many possible outcomes are there?
**b** How many outcomes have three green tickets?
**c** How many outcomes have one green ticket?
**d** How many outcomes have green as the first choice?

After each choice the ticket is **not** replaced.

What is the probability that the tickets chosen will be:

**e** three greens **f** no greens **g** one green
**h** green as the first choice **i** red as the first or second choice?

**5** This tree diagram represents the choosing of three tickets from a bag, at random (without replacement). The bag contains three blue and two yellow tickets.




Use this tree diagram to answer these questions.

**a** How many possible outcomes are there?
**b** Is the number of possible outcomes $5 \times 4 \times 3$?
**c** How many outcomes have three blue tickets?
**d** How many outcomes produce a blue and two yellow tickets?
**e** How many outcomes have one yellow ticket?
**f** How many outcomes have yellow as the first choice?
**g** Explain why it is impossible to choose three yellow tickets.

What is the probability of choosing:

**h** three blues
**i** no yellows
**j** exactly one yellow
**k** yellow as the first choice
**l** blue as the first or second choice?

*Note:*
First choice could be any one of the 5 original tickets.
Second choice will be one of the remaining 4 tickets.
Third choice will be one of the remaining 3 tickets.
There are 18 ways of choosing 'a blue and 2 yellows':

- 6 in the order BYY
- 6 in the order YBY
- 6 in the order YYB.

**6** A bag contains four blue and three red tickets. A ticket is taken at random and not replaced. A second ticket is taken and is not replaced. A third ticket is taken and not replaced.
Use the diagram for Case 3 on page 104 to answer the following questions.

**a** How many possible outcomes are there?

How many ways are there of choosing:

**b** three blue tickets **c** a blue and two red tickets
**d** three red tickets **e** less than three red tickets
**f** a red ticket as the third choice **g** a red and two blue tickets?

What is the probability of choosing:

**h** three blue tickets **i** a blue and two red tickets
**j** three red tickets?

# 4:04 The probability of two- and three-step events

To find the total number of outcomes in a two-step or three-step experiment, multiply the numbers of possible choices in each step of the experiment. For example, if a coin is tossed and a dice is rolled, the 12 possible outcomes could be shown in a table.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| H | H1 | H2 | H3 | H4 | H5 | H6 |
| T | T1 | T2 | T3 | T4 | T5 | T6 |

We could find this result by multiplying the number of outcomes for the coin toss (2) by the number of outcomes for the dice roll (6).

Total number of possible outcomes = 2 × 6
= 12

**Discussion**

Two of these counters are to be chosen at random, without replacement.

- To choose two blue counters, we must choose a blue counter at each choice (i.e. B then B).
  There are 5 ways of choosing a blue as the first choice.
  There are 4 ways of choosing a blue as the second choice once a blue is removed.
  $n$(blue then blue without replacement) = 5 × 4
  This is: the number in choice 1 × the number in choice 2.
- We can choose a blue and a red counter by choosing a blue then a red or by choosing a red then a blue (i.e. B then R, or R then B).
  $n$(blue then red without replacement) = 5 × 2
  $n$(red then blue without replacement) = 2 × 5
  So the number of ways of choosing a blue and a red = (5 × 2) + (2 × 5)
  = 20
- If two coins are tossed at the same time, or two tickets are chosen at once, we can treat these as happening one after the other.

$n(E)$ means the number of ways of choosing event $E$.

## PREP QUIZ 4:04

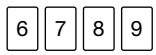
Use the diagram above showing one green, five blue and two red counters to find the number of ways of choosing:

1 any two counters
2 two red counters
3 two blue counters
4 a green then a blue counter
5 a blue then a green counter
6 a green and a blue counter
7 a green then a red counter
8 a blue then a red counter
9 a red then a blue counter
10 a blue and a red counter.

**To find the number of ways an event involving two or three steps can occur, multiply together the number of successful ways of performing each of the steps involved.**

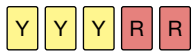
- There are are two square numbers (1 and 4) on a dice. Each roll of a dice has two possible successful outcomes for square numbers. If the dice is rolled three times, the number of ways of obtaining three square numbers = 2 × 2 × 2 = 8

- Four cards marked 6, 7, 8 and 9 are used to show a three-digit number. Cards can be used only once in each number.




  The number of ways of showing a three-digit number = 4 × 3 × 2 = 24

- Two cards are chosen randomly from five cards; three are yellow and two are red. A yellow and a red ticket can be chosen in two ways (YR or RY).




  Note that there may be more than one way to obtain a certain outcome.

  The number of ways of choosing a red and yellow card = (3 × 2) + (2 × 3) = 12

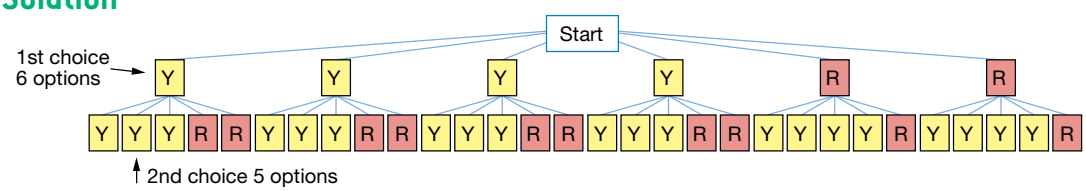
## WORKED EXAMPLE 1

A box contains four yellow and two red tickets. Two tickets are chosen at random without replacement.

**a** What is the probability of choosing two yellow tickets?

**b** What is the probability that the two tickets will be different colours?

### Solution




**a** There are 6 options for the first choice and 5 options for the second choice, after the first ticket has been removed. The total number of outcomes = 6 × 5

To obtain two yellow tickets we must choose a yellow then a yellow.

We multiply the number of ways of choosing yellow as the first choice by the number of ways of choosing yellow as second choice, after the first yellow is removed.

The number of ways of selecting two yellow tickets = 4 × 3

$$P(\text{choosing 2 yellow tickets}) = \frac{\text{number of ways of choosing two yellows}}{\text{total number of outcomes}}$$

$$= \frac{4 \times 3}{6 \times 5}$$

$$= \frac{2}{5} \text{ or } 40\%$$

$P(E) = \dfrac{n(E)}{n(S)}$

The probability of choosing two yellow tickets is $\frac{2}{5}$.

**b** We can choose one ticket of each colour by choosing yellow then red, or red then yellow. So the number of ways of choosing one of each colour = (4 × 2) + (2 × 4) or 16 ways.

$$P(\text{choosing a yellow and a red}) = \frac{\text{number of ways of choosing a yellow and a red}}{\text{total number of outcomes}}$$

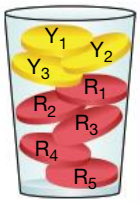
$$= \frac{(4 \times 2) + (2 \times 4)}{6 \times 5}$$

$$= \frac{16}{30} \text{ or } \frac{8}{15}$$

$P(E) = \dfrac{n(E)}{n(S)}$

## WORKED EXAMPLE 2

A glass contains three yellow and five red counters. Three counters are to be chosen at random, without replacement.




**a** What is the probability of choosing three red counters?

**b** What is the probability of choosing two yellows and a red?

### Solution

**a** There are $8 \times 7 \times 6$ ways of selecting three counters, in order, without replacement.

We multiply the 8 options for Choice 1 by the 7 options for Choice 2 and then by the 6 options for Choice 3.

There are $5 \times 4 \times 3$ ways of selecting three red counters, in order, without replacement.

$$P(\text{choosing 3 red counters}) = \frac{\text{number of ways of choosing 3 reds}}{\text{total number of outcomes}}$$

$$= \frac{5 \times 4 \times 3}{8 \times 7 \times 6}$$

$$= \frac{5}{28}$$

$P(E) = \dfrac{n(E)}{n(S)}$

The probability of choosing three red counters is $\frac{5}{28}$.

**b** We can choose two yellows and a red by choosing YYR, YRY or RYY.

For YYR (yellow, yellow, red) there are 3 ways of choosing the first yellow, 2 ways of choosing the second yellow and 4 ways of choosing the red. So there are $3 \times 2 \times 4$ ways of choosing YYR.

So the number of ways of choosing two yellows and a red

$= (3 \times 2 \times 4) + (3 \times 4 \times 2) + (4 \times 3 \times 2)$

$$P(\text{2 yellows and a red}) = \frac{\text{number of ways of choosing YYR, YRY, RYY}}{\text{total number of outcomes}}$$

$$= \frac{(3 \times 2 \times 4) + (3 \times 4 \times 2) + (4 \times 3 \times 2)}{8 \times 7 \times 6}$$

$$= \frac{72}{336} \text{ or } \frac{3}{14}$$

$P(E) = \dfrac{n(E)}{n(S)}$

## Exercise 4:04

**A**

This spinner is spun.

**B**

This coin is tossed.

**C**

This dice is rolled.

**D**

A card is chosen at random.

**1** Find the number of outcomes in the sample space if the experiment involves the two steps:

**a** A and B **b** A and C **c** A and D **d** B and C **e** B and D **f** C and D

Find the probability of:

**g** tossing a head and rolling a 6

**h** choosing the Ace of spades and tossing a tail

**i** spinning a D and rolling a 5

**j** spinning a C and rolling an even number.

**2** Find the number of outcomes in the sample space if the experiment involves the three steps:

**a** A, B and C **b** A, B and D **c** A, C and D

**d** B, C and D **e** B, B and B **f** A, A and A

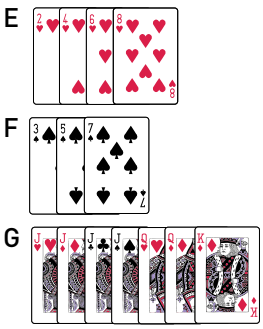
Find the probability of:

**g** tossing a head, rolling a 6 and choosing a red Ace

**h** spinning a C, tossing a head and rolling a 5.

**3** Two cards are to be selected at random from one of the groups shown, **replacing the first card** before the second is chosen. Find the number of ways this can be done using the cards in:

**a** E **b** F **c** G




Find the probability of:

**d** choosing the 2 of hearts followed by the 4 of hearts using Group E

**e** choosing a Jack followed by a Queen using Group G

**f** choosing an odd number followed by a spade using Group F.

**4** Two cards are to be selected at random from one of the groups above, **without replacing the first card**. Find the number of ways this can be done using the cards in:

**a** E **b** F **c** G

Find the probability of:

**d** choosing the 2 of hearts followed by the 4 of hearts from Group E

**e** choosing a Jack followed by a Queen from Group G

**f** choosing an odd number followed by a spade from Group F.

**5** Three cards are to be selected at random from one group above, **without replacement**. Find the number of ways this can be done using the cards in:

**a** E **b** F **c** G

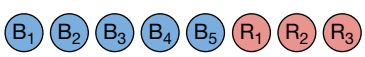
Find the probability of:

**d** choosing Jack, Queen and King in that order from Group G

**e** choosing a diamond then a club then a diamond from Group G

**f** choosing the 2, 4 and 6 in that order from Group E.

**6** Two counters are to be chosen at random from this group of counters, **replacing the first counter** before the second is chosen.




**a** In how many ways can the two counters be chosen? Call this $n(S)$.

**b** **i** In how many ways can we choose two blue counters? Call this $n(E_1)$.

**ii** What is the probability that we will choose two blue counters?

**c** **i** In how many ways can we choose two red counters? Call this $n(E_2)$.

**ii** What is the probability that we will choose two red counters?

**d** **i** In how many ways can we choose a blue counter followed by a red counter? Call this $n(E_3)$.

**ii** What is the probability that we will choose a blue counter followed by a red counter?

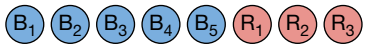
**e** **i** In how many ways can we choose a red counter followed by a blue counter? Call this $n(E_4)$.

**ii** What is the probability that we will choose a red counter followed by a blue counter?

**f** i In how many ways can we choose a blue and a red counter in any order? Call this $n(E_5)$.

ii What is the probability that we will choose a blue and a red counter in any order?

**7** Two counters are to be chosen at random from this group of counters, **without replacement**.




**a** In how many ways can the two counters be chosen? Call this $n(S)$.

**b** i In how many ways can we choose two blue counters? Call this $n(E_1)$.

ii What is the probability that we will choose two blue counters?

**c** i In how many ways can we choose two red counters? Call this $n(E_2)$.

ii What is the probability that we will choose two red counters?

**d** i In how many ways can we choose a blue counter followed by a red counter? Call this $n(E_3)$.

ii What is the probability that we will choose a blue counter followed by a red counter?

**e** i In how many ways can we choose a red counter followed by a blue counter? Call this $n(E_4)$.

ii What is the probability that we will choose a red counter followed by a blue counter?

**f** i In how many ways can we choose a blue and a red counter in any order? Call this $n(E_5)$.

ii What is the probability that we will choose a blue and a red counter in any order?

**8** Two names are to be drawn from a hat at random to determine the captain and vice-captain of our team. Because the same person cannot be both captain and vice-captain, the first name is not replaced before the second is chosen. The names being considered are Hannah, Flynn, Hudson, Alana and Naomi.

**a** In how many ways can the two roles be chosen? Call this $n(S)$.

**b** What is the probability that we will choose Hannah as captain and Hudson as vice-captain?

**c** What is the probability that we will choose Hudson as captain and not Flynn as vice-captain?

**d** What is the probability that we will choose Flynn or Hudson as captain and Alana or Naomi as vice-captain?

**9** Two tickets are to be chosen at random from this group of tickets, **with replacement**.

**a** In how many ways can the two tickets be chosen? Call this $n(S)$.

**b** i In how many ways can we choose two yellow tickets? Call this $n(E_1)$.

ii What is the probability that we will choose two yellow tickets?

**c** i In how many ways can we choose two red tickets? Call this $n(E_2)$.

ii What is the probability that we will choose two red tickets?

**d** i In how many ways can we choose a green ticket followed by a red ticket? Call this $n(E_3)$.

ii What is the probability that we will choose a green ticket followed by a red ticket?

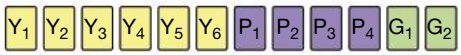
**e** i In how many ways can we choose a red ticket followed by a green ticket? Call this $n(E_4)$.

ii What is the probability that we will choose a red ticket followed by a green ticket?

**f** i In how many ways can we choose a green and a red ticket in any order? Call this $n(E_5)$.

ii What is the probability that we will choose a green and a red ticket in any order?

**10** Three counters are to be chosen at random from this group of counters, **without replacement**.




**a** In how many ways can the three counters be chosen? Call this $n(S)$.

**b** **i** In how many ways can we choose three purple counters? Call this $n(E_1)$.

**ii** What is the probability that we will choose three purple counters?

**c** **i** In how many ways can we choose three yellow counters? Call this $n(E_2)$.

**ii** What is the probability that we will choose three yellow counters?

**d** **i** In how many ways can we choose two green counters followed by a yellow counter? Call this $n(E_3)$.

**ii** What is the probability that we will choose two green counters followed by a yellow counter?

## INVESTIGATION 4:04 WHAT IS THE CHANCE OF A BOY AND A GIRL?

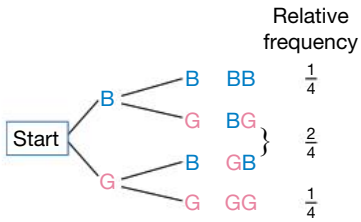
Consider that, overall, there is an even chance of a baby being born a girl or a boy.

We could use this tree diagram to predict that in a family of two children, there is a 25% chance of 2 boys, a 50% chance of a boy and a girl, and a 25% chance of 2 girls.

But is this true?

Could some couples tend to have more children of the same sex, even though about 50% of each sex is born overall?

**The first two children in a family**




This tree diagram shows all possible outcomes.

**1** Survey as many families with two or more children as you can, and record the sex of the first two children born.

| The first two children | Tally | Number | Relative frequency (%) |
|---|---|---|---|
| 2 boys (BB) | | | |
| a boy and a girl (BG or GB) | | | |
| 2 girls (GG) | | | |
| *Total:* | | | |

**2** Combine your data with that of other students, trying not to count the same family more than once. Remember that the more cases we consider, the more accurate our conclusions should be.

- Based on the results of your investigation, do you think that there is a 50% chance that the first two children in the same family will be a boy and a girl?
- If not, what estimate would you give for the chance that the first two children in a family will be a boy and a girl?

**3** Do you think you have collected sufficient data to be sure that your conclusion is correct? Explain why or why not.

# 4:05 The multiplication rule for multi-step events

## Independent events

Two events are independent if the occurrence of the first event does not affect the chance of the occurrence of the second. The probability of the second will not be affected. (One happening or not happening does not influence the chance of the other happening.)

## WORKED EXAMPLES

1 'Tossing a head' on the first toss of a coin does not affect the chance of 'tossing a head' on a second toss of the coin so these events are independent events.

2 'Spinning a 5 on a spinner' and 'rolling a 5 on a dice' are independent events, because the occurrence of the first has no bearing on the occurrence of the second.

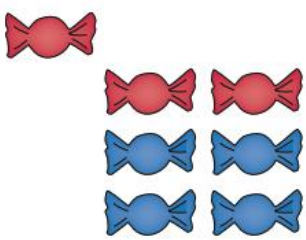
3 Three tickets are to be chosen at random, with replacement, from 4 blue and 4 red tickets. The probabilities involved in any choice will not be affected by the results of earlier choices, because tickets are replaced. 'Choosing a blue ticket as Choice 1', 'choosing a blue ticket as Choice 2' and 'choosing a red ticket as Choice 3' are therefore three independent events.

## Dependent events

Two events are dependent events if the occurrence of the first event affects the chance of the occurrence of the second. The probability of the second will be affected by the outcome of the first. (One happening or not happening influences the chance of the other happening.)

## WORKED EXAMPLE 1

A bag contains three red and four blue lollies. Two lollies are to be chosen at random, without replacement.



The probability of choosing a red lolly as the first choice is $\frac{3}{7}$. If a red is chosen as the first choice and not replaced, only two red and four blue lollies remain for the second choice.

The probability of choosing a red lolly as the second choice is $\frac{2}{6}$.

When the first choice is made, the number of lollies available for the second choice is reduced. The probabilities of the second choice are affected by the first choice, so they are dependent events.

## WORKED EXAMPLE 2

A two-digit number is to be made by randomly choosing two cards in order, without replacement, from five cards marked 6, 6, 7, 7 and 8.

The probability of choosing a 7 as the tens digit is $\frac{2}{5}$. If a 7 is chosen as the tens digit and not replaced, the fours cards remaining are 6, 6, 7 and 8.

The probability of choosing a 7 as the ones digit is $\frac{1}{4}$.

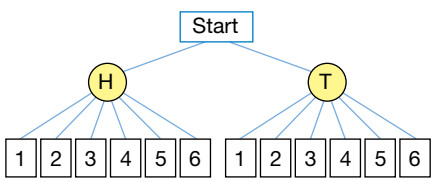
When the first choice is made, the number of cards available for the second choice is reduced. The probabilities of the second choice are affected by the first choice, so they are dependent events.

## The multiplication rule for independent events

When two events $A$ and $B$ are independent, the probability of both occurring is:
$P(A \text{ and } B) = P(A) \times P(B)$

### Explanation

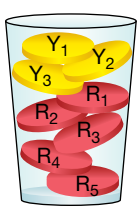
**1** A coin is tossed and a dice is rolled. The tree diagram below shows all possible outcomes. We can see that there are $2 \times 6$ possible outcomes, and that there are $1 \times 3$ ways of obtaining a head and an odd number.




$$P(\text{head and an odd number}) = \frac{n(E)}{n(S)}$$
$$= \frac{1 \times 3}{2 \times 6}$$
$$= \frac{1}{2} \times \frac{3}{6}$$
$$= \left(\begin{matrix}\text{chance of choosing} \\ \text{a head}\end{matrix}\right) \times \left(\begin{matrix}\text{chance of choosing} \\ \text{an odd number}\end{matrix}\right)$$

We can see that the multiplication rule gives the same result as a tree diagram.

**2** Three yellow and five red counters are in a glass. Two counters are chosen at random, with replacement. What is the probability of choosing 2 red counters?




There are 8 ways of choosing the first counter and there are 8 ways of choosing the second counter because the first counter chosen is replaced.
The total number of outcomes $n(S) = 8 \times 8 = 64$

There are 5 ways of choosing red as the first counter and 5 ways of choosing red as the second counter because the first counter chosen is replaced.

$$P(\text{2 red counters}) = \frac{\text{number of ways of choosing 2 reds}}{\text{total number of outcomes}}$$
$$= \frac{5 \times 5}{8 \times 8}$$
$$= \frac{5}{8} \times \frac{5}{8}$$
$$= \left(\begin{matrix}\text{chance of choosing} \\ \text{a red counter}\end{matrix}\right) \times \left(\begin{matrix}\text{chance of choosing} \\ \text{a red counter}\end{matrix}\right)$$

$$P(E) = \frac{n(E)}{n(S)}$$

We can see that the multiplication rule gives the same result as our reasoning.

## WORKED EXAMPLE 1

I have two containers. Container 1 has three green tickets and seven blue tickets. Container 2 has two red tickets and three yellow tickets. If I choose a ticket from each container at random, what is the probability of choosing a green ticket and a yellow ticket?

### Solution

$P(\text{green from container 1}) = \frac{3}{10}$, $P(\text{yellow from container 2}) = \frac{3}{5}$, $P(A \text{ and } B) = P(A) \times P(B)$

$P(\text{green and yellow}) = P(\text{green}) \times P(\text{yellow})$

$= \frac{3}{10} \times \frac{3}{5}$

$= \frac{9}{50}$

The probability of choosing a green ticket and a yellow ticket is $\frac{9}{50}$.

## WORKED EXAMPLE 2

A normal dice is rolled, and a fair spinner showing the letters D, E, F and G is spun.
What is the chance of rolling an odd number on the dice and spinning a G on the spinner?

### Solution

$P(\text{rolling an odd number}) = \frac{3}{6}$, $P(\text{spinning a G}) = \frac{1}{4}$, $P(A \text{ and } B) = P(A) \times P(B)$

$P(\text{odd number and a G}) = P(\text{odd number}) \times P(\text{spinning a G})$

$= \frac{3}{6} \times \frac{1}{4}$

$= \frac{3}{24}$ (or $\frac{1}{8}$)

The probability of rolling an odd number and spinning a G is $\frac{1}{8}$.

## WORKED EXAMPLE 3

Two of the seven marbles in a bag are to be chosen at random, with replacement. Three marbles are red and four are blue. Find the probability of choosing:

**a** two red marbles

**b** a red marble, then a blue marble.

### Solution

**a** $P(\text{red as 1st choice}) = \frac{3}{7}$

$P(\text{red as 2nd choice}) = \frac{3}{7}$

$P(A \text{ and } B) = P(A) \times P(B)$

$P(\text{2 red marbles}) = P(\text{red}) \times P(\text{red})$

$= \frac{3}{7} \times \frac{3}{7}$

$= \frac{9}{49}$

The probability of choosing two red marbles is $\frac{9}{49}$.

**b** $P(\text{red as 1st choice}) = \frac{3}{7}$

$P(\text{blue as 2nd choice}) = \frac{4}{7}$

$P(A \text{ and } B) = P(A) \times P(B)$

$P(\text{red then blue}) = P(\text{red}) \times P(\text{blue})$

$= \frac{3}{7} \times \frac{4}{7}$

$= \frac{12}{49}$

The probability of choosing a red marble, then a blue marble is $\frac{12}{49}$.

# Exercise 4:05

**A**

The coin is tossed.

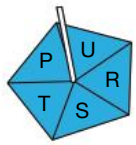
**B**

The dice is rolled.

**C**

A ticket is chosen at random.

**D**




The spinner is spun.

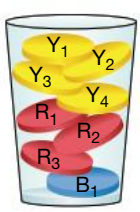
**1** Refer to A to D above, and use the multiplication rule $P(A \text{ and } B) = P(A) \times P(B)$ for two-step events to answer these questions. Find the probability that:

- **a** a head is tossed and a 6 is rolled
- **b** a head is tossed and a red ticket is chosen
- **c** a head is tossed and a T is spun
- **d** a 6 is rolled and a red ticket is chosen
- **e** a 6 is rolled and a T is spun
- **f** a red ticket is chosen and a T is spun
- **g** a yellow ticket is chosen and a number less than 3 is rolled
- **h** a yellow ticket is chosen and a tail is tossed
- **i** a yellow ticket is chosen and P, U or R is spun.

**2** Refer to A to D in Question **1**, and use the multiplication rule $P(A \text{ and } B) = P(A) \times P(B)$ for two-step events to answer these questions. Find the probability that:

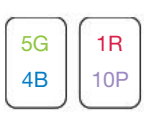
- **a** when a coin is tossed twice, the outcome will be two heads
- **b** when a dice is rolled twice, the outcome will be two sixes
- **c** two red tickets are chosen from two choices, with replacement
- **d** we would obtain two Ts from two spins of the spinner.

**3** One blue, three red and four yellow counters are placed in a glass and two of these counters are chosen at random, with replacement. Find the probability of choosing:




- **a** two yellow counters
- **b** two red counters
- **c** a yellow then a red
- **d** a red then a yellow
- **e** a yellow then a blue
- **f** a blue then a yellow
- **g** a red then a blue
- **h** a blue then a red.

**4** I have two containers. One has five green tickets and four blue tickets. The other has one red ticket and 10 purple tickets. If I choose a ticket from each container at random, what is the probability of my choosing:

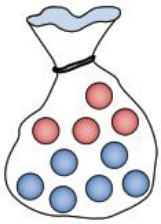



- **a** a green ticket and a purple ticket
- **b** a blue ticket and a purple ticket
- **c** a green ticket and a red ticket
- **d** a blue ticket and a red ticket?

**5** A normal dice is rolled, and a fair spinner showing the letters B, C, D, E, F, G and H is spun. Find the chance of:

- **a** rolling a 2 on the dice and spinning a B on the spinner
- **b** rolling an even number on the dice and spinning a consonant (not a, e, i, o or u) on the spinner
- **c** rolling a 5 or a 6 on the dice and spinning a vowel (a, e, i, o or u) on the spinner.

**6** Two of the 10 marbles in a bag are to be chosen at random, with replacement. Four of the marbles are red and six are blue. Find the probability of choosing:



**a** two red marbles
**b** a red marble, then a blue marble
**c** two blue marbles
**d** a blue marble then a red marble.

**7** I have just finished my term tests. I estimate that the probability of my passing Mathematics is $\frac{4}{5}$, the probability of my passing English is $\frac{2}{3}$, the probability of my passing Science is $\frac{3}{5}$ and the probability of my passing Indonesian is $\frac{1}{2}$. Assuming that these estimates are accurate, find the probability that I will pass both:

**a** Mathematics and English
**b** Mathematics and Science
**c** Indonesian and English
**d** Science and Indonesian.

**8** The probability that I will win a raffle is $\frac{3}{100}$. The probability that I will lose my raffle tickets is $\frac{3}{8}$. Find the probability that:

**a** I will win the raffle and lose my tickets
**b** I will win the raffle and not lose my tickets.

By considering the tree diagrams in Section 4:03, we can see that when three events $A$, $B$ and $C$ are independent, the probability of all three occurring together is:

$$P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C)$$

**9** The probability that Sam will pass in the term tests are: Mathematics $\frac{9}{10}$, English $\frac{4}{5}$, Science $\frac{1}{2}$, Indonesian $\frac{2}{3}$.

Find the probability that Sam will pass in:

**a** Mathematics, English and Science
**b** Mathematics, English and Indonesian
**c** English, Science and Indonesian
**d** Mathematics, Indonesian and Science.

**A**

The coin is tossed.

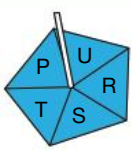
**B**

The dice is rolled.

**C**

A ticket is chosen at random.

**D**




The spinner is spun.

**10** Refer to A to D above, to answer these questions. Find the probability of obtaining:

**a** 3 heads by tossing the coin three times
**b** 'a number less than 6' each time, when rolling the dice three times
**c** three red tickets by choosing three tickets, with replacement
**d** three yellow tickets by choosing three tickets, with replacement
**e** the letters S, U and P in that order by spinning the spinner three times
**f** a head, an even number and a red ticket using A, B and C
**g** a tail, a square number and a yellow ticket using A, B and C
**h** a head, a red ticket and a letter from the word 'SPUR' using A, C and D.

**11** Kai-Lin, Su-Lin, Mark and Chris are all to compete in a different age race tomorrow. Kai-Lin has 1 chance in 4 of winning, Su-Lin has 3 chances in 5, Mark has 1 chance in 6 and Chris has 7 chances in 8. Find the probability that:

- **a** Kai-Lin and Su-Lin both win their races
- **b** Kai-Lin, Su-Lin and Mark all win their races
- **c** Kai-Lin, Su-Lin and Chris all win their races
- **d** Mark, Chris and Su-Lin all win their races
- **e** Kai-Lin and Su-Lin win but Mark does not win
- **f** Kai-Lin and Mark win but Chris does not win
- **g** all four people win their races.

**12** There are two hats. In the first hat there are six 10c coins and four $1 coins. In the second hat there are three 10c coins and seven $1 coins.

- **a** If one coin is selected at random from each hat, what is the probability they will both be 10c coins?
- **b** If one of the hats is selected at random and then a coin is chosen at random from that hat, what is the probability of selecting a $1 coin? (Hint: The $1 coin could be chosen from either hat.)

**13** There are 5 goats and 5 sheep in a paddock. An animal is selected at random. This is done three times. What is the probability of the third animal chosen being a goat if:

- **a** after each selection the animal is returned to the paddock?
- **b** each animal chosen is not returned to the paddock and the first two animals are:
  - **i** both goats
  - **ii** both sheep?

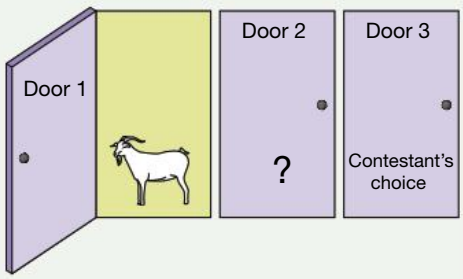
**14** The probability of my spending Christmas Day in Sydney is $\frac{7}{10}$, the probability of my spending it in London is $\frac{2}{10}$ and the probability of my spending it elsewhere is $\frac{1}{10}$. Explain why the probability of my spending the day in both Sydney and London is not $\frac{7}{10} \times \frac{2}{10}$.

## FUN SPOT 4:05 THE MONTY HALL PROBLEM

A quiz show host, Monty Hall, would give his contestants a choice of three doors. Behind one was a car; behind the other two were goats. After the contestant had named a door, Monty, who knew where the car was hidden, would open one of the other doors, to show that it did not contain the car. He would then offer the contestant the opportunity to switch from their original choice to the other door not yet opened.

**Questions**

1. Would the contestant improve his or her chances of winning the car if they changed from their original choice?
2. Should the contestant change their choice or not? Give a reason for your answer.



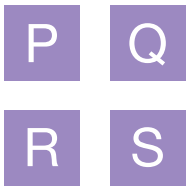
# 4:06 The multiplication rule for dependent events

If events are dependent we can still use the multiplication rule, provided that the second step, and any other steps that follow, are the probabilities of the events happening given that the earlier events have already happened. (These probabilities are **conditional probabilities**.)

If the events $A$ and $B$ are dependent, the probability of both occurring is:
$P(A \text{ and } B) = P(A) \times P(B)$, where $P(B)$ is the chance $B$ occurs given that $A$ has already occurred.

## WORKED EXAMPLE 1

Two movies are to be chosen at random from the movies P, Q, R and S. The first movie chosen will be shown on Monday and not replaced (so the same movie is not shown on both nights). The second movie will be shown on Tuesday. What is the probability that Movie P will be shown on Monday and Movie S will be shown on Tuesday?




### Solution

Once the first movie has been chosen, there will only be three movies left from which to choose the second movie.

$P(A \text{ and } B) = P(A) \times P(B)$

$P(\text{P then S}) = P(\text{P}) \times P(\text{S if P has already been chosen})$

$= \frac{1}{4} \times \frac{1}{3}$ (Once the first is chosen, only 3 movies remain.)

$= \frac{1}{12}$

There is a 1 in 12 chance that Movie P will be shown on Monday and Movie S on Tuesday.

## WORKED EXAMPLE 2

Two counters are chosen at random **without replacement** from three red and two green counters. Find the probability of choosing:

**a** two red counters **b** a red and a green in any order.

### Solutions

**a** $P(A \text{ and } B) = P(A) \times P(B)$

$P(\text{two red}) = P(\text{red}) \times P(\text{red if a red has already been chosen})$

$= \frac{3}{5} \times \frac{2}{4}$ (Once the first red is chosen, 2 of the remaining 4 counters are red.)

$= \frac{6}{20}$ (or $\frac{3}{10}$)

**b** $P(\text{a red and a green in any order}) = P(\text{R then G}) + P(\text{G then R})$

$= (\frac{3}{5} \times \frac{2}{4}) + (\frac{2}{5} \times \frac{3}{4})$

$= \frac{6}{20} + \frac{6}{20}$

$= \frac{12}{20}$ (or $\frac{3}{5}$)

## WORKED EXAMPLE 3

Three counters are chosen at random **without replacement** from three red and two green counters. Find the probability of choosing:

**a** three red counters

**b** two green then a red.

### Solutions

**a** $P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C)$

$P(3 \text{ red})$

$= P(\text{red}) \times P(\text{red if the first choice was red}) \times P(\text{red if the first two choices were red})$

$= \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3}$

$= \frac{6}{60}$ (or $\frac{1}{10}$)

**b** $P(\text{two green then a red})$

$= P(\text{G}) \times P(\text{G given G on the 1st choice}) \times P(\text{R given two greens chosen})$

$= \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3}$

$= \frac{6}{60}$ (or $\frac{1}{10}$)

## Using tree diagrams

Tree diagrams can be used to show the probabilities of events in two- and three-step experiments. We place the probability of each event on the arm leading to that event.

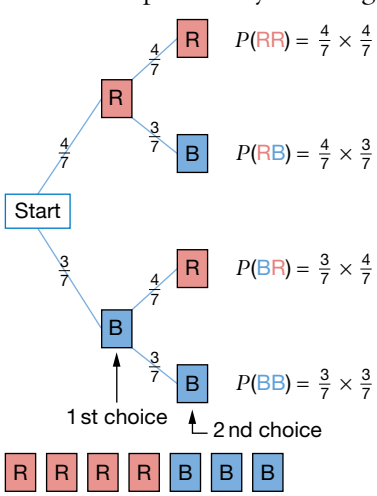
If the probabilities are placed on the arms of a tree diagram, we can find the probability of a multi-step event by multiplying the probabilities on the arms of that event. This is also useful for compound events that can be achieved in more than one way.

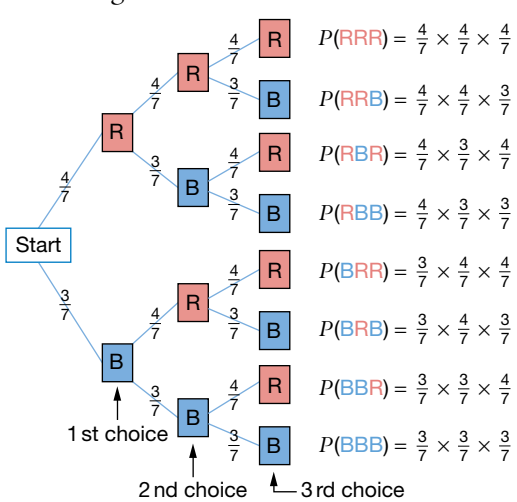
*Note:*

- On each arm of the tree diagram, the probability of that choice is shown. To find the probability of a particular outcome, we multiply the probabilities on the arms of that outcome.
- In **Case A** the choices are independent events, as the ticket chosen is replaced after each choice. The probabilities for each selection do not change from one choice to the next.
- In **Case B** the choices are dependent events, as the chosen ticket is not replaced after each choice. The probabilities for each selection change from one choice to the next, as the number of tickets available to be chosen is decreasing. The probabilities for the second selection are not the same as the probabilities for the first.

**Case A (Independent events)**

Four red and three blue tickets are placed in a box and tickets are chosen at random **with replacement**. The probability tree diagrams for selecting two and three tickets are shown.



- If two tickets are chosen, a blue and a red ticket could be chosen in two ways.

$P(\text{blue and red}) = P(\text{blue then red}) + P(\text{red then blue})$

$= P(\text{BR}) + P(\text{RB})$

$= (\frac{3}{7} \times \frac{4}{7}) + (\frac{4}{7} \times \frac{3}{7})$

$= \frac{24}{49}$

- If three tickets are chosen, two reds and a blue ticket could be chosen in three ways.

$P(\text{2 reds and a blue}) = P(\text{2 reds then blue}) + P(\text{red then blue then red}) + P(\text{blue then 2 reds})$

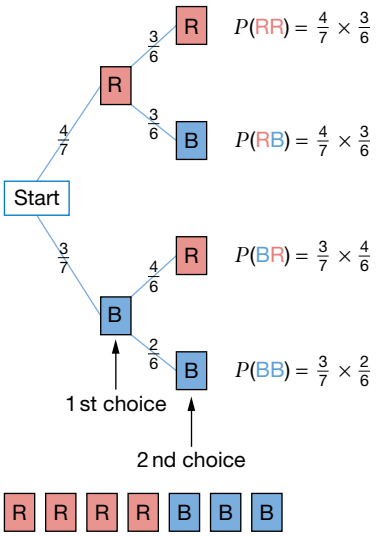
$= P(\text{RRB}) + P(\text{RBR}) + P(\text{BRR})$

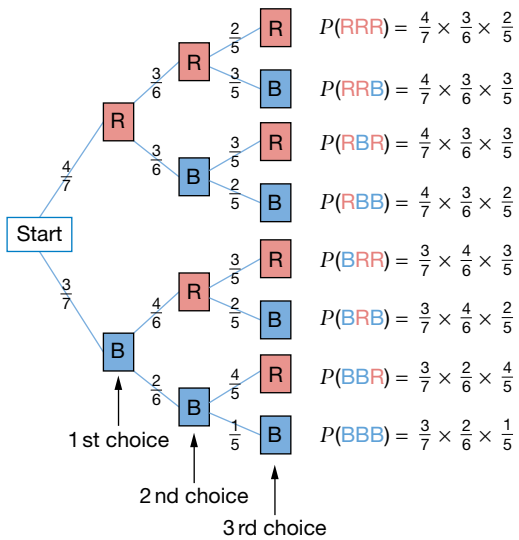
$= (\frac{4}{7} \times \frac{4}{7} \times \frac{3}{7}) + (\frac{4}{7} \times \frac{3}{7} \times \frac{4}{7}) + (\frac{3}{7} \times \frac{4}{7} \times \frac{4}{7})$

$= \frac{144}{343}$

**Case B (Dependent events)**

Four red and three blue tickets are placed in a box and tickets are chosen at random **without replacement**. The probability tree diagrams for selecting two and three tickets are shown.




- If two tickets are chosen, a blue and a red ticket could be chosen in two ways.

$P(\text{blue and red}) = P(\text{blue then red}) + P(\text{red then blue})$

$= P(\text{BR}) + P(\text{RB})$

$= (\frac{3}{7} \times \frac{4}{6}) + (\frac{4}{7} \times \frac{3}{6})$

$= \frac{24}{42}$ (or $\frac{4}{7}$)

- If three tickets are chosen, two reds and a blue ticket could be chosen in three ways.

$P(\text{2 reds and a blue}) = P(\text{2 reds then blue}) + P(\text{red then blue then red}) + P(\text{blue then two reds})$

$= P(\text{RRB}) + P(\text{RBR}) + P(\text{BRR})$

$= (\frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}) + (\frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}) + (\frac{3}{7} \times \frac{4}{6} \times \frac{3}{5})$

$= \frac{108}{210}$ (or $\frac{18}{35}$)

## Exercise 4:06

**1** To select a captain and vice-captain from our team of five players we placed our five names in a hat and drew out two names at random, the first to be captain, the second to be vice-captain. What is the probability that:

**a** I will be captain and my team-mate, Jo, will be vice-captain
**b** Jo will be captain and I will not be vice-captain
**c** neither Jo nor I will be captain or vice-captain?

**2** Two different given names are to be chosen at random from Hannah, Joy, Joanna and Belle. Hannah Joy is a different outcome to Joy Hannah. What is the probability that the outcome:

**a** will be Hannah Joy **b** will not be Hannah Joy **c** will start with Hannah?

**3** Six cards, numbered 1 to 6, are placed in a hat. Two of these are chosen at random, without replacement, to make a two-digit number. The cards chosen are placed in order, the first becoming the tens digit. What is the probability that the number formed:

**a** will be 65 **b** will be 12 or 13 **c** will not be 12 or 13?

Three of these cards are chosen at random, without replacement, to make a three-digit number. The cards chosen are placed in order of choice, the first becoming the hundreds digit. What is the probability that the number formed:

**d** will be 123 **e** will be 123 or 124 **f** will not be 123 or 124?

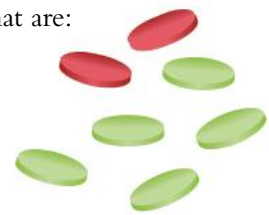
**4** Two tickets are chosen at random **without replacement** from 20 tickets marked 1 to 20. Find the probability of choosing:

**a** the 7 and then the 14 **b** two even numbers
**c** two odd numbers **d** two one-digit numbers
**e** two two-digit numbers **f** an odd then and even number
**g** two three-digit numbers **h** two multiples of 10.

If three of these tickets are chosen at random without replacement, find the probability of choosing:

**i** three one-digit numbers **j** the 5, the 10 and the 15 in that order
**k** three odd numbers **l** three numbers that are not all odd.

**5** Two counters are chosen at random **without replacement** from two red and five green counters. Find the probability of choosing counters that are:



**a** both red **b** not both red
**c** both green **d** a red and a green in either order.

If three counters are chosen at random without replacement, find the probability of choosing:

**e** three green counters **f** three red counters
**g** two green then a red **h** two green and a red in any order.

**6** Tom's grandchildren each chose one jelly bean at random from a jar containing eight jelly beans. Six of these were green and two were red. Find the probability that the first two jelly beans chosen were:

**a** a green then a red
**b** both green
**c** both red
**d** not both red
**e** a red then a green
**f** a green and a red in any order.

Find the probability that the first three jelly beans chosen were:

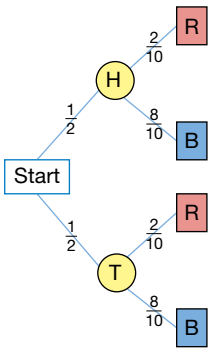
**g** two red and then a green
**h** two green and then a red
**i** all green
**j** not all green.

**7** From a hat containing three yellow and five red tickets, three tickets are chosen at random **without replacement**.

**a** What is the probability of choosing three red tickets?
**b** What is the probability of choosing one red ticket and two yellow tickets in any order?

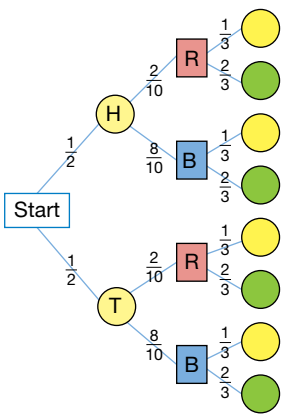
**8** A coin is tossed and a ticket is chosen at random from a hat containing two red and eight blue tickets. The probability of each step has been placed on the appropriate branch of the tree diagram. Find the probability of obtaining:

**a** a head and a red ticket
**b** a head and a blue ticket
**c** a tail and a red ticket
**d** a tail and a blue ticket
**e** a tail and any colour of ticket
**f** a head and any colour of ticket
**g** a blue ticket with either a head or a tail
**h** a red ticket with either a head or a tail.



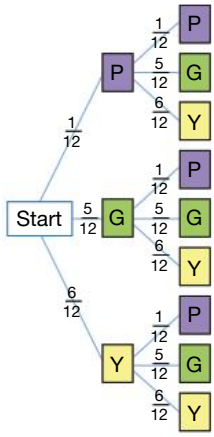

**9** A coin is tossed, a ticket is chosen at random from a hat containing two red and eight blue tickets, and a counter is chosen at random from one yellow and two green counters. The probability of each step has been placed on the appropriate branch of the tree diagram. Find the probability of obtaining:

**a** a head, a red ticket and a yellow counter
**b** a head, a blue ticket and a green counter
**c** a tail, a red ticket and a yellow counter
**d** a tail, a blue ticket and a yellow counter
**e** a head, a blue ticket and a yellow counter
**f** a tail, a red ticket and a green counter
**g** a head, any colour of ticket and a yellow counter
**h** a tail, any colour of ticket and a green counter.



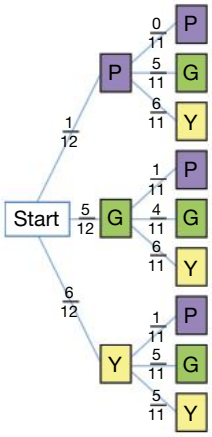
**10** Two tickets are chosen at random from a hat containing one purple, five green and six yellow tickets. The first ticket **is replaced** before the second is chosen. The probability of each step has been placed on the appropriate branch of the tree diagram. Find the probability of choosing:

**a** two purple tickets
**b** two green tickets
**c** a green then a yellow
**d** a yellow then a green
**e** a green and a yellow in any order
**f** no yellow tickets
**g** at least one yellow
**h** two green tickets knowing that the first ticket was green
**i** two yellow tickets knowing that the first ticket was yellow
**j** two tickets that are not the same colour.



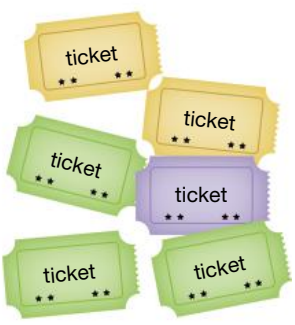

**11** Two tickets are chosen at random from a hat containing one purple, five green and six yellow tickets. The first ticket **is not replaced** before the second is chosen. The probability of each step has been placed on the appropriate branch of the tree diagram. Find the probability of choosing:

**a** two purple tickets
**b** two green tickets
**c** a green then a yellow
**d** a yellow then a green
**e** a green and a yellow in any order
**f** no yellow tickets
**g** at least one yellow
**h** two green tickets knowing that the first ticket was green
**i** two yellow tickets knowing that the first ticket was yellow
**j** two tickets that are not the same colour.



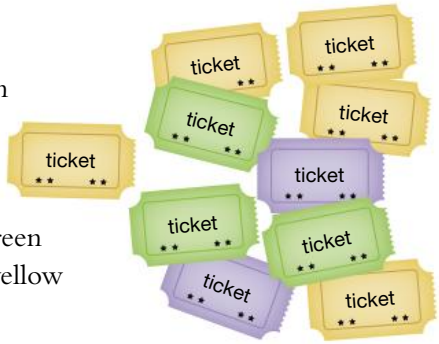

**12** Two tickets are chosen at random from a hat containing one purple, two yellow and three green tickets. The first ticket **is replaced** before the second is chosen. Create a probability tree like the one in Question **10**. Place the probability of each step on the appropriate branch of your tree diagram, and then use your diagram to find the probability of choosing:

**a** two purple tickets
**b** two green tickets
**c** a green then a yellow
**d** a yellow then a green
**e** a green and a yellow in any order
**f** no yellow tickets
**g** at least one yellow ticket
**h** two green tickets knowing that the first ticket was green
**i** two yellow tickets knowing that the first ticket was yellow
**j** two tickets that are not the same colour.



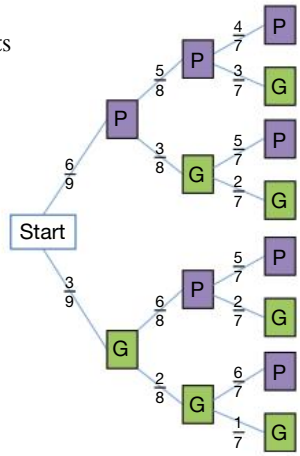
**13** Two tickets are chosen at random from a hat containing two purple, five yellow and three green tickets. The first ticket **is not replaced** before the second ticket is chosen. Create a probability tree like the one in Question **11**. Place the probability of each step on the appropriate branch of your tree diagram, and then use your diagram to find the probability of choosing:

**a** two purple tickets
**b** two green tickets
**c** a green then a yellow
**d** a yellow then a green
**e** a green and a yellow in any order
**f** no yellow tickets
**g** at least one yellow
**h** two green tickets knowing that the first ticket was green
**i** two yellow tickets knowing that the first ticket was yellow
**j** two tickets that are not the same colour.



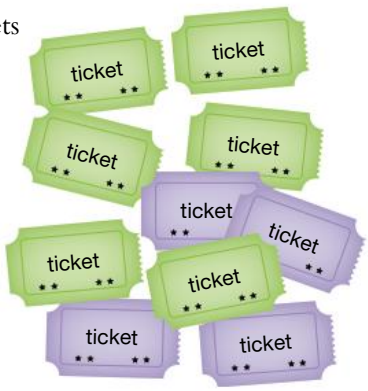

**14** Three tickets are chosen at random **without replacement** from a hat containing six purple and three green tickets. The probability of each step has been placed on the appropriate branch of the tree diagram. Find the probability of choosing:

**a** three purple tickets
**b** fewer than three purple tickets
**c** three green tickets
**d** fewer than three green tickets
**e** three tickets that are the same colour
**f** a green then two purples
**g** two purples then a green
**h** a purple then a green then a purple
**i** two greens and a purple in any order
**j** any outcome where the last ticket chosen is green
**k** any outcome where the last ticket chosen is not green
**l** three green tickets knowing that the first ticket was green
**m** three purple tickets knowing that the first ticket was purple.




**15** Three tickets are chosen at random **without replacement** from a hat containing four purple and six green tickets. Create a probability tree like the one in Question **14**. Place the probability of each step on the appropriate branch of your tree, and then use your diagram to find the probability of choosing:

**a** three purple tickets
**b** fewer than three purple tickets
**c** three green tickets
**d** fewer than three green tickets
**e** three tickets that are not all the same colour
**f** a green then two purples
**g** two purples then a green
**h** a purple then a green then a purple
**i** two greens and a purple in any order
**j** any outcome where the last ticket chosen is green
**k** any outcome where the last ticket chosen is not green
**l** three green tickets knowing that the first ticket was green
**m** three purple tickets knowing that the first ticket was purple.




**16** Find the probability of choosing three consecutive numbers at random, without replacement, from the numbers 1 to 9. (Hint: Use the multiplication rule for dependent events.)

## MATHS TERMS 4

**complementary event**

- the complement of 'an event happening' is that 'the event does not happen',
  e.g. If the event is 'throwing a 6', the complementary event would be 'not throwing a 6'

**dependent events**

- the occurrence of the first event affects the chance of occurrence of the second event. The probability of the second event is affected by the outcome of the first

**experimental probability**

- the predicted chance of occurrence based on the relative frequencies taken from a chance experiment

$$\text{experimental probability} = \frac{\text{number of times event occurred}}{\text{total number of trials}}$$

**independent events**

- the occurrence of the first event does not affect the chance of occurrence of the second event. The probability of the second event is not affected by the outcome of the first

**multiplication rule for multi-step events**

- when two events are independent, the probability of both occurring is:
  $P(A \text{ and } B) = P(A) \times P(B)$
- when two events are dependent, the probability of both occurring is:
  $P(A \text{ and } B) = P(A) \times P(B)$, where $P(B)$ is the chance $B$ occurs given that $A$ has already occurred

**mutually exclusive events**

- events that cannot occur at the same time. They have no elements in common,
  e.g. tossing an odd number on a dice and tossing a 4

**outcomes**

- the possible results when a trial is to be performed,
  e.g. the possible outcomes when tossing a dice are 1, 2, 3, 4, 5, and 6

**probability $P(E)$**

- the chance of an event happening. This will be a number between 0 and 1 or equal to 0 or 1

**random**

- choosing at random does not affect the likelihood of any outcome

**relative frequency**

- this is the same as the experimental probability

$$\text{relative frequency} = \frac{\text{frequency of an event}}{\text{total frequency}}$$

**sample space**

- the set of all possible outcomes,
  e.g. 1, 2, 3, 4, 5 and 6 on a normal dice

**theoretical probability**

- the likelihood of an event occurring under ideal circumstances,
  e.g. the probability of rolling a 6 on a normal dice is $\frac{1}{6}$

**trial**

- one case of carrying out a chance experiment,
  e.g. one spin of a spinner

**two- and three-step chance experiments**

- each step is carried out in turn. The experiment may be undertaken with replacement or without replacement,
  e.g. draw two counters from a bag containing two blue and three red counters without replacement

# DIAGNOSTIC TEST 4 PROBABILITY

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** a A card is chosen at random from a standard pack. Find the probability that the card is:
i a red King ii not a red King.
b Given that an odd number has been rolled on a normal dice, find the probability that the number is:
i 5 ii not 5.

4:01

**2** Using a fair coin, I have tossed five heads in a row.
Is the probability of tossing a head on the next toss:
A less than 50% B 50% C more than 50%?

4:02

**3** Three blue, two red and one green counter are placed in a bag. One is taken at random, replaced and then a second counter is chosen.
The table shows all possible outcomes.

| First choice \ Second choice | $B_1$ | $B_2$ | $B_3$ | $R_1$ | $R_2$ | $G_1$ |
|---|---|---|---|---|---|---|
| $B_1$ | BB | BB | BB | BR | BR | BG |
| $B_2$ | BB | BB | BB | BR | BR | BG |
| $B_3$ | BB | BB | BB | BR | BR | BG |
| $R_1$ | RB | RB | RB | RR | RR | RG |
| $R_2$ | RB | RB | RB | RR | RR | RG |
| $G_1$ | GB | GB | GB | GR | GR | GG |

a What is the probability that both counters are blue?
b What is the probability that both counters will be blue if we already know that the first counter chosen was blue?
c What is the probability that the two counters will be a red and a green (in any order)?

4:03

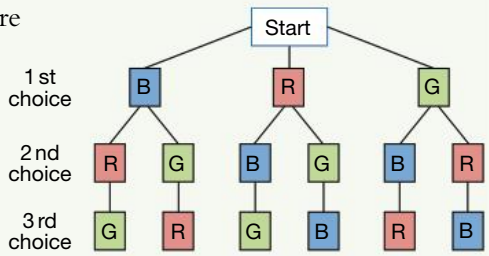
**4** Three blue, two red and one green counter are placed in a bag. One is taken at random. This time, the first counter is not replaced before the second counter is chosen.
The table shows all possible outcomes.

| First choice \ Second choice | $B_1$ | $B_2$ | $B_3$ | $R_1$ | $R_2$ | $G_1$ |
|---|---|---|---|---|---|---|
| $B_1$ | | BB | BB | BR | BR | BG |
| $B_2$ | BB | | BB | BR | BR | BG |
| $B_3$ | BB | BB | | BR | BR | BG |
| $R_1$ | RB | RB | RB | | RR | RG |
| $R_2$ | RB | RB | RB | RR | | RG |
| $G_1$ | GB | GB | GB | GR | GR | |

a What is the probability that both counters are blue?
b What is the probability that both counters will be blue if we already know that the first counter chosen was blue?
c What is the probability that the two counters will be a red and a green (in any order)?

4:03

**5** One blue, one red and one green ticket are placed in a bag. Three tickets are chosen without replacement after each choice is made. The tree diagram shows all possible outcomes.




Find the probability of choosing three tickets so that they are:
a red then blue then green b all blue c three different colours.

4:03

6 Find the number of outcomes in the sample space if: 4:04
- a a fair dice is tossed twice
- b a coin is tossed three times
- c two tickets are chosen from five with replacement
- d two tickets are chosen from five without replacement
- e three tickets are chosen from five with replacement
- f three tickets are chosen from five without replacement.

7 Four yellow and two red tickets are in a box. Two tickets are to be chosen from the box at random without replacement. 4:04
- a What is the probability of choosing two yellow tickets?
- b What is the probability that the two tickets will be different colours?

8 Three yellow and five red counters are in a glass. Three counters are chosen at random, without replacement. 4:04
- a What is the probability of choosing three red counters?
- b What is the probability of choosing two yellows and a red?

9 a I have two containers. One has three green tickets and seven blue tickets. The other has two red tickets and three yellow tickets. If I choose a ticket from each container, what is the probability of my choosing a green ticket and a yellow ticket? 4:05
- b A normal dice is rolled, and a fair spinner showing D, E, F and G is spun. What is the chance of rolling an odd number on the dice and spinning a G on the spinner?

10 Two movies are to be shown at random from the movies A, B, C, D and E. The first movie chosen will be shown on Monday and not replaced, and the second chosen will be shown on Tuesday. What is the probability that movie B will be shown on Monday and movie D will be shown on Tuesday? 4:06

11 a Two counters are chosen at random without replacement from three red and two green counters. Find the probability of choosing: 4:06
- i two red counters
- ii a red and a green in any order.

b If three counters are chosen at random without replacement from three red and two green counters, find the probability of choosing:

- i three red counters
- ii two green then a red.

# ASSIGNMENT 4A Chapter review

**1** From a standard pack of 52 cards, a card is chosen at random. What is the probability that the card chosen is:

**a** a number between 3 and 8
**b** a club that is a picture card
**c** not a red Jack
**d** not a club that is a picture card?

**2** An unknown number of tickets were placed in a hat.
Two tickets were drawn from the hat without replacing the first ticket. This was repeated 100 times to obtain the results in the table. Use these results to find, as a percentage, the experimental probability of choosing:

**a** a blue and a green ticket
**b** two blue tickets
**c** fewer than two blue tickets
**d** exactly one blue ticket
**e** at least one blue ticket
**f** two tickets of the same colour.

Here is the result of 100 trials, taking two tickets from a hat without replacement.

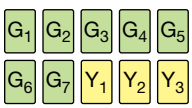
| Outcome | Frequency |
|---|---|
| two blue | 21 |
| two red | 12 |
| two green | 8 |
| blue and red | 30 |
| blue and green | 18 |
| red and green | 11 |
| *Total:* | 100 |

**3** I bought 5 of the 100 tickets in a raffle. Three prizes are to be drawn in order.
Find the probability that:

**a** I will win the first prize drawn
**b** I will win the second prize drawn given that I did not win the first prize drawn
**c** I will win the third prize drawn given that I did not win either of the first two prizes
**d** I will win the third prize drawn given that I have already won one prize.

**4** Use the pictures on the right to answer these questions. Find the probability of:

**a** spinning a C on the spinner then tossing a tail on a coin
**b** rolling a 5 on a dice then spinning a D on the spinner
**c** choosing a green ticket at random from these tickets then tossing a head on a coin
**d** spinning a vowel then tossing a head on a coin
**e** choosing a yellow ticket at random then spinning a letter of the word CAFE
**f** rolling an even number on a dice then spinning a consonant on the spinner.



## ASSIGNMENT 4B Working mathematically

**1** The travel graph shows the journeys of James and Callum between town A and town B. (They travel on the same road.)

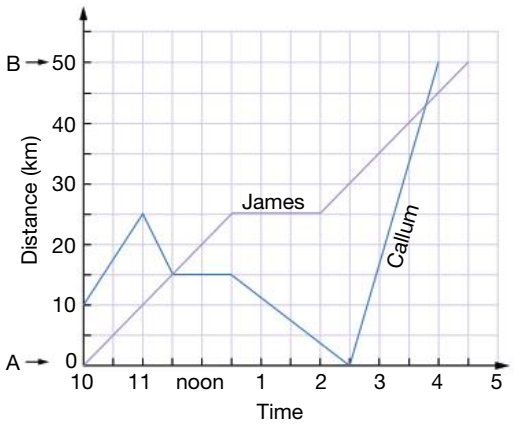
**a** How far from A is Callum when he commences his journey?

**b** How far is James from B at 2:30 pm?

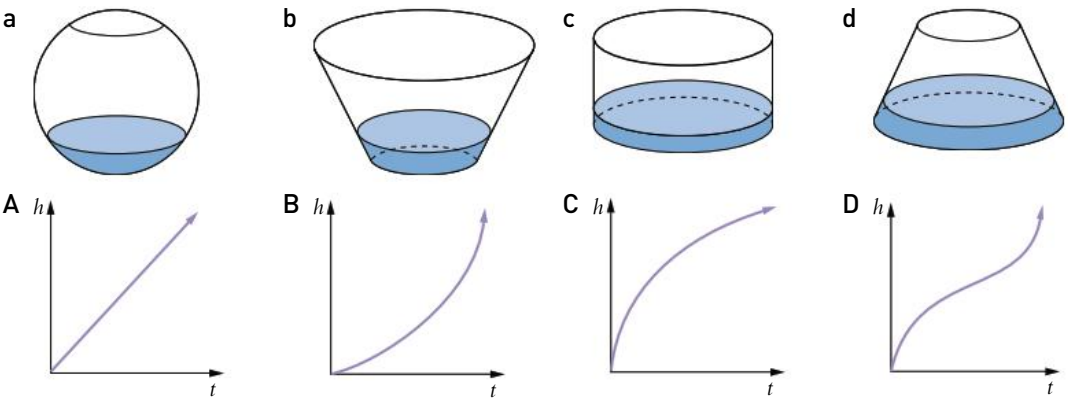
**c** When do James and Callum first meet?

**d** Who reaches town B first?

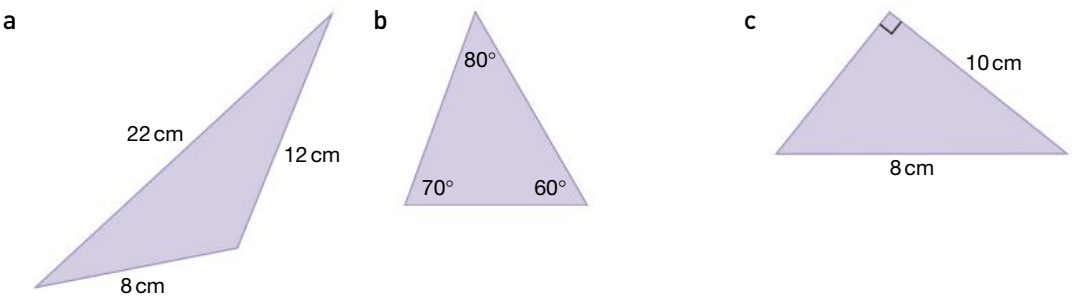
**e** At what time does Callum stop to rest?



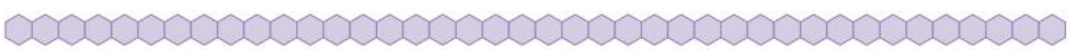

**2** Water is poured into each container at a constant rate. The graphs indicate the height of the water in each container against time. Match each graph with the correct container.




**3** Explain why each of these triangles is impossible.




**4** How many sides are drawn when 40 regular hexagons are joined in a row, as as shown below?



**5** How many two-digit numbers can be made from the digits 0, 1, 2, 3, 4 and 5 if the number is a multiple of 5, and no digit is used more than once in the same number?

# ASSIGNMENT 4C Cumulative revision

**1** **a** Write an expression for the average of $x$, $y$ and $z$. 1:02
**b** Write the square of the sum of $m$ and $n$.
**c** Write the sum of three consecutive odd numbers if the first is $m$.

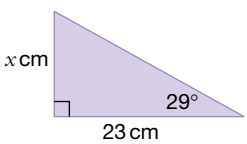
**2** The points $A(-2, 5)$ and $B(4, 7)$ are joined. Find: 1:10
**a** the midpoint of $AB$
**b** the length of $AB$
**c** the gradient of $AB$
**d** the equation of the line $AB$
**e** the equation of the line that passes through $(5, 12)$ and is parallel to $AB$.

**3** **a** Solve the equation $7(y + 3) - 2(8 - 3y) = -21$. 1:07
**b** Solve the simultaneous equations $4x - 3y = 13$ and $2x - y = 9$. 1:09
**c** Solve $x^2 - 8x + 2 = 0$, leaving your answer in surd form. 2:03

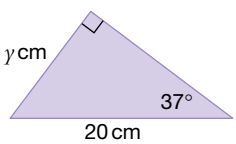
**4** **a** $P$ is directly proportional to $M$, and when $M$ is $0{\cdot}3$, $P$ is 6. Find the value of $P$ when $M$ is $5{\cdot}2$. 1:14
**b** $B$ is inversely proportional to $E$, and when $E$ is 5, $B$ is 8. Find the value of $B$ when $E$ is 2.

**5** Use trigonometry to find the value of the pronumeral, correct to one decimal place. 1:12

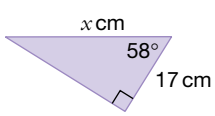
**a**




**b**

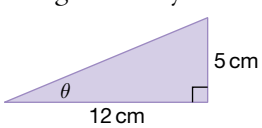



**c**

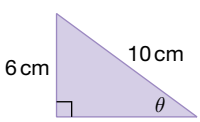



**6** Use trigonometry to find the size of $\theta$, correct to the nearest minute. 1:12

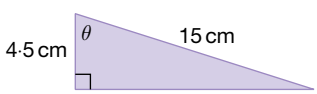
**a**




**b**

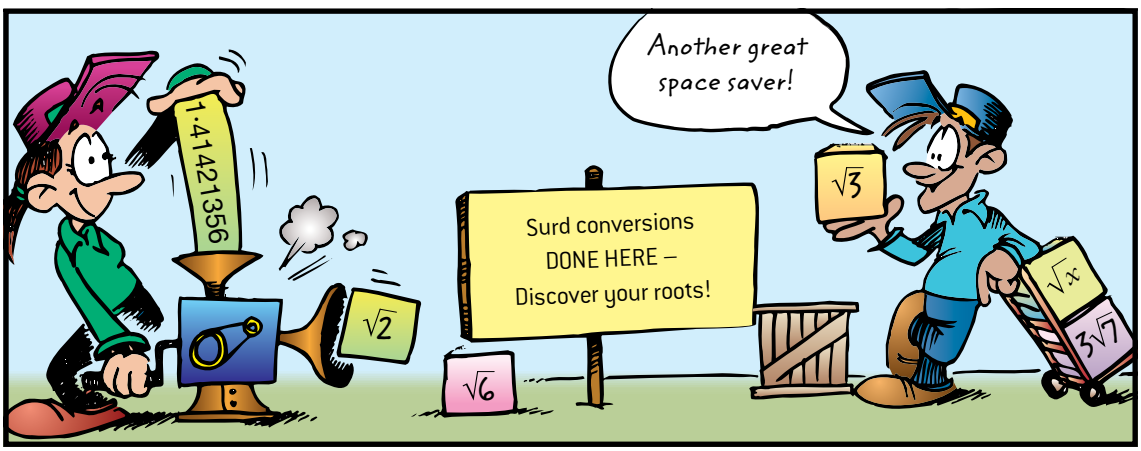



**c**



# SURDS

# 5




## Contents


**5:01** The real number system
Challenge 5:01 Proof that $\sqrt{2}$ is irrational
Investigation 5:01 f-stops and $\sqrt{2}$
**5:02** Surds
**5:03** Addition and subtraction of surds
**5:04** Multiplication and division of surds
Fun spot 5:04 What do Inuits sing at birthday parties?
**5:05** Binomial products
Investigation 5:05 Iteration to find square roots
**5:06** Rationalising the denominator
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

Selections from ***Surds and Indices*** [Stage 5.3§]

- Define rational and irrational numbers and perform operations with surds and fractional indices (ACMNA264)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 5:01 The real number system

## PREP QUIZ 5:01

Evaluate:

**1** $4^2$ **2** $7^2$ **3** $20^2$ **4** $6^2 + 5^2$

**5** $\sqrt{81}$ **6** $\sqrt{64}$ **7** $\sqrt{144}$ **8** $\sqrt{4} + \sqrt{25}$

Use a calculator to find an approximation correct to six decimal places for:

**9** $\sqrt{2}$ **10** $\sqrt{11} + 5$

The real number system is made up of two groups of numbers: rational and irrational numbers.

## Rational numbers

Any number that can be written as a fraction, $\frac{a}{b}$, where $a$ and $b$ are whole numbers and $b \neq 0$, is a rational number. These include integers, fractions, mixed numbers, terminating decimals and recurring decimals.

e.g. $\frac{7}{8}$, $6\frac{3}{5}$, $1{\cdot}25$, $0{\cdot}07$, $0{\cdot}\dot{4}$, $\sqrt{81}$

These examples can all be written as fractions.

$\frac{7}{8}, \frac{33}{5}, \frac{5}{4}, \frac{7}{100}, \frac{4}{9}, \frac{9}{1}$

An integer is a rational number whose denominator is 1.

## Irrational numbers

It follows that irrational numbers cannot be written as a fraction $\frac{a}{b}$ where $a$ and $b$ are whole numbers. We have met a few numbers like this in our study of the circle and Pythagoras' theorem.

e.g. $\pi$, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{3} + 2$

The calculator can only give approximations for these numbers. The decimals continue without terminating or repeating.

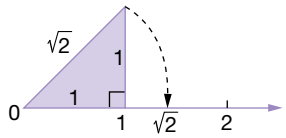
$3{\cdot}141\,592\,65 \ldots$, $1{\cdot}414\,213\,56 \ldots$, $1{\cdot}587\,401\,05 \ldots$, $3{\cdot}732\,050\,80 \ldots$

## Irrational numbers on the number line

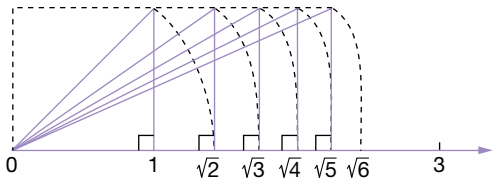
Although irrational numbers cannot be given an exact decimal value, their positions can still be plotted on the number line. As can be seen when using Pythagoras' theorem, a number such as $\sqrt{2}$ can correspond to the length of a side of a triangle, and so this length can be shown on a number line.

Examine the diagram on the right.

The position of $\sqrt{2}$ can be marked on a number line using a compass open to the length of the hypotenuse of the triangle. (This agrees with the decimal approximation from the calculator of 1·414 213 56.)




The previous construction can be extended to give the position of other square roots on the number line.




Another irrational number you have met before is $\pi$. You should know that it has an approximate value of 3·142, so it would lie on the number line in the position shown.

## Exercise 5:01

**1** For each number write *rational* or *irrational*. (A calculator might help you to decide.)

| | | | | |
|---|---|---|---|---|
| **a** $\frac{7}{10}$ | **b** $\sqrt{2}$ | **c** 0·3 | **d** $1\frac{1}{2}$ | **e** 1·6 |
| **f** $\pi$ | **g** $\sqrt{5}$ | **h** $\frac{3}{4}$ | **i** $\sqrt{16}$ | **j** 0·99 |
| **k** $\frac{22}{7}$ | **l** 9 | **m** $\sqrt{625}$ | **n** 0·666 | **o** 70 |
| **p** $\sqrt{13}$ | **q** $1\frac{1}{3}$ | **r** $1+\sqrt{3}$ | **s** $\sqrt{1}$ | **t** $\sqrt{4}+\sqrt{9}$ |
| **u** 0·0005 | **v** $1-\sqrt{9}$ | **w** $-\sqrt{3}$ | **x** $\sqrt{7}+\sqrt{2}$ | **y** $\sqrt[3]{27}$ **z** $\sqrt[3]{10}$ |

**2** Use your calculator to find an approximation correct to one decimal place for the following. Also use these values to show the position of each on a number line.

| | | | | |
|---|---|---|---|---|
| **a** $\sqrt{2}$ | **b** $\sqrt{3}$ | **c** $\sqrt{5}$ | **d** $\sqrt{6}$ | **e** $\sqrt{7}$ |
| **f** $\sqrt{8}$ | **g** $\sqrt{10}$ | **h** $\sqrt{12}$ | **i** $\sqrt{20}$ | **j** $\pi$ |

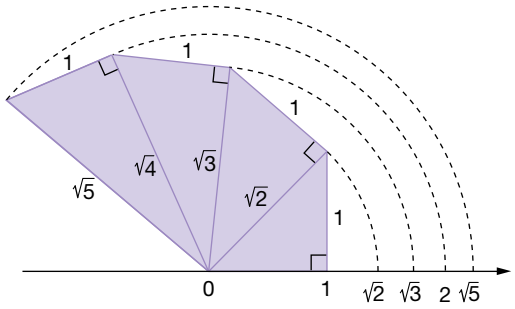
**3** Between which two consecutive integers does each number below lie?

| | | | | |
|---|---|---|---|---|
| **a** $\sqrt{11}$ | **b** $\sqrt{18}$ | **c** $\sqrt{41}$ | **d** $\sqrt{78}$ | **e** $\sqrt{95}$ |
| **f** $\sqrt{125}$ | **g** $\sqrt{180}$ | **h** $\sqrt{250}$ | **i** $\sqrt{390}$ | **j** $\sqrt{901}$ |

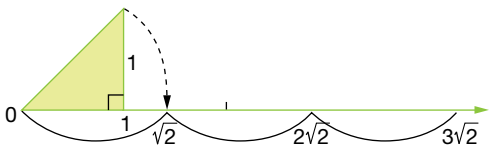
**4** Arrange each set of numbers below in order, from smallest to largest.

| | | |
|---|---|---|
| **a** $\sqrt{5}, 2, \sqrt{3}$ | **b** $\sqrt{8}, 3, \pi$ | **c** $\sqrt{10}, \sqrt{12}, 3$ |
| **d** $7, \sqrt{40}, \sqrt{50}, 6{\cdot}5$ | **e** $\pi, \sqrt{2}, 2{\cdot}1, \sqrt{12}$ | **f** $5{\cdot}6, \sqrt{26}, 6, \sqrt{30}$ |
| **g** $8{\cdot}1, \sqrt{65}, 7{\cdot}9, \sqrt{60}$ | **h** $\sqrt{98}, 10, \sqrt{102}, 10{\cdot}1$ | **i** $3{\cdot}1, \pi, \sqrt{9}, 3{\cdot}2$ |
| **j** $\sqrt{20}, 4{\cdot}1, 4{\cdot}5, \sqrt{21}$ | **k** $20, \sqrt{390}, 21, \sqrt{420}$ | **l** $\sqrt{600}, \sqrt{610}, 24, 25$ |

**5** This diagram shows another construction for locating square roots on the number line. Use a set square to draw the triangles on graph paper, then use a compass to draw the arcs on the number line. Extend your diagram to show $\sqrt{6}$. Check the accuracy of your constructions with your calculator.



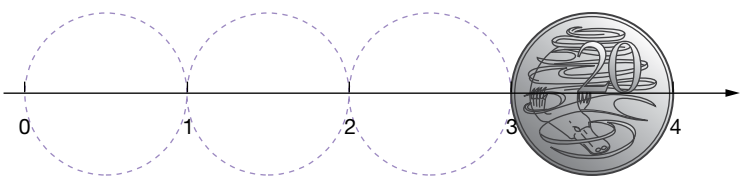

**6** To show multiples of a square root on the number line we can use a compass to mark off equal intervals.




**a** Repeat the constructions in Question **5** to find the position of $\sqrt{2}$ on the number line. Then use a compass to mark the position of $2\sqrt{2}$ and $3\sqrt{2}$.

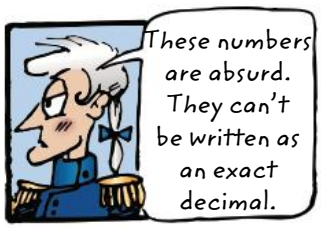
**b** Draw a diagram and show the position of $\sqrt{3}$, $2\sqrt{3}$ and $-\sqrt{3}$ on a number line.

**7** The position of $\pi$ on the number line can be shown by doing the following. Use the diameter of a twenty-cent coin to mark off units on a number line.



Then mark a point on the circumference of the coin, align it with zero on the number line, and roll it along the line carefully until the mark meets the number line again. This will show the position of $\pi$ on the number line.

## CHALLENGE 5:01 PROOF THAT $\sqrt{2}$ IS IRRATIONAL




Let us suppose that $\sqrt{2}$ is rational and, therefore, can be written as a fraction in the form $\frac{p}{q}$ where $p$ and $q$ are positive integers and $q \neq 0$, **with no common factor**. (This assumption is essential.)

$$\sqrt{2} = \frac{p}{q}$$

$$2 = \frac{p^2}{q^2} \quad \text{(squaring both sides)}$$

$$2q^2 = p^2$$

This last step implies that $p^2$ must be divisible by 2 (2 is prime). Therefore 2 must divide into $p$ exactly.

$\therefore$ $p$ can be expressed in the form $2k$ for some integer $k$.

$$\therefore 2q^2 = (2k)^2$$
$$2q^2 = 4k^2$$
$$q^2 = 2k^2$$

Now, as for $p$ above, it can be argued from this last step that $q$ must be divisible by 2. But $p$ and $q$ were said to have no common factor, hence a contradiction exists. So our original assumption was wrong.

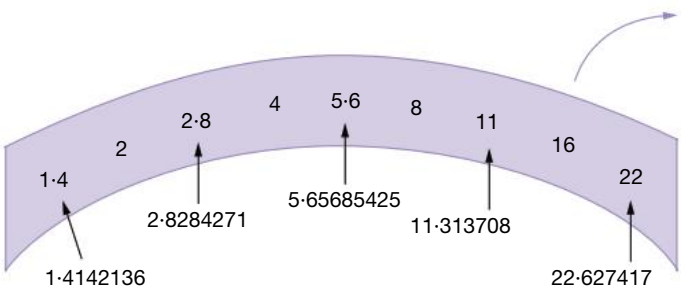
Therefore $p$ and $q$ cannot be found so that $\sqrt{2} = \frac{p}{q}$. Hence, $\sqrt{2}$ must be irrational.

- Try to use the method above to prove that these are irrational.

**1** $\sqrt{3}$ **2** $\sqrt{5}$ **3** $\sqrt{11}$

## INVESTIGATION 5:01 F-STOPS AND $\sqrt{2}$

On many cameras it is possible to change how far the shutter opens, using what are called f-stops.



The f-stops 2, 4, 8 and 16 are accurate. More accurate readings for the rest are given below the scale.

- Find the pattern in the accurate f-stops.
  - Try squaring each accurate f-stop number.
  - Try dividing each f-stop number by the one before it.
- Try to discover how f-stops are used.

# 5:02 Surds

## PREP QUIZ 5:02

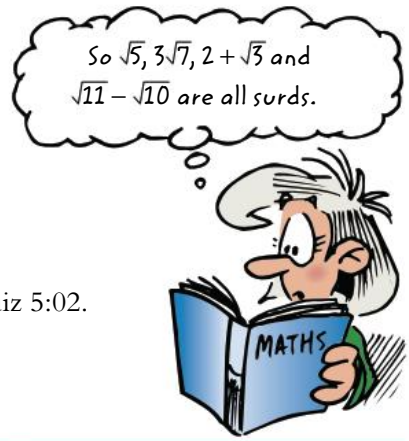
Find the value of:

1 $\sqrt{16}$ 2 $\sqrt{9}$ 3 $\sqrt{36}$ 4 $\sqrt{16+9}$ 5 $\sqrt{16}+\sqrt{9}$

6 $\sqrt{16\times 9}$ 7 $\sqrt{16}\times\sqrt{9}$ 8 $\sqrt{\frac{36}{9}}$ 9 $\frac{\sqrt{36}}{\sqrt{9}}$ 10 $(\sqrt{16})^2$

*Surds* are numerical expressions that involve irrational roots. They are irrational numbers.

Surds are irrational roots of the form $\sqrt[n]{x}$, where $x$ is a rational number, and $n$ is an integer such that $n \geq 2$.




Surds obey the following rules, which are suggested by Prep quiz 5:02.

Rule 1 $\sqrt{xy} = \sqrt{x}\times\sqrt{y}$

## WORKED EXAMPLE 1

**a** $\sqrt{100} = \sqrt{4}\times\sqrt{25}$
$= 2\times 5$
$= 10$ (which is true)

**b** $\sqrt{27} = \sqrt{9}\times\sqrt{3}$
$= 3\times\sqrt{3}$
$= 3\sqrt{3}$

**c** $\sqrt{5}\times\sqrt{7} = \sqrt{5\times 7}$
$= \sqrt{35}$

Rule 2 $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

*Note:* $\sqrt{x}$ means the positive square root of $x$ when $x > 0$.
$\sqrt{x} = 0$ when $x = 0$.

## WORKED EXAMPLE 2

**a** $\sqrt{\frac{16}{4}} = \frac{\sqrt{16}}{\sqrt{4}}$
i.e. $\sqrt{4} = \frac{4}{2}$
$= 2$ (which is true)

**b** $\sqrt{125}\sqrt{5} = \sqrt{125\sqrt{5}}$
$= \sqrt{25}$
$= 5$

**c** $\sqrt{30}\sqrt{5} = \sqrt{30\sqrt{5}}$
$= \sqrt{6}$

Rule 3 $(\sqrt{x})^2 = x$

*Note:* For $\sqrt{x}$ to exist, $x$ cannot be negative.

## WORKED EXAMPLE 3

**a** $(\sqrt{25})^2 = (5)^2$
$= 25$

**b** $(\sqrt{7})^2 = 7$

**c** $(3\sqrt{2})^2 = 3^2 \times (\sqrt{2})^2$
$= 9 \times 2$
$= 18$

A surd is in its simplest form when the number under the square root sign is as small as possible. To simplify a surd, express it as the product of two square roots, one being the root of a square number.

## WORKED EXAMPLE 4

Simplify the following surds.

**a** $\sqrt{18} = \sqrt{9} \times \sqrt{2}$
$= 3 \times \sqrt{2}$
$= 3\sqrt{2}$

**b** $\sqrt{75} = \sqrt{25} \times \sqrt{3}$
$= 5 \times \sqrt{3}$
$= 5\sqrt{3}$

**c** $5\sqrt{48} = 5 \times \sqrt{16} \times \sqrt{3}$
$= 5 \times 4 \times \sqrt{3}$
$= 20\sqrt{3}$

## Exercise 5:02

P Foundation worksheet 5:02 Surds

**1** Simplify:

**a** $\sqrt{3} \times \sqrt{5}$ **b** $\sqrt{5} \times \sqrt{3}$ **c** $\sqrt{7} \times \sqrt{6}$ **d** $\sqrt{6} \times \sqrt{7}$
**e** $\sqrt{10} \times \sqrt{3}$ **f** $\sqrt{23} \times \sqrt{2}$ **g** $\sqrt{13} \times \sqrt{5}$ **h** $\sqrt{11} \times \sqrt{3}$
**i** $\sqrt{5} \times \sqrt{2}$ **j** $\sqrt{7} \times \sqrt{2}$ **k** $\sqrt{11} \times \sqrt{10}$ **l** $\sqrt{13} \times \sqrt{7}$
**m** $\sqrt{26} \checkmark \sqrt{2}$ **n** $\sqrt{55} \checkmark \sqrt{5}$ **o** $\sqrt{77} \checkmark \sqrt{11}$ **p** $\sqrt{34} \checkmark \sqrt{17}$
**q** $\frac{\sqrt{38}}{\sqrt{2}}$ **r** $\frac{\sqrt{57}}{\sqrt{3}}$ **s** $\frac{\sqrt{60}}{\sqrt{10}}$ **t** $\frac{\sqrt{22}}{\sqrt{11}}$

**2** Square each of the surds below (Rule 3).

**a** $\sqrt{16}$ **b** $\sqrt{9}$ **c** $\sqrt{1}$ **d** $\sqrt{100}$
**e** $\sqrt{5}$ **f** $\sqrt{8}$ **g** $\sqrt{15}$ **h** $\sqrt{73}$
**i** $2\sqrt{2}$ **j** $3\sqrt{5}$ **k** $2\sqrt{3}$ **l** $5\sqrt{3}$
**m** $7\sqrt{3}$ **n** $2\sqrt{7}$ **o** $9\sqrt{11}$ **p** $6\sqrt{5}$
**q** $10\sqrt{10}$ **r** $9\sqrt{20}$ **s** $6\sqrt{50}$ **t** $15\sqrt{15}$

**3** Simplify each of these surds.

**a** $\sqrt{8}$ **b** $\sqrt{20}$ **c** $\sqrt{12}$ **d** $\sqrt{50}$
**e** $\sqrt{24}$ **f** $\sqrt{32}$ **g** $\sqrt{45}$ **h** $\sqrt{54}$
**i** $\sqrt{28}$ **j** $\sqrt{90}$ **k** $\sqrt{56}$ **l** $\sqrt{63}$
**m** $\sqrt{44}$ **n** $\sqrt{52}$ **o** $\sqrt{108}$ **p** $\sqrt{40}$
**q** $\sqrt{99}$ **r** $\sqrt{60}$ **s** $\sqrt{96}$ **t** $\sqrt{76}$
**u** $\sqrt{68}$ **v** $\sqrt{126}$ **w** $\sqrt{200}$ **x** $\sqrt{162}$

**4** Simplify each surd and then, taking $\sqrt{2} = 1{\cdot}41$ and $\sqrt{3} = 1{\cdot}73$, give a decimal approximation correct to one decimal place.

| | | | |
|---|---|---|---|
| **a** $\sqrt{18}$ | **b** $\sqrt{27}$ | **c** $\sqrt{8}$ | **d** $\sqrt{12}$ |
| **e** $\sqrt{32}$ | **f** $\sqrt{48}$ | **g** $\sqrt{50}$ | **h** $\sqrt{162}$ |

**5** Write each surd below in its simplest form.

| | | | |
|---|---|---|---|
| **a** $2\sqrt{12}$ | **b** $3\sqrt{8}$ | **c** $2\sqrt{50}$ | **d** $4\sqrt{18}$ |
| **e** $5\sqrt{20}$ | **f** $2\sqrt{75}$ | **g** $10\sqrt{27}$ | **h** $3\sqrt{56}$ |
| **i** $2\sqrt{125}$ | **j** $4\sqrt{45}$ | **k** $3\sqrt{24}$ | **l** $2\sqrt{54}$ |
| **m** $7\sqrt{56}$ | **n** $3\sqrt{72}$ | **o** $3\sqrt{44}$ | **p** $5\sqrt{90}$ |
| **q** $6\sqrt{200}$ | **r** $5\sqrt{98}$ | **s** $9\sqrt{108}$ | **t** $5\sqrt{68}$ |

**6** Simplified surds can be written as an entire root by reversing the above process. Express the following as entire square roots. For example, $4\sqrt{3} = \sqrt{16} \times \sqrt{3} = \sqrt{48}$.

| | | | |
|---|---|---|---|
| **a** $2\sqrt{3}$ | **b** $3\sqrt{2}$ | **c** $2\sqrt{5}$ | **d** $3\sqrt{6}$ |
| **e** $4\sqrt{2}$ | **f** $5\sqrt{3}$ | **g** $3\sqrt{7}$ | **h** $5\sqrt{2}$ |
| **i** $6\sqrt{2}$ | **j** $5\sqrt{6}$ | **k** $3\sqrt{10}$ | **l** $4\sqrt{7}$ |
| **m** $6\sqrt{7}$ | **n** $5\sqrt{10}$ | **o** $7\sqrt{2}$ | **p** $10\sqrt{2}$ |
| **q** $9\sqrt{3}$ | **r** $8\sqrt{4}$ | **s** $7\sqrt{9}$ | **t** $12\sqrt{3}$ |

# 5:03 Addition and subtraction of surds

## PREP QUIZ 5:03

Simplify:

**1** $\sqrt{12}$ **2** $\sqrt{20}$ **3** $\sqrt{32}$ **4** $\sqrt{50}$

Evaluate:

**5** $\sqrt{4} + \sqrt{9}$ **6** $\sqrt{4+9}$ (1 dec. pl.) **7** $\sqrt{16} - \sqrt{4}$

**8** $\sqrt{16-4}$ (1 dec. pl.) **9** $\sqrt{25} + \sqrt{36}$ **10** $\sqrt{25+36}$ (1 dec. pl.)

As can be seen from the Prep quiz, if $x$ and $y$ are two positive numbers:

- $\sqrt{x} + \sqrt{y}$ does not equal $\sqrt{x+y}$
- $\sqrt{x} - \sqrt{y}$ does not equal $\sqrt{x-y}$

$\sqrt{5} + \sqrt{8} \neq \sqrt{13}$
$\sqrt{10} - \sqrt{7} \neq \sqrt{3}$

Only **like** surds can be added or subtracted. For example:

$2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$ and $5\sqrt{2} - 4\sqrt{2} = \sqrt{2}$

However, we can only tell whether surds are like or unlike if each is expressed in its simplest form.

When adding or subtracting surds:
- write each surd in its simplest form
- add or subtract *like* surds only.

## WORKED EXAMPLES

Simplify each of the following.

**1** $4\sqrt{3} + 7\sqrt{3} - 2\sqrt{3}$
$= 9\sqrt{3}$

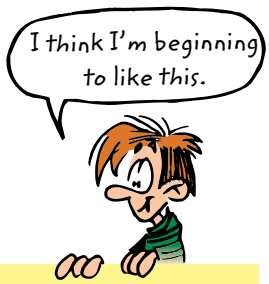
**2** $8\sqrt{2} + \sqrt{5} - \sqrt{2} + 2\sqrt{5}$
$= 7\sqrt{2} + 3\sqrt{5}$

**3** $\sqrt{8} + \sqrt{18}$
$= 2\sqrt{2} + 3\sqrt{2}$
$= 5\sqrt{2}$

**4** $\sqrt{75} + \sqrt{27} - 2\sqrt{3}$
$= 5\sqrt{3} + 3\sqrt{3} - 2\sqrt{3}$
$= 6\sqrt{3}$

**5** $2\sqrt{12} + 3\sqrt{48}$
$= 2(2\sqrt{3}) + 3(4\sqrt{3})$
$= 4\sqrt{3} + 12\sqrt{3}$
$= 16\sqrt{3}$

**6** $2\sqrt{75} - \sqrt{45} + 2\sqrt{20}$
$= 2(5\sqrt{3}) - 3\sqrt{5} + 2(2\sqrt{5})$
$= 10\sqrt{3} - 3\sqrt{5} + 4\sqrt{5}$
$= 10\sqrt{3} + \sqrt{5}$




Remember how in algebra only *like* terms could be added or subtracted.
e.g. $5x + 3x = 8x$
$7a - 6a = a$

## Exercise 5:03

P Foundation worksheet 5:03 Addition and subtraction of surds

**1** Simplify:

**a** $3\sqrt{2} + 2\sqrt{2}$ **b** $4\sqrt{3} + 7\sqrt{3}$ **c** $5\sqrt{6} + 2\sqrt{6}$
**d** $10\sqrt{3} - 7\sqrt{3}$ **e** $9\sqrt{5} - 6\sqrt{5}$ **f** $4\sqrt{2} - 3\sqrt{2}$
**g** $\sqrt{7} + 4\sqrt{7}$ **h** $5\sqrt{3} + \sqrt{3}$ **i** $9\sqrt{6} - \sqrt{6}$
**j** $9\sqrt{5} + 2\sqrt{5} + 3\sqrt{5}$ **k** $4\sqrt{10} + 7\sqrt{10} - 2\sqrt{10}$ **l** $4\sqrt{3} - 3\sqrt{3} + 5\sqrt{3}$
**m** $10\sqrt{2} - 2\sqrt{2} - 5\sqrt{2}$ **n** $\sqrt{3} + 7\sqrt{3} - 5\sqrt{3}$ **o** $2\sqrt{7} + 3\sqrt{7} - 5\sqrt{7}$

**2** Simplify by collecting like surds.

**a** $2\sqrt{5} + 3\sqrt{7} + 4\sqrt{5}$ **b** $\sqrt{7} + 3\sqrt{7} + 2\sqrt{5}$
**c** $2\sqrt{3} + 3\sqrt{5} + 2\sqrt{3} + 5\sqrt{5}$ **d** $9\sqrt{3} + 2\sqrt{2} - 7\sqrt{3} + 3\sqrt{2}$
**e** $6\sqrt{10} + 5\sqrt{7} - 5\sqrt{10} - \sqrt{7}$ **f** $3\sqrt{3} + 6\sqrt{5} - \sqrt{3} - 4\sqrt{5}$
**g** $6\sqrt{11} - 2\sqrt{7} + 2\sqrt{11} + 5\sqrt{7}$ **h** $9\sqrt{2} + 3\sqrt{3} + 9\sqrt{3} - 8\sqrt{2}$
**i** $5\sqrt{2} + 4\sqrt{5} - 3\sqrt{5} - 6\sqrt{2}$ **j** $10\sqrt{7} - 2\sqrt{5} - 8\sqrt{7} - 3\sqrt{5}$

**3** Simplify completely.

**a** $\sqrt{8} + \sqrt{2}$ **b** $\sqrt{12} + 2\sqrt{3}$ **c** $2\sqrt{2} + \sqrt{18}$ **d** $2\sqrt{5} + 2\sqrt{20}$
**e** $\sqrt{27} + 2\sqrt{3}$ **f** $3\sqrt{6} + \sqrt{24}$ **g** $2\sqrt{8} - \sqrt{2}$ **h** $3\sqrt{5} - \sqrt{20}$
**i** $\sqrt{32} - 3\sqrt{2}$ **j** $\sqrt{18} + \sqrt{32}$ **k** $\sqrt{20} + \sqrt{45}$ **l** $2\sqrt{27} - \sqrt{48}$
**m** $\sqrt{75} - 2\sqrt{12}$ **n** $\sqrt{98} + 3\sqrt{50}$ **o** $3\sqrt{50} + 2\sqrt{32}$ **p** $5\sqrt{28} + 2\sqrt{63}$
**q** $4\sqrt{45} - 2\sqrt{20}$ **r** $2\sqrt{75} - 3\sqrt{48}$

**4** Simplify:

**a** $2\sqrt{8} - \sqrt{18} + 3\sqrt{2}$ **b** $2\sqrt{5} + \sqrt{45} - \sqrt{20}$ **c** $\sqrt{27} + 2\sqrt{48} - 5\sqrt{3}$
**d** $5\sqrt{7} - \sqrt{63} + 2\sqrt{28}$ **e** $5\sqrt{3} + \sqrt{50} - \sqrt{12}$ **f** $2\sqrt{45} + \sqrt{20} + \sqrt{32}$
**g** $9\sqrt{8} + 3\sqrt{12} - \sqrt{27}$ **h** $5\sqrt{18} + \sqrt{72} - \sqrt{75}$

# 5:04 Multiplication and division of surds

## WORKED EXAMPLES

Simplify the following.

1 $\sqrt{7} \times \sqrt{3}$

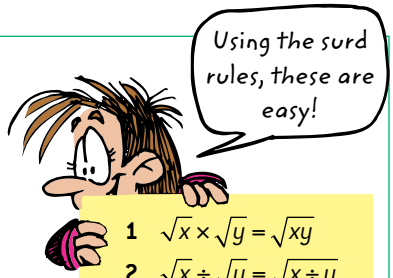
2 $\sqrt{96}\sqrt{12}$

3 $3\sqrt{2} \times 5\sqrt{2}$

4 $5\sqrt{8} \times 3\sqrt{6}$

5 $\dfrac{4\sqrt{3} \times \sqrt{18}}{\sqrt{12}}$

6 $\sqrt{3}(2\sqrt{3} - \sqrt{5})$




1 $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$

2 $\sqrt{x} \div \sqrt{y} = \sqrt{x \div y}$

### Solutions

1 $\sqrt{7} \times \sqrt{3} = \sqrt{7 \times 3}$
$= \sqrt{21}$

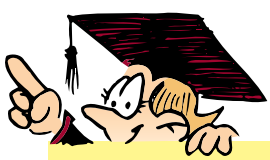
2 $\sqrt{96}\sqrt{12} = \sqrt{96\sqrt{12}}$
$= \sqrt{8}$
$= 2\sqrt{2}$

3 $3\sqrt{2} \times 5\sqrt{2} = 3 \times 5 \times \sqrt{2} \times \sqrt{2}$ $\sqrt{2} \times \sqrt{2} = 2$
$= 15 \times 2$
$= 30$

4 $5\sqrt{8} \times 3\sqrt{6} = 5 \times 3 \times \sqrt{8} \times \sqrt{6}$
$= 15 \times \sqrt{48}$
$= 15 \times 4\sqrt{3}$
$= 60\sqrt{3}$

5 $\dfrac{4\sqrt{3} \times \sqrt{18}}{\sqrt{12}} = \dfrac{4\sqrt{3} \times 3\sqrt{2}}{2\sqrt{3}}$
$= \dfrac{4 \times 3 \times \sqrt{3} \times \sqrt{2}}{2\sqrt{3}}$
$= \dfrac{12 \times \sqrt{6}}{2\sqrt{3}}$
$= 6\sqrt{\dfrac{6}{3}}$
$= 6\sqrt{2}$

6 $\sqrt{3}(2\sqrt{3} - \sqrt{5}) = \sqrt{3} \times 2\sqrt{3} - \sqrt{3} \times \sqrt{5}$
$= 2 \times 3 - \sqrt{3 \times 5}$
$= 6 - \sqrt{15}$



At this point we could cancel like this:

$$\frac{{}^{2}\cancel{4} \times 3 \times {}^{1}\cancel{\sqrt{3}} \times \sqrt{2}}{{}_{1}\cancel{2}\cancel{\sqrt{3}}_{1}} = 6\sqrt{2}$$

# Exercise 5:04

P Foundation worksheet 5:04
Multiplication and division of surds

**1** Simplify these products.

**a** $\sqrt{2} \times \sqrt{3}$ **b** $\sqrt{5} \times \sqrt{7}$ **c** $\sqrt{3} \times \sqrt{11}$

**d** $\sqrt{5} \times \sqrt{2}$ **e** $\sqrt{3} \times \sqrt{7}$ **f** $\sqrt{10} \times \sqrt{7}$

**g** $\sqrt{2} \times \sqrt{8}$ **h** $\sqrt{3} \times \sqrt{12}$ **i** $\sqrt{5} \times \sqrt{20}$

**j** $\sqrt{3} \times \sqrt{6}$ **k** $\sqrt{5} \times \sqrt{10}$ **l** $\sqrt{2} \times \sqrt{10}$

**m** $2\sqrt{3} \times \sqrt{5}$ **n** $3\sqrt{2} \times 4\sqrt{5}$ **o** $7\sqrt{2} \times 2\sqrt{3}$

**p** $5\sqrt{2} \times 2\sqrt{2}$ **q** $3\sqrt{3} \times 2\sqrt{3}$ **r** $4\sqrt{5} \times \sqrt{5}$

**s** $4\sqrt{2} \times 3\sqrt{10}$ **t** $2\sqrt{6} \times \sqrt{8}$ **u** $4\sqrt{3} \times 2\sqrt{15}$

**v** $2\sqrt{x} \times 3\sqrt{x}$ **w** $\sqrt{8x} \times \sqrt{2}$ **x** $a\sqrt{x} \times a\sqrt{x}$

**2** Simplify:

**a** $\sqrt{10} \div \sqrt{2}$ **b** $\sqrt{12} \div \sqrt{4}$ **c** $\sqrt{6} \div \sqrt{3}$

**d** $\sqrt{27} \div \sqrt{3}$ **e** $\sqrt{32} \div \sqrt{8}$ **f** $\sqrt{45} \div \sqrt{5}$

**g** $5\sqrt{2} \div \sqrt{2}$ **h** $6\sqrt{5} \div \sqrt{5}$ **i** $10\sqrt{3} \div 5\sqrt{3}$

**j** $16\sqrt{8} \div 8$ **k** $30\sqrt{5} \div 10$ **l** $24\sqrt{7} \div 24$

**m** $12\sqrt{10} \div 2\sqrt{5}$ **n** $9\sqrt{12} \div 3\sqrt{6}$ **o** $10\sqrt{15} \div 5\sqrt{5}$

**p** $\sqrt{20} \div 2$ **q** $\sqrt{75} \div 5$ **r** $5\sqrt{8} \div 10$

**s** $\sqrt{2x} \div \sqrt{2}$ **t** $\sqrt{5a} \div \sqrt{a}$ **u** $\sqrt{20p} \div 2\sqrt{p}$

**3** Simplify fully.

**a** $\dfrac{2\sqrt{3} \times 2\sqrt{6}}{4}$ **b** $\dfrac{4\sqrt{5} \times 2\sqrt{6}}{\sqrt{10}}$ **c** $\dfrac{2\sqrt{5} \times 3\sqrt{8}}{6\sqrt{20}}$

**d** $\dfrac{\sqrt{15} \times \sqrt{3}}{3\sqrt{5}}$ **e** $\dfrac{2\sqrt{3} \times \sqrt{6}}{\sqrt{12}}$ **f** $\dfrac{3\sqrt{7} \times 2\sqrt{6}}{\sqrt{21}}$

**g** $\dfrac{2\sqrt{6} \times 5\sqrt{5}}{10\sqrt{15}}$ **h** $\dfrac{6\sqrt{2} \times \sqrt{6}}{4\sqrt{3}}$ **i** $\dfrac{\sqrt{12} \times \sqrt{27}}{\sqrt{8} \times 2\sqrt{6}}$

**4** Expand and simplify.

**a** $\sqrt{2}(\sqrt{3} + \sqrt{2})$ **b** $\sqrt{5}(\sqrt{5} + \sqrt{2})$ **c** $\sqrt{7}(2\sqrt{7} - \sqrt{2})$

**d** $\sqrt{3}(5 - \sqrt{3})$ **e** $\sqrt{2}(2\sqrt{3} - 1)$ **f** $\sqrt{10}(5\sqrt{2} - 4)$

**g** $2\sqrt{2}(\sqrt{2} + 1)$ **h** $3\sqrt{5}(\sqrt{5} + 2)$ **i** $4\sqrt{3}(\sqrt{2} - \sqrt{3})$

**j** $3\sqrt{6}(\sqrt{5} + \sqrt{6})$ **k** $2\sqrt{7}(\sqrt{7} - \sqrt{2})$ **l** $\sqrt{3}(7 - 3\sqrt{3})$

**m** $2\sqrt{2}(\sqrt{3} + 2\sqrt{2})$ **n** $4\sqrt{5}(\sqrt{2} - 2\sqrt{5})$ **o** $5\sqrt{6}(2\sqrt{6} - 3\sqrt{2})$

**p** $\sqrt{a}(\sqrt{a} + 1)$ **q** $\sqrt{x}(2\sqrt{x} + 3)$ **r** $2\sqrt{y}(3\sqrt{y} + 2\sqrt{x})$

## FUN SPOT 5:04 — WHAT DO INUITS SING AT BIRTHDAY PARTIES?

Answer each question and write the letter for that question in the box above the correct answer.

*Solve:*

**A** $\frac{x}{2} = 5$ **E** $\frac{x}{5} = 3$ **E** $\frac{x}{10} = 1{\cdot}86$

**L** $\frac{x}{7} = 0{\cdot}185$ **E** $\frac{x}{5} = 0{\cdot}5$ **O** $0{\cdot}560 = \frac{x}{5}$

**W** $5x + 3 = 21$ **Y** $\frac{14}{x} = 1$ **Z** $2x - 5 = 7 - 6x$

*Simplify:*

**L** $5x - x$ **O** $5x \surd x$ **Y** $5x \times x$ **E** $2x \times 3y$

**L** $x^3 \times x^5$ **O** $x^6 \surd x^2$ **F** $(2x^5)^3$ **G** $5x^0 \times 4$

**D** $35 \times 10^{-2}$ **E** $\sqrt{169}$ **F** $\sqrt{7} \times \sqrt{7}$ **G** $(3\sqrt{3})^2$

**H** $\sqrt{2} + \sqrt{2}$ **I** $\sqrt{50}$ **J** $\sqrt{3} \times \sqrt{6}$ **L** $3\sqrt{7} - \sqrt{7}$

**N** $\sqrt{50} + \sqrt{18}$ **O** $5\sqrt{2} \times 3\sqrt{2}$ **R** $\sqrt{32} \surd \sqrt{2}$ **S** $\sqrt{6} \surd \sqrt{2}$

**T** $\sqrt{3} + 2\sqrt{3}$

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $3\sqrt{3}$ | $2\sqrt{2}$ | $x = 18{\cdot}6$ | $x = 14$ | $\sqrt{3}$ | $5\sqrt{2}$ | $8\sqrt{2}$ | 20 | 7 | 4 | $x = 15$ | 13 | $x = 1{\cdot}5$ | $x = 2{\cdot}5$ | $x = 10$ |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $3\sqrt{2}$ | 30 | $x^8$ | $4x$ | $5x^2$ | 27 | $x = 2{\cdot}8$ | $x^4$ | 0·35 | $8x^{15}$ | $6xy$ | $x = 1{\cdot}295$ | $2\sqrt{7}$ | 5 | $x = 3{\cdot}6$ |

The design of the Parthenon is said to be based on the golden ratio, $1 : \varphi$

where $\varphi = \frac{1+\sqrt{5}}{2}$ $(\varphi \approx 1{\cdot}618)$

# 5:05 Binomial products

## PREP QUIZ 5:05

Simplify the following.

**1** $\sqrt{3} \times \sqrt{5}$ **2** $\sqrt{18} \times \sqrt{2}$ **3** $(\sqrt{5})^2$ **4** $(3\sqrt{2})^2$ **5** $3\sqrt{2} + 7\sqrt{2}$

**6** $9\sqrt{3} - \sqrt{3}$ **7** $3\sqrt{3} - 3\sqrt{3}$

Expand and simplify where possible.

**8** $5(\sqrt{7} - 4)$ **9** $\sqrt{10}(\sqrt{10} + \sqrt{3})$ **10** $2\sqrt{3}(3\sqrt{2} - \sqrt{3})$

In Year 9, we saw how to expand a binomial product.

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

The same procedure is used if some of the terms are not pronumerals but surds. Examine the following examples.

## WORKED EXAMPLES

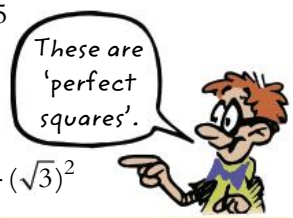
Expand and simplify:

**1 a** $(\sqrt{2} + 3)(\sqrt{2} - 5)$
$= \sqrt{2}(\sqrt{2} - 5) + 3(\sqrt{2} - 5)$
$= (\sqrt{2})^2 - 5\sqrt{2} + 3\sqrt{2} - 15$
$= 2 - 2\sqrt{2} - 15$
$= -13 - 2\sqrt{2}$

**2 a** $(2\sqrt{3} + 5)^2$
$= (2\sqrt{3})^2 + 2 \times 2\sqrt{3} \times 5 + (5)^2$
$= 12 + 20\sqrt{3} + 25$
$= 37 + 20\sqrt{3}$

*Remember!*
$(a + b)(a - b) = a^2 - b^2$

**b** $(3\sqrt{2} - \sqrt{5})(2\sqrt{2} + 3\sqrt{5})$
$= 3\sqrt{2}(2\sqrt{2} + 3\sqrt{5}) - \sqrt{5}(2\sqrt{2} + 3\sqrt{5})$
$= 12 + 9\sqrt{10} - 2\sqrt{10} - 15$
$= 7\sqrt{10} - 3$




**b** $(\sqrt{7} - \sqrt{3})^2$
$= (\sqrt{7})^2 - 2 \times \sqrt{7} \times \sqrt{3} + (\sqrt{3})^2$
$= 7 - 2\sqrt{21} + 3$
$= 10 - 2\sqrt{21}$

*Remember!*
$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$

**3 a** $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2}) = (\sqrt{5})^2 - (\sqrt{2})^2$
$= 5 - 2$
$= 3$

**b** $(5\sqrt{2} - \sqrt{7})(5\sqrt{2} + \sqrt{7}) = (5\sqrt{2})^2 - (\sqrt{7})^2$
$= 50 - 7$
$= 43$

# Exercise 5:05

P Foundation worksheet 5:05
Binomial products—surds

Expand and simplify the following.

**1** **a** $(\sqrt{2}+3)(\sqrt{2}+1)$ **b** $(\sqrt{3}+5)(\sqrt{3}-1)$ **c** $(\sqrt{7}-2)(\sqrt{7}-5)$

**d** $(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{5})$ **e** $(\sqrt{7}-\sqrt{2})(\sqrt{5}-\sqrt{2})$ **f** $(\sqrt{10}+\sqrt{2})(\sqrt{5}+\sqrt{3})$

**g** $(5+\sqrt{2})(\sqrt{3}+\sqrt{2})$ **h** $(2-\sqrt{3})(2-\sqrt{5})$ **i** $(\sqrt{6}-\sqrt{2})(5-\sqrt{6})$

**j** $(\sqrt{2}+\sqrt{3})(2\sqrt{2}+1)$ **k** $(\sqrt{3}+2\sqrt{5})(\sqrt{3}+\sqrt{5})$ **l** $(2\sqrt{3}+\sqrt{2})(\sqrt{3}+2\sqrt{2})$

**m** $(5\sqrt{2}-\sqrt{7})(\sqrt{7}-2\sqrt{2})$ **n** $(2\sqrt{5}-\sqrt{3})(5\sqrt{3}+\sqrt{5})$ **o** $(\sqrt{7}-5\sqrt{2})(\sqrt{2}-5\sqrt{7})$

**p** $(5\sqrt{3}+2\sqrt{7})(2\sqrt{3}-5\sqrt{7})$ **q** $(10\sqrt{10}+\sqrt{7})(2\sqrt{7}+\sqrt{10})$ **r** $(5\sqrt{3}+7\sqrt{2})(7\sqrt{3}-2\sqrt{2})$

**s** $(9\sqrt{2}-\sqrt{3})(4\sqrt{2}+2\sqrt{3})$ **t** $(5\sqrt{7}+4)(2\sqrt{7}-7)$ **u** $(6\sqrt{3}+5)(7-2\sqrt{3})$

**v** $(\sqrt{x}+3)(\sqrt{x}+2)$ **w** $(\sqrt{m}+\sqrt{n})(2\sqrt{m}+\sqrt{n})$ **x** $(3\sqrt{a}-2\sqrt{b})(2\sqrt{a}+3\sqrt{b})$

**2** **a** $(\sqrt{2}+1)^2$ **b** $(\sqrt{3}-5)^2$ **c** $(\sqrt{5}+2)^2$

**d** $(\sqrt{3}+\sqrt{2})^2$ **e** $(\sqrt{5}-\sqrt{2})^2$ **f** $(\sqrt{3}+\sqrt{10})^2$

**g** $(2\sqrt{3}+1)^2$ **h** $(3\sqrt{2}-4)^2$ **i** $(5+2\sqrt{5})^2$

**j** $(2\sqrt{2}+\sqrt{7})^2$ **k** $(3\sqrt{5}+\sqrt{10})^2$ **l** $(\sqrt{7}-3\sqrt{5})^2$

**m** $(5\sqrt{3}+2\sqrt{2})^2$ **n** $(7\sqrt{3}-2\sqrt{5})^2$ **o** $(5\sqrt{10}-10\sqrt{3})^2$

**p** $(\sqrt{x}+\sqrt{y})^2$ **q** $(2\sqrt{m}+5)^2$ **r** $(3\sqrt{p}-2\sqrt{q})^2$

**3** **a** $(\sqrt{2}+1)(\sqrt{2}-1)$ **b** $(5+\sqrt{3})(5-\sqrt{3})$

**c** $(\sqrt{10}-7)(\sqrt{10}+7)$ **d** $(4-\sqrt{2})(4+\sqrt{2})$

**e** $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **f** $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$

**g** $(\sqrt{10}-\sqrt{8})(\sqrt{10}+\sqrt{8})$ **h** $(\sqrt{11}+\sqrt{7})(\sqrt{11}-\sqrt{7})$

**i** $(2\sqrt{3}-5)(2\sqrt{3}+5)$ **j** $(6-3\sqrt{2})(6+3\sqrt{2})$

**k** $(\sqrt{7}+2\sqrt{3})(\sqrt{7}-2\sqrt{3})$ **l** $(3\sqrt{5}-\sqrt{3})(3\sqrt{5}+\sqrt{3})$

**m** $(5\sqrt{2}-2\sqrt{3})(5\sqrt{2}+2\sqrt{3})$

**n** $(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})$

**o** $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$

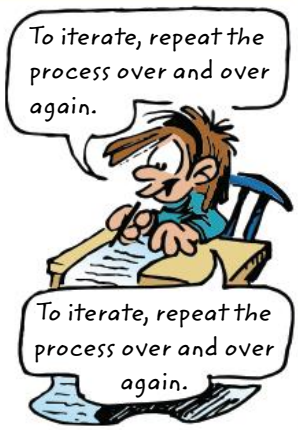
**p** $(2\sqrt{a}+3\sqrt{b})(2\sqrt{a}-3\sqrt{b})$

Important notice!
The two binomials in each part of Question **3** are *conjugate surds*.
Note that when a binomial surd is multiplied by its 'conjugate', the answer is always a rational number.

## INVESTIGATION 5:05 ITERATION TO FIND SQUARE ROOTS

Iteration is the repetition of a process.

We can use a simple process to find square roots.




### Example

Without using a calculator find $\sqrt{3}$ correct to four decimal places.

**Step 1** Estimate the value of $\sqrt{3}$.
Let $E = 1{\cdot}6$ ($E^2$ should be close to 3)

**Step 2** Divide by your estimate.
$3 \div 1{\cdot}6 = 1{\cdot}875$
$1{\cdot}6 \times 1{\cdot}875 = 3$
Therefore, $\sqrt{3}$ must lie between 1·6 and 1·875.

**Step 3** Average these two numbers to get a better estimate.

$$\frac{1{\cdot}6 + 1{\cdot}875}{2} = 1{\cdot}7375$$

$$\therefore \sqrt{3} \doteqdot 1{\cdot}7375$$

- Use 1·7375 as the new estimate and repeat the steps above (iterate).
  $E = 1{\cdot}7375$
  $3 \div 1{\cdot}7375 \doteqdot 1{\cdot}72662$
  $$\frac{1{\cdot}7375 + 1{\cdot}72662}{2} = 1{\cdot}73206$$
  $\therefore \sqrt{3} \doteqdot 1{\cdot}73206$

- If we use 1·732 06 as our next estimate we get a better approximation (i.e. $\sqrt{3} \doteqdot 1{\cdot}732051$).
  Now $\sqrt{3}$ lies between 1·732 06 and 1·732 051.
  $\therefore \sqrt{3} = 1{\cdot}7321$ (4 dec. pl.)

### Questions

1 Use iteration to find each of the following, correct to four decimal places.
**a** $\sqrt{2}$ **b** $\sqrt{5}$ **c** $\sqrt{70}$ **d** $\sqrt{110}$

2 Investigate finding the square root of a number $n$, using iteration of the formula:

$$\text{New estimate} = \frac{x^2 + n}{2x}$$

where $x$ is your last estimate and we wish to find $\sqrt{n}$.

# 5:06 Rationalising the denominator

## PREP QUIZ 5:06

Simplify the following:

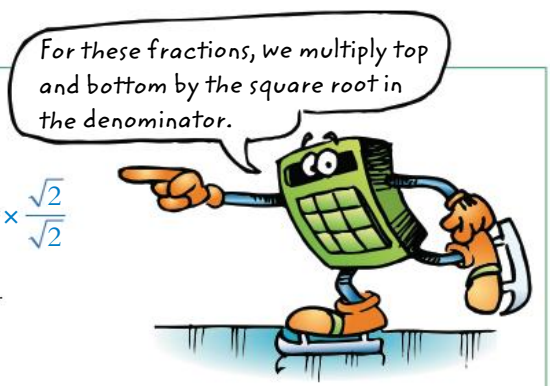
1 $\sqrt{5} \times \sqrt{5}$
2 $\sqrt{10} \times \sqrt{10}$
3 $2\sqrt{3} \times \sqrt{3}$
4 $5\sqrt{2} \times \sqrt{2}$
5 $2\sqrt{6} \times \sqrt{6}$
6 $(\sqrt{2}+1)(\sqrt{2}-1)$
7 $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$
8 $(5-\sqrt{2})(5+\sqrt{2})$
9 $(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})$
10 $(5\sqrt{2}-3\sqrt{3})(5\sqrt{2}+3\sqrt{3})$

If a fraction has a surd (i.e. an irrational number) in its denominator, we generally rewrite the fraction with a 'rational' denominator by using the methods shown below.

### Monomial denominators

## WORKED EXAMPLES

Rewrite with rational denominators.




**1** $\dfrac{3}{\sqrt{3}} = \dfrac{3}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{3\sqrt{3}}{3}$

$= \sqrt{3}$

**2** $\dfrac{1}{5\sqrt{2}} = \dfrac{1}{5\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{\sqrt{2}}{5 \times 2}$

$= \dfrac{\sqrt{2}}{10}$

**3** $\dfrac{\sqrt{5}}{\sqrt{12}} = \dfrac{\sqrt{5}}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{15}}{2 \times 3}$

$= \dfrac{\sqrt{15}}{6}$

**4** $\dfrac{2+\sqrt{3}}{2\sqrt{3}} = \dfrac{2+\sqrt{3}}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{3}(2+\sqrt{3})}{2 \times 3}$

$= \dfrac{2\sqrt{3}+3}{6}$

*Note:*
Multiplying by $\dfrac{\sqrt{3}}{\sqrt{3}}$ is the same as multiplying by 1.

## Binomial denominators

### WORKED EXAMPLES

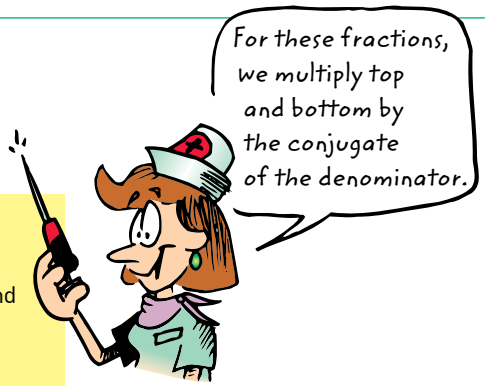
Rationalise the denominators for each expression.

**1** $\dfrac{5}{5-\sqrt{2}} = \dfrac{5}{5-\sqrt{2}} \times \dfrac{5+\sqrt{2}}{5+\sqrt{2}}$

$= \dfrac{5(5+\sqrt{2})}{25-2}$

$= \dfrac{25+5\sqrt{2}}{23}$

*Note:*
The product of a binomial surd and its conjugate is always rational.




**2** $\dfrac{1}{\sqrt{3}+\sqrt{2}} = \dfrac{1}{\sqrt{3}+\sqrt{2}} \times \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

$= \dfrac{\sqrt{3}-\sqrt{2}}{3-2}$

$= \sqrt{3}-\sqrt{2}$

**3** $\dfrac{2\sqrt{3}+\sqrt{5}}{2\sqrt{3}-\sqrt{5}} = \dfrac{2\sqrt{3}+\sqrt{5}}{2\sqrt{3}-\sqrt{5}} \times \dfrac{2\sqrt{3}+\sqrt{5}}{2\sqrt{3}+\sqrt{5}}$

$= \dfrac{(2\sqrt{3}+\sqrt{5})^2}{12-5}$

$= \dfrac{17+4\sqrt{15}}{7}$

## Exercise 5:06

**1** Rationalise the denominator for each of the following:

**a** $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{1}{\sqrt{5}}$ **c** $\dfrac{2}{\sqrt{3}}$ **d** $\dfrac{5}{\sqrt{10}}$ **e** $\dfrac{3}{\sqrt{2}}$ **f** $\dfrac{6}{\sqrt{3}}$

**g** $\dfrac{10}{\sqrt{5}}$ **h** $\dfrac{2}{\sqrt{11}}$ **i** $\dfrac{\sqrt{2}}{\sqrt{3}}$ **j** $\dfrac{\sqrt{3}}{\sqrt{5}}$ **k** $\dfrac{\sqrt{5}}{\sqrt{10}}$ **l** $\dfrac{\sqrt{3}}{\sqrt{15}}$

**m** $\dfrac{1}{2\sqrt{2}}$ **n** $\dfrac{2}{5\sqrt{3}}$ **o** $\dfrac{7}{2\sqrt{5}}$ **p** $\dfrac{10}{2\sqrt{3}}$ **q** $\dfrac{\sqrt{6}}{2\sqrt{3}}$ **r** $\dfrac{\sqrt{5}}{5\sqrt{2}}$

**s** $\dfrac{2\sqrt{3}}{3\sqrt{2}}$ **t** $\dfrac{5\sqrt{7}}{3\sqrt{5}}$ **u** $\dfrac{2+\sqrt{3}}{\sqrt{3}}$ **v** $\dfrac{1+\sqrt{5}}{\sqrt{2}}$ **w** $\dfrac{\sqrt{7}+\sqrt{3}}{2\sqrt{7}}$ **x** $\dfrac{\sqrt{10}-\sqrt{5}}{5\sqrt{10}}$

**2** Express with a rational denominator.

**a** $\dfrac{1}{1+\sqrt{2}}$ **b** $\dfrac{1}{\sqrt{3}-1}$ **c** $\dfrac{1}{\sqrt{7}-\sqrt{5}}$ **d** $\dfrac{1}{\sqrt{10}+\sqrt{2}}$

**e** $\dfrac{3}{\sqrt{3}+2}$ **f** $\dfrac{5}{5-\sqrt{2}}$ **g** $\dfrac{10}{\sqrt{5}-\sqrt{2}}$ **h** $\dfrac{12}{\sqrt{7}-\sqrt{3}}$

**i** $\dfrac{1}{2\sqrt{3}+5}$ **j** $\dfrac{2}{5-2\sqrt{2}}$ **k** $\dfrac{3}{3\sqrt{2}+2\sqrt{3}}$ **l** $\dfrac{1}{4\sqrt{3}-3\sqrt{2}}$

**m** $\dfrac{5+\sqrt{2}}{5-\sqrt{2}}$ **n** $\dfrac{4+\sqrt{3}}{4-\sqrt{3}}$ **o** $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ **p** $\dfrac{3\sqrt{2}-\sqrt{3}}{3\sqrt{2}+\sqrt{3}}$

**3** Evaluate each fraction correct to three significant figures (using your calculator). Then rationalise the denominator and evaluate the fraction again. Compare this answer with your first calculation.

**a** $\frac{2}{\sqrt{5}}$ **b** $\frac{3}{\sqrt{7}}$ **c** $\frac{\sqrt{3}}{2\sqrt{2}}$ **d** $\frac{\sqrt{7}}{3\sqrt{5}}$

**e** $\frac{1}{\sqrt{7}-\sqrt{5}}$ **f** $\frac{\sqrt{3}}{\sqrt{3}+\sqrt{2}}$ **g** $\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}-\sqrt{7}}$ **h** $\frac{2\sqrt{3}-\sqrt{7}}{2\sqrt{3}+\sqrt{7}}$

**4** Rationalise each denominator, then express as a single fraction.

**a** $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{5}}$

**d** $\frac{2}{\sqrt{10}}-\frac{3}{\sqrt{5}}$ **e** $\frac{3}{\sqrt{8}}+\frac{5}{\sqrt{2}}$ **f** $\frac{2}{2\sqrt{3}}-\frac{1}{3\sqrt{2}}$

**g** $\frac{2}{5\sqrt{2}}+\frac{5}{\sqrt{10}}$ **h** $\frac{\sqrt{5}}{\sqrt{2}}+\frac{\sqrt{5}}{\sqrt{3}}$ **i** $\frac{\sqrt{2}}{\sqrt{5}}-\frac{\sqrt{3}}{\sqrt{2}}$

**j** $\frac{1}{2-\sqrt{3}}+\frac{1}{2+\sqrt{3}}$ **k** $\frac{1}{\sqrt{5}-\sqrt{3}}-\frac{1}{\sqrt{7}+\sqrt{5}}$ **l** $\frac{5}{6-\sqrt{3}}+\frac{3}{5+\sqrt{3}}$

## MATHS TERMS 5

**conjugate**

- binomials that multiply to give the difference of two squares are the 'conjugate' of each other,
  e.g. $(a-b)$ and $(a+b)$
  $(\sqrt{3}+5)$ and $(\sqrt{3}-5)$
  are conjugate pairs

**irrational numbers**

- numbers that *cannot* be expressed in the form $\frac{a}{b}$, where $a$ and $b$ are integers
- they cannot be given an exact decimal value,
  e.g. $\pi$, $\sqrt{3}$, $3\sqrt{2}+1$

**rational numbers**

- numbers that can be written in the form $\frac{a}{b}$ where $a$ and $b$ are integers $(b \neq 0)$
- they can be expressed as a terminating or repeating decimal
  e.g. integers, fractions, percentages

**real numbers**

- the combination of rational and irrational numbers

**surds**

- numerical expressions that involve irrational roots,
  e.g. $\sqrt{3}$, $\sqrt[3]{5}$, $2\sqrt{7}+5$

## DIAGNOSTIC TEST 5 SURDS

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

| | | |
|---|---|---|
| 1 | For each expression, state whether it is *rational* or *irrational*.<br>a $\sqrt{50}$ b $0\dot{4}$ c $\sqrt{2}$ d $\sqrt{16}$ | 5:01 |
| 2 | Evaluate correct to three decimal places:<br>a $\sqrt{5}$ b $\sqrt{13}$ c $\sqrt{21}$ d $\sqrt{47}$ | 5:01 |
| 3 | Simplify each surd.<br>a $\sqrt{20}$ b $\sqrt{27}$ c $3\sqrt{8}$ d $2\sqrt{75}$ | 5:02 |
| 4 | Express each of the following as an entire square root.<br>a $2\sqrt{5}$ b $3\sqrt{2}$ c $5\sqrt{7}$ d $4\sqrt{5}$ | 5:02 |
| 5 | Simplify completely.<br>a $2\sqrt{3}+4\sqrt{3}$ b $6\sqrt{5}-\sqrt{5}$ c $\sqrt{8}-\sqrt{2}$ d $\sqrt{27}+2\sqrt{3}$ | 5:03 |
| 6 | Simplify:<br>a $\sqrt{5}\times\sqrt{6}$ b $\sqrt{3}\times\sqrt{12}$ c $2\sqrt{3}\times\sqrt{5}$ d $3\sqrt{8}\times2\sqrt{2}$ | 5:04 |
| 7 | Simplify:<br>a $\sqrt{12}\div\sqrt{2}$ b $\sqrt{32}\div\sqrt{8}$ c $5\sqrt{3}\div\sqrt{3}$ d $10\sqrt{10}\div2\sqrt{5}$ | 5:04 |
| 8 | Expand and simplify:<br>a $(2+\sqrt{5})(3+\sqrt{5})$ b $(2\sqrt{3}+\sqrt{2})(\sqrt{3}-3\sqrt{2})$<br>c $(\sqrt{7}+\sqrt{3})^2$ d $(5-\sqrt{3})(5+\sqrt{3})$ | 5:05 |
| 9 | Rationalise the denominator of:<br>a $\frac{3}{\sqrt{2}}$ b $\frac{5}{\sqrt{5}}$ c $\frac{\sqrt{3}+1}{2\sqrt{3}}$ d $\frac{5-\sqrt{2}}{5\sqrt{2}}$ | 5:06 |
| 10 | Express with a rational denominator.<br>a $\frac{1}{5+\sqrt{2}}$ b $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ c $\frac{\sqrt{10}-2}{\sqrt{10}+2}$ d $\frac{\sqrt{5}+\sqrt{7}}{3\sqrt{7}-2\sqrt{5}}$ | 5:06 |

Can you use your calculator to find the value of $2^{500}$? What is the largest power of 2 that can be calculated using your calculator?

## ASSIGNMENT 5A Chapter review

**1** Between which two consecutive integers does each square root lie?

**a** $\sqrt{7}$ **b** $\sqrt{30}$ **c** $\sqrt{90}$

**d** $\sqrt{150}$ **e** $\sqrt{420}$ **f** $\sqrt{899}$

**2** Simplify each surd.

**a** $\sqrt{18}$ **b** $\sqrt{54}$ **c** $2\sqrt{32}$

**d** $5\sqrt{24}$ **e** $10\sqrt{40}$ **f** $9\sqrt{75}$

**3** Rewrite as an entire surd. (e.g. $3\sqrt{5} = \sqrt{45}$)

**a** $4\sqrt{3}$ **b** $3\sqrt{7}$

**c** $5\sqrt{6}$ **d** $10\sqrt{5}$

**4** Simplify these expressions.

**a** $6\sqrt{5} + 3\sqrt{5}$

**b** $6\sqrt{5} - 3\sqrt{5}$

**c** $6\sqrt{5} \times 3\sqrt{5}$

**d** $6\sqrt{5} \surd 3\sqrt{5}$

**5** Simplify completely.

**a** $3\sqrt{5} + \sqrt{20}$

**b** $\sqrt{18} + 3\sqrt{8}$

**c** $2\sqrt{50} - \sqrt{32}$

**d** $3\sqrt{24} - 2\sqrt{54}$

**e** $4\sqrt{12} + 2\sqrt{27}$

**f** $\sqrt{98} - \sqrt{72}$

**g** $3\sqrt{50} + 4\sqrt{48} - 5\sqrt{8}$

**h** $3\sqrt{28} - 4\sqrt{27} + 2\sqrt{18} - \sqrt{63}$

**6** Simplify:

**a** $2\sqrt{5} \times \sqrt{20}$

**b** $2\sqrt{7} \times 3\sqrt{2}$

**c** $3\sqrt{6} \times \sqrt{12}$

**d** $4\sqrt{3} \times 2\sqrt{6}$

**e** $6\sqrt{50} \surd 3\sqrt{10}$

**f** $4\sqrt{40} \surd 8\sqrt{10}$

**g** $15\sqrt{12} \surd 5\sqrt{6}$

**h** $3\sqrt{30} \surd 6\sqrt{6}$

**7** Expand and simplify:

**a** $\sqrt{3}(\sqrt{3} + \sqrt{15})$

**b** $(\sqrt{5} + \sqrt{2})(\sqrt{5} + \sqrt{3})$

**c** $(2\sqrt{6} + \sqrt{3})(\sqrt{6} - 2\sqrt{3})$

**d** $(\sqrt{7} - \sqrt{2})^2$

**e** $(2\sqrt{5} - 4\sqrt{3})(2\sqrt{5} + 4\sqrt{3})$

**f** $(\sqrt{m} + \sqrt{n})^2$

**g** $(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n})$

**8** Rationalise the denominator of each expression.

**a** $\dfrac{5}{2\sqrt{5}}$ **b** $\dfrac{2\sqrt{3}}{3\sqrt{2}}$

**c** $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$ **d** $\dfrac{3}{2\sqrt{5}} - \dfrac{2}{3\sqrt{2}}$

**9** Rationalise the denominator of each expression.

**a** $\dfrac{2}{\sqrt{5} - 1}$ **b** $\dfrac{1}{\sqrt{7} + \sqrt{2}}$

**c** $\dfrac{\sqrt{5} - 2}{\sqrt{5} + 2}$ **d** $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}}$

**10** Simplify each expression, writing the answer with a rational denominator.

**a** $\dfrac{5}{\sqrt{3}} - \dfrac{2}{\sqrt{3} - 1}$

**b** $\dfrac{3}{\sqrt{5} + 2} \times \dfrac{2}{\sqrt{5} + 1}$

**c** $\dfrac{\sqrt{7} + 2}{\sqrt{3} + 1} \div \dfrac{2\sqrt{3} + 1}{2\sqrt{7} - 1}$

## ASSIGNMENT 5B Working mathematically

**1** How many digits are needed to number the pages of:
- **a** a 50-page book
- **b** a 500-page book
- **c** a 5000-page book?

**2** **a** A cube made up of 27 smaller cubes is painted on the outside. Some of the smaller cubes are painted on 3 faces, some on 2 faces, some on 1 face and some on no faces at all. How many small cubes are painted on:

**i** 3 faces **ii** 2 faces **iii** 1 face **iv** no faces?

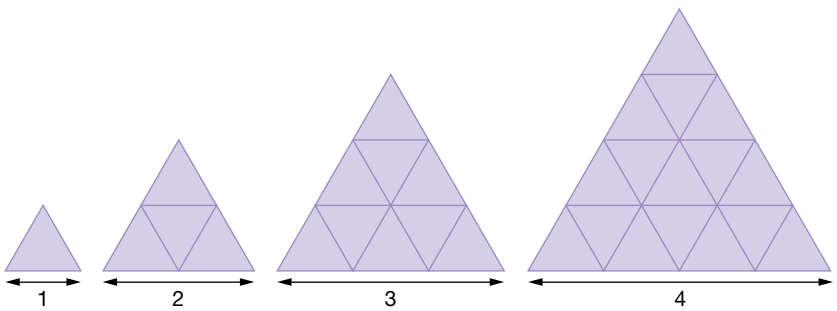
**b** A larger cube with 4 smaller cubes along each edge (i.e. 64 cubes altogether) was painted.
How many small cubes would be painted on:

**i** 3 faces **ii** 2 faces
**iii** 1 face **iv** no faces?

**3** **a** **i** How many small equilateral triangles are contained within each figure below?

**ii** How many small triangles would be contained in a similar large triangle that had a side length of 10 units?

**b** The second figure is a triangle with a side length of 2 units. How many of this size triangle can be found in the next two figures?




**4** **a** How many terms are in the sequence of numbers: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?
- **b** If this sequence were to be continued up to, and including, nine 9s, how many terms would there be?
- **c** Determine a rule that would give the number of terms if the sequence were continued to include $n$ lots of the number $n$.
- **d** Find the value of the:

**i** 100th term **ii** 1000th term.

**5** Hawker and Saines both started driving at 9:00 am. They drove the same distance and arrived at their destination at the same time. Hawker drove for twice as long as Saines rested and Saines drove for three times as long as Hawker rested. Who drove the fastest?

## ASSIGNMENT 5C Cumulative revision

**1** Rewrite each of these numbers correct to two significant figures. 1:01G

**a** 2·357 **b** 0·023 57 **c** 2 357 000

**2** Expand and simplify the following expressions. 1:02

**a** $4(3x - 5) - 3(2 - 5x)$ **b** $(2a - 5)(3a + 7)$ **c** $(4y + 1)(4y - 1)$

**d** $(3m + 2)^2$ **e** $(a + b)^2 - (a - b)^2$

**3** Make the subject of each formula the variable $x$. 1:06

**a** $y = 3x - 5$ **b** $\frac{x}{m} + \frac{y}{n} = 1$ **c** $Y = \sqrt{\frac{a}{x}}$

**4** Find the equation of the line in the number plane which: 1:09

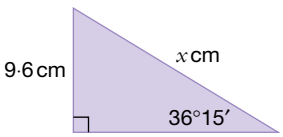
**a** has a gradient of 5 and a $y$-intercept with the coordinates $(0, 4)$

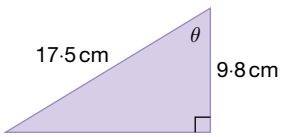
**b** has a gradient of 3 and passes through the point $(2, -1)$

**c** passes through the two points $(3, 5)$ and $(6, -1)$

**d** intersects the $x$-axis at $(6, 0)$ and the $y$-axis at $(0, 3)$.

**5** **a** Find $x$ correct to one decimal place. **b** Find $\theta$ correct to the nearest minute. 1:11




**6** It takes 5 people 6 days to paint a house, working 8-hour shifts. 1:13

**a** How long would it take 4 people to complete the same job?

**b** How many people would be needed to complete the same job in less than 4 days?

**7** Solve these quadratic equations. 2:04

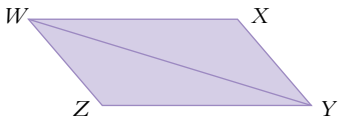
**a** $(2x - 1)(3x + 2) = 0$ **b** $x^2 - 5x - 24 = 0$ **c** $4x^2 - 4x + 1 = 0$

**d** $18 + 7x - x^2 = 0$ **e** $x^2 + 5x + 2 = 0$ **f** $2x^2 - 3x - 1 = 0$

**8** $WXYZ$ is a quadrilateral where $WX = YZ$ and $WZ = YX$. 3:02

**a** Prove that triangles $WYX$ and $YZW$ are congruent.

**b** Hence prove that $WX$ is parallel to $YZ$.




**9** Four cards marked 4, 5, 6 and 8 are placed in a hat. One card is drawn out and placed on a table. This is to be the tens digit of a two-digit number. Another card is then drawn out and placed beside the first card to complete the number. 4:02

**a** How many two-digit numbers can be formed? That is, how many are in the sample space?

**b** What is the probability of forming:

**i** a number greater than 65 **ii** an odd number?

**10** Four black marbles and six white marbles are placed in a bowl. Three marbles are chosen at random, without replacement. What is the probability of choosing: 4:06

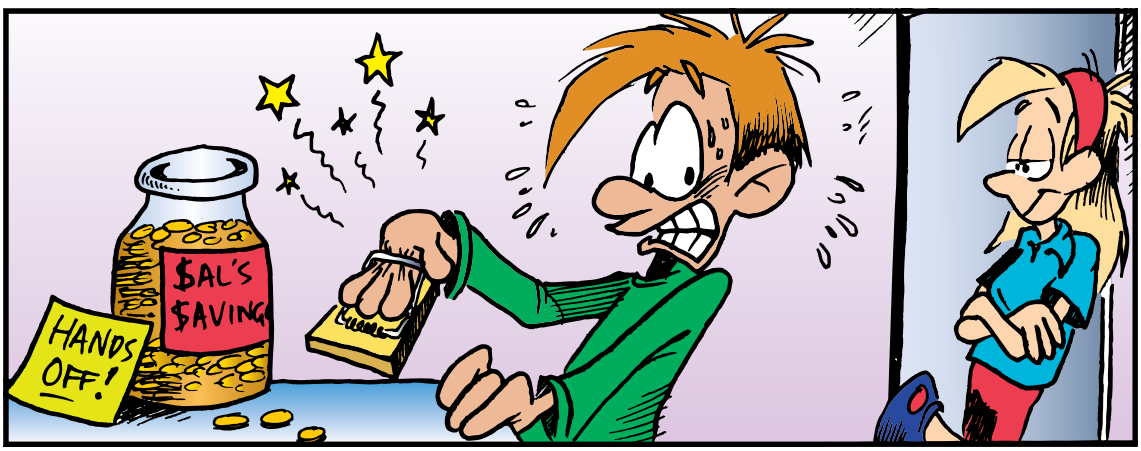
**a** three black marbles

**b** three white marbles

**c** at least one of each colour?

# FINANCIAL MATHEMATICS

6




## Contents


6:01 Saving money
Investigation 6:01 Who can you bank on?
6:02 Simple interest
6:03 Solving simple interest problems
Investigation 6:03 Financial spreadsheets
6:04 Compound interest
Fun spot 6:04 What is the difference between a book and a bore?
6:05 Depreciation
6:06 Compound interest and depreciation formulas
Investigation 6:06 Compound interest tables
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

Selections from ***Financial Mathematics*** [Stages 5.1, 5.2◊]

- Solve problems involving simple interest (ACMNA211)
- Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies (ACMNA229)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

In Year 9 we concentrated on aspects of earning and spending money. Now we will consider aspects of saving and investing money.

# 6:01 Saving money

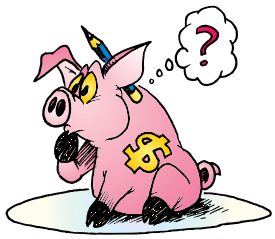
| Savings accounts | Target and award saver accounts | Term deposit accounts |
|---|---|---|
| **Meaning** | | |
| Usually a transaction card is used to deposit or withdraw your money. | A savings account from which you cannot withdraw money without forfeiting interest. | Your money is invested for a fixed period of time, usually at an agreed interest rate. |
| **Advantages** | | |
| You can deposit and withdraw without notice.<br>Safe, encourages saving and may help you get a loan. | Better interest rate than savings account. Encourages saving by reducing the temptation to withdraw your money. | Higher interest rates than savings accounts and this rate is usually fixed for the period of the investment. No extra bank charges. |
| **Disadvantages** | | |
| Lower interest rates are offered for savings accounts. Bank fees apply. | If you withdraw your money early you do not receive the higher rate of interest, but receive a lower rate instead. This causes inconvenience if some of the money is needed urgently. | It usually requires the investment of a minimum amount. The fixed amount invested cannot be withdrawn before the end of the agreed period (or term) without reducing the interest rate for the investment. |
| | | |

Follow these steps when saving or investing money.

1. Compare rates of interest (they are continually changing).
2. Consider the terms of the investment carefully.
3. Ask questions about terms, services and safety.
4. When making your choice, remember that higher interest rates are often given where there is inconvenience or some risk to your money.

## Exercise 6:01

**1** Write answers in your own words. Refer to the table on the previous page if necessary.



- **a** What are the advantages of a normal *savings* account?
- **b** What is a *term deposit* account?
- **c** What are the advantages of a *term deposit* account?
- **d** What are the disadvantages of a *term deposit* account?
- **e** What are the disadvantages of a normal *savings* account?
- **f** What are the advantages of a *target savings* account?
- **g** What are the disadvantages of an *award saver* account?

**2** Angus decided to open an *award saver* account. He chose to bank $200 each month, starting with a balance of $200. The money will earn 2·65% p.a. interest calculated daily and paid monthly as long as no withdrawals are made during that month. If a withdrawal is made, the interest rate for that month falls to 0·5% p.a.

- **a** What rate of interest applies if a withdrawal is made?
- **b** What rate of interest applies if no withdrawal is made?
- **c** How often is interest calculated?
- **d** How often is interest paid?
- **e** If 2·65% interest is given for 1 year (365 days), what percentage interest would be given for one day? (Give your answer correct to three decimal places.)

**3** Tomeka was given the following information when asking about a *term deposit* account.

| Term rates—Interest paid at maturity—Interest rates p.a. | | | | | | |
|---|---|---|---|---|---|---|
| | 3–<6 months | 6–<8 months | 8–<10 months | 10–<12 months | 12–<24 months | 24 months |
| $500–$4999 | 4·10% | 4·10% | 4·15% | 4·20% | 4·25% | 4·35% |
| $5000–$9999 | 4·60% | 4·60% | 4·65% | 4·70% | 4·75% | 4·85% |
| $10 000–$19 999 | 4·60% | 4·65% | 4·70% | 4·75% | 4·85% | 4·92% |
| $20 000–$49 999 | 4·65% | 4·70% | 4·75% | 4·80% | 4·87% | 4·95% |

- **a** If Tomeka has $4000 to invest, what is the shortest term necessary to receive 4·20% p.a. for the investment?
- **b** What interest rate is given on:
  - **i** $12 000 invested for 10 months
  - **ii** $9000 invested for 7 months
  - **iii** $45 000 invested for 18 months?

**4** Emma opened a special cash management account that offered interest of 2·5% p.a. for a balance under $5000, 4·25% p.a. for a balance between $5000 and $10 000, 4·75% p.a. for a balance between $10 000 and $20 000, and 6% p.a. for a balance over $20 000. Her first five cash or personal cheque withdrawals each month are free of bank transaction fees. After that she pays $3 per transaction. She needed $5000 to open this account.

**a** How many transactions are allowed each month before bank transaction fees are charged? What are these transaction fees?

**b** What interest rate would Emma receive if her balance is:

**i** $8150 **ii** $1809 **iii** $17 080?

**c** How much did Emma need to open this account?

**5** A major Australian bank was offering these interest rates for their ordinary savings accounts in 2012.

| Balance | Interest rate |
|---|---|
| Balances from $0 to less than $10 000 | 0·01% p.a. |
| Balances between $10 000 and $19 999 | 0·01% p.a. |
| Balances between $20 000 and $49 999 | 0·01% p.a. |
| Balances between $50 000 and $99 999 | 0·05% p.a. |
| Balances over $100 000 | 0·20% p.a. |

*Note:* This is 0·01% i.e. $\frac{1}{10\,000}$ [Not 1%]

For these savings accounts a monthly account-keeping fee of $4 applied, unless you satisfied a condition that granted an exemption. One such condition was being under 21 years of age.

**a** Calculate the interest a balance of $1000 would earn in an ordinary savings account for:

**i** 1 year **ii** 1 month (nearest cent)

**b** How much would have to be in the account each month to earn enough interest to cover the $4 account-keeping fee?

**6** The bank in Question **5** also had special youth savings accounts for those under 18. These accounts paid 4·00% p.a. interest on top of the rates shown in the table. However, this extra interest is only paid at the end of each month if there is at least one deposit during the month and no withdrawals are made.

**a** What interest rate applies for a youth savings account with a balance of $1000?

**b** If the extra conditions did not apply, calculate the interest a balance of $1000 would earn in a youth savings account for:

**i** 1 year

**ii** 1 month (Give your answer to the nearest cent.)

**c** Henry opened a youth savings account on his 16th birthday on 31 July with a deposit of $5000 given to him by his grandparents. He deposited a further $100 on 1 September.

**i** How much interest did Henry earn for the month of August?

**ii** How much interest did Henry earn for the month of September?

## INVESTIGATION 6:01 WHO CAN YOU BANK ON?

Most people have some type of savings account with a bank. They all offer a range of accounts that have different terms and conditions. The interest rates may vary as well as the fees and charges that often go unnoticed by customers when they sign up.

Go to the websites of the major banks and community banks, and look for the following information.

- The types of savings accounts on offer.
- Interest rates paid on deposits, including how often the interest is paid.
- Fees that are charged, including monthly fees, withdrawal fees, service fees and ATM fees.
- Term deposits on offer and the interest rates that apply.

You could also seek this information from other financial institutions such as credit unions and compare with the major banks.

# 6:02 Simple interest

## PREP QUIZ 6:02

What fraction of a year is:

**1** 1 month **2** 5 months **3** 11 months

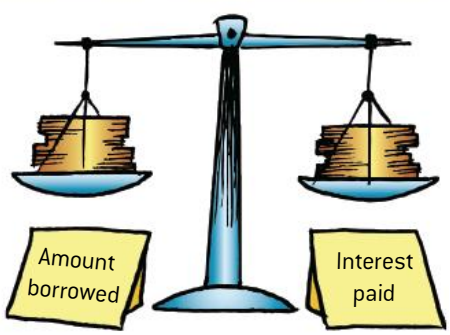
**4** 1 day **5** 7 days **6** 128 days?

To find 9·5% of \$800 press:
9·5 ÷ 100 × 800 =

Find:

**7** 9% of \$650 **8** 13·6% of \$800 **9** $37\frac{1}{2}$% of \$40 000 **10** 95% of \$7.80

- **Interest** is the payment made for the use of money invested (or borrowed).
- The money invested (or borrowed) is called the **principal**.
- **Simple interest** is interest paid on the original principal. The same interest is paid for each time period. This is also called a flat rate of interest.




What is the simple interest on \$750 invested at 8% p.a. for 3 years?

Interest for 1 year is 8% of \$750 = \$60

For simple interest, the same amount of interest is earned each year.
Simple interest = (interest for 1 year) × (number of years)

p.a. means per annum or per year

Interest for 3 years = (8% of \$750) × 3
= \$60 × 3
= \$180

Interest on \$100 at a rate of 15% p.a. simple interest for 1 year
= \$100 × 15% × 1
= \$100 × 0·15 × 1 or $\$\frac{100}{1} \times \frac{15}{100} \times \frac{1}{1}$

Interest on \$100 at a rate of 15% p.a. simple interest for 7 months
= \$100 × 15% × $\frac{7}{12}$ (7 months = $\frac{7}{12}$ of a year)

**Simple interest**

$$I = PRN$$

- If $R$ is given as % per annum, then $N$ must be in years.
- If $R$ is given as % per month, then $N$ must be in months.

where $I$ is the simple interest
$P$ is the principal invested
$R$ is the rate of interest
$N$ is the number of time periods

## WORKED EXAMPLES

1 Find the simple interest paid on \$860 invested at $6\frac{1}{2}$% p.a. for 5 years.

2 What is the simple interest on \$2400 at 8% p.a. for 5 months?

3 Find the simple interest on \$900 for 240 days at a rate of 11% p.a.

4 What is the simple interest paid on \$1950 invested for 7 months at 0·75% per month?

### Solutions

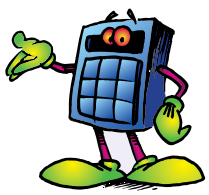
1 $I = PRN$
$= 860 \times 6\frac{1}{2}\% \times 5$
$= 860 \times 0{\cdot}065 \times 5$
$= \$279.50$
∴ The simple interest is \$279.50.

2 $I = PRN$
$= 2400 \times 8\% \times \frac{5}{12}$
$= 2400 \times 0{\cdot}08 \times 5 \div 12$
$= \$80$
∴ The simple interest is \$80.

5 months is $\frac{5}{12}$ years

3 $I = PRN$
$= 900 \times 11\% \times \frac{240}{365}$
$= 900 \times 0{\cdot}11 \times 240 \div 365$
$\doteqdot \$65.10$
∴ The simple interest is \$65.10.

4 $I = PRN$
$= 1950 \times 0{\cdot}75\% \times 7$
$= 1950 \times (0{\cdot}75 \div 100) \times 7$
$= 1950 \times 0{\cdot}0075 \times 7$
$\doteqdot \$102.38$
∴ The simple interest is \$102.38.



To change 0·75% to a decimal, press ·75 ÷ 100 = on your calculator.
e.g. 1950 × 0·75% × 7 becomes 1950 × ·75 ÷ 100 × 7 =

# Exercise 6:02

P Foundation worksheet 6:02
Simple interest

**1** Find the simple interest charged for a loan of:

**a** \$620 at 18% p.a. for 4 years
**b** \$4500 at 26% p.a. for 5 years
**c** \$15.60 at 15% p.a. for 3 years
**d** \$391 at 9% p.a. for 7 years
**e** \$90 000 at 7% p.a. for 25 years
**f** \$90 000 at 11% p.a. for 25 years
**g** \$500 for 2 years at 14% p.a.
**h** \$25 000 for 6 years at 8% p.a.
**i** \$16 860 for 10 years at 9% p.a.
**j** \$2758 for 5 years at 10% p.a.

**2** Find the simple interest earned on:

**a** \$98 at 6·4% p.a. for 5 years
**b** \$9500 at 7·2% p.a. for 3 years
**c** \$120.80 at $6\frac{1}{2}$% p.a. for 2 years
**d** \$8.40 at $5\frac{1}{2}$% p.a. for 7 years
**e** \$600 at 5·8% p.a. for 4 years
**f** \$850 at 7·1% p.a. for 6 years
**g** \$138 at $8\frac{1}{4}$% p.a. for 2 years
**h** \$140 at $10\frac{3}{4}$% p.a. for 14 years

**3** Find the simple interest charged for a loan of:

**a** \$2000 for 3 months at 1·1% per month
**b** \$8550 for 2 months at 1·6% per month
**c** \$96 for 7 months at 0·9% per month
**d** \$900 for 18 months at 0·7% per month
**e** \$280 000 for 6 months at 1·2% per month
**f** \$89.75 for 1 month at 0·9% per month
**g** \$697.80 for 5 months at 1·25% per month
**h** \$896.40 for 3 months at 0·25% per month
**i** \$30 465 for 7 months at 1·3% per month
**j** \$806 for 9 months at 1·125% per month

**4** Find the simple interest on:

11 months is $\frac{11}{12}$ years

**a** \$6200 at 12% p.a. for 6 months
**b** \$1150 at 10% p.a. for 1 month
**c** \$1150 at 10% p.a. for 7 months
**d** \$20 000 at 7% p.a. for 11 months
**e** \$8400 at 17% p.a. for 9 months
**f** \$9516 at 13% p.a. for 11 months
**g** \$28.20 at 12% p.a. for 5 months
**h** \$708 at 8% p.a. for 1 month
**i** \$1848 for 5 months at 14% p.a.
**j** \$540 for 7 months at 4% p.a.

**5** Find the simple interest (to the nearest cent) earned on:

**a** \$490 for 76 days at 16% p.a.
**b** \$1096 for 207 days at 26% p.a.
**c** \$740 for 9 days at 37% p.a.
**d** \$9700 for 304 days at 9% p.a.
**e** \$6000 for 40 days at $10\frac{1}{4}$% p.a.
**f** \$50 000 for 30 days at $6\frac{1}{2}$% p.a.
**g** \$3865 for 23 days at $9\frac{3}{4}$% p.a.
**h** \$65 for 203 days at $3\frac{1}{2}$% p.a.
**i** \$961.80 for 407 days at 11·8% p.a.
**j** \$4150 for 15 days at 12·75% p.a.

$10\frac{1}{4}\% = 10{\cdot}25 \div 100$
$= 0{\cdot}1025$

# 6:03 Solving simple interest problems

## PREP QUIZ 6:03

Write each of these percentages as a decimal.

**1** 18% **2** 5% **3** 5·4% **4** $5\frac{1}{2}\%$ **5** $5\frac{1}{4}\%$

If $I = PRN$:

**6** find $I$ when $P = 2000$, $R = 5\%$ and $N = 2$

**7** find $I$ when $P = 100$, $R = 0{\cdot}1$ and $N = 3$

**8** find $P$ when $I = 5$, $R = 0{\cdot}1$ and $N = 10$

**9** find $R$ when $I = 32$, $P = 80$ and $N = 4$

**10** find $N$ when $I = 120$, $P = 500$ and $R = 0{\cdot}08$.

Simple interest formula: $I = PRN$

## WORKED EXAMPLE 1

Rhonda borrowed $78 000 for 4 months at 12·75% p.a. simple interest to pay the deposit for her new house while her old one was being sold. How much interest did she pay?

### Solution

$P = \$78\,000$

$N = 4$ months or $\frac{4}{12}$ years

$R = 12{\cdot}75\%$ or $0{\cdot}1275$

$I = \ldots$

$I = PRN$

$= 78\,000 \times 12{\cdot}75\% \times \frac{4}{12}$

$= 78\,000 \times 0{\cdot}1275 \times \frac{4}{12}$

$= \$3315$

$\therefore$ Rhonda paid $3315 in interest.

## WORKED EXAMPLE 2

I paid $8000 to borrow $50 000 for 2 years. What was the rate of simple interest charged?

### Solution

$I = \$8000$

$P = \$50\,000$

$N = 2$ years

$R = \ldots$

$I = PRN$

$8000 = 50\,000 \times R \times 2$

$= 100\,000\,R$

Divide both sides by 100 000.

$0{\cdot}08 = R$

$R = 8\%$

$\therefore$ I was charged interest at a rate of 8% p.a.

## WORKED EXAMPLE 3

Greg borrowed \$1200 at 1·5% simple interest per month. Which is the best estimate of the interest charged for 6 months: \$20, \$100, \$200 or \$1800?

### Solution

Here we are asked to make an estimate. Your reasoning could be as follows.

Each month Greg pays \$1.5 for every \$100 borrowed (i.e. 1·5%).
That's \$15 for \$1000, so it's more than \$15 for \$1200.
He'd pay 6 times as much for 6 months, so the interest must be more than \$90.
The best of these estimates is \$100.

## Exercise 6:03

**1** **a** Jack borrowed \$14 695 at 8% p.a. simple interest for 5 years so that he could buy a car. At the end of the 5 years, both interest and loan had been paid.

**i** How much interest was charged?

**ii** How much was paid back altogether?

**b** Scott borrowed \$4000 from Mona to be repaid after 3 years, along with 9% p.a. simple interest. How much will Mona receive after 3 years?

**c** Erika invested \$8400 in the credit union for 11 months at 1·2% per month simple interest. How much interest did she earn?

**d** Jenny invests \$4000 for 3 years at 6·5% p.a. simple interest, while Robert invests \$4000 for 3 years at 5·8% p.a. simple interest. How much more interest does Jenny receive than Robert?

**e** Sam borrowed \$60 000 and paid it back over 5 years. During this time he was charged 9·5% p.a. simple interest on the original amount borrowed. How much interest did he pay?

**f** What simple interest would \$8000 earn in 3 months if the rate of interest is $11\frac{1}{4}$% p.a?

**g** A building society pays interest daily. If an amount of \$15 640 is invested at 5·5% p.a, how much simple interest would be earned in:

**i** 1 day **ii** 3 days **iii** 29 days?

**2** **a** Each year 350 people are allowed to join a club. Of these an average of 46% are women. How many women would you expect to join the club in 4 years?

**b** Twenty-five loads of soil, each 9·6 tonnes when loaded, were taken to the sports complex to top-dress the new oval. An average of 3·5% of each load was lost in transit. How much soil was lost?

**c** When James got a job, he decided to give his parents 22% of his weekly net pay. This was his contribution to family expenses. If his weekly net pay was \$316.80, how much did he give his parents in the first 15 weeks?

**d** Julia heard that a famine was causing great suffering to many people in Africa. She decided that for the next 7 months she would give 15% of her savings of \$248 per month towards relief for famine victims. How much did she give?

Do not use a calculator for Question **3**.

**3** **a** Morgan borrowed \$7000 at 8·25% p.a. simple interest. Which is the best estimate of the interest charged for 6 years: \$35, \$350, \$550 or \$3500?

**b** Luke borrowed \$880 at 11% p.a. simple interest for 9 months. Which is the best estimate of the interest charged: \$8, \$80, \$800 or \$8000?

**c** Tess borrowed \$140 000 at 9·75% p.a. interest over 20 years. Which is the best estimate of the interest charged: \$2800, \$28 000, \$280 000 or \$2 800 000?

**d** Rajiv invested \$40 000 at 4·9% p.a. interest for 7 months. Which is the best estimate of the interest earned: \$100, \$1000, \$10 000 or \$100 000?

**4** When I invest money, at the end of the time of investment I am given the principal invested plus the interest. How much will I receive after:

**a** 3 years if I invest \$10 000 at 7% p.a. simple interest

**b** 18 months if I invest \$3700 at 10% p.a. simple interest

**c** 7 months if I invest \$30 000 at 1·1% per month simple interest

**d** 10 months if I invest \$14 600 at 9·5% p.a. simple interest?

**5** When an amount of money is borrowed over a period of time, the amount to be repaid is equal to the amount borrowed plus the interest charged. How much must I pay back if I borrow:

**a** \$10 000 for 3 years at 10% p.a. simple interest

**b** \$108 000 for 25 years at 6·25% p.a. simple interest

**c** \$5000 at $8\frac{1}{2}$% simple interest for 6 years

**d** \$30 000 at 12% p.a. simple interest for 3 months?

**6** **a** Find the simple interest rate (to the nearest per cent) that will allow \$2500 to earn \$675 interest in 3 years.

**b** What sum of money would you have to invest for 5 years at 11% simple interest to produce interest at \$3080?

**c** For how many years would \$8000 have to be invested at $11\frac{1}{2}$% simple interest to produce \$5520 interest?

**d** What simple interest rate would allow \$6000 to grow to an amount of \$14 550 in 10 years?

**Discussion**

If the interest paid on \$50 000 is \$8000, what might be the interest rate and time period?

## INVESTIGATION 6:03 FINANCIAL SPREADSHEETS

Below are two versions of a spreadsheet.

- The first table shows the income earned by a student over 4 years.
- The numbers down the left and the letters across the top allow us to use coordinates to name any cell of the spreadsheet.
- The second table shows the formulas used to obtain the values in column F and row 10.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Income** | **2003 (Y7)** | **2004 (Y8)** | **2005 (Y9)** | **2006 (Y10)** | **Total** |
| 2 | Odd jobs | 212 | 264 | 220 | 160 | $856 |
| 3 | Selling newspapers | 364 | 380 | 0 | 0 | $744 |
| 4 | Mowing lawns | 60 | 260 | 180 | 45 | $545 |
| 5 | Baby-sitting | 0 | 140 | 235 | 380 | $755 |
| 6 | Typing | 0 | 0 | 20 | 110 | $130 |
| 7 | Cashier | 0 | 0 | 1654 | 1840 | $3,494 |
| 8 | Washing cars | 104 | 260 | 86 | 52 | $502 |
| 9 | | | | | | |
| 10 | **Total** | **$740** | **$1,304** | **$2,395** | **$2,587** | **$7,026** |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Income** | **2003 (Y7)** | **2004 (Y8)** | **2005 (Y9)** | **2006 (Y10)** | **Total** |
| 2 | Odd jobs | 212 | 264 | 220 | 160 | =SUM(B2:E2) |
| 3 | Selling newspapers | 364 | 380 | 0 | 0 | =SUM(B3:E3) |
| 4 | Mowing lawns | 60 | 260 | 180 | 45 | =SUM(B4:E4) |
| 5 | Baby-sitting | 0 | 140 | 235 | 380 | =SUM(B5:E5) |
| 6 | Typing | 0 | 0 | 20 | 110 | =SUM(B6:E6) |
| 7 | Cashier | 0 | 0 | 1654 | 1840 | =SUM(B7:E7) |
| 8 | Washing cars | 104 | 260 | 86 | 52 | =SUM(B8:E8) |
| 9 | | | | | | |
| 10 | **Total** | **=SUM(B2:B8)** | **=SUM(C2:C8)** | **=SUM(D2:D8)** | **=SUM(E2:E8)** | **=SUM(F2:F8)** |

**1** What is referred to in cell:
**a** D6 **b** B3 **c** A5 **d** F8 **e** C10?

**2** What is meant by:
**a** =SUM(B2:E2) **b** =SUM(C2:C8)?

**3** What is the value of '=SUM(B10:E10)'?

**4** Why do '=SUM(B10:E10)' and '=SUM(F2:F8)' have the same value?
Make a spreadsheet of your own on a subject of your choice.

# 6:04 Compound interest

## PREP QUIZ 6:04

What amount would be returned after 1 year if the rate of simple interest is 10% p.a. and the principal invested is:

**1** $10 000 **2** $11 000 **3** $12 100 **4** $13 310 **5** $14 641?

What amount would be returned after 1 month if the rate of simple interest is 12% p.a. and the principal invested is:

**6** $100 000 **7** $101 000 **8** $102 010 **9** $103 030.10?

**10** $2000 is invested for 1 year at 10% p.a. simple interest. The money received is then invested for 1 year at 10% p.a. simple interest. How much is received at the end of the second year?

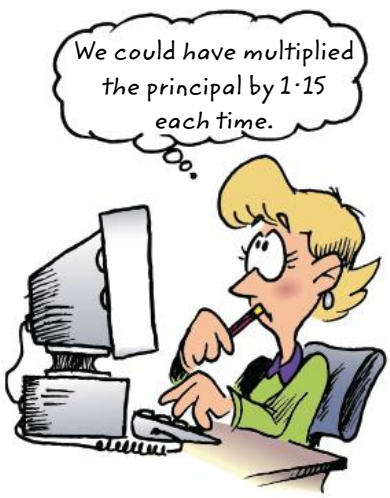
- When *simple interest* is applied, the interest earned in one year (or time period) does not itself earn interest in following years. The interest in any time period is calculated using the original principal and so is the same for each time period.
- When *compound interest* is applied, the interest earned in one year (or time period) itself earns interest in following years. The interest is added to the previous principal and so the amount of interest in each time period increases from one period to the next.

> For compound interest, interest earned in previous time periods also earns interest.

Consider what happens when $40 000 is invested for 4 years at 15% p.a. compound interest. Find the total amount of interest earned.

Amount after 1 year = principal + interest

| |
|---|
| Amount after 1 year = $40 000 + 15% × $40 000<br>= $40 000 + $6000<br>= $46 000 |
| Amount after 2 years = $46 000 + 15% × $46 000<br>= $46 000 + $6900<br>= $52 900 |
| Amount after 3 years = $52 900 + 15% × $52 900<br>= $52 900 + $7935<br>= $60 835 |
| Amount after 4 years = $60 835 + 15% × $60 835<br>= $60 835 + $9125.25<br>= $69 960.25 |




∴ Total interest = final amount − original principal
= $69 960.25 − $40 000
= $29 960.25

## WORKED EXAMPLE 1

Find the compound interest earned if $9000 is invested for 3 years at 13% p.a. and interest is compounded yearly. Answer to the nearest cent.

### Solution

In each year, the principal will increase by 13%.

**Method 1**

Amount after 1 year
= $9000 + 13% of $9000
= $10 170
Amount after 2 years
= $10 170 + 13% of $10 170
= $11 492.10
Amount after 3 years
= $11 492 + 13% of $11 492.10
≒ $12 986.07

**Method 2**

Amount after 1 year
= $9000 × 1·13
= $10 170
Amount after 2 years
= $10 170 × 1·13
= $11 492.10
Amount after 3 years
= $11 492.10 × 1·13
≒ $12 986.07

∴ Interest earned in 3 years = final amount − original principal
= $12 986.07 − $9000
= $3986.07

## WORKED EXAMPLE 2

$12 000 is invested at a compound interest rate of 9% p.a. Interest, however, is compounded monthly. Calculate the amount by which the investment will grow in 2 months.

### Solution

9% p.a. is the same as $\frac{9}{12}$% per month.
$\frac{9}{12}$% = $\frac{3}{4}$% = 0·75%

0·75% = 0·75 ÷ 100
= 0·0075

**Method 1**

Amount after 1 month
= $12 000 + 0·75% of $12 000
= $12 090
Amount after 2 months
= $12 090 + 0·75% of $12 090
≒ $12 180.68

**Method 2**

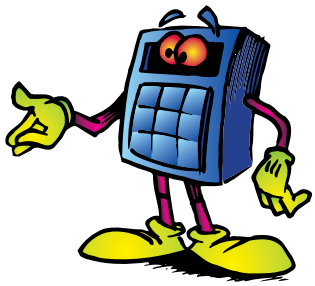
$A_1$ = $12 000 × 1·0075
= $12 090

$A_2$ = $12 090 × 1·0075
≒ $12 180.68

∴ Interest earned in 2 months = final amount − original principal
= $12 180.68 − $12 000
= $180.68

# Finding compound interest by repeated multiplication using a calculator

In Method 2 in the previous examples, we can see that to calculate the amount accumulated after $n$ time periods, we simply have to multiply the principal by (1 + interest rate) $n$ times. The amount \$9000 accumulates to, at 13% p.a., after 3 years is given by:



$$\text{Amount} = \$9000 \times 1{\cdot}13 \times 1{\cdot}13 \times 1{\cdot}13$$
$$= \$12\,986.07$$

$$\therefore \text{ Interest earned} = \$12\,986.07 - \$9000$$
$$= \$3986.07$$

## Exercise 6:04

P Foundation worksheet 6:04 Compound interest

**1** Compare the money earned at simple interest and at compound interest in the tables below.

| Rate | Simple interest on \$100 for: | | | | |
|---|---|---|---|---|---|
| | 2 years | 4 years | 6 years | 8 years | 10 years |
| 10% p.a. | \$20 | \$40 | \$60 | \$80 | \$100 |

| Rate | Compound interest (monthly) on \$100 for: | | | | |
|---|---|---|---|---|---|
| | 2 years | 4 years | 6 years | 8 years | 10 years |
| 10% p.a. | \$22.04 | \$48.94 | \$81.76 | \$121.82 | \$170.70 |

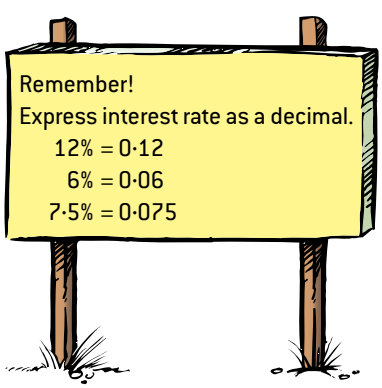
How much more interest is earned on \$100 invested at 10% p.a. compound interest (compounded monthly) than is earned on \$100 invested at 10% p.a. simple interest invested for a period of:

**a** 2 years **b** 6 years **c** 8 years **d** 10 years?

In Questions **2** to **5**, interest is compounded annually.

**2** By repeated multiplication by (1 + interest rate) find how much each principal accumulates to, for the given number of years.

**a** \$2000 at 10% p.a. for 2 years
**b** \$5000 at 12% p.a. for 3 years
**c** \$10 000 at 7% p.a. for 2 years
**d** \$50 000 at $6\frac{1}{2}$% p.a. for 3 years
**e** \$25 000 at 7·5% p.a. for 4 years



**3** Find the amount received and the compound interest earned if:

**a** \$400 is invested for 2 years at 12% p.a.
**b** \$1900 is invested for 3 years at 13% p.a.
**c** \$5350 is invested for 2 years at 9% p.a.
**d** \$100 is invested for 3 years at 8% p.a.
**e** \$2874 is invested for 2 years at 7·5% p.a.
**f** \$650 000 is invested for 2 years at 6% p.a.
**g** \$85 700 is invested for 3 years at $12\frac{1}{2}$ p.a.

**4** Naomi invested \$5000 for 4 years at a rate of 10% p.a. compound interest. To what amount did the investment grow in this time, and what was the total interest earned?

**5** Joni was given 8% p.a. compound interest on an investment of \$80 000 over 3 years. What interest did she earn altogether?

**6** Find the amount of compound interest earned (to the nearest cent) if:

**a** \$400 is invested for 3 months at 1% per month, compounded monthly
**b** \$53 000 is invested for 2 months at 0·8% per month, compounded monthly
**c** \$8000 is invested at 12% p.a. for 2 months, compounded monthly
**d** \$2870 is invested at 6% p.a. for 3 months, compounded monthly.

**7** Luke could invest \$1000 at a simple interest rate of 11% p.a. for 4 years or at 10% p.a. compounded interest for the same period of time. Which is the better investment and by how much?

**8** The population of Kangaroo Creek is 200 and is expected to increase each year by 80% of the previous year's population. What is the expected population in 3 years' time?

**9** The value of a house has increased by 20% of the previous year's value in each of the last 4 years. If, 4 years ago, the value of the house was \$100 000, what is its value now?

**10** If we assume an inflation rate of 8% p.a, what would you expect to pay in 3 years' time for:

**a** a garage that now costs \$30 000
**b** a container of ice-cream that now costs \$4.50
**c** a toothbrush that now costs \$2.85?

## FUN SPOT 6:04

## WHAT IS THE DIFFERENCE BETWEEN A BOOK AND A BORE?

Answer each question and write the letter for that question in the box above the correct answer.

*Write as a percentage:*

**H** $\frac{1}{20}$ **O** $\frac{3}{4}$

**T** 0·85 **U** 0·02

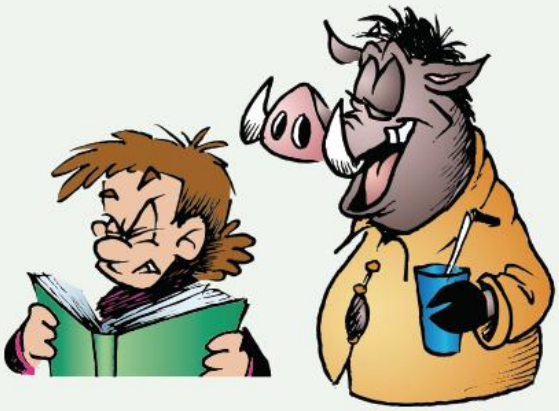
*Write as a decimal:*

**O** 3% **U** 12·5%

*Calculate:*

**A** 1% of \$1 **B** 15% of \$1

**C** 4% of \$810 **E** $5\frac{1}{2}$% of \$8000



*Find the simple interest earned by investing \$100 at 12% p.a. for:*

**H** 1 year **K** 4 years **N** 6 months

*Interest of 10·25% p.a. was charged on a loan, where interest was calculated daily. What percentage interest (given to three decimal places) would be charged for:*

**O** 1 day

**P** the month of April? (Take 1 year to be 365 days.)

**S** Given that $I = PRN$, find the value of $R$ if $I = 180$, $P = 600$ and $N = 3$.

*A watch was bought for \$200. Find its value after 1 year if it depreciated:*

**T** 20% **U** 11% **Y** 1·5%

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \$197 | 0·03 | 2% | \$32·40 | \$0·01 | \$6 | 10% | \$12 | 0·125 | 85% | \$178 | 0·842% | \$160 | 5% | \$440 | \$0·15 | 0·028% | 75% | \$48 | |

# 6:05 Depreciation

## PREP QUIZ 6:05

**1** 100% – 20% **2** 100% – 7% **3** 20% + 80%

**4** 7% + 93% **5** Reduce \$350 by 20%. **6** Find 80% of \$350.

**7** Reduce \$800 by 7%. **8** Find 93% of \$800.

**9** Is reducing an amount by 20% the same as finding 80% of the amount?

**10** Is reducing an amount by 7% the same as finding 93% of the amount?

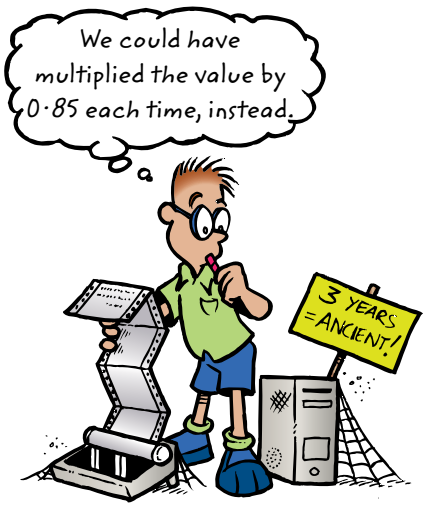
- When the value for an item decreases year by year it is said to be **depreciating**. Many of the things we own depreciate at more or less a constant rate. This may be caused by age or by the availability of new models.
- **Depreciation** is the loss in value of an object over a period of time.

Consider what happens when machinery worth $40 000 depreciates for 4 years at a rate of 15% p.a. Find the total amount of depreciation.

Value after 1 year = original value − depreciation for that year

| |
|---|
| Value after 1 year = $40 000 − 15% × $40 000<br>= $40 000 − $6000<br>= $34 000 |
| Value after 2 years = $34 000 − 15% × $34 000<br>= $34 000 − $5100<br>= $28 900 |
| Value after 3 years = $28 900 − 15% × $28 900<br>= $28 900 − $4335<br>= $24 565 |
| Value after 4 years = $24 565 − 15% × $24 565<br>= $24 565 − $3684.75<br>= $20 880.25 |

∴ Total depreciation = original value − final value
= $40 000 − $20 880.25
= $19 119.75




## WORKED EXAMPLE 1

A new computer depreciates by 30% per year. If it costs $4000 new, what would it be worth in 3 years?

### Solution

Each year the value decreases by 30%.

**Method 1**

Value after 1 year
= $4000 − 30% of $4000
= $2800
Value after 2 years
= $2800 − 30% of $2800
= $1960
Value after 3 years
= $1960 − 30% of $1960
= $1372

**Method 2**

Value after 1 year
= 70% of $4000
= $2800
Value after 2 years
= 70% of $2800
= $1960
Value after 3 years
= 70% of $1960
= $1372

∴ The value of the computer after 3 years would be $1372.

## WORKED EXAMPLE 2

If the population of Bilby Downs is decreasing by 20% every year, what would be the population in 2 years if it is now 700?

### Solution

After a 20% decrease, 80% of the population remains.

**Method 1**

Population after 1 year
= 700 – 20% of 700
= 560
Population after 2 years
= 560 – 20% of 560
= 448

**Method 2**

Population after 1 year
= 80% of 700
= 560
Population after 2 years
= 80% of 560
= 448

∴ The population of Bilby Downs after 2 years would be 448.

# Exercise 6:05

**1** **a** In the first year after purchase, a car costing $24 300 depreciated 18%. What was its value after 1 year?

**b** Find the depreciation during the first year if a ring bought for $2300 depreciated at a rate of 9·3% p.a.

**2** **a** Find the value of a yacht after 2 years if its original value was $136 000 and the rate of depreciation is 12% p.a.

**b** A machine is purchased for $81 600. If it depreciates at a rate of 8% p.a, what will be its value after 2 years?

**c** Furniture purchased for $4350 depreciates at a rate of 11% p.a. What would be its value after 3 years?

**d** Find the value of a teacher's library after 3 years if its original value was $9800 and it depreciated at a rate of 10% p.a.

**e** If the wallaby population of our district is 840 and is dropping by 12% p.a, how many wallabies would you expect to be in the district in 2 years?

**f** The population of Kangaburra is dropping at a rate of 4% p.a. If its population in 2000 was 56 700, what was its likely population in the year 2003?

**3** Find the depreciation (i.e. loss) for each of the examples in Question **2**.

**4** **a** Elena bought a video camera for $1350. Find the value of the video camera after 3 years if it depreciated at a rate of 16% p.a.

**b** If the $1350 had been invested for 3 years at 6% p.a. compound interest, what amount would she have received?

**5** **a** Tim bought a second-hand car for $35 000. If it depreciates at a rate of 3% p.a, what would be its value in 2 years?

**b** If $35 000 is invested at 5% p.a. compound interest (just covering the inflation rate), what would it grow to in 2 years?

**6** Julia and Lillian each invested $5000. Julia's investment depreciated at a rate of 3% p.a. for 3 years, while during that time Lillian's investment grew at a rate of 9% p.a. (compounded yearly). Find the difference in the value of their investments at the end of the 3 years.

**7** Which of the rates 5% p.a, 6% p.a, 7% p.a. or 8% p.a. would cause a car valued at $6000 to drop in value to $4826 in 3 years?

**8** Calculate the number of years it would take for a machine valued at $11 000 to drop below $7000 in value if the rate of depreciation is 14% p.a.

# 6:06 Compound interest and depreciation formulas

## PREP QUIZ 6:06

Write each percentage as a decimal.

**1** 9% **2** $6\frac{1}{2}\%$ **3** 11·25%

Give the rate per month correct to two decimal places:

**4** 9% p.a. **5** 8% p.a. **6** 6·5% p.a.

Evaluate:

**7** $1 + R$, if $R = 0{\cdot}15$ **8** $1 - R$, if $R = 0{\cdot}15$

**9** $P(1 + R)^n$, if $P = \$40\,000$, $R = 0{\cdot}15$ and $n = 4$

**10** $P(1 - R)^n$, if $P = \$40\,000$, $R = 0{\cdot}15$ and $n = 4$

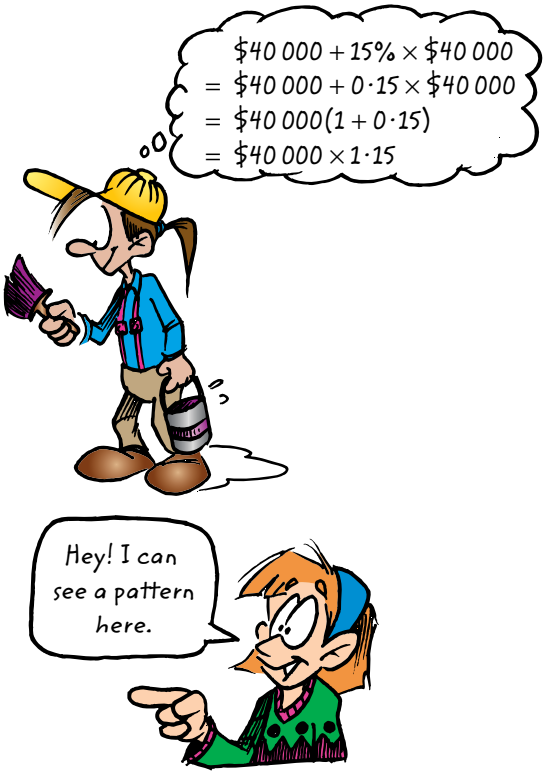
## Compound interest

On page 165 we saw what happens when $40 000 is invested for 4 years at 15% p.a. compound interest.

We can find the amount to which the investment grows in each year by multiplying the amount of the previous year by 1·15 (i.e. 1 + 15%).

Amount after 1 year = principal + interest

| | |
|---|---|
| Amount after 1 year | = \$40 000 + 15% of \$40 000<br>= \$40 000 × 1·15<br>= \$46 000 |
| Amount after 2 years | = \$46 000 + 15% of \$46 000<br>= \$46 000 × 1·15<br>= \$52 900 |
| Amount after 3 years | = \$52 900 + 15% of \$52 900<br>= \$52 900 × 1·15<br>= \$60 835 |
| Amount after 4 years | = \$60 835 + 15% of \$60 835<br>= \$60 835 × 1·15<br>= \$69 960.25 |




The working above could have been shown as:

Amount after 1 year $= \$40\,000(1 + 0{\cdot}15)^1$
Amount after 2 years $= \$40\,000(1 + 0{\cdot}15)^2$
Amount after 3 years $= \$40\,000(1 + 0{\cdot}15)^3$
Amount after 4 years $= \$40\,000(1 + 0{\cdot}15)^4$
$= \$69\,960.25$

Clearly if \$$P$ were to be invested at a rate of $r$ per year, then the amount $A$ to which the investment would grow in $n$ years would be given by $A = P(1 + R)^n$.

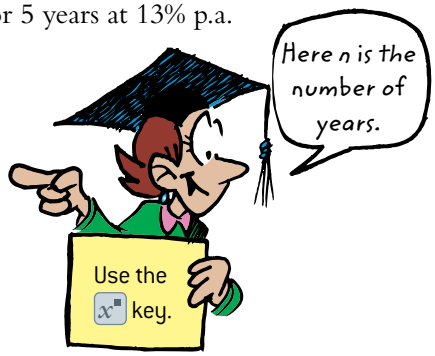
Compound interest formula

$$A = P(1 + R)^n$$

where $P$ is the principal invested
$n$ is the number of time periods
$R$ is the rate of compound interest for one time period as a decimal
$A$ is the amount after $n$ time periods.

## WORKED EXAMPLE 1

Find the compound interest earned if \$9000 is invested for 5 years at 13% p.a. (Answer to the nearest cent.)




### Solution

$P = \$9000$
$n = 5$
$R = 13\%$
$= 0{\cdot}13$

$A = P(1 + R)^n$
$= \$9000(1 + 0{\cdot}13)^5$
$= \$9000(1{\cdot}13)^5$
$= \$16\,581.92$
$\therefore$ Interest $= \$16\,581.92 - \$9000$
$= \$7581.92$

$\therefore$ The compound interest earned in 5 years is \$7581.92.

## WORKED EXAMPLE 2

$12 500 is invested at a compound interest rate of 9% p.a. Interest, however, is compounded monthly. Calculate the amount to which the investment will grow in 4 years.

### Solution

Here:

$P = \$12\,500$

Time = 4 years

= 48 months

$\therefore n = 48$

Rate = 9% p.a.

= 9 ÷ 12% per month

= 0·75% per month

$\therefore R = 0{\cdot}0075$

$A = P(1 + R)^n$

$= \$12\,500(1 + 0{\cdot}0075)^{48}$

$= \$12\,500(1{\cdot}0075)^{48}$

$= \$17\,892.567$

$\doteqdot \$17\,892.57$

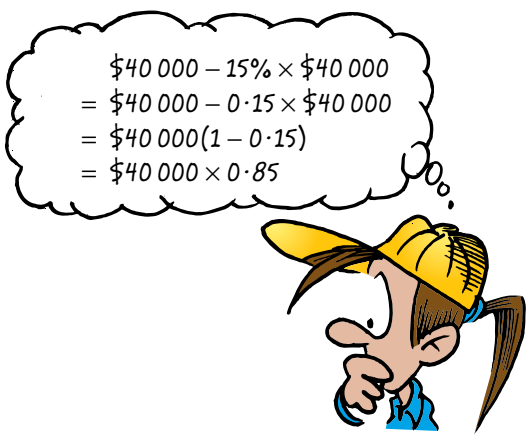
$\therefore$ The amount to which the investment will grow in 4 years is $17 892.57.

## Depreciation

On page 170 we saw what happens to the value of machinery worth $40 000 when it depreciates for 4 years at a rate of 15% p.a. We found the value of the machinery each year by multiplying the value of the previous year by 0·85 (i.e. 1 – 15%).

Value after 1 year = original value – depreciation for that year

| | |
|---|---|
| Value after 1 year | = $40 000 – 15% of $40 000<br>= $40 000 × 0·85<br>= $34 000 |
| Value after 2 years | = $34 000 – 15% of $34 000<br>= $34 000 × 0·85<br>= $28 900 |
| Value after 3 years | = $28 900 – 15% of $28 900<br>= $28 900 × 0·85<br>= $24 565 |
| Value after 4 years | = $240 565 – 15% of $24 565<br>= $24 565 × 0·85<br>= $20 880.25 |




The working above could have been shown as:

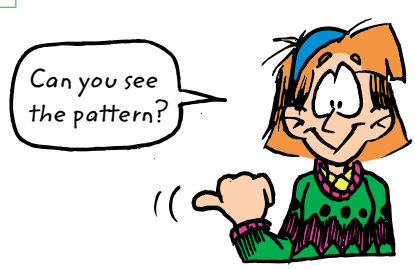
Value after 1 year $= \$40\,000(1 - 0{\cdot}15)^1$

Value after 2 years $= \$40\,000(1 - 0{\cdot}15)^2$

Value after 3 years $= \$40\,000(1 - 0{\cdot}15)^3$

Value after 4 years $= \$40\,000(1 - 0{\cdot}15)^4$

$= \$20\,880.25$




Clearly, if an item of value $\$P$ were to depreciate at a rate $R$ per year, then the value $A$, after $n$ years, would be given by $A = P(1 - R)^n$.

Depreciation formula

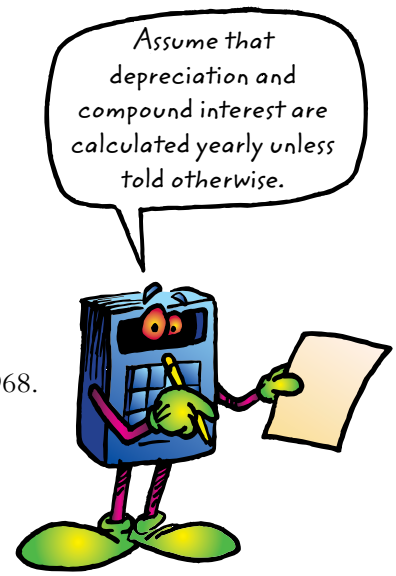
$$A = P(1 - R)^n$$

where $P$ is the original value
$n$ is the number of time periods
$R$ is the rate of depreciation for one time period as a decimal
$A$ is the value after $n$ time periods.

## WORKED EXAMPLES

**1** Adam buys a second-hand car for \$13 400. What will the car be worth in 5 years, to the nearest dollar, if each year it depreciates 18%?

**2** Carol paid \$2200 for a new computer. Find the value of the unit in 6 years, to the nearest dollar, if the rate of depreciation is 28% p.a.




### Solutions

**1** $P = \$13\,400$
$n = 5$
$R = 18\%$
$= 0{\cdot}18$

$A = P(1 - R)^n$
$= 13\,400(1 - 0{\cdot}18)^5$
$= 13\,400(0{\cdot}82)^5$
$\fallingdotseq \$4967.91$

$\therefore$ The value of the car in 5 years is \$4968.

**2** $P = \$2200$
$n = 6$
$R = 28\%$
$= 0{\cdot}28$

$A = P(1 - R)^n$
$= \$2200(1 - 0{\cdot}28)^6$
$= \$2200(0{\cdot}72)^6$
$\fallingdotseq \$306.49$

$\therefore$ After 6 years, the value of the computer is \$306.

## Exercise 6:06

P Foundation worksheet 6:06
Compound interest formula

**1** Use the formula $A = P(1 + R)^n$ to find the amount received if:
- **a** \$4000 is invested for 5 years at 12% p.a.
- **b** \$860 is invested for 7 years at 7% p.a.
- **c** \$17 360 is invested for 20 years at 6·25% p.a.
- **d** \$476.50 is invested for 6 years at $5\frac{1}{2}\%$ p.a.

**2** Find the interest earned for each of the investments in Question **1**.

**3** Use the formula $A = P(1 - R)^n$ to find the value of an item worth:
- **a** \$3000 after it depreciates 9% p.a. for 12 years
- **b** \$35 000 after it depreciates 25% p.a. for 4 years
- **c** \$465 after it depreciates 30% p.a. for 7 years.

**4** Find the amount of compound interest earned (to the nearest cent) if:

**a** $400 is invested for 5 years at 12% p.a. compounded monthly

**b** $5350 is invested for 10 years at 9% p.a. compounded monthly

**c** $2874 is invested for 20 years at 7·5% p.a. compounded monthly

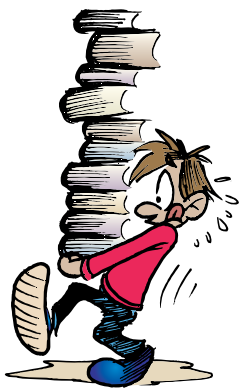
**d** $500 is invested for 6 years at 8% p.a. compounded half-yearly.

**5** Jenny Chew discovered that she was to inherit the amount to which an investment of her great-grandfather had grown. He had invested 50 pounds at 9% p.a. compound interest, 100 years ago. Find how much she will receive if one pound is equal to two dollars. (Assume that interest is compounded annually.)

**6** Luke Travers had to decide between investing his $1000 at a simple interest rate of 16% p.a. for 8 years or investing it at 11% p.a. compound interest for the same period of time. Which would be the better investment and by how much?

**7** A library depreciates at a rate of 15% p.a. If its value now is $18 700, what will be its value in:

**a** 8 years **b** 15 years **c** 21 years?



**8** Find the value of a $20 million ship after 10 years if its rate of depreciation is:

**a** 9% p.a. **b** 18% p.a. **c** $17\frac{1}{2}$% p.a.

**9** After having depreciated at a rate of 12% p.a, a printing machine is now worth $2000. What was the machine's value 7 years ago?

**10** A library now worth $17 600 has been depreciating at a rate of 9% p.a. for the last 8 years. What was its value 8 years ago?

**11** $1000 is invested for 5 years at 12% p.a. compound interest. To what does this investment grow if interest is compounded:

**a** yearly **b** 6 monthly **c** 3 monthly

**d** monthly **e** fortnightly **f** daily?

Is there a limit to which the investment can grow as we reduce the time period for compounding the interest? (Consider compounding the interest each hour, minute and second.)

**12** Adam invests $10 000 at 9% p.a. simple interest while Su-Lin invests $10 000 at 9% p.a. compounded monthly. What is the difference in the value of their investments after:

**a** 1 year **b** 5 years **c** 10 years?

If this car depreciates at 15% a year, how long will it take to halve its price if its original price is $150 000?

## INVESTIGATION 6:06 COMPOUND INTEREST TABLES

If a calculator with a power key or a computer is not available, a table could be used that shows the effects of compound interest. The sample table below shows the amount $1 will grow to for various interest rates and time periods.

| Accumulated value of $1 (to four decimal places) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Number of time periods | Interest rate | | | | | | | | |
| | 0·5% | 1% | 5% | 6% | 7% | 8% | 10% | 12% | 15% |
| 1 | 1·0050 | 1·0100 | 1·0500 | 1·0600 | 1·0700 | 1·0800 | 1·1000 | 1·1200 | 1·1500 |
| 2 | 1·0100 | 1·0201 | 1·1025 | 1·1236 | 1·1449 | 1·1664 | 1·2100 | 1·2544 | 1·3225 |
| 3 | 1·0151 | 1·0303 | 1·1576 | 1·1910 | 1·2250 | 1·2597 | 1·3310 | 1·4049 | 1·5209 |
| 4 | 1·0202 | 1·0406 | 1·2155 | 1·2625 | 1·3108 | 1·3605 | 1·4641 | 1·5735 | 1·7490 |
| 5 | 1·0253 | 1·0510 | 1·2763 | 1·3382 | 1·4026 | 1·4693 | 1·6105 | 1·7623 | 2·0114 |
| 6 | 1·0304 | 1·0615 | 1·3401 | 1·4185 | 1·5007 | 1·5869 | 1·7716 | 1·9738 | 2·3131 |
| 7 | 1·0355 | 1·0721 | 1·4071 | 1·5036 | 1·6058 | 1·7138 | 1·9487 | 2·2107 | 2·6600 |
| 8 | 1·0407 | 1·0829 | 1·4775 | 1·5938 | 1·7182 | 1·8509 | 2·1436 | 2·4760 | 3·0590 |
| 9 | 1·0459 | 1·0937 | 1·5513 | 1·6895 | 1·8385 | 1·9990 | 2·3579 | 2·7731 | 3·5179 |
| 10 | 1·0511 | 1·1046 | 1·6289 | 1·7908 | 1·9672 | 2·1589 | 2·5937 | 3·1058 | 4·0456 |
| 11 | 1·0564 | 1·1157 | 1·7103 | 1·8983 | 2·1049 | 2·3316 | 2·8531 | 3·4785 | 4·6524 |
| 12 | 1·0617 | 1·1268 | 1·7959 | 2·0122 | 2·2522 | 2·5182 | 3·1384 | 3·8960 | 5·3503 |
| 18 | 1·0939 | 1·1961 | 2·4066 | 2·8543 | 3·3799 | 3·9960 | 5·5599 | 7·6900 | 12·3755 |
| 24 | 1·1272 | 1·2697 | 3·2251 | 4·0489 | 5·0724 | 6·3412 | 9·8497 | 15·1786 | 28·6252 |

### Exercises

Find the accumulated value and hence the compound interest earned when:

1 $10 000 is invested for 10 years at 8% p.a. compounded annually

2 $50 000 is invested for 18 months at 12% p.a. compounded annually.

### Solutions

1 If interest rate = 8% and time periods = 10, then $1 will grow to $2.1589.
∴ $10 000 will grow to:
$2.1589 × 10 000 = $21 589
∴ Interest = $21 589 − $10 000
= $11 589

2 If interest rate = 12% p.a, then interest rate per month = 1%.
So for 18 months, $1 will grow to $1.1961.
∴ $50 000 will grow to:
$1.1961 × 50 000 = $59 805
∴ Interest = $59 805 − $50 000
= $9805

## Exercises

1 Find the accumulated amount and the interest earned when:
   **a** $20 000 is invested for 12 years at 7% p.a. compounded annually
   **b** $150 000 is invested for 24 years at 10% p.a. compounded annually
   **c** $7500 is invested for 8 years at 6% p.a. compounded annually.

2 Find an accumulated amount when $100 000 is invested at:
   **a** 12% p.a. for 2 years, interest compounded annually
   **b** 1% per month for 24 months, interest compounded monthly.

3 Why are the answers different for parts **a** and **b** of Question **2** when the interest rate and terms are the same? (2 years = 24 months, 12% p.a. = 1% per month)

## MATHS TERMS 6

**compound interest**
- interest earned is added to the principal so that the interest earned for the next time period is calculated on this greater amount
- the interest earned in previous time periods also earns interest
- the formula for compound interest is:
  $A = P(1 + R)^n$
  where $A$ = amount after $n$ time periods
  $P$ = principal invested
  $n$ = number of time periods
  $R$ = rate of interest for one time period as a decimal

**depreciation**
- the loss in value of an object over a period of time
- the formula for calculating the reduced value of an object is:
  $A = P(1 - R)^n$
  where $A$ = value after $n$ time periods
  $P$ = original value
  $n$ = number of time periods
  $R$ = rate of depreciation for one time period as a decimal

**flat rate**
- the same as a simple interest rate

**instalment**
- a regular payment made to reduce the amount owing on a loan

**interest**
- the payment made for the use of money invested or borrowed
- usually expressed as a percentage of the amount borrowed

**principal**
- the amount of money invested or borrowed

**repayment**
- a payment made to reduce the amount owing on a loan

**simple interest**
- interest is paid on the original principal throughout the term of the loan
- the formula for calculating simple interest is:
  $I = PRN$
  where $I$ = simple interest
  $P$ = principal invested
  $R$ = rate of interest
  $N$ = number of time periods

## DIAGNOSTIC TEST 6 FINANCIAL MATHEMATICS

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

| | |
|---|---|
| 1 a Find the simple interest charged on \$600 at 12% p.a. for 5 years.<br>b Find the simple interest earned by \$14 260 invested for 3 years at 4% p.a.<br>c Find the interest charged on a loan of \$85 000 taken over 15 years if a simple interest rate of 8% p.a. is charged. | 6:02 |
| 2 a Find the simple interest charged on \$860 at $6\frac{1}{2}$% p.a. for 5 years.<br>b Find the interest paid on \$25 000 invested for 6 years at a simple interest rate of 9·3% p.a.<br>c Find the simple interest earned if \$9000 is invested for 3 years at a rate of $6\frac{3}{4}$% p.a. | 6:02 |
| 3 a What is the simple interest on \$2400 at 8% p.a. for 5 months?<br>b Find the simple interest on \$900 for 240 days at a rate of 11% p.a.<br>c What is the simple interest paid on \$19 500 invested for 7 months at 0·75% per month? | 6:02 |
| 4 a Jodie borrowed \$78 000 for 4 months at 12·75% p.a. simple interest to pay for her new house while her old one was being sold. How much interest did she pay?<br>b I paid \$8000 to borrow \$50 000 for 2 years. What was the rate of simple interest charged?<br>c Troy borrowed \$2400 at 1·5% per month simple interest. Which is the best estimate of the interest charges for 3 months:<br>\$20, \$100, \$200 or \$1800? | 6:03 |
| 5 a Find the compound interest earned if \$9000 is invested for 3 years at 13% p.a. if interest is compounded yearly. Answer correct to the nearest cent.<br>b \$12 000 is invested at a compound interest rate of 9% p.a. Interest, however, is compounded monthly. Calculate the amount to which the investment will grow in 2 months.<br>c \$12 500 is invested at a compound interest rate of 9% p.a. Interest, however, is compounded monthly. Use the formula to calculate the amount to which the investment will grow in 4 years. | 6:04 |
| 6 a A new computer depreciates by 30% per year. It costs \$4000 new. What will it be worth in 4 years?<br>b If the population of Bilby Downs is decreasing by 20% every year, what would be the population in 2 years if it is now 800?<br>c Carol paid \$2200 for a new bicycle. Use the depreciation formula to find the value of the bicycle in 6 years if its rate of depreciation is 15% p.a. | 6:05 |

## ASSIGNMENT 6A Chapter review

**1** A bank pays 4·5% p.a. for a target savings account.
- **a** What would be the interest earned on a balance of $3240 for one month?
- **b** If this interest is added to the account, how much interest would be earned in the next month?

**2** Annabelle invested $60 000 in a term deposit that paid interest at a rate of 4·35% p.a. for 6 months or 4·45% p.a. for 12 months.
- **a** Annabelle chose to invest for 6 months. How much interest did she earn?
- **b** At the end of the 6 months she added the interest to the original $60 000 and invested it again for another 6 months at the same rate. How much interest did she earn for this 6 months?
- **c** If Annabelle had chosen to invest the $60 000 initially for 12 months at the higher rate, would she have earned more or less interest, and what would have been the difference?

**3** Determine the amount of simple interest earned by the following:
- **a** $1500 at 7% p.a. over 5 years
- **b** $9500 at 9·5% p.a. over $3\frac{1}{2}$ years
- **c** $2200 at 8% p.a. over 5 months
- **d** $660 at 1·5% per month over 9 months

**4** Kate's savings account pays interest of 5% p.a. Interest is calculated and added to the account every 6 months. If she starts with $2000 in the account and does not use the account for 2 years, what will her new balance be at the end of the 2 years?

**5** $24 000 is invested for 2 years. Interest is paid at a rate of 6% p.a. and is compounded annually. Interest is not withdrawn. How much interest will have been earned after 2 years?

**6** Machinery valued at $140 000 depreciates at a rate of 16% p.a. Find the value of the machinery after 3 years.

**7** Terry invested $250 000 in an account that earned 6% p.a. compound interest.
- **a** How much would be in the account at the end of 10 years if:
  - **i** interest was calculated annually
  - **ii** interest was calculated every 3 months
  - **iii** interest was calculated monthly?
- **b** How much more interest is earned by calculating the interest monthly rather than annually?

**8** Margaret decided to invest $50 at 5% p.a. compound interest for 200 years, with the interest to be added annually. Her nearest living relative after 200 years would inherit the proceeds.
- **a** Find the amount the relative should inherit.
- **b** Find the amount if the interest rate had been 5% p.a. simple (flat) interest.

**9** A printing machine bought for $245 000 will depreciate at the rate of 17·5% p.a. When the value of the machine falls below $100 000, it is replaced by a new model. After how many years will the printing machine be replaced?

**10** Bilbo invested $2000 in an account that will earn 5·5% p.a. compound interest calculated monthly. The interest is added to the balance each month. At the end of the year, for the next two years, he deposited another $2000 into the account. How much will Bilbo have in the account at the end of 3 years?

## ASSIGNMENT 6B Working mathematically

**1** Find the number of three-digit numbers possible if only the digits 0, 1, 2, 3, 4 and 5 may be used, the number must be a multiple of 5, and no digit may be used more than twice in the same number.

**2** Of the 30 students in 10M, 13 love surfing, 15 love hiking and 7 love neither.

**a** How many students love both surfing and hiking?

**b** How many love surfing but not hiking?

**c** How many love hiking but not surfing?

**3** Roger started a trip to the country between 8 am and 9 am when the hands of the clock were together. He arrived at his destination between 2 pm and 3 pm, when the hands of the clock were exactly 180° apart. For how long did he travel?

**4** $3! = 3 \times 2 \times 1$ (pronounced 3 factorial)
$5! = 5 \times 4 \times 3 \times 2 \times 1$
$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
How many zeros are there at the end of the answer to 20!? (A calculator will be of no help.)

**5** **a** Use the data for Australia in 2007 in the table on the right to answer the following questions.

**i** How many people were killed on the road?

**ii** How many fatalities were there per 100 000 people?

**iii** Therefore, what is an estimation of Australia's population according to these figures?

**b** Where did Australia rank among these 20 OECD countries in terms of least fatalities per 100 000 population?

**c** Compared to 2005, which country in 2007 had:

**i** the greatest decrease in fatalities

**ii** the greatest increase in fatalities?

**Road safety comparisons: fatalities OECD countries 2007**

| | Number of fatalities in 2007 | | |
|---|---|---|---|
| | 2007 | % compared to 2005 | Fatalities per 100 000 persons |
| Australia | 1603 | −1·5% | 7·6 |
| Austria | 691 | −10·0% | 8·3 |
| Czech Republic | 1222 | −5·0% | 11·9 |
| Denmark | 406 | 22·7% | 7·4 |
| Finland | 380 | 0·0% | 7·2 |
| France | 4620 | −13·1% | 7·5 |
| Germany | 4949 | −7·7% | 6·0 |
| Hungary | 1232 | −3·6% | 12·2 |
| Iceland | 15 | −21·1% | 4·9 |
| Japan | 6639 | −16·3% | 5·2 |
| Netherlands | 709 | −5·5% | 4·3 |
| New Zealand | 422 | 4·2% | 10·0 |
| Norway | 233 | 4·0% | 5·0 |
| Poland | 5583 | 2·6% | 14·7 |
| Portugal | 974 | −21·9% | 9·2 |
| Slovenia | 293 | 13·6% | 14·6 |
| Spain | 3823 | −13·9% | 8·6 |
| Sweden | 471 | 7·0% | 5·2 |
| Switzerland | 384 | −6·1% | 5·1 |
| United States | 41 059 | −5·5% | 13·6 |
| *Total* | 75 708 | −8·8% | 7·8 |

## ASSIGNMENT 6C Cumulative revision

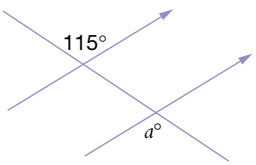
**1** **a** Find the GST that needs to be added to a price of $43.50 before a sale. 1:08
**b** How much GST is contained in a selling price of $57.20?

**2** Solve each pair of simultaneous equations. 1:09
**a** $2x + 5y = 3$, $3x - 4y = 16$ **b** $5a - 3b = 2$, $b = 6a + 8$

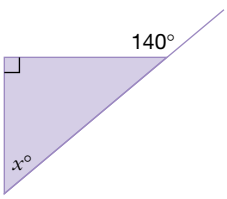
**3** Solve these quadratic equations. 2:01
**a** $x^2 - 64 = 0$ **b** $(x - 9)(x + 2) = 0$
**c** $x^2 - x - 12 = 0$ **d** $6x^2 + 13x - 5 = 0$

**4** Use the quadratic formula to solve these equations correct to three decimal places. 2:03
**a** $x^2 - 3x - 1 = 0$ **b** $2x^2 + 5x + 1 = 0$ **c** $x^2 - 4x + 5 = 0$

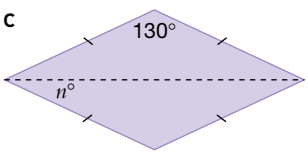
**5** Find the value of the pronumeral in each figure. 3:01
**a**



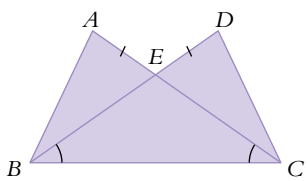

**b**



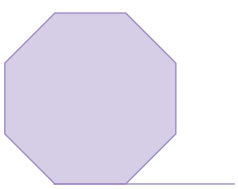

**c**




**6** In the diagram, $\angle ACB = \angle DBC$ and $AE = DE$. 3:02
Show that $\Delta ABC \equiv \Delta DCB$ and hence that $AB = DC$.




**7** For a regular octagon, what is the size of each: 3:06
**a** interior angle
**b** exterior angle?



**8** There are 10 red and 10 blue marbles in a bag. A marble is selected and its colour noted. This is done three times. Find the probability of the third marble chosen being blue if: 4:02
**a** after each selection the marble is put back in the bag
**b** each marble is not put back in the bag and the first two marbles are:
**i** blue **ii** red

**9** There are two boxes. In one box there are ten $10 notes and ten $20 notes. 4:05
In the second box there are fifteen $10 notes and five $20 notes.
**a** If one note is selected at random from each box, what is the probability they will both be $20?
**b** If only one box is chosen at random and a note is selected, what is the probability of selecting a $20 note?

**10** Write these fractions with rational denominators: **a** $\frac{3}{2\sqrt{6}}$ **b** $\frac{3}{4 - \sqrt{10}}$ 5:06

# LINEAR AND NON-LINEAR RELATIONSHIPS

7

## Contents


7:01 Review of coordinate geometry
7:02 Parallel and perpendicular lines
GeoGebra activity 7:02A Parallel lines
GeoGebra activity 7:02B Perpendicular lines
7:03 Using coordinate geometry
7:04 Special forms of the parabola
GeoGebra activity 7:04A Parabolas of the form $y = ax^2$
GeoGebra activity 7:04B Parabolas of the form $y = ax^2 + k$
GeoGebra activity 7:04C Parabolas of the form $y = (x + a)^2$ and $y = (x + a)^2 + k$
Investigation 7:04 The graphs of parabolas
7:05 Parabolas of the form $y = ax^2 + bx + c$
Fun spot 7:05 Why didn't the bald man need his keys?
Investigation 7:05 Describing number patterns
GeoGebra activity 7:05 Parabolas of the form $y = ax^2 + bx + c$
7:06 Exponential graphs
GeoGebra activity 7:06 Exponential graphs
7:07 The hyperbola
GeoGebra activity 7:07 Hyperbolas of the form $y = \frac{k}{x}$, $y = \frac{k}{x} + c$ and $y = \frac{k}{x - b}$
7:08 Circles and their equations
7:09 Curves of the form $y = ax^3$ and $y = ax^3 + d$
7:10 Curves of the form $y = ax^n$ and $y = ax^n + d$
7:11 Curves of the form $y = ax^n$ and $y = a(x - r)^n$
GeoGebra activity 7:11 Curves involving powers
7:12 Curves of the form $y = (x - r)(x - s)(x - t)$
7:13 The intersection of graphs
Investigation 7:13 A parabola and a circle
GeoGebra activity 7:13 The intersection of graphs
7:14 Miscellaneous equations and graphs
GeoGebra activity 7:14 Curves and their equations
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

Selections from ***Linear Relationships*** [Stages 5.2, 5.3[§]] and ***Non-Linear Relationships*** [Stages 5.1, 5.2[◊], 5.3[§]]

- Solve problems involving parallel and perpendicular lines (ACMNA238)
- Solve problems using various standard forms of the equation of a straight line (NSW)
- Explore the connection between algebraic and graphical representations of relationships such as simple quadratics, circles and exponentials using digital technologies as appropriate (ACMNA239)
- Graph simple non-linear relationships, with and without the use of digital technologies, and solve simple related equations (ACMNA296)
- Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations (ACMNA267)
- Describe, interpret and sketch cubics, other curves and their transformations. (NSW)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 7:01 Review of coordinate geometry

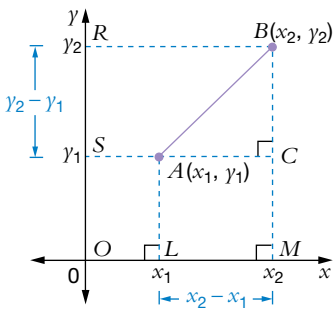
The techniques listed below were studied in Year 9 and were reviewed in Exercise 1:08. These skills will be used in the next two sections.

## Length, gradient and midpoint

The following formulas are used to find the length, gradient and midpoint of the interval joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Length of $AB$ (or distance between $A$ and $B$)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$




Gradient (or slope) of $AB$

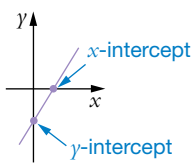
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Midpoint of $AB$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

## Graphing straight lines

To graph a straight line we use the equation to find the $x$-and $y$-coordinates for two (or more) points on the line. These points are then plotted on a number plane. Two of the easiest points to use are the $x$- and $y$-intercepts. The $x$-intercept is found by substituting $y = 0$ into the line's equation and then solving for $x$. The $y$-intercept is found by substituting $x = 0$ into the equation and then solving for $y$.




## Equations of lines

The equation of a line is uniquely determined by certain sets of conditions. Three sets of conditions lead to the following equations, which are used to find the equation of a straight line.

**Gradient-intercept form**

A line with gradient $m$ and $y$-intercept $b$ has the equation $y = mx + b$.

**Point-gradient form**

A line with gradient $m$ that passes through the point $(x_1, y_1)$ has the equation $y - y_1 = m(x - x_1)$.

**Two-point form**

The equation of a line that passes through the points $(x_1, y_1)$ and $(x_2, y_2)$ is:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

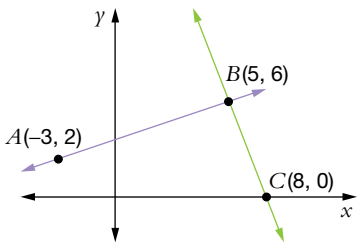
### General form

There are many different ways of writing the equation of a line. When an equation is written in the form $ax + by + c = 0$ where $a$, $b$ and $c$ are integers and $a > 0$, it is said to be in general form. There is only one way to write the equation of a line in general form.

## WORKED EXAMPLES

**1** Find the equation of $AB$ in general form. What are its $x$- and $y$-intercepts?

**2** Find the equation of $BC$ and its $y$-intercept.




### Solutions

**1** Use the point-gradient form:

$$y - y_1 = m(x - x_1)$$

$$m = \frac{6 - 2}{5 - -3} = \frac{4}{8} = \frac{1}{2}$$

$$(x_1, y_1) = (5, 6)$$

$$\therefore\ y - 6 = \tfrac{1}{2}(x - 5)$$

$$2y - 12 = x - 5$$

$$x - 2y + 7 = 0$$

When $y = 0$, $x = -7$

$\therefore$ $x$-intercept $= -7$

When $x = 0$, $y = 3\frac{1}{2}$

$\therefore$ $y$-intercept $= 3\frac{1}{2}$

Using $(x_1, y_1) = (-3, 2)$ will give the same answer.

**2** Use the two-point form:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$(x_1, y_1) = (5, 6),\ (x_2, y_2) = (8, 0)$$

$$y - 6 = \frac{0 - 6}{8 - 5}(x - 5)$$

$$y - 6 = -2(x - 5)$$

$$y - 6 = -2x + 10$$

$$y = -2x + 16$$

$\therefore$ $y$-intercept $= -16$

This can also be done by using the two points to calculate the gradient and then using the point-gradient form.

## Exercise 7:01

**1** If $A$ is the point $(-2, 2)$ and $B$ is the point $(4, 5)$ find:

**a** the length $AB$ **b** the gradient of $AB$ **c** the midpoint of $AB$.

**2** Find the $x$- and $y$-intercepts of the following lines and sketch the graph of each line.

**a** $x + y = 5$ **b** $4x - y = 2$ **c** $2x + 3y = 6$ **d** $4x - 5y = 12$

**3** Rearrange each equation into $y = mx + b$ form to find the gradient and $y$-intercept of the line.

**a** $x + y = -4$ **b** $x - 3y - 9 = 0$ **c** $5x + 2y - 10 = 0$ **d** $4x - 5y + 37 = 0$

**4** Find the equation of the line in gradient-intercept form that:

**a** passes through $(1, 4)$ and has a gradient of $-2$

**b** has a gradient of $\frac{1}{2}$ and passes through the point $(-1, -2)$

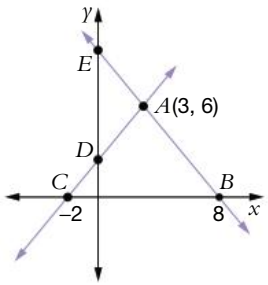
**c** has a $y$-intercept of 6 and a gradient of $-\frac{2}{3}$

**d** passes through the points $(3, 0)$ and $(-2, 4)$

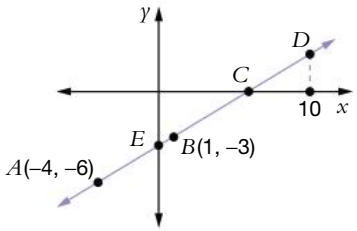
**e** has $x$- and $y$-intercepts of 3 and $-4$ respectively.

**5** Write each of the answers in Question **4** in general form.

**6** **a** Find the lengths of $AB$ and $AC$.
**b** Find the equations of $AC$ and $AB$.
**c** Use your answers in **b** to find the coordinates of $D$ and $E$.

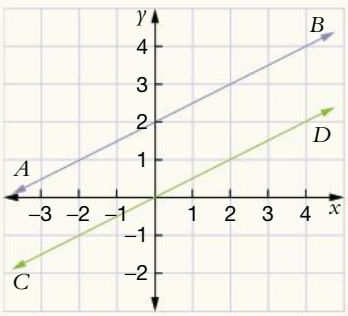



**7** **a** Find the slope of the line $AB$.
**b** Find the equation of $AB$.
**c** Write the coordinates of:
**i** $C$
**ii** $D$
**iii** $E$




# 7:02 Parallel and perpendicular lines

## PREP QUIZ 7:02

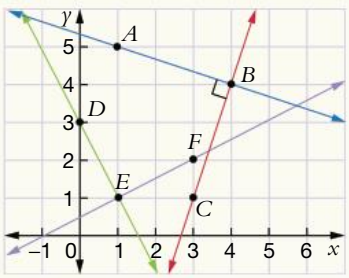



**1** What is the gradient of each line?

**2** Are the lines parallel?

**3** If $EF$ was drawn parallel to $AB$, what would its gradient be?

**4** Is it possible for two lines with different gradients to be parallel?




In the diagram $AB \perp BC$ and $DE \perp EF$.

**5** Find the gradient of $AB$. Call this $m_1$.

**6** Find the gradient of $BC$. Call this $m_2$.

**7** Using your answers to Questions **5** and **6**, find the product of the gradients, $m_1 m_2$.

**8** Find the gradient of $DE$. Call this $m_3$.

**9** Find the gradient of $EF$. Call this $m_4$.

**10** Using your answers to Questions **8** and **9**, find the product of the gradients, $m_3 m_4$.

Questions **1** to **4** in the Prep quiz remind us that:
- two straight lines are parallel if their gradients are equal
- the gradients of two lines are equal if the lines are parallel.

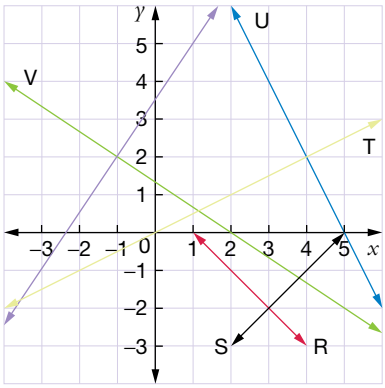
Questions **5** to **10** in the Prep quiz suggest that a condition for two lines to be perpendicular might be that the product of their gradients is equal to −1.

We do not intend to prove this here, but let us look at several pairs of lines where the product of the gradient is −1 to see if the angle between the lines is 90°.

**A** Line R has gradient −1 $\quad \therefore m_1 = -1$
Line S has gradient 1 $\quad \therefore m_2 = 1$
Note that $m_1 m_2 = -1$

**B** Line T has gradient $\frac{1}{2}$ $\quad \therefore m_1 = \frac{1}{2}$
Line U has gradient −2 $\quad \therefore m_2 = -2$
Note that $m_1 m_2 = -1$

**C** Line V has gradient $-\frac{2}{3}$ $\quad \therefore m_1 = -\frac{2}{3}$
Line W has gradient $\frac{3}{2}$ $\quad \therefore m_2 = \frac{3}{2}$
Note that $m_1 m_2 = -1$



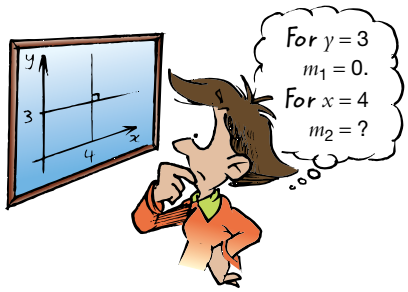

Using measurement or Pythagoras' theorem, we can show that the angle between each pair of lines is 90°.

- If two lines are perpendicular then the product of their gradients is −1.
- $m_1 m_2 = -1$ (where neither gradient is zero).
- If the product of the gradients of two lines is −1, then the lines are perpendicular.

Two lines with gradients of $m_1$ and $m_2$ are:

- parallel if $m_1 = m_2$
- perpendicular if $m_1 m_2 = -1$
  (or $m_1 = -\dfrac{1}{m_2}$) where neither $m_1$ nor $m_2$ can equal zero.

The symbol for parallel lines is $\|$.
The symbol for perpendicular lines is $\perp$.




## WORKED EXAMPLE 1

Which of the lines $y = 4x$, $y = 3x + 2$ and $y = x$ is perpendicular to $x + 4y + 2 = 0$?

### Solution

**Step 1** Find the gradient of $x + 4y + 2 = 0$.
Writing this in gradient form gives: $y = -\frac{1}{4}x - 2$
$\therefore$ The gradient of this line is $-\frac{1}{4}$.

**Step 2** Find the gradients of the other lines.
The gradient of $y = 4x$ is 4.
The gradient of $y = 3x + 2$ is 3.
The gradient of $y = x$ is 1.

**Step 3** Find which gradient in Step 2 will multiply $-\frac{1}{4}$ to give −1.
Conclusion: $-\frac{1}{4} \times 4 = -1$
$\therefore$ $x + 4y + 2 = 0$ is perpendicular to $y = 4x$.

## WORKED EXAMPLE 2

Find the equation of the line that passes through the point (2, 4) and is perpendicular to $y = 3x - 2$.

### Solution

Let the equation of the line be $y = mx + b$.
Now, the gradient of $y = 3x - 2$ is 3.

$\therefore\ m = -\frac{1}{3}$ $\quad -\frac{1}{3} \times 3 = -1$

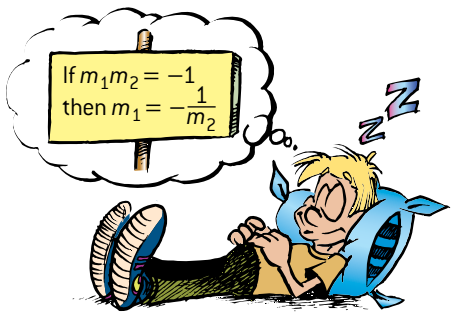
$\therefore\ y = -\frac{1}{3}x + b$

$4 = -\frac{1}{3}(2) + b$ $\quad$ (2, 4) lies on line

$4 = -\frac{2}{3} + b$

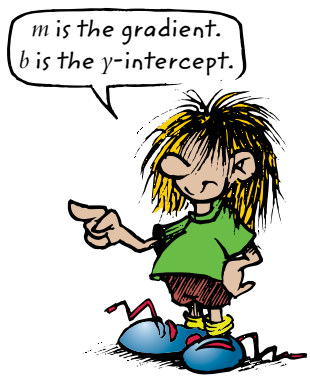
$\therefore\ b = 4\frac{2}{3}$

$\therefore$ The equation of the line is $y = -\frac{1}{3}x + 4\frac{2}{3}$.




## WORKED EXAMPLE 3

Find the equation of the line that passes through the point (1, 4) and is parallel to $y = 3x - 2$.




### Solution

Let the equation of the line be $y = mx + b$.

$y = 3x - 2$ has gradient 3

$\therefore\ m = 3$ $\quad$ parallel lines have equal gradients

$\therefore\ y = 3x + b$

$4 = 3(1) + b,$ $\quad$ (1, 4) lies on the line

$\therefore\ b = 1$

$\therefore$ The equation of the line is $y = 3x + 1$.

## Exercise 7:02

P Foundation worksheet 7:02
Parallel and perpendicular lines

**1** Are the following pairs of lines parallel?

**a** $y = 3x + 2$ and $y = 3x - 1$
**b** $y = 5x - 2$ and $y = 2x - 5$
**c** $y = x + 7$ and $y = x + 1$
**d** $y = x - 3$ and $y = 1x + 2$
**e** $y = 3x + 2$ and $2y = 6x - 3$
**f** $y = 2x + 1$ and $2x - y + 3 = 0$
**g** $3x + y - 5 = 0$ and $3x + y + 1 = 0$
**h** $x + y = 6$ and $x + y = 8$

**2** Are the following pairs of lines perpendicular?

**a** $y = \frac{1}{5}x + 3,\ y = -5x + 1$
**b** $y = 3x - 2,\ y = -\frac{1}{3}x + 7$
**c** $y = 2x - 1,\ y = -\frac{1}{2}x + 3$
**d** $y = \frac{2}{3}x + 4,\ y = -\frac{3}{2}x - 5$
**e** $y = 4x,\ y = \frac{1}{4}x - 3$
**f** $y = \frac{3}{4}x - 1,\ y = -\frac{4}{3}x$
**g** $y = 3x - 1,\ x + 3y + 4 = 0$
**h** $x + y = 6,\ x - y - 3 = 0$

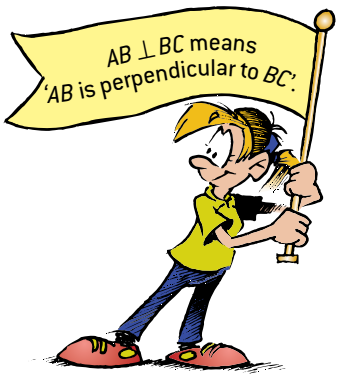
**3** **a** Which of the following lines are parallel to $y = 2x + 3$?

$y = 3x + 2$ $\quad 2x - y + 6 = 0$ $\quad 2y = x + 3$ $\quad y = 2x - 3$

**b** Two of the following lines are parallel. Which are they?

$y = x - 3$ $\quad x + y = 3$ $\quad y = 3x$ $\quad 3y = x$ $\quad y = -x + 8$

**c** $A$ is the point $(1, 3)$, $B$ is $(3, 4)$, C is $(6, 7)$ and $D$ is $(8, 8)$. Which of the lines $AB$, $BC$, $CD$ and $DA$ are parallel?




**4** **a** Which of the following lines are perpendicular to $y = 2x$?

$y = 3x$ $\quad y = 2x - 3$ $\quad x + 2y = 4$ $\quad y = -0{\cdot}5x + 5$

**b** Two of the following lines are perpendicular. Which are they?

$y = -1\frac{1}{2}x + 2$ $\quad y = \frac{1}{2}x - 1$ $\quad y = \frac{2}{3}x$

**c** $A$ is the point $(2, -1)$, $B$ is the point $(3, -2)$ and C is $(4, -1)$. Prove that $AB \perp BC$.

**5** **a** Find the equation of the line that has $y$-intercept 3 and is parallel to $y = 5x - 1$.

**b** Line $AB$ is parallel to $y = 3x - 4$. Find the equation of $AB$ if its $y$-intercept is $-1$.

**c** Line $EF$ is parallel to $y = x + 5$. Its $y$-intercept is 3. What is the equation of $EF$?

**d** A line has a $y$-intercept of 10 and is parallel to the line $x + y = 4$. What is the equation of this line?

**6** **a** Find the equation of the line that has a $y$-intercept of 5 and is perpendicular to $y = -\frac{1}{3}x + 1$.

**b** The line $AB$ is perpendicular to $y = -x + 4$. Its $y$-intercept is 3. What is the equation of $AB$?

**c** Find the equation of $CD$ if it is perpendicular to the line $y = -\frac{1}{2}x$ and has a $y$-intercept of 0.

**d** A line has a $y$-intercept of 1·5 and is perpendicular to the line $y = -2x + 1$. Find the equation of the line.

**7** **a** $AB$ is a line that passes through the point $(2, 3)$. What is the equation of $AB$ if it is parallel to $y = 5x + 2$?

**b** Find the equation of the line that passes through $(1, 0)$ and is parallel to $y = -3x - 1$.

**c** A is the point $(0, 0)$ and $B$ is the point $(1, 3)$. Find the equation of the line that has $y$-intercept 5 and is parallel to $AB$.

**d** Find the equation of the line that has $y$-intercept $-3$ and is parallel to the $x$-axis.

**e** What is the equation of the line that is parallel to the $x$-axis and passes through the point $(-2, -3)$?

**8** **a** If $AB$ passes through the point $(2, 3)$ and is perpendicular to $y = 2x - 7$, find the equation of $AB$ in general form.

**b** Find the equation of the line that passes through $(1, 0)$ and is perpendicular to $y = -3x - 1$. Write your answer in general form.

**c** $A$ is the point $(0, 0)$ and $B$ is the point $(1, 3)$. Find the equation of the line that has $y$-intercept 5 and is perpendicular to $AB$. Give the answer in general form.

**d** Find the equation of the line that has $y$-intercept $-3$ and is perpendicular to the $y$-axis.

**e** What is the equation of a line that is perpendicular to the $x$-axis and passes through $(3, -2)$?

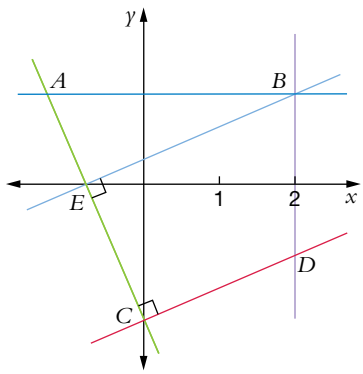
**9** **a** Find the equation of the line that is parallel to the line $2x - 3y + 6 = 0$ and passes through the point $(3, -4)$. Give the answer in general form.

**b** A line is drawn through $(-1, 2)$, perpendicular to the line $4x + 3y - 6 = 0$. Find its equation in general form.

**c** A line is drawn through the point $(-1, -1)$, parallel to the line $2x - 3y + 9 = 0$. Where will it cross the $x$-axis?

**d** A line is drawn parallel to $4x - 3y + 1 = 0$, through the points $(1, 3)$ and $(6, a)$. What is the value of $a$?

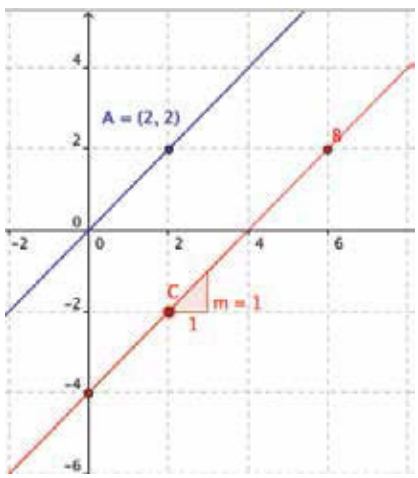
**10** In the diagram, the line $5x + 2y + 5 = 0$ cuts the $x$-axis and $y$-axis at $E$ and $C$ respectively. $BD$ is the line $x = 2$, $AB$ is parallel to the $x$-axis and $BE$ and $CD$ are perpendicular to $AC$. Find the coordinates of the points $A$, $B$, $C$, $D$ and $E$.




## P GEOGEBRA ACTIVITY 7:02A PARALLEL LINES

1 Use the sliders to change the slopes of the lines and investigate when the lines will be parallel. How are the equations of parallel lines similar? How are they different?

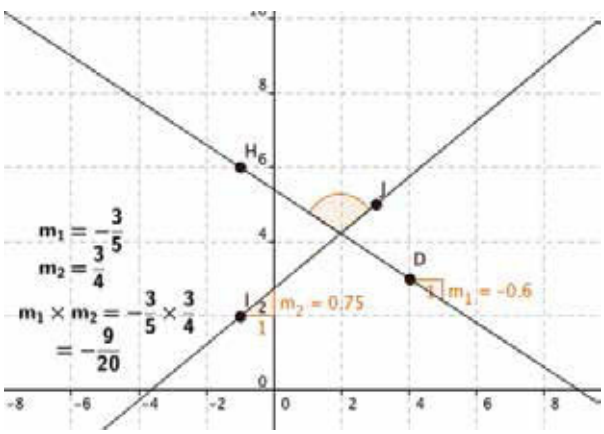
2 This activity uses a variable line and point. Move the point and work out the equation of the line through the point parallel to the given line. Use the checkbox to see if you are correct.




## P GEOGEBRA ACTIVITY 7:02B PERPENDICULAR LINES

1 Compare the gradients of two lines in this activity to investigate the condition for two lines to be perpendicular.

2 This activity uses a variable line and point. Move the point and work out the equation of the line through the point perpendicular to the given line. Use the checkbox to see if you are correct.



# 7:03 Using coordinate geometry

The techniques of coordinate geometry can be used to study normal geometric problems. In the following examples we look at some simple uses of these techniques.

## WORKED EXAMPLE 1

A triangle is formed by the points $O(0, 0)$, $A(2, 3)$ and $B(4, 0)$. $E$ and $F$ are the midpoints of the sides $OB$ and $AB$. Show:

**a** that $\Delta OAB$ is isosceles **b** that $EF$ is parallel to $OB$.

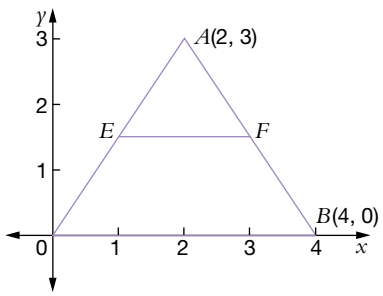
### Solution

**a** $OA = \sqrt{(2-0)^2 + (3-0)^2}$ $\quad AB = \sqrt{(4-2)^2 + (0-3)^2}$

$= \sqrt{4+9}$ $\quad = \sqrt{4+9}$

$= \sqrt{13}$ $\quad = \sqrt{13}$

$\therefore \Delta OAB$ is isosceles (two equal sides).




**b** Now, $E$ is $(1, 1\frac{1}{2})$ and $F$ is $(3, 1\frac{1}{2})$.

$\therefore EF$ is horizontal ($E$ and $F$ have same $y$-coordinates).

$OB$ is horizontal.

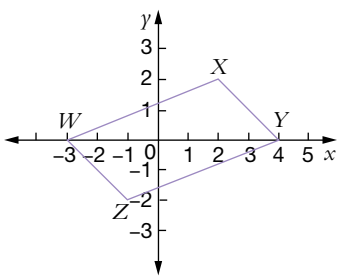
$\therefore EF$ is parallel to $OB$.

## WORKED EXAMPLE 2

$W(-3, 0)$, $X(2, 2)$, $Y(4, 0)$ and $Z(-1, -2)$ are the vertices of a quadrilateral.

**a** Show that $WXYZ$ is a parallelogram.

**b** Show that the diagonals bisect each other.




### Solution

**a** Calculating the slopes of the four sides gives the following.

Slope of $WX = \dfrac{2-0}{2-(-3)}$ $\quad$ Slope of $ZY = \dfrac{0-(-2)}{4-(-1)}$

$= \dfrac{2}{5}$ $\quad = \dfrac{2}{5}$

Slope of $WZ = \dfrac{0-(-2)}{-3-(-1)}$ $\quad$ Slope of $XY = \dfrac{2-0}{2-4}$

$= -1$ $\quad = -1$

$\therefore WX \parallel ZY$ (equal slopes) and $WZ \parallel XY$ (equal slopes)

$\therefore WXYZ$ is a parallelogram (opposite sides are parallel).

**b** Midpoint of $XZ = \left(\dfrac{2+(-1)}{2}, \dfrac{2+(-2)}{2}\right) = (\frac{1}{2}, 0)$ $\quad$ Midpoint of $WY = (\frac{1}{2}, 0)$

$\therefore (\frac{1}{2}, 0)$ is the midpoint of both diagonals.

$\therefore$ The diagonals $XZ$ and $WY$ bisect each other.

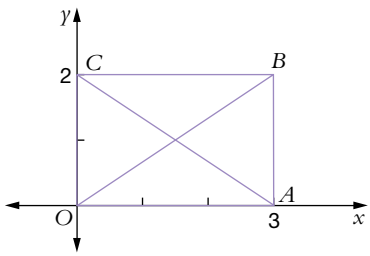
## Exercise 7:03

P Foundation worksheet 7:03
Coordinate geometry

**1** **a** Show that the triangle formed by the points $O(0, 0)$, $A(3, 1)$ and $B(1, 3)$ is isosceles.
**b** Show that the triangle formed by the points $(0, 0)$, $(1, 3)$ and $(7, 1)$ is right-angled.
**c** Show that the triangle with vertices at $(0, 0)$, $(-2, 2)$ and $(2, 2)$ is both right-angled and isosceles.

**2** **a** Show that the quadrilateral with vertices at $A(0, 2)$, $B(3, 0)$, $C(0, -2)$ and $D(-3, 0)$ is a rhombus.
**b** A quadrilateral is formed by joining the points $O(0, 0)$, $B(1, 2)$, $C(5, 0)$ and $D(4, -2)$. Show that it is a rectangle.
**c** The points $A(0, 2)$, $B(2, 0)$, $C(0, -2)$ and $D(-2, 0)$ are joined to form a quadrilateral. Show that it is a square.

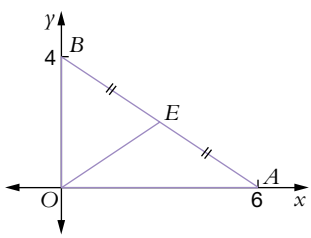
**3** **a** If $OABC$ is a rectangle, what are the coordinates of $B$?
**b** Find the length of $OB$ and $AC$. What property of a rectangle have you proved?
**c** Find the midpoint of $OB$ and $AC$. What does your answer tell you about the diagonals of a rectangle?




**4** The points $A(0, 0)$, $B(6, 4)$ and $C(4, -2)$ form a triangle.
**a** Find the midpoints of $AB$ and $AC$.
**b** Find the slope of the line joining the midpoints in **a**.
**c** What is the slope of $BC$?
**d** What do your answers to parts **b** and **c** tell you?

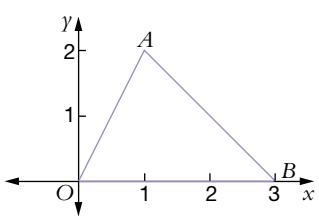
**5** The points $A(4, 0)$, $B(4, 4)$, $C(0, 4)$ and $D(0, 0)$ form a square. Find the slopes of $BD$ and $AC$. What does your result say about the diagonals $BD$ and $AC$?

**6** A right-angled triangle $OAB$ is shown.
**a** Find the coordinates of $E$, the midpoint of $AB$.
**b** Find the length of $OE$.
**c** Find the length of $EA$.
**d** What can you say about the distance of $E$ from $O$, $A$ and $B$?




**7** A triangle has its vertices at the points $A(-1, 3)$, $B(2, 4)$ and $C(1, 1)$.
**a** Show that the triangle is isosceles.
**b** Find $E$, the midpoint of $AC$.
**c** Find the slope of the line joining $E$ to $B$.
**d** Show that $EB$ is perpendicular to $AC$.
**e** Describe how you could find the area of $\Delta ABC$.

**8** **a** Find the midpoints of $OA$ and $AB$.
**b** Find the length of the line joining the midpoints in **a**.
**c** Show that your answer in **b** is half the length of $OB$.



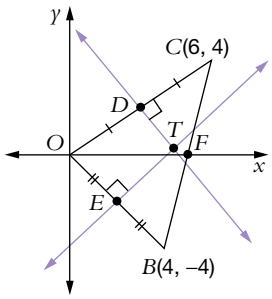
**9** In the diagram $D$, $E$ and $F$ are the midpoints of the sides of the triangle $OBC$. $DT$ and $ET$ are the perpendiculars of the sides $OC$ and $OB$.

**a** Find the equation of the perpendicular bisectors $DT$ and $ET$.

**b** Use simultaneous equations to find the point of intersection, $T$.

**c** Find the equation of the perpendicular bisector of $BC$ and show that it passes through $T$.

**d** How far is $T$ from the points $C$, $B$ and $O$?

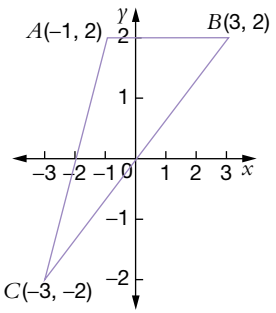



**10** A median is a line joining a vertex of a triangle to the midpoint of the opposite side.

**a** Find the equations of the medians.

**b** Find the point of intersection of two of the medians.

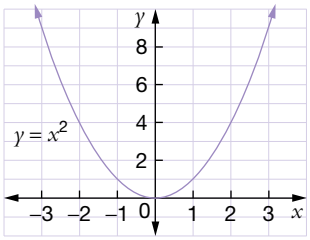
**c** Show that the third median passes through the point of intersection of the other two.



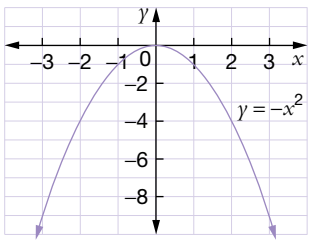

# 7:04 Special forms of the parabola

In Year 9 you were introduced to a curve called the parabola.

The simplest parabolas are described by the equations $y = x^2$ and $y = -x^2$ and their graphs can be obtained by completing a table of values.




| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |




| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | –9 | –4 | –1 | 0 | –1 | –4 | –9 |

Features of parabolas:

- they are symmetrical about an axis of symmetry
- they have a turning point called the vertex
- they are either concave up or concave down.

In the following exercise we will look at some special forms of the parabola and how they are related. The connection between the form of the equation and the graph of the parabola will also be investigated.

# Exercise 7:04

*Note:* A graphics calculator or graphing software could be used in the following exercises as an alternative to plotting points.

**1** Complete the following tables and then graph all four curves on one number plane.
Hint: On the $y$-axis, use values from 0 to 12.

**a** $y = x^2$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** $y = 2x^2$

| $x$ | −2 | −1 | −0·5 | 0 | 0·5 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**c** $y = 3x^2$

| $x$ | −2 | −1 | −0·5 | 0 | 0·5 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

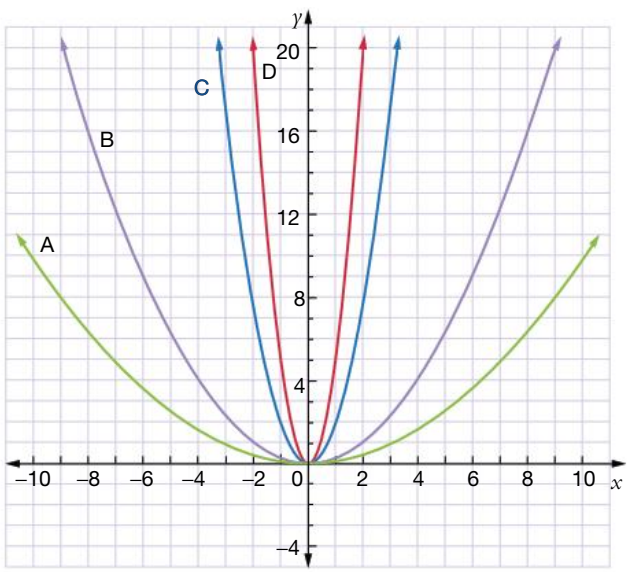
**d** $y = 0{\cdot}5x^2$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

- For the equation $y = ax^2$, what is the effect on the graph of varying the value of $a$?

**2** Match each of the parabolas A to D with the equations below.

**a** $y = 0{\cdot}25x^2$
**b** $y = 5x^2$
**c** $y = 2x^2$
**d** $y = 0{\cdot}1x^2$




**3** Complete the following tables and then graph all four curves on one number plane.
Hint: On the $y$-axis, use values from −2 to 13.

**a** $y = x^2$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** $y = x^2 + 2$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**c** $y = x^2 + 4$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

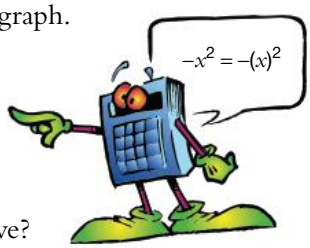
**d** $y = x^2 - 2$

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

- What is the difference in the curves $y = x^2$ and $y = x^2 + 2$?
- Can you see that the shape of the curve is the same in each case?
- For the equation $y = x^2 + c$, what is the effect on the graph of varying the value of $c$?

**4** **a** Complete the table of values for $y = -x^2$ and sketch its graph.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |




**b** Sketch the graph of $y = -2x^2$.

**c** For $y = ax^2$, what does the graph look like if $a$ is negative?

**5** On the same number plane, sketch the graphs of $y = -x^2 - 1$ and $y = -x^2 + 2$.

**6** Match each of the following equations with the graphs A to E.

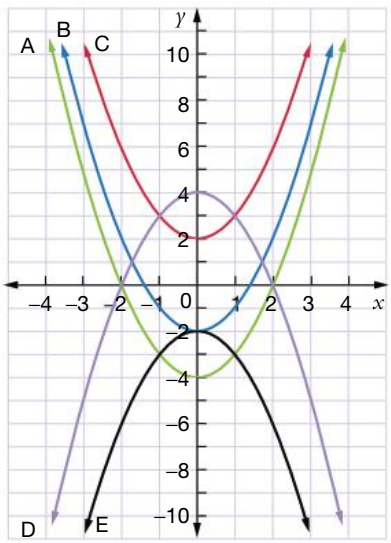
**a** $y = -x^2 - 2$

**b** $y = x^2 + 2$

**c** $y = x^2 - 2$

**d** $y = x^2 - 4$

**e** $y = 4 - x^2$




**7** The graphs of $y = x^2$ and $y = (x - 2)^2$ are shown on the diagram.

**a** How are the two graphs related?

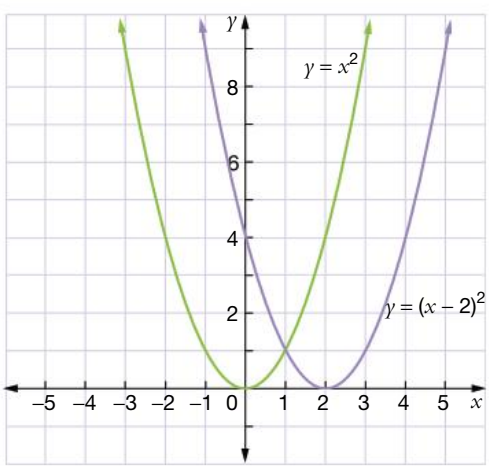
**b** Graph the parabola $y = (x + 2)^2$.
How is it related to the graph of $y = x^2$?

**c** Sketch the graphs of $y = (x - 4)^2$ and $y = (x + 1)^2$ on different number planes.

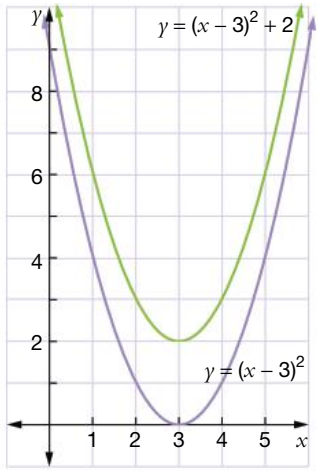
**d** How is the graph of $y = (x + a)^2$ related to the graph of $y = x^2$ when:

**i** $a$ is positive

**ii** $a$ is negative?



**8** The graphs of $y = (x - 3)^2$ and $y = (x - 3)^2 + 2$ are shown on the diagram.




- **a** How is the graph of $y = (x - 3)^2 + 2$ related to the graph of $y = (x - 3)^2$?
- **b** How would the graph of $y = (x - 3)^2 - 2$ be obtained from the graph of $y = (x - 3)^2$?
- **c** Sketch the graph of $y = (x + 3)^2 + 2$.
- **d** Use your answers for **a** to **c** to explain the connection between the equation of the parabola and the coordinates of its vertex.
- **e** Sketch the graph of each parabola on a separate number plane.
  - **i** $y = (x - 2)^2 + 3$
  - **ii** $y = (x + 2)^2 + 3$
  - **iii** $y = (x - 4)^2 - 2$
  - **iv** $y = (x + 1)^2 - 2$

**9** On separate number planes, sketch the following parabolas.

**a** $y = (x + 4)^2$ **b** $y = -(x + 4)^2$ **c** $y = -(x + 4)^2 + 3$

**10** Find the equation of the parabola that results from performing the following transformations on the parabola $y = x^2$.

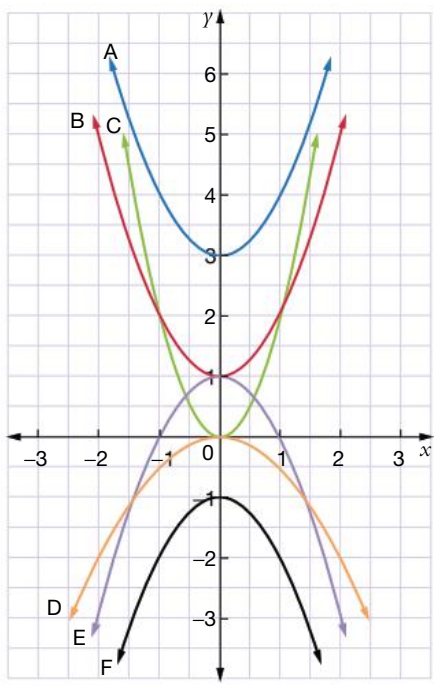
- **a** moving it up 2 units
- **b** moving it down 2 units
- **c** moving it 2 units to the right
- **d** moving it 2 units to the left
- **e** turning it upside down and then moving it up 4 units
- **f** turning it upside down and then moving it down 2 units
- **g** moving it up 2 units and then reflecting it in the $x$-axis
- **h** moving it 2 units to the right and then turning it upside down
- **i** moving it up 2 units and then moving it 2 units to the left
- **j** turning it upside down, moving it 3 units to the left and then moving it down 2 units

**11** Match each equation with the corresponding parabola in the diagram.

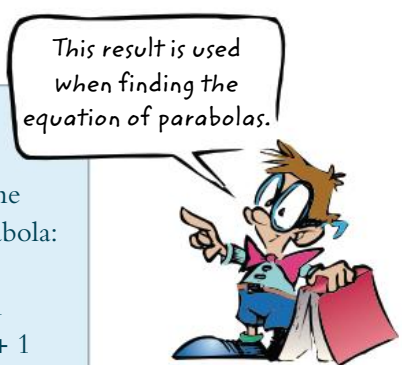
**a** $y = 2x^2$ **b** $y = -\frac{1}{2}x^2$

**c** $y = x^2 + 1$ **d** $y = x^2 + 3$

**e** $y = -x^2 - 1$ **f** $y = 1 - x^2$




A point lies on a parabola if its coordinates satisfy the equation of the parabola: e.g. $(1, 3)$ lies on the parabola $y = 2x^2 + 1$ because $3 = 2 \times 1^2 + 1$



**12** Match each equation with the corresponding parabola A to F in the diagram.

**a** $y = (x - 3)^2$

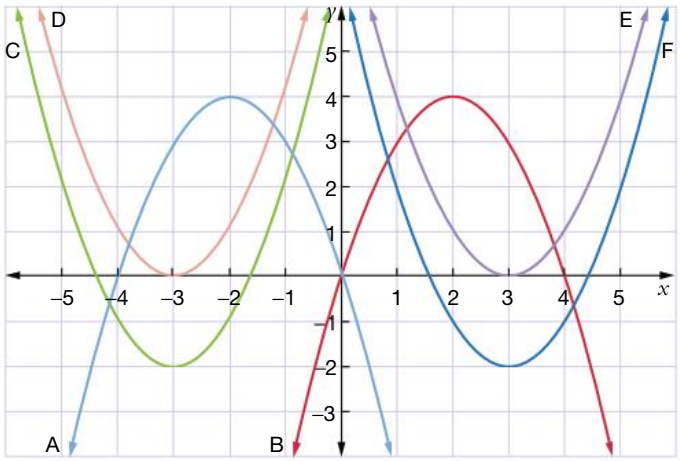
**b** $y = (x + 3)^2$

**c** $y = -(x + 2)^2 + 4$

**d** $y = -(x - 2)^2 + 4$

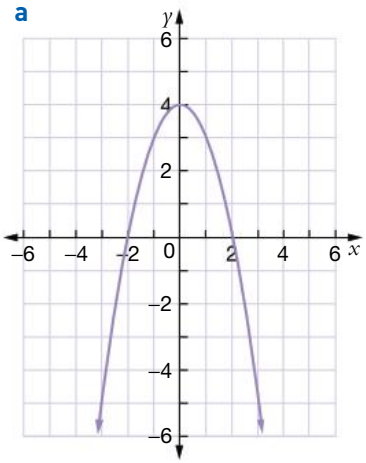
**e** $y = (x + 3)^2 - 2$

**f** $y = (x - 3)^2 - 2$

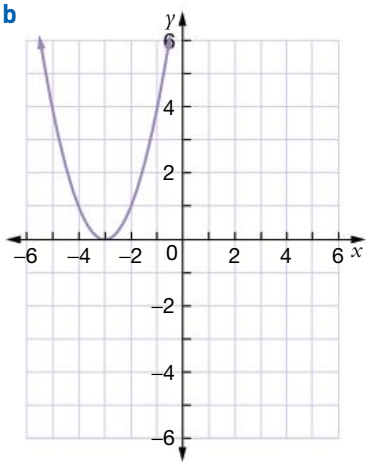



**13** The parabolas shown are the result of translating and/or reflecting the parabola $y = x^2$. Find the equation of each parabola.

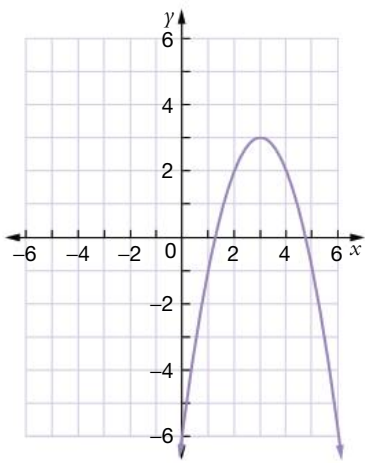
**a**



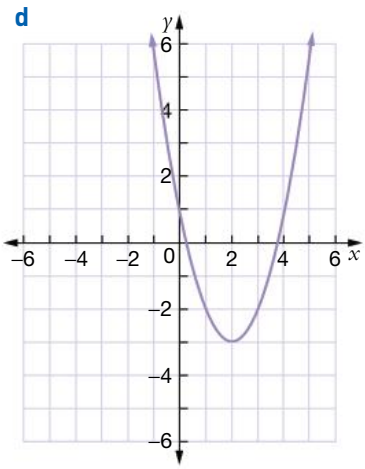

**b**



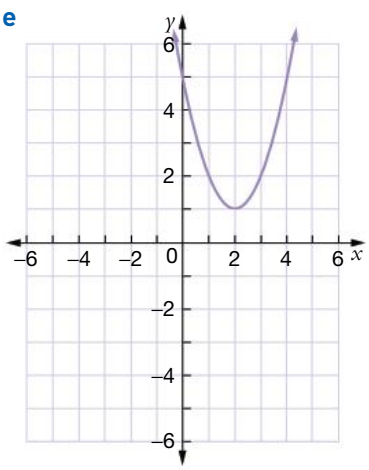

**c**



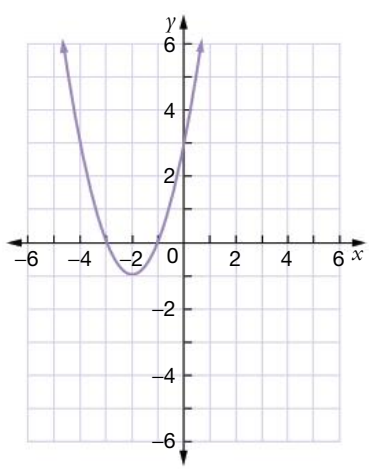

**d**




**e**



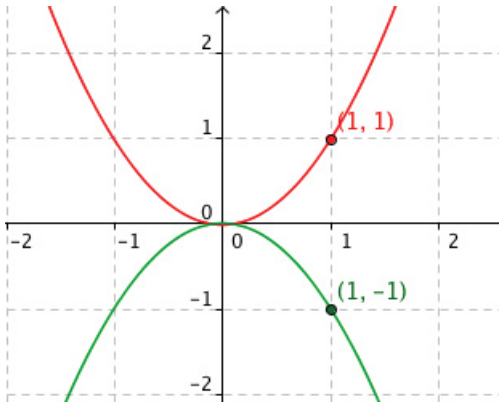

**f**



## GEOGEBRA ACTIVITY 7:04A — PARABOLAS OF THE FORM $y = ax^2$

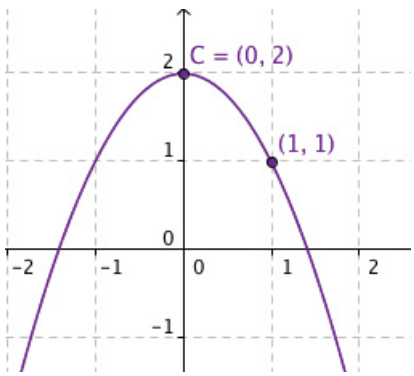
This activity uses sliders to change the value of $a$. You can see instantly what changing the value of $a$ does to the shape of the graph.

You can then choose to find the equation of an unknown parabola that is randomly generated.




## GEOGEBRA ACTIVITY 7:04B — PARABOLAS OF THE FORM $y = ax^2 + k$

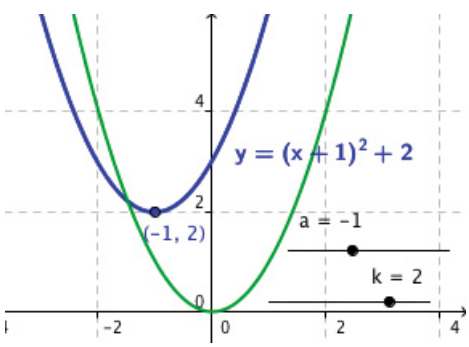
Investigate how changing the values of $a$ and $c$ change the shape of the parabola. Test your skills by finding the equation of a randomly generated parabola.




## GEOGEBRA ACTIVITY 7:04C — PARABOLAS OF THE FORM $y = (x + a)^2$ AND $y = (x + a)^2 + k$

Use the checkboxes and sliders to change the values of $a$ and $k$. How are the curves related to $y = x^2$?

Test your knowledge of the relationship between an equation and its graph by finding the equation of a randomly generated curve.



## INVESTIGATION 7:04 THE GRAPHS OF PARABOLAS

Use the GeoGebra activities on page 198 or other graphing technologies to investigate graphs of parabolas given in the following forms for varying values of $a$ and $k$. Consider both positive and negative values of $a$ and $k$.

1 $y = ax^2$

2 $y = ax^2 + k$

3 $y = (x + a)^2$

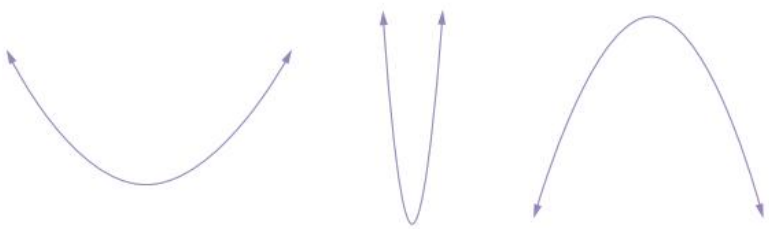
4 $y = (x + a)^2 + k$

Write a report on each of the forms, explaining how the features of the graph such as the concavity, the axis of symmetry, the position of the vertex, the coordinates of the $y$-intercept and the number of $x$-intercepts are related to the values of $a$ and $k$.

# 7:05 Parabolas of the form $y = ax^2 + bx + c$

All parabolas have the same basic shape.

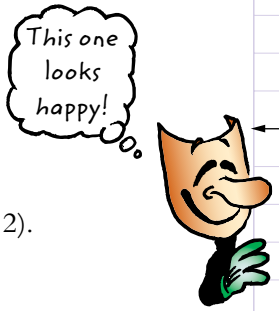
- They are all concave up or concave down with a single vertex or turning point.
- They are symmetrical about an axis of symmetry.



We have looked at what numbers in these equations influence the steepness, the concavity and the position of the graph on the number plane.

In this section we look at how to sketch a parabola when its equation is given in the form $y = ax^2 + bx + c$. We will also look at how to find features of the parabola, such as the $x$- and $y$-intercepts, the axis of symmetry and the vertex.

## Finding the *y*-intercept



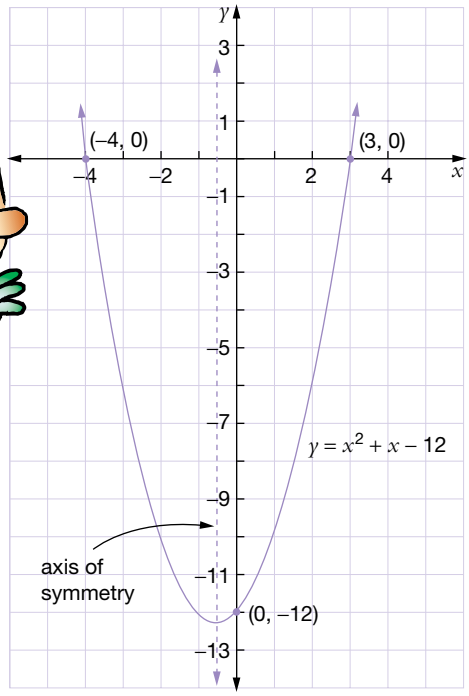

To find the $y$-intercept of $y = x^2 + x - 12$, we let $x = 0$.

$y = x^2 + x - 12$

When $x = 0$, $y = -12$

$\therefore$ The $y$-intercept is $-12$.

$\therefore$ The curve cuts the $y$-axis at $(0, -12)$.




## Finding the *x*-intercepts

To find the $x$-intercepts of $y = x^2 + x - 12$, we let $y = 0$.

$y = x^2 + x - 12$

When $y = 0$, $\quad 0 = x^2 + x - 12$

Solving this, $\quad 0 = (x + 4)(x - 3)$

$\therefore\ x = -4$ or $3$

$\therefore$ The $x$-intercepts are $-4$ and $3$.

$\therefore$ The curve cuts the $x$-axis at $(-4, 0)$ and $(3, 0)$.

*Note:* If $x^2 + x - 12 = 0$ was difficult to factorise, then the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ could have been used to find the $x$-intercepts.

## Finding the axis of symmetry

The parabola $y = x^2 + x - 12$ has a vertical axis of symmetry that cuts the $x$-axis half-way between $(-4, 0)$ and $(3, 0)$, which are the two $x$-intercepts.

The axis of symmetry has the equation:

$$x = \frac{-4 + 3}{2}$$

$$x = -\tfrac{1}{2}$$

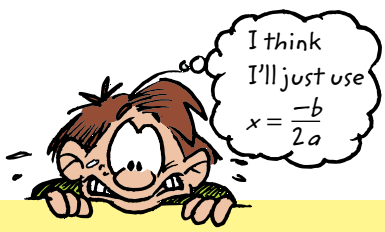
For the parabola $y = x^2 + x - 12$, $a = 1$, $b = 1$, $c = -12$.

$\therefore$ The axis of symmetry is:

$$x = \frac{-1}{2(1)}$$

$$x = -\tfrac{1}{2}$$

> The axis of symmetry of the parabola $y = ax^2 + bx + c$ is given by the equation:
> $$x = -\frac{b}{2a}$$




For the parabola $y = ax^2 + bx + c$, the $x$-intercepts are found by solving $0 = ax^2 + bx + c$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The axis of symmetry will cut the $x$-axis half-way between these two values.

$$x = \frac{\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}}{2}$$

$$x = \frac{\left(\dfrac{-2b}{2a}\right)}{2} = \frac{-2b}{2a} \div 2$$

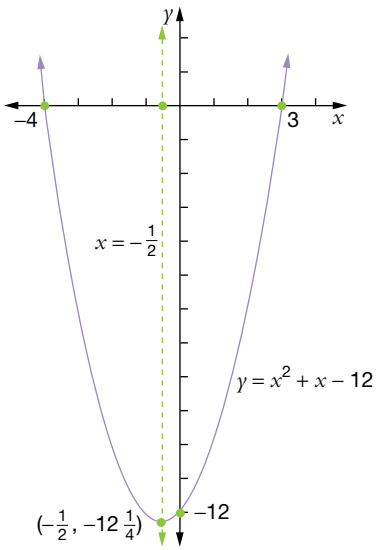
$$x = \frac{-b}{2a}$$

$\therefore$ The axis of symmetry is $x = -\dfrac{b}{2a}$.

## Finding the vertex (or turning point)

**Method 1**

As the vertex lies on the axis of symmetry, its $x$-coordinate will be the same as that of the axis of symmetry. The $y$-coordinate can be found by substituting this $x$-value into the equation of the parabola.




For $y = x^2 + x - 12$, the axis of symmetry is $x = -\frac{1}{2}$.

Now, when $x = -\frac{1}{2}$,

$y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$

$y = -12\frac{1}{4}$

$\therefore$ The vertex of the parabola is $(-\frac{1}{2}, -12\frac{1}{4})$.

**Method 2**

The vertex can also be found by finding the midpoint of the interval joining the x-intercepts.

Here the $x$-intercepts are $-4$ and 3. The midpoint is $x = \dfrac{-4 + 3}{2}$ or $x = -\dfrac{1}{2}$.

Substituting $x = -\frac{1}{2}$ into $y = x^2 + x - 12$ gives $y = -12\frac{1}{4}$.

Hence the vertex is $(-\frac{1}{2}, -12\frac{1}{4})$.

**Method 3**

The method of completing the square can also be used to find the vertex.

$y = x^2 + x - 12$

$y = (x^2 + x + \frac{1}{4}) - \frac{1}{4} - 12$

$y = (x + \frac{1}{2})^2 - 12\frac{1}{4}$

This parabola is the result of moving the parabola $y = x^2$ horizontally $\frac{1}{2}$ half a unit to the left and then vertically down $12\frac{1}{4}$ units. Hence the vertex of the parabola is $(-\frac{1}{2}, -12\frac{1}{4})$.

The shape of the parabola is clearly demonstrated by the water arcs of this fountain.

## WORKED EXAMPLE

For each equation, find:

**a** the $y$-intercept
**b** the $x$-intercept
**c** the axis of symmetry
**d** the vertex (turning point).

Use these results to sketch each graph.

**1** $y = x^2 + 2x - 3$ **2** $y = 2x^2 + 4x + 3$ **3** $y = 4x - x^2$

### Solutions

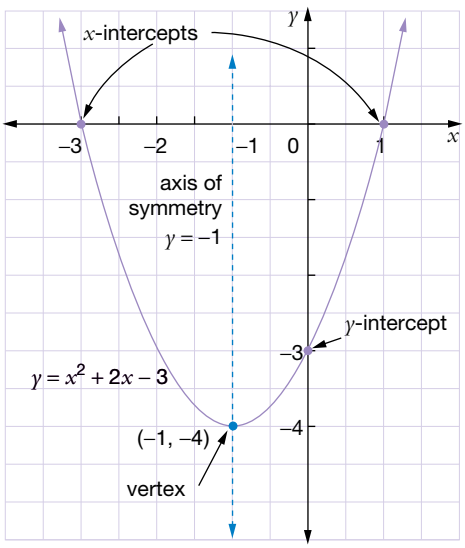
**1** $y = x^2 + 2x - 3$

**a** For the $y$-intercept, let $x = 0$.
$y = (0)^2 + 2(0) - 3$
The $y$-intercept is $-3$.

**b** For the $x$-intercepts, let $y = 0$.
$0 = x^2 + 2x - 3$
$(x + 3)(x - 1) = 0$
The $x$-intercepts are $-3$ and $1$.

**c** Axis of symmetry:
$x = \dfrac{-3 + 1}{2}$ (midpoint of $x$-intercepts)
$\therefore\ x = -1$ is the axis of symmetry.

**d** To find the vertex, substitute $x = -1$ into the equation to find the $y$-value.
$y = (-1)^2 + 2(-1) - 3$
$y = -4$
$\therefore$ The vertex is $(-1, -4)$.




We now plot the above information on a number plane and fit the parabola to it.

**2** $y = 2x^2 + 4x + 3$

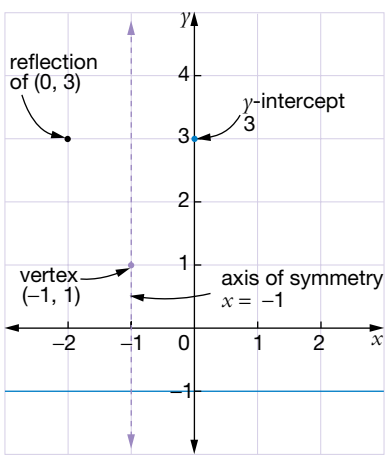
**a** To find the $y$-intercept of $y = 2x^2 + 4x + 3$, let $x$ be zero.
$y = 2(0)^2 + 4(0) + 3$
$\therefore$ The $y$-intercept is 3.

**b** For the $x$-intercepts, solve $2x^2 + 4x + 3 = 0$.
Use the formula:

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$= \frac{-4 \pm \sqrt{-8}}{4}$$

This gives us a negative number under the square root sign.
You can't find the square root of a negative number!
Hence there are no real solutions, so the parabola does not cut or touch the $x$-axis.



**c** $y = 2x^2 + 4x + 3$

$a = 2, b = 4, c = 3$

$\therefore$ Axis of symmetry:

$$x = -\frac{b}{2a}$$

$$x = \frac{-4}{2 \times 2}$$

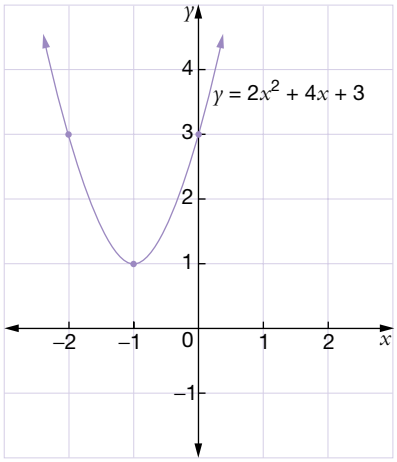
$\therefore$ The axis of symmetry is $x = -1$.

**d** The vertex is the turning point of the curve, and is on the axis of symmetry $x = -1$.

When $x = -1$, $y = 2(-1)^2 + 4(-1) + 3$

$= 1$

$\therefore$ The vertex is $(-1, 1)$.




**3** $y = 4x - x^2$

**a** For $y = 4x - x^2$, if $x = 0$, then $y = 0$.

$\therefore$ The curve cuts the $y$-axis at the origin.

**b** When $y = 0$, $4x - x^2 = 0$

$x(4 - x) = 0$

$\therefore$ The $x$-intercepts are 0 and 4.

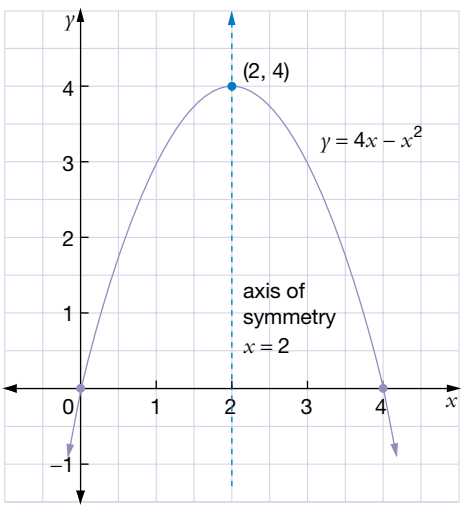
**c** Axis of symmetry:

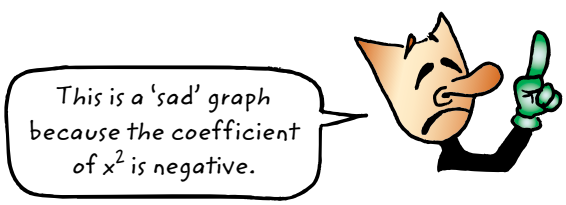
$x = 2$ (midpoint of $x$-intercepts)

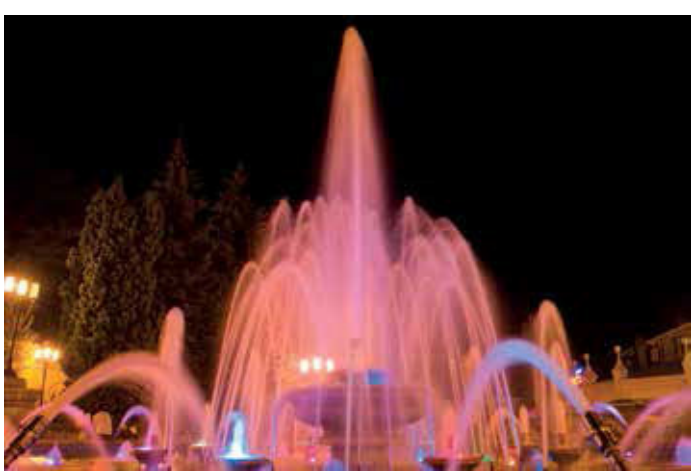
**d** When $x = 2$, $y = 4(2) - (2)^2$

$= 4$

$\therefore$ The vertex is $(2, 4)$.



Water spurts are in the shape of parabolas.

# Exercise 7:05

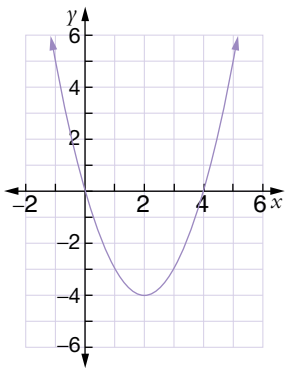
P Foundation worksheet 7:05
The parabola $y = ax^2 + bx + c$

**1** For each of the graphs, find:

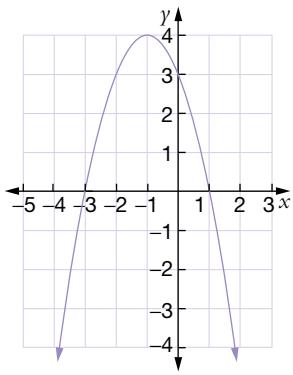
**i** the $y$-intercept **ii** the $x$-intercepts

**iii** the equation of the axis of symmetry **iv** the coordinates of the vertex.

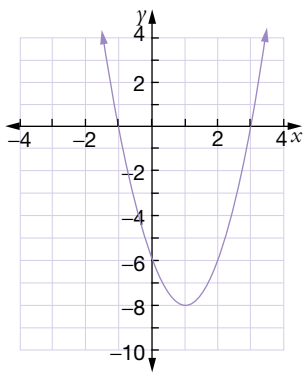
**a**




**b**




**c**




**2** Find the $y$-intercepts of the following parabolas.

**a** $y = x^2 - 6x + 5$ **b** $y = 2x^2 - 8$ **c** $y = (x - 2)(x + 3)$

**3** Find the $x$-intercepts of the following parabolas.

**a** $y = x^2 - 2x - 8$ **b** $y = 3x^2 + 10x - 8$ **c** $y = (x - 3)(4x + 7)$

**4** Find the equation of the axis of symmetry and the coordinates of the vertex of the following parabolas.

**a** $y = (x - 3)(x - 5)$ **b** $y = 3(x - 2)(x + 6)$

**c** $y = -\frac{1}{2}(x + 4)(2 - x)$ **d** $y = x^2 - 6x + 7$

**e** $y = 3x^2 - 9x + 14$ **f** $y = 4 - 3x - x^2$

**5** For the parabola $y = x^2 + 2x - 8$, find:

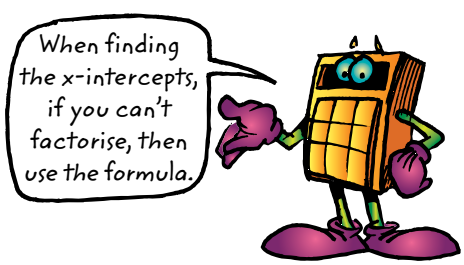
**a** the $y$-intercept

**b** the $x$-intercepts

**c** the axis of symmetry

**d** the vertex.

**e** Hence, sketch its graph.



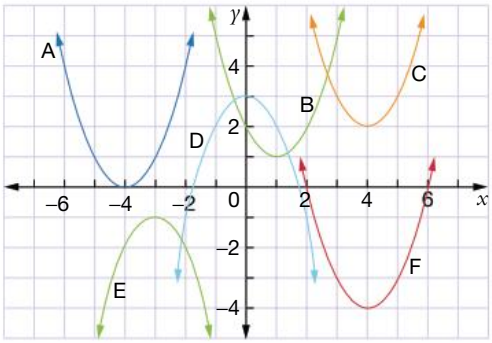

**6** Repeat the steps in Question **5** to graph the following equations, showing all the relevant features.

**a** $y = x^2 - 6x + 5$ **b** $y = x^2 - 6x$

**c** $y = 2x^2 - 8x - 10$ **d** $y = -x^2 + 6x - 9$

**e** $y = x^2 - 3x - 4$ **f** $y = -2x^2 - 3x - 1$

**7** Match each parabola with one of the equations written below the diagram.




**a** $y = -x^2 + 3$ **b** $y = x^2 - 2x + 2$
**c** $y = x^2 - 8x + 12$ **d** $y = x^2 + 8x + 16$
**e** $y = -x^2 - 6x - 10$ **f** $y = x^2 - 8x + 18$

1 Find the turning point.
2 Is it happy ∪ ($a$ is $+ve$) or sad ∩ ($a$ is $-ve$)?
3 Visualise the graph before you sketch.

**8** Sketch each set of three parabolas on the same number plane.

| | | | |
|---|---|---|---|
| **a** | **i** $y = x^2 - 4$ | **ii** $y = x^2 - 4x$ | **iii** $y = x^2 - 4x + 4$ |
| **b** | **i** $y = 9 - x^2$ | **ii** $y = 9x - x^2$ | **iii** $y = 10 + 9x - x^2$ |
| **c** | **i** $y = (x - 3)(x + 5)$ | **ii** $y = 2(x - 3)(x + 5)$ | **iii** $y = (3 - x)(5 + x)$ |
| **d** | **i** $y = x^2 - 2x - 8$ | **ii** $y = 2x^2 - 4x - 16$ | **iii** $y = 8 + 2x - x^2$ |

**9** Sketch the graph of each quadratic relationship, showing all relevant features.

**a** $y = 2x^2 - 8$ **b** $y = 16 - x^2$ **c** $y = x^2 - 8x + 7$
**d** $y = (3 - x)(7 + x)$ **e** $y = 24 - 2x - x^2$ **f** $y = 4x^2 + 16x + 7$

**10** **a** Sketch the graphs of $y = x^2 + 3x$ and $y = 2x^2 + 6x$. Compare the two graphs and describe the differences between them.

**b** Sketch the graphs of $y = (x - 3)(x + 2)$ and $y = 3(x - 3)(x + 2)$. Compare the two graphs and describe the differences between them.

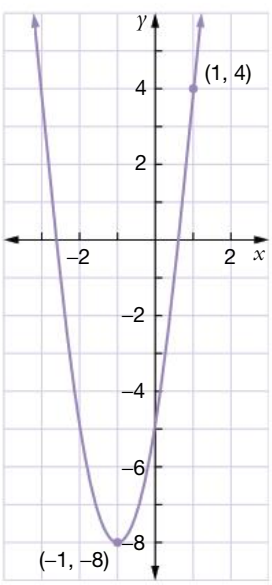
**11** The parabola in the diagram has its vertex at $(-1, -8)$ and passes through the point $(1, 4)$.
The equation of the parabola has the form $y = ax^2 + bx + c$.

**a** Use the $y$-intercept to show that $c = -5$.

**b** Use the equation of the axis of symmetry to show that $b = 2a$ and that the equation of the parabola is of the form $y = ax^2 + 2ax - 5$.

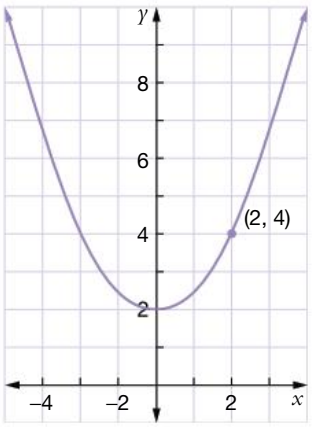
**c** Substitute the coordinates of the vertex or the point $(1, 4)$ to find the value of $a$.

**d** What is the equation of the parabola?

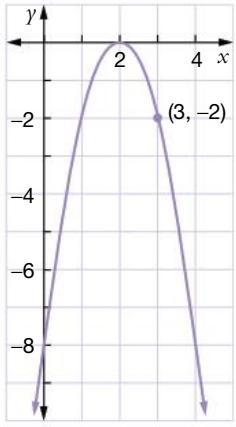


**12** Use the method in Question **11** to find the equation of each of the following parabolas.

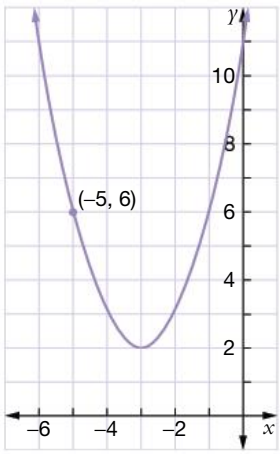
**a**



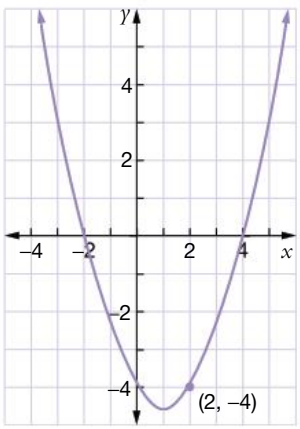

**b**



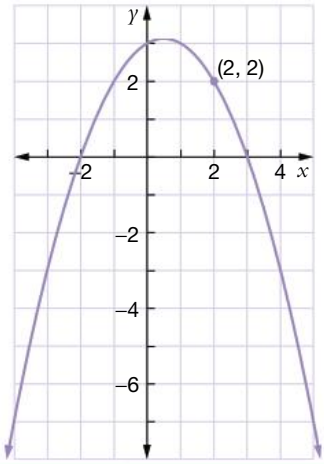

**c**



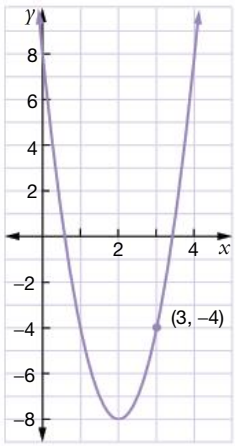

**d**




**e**




**f**




## FUN SPOT 7:05 — WHY DIDN'T THE BALD MAN NEED HIS KEYS?

Work out the answer to each question and write the letter for that part in the box that is above the correct answer.

*Factorise:*

**E** $x^2 - 3x - 4$ **S** $x^2 - 16$

**O** $x^2 - 4x$ **S** $x^2 + 3x - 4$

*What is the axis of symmetry for each parabola?*

**H** $y = x^2 - 4x + 4$ **S** $y = x^2 + 4x$

**K** $y = x^2 - 4$ **O** $y = x^2 - 3x - 4$

*Where does each parabola below cut the y-axis?*

**I** $y = x^2 - 4x + 4$

**H** $y = x^2 - 4x$

**T** $y = x^2 - 4$

*What is the vertex for each parabola?*

**L** $y = x^2 - 4x + 4$

**C** $y = x^2 + 4x$

**L** $y = x^2 + 2x - 3$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | |
| (0, 0) | $(x-4)(x+1)$ | $(-1, -4)$ | $x = \frac{3}{2}$ | $(x-4)(x+4)$ | $(0, -4)$ | $x = 2$ | $(0, 4)$ | $x = -2$ | $(2, 0)$ | $x(x-4)$ | $(-2, -4)$ | $x = 0$ | $(x+4)(x-1)$ |

## INVESTIGATION 7:05 DESCRIBING NUMBER PATTERNS

If we know that a table represents a parabola, we can generate its equation from the table using the general equation of the parabola and simultaneous equations.

Consider the table below.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 3 | 6 | 11 | 18 | 27 | 38 |

If the values in the table are thought of as points on a parabola, we can substitute the $x$- and $y$-values of three points into the equation $y = ax^2 + bx + c$ to produce a set of three simultaneous equations.

To find the equation of the parabola we need to find the values of $a$, $b$ and $c$.

$y = ax^2 + bx + c$

$\therefore c = 3$ substituting (0, 3)

(Substituting $x = 0$ will always give the value of the constant term $c$.)

We now know that the parabola has an equation of the form $y = ax^2 + bx + 3$.

$y = ax^2 + bx + 3$

$6 = a + b + 3$ substituting (1, 6)

$\therefore a + b = 3$ 1

$y = ax^2 + bx + 3$

$11 = 4a + 2b + 3$ substituting (2, 11)

$\therefore 2a + b = 4$ 2

Solving the simultaneous equations gives the solution:

$a + b = 3$ 1

$2a + b = 4$ 2

$a = 1, b = 2$

Substituting $a = 1, b = 2, c = 3$ into $y = ax^2 + bx + c$ gives the rule $y = x^2 + 2x + 3$.

This method will always work, but solving the simultaneous equations is difficult if the table does not contain a point with an $x$-value of 0.

### Exercise

Find the equation of the parabola that would produce each of the following tables of values.

1

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 2 | 6 | 12 | 20 | 30 | 42 |

2

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 16 | 7 | 0 | −5 | −8 | −9 |

3

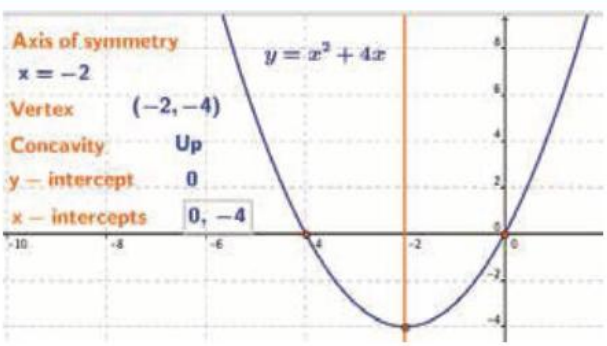
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | −4 | −1 | 6 | 17 | 32 | 51 |

4

| $x$ | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $y$ | 19 | 10 | 5 | 4 | 7 | 14 |

## GEOGEBRA ACTIVITY 7:05 PARABOLAS OF THE FORM $y = ax^2 + bx + c$

Change the sliders for $a$, $b$ and $c$ in this activity to explore parabolas of the form $y = ax^2 + bx + c$. You can also use the equation to determine the features of the parabola and check your skills by using a checkbox.



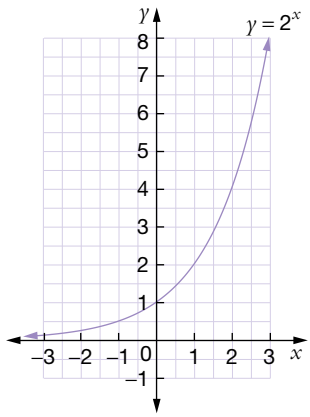
# 7:06 Exponential graphs

When equations such as $y = 2^x$, $y = -2^x$ and $y = 2^{-x} + 1$ are graphed, the resulting curves are called exponential curves. Exponential equations involve calculating varying powers of a fixed positive number, often called the base. In the examples above the base is 2.

Exponential graphs can be drawn by producing a table of values and plotting the resulting points on a number plane. This is shown for the curves $y = 2^x$ and $y = 1 + 3^{-x}$.

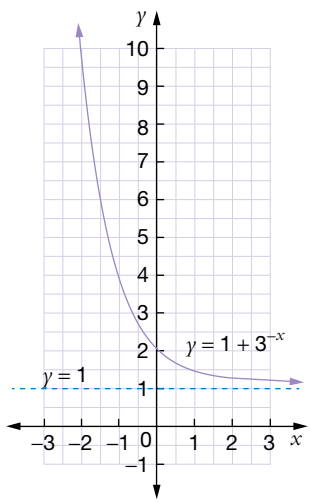
### $y = 2^x$

- The curve increases as the $x$-values increase.
- The curve rises steeply for positive values of $x$.
- The curve flattens out for negative values of $x$ and the smaller $x$ becomes the closer the curve gets to the $x$-axis. The $x$-axis is an asymptote for this part of the curve.
- Because $2^x$ is always positive the curve is always above the $x$-axis.
- The curve passes through $(0, 1)$ on the $y$-axis because $2^0 = 1$. The $y$-intercept is 1.

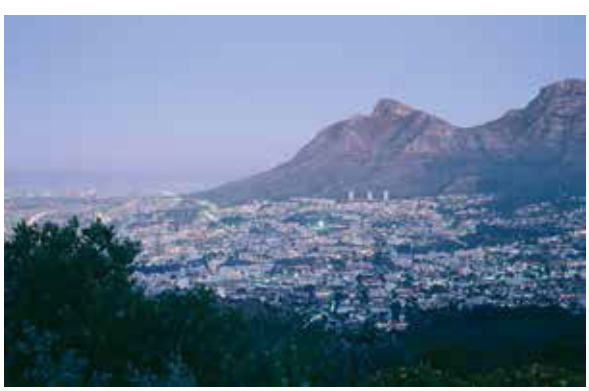



### $y = 1 + 3^{-x}$

- The curve decreases as the $x$-values increase.
- As $x$ increases, the curve gets closer to the line $y = 1$. We say that $y = 1$ is an asymptote and that the curve is asymptotic.
- The curve is always above the line $y = 1$ because $3^{-x}$ is always positive.
- The curve passes through $(0, 2)$. The $y$-intercept is 2.



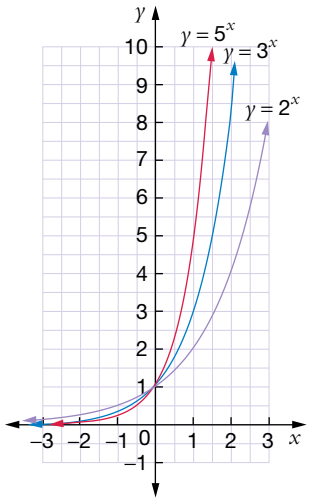

In the following exercise the similarities and differences between different curves will be investigated. Analysing the equation of the curve will enable you to predict the shape of the curve.



Exponential curves are used in the study of population growth.

# Exercise 7:06

**1** The graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$ are shown on the number plane.




Compare the graphs.

**a** Where do the curves cross?

**b** Is $y = 5^x$ always above $y = 3^x$?

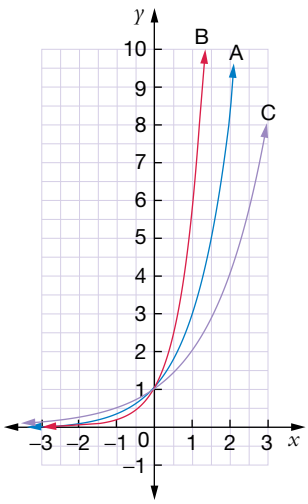
**c** Which graph increases the fastest?
Which graph increases the slowest?

**d** If you were asked to graph $y = 10^x$ on this graph, where would it be in relation to the other curves?

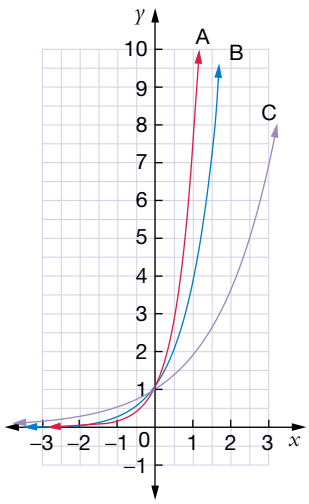
**e** If you were asked to graph $y = 1{\cdot}8^x$ on this graph, where would it be in relation to the other curves?

**2** In each of the following, match each curve with its equation.

**a** $y = 2^x$, $y = 3^x$, $y = 6^x$



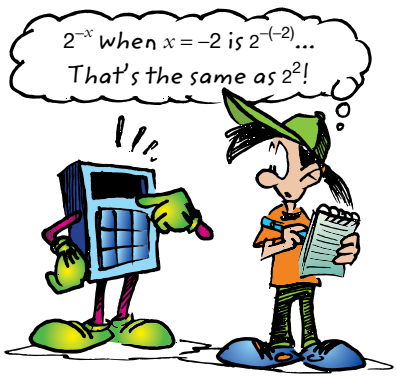

**b** $y = 4^x$, $y = 1{\cdot}9^x$, $y = 8^x$




**3** **a** Complete the table of values for $y = 2^{-x}$ and graph the curve on a number plane.

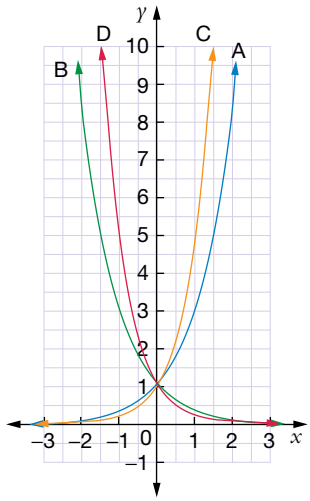
| $x$ | −2 | −1·5 | −1 | −0·5 | 0 | 0·5 | 1 | 1·5 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

**b** Compare your graph with $y = 2^x$. What do you notice?



**4** Match each of the graphs A to D to the following equations.

**a** $y = 3^x$ **b** $y = 5^x$ **c** $y = 5^{-x}$ **d** $y = 3^{-x}$




**5** If each of the following curves is reflected in the $y$-axis, what is the equation of the curve that is obtained?

**a** $y = 4^x$ **b** $y = 6^x$ **c** $y = 8^{-x}$

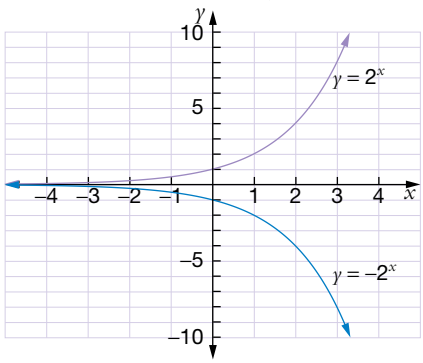
**d** $y = 1 \cdot 2^x$ **e** $y = 3 \cdot 6^{-x}$

**6** The graphs $y = 2^x$ and $y = -2^x$ have been graphed on the number plane.

**a** Look at each pair of points with the same $x$-coordinate e.g. $(1, 2)$ and $(1, -2)$. Is each pair of points equidistant from the $x$-axis?

**b** Are these curves reflections of each other in the $x$-axis?

**c** Draw on the same number plane the graphs of $y = 2^{-x}$ and $y = -2^{-x}$? Are these curves reflections of each other in the $x$-axis?




**7** If each of the following curves is reflected in the $x$-axis, what is the equation of the curve that is obtained?

**a** $y = 3^x$ **b** $y = 4^{-x}$ **c** $y = 1 \cdot 8^{-x}$ **d** $y = -(1 \cdot 2^x)$ **e** $y = -6^{-x}$

**8** **a** How are the graphs of $y = a^x$ and $y = a^{-x}$ related?

**b** How are the graphs of $y = a^x$ and $y = -a^x$ related?

**9** Study the graphs of the curves $y = 2^x$, $y = 2^x + 1$ and $y = 2^x - 1$ on the diagram.

**a** What transformation will change:

**i** $y = 2^x$ into $y = 2^x + 1$

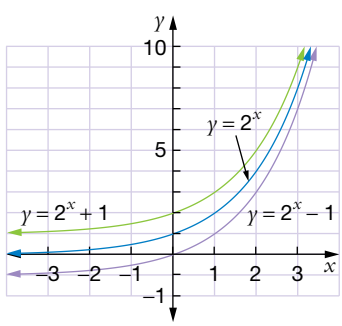
**ii** $y = 2^x$ into $y = 2^x - 1$?

**b** What is the equation of the curve that results if $y = 2^x$ is translated:

**i** 3 units vertically upwards

**ii** 4 units vertically downwards?

**c** Sketch on the same number plane the curves $y = 2^x + 2$ and $y = 2^x - 3$.




**10** Find the equation of the curve that results when the curve $y = 3^x$ is:

**a** reflected in the $x$-axis

**b** reflected in the $y$-axis

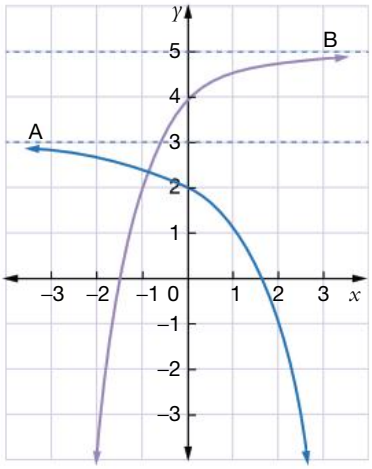
**c** translated 1 unit vertically upwards

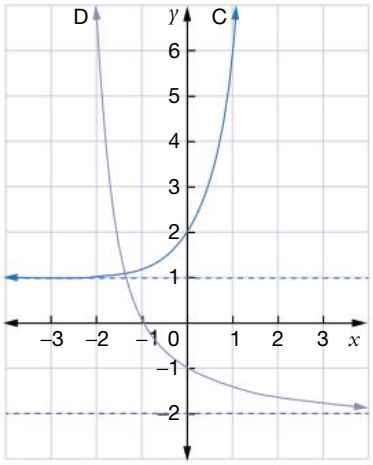
**d** translated 1 unit vertically downwards

**e** reflected in the $y$-axis and then translated 1 unit vertically downwards

**f** reflected in the $x$-axis and then translated 1 unit vertically upwards.

**11** For each of the curves A to D answer the following questions.



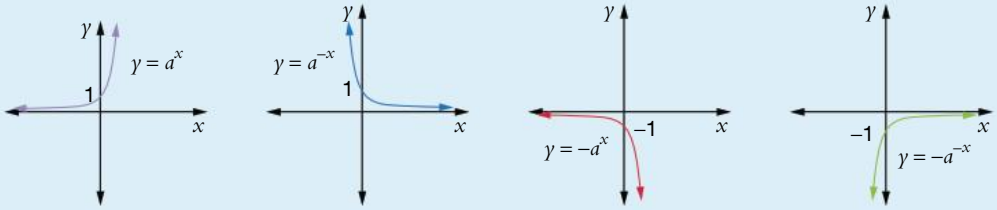

**a** Is the curve increasing or decreasing?
**b** Is the curve concave up or concave down?
**c** What happens to the $y$-values as the $x$-values become:
  **i** very large  **ii** very small?
**d** What is the $y$-intercept?

**12** From the list of equations below, select those that would fit into each of the categories **a** to **f**.

| | | | |
|---|---|---|---|
| **A** $y = 3^x$ | **B** $y = 2^{-x}$ | **C** $y = -4^x$ | **D** $y = 3^{-x}$ |
| **E** $y = 2^x + 1$ | **F** $y = 1 - 3^x$ | **G** $y = 4^x$ | **H** $y = -2^{-x}$ |
| **I** $y = 1 - 2^x$ | **J** $y = 4^{-x} + 3$ | **K** $y = -3^x$ | **L** $y = 4^x - 1$ |

**a** The curves that can be formed from $y = 3^x$ using reflections and/or translations.
**b** The curves that are increasing.
**c** The curves that are decreasing.
**d** The curves that are increasing and concave up.
**e** The curves that are increasing and concave down.
**f** The curves that have a $y$-intercept of:
  **i** 1  **ii** –1

The basic exponential curve shapes ($a > 1$)



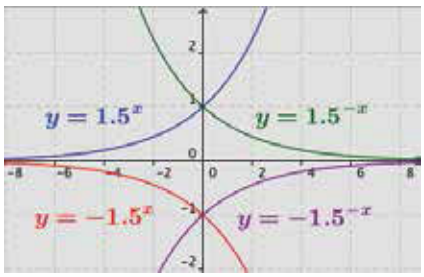
## P GEOGEBRA ACTIVITY 7:06 EXPONENTIAL GRAPHS

Investigate four basic types of exponential graphs:

$y = a^x + c$, $y = a^{-x} + c$, $y = -a^x + c$ and $y = -a^{-x} + c$

Change the values of $a$ and $c$ to observe the changes to the graph.

Test your skills by finding the equation of a randomly generated parabola.

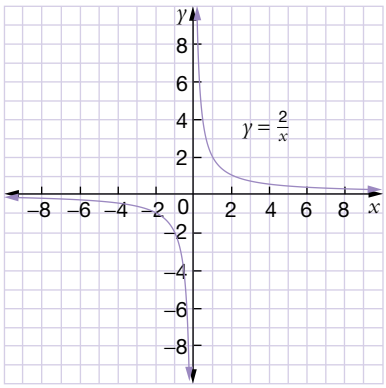



# 7:07 The hyperbola

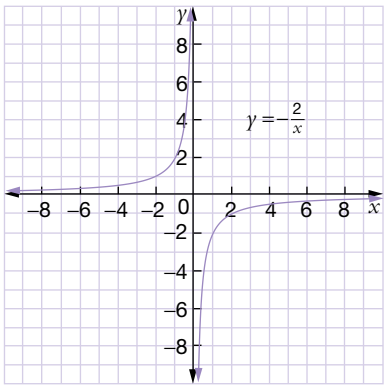
## The basic form $y = \frac{k}{x}$

The graphs below have been produced using $k = 2$ and $k = -2$.

$y = \frac{2}{x}$




$y = -\frac{2}{x}$



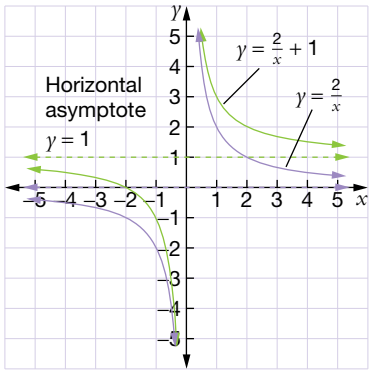

**Features of $y = \frac{k}{x}$**

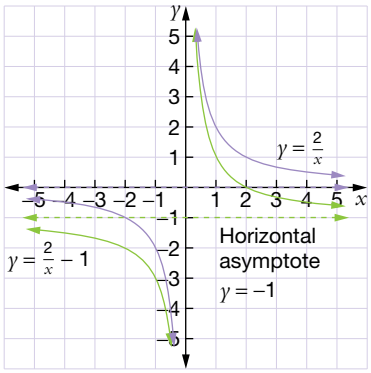
- No value for $y$ exists when $x = 0$. This is because no number can be divided by zero.
- The graph has two parts. These parts are in opposite quadrants and are the same shape and size.
- If $k$ is positive then the curve is in the 1st and 3rd quadrants.
  If $k$ is negative it is in the 2nd and 4th quadrants.
- The curve is symmetrical.
- The curve approaches two lines (the $x$- and $y$-axes) but will never touch them.
  These lines are called asymptotes.

## Graphs of the form $y = \frac{k}{x} + c$

As we have seen with the parabola, the addition of a constant translates the curve vertically.

- Moving $y = \frac{2}{x}$ up 1 unit produces the curve $y = \frac{2}{x} + 1$.
- Moving $y = \frac{2}{x}$ down 1 unit produces the curve $y = \frac{2}{x} - 1$.




The curve $y = \frac{k}{x} + c$ is obtained by translating the curve $y = \frac{k}{x}$:

- $c$ units up if $c$ is positive
- $c$ units down if $c$ is negative.

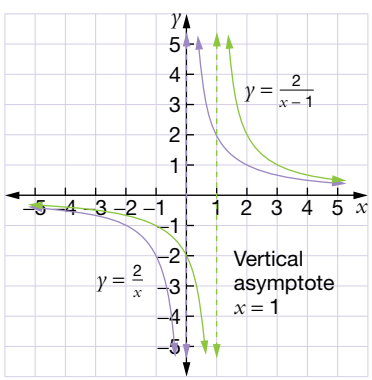
The asymptotes are $x = 0$ and $y = c$.

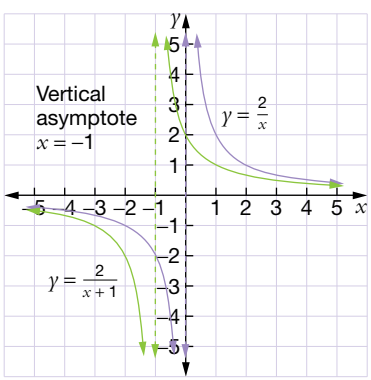
Moving the curve vertically will also change the horizontal asymptote.

## Graphs of the form $y = \frac{k}{x - b}$

The following diagrams show the relationships between the curve $y = \frac{2}{x}$ and the curves $y = \frac{2}{x - 1}$ and $y = \frac{2}{x + 1}$.

- Moving $y = \frac{2}{x}$ right 1 unit produces the curve $y = \frac{2}{x - 1}$.
- Moving $y = \frac{2}{x}$ left 1 unit produces the curve $y = \frac{2}{x + 1}$.




The hyperbola $y = \frac{k}{x - b}$ is obtained by translating the curve $y = \frac{k}{x}$:

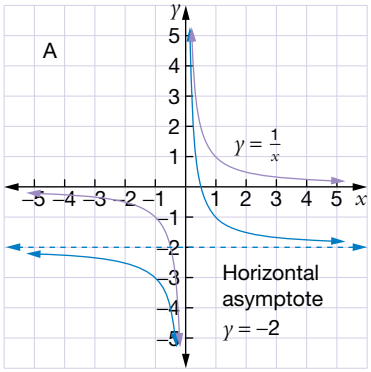
- $b$ units to the right if $b$ is positive
- $b$ units to the left if $b$ is negative.

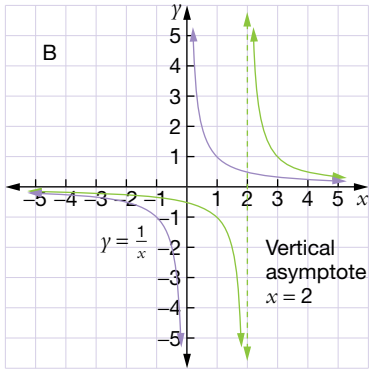
The asymptotes are $x = b$ and $y = 0$.

Notice that when the curve is moved horizontally, the vertical asymptote moves as well.

## WORKED EXAMPLE 1

The curves A and B have been produced by translating the curve $y = \frac{1}{x}$. What are the equations of A and B?




### Solution

Curve A has been produced by moving $y = \frac{1}{x}$ down 2 units.

See points (1, 1) and (1, –1).

Hence the equation of curve A is $y = \frac{1}{x} - 2$.

Curve B has been produced by moving $y = \frac{1}{x}$ right 2 units.

See the points (1, 1) and (3, 1).

Hence the equation of curve B is $y = \frac{1}{x - 2}$.

## WORKED EXAMPLE 2

Sketch the following curve and label the asymptote and the $x$- and $y$-intercepts.

$y = -\frac{1}{x} + 2$

### Solution

$y = -\frac{1}{x} + 2$ has come from moving

$y = -\frac{1}{x}$ up 2 units.

As $x$ increases, $y$ approaches 2.

$\therefore$ $y = 2$ is an asymptote.

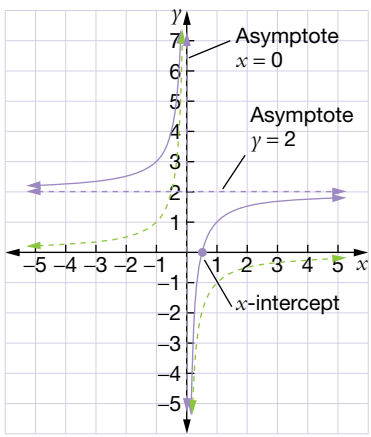
Now, $x \neq 0$ because you cannot divide by 0.

$\therefore$ $x = 0$ is an asymptote and there is no $y$-intercept.

If $y = 0$, $\quad 0 = -\frac{1}{x} + 2$

$$\frac{1}{x} = 2$$

$\therefore$ $x = \frac{1}{2}$ is the $x$-intercept.



## WORKED EXAMPLE 3

Sketch the following curve and label the asymptote and the $x$- and $y$-intercepts.

$$y = \frac{2}{x-1}$$

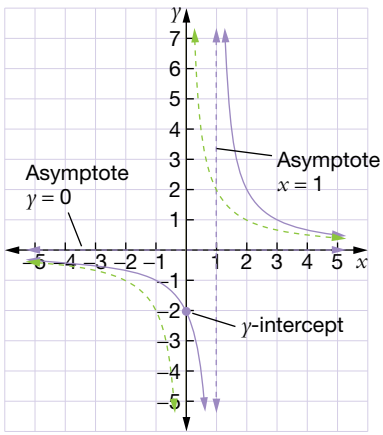
### Solution

$y = \frac{2}{x-1}$ has come from moving $y = \frac{2}{x}$ 1 unit to the right.

Now $x \neq 1$ because you cannot divide by 0.
$\therefore$ There is an asymptote at $x = 1$.

If $x = 0$, $y = -2$.
$\therefore$ $y$-intercept $= -2$

Now, as $x$ becomes large $y$ approaches 0.
$\therefore$ $y = 0$ is an asymptote.




## Exercise 7:07

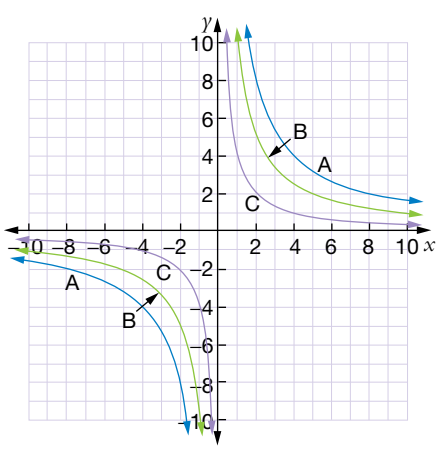
**1** Complete the table below (correct to one decimal place) for each of the following hyperbolas. Use the points obtained to graph the curve.

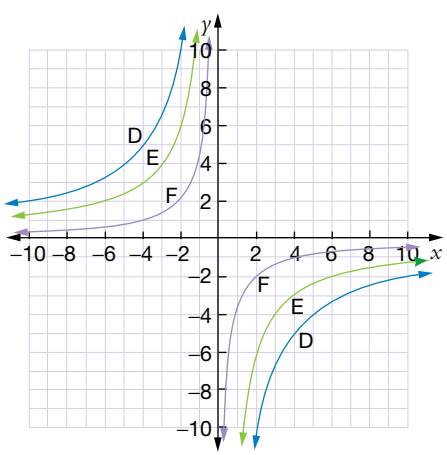
**a** $y = \frac{4}{x}$ **b** $y = -\frac{1}{x}$ (*Note:* $y = -\frac{1}{x}$ is the same as $y = \frac{-1}{x}$)

| $x$ | −8 | −4 | −2 | −1 | −0·5 | 0·5 | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | | |

**2** Match each of the graphs A to F with the following equations.

**a** $y = \frac{4}{x}$ **b** $y = -\frac{4}{x}$ **c** $y = -\frac{20}{x}$ **d** $y = \frac{10}{x}$ **e** $y = -\frac{12}{x}$ **f** $y = \frac{16}{x}$



**3** **a** What does the sign of $k$ indicate about the graph of $y = \frac{k}{x}$?

**b** What happens to the graph of $y = \frac{k}{x}$ as the value of k increases?

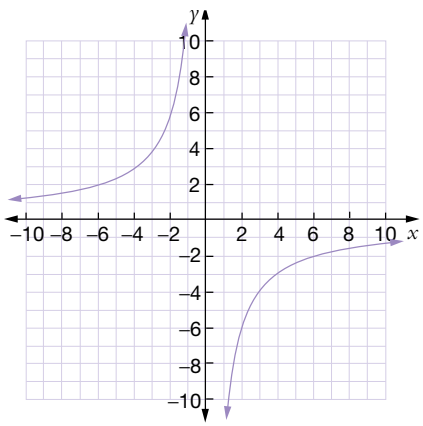
**4** **a** Does the point $(4, 2)$ lie on the hyperbola $y = \frac{8}{x}$?

**b** If the point $(3, -6)$ lies on the hyperbola $y = \frac{k}{x}$, what is the value of $k$?

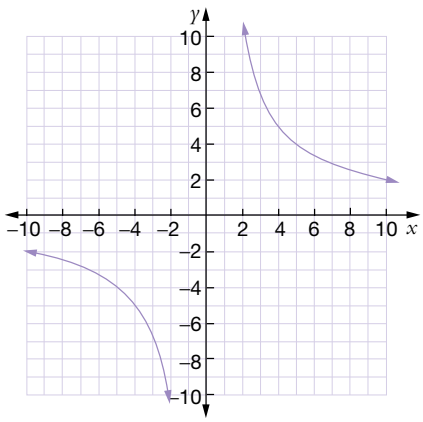
**c** The hyperbola $y = \frac{k}{x}$ passes through the point $(10, 2)$. What is the value of $k$?

**5** For each of the following, find a point that the hyperbola passes through and, by substituting this into the equation $y = \frac{k}{x}$, find the equation of the hyperbola.

**a**



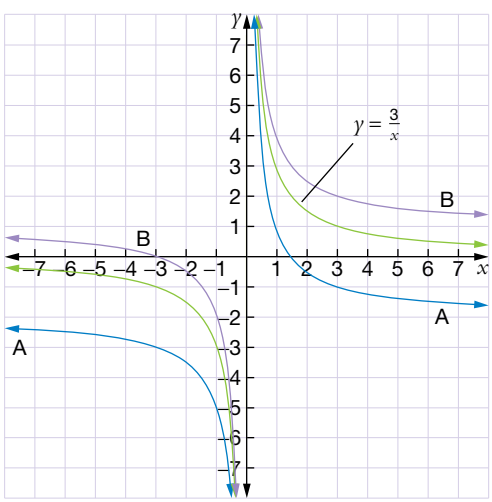

**b**

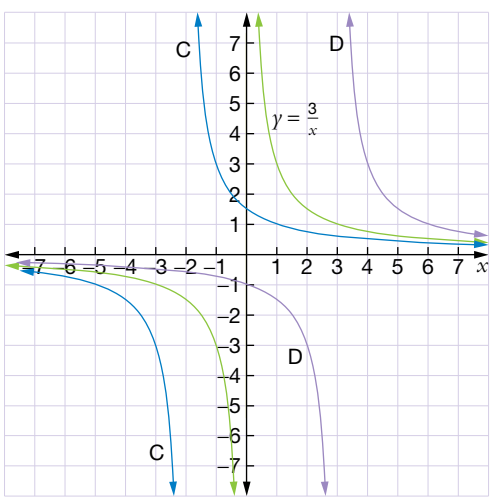



**6** Use a table of values to sketch each of the following hyperbolas. Find the asymptotes for each curve and the $x$- and $y$-intercepts where they exist.

**a** $y = \frac{1}{x} + 2$ **b** $y = \frac{3}{x} - 1$ **c** $y = \frac{-1}{x - 2}$

**7** The curves A, B, C and D have been produced by translating the curve $y = \frac{3}{x}$. What are the equations of A, B, C and D?



**8** What is the equation of the curve that results from translating the curve $y = \frac{1}{x}$:

**a** 1 unit up **b** 1 unit down **c** 1 unit to the right **d** 1 unit to the left?

**9** Write the translation that changes $y = \frac{4}{x}$ into:

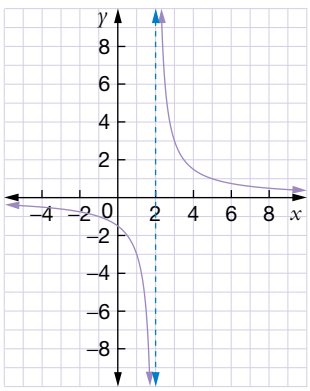
**a** $y = \frac{4}{x} - 3$ **b** $y = \frac{4}{x} + 2$ **c** $y = \frac{4}{x - 3}$ **d** $y = \frac{4}{x + 2}$

**10** Match each of the graphs A to F with the following equations.

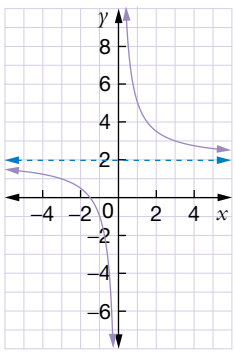
**a** $y = \frac{3}{x} + 2$ **b** $y = \frac{-3}{x + 2}$ **c** $y = -\frac{3}{x} - 2$

**d** $y = \frac{-3}{x - 2}$ **e** $y = -\frac{3}{x} + 2$ **f** $y = \frac{3}{x - 2}$

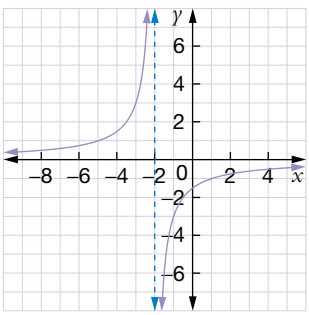
**A**



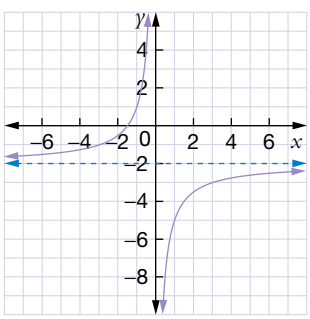

**B**



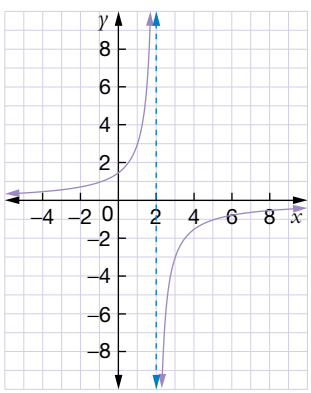

**C**



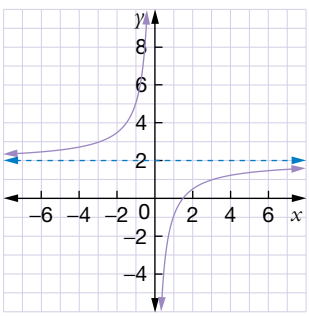

**D**




**E**




**F**

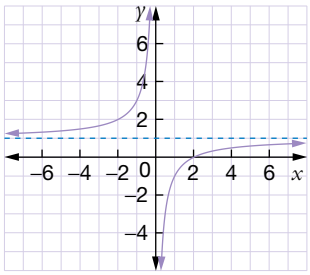



**11** Graph the following curves, showing the asymptotes and the $x$- and $y$-intercepts if they occur.

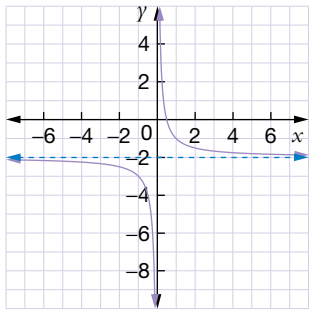
**a** $y = \frac{-2}{x - 1}$ **b** $y = \frac{2}{x} - 1$ **c** $y = \frac{1}{x + 2}$

**12** Find the equation of each of the following hyperbolas.

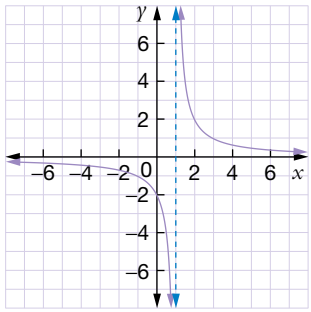
**a**




**b**



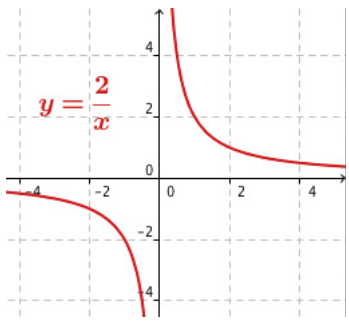

**c**




## P GEOGEBRA ACTIVITY 7:07

### HYPERBOLAS OF THE FORM $y = \frac{k}{x}$, $y = \frac{k}{x} + c$ AND $y = \frac{k}{x - b}$

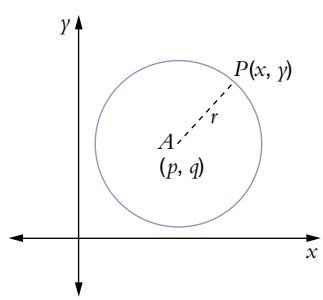
Use the checkboxes and the sliders to investigate the effect that changing the values of $k$, $c$ and $b$ has on hyperbolas of the form $y = \frac{k}{x}$, $y = \frac{k}{x} + c$ and $y = \frac{k}{x - b}$. You can then test your knowledge by finding the equation of a randomly generated curve.




# 7:08 Circles and their equations

The fact that every point on a circle is the same distance away from the centre allows us to derive its equation.

Let $P(x, y)$ be any point on a circle that has its centre at $A(p, q)$ and a radius of $r$ units.




Now, $AP = r$

Using coordinate geometry:

$$AP = \sqrt{(x - p)^2 + (y - q)^2}$$

$$\therefore \sqrt{(x - p)^2 + (y - q)^2} = r$$

$$(x - p)^2 + (y - q)^2 = r^2$$

The equation of a circle that has centre $(p, q)$ and radius $r$ is given by the equation:

$$(x - p)^2 + (y - q)^2 = r^2$$

## WORKED EXAMPLE 1

**a** What is the equation of a circle that has centre $(-3, 4)$ and radius 5?

**b** Find the centre and radius of the circle $(x - 6)^2 + (y + 7)^2 = 9$.

### Solutions

**a** The circle formula is $(x - p)^2 + (y - q)^2 = r^2$.

Here, $(p, q)$ is $(-3, 4)$ and $r = 5$.

$\therefore (x - -3)^2 + (y - 4)^2 = 5^2$

The equation of the circle is $(x + 3)^2 + (y - 4)^2 = 25$.

**b** $(x - 6)^2 + (y + 7)^2 = 9$ can be written as $(x - 6)^2 + (y - -7)^2 = 3^2$

This circle has centre $(6, -7)$ and radius 3.

## WORKED EXAMPLE 2

Find the centre and radius of the circle $x^2 + y^2 + 6x - 2y + 6 = 0$.

### Solution

Here, we write the equation in the form

$(x - p)^2 + (y - q)^2 = r^2$ by completing the squares.

$x^2 + y^2 + 6x - 2y + 6 = 0$

$x^2 + 6x + y^2 - 2y = -6$

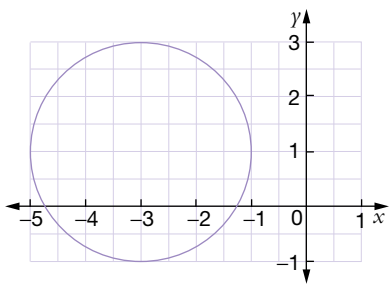
Completing each square.

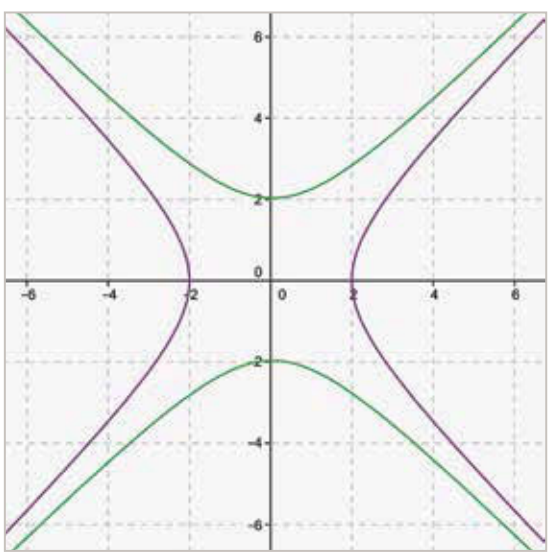
$(x^2 + 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$

$(x + 3)^2 + (y - 1)^2 = 4$

$(x - -3)^2 + (y - 1)^2 = 2^2$

This circle has centre $(-3, 1)$ and radius 2.




Changing the equation produces many interesting curves.

The curves shown here are both different types of hyperbolas. Their equations are $x^2 - y^2 = 4$ and $y^2 - x^2 = 4$.

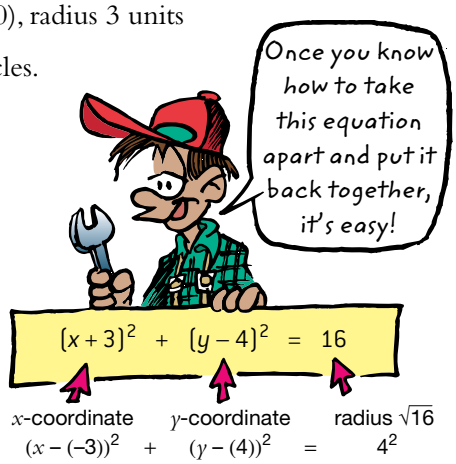
## Exercise 7:08

**1** Find the equation of the following circles.

**a** centre (1, 1), radius 7 units
**b** centre (5, 0), radius 2 units
**c** centre (−3, −5), radius 4 units
**d** centre (2, −5), radius 1 unit
**e** centre (0, 2), radius $\frac{1}{2}$ unit
**f** centre (0, 0), radius 3 units

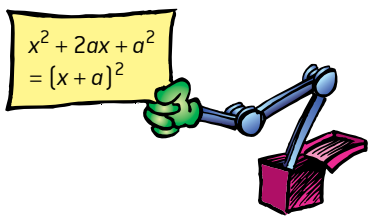
**2** Find the centre and radius of each of the following circles.

**a** $(x-2)^2 + (y-3)^2 = 64$
**b** $(x+4)^2 + (y-1)^2 = 4$
**c** $(x+3)^2 + (y+3)^2 = 9$
**d** $(x-6)^2 + (y-5)^2 = 100$
**e** $x^2 + (y+5)^2 = 16$
**f** $(x-3)^2 + y^2 = 1$
**g** $x^2 + y^2 = 81$
**h** $x^2 + y^2 = 49$
**i** $x^2 + y^2 = 11$
**j** $(x-7)^2 + (y-8)^2 = 2$




**3** Find the centre and radius of each of the following circles.

**a** $x^2 - 10x + y^2 + 8y + 32 = 0$
**b** $x^2 + y^2 + 8x - 14y = 35$
**c** $x^2 + y^2 - 18x - 20y + 60 = 0$
**d** $x^2 + y^2 - 9x + 13\frac{1}{4} = 0$




**4** On the one number plane, draw the graphs of:

**a** $x^2 + y^2 = 4$
**b** $(x+3)^2 + (y+3)^2 = 9$
**c** $(x-4)^2 + y^2 = 1$
**d** $(x-1)^2 + (y-1)^2 = 16$

**5** Find the $x$- and $y$-intercepts of the circles:

**a** $x^2 + (y-2)^2 = 9$
**b** $(x-2)^2 + (y-1)^2 = 16$

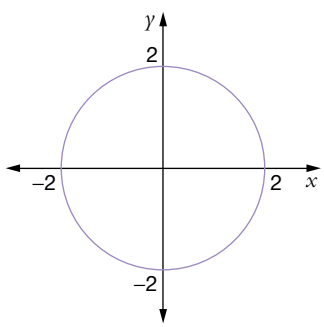
**6** **a** A circle with its centre at (3, 4) passes through the origin. What is its equation?

**b** Which of the following represents the equation of a circle?

**i** $x^2 - y^2 = 16$
**ii** $4x^2 + 4y^2 = 8$
**iii** $6x^2 + 3y^2 = 6$

**7** The circle $x^2 + y^2 = 4$ is translated to new positions. What is the equation of the circle if it is translated:

**a** (horizontally) 2 units to the right
**b** (horizontally) 2 units to the left
**c** (vertically) 2 units up
**d** (vertically) 2 units down?




**8** Write the new equation if the circle with equation $(x-2)^2 + (y+2)^2 = 4$ is translated 2 units to the right and then 1 unit down.

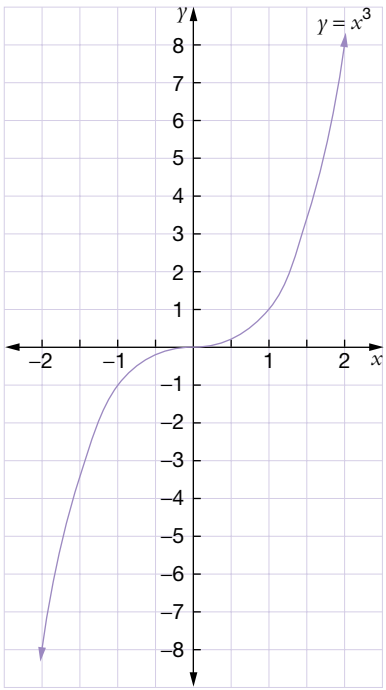
# 7:09 Curves of the form $y = ax^3$ and $y = ax^3 + d$

Curves of the form $y = ax^3 + d$ are called *cubics* because of the $x^3$ term. The simplest cubic graph is $y = x^3$.

As with other graphs, a table of values can be used to produce the points on the curve.

$y = x^3$

| $x$ | −2 | −1·5 | −1 | −0·5 | 0 | 0·5 | 1 | 1·5 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | −8 | −3·4 | −1 | −0·1 | 0 | 0·1 | 1 | 3·4 | 8 |




Features of $y = x^3$:

- It is an increasing curve.
- As $x$ increases, the value of $x^3$, and hence $y$, becomes large very quickly. This means it is difficult to fit the points on a graph.
- When $x$ is positive, $x^3$, and hence $y$, is positive.
  When $x$ is negative, $x^3$, and hence $y$, is negative.
  When $x$ is zero, $x^3$ is zero.
- The $x$- and $y$-intercepts are both zero.

In this section, the relationship between the curve $y = x^3$ and the curves $y = ax^3$ and $y = ax^3 + d$ for various values of $a$ and $d$ will be investigated.

## Exercise 7:09

**1** **a** Match each of the equations below with the graphs A, B and C.

**i** $y = 2x^3$

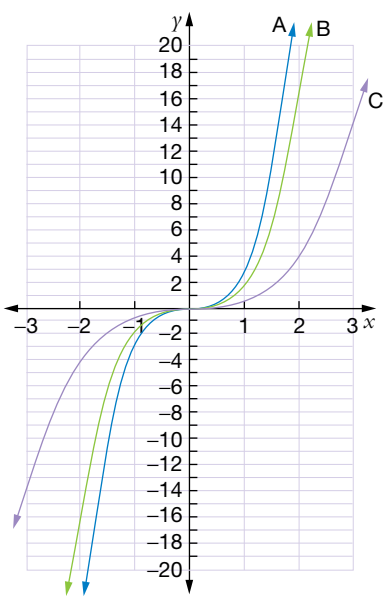
**ii** $y = \frac{1}{2}x^3$

**iii** $y = 3x^3$

**b** Which graph increases the fastest? (Which is the steepest?)

**c** Which graph increases the slowest?

**d** How can you tell which graph is the steepest by looking just at the equations?



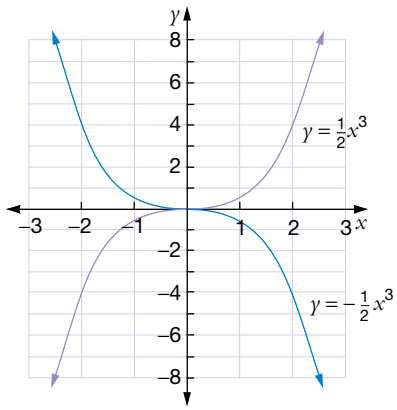
**2** State which curve in each pair is steeper.

**a** $y = x^3$ or $y = 3x^3$ **b** $y = x^3$ or $y = \frac{1}{2}x^3$ **c** $y = 2x^3$ or $y = 3x^3$

**3** The graphs of $y = \frac{1}{2}x^3$ and $y = -\frac{1}{2}x^3$ are shown.

**a** How are the graphs related?

**b** What is the effect on $y = ax^3$ of the sign of $a$?



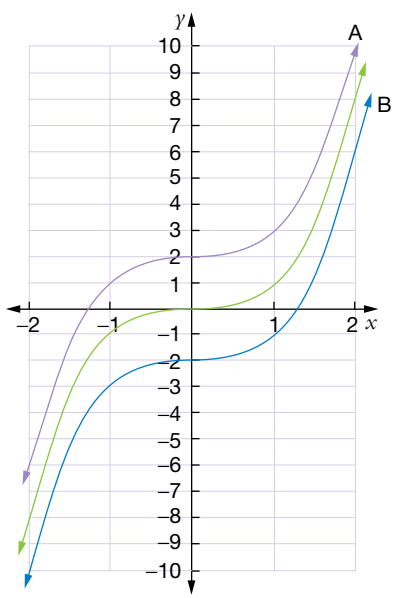

**4** From your results so far, you should have noticed that all the curves are either *decreasing* or *increasing*. Without sketching, state whether the following are increasing or decreasing.

**a** $y = 4x^3$ **b** $y = -10x^3$ **c** $y = 0{\cdot}25x^3$

**d** $y = \frac{1}{5}x^3$ **e** $y = -\frac{1}{3}x^3$ **f** $y = -\frac{x^3}{5}$

**5** **a** Copy and complete the tables of values for the three curves $y = x^3$, $y = x^3 + 2$ and $y = x^3 - 2$.

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $x^3$ | −8 | −1 | 0 | 1 | 8 |
| $x^3 + 2$ | | | | | |
| $x^3 - 2$ | | | | | |




**b** What are the equations of curves A and B?

**c** How is the graph of $y = x^3 + 2$ related to the graph of $y = x^3$?

**d** How is the graph of $y = x^3 - 2$ related to the graph of $y = x^3$?

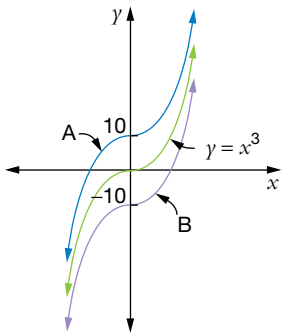
**6** Given the graph of $y = \frac{1}{2}x^3$, describe how you would obtain the graphs of:

**a** $y = \frac{1}{2}x^3 + 1$ **b** $y = \frac{1}{2}x^3 - 1$ **c** $y = \frac{1}{2}x^3 + 2$

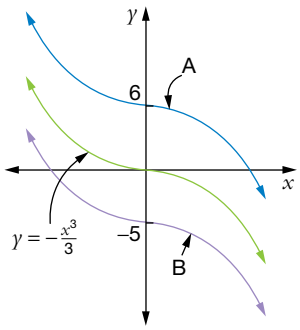
**d** $y = \frac{1}{2}x^3 - 2$ **e** $y = -\frac{1}{2}x^3$ **f** $y = -\frac{1}{2}x^3 + 1$

**7** In each diagram the two curves A and B were obtained by moving the other curve up or down. Give the equations of the curves A and B.

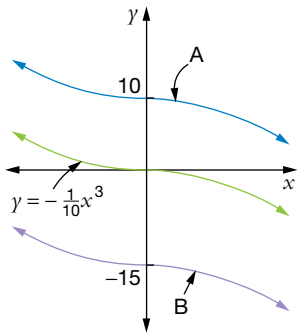
**a**




**b**




**c**




**8** Sketch each pair of graphs on the same number plane.

**a** $y = 2x^3$
$y = 2x^3 - 2$

**b** $y = -x^3$
$y = -x^3 + 2$

**c** $y = \frac{1}{4}x^3$
$y = \frac{1}{4}x^3 + 4$

**9** Sketch each pair of graphs on the same number plane.

**a** $y = x^3 + 1$
$y = -x^3 + 1$

**b** $y = 2x^3 + 1$
$y = x^3 + 1$

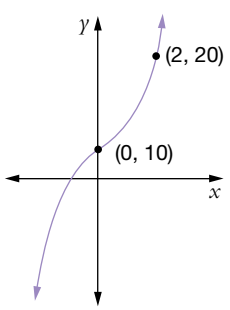
**10** From the list of equations, write the letter or letters corresponding to the equations of the curves:

**a** that can be obtained by moving $y = x^3$ up or down
**b** that are the same shape as A
**c** that are decreasing
**d** that pass through $(0, 0)$
**e** that can be obtained from the curve $y = -\frac{1}{3}x^3$ by reflection in the $y$-axis
**f** that have the largest $y$-intercepts.

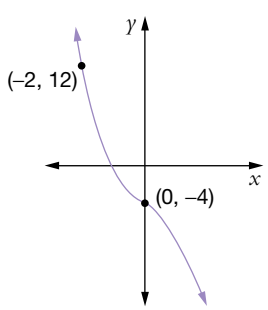
**A** $y = 2x^3$
**B** $y = x^3 - 1$
**C** $y = 3 - \frac{1}{3}x^3$
**D** $y = 2x^3 - 1$
**E** $y = 3 + 2x^3$
**F** $y = -\frac{1}{3}x^3$
**G** $y = \frac{1}{3}x^3$
**H** $y = 2 + x^3$
**I** $y = -\frac{1}{3}x^3 - 2$

**11** Given that each of the graphs is of the form $y = ax^3 + d$, find its equation.

**a**




**b**

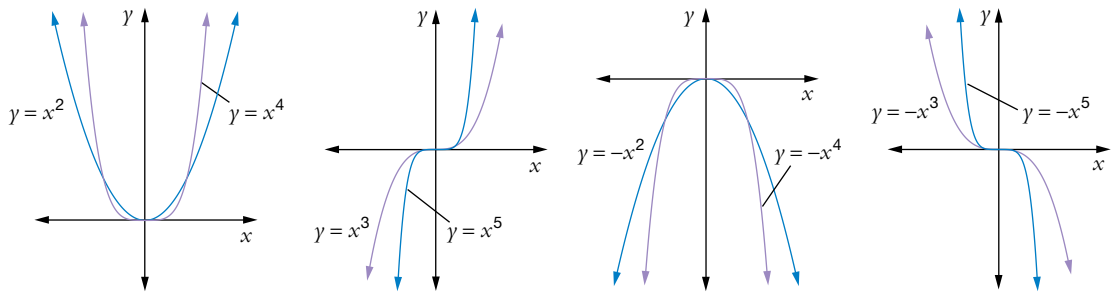


**12** For equations of the form $y = ax^3 + d$, describe the effect on the graph of different values of $a$ and $d$.

# 7:10 Curves of the form $y = ax^n$ and $y = ax^n + d$

## Curves of the form $y = ax^n$

Curves of the form $y = ax^n$ include these basic shapes. You have already graphed the parabolas and cubics.




Graphing curves of the form $y = ax^n$ is simple if you recognise the relationship between the signs of $x$ and $y$ for different values of $a$ and $n$.

- To find the sign of $y$ when $y = ax^n$ we need to realise that:
  If $n$ is even: $x^n$ is always positive (except for $x = 0$).
  If $n$ is odd: $x^n$ is positive when $x$ is positive.
  $x^n$ is negative when $x$ is negative.
  The signs of $a$ and $x^n$ will then determine the sign of $y$.
- As $x$ becomes smaller, $ax^n$ (and therefore $y$) becomes smaller.
  As $x$ becomes larger, $ax^n$ (and therefore $y$) becomes larger.
- If $x = 0$, $y = 0$; and if $y = 0$, $x = 0$. Hence, the curve only crosses the $x$-axis once, at the point $(0, 0)$.

When considering whether a number is large or small, we are concerned only with its size and not its sign.

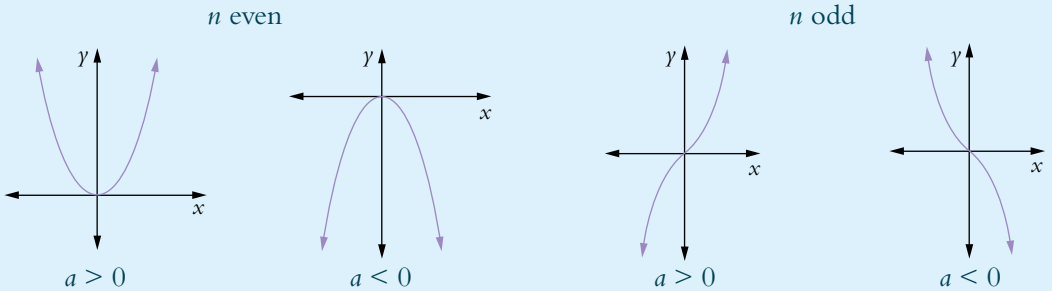
Numbers close to zero are small in size.

Numbers far from zero are large in size.

From this information we can deduce the following.

- Curves with an even value of $n$ will have shapes like parabolas ($y = ax^2$).
- Curves with an odd value of $n$ will have shapes like cubics ($y = ax^3$).

Curves of the form $y = ax^n$



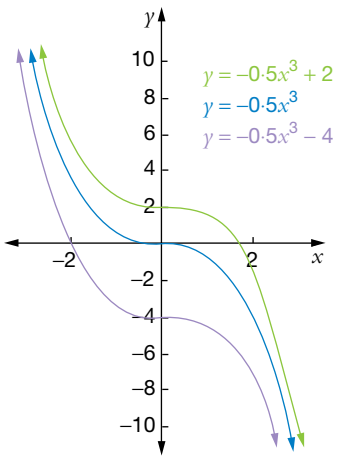
- As $n$ increases, the steepness of the curve increases, e.g. $y = x^4$ will be steeper than $y = x^2$. The curve with the higher value of $n$ will be above the other curve for all values of $x$ except from $-1$ to 1.
- As we have already seen in 7:09 for the curve $y = ax^3$, changing $a$ changes the 'steepness' of the curve. Hence, for the same value of $n$, a larger value of a will result in a steeper curve. e.g. $y = 2x^5$ will be steeper than $y = x^5$.

## Curves of the form $y = ax^n + d$

Curves of the form $y = ax^n + d$ can be obtained by translating the curve $y = ax^n$ up or down.

- If $d$ is positive, translate it up $d$ units.
- If $d$ is negative, translate it down $d$ units.

This is illustrated in the diagram on the right for the curves $y = -0{\cdot}5x^3$, $y = -0{\cdot}5x^3 + 2$ and $y = -0{\cdot}5x^3 - 4$.



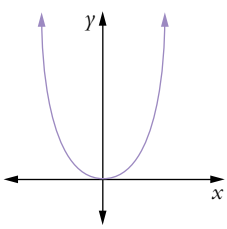

## WORKED EXAMPLES

Sketch each of the following curves.

**1** $y = \frac{1}{2}x^4$ **2** $y = -2x^5$ **3** $y = -1 - 2x^6$

### Solutions

**1**

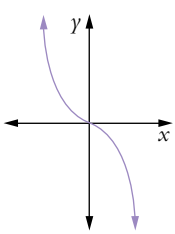



As $\frac{1}{2}x^4$ is positive for all $x$ (except $x = 0$), then $y$ is positive (except when $x = 0$).

When $x$ is small (close to 0), $y$ is small.

As $x$ becomes larger (moves away from 0), $\frac{1}{2}x^4$, and hence $y$, becomes larger.

**2**



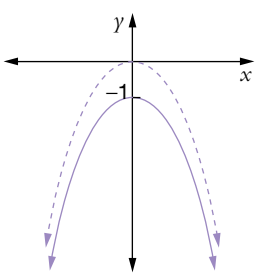

When $x$ is positive, $-2x^5$ is negative and $y$ is negative.

When $x$ is negative, $-2x^5$ is positive and $y$ is positive.

When $x$ is small (close to 0), $y$ is small.

As $x$ becomes larger, $y$ becomes larger.

**3**




$y = -1 - 2x^6$

$\therefore\ y = -2x^6 - 1$

This is the curve $y = -2x^6$ (shown as a dotted line) moved down 1 unit to get $y = -2x^6 - 1$.

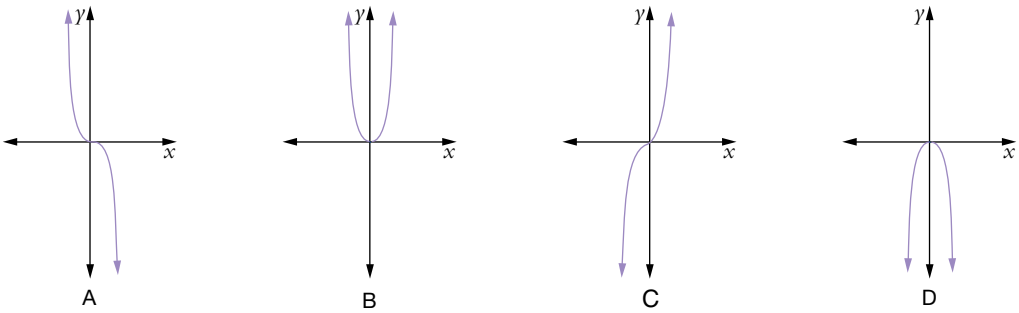
# Exercise 7:10

P Foundation worksheet 7:10
Curves of the form $y = ax^n$ and $y = ax^n + d$

**1** Choose which of the curves A, B, C or D below is the best sketch for each of the following.

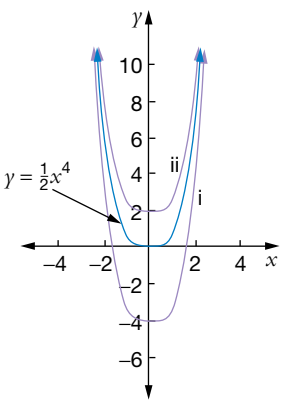
**a** $y = x^4$ **b** $y = -x^5$ **c** $y = 2x^7$ **d** $y = -3x^6$ **e** $y = 4x^4$

**f** $y = -4x^5$ **g** $y = \frac{x^5}{10}$ **h** $y = -\frac{1}{10}x^7$ **i** $y = -2x^8$

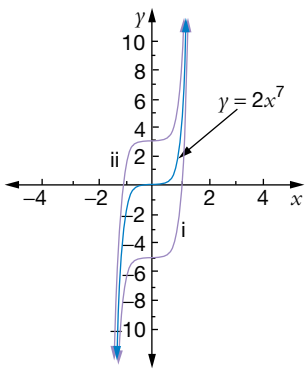



**2** In each diagram, the two curves labelled **i** and **ii** were obtained by moving the blue curve up or down. Give the equation of each curve.

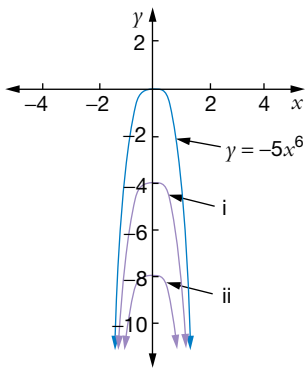
**a**




**b**

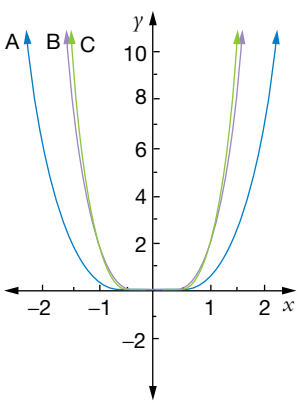



**c**



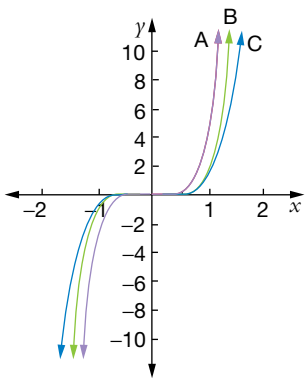

**3** In each of the following, match the curves A, B and C with the equations below.

**a**



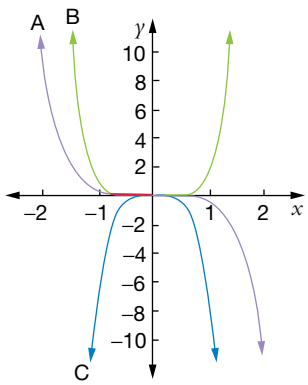

i $y = 2x^4$
ii $y = 0{\cdot}5x^4$
iii $y = 2x^6$

**b**




i $y = x^5$
ii $y = x^7$
iii $y = 4x^5$

**c**




i $y = -5x^4$
ii $y = x^6$
iii $y = -0{\cdot}25x^5$

**4** Sketch the following on separate number planes.

**a** $y = \frac{1}{2}x^5$ **b** $y = 2x^4$

**c** $y = -2x^6$ **d** $y = x^5 - 1$

**e** $y = x^6 + 1$ **f** $y = -2x^4 + 4$

When graphing $y = ax^n + d$, look first at $n$, then $a$, then $d$.

**5** Sketch each pair of graphs on the same number plane.

**a** $y = x^4$, $y = x^6$ **b** $y = x^4$, $y = -x^4$ **c** $y = x^5$, $y = 1 + x^5$

**d** $y = x^6 + 1$, $y = x^6 - 1$ **e** $y = 2x^5 + 1$, $y = 2x^3 + 1$ **f** $y = 1 - x^6$, $y = -1 - x^6$

**6** **a** How do each of the numbers in the equation $y = -2x^4 + 1$ determine the features of the curve?

**b** Describe the shape and features of the curve $y = 2x^3 - 3$.

**c** What is the relationship between the curves $y = 2x^5$ and $y = -2x^5$?

**7** From the curves of the form $y = ax^n$, sketch a possible curve, given that:

**a** $a$ is positive and $n$ is even **b** $a$ is negative and $n$ is even

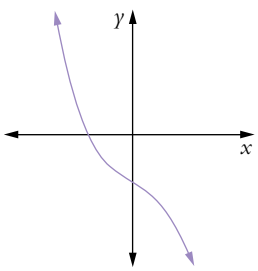
**c** $a$ is positive and $n$ is odd **d** $a$ is negative and $n$ is odd.

**8** From the curves of the form $y = ax^n + d$, sketch a possible curve, given that:

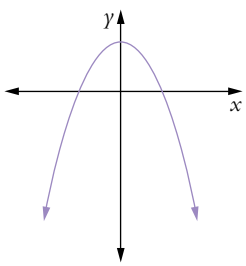
**a** $a$ is positive, $n$ is odd and $d$ is negative **b** $a$ is negative, $n$ is even and $d$ is positive.

**9** Each of the curves below has the form $y = ax^n + d$. For each curve, state the signs of $a$ and $d$, and state whether $n$ is even or odd.

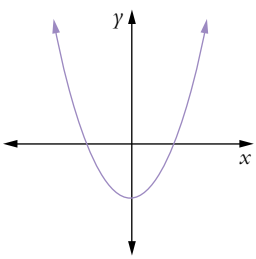
**a**

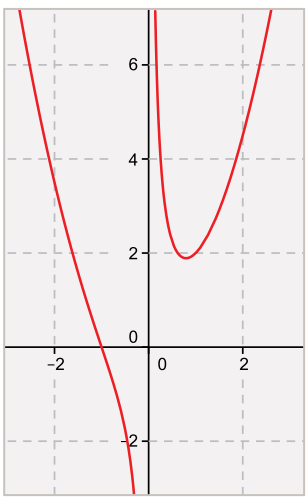



**b**




**c**




The curve shown has the equation $y = x^2 + \frac{1}{x}$. It is a hybrid of a parabola and an hyperbola.

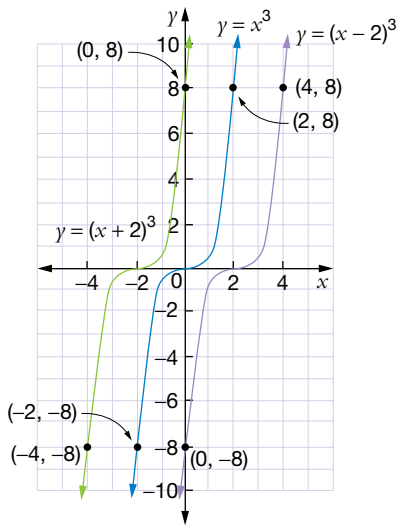
It is called a trident and was studied by Isaac Newton.

# 7:11 Curves of the form $y = ax^n$ and $y = a(x - r)^n$

The diagram shows the graphs of:
$y = x^3$, $y = (x - 2)^3$ and $y = (x + 2)^3$
The curves are all the same shape.

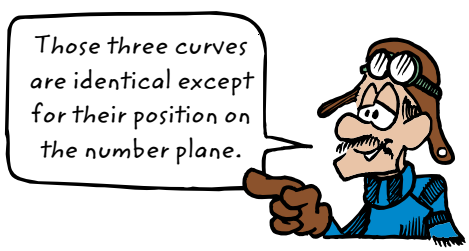
- If $y = x^3$ is translated 2 units to the right the curve $y = (x - 2)^3$ is obtained.
- If $y = x^3$ is translated 2 units to the left the curve $y = (x + 2)^3$ is obtained.

The above case is an example of the general result.




If the curve $y = ax^n$ is moved (horizontally):

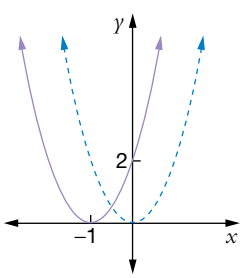
- $r$ units to the right, the equation of the new curve is $y = a(x - r)^n$
- $r$ units to the left, the equation of the new curve is $y = a(x + r)^n$.




## WORKED EXAMPLES

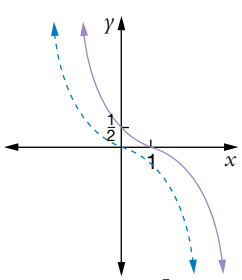
Sketch each of the following curves.

**1** $y = 2(x + 1)^4$




$y = 2(x + 1)^4$ has been produced by moving the curve $y = 2x^4$ 1 unit to the left.

**2** $y = -\frac{1}{2}(x - 1)^5$




$y = -\frac{1}{2}(x - 1)^5$ has been produced by moving the curve $y = -\frac{1}{2}x^5$ 1 unit to the right.

You must know how to graph $y = ax^n$ before you can graph $y = a(x - r)^n$ or $y = a(x + r)^n$.

# Exercise 7:11

P Foundation worksheet 7:11
Curves of the form $y = ax^n$ and $y = a(x - r)^n$

**1**

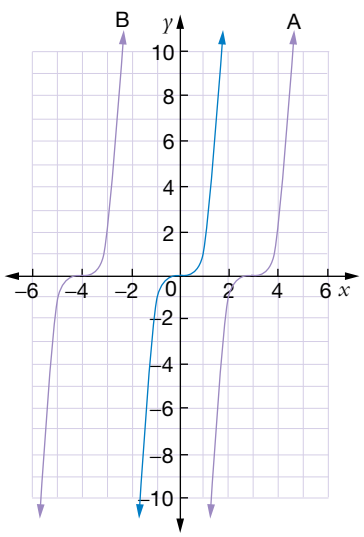



Diagram 1

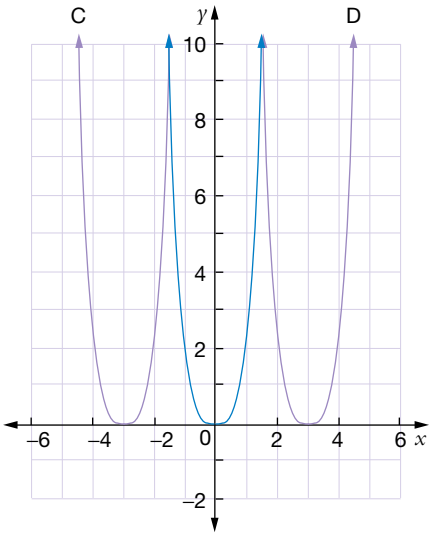



Diagram 2

**a** The blue curve in Diagram 1 is $y = x^5$. It has been translated horizontally to produce curves A and B. What are their equations?

**b** The blue curve in Diagram 2 is $y = 2x^4$. What curve has the equation $y = 2(x - 3)^4$?

**2** Write the equation of the curve that is obtained by translating the curve $y = x^4$ (horizontally):

**a** 4 units to the left **b** 4 units to the right **c** 1 unit to the right **d** $\frac{1}{2}$ unit to the left.

**3** Describe how $y = -2x^5$ must be translated to produce curves with the equations:

**a** $y = -2(x - 1)^5$ **b** $y = -2(x + 3)^5$ **c** $y = -2(x - \frac{1}{2})^5$

**4** Make sketches on separate number planes of:

**a** $y = -(x + 1)^4$ **b** $y = (x + 1)^4$ **c** $y = (x - 1)^4$ **d** $y = -(x - 1)^4$

**5** Sketch on separate number planes the following curves.

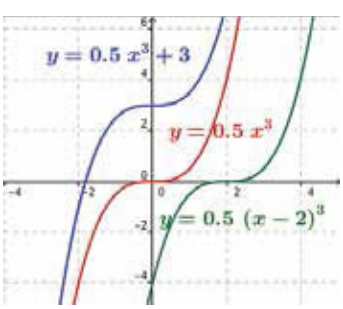
**a** $y = 3(x + 2)^2$ **b** $y = 2(x - 3)^5$ **c** $y = -2(x - 4)^4$ **d** $y = -(x + 1)^6$

**6** Make sketches of each pair of curves on the same number plane.

**a** $y = -(x - 1)^2$, $y = (x + 1)^2$ **b** $y = (x - 3)^3$, $y = -(x - 3)^3$ **c** $y = 2(x - 3)^3$, $y = 2(x - 3)^5$ **d** $y = 2(x + 1)^2$, $y = 2(x + 1)^4$

## P GEOGEBRA ACTIVITY 7:11 CURVES INVOLVING POWERS

Use the checkboxes and the sliders to investigate the effect that changing the values of $a$, $n$, $k$ and $r$ has on curves of the form $y = ax^n$, $y = ax^n + k$ and $y = a(x - r)^n$. Then you can test your knowledge by finding the equation of a randomly generated curve.



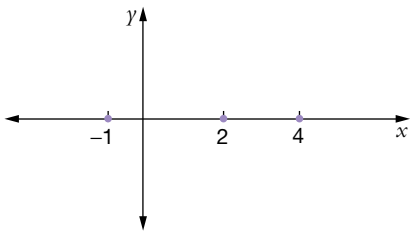
# 7:12 Curves of the form $y = a(x - r)(x - s)(x - t)$

In the work so far, the sign analysis for $x$ and $y$ has been of great importance. In this section, it is also important, though not as simple.

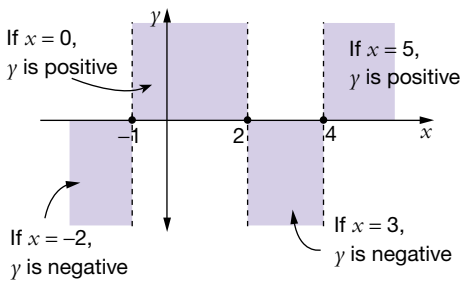
## Discussion

Consider the example $y = 3(x - 2)(x + 1)(x - 4)$.

- Now, $x$ can take any value and for every $x$ value there is a corresponding $y$-value. Hence, the curve has no gaps.
- First, find the $x$–intercepts of the curve.
  If $y = 0$ then:
  $3(x - 2)(x + 1)(x - 4) = 0$
  $\therefore\ x - 2 = 0$ or $x + 1 = 0$ or $x - 4 = 0$
  $\therefore\ x = 2$ or $-1$ or $4$
  Plot these $x$–intercepts.

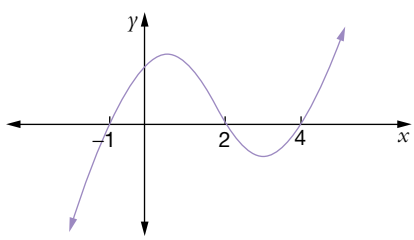



- Now, for all the other values of $x$, $y$ is either positive or negative. The three $x$-intercepts divide the $x$-axis into four sections. In each of these sections the $y$ values will be either positive or negative. Testing a point in each of the sections will determine whether that section is positive or negative.




- Also, as $x$ increases past 4, $y$ increases, and as $x$ decreases below $-1$, $y$ decreases.
- Putting all the information together allows us to sketch the curve.

*Testing a value:*
When $x = 5$, $y = 3(+)(+)(+)$ positive
When $x = 3$, $y = 3(+)(+)(-)$ negative




$y = 3(x - 2)(x + 1)(x - 4)d$

These ideas are explored further in the following examples.

## WORKED EXAMPLES

Make sketches of the following curves.

1 $y = 2(x - 1)(x - 4)(x + 2)$

2 $y = -2(x - 1)(x - 4)(x + 2)$

3 $y = (x - 1)^2(x - 4)$

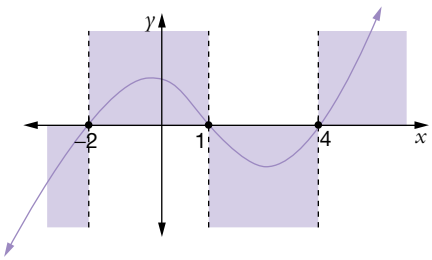
### Solutions

1 $y = 2(x - 1)(x - 4)(x + 2)$

- The curve has $x$-intercepts at $x = 1$, $x = 4$ and $x = -2$.
- Testing values of $x$ shows that the curve is in the coloured regions.

| $x$ | −3 | −2 | 0 | 1 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| **sign of $y$** | (−) | 0 | (+) | 0 | (−) | 0 | (+) |

- Considering the size of $y$ when $x$ is greater than 4 and less than −2 gives the correct shape.

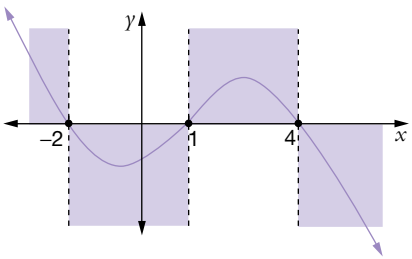



2 $y = -2(x - 1)(x - 4)(x + 2)$

- The curve has $x$-intercepts at $x = 1$, $x = 4$ and $x = -2$.
- Testing values of $x$ shows that the curve is in the coloured regions.

| $x$ | −3 | −2 | 0 | 1 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| **sign of $y$** | (+) | 0 | (−) | 0 | (+) | 0 | (−) |

- Considering the size of $y$ when $x$ is greater than 4 and less than −2 gives the correct shape.

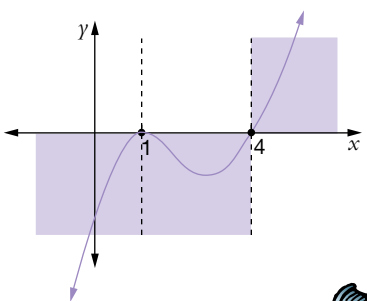


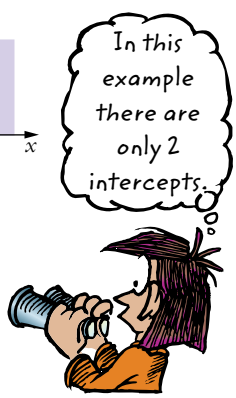

3 $y = (x - 1)^2(x - 4)$

- $x$–intercepts occur at $x = 1$ and $x = 4$.
- Testing values shows that the curve is in the coloured regions.

| $x$ | 0 | 1 | 2 | 4 | 5 |
|---|---|---|---|---|---|
| **sign of $y$** | (−) | 0 | (−) | 0 | (+) |

- Considering the size of $y$ when $x$ is greater than 4 and less than 1 gives the correct shape.



# Exercise 7:12

P Foundation worksheet 7:12
Curves of the form
$y = a(x - r)(x - s)(x - t)$

**1** Complete the sign analysis tables for each of the given curves.

**a** $y = 2(x + 4)(x + 2)(x + 1)$

| $x$ | −5 | −4 | −3 | −2 | −1·5 | −1 | 0 |
|---|---|---|---|---|---|---|---|
| $y$ | | 0 | | 0 | | 0 | |

**b** $y = -4(x + 2)(x - 2)(x - 5)$

| $x$ | −3 | −2 | 0 | 2 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $y$ | | 0 | | 0 | | 0 | |

**c** $y = \frac{1}{2}(x - 1)^2(x - 4)$

| $x$ | 0 | 1 | 2 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | | 0 | | 0 | |

**d** $y = -(x + 1)(x - 2)^2$

| $x$ | −2 | −1 | 0 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | | 0 | | 0 | |

**2** Find the $x$-intercepts for each of the following curves.

**a** $y = (x - 1)(x - 2)(x - 4)$
**b** $y = (1 - x)(x + 2)(4 - x)$
**c** $y = (x - 2)^2(x + 5)$
**d** $y = 2x(x + 4)(x - 2)$

To find where a curve crosses the $x$-axis (i.e. $x$-intercepts), substitute $y = 0$.

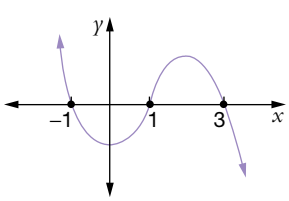
**3** Make sketches of the following curves on separate number planes.

**a** $y = (x - 2)(x + 2)(x - 4)$
**b** $y = -(x - 2)(x + 2)(x - 4)$
**c** $y = 2(x - 1)(x - 3)^2$
**d** $y = -2(x - 1)(x - 3)^2$

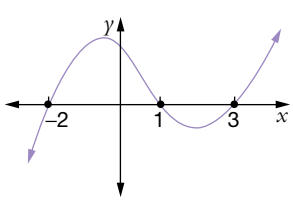
**4** **a** Find the $y$-intercepts of the curves in Question **3**.
**b** How can the $y$-intercepts be used to check your sketches?

**5** Write a possible equation for each of the curves shown.

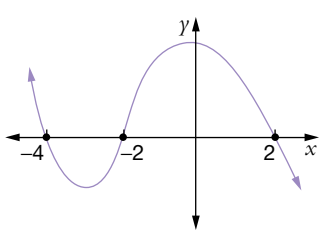
**a**



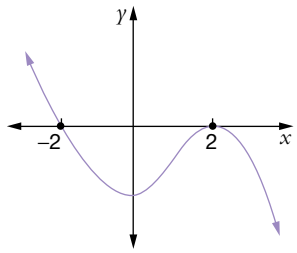

**b**



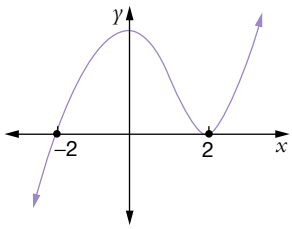

**c**



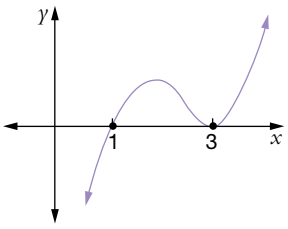
**d**




**e**




**f**




**6** What additional information would have been needed in Question **5** to find the actual equation of the curve?

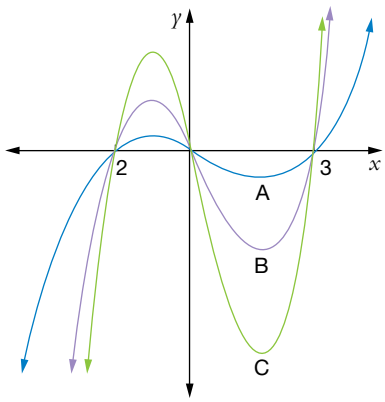
**7** **a** Match each of the curves to the following equations.

**i** $y = x(x - 3)(x + 2)$

**ii** $y = 2x(x - 3)(x + 2)$

**iii** $y = 0{\cdot}25x(x - 3)(x + 2)$

**b** On the same number plane, sketch the curves:
$y = -x(x - 3)(x + 2)$ and $y = -2x(x - 3)(x + 2)$



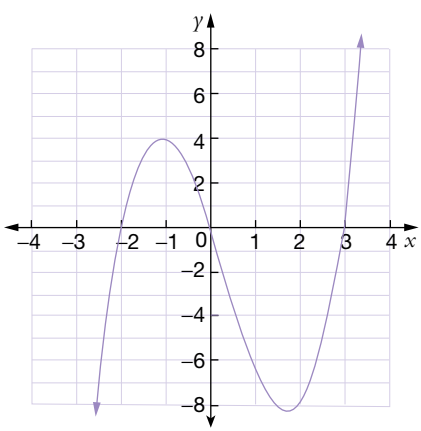

**8** The equation of the curve shown is $y = x(x + 2)(x - 3)$. Give the equation of the curve that would result if this curve was:

**a** translated 1 unit to the right

**b** translated 2 units to the left

**c** reflected in the $x$-axis

**d** reflected in the $y$-axis.




**9** **a** A cubic curve is known to have $x$-intercepts of $-1$, 1 and 4 and $y$-intercept of 8. What is its equation?

**b** A cubic curve has $x$-intercepts of 2 and $-2$ and passes through the point $(3, -10)$. What is its equation?

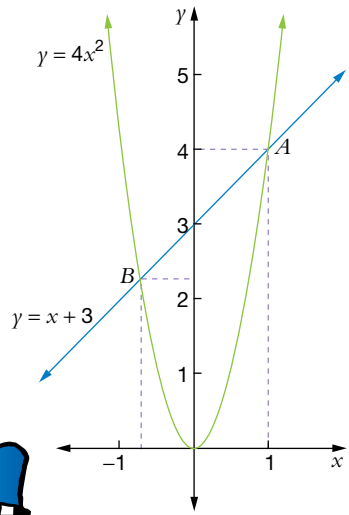
**10** Sketch the graph of $y = (x - r)(x - s)(x - t)$ if $r = s = t$.

# 7:13 The intersection of graphs

In this section, both graphical and algebraic methods will be used to find the point or points of intersection of a line with a parabola, circle or hyperbola.

### Graphical method

- In the diagram, the points of intersection of the parabola $y = 4x^2$ and the line $y = x + 3$ are $A$ and $B$. It would appear from the graph that: $A$ is the point $(1, 4)$ and $B$ is the point $(-0{\cdot}7, 2{\cdot}2)$.
- However, the only way to be certain that these are the exact points of intersection is to substitute the coordinates into the original equations and see if true statements result.




Check $A(1, 4)$:

For $y = 4x^2$ and $y = x + 3$

$4 = 4 \times 1^2$ $\quad$ $4 = 1 + 3$

Both of these statements are true.

$\therefore$ $A(1, 4)$ is the exact point of intersection.

Check $B(-0{\cdot}7, 2{\cdot}2)$:

For $y = 4x^2$ and $y = x + 3$

$2{\cdot}2 = 4 \times (-0{\cdot}7)^2$ $\quad$ $2{\cdot}2 = -0{\cdot}7 + 3$

Both of these statements are false.

$\therefore$ $B(-0{\cdot}7, 2{\cdot}2)$ only approximates the point of intersection. The accuracy of the estimate can be improved by trial and error.

Even though the graphical method is slow and often cannot give us an exact solution, it is still a valuable technique.

To find points of intersection graphically:

1. Draw the graphs.
2. Read off coordinates.
3. Check the solutions by substitution.

### Algebraic method

This method is based on the fact that at a point of intersection the $x$-coordinates on both curves are equal and the $y$-coordinates on both curves are equal.

So, for the curves $y = 4x^2$ and $y = x + 3$:

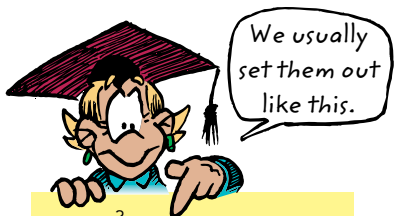
If the $y$-coordinates are equal, then:

$$4x^2 = x + 3$$
$$4x^2 - x - 3 = 0$$
$$(4x + 3)(x - 1) = 0$$
$$x = -\tfrac{3}{4} \text{ or } 1$$

Substitution of these values into either $y = 4x^2$ or $y = x + 3$ gives:

$y = 4$ (when $x = 1$)

$y = 2\frac{1}{4}$ (when $x = -\frac{3}{4}$)




$y = 4x^2$ 1

$y = x + 3$ 2

Subsitute 1 in 2

$4x^2 = x + 3$

$4x^2 - x - 3 = 0$

$(4x + 3)(x - 1) = 0$

$x = \frac{-3}{4}$ or 1

Substitute in 2

When $x = \frac{-3}{4}$, $y = 2\frac{1}{4}$

$x = 1$, $y = 4$

$\therefore$ Points of intersection are:

$(1, 4)$ and $(-\frac{3}{4}, 2\frac{1}{4})$

$\therefore$ Points of intersection are:

$(1, 4)$ and $(-\frac{3}{4}, 2\frac{1}{4})$

These should be checked by substituting each point into both equations.

The points of intersection can also be found using the fact that the $x$-coordinates are equal, but in this case it requires more work.

$y = 4x^2$ 1

$y = x + 3$ 2

Rearranging these equations gives:

$x^2 = \frac{y}{4}$ 3

$x = y - 3$ 4

Substituting 4 into 3 gives:

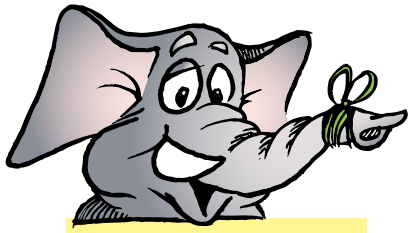
$$(y - 3)^2 = \frac{y}{4}$$
$$4(y - 3)^2 = y$$
$$4(y^2 - 6y + 9) = y$$
$$4y^2 - 24y + 36 = y$$
$$y^2 - 25y + 36 = 0$$
$$(4y - 9)(y - 4) = 0$$
$$y = \frac{9}{4} \text{ or } 4$$

Substituting in 1 gives:

When $y = \frac{9}{4}$, $x = \pm\frac{3}{4}$

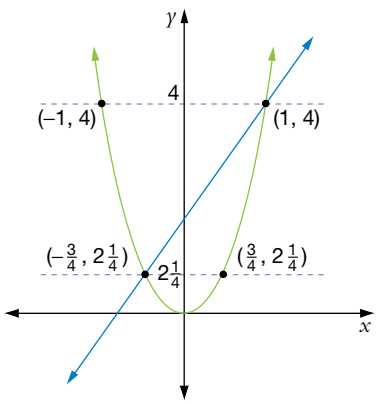
$y = 4$, $x = \pm 1$

Checking in both 1 and 2 shows that the points of intersection are $(1, 4)$ and $(-\frac{3}{4}, 2\frac{1}{4})$.



It is important to check all possible solutions in both of the original equations.

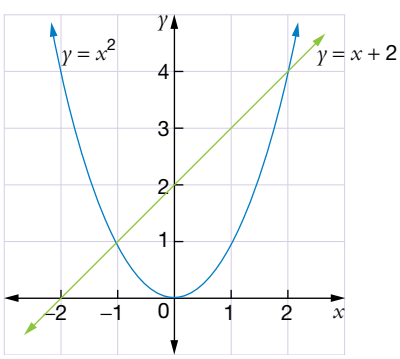
The sketch below shows that there are only two solutions.



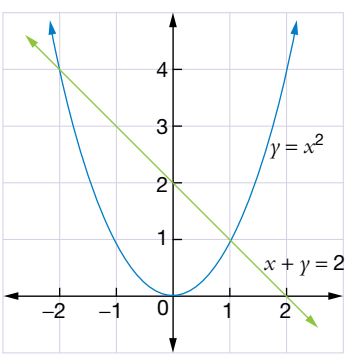

## Exercise 7:13

**1** Find the points of intersection of the graphs in each part. Check that the coordinates of each point of intersection satisfy both equations.

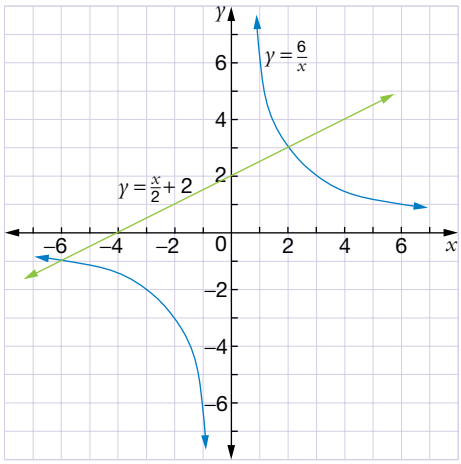
**a**



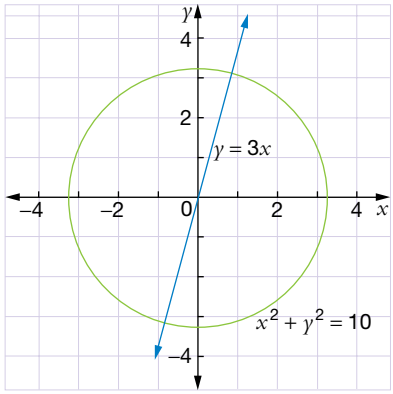

**b**



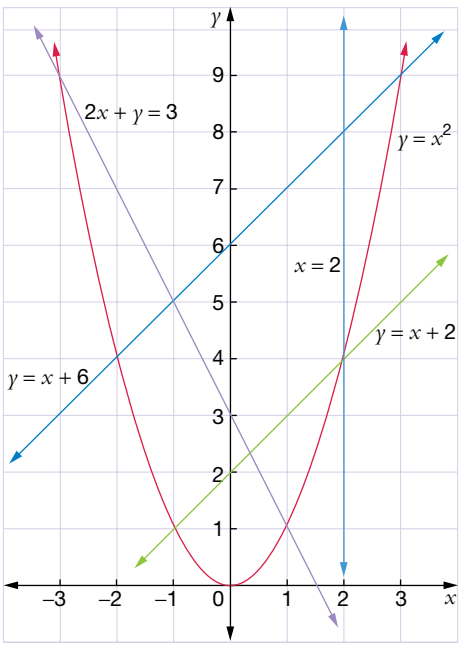
**c**




**d**




**2**



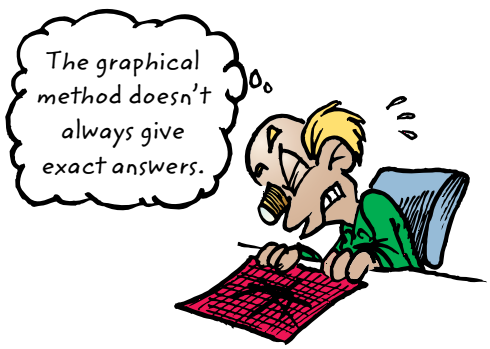

Use this diagram to find the point or points of intersection of the parabola $y = x^2$ with the line:

**a** $y = x + 6$

**b** $x = 2$

**c** $y = x + 2$

**d** $2x + y = 3$

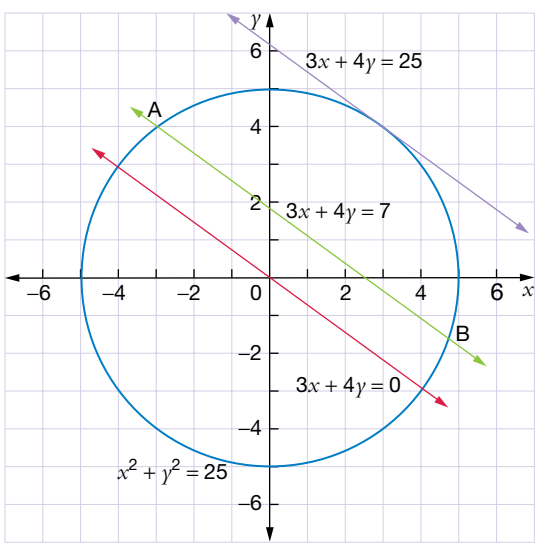



**3** Use the diagram to answer the following questions.

**a** What are the points of intersection of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 0$?

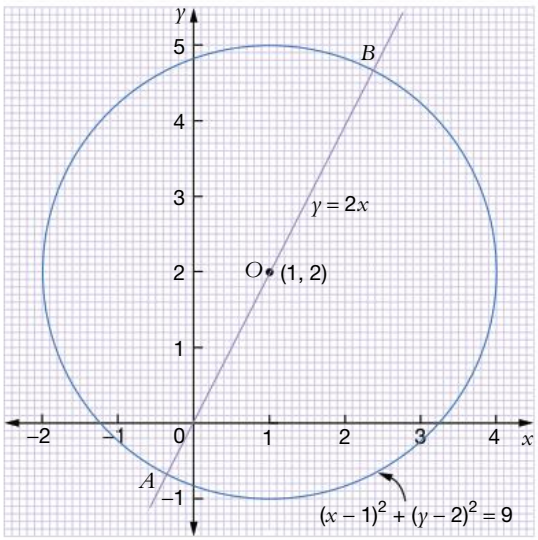
**b** The line $3x + 4y = 25$ is a tangent to the circle $x^2 + y^2 = 25$. What is the point of intersection?

**c** The line $3x + 4y = 7$ meets the circle at $A(-3, 4)$ and at $B$. Use the graph to estimate the coordinates of $B$.

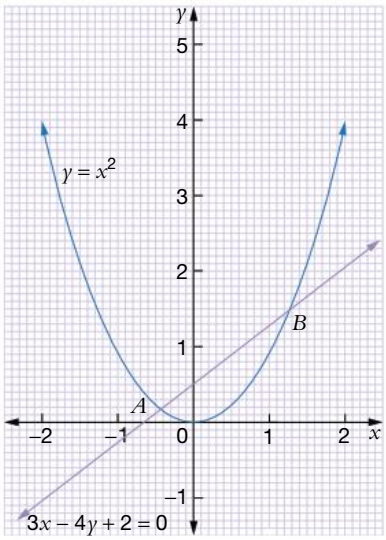


**4** In each, estimate the coordinates of the points of intersection, $A$ and $B$.

**a**




**b**




**5** Find the points of intersection of the following pairs of equations using the graphical method.

**a** $y = x^2$, $y = 2x$

**b** $y = x^2$, $y - 2x = 3$

**c** $y = x$, $y = \frac{1}{x}$

**d** $y = x - 2$, $y = \frac{1}{x}$

**e** $x^2 + y^2 = 4$, $y = x$

**f** $x^2 + y^2 = 4$, $2x - y = 2$

**6** Find the points of intersection of the equations in Question **5** using the algebraic method.

**7** Find the points of intersection of the equations:

**a** $y = x + 1$ and $y = x^2 - 1$ **b** $xy = 2$ and $x + y = 4$

**8** Use the algebraic method to find the coordinates of $A$ and $B$ in Question **4**.

## INVESTIGATION 7:13 A PARABOLA AND A CIRCLE

Consider the parabola $y = k - x^2$ and the circle $x^2 + y^2 = 4$.

**1** Use the graphical method to find the number of points of intersection if:

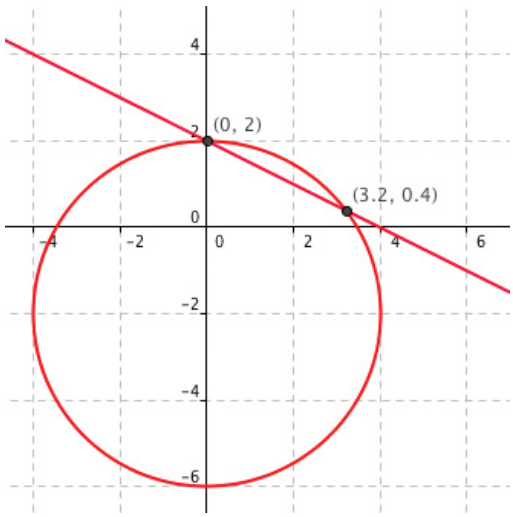
**a** $k = -2$ **b** $k = 0$ **c** $k = 2$ **d** $k = 4$

**2** Use the algebraic method to find the points of intersection of $y = k - x^2$ and $x^2 + y^2 = 4$ when $k = 4$.

**3** Determine the maximum value of $k$ for which there is at least one point of intersection.

## GEOGEBRA ACTIVITY 7:13 THE INTERSECTION OF GRAPHS

Use graphical means to find the point or points of intersection of a straight line with a parabola, hyperbola or circle. Sliders can be used to vary the equations and you can use a checkbox to test the accuracy of your solution.

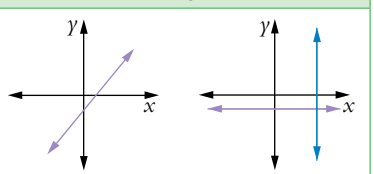
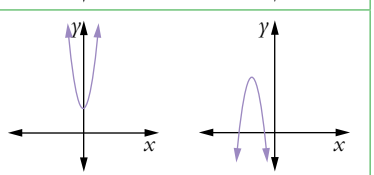
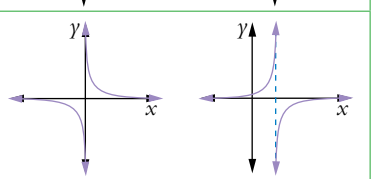


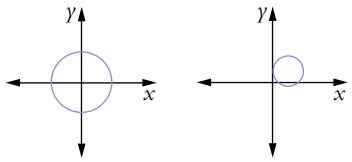
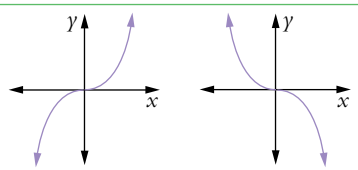
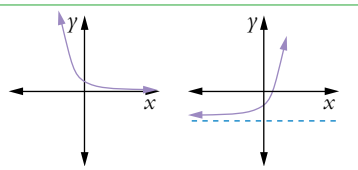
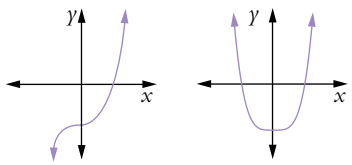

# 7:14 Miscellaneous equations and graphs

For all the graphs met in this chapter it is important that you can:

- identify different types of graphs from their equations
- determine a possible equation of a given graph.

Information about the features of graphs can be obtained from the equation. You need to know how to find these features from the equation before attempting the next exercise.

| Type of Graph | Features | Graph |
|---|---|---|
| Straight line | • $x$- and $y$-intercepts<br>• slope<br>• horizontal and vertical lines |  |
| Parabola | • $x$- and $y$-intercepts (when they exist)<br>• vertex<br>• axis of symmetry<br>• concavity |  |
| Hyperbola | • $x$- and $y$-intercepts (where they exist)<br>• asymptotes<br>• orientation (Quadrants 1 and 3 or Quadrants 2 and 4) |  |

| Type of Graph | Features | Graph |
|---|---|---|
| Circle | • centre<br>• radius |  |
| Cubic | • basic shape (Is it an increasing or decreasing curve?) |  |
| Exponential | • $x$- and $y$-intercepts (where they exist)<br>• concavity<br>• Is it increasing or decreasing?<br>• asymptote |  |
| $y = ax^n$ | • $x$- and $y$-intercepts<br>• basic shape (Does it look like a parabola or a cubic?)<br>• concavity |  |

**Translation:** Terms in an equation indicate that the curve has been formed by translating a simpler curve in either a vertical or horizontal direction.

$y = ax^3 + d$ has been formed by translating $y = ax^3$, $d$ units in a vertical direction.

$y = a(x - r)^2$ has been formed by translating $y = x^2$, $r$ units in a horizontal direction.

**Symmetry:** Graphs that contain an even power of $x$ have an axis of symmetry. If the $y$-axis is an axis of symmetry then the equation will be identical when $x$ is replaced by $-x$.

e.g. $y = x^2 + 4$, $y = 1 + x^6$, $x^2 + y^2 = 1$

## Exercise 7:14

**1** From the list of equations given below, choose those that represent:

**a** a straight line **b** a circle **c** a parabola **d** a hyperbola **e** an exponential.

**A** $x^2 + y^2 = 4$ **B** $y = (x - 4)^2$ **C** $y = \dfrac{1}{x - 4}$ **D** $y = 4^x - 1$

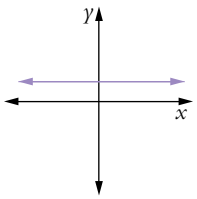
**E** $y = 4$ **F** $y = x - 4$ **G** $x = 4$ **H** $y = 4 - x^2$

**I** $y = \dfrac{4}{x}$ **J** $y = (x + 4)^2$ **K** $y = x^4$ **L** $(x - 4)^2 + y^2 = 4$

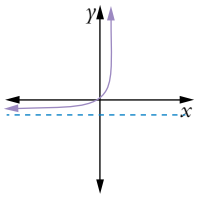
**M** $y = -x^4$ **N** $y = 4^{-x} + 1$ **O** $y = -\dfrac{1}{x} + 4$ **P** $y = \dfrac{1}{x} - 4$

**2** Select the equation from the list in Question **1** that matches each of the graphs below.

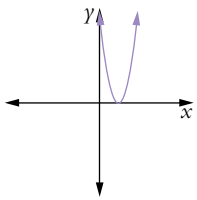
**a**



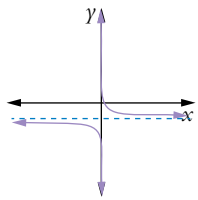

**b**



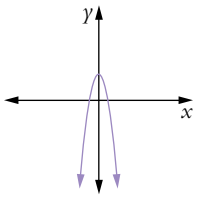

**c**



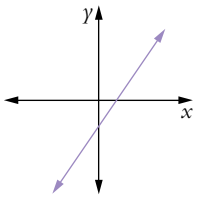

**d**



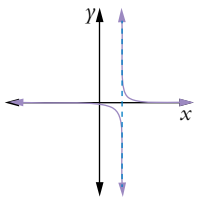

**e**



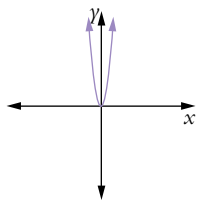

**f**



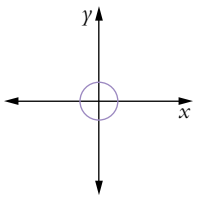

**g**



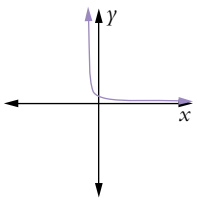

**h**



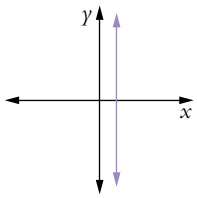

**i**



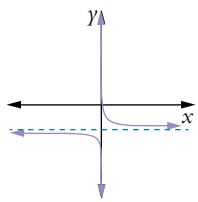

**j**



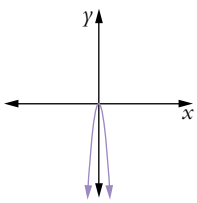

**k**



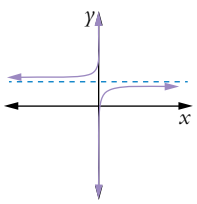

**l**



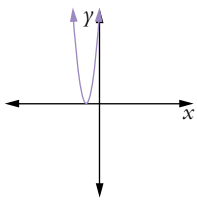

**m**



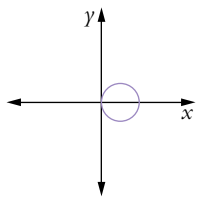

**n**




**o**

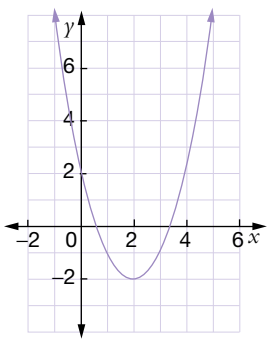



**p**

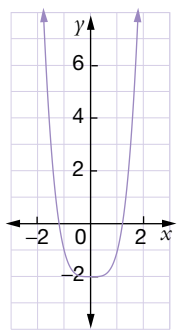



**3** Determine the equation of each graph.

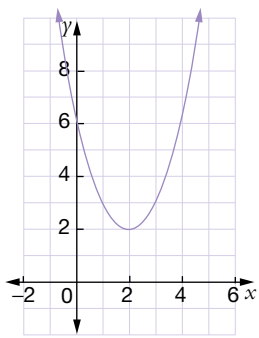
**a**



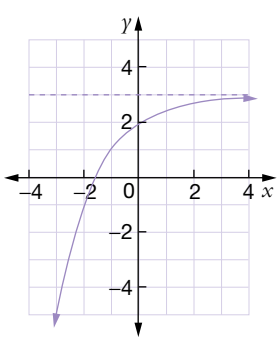

**b**



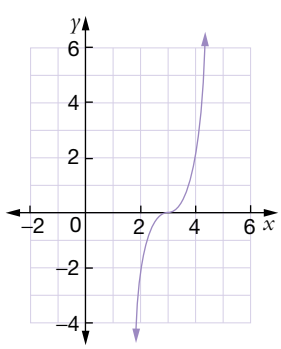

**c**



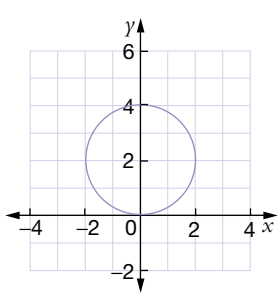

**d**



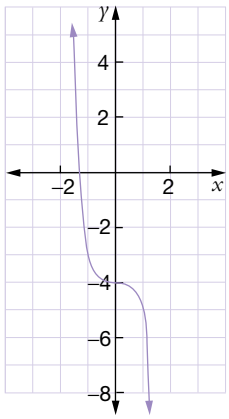

**e**



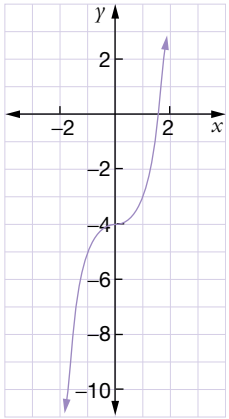

**f**



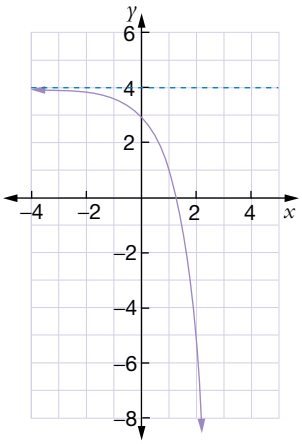
g



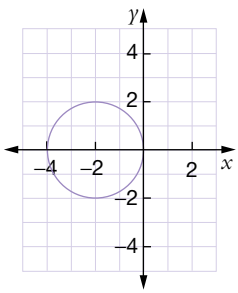

h



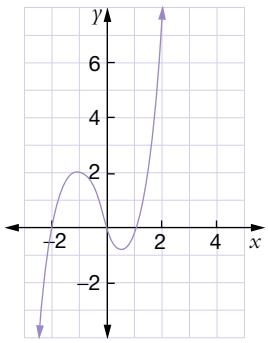

i



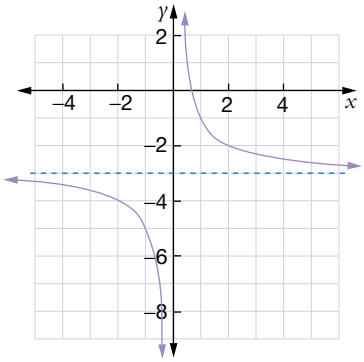

j



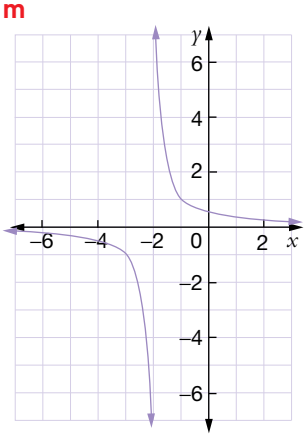

k



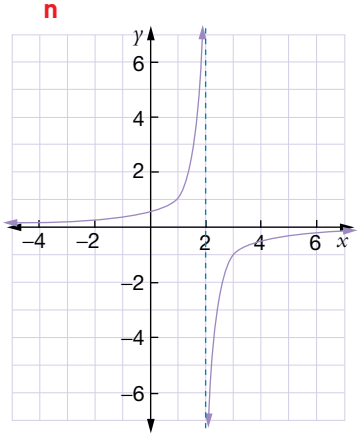

l



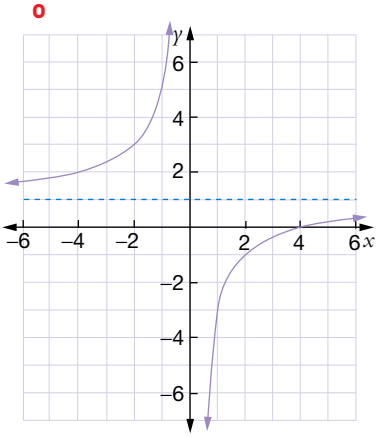

m




n

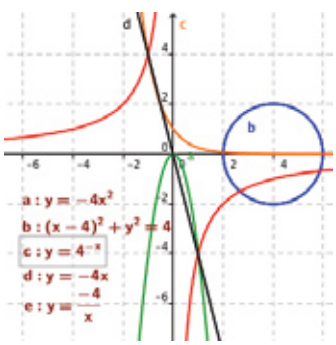



o




## P GEOGEBRA ACTIVITY 7:14 CURVES AND THEIR EQUATIONS

Find the equations of five randomly generated graphs. Tick the checkboxes to check your answers. A suggestion box gives information about each graph.



## MATHS TERMS 7

**circle (equation of)**

- the equation of a circle with its centre at the point $(p, q)$ and a radius of $r$ is $(x - p)^2 + (y - q)^2 = r^2$

**cubic**

- a curve that contains an $x^3$ term as its highest power

**equation**

- an algebraic statement that expresses the relationship between the $x$- and $y$-coordinates of every point $(x, y)$ on the curve

**exponential curve**

- a curve whose basic equation has one of the forms: $y = a^x$, $y = -a^x$, $y = a^{-x}$ or $y = -a^{-x}$, where $a > 0$

**graph (of a curve)**

- the line that results when the points that satisfy a curve's equation are plotted on a number plane

**horizontally**

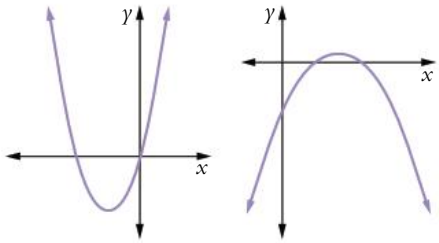
- in the direction of the $x$-axis

**hyperbola**

- a curve with an equation of the form $y = \frac{k}{x}$, $y = \frac{k}{x} + c$ or $y = \frac{k}{x - b}$, where $k$, $b$ and $c$ are constants. It has two asymptotes which are lines that the curve approaches but never reaches

**parabola**

- a curve with the equation $y = ax^2 + bx + c$




- the line of symmetry of the parabola is its axis of symmetry. The equation of the axis of symmetry is $x = -\frac{b}{2a}$
- parabolas can be concave up or concave down
- the point where the parabola turns around is its vertex (or turning point)

**reflect (a curve)**

- to flip a curve about a line, usually the $x$- or $y$-axis

**sketch (a curve)**

- an approximation of a graph

**translate (a curve)**

- to slide the curve in the direction of a line, usually the $x$- or $y$-axis

**vertically**

- in the direction of the $y$-axis

**$x$-intercept**

- the point where a line or curve crosses the $x$-axis

**$y$-intercept**

- the point where a line or curve crosses the $y$-axis

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** Find the equation of the line that: 7:02
- **a** has a $y$-intercept of 2 and is parallel to $y = 4x - 1$
- **b** passes through $(1, 7)$ and is parallel to $y = -3x + 4$
- **c** is perpendicular to $y = \frac{2}{3}x + 1$ and passes through $(-1, 4)$
- **d** is perpendicular to $y = 1 - 2x$ and passes through $(-1, 4)$.

**2** On the same number plane, sketch the graphs of: 7:04
**a** $y = x^2$ **b** $y = x^2 - 4$ **c** $y = x^2 + 2$

**3** On the same number plane, sketch the graph of: 7:04
**a** $y = 2x^2$ **b** $y = \frac{1}{2}x^2$ **c** $y = -x^2$

**4** Sketch the graphs of: 7:04
**a** $y = (x - 1)^2$ **b** $y = (x + 3)^2$ **c** $y = (x - 2)^2 + 1$

**5** Find the $y$-intercept of the parabolas: 7:05
**a** $y = (x - 1)(x + 3)$ **b** $y = x^2 - 6x$
**c** $y = 8 - 2x - x^2$ **d** $y = 4x^2 + 8x - 5$

**6** Find the $x$-intercepts for each of the parabolas in Question 5. 7:05

**7** Find the equation of the axis of symmetry for each of the parabolas in Question 5. 7:05

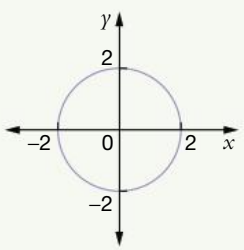
**8** Find the vertex of each of the parabolas in Question 5. 7:05

**9** Sketch each of the parabolas in Question 5. 7:05

**10** Sketch, on the same number plane, the graphs of: 7:06
**a** $y = 2^x$ **b** $y = 3^x$ **c** $y = 2^{-x} - 1$

**11** Sketch the graphs of: 7:07
**a** $y = \frac{2}{x}$ **b** $y = -\frac{2}{x}$ **c** $y = \frac{2}{x - 2}$

**12** 7:08
- **a** What is the equation of this circle?
- **b** What is the equation of a circle that has its centre at the origin and a radius of 7 units?
- **c** State the centre and radius of the circle: $(x - 2)^2 + (y + 1)^2 = 16$




**13** Sketch the graphs of: 7:09
**a** $y = x^3 + 1$ **b** $y = 1 - x^3$ **c** $y = 2x^3 - 1$

**14** Sketch the curves: 7:10
**a** $y = 2x^5$ **b** $y = \frac{1}{2}x^6$ **c** $y = -2x^4$

**15** Sketch the curves: 7:10

a $y = -\frac{1}{2}x^3 - 1$ b $y = -x^4 + 1$ c $y = -x^5 + 1$

**16** Sketch the curves: 7:11

a $y = (x - 3)^3$ b $y = 2(x + 1)^4$ c $y = -\frac{1}{2}(x - 1)^5$

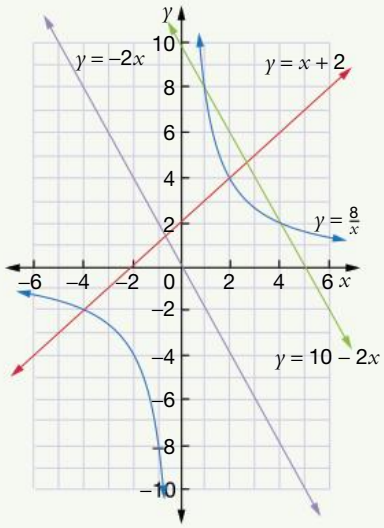
**17** Sketch the curves: 7:12

a $y = x(x - 2)(x - 3)$

b $y = 2(x + 1)(x - 2)^2$

c $y = -(x + 1)(x - 2)(x + 3)$

**18** Use the graph to find the points of intersection of: 7:13




a $y = \frac{8}{x}$ and $y = x + 2$

b $y = \frac{8}{x}$ and $y = -2x$

c $y = \frac{8}{x}$ and $y = 10 - 2x$

**19** Find the points of intersection of: 7:13

a $y = x^2$ and $y = 12 - x$

b $y = x - 3$ and $y = -\frac{2}{x}$

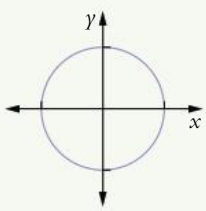
c $y = x + 2$ and $x^2 + y^2 = 10$

**20** Match each graph with its equation from the list. 7:14

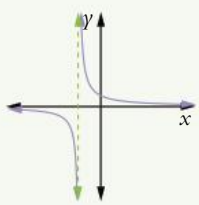
A $y = x^2 - 2$ B $y = 2^x - 2$ C $y = \frac{2}{x + 2}$

D $y = 1 - 2x^2$ E $x^2 + y^2 = 2$ F $y = -2(x - 2)^3$

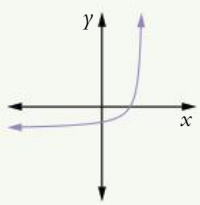
a



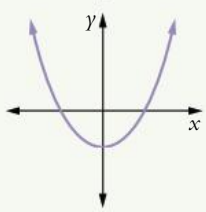

b



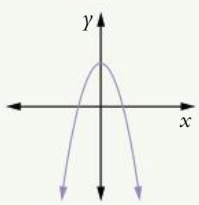

c



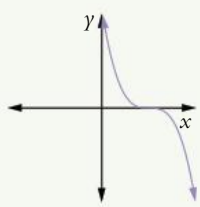

d




e



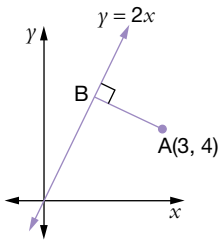

f



## ASSIGNMENT 7A Chapter review

**1** Find the equation of the line that passes through the point (3, 4) if it is:
- **a** parallel to the line $y = 2x + 10$
- **b** perpendicular to the line $y = 2x + 10$.

**2** Find the coordinates of $B$.




**3** Describe the relationship between the graph of $y = x^2$ and the graph of:
- **a** $y = x^2 + 5$
- **b** $y = x^2 - 5$
- **c** $y = (x - 5)^2$
- **d** $y = (x + 5)^2$
- **e** $y = (x - 5)^2 + 5$

**4** A parabola has an equation of $y = 4x^2 - 4x - 3$.
- **a** Find:
  - **i** the $y$-intercept
  - **ii** the $x$-intercepts
  - **iii** the equation of the axis of symmetry
  - **iv** the coordinates of the vertex.
- **b** Sketch the graph of the parabola.

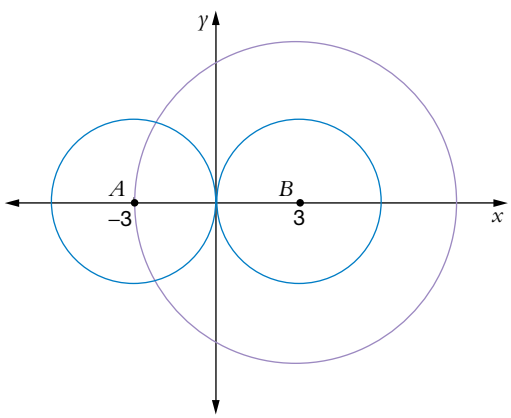
**5** State the equation of each curve that would result if the curve $y = \frac{4}{x}$ was:
- **a** translated 2 units up
- **b** translated 2 units to the right
- **c** translated 2 units to the left
- **d** reflected in the $y$-axis
- **e** reflected in the $y$-axis and translated 2 units up.

**6** What is the equation of the curve that results when the following curves are reflected in:
- **i** the $x$-axis
- **ii** the $y$-axis?
  - **a** $y = x^5$
  - **b** $y = x^5 - 2$
  - **c** $y = \frac{1}{2}x^6$
  - **d** $y = -3(x - 2)^3$

**7** Sketch the following curves on the same number plane.
- **a** $y = (x - 2)^2(x - 4)$
- **b** $y = 2(x - 2)^2(x - 4)$
- **c** $y = (x - 2)^2(4 - x)$

**8** Make sketches of the curves:
- **a** $y = 2x^7$
- **b** $y = -2x^7$
- **c** $y = 2x^7 - 1$
- **d** $y = 1 - 2x^7$
- **e** $y = 2(x - 1)^7$

**9**




The two small circles have their centres at $A$ and $B$. The larger circle's centre is also at $B$ and it passes through $A$. What is the equation of:
- **a** the circle centred at $A$
- **b** the larger circle with its centre at $B$?

**10** Find the points of intersection of the line $y = 2x + 3$ with:
- **a** the parabola $y = x^2$
- **b** the hyperbola $y = \frac{9}{x}$.

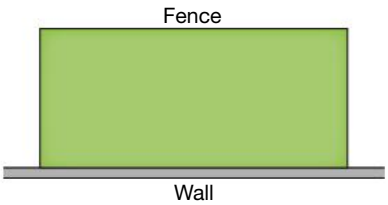
## ASSIGNMENT 7B Working mathematically

**1** You are given an equation of the form $y = ax^n + d$. Describe how you would sketch the curve.

**2** Describe briefly the features of the curves with the following equations.

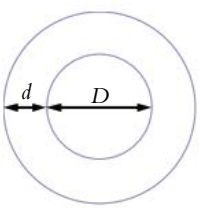
$y = -4x$
$y = 4x^2$
$y = -4x^2 - 4$
$y = (x - 4)^3$
$y = 4^x$
$y = x^4$
$x^2 + y^2 = 4$

**3**




A rectangular pen is to be fenced on three sides with 20 m of fencing material. What is the largest area that can be enclosed and what are its dimensions?

**4**




A pair of concentric circles is as shown. If you are given the circumference, $C$, of the outer circle and the difference between the radii, $d$, find a formula for the inside diameter $D$ in terms of $C$ and $d$.

**5** A new number plate consists of two letters, followed by two digits, followed by two letters as shown.

How many different number plates are possible?

**6** The concentration of an acid is measured by its pH. An acid with a concentration of:

- 0·1 mole/L has a pH of 1
- 0·01 mole/L has a pH of 2
- 0·001 mole/L has a pH of 3.

**a** Write each of 0·1, 0·01, and 0·001 in index notation.

**b** Explain how the pH is related to the concentration of the acid.

**c** If acid had a concentration of 0·000 01 mole/L, what is its pH?

**d** What is the concentration of an acid with a pH of 4·5?

**e** One acid has a pH of 2 and the other has a pH of 4. Which is the more concentrated acid? How many times is it more concentrated?

# ASSIGNMENT 7C Cumulative revision

**1** Solve the quadratic equations. 2:01, 2:02

**a** $x^2 = 6x$ **b** $(x + 6)^2 = 16$

**c** $(x + 6)(x - 16) = 0$ **d** $(x + 6)(x - 6) = 6x$

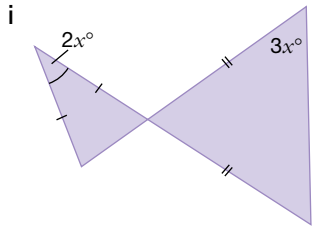
**2** **a** A regular polygon has 60 sides. Find the size of its: 3:06, 3:01

**i** interior angle

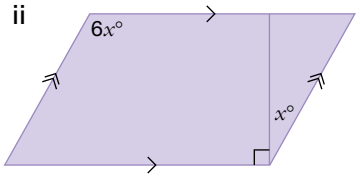
**ii** exterior angle.

**b** Find the value of $x$ in the following diagrams.

**i**




**ii**




**3** If three normal dice are rolled, find the probability of obtaining: 4:05

**a** three 6s **b** no 6s **c** one 6 **d** at least one 6.

**4** Simplify 5:03, 5:04

**a** $3\sqrt{2} - \sqrt{2} + 5\sqrt{2}$ **b** $\sqrt{27} + 2\sqrt{3}$

**c** $\left(3\sqrt{5}\right)^2$ **d** $\dfrac{\sqrt{5} \times 2\sqrt{3} \times \sqrt{2}}{4\sqrt{15}}$

**5** Zara wished to invest \$10 000. Use a 'guess and refine' strategy to find the minimum number of years it would take to more than double her investment if interest is compounded yearly at a rate of: 6:06

**a** 3·6% p.a. **b** 4·8% p.a.

**6** The value of a library is depreciated at a rate of 17·5% p.a. If the library is presently valued at \$600 000, what will its value be after 4 years? 6:06

**7** Solve the following pairs of simultaneous equations. 1:09

**a** $2x - y = 6$
$x - y = 4$

**b** $2a - 3b = -2$
$a + 2b = 6$

**c** $3m - n = 6$
$2m + 5n = 8$

# SURFACE AREA AND VOLUME

8

## Contents


8:01 Review of surface area and volume
8:02 Surface area of a pyramid
8:03 Surface area of a cone
Investigation 8:03 The surface area of a cone
8:04 Surface area of a sphere
Investigation 8:04 The surface area of a sphere
Fun spot 8:04 How did the raisins win the war against the nuts?
8:05 Volume of a pyramid
Investigation 8:05 The volume of a pyramid
8:06 Volume of a cone
8:07 Volume of a sphere
Investigation 8:07 Estimating your surface area and volume
8:08 Practical applications of surface area and volume
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Measurement and Geometry**

Selections from: ***Area and Surface Area*** [Stages 5.2, 5.3] and ***Volume*** [Stages 5.2, 5.3]

- Solve problems involving surface area for a range of prisms, cylinders and composite solids (ACMMG242)
- Solve problems involving the surface areas of right pyramids, right cones, spheres and related composite solids (ACMMG271)
- Solve problems involving volume for a range of prisms, cylinders and composite solids (ACMMG242)
- Solve problems involving the volumes of right pyramids, right cones, spheres and related composite solids (ACMMG271)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 8:01 Review of surface area and volume

In Year 9 the surface area and volume of prisms, cylinders, prism-like solids and composite solids were calculated.

The following formulas may be needed.

**Area formulas**

**1** square: $A = s^2$

**2** rectangle: $A = LB$

**3** triangle: $A = \frac{1}{2}bh$

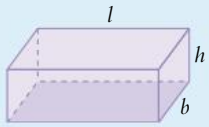
**4** trapezium: $A = \frac{1}{2}h(a + b)$

**5** parallelogram: $A = bh$

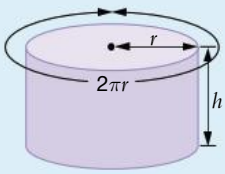
**6** rhombus: $A = \frac{1}{2}xy$

**7** circle: $A = \pi r^2$

**8** Surface area of a rectangular prism:
$A = 2lb + 2lh + 2bh$



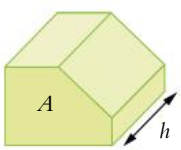

**9** Surface area of a cylinder:
$A = 2\pi rh + 2\pi r^2$

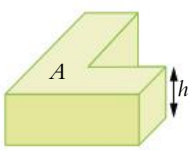


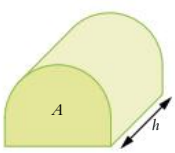

Use this formula to calculate the volumes of all prisms and prism-like solids with uniform cross-sections:

$$V = Ah$$

where $A$ = area of the cross-section of the base
$h$ = height of the solid.



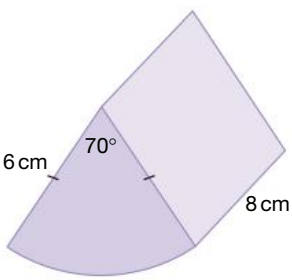

## WORKED EXAMPLE 1

Calculate the surface area and volume of the prism-like solid shown.

Surface area = 2 × area of the sector + area of the rectangles + area of curved surface.

$$= 2 \times \frac{70}{360} \times \pi \times 6^2 + (2 \times 6 \times 8) + \frac{70}{360} \times 2\pi \times 6 \times 8$$

$= 198{\cdot}6\,\text{cm}^2$ (1 dec. pl.)




Volume = $Ah$

$$= \frac{70}{360} \times \pi \times 6^2 \times 8$$

$= 175{\cdot}9\,\text{cm}$ (1 dec. pl.)

## WORKED EXAMPLE 2

Find the surface area and volume of this composite solid.

Surface area $= 2 \times (14 \times 14) - (\pi \times 4^2) + 4 \times (14 \times 6) + (2\pi \times 4 \times 4) + (\pi \times 4^2)$

$= 728 + 32\pi$

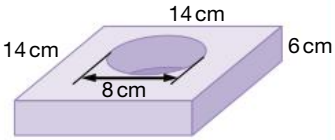
$= 829\,\text{cm}^2$ (nearest $\text{cm}^2$)

Volume = volume of rectangular prism − volume of cylinder

$= (14 \times 14 \times 6) - (\pi \times 4^2 \times 4)$

$= 1176 - 64\pi$

$= 975\,\text{cm}^2$ (nearest $\text{cm}^3$)



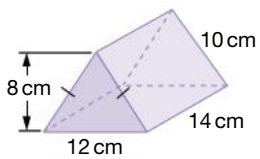

The hole is 4 cm deep.

## Exercise 8:01

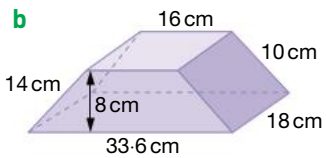
P Foundation worksheet 8:01
Surface area review

**1** Find the surface area of the following prisms.

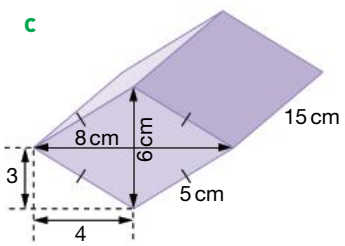
**a**




**b**

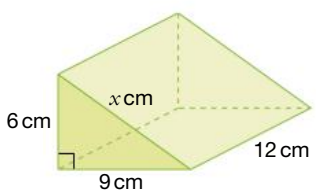



**c**

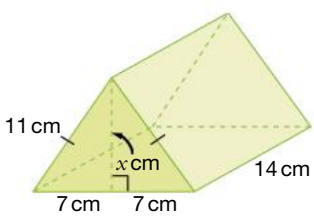



**2** In each of the following questions use Pythagoras' theorem to calculate the unknown length $x$, correct to two decimal places, and then calculate the surface area.

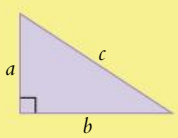
**a**




**b**



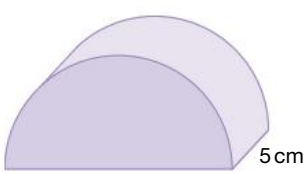

Caution!
$c^2 = a^2 + b^2$



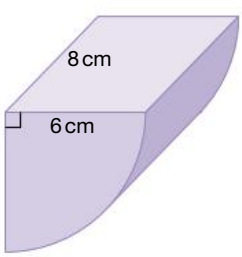

Right-angled triangles

**3** The following solids have uniform cross-sections that are sectors. Find the volumes of the solids correct to the nearest cubic centimetre.

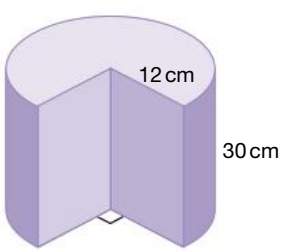
**a**




**b**



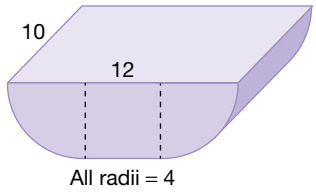

**c**



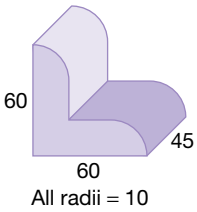

**4** Find the surface area of each of the solids in Question 3 correct to the nearest square centimetre.

**5** Calculate the volumes of the following solids correct to the nearest cubic centimetre. All measurements are in centimetres.

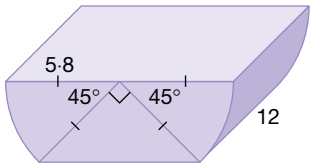
**a**




**b**



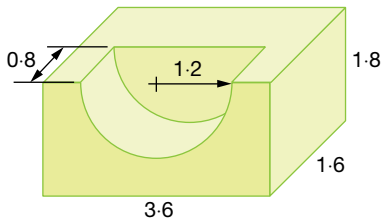

**c**



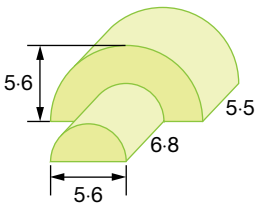

**6** Find the surface area of each of the solids in Question **5** to the nearest square centimetre.

**7** Calculate the volumes of the following composite solids. All measurements are in metres. Give the answers correct to one decimal place.

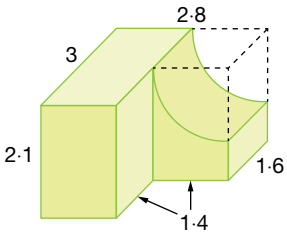
**a**




**b**

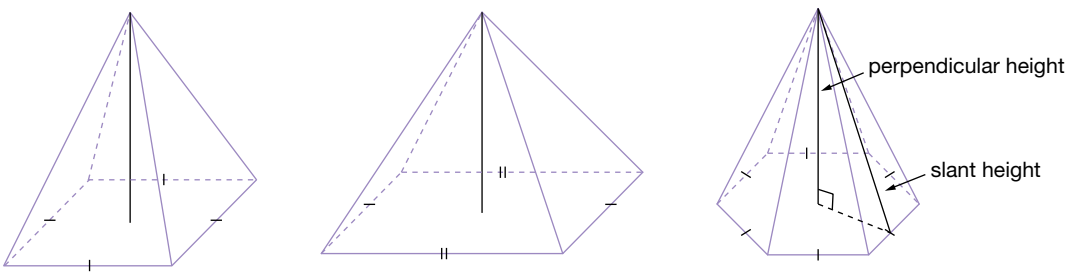



**c**




**8** Calculate the surface areas of the solids in Question **7**. Give the answers correct to one decimal place.

# 8:02 Surface area of a pyramid




The diagrams above show a square pyramid, a rectangular pyramid and an hexagonal pyramid. These pyramids are all right pyramids because the apex is directly above the centre of the base. The interval joining the apex to the centre is called the **perpendicular height** of the pyramid.

> The surface area of a right pyramid is found by adding the area of the base to the area of the triangular faces.

To find the area of the triangular faces we need to find the perpendicular height of each face. This is called the **slant height**. The calculation of the slant height uses Pythagoras' theorem. This is shown on the following page for both the square and rectangular pyramids.

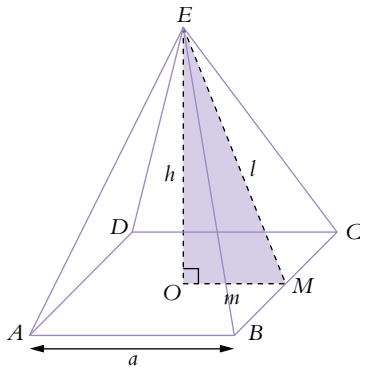
## Square pyramid

*ABCDE* is a square pyramid with a base edge of $a$ units and a perpendicular height of $h$ units. $M$ is the midpoint of the base edge, *BC*. *EO* is the perpendicular height of the pyramid.

*EM* is the slant height of the triangular face *EBC*.

*OM* is half of the side length of the square.

Using Pythagoras' theorem in $\Delta EMO$ gives $l^2 = h^2 + m^2$, where $h$ = perpendicular height of the prism, $l$ = slant height and $m = \frac{1}{2}a$.



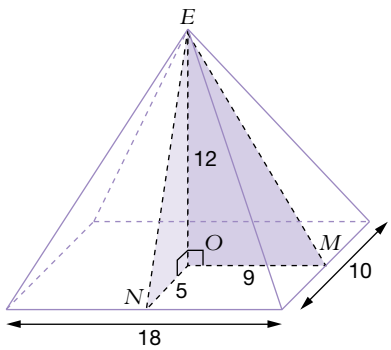

## Rectangular pyramid

This diagram shows a pyramid with a rectangular base 18 units by 10 units and a perpendicular height of 12 units. *EN* and *EM* are slant heights of the pyramid.

In $\Delta ENO$,
$EN^2 = 5^2 + 12^2$
$= 169$
$\therefore EN = 13$

In $\Delta EMO$,
$EM^2 = 9^2 + 12^2$
$= 225$
$\therefore EM = 15$

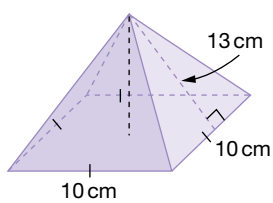
As $EN \neq EM$, rectangular prisms have two different slant heights.




When calculating the surface area of a pyramid, Pythagoras' theorem is used to calculate the slant height of the triangular faces.

## WORKED EXAMPLE 1

Calculate the surface area of the right square pyramid.




### Solution

Surface area = (area of square) + 4 × (area of a triangular face)

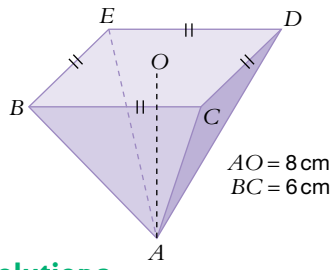
$= (10 \times 10) + 4 \times \frac{10 \times 13}{2}$
$= 100 + 260$
$= 360\,\text{cm}^2$

In right square pyramids, all the triangular faces are congruent. This simplifies the calculation of the surface area.

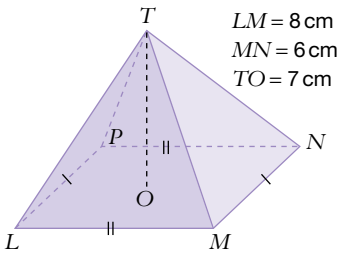
## WORKED EXAMPLE 2

Calculate the surface areas of the following right square and rectangular pyramids.

**a**

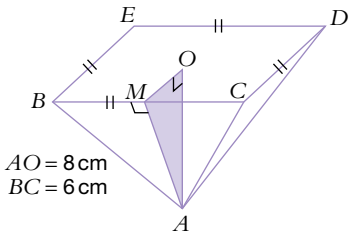



**b**




### Solutions

**a** As the slant height of the triangular face is not given, this must be calculated.
In the diagram, $AM$ is the slant height of the face.




In $\Delta AOM$

$AM^2 = AO^2 + OM^2$ (Pythagoras' theorem)

$= 8^2 + 3^2$ (*Note:* $OM = \frac{1}{2}CD$)

$= 64 + 9$

$= 73$

$AM = \sqrt{73}$

Surface area = (area of square) + 4 × (area of a triangular face)

$$= (6 \times 6) + 4 \times \frac{6 \times \sqrt{73}}{2}$$

$= 36 + 12\sqrt{73}$

$= 138{\cdot}5 \text{ cm}^2$ (1 dec. pl.)

**b** The slant heights $TA$ and $TB$ must be calculated, as these are not given.

In $\Delta TOA$,

$TA^2 = AO^2 + OT^2$

$= 3^2 + 7^2$

$= 58$

$\therefore\ TA = \sqrt{58}$ cm

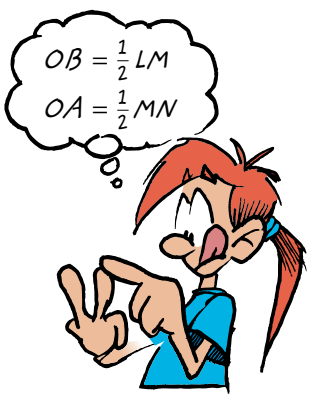
In $\Delta TOB$,

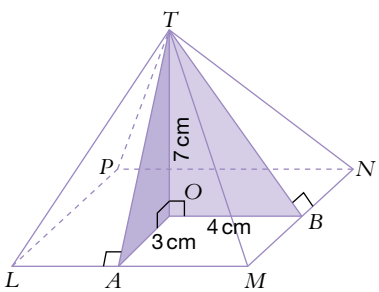
$TB^2 = BO^2 + OT^2$

$= 4^2 + 7^2$

$= 65$

$\therefore\ TB = \sqrt{65}$ cm




Now, Surface area = (area of rect. $LMNP$) + 2 × (area of $\Delta TMN$) + 2 × (area of $\Delta TLM$)

$$= (LM \times MN) + 2 \times \frac{MN \times TB}{2} + 2 \times \frac{LM \times TA}{2}$$

$$= 8 \times 6 + \frac{2 \times 6 \times \sqrt{65}}{2} + \frac{2 \times 8 \times \sqrt{58}}{2}$$

$= 48 + 6\sqrt{65} + 8\sqrt{58}$

$= 157{\cdot}3 \text{ cm}^2$ (1 dec. pl.)

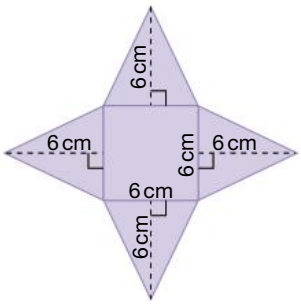
In right rectangular pyramids, the opposite triangular faces are congruent.

# Exercise 8:02

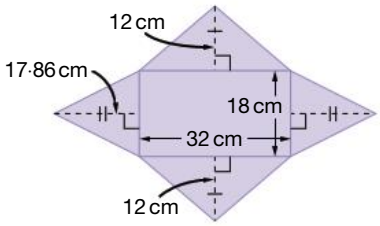
Foundation worksheet 8:02
Surface area of a pyramid

**1** The following diagrams represent the nets of pyramids. Calculate the area of each net.

**a**

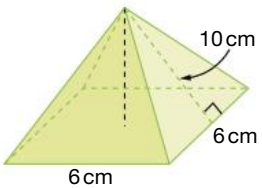



**b**

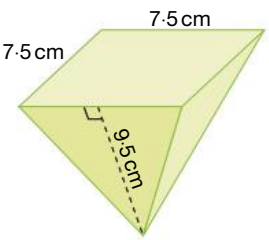



**2** Calculate the surface areas of the following right square and rectangular pyramids.

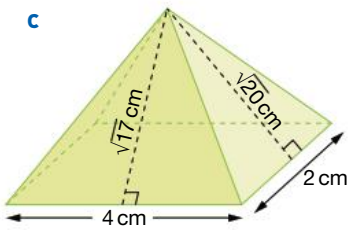
**a**




**b**

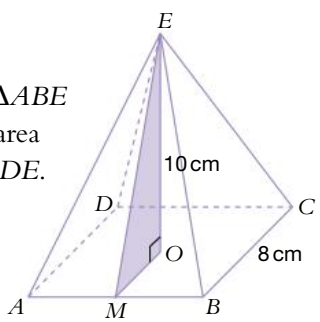



**c**



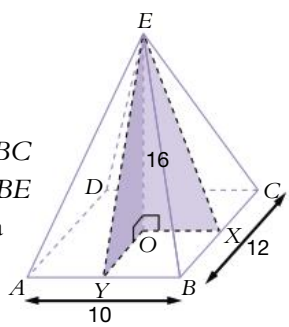

**3** **a** $ABCDE$ is a square pyramid.
Find:

- **i** $OM$
- **ii** $EM$
- **iii** area of $\Delta ABE$
- **iv** surface area of $ABCDE$.



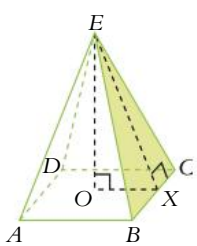

**b** $ABCDE$ is a rectangular pyramid.
Find:

- **i** $OX$
- **ii** $EX$
- **iii** $OY$
- **iv** $EY$
- **v** the area of $\Delta EBC$
- **vi** the area of $\Delta ABE$
- **vii** the surface area of $ABCDE$.



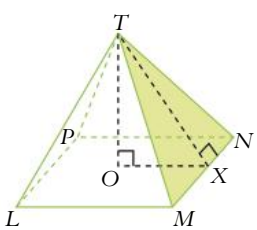

**4** Use Pythagoras' theorem to calculate the perpendicular height of each face and then calculate the surface area of each square pyramid. Give the answers in surd form.

**a**




$AB = 10$ cm
$EO = 12$ cm

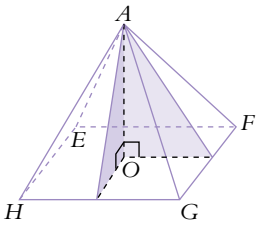
**b**




$LM = 10$ cm
$TO = 10$ cm

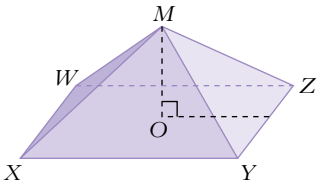
**5** Calculate the surface areas of the following pyramids. Give all answers correct to one decimal place where necessary.

**a**



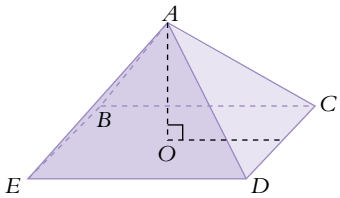

*EFGH* is a rectangle
$AO = 8$ cm, $HG = 12$ cm
$GF = 9$ cm

**b**




*WXYZ* is a square.
$MO = 3$ cm, $XY = 11$ cm

**c**




*BCDE* is a rectangle
$AO = 4$ cm, $ED = 10$ cm
$DC = 6$ cm

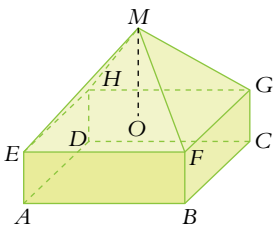
**6** Find the surface areas of the following right pyramids:

**a** a square pyramid, base edge 6 cm, height 5 cm

**b** a rectangular pyramid, base 7 cm by 5 cm, height 10 cm

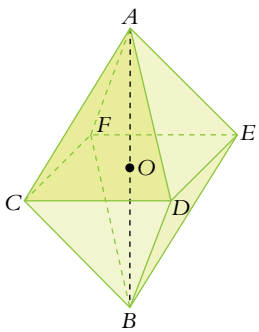
**7** Find the surface areas of the following solids. Give all answers correct to three significant figures.

**a**



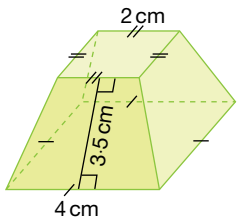

*ABCD* is a square
$AB = 5$ cm, $MO = 3$ cm
$CG = 2$ cm

**b**

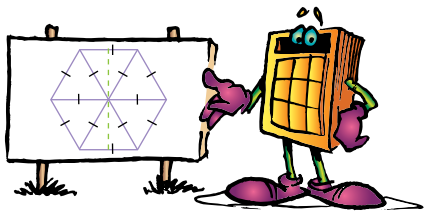



$AO = OB = 9$ cm
$CD = DE = FE = FC = 6$ cm

**c**



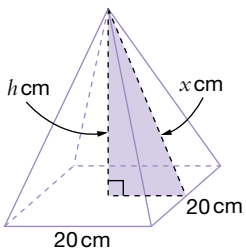

**8** Find the surface area of a right pyramid that has a regular hexagonal base of edge 6 cm and a height of 8 cm.



**9** A square pyramid has a surface area of 2000 $cm^2$.
If the base edge is 20 cm, calculate:

**a** the perpendicular height, $x$ cm, of one of the triangular faces

**b** the perpendicular height, $h$ cm, of the pyramid.



# 8:03 Surface area of a cone

## PREP QUIZ 8:03

1 Area = …

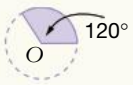
2 Circumference = …

3 Simplify: $\frac{2\pi r}{2\pi l}$

4 Simplify: $\frac{r}{l} \times \pi l^2$

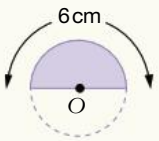
What fraction of a circle is each of the following sectors?

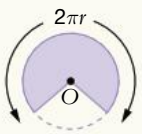
5


6 


7

circumference = 9 cm

8 


9 

circumference = $2\pi l$

10 Evaluate $\pi rl$ if $r = 3{\cdot}5$ and $l = 6{\cdot}5$, correct to one decimal place.

## INVESTIGATION 8:03 THE SURFACE AREA OF A CONE

The surface area of a cone comprises two parts: a circle and a curved surface.
The curved surface is formed from a sector of a circle.

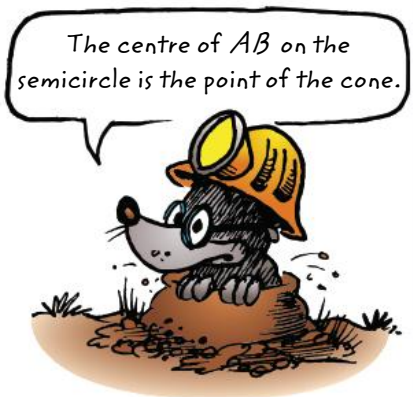
- This investigation involves making two cones and calculating their surface areas.

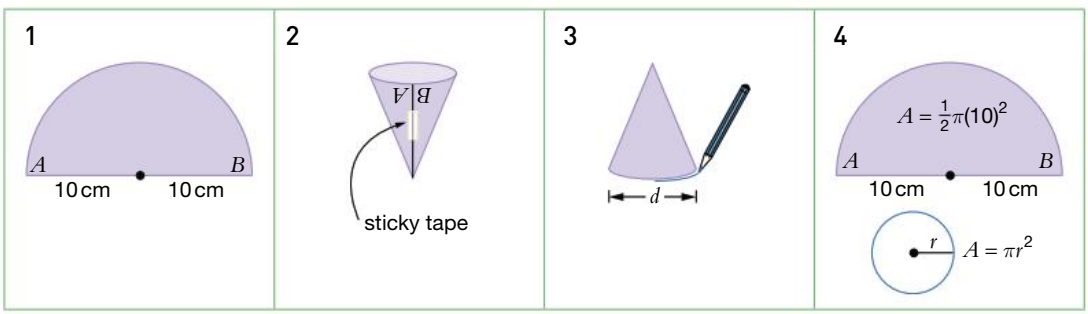
**Step 1** Draw a semicircle of radius 10 cm.

**Step 2** Make a cone by joining opposite sides of the semicircle, as shown below.

**Step 3** Place the cone face down and trace the circular base. Measure the diameter of this base.

**Step 4** Calculate the area of the original semicircle plus the area of the circular base. This would be the total surface area of the cone if it were closed.

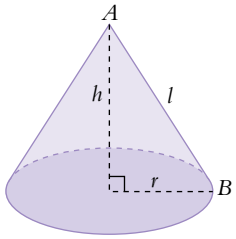


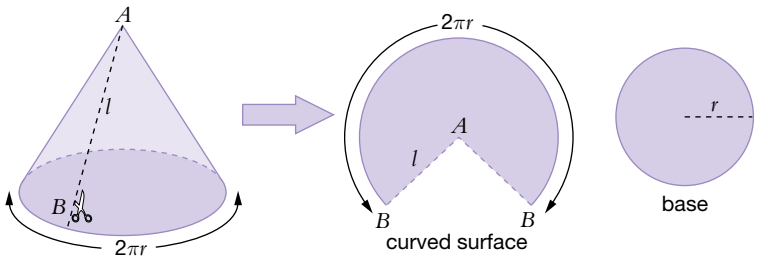



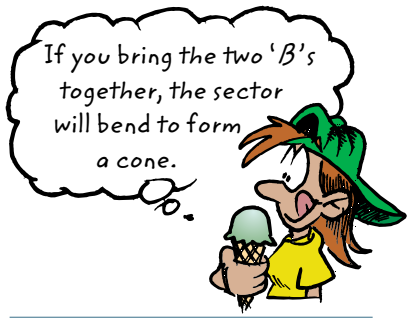

- Repeat the steps above, making the original sector a quarter circle of radius 10 cm. What is the surface area of a closed cone of these dimensions?

A cone may be thought of as a pyramid with a circular base.
Consider a cone of slant height $l$ and base radius $r$.

- Imagine what would happen if we cut along a straight line joining the vertex to a point on the base.
- By cutting along this line, which is called the **slant height**, we produce the net of the curved surface. The net of the curved surface is a sector of a circle, radius $l$.




To calculate the area of a sector, we must find what fraction it is of the complete circle. Normally this is done by looking at the sector angle and comparing it to 360°, but it can also be done by comparing the length of the sector's arc to the circumference of the circle. Hence, if the sector's arc length is half the circumference of the circle, then the sector's area is half the area of the circle. (See Prep quiz 8:03.)

$$\therefore \text{Area of sector} = \frac{\text{length of sector's arc}}{\text{circumference of circle}} \times \text{area of circle}$$

$$= \frac{2\pi r}{2\pi l} \times \pi l^2$$

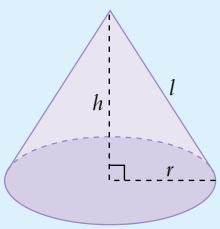
$$= \pi r l$$

Now, since the area of the sector = area of the curved surface,

curved surface area = $\pi r l$

**Surface area of a cone:**

Surface area = $\pi r l + \pi r^2$




where
$r$ = radius of the cone
$l$ = slant height of the cone
*Note:* $l = \sqrt{h^2 + r^2}$

## WORKED EXAMPLE

Find the surface area of a cone with a radius of 5 cm and:

**a** a slant height of 8 cm **b** a height of 12 cm.

### Solutions

**a** Surface area = $\pi r l + \pi r^2$

$= \pi \times 5 \times 8 + \pi \times 5^2$

$= 40\pi + 25\pi$

$= 65\pi \text{ cm}^2$

$= 204{\cdot}2 \text{ cm}^2$ (1 dec. pl.)

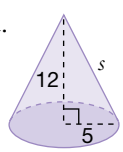
The height of a cone is the perpendicular height.

**b** First the slant height must be calculated.

$l^2 = 5^2 + 12^2$ (Pythagoras' theorem)

$= 169$

$\therefore l = 13$




Surface area = $\pi r s + \pi r^2$

$= 65\pi + 25\pi$

$= 90\pi \text{ cm}^2$

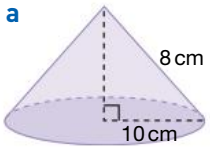
$= 282{\cdot}7 \text{ cm}^2$ (1 dec. pl.)

# Exercise 8:03

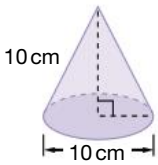
P Foundation worksheet 8:03
Surface area of a cone

**1** Find the curved surface areas of the following cones, giving answers in terms of $\pi$.

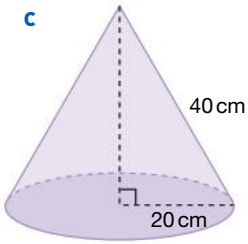
**a**



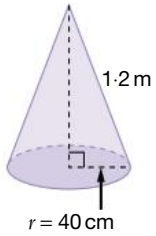

**b**




**c**




**d**




**2** Find the surface area of each of the cones in Question **1**, giving all answers in terms of $\pi$.

**3** Calculate the surface areas of the following cones, giving all answers in terms of $\pi$.

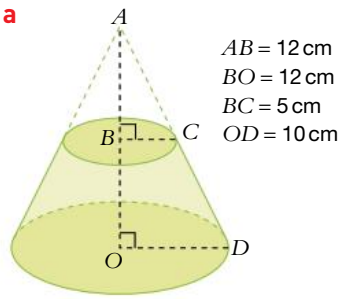
**a** radius 8 cm and height 6 cm
**b** radius 1·6 m and height 1·2 m
**c** diameter 16 cm and height 15 cm
**d** diameter 1 m and height 1·2 m

**4** In each of the following, find the surface area of the cone correct to four significant figures.

**a** radius 16 cm and height 20 cm
**b** radius 5 cm and slant height 12 cm
**c** radius 12·5 cm and height 4·5 cm
**d** diameter 1·2 m and height 60 cm
**e** diameter 3·0 m and slant height 3·5 m

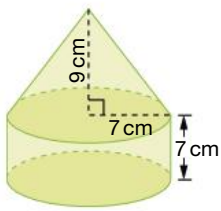
**5** Find the surface areas of the following solids. Give all answers correct to one decimal place.

**a**

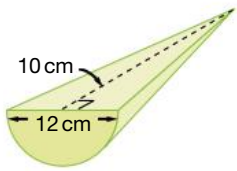



$AB = 12$ cm
$BO = 12$ cm
$BC = 5$ cm
$OD = 10$ cm

**b**



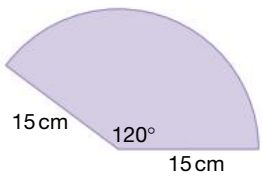

**c**




*Note:* This is half a cone.

**6** A cone is to be formed by joining the radii of the sector shown. In the cone that is formed, find:

**a** the slant height
**b** the radius
**c** the perpendicular height.



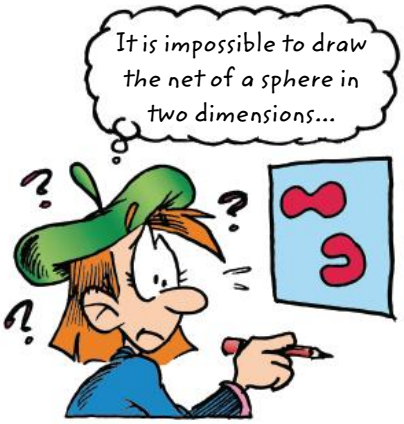

**7** **a** A cone with a radius of 5 cm has a surface area of $200\pi$ cm$^2$. What is the perpendicular height of the cone?

**b** A particular cone cannot have a surface area greater than $1000\pi$ cm$^2$. What is the largest radius, correct to one decimal place, that will achieve this if the slant height is 20 cm?

# 8:04 Surface area of a sphere

## INVESTIGATION 8:04 THE SURFACE AREA OF A SPHERE

Carry out the experiment outlined below to demonstrate that the reasoning is correct.



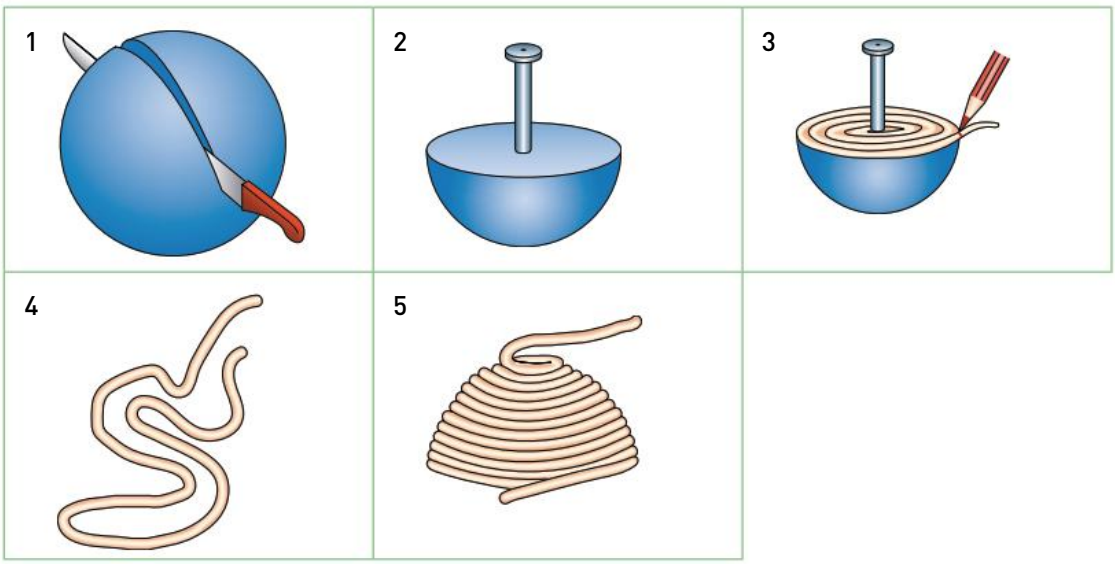

**Step 1** Cut a solid polystyrene ball or an orange into two halves. The faces of the two hemispheres are identical circles.

**Step 2** Push a long nail through the centre of a hemisphere, as shown.

**Step 3** Using thick cord, cover the circular face of one of the hemispheres as shown, carefully working from the inside out. Mark the length of cord needed. Call this length $A$. Length $A$ covers the area of the circle, ie $\pi r^2$.

**Step 4** Put a second mark on the cord at a point that is double the length $A$. The length of the cord to the second mark is $2A$, which covers the area of two identical circles, i.e. $2\pi r^2$.

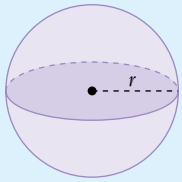
**Step 5** Turn the other hemisphere over and use the cord of length $2A$ to cover the outside of the hemisphere.
It should fit nicely.
It seems that $2A$ covers half of the sphere.
It would take $4A$ to cover the whole sphere.
Surface area of a sphere = $4A$
Surface area = $4\pi r^2$



The surface area of a sphere is given by the formula:

$$SA = 4\pi r^2$$

where $SA$ is the surface area and $r$ is the radius.

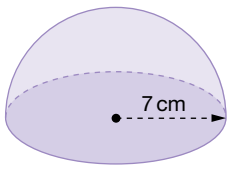



The proof of the formula for the surface area of a sphere is beyond the scope of this course.

## WORKED EXAMPLES

**1** Find the surface area of a sphere of diameter 12 cm.

**2** Find the surface area of the hemisphere shown here.




### Solutions

**1** Diameter = 12 cm

$\therefore$ Radius = 6 cm

Now, $SA = 4\pi r^2$

$= 4 \times \pi \times 6^2$

$= 144\pi \text{ cm}^2$

$= 452 \text{ cm}^2$ (3 sig. fig.)

**2** $SA$ = area of curved surface + area of circle

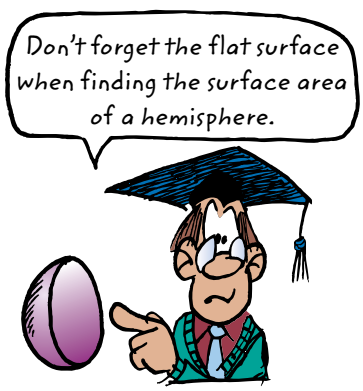
$$= \frac{4\pi r^2}{2} + \pi r^2$$

$= 3\pi r^2$

$= 3\pi \times 7^2$

$= 147\pi \text{ cm}^2$

$= 461{\cdot}8 \text{ cm}^2$ (1 dec. pl.)




## Exercise 8:04

**1** Find the surface area of a sphere with:

**a** radius = 5 cm **b** radius = 7·6 cm **c** radius = 3·2 m

**d** diameter = 18 cm **e** diameter = 1·6 m **f** diameter = 8000 km

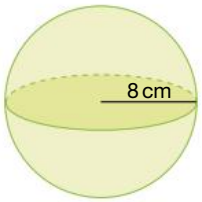
Give all answers as exact answers (multiples of $\pi$) and also as approximations correct to three significant figures.

**2** Calculate the exact surface area of a hemisphere with:

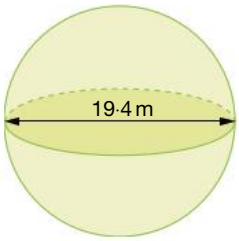
**a** radius = 12 cm **b** diameter = 12 cm

**3** Calculate the surface area of each solid correct to two decimal places.

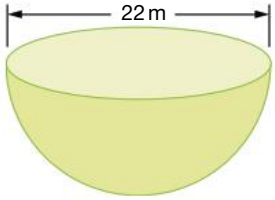
**a**




**b**




**c**

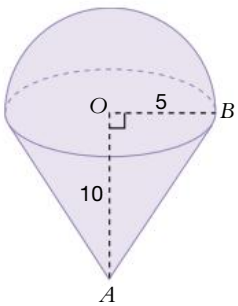



**4** A cylinder is 3 m long and has a radius of 80 cm. A hemispherical cap is placed on each end of the cylinder. Calculate the surface area of the solid correct to four significant figures.

**5** A sphere has a surface area of 200 $cm^2$. Find its radius correct to one decimal place.

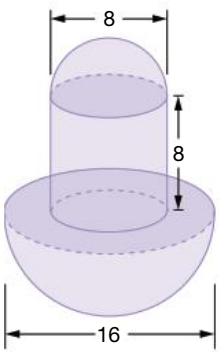
**6** Calculate the surface areas of each of the following solids, correct to three significant figures. (All measurements are in centimetres unless stated otherwise.)

**a**

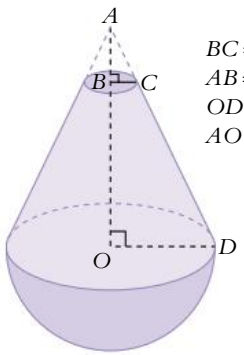



$O$ is the centre of the hemisphere.

**b**




**c**




$BC = 0{\cdot}6$ m
$AB = 0{\cdot}8$ m
$OD = 3{\cdot}0$ m
$AO = 4{\cdot}0$ m

The dimetrodon was a dinosaur that had a large sail to absorb and dissipate heat efficiently. Dimetrodons of different sizes had a slightly different shape. Larger specimens had proportionally larger sails. These sails varied according to the volume rather than the surface area or the length of the creature. Being cold blooded, they needed to remove or absorb heat according to their body mass.

## FUN SPOT 8:04 — HOW DID THE RAISINS WIN THE WAR AGAINST THE NUTS?

Work out the answer to each question and write the letter for that part in the box that is above the correct answer.

*Simplify where possible:*

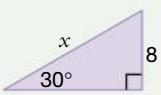
**B** $8x + x$
**C** $8x \div x$
**D** $8x \times x$
**E** $8x - x$
**H** $xy + yx$
**E** $2x - 2$
**E** $3x - 4 + 3x$
**E** $3(2x - 4) - 6x$
**T** $3x \times 3x$
**T** $-2x - 5x$

*Solve:*

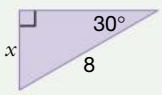
**G** $\frac{x}{3} = 3$
**H** $\frac{x}{3} = \frac{1}{3}$
**E** $3x = \frac{1}{3}$
**H** $\frac{3}{4}x = \frac{3}{8}$
**I** $x^2 = 9$
**I** $0{\cdot}3x = 3$
**J** $\sqrt{x} = 9$
**L** $x = \frac{x}{2}$

*Find the value of the pronumeral in each of the following:*

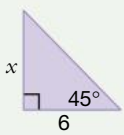
**L**




**M**




**N**




*Find the value of $b^2 - 4ac$ if:*

**N** $a = 2, b = 10, c = 5$
**N** $a = 3, b = -2, c = 4$
**O** $a = 1, b = -1, c = 4$
**S** $a = 6, b = -3, c = -1$

*Simplify:*

**S** $(a^{-1})^{-2}$
**S** $a^{-1} \times a^1$
**E** $(2a^2)^3$
**T** $\sqrt{x^{16}}$
**T** $\frac{a}{4} + \frac{a}{5}$
**E** $\frac{a^2}{5} \times \frac{5}{a}$
**T** $3a\sqrt{\frac{a}{4}}$
**U** $a - \frac{3a}{16}$
**Y** $6a + 3a \div 3$

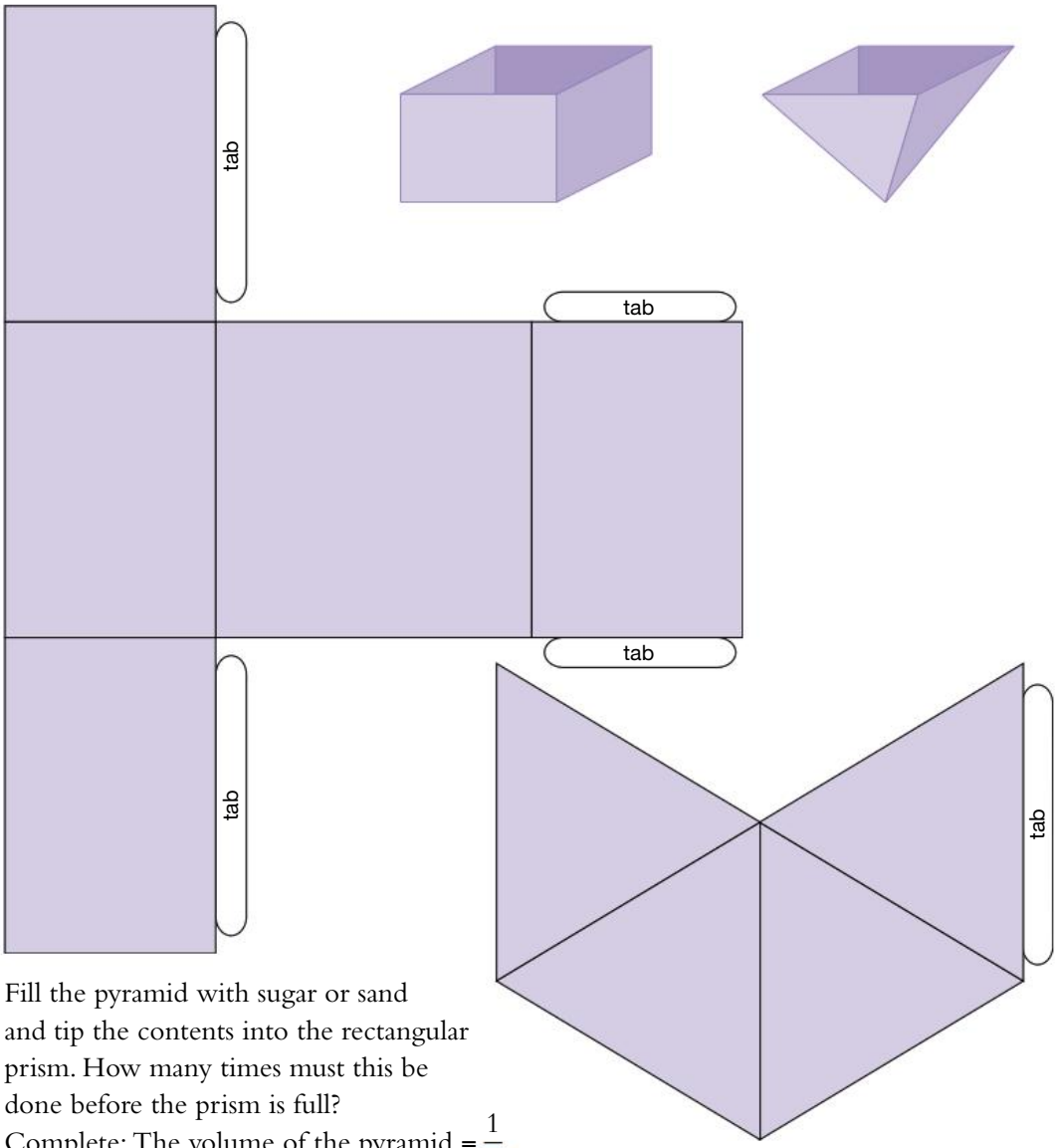
| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x^8$ | $x = \frac{1}{2}$ | $7x$ | $7a$ | $a^2$ | $\frac{13a}{16}$ | $9x$ | $x = 81$ | $a$ | $8$ | $9x^2$ | $-12$ | $8x^2$ | $12$ | $x = 1$ | $x = \frac{1}{9}$ | $4$ |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{9a}{20}$ | $-15$ | $x = 10$ | $60$ | $-7x$ | $2x - 2$ | $-44$ | $1$ | $8a^6$ | $33$ | $2xy$ | $6x - 4$ | $x = 0$ | $16$ | $x = \pm 3$ | $6$ | $x = 9$ |

# 8:05 Volume of a pyramid

## INVESTIGATION 8:05 THE VOLUME OF A PYRAMID

- Photocopy the nets below onto light cardboard.
- Use these nets (with tabs) to construct an open rectangular prism of length 4·2 cm, breadth 4·2 cm and height 2·8 cm and an open pyramid of length 4·2 cm, breadth 4·2 cm and height 2·8 cm.
  (*Note:* To produce this pyramid, each triangle must have a height of 3·5 cm.)



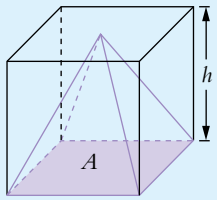

- Fill the pyramid with sugar or sand and tip the contents into the rectangular prism. How many times must this be done before the prism is full?
- Complete: The volume of the pyramid = $\frac{1}{\square}$ of the volume of the prism.

Investigation 8:05 demonstrates that the volume of a pyramid is one-third of the volume of a prism with the same base area and height.

The volume of a pyramid is given by the formula:

$$V = \frac{1}{3}Ah$$

where $V$ = volume, $A$ = area of base and $h$ is the height of the pyramid.




## WORKED EXAMPLE 1

**a** Find the volume of a rectangular pyramid that has a base 6·2 cm long and 4·5 cm wide and a height of 9·3 cm.

**b** A square pyramid has a volume of 625 $cm^2$ and a height of of 12 cm. What is the length of the side of the square?

### Solutions

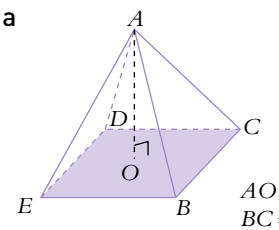
**a**
$V = \frac{1}{3}Ah$
$A = 6{\cdot}2 \times 4{\cdot}5\ cm^2$
$h = 9{\cdot}3\ cm$
$\therefore\ V = \frac{1}{3} \times 6{\cdot}2 \times 4{\cdot}5 \times 9{\cdot}3\ cm^3$
$= 86{\cdot}49\ cm^3$

**b** Let the side length of the square be $x$ cm.
$V = \frac{1}{3}Ah$
$625 = \frac{1}{3} \times x^2 \times 12$
$x^2 = 156{\cdot}25$
$\therefore\ x = 12{\cdot}5\ cm^2$

## WORKED EXAMPLE 2

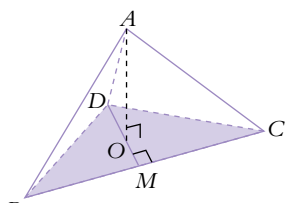
Calculate the volumes of the pyramids pictured.

**a**

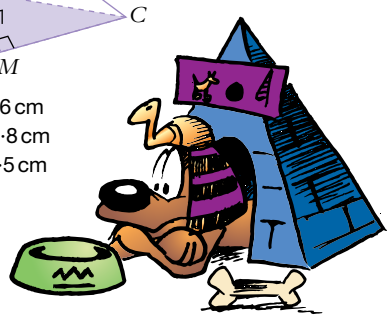



$AO = 12\ cm$
$BC = 6{\cdot}5\ cm$
$EB = 6{\cdot}5\ cm$

**b**




$BC = 26{\cdot}6\ cm$
$DM = 16{\cdot}8\ cm$
$AO = 10{\cdot}5\ cm$



### Solutions

**a**
$V = \frac{1}{3}Ah$
$h = 12\ cm$
$A = 6{\cdot}5 \times 6{\cdot}5\ cm^2$
$\therefore\ V = \frac{1}{3} \times 6{\cdot}5 \times 6{\cdot}5 \times 12$
$= 169\ cm^3$

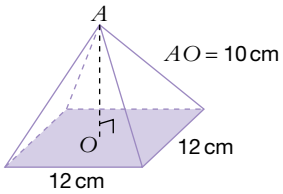
**b**
$V = \frac{1}{3}Ah$
$A = \dfrac{BC \times DM}{2}\ cm^2$
$= \dfrac{26{\cdot}6 \times 16{\cdot}8}{2}\ cm^2$
$h = 10{\cdot}5\ cm$
$\therefore\ V = \frac{1}{3} \times \dfrac{6{\cdot}6 \times 16{\cdot}8}{2} \times 10{\cdot}5$
$= 782{\cdot}04\ cm^3$

# Exercise 8:05

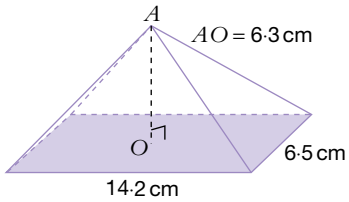
P Foundation worksheet 8:05
Volume of a pyramid

**1** Calculate the volumes of the square and rectangular pyramids drawn below.

**a**




**b**




**2** **a** Calculate the volume correct to two decimal places of a square pyramid with a base edge of 3·25 m and a height of 6·3 m.

**b** A rectangular pyramid is 16·2 cm high. Its base is 5·8 cm long and 7·5 cm wide. Find its volume.

**3** **a** A pyramid has a triangular base with an area of 4·32 $m^2$. Find the volume of the pyramid if it has a height of 2·5 m.

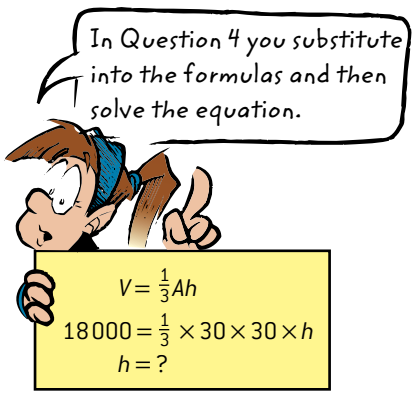
**b** Find the volume of a hexagonal pyramid of height 15 cm if the hexagonal base has an area of 6·2 $cm^2$.

**4** **a** A square pyramid has a volume of 18 000 $cm^3$. If the side length of the square is 30 cm, what is its height?

**b** A pyramid has a 12 cm square as its base. How high must the pyramid be if it is to have a volume of 500 $cm^3$?

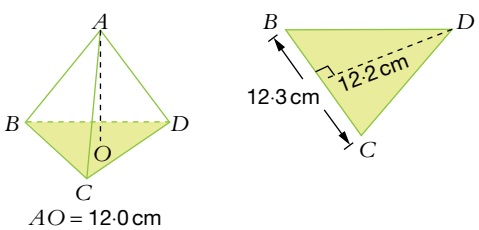
**c** A square pyramid with a height of 120 cm has a volume of 64 000 $cm^3$. What is the area of its base and the length of the side of the square?

**d** A rectangle that is twice as long as it is wide forms the base of a rectangular pyramid that is 60 cm high. If the volume of the pyramid is 4000 $cm^3$, what are the dimensions of the rectangle?

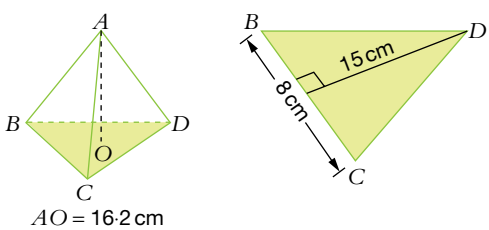



**5** Calculate the volumes of the triangular pyramids below. The base of each pyramid has been drawn separately alongside it.

**a**

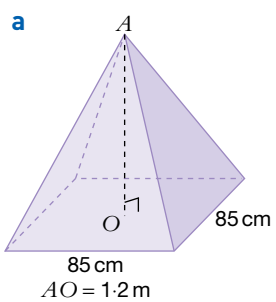



**b**

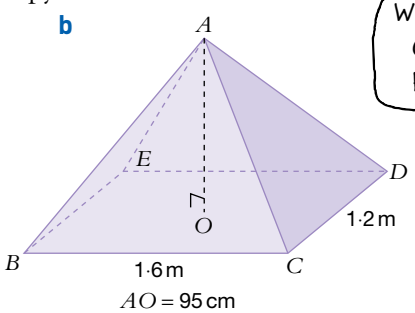


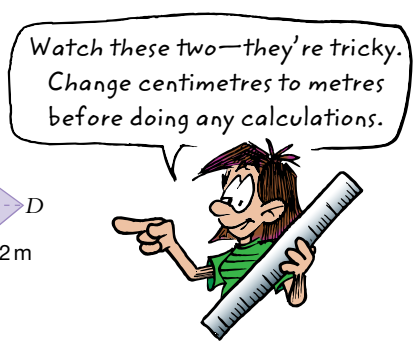

**6** Calculate the volumes of the pyramids drawn below.

**a**



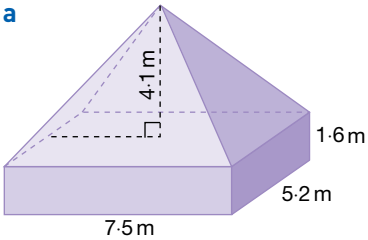

**b**



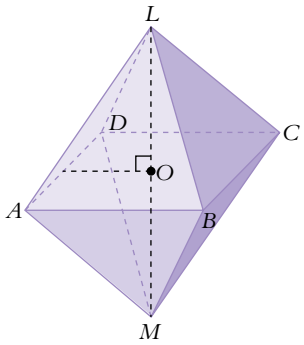



**7** Calculate the volumes of the following solids.

**a**

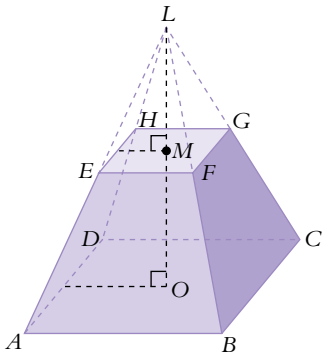



**b**



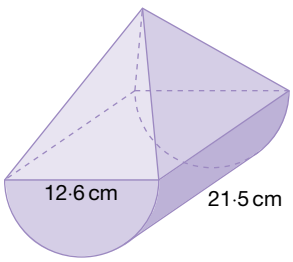

$ABCD$ is a square.
$AB = 12$ cm
$LO = MO = 10$ cm

**c**



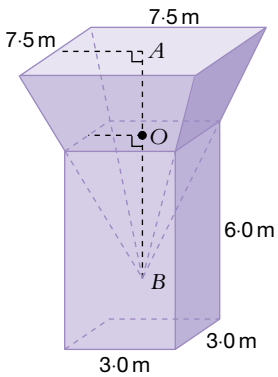

$ABCD$ and $EFGH$ are squares.
$AB = 20$ cm, $EF = 10$ cm
$MO = 15$ cm, $LM = 15$ cm

**d**

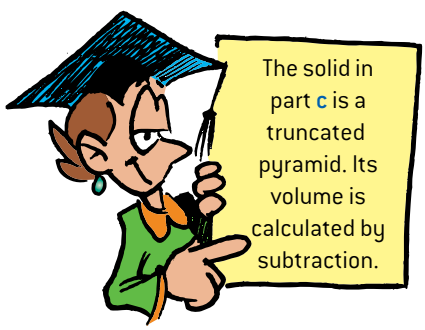



The height of the pyramid is 8·5 cm. (Answer correct to 3 sig. fig.)

**e**

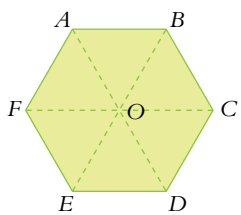



$AO = 4·5$ m
$BO = 3·0$ m



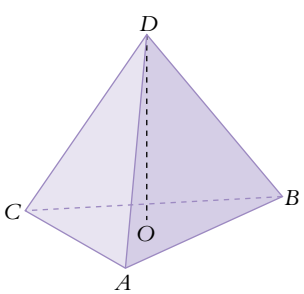

**8** **a** Calculate the volume of a right square pyramid if it has a base area of 64 cm$^2$ and the distance from the apex to a corner of the base is 15 cm.

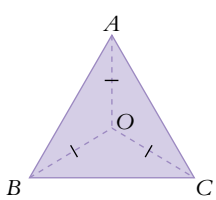
**b** Calculate the volume of a pyramid that has a height of 8 cm. The base of the pyramid is a regular hexagon with a side length of 6 cm. The base is shown in the diagram.




$O$ is the centre of the base.

**9** A tetrahedron is a triangular pyramid in which each face is an equilateral triangle. Calculate the volume of a tetrahedron that has all its edges 6 cm in length. *Hint:* You will need to know some geometry and trigonometry. Diagrams of the pyramid and its base are drawn below.



# 8:06 Volume of a cone

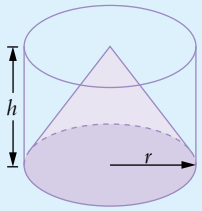
Just as the cylinder could be thought of as a 'circular prism', so the cone can be thought of as a 'circular pyramid'.

The volume of a cone is one-third of the volume of a cylinder with the same base area and height.

The volume of a cone is given by the formula:

$$V = \frac{1}{3}\pi r^2 h$$

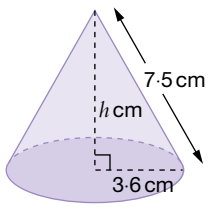
where $r$ is the radius of the circular base and $h$ is the height.




## WORKED EXAMPLES

**1** Find the volume of a cone with a radius of 6·2 cm and a height of 5·8 cm. Give your answer correct to three significant figures.

**2** Use Pythagoras' theorem to calculate the height, $h$, correct to three decimal places and then use this value to calculate the volume correct to three significant figures.




### Solutions

**1** $V = \frac{1}{3}\pi r^2 h$

$r = 6{\cdot}2$ cm

$h = 5{\cdot}8$ cm

$\therefore V = \frac{1}{3} \times \pi \times (6{\cdot}2)^2 \times 5{\cdot}8$

$= 233\text{ cm}^3$ (3 sig. fig.)

**2** By Pythagoras' theorem

$h^2 + 3{\cdot}6^2 = 7{\cdot}5^2$

$h^2 = 7{\cdot}5^2 - 3{\cdot}6^2$

$h = \sqrt{7{\cdot}5^2 - 3{\cdot}6^2}$

$= 6{\cdot}580$ (3 dec. pl.)

$V = \frac{1}{3}\pi r^2 h$

$r = 3{\cdot}6$

$h = 6{\cdot}580$

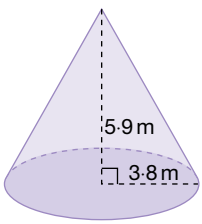
$\therefore V = \frac{1}{3} \times \pi \times 3{\cdot}6^2 \times 6{\cdot}580$

$= 89{\cdot}3\text{ cm}^3$ (3 sig. fig.)

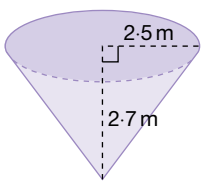
## Exercise 8:06

**1** Find the volumes of the following cones correct to two decimal places.

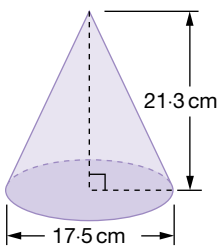
**a**




**b**




**c**



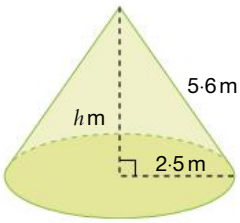
**2** **a** A cone has a base radius of 5·2 cm and a height of 7·8 cm. Calculate its volume correct to two significant figures.

**b** The diameter of the base of a cone is 12·6 cm. If the cone has a height of 15·3 cm, find its volume correct to three significant figures.

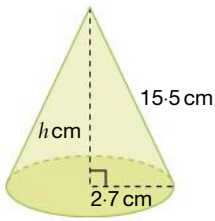
**c** A cone has a base diameter of 2·4 m and a height of 45 cm. Calculate its volume to the nearest tenth of a cubic metre.

**3** Use Pythagoras' theorem to calculate $h$, correct to three decimal places, and then use this value to calculate the volume correct to three significant figures.

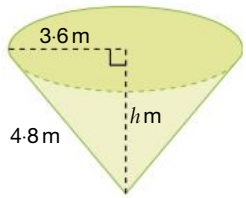
**a**




**b**



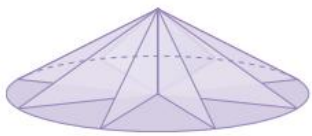

**c**




**4** A right-angled triangle with sides of 5, 12 and 13 cm is rotated to form a cone. What is the volume of the cone if it is rotated about:

**a** the 12 cm side

**b** the 5 cm side?



**5** **a** A cone has a radius of 10 cm. It has a volume of 1000 cm$^3$. Calculate the height of the cone correct to one decimal place.

**b** A cone with a height of 20 cm has a volume of 1500 cm$^3$. Calculate the radius of the cone correct to one decimal place.

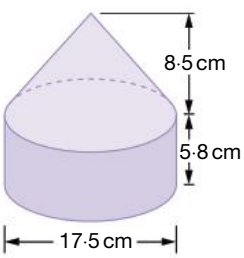
In Question 5, substitute into the formula $V = \frac{1}{3}\pi r^2 h$ and then solve the equation.

**c** The height of a cone is twice its radius and the cone's volume is exactly 1000 cm$^3$. Calculate the dimensions of the cone correct to one decimal place.

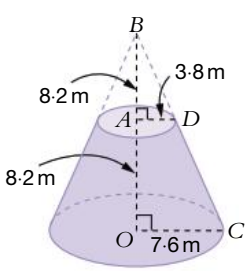
**d** A conical flask filled with liquid has a radius of 10 cm and a height of 10 cm. How high must a cylinder be to hold the same amount of liquid if its radius is 5 cm?

**6** Calculate the volumes of the following solids. Give all answers correct to three significant figures.

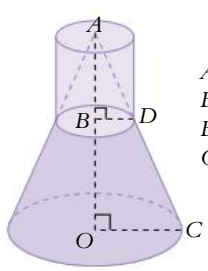
**a**




**b**

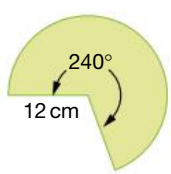



**c**




$AB = 4{\cdot}8$ cm
$BO = 6{\cdot}4$ cm
$BD = 3{\cdot}6$ cm
$OC = 4{\cdot}8$ cm

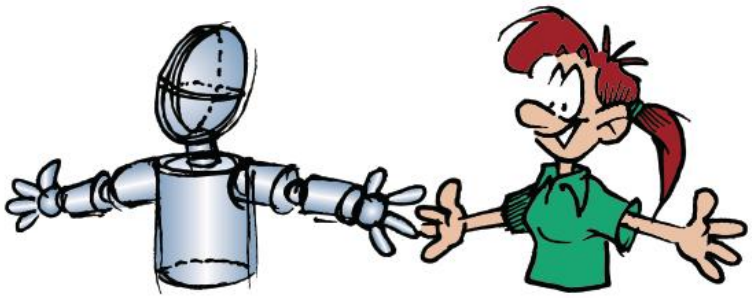
**7** The sector shown is formed into a cone by joining its two radii. Calculate the volume of the cone correct to the nearest cubic centimetre.



# 8:07 Volume of a sphere

## INVESTIGATION 8:07 ESTIMATING YOUR SURFACE AREA AND VOLUME

- Describe how you could estimate your surface area using a suitable roll of material.



- The volume of a solid can be established by placing it in a container of water. The amount of water that it displaces is equivalent to the volume of the solid. Describe how you could use this principle, which was discovered by Archimedes, to calculate your volume.

As with the pyramid and cone, the formula for the volume of a sphere is hard to prove.

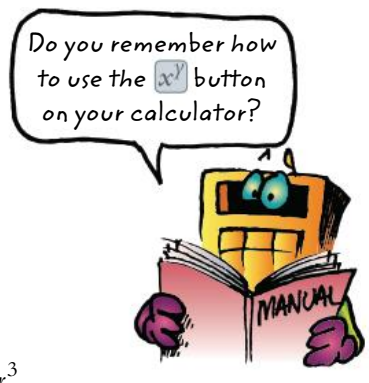
Here the formula is just stated without proof.

The volume of a sphere is given by the formula:

$$V = \frac{4}{3}\pi r^3$$

where $r$ is the radius of the sphere.

## WORKED EXAMPLES




1 Find the volume of a sphere that has a radius of 5·20 cm. Give your answer correct to three significant figures.

2 A sphere has a volume of 1000 m$^3$. Calculate its radius correct to the nearest centimetre.

3 If the Earth is considered to be a sphere of radius 6378 km, find its volume correct to four significant figures.

### Solutions

1 $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times (5{\cdot}20)^3$

$= 589\,\text{cm}^3$ (3 sig. fig.)

3 $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 6378^3$

$= 1{\cdot}087 \times 10^{12}\,\text{km}^3$ (4 sig. fig.)

2 $V = \frac{4}{3}\pi r^3$

$1000 = \frac{4}{3}\pi r^3$

$r^3 = \frac{3000}{4\pi}$

$r = \sqrt[3]{\frac{3000}{4\pi}}$

$\therefore\ r = 6{\cdot}20\,\text{m}$ (nearest cm)

## Exercise 8:07

**1** Find the exact volumes of the following spheres:

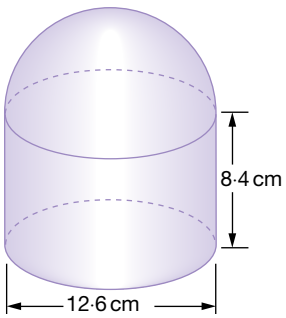
**a** radius = 3 cm
**b** diameter = 4 m
**c** radius = 3·6 m
**d** diameter = 4·8 m

**2** Find the volumes of the following solids. Give the answers correct to three significant figures.

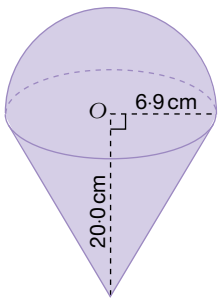
**a** sphere, radius 1·2 m
**b** sphere, diameter = 25·6 cm
**c** hemisphere, radius 3·15 cm
**d** hemisphere, diameter = 2·40 m

**3** Calculate the volumes of the following solids, correct to three significant figures.

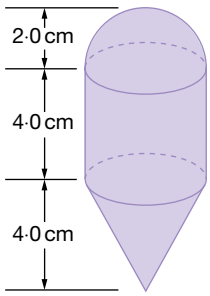
**a**




**b**




**c**




**4** Calculate the volume of a spherical shell that has an inner radius of 5 cm and an outer radius of 6 cm. Give the answer in terms of $\pi$.

**5** **a** A sphere has a volume of 2000 $cm^3$. Calculate its radius correct to one decimal place.

**b** What is the smallest radius a sphere can have if it has to have a volume greater than 5000 $cm^3$? Round your answer up to the nearest whole number.

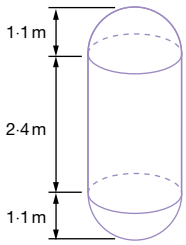
**c** A sphere has to have a volume of 1 $m^3$. What is the radius of the sphere? Round your answer up to the nearest centimetre.

Why is gas stored in spherical tanks? Estimate the volume of gas held in one of these tanks.

# 8:08 Practical applications of surface area and volume

## WORKED EXAMPLE

A buoy consists of a cylinder with two hemispherical ends, as shown in the diagram. Calculate the volume and surface area of this buoy, correct to one decimal place.




### Solution

The two hemispheres have the same radius, so they will form a sphere if joined.

Volume = volume of sphere + volume of cylinder

$= \frac{4}{3}\pi r^3 + \pi r^2 h$

$= \frac{4}{3}\pi(1{\cdot}1)^3 + \pi \times 1{\cdot}1^2 \times 2{\cdot}4$

$= 14{\cdot}7\,\text{m}^3$ (1 dec. pl.)

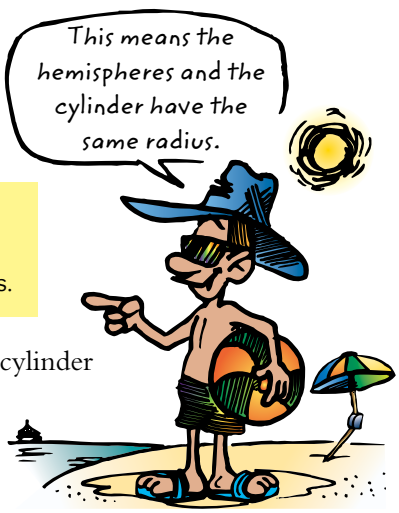
The height of the hemisphere is equal to its radius.

Surface area = surface area of sphere + curved surface area of cylinder

$= 4\pi r^2 + 2\pi rh$

$= 4\pi \times 1{\cdot}1^2 + 2 \times \pi \times 1{\cdot}1 \times 2{\cdot}4$

$= 31{\cdot}8\,\text{m}^2$ (1 dec. pl.)

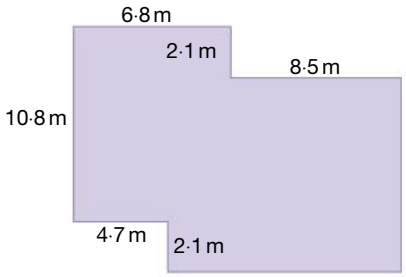



## Exercise 8:08

**1** Assuming that the Earth is a sphere of radius 6400 km, find correct to two significant figures:

- **a** the volume of the Earth in cubic metres
- **b** the mass of the Earth if the average density is 5·4 tonnes/$\text{m}^3$
- **c** the area of the Earth's surface covered by water if 70% of the Earth is covered by water.

**2** A house is to be built on a concrete slab that is 20 cm thick. The cross-sectional shape of the slab is shown in the diagram. Calculate the volume of concrete needed for the slab.




**3** A glasshouse is in the shape of a square pyramid. Calculate the area of the four triangular faces to the nearest square metre if the side of the square is 20 m and the height of the pyramid is 17 m.

**4** A special chocolate is made in the shape of a spherical shell. The outside diameter of the shell is 4 cm while the inside diameter is 3·7 cm. Calculate the volume of chocolate required.

**5** A gas tank consists of a cylinder with a hemisphere on top. It is made of metal that is 1 cm thick. It has an overall height of 90 cm and an external diameter of 20 cm. Calculate:

**a** the surface area of the tank

**b** the internal volume of the tank.

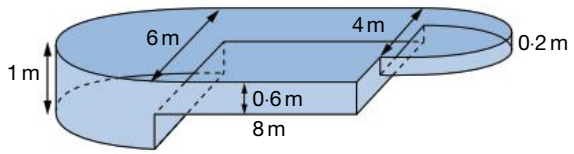
(Give answers correct to two significant figures.)

**6** The cross-section of a swimming pool consists of a rectangle with two semicircular ends. The depth of the water in the smaller semicircular end is 0·2 m while the depth in the other semicircular end is 1m. The rectangular part has a depth of 0·6 m. Calculate:

**a** the capacity of the pool to the nearest kilolitre

**b** the cost of tiling the walls and floor of the pool, to the nearest $100, at $100 per $m^2$.

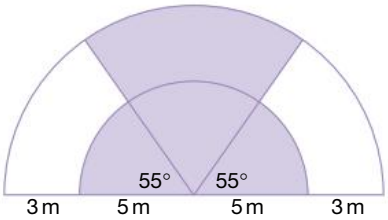
*Remember:* $1\text{ kL} = 1\text{ m}^3$



**7** The diagram shows the cross-sectional area of part of a landscape design. The shaded area is to be concreted while the remaining area is to form two fish ponds.

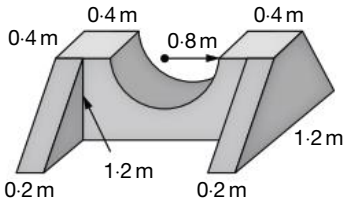
**a** Calculate the volume of concrete needed if it is to have a uniform depth of 100 mm. (Answer correct to one decimal place.)

**b** Calculate the capacity of the fish ponds in kilolitres if the depth of water is 60 cm.



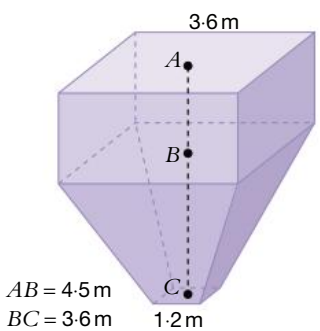

*Remember:* 1 L = 1000 $cm^3$

**8** The structure shown is made to support a water pipe. The two identical ends are trapezoidal prisms that join a rectangular prism with a half-cylinder removed. Calculate the volume of concrete needed to make the structure, correct to two decimal places.

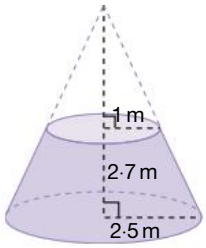



**9** The large tank in the photo consists of two cones and a cylinder. If the diameter of the cylinder is 5·2 m and the heights of the bottom cone, cylinder and top cone are 2·8 m, 8·5 m and 1·8 m respectively, calculate the volume of the tank correct to one decimal place.

**10** A storage bin has been made from a square prism and a square pyramid. The top 1·8 m of the pyramid has been removed. Calculate the volume of the bin.

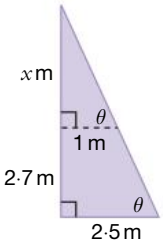


**11** **a** The solid shown is known as a frustrum.
It is formed by removing the top part of the cone.

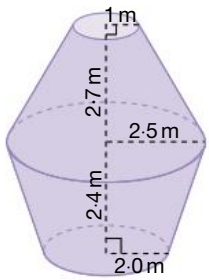



**i** By comparing the values of tan $\theta$ in two different triangles, find the value of $x$.

**ii** Find the volume of the frustrum.

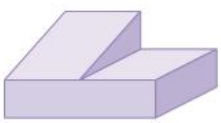



**b** A storage bin for mixing cement is formed from two truncated cones (frustrums). Calculate the volume of this bin.




## MATHS TERMS 8

**composite solid**
- a solid that is formed by joining simple solids

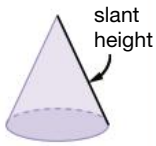


**prism (right)**
- a solid that has two identical ends joined by rectangular faces

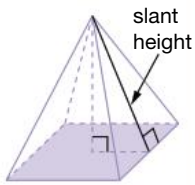
**pyramid**
- a solid that has a base from which triangular faces rise to meet at a point (called the apex)

**perpendicular height (of a cone or a pyramid)**
- the distance from the apex of a cone or a right pyramid to the centre of the base

**slant height (of a cone)**
- the distance from a point on the circumference of the circular base to the apex of the cone




**slant height (of a pyramid)**
- the perpendicular height of the triangular face of a pyramid. (i.e. the distance from the apex of the pyramid to the midpoint of the base edge)



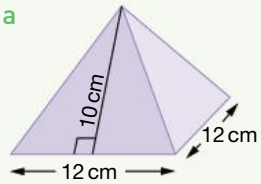

**surface area**
- the sum of the areas of the faces or surfaces of a three-dimensional figure (or solid)

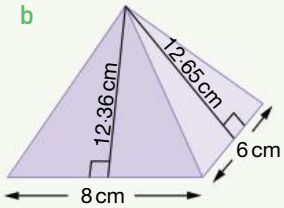
**volume**
- the amount of space (cubic units) inside a three-dimensional shape

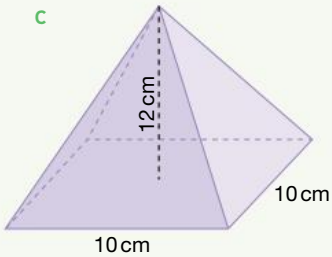
Each part of this test has similar items that test a certain question type.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

8:02

1 Calculate the surface areas of the following right pyramids.

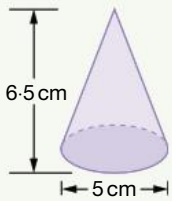
a 

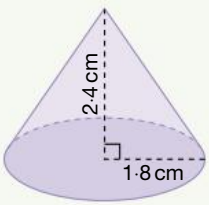

b 

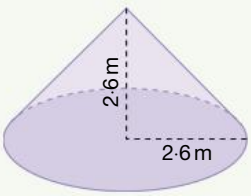

c 


8:03

2 Calculate the surface areas of the following cones correct to one decimal place.

a 


b 


c 

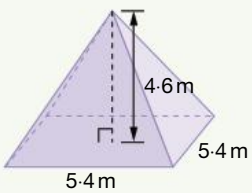

8:04

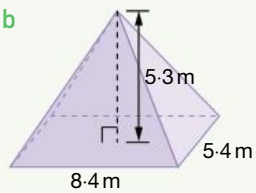
3 Calculate the surface areas of the following solids correct to two decimal places.

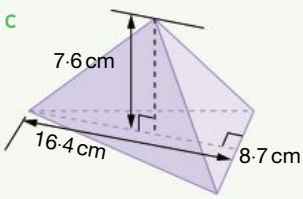
a a sphere of radius 5 cm
b a sphere of diameter 16·6 cm
c a hemisphere of radius 3 cm

8:05

4 Calculate the volumes of the following solids.

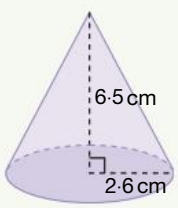
a 

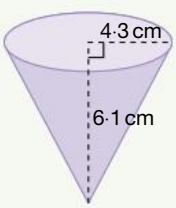

b 

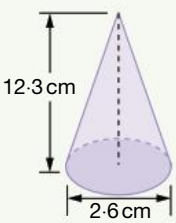

c 


8:06

5 Calculate the volumes of the following solids correct to one decimal place.

a 


b 


c 

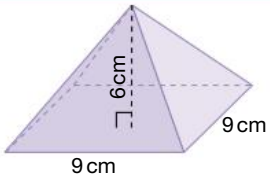

8:07

6 Calculate the volumes of the following solids correct to one decimal place.

a a sphere of radius 5 cm
b a sphere of diameter 8·6 cm
c a hemisphere of diameter 15 cm

## ASSIGNMENT 8A Chapter review

**1** Calculate the volume and surface area of the pyramid shown.

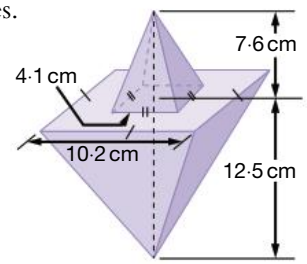



**2** A cone has a diameter of 16 cm and a height of 15 cm. Calculate the surface area of the cone in terms of $\pi$.

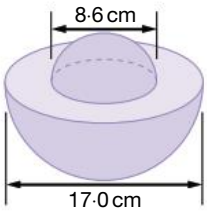
**3** Calculate the surface area of a hemisphere that has a diameter of 16 cm. Give your answer correct to two significant figures.

**4** A spherical tank is to hold $100\,m^3$. What radius (to the nearest millimetre) will give a volume closest to $100\,m^3$?

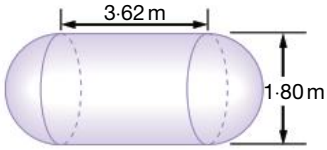
**5** Calculate the volume of the solid pictured. Give the answer correct to three significant figures.



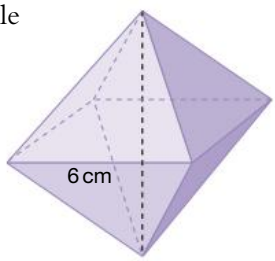

**6** Calculate the surface area of the solid shown. Give the answer correct to three significant figures.




**7** A tank for holding liquid chemicals consists of a cylinder with two hemispherical ends as shown in the diagram. Calculate its volume correct to three significant figures.




**8** An octahedron is a double pyramid with all its edges equal in length. Calculate the surface area and volume of an octahedron with all its edges 6 cm in length.



Engineers solve many surface area and volume problems in the design and construction of buildings.

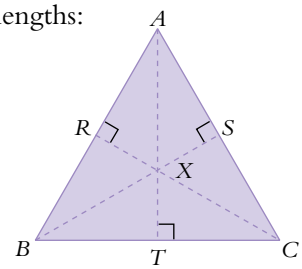
## ASSIGNMENT 8B Working mathematically

**1** The value of a library is depreciated at a rate of 15% pa. If the library is presently valued at $800 000, what will its value be after 4 years?

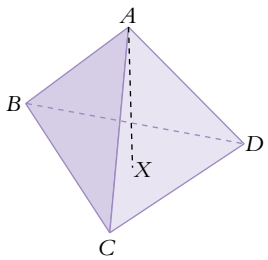
**2** $\Delta ABC$ is equilateral with a side of length $a$ units. $AT$, $BS$ and $RC$ are perpendicular heights of the triangle that meet at $X$. Find the lengths:

**a** $BX$

**b** $XT$

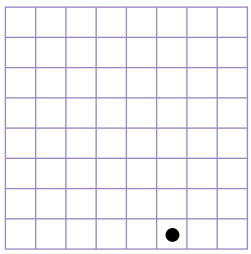



**3** $ABCD$ is a right triangular pyramid. Its base is an equilateral triangle of side $\sqrt{72}$ units, and the edges $AB$, $AC$ and $AD$ are all 6 units long. Find the height, $AX$ and the volume of the pyramid.

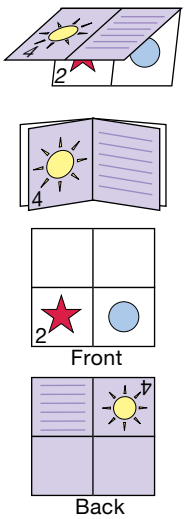



**4** Three positive whole numbers are multiplied in pairs. The answers obtained are 756, 1890 and 4410. What are the numbers?

**5** The dot indicates the position of one chess queen on a chess board. How many more queens can you place on the board so that none of the queens threatens another?



**6** One large sheet of paper was ruled up and folded. It was then cut along the fold shown in the diagram on the top to form an 8-page booklet. On the diagrams to the right, write the page number on each quarter (as has been done for pages 2 and 4).



A glass and steel pyramid forms the entrance to the Louvre in Paris.

# ASSIGNMENT 8C Cumulative revision

**1** **a** Find the equation of the line that is parallel to the line $y = 3x - 5$ and passes through the point $(1, 7)$.
**b** A line is perpendicular to the line $y = -2x + 4$ and has a $y$-intercept of $-3$. What is the equation of the line?

7:02, 7:03

**2** From the following list of equations choose those that represent:
**a** lines **b** parabolas **c** circles **d** exponentials

| | | | |
|---|---|---|---|
| $y = x^2 - 2$ | $y = 2^x$ | $y = 1 - 2x^2$ | $y = x - 2$ |
| $y = -2x^2$ | $y = -x - 2$ | $y = 2^{-x} - 2$ | $y = -2$ |
| $y = 2 - 2^x$ | $x^2 + y^2 = 1$ | $y = -2x$ | $x = 2$ |

Chapter 7

**3** Expand and simplify:
**a** $(2 + \sqrt{5})(3 - \sqrt{5})$ **b** $(\sqrt{7} - \sqrt{5})(\sqrt{7} + \sqrt{5})$ **c** $(2\sqrt{5} - \sqrt{3})^2$

5:05

**4** Express each of the following with a rational denominator.
**a** $\frac{6}{\sqrt{3}}$ **b** $\frac{6\sqrt{2}}{\sqrt{3}}$ **c** $\frac{\sqrt{2}}{2\sqrt{3}}$ **d** $\frac{6\sqrt{2}}{4\sqrt{3}}$

5:06

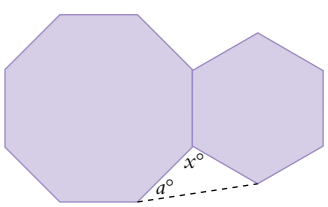
**5** Hyun buys 10 tickets in a charity raffle in which 1000 tickets are sold. Three tickets will be selected at random, with each ticket winning a prize. What is the probability (as a percentage correct to one decimal place) that Hyun:
**a** does not win any prizes **b** wins at least one prize?

4:05

**6** **a** Given that $s = ut + \frac{1}{2}at^2$, find $t$ given that $s = 1000$, $u = 5$ and $a = 10$.
**b** The sum of the squares of three consecutive numbers is 20 669. What are the numbers? (Hint: let the middle number be $x$.)

2:05

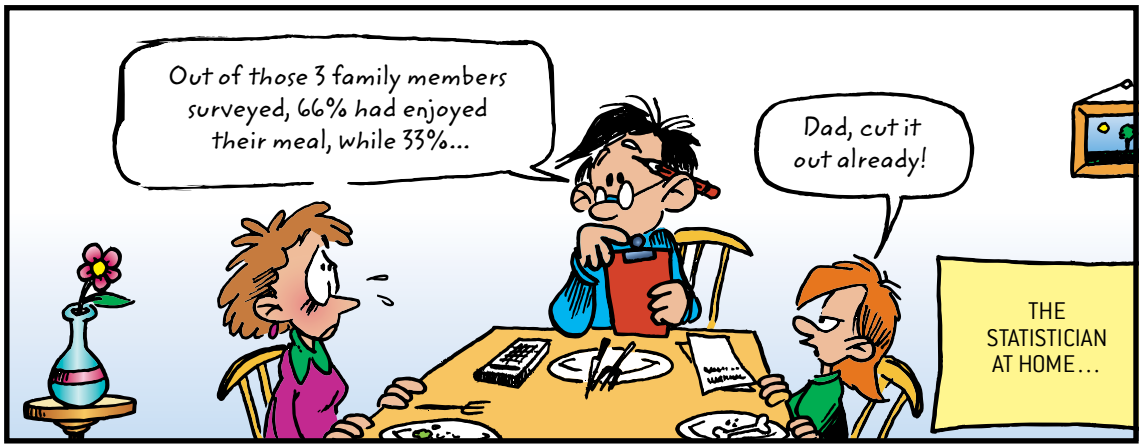
**7** The diagram shows a regular hexagon and octagon. Find the values of the pronumerals in the diagram.




3:06

# STATISTICS

# 9




## Contents


**9:01** Statistics: A review
Fun spot 9:01 Why did the teacher wear sunglasses?
**9:02** Measures of spread: Standard deviation
**9:03** Bivariate data
**9:04** Scatter plots
Challenge 9:04 Further scatter plots
Investigation 9:04 Finding the line of best fit using a spreadsheet
**9:05** Statistics all around us
**9:06** Statistics: Getting it together
Investigation 9:06 Opinion polls: How accurate are they?
Challenge 9:06 Use of Australian census data
**9:07** Statistics: Can they be trusted?
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Statistics and Probability**

Selections from ***Single Variable Data Analysis*** [Stages 5.1, 5.2◊, 5.3] ***Bivariate Data Analysis*** [Stages 5.2, 5.3]

- Identify everyday questions and issues involving at least one numerical and at least one categorical variable, and collect data directly from secondary sources (ACMSP228)
- Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data (ACMSP253)
- Investigate reports of surveys in digital media and elsewhere for information on how data was obtained to estimate population means and medians (ACMSP227)
- Calculate and interpret the mean and standard deviation of data and use these to compare data sets (ACMSP278)
- Investigate and describe bivariate numerical data where the independent variable is time (ACMSP252)
- Use scatter plots to investigate and comment on relationships between two numerical variables (ACMSP251)
- Use information technologies to investigate bivariate numerical data sets; where appropriate, students use a straight line to describe the relationship, allowing for variation (ACMSP279)
- Investigate reports of studies in digital media and elsewhere for information on their planning and implementation (ACMSP277)

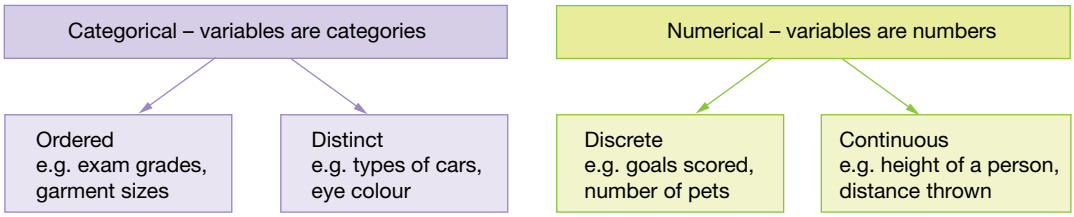
**Working Mathematically**

- Communicating • Problem Solving • Reasoning • Understanding • Fluency

# 9:01 Statistics: A review

## Types of data

The data we collect is made up of variables. These are pieces of information like a quantity or a characteristic that can be observed or measured. They may change either over time or between individual observations. The main types of data are:




## Collecting data

Three main ways of collecting data are shown in the following table.

| Census | Sample | Observation |
|---|---|---|
| A whole population is surveyed, e.g. every student in the school is questioned. | A selected group of a population is surveyed, e.g. a small number in each class is questioned. | Numerical facts are collected and tabulated, e.g. sports data, weather, sales figures, etc. |

A **sample** is usually **random** to limit the chance of bias occurring. However, it may be **systematic** if the members of the sample are chosen according to a rule, such as every 10th member of a population. If a population is composed of various sub-groups, the sample could be **stratified** to ensure a proportionate representation of each group in the sample.

**Primary source** data is collected first-hand by observation or survey.

**Secondary source** data is obtained from an external source such as a newspaper, website or another person's research.

## Sorting data

A large amount of data needs to be tabulated (organised into a table) so that it can be analysed. A common form of table is the frequency distribution table. The first table on the following page is for discrete data. The second is for grouped data that may be used to cluster discrete data into groups or to divide continuous data into adjoining groups.

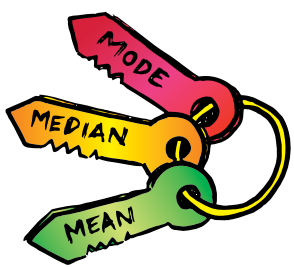
| Outcome ($x$) | Tally | Frequency ($f$) | $f \times x$ | Cumulative Frequency ($c.f.$) |
|---|---|---|---|---|
| 1 | \|\|\| | 3 | 3 | 3 |
| 2 | \|\|\|\| | 4 | 8 | 7 |
| 3 | 𝍸 \|\| | 7 | 21 | 14 |
| 4 | 𝍸 \|\|\|\| | 9 | 36 | 23 |
| 5 | 𝍸 | 5 | 25 | 28 |
| 6 | \|\| | 2 | 12 | 30 |
| | *Total:* | 30 | 105 | |

| Class | Class Centre ($c.c.$) | Tally | Frequency ($f$) | $f \times c.c.$ | Cumulative Frequency ($c.f.$) |
|---|---|---|---|---|---|
| $1 - <5$ | 3 | \|\|\| | 3 | 9 | 3 |
| $5 - <9$ | 7 | \|\|\|\| | 4 | 28 | 7 |
| $9 - <13$ | 11 | 𝍸 \|\| | 7 | 77 | 14 |
| $13 - <17$ | 15 | 𝍸 \|\|\|\| | 9 | 135 | 23 |
| $17 - <21$ | 19 | 𝍸 | 5 | 95 | 28 |
| $21 - <25$ | 23 | \|\| | 2 | 46 | 30 |
| | | *Total:* | 30 | 390 | |

After the data has been sorted, certain key numbers can be determined. Some measure how the data clusters around the 'centre'. These are called **measures of central tendency** (or measures of location). Others measure how the data spreads from the centre. These are called **measures of spread**.

## Measures of central tendency

### Mode




- The score or outcome that occurs the most, i.e. with the highest frequency. For the data in the table above the mode is 4 (highest frequency of 9).
- For grouped data this becomes the modal class. For the table above, the modal class is $13 - <17$.

### Mean

- The sum of the scores divided by the number of scores, i.e. the usual definition of 'average'.
- For tabulated data: $\text{mean} = \dfrac{\text{sum of } f \times x \text{ column}}{\text{sum of freq. column}}$ or $\text{mean} = \dfrac{\text{sum of } f \times c.c. \text{ column}}{\text{sum of freq. column}}$

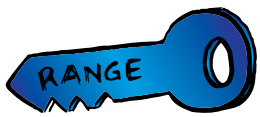
  For the tables above: $\text{mean} = \frac{105}{30} = 3{\cdot}5$ $\text{mean} = \frac{390}{30} = 13{\cdot}0$

**Median**

- The 'middle' score when the scores are placed in order. If there is an even number of scores the median is the average of the two middle scores. The frequency table on the previous page has 30 scores, so the median is the average of the 15th and 16th scores. Looking down the c.f. column, the 15th and 16th scores are both 4. So the median is 4.
- For grouped data this becomes the median class. In the grouped frequency table on the previous page, the 15th and 16th scores both lie in the class 13 – <17. So this is the median class.

## Measures of spread

**Range**




- The highest score minus the lowest score. In the first table on the previous page, the highest score is 6 and the lowest score is 1.
  Range = 6 – 1 = 5
- For grouped data, unless the original scores are known, the maximum possible range can be determined by using the class groupings. In the second table on the previous page, the highest possible score is close to 25 and the lowest is 1.
  Range = 25 – 1 = 24

**Interquartile range (IQR)**

- The range of the middle 50% of scores, i.e. the difference between the first and third quartiles: IQR = $Q_3 - Q_1$
- For the 30 scores shown here:

1 2 2 3 3 3 4 (4) 5 5 5 6 6 6 7 7 | 7 8 8 8 9 9 (9) 10 10 10 10 11 11 12

$Q_1$ median ($Q_2$) $Q_3$

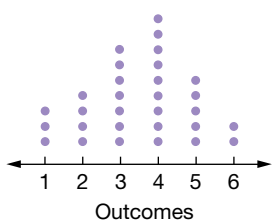
$\therefore$ IQR = 9 – 4
= 5

- The interquartile range is not very meaningful for a small set of scores.
- Associated with these measures of spread is the 'five-number summary' of a set of data that is defined as: minimum score, first quartile $Q_1$, median $Q_2$, third quartile $Q_3$, maximum score.
- The five-number summary for the scores above is (1, 4, 7, 9, 12).

## Displaying data

A visual display of data is useful in giving an overall impression. Some are reviewed here, the first three using the data in the previous tables.

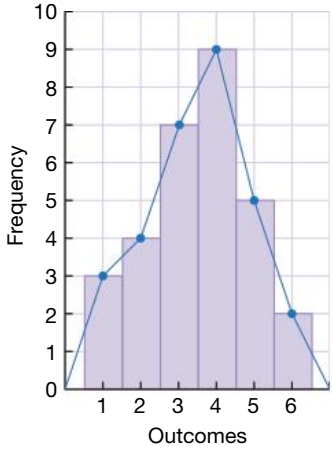
**Dot plot**




- A simple display where each score is represented by a dot.
- The mode is easy to identify as the highest column of dots.
- The highest and lowest scores determine the range.
- A clear impression of the spread of the scores is given.
- Any outliers are easily identified.

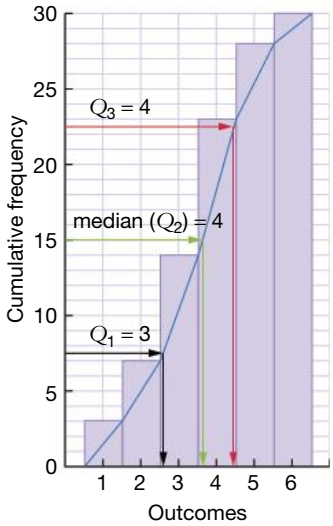
### Frequency histogram and polygon

- The frequency of each score is represented by a column in a histogram and a dot in a polygon. These dots coincide with the centre top of each column. The dots are joined to form the polygon, which is completed by joining to the axis as shown.
- The mode is identified by the highest column.
- A clear impression of the spread of the scores is given.
- Any outliers are easily identified.
- For grouped data the classes can be represented on the horizontal axis by the class centres.




### Cumulative frequency histogram and polygon

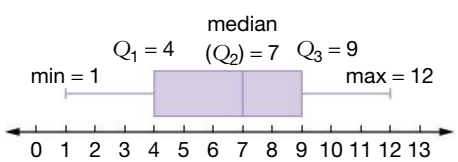
- Graphing the cumulative frequency results in columns of increasing height, the last column representing the total frequency.
- The polygon is formed by joining the corners of adjoining columns as shown.
- This polygon is useful for indicating the median and quartiles, as shown.




### Box plot

- This is drawn using the five-number summary for a set of data.
- It gives an impression of the spread of the data and also whether it is **symmetrical** or **skewed** from its centre. This will be indicated by the box being nearer to one end than the other.
- If there are more low scores the skew is said to be positive; more high scores would mean the data is negatively skewed.

Using the five-number summary (1, 4, 7, 9, 12):




This set of data is fairly symmetrical. It is only slightly negatively skewed.

### Stem-and-leaf plot

- A stem-and-leaf plot resembles a histogram (on its side) in which the data has been grouped.
- The individual scores can still be identified.
- The data may be unordered, as in the plot to the right.
- Two sets of data can be compared using a back-to-back stem-and-leaf plot like the one below. The data in this plot has been ordered.
- The range and mode are easily identified.
- The scores are ordered so the median and quartiles can be determined by counting.

1/7 represents 17

| Stem | Leaf |
|---|---|
| 0 | |
| 1 | 7 6 6 9 |
| 2 | 4 0 8 |
| 3 | 6 2 2 2 0 9 |
| 4 | 9 5 5 6 1 7 8 8 |
| 5 | 9 9 2 9 |
| 6 | 0 |

**Range** = 55 − 8
= 47
**Mode** = 12 (frequency of 3)
There are 26 scores.
**Median** = average of 13th and 14th scores (27, 27)
= 27
$Q_1$ = 12 (7th score)
$Q_3$ = 36 (20th score)
**IQR** = 36 − 12
= 24

| Stem | Leaf | Stem |
|---|---|---|
| 9 8 8 | 0 | |
| 7 4 3 2 2 2 0 | 1 | 6 6 7 9 |
| 9 8 8 7 7 3 1 | 2 | 0 4 8 |
| 9 6 6 4 4 | 3 | 0 2 2 2 6 9 |
| 3 2 2 | 4 | 1 5 5 6 7 8 8 9 |
| 5 | 5 | 2 9 9 9 |
| | 6 | 0 |

**Range** = 60 − 16
= 44
**Mode** = 32 (frequency of 3)
There are 26 scores.
**Median** = average of 13th and 14th scores (39, 41)
= 40
$Q_1$ = 28 (7th score)
$Q_3$ = 48 (20th score)
**IQR** = 48 − 28
= 20

## Exercise 9:01

**1** Find the following measures for each set of discrete numerical data.

**i** range **ii** mode **iii** mean **iv** median

**a** 3, 4, 4, 5, 7, 9, 10

**b** 13, 14, 17, 16, 14, 19

**c** 1·7, 1·9, 1·6, 1·7, 1·4, 1·9, 2·0, 1·5

**d**

| Outcome | $f$ | $fx$ | $c.f.$ |
|---|---|---|---|
| 6 | 2 | 12 | 2 |
| 7 | 3 | 21 | 5 |
| 8 | 6 | 48 | 11 |
| 9 | 7 | 63 | 18 |
| 10 | 2 | 20 | 20 |
| *Total:* | 20 | 164 | |

**e**

| Outcome | $f$ | $fx$ | $c.f.$ |
|---|---|---|---|
| 5 | 4 | 20 | 3 |
| 10 | 6 | 60 | 9 |
| 15 | 8 | 120 | 17 |
| 20 | 5 | 100 | 21 |
| 25 | 2 | 50 | 25 |
| *Total:* | 25 | 350 | |

**2** Which of the following can be applied to this categorical data: the mode, the mean or the median? Find its value for this data.

| Type of car | Frequency |
|---|---|
| Toyota | 250 |
| Mazda | 185 |
| Nissan | 146 |
| Hyundai | 127 |

**3** **a** Determine the mode, mean and median for the scores: 5, 8, 7, 5, 5, 4, 8

**b** Another score of 6 is added to this data set. What are the new values for the mean and median?

**4** **a** Determine the median for the scores: 12, 9, 13, 1, 16, 10, 15

**b** Find the new median caused by an additional score of:

**i** 15 **ii** 9

**5** For the data in this table find the:

**a** range **b** mode **c** mean **d** median

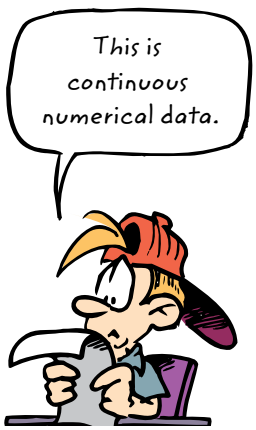
| Score | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| Frequency | 5 | 7 | 10 | 4 | 2 | 2 |

**6** For this grouped data, find the:

**a** maximum range **b** modal class

**c** median class **d** mean

| Class | *c.c.* | $f$ | $f \times c.c.$ | *c.f.* |
|---|---|---|---|---|
| 0–<10 | 5 | 4 | 20 | 4 |
| 10–<20 | 15 | 8 | 120 | 12 |
| 20–<30 | 25 | 10 | 250 | 22 |
| 30–<40 | 35 | 11 | 385 | 33 |
| 40–<50 | 45 | 7 | 315 | 40 |
| | *Total:* | 40 | 1090 | |




**7** **a** Copy and complete these frequency distribution tables and for each find the mean and median.

**i**

| Score | Tally | $f$ | $fx$ | *c.f.* |
|---|---|---|---|---|
| 1 | \|\|\| | | | |
| 2 | 𝍸 | | | |
| 3 | 𝍸 \|\|\|\| | | | |
| 4 | 𝍸 \|\| | | | |
| 5 | \|\|\|\| | | | |
| 6 | \|\| | | | |

*Total:* ___ ___

**ii**

| Score | Tally | $f$ | $fx$ | *c.f.* |
|---|---|---|---|---|
| 0 | 𝍸 \|\| | | | |
| 1 | 𝍸 𝍸 \|\| | | | |
| 2 | 𝍸 𝍸 | | | |
| 3 | 𝍸 \|\|\| | | | |
| 4 | 𝍸 \|\| | | | |
| 5 | \|\|\|\| | | | |
| 6 | \|\|\| | | | |
| 7 | | | | |
| 8 | \| | | | |

*Total:* ___ ___

**b** Construct a frequency histogram and polygon for the data in each table.

**8** **a** Complete this table of grouped data.
**b** What is the maximum possible value for the range?
**c** What is the modal class?
**d** Find the median class.
**e** Use the total of the $f \times c.c.$ column to calculate an estimate for the mean.
**f** Draw a cumulative frequency histogram and polygon for this data. Identify the median class.

| Class | *c.c.* | Tally | *f* | $f \times c.c.$ | *c.f.* |
|---|---|---|---|---|---|
| 0–4 | 2 | \|\|\|\| | | | |
| 5–9 | 7 | 𝍸 \|\|\|\| | | | |
| 10–14 | | 𝍸 𝍸 𝍸 | | | |
| 15–19 | | 𝍸 𝍸 \| | | | |
| 20–24 | | 𝍸 \|\| | | | |
| 25–29 | | \|\|\| | | | |

*Total:* ___ ___

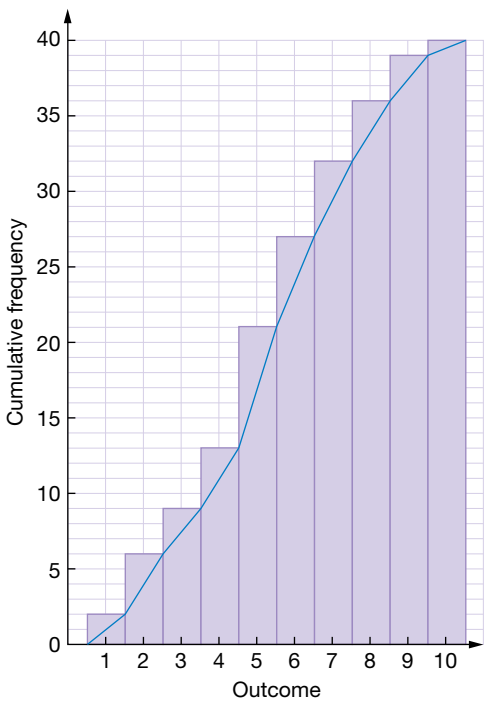
**9** **a** For each set of data, identify the median first, then the first quartile ($Q_1$) and the third quartile ($Q_3$). Also state the interquartile range. (*Note:* The scores must be in order.)

**i** 2 4 5 7 8 8 9 9 10 12
**ii** 3 3 4 5 6 6 7 7 8 9 10 10
**iii** 5 6 3 2 3 6 7 1 8
**iv** 9 4 3 3 6 2 7 8 9 1 2 5 5

**b** Use your answers to **a** to write the five-number summary for each set of data.

**10** **a** Use this cumulative frequency histogram and polygon to determine the following:

**i** the total number of scores
**ii** the range
**iii** the median
**iv** the value of $Q_1$
**v** the value of $Q_3$
**vi** the interquartile range
**vii** the five-number summary.




**b** Use the graph to complete this frequency distribution table. First enter the data in the cumulative frequency column.

| Outcome | *f* | *fx* | *c.f.* |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |
| 8 | | | |
| 9 | | | |
| 10 | | | |

*Total:* ___ ___

**c** Use the table to:

**i** identify the mode
**ii** calculate the mean.

**11** **a** For the data shown in this dot plot find:

**i** the range

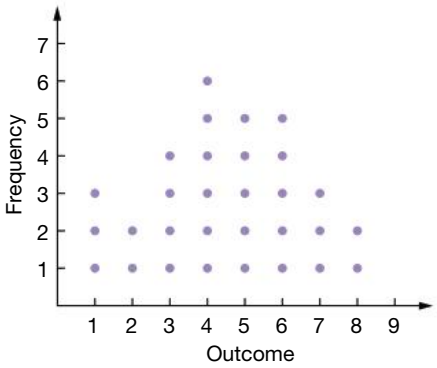
**ii** the median

**iii** the first quartile

**iv** the third quartile.

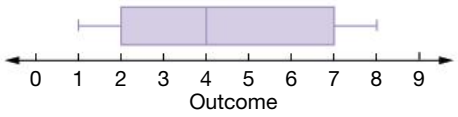
**b** Construct a cumulative frequency histogram and polygon and indicate on it the quartiles and median.

**c** Draw a box plot for this data.

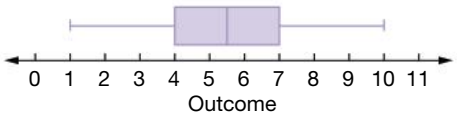



**12** **a** Write the five-number summary for each of these box plots.

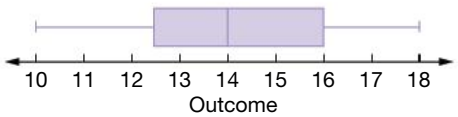
**i**



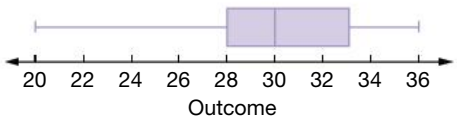

**ii**




**iii**




**iv**




**b** Which box plot shows symmetrical data?

**c** Which box plot shows the data that is most skewed? Is it negatively or positively skewed?

**13** **a** For this ordered stem-and-leaf plot, determine:

**i** the range

**ii** the mode

**iii** the median

**iv** the interquartile range.

**b** Construct a box plot for this set of data.

| Stem | Leaf |
|---|---|
| 0 | 4 8 9 |
| 1 | 1 2 8 |
| 2 | 2 5 5 5 7 9 |
| 3 | 0 4 4 7 7 |
| 4 | 1 2 6 |
| 5 | 1 4 7 9 |
| 6 | 5 |

**14** This back-to-back stem-and-leaf plot shows the percentage test results for two classes.

**a** How many students are in each class?

**b** What is the modal score for each class?

**c** Determine the median for each class.

**d** Calculate the mean for each class.

**e** Determine the values of the first and third quartiles for each class.

**f** Construct a box plot for each class against the same axis.

**g** For which class is the data more skewed?

**h** Class A has four more students than Class B. If the lowest four scores were removed from class A, which class would then have the:

**i** lower median

**ii** lower mean?

| Class A | | Class B |
|---|---|---|
| 6 4 | 3 | 9 |
| 9 9 5 0 | 4 | 6 8 8 |
| 9 7 3 3 1 | 5 | 3 5 6 7 9 |
| 9 8 8 7 7 3 0 | 6 | 4 7 7 7 9 |
| 7 6 1 1 1 1 | 7 | 2 2 4 4 9 |
| 8 5 2 | 8 | 3 4 9 |
| 2 | 9 | 1 5 |

**15** The results of a random survey of adult residents in a particular suburb are shown in this table, indicating their level of income.

| Income (× $1000) | *c.c.* | *f* | *f* × *c.c.* | *c.f.* |
|---|---|---|---|---|
| 30–<40 | | 5 | | |
| 40–<50 | | 8 | | |
| 50–<60 | | 14 | | |
| 60–<70 | | 13 | | |
| 70–<80 | | 10 | | |
| 80–<90 | | 4 | | |
| 90–<100 | | 2 | | |
| | | *Total:* 56 | | |

**a** Copy and complete this frequency distribution table.

**b** Calculate an approximate value for the mean.

**c** Construct a cumulative frequency histogram and polygon.

**d** Use the graph to find the median class and the first and third quartile classes.

**e** Using the class centres to represent each class, write the five-number summary for this grouped data.

**f** Use the five-number summary to draw a box plot.

**g** Does the box plot show the data to be symmetrical or skewed?

## FUN SPOT 9:01 WHY DID THE TEACHER WEAR SUNGLASSES?

Answer each question and write the letter for that question in the box above the correct answer.

*For this set of data 3 4 5 5 7 8 9 11 find:*

**D** the range **G** the mode **S** the mean

**A** the median **I** $Q_1$ **E** $Q_3$

**H** interquartile range

*For the data in this table find:*

| Score | *f* | *fx* | *c.f.* |
|---|---|---|---|
| 10 | 4 | 40 | 4 |
| 12 | 5 | 60 | 9 |
| 14 | 8 | 112 | 17 |
| 16 | 7 | 112 | 24 |
| 18 | 6 | 108 | 30 |
| 20 | 5 | 100 | 35 |
| *Total:* | 35 | 532 | |

**R** the range

**T** the mode

**L** the mean

**S** the median

**H** $Q_1$

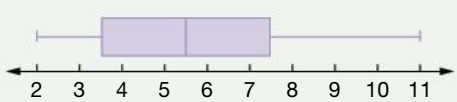
**C** $Q_3$

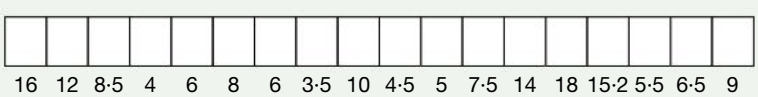
**A** interquartile range

*For this box plot find:*

**S** the range **A** the median

**B** $Q_1$ **H** $Q_3$



# 9:02 Measure of spread: Standard deviation

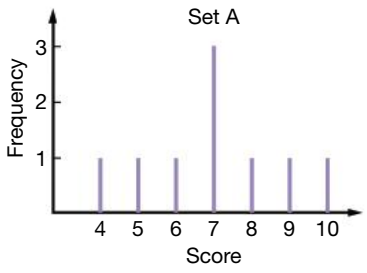
The previous review section reminded you of two measures of spread, the **range** and the **interquartile range**.

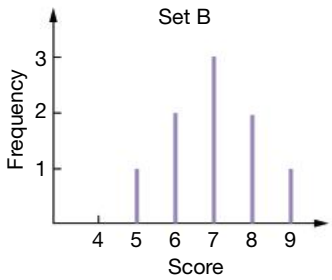
Another measure of spread is **standard deviation**.

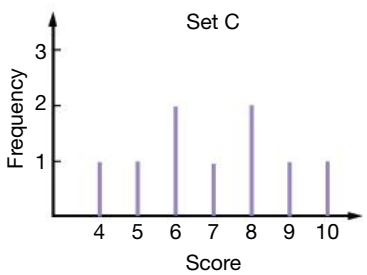
> The standard deviation is a measure of how far the scores are spread about the mean.
> It can be thought of as the average distance of the scores from the mean.

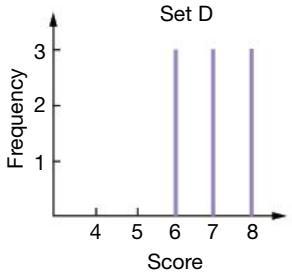
The smaller the standard deviation, the less the spread of the scores.

The graphs below show the distribution of four sets of nine scores. Each set of scores has a mean of 7, but clearly the spread of each set of scores is different.



## How do the standard deviations for the sets vary?

- Set D clearly has the smallest spread of scores about the mean. Three of the scores are on the mean and the other six scores are only one from the mean. It will have the smallest standard deviation.
- Sets A and C are identical except for the distribution of the scores from six to eight. In this range, it can be seen that set A has three scores on the mean and two scores that are one unit from the mean, while set B has only one score on the mean and four scores that are one unit from the mean. So the scores in set C are spread further from the mean than those of set A. Hence, the standard deviation of set C is greater than the standard deviation of set A.
- The scores in set B are not as spread as the scores of sets A and C, but are further spread than those of set D.
- The standard deviation for set B will be higher than that of set D, but lower than that of set A.

## Calculating the standard deviation ($\sigma_x$) using a formula

The standard deviation is defined by the formula below.

$$\text{Standard deviation} = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$

where $x$ is each score
$\bar{x}$ is the mean
$n$ is the number of scores

$\Sigma$ means 'the sum of'.

The squares of the distances from the mean of each score are averaged, and then the square root of the answer is taken to, in a way, compensate for using the squares of the deviations rather than the deviations themselves.

**Step 1** Calculate the mean $\bar{x}$.

**Step 2** Find the difference between each score and the mean.

**Step 3** Square each difference.

**Step 4** Find the sum of these squares, and divide by $n$.

**Step 5** Take the square root of this result.

## WORKED EXAMPLE 1

Find the standard deviation of these scores: 9, 6, 8, 5, 2, 7, 4, 7.

### Solution

**Step 1** The mean, $\bar{x} = 6$

**Step 2** The difference between each score and the mean is shown in the table.

**Step 3** Each difference in the table has been squared.

| $x$ | 9 | 6 | 8 | 5 | 2 | 7 | 4 | 7 |
|---|---|---|---|---|---|---|---|---|
| $x - \bar{x}$ | 3 | 0 | 2 | −1 | −4 | 1 | −2 | 1 |
| $(x - \bar{x})^2$ | 9 | 0 | 4 | 1 | 16 | 1 | 4 | 1 |

**Step 4**
$$\frac{\Sigma(x - \bar{x})^2}{n} = \frac{9 + 0 + 4 + 1 + 16 + 1 + 4 + 1}{8}$$
$$= 4{\cdot}5$$

**Step 5**
$$\sqrt{\frac{\Sigma(x - \bar{x})^2}{n}} = \sqrt{4{\cdot}5}$$
$$\therefore \sigma_x = 2{\cdot}12 \text{ (2 dec. pl.)}$$

$\sigma_x$ is the symbol for population standard deviation.

## Calculating the standard deviation ($\sigma_x$) using a calculator

The instructions for the example below are for a Casio fx-82AU PLUS. Consult your calculator's manual if you have a different brand or model.

To find the standard deviation for the scores in the table shown:

- Press SHIFT then SET UP and toggle to the second screen of functions. Select 3: STAT. Then select 1:ON to choose a frequency column. (Select 2:OFF for a set of individual scores)
- Press MODE then select 2: STAT to enter statistics mode.
- Select 1: 1–VAR and two columns should appear on the screen.
- Enter the scores in the 'Outcome' column labelled '$x$' using the equals key: 0 = 1 = 2 = etc.
- Toggle across to the second column labelled 'FREQ' and up to the top of the column and enter the frequency for each score: 2 = 3 = 4 = etc.

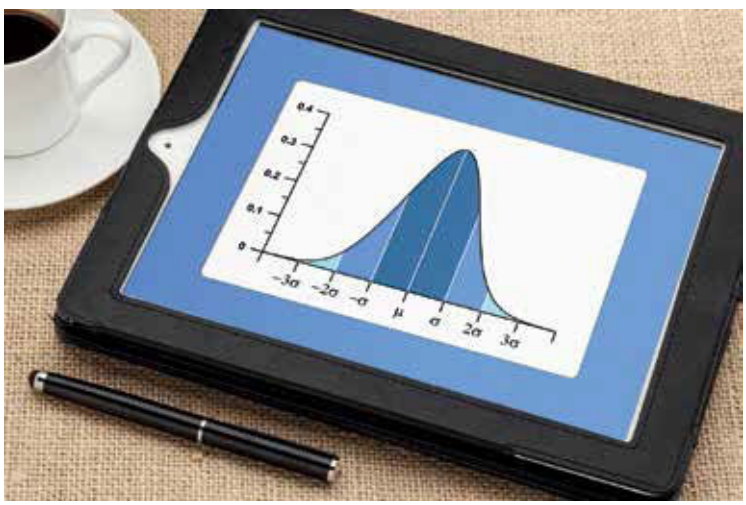
| Outcome | Frequency |
|---|---|
| 0 | 2 |
| 1 | 3 |
| 2 | 4 |
| 3 | 7 |
| 4 | 3 |
| 5 | 1 |

Now this step is IMPORTANT!
Press the AC (All Clear) key to remove the table from the screen.

*Note:* If you don't enter any numbers in the frequency column the calculator assumes there is only 1 of each score.

- Press SHIFT then STAT (located above the 1 key).
- Select 4: VAR. You will be presented with four choices.
- Select 2: $\bar{x}$ and then press the equals key. This will give the mean of 2·45.
- Again: Press SHIFT then STAT. Select 4: VAR.
- Select 3: $\sigma x$ and then press the equals key. This will give you the standard deviation of 1·32, correct to two decimal places.

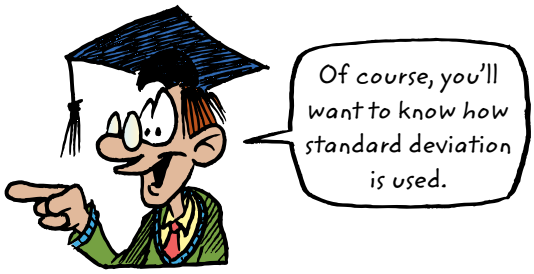
The $\sigma_x$ or $\sigma_n$ symbol is used to represent the population standard deviation.
*Note:* There are two standard deviations: the population standard deviation ($\sigma_x$) and the sample standard deviation ($\sigma_{x-1}$ or $s_x$). In this course we will only use the population standard deviation.



## Comparing performances using the standard deviation

- The standard deviation may be used to compare performances on different tests.
- We compare marks by seeing how many standard deviations each mark is above or below the mean.
- Consider the results below.

| | Score | $x$ | $\sigma_n$ |
|---|---|---|---|
| Maths | 72 | 60 | 12 |
| English | 72 | 60 | 4 |




The scores seem the same until we notice that the English marks have a much smaller standard deviation.

- A small standard deviation means that the scores are clustered together, so an English mark of 72 is probably the highest English mark.
- A large standard deviation means that the scores are well spread. The Maths mark of 72 is above average but nowhere near the top of the class.

For Maths, 72 is 1 standard deviation above the mean.
For English, 72 is 3 standard deviations above the mean.
The English mark is far more impressive.

## WORKED EXAMPLE

The marks from two tests are shown below. In which test would a result of 15 be more impressive?

**Test A:** 9, 11, 12, 12, 13, 13, 14, 15, 15, 16
**Test B:** 5, 7, 9, 10, 12, 13, 13, 15, 17, 19

### Solution

Using the calculator, we can find the mean and standard deviation for each test.

**Test A:** $\bar{x} = 13$, $\sigma_n = 2{\cdot}0$
**Test B:** $\bar{x} = 12$, $\sigma_n = 4{\cdot}1$ (1 dec. pl.)

For test A, 15 is 1 standard deviation above the mean. $(13 + 2{\cdot}0 \times 1 = 15)$
For test B, 15 is 0·73 standard deviations above the mean. $(12 + 4{\cdot}1 \times 0{\cdot}73 = 15$ *Note:* $\frac{3}{4{\cdot}1} = 0{\cdot}73)$
This means that a result of 15 on test A is more impressive.

## Exercise 9:02

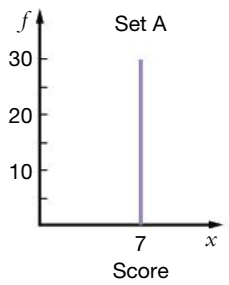
P Foundation worksheet 9:02 Standard deviation

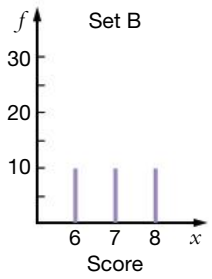
**1** Determine the mean of the scores 5, 9, 7, 6, 3 and then find their standard deviation using the formula: $\sigma_x = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n}}$

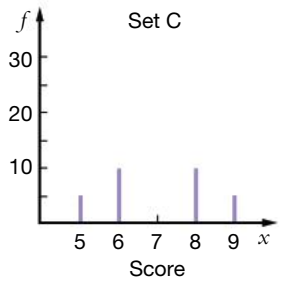
**2** Each of the following sets of scores has a mean of 7. Which set of scores has:

**i** the smallest standard deviation **ii** the largest standard deviation?

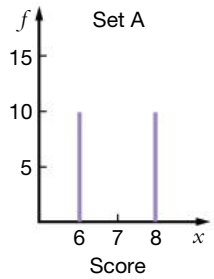
**a**

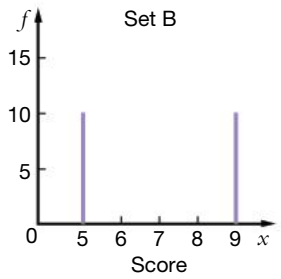


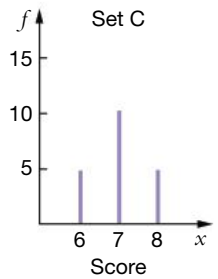



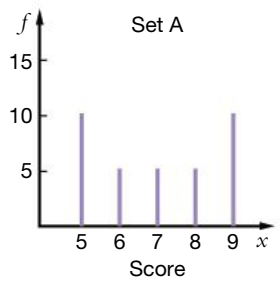

**b**

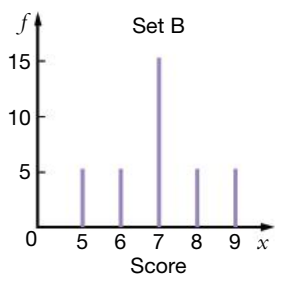


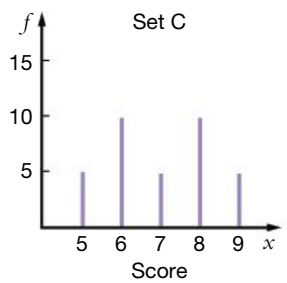



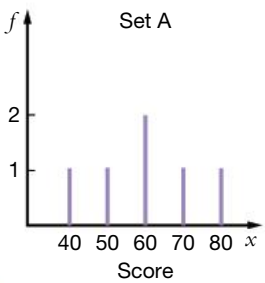

**c**

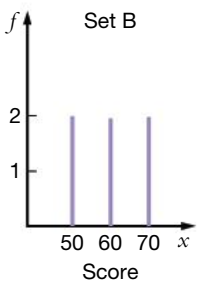


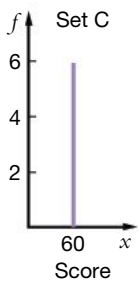



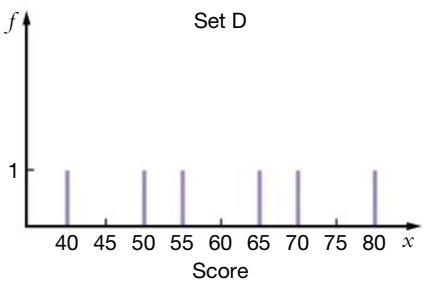




**3** Each of the sets of scores A to D has a mean of 60. Without calculating the standard deviations, arrange the sets in order of increasing standard deviation.



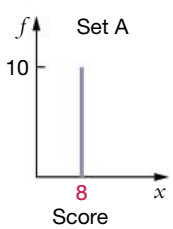



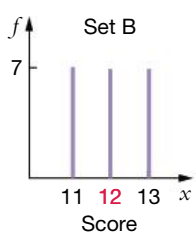



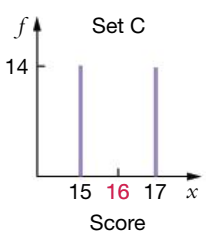



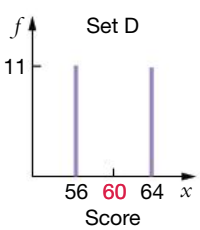

**4** Which of the sets of scores A to D has a standard deviation of:

**A** 0 **B** 1 **C** 4 **D** 0·8 (1 dec. pl.)?




The mean for each set is coloured red.

**5** Use your calculator to find the mean ($x$) and standard deviation ($\sigma_n$) for each set of scores (correct to one decimal place).

**a** 5, 3, 7, 9, 8, 8, 6, 3
2, 1, 9, 7, 6, 6, 5, 2

**b** 11, 9, 15, 16, 8, 14, 14, 10
12, 13, 17, 17, 10, 8, 19, 11

**c** 9, 7, 2, 5, 6, 3, 9, 7
3, 9, 4, 8, 8, 4, 5, 2
4, 6, 3, 2

**d** 61, 92, 47, 32, 56, 72, 39, 51
27, 32, 84, 90, 67, 59, 77, 66
53, 67, 75, 81, 49, 72, 36, 29

**6** **Set A:** 20, 60, 65, 68, 73
**Set B:** 58, 60, 65, 68, 73

**a** Calculate the standard deviation for both sets of scores.
**b** What is the only difference between the sets of scores?
**c** Do you think that the standard deviation is influenced by an outlier?

**7** Use your calculator to determine the mean and standard deviation of each set of scores represented in the tables below. Give answers correct to two decimal places.

**a**

| Score | Frequency |
|---|---|
| 1 | 2 |
| 2 | 5 |
| 3 | 7 |
| 4 | 4 |
| 5 | 2 |

**b**

| Score | Frequency |
|---|---|
| 8 | 1 |
| 9 | 5 |
| 10 | 9 |
| 11 | 11 |
| 12 | 3 |
| 13 | 2 |

**c**

| Score | Frequency |
|---|---|
| 10 | 5 |
| 20 | 9 |
| 30 | 14 |
| 40 | 11 |
| 50 | 8 |
| 60 | 3 |

**d**

| Score | 47 | 48 | 49 | 50 | 51 | 52 |
|---|---|---|---|---|---|---|
| Frequency | 8 | 11 | 16 | 21 | 14 | 10 |

**e**

| Score | 5 | 10 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| Frequency | 3 | 9 | 11 | 16 | 22 | 17 | 12 | 8 | 2 |

**8** Sets A and B are identical except for the minimum score.
**Set A:** 15, 48, 50, 58, 60, 60, 70, 70, 75, 80, 82, 84, 85, 86
**Set B:** 48, 48, 50, 58, 60, 60, 70, 70, 75, 80, 82, 84, 85, 86

**a** Calculate the mean and standard deviation for both sets of scores.
**b** Do you think the effect of an outlier is as great when there are a larger number of scores?

**9** A building supervisor measured the time it took in days for two builders to build the same houses under similar conditions. Both builders had the same mean of 190 days, but the standard deviation for builder A was 21 days, while for builder B it was 12 days.
On the basis of this data, which builder is more likely to finish a house:

**a** in 170 days
**b** in 210 days
**c** in closer to 190 days?

**10** Rick and George have played 20 rounds of golf on their local golf course. The mean and standard deviation for each was:

| | Mean | Standard deviation |
|---|---|---|
| Rick | 105 | 5·6 |
| George | 105 | 9·2 |

In golf, low scores are better than high scores.

**a** On the basis of these results, in their next game, who is most likely to have:
- **i** a score of 87
- **ii** a score of 125
- **iii** a score closer to 105?

**b** Who do you think is the better golfer? Give a reason for your answer.

**11** Bronwyn was considering working for a year in two different cities. After researching the internet, she found the following information about their temperatures.

| | Temperatures | | | |
|---|---|---|---|---|
| | Summer | | Winter | |
| | $x$ | $\sigma_n$ | $x$ | $\sigma_n$ |
| City A | 32° | 1° | 24° | 2° |
| City B | 32° | 4° | 24° | 6° |

**a** Which city is more likely to experience the hottest summer temperatures?

**b** Which city is more likely to experience the coldest winter temperatures?

**c** Bronwyn prefers a climate where the temperature does not fluctuate greatly. Which city should she choose to work in?

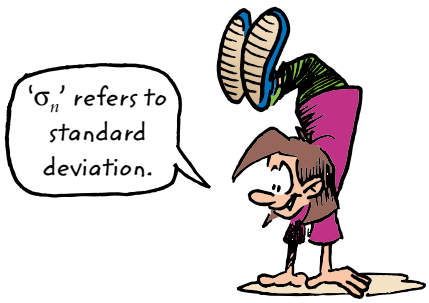
**12** For a class of 20 students, the percentage Maths and English marks were:

| | |
|---|---|
| Maths | 72, 63, 87, 94, 55, 46, 66, 81, 62, 84<br>97, 59, 75, 77, 49, 57, 68, 77, 51, 70 |
| English | 61, 39, 52, 45, 79, 59, 51, 63, 71, 75<br>66, 60, 53, 48, 59, 68, 61, 72, 46, 59 |

For each set of marks, calculate the mean and standard deviation correct to one decimal place. If Tom scored 75 in Maths and 70 in English, which is the better mark relative to the class results?

**13** For each of the following, indicate which you think is the better score.

| | | Score | Mean | $\sigma_n$ |
|---|---|---|---|---|
| **a** | Andrew | 70 | 60 | 5 |
| | Tom | 75 | 60 | 10 |
| **b** | Klare | 62 | 55 | 7 |
| | Jane | 74 | 70 | 8 |
| **c** | James | 63 | 57 | 12 |
| | Cate | 63 | 59 | 4 |
| **d** | Allyson | 18 | 20 | 4 |
| | John | 16 | 22 | 6 |
| **e** | Kylie | 41 | 50 | 6 |
| | Adam | 40 | 49 | 9 |
| **f** | Michelle | 50 | 62 | 8 |
| | Alan | 48 | 55 | 7 |

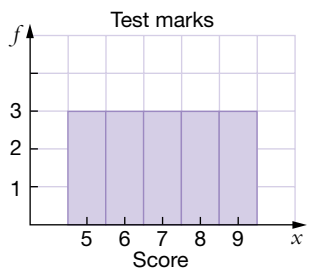



**14** The mean of a set of 20 scores is 10 and the standard deviation of the scores is 2·5. Describe the effect that an additional score will have on the mean and the standard deviation if the score is:

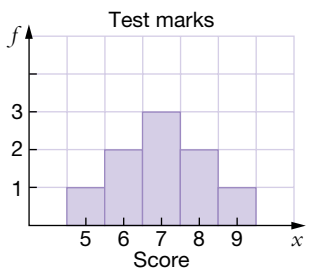
**a** 6 **b** 11 **c** 13

**15** **a** Find the range, mean and standard deviation for the data shown in the histograms below.

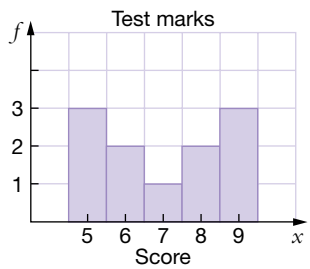
**i**




**ii**




**iii**




**b** Compare the range, mean and standard deviation for the histograms. Give $\sigma_n$ correct to one decimal place.

**c** Investigate what happens to the range, mean and standard deviation of each set of marks in part **a**, when 10 is added to each mark.

**d** Investigate what happens to the range, mean and standard deviation of each set of scores in part **a**, when each mark is multiplied by 10.

# 9:03 Bivariate data

Statistics met in previous years involved analysing data for a single variable. Measuring heights, counting types of vehicles and tabulating exam results are all examples of collecting single-variable data (univariate data). Sometimes data may be obtained about two different variables that may, or may not, be related. For example, measuring the arm span and height of a group of people or the English marks and Mathematics marks for a class of students. This is called **bivariate** data, and we can analyse this data to see if a relationship exists between the two variables.

In some cases one of the variables is time, and we measure another variable against it to determine if any pattern exists. For example, we could measure the height of a young child each month. In this case:

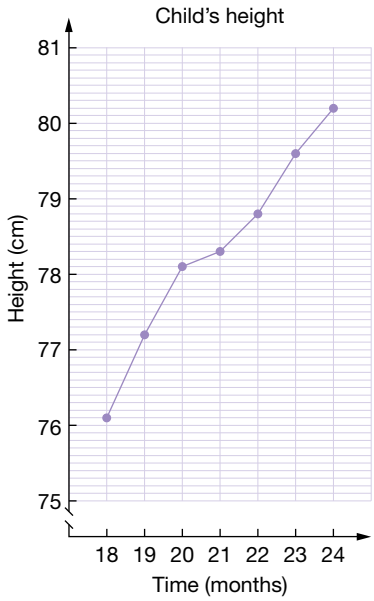
- the **dependent variable** is height (in cm) because each measurement depends upon the month in which it was taken
- the **independent variable** is age (in months).

This table shows the average height of a particular young child from 18 months to 24 months.

| Age (months) | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|
| Height (cm) | 76·1 | 77·2 | 78·1 | 78·3 | 78·8 | 79·7 | 80·2 |

Notice that the height is increasing each month. For this 6-month period the child has grown 4·1 cm. This is an average increase of approximately 0·7 cm per month.

This data is best viewed as a line graph, which gives a visual impression of the relationship.




Conclusions might be drawn from any observed pattern in bivariate data. Take care not to make assumptions that cannot be supported.

For example, it might be assumed from this data that the child will gain 0·7 cm in height every month. Assuming this is true, the newborn baby's length would have been:

Height = 76·1 − 18 × 0·7
= 76·1 − 12·6
= 63·5 cm

Does this seem a reasonable length for a newborn baby?

At age 10, the child's height would be:

Height = 80·2 + 0·7 × 12 months × 8 years
= 147·4 cm

At age 21, the height would be:

Height = 80·2 + 0·7 × 12 months × 19 years
= 239·8 cm

Do these heights seem reasonable?

To extend the data beyond the data that has actually been measured is to 'extrapolate' the data.

It is obvious that an incorrect assumption has been made. You cannot expect the growth rate to be constant throughout the child's life. In other examples, incorrect assumptions may not be so obvious. Ensure that conclusions you draw can be supported by the data.

## WORKED EXAMPLE 1

This table shows two sets of data measured against the time of year. For each month the highest temperature (°C) for the 30-year period 1981–2010, and the single year 2011, was recorded at Observatory Hill in Sydney.

| Month | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Max. temp (°C) 1981–2010 | 44·2 | 40·8 | 39·8 | 33·9 | 29·0 | 25·2 | 25·9 | 31·3 | 34·5 | 38·2 | 41·8 | 40·9 |
| Max. temp (°C) 2011 | 35·4 | 41·5 | 33·6 | 27·0 | 23·3 | 21·0 | 21·4 | 25·7 | 32·5 | 34·3 | 37·2 | 26·8 |

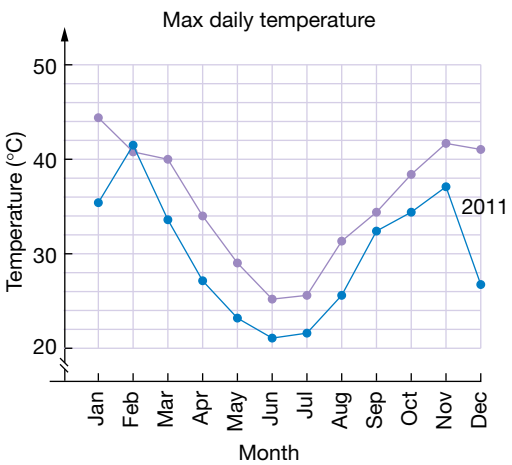
**a** Graph these sets of data on the same diagram. Join the plotted points to form continuous line graphs. Does the line joining adjacent points have meaning?

**b** For which month was a maximum temperature recorded in 2011 that was higher than for the previous 30 years?

**c** What feature is unusual for 2011 compared to the previous 30 years?

### Solution

**a** The graph has been constructed with months on the horizontal axis and temperature on the vertical axis, which has been 'truncated' from 0 to 20 because there is no data in this range. Time is usually shown on the horizontal axis. Points along the line segments cannot be read as data points but the line does indicate a trend between the data points.




**b** February recorded a maximum temperature greater than any in the past 30 years during February.

**c** The maximum temperatures recorded during the summer months of January and December seem markedly low. Note these are at opposite ends of 2011. They are not part of the same summer season. This should evoke interest in data for the following years.

## WORKED EXAMPLE 2

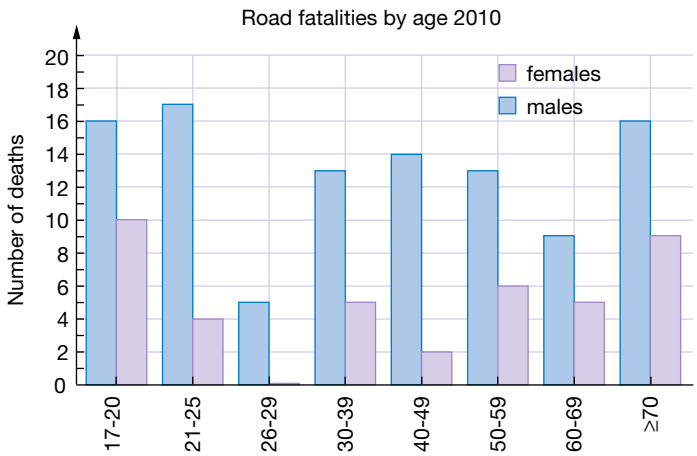
The NSW Centre for Road Safety published these figures for car driver deaths in 2010 showing the age and sex of the driver.

| Age (years) | 17–20 | 21–25 | 26–29 | 30–39 | 40–49 | 50–59 | 60–69 | ≥70 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Male | 16 | 17 | 5 | 13 | 14 | 13 | 9 | 16 | 103 |
| Female | 10 | 4 | 0 | 5 | 2 | 6 | 5 | 9 | 41 |
| *Sub-total:* | 26 | 21 | 5 | 18 | 16 | 19 | 14 | 25 | 144 |

**a** Draw a column graph showing both male and female data on the same graph.
**b** Why is a line graph not suitable for this data?
**c** Which group appears to be the safest drivers?
**d** Which group appears to be most at risk?
**e** Has the data been organised to give a fair picture of the driving risk of various ages?
**f** How would data such as this be useful?
**g** Has the number of drivers in each age category been given?

### Solutions

**a** The column graph has been drawn with columns for males and females placed together for comparison.




**b** The age groups on the horizontal axis are 'categories'. The scale is not continuous.
**c** For 2010 the group with the safest driving record appears to be female drivers in the category 26–29 years.
**d** The group with the worst record appears to be males in the 21-25 years category. However, it should be noted that this category spans 5 years but the 17–20 category spans only 4 years. This represents a worse average.
**e** The data is distorted because the age groups are not of equal spreads. The ≥70 group represents more than 10 years. If the 17–20 and 21–25 categories were combined it would represent a 9-year spread and the data would show a more dramatic picture.

**f** Insurance companies rely on this type of data to assess the risk when setting the premiums. Also governments and road safety groups need to know where to target their efforts, and funding, in order to improve road safety.

**g** No. Knowing the number in each category would help to draw sensible conclusions.

## Exercise 9:03

**1** This chart shows Henry's weight for the first 2 years of his life.

**a** What was his weight at birth?

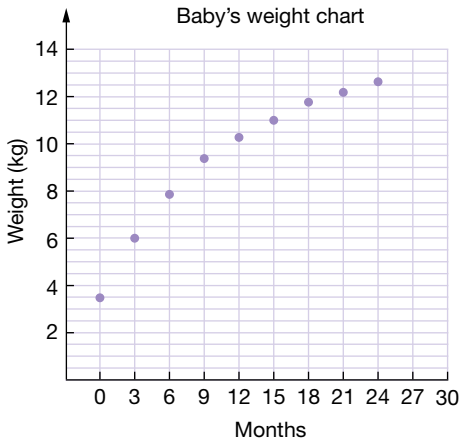
**b** What was his approximate weight when he was 1 year old?

**c** What percentage increase was this of his birth weight?

**d** What was Henry's weight on his second birthday? What percentage increase was this of his weight at 1 year old?

**e** What would be his approximate weight at 27 months?

**f** Can you predict Henry's weight when he is 10 years old?




**2** This table shows Henry's height for the first 24 months of his life, measured at 3 monthly intervals to the nearest cm.

| Age (months) | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| Height (cm) | 50 | 61 | 67 | 72 | 76 | 79 | 82 | 85 | 87 |

**a** Plot this data on a graph similar to the one shown in Question 1. (The vertical axis showing the height in cm could be started from 40.)

**b** Join the plotted points to form a line graph. Does the shape of the graph appear similar to the graph in Question 1?

**c** Do you think this shows that the age and height of a baby are related variables?

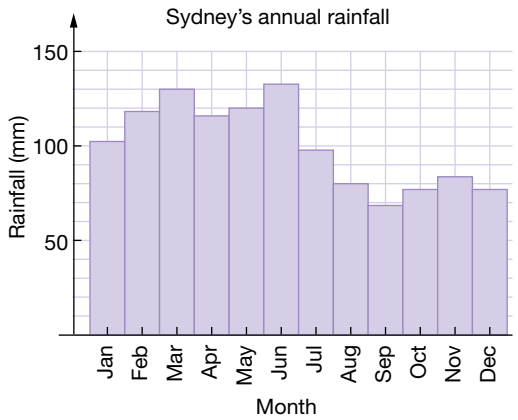
**d** Use the line graph to estimate Henry's height at 14 months.

**e** Would you expect the shape of the graph to continue until Henry reaches adulthood? Give a reason for your answer.

**3** This table shows the mean rainfall for each month of the year measured from 1858 to 2012 for Sydney. It also shows the rainfall for the single year 2012. Note that the yearly total for 2012 is very close to the mean.

| Month | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec | Year |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean rainfall (mm) 1858–2012 | 101·5 | 118·0 | 130·2 | 126·7 | 120·6 | 131·2 | 98·3 | 80·3 | 68·6 | 77·1 | 83·6 | 77·6 | 1213·0 |
| Rainfall (mm) 2012 | 138·8 | 111·0 | 269·8 | 187·0 | 37·2 | 244·2 | 56·2 | 19·0 | 23·8 | 29·4 | 52·0 | 45·2 | 1213·6 |

The histogram shows the mean rainfall data from the table.




**a** For which month is the mean rainfall the:
  **i** lowest? **ii** highest?

**b** Which appears to be wetter: the first half of the year or the second?

**c** Draw a similar column graph showing the monthly rainfall for 2012. (Note the maximum value needed on the vertical axis.)

**d** Compare the two graphs. Comment on similarities and differences.

**4** This table shows the final position of each team in the 2012 ANZ Netball Competition as well as the goals scored for and against each team.

| Pos | Team | GF | GA |
|---|---|---|---|
| 1 | Vixens | 645 | 569 |
| 2 | Mystics | 667 | 633 |
| 3 | Magic | 699 | 594 |
| 4 | T'birds | 670 | 589 |
| 5 | Swifts | 624 | 638 |
| 6 | Firebirds | 686 | 640 |
| 7 | Pulse | 585 | 626 |
| 8 | Fever | 608 | 673 |
| 9 | Steel | 639 | 728 |
| 10 | Tactix | 634 | 767 |

**a** Draw a column graph showing the goals scored for and against on the same graph.

**b** Use the graph to determine which variable seems to be the greater indicator of competition success: goals for (attack) or goals against (defence)? How might a coach be interested in the result?

**c** Why is either variable not a perfect predictor of where a team will finish in the competition?

**5** The table shows the life expectancy for people in 20 countries as well as a measure of their wealth, the Gross Domestic Product (GDP) per person in US dollars. Below is a column graph for each of these variables.

The difference in wealth is vast, with some columns barely visible.

**a** List the top 8 countries in order according to their GDP per person.

**b** List, in order, the 8 countries with the greatest life expectancy.

**c** How many countries appear in the 'top 8' of both lists?

**d** Identify the 'bottom 8' from each graph.

**e** Which country is in the bottom 8 for GDP but not for life expectancy?

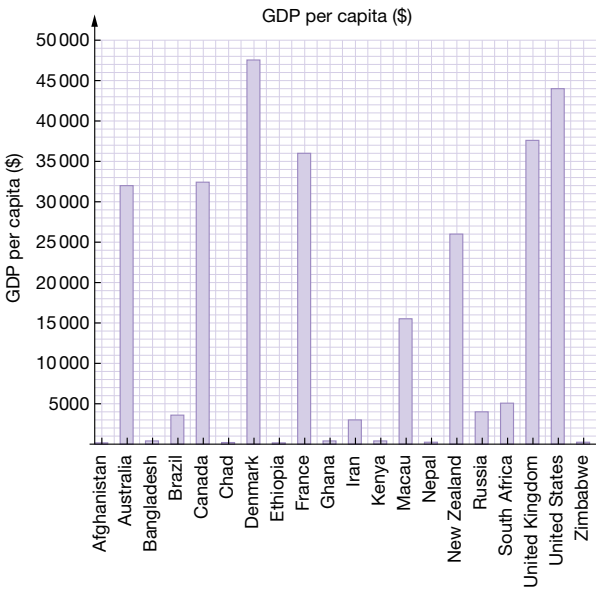
**f** Which country is in the bottom 8 for life expectancy but not for GDP?

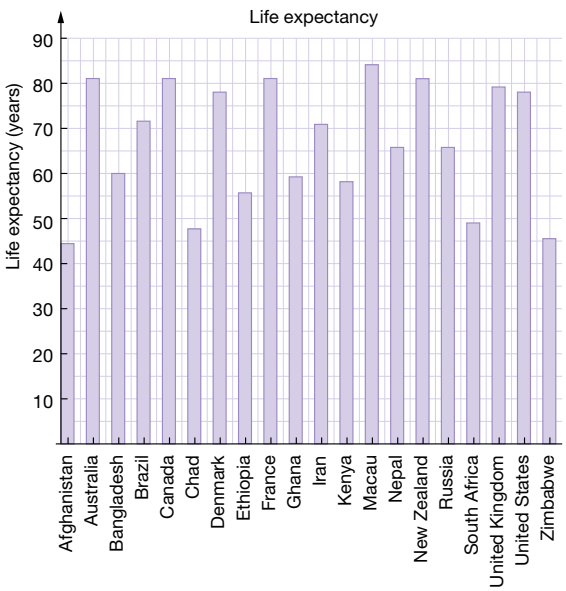
**g** This data would suggest a strong link between these two variables. How could this be further investigated?

**h** What other factors might be investigated to determine their effect on life expectancy?

**Note:* Macau is a self-governing region of China.*

| Country | Life Expectancy (years) | GDP per person ($) |
|---|---|---|
| Afghanistan | 44·6 | 222 |
| Australia | 81·6 | 32 170 |
| Bangladesh | 60·2 | 467 |
| Brazil | 72.0 | 3455 |
| Canada | 81·2 | 32 546 |
| Chad | 47·7 | 396 |
| Denmark | 78·3 | 47 363 |
| Ethiopia | 55·4 | 156 |
| France | 81.0 | 35 854 |
| Ghana | 59·9 | 452 |
| Iran | 71·1 | 2771 |
| Kenya | 57·9 | 527 |
| Macau* | 84·4 | 15 413 |
| Nepal | 65·5 | 268 |
| New Zealand | 80·4 | 25 942 |
| Russia | 66.0 | 4466 |
| South Africa | 49.0 | 4959 |
| United Kingdom | 79.0 | 37 632 |
| United States | 78·1 | 43 743 |
| Zimbabwe | 45·8 | 343 |



**6** The table shows the road fatalities for 2010 against the number of registered vehicles and the population of each state or territory. Column graphs have been drawn below comparing the number of fatalities with each of the other variables.

**a** Calculate the number of fatalities per 10 000 vehicles for NT by dividing 49 by 14. How is this figure shown in graph A?

**b** By looking at graph A, which state or territory has the next worst rate of fatalities per 10 000 vehicles?

**c** Which state or territory has the lowest rate of fatalities per 10 000 vehicles? Use the table to calculate this rate correct to two decimal places.

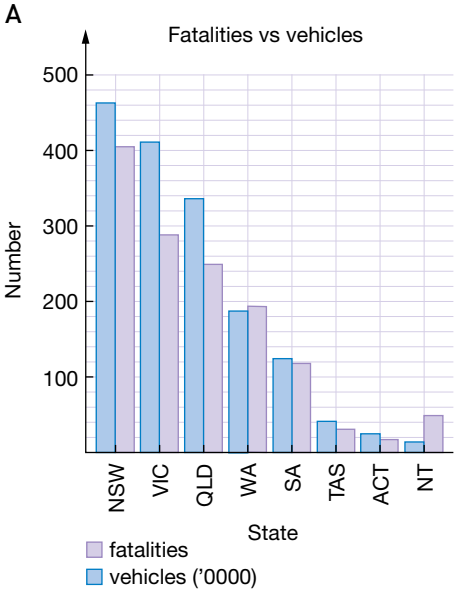
**d** Looking at graph B, determine the states or territories with the two lowest rates of fatalities per 10 000 population.

**e** Which variable would you expect to have a greater influence on the number of fatalities: the number of vehicles or the population? Does the data support your answer?

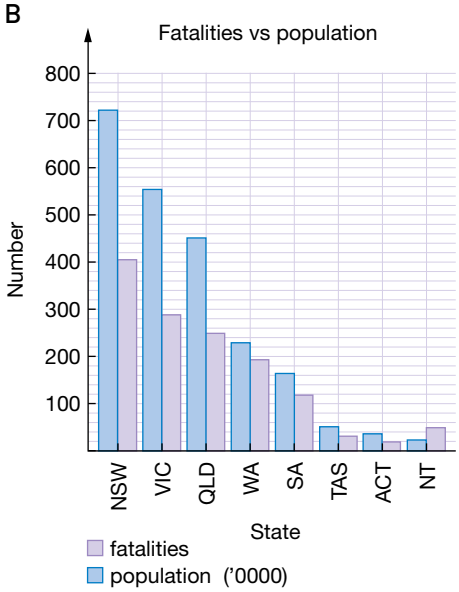
**f** Can you tell which state or territory has the best or worst drivers from this data? What other factors might be considered?

| State | Fatalitites | Vehicles ('0000) | Pop'n ('0000) |
|---|---|---|---|
| NSW | 405 | 463 | 722 |
| Vic | 288 | 411 | 554 |
| Qld | 249 | 336 | 451 |
| WA | 193 | 187 | 229 |
| SA | 118 | 124 | 164 |
| Tas | 31 | 41 | 51 |
| ACT | 19 | 25 | 36 |
| NT | 49 | 14 | 23 |

A




B



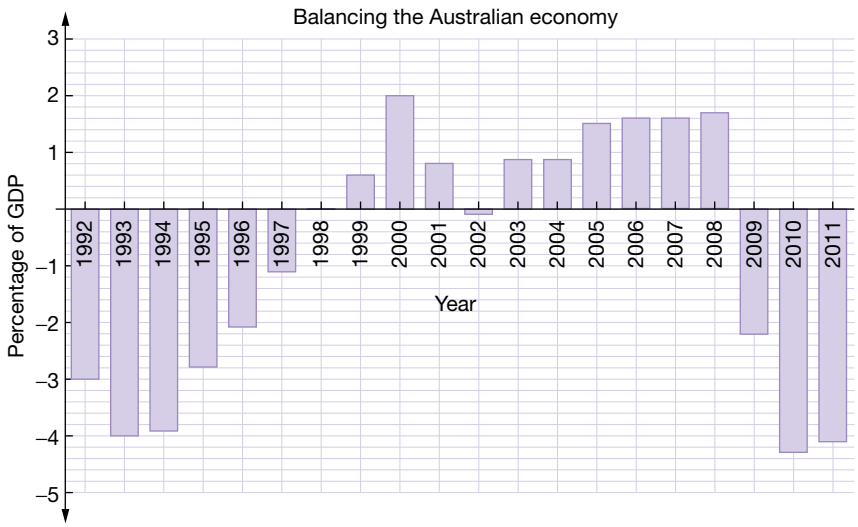
**7** The column graph below shows a surplus for the Australian economy when the column is above the horizontal axis, and a deficit when it is below. It is expressed as a percentage of the Gross Domestic Product (GDP). A surplus can be viewed as a saving and a deficit as an overspending.

**a** For the 20 years shown, in what year did the greatest surplus occur?

**b** What year registered the greatest deficit?

**c** What world event caused the large 'drop' in the economy from 2008 to 2009?

**d** These results are important to governments as they try to 'manage' the economy. How might their policies affect these figures?




# 9:04 Scatter plots

Sometimes two sets of data are collected from the same population or sample that is representing two different variables. A question often asked is: 'Are these two variables related, and if so, how strongly does the value of one variable reflect the value of the other?'

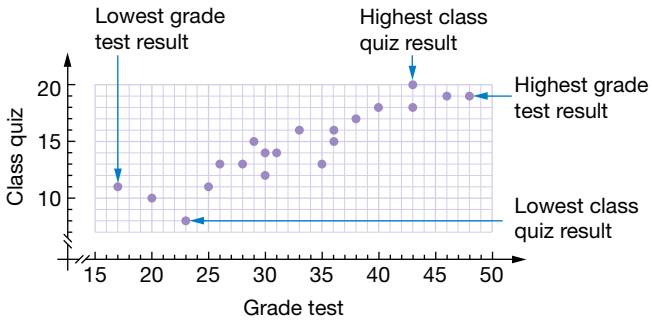
Before investigating any such relationship a convenient method of analysis is needed. A very useful and easily constructed tool is a **scatter plot** or **scatter diagram**.

For example, the Grade test results out of 50, and Class quiz marks out of 20 for a class of 20 students are shown in the table below.

| Student | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Test | 23 | 20 | 17 | 25 | 30 | 28 | 35 | 26 | 30 | 31 | 36 | 29 | 33 | 36 | 38 | 42 | 40 | 48 | 46 | 43 |
| Quiz | 8 | 10 | 11 | 11 | 12 | 13 | 13 | 13 | 14 | 14 | 15 | 15 | 16 | 16 | 17 | 18 | 18 | 19 | 19 | 20 |

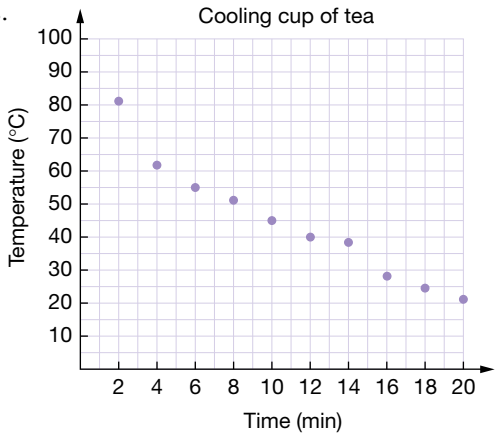
The test results and the quiz marks for each student are plotted as an ordered pair to produce a scatter plot, which is shown on the following page.

Looking at this scatter plot gives the general impression that as one variable increases, so does the other. This is said to be a **positive** relationship. But the relationship is not exact. The student with the lowest quiz mark did not get the lowest test result. If there was an exact relationship between the two variables the points would lie along a straight line.

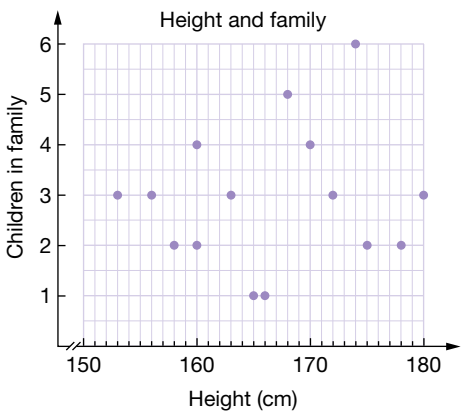



Consider these scatter plots that show different patterns.

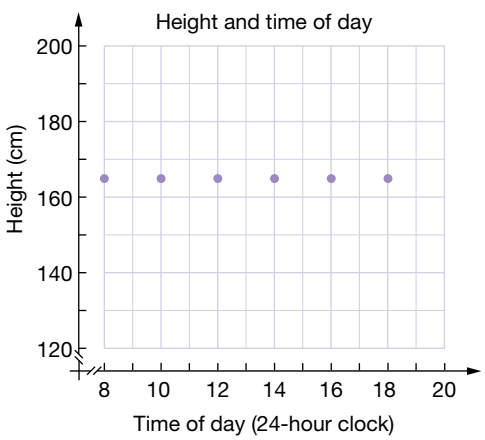
This scatter plot indicates that as time increases the temperature of a cup of tea decreases. This is said to be a **negative** relationship between these two variables. If a line was drawn connecting any pair of points it would have a negative slope.




This scatter plot seems totally random. It would suggest there is no direct relationship between the height of students and the number of children in their families.



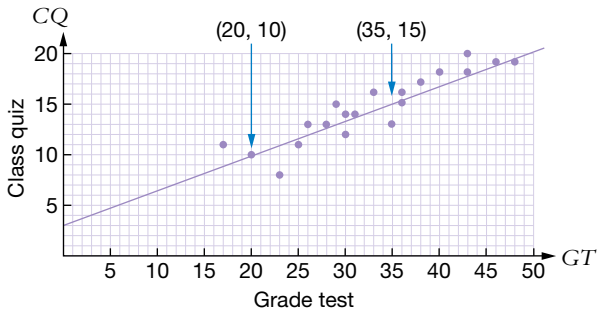

This scatter plot shows a linear pattern, but because the 'line' of points is horizontal it suggests that the time of day has no bearing on the height of a person. So as with the scatter plot above, there is no relationship linking the variables.



## Line of best fit

For scatter plots that appear to show a relationship between the two variables, a line can be drawn that runs through the 'middle' of the plotted points. In this chapter only linear relationships will be examined but it is possible for other relationships, such as parabolic or hyperbolic, to exist between variables. As an example, consider the scatter plot drawn previously.

This time the full axes have been shown and a line that seems to 'best fit' the data has been drawn and extended to the vertical axis.



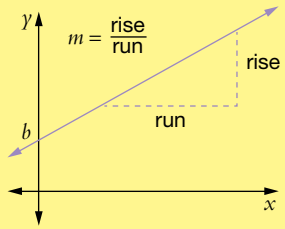

**Equation of a straight line**

gradient-intercept form

$y = mx + b$

where: $m$ = gradient

$b = y$-intercept




The gradient of the line is calculated by selecting two convenient points through which the line passes. Using the points (20, 10) and (35, 15): $m = \frac{15-10}{35-20} = \frac{1}{3}$

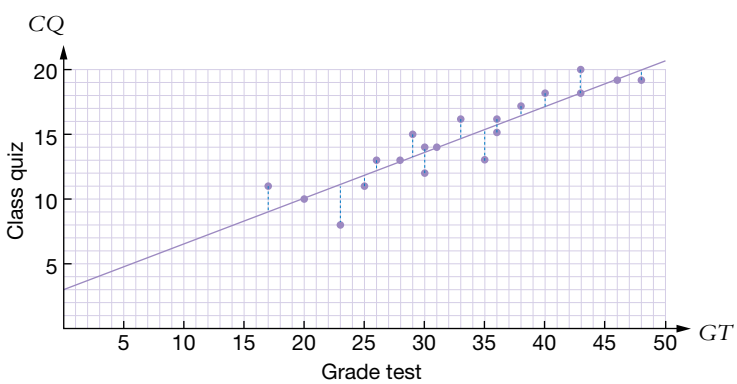
The line intersects the vertical axis close to 3, so using this as a convenient approximation for the '$y$-intercept', the equation of this 'line of best fit' on a number plane would be: $y = \frac{1}{3}x + 3$.
So in terms of the variables in this example: $CQ = \frac{1}{3}GT + 3$

Estimating the line of best fit by 'eyesight' is going to be of limited accuracy. The method of locating the most accurate line of best fit involves limiting the total distance the data points are from the line. It is called the *method of least squares*. The vertical distance of each point from the line is measured and squared. (Squaring eliminates the need to measure above and below the line as positive and negative.) These squared measurements are then summed. The aim is to find the line that minimises this sum. This calculation is beyond the scope of this course but the result can be obtained using technology.

This diagram shows the vertical distances from each point to the line of best fit. Using a graphics program, these distances have been squared and summed. The line drawn shows the case when this is a minimum. Using this program:

$m = 0{\cdot}352\,235$

$y$-intercept $= 3{\cdot}029\,089$



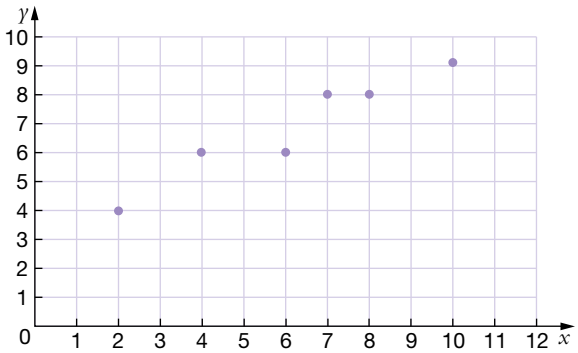

Therefore line is:

$CQ = 0{\cdot}35GT + 3{\cdot}03$ (2 dec. pl.)

Finding the line of best fit using an Excel spreadsheet is shown in Investigation 9:04.

## WORKED EXAMPLE 1

**a** Add a line of best fit to this scatter plot.
**b** Determine the gradient of the line.
**c** Extend the line to the $y$-axis and find the $y$-intercept.
**d** Write the equation of the line.




### Solution

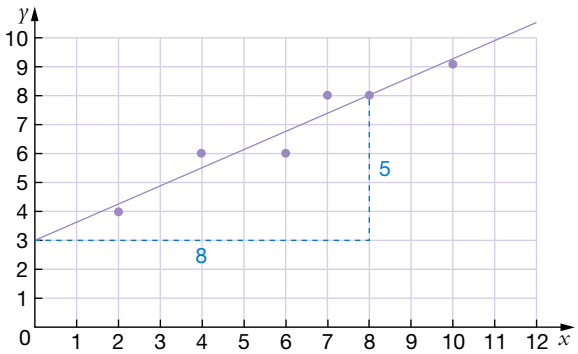
**a** The line has been drawn as shown.

**b** Using the points $(8, 8)$ and $(0, 3)$ to find the gradient: $m = \frac{8 - 3}{8 - 0} = \frac{5}{8}$

**c** The line has been extended to cut the $y$-axis at 3.

**d** Using the gradient and the $y$-intercept, the equation of the line of best fit is:

$y = \frac{5}{8}x + 3$




## WORKED EXAMPLE 2

The table shows the number of a product sold as its price increased.

| Price ($) | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| Number sold | 25 | 24 | 20 | 19 | 16 | 13 | 12 | 12 | 10 |

**a** Plot the data on a scatter plot.
**b** Does a pattern exist between the variables?
**c** Describe the relationship between the variables.
**d** Draw the line of best fit and estimate its equation.

### Solution

**a**

**b** The scatter plot shows that as the price increased the sales decreased.

**c** This is a negative relationship. As the data points are close to lying on a straight line, this relationship would be classified as a strong negative relationship.

**d** A line of best fit has been drawn on the graph. However, this graph has a truncated $x$-axis, so we cannot see the $y$-intercept. The two convenient points, (10, 25) and (15, 15) will be used to determine its equation.

Gradient $= \frac{25 - 15}{10 - 15} = -2$

Using point-gradient form: $y - y_1 = m(x - x_1)$

i.e. $N - 25 = -2(P - 10)$

$= -2P + 20$

So the equation is: $N = -2P + 45$

## WORKED EXAMPLE 3

This table shows the average monthly temperature and rainfall for the town of Sidnee.

| Month | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Temp (°C) | 25·8 | 25·7 | 24·7 | 22·4 | 19·3 | 16·9 | 16·2 | 17·7 | 19·9 | 22·0 | 23·6 | 25·1 |
| Rainfall (mm) | 103 | 117 | 131 | 127 | 123 | 128 | 98 | 82 | 69 | 77 | 83 | 78 |

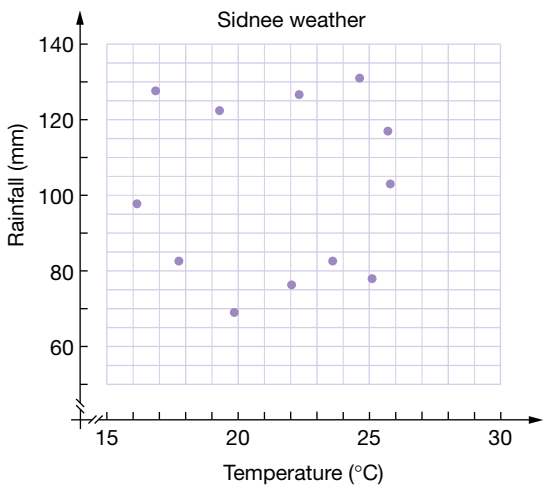
**a** Plot this data on a scatter plot with axes that are applicable for the data shown.

**b** Determine the nature of any relationship between these variables.

### Solution

**a** Appropriate scales would start at 60 mm for rainfall and 15°C for temperature as no measurement falls below these values.

**b** The pattern shown by the scatter plot appears random. There is certainly no increasing or decreasing pattern. According to this set of data there is no relationship between average monthly temperature and rainfall for Sidnee.



## WORKED EXAMPLE 4

Suvir's mother measured his height every year on his birthday. His height for the even numbered years is shown in the table.

| Age (years) | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|
| Height (cm) | 103 | 114 | 129 | 140 | 152 | 169 | 178 |

**a** Show this bivariate data on a scatter plot.

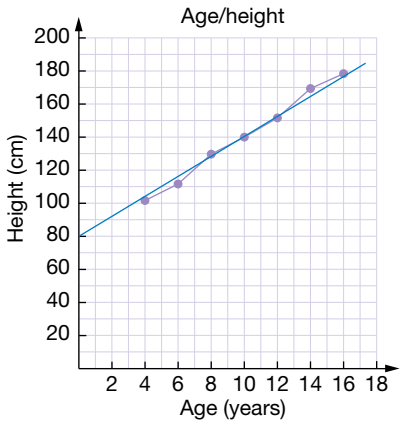
**b** Draw a line of best fit and determine its equation.

**c** Can you estimate Suvir's height when he was 11 years of age?

**d** Can you estimate Suvir's height when he was born and at 25 years of age?

### Solution

**a**




**b** The plotted data points have been joined showing a pattern close to a straight line. Therefore the variables exhibit a strong positive relationship **for the measured data**.

An approximate line of best fit has been drawn that has a $y$-intercept of 80 and passes through the point (10, 140).

$$\text{The gradient} = \frac{140 - 80}{10 - 0}$$
$$= \frac{60}{10}$$
$$= 6$$

The equation of the line is:

$H = 6A + 80$,

where $H$ is Suvir's height (in cm) and

$A$ is Suvir's age (in years).

**c** It is reasonable to assume that the pattern continues between the data points. So an approximation for Suvir's height at age 11 would be the average of his height at ages 10 and 12, which is 146 cm.

This is also obtained by substituting $A = 11$ into the equation of the line of best fit.

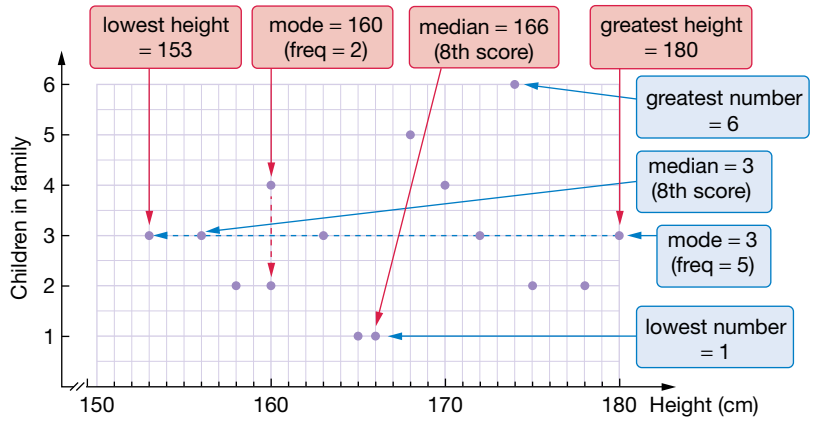
$H = 6(11) + 80$

$= 146$ cm

**d** If this pattern continued beyond the measured data Suvir's height at birth would have been about 78 cm and his height at 25 years would be over 230 cm. Both of these estimates are unreasonable and show the danger of assuming all patterns will continue, without supporting evidence. Clearly, for this example, growth rates change with age.

- **Interpolation** is the process of estimating data that lie within the domain of the values given. This is valid when a definite relationship exists between the two variables.
- **Extrapolation** is the process of predicting data beyond the values given. This is often not useful and can lead to false results as there is no guarantee that an observed pattern will continue beyond the data presented.

The **range, mode and median** can be determined for each of the variables from a scatter plot by observation and counting. For this scatter plot there are 15 data points, so the median would be represented by the 8th data point.



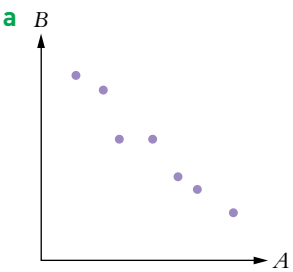

For height:
Range = 180 – 153
= 27 cm
Median = 166 cm
Mode = 160 cm

For children in family:
Range = 6 – 1
= 5
Mode = 3
Median = 3

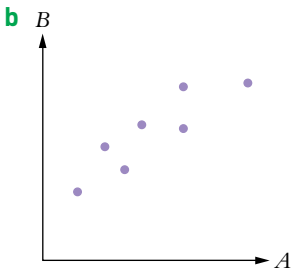
## Exercise 9:04

**1** State whether the variables $A$ and $B$ plotted against each other in each scatter plot shows a linear relationship or not. If so, is it a positive or negative relationship?

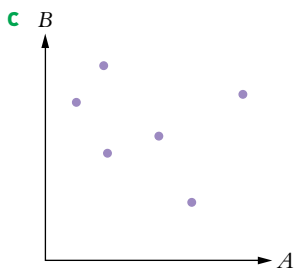
**a**



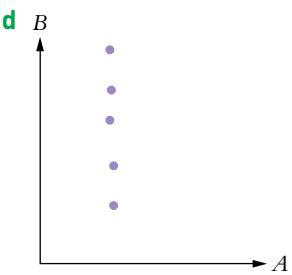

**b**



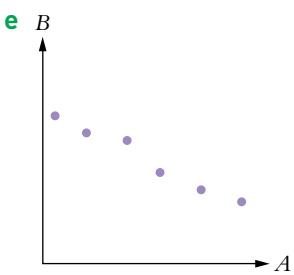

**c**



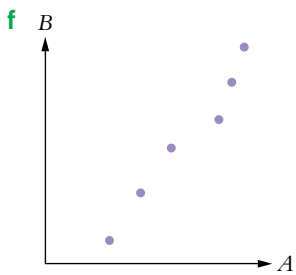

**d**




**e**




**f**




**2** Plot each set of data on a scatter plot and determine the type of relationship that exists between the two variables.

**a**

| $x$ | 3 | 4 | 5 | 5 | 6 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 9 | 7 | 6 | 7 | 6 | 5 | 5 | 3 |

**b**

| $t$ | 4 | 5 | 7 | 7 | 8 | 10 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|
| $N$ | 6 | 9 | 9 | 12 | 11 | 12 | 14 | 15 |

**c**

| $t$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $S$ | 22 | 23 | 23 | 24 | 25 | 24 | 25 | 26 |

**d**

| $x$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|---|
| $n$ | 9 | 6 | 10 | 3 | 9 | 5 | 9 | 4 |

**e**

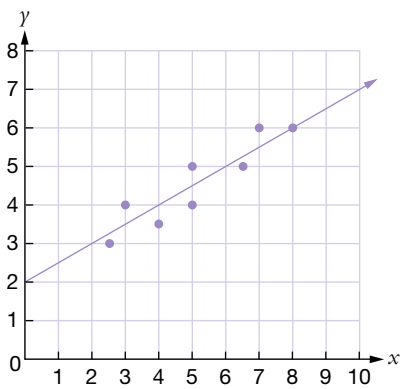
| $K$ | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 |
|---|---|---|---|---|---|---|---|---|
| $P$ | 50 | 45 | 38 | 36 | 32 | 24 | 19 | 15 |

**f**

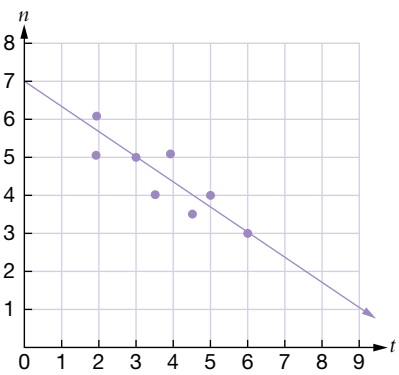
| $P$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $Q$ | 7 | 6 | 6 | 7 | 7 | 6 | 7 | 6 |

**3** Determine the equation of the line of best fit shown on each scatter plot.

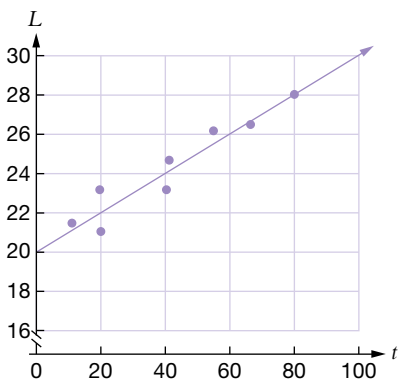
**a**



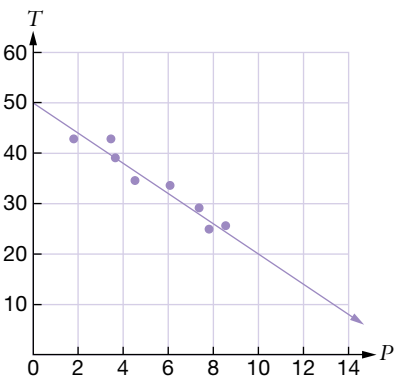

**b**




**c**

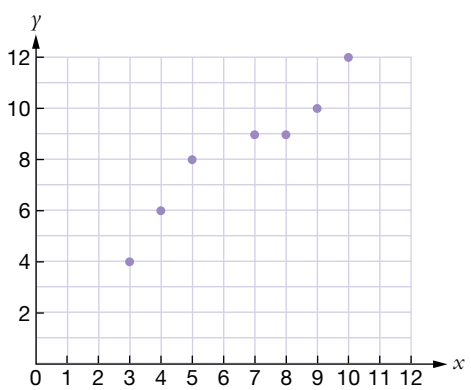



**d**

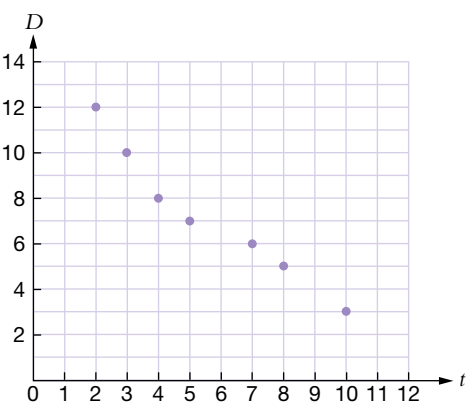



**4** Draw a line of best fit for each set of data. Find an approximation for the gradient and the $y$-intercept, and then determine the equation for each line.

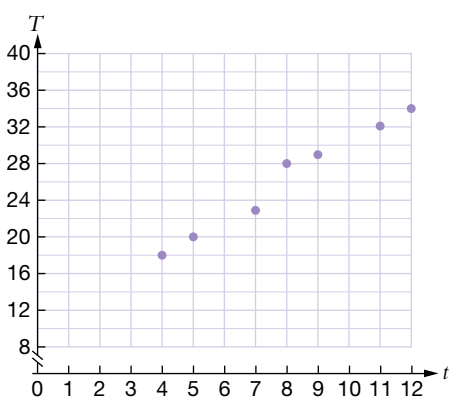
**a**



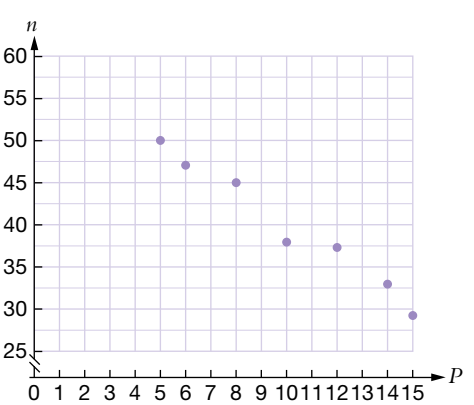

**b**




**c**

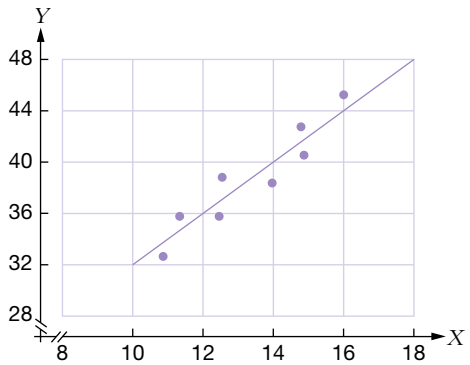



**d**

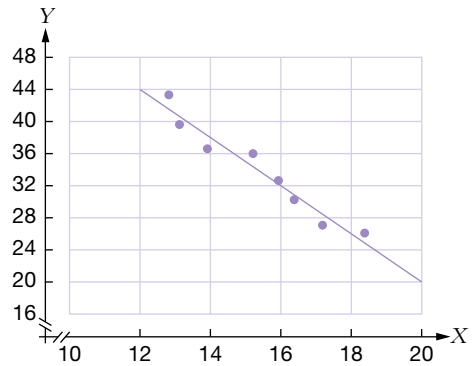


**5** These scatter plots have truncated axes. Use the lines of best fit drawn to determine an equation that represents the relationship between the variables $X$ and $Y$. (See Worked Example 2.)

**a**




**b**




**6** Mark drove in a rally from Penrith to Coffs Harbour and checked his trip meter every hour. The total distance travelled at the end of each hour is shown in this table.

| Time (hours) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Distance (km) | 82 | 191 | 295 | 369 | 487 | 561 |

**a** Plot this data on a scatter plot.

**b** Describe the relationship between time and the distance travelled.

**c** What was Mark's average speed for journey?

**d** What does the gradient represent in this example?

**7** This shows the points table after 10 rounds in the A-League 2012 football competition as well as the goals for and against each team.

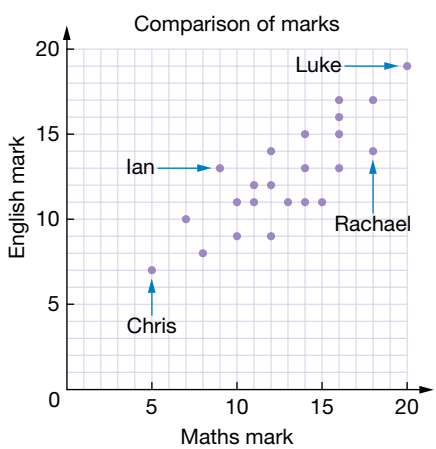
| Team | G. For | G. Agt | Points |
|---|---|---|---|
| C.C. Mariners | 19 | 8 | 23 |
| Adelaide United | 17 | 9 | 22 |
| Melb. Victory | 17 | 20 | 16 |
| W.S. Wanderers | 6 | 8 | 13 |
| Newcastle Jets | 14 | 19 | 13 |
| Wel. Phoenix | 13 | 12 | 12 |
| Melb. Heart | 12 | 12 | 12 |
| Perth Glory | 9 | 10 | 11 |
| Brisbane Roar | 13 | 13 | 10 |
| Sydney F.C. | 14 | 23 | 10 |

**a** Draw a scatter plot with points on the horizontal axis and goals for on the vertical axis.

**b** Draw a similar scatter plot plotting points on the horizontal axis and goals against on the vertical axis.

**c** Compare the two scatter plots. Which appears to have a greater relationship with the competition points: goals for or goals against?

**8** This scatter plot shows the Maths marks plotted against the English marks for a class of students.




**a** How many students are in the class?

**b** Write the Maths and English marks scored by:

**i** Chris **ii** Luke **iii** Ian **iv** Rachael

**c** What is the mode Maths mark?

**d** What is the median Maths mark?

**e** What is the mode English mark?

**f** What is the median English mark?

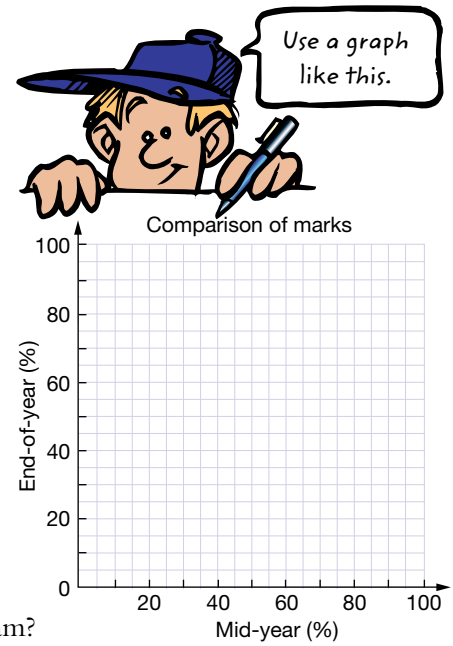
**g** What relationship is suggested by this scatter plot?

**h** If a line of best fit was drawn connecting Chris's marks with Luke's marks, what would be the equation of this line in terms of the variables $M$ (Maths marks) and $E$ (English marks)?

**9** The exam results in mid-year and end-of-year exams for Mr Justin's class are given below.

| Name | Mid-year | Year |
|---|---|---|
| Achnid, Rex | 27 | 45 |
| Amine, Xavier | 69 | 72 |
| Bandon, Alice | 45 | 42 |
| Beese, Owen | 66 | 69 |
| Care, Ivy | 96 | 90 |
| Deals, Ian | 93 | 90 |
| Deavour, Nell | 90 | 78 |
| Fend, Dina | 54 | 57 |
| Hance, Nina | 84 | 78 |
| Honest, Ban | 75 | 63 |
| Ickle, Vicki | 39 | 48 |
| Loyal, Betty | 93 | 84 |

| Name | Mid-year | Year |
|---|---|---|
| Madeit, Ena | 48 | 54 |
| Merit, David | 63 | 69 |
| Night, Unice | 60 | 66 |
| Press, Mark | 69 | 63 |
| Reason, Chris | 84 | 69 |
| Share, Iris | 96 | 87 |
| Swerthy, Eve | 87 | 72 |
| Team, Sue | 75 | 75 |
| Tist, Ross | 75 | 81 |
| Vee, Tony | 30 | 36 |
| Walker, Joy | 57 | 51 |
| Wander, Yin | 78 | 72 |




**a** Draw a scatter plot to show these results.
**b** What relationship is there between the results?
**c** What was the position of O. Beese in the end-of-year exam?
**d** What was the position of D. Merit in the mid-year exam?
**e** What was the mode in the mid-year exam?
**f** What was the median in the end-of-year exam?
**g** Draw a line of best fit and estimate its equation in terms of MY (mid-year) and Y (end-of-year) marks.

**10** There are 16 teams in the NRL Rugby League Competition. This table shows the position each team finished as well as points gained on the competition table. The points scored for each team as well as those scored against each team are also shown for the 2012 season.

| Team | Points | Pts for | Pts against |
|---|---|---|---|
| Bulldogs | 40 | 568 | 369 |
| Storm | 38 | 579 | 361 |
| Rabbitohs | 36 | 559 | 428 |
| Sea Eagles | 36 | 497 | 403 |
| Cowboys | 34 | 597 | 445 |
| Raiders | 30 | 545 | 536 |
| Sharks | 29 | 445 | 441 |
| Broncos | 28 | 481 | 447 |
| Dragons | 26 | 405 | 438 |
| West Tigers | 26 | 506 | 551 |
| Titans | 24 | 449 | 477 |
| Knights | 24 | 448 | 488 |
| Roosters | 21 | 462 | 626 |
| Warriors | 20 | 497 | 609 |
| Panthers | 20 | 409 | 575 |
| Eels | 16 | 431 | 674 |

**a** Construct three scatter plots using:
  **i** points and points for
  **ii** points and points against
  **iii** points for and points against.
**b** Which scatter plot shows a positive relationship between the variables?
**c** Which scatter plots show a negative relationship between the variables?
**d** Between which two variables does the relationship appear the strongest?

**11** **a** Draw a scatter plot for the data below using a scale of 1 mm = 2 marks on each axis.

**b** Determine the range and median for both sets of results.

**c** Describe the relationship that exists between the sets of data.

**d** Draw a line of best fit and determine an approximate equation for the relationship between English mark $E$ and Mathematics mark $M$.

**e** If an additional student sat for the Maths exam and scored 70, what would you expect her English mark to be?

| Name | Maths | English |
|---|---|---|
| Able, Nancy | 76 | 64 |
| Argue, Youst | 82 | 78 |
| Beaudy, Ungwin | 50 | 70 |
| Cycle, Ivan | 63 | 74 |
| Deer, Ivy | 46 | 50 |
| Evator, Lois | 38 | 38 |
| Eye, Graham | 68 | 54 |
| Force, Neil | 98 | 86 |
| Hale, Xavier | 60 | 60 |
| Hance, Neville | 78 | 68 |
| Iting, Xavier | 82 | 76 |
| Leave, Ben | 38 | 60 |
| Mitt, Ona | 66 | 50 |
| Muse, Betty | 78 | 78 |
| Pand, Xavier | 20 | 50 |

| Name | Maths | English |
|---|---|---|
| Pea, Tom | 58 | 62 |
| Pen, Owen | 48 | 46 |
| Podd, Peter | 80 | 74 |
| Pott, Timothy | 65 | 60 |
| Saidit, Unice | 72 | 78 |
| Saw, Carol | 56 | 62 |
| Shooter, Petra | 68 | 68 |
| Side, Cicile | 50 | 46 |
| Snore, Irwin | 60 | 80 |
| String, Barry | 46 | 56 |
| Table, Theresa | 84 | 76 |
| Tee, Mandy | 52 | 56 |
| Tension, Dennis | 74 | 82 |
| Toteler, Tina | 66 | 56 |
| Tsa, Paul | 70 | 80 |

## CHALLENGE 9:04 FURTHER SCATTER PLOTS

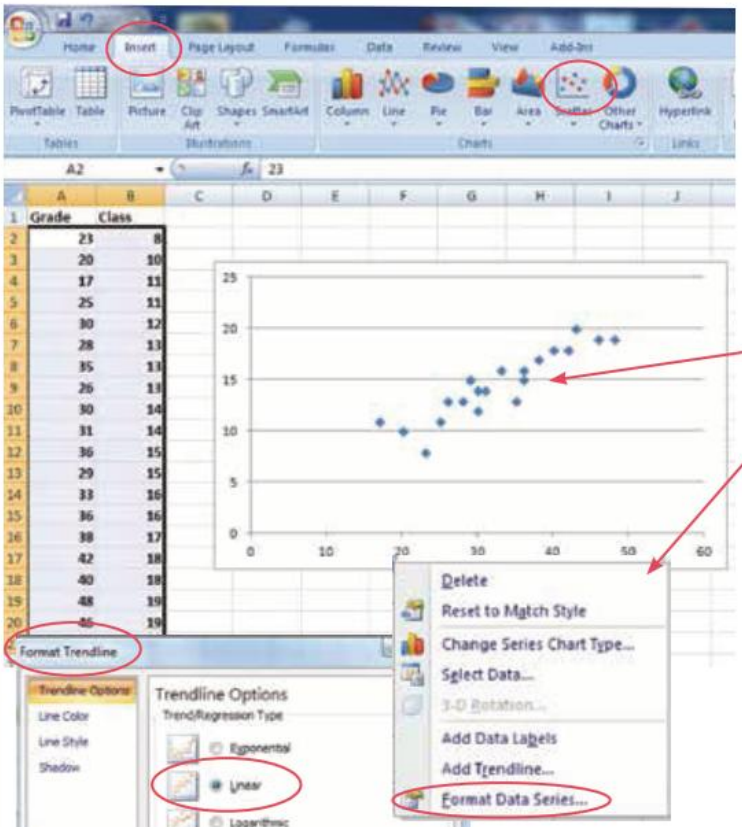
Gather data from students in your class or year group to investigate the relationship between other variables. Choose variables that you suspect of having a positive or negative relationship as well as some you would not expect to be related.

You could select two sets of data from this suggested list.

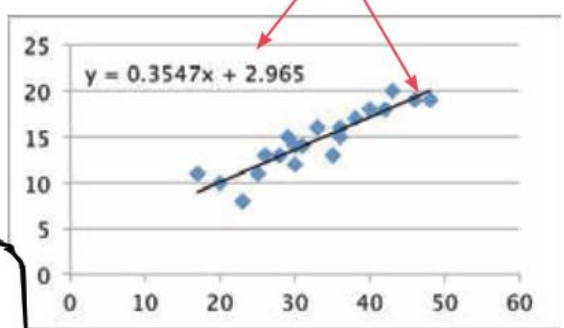
- Age in months
- Day of the month you were born
- Weight in kg
- Height in cm
- Number of pets
- Shoe size
- Arm length
- Weight at birth
- Head circumference
- Time to get to school
- Number of tunes on a phone
- Exam or test marks
- Number of CDs owned
- Letters in your name
- Hours of TV watched

## INVESTIGATION 9:04

## FINDING THE LINE OF BEST FIT USING A SPREADSHEET




- Enter the data shown for the Grade test/Class quiz on page 304. This data is shown in columns A and B.
- Highlight this data, select the **Insert** tab in the menu as shown, and then choose **Scatter**. A scatter plot will appear as shown.
- 'Right click' on any data point in the scatter plot. The menu as shown here will appear.
  Choose 'Add Trendline'. (This is the same as a line of best fit.)
- The menu 'Format Trendline' will appear. Ensure the 'Linear' Trend Type is selected. At the bottom of the menu tick the box for 'Display Equation on chart'.
- The trendline (line of best fit) will appear on the scatter plot as well as the equation of the line.



This is pretty trendy! Use data from Exercise 9:04 to draw another line of best fit (trendline) using a spreadsheet.

*Note:* Right clicking on the axes or the points will provide menus to refine the graph. Experiment with the options to see their various effects.

# 9:05 Statistics all around us

We are surrounded by statistics; technology has enabled the collection, storage and analysis of data to be much more accessible. Statistics help us to understand the world around us. They are a key to producing summaries of data, drawing inferences and predicting the future.

Meaningful and useful statistics depend upon reliable and adequate data. Governments rely on the use of statistics to run their country. It was this use of such data, 'data about the State' that led to the use of the word 'statistics'.

Statistics are crucial to scientific discovery, business operations, medicine, sociology, engineering, sport and many other forms of human endeavour.

This exercise will highlight some secondary sources of statistics that are used in our society today.

## Exercise 9:05

**1** Every sport is a source of statistics; none more so than cricket. The table below shows the test match record of Michael Clarke after 87 tests. Often you have to be familiar with the sport to understand the data. Do you know what each piece of data means or how it is calculated?

**a** A match normally has two innings. If he played in 87 matches, in how many did Michael Clarke only bat once?

**b** For the 145 times he batted, how many times was he given 'Out'?

**c** The 'Aggregate' is the total runs scored. Divide this number by your answer for part **b**. What piece of data does this give?

**d** As a bowler, Michael Clarke had 1098 runs scored off his 374 overs. Dividing 1098 by 374 gives which statistic?

**e** Dividing the number of balls bowled by the number of wickets taken yields which statistic?

If there are any other terms you are not familiar with, ask a cricket fan in your class.

| Batting | |
|---|---|
| Innings | 145 |
| Not outs | 15 |
| Aggregate | 6804 |
| Average | 52·34 |
| Highest score | 329 |
| 50s | 24 |
| 100s | 21 |
| 200s | 4 |
| 300s | 1 |
| Ducks | 6 |
| Pairs | 0 |
| Scoring rate | 55·98 |

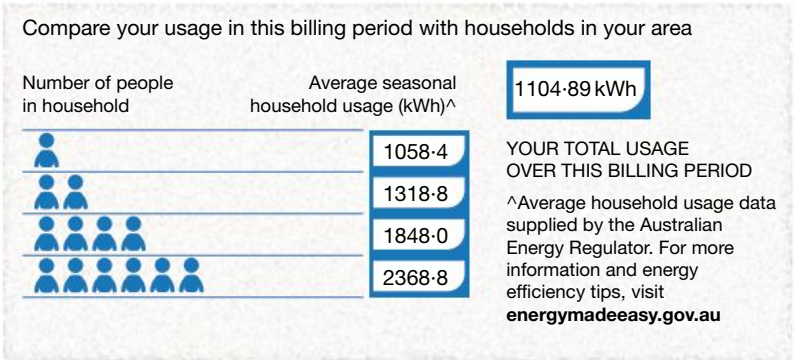
| Bowling | |
|---|---|
| Overs | 374 |
| Balls | 2244 |
| Maidens | 56 |
| Runs | 1098 |
| Wickets | 30 |
| Average | 36·60 |
| 5 wicket/innings | 2 |
| 10 wicket/innings | 0 |
| Best (Inns) | 6/9 |
| Best (Match) | 6/9 |
| Economy rate | 2.94 |
| Strike rate | 74·80 |

**2** The ratings for television networks are shown in this table. The figures are percentages but what do they mean?

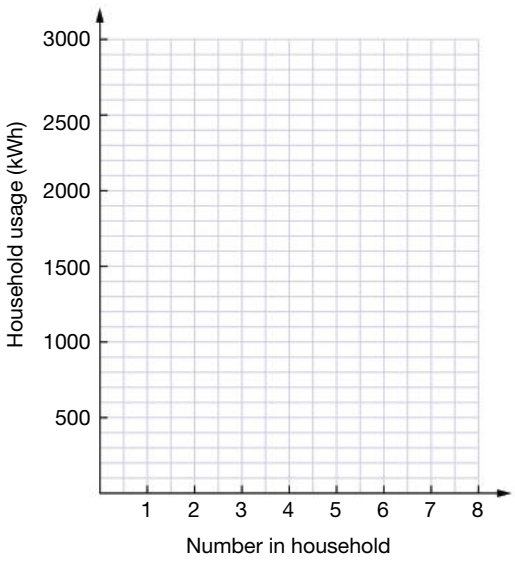
| Year | ABC | Seven | Nine | Ten | SBS |
|---|---|---|---|---|---|
| 2003 | 15·6% | 25·6% | 31·2% | 23·2% | 4·5% |
| 2004 | 16·9% | 24·9% | 29·9% | 23·2% | 4·6% |
| 2005 | 15·7% | 27·1% | 29·2% | 23·8% | 6·1% |
| 2006 | 15·4% | 27·8% | 29·1% | 22·5% | 5·4% |
| 2007 | 16·3% | 29·0% | 27·4% | 21·6% | 5·6% |
| 2008 | 17·5% | 28·7% | 27·2% | 21·0% | 5·6% |
| 2009 | 16·4% | 28·8% | 26·8% | 22·2% | 5·8% |
| 2010 | 16·9% | 28·6% | 27·7% | 21·2% | 5·6% |

**a** Add the percentages for 2010 for each network. What is the total?
**b** Each figure is a percentage of what quantity?
**c** From 2003 to 2010, for which networks did the ratings:
  **i** increase
  **ii** decrease?
**d** Why are these ratings important?
**e** How could a network improve its ratings?

**3** The cost of electricity is of concern to many households. On the bill the electricity provider indicates the power used for the three-month period as well as a means of comparing the usage to similar households. This is shown in the following diagram.




**a** The bill is for a two person household. What is the total usage for this period?
**b** How does the usage compare with other similar households?
**c** Construct a scatter plot using the axes shown, by plotting the usage for households with 1, 2, 4 and 6 people.
**d** Is there a relationship between these variables? How would you describe it?
**e** By drawing a line between the dots in this plot, estimate the average usage for a household with 5 people.
**f** Extrapolate this pattern to estimate the usage for a household with:
  **i** no one
  **ii** 8 people
  **iii** 20 people.
Are these estimates reasonable?



**4** The Department of Immigration and Citizenship published these figures for the Financial Year 2010–11 showing the change in the number of permanent residents for each state and territory. Also shown is the population for each state and territory as at June 2011.

| Resident | NSW | Vic | Qld | SA | WA | Tas | NT | ACT | Australia* |
|---|---|---|---|---|---|---|---|---|---|
| Arrivals | 38 816 | 33 632 | 25 827 | 7 204 | 18 441 | 949 | 960 | 1 623 | 127 458 |
| Departures | 34 802 | 19 320 | 17 851 | 3 636 | 10 256 | 803 | 526 | 1 259 | 88 461 |
| Population | 7 211 468 | 5 534 526 | 4 474 098 | 1 638 232 | 2 352 215 | 511 195 | 231 331 | 367 752 | 22 323 933 |

* Includes Territory of Christmas Island, Territory of Cocos (Keeling) Islands and Jervis Bay.

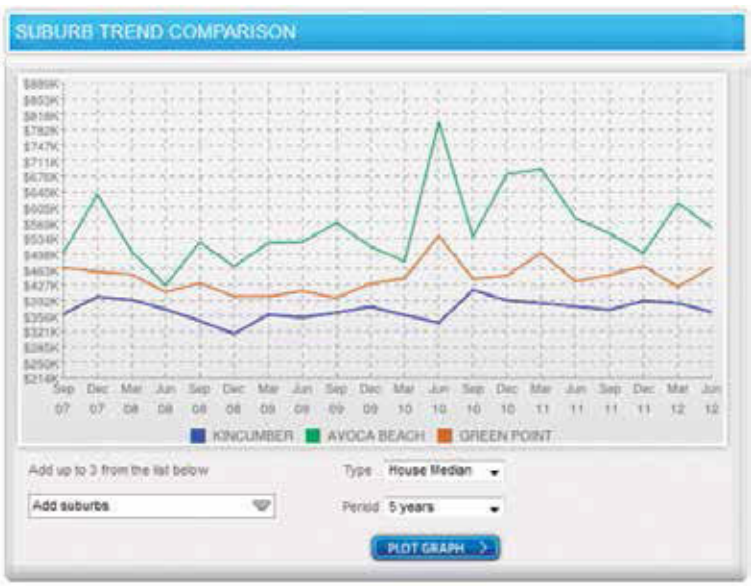
**a** Calculate the net increase in Australia's population due to migration. What percentage of the total population is this?

**b** Which State or territory experienced the greatest increase in population due to migration?

**c** Which State or territory experienced the greatest percentage increase in its population?

**5** The price of housing can fluctuate with economic conditions and be determined by location. To compare prices the median is usually used as a measure. The explanation on the right was given on the Property Data website.

The figures for three adjoining suburbs are shown in this table and a graph comparing the median prices is shown below.




**What is a median price?**

The median price is the middle price in a series of sales. For example, if 15 sales are recorded in a suburb and arranged in order from lowest to highest value, the eighth sale price is the median price. In the case where there are an even number of sales in a series, the median is the average of the middle two prices.

Median prices are used rather than average prices because median prices are unaffected by a few unusually high or low prices, making them a more accurate indicator of true market activity. The lower quartile is the point where one quarter of the sales are of lesser value and the upper quartile is the point where one quarter of the sales are of a higher value.

| Location | Lower Quartile | Median Price | Upper Quartile |
|---|---|---|---|
| Avoca Beach | $512 500 | $557 500 | $845 500 |
| Green Point | $422 500 | $467 500 | $519 250 |
| Kincumber | $295 500 | $364 925 | $417 775 |

**a** The median price of which suburb seems to fluctuate the most?

**b** The figures shown in the table are based on quarterly sales figures for the 3 months ending June 2012. Can you tell how many houses were sold? Does it matter?

**c** The median indicates the 'middle' house price. If the mean was used instead, which suburb's 'average' figure would change the most?

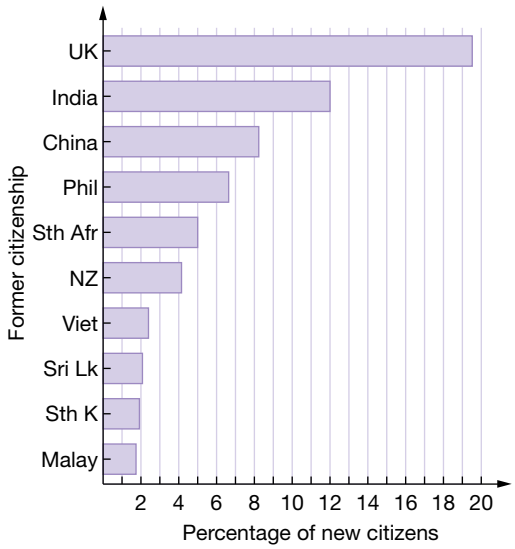
**d** Which suburb's house prices seem symmetrical, and which are skewed?

**6** According to the Department of Immigration and Citizenship more than 4·5 million people have become citizens since Australian citizenship was introduced in 1949.

In 1949, during the inaugural year of the *Nationality and Citizenship Act 1948*, Australian citizenship was granted to 2493 people from just over 35 different nationalities. The top 5 nationalities (and number of grants) at that time according to the *Australian Yearbook 1953, ABS* were: Italian (708) Polish (597) Greek (276) German (225) and Yugoslav (80)

In 2011-12, 84 183 people from at least 180 different countries became Australian citizens by conferral. The following table shows the number of people who were conferred as Australian citizens at ceremonies in 2011–12, according to their former nationality/citizenship.

| Former citizenship | Persons | Per cent |
|---|---|---|
| United Kingdom | 16 401 | 19·5 |
| India | 10 076 | 12·0 |
| China | 6876 | 8·2 |
| Philippines | 5592 | 6·6 |
| South Africa | 4206 | 5·0 |
| New Zealand | 3458 | 4·1 |
| Vietnam | 1929 | 2·3 |
| Sri Lanka | 1671 | 2·0 |
| South Korea | 1570 | 1·9 |
| Malaysia | 1487 | 1·8 |
| Other | 30 917 | 36·7 |
| *Total:* | 84 183 | 100·0 |




**a** The graph shows the 'top 10' sources of migrants for 2011–12. What group listed in the table is not shown on the graph?

**b** At least how many countries, apart from the 'top 10', are a country of origin for Australian migrants?

**c** Why were the top 5 nationalities in 1949 European?

**d** Why do you think the United Kingdom was not among the top 5?

**e** Apart from the United Kingdom, how many of the 2011–12 top 10 are European?

**f** How many of the top 10 nationalities are Asian?

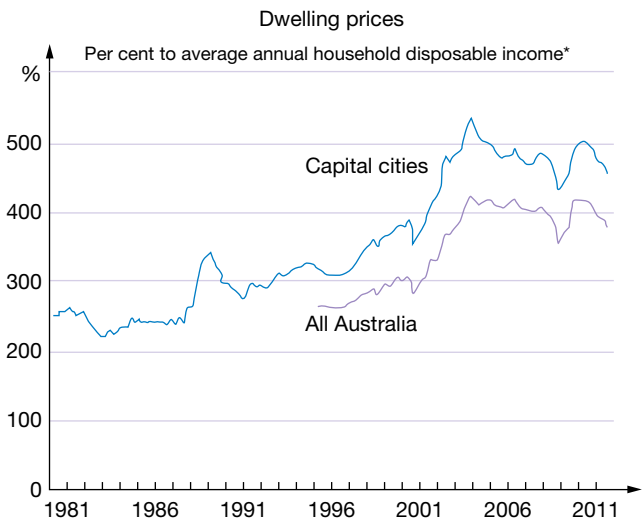
**7** In February 2012, the head of the Reserve Bank's financial stability department reckoned that Australia would avoid a US-style house price meltdown. The chart produced shows house prices as a percentage of annual household disposable income. The ratio peaked in 2003 due to boosted incomes. The spike down in 2008 relates to the GFC (Global Financial Crisis). The bounce back reflects the government's attempts to reinflate the market with the first homeowner's grant.

**a** What year does the graph indicate recorded the lowest percentage of annual income needed to purchase a house?

**b** Why is the graph higher for capital cities?

**c** In 2011, was the percentage going up or down?

**d** If houses are going to be more affordable in the future, at least one of two things must happen. What are they?




* Household disposable income excludes unincorporated enterprises and is before interest payments.
Source: ABS; APM; RBA; REIA

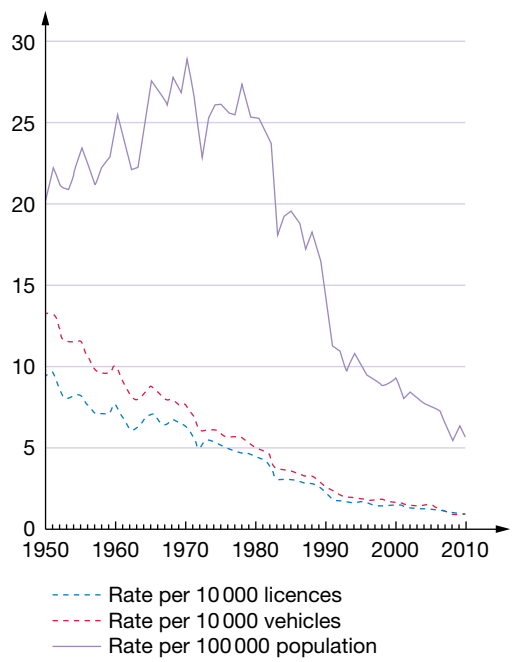
**8** This graph shows the fatality rate per 10 000 vehicles, 10 000 licence holders and 100 000 population for years 1950 to 2010 in NSW.

**a** The line graphs showing the fatality rate per 10 000 vehicles and 10 000 licences are similar. They are progressively falling, but until the early 1980s the rate per 100 000 people was increasing. Explain how this could happen.

**b** The blue and red lines merge in the 2000s. Why do you think they were so different in 1950?

**c** What caused the significant decrease in the 1980s for the purple graph?

**d** Road Safety authorities have tried to introduce measures to ensure the rate keeps decreasing. What measures have contributed to safer road travel?

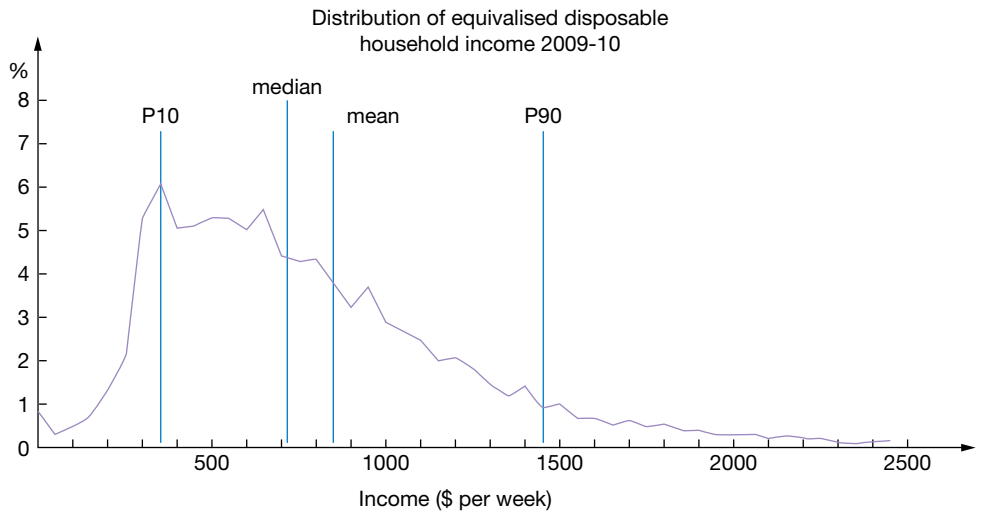



Statistical statement for the year ended 31 December 2010
Prepared by the Centre for Road Safety, Transport for NSW

**9** The Australian Bureau of Statistics (ABS) collected the data below showing the 'average' income for the States and Territories of Australia. ('Equivalised' means that adjustments have been made to take into account the number of people living in each household.)

| Equivalised disposable household income per week | NSW $ | Vic $ | Qld $ | SA $ | WA $ | Tas $ | NT $ | ACT $ | Aust $ |
|---|---|---|---|---|---|---|---|---|---|
| Mean income | 859 | 818 | 817 | 798 | 966 | 708 | 938 | 1101 | 848 |
| Median income | 718 | 695 | 705 | 685 | 770 | 613 | 885 | 1009 | 715 |

'While the mean equivalised disposable household income of all households in Australia in 2009–10 was $848 per week, the median (i.e. the midpoint when all people are ranked in order of income) was lower at $715 per week. This difference reflects the typically asymmetric distribution of income where a relatively small number of people have relatively high household incomes, and a large number of people have relatively lower household incomes.'

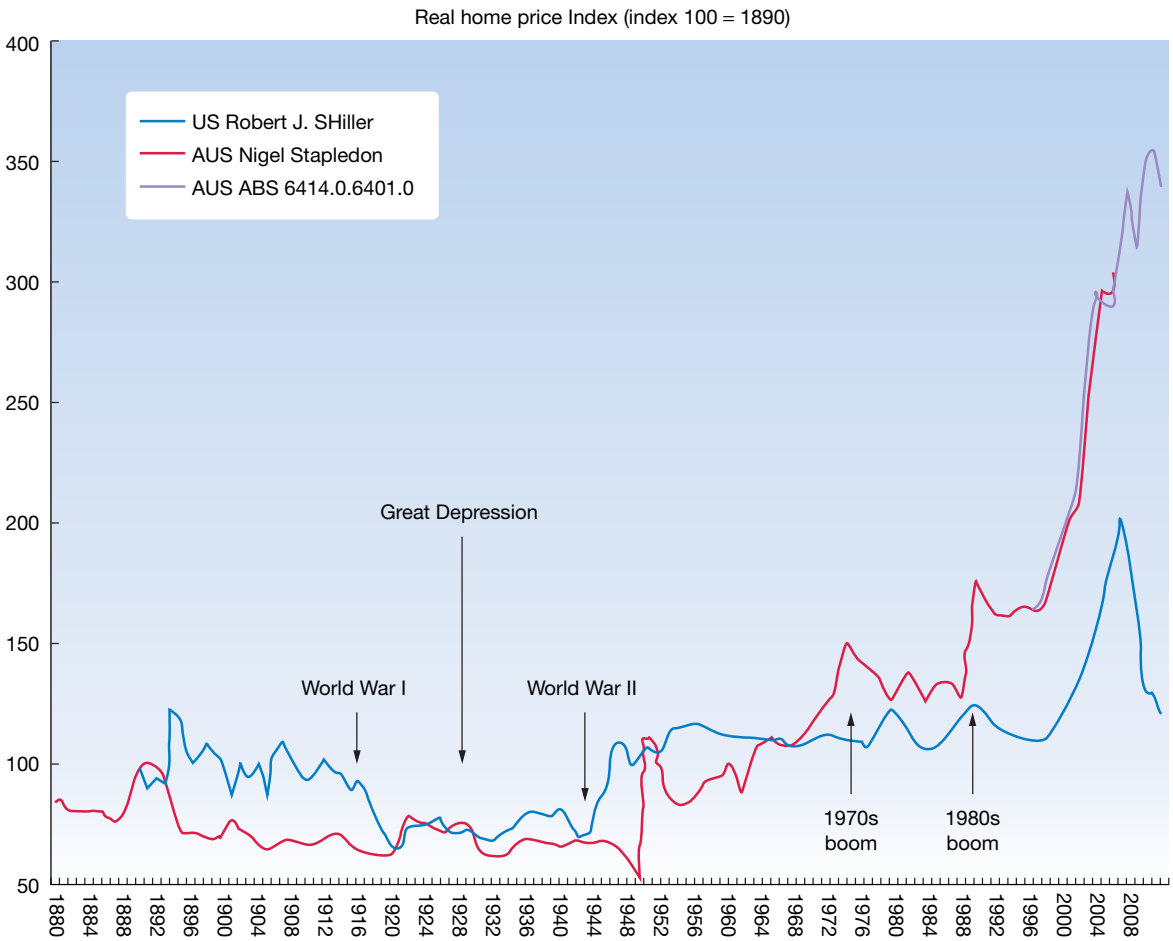



Source: Household Income and Income Distribution, Australia (6523.0)

**a** Which state or territory has the highest average incomes? Why do you think this is the case?
**b** Which state or territory has the greatest difference between the median and mean incomes? Can you think of a reason for this large difference?
**c** The quote above says the graph shows an 'asymmetric' distribution. This means it is skewed. Is it skewed positively or negatively?
**d** On the graph are lines marked as P10 and P90. What do you think they indicate?
**e** Which line could be marked as P50: the median or the mean?

**10** In January 2012, Dr Alex Cowie, Editor of 'Diggers & Drillers' wrote:
Property makes up a cornerstone of many Australian's wealth. And Aussie property has had a bull market like no other. It started in the 1970s, decades before things got going in the United States. Over 40 years, Aussie property has gained an average of about 3% a year. Doesn't sound like much, but it adds up quickly. From the 1990s onwards, Australian house prices went up a few gears, and never looked back. Prices rose at an average rate of 6% a year at that time.

Is the party over? Australian property prices pulled back from gravity-defying heights by at least 3·7% last year. Melbourne houses are down 9%. Many home owners are nervous. Is this pullback the start of something bigger? Nearly every property market elsewhere in the world has spent the last few years imploding. US property prices are down by 40% in six years. Compare US prices in blue on the chart below against Australian house prices (in red)
Is the 40-year run up in Aussie property setting us up for the crash of all crashes?

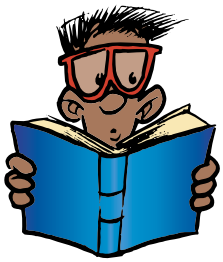



**a** What is measured on the vertical axis in this graph and how is it measured?
**b** When were houses relatively cheapest in Australia?
**c** The article finishes with a question. Can you answer it using this graph?
**d** What factors might cause a property 'crash'?

# 9:06 Statistics: Getting it together

Over the years, various forms of data collection and recording have been used. This information has then been used to change societies and justify actions. It is important that the processes of information collection are open to scrutiny in order for the results to be accepted as valid. We can formulate key questions in order to investigate issues concerned with the collection, organisation, display and analysis of data.

## 1 Design an experiment



**Ask:**

- What question do I wish to answer?
- What data is needed?
- How could this data be best obtained?
- What results do I expect?

**Questionnaires:**

- Bias should not be present. For example, do we ask: 'What do you think of the mayor?' or 'What don't you like about the mayor?'
- If the questions are poorly worded, the information collected is inaccurate.
- Be prepared to refine questions, if necessary.
- Do the responses truly reflect the person's viewpoint?

## 2 Plan the collection of data

**Ask:**

- How much data should be collected? Should a sample be used or the whole population?
- How should the data be collected?
- Is the method of collection practical and relevant?
- Am I collecting fact or opinion?
- Can I easily display the information being collected?

**Sampling:**

- Decide between a census (everyone) and a sample.
- Does each person have the same chance of being selected in a random sample?
- Are there subgroups in a population that requires **stratified random sampling**?
- Will the size of the sample ensure a fair investigation with unbiased results?

## 3 Collect the data

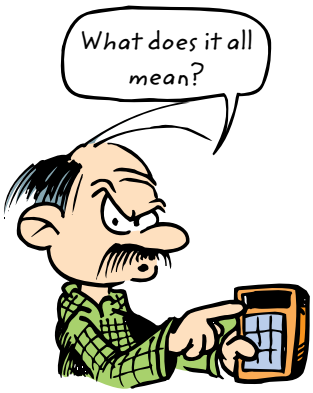
**Ask:**

- Have I collected enough information to make it useful?
- How can I check the data for possible errors or bias?
- How should I tabulate the data?
- Is the information **discrete** or **continuous** (where rounding needs to be done during measurement)?

## 4 Describe the methods used

**Report:**

- Explain the methods used to collect the data.
- Explain the sampling methods used.




## 5 Analyse and display the data

**Choose techniques:**

- Classify, sequence and tabulate data, using grouped intervals where appropriate.
- Carry out appropriate statistical calculations.
- Choose suitable graphs to display the data.
- Organise data into tables. Choose scales and draw appropriate graphs.

## 6 Discuss your observations

**Consider the following:**

- Make inferences from tables and graphs.
- Be wary of jumping to conclusions.
- Does any relationship exist, and if so, what type?
- How strong is any relationship?
- How reliable are the findings?
- How can the results be summarised?
- Can any results be generalised to other situations?
- Test conclusions by discussing them with others.

## Example: How do the heights of males and females compare in secondary schools?

**1 Design an experiment**

- 'Let's compare the heights of males and females.'
- Measure the heights of equal numbers of males and females in each year group. Provide measuring instruments. Decide how heights will be measured (shoes on or off, back to the wall, etc).

**2 Plan the collection of data**

- Collect information from both sexes and a variety of age groups. Stratified random sampling should be used.
- For each year group, keep male and female results separate, and examine the data in separate year groups.

**3 Collect the data**

- Choose at least 10 males and 10 females at random from each year group.
- Clear directions on how to measure must be given.
- Measurements would be continuous, so data would need to be rounded to the nearest centimetre.

**4 Describe the methods used**

- Every tenth male and female could be chosen from class lists of both boys and girls.
- Heights will be measured without shoes and with students standing with their backs against the wall, where a measuring tape will be fastened.
- Use a ruler, placed on the student's head, to read the measurement.

**5 Analyse and display the data**

- Check the raw data for obvious errors (e.g. a height of 12 cm).
- Dot plots and box plots should be used to show the spread of scores for each sex in each year group, and to compare them.
- The mean, range and standard deviation should be calculated for each group and any trends in the data determined.
- Back-to-back stem-and-leaf displays would be useful to compare the heights of boys and girls in each year group.
- The information and graphs would be studied to determine trends.

**6 Discuss the observations**

- Conclusions would be drawn from the displays and the statistical results.
- The results and observations would be recorded and from these, new questions might arise.
- The results would be shared with others and discussed.

## Exercise 9:06

**1** Care must be given to possible responses to a particular question. Consider these questions.

**X** Do you think school should start 30 minutes early? (Yes/No)

**Y** Do you think school should start 30 minutes earlier if you will still finish at the same time? (Yes/No)

**Z** If you think it is a good idea to start school 30 minutes earlier, state why.

**a** Which question asks for an opinion?

**b** Which question is the clearest in what it is asking?

**2** **a** Trial the following questionnaire with three people. It is investigating people's attitudes towards changing our flag.

**A** Do you like the colours red, white and blue? (Yes/No)

**B** Do you like the Union Jack? (Yes/No)

**C** Do you like our national flag? (Yes/No)

**b** Ask each of the three people if they felt that the questions allowed them to express an opinion about changing Australia's flag in the future.

**c** Write a better set of questions for this purpose and trial it on the same people. Does your questionnaire allow them to better express their views?

**3** There are different ways of presenting questions in order to collect information. You are given the task of surveying Year 12 students to find out what they hoped to do in the year following their HSC. Make up a short questionnaire to obtain this information using:

**a** open questions

**b** yes/no questions

**c** tick-a-box questions

**d** responses using a 1 to 5 scale.

**4** Compile a questionnaire that could be used to collect views from students concerning:

- the legal driving age
- the age for compulsory education
- the time they spend using a computer
- the time they spend doing some form of exercise or sport.

**5** Using the outline suggested in the example before this Exercise, investigate one of the following:

- how the heights of males and females compare in each year in your school
- the number of books read in the past 12 months by students in each year
- the number of online friends of students in Years 10, 11 and 12
- the views of students with reference to some relevant political or social issue

**6** Insurance companies use statistics to determine the premium (payment) it will charge to insure a vehicle. Some factors are listed below. What data do you think the insurance company assesses for each of these factors and how would you expect it to affect the premium?

**a** location; where you live
**b** use of vehicle
**c** previous insurance claims history
**d** type of vehicle
**e** extra coverage

**7** Are males more likely to be killed on the roads than females? Consider the data in the table below supplied by the Department of Infrastructure and Transport.

**a** What conclusions, if any, can be drawn from the data?
**b** What other information would be useful? Do any other factors need to be taken into account?

| Annual road deaths per 100 000 population by gender and age group | | | | | | | |
|---|---|---|---|---|---|---|---|
| Age in years | 0–16 | 17–25 | 26–39 | 40–59 | 60–69 | ≥70 | All deaths |
| Male | 2·2 | 14·0 | 9·6 | 8·6 | 7·1 | 12·1 | 8·3 |
| Female | 1·5 | 5·1 | 2·7 | 2·8 | 2·7 | 6·1 | 3·1 |
| *Total:* | 1·9 | 9·7 | 6·2 | 5·7 | 4·9 | 8·7 | 5·7 |

**8** Critically comment on each of these statements.

**a** Home is a dangerous place because more accidents occur at home than away from home.
**b** Most accidents at home occur in the kitchen, so houses should be built without one.
**c** Many deaths occur in hospitals, therefore it is not a good place to go when you are sick.
**d** More road accidents occur close to home, so short trips should be avoided.

## INVESTIGATION 9:06 OPINION POLLS: HOW ACCURATE ARE THEY?

We are bombarded by the media with opinion polls and surveys of all kinds, but few people recall being asked for their opinion. Political polls are of concern to politicians to see if their policies and performance are favourable. Market polls are important to business so they can see where their advertising needs to be focused or their products improved. If only a sample of a population is surveyed, how can its accuracy be ensured? The following example is cited by Data Analysis Australia, a group of strategic information consultants.

> 'Modern political polling began in 1936, with two polls attempting to predict the outcome of the American presidential election. The *Literary Digest* conducted its poll by sending out 10 million post cards asking people how they would vote. They received almost 2·3 million back and said that Alfred Landon was leading Franklin Roosevelt by 57–43 per cent. In contrast, market researcher George Gallup employed a much smaller sample of only 5,000, but because he ensured that it was representative of the American voting public, he predicted Roosevelt to win by a landslide. In the event, Roosevelt won 60% and Landon just 37%. The *Literary Digest* lost credibility and subsequently merged with *Time* magazine in 1938.'

The *Literary Digest* thought it would be accurate because of its large sample size but it was not representative. It chose its sample from its owner reader lists, telephone subscribers and car owners. Today these lists might be more representative but in 1936 these were indicators of wealth and so many who were poor due to the Depression were not included.

For a sample to be a reliable representation of the population every person must have a chance of being surveyed.

- It must be random.
- It must match the make-up of the population in gender, age groups and location.
- It must consider other factors when the survey is completed such as the time of day, the day of the week, the time of the year or even the weather which, depending on the subject of the survey, may have an effect.

An example of a market survey is shown in this table. It lists the top 10 TV programs watched in Sydney during July 2012, according to the Roy Morgan research group. Only a sample of households was surveyed, but those asked would have been carefully selected to represent the viewing population.

The percentage shown is the percentage of those watching TV at that time.

**Top 10 TV results for Sydney: July 2012**

| Rank | Channel | Program | % |
|---|---|---|---|
| 1 | 9 | Rugby League: State of Origin (Wed) | 19·5 |
| 2 | 9 | NRL (Fri) | 12·4 |
| 3 | 7 | Downton Abbey (Sun) | 12·0 |
| 4 | 7 | Seven News (Mon–Sun) | 11·1 |
| 5 | 9 | Nine News (Mon–Sun) | 10·1 |
| 6 | 7 | Revenge (Mon) | 9·5 |
| 7 | 7 | Sunday Night (Sun) | 8·7 |
| 8 | 7 | Home and Away (Mon–Fri) | 8·7 |
| 9 | 7 | Today Tonight (Mon–Fri) | 8·1 |
| 10 | 2 | ABC News (Mon–Sun) | 8·0 |

### Suggested research

- Use the internet to research polling organisations such as market research companies and newspapers.
- Gain further insights into polling methods from sites such as Data Analysis Australia.
- Investigate the accuracy of previous political polls in predicting the result of an election.

## CHALLENGE 9:06 USE OF AUSTRALIAN CENSUS DATA

Australian census data can be, and should be, used by government at national, state or local levels to identify issues that need attention, or are likely to need attention in the future. It should guide government and other community organisations in the allocation of resources.

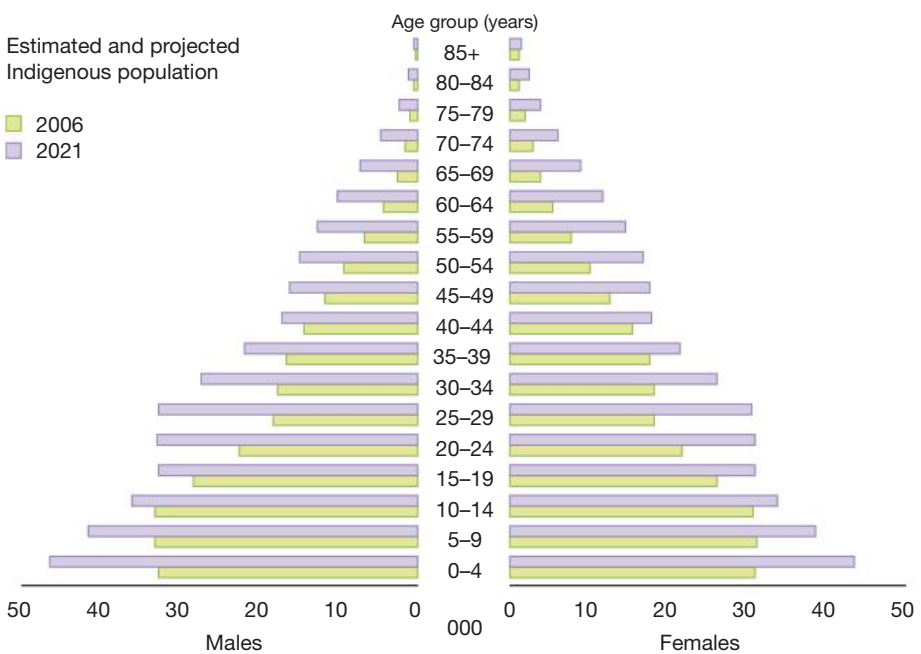
The **Australian Bureau of Statistics** website could be used to examine many sets of data that would promote discussion of government policy. A statistical overview of Aboriginal and Torres Strait Islander peoples has been chosen here as an example.

The estimated resident Indigenous population of Australia at 30 June 1991 was 351 000 people. In 2006, there were 517 000 people, representing 2·5% of the total Australian population. Between 1991 and 2006 the Indigenous population increased by 2·6% per year on average, compared with 1·2% for the total Australian population. The population of Indigenous Australians is projected to increase to between 713 300 and 721 100 people in 2021, at an average growth rate of 2·2% per year. In comparison, the average growth rate of the total Australian population is projected to be between 1·2% and 1·7% per year over the same period.

The Indigenous population is projected to increase across all age groups between 2006 and 2021. The number of Indigenous children (0–14 years) is projected to increase from 194 200 in 2006 to between 242 600 and 243 400 in 2021. This equates to an increase of 25% over the period. The number of young adults (15–24 years) is expected to increase by a similar proportion (29%), from 99 700 to between 128 600 and 128 800.

**Estimated and projected Indigenous population, 30 June**

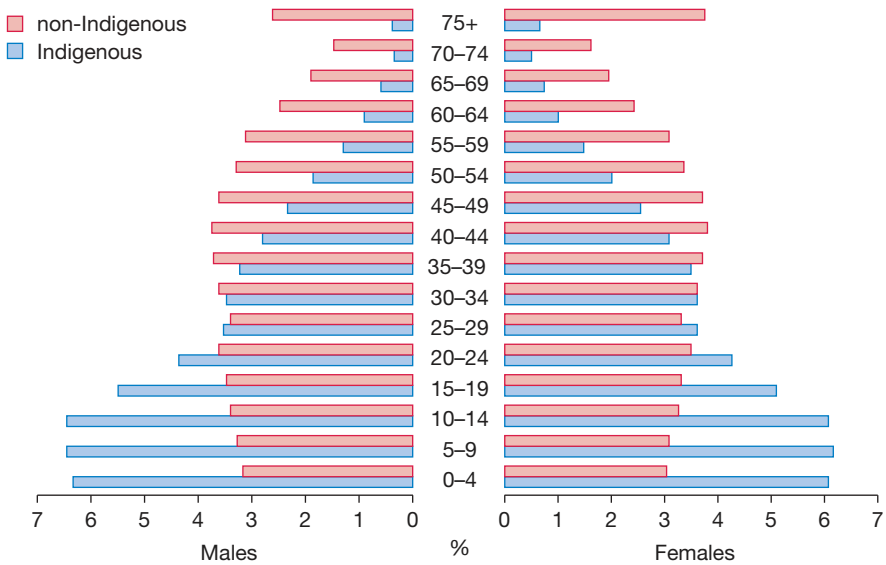
| Age group (years) | 1991 | 2006 | 2021 |
|---|---|---|---|
| 0–14 | 137 749 | 194 249 | 243 429 |
| 15–24 | 73 355 | 99 722 | 128 781 |
| 25–34 | 59 091 | 73 149 | 117 515 |
| 35–44 | 38 742 | 65 011 | 78 599 |
| 45–54 | 20 431 | 44 885 | 66 190 |
| 55–64 | 11 831 | 24 112 | 49 027 |
| 65–74 | 6706 | 10 768 | 26 704 |
| 75 and over | 3080 | 5147 | 10 819 |
| *Total:* | 350 985 | 517 043 | 721 064 |



The Indigenous population has a different age structure to the rest of the Australian population. In common with many other developed countries, the non-Indigenous population of Australia is ageing, whereas Indigenous peoples are facing increased growth in younger age groups. This can be seen in the graph below.

In 2006, the median age was 21 years for Indigenous Australians, and 37 years for the non-Indigenous population. Of the Indigenous population, 3% were aged 65 years and over, and 38% were under 15 years of age (compared with 19% of the non-Indigenous population).

Estimated resident population, comparing Indigenous and non-Indigenous age structures, 2006



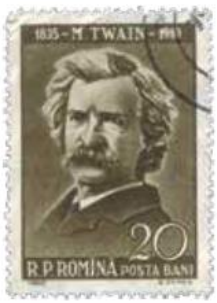

Sources: Australian Human Rights Commission website—'A statistical overview of Aboriginal and Torres Straight Islander peoples in Australia'
ABS—Experimental estimates and projections, Aboriginal and Torres Straight Islander Australians, 1991 to 2021 (cat no. 3238.0)

## Discussion

- How would this data prove useful to government?
- What further data should be considered in light of the information presented here?
- What factors may be causing the differences in the population pyramids shown above?
- Why is the Aboriginal population growing at a faster rate than the rest of the Australian population?

Consider studying the ABS website for further population data and information. It is amazing what lies within!

# 9:07 Statistics: Can they be trusted?




'There are three kinds of lies: lies, damned lies, and statistics.' This statement was made popular by Mark Twain, who attributed it to the British Prime Minister Benjamin Disraeli. It points to the suspicion that statistics are used to strengthen weak arguments or even promote an untruth.

In his book *The Dictionary of Misinformation*, Tom Burnan states:
'No matter how much reverence is paid to anything purporting to be statistics, the term has no meaning unless the source, relevance, and truth are checked … The worship of statistics has had the particularly unfortunate result of making the job of the plain, outright liar that much easier.'

And a quote reportedly by Charles H Grosvenor:
'Figures don't lie, but liars do figure.'

Suspicion is also caused by those who selectively choose statistics to support a particular argument when the same set of data can be used to support an alternative argument. Of course, the problem is not with the statistics but with how they are used, how they were produced and what they truly represent. Problems occur when people misunderstand what the data is saying or wilfully misrepresent it.

Incorrect conclusions can be drawn by not looking carefully at how the data was collected and how it has been presented. In previous years some examples of how statistics and their graphs can be misused have been shown. This exercise will review some of those and encourage the questioning of conclusions presented in the media and other sources.

## Exercise 9:07

**1** Graphs are often used to grab our attention. This graph is showing the movement of the 'All Ordinaries' index in the business section of the newspaper.

- **a** What initial impression is given by this graph?
- **b** The horizontal axis is labelled Dec 03 – Dec 07. What is the scale on this axis? To what do the points on the graph correspond?
- **c** What is measured by the vertical axis?
- **d** Looking at the high and low figures, what do you think the scale is on the vertical axis?
- **e** The difference of 37·9 (0·8%) is not the difference between the high and low figures. Subtract 37·9 from 4555·9. What point on the graph does this figure represent?

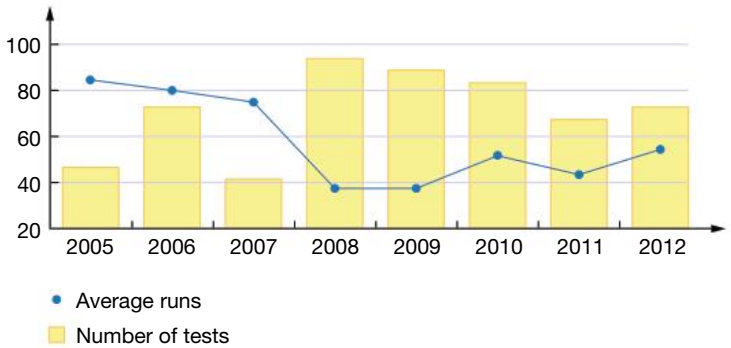
Source: *Sydney Morning Herald* 8–9 Dec 2007

**2** The diagram on the right indicates the recent gains and losses on the stock market. The black bars indicate gains, the red bars indicate losses.

- **a** Do the lengths of the bars for Energy World and Imdex look the same?
- **b** The length of the bar for Intrepid Mines looks twice the length of the bar for BOQ Peps. Do the percentage figures support this?
- **c** Are the lengths of the bars in the diagram proportional to the percentages shown?
- **d** Why do you think the information has been presented in this way?

Source: *Daily Telegraph* 29 Nov 2012

**3** The graph below represents the cricket career of Michael Hussey in test matches. The line graph connects the points showing his average number of runs for each year.




Source: *Daily Telegraph* Dec 2012

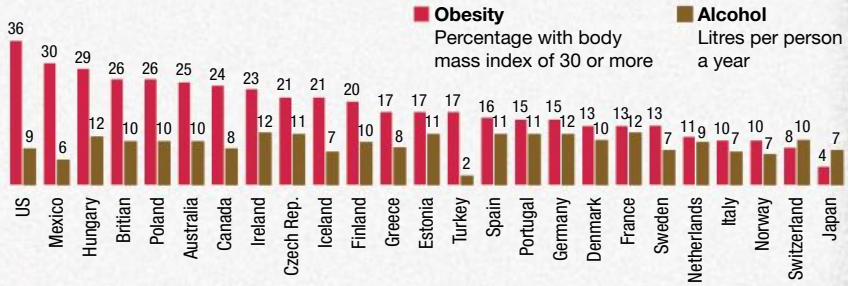
- **a** What does the yellow column graph show?
- **b** Can you say how many test matches Michael Hussey played in 2005? What is missing from the graph?
- **c** Michael Hussey played four tests in 2007 and 14 in 2008. If the columns have been drawn to scale, how many tests did he play in 2010?
- **d** In a cricket match there are two 'innings' but a batsman may not always bat twice. In 2007 Michael Hussey played in four tests but only batted six times. In 2009, he played in 13 tests and batted 23 times. Is it easier to have a higher average if you bat less? Does this graph support your answer?

**4** The article below is titled 'Health v wealth'.

## Data mine

### Health v wealth

More than a third of US citizens and a quarter of Australians are obese, says the Organisation for Economic Co-operation and Development, the industrialised countries' body. But the average Australian drinks a litre a year more than the average American.




Rounded to the nearest whole number

Sources: OECD. MCT

Source: *Sun-Herald* 16 Dec 2012

**a** How is the health of each nation being represented?

**b** How is the wealth of each nation being represented?

**c** What do the figures on the top of each column represent?

**d** The heights of the columns seem to be in proportion to these numbers. That is, the red column for the US is four times the height of the brown column. Does this mean that if people in a country drank 1 litre of alcohol per year on average, this should equate to 1% of the population being obese?

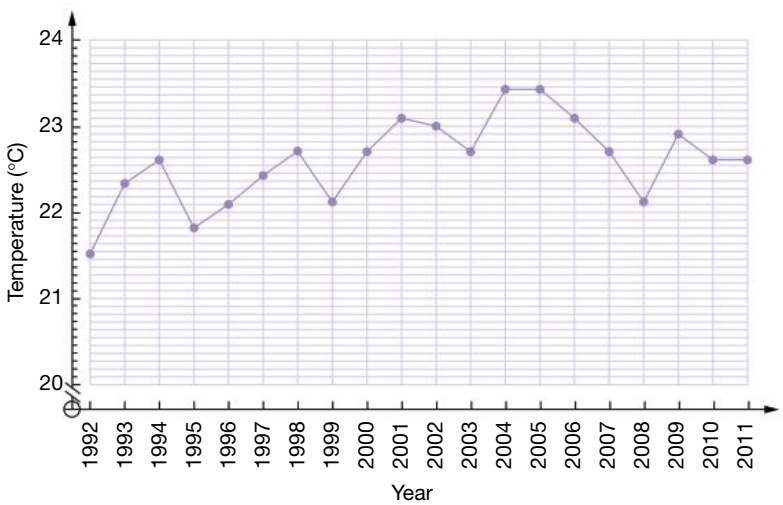
**e** Does this dual column graph suggest the two variables are related?

**f** Draw a scatter plot for these variables.

**g** Does any relationship exist?

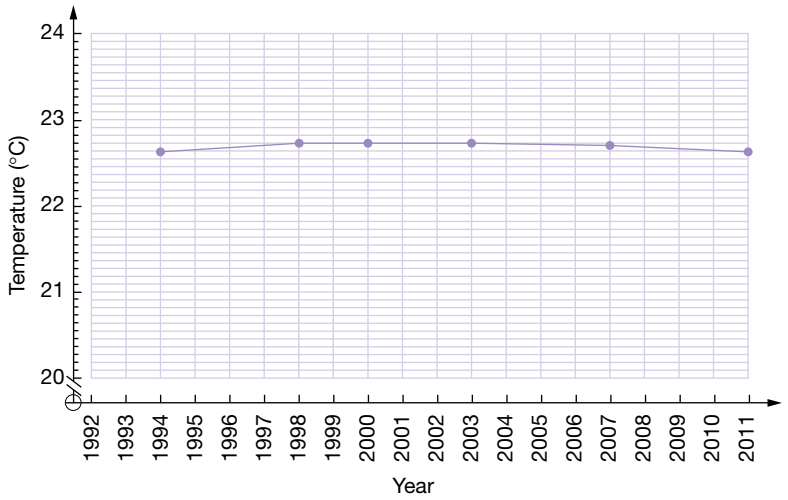
**h** Why is this not a fair comparison of the wealth and health of a nation?

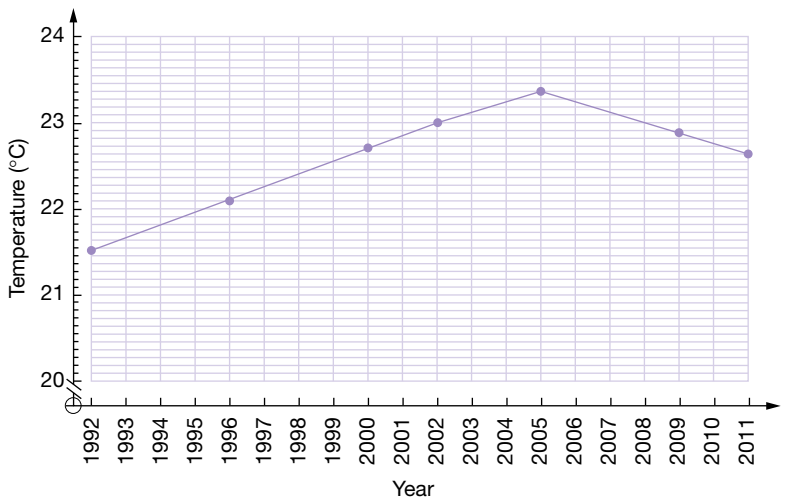
**5** Sometimes an argument will be supported by the selective use of data. The graph and table below show the mean maximum temperature for Sydney for the years 1992–2011.




| Year | Mean max temp |
|---|---|
| 1992 | 21·5 |
| 1993 | 22·3 |
| 1994 | 22·6 |
| 1995 | 21·8 |
| 1996 | 22·1 |
| 1997 | 22·4 |
| 1998 | 22·7 |
| 1999 | 22·1 |
| 2000 | 22·7 |
| 2001 | 23·1 |
| 2002 | 23·0 |
| 2003 | 22·7 |
| 2004 | 23·4 |
| 2005 | 23·4 |
| 2006 | 23·1 |
| 2007 | 22·7 |
| 2008 | 22·1 |
| 2009 | 22·9 |
| 2010 | 22·6 |
| 2011 | 22·6 |

A general trend upwards might be detected but the line graph joining the points is quite 'saw-toothed', i.e. up and down, with a peak showing in the years 2004–2005. By selecting only some of the data, two very different trends can be shown, as in the graphs below.




The same data points have been used from the data, but only some of them, for each graph.

Further statistics can be found on the Bureau of Meteorology website. The site used is Observatory Hill in Sydney.

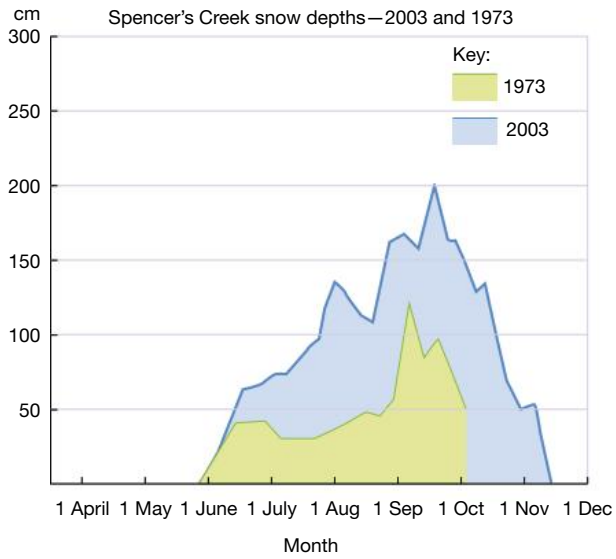
**a** What trend for the temperature of Sydney for the 20 years shown is suggested by:
  **i** graph A  **ii** graph B?

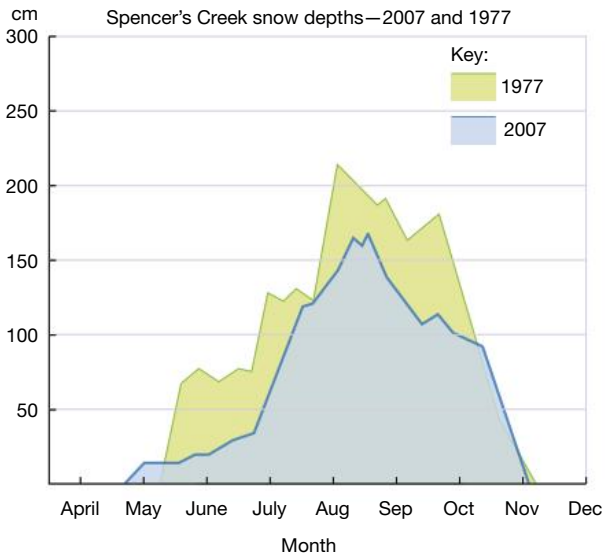
**b** Using interpolation, estimate a mean maximum temperature for 2001, using:
  **i** graph A  **ii** graph B.

**c** Do either of the answers to **b** match the figure in the table?

**d** Extrapolate each graph to predict the temperature for years after 2011. How confident could you be of your prediction?

**6** Another example of using selective or inadequate data can be seen in the graphs below. The original graph is shown on the left. The later graph is shown on the right.



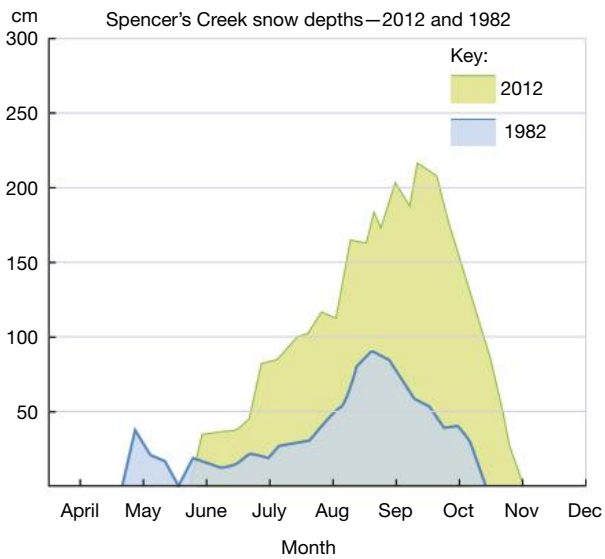




**a** The question for the original graph was: 'Looking at the snow depths for these 2 years it would seem that the depths have significantly increased over this 30-year period. Is this a reasonable conclusion?' What would have been your answer?

**b** The question for the later graph was exactly the same, except the word 'increased' was changed to 'decreased'. How would this have changed your answer?

**c** The graph on the right shows the data again collected in a further 5 years' time. Using this 30-year comparison, has the snow depth increased or decreased over time?

**d** How could a more reliable view be obtained to determine if there is any pattern for the snow depth over time?



**7** This first table shows the annual carbon dioxide ($CO_2$) emissions, in millions of tonnes, for the top five countries plus Australia for 2009.

| Rank | 15 | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|
| Country | Australia | Japan | Russia | India | USA | China |
| $CO_2$ emissions | 418 | 1098 | 1572 | 1602 | 5425 | 7711 |
| % of world emissions | 1·4 | 3·6 | 5·2 | 5·3 | 17·8 | 25·4 |

The second table shows $CO_2$ emissions, in tonnes, per capita (per person) for these countries in order.

| Country | India | China | Japan | Russia | USA | Australia |
|---|---|---|---|---|---|---|
| $CO_2$ emissions per capita | 1·38 | 5·83 | 8·64 | 11·23 | 17·67 | 19·64 |

**a** If you wanted to argue that Australia needs to reduce its $CO_2$ emissions, which table of data are you more likely to present to support your case?

**b** If you wanted to argue that any reduction in Australia's $CO_2$ emissions will have little global effect, which table of data would you most likely use?

**c** The second table takes into account each country's population size. Consider the countries in the list. What other factors might be relevant to your argument apart from population size?

**d** The table below lists the ranking of each country in the world according to its per capita emissions. How might this table influence your argument?

| Country | India | China | Japan | Russia | USA | Australia |
|---|---|---|---|---|---|---|
| $CO_2$ emissions per capita rank | 143 | 75 | 53 | 32 | 17 | 13 |

A snow depths calculator can be found at the Snowy Hydro website.

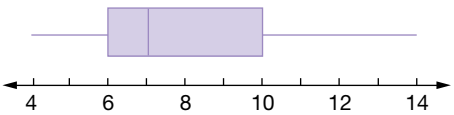
## MATHS TERMS 9

**bivariate data**

- data that has two variables

**box plot (box-and-whisker plot)**

- a diagram obtained from the five-number summary
- the box shows the middle 50% of scores (the interquartile range)
- the whiskers show us the extent of the bottom and top quartiles as well as the range




**census**

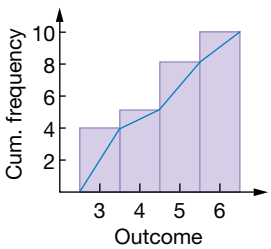
- a survey of a whole populations

**cumulative frequency**

- the number of scores less than or equal to a particular outcome,
  e.g. for the data 3, 6, 5, 3, 5, 5, 4, 3, 3, 6
  the cumulative frequency of 5 is 8
  (there are 8 scores of 5 or less)

**cumulative frequency histogram (and polygon)**

- these show the outcomes and their cumulative frequencies

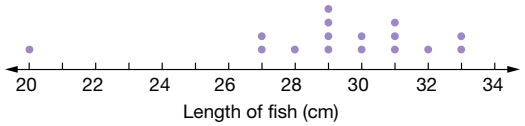



**data**

- the pieces of information (or 'scores') to be examined
- *categorical*: data that uses non-numerical categories
  - *ordered* data involves a ranking, e.g. exam grades, garment sizes
  - *distinct* data has no order, e.g. colours, types of cars
- *numerical:* data that uses numbers to show 'how much'
  - *continuous* data can have any numerical value within a range, e.g. height
  - *discrete* data is restricted to certain numerical values, e.g. number of pets

**dot plot**

- a type of graph that uses one axis and a number of dots above the axis




**extrapolation**

- predicting data beyond the range of values given

**five-number summary**

- a set of numbers consisting of the minimum score, the three quartiles and the maximum score

**frequency**

- the number of times an outcome occurs in the data,
  e.g. for the data 3, 6, 5, 3, 5, 5, 4, 3, 3, 6
  the outcome 5 has a frequency of 3

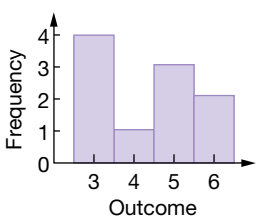
**frequency distribution table**

- a table that shows all the possible outcomes and their frequencies (it usually is extended by adding other columns such as the cumulative frequency),
  e.g.

| Outcome | Frequency | Cumulative frequency |
|---|---|---|
| 3 | 4 | 4 |
| 4 | 1 | 5 |
| 5 | 3 | 8 |
| 6 | 2 | 10 |

**frequency histogram**

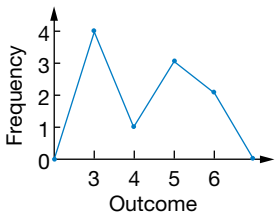
- a type of column graph showing the outcomes and their frequencies



**frequency polygon**

- a type of line graph showing outcomes and their frequencies
- to complete the polygon, the outcomes immediately above and below the actual outcomes are used (the height of these columns is zero),

e.g.




**grouped data**

- data that is organised into groups or classes
- *class intervals:* the size of the groups into which the data is organised,
  e.g. 1–5 (5 scores); 11–20 (10 scores)
- *class centre:* the middle outcome of a class,
  e.g. the class 1–5 has a class centre of 3

**interpolation**

- estimating data that lie within the domain of the values given

**interquartile range**

- the range of the middle 50% of scores
- the difference between the median of the upper half of scores and the median of the lower half of scores

$\text{IQR} = Q_3 - Q_1$

**line of best fit**

- a line that 'best fits' the data on a scatter plot mean

**mean**

- the number obtained by 'evening out' all the scores until they are equal,
  e.g. if the scores 3, 6, 5, 3, 5, 5, 4, 3, 3, 6 were 'evened out', the number obtained would be 4·3
- to obtain the mean, use the formula:

$$\text{mean} = \frac{\text{sum of the scores}}{\text{total number of scores}}$$

**median**

- the middle score for an odd number of scores or the mean of the middle two scores for an even number of scores
- the *median class* in grouped data contains the median

**mode (modal class)**

- the outcome or class that contains the most scores

**ogive**

- this is another name for the cumulative frequency polygon

**outcome**

- a possible value of the data

**outlier**

- a score that is separated from the main body of scores

**quartiles**

- the points that divide the scores up into quarters
- the second quartile, $Q_2$, divides the scores into halves ($Q_2$ = median)
- the first quartile, $Q_1$, is the median of the lower half of scores
- the third quartile, $Q_3$, is the median of the upper half of scores

4 5 6 | 6 7 7 | 7 9 9 | 11 12 15

$Q_1 = 6 \quad Q_2 = 7 \quad Q_3 = 10$

**range**

- the difference between the highest and lowest scores

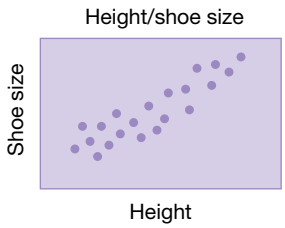
**sample**

- a part (usually a small part) of a large population
- *random sample:* a sample taken so that each member of the population has the same chance of being included
- *systematic sample:* a sample selected according to some ordering scheme, e.g. every tenth member
- *stratified sample:* a sample is proportionally taken from each subgroup in a population

**scatter plot**

- a graph that uses points on a number plane to show the relationship between two categories

e.g.




**shape (of a distribution)**

- a set of scores can be symmetrical or skewed

**sources of data**

- *primary:* the data has been collected by yourself
- *secondary:* the data has come from an external source, e.g. newspapers, internet

**standard deviation**

- a measure of spread that can be thought of as the average distance of scores from the mean
- the larger the standard deviation, the larger the spread

**statistics**

- the collection, organisation and interpretation of numerical data

**stem-and-leaf plot**

- a graph that shows the spread of scores without losing the identity of the data,
  e.g. Test results, 8M
  (6/8 represents 68)

| Stem | Leaf |
|---|---|
| 4 | 1 8 |
| 5 | 9 3 7 0 4 |
| 6 | 8 5 6 7 2 4 8 6 |
| 7 | 3 7 3 1 9 3 5 |
| 8 | 5 4 8 0 4 |
| 9 | 3 7 1 |

- *ordered stem-and-leaf plot:*
  the *leaves* are placed in order,
  e.g.

| Stem | Leaf |
|---|---|
| 4 | 1 8 |
| 5 | 0 3 4 7 9 |
| 6 | 2 4 5 6 6 7 8 8 |
| 7 | 1 3 3 3 5 7 9 |
| 8 | 0 4 4 5 8 |
| 9 | 1 3 7 |

- *back-to-back stem-and-leaf plot:*
  this can be used to campare two sets of scores, one set on each side,
  e.g. Test results
  (4/1 represents 41)

| | Stem | Leaf | Stem |
|---|---|---|---|
| 9 8 6 6 0 | | 3 | |
| 9 7 7 3 1 1 1 | | 4 | 3 6 6 8 |
| 8 8 5 3 3 0 | | 5 | 1 7 9 9 |
| 9 8 7 5 3 | | 6 | 3 8 9 |
| | | 7 | 0 5 5 5 8 9 |
| | 0 | 8 | 2 6 7 7 |
| | | 9 | 0 0 1 3 |

**variable**

- something that can be observed, measured or counted to provide data

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** For each data set find the:

i range ii mode iii mean iv median
v first quartile $Q_1$ vi third quartile $Q_3$ vii interquartile range.

a 2 3 3 7 8 9 10
b 6 4 3 9 5 9 4 8

9:01

**2** a Complete this frequency distribution table.
b Find the:
i range
ii mode
iii mean
iv median
v first quartile $Q_1$
vi third quartile $Q_3$.

| Outcome | Tally | $f$ | $f \times x$. | c.f. |
|---|---|---|---|---|
| 6 | 𝍸 \|\| | | | |
| 7 | 𝍸 𝍸 \| | | | |
| 8 | 𝍸 \|\|\| | | | |
| 9 | 𝍸 | | | |
| 10 | \|\|\|\| | | | |
| 11 | \|\|\| | | | |
| 12 | \|\| | | | |

*Total:* ___ ___

9:01

**3** a For this table of grouped continuous data find:
i the modal class
ii the median class
iii an estimate for the mean.
b Construct a cumulative frequency histogram and polygon and show on it the position of the median and quartiles.
c Find an estimate for the interquartile range.

| Class | c.c. | $f$ | $f \times$ c.c. | c.f. |
|---|---|---|---|---|
| 0–<10 | 5 | 4 | 20 | 4 |
| 10–<20 | 15 | 6 | 90 | 10 |
| 20–<30 | 25 | 9 | 225 | 19 |
| 30–<40 | 35 | 15 | 525 | 34 |
| 40–<50 | 45 | 10 | 450 | 44 |
| 50–<60 | 55 | 6 | 330 | 50 |
| | *Total:* | 50 | 1640 | |

9:01

**4** a For this stem-and-leaf plot find the:
i range
ii mode
iii median
iv lower (first) quartile $Q_1$
v upper (third) quartile $Q_3$
vi interquartile range.
b Write the five-number summary for this set of data.
c Using the five-number summary, construct a box plot.

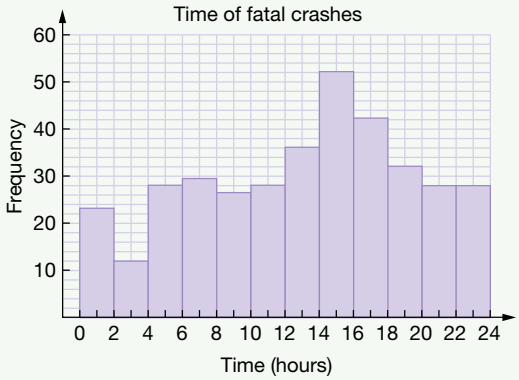
| Stem | Leaf |
|---|---|
| 4 | 3 7 |
| 5 | 1 4 8 |
| 6 | 2 5 5 7 9 |
| 7 | 1 3 6 6 6 8 |
| 8 | 4 5 7 7 |
| 9 | 0 2 5 |

9:01

5 Use a calculator to find the standard deviation for each of these tests (correct to two decimal places). In which test would the result of 15 be more impressive? 9:02

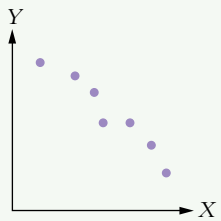
**Test A:** 9, 11, 12, 12, 13, 13, 14, 15, 15, 16
**Test B:** 5, 7, 9, 10, 12, 13, 13, 15, 17, 19
**Test C:** 10, 15, 16, 8, 13, 3, 6, 16, 8, 5

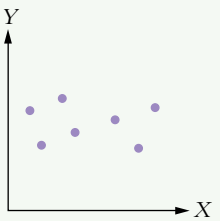
6 This histogram shows the number of fatal crashes that occurred during a year according to the time of day, organised in 2-hour periods. This is an example of bivariate data. 9:02

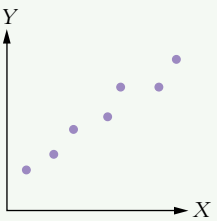



a What is the independent variable?
b What is the dependent variable?
c Which two-hour period registered the highest number of fatalities?
d Which two-hour period registered the lowest number of fatalities?
e Does this data show the time when it is safest to drive? What other factors need to be considered?

7 Which of these scatter plots show a linear relationship between the variables $X$ and $Y$. State whether the linear relationship is positive or negative. 9:03

a 


b 


c 


8 Plot each set of data on a scatter plot and determine what type of relationship exists between the two variables. 9:03

a

| $A$ | 1 | 2 | 4 | 6 | 7 | 9 |
|---|---|---|---|---|---|---|
| $B$ | 8 | 10 | 18 | 23 | 26 | 31 |

b

| $T$ | 2 | 3 | 5 | 6 | 7 | 9 |
|---|---|---|---|---|---|---|
| $N$ | 3 | 1 | 6 | 4 | 2 | 5 |

c

| $P$ | 2 | 3 | 4 | 5 | 7 | 9 |
|---|---|---|---|---|---|---|
| $Q$ | 23 | 20 | 18 | 14 | 10 | 3 |

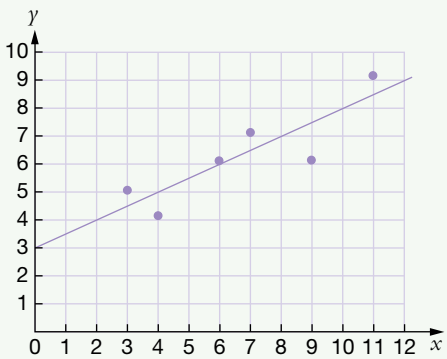
d

| $X$ | 1 | 3 | 4 | 6 | 8 | 9 |
|---|---|---|---|---|---|---|
| $Y$ | 5 | 6 | 6 | 5 | 6 | 5 |

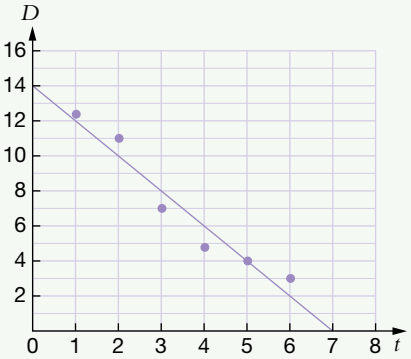
9:04

**9** Determine the equation of the line of best fit shown on each scatter plot.

**a**

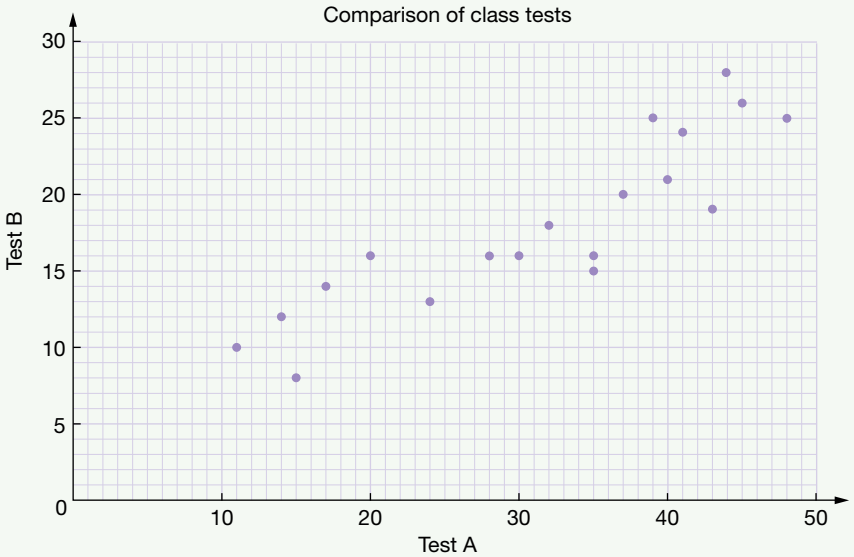



**b**




9:04

**10** This scatter plot shows the result for two class tests, one out of 50 marks and the other out of 30 marks. There are 19 in the class.




**a** What did the student who scored 32 for Test A receive as a result for Test B?

**b** What did the student who scored 24 for Test B receive as a result for Test A?

**c** For Test A determine:

**i** the range **ii** the median **iii** $Q_1$ **iv** $Q_3$

**d** Write the five-number summary for Test A.

**e** For Test B determine:

**i** the range **ii** the median **iii** $Q_1$ **iv** $Q_3$

**f** Write the five-number summary for Test B.

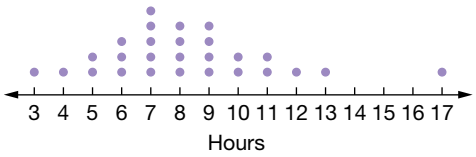
**g** Construct a box plot for each test.

**h** Which test has results that are:

**i** negatively skewed **ii** positively skewed?

## ASSIGNMENT 9A Chapter review

**1** This dot plot shows the number of whole hours spent doing homework by a Year 10 class in a particular week.




- **a** How many students are in the class?
- **b** What is the most common number of hours spent (the mode)?
- **c** Calculate the mean number of hours spent.
- **d** Find the median and quartiles for this data and hence write its five-number summary.
- **e** Draw a box plot for this data.
- **f** Looking at the dot plot and box plot, is this data symmetrical or skewed?

**2** **a** Use the data shown in the stem-and-leaf plot to complete the frequency distribution table using the classes shown.

- **b** Use the discrete data in the stem-and-leaf plot to calculate the mean.
- **c** Calculate a value for the mean using the totals of the frequency and $f \times c.c.$ columns from the table.
  Compare it to the answer for part **b**.

| Stem | Leaf |
|---|---|
| 1 | 6 9 |
| 2 | 2 4 7 7 9 |
| 3 | 0 4 5 5 5 9 9 |
| 4 | 1 3 6 8 |
| 5 | 2 4 6 6 7 8 8 9 |
| 6 | 3 3 5 6 8 8 |
| 7 | 0 1 3 6 9 |
| 8 | 0 2 5 7 |
| 9 | 1 4 |

| Class | *c.c.* | $f$ | $f \times c.c.$ | *c.f.* |
|---|---|---|---|---|
| 10–19 | 14·5 | | | |
| 20–29 | 24·5 | | | |
| 30–39 | | | | |
| 40–49 | | | | |
| 50–59 | | | | |
| 60–69 | | | | |
| 70–79 | | | | |
| 80–89 | | | | |
| 90–99 | | | | |
| *Totals:* | | ___ | ___ | |

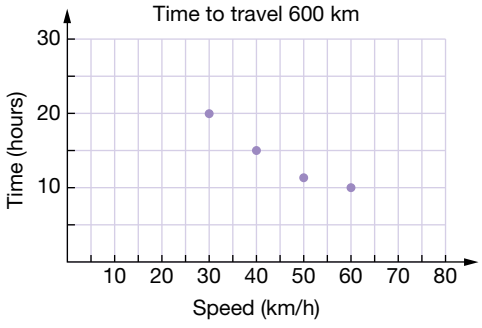
- **d** Find the mode from the stem-and-leaf plot. Does it fall in the modal class from the table?
- **e** Find the median from the stem-and-leaf plot. Does this fall in the median class from the table?

**3** A class performed well in a test, all scoring over 60% except Sammy, who had been away sick before the test. He scored 31%. The teacher decided not to include Sammy's result when recording the class statistics for this test. What effect would this have on the mean and standard deviation?

**4** Nick and Dylan are golfers. Each has played 10 rounds of golf on the same course and their scores have been recorded below.
Nick's scores: 73, 81, 77, 85, 76, 76, 84, 73, 80, 75
Dylan's scores: 70, 84, 82, 78, 83, 73, 73, 74, 85, 78

- **a** Calculate the mean and standard deviation for each set of scores.
- **b** Considering these calculations, who is the more consistent golfer?

**5** This scatter plot shows the time calculated by Liam to drive a distance of 600 km at average speeds of 30, 40, 50 and 60 km/h. He concluded that a negative linear relationship existed between these two variables.

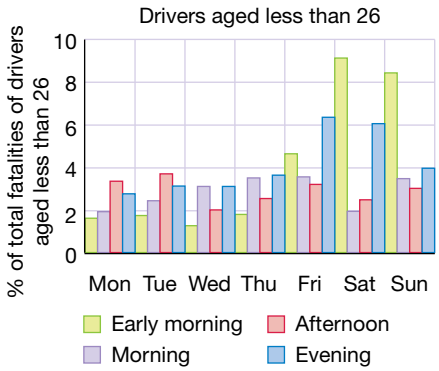


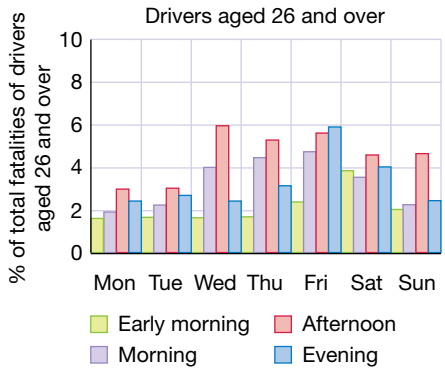

- **a** How long would Liam take for the trip at 40 km/h?
- **b** To complete the trip in 10 hours, what average speed is required?

**c** Increasing the average speed from 30 km/h to 60 km/h reduces the time taken by 10 hours. If the relationship is linear then if Liam could increase the average speed to 90 km/h, he could save another 10 hours and take no time at all. What is wrong with this reasoning?

**d** The relationship is negative but is it linear?

**6** The graphs below show the fatalities by time of day and day of week of crashes for major road user categories, Australia, 2005–2007.




**a** What do the graphs above suggest is the most dangerous time of day for:

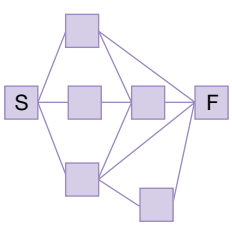
**i** drivers aged less than 26 **ii** drivers aged 26 and over?

**b** Suggest a reason why the times in **a** are the most dangerous.

## ASSIGNMENT 9B Working mathematically

**1** You have forgotten the last three digits of your PIN. You do remember, however, that the digits are different and they add up to 6. How many possible endings are there to your PIN?

**2** Find the number of routes through the network (S to F), assuming that travel must always be from left to right.

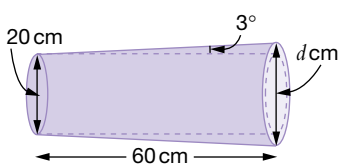



**3** Some numbers are the sum of two squared whole numbers. For example, $13 = 2^2 + 3^2$ and $37 = 1^2 + 6^2$.

**a** How many numbers from 100 to 120 are the sum of two squared whole numbers?

**b** The number 685 can be written as the sum of two squared whole numbers in two ways. What are they?

**4** A tapered shaft has a length of 60 cm. The diameter at the smaller end is 20 cm. If the taper is 3° (as shown), find the diameter, $d$ cm, of the larger end of the shaft.




**5** Max's best time for 50 m freestyle is 40 seconds. Joshua's best time is 25 seconds. If they both swam in the same 50 m freestyle race, how far behind Joshua would Max be when Joshua finished? (Assume that they both swim at constant rates.)

# ASSIGNMENT 9C Cumulative revision

**1** Using the formula $S = ut + \frac{1}{2}at^2$, find the value of: 1:07

**a** $S$ when $u = 9$, $a = 6$ and $t = 5$ **b** $S$ when $u = 3{\cdot}6$, $a = 9{\cdot}8$ and $t = 2{\cdot}5$

**c** $u$ when $S = 180$, $a = 20$ and $t = 4$ **d** $a$ when $S = 123$, $u = 10$ and $t = 6$

**2** Five bottles of drink and seven ice-creams cost \$35 and seven bottles of drink and five ice-creams cost \$37. Write equations to represent this and solve them simultaneously to find the price of each bottle of drink and each ice-cream. 1:09

**3** For the two points $A(3, 5)$ and $B(9, -3)$, find: 1:10

**a** the length of the interval $AB$ **b** the midpoint of $AB$

**c** the gradient of line $AB$.

**4** For the line $y = 4x - 5$, find the line that is: 1:10

**a** parallel to it and passes through the point $(1, 3)$

**b** perpendicular to it and passes through the point $(4, 3)$.

**5** Solve the equations: 2:04

**a** $x^2 - 100 = 0$ **b** $(3x - 5)(2x + 3) = 0$

**c** $x^2 - x - 56 = 0$ **d** $2x^2 - x - 5 = 0$

**6** Two six-sided dice have on their faces the letters $A$, $B$, $C$, $D$, $E$ and $F$. When the dice are rolled, what is the probability that the two uppermost faces will: 4:04

**a** both show an $A$ **b** show the same letter

**c** show different letters? (*Hint:* A tree diagram might help.)

**7** Simplify these surds: 5:03, 5:05

**a** $\sqrt{75}$ **b** $\sqrt{18} - \sqrt{8}$

**c** $(\sqrt{5} - 2)(\sqrt{5} + 2)$ **d** $(\sqrt{7} - \sqrt{5})^2$

**8** A sum of \$12 000 is invested at 5·4% p.a. interest for 10 years. 6:02, 6:06

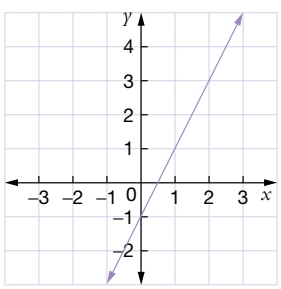
**a** Find the amount of interest earned if this is simple interest.

**b** Find the interest earned if it is compound interest and the interest is calculated:

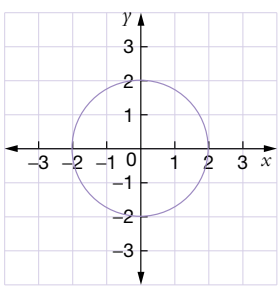
**i** annually **ii** monthly

**9** Write the equations of these graphs. 7:14

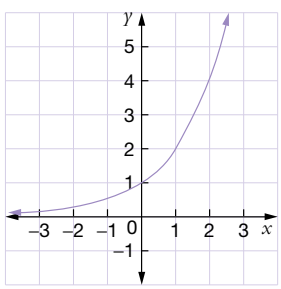
**a**




**b**

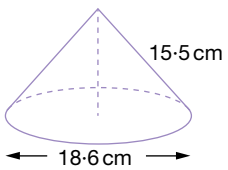



**c**




**10** **a** Find the surface area of this closed cone to the nearest square centimetre. 8:03, 8:06

**b** Find the volume of this cone to the nearest cubic centimetre.



# SIMILARITY

# 10

## Contents


10:01 Similar triangle proofs
10:02 Sides and areas of similar figures
10:03 Sides and volumes of similar solids
Investigation 10:03 King Kong: Could he have lived?
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Measurement and Geometry**

Selections from ***Properties of Geometrical Figures*** [Stage 5.3§]

- Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes (ACMMG244)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

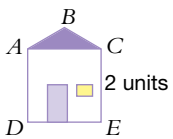
# 10:01 Similar triangle proofs

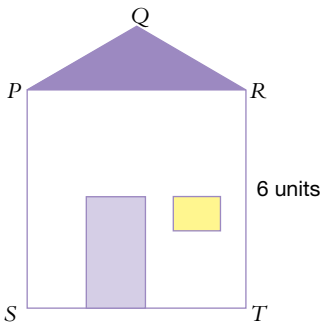
- Two figures are similar when one figure can be enlarged and superimposed on the other so that they coincide exactly.
- Similar figures have the same shape, but different size.

||| means 'is similar to'.

If two figures are similar then:
- matching angles are equal, and
- the ratios of matching sides are equal.

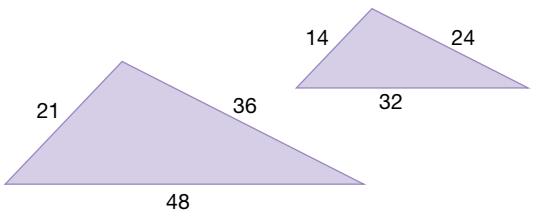
The second figure on the right is an enlargement of the first figure. The enlargement factor is 3. Each length in the second figure is 3 times the matching length in the first figure. Matching angles are equal.



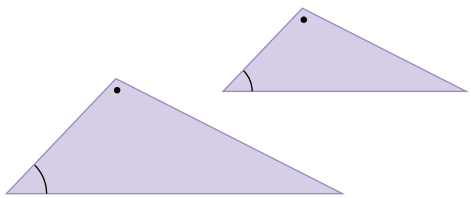

The conditions needed for two triangles to be similar are summarised below. Conditions **1** to **3** were investigated in Year 9. The fourth condition is a special case of Condition **1** when applied to right-angled triangles.

**1** Two triangles are similar if the lengths of matching sides are in the same ratio.



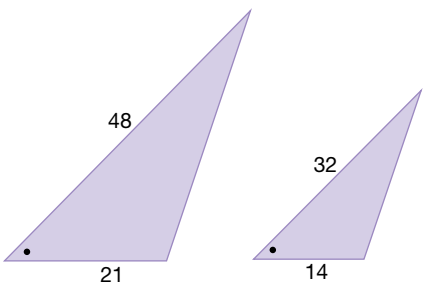

$\frac{21}{14} = \frac{36}{24} = \frac{48}{32}$

**2** Two triangles are similar if two angles of one are equal to two angles of the other.



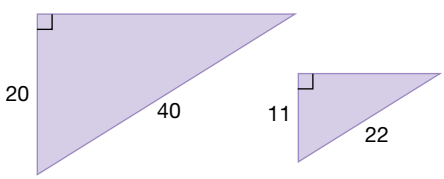
(The third angles are also equal.)

**3** Two triangles are similar if the lengths of two pairs of sides are in the same ratio and the angles included by these sides are equal.




$\frac{48}{32} = \frac{21}{14}$

**4** Two triangles are similar if both triangles are right-angled and the ratio of the hypotenuse to one side in one triangle equals the ratio of the hypotenuse to one side in the other triangle.




$\frac{40}{20} = \frac{22}{11}$

In the following exercise, the similarity conditions will be used to:
- write formal proofs of similarity of triangles and hence find unknown lengths and angles
- prove and apply further theorems.

## WORKED EXAMPLE 1

$AB \parallel DE$. Prove that $\Delta ABC \,|||\, \Delta EDC$. Hence, find the value of $x$. Units are in metres.

### Solution

In $\Delta$s $ABC$ and $EDC$

$\angle ABC = \angle EDC$ (alt. angles $AB \parallel DE$)

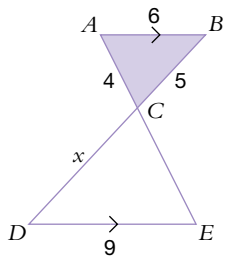
$\angle ACB = \angle ECD$ (vert. opp. angles)

$\therefore \Delta ABC \,|||\, \Delta EDC$ (equiangular)

$\therefore \dfrac{x}{5} = \dfrac{9}{6}$ (ratio of matching sides are equal)

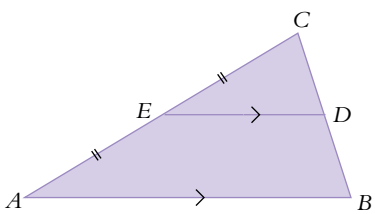
$x = \dfrac{5 \times 9}{6}$

$\therefore x = 7\frac{1}{2}$




## WORKED EXAMPLE 2

In the diagram shown, E is the midpoint of $AC$ and $ED$ is parallel to $AB$.
Prove that $CD = DB$.




### Solution

In $\Delta$s $CED$ and $CAB$

$\angle CED = \angle CAB$ (corr. $\angle$s, $ED \parallel AB$)

$\angle CDE = \angle CBA$ (corr. $\angle$s, $ED \parallel AB$)

$\therefore \Delta CED \,|||\, \Delta CAB$ (2 pairs of equal angles)

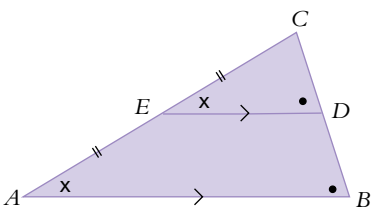
$\therefore \dfrac{CB}{CD} = \dfrac{CA}{CE}$ (ratios of matching sides are equal)

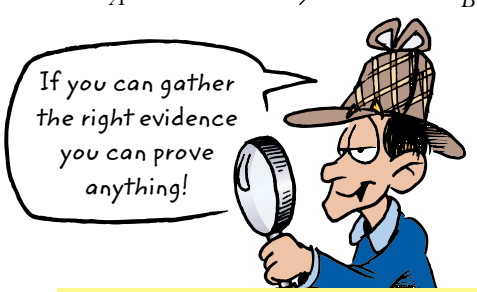
$\therefore \dfrac{CB}{CD} = \dfrac{2}{1}$ ($CA = 2 \times CE$)

$\therefore CB = 2 \times CD$

But $CB = CD + DB$

$\therefore CD = DB$




This example proves the following theorem:

A line drawn through the midpoint of a side of a triangle parallel to another side bisects the third side.

*Reminder:*

$\parallel$ is congruent to

$|||$ is similar to

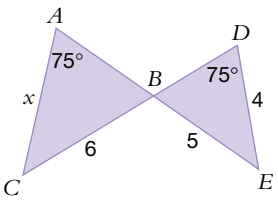
A table of commonly used abbreviations and symbols can be found on page 56.

# Exercise 10:01

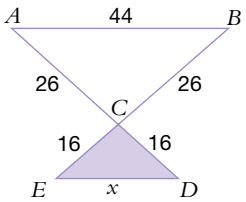
P Foundation worksheet 10:01
Similar triangle proofs

**1** For each part, prove that the triangles are similar and then find the value of the pronumeral.

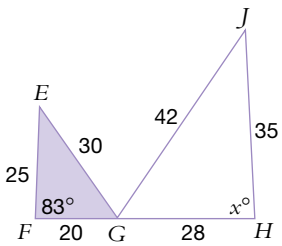
**a**



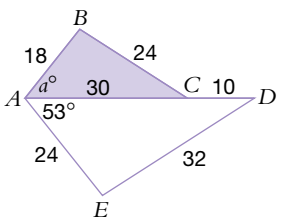

**b**



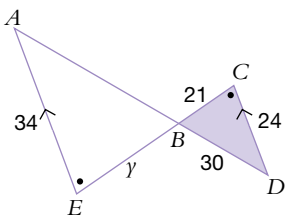

**c**




**d**

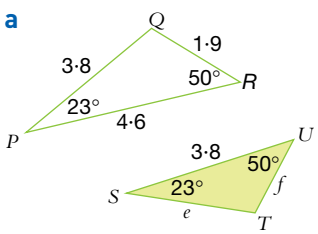



**e**

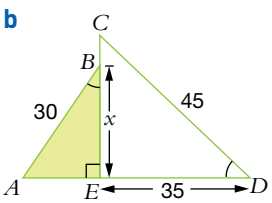



**2** State what condition could be used to prove that the triangles are similar, and then find the value of the pronumerals.

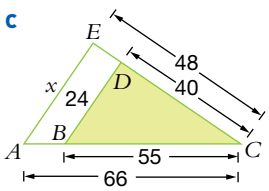
**a**



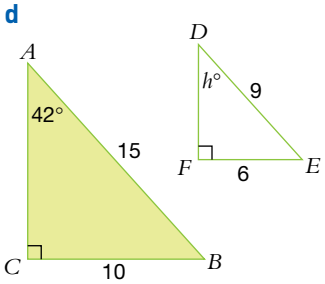

**b**



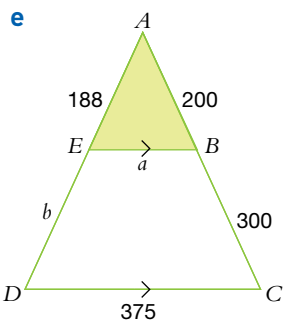

**c**



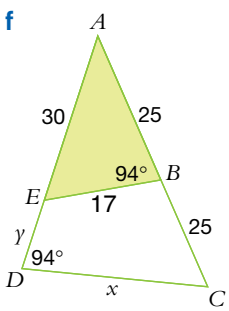

**d**



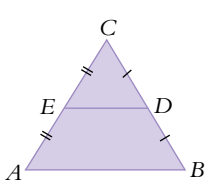

**e**




**f**




**3** In the diagram, $E$ and $D$ are the midpoints of $AC$ and $BC$ respectively. Prove that $ED \parallel AB$ and $ED = \frac{1}{2}AB$.

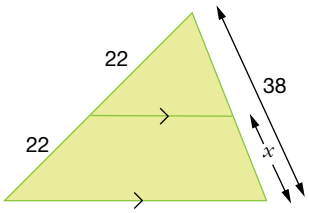



Question **3** proves the following theorem:

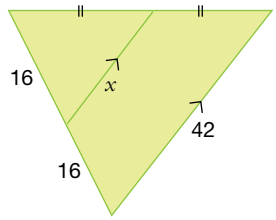
The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

**4** Use the theorems on pages 348 and 349 to find the value of the pronumerals in each of the following.

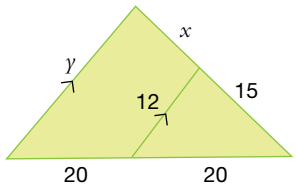
**a**



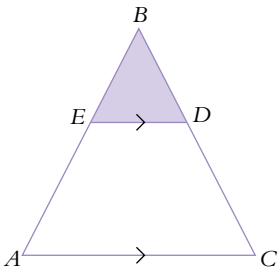

**b**




**c**



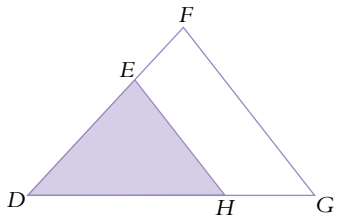

**5** **a**




$D$ divides $BC$ in the ratio 4 : 5.
$AC \parallel ED$, $BA = 35$ m

- **i** Prove that $\Delta EBD \,|||\, \Delta ABC$, giving reasons.
- **ii** Find the ratio of $BD$ to $BC$.
- **iii** Find the length of $BE$.
- **iv** Find the length of $EA$.
- **v** Find the ratio of $BE$ to $EA$.
- **vi** Show that $ED$ divides $BA$ and $BC$ in the same ratio, i.e. show that $\frac{BE}{EA} = \frac{BD}{DC}$.

**b**




$DE : EF = 3 : 2$, $DH : HG = 3 : 2$

- **i** Find $DE : DF$.
- **ii** Find $DH : DG$.
- **iii** Prove that $\Delta EDH \,|||\, \Delta FDG$.
- **iv** Is $\angle DEH$ equal to $\angle DFG$? Why?
- **v** Is $\angle DHE$ equal to $\angle DGF$? Why?
- **vi** Is $EH \parallel FG$? Why?

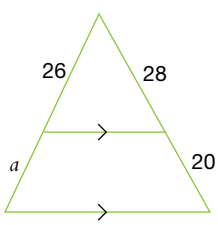
You have shown that if a line divides two sides of a triangle in the same ratio it must be parallel to the third side.

From Question **5a**, we see that:

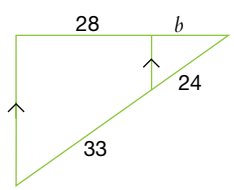
A line parallel to one side of a triangle divides the other two sides in the same ratio.

**6** Find the value of the pronumeral in each of the following. Lengths are in metres.

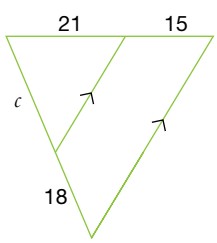
**a**



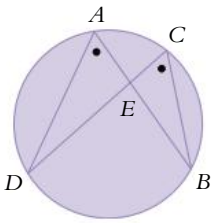

**b**




**c**

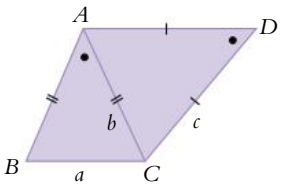


**7 a**



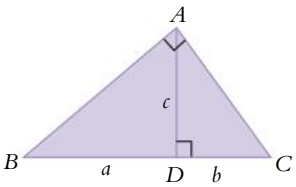

Prove that $\Delta ADE \,|||\, \Delta CBE$ and hence that $AE \times EB = CE \times ED$.

**b**




$\Delta$s $ABC$ and $ADC$ are isosceles with $\angle BAC = \angle ADC$. Prove that $ac = b^2$.

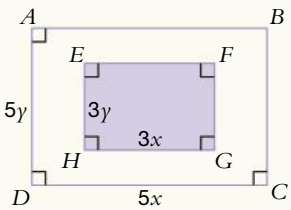
**c**




$\angle BAC = 90°$ and $AD \perp BC$. Prove that $ab = c^2$.

# 10:02 Sides and areas of similar figures

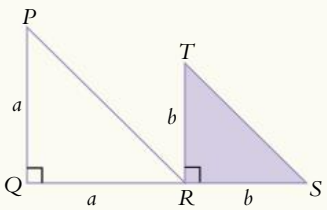
## PREP QUIZ 10:02




The rectangles are similar. Find the ratio:

**1** $AD : EH$

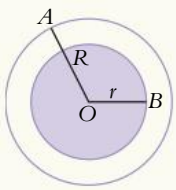
**2** area $ABCD$ : area $EFGH$.




The triangles are similar. Find the ratio:

**3** $PQ : TR$

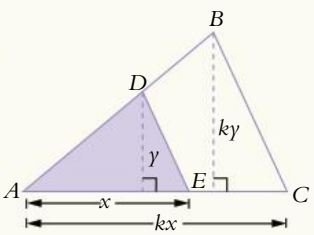
**4** area $\Delta PQR$ : area $\Delta TRS$.




The circles are similar. Find the ratio:

**5** $OA : OB$

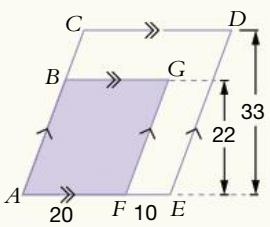
**6** area of large circle to area of small circle.




$\Delta ABC \,|||\, \Delta ADE$. Find the ratio:

**7** $AC : AE$

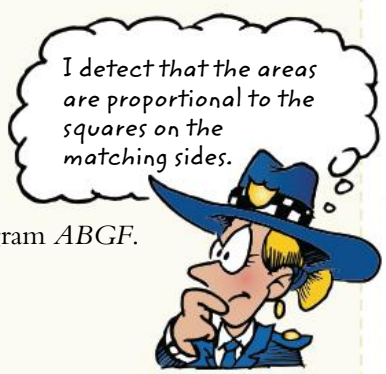
**8** area $\Delta ABC$ : area $\Delta ADE$.




Parallelogram $ACDE \,|||$ parallelogram $ABGF$. Find the ratio:

**9** $AE : AF$

**10** area $ACDE$ : area $ABGF$.



In the Prep quiz, we investigated the relationship between the sides of similar figures and the areas of those same figures.

Results:

| Similar figures | Ratio of sides | Ratio of areas |
|---|---|---|
| Rectangles | $5:3$ | $5^2:3^2$ |
| Triangles | $a:b$ | $a^2:b^2$ |
| Circles | $R:r$ | $R^2:r^2$ |
| Triangles | $k:1$ | $k^2:1^2$ |
| Parallelograms | $3:2$ | $3^2:2^2$ |

In similar figures:

If the ratio of matching sides is $a:b$,
then the ratio of their areas is $a^2:b^2$.

The areas of similar figures are proportional to the squares on matching sides.

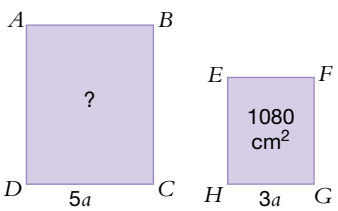
## WORKED EXAMPLES

1 The pages of two morning newspapers are similar in shape, and the widths are in the ratio $3:5$. Find the area of the larger page if the smaller one has an area of $1080\,\text{cm}^2$.

2 Two similar decorative stickers of boomerangs were produced for the Australia Day celebrations. The area of the larger boomerang sticker is $108\,\text{cm}^2$ and the area of the smaller is $48\,\text{cm}^2$. If the length of the larger is $18\,\text{cm}$, find the length of the smaller.

### Solutions

1



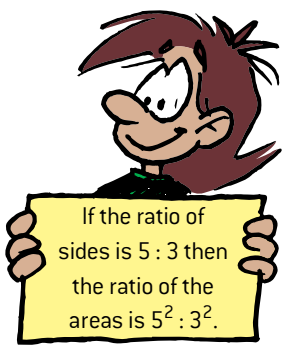

$$\frac{\text{area of } ABCD}{\text{area of } EFGH} = \frac{5^2}{3^2}$$

$$\therefore \frac{\text{area of } ABCD}{1080} = \frac{25}{9}$$

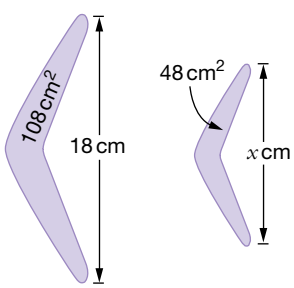
$$\text{area of } ABCD = \frac{25 \times 1080}{9}$$

$$= 3000\,\text{cm}^2$$

$\therefore$ The area of the larger page is $3000\,\text{cm}^2$.




2




Let the length of the smaller sticker be $x\,\text{cm}$.

$$\therefore \frac{x^2}{18^2} = \frac{48}{108}$$

$$x^2 = \frac{48 \times 18^2}{108}$$

$$x^2 = 144$$

$$\therefore x = \pm 12$$

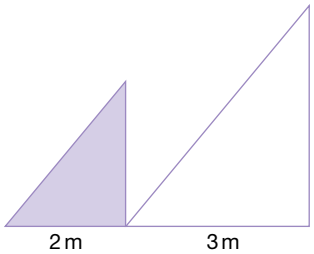
$\therefore$ The length of the smaller sticker is $12\,\text{cm}$.

A length must be positive so we ignore the $-12$.

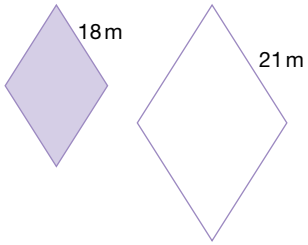
# Exercise 10:02

**1** In each of the following, the figures are similar. Find the ratio of the smaller area to the larger.

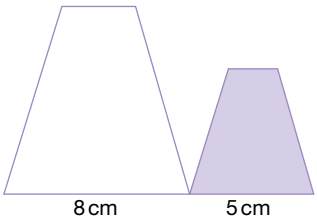
**a**



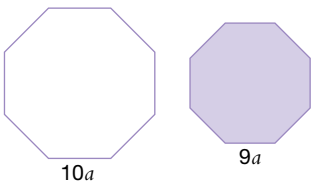

**b**



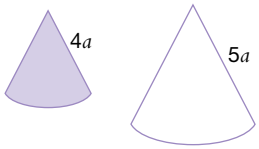

**c**



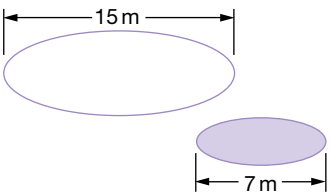

**d**




**e**




**f**




**2** **a** In **1a**, the area of the larger triangle is $5{\cdot}85\,m^2$. Find the area of the smaller triangle.

**b** In **1b**, the area of the larger rhombus is $367{\cdot}5\,cm^2$. Find the area of the smaller rhombus.

**c** In **1c**, the area of the larger trapezium is $40\,cm^2$. Find the area of the smaller trapezium.

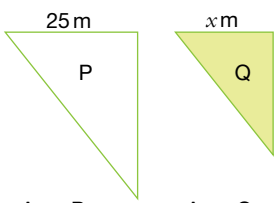
**d** In **1d**, the area of the smaller octagon is $162\,m^2$. Find the area of the larger octagon.

**e** In **1e**, the area of the smaller sector is $86{\cdot}4\,cm^2$. Find the area of the larger sector.

**f** In **1f**, the area of the smaller ellipse is $17{\cdot}0\,m^2$ (to 1 dec. pl.). Find the area of the larger ellipse correct to one decimal place.

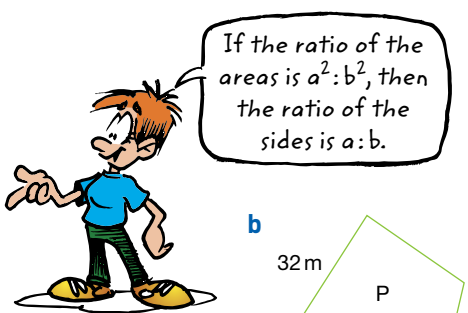
**3** In each part below, the figures are similar. Find the ratio of a side on the smaller figure to the matching side on the larger, then find the value of the pronumeral.

**a**

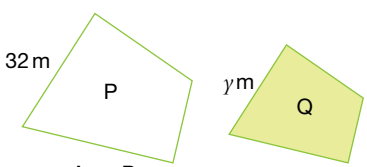




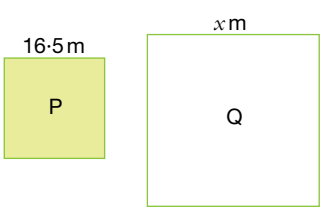




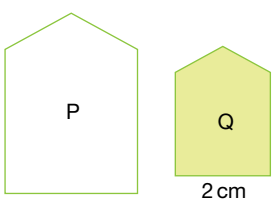

**b**



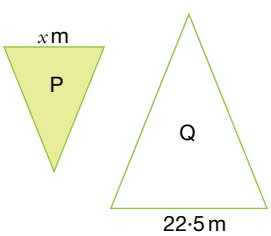

**c**




**d**




**e**



**4** **a** If the dimensions of an equilateral triangle are trebled, how is the area affected?

**b** If the areas of similar quadrilaterals are in the ratio 16 : 25, what is the ratio of matching sides?

**c** The ratio of a length in a house to the matching length on the house plan is 150 : 1. Find the ratio of the area of the house to its area on the plan.

**d** The ratio of matching sides of two similar figures is 4 : 7. What is the ratio of the perimeters of the similar figures?

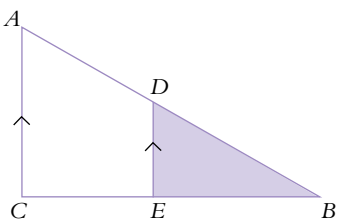
**5** The matching sides of two similar kites are in the ratio 11 : 16. Find the area of the smaller if the larger has an area of $1{\cdot}44\,m^2$.

**6** **a** A photograph has a width of 9·5 cm and an area of $104{\cdot}5\,cm^2$. An enlargement is to be made that has a width of 19 cm. What will be the area of the enlargement?

**b** A postcard is 8 cm wide and has an area of $96\,cm^2$. If the postcard is enlarged so that its area is $1536\,cm^2$, what is the breadth of the enlargement?

**7** My neighbour, whose lawn is a similar shape to mine but has dimensions $\frac{2}{3}$ as large, was surprised to hear that I used 90 kg of ammonium sulfate to treat my lawn. How much would he need to treat his lawn in the same way?

**8**




$DE \parallel AC$ and $CE : EB = 2 : 3$. Find:

**a** $AD : DB$

**b** $AC : DE$

**c** area $DBE$ : area $ABC$

**d** area $DBE$ : area $ADEC$

**e** area $ABC$ : area $ADEC$

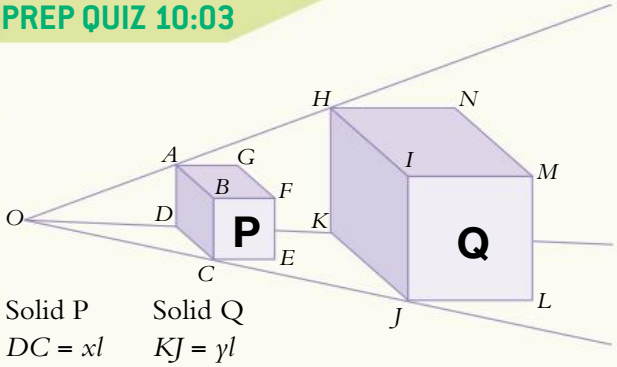
A doll twice as high as another would have four times the surface area.

A doll twice as high as another would have eight times the volume.

# 10:03 Sides and volumes of similar solids

Similar solids:
- have the same shape
- have matching angles of equal size
- have matching side lengths in the same ratio.

## PREP QUIZ 10:03




Solid P
$DC = xl$
$CE = xb$
$CB = xh$

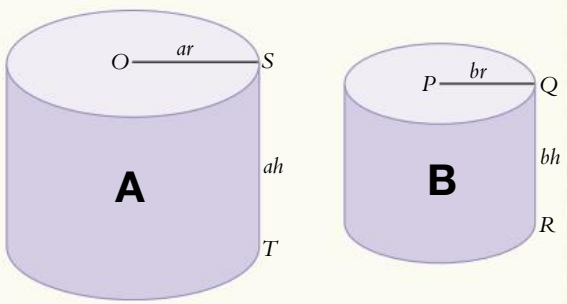
Solid Q
$KJ = yl$
$JL = yb$
$JI = yh$

These rectangular prisms are similar.

1 What is the ratio $DC : KJ$?
2 What is the ratio $CE : JL$?
3 What is the ratio area $ABCD$ : area $HIJK$?
4 What is the ratio area $BCEF$ : area $IJLM$?
5 What is the ratio volume of P : volume of Q?

These two cylinders are similar.

6 What is the ratio $ST : QR$?
7 What is the ratio $OS : PQ$?
8 What is the ratio of the cross-section of A to the cross-section of B?
9 What is the ratio of the curved surface area of A to the curved surface area of B?
10 What is the ratio volume of A : volume of B?



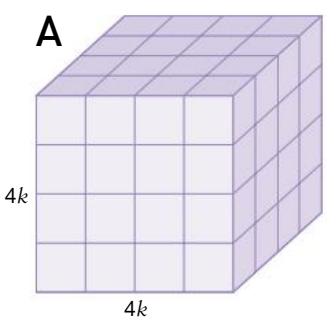

$O$ and $P$ are centres.

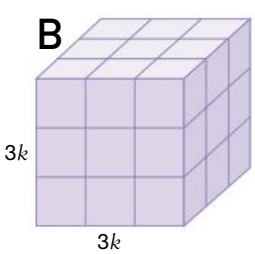
## WORKED EXAMPLE 1

Two cubes have the lengths of matching edges in the ratio 4 : 3. Find the ratio of:

**a** the areas of matching faces

**b** their volumes.




### Solutions

All cubes are similar.

**a** Areas are proportional to the squares of sides.

$$\therefore \frac{\text{area of A}}{\text{area of B}} = \frac{(4k)^2}{(3k)^2}$$

$$= \frac{16}{9}$$

(*Note:* The area of each face of A is $16k^2$. The area of each face of B is $9k^2$.)

**b** Volumes are proportional to the cubes of sides.

$$\therefore \frac{\text{volume of A}}{\text{volume of B}} = \frac{4^3}{3^3}$$

$$= \frac{64}{27}$$

(*Note:* The volume of A is $64k^3$. The volume of B is $27k^3$.)

## WORKED EXAMPLE 2

Two similar statues were to be made at Naomi's studio. She first completed the smaller one and painted it with a special gold paint. The smaller statue was 1·6 m tall, had a volume of 0·384 $m^2$ and needed 400 mL of gold paint to complete the required two coats.

**a** What volume will the larger statue have if it is to be 2 m tall?

**b** How much gold paint will be required to give the larger statue two coats?

### Solutions

The statues are similar solids.

**a** Volumes are proportional to the cubes of lengths.

$$\therefore \frac{\text{volume L}}{\text{volume S}} = \frac{2^3}{1{\cdot}6^3}$$

$$\therefore \frac{\text{volume L}}{0{\cdot}384\ \text{m}^3} = \frac{2^3}{1{\cdot}6^3}$$

$$\therefore \text{volume L} = \frac{2^3 \times 3{\cdot}384}{1{\cdot}6^3}\ \text{m}^3$$

$$= 0{\cdot}75\ \text{m}^3$$

**b** Paint required is proportional to area to be covered.
Areas are proportional to the squares of length.

$$\therefore \frac{\text{paint for L}}{\text{paint for S}} = \frac{2^2}{1{\cdot}6^2}$$

$$\therefore \frac{\text{paint for L}}{400\ \text{mL}} = \frac{2^2}{1{\cdot}6^2}$$

$$\therefore \text{paint for L} = \frac{2^2 \times 400}{1{\cdot}6^2}\ \text{mL}$$

$$= 625\ \text{mL}$$

In similar objects, if the ratio of matching sides is $a : b$ then:

- the ratio of matching areas is $a^2 : b^2$
- the ratio of matching volumes is $a^3 : b^3$.

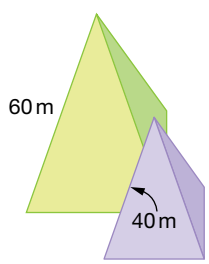
## Exercise 10:03

**1** In each, the solids are similar. Find:

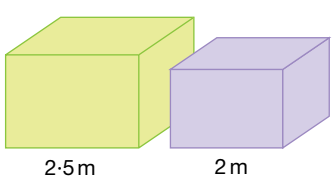
**i** the ratio of matching areas (smaller to larger)

**ii** the ratio of their volumes (smaller to larger).

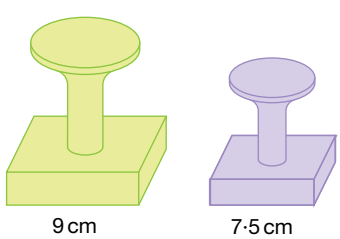
**a**



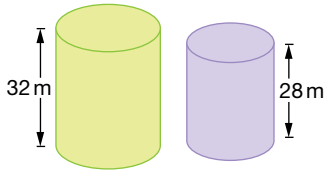

**b**



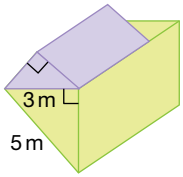

**c**



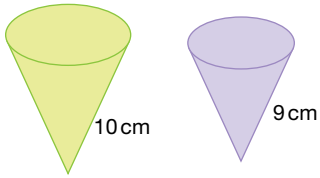
**d**




**e**




**f**




**2** **a** In **1a**, the area of the base of the smaller solid is $780\,m^2$. Find the area of the base of the larger solid.

**b** In **1a**, the volume of the larger solid is $30\,564\,m^3$. Find the volume of the smaller solid.

**c** In **1b**, the volume of the smaller solid is $10{\cdot}08\,m^3$. What is the volume of the larger solid?

**d** In **1c**, the volume of the larger solid is $243\,cm^3$. What is the volume of the smaller solid?

**e** In **1d**, the cross-section of the larger cylinder has an area of $400\,m^2$. Find the cross-sectional area of the smaller cylinder.

**f** In **1e**, the surface area of the smaller triangular prism is $27\,m^2$. What is the surface area of the larger triangular prism?

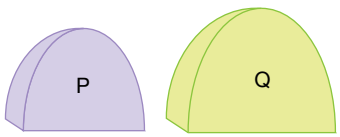
**g** In **1e**, the volume of the smaller prism is $8{\cdot}1\,m^3$. Find the volume of the larger prism.

**h** In **1f**, the surface area of the larger cone is $226\,cm^2$. Find the surface area of the smaller cone correct to the nearest square centimetre.

**i** In **1f**, the volume of the larger solid is $287\,cm^3$. What is the volume of the smaller solid correct to the nearest cubic centimetre?

**3** In each part, the solids are similar. Find the ratio of a side on the smaller solid to the matching side on the larger solid. Hence, find the ratio of their surface areas.

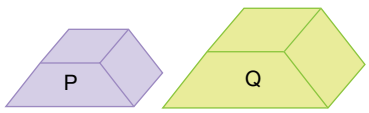
**a**




Volume P $= 125\,cm^3$

Volume Q $= 343\,cm^3$

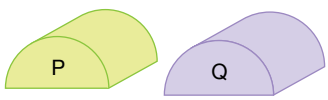
**b**




Volume P $= 4{\cdot}096\,cm^3$

Volume Q $= 8\,m^3$

**c**




Volume P $= 27\,m^3$

Volume Q $= 32{\cdot}768\,m^3$

If the ratio of matching volumes $= m : n$, the ratio of matching sides $= \sqrt[3]{m} : \sqrt[3]{n}$.

**4** **a** In **3a**, the surface area of solid P is $63\,cm^2$. Find the surface area of solid Q.

**b** In **3b**, the length from end to end (i.e. the height) on prism P is $0{\cdot}7\,m$. What is the matching length on prism Q?

**c** In **3b**, the area of the bottom face of solid Q is $5{\cdot}04\,m^2$. Find the area of the matching face of solid P.

**d** In **3c**, the diameter of the cross-section of prism P is 16 m. Find the diameter of the cross-section of prism Q.

**e** In **3c**, the curved surface area of solid Q is $72\pi\,m^2$. What is the curved surface area of solid P?

**5** Two similar vases were to be made in Naomi's studio. She first completed the smaller one and gave it two coats of lacquer. The smaller vase was 40 cm tall, had a volume of 2625 $cm^3$ and needed 100 mL of lacquer to complete the two coats.

**a** What volume will the larger vase have if it is to be 60 cm tall?

**b** How much lacquer will be required to give the larger vase two coats?

**6** The volume of metal in a scale model of a metal high-tension tower is 54 $cm^3$. The scale used was 1 : 50. It required 50 mL of paint to give the model one coat.

**a** What volume of metal would be required to build the high-tension tower?

**b** How much paint would be required to give the real tower one coat?

**7** Let us assume that a boy and his father are similar solids. Each is wearing similar swimming costumes. The boy is one-quarter as tall as his father.

**a** If the boy loses 1 unit of heat from his exposed skin, how many units of heat would you expect his father to lose?

**b** If the father displaces 72 000 $cm^3$ of water when he dives into the pool, what volume of water would the son displace?

**8** Two similar glasses have heights in the ratio 3 : 5. If, when using the larger glass, it takes 54 glasses of water to fill a fish tank, how many glasses of water would be needed to fill it using the smaller glass?

**9** This Giant Ram in Wagin, Western Australia, took the sculptor, David Chappell, 8 months to complete and is 9 times as tall as a live ram. Its height is 9 m.

Complete:

**a** The surface area is … times greater than that of a live ram.

**b** The volume is … times greater than that of a live ram.

**10** A model is $\frac{1}{32}$ of the real object's height. Complete:

**a** The surface area of the model is … of the surface area of the real object.

**b** The volume of the model is … of the volume of the real object.

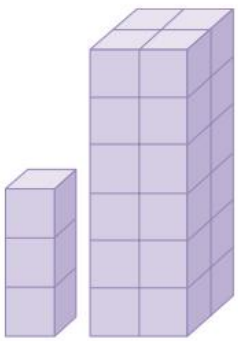
**c** If the real object has a volume of 90 000 $cm^3$, what would be the volume of the model? (Answer correct to two decimal places.)

## INVESTIGATION 10:03 KING KONG: COULD HE HAVE LIVED?

Could an ape be 10 m, 20 m, 30 m or even 40 m high? Would its enormous size cause it problems? The relationship between area and volume gives us the answer.

### Model of King Kong

Let an ordinary ape be represented by a square prism with a height 3 times the width as shown. Its base of one square unit (feet) can comfortably carry the 3 units of mass above it. (Each unit of volume is one unit of mass.)



1 The model of the giant ape is twice as high as the model of a normal ape.
   - **a** What is its volume?
   - **b** What is its base area?
   - **c** How many units of mass must each unit of area in the base support now?

2 Imagine an ape 50 times as high as a normal ape.
   - **a** What is its volume?
   - **b** What is its base area?
   - **c** How many units of mass must each unit of area in the base support now?

• *Note:*
   - **a** The area of a scaled-up object goes up with the square of the enlargement factor. The volume of a scaled-up object goes up with the cube of the enlargement factor.
   - **b** In general, if the enlargement factor is $F$, the pressure on the bottom face will be $F$ times as much.
   - **c** At some point, the bottom face (the skin and bone) will exceed its ability to withstand that pressure. Using this concept, Galileo was able to give a good estimate for the maximum height of trees. His estimate of 90 m was close to the world's highest of 110 m.

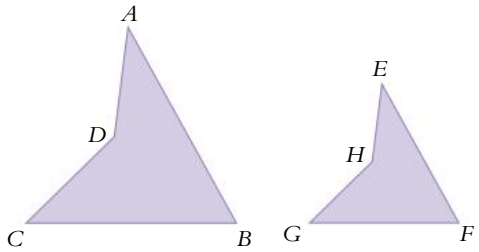
3 Does the above suggest that there may be a maximum height for mountains and buildings? Write your opinion and give your reasons.

4 Consider the effect of enlarging a model aeroplane. The model plane might fly, but when it is enlarged the increase in mass (proportional to the cube of the dimensions) is more than the increase in wing-surface area (proportional to the square of the dimensions). How would you change the design of the enlargement so that the larger aeroplane might fly? (The ideas here are taken from the ABC series 'For All Practical Purposes: Growth and Form' from the NSW Mathematics Syllabus.)

## MATHS TERMS 10

**matching angles (or sides)**

- sides (or angles) that are in the same (or corresponding) positions




Matching sides
- $AB$ and $EF$
- $BC$ and $FG$
- $CD$ and $GH$
- $DA$ and $HE$

Matching angles
- $\angle A$ and $\angle E$
- $\angle B$ and $\angle F$
- $\angle C$ and $\angle G$
- $\angle D$ and $\angle H$

**scale drawing**

- a drawing that is similar to the original

**scale factor**

- the ratio of the lengths of matching sides on a pair of similar figures
- it can be given as either an enlargement factor or a reduction factor

**similar figures**

- figures that have the same shape but a different size
- they have matching angles equal and matching sides in the same ratio
- the ratio of matching sides gives us the scale factor

**similar triangles (tests for)**

- a set of four tests that can be used to prove that two triangles are similar

**superimpose**

- the placement of one figure upon another in such a way that the parts of one coincide with the parts of the other

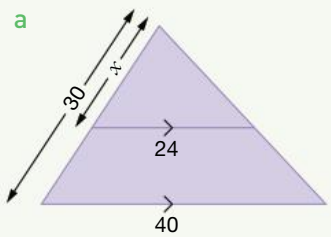
## DIAGNOSTIC TEST 10 SIMILARITY

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

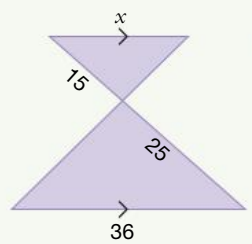
10:01

1 Use similar triangles to find the value of the pronumeral in each case.

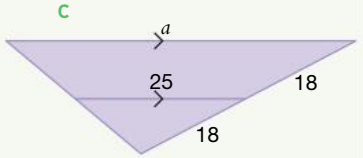
a




b

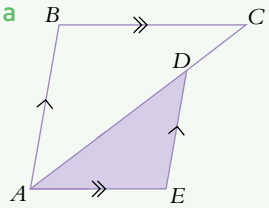



c



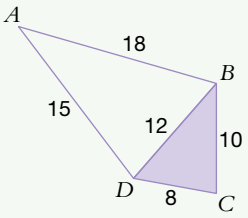
10:01

**2 a**



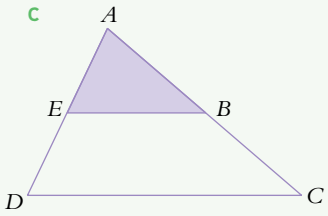

Prove that
$\Delta ABC \,|||\, \Delta DEA$.

**b**




Prove that
$\Delta ABD \,|||\, \Delta BDC$.

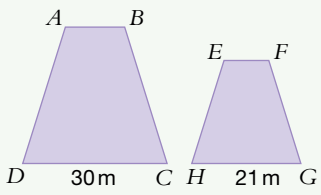
**c**




$AE = ED$, $AB = BC$
Prove that
$\Delta AEB \,|||\, \Delta ADC$.

10:02

**3 a**

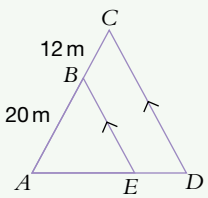



$ABCD \,|||\, EFGH$
Find:

i area $ABCD$ : area $EFGH$

ii area $ABCD$ if area $EFGH$ is $269{\cdot}5\,m^2$.

**b**

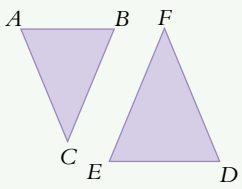



$BE \,||\, CD$
Find:

i area $ABE$ : area $ACD$

ii area $ABE$ if area $ACD$ is $512\,m^2$.

**c**




Area $ABC = 243\,mm^2$
Area $EDF = 300\,mm^2$

i Find $AB : ED$.

ii Find the length of $EF$ if $AC = 18\,mm$.

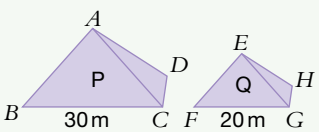
10:03

**4 a** Solids P and Q are similar.
Find:

i area $ABC$ : area $EFG$

ii volume P : volume Q

iii the volume of P if the volume of Q is $2000\,m^3$.




**b** The solid models of fish P and Q are similar.
Surface area of P = $44{\cdot}8\,cm^2$
Surface area of Q = $179{\cdot}2\,cm^2$

i Find the ratio, length of P : length of Q.

ii Find the ratio, volume P : volume Q.

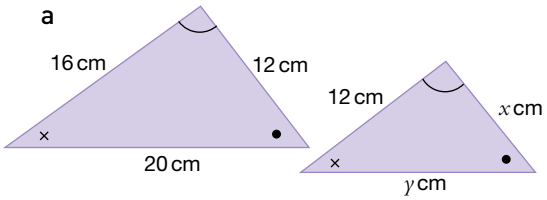
iii Find the volume of Q if the volume of P is $18{\cdot}2\,cm^3$.

**c** The heights of two similar statues are 3·2 m and 3·6 m. If the volume of the smaller is $80{\cdot}64\,m^3$, find the volume of the larger statue.

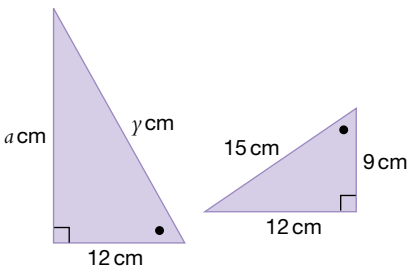
# ASSIGNMENT 10A Chapter review

**1** Each pair of triangles is similar. Find the value of the pronumerals.

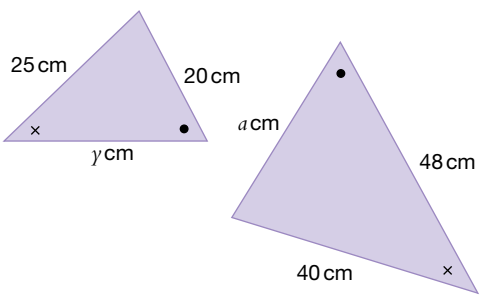
**a**




**b**

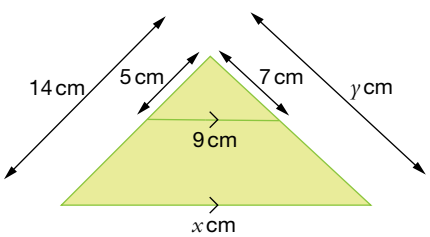



**c**

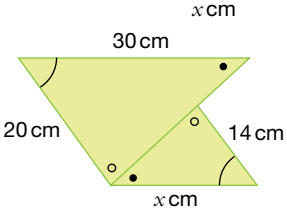



**2** Find the value of the pronumerals in each of the following.

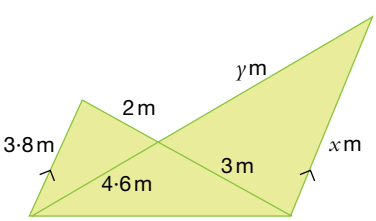
**a**



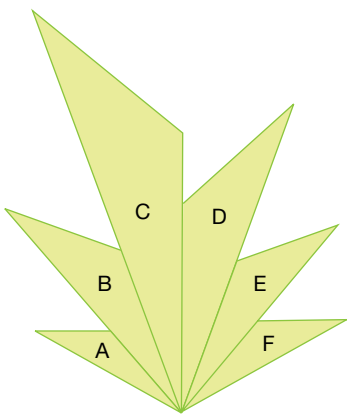

**b**



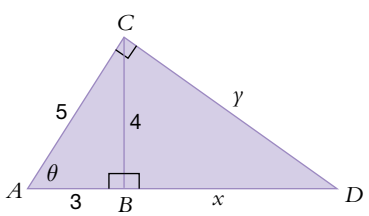

**c**




**3** The diagram below shows a piece of jewellery based on similar triangles.
Triangle A has sides of length 10 mm, 15 mm and 22 mm.
Triangles B and C have been produced from triangle A by successive enlargements using an enlargement factor of 1·6.
Triangle C has been reduced and reflected to produce triangle D. The reduction factor is 0·75.
Triangles E and F have been produced from triangle D by successive reductions using the same reduction factor of 0·75.
Calculate the side lengths of triangles C and F.




**4** Prove that $\Delta ABC \,|||\, \Delta CBD$ and hence find the values of $x$ and $y$.

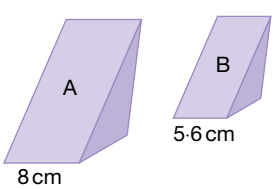



**5** Solids A and B are similar. Find the ratio of their:

**a** matching sides

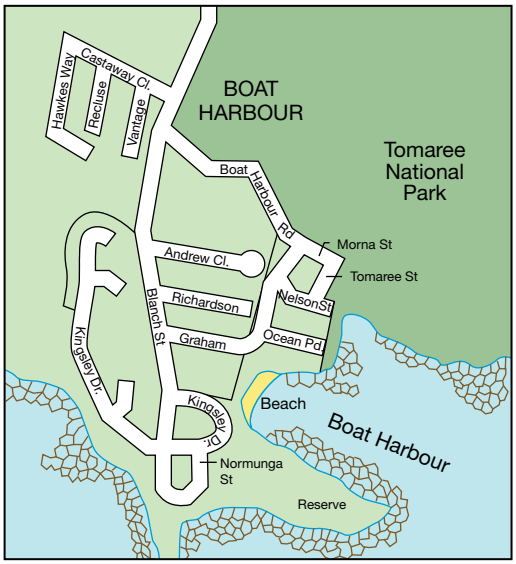
**b** surface areas

**c** volumes



# ASSIGNMENT 10B Working mathematically

**1**




Scale = 1:17 000

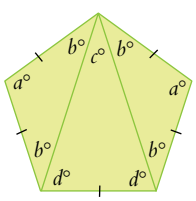
**a** What is the scale of this map?

**b** What distance would 1 cm on this map represent?

**c** What is the real length of Richardson Avenue?

**d** What is the real length of Ocean Parade?

**2** Sharon makes a regular pentagon from three isosceles triangles as shown. Find the sizes of the angles in the triangles.

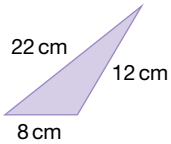



**3** **a** If 6 people can do a piece of work in 8 days, in what time will 18 people do it, working at the same rate?

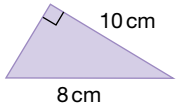
**b** If 14 people can do a piece of work in 12 days, how many people will be needed to do the work in 21 days, working at the same rate?

**4** Why are each of these triangles impossible to draw to scale?

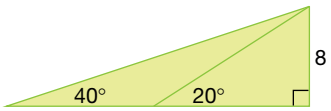
**a**



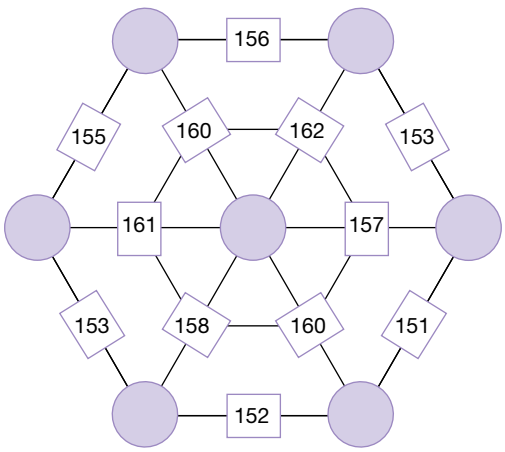

**b**




**5** Why is the diagram shown impossible to draw to scale?




**6** The numbers 74, 75, 77, 77, 78, 79 and 83 must be placed in the coloured circles so that the number in each rectangle is the sum of the numbers in the circles on either side of it. Which of these numbers must be placed in the centre circle?



## ASSIGNMENT 10C Cumulative revision

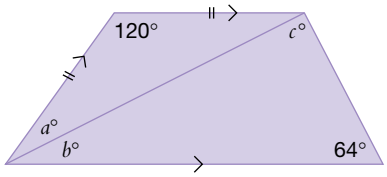
**1** Solve: 1:06

**a** $3(2x + 5) \leq 8$ **b** $5 - 3x > x - 10$ **c** $5(1 - x) - 2(x - 3) \geq 10$

**2** Find the $x$- and $y$-intercepts of each parabola. 2:02, 7:05

**a** $y = x^2 - 3x - 4$ **b** $y = 2x^2 + 7x - 15$

**3** Find the values of the pronumerals. 3:01




**4** Simplify each of the following. 5:04, 5:05

**a** $2\sqrt{3}(2\sqrt{3} - 5)$ **b** $(2\sqrt{3} - 5)(2\sqrt{3} + 5)$ **c** $(2\sqrt{3} - 5)^2$

**5** Express with a rational denominator. 5:06

**a** $\dfrac{1}{\sqrt{3} - 1}$ **b** $\dfrac{\sqrt{3}}{\sqrt{3} - 1}$ **c** $\dfrac{\sqrt{3} + 1}{\sqrt{3} - 1}$

**6** What is the equation of the curve that results when the curve $y = -2x^8$ is translated: 7:10

**a** up 2 units **b** down 2 units

**c** left 2 units **d** right 2 units?

**7** Calculate the volume and surface area of a square pyramid that has a base edge of 12 cm and a perpendicular height of 8 cm. 8:02, 8:05

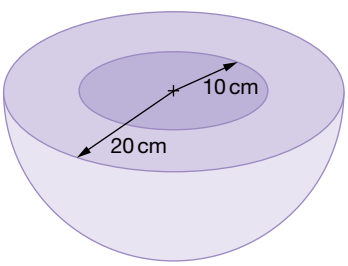
**8** The solid consists of a hemisphere that has a hemispherical bowl cut from it. 8:04, 8:07

**a** Find the volume of water that the bowl could hold.

**b** Find the volume of the complete solid.

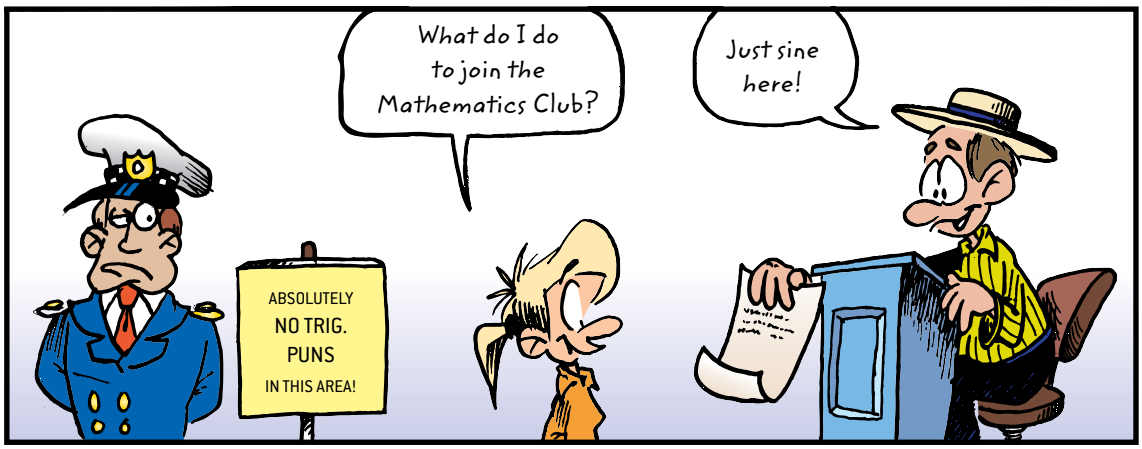
**c** Find the surface area of the solid.

(Give the answers correct to two significant figures.)



# TRIGONOMETRY

# 11




## Contents


**11:01** Trigonometric ratios of other angles
Investigation 11:01 Trigonometric graphs
GeoGebra activity 11:01A Using the unit circle
GeoGebra activity 11:01B Trigonometric graphs and the unit circle
GeoGebra activity 11:01C Features of trigonometric graphs
**11:02** Trigonometric relationships between acute and obtuse angles
Fun spot 11:02 Why are camels terrible dancers?
**11:03** The sine rule
GeoGebra activity 11:03 The sine rule
**11:04** The sine rule: the ambiguous case
**11:05** The cosine rule
Fun spot 11 :05 Why did Tom's mother feed him Peter's ice-cream?
**11:06** Area of a triangle
**11:07** Miscellaneous problems
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Measurement and Geometry**

Selections from ***Trigonometry and Pythagoras' Theorem*** [Stage 5.3§]

- Use the unit circle to define trigonometric functions, and graph them, with and without the use of digital technologies (ACMMAG274)
- Solve simple trigonometric equations (ACMMG275)
- Establish the sine, cosine and area rules for any triangle and solve related problems (ACMMG273)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

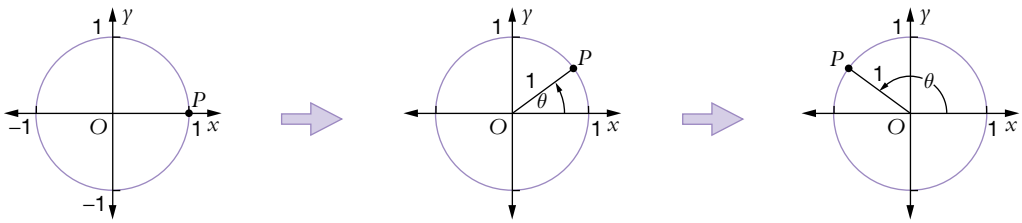
# 11:01 Trigonometric ratios of other angles

To this point we have only needed to know the trigonometric ratios for acute angles. The way in which the trigonometric ratios were defined for acute angles needs to be refined so that the trigonometric ratios of larger angles can be calculated.

## The unit circle

Sine, cosine and tangent ratios can be defined in terms of a circle of radius 1 unit.

- We refer to a **unit circle** that has its centre at (0, 0) on the number plane. If we take a radius $OP$ initially along the $x$-axis and rotate it anticlockwise about $O$, of course point $P$ will still lie on the circle.




The coordinates of $P$ on the number plane will depend on the size of $\theta$, the angle of rotation.

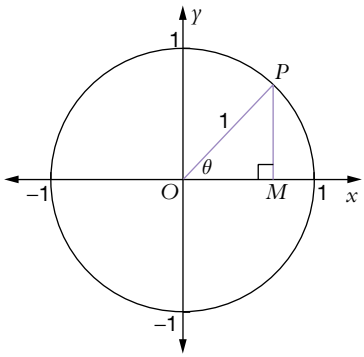
If a perpendicular $PM$ is drawn as in this diagram, the method of defining the trig ratios that we met in Year 9 would suggest that:

$\cos\theta = \dfrac{OM}{1}$ (i.e. $\dfrac{\text{adj.}}{\text{hyp.}}$) $\quad \sin\theta = \dfrac{PM}{1}$ (i.e. $\dfrac{\text{opp.}}{\text{hyp.}}$)

$= OM$ $\quad = PM$

$=$ $x$-coordinate of $P$ $\quad =$ $y$-coordinate of $P$

$\therefore$ $P$ is the point $(\cos\theta, \sin\theta)$.



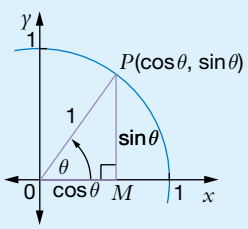

Hence, the previous method of defining the trig ratios using side ratios suggests a new way of defining the trig ratios as the coordinates of the point $P$.

This method is used to give the trig ratios for acute angles and, in fact, angles of any size.

For a point $P$ on a unit circle that has been rotated through an angle $\theta$ about $O$ from the positive $x$-axis:

$\cos\theta$ = the $x$-coordinate of $P$

$\sin\theta$ = the $y$-coordinate of $P$.



• The triangle on the previous page can also be used to redefine the tangent function.

$$\tan\theta = \frac{PM}{OM} \text{ (i.e. } \frac{\text{opp.}}{\text{adj.}}\text{)}$$
$$= \frac{y\text{-coordinate of } P}{x\text{-coordinate of } P}$$
$$= \frac{\sin\theta}{\cos\theta}$$

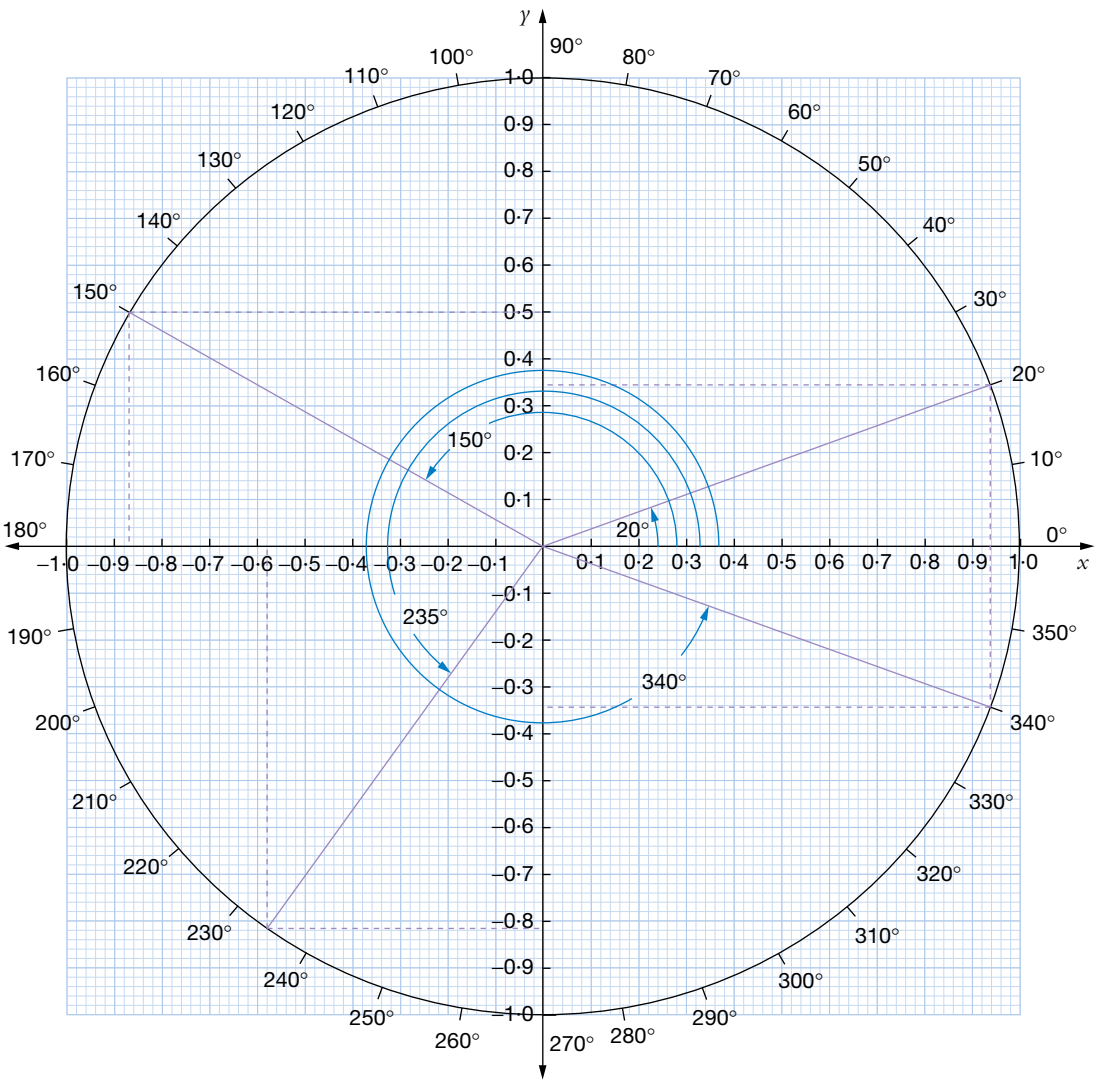
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The values of the trig ratios are calculated using a calculator, but it is important to understand the unit circle definitions. The unit circle explains:

- why the ratios vary between positive and negative
- how the ratios change as the angle increases from 0° to 360°.

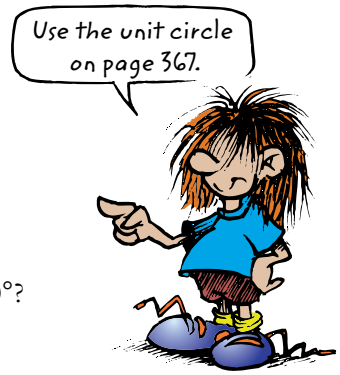
Many relationships exist between the trig ratios, and some of these will be investigated in the following exercise.

*Remember:* Each point on the circle has coordinates $(\cos\theta, \sin\theta)$.




Refer to this unit circle in the following examples and in Exercise 11:01.

## WORKED EXAMPLES




**1** State the sign of:
- **a** $\cos 150°$ and $\sin 150°$
- **b** $\sin 235°$ and $\cos 235°$
- **c** $\sin 340°$ and $\cos 340°$

**2** What are the coordinates of $P$ when $\theta$ is 180°? Use these to state the value of $\cos 180°$.

**3** What can you say about the sign of $\sin\theta$ for $180° < \theta < 360°$?

**4** Find two values of $\theta$ for which $\sin\theta = 0{\cdot}34$.

### Solutions

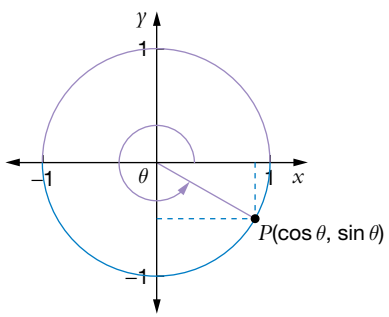
**1** **a** When $\theta = 150°$, the $x$-coordinate of $P$ is negative and the $y$-coordinate is positive.
$\therefore$ $\cos 150°$ is negative; $\sin 150°$ is positive

**b** When $\theta = 235°$, the $x$-coordinate of $P$ is negative and the $y$-coordinate is negative.
$\therefore$ $\cos 235°$ is negative; $\sin 235°$ is negative

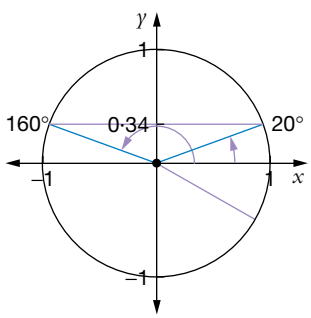
**c** When $\theta = 340°$, the $x$-coordinate of $P$ is positive and the $y$-coordinate is negative.
$\therefore$ $\cos 340°$ is positive; $\sin 340°$ is negative

**2** When $\theta = 180$, $P$ is the point $(-1, 0)$.
$\therefore$ $\cos 180° = -1$ ($\cos\theta$ = $x$-coordinate of $P$)

**3** When $\theta$ is between 180° and 360°, $P$ lies on the unit circle in either the 3rd or 4th quadrant, (on the blue part of the unit circle).
Hence the $y$-coordinate of $P$ is negative.
$\therefore$ $\sin\theta$ is negative for $180° < \theta < 360°$.



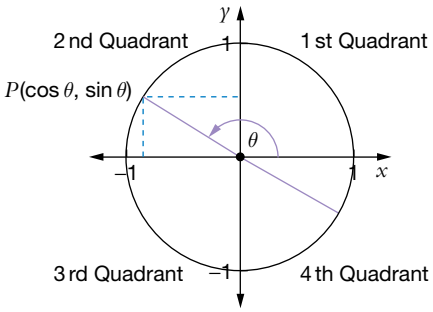

**4** If $\sin\theta = 0{\cdot}34$, we need to find two points on the unit circle with a $y$-coordinate of 0·34. Draw a horizontal line through 0·34 on the $y$-axis to meet the unit circle. It will give the two answers as 20° and 160°.



# Exercise 11:01

**1** In which quadrant of the number plane will $P$ be when $\theta$ is:

**a** 120° **b** 240° **c** 330° **d** 70°?




**2** Use the coordinates of $P$ to find the sign of:

**a** $\sin 120°$ **b** $\sin 240°$ **c** $\sin 330°$ **d** $\sin 70°$

**e** $\cos 120°$ **f** $\cos 240°$ **g** $\cos 330°$ **h** $\cos 70°$

**3** In which quadrants of the number plane could a point $P$ be if:

**a** the $x$-coordinate is positive **b** the $x$-coordinate is negative

**c** the $y$-coordinate is positive **d** the $y$-coordinate is negative?

**4** $P$ is a point on the unit circle. State in which quadrants $P$ could be if it is known that:

**a** $\sin\theta$ is positive **b** $\sin\theta$ is negative

**c** $\cos\theta$ is positive **d** $\cos\theta$ is negative

**e** $\sin\theta$ is negative and $\cos\theta$ is positive.

**5** Describe what happens to $\sin\theta$ as $\theta$ increases from:

**a** 0° to 90° **b** 90° to 180° **c** 180° to 270° **d** 270° to 360°

**6** Describe what happens to $\cos\theta$ as $\theta$ increases from:

**a** 0° to 90° **b** 90° to 180° **c** 180° to 270° **d** 270° to 360°

**7** **a** If $x$ is negative and $y$ is positive what is the sign of $\frac{y}{x}$?

**b** In which quadrants do the $x$-coordinate and the $y$-coordinate have opposite signs?

**c** In which quadrants is $\tan\theta$ negative?

**d** In which quadrants is $\tan\theta$ positive?

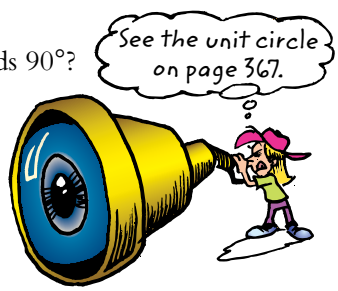
**8** **a** What is the largest and smallest $y$-coordinate of any point on the unit circle?

**b** Given your answer to **a**, what are the largest and smallest values of $\sin\theta$ that are possible?

**c** What are the largest and smallest possible values of $\cos\theta$?

**9** **a** What is happening to the value of $\tan\theta$ as $\theta$ increases from 0° towards 90°?

**b** Why can there be no value for $\tan 90°$?




**10** Estimate each of the following pairs of values. What do you notice?

**a** $\sin 75°$ and $\sin 105°$ **b** $\sin 80°$ and $\sin 100°$

**c** $\cos 70°$ and $\cos 110°$ **d** $\cos 40°$ and $\cos 140°$

**11** Use the unit circle to find two values of $\theta$ to the nearest degree if:

**a** $\sin\theta = 0{\cdot}5$ **b** $\sin\theta = 0{\cdot}77$ **c** $\sin\theta = 0{\cdot}17$

**d** $\sin\theta = -0{\cdot}5$ **e** $\sin\theta = -0{\cdot}77$ **f** $\sin\theta = -0{\cdot}17$

**12** Use the unit circle to find two values of $\theta$ to the nearest degree if:

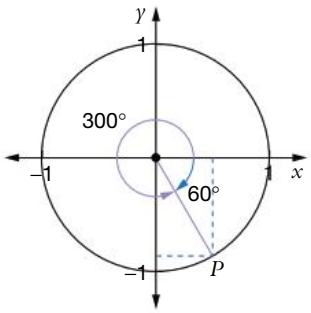
**a** $\cos\theta = 0{\cdot}5$ **b** $\cos\theta = 0{\cdot}64$ **c** $\cos\theta = 0{\cdot}87$

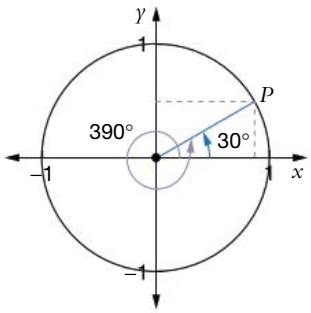
**d** $\cos\theta = -0{\cdot}5$ **e** $\cos\theta = -0{\cdot}64$ **f** $\cos\theta = -0{\cdot}87$

## INVESTIGATION 11:01 TRIGONOMETRIC GRAPHS

The unit circle can be used to define:

- angles that are greater than 360°
- negative angles (i.e. angles that are measured in a clockwise direction).



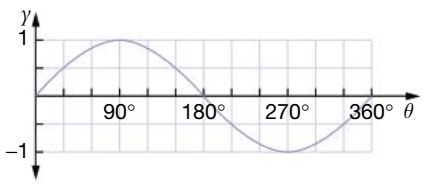

**1** **a** Use a calculator to complete the table correct to three decimal places.

**b** Are the trig ratios for each of the angles equal?

| $\theta$ | 300° | 660° | –60° |
|---|---|---|---|
| $\sin\theta$ | | | |
| $\cos\theta$ | | | |
| $\tan\theta$ | | | |

**2** Use the unit circle to explain why angles that differ by 360° have the same trig ratios.

**3** The graph shows $y = \sin\theta$ for $0° \leq \theta \leq 360°$.




**a** If you added 360° to all the values on the horizontal axis would the shape of the curve change?

**b** If you subtracted 360° from all the values on the horizontal axis would the shape of the curve change?

**c** How could you use the graph shown to obtain the graph of the following curves?

**i** $y = \sin\theta$ for $360° \leq \theta \leq 720°$

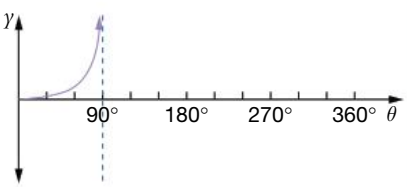
**ii** $y = \sin\theta$ for $-360° \leq \theta \leq 0°$

**4** Sketch the graph of $y = \cos\theta$ for $0° \leq \theta \leq 360°$. Is the graph symmetrical about the $y$-axis?

**5** How could you use the graph in Question 4 to obtain the graph of $y = \cos\theta$ for the following sets of values?

**a** $360° \leq \theta \leq 720°$ **b** $-360° \leq \theta \leq 0°$

**6** Part of the graph of $y = \tan\theta$ is shown.




**a** Complete the graph for values of $\theta$ between 90° and 360°.

**b** Why do vertical asymptotes appear at $\theta = 90°$ and $\theta = 270°$?

**c** Sketch $y = \tan\theta$ for $360° \leq \theta \leq 720°$.

**d** How long does it take for $y = \tan\theta$ to repeat its shape?

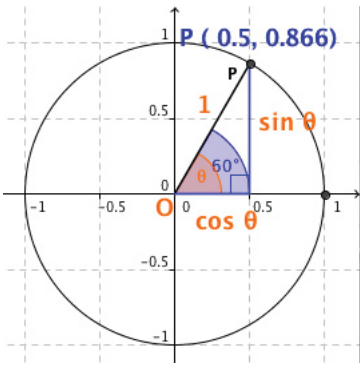
**7** Curves that repeat their shapes are said to be periodic. The period (or periodicity) of the curve is the distance that the curve needs to be translated horizontally so that it is superimposed on itself. State the period of each of the following graphs.

**a** $y = \sin\theta$ **b** $y = \cos\theta$ **c** $y = \tan\theta$

## P GEOGEBRA ACTIVITY 11:01A USING THE UNIT CIRCLE

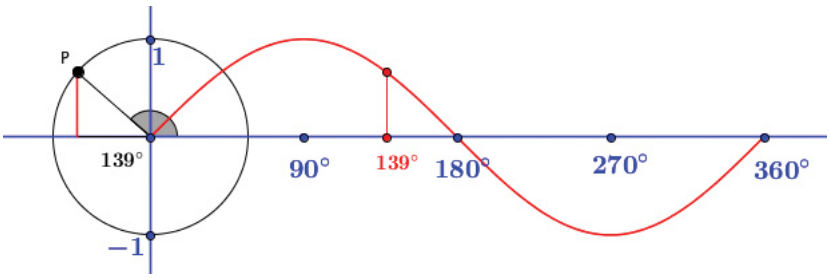
Use the slider to change the position of $P$ on the unit circle. As $P$ changes position the coordinates of $P$ and hence the values of the trig ratios for the given angle are displayed.

Checkboxes can be used to show positive angles, negative angles, the definitions and the signs of the trig ratios.



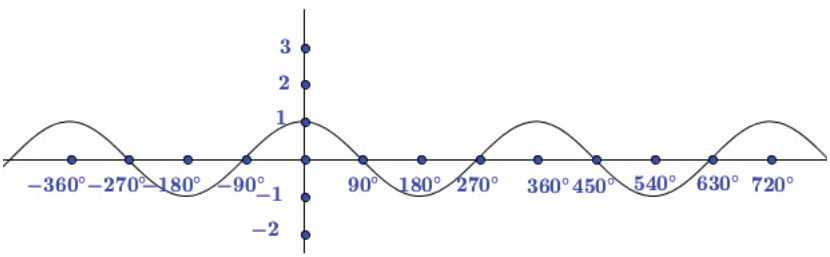

## P GEOGEBRA ACTIVITY 11:01B TRIGONOMETRIC GRAPHS AND THE UNIT CIRCLE

Drag a point around the unit circle and observe how the three trigonometric graphs are generated.




## P GEOGEBRA ACTIVITY 11:01C FEATURES OF TRIGONOMETRIC GRAPHS

Choose a graph and look at the curve from −360° to 720°. Use a copy of the curve to study its symmetry and periodicity.



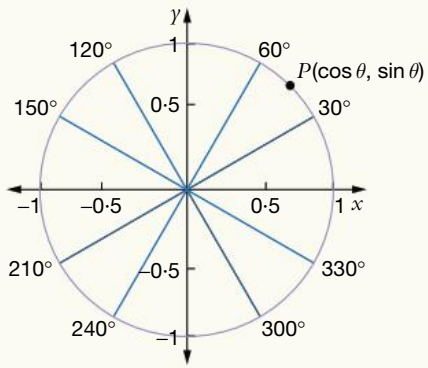
# 11:02 Trigonometric relationships between acute and obtuse angles

## PREP QUIZ 11:02

Evaluate the following correct to two decimal places.

**1** $\sin 40°$ **2** $\sin 140°$ **3** $\cos 40°$ **4** $\cos 140°$

**5** Can an acute angle and an obtuse angle have equal sine ratios?




What is the sign of $\sin\theta$ if:

**6** $\theta$ is acute

**7** $\theta$ is obtuse?

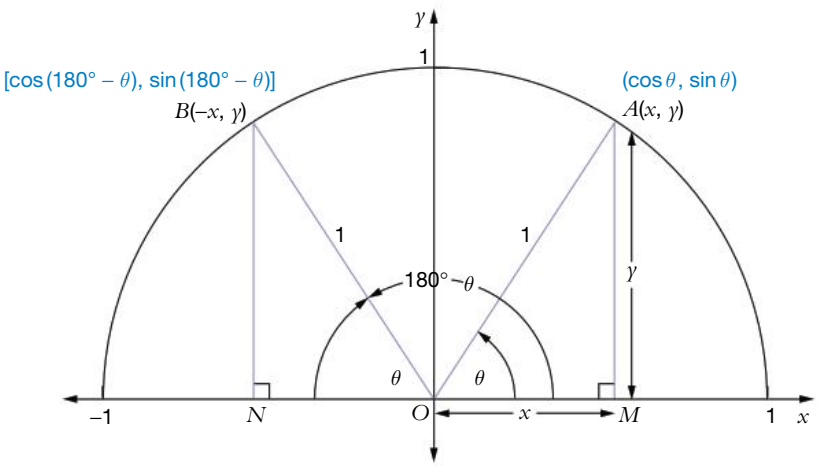
**8** Can an acute angle and an obtuse angle have equal cosine ratios?

Use the unit circle on the right to find $\theta$ if:

**9** $\sin\theta = 0{\cdot}5$

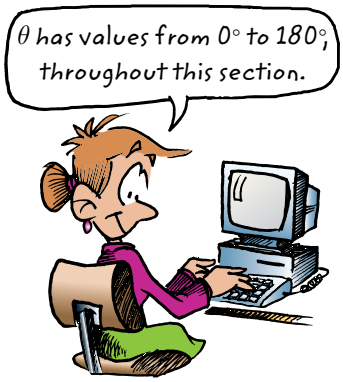
**10** $\cos\theta = 0{\cdot}5$

- It is possible for an acute angle and an obtuse angle to have the same sine value.




- There is a relationship between the trigonometric ratios of acute and obtuse angles. To investigate this, $OA$ and $OB$ have been drawn at an angle $\theta$ to the positive arm of the $x$-axis and the negative arm of the $x$-axis respectively. Hence, $\angle BOM = 180° - \theta$

$\therefore$ The coordinates of $A$ are $(\cos\theta, \sin\theta)$.
The coordinates of $B$ are $[\cos(180° - \theta), \sin(180° - \theta)]$.




Now, $\Delta$s $AMO$ and $BNO$ are congruent (AAS).
$\therefore$ $AM = BN$ and $OM = ON$.
If $A$ is the point $(x, y)$ then $B$ is the point $(-x, y)$.

$A$ has coordinates $(\cos\theta, \sin\theta)$ or $(x, y)$, so:

$\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{y}{x}$.

$B$ has coordinates $[\cos(180° - \theta), \sin(180° - \theta)]$ or $(-x, y)$, so:

$\cos(180° - \theta) = -x$, $\sin(180° - \theta) = y$ and $\tan(180° - \theta) = \frac{y}{-x}$.

Equating the two sets of ratios gives these results.

$$\sin(180° - \theta) = \sin\theta, \quad \cos(180° - \theta) = -\cos\theta, \quad \tan(180° - \theta) = -\tan\theta$$

This means, for example, that:
$\sin 120° = \sin(180° - 60°) = \sin 60°$
$\cos 160° = \cos(180° - 20°) = -\cos 20°$
$\tan 141° = \tan(180° - 39°) = -\tan 39°$

For acute angles the trig ratios are positive.

For obtuse angles only sine is positive.

Because $\theta$ and $180 - \theta$ are supplementary angles we can say that:

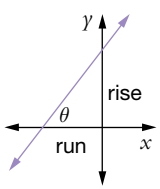
- the sines of supplementary angles are equal
- the cosines and tangents of supplementary angles have equal numerical values but they have the opposite sign.

## Finding the angle of inclination of a line

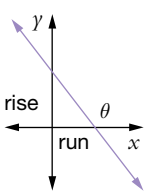
- In coordinate geometry we have seen that the slope of a line is given by the formula $m = \frac{\text{rise}}{\text{run}}$.

An angle can also be used to measure the slope of a line. If $\theta$ is the angle between the line and the positive direction of the $x$-axis then two cases arise.

Case 1: $m$ is positive and $\theta$ is acute.




Case 2: $m$ is negative and $\theta$ is obtuse.




In both of these cases $m$ and $\theta$ are connected by the formula $m = \tan\theta$.

## WORKED EXAMPLE 1

Which acute angle has the same sine as 150°?

$\sin(180° - \theta) = \sin\theta$

$180° - \theta = 150°$

$\theta = 30°$

$\therefore \sin 30° = \sin 150°$

## WORKED EXAMPLE 2

If $\theta$ is an angle between 0° and 180°, find $\theta$ to the nearest degree if:

**a** $\cos\theta = 0{\cdot}6$ ($\therefore \theta$ is acute)

Press: SHIFT cos 0.6 =

$\theta = 53{\cdot}13°$ (from the calculator)

$\theta = 53°$ (nearest degree)

**b** $\tan\theta = -0{\cdot}8$ ($\therefore \theta$ is obtuse)

Let $\theta = 180° - \alpha$ ($\alpha$ is acute)

$\tan(180° - \alpha) = -0{\cdot}8$

$-\tan\alpha = -0{\cdot}8$ ($\tan(180° - \alpha) = -\tan\alpha$)

$\tan\alpha = 0{\cdot}8$

Press: SHIFT tan 0.8 = ° ' "

$\alpha = 38°40'$ (nearest minute)

But $\theta = 180° - \alpha$

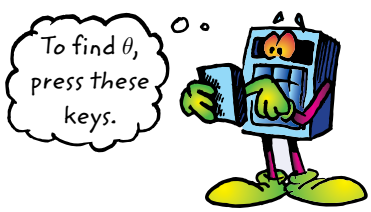
$\therefore \theta = 180° - 38°40'$

$= 141°$ (nearest degree)

If $\theta$ is obtuse, solve $\tan\theta = 0{\cdot}8$ and subtract the answer from 180°.

## WORKED EXAMPLE 3

Give an acute and obtuse value of $\theta$ (to the nearest minute) for which $\sin\theta = 0{\cdot}354$.




$\sin\theta = 0{\cdot}354$

Press: SHIFT sin 0.354 = ° ' "

$\theta = 20°44'$ (nearest minute)

Now $\sin(180° - \theta) = \sin\theta$

If $\theta = 20°44'$ then $180° - \theta = 159°16'$.

$\therefore \theta = 20°44'$ or $159°16'$

## WORKED EXAMPLE 4

Find the angle of inclination of the line $y = 2x - 3$ to the nearest degree.

$m = \tan\theta$

$m = 2$ (using $y = mx + b$ form)

$\tan\theta = 2$

Press: SHIFT tan 2 = ° ' "

$\therefore \theta = 63°$ (nearest degree)

# Exercise 11:02

P Foundation worksheet 11:02
Trig ratios of obtuse angles

**1** Use a calculator to evaluate the following correct to three decimal places.

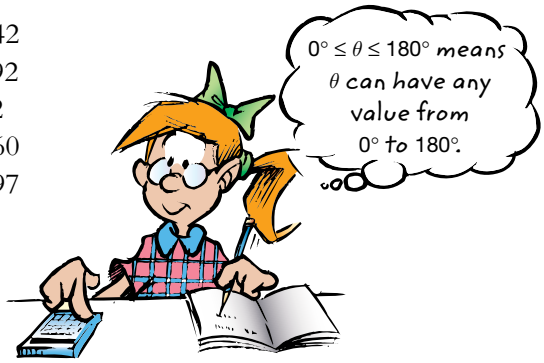
**a** $\sin 167°30'$ **b** $\cos 140°20'$ **c** $\tan 150°19'$

**2** Use the results on page 373 to express each of the following as a ratio of an acute angle.

**a** $\sin 150°$ **b** $\cos 100°$ **c** $\tan 140°$
**d** $\sin 125°$ **e** $\sin 140°32'$ **f** $\cos 167°31'$
**g** $\tan 101°03'$ **h** $\cos 92°50'$

**3** If $\theta$ is an angle between $0°$ and $180°$, find $\theta$ to the nearest minute if:

**a** $\cos\theta = 0{\cdot}716$ **b** $\cos\theta = -0{\cdot}716$
**c** $\tan\theta = 8{\cdot}215$ **d** $\tan\theta = -8{\cdot}215$
**e** $\cos\theta = -0{\cdot}5$ **f** $\tan\theta = -1$
**g** $\cos\theta = 0{\cdot}906$ **h** $\cos\theta = -0{\cdot}342$
**i** $\tan\theta = 1{\cdot}881$ **j** $\tan\theta = -1{\cdot}192$
**k** $\cos\theta = -0{\cdot}966$ **l** $\cos\theta = 0{\cdot}602$
**m** $\tan\theta = -0{\cdot}754$ **n** $\cos\theta = -0{\cdot}760$
**o** $\tan\theta = 3{\cdot}323$ **p** $\cos\theta = -0{\cdot}997$




**4** If $0° \leq \theta \leq 180°$, give two possible values for $\theta$ (to the nearest minute) if $\sin\theta$ is equal to:

**a** 0·5 **b** 0·73
**c** 0·36 **d** 0·453
**e** 0·990 **f** 0·3665
**g** 0·7083 **h** 0·0567

**5** Complete the table below and use it to sketch $y = \sin x°$ for $0° \leq x° \leq 180°$.

| $x°$ | 0° | 20° | 40° | 60° | 80° | 90° | 100° | 120° | 140° | 160° | 180° |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin x°$ | | | | | | | | | | | |

**6** Make a sketch of $y = \cos x°$ for $0° \leq x° \leq 180°$.

**7** Find the angle of inclination to the $x$-axis of each line to the nearest degree.

**a** $y = 3x$ **b** $y = x + 2$ **c** $y = \frac{2}{5}x + 5$
**d** $y = 3 - x$ **e** $y = -\frac{x}{3} + 1$ **f** $x - \sqrt{3}y = 10$

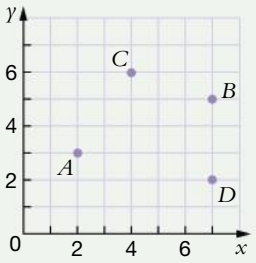
**8** Find the angle of inclination (to the nearest degree) of the line that passes through the points:

**a** $(1, 2)$ and $(6, 9)$ **b** $(-3, 6)$ and $(0, 1)$.

## FUN SPOT 11:02 WHY ARE CAMELS TERRIBLE DANCERS?

Answer each question and write the letter for that question in the box above the correct answer.

*Find:*




**E** the midpoint of $AB$
**T** the distance from $A$ to $B$
**L** the gradient of $CB$
**E** the equation of $BD$
**T** the equation of the $x$-axis
**A** the equation of $AC$
**F** the area of $\Delta ABD$.
**E** Through which of the points $A$, $B$, $C$ or $D$ does the line $2x - 5y = 4$ pass?

*Find the equation linking $x$ and $y$ for each table.*

**H**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $y$ | 3 | 4 | 5 | 6 |

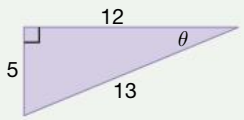
**L**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $y$ | 2 | 5 | 8 | 11 |

**T**

| $x$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $y$ | 4 | 9 | 16 | 25 |

*Use the triangle to find the value of:*




**E** $\sin\theta$ **A** $\cos\theta$ **F** $\tan\theta$

**H** $\dfrac{1}{\sin\theta}$ **L** $\dfrac{1}{\cos\theta}$ **O** $\dfrac{1}{\tan\theta}$

**E** $(\sin\theta)^2 + (\cos\theta)^2$

**T** $(\tan\theta)^2 - \left(\dfrac{1}{\cos\theta}\right)^2$

*Solve:*

**V** $x^2 = 49$ **W** $\dfrac{1}{x} = \dfrac{7}{2}$ **Y** $x^2 - 6x + 9 = 0$

| $\sqrt{29}$ | $\frac{13}{5}$ | $(4\frac{1}{2}, 4)$ | $x = 3$ | $\frac{12}{13}$ | $\frac{13}{12}$ | $\frac{1}{3}$ | $y = x + 3$ | $y = \frac{3}{2}x$ | $y = \pm 7$ | $x = 7$ | $y = x^2$ | $x = \frac{2}{7}$ | $\frac{12}{5}$ | $y = 3x + 2$ | $\frac{5}{13}$ | $\frac{5}{12}$ | $y = 0$ | 7·5 units$^2$ | 1 | D | $-1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

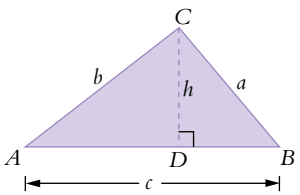
# 11:03 The sine rule

Now that trig ratios can be calculated for obtuse angles, trigonometry can be applied to any type of triangle.

In the triangles below:

- side $a$ is opposite $\angle A$
- side $b$ is opposite $\angle B$
- side $c$ is opposite $\angle C$.

*Note:* $\sin(180° - B) = \sin B$




$ABC$ is an acute-angled triangle.
$CD$ is perpendicular to $AB$.

In $\Delta ACD$, $\sin A = \dfrac{h}{b}$

$\therefore\ h = b \sin A$ 1

In $\Delta BCD$, $\sin B = \dfrac{h}{a}$

$\therefore\ h = a \sin B$ 2

Equating 1 and 2 from either triangle gives:

$$a \sin B = b \sin A$$

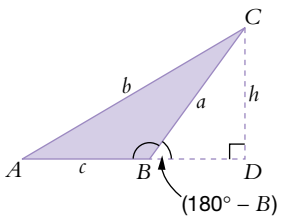
$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, it can be shown that:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

*Note:* To use the sine rule, at least one angle and the side opposite it must be known.

Thus, we obtain:




$ABC$ is an obtuse-angled triangle.
$CD$ is perpendicular to $AB$ (produced to $D$).

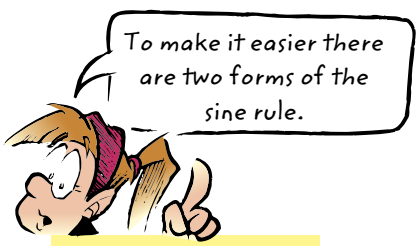
In $\Delta ACD$, $\sin A = \dfrac{h}{b}$

$\therefore\ h = b \sin A$ 1

In $\Delta BCD$, $\sin(180° - B) = \dfrac{h}{a}$

$\therefore\ h = a \sin(180° - B)$

$\therefore\ h = a \sin B$ 2

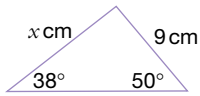



- To find a side use the first form.
- To find an angle use the second form.

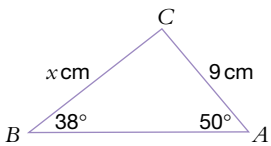
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

## WORKED EXAMPLE 1

Find the value of $x$ in this triangle. Answer correct to one decimal place.




Here, two angles and two sides are involved. When finding a side, it is best to use the first form of the sine rule.




*Remember:*
side $a$ is opposite $\angle A$
side $b$ is opposite $\angle B$

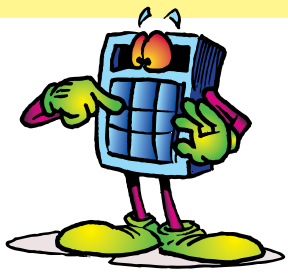
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 50^\circ} = \frac{9}{\sin 38^\circ}$$

$$\therefore\ x = \frac{9 \sin 50^\circ}{\sin 38^\circ}$$

$$\doteqdot 11{\cdot}2 \text{ cm}$$

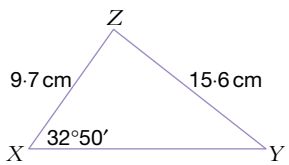
Press: 9 × sin 50 ) ÷ sin 38 =



## WORKED EXAMPLE 2

$\angle X = 32°50'$, $x = 15{\cdot}6$ cm, $y = 9{\cdot}7$ cm. Find $\angle Y$ correct to the nearest minute.

Always draw a diagram if one is not given with the question. When finding an angle, the second form of the sine rule is easier to use.




$$\frac{\sin Y}{y} = \frac{\sin X}{x}$$

$$\therefore\ \frac{\sin Y}{9{\cdot}7} = \frac{\sin 32°50'}{15{\cdot}6}$$

$$\sin Y = \frac{9{\cdot}7 \sin 32°50'}{15{\cdot}6}$$

$$\doteqdot 0{\cdot}3371$$

Press: SHIFT sin 0.3371 = ° ' "

$$\therefore\ \angle Y = 19°42'$$

## WORKED EXAMPLE 3

Town $B$ is 20 km due east of $A$. If the bearing of town $C$ is N35°E from $A$ and N65°W from $B$ find the distance from $A$ to $C$ and from $B$ to $C$.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{AC}{\sin 25^\circ} = \frac{20}{\sin 100^\circ}$$

$$AC = \frac{20 \sin 25}{\sin 100^\circ}$$

$$\doteqdot 8{\cdot}6 \text{ km}$$

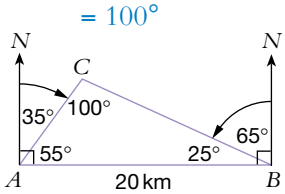
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{BC}{\sin 55^\circ} = \frac{20}{\sin 100^\circ}$$

$$BC = \frac{20 \sin 55^\circ}{\sin 100^\circ}$$

$$\doteqdot 16{\cdot}6 \text{ km}$$

$\angle CAB = 90° - 35° = 55°$
$\angle CBA = 90° - 65° = 25°$
$\therefore\ \angle ACB = 180° - (55° + 25°)$
$= 100°$




The distance $AC$ is 8·6 km and the distance $BC$ is 16·6 km (1 dec. pl.).

# Exercise 11:03

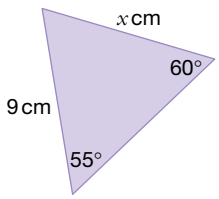
Foundation worksheet 11:03
The sine rule

**1** Solve the following equations.

**a** $\dfrac{a}{0{\cdot}866} = \dfrac{12{\cdot}5}{0{\cdot}48}$ **b** $\dfrac{a}{0{\cdot}4356} = \dfrac{7{\cdot}8}{0{\cdot}648}$ **c** $\dfrac{a}{\sin 20°} = \dfrac{12}{\sin 60°}$ **d** $\dfrac{a}{\sin 70°} = \dfrac{20{\cdot}6}{\sin 50°}$

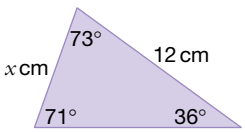
**2** For each triangle, part of the sine rule has been written. Copy and complete the rule, then find $x$ correct to one decimal place.

**a**



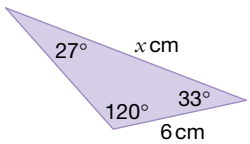

$\dfrac{x}{\sin 55°} = \dfrac{9}{\ldots}$

**b**




$\dfrac{x}{\ldots} = \dfrac{12}{\sin 71°}$

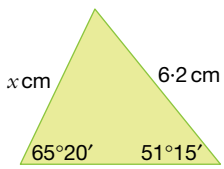
**c**



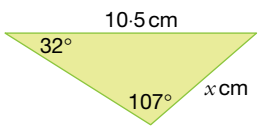

$\dfrac{x}{\ldots} = \dfrac{6}{\ldots}$

**3** Find $x$ in the following diagrams, correct to one decimal place.

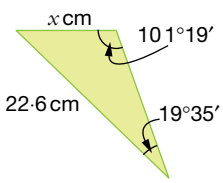
**a**




**b**




**c**




**4** In each of the following, find the value of $\theta$:

**i** correct to the nearest degree

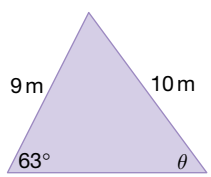
**ii** correct to the nearest minute.

**a** $\sin\theta = 0{\cdot}7178$ **b** $\sin\theta = 0{\cdot}8164$ **c** $\sin\theta = 0{\cdot}2$

**d** $\sin\theta = \dfrac{14{\cdot}6\sin 48°}{12{\cdot}6}$ **e** $\dfrac{\sin\theta}{11{\cdot}4} = \dfrac{\sin 63°}{16{\cdot}2}$

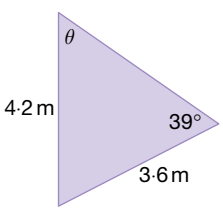
**5** For each triangle, part of the sine rule has been written. Copy and complete the rule and then find the angle $\theta$, correct to the nearest degree.

**a**



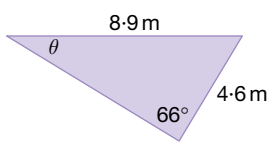

$\dfrac{\sin\theta}{9} = \dfrac{\sin 63°}{\ldots}$

**b**




$\dfrac{\sin\theta}{\ldots} = \dfrac{\sin 39°}{\ldots}$

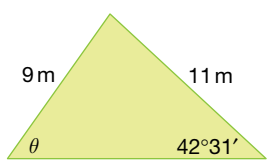
**c**



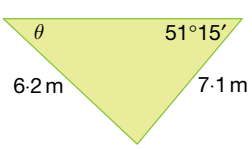

$\dfrac{\sin\theta}{\ldots} = \dfrac{\sin 66°}{\ldots}$

**6** Find the value of $\theta$ to the nearest minute.

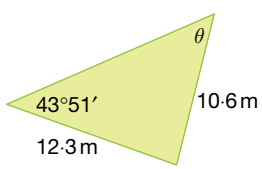
**a**




**b**

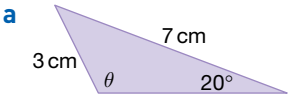



**c**

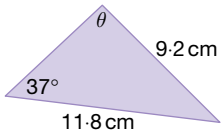


**7** Find $\theta$ to the nearest degree, noting that in each triangle $\theta$ is obtuse.

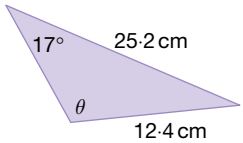
**a**




**b**




**c**




**8** Answer the following by first drawing a diagram.

**a** In $\Delta ABC$, $\angle A = 30°$, $\angle B = 52°$, $a = 9$ cm; find $b$ (1 dec. pl.).

**b** In $\Delta PQR$, $\angle P = 51°$, $\angle R = 77°$, $r = 10{\cdot}2$ cm; find $p$ (1 dec. pl.).

**c** In $\Delta XYZ$, $\angle Y = 19°$, $y = 19{\cdot}1$ cm; $x = 15{\cdot}7$ cm; find $\angle X$ (nearest minute).

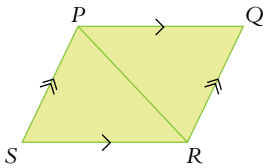
**d** In $\Delta LMN$, $\angle M = 37°$, $m = 8{\cdot}7$ m; $n = 4{\cdot}6$ m; find $\angle N$ (nearest minute).

**9** $PQRS$ is a parallelogram, where $PQ = 8$ cm, $QR = 5$ cm, and $\angle PRQ = 68°$.
Find:

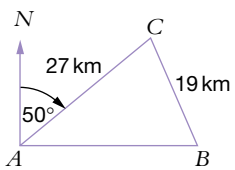
**a** $\angle RPQ$ (nearest degree)

**b** $\angle PQR$ (nearest degree)

**c** the diagonal $PR$ (1 dec. pl.).




**10**



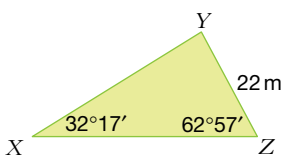

Three towns are situated so that the distance from $A$ to $C$ is 27 km, the distance from $B$ to $C$ is 19 km and the bearing of $C$ from $A$ is N50°E. If $B$ is due east of $A$, find:

**a** $\angle ABC$ (nearest degree)

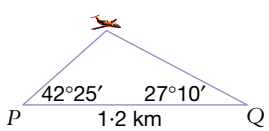
**b** $\angle ACB$ (nearest degree)

**c** the distance of $B$ from $A$ (2 sig. fig.).

**11** Three posts $X$, $Y$ and $Z$ are situated so that $\angle X$ is 32°17′ and $\angle Z$ is 62°57′. If the distance between $Y$ and $Z$ is 22 m, find the distance between $X$ and $Y$ to the nearest metre.



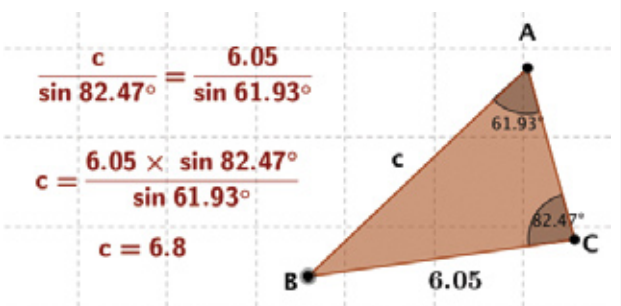

**12**




A man at $P$ observes a plane to the south at an angle of elevation of 42°25′, while a second man at $Q$ observes the same plane to the north at an angle of elevation of 27°10′. If the distance between the observers is 1·2 km, find the distances of each observer from the plane, to the nearest metre.

## P GEOGEBRA ACTIVITY 11:03 THE SINE RULE

Click and drag the vertices of a triangle and verify the sine rule for varying angle sizes and side lengths. Then use the sine rule to calculate the values of either a side or an angle. Tick the checkbox to see if you are correct.



# 11:04 The sine rule: The ambiguous case

When an equation like $\sin\theta = 0{\cdot}5$ is solved, there is an acute and an obtuse answer.
Hence, when the sine rule is used to find an angle, it is possible to have two solutions.
When this occurs, we have what is called the ambiguous case.

- The acute angle will always be a solution.
- The obtuse angle will only be a solution if it gives an angle sum less than 180° when added to the other angles in the triangle.

Both the one-solution and two-solution cases are shown in the examples below.

## WORKED EXAMPLES

**1** Use the sine rule to find the size of angle $C$.

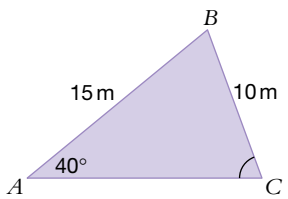
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{15} = \frac{\sin 40°}{10}$$

$$\sin C = \frac{15\sin 40°}{10}$$

$$\doteqdot 0{\cdot}964$$

$\therefore C = 74{\cdot}6°$ (from the calculator)



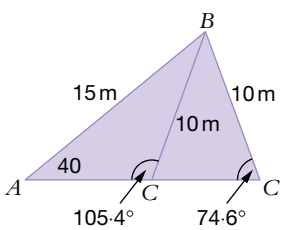

However:

$\sin(180° - 74{\cdot}6°)$ is also equal to $0{\cdot}964$.

$\therefore C = 74{\cdot}6°$ or $105{\cdot}4°$

$105{\cdot}4° + 40° < 180°$, so $105{\cdot}4°$ is also a solution.

The diagram shows the two possible solutions.



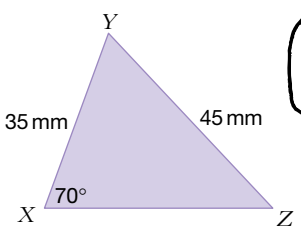

**2** Find the size of angle $Z$.

$$\frac{\sin Z}{35} = \frac{\sin 70°}{45}$$

$$\sin Z = \frac{35\sin 70°}{45}$$

$$\doteqdot 0{\cdot}731$$

$\therefore Z \doteqdot 47°$




$\sin(180° - 47°)$ is also equal to $0{\cdot}73$

$\therefore Z = 47°$ or $133°$ (to the nearest degree)

$70° + 133° > 180°$, so it is impossible for $Z$ to be $133°$.

We therefore reject the answer $133°$.

$\therefore Z = 47°$

# Exercise 11:04

P Foundation worksheet 11:04
The sine rule: The ambiguous case

**1** Find the acute and obtuse angles that are solutions to the equations.

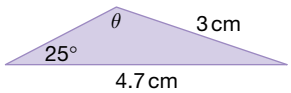
**a** $\sin\theta = 0{\cdot}8563$

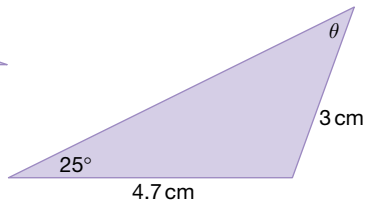
**b** $\sin\theta = 0{\cdot}216$

**c** $\sin\theta = \dfrac{15\sin 60^\circ}{20}$

**d** $\sin\theta = \dfrac{20\sin 15^\circ}{15}$

**2**



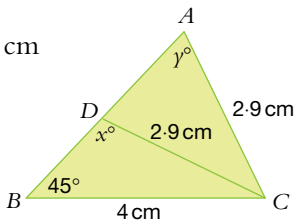

In both triangles, $\sin\theta = \dfrac{4{\cdot}7\sin 25^\circ}{3}$. Find $\theta$ in both triangles.

Give answers to the nearest degree.

**3** Triangles $ABC$ and $DBC$ both have sides 4 cm and 2·9 cm in length and a non-included angle of 45°.

**a** Show that $\sin x^\circ$ and $\sin y^\circ$ both equal $\dfrac{4\sin 45^\circ}{2{\cdot}9}$.

**b** Find $x^\circ$ and $y^\circ$ to the nearest degree.




**4** Find (to the nearest minute) the angle required in each of the following.

**a** In $\Delta XYZ$, $\angle X = 30^\circ$, $x = 5$ cm, $y = 9$ cm; find $\angle Y$.

**b** In $\Delta PQR$, $\angle Q = 19^\circ 20'$, $q = 2{\cdot}9$ cm, $r = 3{\cdot}7$ cm; find $\angle R$.

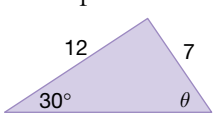
**c** In $\Delta ABC$, $\angle B = 32^\circ 17'$, $b = 10{\cdot}7$ cm, $a = 12{\cdot}1$ cm; find $\angle A$.

**d** In $\Delta KLM$, $\angle M = 27^\circ 51'$, $m = 8{\cdot}7$ cm, $k = 9{\cdot}8$ cm; find $\angle K$.

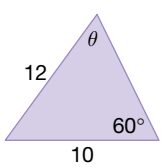
Naming sides and angles in triangles:
$x$ is opposite $\angle X$, $y$ is opposite $\angle Y$
$m$ is opposite $\angle M$, $k$ is opposite $\angle K$

**5** Find the possible values of $\theta$ in each of the following.

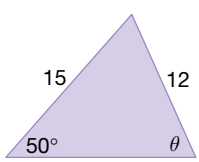
**a**




**b**




**c**



# 11:05 The cosine rule

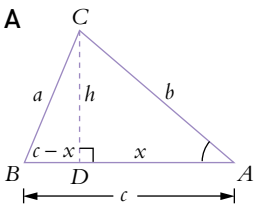
Like the sine rule, the cosine rule can be used to find unknown sides or angles in triangles. It is applied to situations where three sides and one angle are involved.

In the triangles below:

- $CD$ is perpendicular to $AB$.
- Let $AD$ be equal to $x$.
- In triangle A, $BD = c - x$.

Using Pythagoras' theorem:

**A**




From $\Delta BCD$:

$$a^2 = (c - x)^2 + h^2$$
$$= c^2 - 2cx + x^2 + h^2 \quad 1$$

From $\Delta ACD$:

$$b^2 = x^2 + h^2$$

Substituting in 1 gives:

$$a^2 = c^2 - 2cx + b^2$$
$$a^2 = b^2 + c^2 - 2cx \quad 2$$

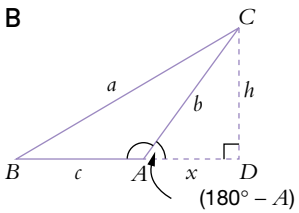
From $\Delta ACD$:

$$\frac{x}{b} = \cos A$$
$$x = b \cos A$$

Substituting in 2 gives:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

**B**




From $\Delta BCD$:

$$a^2 = (c + x)^2 + h^2$$
$$= c^2 + 2cx + x^2 + h^2 \quad 3$$

From $\Delta ACD$:

$$b^2 = x^2 + h^2$$

Substituting in 3 gives:

$$a^2 = c^2 + 2cx + b^2$$
$$a^2 = b^2 + c^2 + 2cx \quad 4$$

From $\Delta ACD$:

$$\frac{x}{b} = \cos(180° - A)$$
$$x = b \cos(180° - A)$$
$$= -b \cos A$$

Substituting in 4 gives:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

For any triangle $ABC$:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

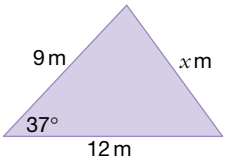
To find an angle, this rule can be rearranged to give:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

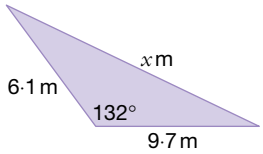
## WORKED EXAMPLE 1

Find $x$ in each triangle, correct to one decimal place.

**a**




**b**




### Solution

**a** $a^2 = b^2 + c^2 - 2bc\cos A$

Applying this formula gives:

$$x^2 = 12^2 + 9^2 - 2 \times 12 \times 9 \cos 37°$$
$$= 144 + 81 - 216\cos 37°$$
$$= 52{\cdot}494\,729$$
$$\therefore\ x = \sqrt{52{\cdot}494\,729}$$
$$= 7{\cdot}2 \quad \text{(1 dec. pl.)}$$

**b** $a^2 = b^2 + c^2 - 2bc\cos A$

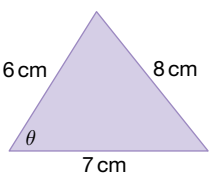
That is:

$$x^2 = 6{\cdot}1^2 + 9{\cdot}7^2 - 2 \times 6{\cdot}1 \times 9{\cdot}7 \cos 132°$$
$$= 37{\cdot}21 + 94{\cdot}09 - 118{\cdot}34\cos 132°$$
$$= 210{\cdot}484\,91$$
$$x = \sqrt{210{\cdot}484\,91}$$
$$= 14{\cdot}5 \quad \text{(1 dec. pl.)}$$

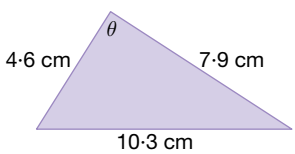
## WORKED EXAMPLE 2

Evaluate $\theta$, correct to the nearest minute.

**a**




**b**




### Solution

**a** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Applying this formula gives:

$$\cos\theta = \frac{6^2 + 7^2 - 8^2}{2 \times 6 \times 7}$$
$$= \frac{36 + 49 - 64}{84}$$
$$= 0{\cdot}25$$
$$\therefore\ \theta = 75°31' \text{ (nearest minute)}$$

**b** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

That is:

$$\cos\theta = \frac{4{\cdot}6^2 + 7{\cdot}9^2 - 10{\cdot}3^2}{2 \times 4{\cdot}6 \times 7{\cdot}9}$$
$$= \frac{21{\cdot}16 + 62{\cdot}41 - 106{\cdot}09}{72{\cdot}68}$$
$$\doteqdot -0{\cdot}3098$$
$$\therefore\ \theta = 108°03' \quad \text{(nearest minute)}$$

The cosine rule can be used in any triangle to find:

- the third side, given the other sides and their included angle
- any angle, given the three sides.

# Exercise 11:05

P Foundation worksheet 11:05
The cosine rule

**1** Find $a$ to one decimal place if $a^2 = b^2 + c^2 - 2bc\cos A$ and:

**a** $b = 10, c = 12, A = 60°$

**b** $b = 6{\cdot}8, c = 9{\cdot}4, A = 120°$

**c** $b = 15, c = 12, A = 75°$

**d** $b = 24{\cdot}5, c = 18{\cdot}2, A = 45°$

**2** Find $A$ to the nearest degree if:

**a** $\cos A = 0{\cdot}6$

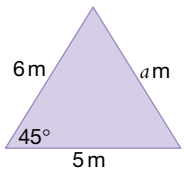
**b** $\cos A = -0{\cdot}6$

**c** $\cos A = \dfrac{6^2 + 10^2 - 8^2}{2 \times 6 \times 10}$

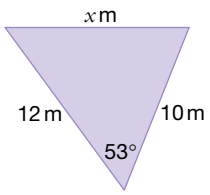
**d** $\cos A = \dfrac{6^2 + 8^2 - 12^2}{2 \times 6 \times 8}$

**3** Find the value of the pronumeral in the following, correct to one decimal place.

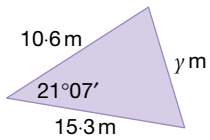
**a**



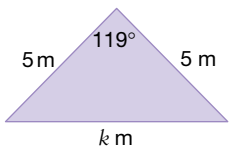

**b**



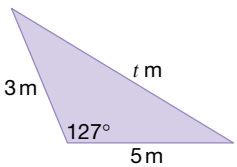

**c**



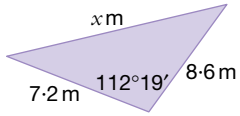

**d**




**e**

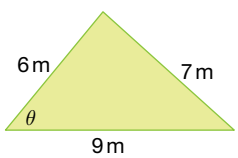



**f**

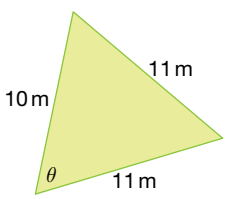



**4** Find the size of angle $\theta$, correct to the nearest minute.

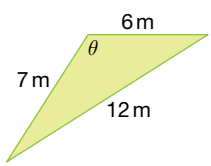
**a**



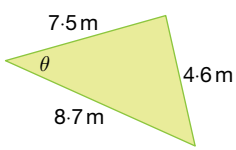

**b**



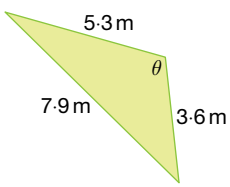

**c**



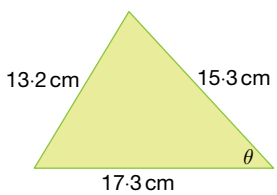

**d**




**e**




**f**




**5** Answer the following by first drawing a diagram.

**a** In $\Delta ABC$, $\angle A = 75°$, $b = 9$ cm, $c = 5$ cm; find $a$ (1 dec. pl.).

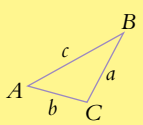
**b** In $\Delta DEF$, $\angle F = 61°$, $d = 2{\cdot}3$ m, $e = 3{\cdot}1$ m; find $f$ (1 dec. pl.).

**c** In $\Delta LMN$, $\angle M = 163°$, $l = 9{\cdot}4$ cm, $n = 8{\cdot}2$ cm; find $m$ (1 dec. pl.).

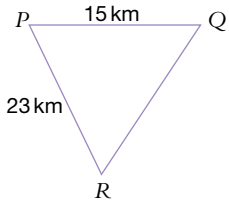
**d** In $\Delta PQR$, $p = 9$ m, $q = 7$ m, $r = 8$ m; find $\angle P$ (nearest degree).

**e** In $\Delta TUV$, $t = 2{\cdot}3$ cm, $u = 1{\cdot}9$ cm, $v = 1{\cdot}7$ cm; find $\angle V$ (nearest minute).

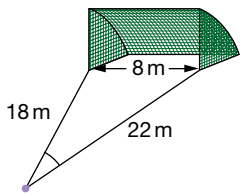
Angle $A$ is opposite side $a$.



**6** Three towns, $P$, $Q$ and $R$, are connected by straight roads. The distance from $P$ to $Q$ is 15 km and the distance from $P$ to $R$ is 23 km. If the roads meet at $P$ at an angle of 50°, how far is town $Q$ from town $R$, to the nearest kilometre?

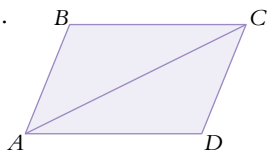



**7**

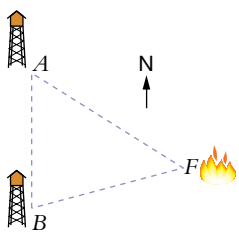



If a soccer goal is 8 m wide and a player shoots for goal when he is 18 m from one post and 22 m from the other, within what horizontal angle must the shot be made to score the goal? Give your answer correct to the nearest degree.

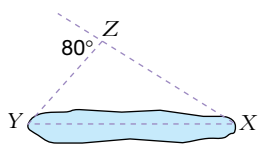
**8** In a parallelogram $ABCD$, $AB$ is 9 cm, $BC$ is 15 cm, and $\angle ABC$ is 130°. Find the length of the diagonal $AC$, correct to the nearest millimetre.




**9** An observer at fire tower $A$ observes a fire 18 km away to have a bearing S67°E. If tower B is 12 km due south of tower $A$, how far is tower $B$ from the fire?

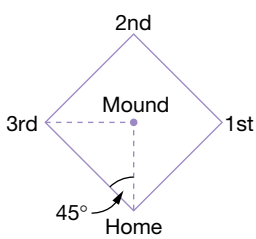



**10**

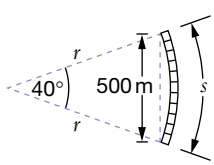



To find the length of a pond, a surveyor walks 350 m from point $X$ to point $Z$, then turns 80° and walks 290 m to point $Y$. Find the length of the pond to the nearest metre.

**11** In a (square) baseball diamond with sides of length 27·4 m, the centre of the pitcher's mound is 18·4 m from home plate. How far is it from the centre of the pitcher's mound to third base? Give your answer correct to one decimal place.

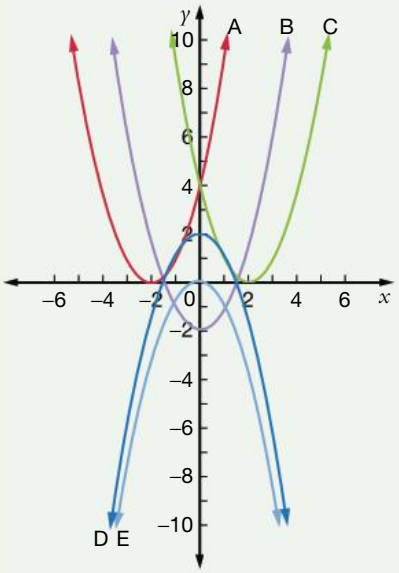



**12** The section of railway track shown is a circular arc that has a chord length of 500 m and subtends an angle of 40° at the centre of the circle. Find the radius $r$ of the circular arc and the length $s$ of that arc.



## FUN SPOT 11:05 WHY DID TOM'S MOTHER FEED HIM PETER'S ICE-CREAM

Answer each question and write the letter for that question in the box above the correct answer.

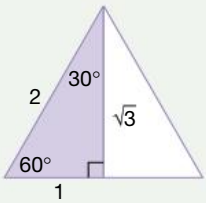


*Match each of the parabolas A to E with its equation.*

**E** $y = (x - 2)^2$ **T** $y = -x^2$ **F** $y = (x + 2)^2$ **H** $y = x^2 - 2$
**O** $y = 2 - x^2$

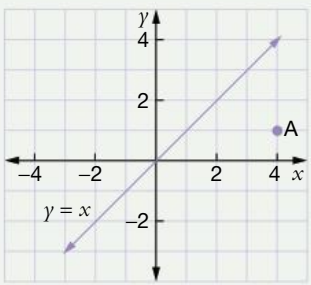
*Use the diagrams on the right to give the exact value of:*

**E** $\cos 60°$ **T** $\sin 60°$ **S** $\tan 60°$
**F** $(\sin 30°)^2$ **H** $(\cos 30°)^2$ **I** $(\tan 30°)^2$
**E** $(\sin 30°)^2 + (\cos 30°)^2$




*Give the resulting coordinates if the point A is:*

**E** reflected in the $x$-axis
**T** translated 4 units left
**K** rotated 90° anticlockwise about (0, 0)
**M** reflected in the $y$-axis
**O** reflected in the line $y = x$
**P** translated two units left, then 3 units up.




*For each point below, what is its reflection in the line $y = x$?*

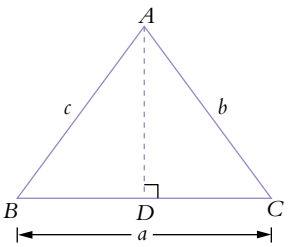
**E** (2, 0) **R** (5, −2) **S** (−1, −2) **T** (0, −1)

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E | (1, 4) | (−1, 4) | C | 1 | (2, 4) | $\frac{3}{4}$ | $\frac{1}{3}$ | (−4, 1) | D | $\frac{1}{4}$ | A | (0, 1) | B | (4, −1) | (−2, −1) | $\frac{\sqrt{3}}{2}$ | (−2, 5) | (0, 2) | $\frac{1}{2}$ | (−1, 0) | $\sqrt{3}$ |

# 11:06 Area of a triangle

To calculate the area of a triangle, the formula $A = \frac{1}{2}bh$ is used, where $b$ is the length of the base and $h$ is the perpendicular height. So, to calculate the area of any triangle, these two measurements need to be known.

Now, consider the triangle $ABC$ below.




$AD$ is perpendicular to $BC$.
The area of $\Delta ABC$ is given by:

Area $= \frac{1}{2}BC \times AD$ 1

From $\Delta ACD$,

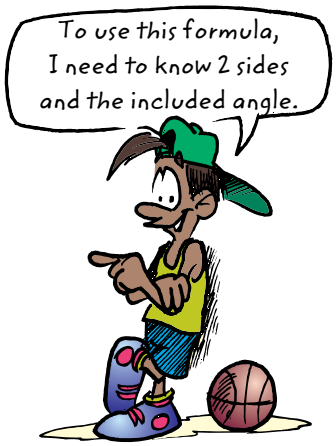
$$\frac{AD}{AC} = \sin C$$

$$AD = AC \sin C$$

Substituting this into 1,

Area $= \frac{1}{2}BC \times AC \sin C$

$= \frac{1}{2}ab \sin C$



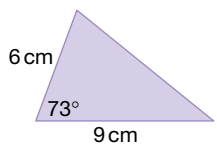

Area $= \frac{1}{2}ab \sin C$

where $a$ and $b$ are two sides and $C$ is the included angle.

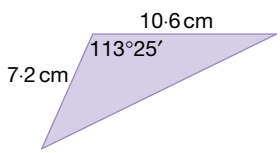
## WORKED EXAMPLES

Find the area of each triangle correct to the nearest square centimetre.

1




2




### Solutions

1 $A = \frac{1}{2}ab \sin C$

That is:

$A = \frac{1}{2} \times 6 \times 9 \times \sin 73°$

$= 25{\cdot}820\,228 \text{ cm}^2$

$= 26 \text{ cm}^2$ (to nearest $\text{cm}^2$)

2 $A = \frac{1}{2}ab \sin C$

That is:

$A = \frac{1}{2} \times 10{\cdot}6 \times 7{\cdot}2 \times \sin 113°25'$

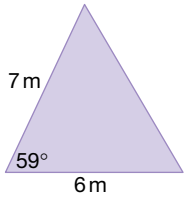
$= 35{\cdot}017\,106 \text{ cm}^2$

$= 35 \text{ cm}^2$ (to nearest $\text{cm}^2$)

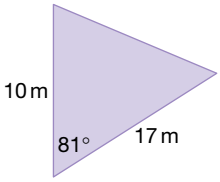
# Exercise 11:06

**1** Find the area of each triangle correct to one decimal place.

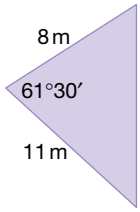
**a**



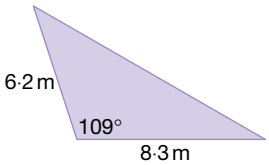

**b**



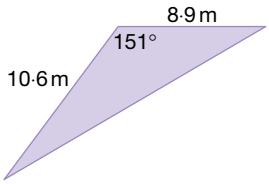

**c**



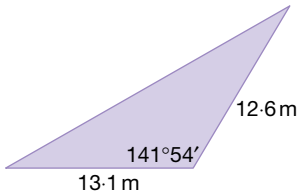

**d**




**e**

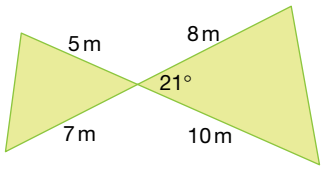



**f**

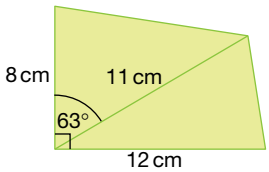



**2** Find the area enclosed by each figure to the nearest square centimetre.

**a**




**b**




**3** For $\Delta XYZ$, find its area, to the nearest square unit, if:

**a** $x = 7$, $y = 10$, $Z = 47°$

**b** $x = 14{\cdot}6$, $y = 17{\cdot}2$, $Z = 72°31'$

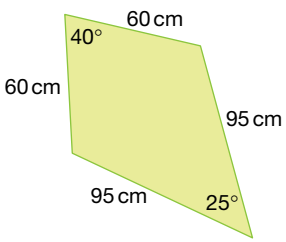
**c** $y = 2{\cdot}3$, $z = 3{\cdot}9$, $X = 62°$

**d** $y = 52$, $z = 63$, $X = 127°55'$

**e** $x = 20$, $z = 31$, $Y = 53°24'$

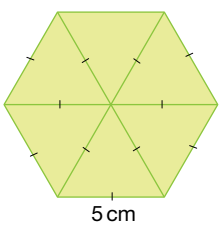
**f** $x = 72{\cdot}3$, $z = 91{\cdot}6$, $Y = 142°07'$

**4** **a**



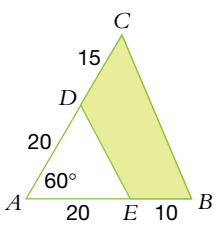

Calculate the area of the kite correct to the nearest square centimetre.

**b**



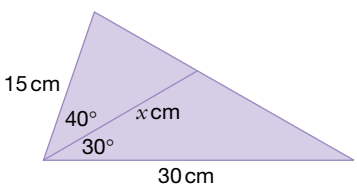

Calculate the area of the hexagon correct to one decimal place.

**c**




Calculate the area of the quadrilateral *BCDE*. All measurements are in centimetres.

**5** Find the value of $x$ correct to one decimal place.

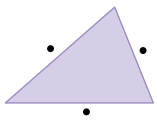


# 11:07 Miscellaneous problems

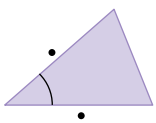
When solving miscellaneous problems, we need to identify which trigonometry rule should be used. The standard congruence tests for triangles can be used to help decide whether the information in a question applies to the sine rule or cosine rule.

**To use the cosine rule you need:**

- Three sides (SSS). This allows you to find all the angles.

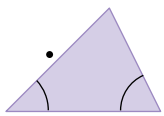


- Two sides and an included angle (SAS). This allows you to find all other angles and the other side.

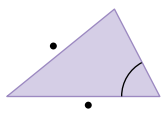


**To use the sine rule you need:**

- Two angles and a side (AAS). This allows you to find the other angle and the other sides.



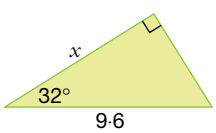
- Two sides and a non-included angle (SSA). This allows you to find the unknown angle opposite the known side. There could be two solutions for the angle, so this is the ambiguous case. (Hence, SSA is *not* a congruence test.)



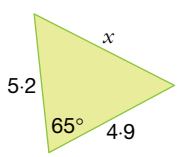
## Exercise 11:07

**1** Use the information above to identify whether the sine rule or the cosine rule is needed and then find the value of the pronumeral correct to one decimal place.

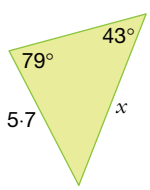
**a**



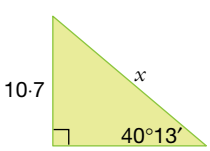

**b**



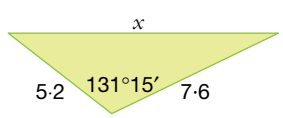

**c**




**d**

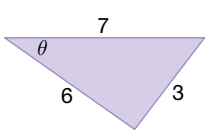



**e**

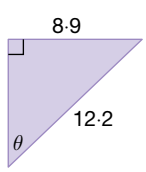



**2** Evaluate $\theta$, correct to the nearest minute.

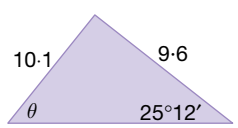
**a**



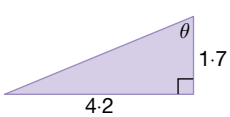

**b**



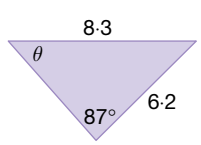

**c**



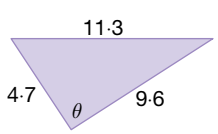

**d**




**e**



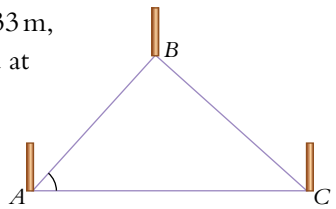

**f**



**3** For a triangle $XYZ$:

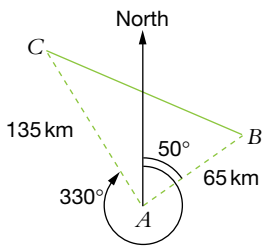
- **a** Find $\angle X$ correct to the nearest minute if $x = 9{\cdot}2$, $y = 5{\cdot}6$, $\angle Z = 90°$.
- **b** Find $x$ correct to one decimal place if $\angle X = 36°$, $\angle Y = 47°$, $y = 9{\cdot}6$.
- **c** Find $z$ correct to one decimal place if $\angle Z = 14°25'$, $x = 4{\cdot}2$, $y = 3{\cdot}7$.
- **d** Find $x$ correct to one decimal place if $\angle Y = 95°17'$, $\angle Z = 47°05'$, $y = 11{\cdot}3$.
- **e** Find $x$ correct to one decimal place if $\angle X = 90°$, $\angle Y = 23°17'$, $y = 5{\cdot}2$.
- **f** Find $\angle Y$ correct to the nearest minute if $x = 10{\cdot}3$, $y = 9{\cdot}6$, $z = 8{\cdot}7$.

**4** Three posts, $A$, $B$ and $C$ are positioned so that $AB$ is 33 m, $BC$ is 45 m and $AC$ is 51 m. Find the angle subtended at post $A$ by $BC$ to the nearest minute.

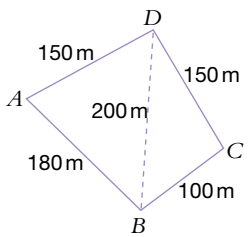



**5** $P$ and $Q$ are two points on the shore 950 m apart. $R$ is a buoy out at sea so that the angles $RPQ$ and $RQP$ are 73°19′ and 68°32′, respectively. Find the distance of $P$ from $R$.

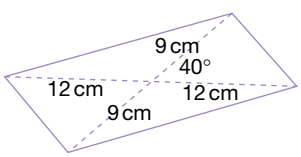
**6** From town $A$, the bearings of towns $B$ and $C$ are 050° and 330° respectively. If $A$ is 65 km from $B$ and 135 km from $C$, how far is town $B$ from town $C$?




**7** A farmer has a piece of land as shown in the diagram. What is the area of the land, to the nearest square metre?




**8** The diagonals of a parallelogram bisect each other at an angle of 40°.

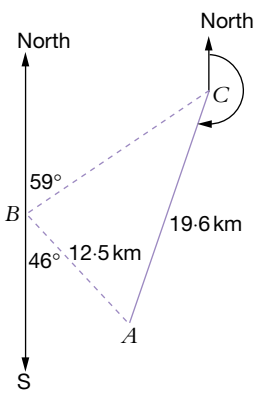



If the diagonals are 24 cm and 18 cm long, find:

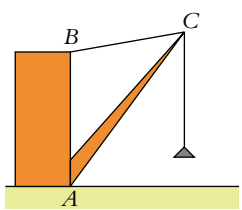
- **a** the area of the parallelogram
- **b** the lengths of the sides of the parallelogram.

Give answers correct to one decimal place.

**9** Town $A$ is 12·5 km S46°E of town $B$; town $C$ is 19·6 km from town $A$ and $C$ bears N59°E from $B$. Find the bearing of $A$ from $C$.



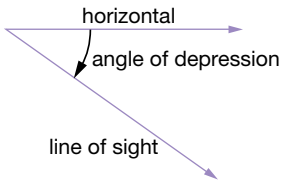

**10** The diagram represents a crane where $AB$ is 7·5 m, $AC$ is 12·6 m and $BC$ is 9·6 m. Find the angle $ABC$ and the height of $C$ above the ground.



## MATHS TERMS 11

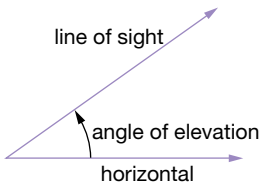
### angle of depression

- when looking down, the angle between the line of sight and the horizontal



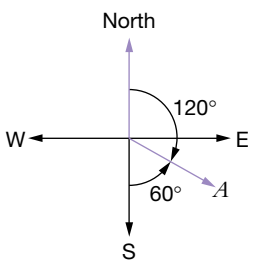

### angle of elevation

- when looking up, the angle between the line of sight and the horizontal




### bearing

- an angle used to measure the direction of a line from north
- bearings can be recorded in two ways, e.g. 120° or S60°E




### cosine rule

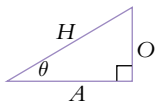
- a rule that is used to find either the third side of a triangle when the other two sides and the included angle (SAS) are known, or an angle when the three sides are known (SSS)

### sine rule

- a rule used to find an angle or side of a triangle when either two angles and a side (AAS), or two sides and a non-included angle (SSA), are known

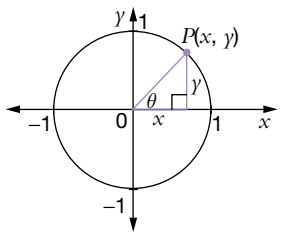
### trigonometric (trig) ratios

- a set of ratios (sine, cosine and tangent) that have constant values for any particular angle
- for acute angles, these can be defined in terms of the side lengths of a right-angled triangle




$$\sin\theta = \frac{O}{H}, \cos\theta = \frac{A}{H}, \tan\theta = \frac{O}{A}$$

- for acute or obtuse angles, they are defined in terms of the coordinates of a point $P$ that has its position on a unit circle determined by a radius drawn at an angle of $\theta$ to the horizontal




$$\sin\theta = y, \cos\theta = x, \tan\theta = \frac{y}{x}$$

# DIAGNOSTIC TEST 11 TRIGONOMETRY

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** Which obtuse angle has the same sine ratio as: 11:02

a $30°$ b $50°$ c $80°$

**2** Write as the trig ratio of an acute angle: 11:02

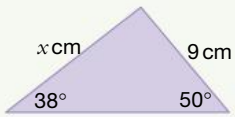
a $\sin 140°$ b $\cos 140°$ c $\tan 140°$

**3** If $90° < \theta < 180°$, find $\theta$ correct to the nearest minute, if: 11:02

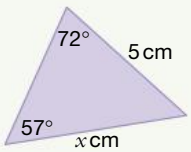
a $\cos\theta = -0{\cdot}625$ b $\sin\theta = 0{\cdot}257$ c $\cos\theta = -0{\cdot}018$

**4** Find the value of $x$, correct to one decimal place. 11:03

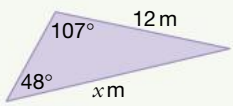
a




b

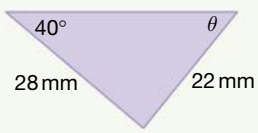



c

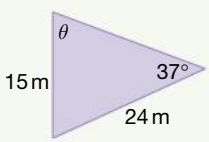



**5** Find the value of $\theta$, correct to the nearest degree. 11:03, 11:04

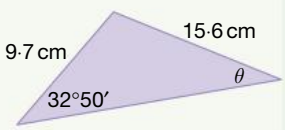
a




b

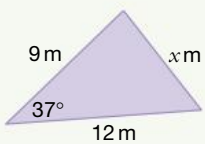



c

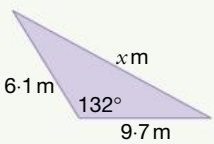



**6** Find the value of $x$, correct to one decimal place. 11:05

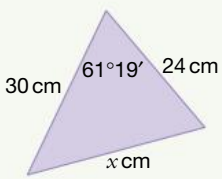
a




b

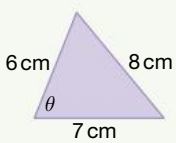



c

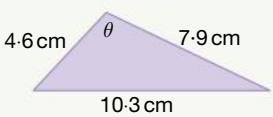



**7** Evaluate $\theta$, correct to the nearest minute. 11:05

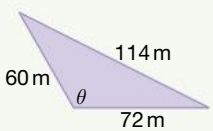
a




b

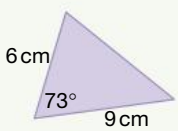



c

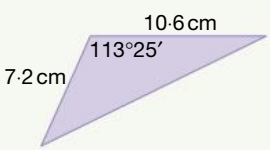



**8** Find the area of each triangle, correct to the nearest square centimetre. 11:06

a




b




c

## ASSIGNMENT 11A Chapter review

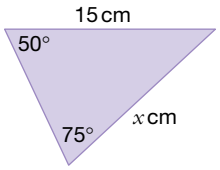
**1** Find $\theta$ if $\theta$ is obtuse and:

**a** $\sin\theta = 0{\cdot}5$ **b** $\cos\theta = -0{\cdot}5$

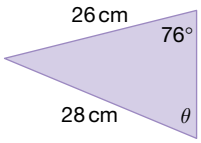
**c** $\tan\theta = -1$

**2** Use the sine rule to find the value of the pronumeral in each of the following.

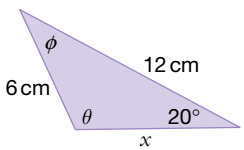
**a**




**b**




**3**



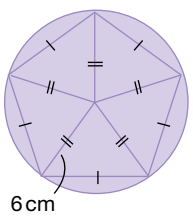

Find $\theta$, $\phi$ and $x$ if:

**a** $\theta$ is acute

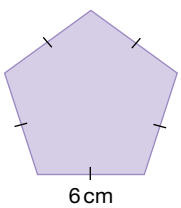
**b** $\theta$ is obtuse.

**4** Find the areas of the following pentagons.

**a**

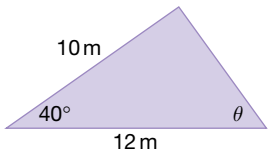



**b**

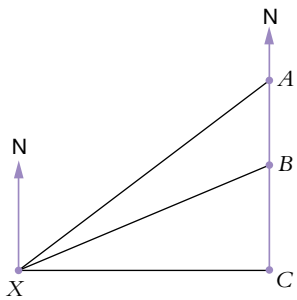



**5** **a** A triangle has sides that are 8 cm, 12 cm and 10 cm in length. Find the size of the largest angle.

**b** Find the size of $\theta$ to the nearest degree.

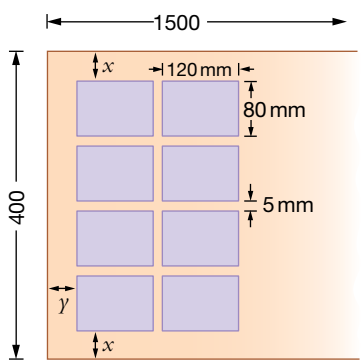



**6** From a position $X$ on a map, $A$ has a bearing of N27°E and $B$ has a bearing of N67°E. Town $C$ is 7·5 km due east of $X$. Find the distance between $A$ and $B$, which are both north of $C$ (1 dec. pl.).



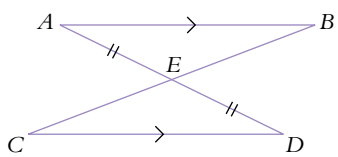

## ASSIGNMENT 11B Working mathematically

**1**

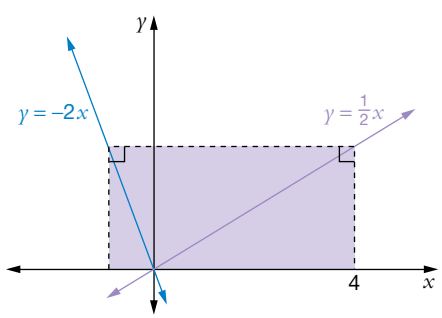



A rectangular table top has a length of 1500 mm and a breadth of 400 mm. The top has 4 rows of 11 tiles in the centre. Each rectangular tile is 120 mm by 80 mm. There is a 5 mm gap between each row of tiles. Find $x$ and $y$ in the diagram.

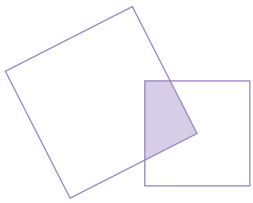
**2** Study the diagram and give three different descriptions of how it could be drawn.



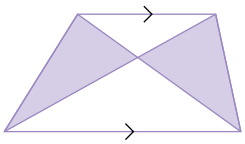

**3** Use the information in the diagram to find the area of the shaded rectangle.



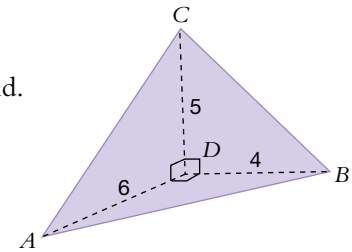
**4** In the diagram, the side length of the smaller square is 40 cm and the side length of the larger square is 60 cm. The large square intersects the smaller square $\frac{1}{4}$ of the way along the two sides of the smaller square, as shown. What is the area of the overlapping region if the vertex of the large square is at the centre of the small square?



**5** The diagram shows a trapezium divided into four triangles by its diagonals. Prove that the areas of the coloured triangles are equal.



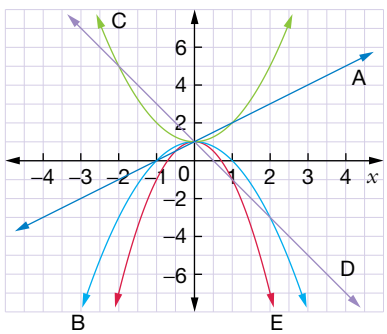
**6** Calculate the surface area of the pyramid.




## ASSIGNMENT 11C Cumulative revision

**1** Match each of graphs A to E with an equation:

**a** $y = x^2 + 1$
**b** $y = 1 - 2x^2$
**c** $y = 1 - x^2$
**d** $y = 1 - 2x$
**e** $y = x + 1$




1:08, 7:04

**2** Find the simple interest on \$18 000 at 5% pa. if it is invested for 120 days. (Assume that 1 year = 365 days.)

6:02

**3** Two similar trapeziums have matching sides 3·6 m and 1·2 m long.

**a** What is the ratio of their areas?
**b** If the smaller trapezium has an area of 2·8 $m^2$, what is the area of the larger trapezium?

10:02

**4** The table on the right gives the $x$ and $y$-coordinates of three points on a line. Use two of the first three points to find the equation of the line and then find the value of the missing number.

| $x$ | 6 | 10 | 4 | 47 |
|---|---|---|---|---|
| $y$ | 4 | 10 | 1 | ? |

1:08

**5** Simplify:

**a** $6\sqrt{8} + 3\sqrt{2}$ **b** $6\sqrt{8} \times 3\sqrt{2}$ **c** $6\sqrt{8} \div 3\sqrt{2}$ **d** $(3\sqrt{5})^3$

5:03, 5:04

**6** Solve:

**a** $x^2 + 5x + 2 = 0$ **b** $4x^2 - 3x - 5 = 0$

2:03

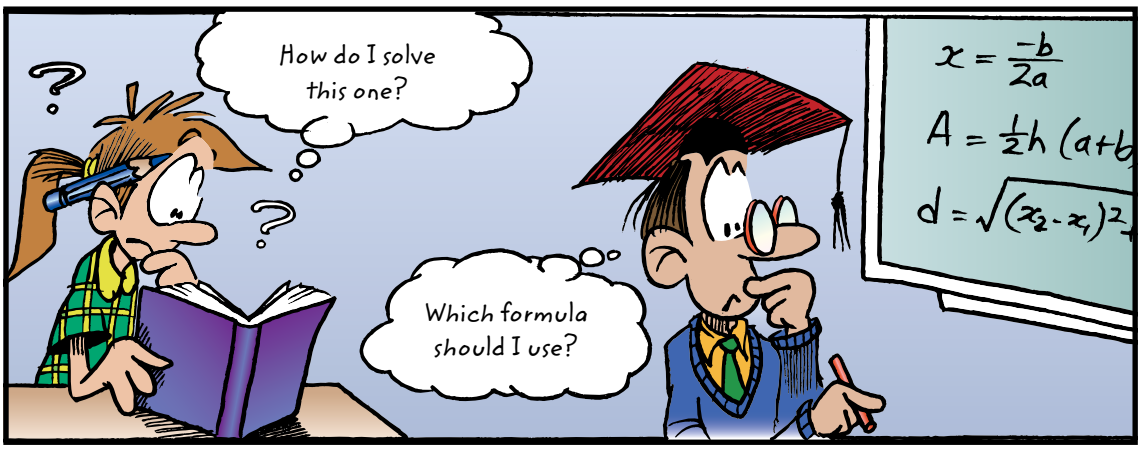
**7** On the same number plane sketch the graphs of:

**a** $y = 2x^3$ **b** $y = -2x^3$

7:09

# FURTHER EQUATIONS

12




## Contents


12:01 Simultaneous equations involving a non-linear equation
Investigation 12:01 Number patterns and algebra
12:02 Simple cubic equations: $ax^3 = k$
12:03 Literal equations: Restrictions on variables
Fun spot 12:03 What small rivers flow into the Nile?
Challenge 12:03 Fibonacci formula
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

Selections from ***Equations*** [Stage 5.3$^{\S}$]

- Solve simple cubic equations (NSW)
- Rearrange literal equations (NSW)
- Solve simultaneous equations, where one equation is non-linear, using algebraic and graphical techniques, including the use of digital technologies (NSW)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 12:01 Simultaneous equations involving a non-linear equation

## PREP QUIZ 12:01

Solve these quadratic equations.

**1** $(x + 3)(x - 2) = 0$ **2** $(2x - 1)(x + 7) = 0$ **3** $5x(x + 4) = 0$

**4** $x^2 - 4 = 0$ **5** $x^2 - 3x + 2 = 0$ **6** $2x^2 + 7x - 4 = 0$

Solve these equations.

**7** $x^2 - x + 5 = 3x + 2$ **8** $2x^2 - x = 3 - 2x$

Solve these simultaneous equations by substitution.

**9** $x + y = 5, y = 2x - 1$ **10** $3x - y - 13 = 0, y = 7 - 2x$

In Year 9 we saw how to solve two simultaneous equations when both equations were linear, such as $y = 3x + 2$ and $3x - 2y = 6$. Now we shall see how to find the common or simultaneous solutions when one equation is linear but the other is non-linear.

## WORKED EXAMPLE 1

Solve the simultaneous equations $y = x^2$ and $y = x + 2$ using the substitution method.

### Solution

$y = x^2$ 1

$y = x + 2$ 2

From 1 we see that $x^2$ is equal to $y$.

If we substitute $x^2$ for $y$ in 2 we have:

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$\therefore x = -1$ or $2$

These values for $x$ can now be substituted into either equation 1 or 2 to find corresponding values for $y$.

Substitute $x = -1$ in 1

$y = (-1)^2$

$= 1$

$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases}$ is a solution.

Substitute $x = 2$ in 1

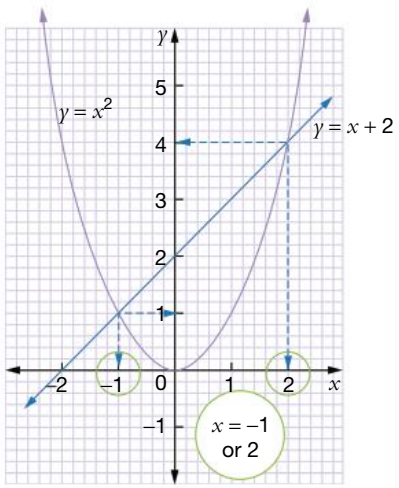
$y = (2)^2$

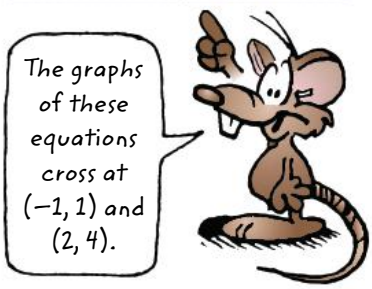
$= 4$

$\therefore \begin{cases} x = 2 \\ y = 4 \end{cases}$ is a solution.

(Check answers by substituting in 2.)

The solutions are $\begin{cases} x = -1 \\ y = 1 \end{cases}$ and $\begin{cases} x = 2 \\ y = 4 \end{cases}$



## WORKED EXAMPLE 2

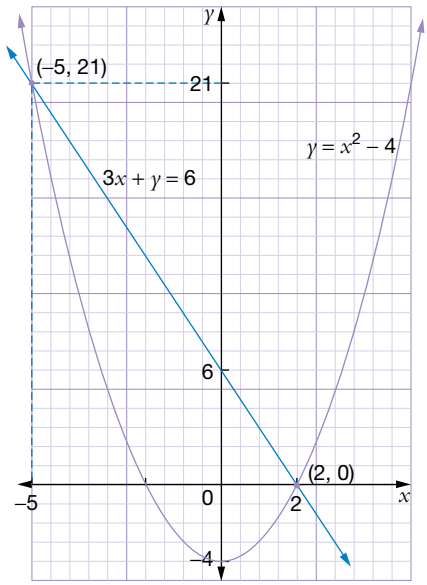
Use the substitution method to find the common solutions to the equations $y = x^2 - 4$ and $3x + y = 6$.

### Solution

$y = x^2 - 4$ 1
$3x + y = 6$ 2

From 1 we see that $x^2 - 4$ is equal to $y$.
If we substitute $x^2 - 4$ for $y$ in 2, we have:
$3x + (x^2 - 4) = 6$
$x^2 + 3x - 10 = 0$
$(x + 5)(x - 2) = 0$
$\therefore x = -5$ or $2$

Substituting these values into 1 or 2 we have:
If $x = -5$, $y = 21$
$x = 2$, $y = 0$
$\therefore$ The solutions are $\begin{cases} x = -5 \\ y = 21 \end{cases}$ and $\begin{cases} x = 2 \\ y = 0 \end{cases}$




Graphical solutions to $y = x^2 - 4$ and $3x + y = 6$.

## WORKED EXAMPLE 3

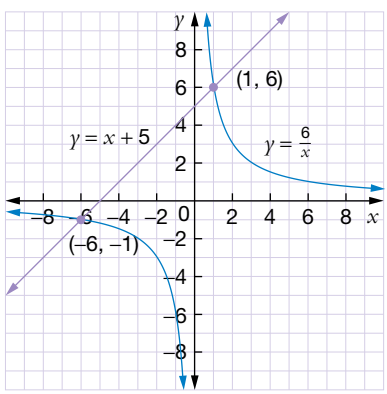
Find the solutions common to the equations $y = x + 5$ and $y = \frac{6}{x}$.

### Solution

$y = x + 5$ 1
$y = \frac{6}{x}$ 2

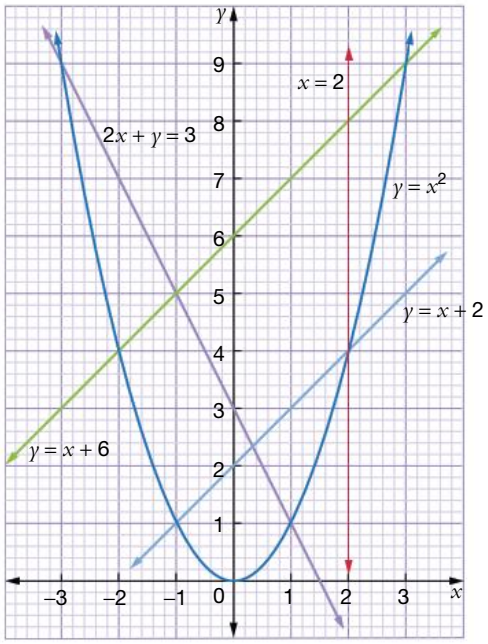
From 1 we see that $x + 5$ is equal to $y$.
If we substitute $x + 5$ for $y$ in 2,
we have:
$x + 5 = \frac{6}{x}$ (*Note:* $x \neq 0$)
$x(x + 5) = 6$
$x^2 + 5x - 6 = 0$
$(x + 6)(x - 1) = 0$
$\therefore x = -6$ or $x = 1$

Substituting these values into 1 or 2 we have:
If $x = -6$, then $y = -1$
$x = 1$, then $y = 6$
$\therefore$ The solutions are $\begin{cases} x = -6 \\ y = -1 \end{cases}$ and $\begin{cases} x = 1 \\ y = 6 \end{cases}$



# Exercise 12:01

**1** Use this diagram to solve these simultaneous equations.




**a** $y = x + 6$
$x = 2$

**b** $y = x + 6$
$2x + y = 3$

**c** $y = x^2$
$x = 2$

**d** $y = x^2$
$y = x + 6$

**e** $y = x^2$
$y = x + 2$

**f** $y = x^2$
$2x + y = 3$

**2** Solve each pair of equations in Question **1** using the substitution method and check your solutions with those for Question **1**.

**3** Use the substitution method to solve these simultaneous equations.

**a** $y = x^2$
$y = -4x - 3$

**b** $y = x^2$
$y = 49$

**c** $y = x^2$
$y = 3x$

**d** $y = x^2$
$y = x + 56$

**e** $y = x^2$
$y = 10x - 21$

**f** $y = x^2$
$y = -3x - 2$

**g** $y = x^2 + 5$
$y = 4x + 50$

**h** $y = x^2 + 7$
$y = 8x$

**i** $y = x^2 - 10$
$y = -9x$

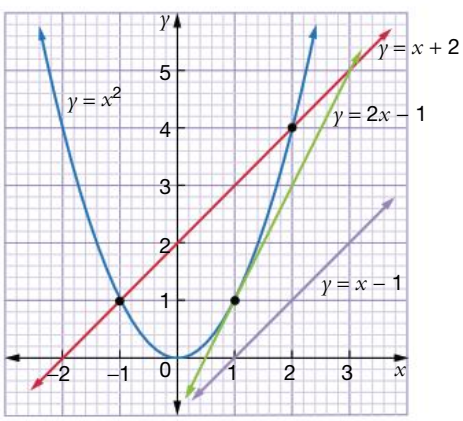
**4** Solve each pair of simultaneous equations using the substitution method.

**a** $y = x^2$ and $y = x + 2$

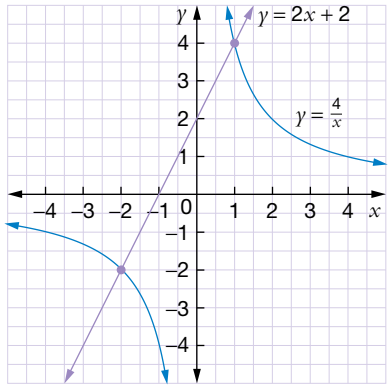
**b** $y = x^2$ and $y = 2x - 1$

**c** $y = x^2$ and $y = x - 1$

Looking at the graphs of these equations, explain why **a** has two solutions, **b** has one solution, and **c** has no solutions.



**5** **a** Solve the equations $y = 2x + 2$ and $y = \frac{4}{x}$ simultaneously. Check your solution with the graphs of these equations.




**b** **i** Solve the simultaneous equations $y = 4 - x$ and $y = \frac{4}{x}$?

**ii** How many solutions are there?

**iii** What does this say about the relationship between the graphs of $y = 4 - x$ and $y = \frac{4}{x}$?

**6** Solve these pairs of simultaneous equations.

**a** $y = 5x + 6$, $y = x^2$

**b** $y = x + 3$, $y = x^2 + 1$

**c** $y = 4x + 9$, $y = x^2 - 3$

**d** $y = 2x + 4$, $y = x^2 - x$

**e** $y = 2x + 14$, $y = x^2 - 3x$

**f** $y = 7 - 3x$, $y = x^2 + 3x$

**g** $y = 4x$, $y = \frac{4}{x}$

**h** $y = 2x + 6$, $y = \frac{8}{x}$

**i** $y = \frac{1}{2}x + 2$, $xy = 6$

**7** By solving these simultaneous equations, find the point(s) of intersection of their graphs.

**a** $y = x^2 - 2$, $y = x$

**b** $y = x^2 - 10$, $y = 10 - x$

**c** $y = x^2 + 2$, $y = 4x - 2$

**d** $y = x^2 + 2x - 20$, $x + y = 8$

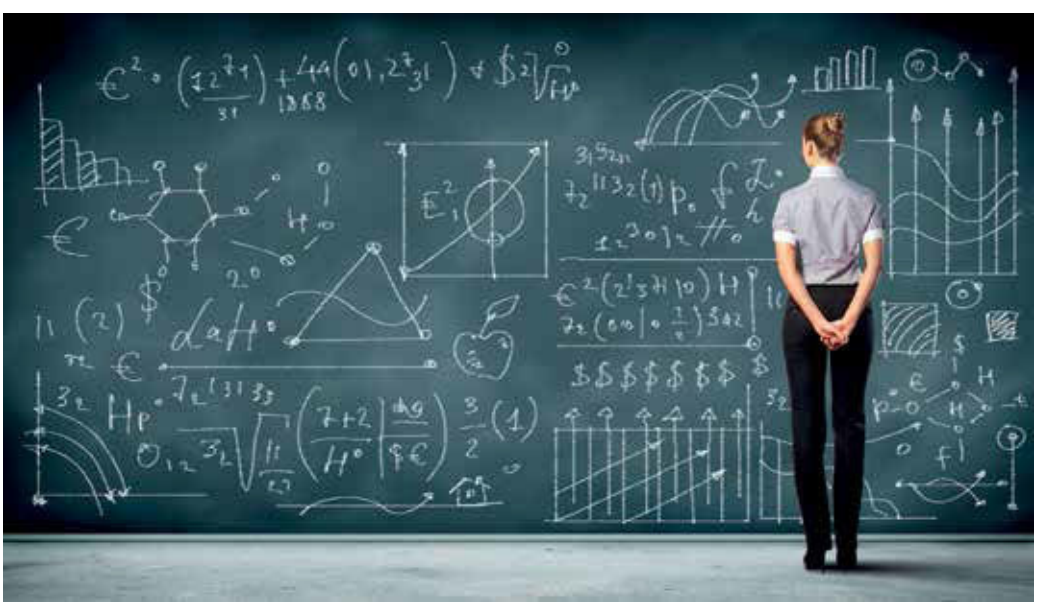
**e** $y = x^2 + 4x + 7$, $2x + y + 2 = 0$

**f** $y = x^2 + x - 2$, $y = x - 3$

**g** $y = x^2 + x - 10$, $y = 2x + 10$

**h** $y = x^2 + 2x - 8$, $3x + y = 6$

**i** $y = 4x + 1$, $y = 2x^2 - x - 2$



## Football scores

In Australian Rules Football, 6 points are awarded for a *goal*, and 1 point for a *behind*.

- Investigate scores of the form 2.12.24, where the product of the number of goals and the number of behinds equals the number of points.
- Note that $xy = 6x + y$, where $x$ is the number of goals and $y$ is the number of behinds. The solutions for $x$ and $y$ must, of course, be positive integers.

## Number patterns

**1** Find three consecutive integers such that the first number plus the product of the other two is equal to the first number times the sum of the other two.
*Hint:* Let the smallest integer be $n$. The other two integers would be $n + 1$ and $n + 2$. Form an equation and solve for $n$. (*Note:* Only integers can be considered to be consecutive numbers.)

**2** Show that it is not possible to find three integers consecutively differing by 2 that would satisfy the condition given in Question **1**. (Such integers would be either consecutive even numbers or consecutive odd numbers.)

**3** Would it be possible to find integers consecutively differing by 3, 4, 5 or 6 that would satisfy the condition? If so, find them.
*Note:* The condition can be expressed as:
$n + (n + a)(n + b) = n(n + a + n + b)$
where $a = 3$ and $b = 6$ if the numbers differ by 3
$a = 4$ and $b = 8$ if the numbers differ by 4
$a = 5$ and $b = 10$ if the numbers differ by 5
$a = 6$ and $b = 12$ if the numbers differ by 6.

# 12:02 Simple cubic equations: $ax^3 = k$

## PREP QUIZ 12:02

Solve these equations:

**1** $x - 1 = 0$ **2** $x^2 - 1 = 0$ **3** $x^3 - 1 = 0$

**4** $x + 1 = 0$ **5** $x^2 + 1 = 0$ **6** $x^3 + 1 = 0$

Evaluate:

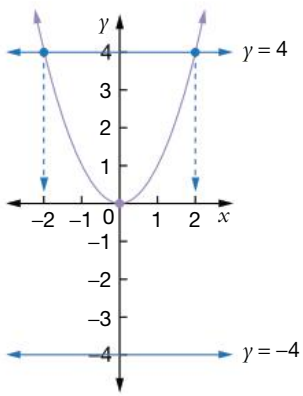
**7** $\sqrt{64}$ **8** $\sqrt{-64}$ **9** $\sqrt[3]{64}$ **10** $\sqrt[3]{-64}$

In previous studies you have seen that a simple linear equation such as $x - 3 = 5$ or $2y = 7$ has one solution. Simple quadratic equations, however, could have two, one or no real solutions.

$x^2 = 4$
$\therefore x = +2$ or $-2$

$x^2 = 0$
$\therefore x = 0$

$x^2 = -4$
($x^2 \neq$ a negative number)
$\therefore$ no real solutions




***Remember:*** (real number)$^2 \neq$ negative number

Looking at the graph of $y = x^2$ it can be seen that:

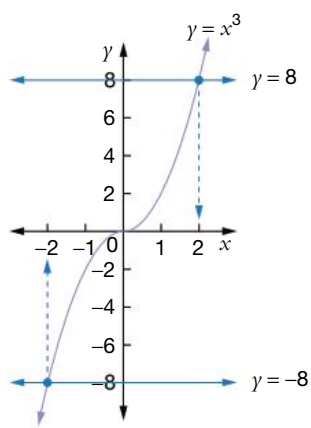
- it cuts the line $y = 4$ twice, indicating the solutions of $x = 2$ and $x = -2$
- it touches the line $y = 0$ once, indicating the solution $x = 0$
- it does not touch the line $y = -4$ at all, indicating no real solutions.

Now consider some simple cubic equations.

It can be seen from the graph that the answer to any simple cubic equation $ax^3 = k$ will only have one real solution, but unlike a quadratic equation *it will always have a solution.*

$x^3 = 8$
$x = \sqrt[3]{8}$
$\therefore x = 2$

$x^3 = -8$
$x = \sqrt[3]{-8}$
$\therefore x = -2$



## WORKED EXAMPLES

Solve each equation:

**1** $2x^3 = 54$

$x^3 = 27$

$x = \sqrt[3]{27}$

$\therefore x = 3$

**2** $3n^3 = 15$

$n^3 = 5$

$\therefore n = \sqrt[3]{5}$

$[n = 1{\cdot}71$ (2 dec. pl.)$]$

**3** $5z^3 + 320 = 0$

$5z^3 = -320$

$z^3 = -64$

$z = \sqrt[3]{-64}$

$\therefore z = -4$

# Exercise 12:02

**1** Solve these cubic equations, leaving the answers in exact surd form if necessary.

**a** $x^3 = 1000$ **b** $a^3 = -125$ **c** $p^3 = 1$

**d** $k^3 = -27$ **e** $y^3 = 10$ **f** $t^3 = -7$

**g** $q^3 = 2$ **h** $n^3 = -20$ **i** $z^3 = -216$

**j** $2x^3 = 16$ **k** $3z^3 = -3$ **l** $5p^3 = 20$

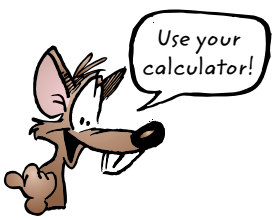
$\sqrt[3]{-5} = -\sqrt[3]{5}$

**2** Solve these cubic equations, leaving the answers as an approximation, correct to 2 decimal places, when necessary.

**a** $x^3 = 20$ **b** $m^3 = -20$ **c** $y^3 = 100$

**d** $a^3 = 2$ **e** $x^3 = 3{\cdot}375$ **f** $t^3 = -13{\cdot}824$

**g** $z^3 = 343$ **h** $n^3 = -6{\cdot}5$ **i** $x^3 = -1728$

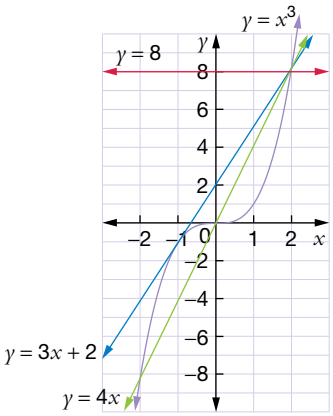



**3** Solve these cubic equations, leaving the answers in exact form.

**a** $8x^3 - 8 = 0$ **b** $5m^3 + 625 = 0$ **c** $3t^3 - 3000 = 0$ **d** $8z^3 + 1 = 0$

**e** $2x^3 - 6 = 0$ **f** $3n^3 + 30 = 0$ **g** $5t^3 - 100 = 0$ **h** $6z^3 + 2 = 0$

**4** The intersection of the curve $y = x^3$ and the line $y = 8$ shows the solution to the equation $x^3 = 8$ as $x = 2$.
(The $x$-value of the point of intersection.)




**a** What are the points of intersection of the curve $y = x^3$ with the line $y = 4x$?

**b** Use your answer to part **a** to determine the solutions to the equation $x^3 = 4x$.

**c** How many solutions are there to the equation in part **b**?

**d** What are the points of intersection of the curve $y = x^3$ with the line $y = 3x + 2$?

**e** Use your answer to part **d** to determine the solutions to the equation $x^3 = 3x + 2$.

**f** How many solutions are there to the equation in part **e**?

**g** What is the maximum number of points of intersection a straight line can have with the curve $y = x^3$ ?

**5** Use the graph to solve each equation.

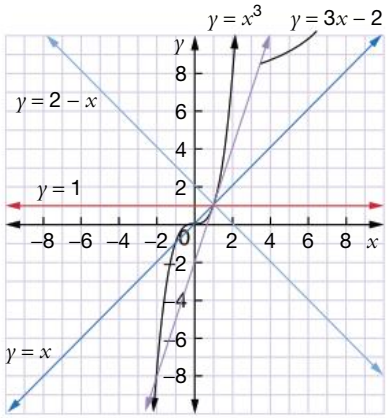
**a** $x^3 = 1$

**b** $x^3 = 2 - x$

**c** $x^3 - x = 0$

(*Note:* This is the same equation as $x^3 = x$.)

**d** $x^3 - 3x + 2 = 0$




# 12:03 Literal equations: Restrictions on variables

## PREP QUIZ 12:03

Complete the following:

**1** $a + 15 = 27$

$\therefore a = 27 - \ldots$

**2** $m + n = p$

$\therefore m = p - \ldots$

**3** $5x = 35$

$\therefore x = \frac{35}{\ldots}$

**4** $ab = c$

$\therefore b = \frac{c}{\ldots}$

Solve:

**5** $3x + 13 = 22$

**6** $6 - 5n = 21$

**7** $\frac{2m}{5} = 3$

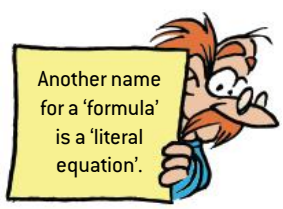
**8** $\frac{x - 3}{2} = 5$

Find $x$ if:

**9** $\sqrt{x} = 9$

**10** $x^2 = 9$

In Year 9 you were shown how to change the 'subject' of a formula to a specified variable. This was the same as 'solving' the literal equation for that variable.




For example, for $v = u + at$ we could say:
'Change the subject of this formula to $a$.'
or 'Solve this literal equation for $a$.'

In each case, the answer is $a = \frac{v - u}{t}$.

A pronumeral is a letter or symbol that represents a variable.

Follow these steps when solving a literal equation.

1. Remove fractions.
2. Expand grouping symbols.
3. Use inverse operations to isolate the pronumeral required.
4. If the required pronumeral appears in more than one term in the equation, gather the terms together and factorise.

## WORKED EXAMPLES

Solve each equation for $a$.

**1** $v^2 = u^2 + 2aS$

$v^2 - u^2 = 2aS$

$\dfrac{v^2 - u^2}{2S} = a$

$\therefore\ a = \dfrac{v^2 - u^2}{2S}$

**2** $R = \sqrt{\dfrac{ax}{b}}$

$R^2 = \dfrac{ax}{b}$

$bR^2 = ax$

$\dfrac{bR^2}{x} = a$

$\therefore\ a = \dfrac{bR^2}{x}$

**3** $y = \dfrac{a}{a+2}$

$y(a+2) = a$

$ay + 2y = a$

$2y = a - ay$

$2y = a(1 - y)$

$\dfrac{2y}{1-y} = a$

$\therefore\ a = \dfrac{2y}{1-y}$

Inverse operations

$+ \leftrightarrow -$

$\times \leftrightarrow \div$

$(\ )^2 \leftrightarrow \sqrt{\ }$

**4** $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$ Sometimes a different set of steps might be followed to arrive at the same correct result.

**Method 1**

$\dfrac{1}{x} - \dfrac{1}{b} = \dfrac{1}{a}$ subtract $\dfrac{1}{b}$ from both sides

$\dfrac{b-x}{bx} = \dfrac{1}{a}$ rewrite LHS with a common denominator

$\dfrac{bx}{b-x} = \dfrac{a}{1}$ invert both sides

$\therefore\ a = \dfrac{bx}{b-x}$

**Method 2**

$\dfrac{1}{x} = \dfrac{b+a}{ab}$ rewrite LHS with a common denominator

$ab = x(b+a)$ cross-multiply

$ab = bx + ax$ expand grouping symbols

$ab - ax = bx$ subtract $ax$ from both sides

$a(b-x) = bx$ factorise LHS

$\therefore\ a = \dfrac{bx}{b-x}$ divide both sides by $(b-x)$

When using formulas or literal equations, we should be aware that there may be restrictions on the values a variable may take. Examine the following formulas and their restrictions.

| Formula | Restriction |
|---|---|
| $y = \sqrt{x-a}$ | $x \geq a$, because expressions under a square root sign cannot be negative.<br>$y \geq 0$, because $\sqrt{\ }$ by definition is positive. |
| $y = 5(x-a)^2$ | $y \geq 0$, because $(x-a)^2$ must be greater than or equal to zero, i.e. $(x-a)^2 \geq 0$. |
| $y = \dfrac{a}{1-r}$ | $r \neq 1$, because the denominator can't equal zero.<br>(The fraction would be undefined.) |

Also, there may be assumptions implicit in the formula that will restrict the values a variable can take. For example, in a formula such as $v = u + at$ where $t$ represents time, it is assumed that $t \geq 0$. (Assume that time is not negative.)

# Exercise 12:03

P Foundation worksheet 12:03
Literal equations

**1** Make $S$ the subject of each formula.

**a** $2S + k = 1$ **b** $a = b + cS$ **c** $n = aS - m$ **d** $Sx - y = z$ **e** $x = y - aS$

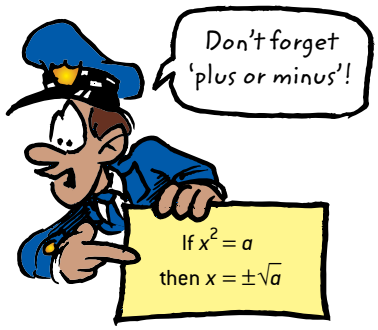
**f** $g = Sp - q$ **g** $a = \frac{b}{S}$ **h** $x = \frac{aS}{b}$ **i** $\frac{St}{u} = v$

**2** Expand the grouping symbols, then solve each equation for $x$.

**a** $3(x - a) = b$ **b** $5(m + x) = n$ **c** $a(x + p) = q$

**d** $4(2 - x) = y$ **e** $t(v - x) = w$ **f** $m = 3(x + 4)$

**g** $t = a(2 + x)$ **h** $v = u(1 + x)$ **i** $h = k(x + m)$




**3** Solve each literal equation for $x$.

**a** $mx^2 = n$ **b** $a = bx^2$ **c** $5x^2 = p$

**d** $x^2 - a = b$ **e** $t = x^2 + u$ **f** $w = v - x^2$

**g** $\frac{x^2}{a} = y$ **h** $z = \frac{x^2}{k}$ **i** $m = \frac{nx^2}{3}$

**4** Make $a$ the subject of each formula.

**a** $\sqrt{ab} = c$ **b** $t = \sqrt{5a}$ **c** $c = \sqrt{a - b}$

**d** $y = \sqrt{a + 5}$ **e** $r = \sqrt{b - a}$ **f** $m = n + \sqrt{a}$

**g** $p = \sqrt{a} - q$ **h** $x = y\sqrt{a}$ **i** $u = r\sqrt{a} + t$

**5** Solve each for $x$. (Rearrange to make $x$ the subject.)

**a** $x + a = b - x$ **b** $ax = px + q$

**c** $x + a = ax + b$ **d** $bx - c = e - dx$

**e** $\frac{x}{3} + \frac{x}{5} = a$ **f** $\frac{ax}{b} - \frac{bx}{4} = 1$

**g** $\frac{x}{x + 2} = a$ **h** $a = \frac{x + 2}{x - 3}$

Gather the $x$ terms together and factorise, taking $x$ out as a common factor.

**6** Change the subject of each formula to the letter shown in brackets.

| | A | | B | | C | |
|---|---|---|---|---|---|---|
| **a** | $P = xy$ | $[x]$ | $p = q + r$ | $[q]$ | $P = Q - R$ | $[R]$ |
| **b** | $b = \frac{xy}{a}$ | $[y]$ | $A = \frac{3r}{s}$ | $[r]$ | $n = \frac{2a}{m}$ | $[m]$ |
| **c** | $a = 3x + y$ | $[x]$ | $m = n + at$ | $[a]$ | $v^2 = u^2 + 2as$ | $[a]$ |
| **d** | $x = 5(a + b)$ | $[b]$ | $m = 3(n - m)$ | $[n]$ | $P = Q(r + t)$ | $[t]$ |
| **e** | $m = a + n^2$ | $[n]$ | $t = v^2 - u^2$ | $[u]$ | $A = x^2 + y^2$ | $[x]$ |
| **f** | $a = \sqrt{bc}$ | $[c]$ | $Y = a\sqrt{x}$ | $[x]$ | $x = \sqrt{a + y}$ | $[a]$ |
| **g** | $x = \sqrt{\frac{Y}{a}}$ | $[Y]$ | $R = \sqrt{\frac{ax}{b}}$ | $[x]$ | $V = \frac{at^2}{2}$ | $[a]$ |
| **h** | $y = \frac{a}{3} + \frac{b}{2}$ | $[a]$ | $x = \frac{m}{5} - \frac{n}{3}$ | $[m]$ | $\frac{A + x}{3} = \frac{A + y}{2}$ | $[A]$ |
| **i** | $m = \sqrt{\frac{a + b}{n}}$ | $[b]$ | $h = \frac{k}{2k + 1}$ | $[k]$ | $T = \frac{1}{a}\sqrt{\frac{m}{n}}$ | $[m]$ |

**7** In these formulas, what values can $x$ possibly take?

**a** $y = \sqrt{x-4}$ **b** $A = \sqrt{x+3}$ **c** $M = \sqrt{x-N}$

**d** $y = \sqrt{4-x}$ **e** $K = \sqrt{10-x}$ **f** $Z = \sqrt{Y-x}$

**g** $y = \sqrt{3x+1}$ **h** $A = \sqrt{5x-2}$ **i** $P = \sqrt{Q-2x}$

**j** What is the smallest value that each of the subjects of these formulas can have?

**8** What value (or values) can $r$ *not* take in these equations?

**a** $A = \dfrac{m}{r-3}$ **b** $M = \dfrac{3}{10+r}$ **c** $S = \dfrac{1+r}{1-r}$

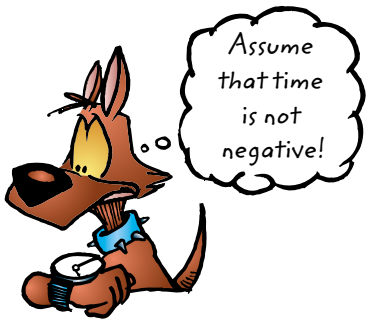
**d** $P = \dfrac{a}{r-q}$ **e** $K = \dfrac{5}{r+s}$ **f** $Y = \dfrac{a+r}{a-r}$

**g** $M = \dfrac{1}{(r-2)(r+2)}$ **h** $Z = \dfrac{Y}{r^2-16}$ **i** $R = \dfrac{3}{r^2-3r+2}$

**9** The equation $H = 10 + 9t - t^2$ gives the height $H$ of a ball above the ground at a time $t$.

**a** At what height above the ground did the ball start? (i.e. $t = 0$)

**b** At what time will the ball hit the ground? (i.e. $H = 0$)




**10** **a** Mr Saines earns \$350 plus \$20 commission on each item $I$ that he sells. His wage $W$ could be given by the formula: $W = 350 + 20I$

What is the smallest possible value for $I$ and hence the minimum wage, $W$?

**b** The surface area of a sphere is given by $S = 4\pi r^2$, where $r$ is the radius of the sphere.

What can be said about the value of $r$?

Hence, change the subject of the formula to $r$.

**c** The area of a rectangle is given by $A = LB$.

**i** Change the subject of the formula to $B$.

**ii** What can be said about the value of $L$?

**iii** For a given area $A$, if the value of $L$ increases, what will happen to $B$?

**d** The time $T$ taken by a pendulum for one swing is given by $T = 2\pi\sqrt{\dfrac{l}{g}}$, where $g = 9{\cdot}8$ m/s$^2$ and $l$ is the length of the string in metres.

**i** What can be said about the value of $l$?

**ii** What will happen to $T$ as the value of $l$ increases?

**iii** Solve this literal equation for $l$.

This giant pendulum swings in an office building in Vancouver, Canada.

## FUN SPOT 12:03 WHAT SMALL RIVERS FLOW INTO THE NILE?

Work out the answer to each question and write the letter for that part in the box that is above the correct answer.

*Solve these equations for $y$.*

| | | |
|---|---|---|
| I $2x + y = 5$ | E $4x - 2y = 7$ | S $2x = 5 + y$ |
| L $\frac{x + 3y}{2} = 4$ | J $\frac{x}{3} + \frac{y}{2} = 1$ | E $2(x + 1) = 5(y - 2)$ |
| V $x = \frac{y}{y + 1}$ | N $2x = \frac{y + 2}{y - 2}$ | U $\frac{y + 3}{x + 1} = \frac{5 + y}{-5 - y}$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| $y = \frac{6 - 2x}{3}$ | $y = -x - 4$ | $y = \frac{x}{1 - x}$ | $y = \frac{4x - 7}{2}$ | $y = \frac{4x + 2}{2x - 1}$ | $y = 5 - 2x$ | $y = \frac{8 - x}{3}$ | $y = \frac{2x + 12}{5}$ | $y = 2x - 5$ |

## CHALLENGE 12:03 FIBONACCI FORMULA

The interesting set of numbers below is known as the Fibonacci sequence.

1, 1, 2, 3, 5, 8, 13, 21, 34 …

Library research:
Who was Fibonacci?
What was his interest in these numbers?

**a** The first nine terms are given; what are the next three terms?

**b** This sequence has many curious properties. One curiosity is that the ratio of successive terms gets closer to a particular value.
To find this value, evaluate the ratios below correct to four decimal places for the first 12 Fibonacci numbers.

$\frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21} \cdots$

The ratios of the terms are approaching the decimal value of the number $T = \frac{1 + \sqrt{5}}{2}$.

This number is usually denoted by the Greek letter $\varphi$ (phi). $\varphi$: 1 is called the golden ratio.

**c** It is easy to see that each term is being generated by adding the previous two together.
We could represent this by the statement:

$F_n = F_{n-1} + F_{n-2}$

A formula that will calculate the $n$th Fibonacci number is reasonably complex.
Curiously, a fairly simple formula contains the expression for $\varphi$ given above.

The value of $F_n$ is given by the nearest integer to the expression: $\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n$

Use this expression to verify the values of $F_8$ and $F_{12}$ above.
Then use the expression to find:

i $F_{15}$ ii $F_{20}$ iii $F_{30}$

## MATHS TERMS 12

**cubic equation**

- an equation in which the highest power is three
- $x^3 = 8$ and $3x^3 - 12 = 0$ are simple cubic equations

**formula (plural: formulas)**

- a formula represents a relationship between physical quantities
- it will always have more than one pronumeral,
  e.g. $A = l \times b$ represents the relationship between the area ($A$) of a rectangle and its length ($l$) and breadth ($b$)

**literal equation**

- another name for a formula
- it will always have more than one pronumeral,
  e.g. $v = u + at$

**quadratic equation**

- an equation in which the highest power of the unknown pronumeral is 2,
  e.g. $x^2 - 16 = 0$, $x^2 + 5x + 6 = 0$
- a quadratic equation may have two solutions

**simultaneous equations**

- two or more equations that have one or more identical (simultaneous) solutions
- the equations can be solved to find the simultaneous solution or solutions,
  e.g. the equations $x + y = 10$ and $x - y = 6$ have many solutions but the only simultaneous solution is $x = 8$ and $y = 2$

**subject**

- the subject of a formula is the pronumeral by itself, on the left-hand side,
  e.g. in the formula $v = u + at$, the subject is $v$

**substitution**

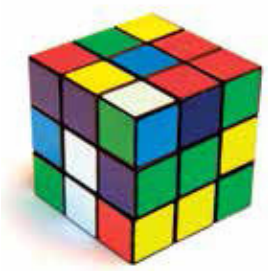
- the replacement of a pronumeral with a numeral or another variable in a formula or expression,
  e.g. to substitute 3 for $a$ in the expression $4a - 2$ would give:
  $4(3) - 2 = 12 - 2$
  $= 10$
  To substitute $(a + 3)$ for $x$ in the expression $2x + 1$ would give:
  $2(a + 3) + 1 = 2a + 6 + 1$
  $= 2a + 7$

**substitution method**

- solving simultaneous equations by substituting an equivalent expression for one pronumeral in terms of another, obtained from another equation,
  e.g. If $y = x + 3$ and $x + y = 7$, then the second equation could be written as $x + (x + 3) = 7$ by substituting for $y$ using the first equation

**variable**

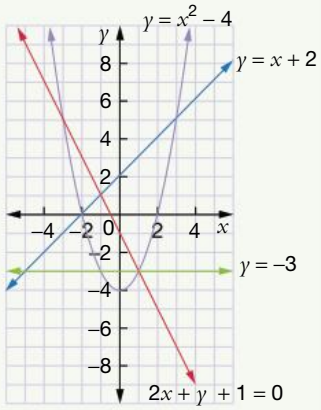
- another name for a pronumeral in an algebraic expression or equation
- variables can be given specific values in order to evaluate an expression



Rubik's cube is a famous mathematical puzzle. The squares on each face can be rotated so that each face is composed of nine squares of the same colour.

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** Use the diagram to solve these pairs of simultaneous equations. 12:01




a $y = x^2 - 4$ and $y = x + 2$
b $y = x^2 - 4$ and $y + 3 = 0$
c $y = x^2 - 4$ and $2x + y + 1 = 0$

**2** Solve each pair of simultaneous equations. 12:01

a $y = 7x + 8$, $y = x^2$
b $y = x + 10$, $y = x^2 - 10$
c $y = x^2 + 4x - 5$, $x - y = 5$

**3** Solve these cubic equations, correct to two decimal places if necessary. 12:02

a $x^3 - 27 = 0$
b $5a^3 = 40$
c $n^3 - 12 = 0$

**4** Solve these literal equations for $a$. 12:03

a $S = m - an$
b $P = \dfrac{a^2 + b^2}{2}$
c $K = n\sqrt{\dfrac{a}{p}}$
d $L = \dfrac{3a}{a - m}$

**5** In these formulas, what values can $x$ *not* take? 12:03

a $y = \sqrt{x - 1}$
b $M = \sqrt{5 + 2x}$
c $A = \dfrac{3x}{x - 4}$
d $P = \dfrac{2}{x^2 - 9}$

# ASSIGNMENT 12A Chapter review

**1** Solve these pairs of simultaneous equations.

**a** $y = x + 6$
$y = x^2 + 6x$

**b** $y = 2x(x + 4)$
$3x - y + 3 = 0$

**c** $y = x^2 - 4x + 3$
$y = 3x - x^2$

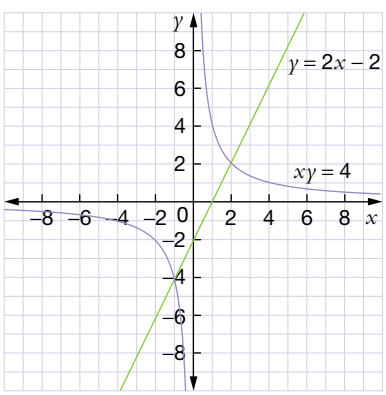
**2** **a** Find the point where the two parabolas, $y = x^2 - 4x$ and $y = x^2 - 8x + 12$, intersect.

**b** Sketch these parabolas on the same number plane, labelling this point of intersection.

**3** **a** The diagram shows the graphs of $xy = 4$ and $y = 2x - 2$. What are the simultaneous solutions for the equations $xy = 4$ and $y = 2x - 2$?

**b** Find algebraically the simultaneous solutions to the equations $xy = 4$ and $x + 2y - 6 = 0$.

**c** Using your answers to parts **a** and **b**, find the solution to the simultaneous equations $y = 2x - 2$ and $x + 2y - 6 = 0$.




**4** Solve these equations, giving the answers correct to 3 dec. pl.

**a** $8x - 3 = 0$

**b** $8x^2 - 3 = 0$

**c** $8x^3 - 3 = 0$

**5** Solve these equations.

**a** $y + 64 = 0$

**b** $y^2 + 64 = 0$

**c** $y^3 + 64 = 0$

**6** Solve these literal equations for the variable indicated.

**a** $P = ab - bc$ $[a]$

**b** $P = ab - bc$ $[c]$

**c** $P = ab - bc$ $[b]$

**d** $h = \sqrt{a^2 + b^2}$ $[a]$

**e** $Z = \dfrac{p + q}{p - q}$ $[p]$

**7** What values can $a$ take in these expressions?

**a** $\sqrt{a - 4}$ **b** $\sqrt{4 - a}$

**c** $\sqrt{a + 4}$ **d** $\sqrt{4a}$

**8** What values can $n$ *not* take in these expressions?

**a** $\dfrac{1}{n - 3}$ **b** $\dfrac{1}{n + 3}$

**c** $\dfrac{1}{3n}$ **d** $\dfrac{1}{n^2 - 9}$

**9** The equation $h = 60t - 10t^2$ gives the height, in metres, of a ball above the ground at time $t$ seconds.

**a** Find when $h = 0$, and therefore the time taken for the ball to return to the ground.

**b** Determine the maximum height of the ball. (Hint: Find the vertex of the parabola.)

**c** Hence, what are the possible values for $h$ and $t$ in this formula?

**10** Rearrange this formula to make $Z$ the subject.

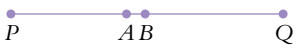
$$X = \frac{5}{Z^2 + 4Z + 4}$$

## ASSIGNMENT 12B Working mathematically

**1** Natasha had a large bag of lollies. After counting them, she realised that when the total was divided by 2 or 3 or 4 or 5 or 6 there was a remainder of one each time, but when the total was divided by 7 there was no remainder. What is the smallest number of lollies that could have been in the bag?

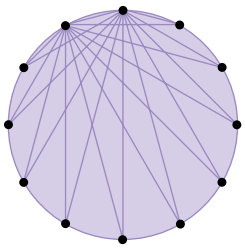
**2** A ball is known to rebound to half the height that it drops. If the ball is initially dropped from a height of 12 m, how far will it have travelled by the time it hits the ground for the 6th time?

**3** The distance from $P$ to $Q$ is 36 cm. How far is the point $A$ from $P$ if it is four times as far from $Q$ as it is from $B$, the midpoint of $PQ$?

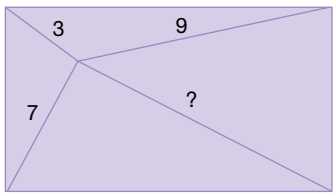



**4** A snail begins to climb a wall. Every hour it manages to climb up 30 cm but it gets tired and slides back 10 cm. How long will it take for the snail to climb up the wall if the wall is 2·98 m high?

**5** The partly completed diagram shows how two of the points on the circle have been connected to ten other points around the circle by straight line segments. If every point has to be connected to every other point, how many straight line segments will there be?



**6** A point inside a rectangle is 3 units from one corner, 7 units from another and 9 units from another. How far is the point from the fourth corner?



Does this car go further on a litre of petrol?

# ASSIGNMENT 12C Cumulative revision

**1** Solve these equations. 1:06, 2:01, 2:06

**a** $\frac{x-3}{2} - \frac{x+1}{5} = 7$ **b** $6x^2 - 13x - 5 = 0$ **c** $x^4 - 37x^2 + 36 = 0$

**2** Simplify: 5:03, 5:04, 5:05

**a** $\sqrt{32} + 2\sqrt{18} - 4\sqrt{8}$ **b** $3\sqrt{6} \times 2\sqrt{12}$ **c** $(5\sqrt{3} - 3\sqrt{5})(5\sqrt{3} + 3\sqrt{5})$

**3** Sketch the graph of the parabola $y = 7 + 6x - x^2$ and answer the following. 7:05

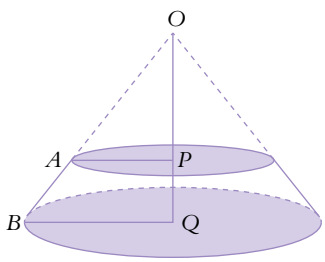
**a** What are the $x$-intercepts?

**b** Where does it cut the $y$-axis?

**c** What are the coordinates of the vertex?

**d** What is the minimum or maximum value of $7 + 6x - x^2$?

**4** Determine the volume *and* surface area of this solid, Given that $OP = PQ = 8$ cm, $AP = 6$ cm and $BQ = 12$ cm. (Answer correct to one decimal place.) 8:03, 8:06




**5** A class of 15 students received the following scores for a test. 9:02

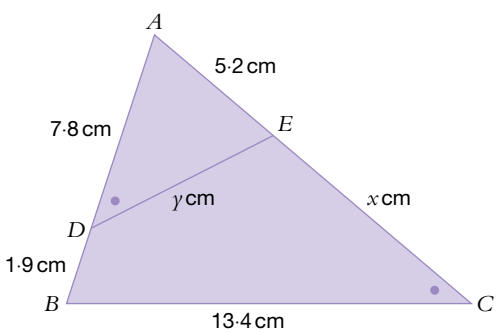
72 67 81 84 59 42 66 76 62 73 75 53 89 48 63

**a** Calculate the mean and standard deviation for these scores. (Answer correct to one decimal place.)

**b** Jackie was away for the test and so completed it later. What must she score for the class mean to be exactly 68?

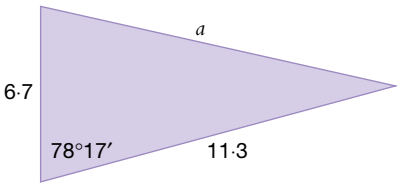
**c** If she achieves exactly this score, how will it affect the standard deviation?

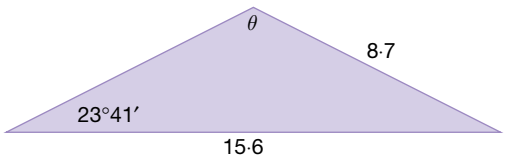
**6** In the diagram shown, $\angle ADE = \angle ACB$. Prove that $\Delta ABC \,|||\, \Delta AED$ and hence find the values of $x$ and $y$. 10:01



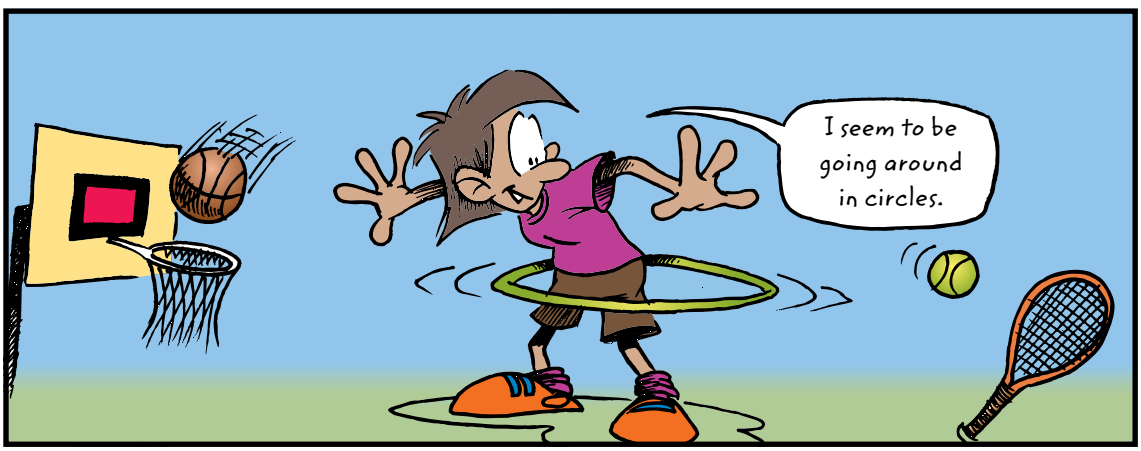

**7** Determine the unknown pronumerals in these triangles. Answer correct to: 11:03

**a** one decimal place **b** the nearest minute.



# CIRCLE GEOMETRY 13




## Contents


**13:01** Circles
- Investigation 13:01 Circles in space
- GeoGebra activity 13:01 Terms used in circle geometry

**13:02** Chord properties of circles (1)
- GeoGebra activity 13:02 Chord properties
- Investigation 13:02 Locating the epicentre of earthquakes

**13:03** Chord properties of circles (2)

**13:04** Angle properties of circles (1)
- GeoGebra activity 13:04 Angle properties (1)

**13:05** Angle properties of circles (2)
- GeoGebra activity 13:05 Angle properties (2)
- Investigation 13:05 Diameter of a circumcircle

**13:06** Tangent properties of circles

**13:07** Further circle properties
- GeoGebra activity 13:07 Intersecting chords and secants
- Fun spot 13:07 How do you make a bus stop?

**13:08** Deductive exercises involving the circle
- Fun spot 13:08 How many sections?

Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Measurement and Geometry**

Selections from ***Circle Geometry*** [Stage 5.3#]

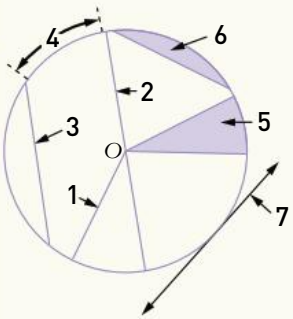
- Prove and apply angle and chord properties of circles (ACMMG272)
- Prove and apply tangent and secant properties of circles (NSW)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 13:01 Circles

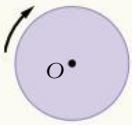
## PREP QUIZ 13:01




*O* is the centre.

For **1** to **9**, give the name of that part.

**8**




The distance around the circle.

**9**

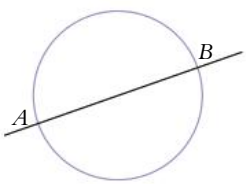
**10**

How many sectors are shown?

The Prep quiz above has reviewed the terms associated with circles with which you should be familiar. In later sections we will investigate some relationships between angles in circles. Before we can do this we need to define some new terms.

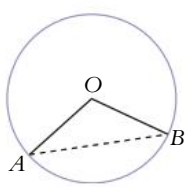
**Secant**

A line that intersects a curve in two places.



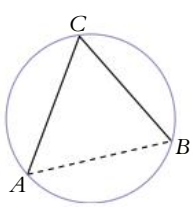

**Angle at the centre**

An angle formed by joining the ends of an arc or chord to the centre of a circle. (We say $\angle AOB$ is an angle at the centre standing on the arc or chord $AB$.)



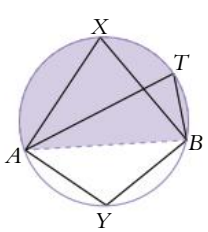

**Angle at the circumference**

An angle formed by joining the ends of an arc or chord to another point on the circumference. (We say $\angle ACB$ is an angle at the circumference standing on the arc or chord $AB$.)



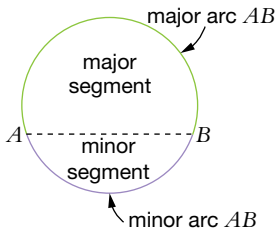

**Angles in the same segment**

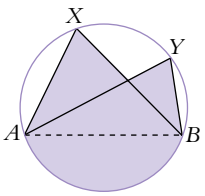
A chord divides a circle into two segments. The larger segment is called the **major segment** and the smaller is called the **minor segment**. In the diagram, $\angle AXB$ is in the major segment and $\angle AYB$ is in the minor segment. Angles $AXB$ and $ATB$ are in the same segment.



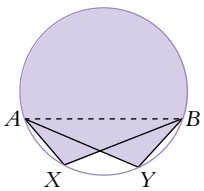
### Angles standing on the same arc

Two points $A$ and $B$ divide a circle into two arcs. The larger arc is called the **major arc** and the smaller is called the **minor arc.** When the ends of an arc or chord are joined to two different points in the same segment, the angles are said to be standing on the same arc.

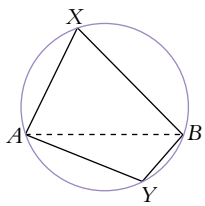



$\angle$s $AXB$ and $AYB$ are standing on the minor arc $AB$.



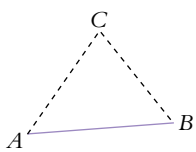

$\angle$s $AXB$ and $AYB$ are standing on the major arc $AB$.




$\angle$s $AXB$ and $AYB$ are not standing on the same arc.

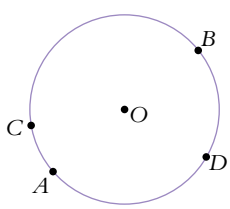
### Subtend

If the ends of an interval $AB$ are joined to a point $C$, the angle formed ($\angle ACB$) is the angle subtended at $C$ by the interval $AB$. In circle geometry, we speak about angles subtended at the circumference by the arc (or chord) $AB$ or angles subtended at the centre by the arc (or chord) $AB$.



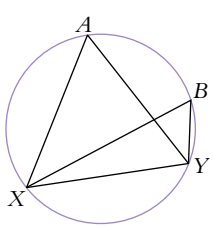

## Exercise 13:01

**1** Copy the diagram and show on separate drawings:




- **a** the angle at the centre standing on the arc $AC$
- **b** the angle at the centre subtended by the arc $AD$
- **c** the angle subtended at $D$ by the arc $BC$
- **d** the angle at $C$ standing on the arc $AD$.

**2** **a** Name an angle at the circumference that is standing on the arc:

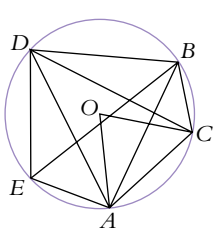



**i** $AX$ **ii** $BY$

**b** Name two angles at the circumference that are standing on the arc:

**i** $XY$ **ii** $AB$

**3** **a** Which two angles are standing on the arc $EA$?




- **b** Which three angles are standing on the arc $AC$?
- **c** How many angles does the arc $DB$ subtend at the circumference?
- **d** The chord $DA$ divides the circle into minor and major segments. Name the angle in the minor segment.
- **e** Only one angle is subtended at the centre. On which chord is it standing?

**4** **a** Name two angles standing on:

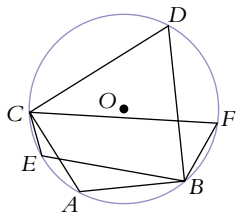
**i** the minor arc $BC$

**ii** the major arc $BC$.

**b** The chord $BC$ divides the circle into major and minor segments. Name two angles in:

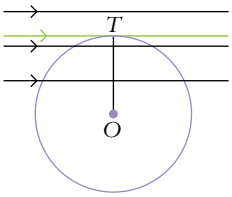
**i** the minor segment

**ii** the major segment.

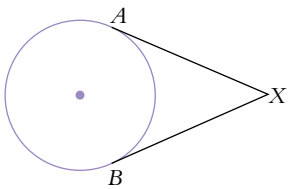



**5** The diagram shows a number of parallel lines and a circle. The tangent is coloured green. $T$ is the point of contact (where the tangent and the circle meet) and $O$ is the centre of the circle.

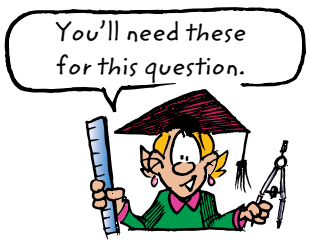
**a** Use the diagram to complete the following. The angle between a tangent and the radius drawn to the point of contact is …



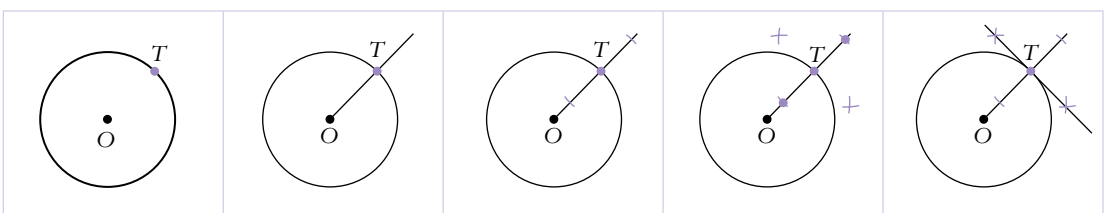

**b** Draw a circle and mark a point, $X$, outside the circle. Use a ruler to draw two tangents to the circle through $X$. Join the points of contact to the centre and measure the angles between the tangents and the radii. Are they right angles?



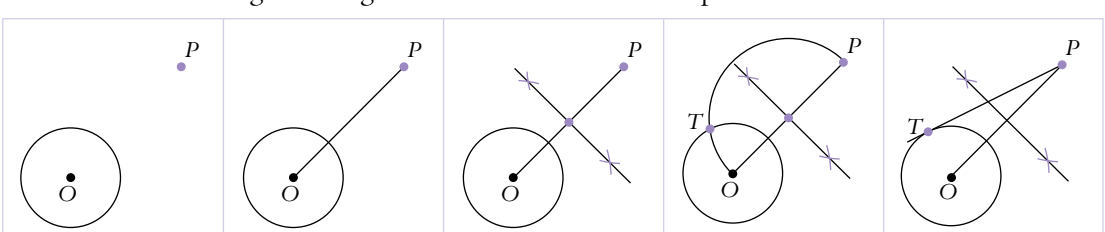

**6** Use a ruler and compass to perform the following constructions involving tangents.



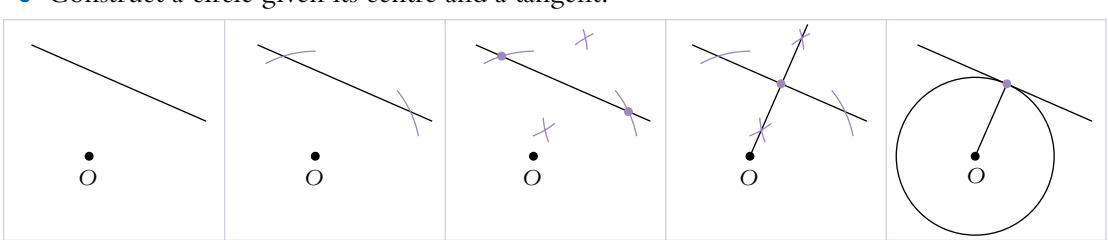

**a** Construct a tangent to a circle at a given point on the circle.




**b** Construct the tangent to a given circle from an external point.

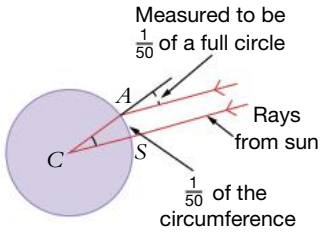



**c** Construct a circle given its centre and a tangent.



## INVESTIGATION 13:01 CIRCLES IN SPACE

The properties of circles have been used to discover measurements related to the Earth and the Moon.



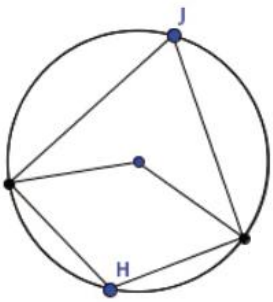

- *Eratosthenes*, in 200 BCE, noticed that the Sun was directly overhead at a certain time in Syene in Egypt. At exactly the same time in Alexandria, which was due north of Syene, the Sun was about $\frac{1}{50}$ of a circle south of the zenith (i.e. $\frac{1}{50}$ of a full circle away from being directly overhead). He reasoned therefore that the distance between the cities must be about $\frac{1}{50}$ of the Earth's circumference.
- *Aristarchus* devised an ingenious method for determining the ratio between the distance to the Moon and the radius of the Earth using the average duration of a lunar eclipse and the length of the month. A simple property of circles was used in his derivation, namely that the arc of a circle subtended by an angle at its centre is proportional to the radius of the circle. That is, if you double the radius, you double the arc length.
- Aristarchus' method, involving simple triangle and circle geometry, resulted in an estimate for the Moon's distance from Earth of 80 Earth radii. The actual distance is in fact about 60 Earth radii.

**Modern measurements of astronomical distances**

| Measure | Measurement (km) |
|---|---|
| Radius of the Sun | 695 000 |
| Radius of the Earth | 6378 |
| Radius of the Moon | 1738 |
| Earth to the Sun (centre to centre, average distance) | 149 594 000 |
| Earth to the Moon (centre to centre, average distance) | 384 393 |

## GEOGEBRA ACTIVITY 13:01 TERMS USED IN CIRCLE GEOMETRY

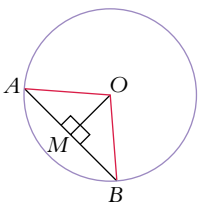
Click and drag points to learn about major and minor arcs and segments as well as angles standing on arcs or angles at the centre. Then choose the exercise to test your knowledge of angles at the centre standing on various arcs.




Angles can stand on either a major arc or a minor arc.
Angles are said to be subtended by an arc.

# 13:02 Chord properties of circles (1)

A perpendicular drawn to a chord from the centre of a circle bisects the chord, and the perpendicular bisector of a chord passes through the centre.




⊥ means 'is perpendicular to'.

We can use congruent triangles to prove this result.

Prove that $AM = MB$ if $O$ is the centre and $OM$ is perpendicular to the chord $AB$.

Construction: Draw in $OA$ and $OB$.

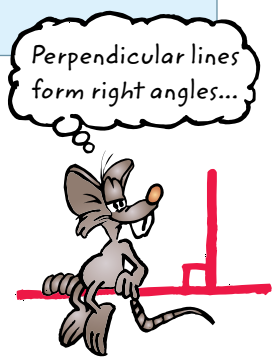
Proof: In Δs $OAM$ and $OBM$:

**1** $\angle AMO = \angle BMO = 90°$ ($OM \perp AB$)

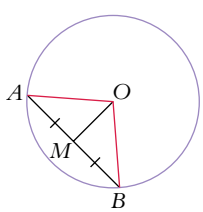
**2** $AO = BO$ (radii of the circle)

**3** $OM$ is common.
$\therefore \Delta OAM \equiv \Delta OBM$ (RHS)
$\therefore AM = MB$ (matching sides of congruent Δs)

$\therefore$ The perpendicular bisects the chord. QED
There is only one perpendicular bisector of a chord, so it must be the one that passes through the centre.




The line from the centre of a circle to the midpoint of the chord meets the chord at right angles.




*Note:* Unless otherwise stated, *O* will be the centre of the circle.

We can use congruent triangles to prove this result.

Data: $O$ is the centre of the circle and $M$ is the midpoint of the chord $AB$.

Aim: To prove that $OM$ is perpendicular to $AB$.

Construction: Draw $OA$ and $OB$.

Proof: In Δs $OAM$ and $OBM$:

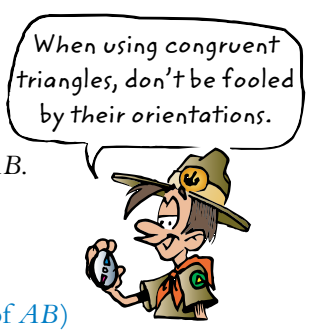
**1** $AM = MB$ ($M$ is the midpoint of $AB$)

**2** $AO = BO$ (radii of the circle)

**3** $OM$ is common.
$\therefore \Delta OAM \equiv \Delta OBM$ (SSS)
$\therefore \angle OMA = \angle OMB$ (matching angles of congruent Δs)

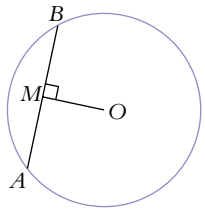
But $\angle OMA + \angle OMB = 180°$ (adjacent angles on a straight line)
$\therefore \angle OMA = 90°$
$\therefore OM \perp AB$ QED

QED: *quod erat demonstrandum* 'which was to be demonstrated'



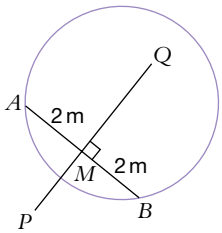
## WORKED EXAMPLES

**1**



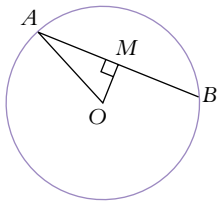

$AB = 15$ cm
Find the length of $MB$, giving reasons.

**2**




Give reasons why $PQ$ must pass through the centre of the circle.

**3**




$AO = 26$ cm, $OM = 10$ cm
Find the length of $AB$, giving reasons.

### Solutions

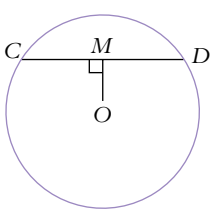
**1** $MB = \frac{1}{2}$ of $AB$
(the perpendicular from $O$ bisects chord $AB$)
$\therefore\ MB = 7{\cdot}5$ cm

**2** $PQ$ passes through the centre because it is the perpendicular bisector of chord $AB$.

**3** $OA^2 = AM^2 + OM^2$
(Pythagoras' theorem)
$26^2 = AM^2 + 10^2$
$\therefore\ AM^2 = 576$
$\therefore\ AM = \sqrt{576}$
$= 24$ cm
Now $AB = 2 \times AM$
($OM$ is the perpendicular bisector of $AB$)
$\therefore\ AB = 48$ cm

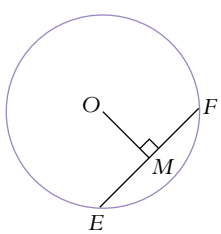
## Exercise 13:02

**1 a**



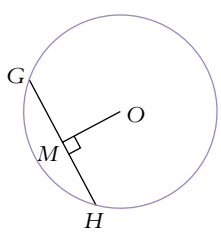

$CD = 35$ cm
Find the length of $MD$, giving reasons.

**b**



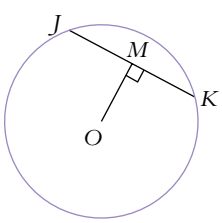

$EM = 27$ cm
Find the length of $MF$, giving reasons.

**c**



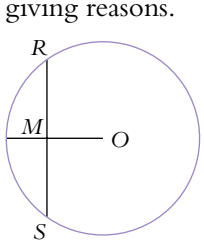

$GM = 18$ cm
Find the length of $GH$, giving reasons.

**d**



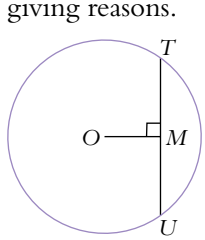

$JM = 9$ cm
Find the length of $MK$ and $JK$.

**e**




$MS = MR$
Find the size of $\angle RMO$.

**f**

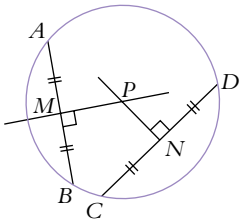



$TU = 18{\cdot}6$ cm
Find the length of $TM$ and $MU$.

**2** **a** **i** Give reasons why $MP$ passes through the centre of the circle.

**ii** Give reasons why $NP$ passes through the centre of the circle.

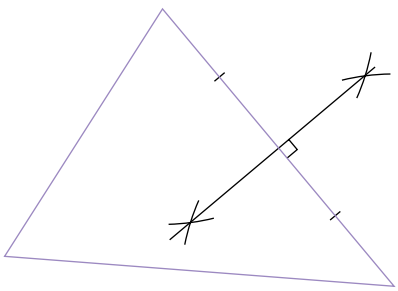
**iii** Which point is the centre here? Why?




**b** Draw any circle. Use the method in **2a** to find the centre of your circle.

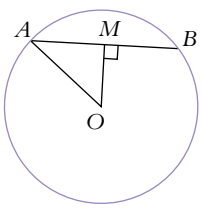
**c** The same method can be used to draw a circle that passes through any three non-collinear points. Choose any three non-collinear points, and by constructing two perpendicular bisectors, locate the centre and then draw the circle that passes through these points.

**3** Draw any triangle and then construct the perpendicular bisectors of each side. These lines should be concurrent. The point of intersection is called the circumcentre. A circle (the circumcircle) can be drawn, with this point as its centre, that will pass through the three vertices of the triangle.



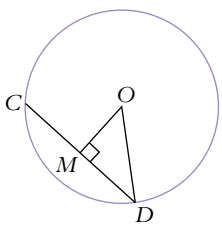
**4** Use Pythagoras' theorem in each of the following. Answer correct to one decimal place.

**a**



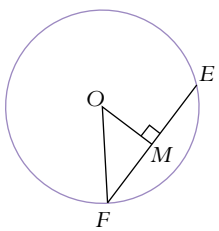

$OM \perp AB$, $AO = 5$ cm, $OM = 4$ cm
Find the length of $AM$ and $AB$.

**b**



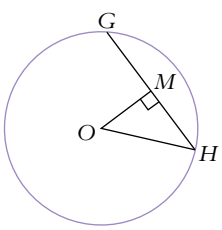

$OM \perp CD$, $MO = 6$ m, $OD = 8$ m
Find the length of $MD$ and $CD$.

**c**



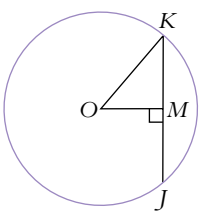

$OM \perp EF$, $OM = 12$ mm, $OF = 13$ mm
Find the length of $FM$ and $FE$.

**d**



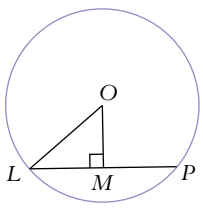

$OM \perp GH$, $MH = 8$ cm, $OM = 6$ cm
Find the length of $OH$ and $GH$.

**e**




$OM \perp JK$, $KJ = 14$ cm, $OM = 3$ m
Find the length of $KM$ and $OK$.

**f**



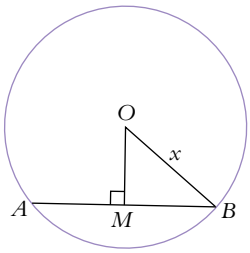

$OM \perp LP$, $OL = 10$ m, $LP = 18$ m
Find the length of $LM$ and $OM$.

**g** A chord of length 12 cm is drawn on a circle of radius 8 cm. How far is this chord from the centre of the circle?

**h** A chord of length 10 cm has a perpendicular distance of 4 cm from the centre of the circle. What is the radius of the circle?

**5** **a** Use trigonometry to find $x$ in each diagram, correct to one decimal place.

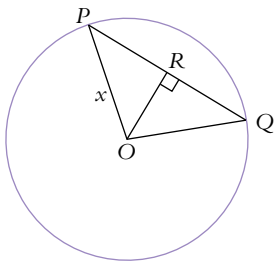
**i**




$\angle MOB = 50°$

$AB = 10$ cm

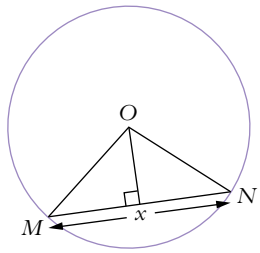
**ii**




$\angle POQ = 140°$

$PQ = 12{\cdot}6$ cm

**iii**



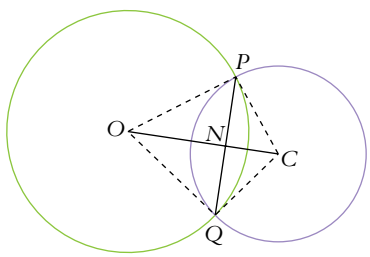

$\angle MON = 126°$

$OM = 7{\cdot}3$ cm

**b** Find the radius of a circle in which a chord of length 14 cm subtends an angle of 70° at the centre. Give the answer correct to one decimal place.

**c** A chord subtends an angle of 110° at the centre of a circle of radius 5·6 cm. Find the length of the chord correct to one decimal place.

**6** These two circles have as their centres points $O$ and $C$. $PQ$ is the common chord joining the points of intersection of the two circles. $N$ is the point where $PQ$ intersects the line $OC$, which joins the centres.




**a** Prove that the triangles $POC$ and $QOC$ are congruent.

**b** Hence, show that $\angle POC = \angle QOC$.

**c** Now, prove that the triangles $PON$ and $QON$ are congruent.

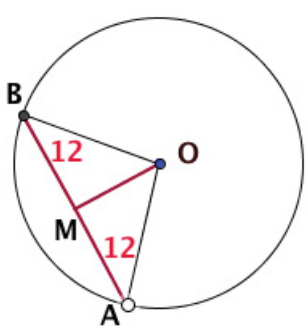
**d** Hence, show that $N$ bisects $PQ$ and that $PQ \perp OC$.

> When two circles intersect, the line joining their centres bisects their common chord at right angles.

## P GEOGEBRA ACTIVITY 13:02 CHORD PROPERTIES

Five different chord properties are investigated in this activity. By moving points and changing the length of chords you can investigate different chord properties.

You can also see how the chord properties can be used to draw a circle through any three non-collinear points.

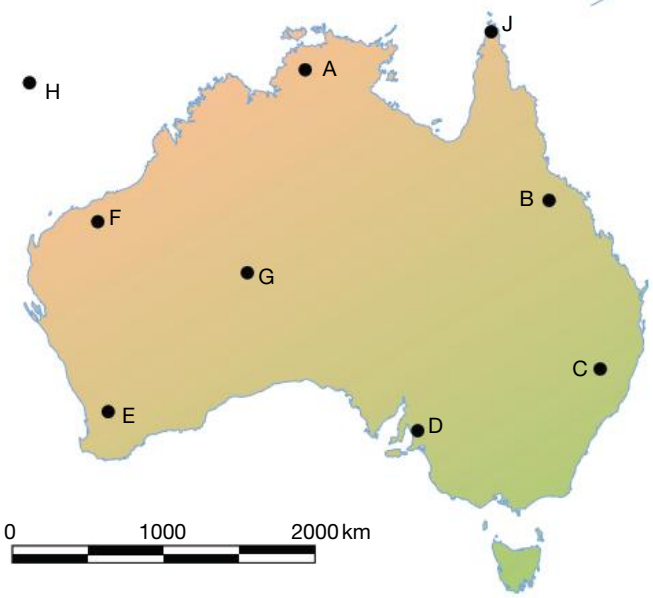



**The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.**

## INVESTIGATION 13:02 LOCATING THE EPICENTRE OF EARTHQUAKES

The epicentre of an earthquake can be located using three intersecting circles.

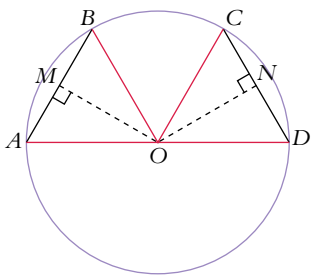
- At seismograph stations, the distance to the epicentre of earthquakes is calculated by examining the graphs of waves detected.
- The position of the epicentre can be determined by drawing circles with radii equal to the distances to the epicentre from three different stations.
- The point of intersection of the three circles is the epicentre.




- Use the scale of the map and the distances from stations given below, to find the point that is the epicentre of each earthquake.
  1. The epicentre is 1770 km from A, 1140 km from C and 3140 km from E.
  2. The epicentre is 1450 km from D, 1020 km from F and 1410 km from A.
  3. The epicentre is 1750 km from A, 4070 km from C and 2160 km from E.
  4. The epicentre is 2640 km from D, 2290 km from C and 2770 km from F.
- What is the Richter scale? Find out what you can about the scale and its use.

# 13:03 Chord properties of circles (2)

Equal chords of a circle are the same distance from the centre and subtend equal angles at the centre.




The altitude (height) of a triangle is perpendicular to the base. Each triangle has three possible altitudes.

Once again we use congruent triangles to prove these properties.

Data: $AB$ and $CD$ are equal chords of the same circle.

Aim: To prove that equal chords subtend equal angles at the centre of the circle and that these chords are the same distance from the centre.
($\angle AOB = \angle COD$ and $OM = ON$)

Proof: In $\Delta$s $ABO$ and $CDO$:

**1** $AB = CD$ (given)

**2** $OA = OC$ (radii of the circle)

**3** $OB = OD$ (radii of the circle)

$\therefore \Delta ABO \equiv \Delta CDO$ (SSS)

$\therefore \angle AOB = \angle COD$ (matching angles of congruent $\Delta$s)

$\therefore$ Equal chords subtend equal angles at the centre. QED

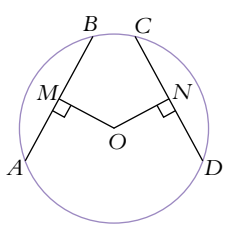
Now, $\Delta$s $ABO$ and $CDO$ are congruent (the same shape and size) so the height of each triangle (dotted line on the figure) must be the same.

$\therefore OM = ON$

$\therefore$ The chords are the same distance from the centre. QED

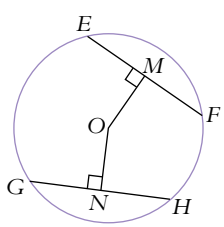
## WORKED EXAMPLES

**1**



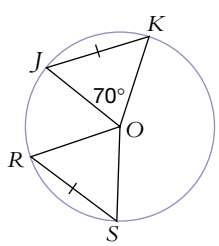

$AB = CD$, $OM = 6$ cm
Find the length of $ON$, giving reasons.

**2**




$OM = ON$, $EF = 13$ m
Find the length of $GH$, giving reasons.

**3**




$JK = RS$, $\angle JOK = 70°$
Find the size of $\angle ROS$, giving reasons.

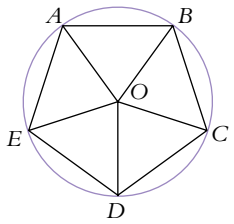
### Solutions

**1** $ON = 6$ cm, as equal chords of a circle are the same distance from the centre.

**2** $GH = 13$ m, as chords that are equidistant from the centre are equal in length.

**3** $\angle ROS = 70°$, as equal chords subtend equal angles at the centre of the circle.

To construct regular figures within a circle, construct the required number of equal sides by measuring equal angles at the centre of the circle.




The regular pentagon $ABCDE$ has been drawn within the circle.

As the angles at the centre are subtended by equal chords, the five angles are equal.
$\therefore$ 5 × angle size = 360° (angles at point)
$\therefore$ angle size = 72°

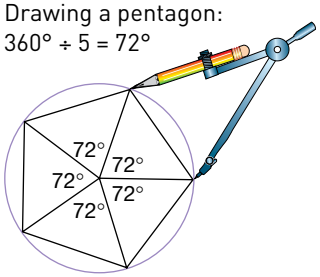
The term 'subtended by' means 'standing on'.

A regular figure has sides of equal length.

## Steps for constructing regular figures within a circle

- Divide 360° by the number of sides in the regular figure to find the size of the angle subtended at the centre by each side.
- Use a protractor to draw the angles at the centre and extend the arms until they meet the circle.
- Join these points on the circle to form the regular figure.

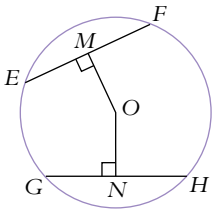
Drawing a pentagon:
360° ÷ 5 = 72°




A compass can be used when one chord length is known.

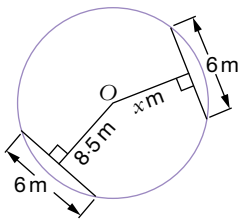
# Exercise 13:03

**1 a**



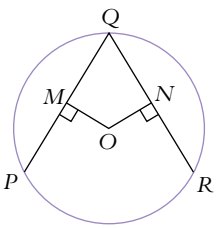

$EF = GH$, $ON = 8$ m
Find the length of $OM$, giving reasons.

**b**



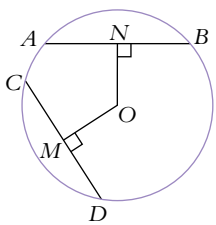

Find the value of $x$, giving reasons.

**c**



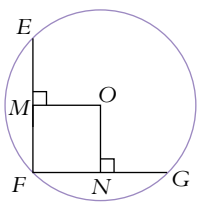

$PQ = QR = 15$ cm
$OM = 3{\cdot}5$ cm
Find the length of $ON$, giving reasons.

**d**



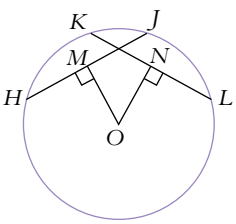

$OM = ON$, $AB = 11{\cdot}5$ m
Find the length of $CD$, giving reasons.

**e**



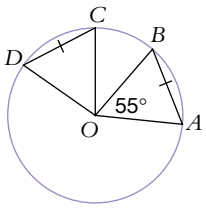

$OM = ON = 12$ m,
$EF = 20$ m
Find the length of $FG$, giving reasons.

**f**



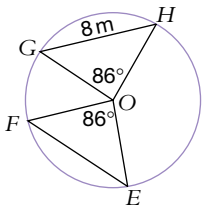

$OM = ON$, $HJ = 13{\cdot}8$ m
Find the length of $KL$, giving reasons.

**2** **a**



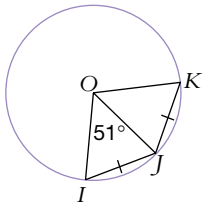

$AB = CD$, $\angle AOB = 55°$
Find the size of $\angle COD$, giving reasons.

**b**



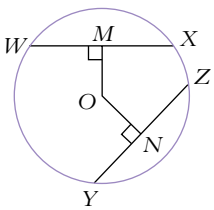

$\angle GOH = \angle FOE$,
$GH = 8$ m
Find the length of $FE$, giving reasons.

**c**



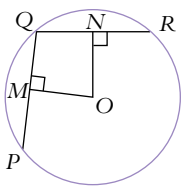

$IJ = JK$, $\angle IOJ = 51°$
Find the size of $\angle JOK$, giving reasons.

**d**



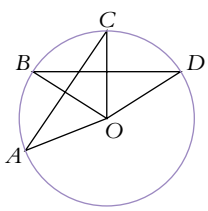

$WM = 6$ m, $YZ = 12$ m,
$OM = 5$ m
Find the length of $ON$, giving reasons.

**e**




$OM = ON$,
$QR = 14$ m
Find the length of $QM$, giving reasons.

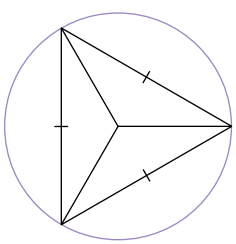
**f**



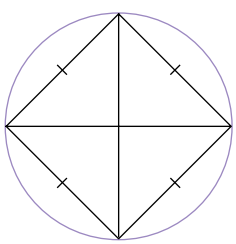

$AC = BD$, $\angle AOC = 125°$
Find the size of $\angle BOD$, giving reasons.

**3** Find the size of the angle subtended at the centre by one side of each of these regular figures.

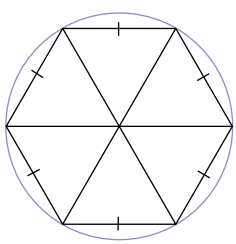
**a**



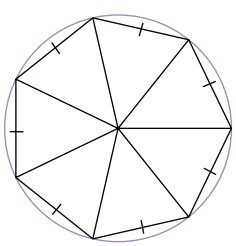
**b**



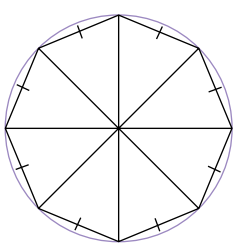
**c**



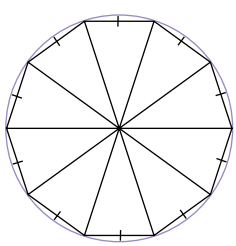
**d**



**e**



**f**

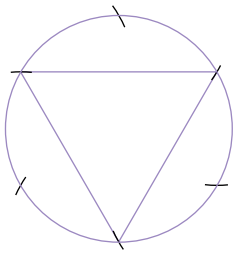


**4** In circles of radius 3 cm, construct:

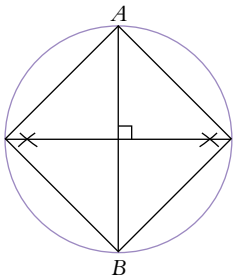
**a** an equilateral triangle
**b** a square
**c** a regular hexagon
**d** a regular octagon.

**5** Use a ruler and compass to copy the constructions of the following regular figures. $AB$ is a diameter in **b** and **d**.

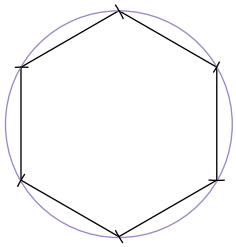
**a**



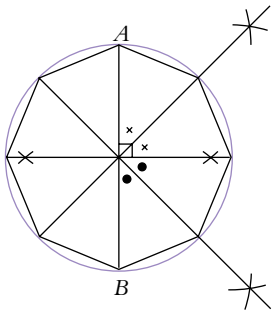
**b**

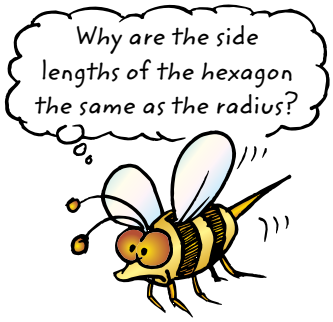



**c**



**d**



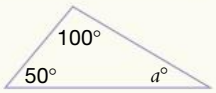




The method of constructing a regular pentagon with a ruler and compass is very complicated, and the construction of a regular heptagon (7 sides) and a regular nonagon (9 sides) with a ruler and compass is impossible.

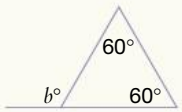
An arc illuminated on the circumference of the Earth.

# 13:04 Angle properties of circles (1)

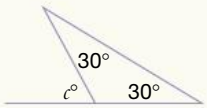
## PREP QUIZ 13:04

1 

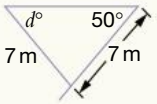
$a$ = …

2 

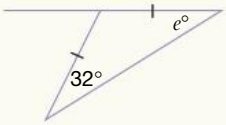
$b$ = …

3 

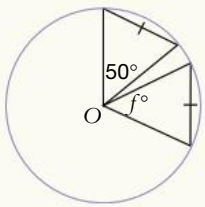
$c$ = …

4 

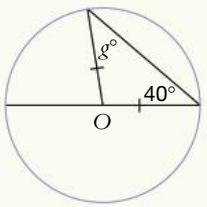
$d$ = …

5 

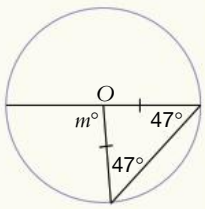
$e$ = …

6 

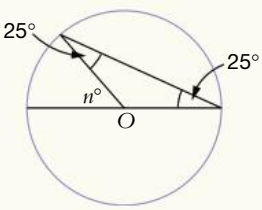
$f$ = …

7 

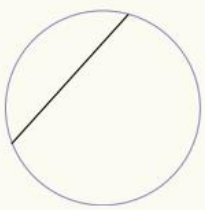
$g$ = …

8 

$m$ = …

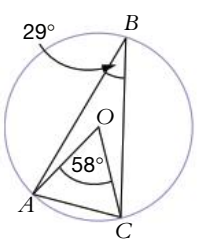
9 

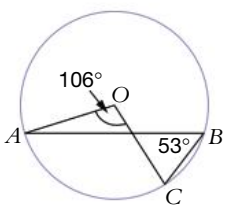
$n$ = …

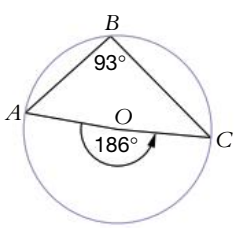
10 

Into how many segments does a chord divide a circle?

The angle subtended by an arc (or chord) at the centre of a circle is double the angle subtended by the arc (or chord) at the circumference.




More than one proof will be required in order to cover all possibilities in establishing the above result. (Refer to the following Figures 1, 2 and 3 on the following page.)

Data: $A$, $B$, $C$ and $P$ are points on the circumference of a circle with centre $O$. $\angle AOC$ is subtended at centre $O$ by arc $APC$, $\angle ABC$ is subtended at the circumference by arc $APC$. In Figure 2, $A$, $O$ and $B$ are collinear.

Aim: To prove that $\angle AOC = 2 \times \angle ABC$.

Collinear points lie in a straight line.

Construction: In Figures 1 and 3, join $BO$ and produce to $D$.

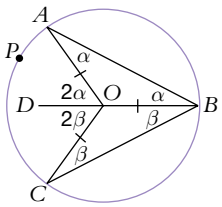


Figure 1

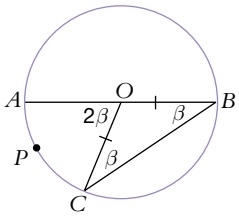


Figure 2

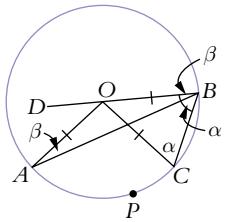


Figure 3

Proof:

In Figure 1:

Let $\angle OBA$ be equal to $\alpha$ and $\angle OBC$ be equal to $\beta$.

Now, $\Delta AOB$ is isosceles
($OA = OB$, radii of circle)
$\therefore \angle OAB = \alpha$ (base angles of isosceles $\Delta$)
$\therefore \angle AOD = \alpha + \alpha$ (exterior angle of $\Delta AOB$)
$= 2\alpha$

Similarly,
$\angle COD = 2\beta$
$\therefore \angle AOC = 2\alpha + 2\beta$
$= 2(\alpha + \beta)$

and $\angle ABC = \alpha + \beta$
$\therefore \angle AOC = 2 \times \angle ABC$

In Figure 3:

Let $\angle OBC$ be equal to $\alpha$ and $\angle OBA$ be equal to $\beta$.

Now, $\Delta AOB$ is isosceles
($OB = OC$, radii of circle)
$\therefore \angle OCB = \alpha$ (base angles of isosceles $\Delta$)
$\therefore \angle DOC = 2\alpha$ (exterior angle of $\Delta OBC$)

Similarly,
$\angle DOA = 2\beta$
$\therefore \angle AOC = \angle DOC - \angle DOA$
$= 2\alpha - 2\beta$
$= 2(\alpha - \beta)$
$\angle ABC = \angle OBC - \angle OBA$
$= \alpha - \beta$
$\therefore \angle AOC = 2 \times \angle ABC$

In Figure 2:

Let $\angle ABC$ be equal to $\beta$.

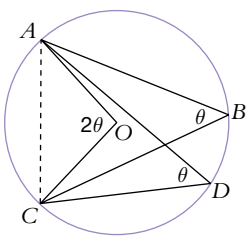
Now, $\Delta BOC$ is isosceles ($OB = OC$, radii of circle)
$\therefore \angle OCB = \beta$ (base angle of isosceles $\Delta$)
$\therefore \angle AOC = 2\beta$ (exterior angle of $\Delta BOC$)
$\therefore \angle AOC = 2 \times \angle ABC$

$\therefore$ The angle subtended at the centre is twice the angle subtended at the circumference. QED

> Angles subtended at the circumference by the same or equal arcs (or chords) are equal.

The angle subtended at the centre is twice the angle subtended at the circumference:

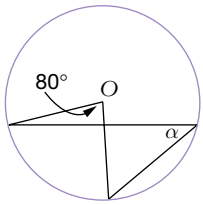
If $\angle ABC = \theta$
then $\angle AOC = 2\theta$ (angle at centre)
and $\angle ADC = \theta$ (angle at circumference)
$\therefore \angle ABC = \angle ADC$

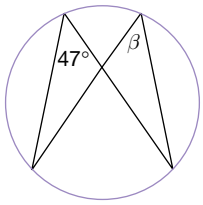


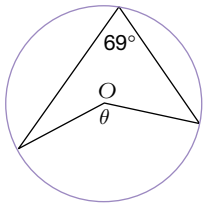

$\therefore$ Angles subtended at the circumference by the same or equal arcs (or chords) are equal.

## WORKED EXAMPLES

Find the value of the pronumerals, giving reasons.

1 


2 


3 


### Solutions

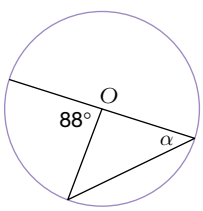
1 $\alpha = 40°$
(Angle at the circumference is half the angle at the centre.)

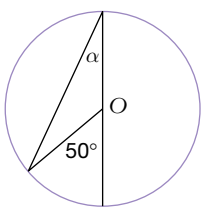
2 $\beta = 47°$
(Angles subtended at the circumference by the same arc are equal.)

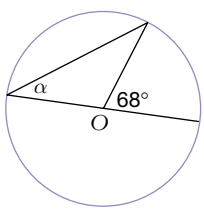
3 $\theta = 138°$
(Angle at the centre is twice the angle at the circumference.)

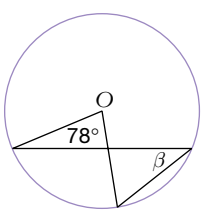
## Exercise 13:04

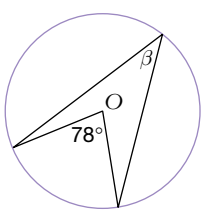
**1** Find the value of the pronumerals in each part.

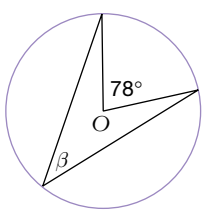
**a** 

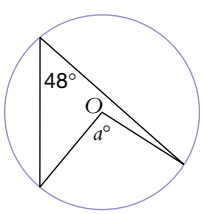

**b** 

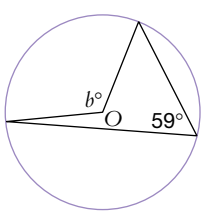

**c** 

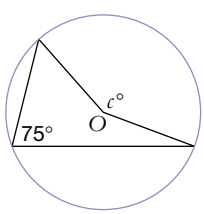

**d** 


**e** 

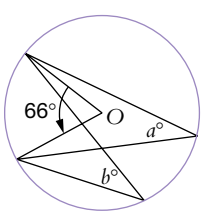

**f** 

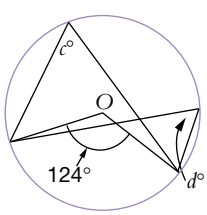

**g** 

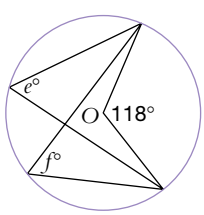

**h** 


**i** 

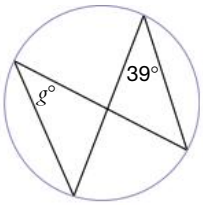

**2** Find the value of each pronumeral.

**a** 

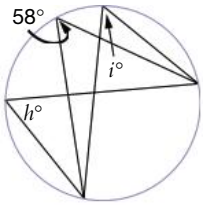

**b** 


**c** 

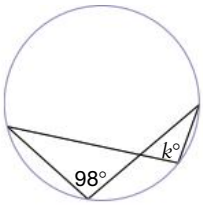
**d**



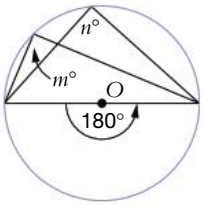

**e**



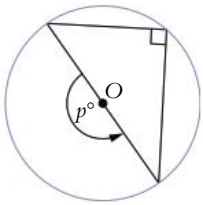

**f**



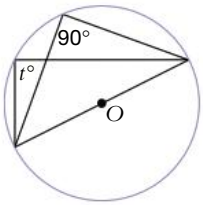

**g**




**h**

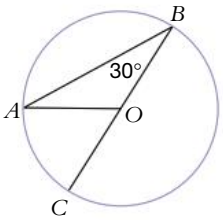


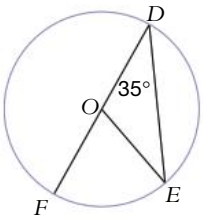

**i**




**3** **a** Using the figures to the right, find the size of:

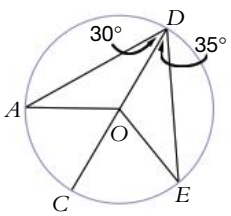
- **i** $\angle BAO$
- **ii** $\angle AOC$
- **iii** $\angle DEO$
- **iv** $\angle EOF$
- **v** $\angle ABO + \angle ODE$
- **vi** $\angle AOC + \angle EOF$



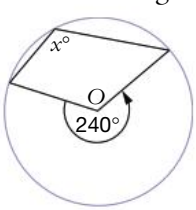




**b** Using the figure to the right, find the size of:

- **i** $\angle DAO$
- **ii** $\angle AOC$
- **iii** $\angle DEO$
- **iv** $\angle EOC$
- **v** $\angle ADE$
- **vi** obtuse $\angle AOE$
- **vii** Is the angle at the centre ($\angle AOE$) twice the angle at the circumference ($\angle ADE$)?

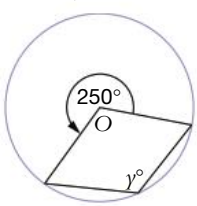



**c**



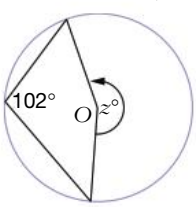

$x = \ldots$

**d**




$y = \ldots$

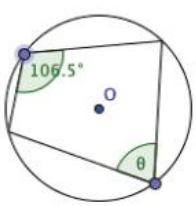
**e**

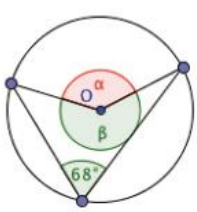


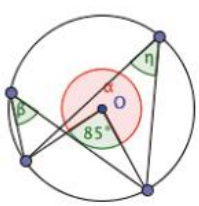

$z = \ldots$

## P GEOGEBRA ACTIVITY 13:04 ANGLE PROPERTIES (1)

Use a slider to change the size of the angle at the centre of the circle. At the same time the size of an angle at the circumference is given. Click and drag points to investigate the relationship between angles subtended by the same arc.



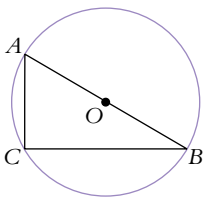



# 13:05 Angle properties of circles (2)

An angle subtended by a diameter at the circumference of a circle is called the angle in a semicircle.

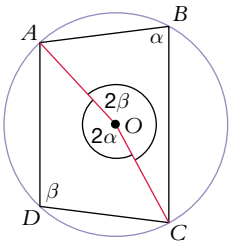
**The angle in a semicircle is a right angle.**

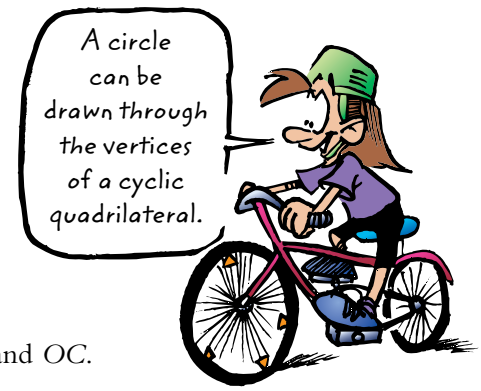



This result is easy to prove, as the diameter makes an angle of 180° at the centre.

Proof: $\angle AOB = 180°$ ($AOB$ is a straight line)
$\therefore \angle ACB = 90°$ (The angle at the centre is twice the angle at the circumference.)
$\therefore$ The angle in a semicircle is a right angle. QED

**Opposite angles of a cyclic quadrilateral are supplementary. (They add up to 180°.)**




Data: $ABCD$ is any cyclic quadrilateral.

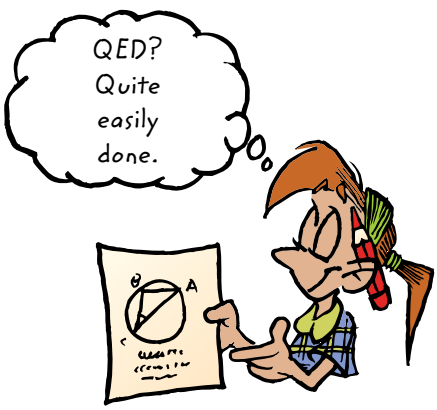
Aim: To prove that opposite angles add up to 180°.

Construction: Draw in the radii $AO$ and $OC$.

Proof: Let $\angle ABC$ be $\alpha$ and $\angle ADC$ be $\beta$.
Obtuse $\angle AOC = 2\alpha$ (angle at centre is twice $\angle ABC$)
Reflex $\angle AOC = 2\beta$ (angle at centre is twice $\angle ADC$)

Now, $2\alpha + 2\beta = 360°$ (angles at a point make 1 revolution)
$\therefore \alpha + \beta = 180°$
$\therefore \angle ABC + \angle ADC = 180°$

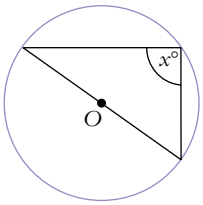
Similarly, $\angle BAD + \angle BCD = 180°$
$\therefore$ Opposite angles of a cyclic quadrilateral are supplementary. QED



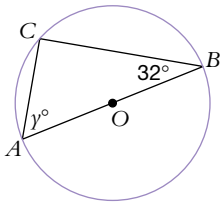
## WORKED EXAMPLES

Find the value of the pronumerals, giving reasons.

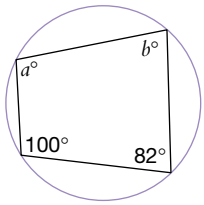
**1**




**2**




**3**




### Solutions

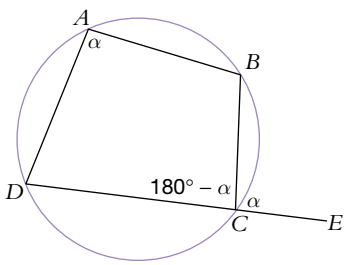
**1** $x = 90$
(angle in a semicircle)

**2** $\angle ACB = 90°$
(angle in a semicircle)
$\therefore \gamma = 58$
(angle sum of a $\Delta$)

**3** $a + 82 = 180$
$\therefore a = 98$
(opposite angles in a cyclic quadrilateral are supplementary)

$b + 100 = 180$
$\therefore b = 80$

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.




Data: $ABCD$ is a cyclic quadrilateral, with $DC$ produced to $E$, forming the exterior angle $BCE$.

Aim: To prove that $\angle BCE = \angle BAD$.

Proof: Let $\angle BAD = \alpha$

Then, $\angle BCD = 180° - \alpha$ (opp. $\angle$s in cyclic quad.)

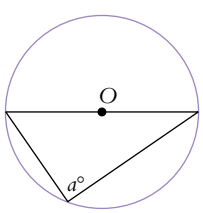
Now, $\angle BCD + \angle BCE = 180°$ ($\angle$s on a straight line)
$\therefore \angle BCE = \alpha$

$\therefore$ The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

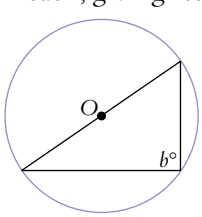
## Exercise 13:05

**1** Find the value of the pronumeral in each, giving reasons.

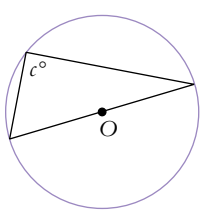
**a**



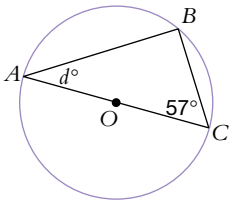

**b**



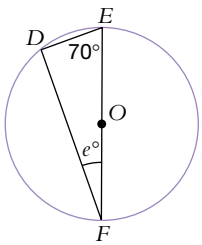

**c**



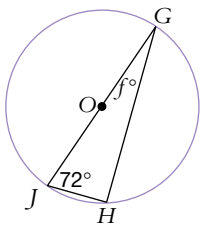
**d**



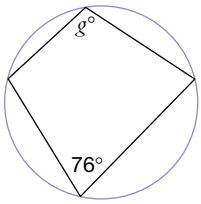

**e**



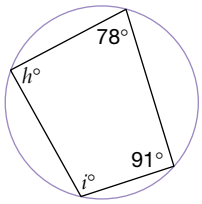

**f**



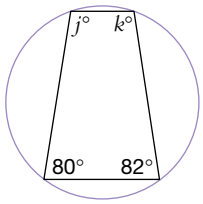

**g**




**h**

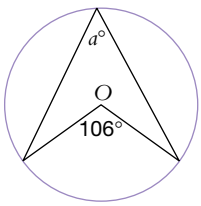



**i**

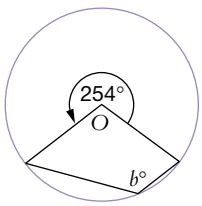



**2** Find the value of the pronumerals in each part.

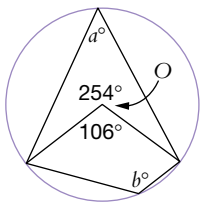
**a**



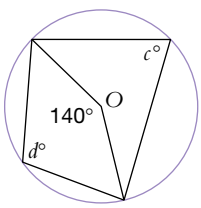

**b**



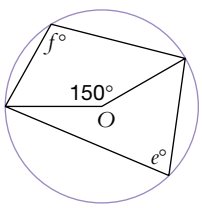

**c**



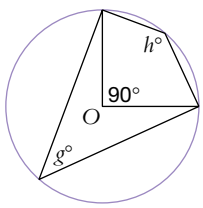

**d**



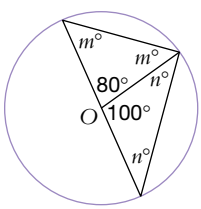

**e**



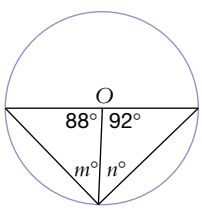

**f**




**g**



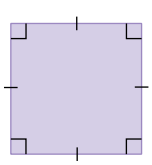

**h**



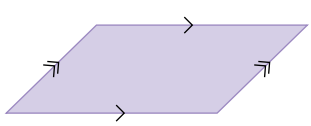

In **g** and **h**, $m + n = 90$.

**3** Can each quadrilateral have a circle drawn through its vertices? Explain why it is possible in each such case.

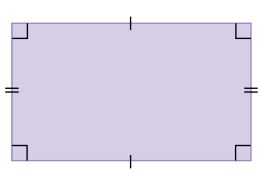
**a**



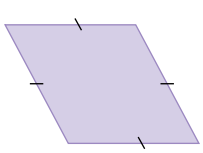
**b**



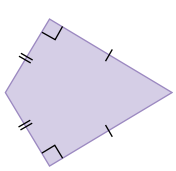
**c**



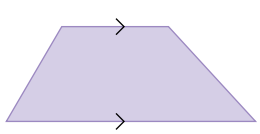
**d**



**e**

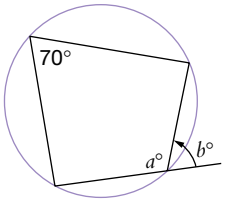


**f**

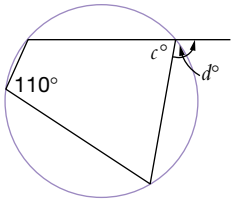


**4** Find the value of the pronumeral, giving reasons.

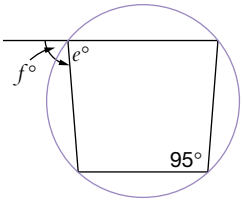
**a**



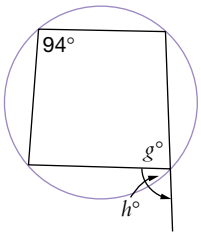

**b**



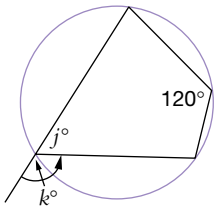

**c**




**d**




**e**

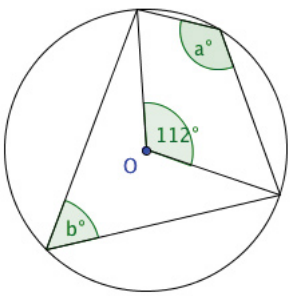



The exterior angle is always equal to the interior opposite angle.

## P GEOGEBRA ACTIVITY 13:05 ANGLE PROPERTIES (2)

In this activity you can move a point around a semicircle and investigate the size of the angle that is formed when this is joined to the endpoints of a diameter.

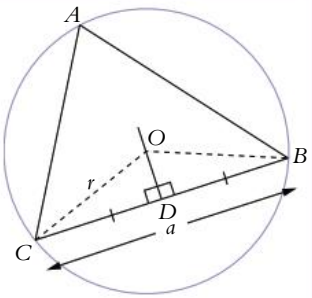
You can also change the angles of a cyclic quadrilateral and look for the relationship that exists between the angles.




**a = 124, b = 56**

## INVESTIGATION 13:05 DIAMETER OF A CIRCUMCIRCLE

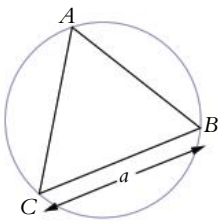
The circumcircle of a triangle is the circle that passes through all three vertices, as in the diagram. But for a particular triangle $ABC$, can we find a measure for the diameter of its circumcircle?




Follow carefully the steps of this investigation.

- Let $\angle A$ in triangle $ABC$ be $A°$. What is the size of $\angle BOC$ and why?
- The centre $O$ will lie on the perpendicular bisector of $BC$. What can be said about $\angle DOC$ and $\angle DOB$ and why?
- What is the size of $\angle DOC$?
- Noting that $DC = \frac{1}{2}a$ and letting $OC = r$ (radius), write down an expression for $\sin(\angle DOC)$.
- Rearrange this expression to make $r$ the subject and hence find an expression for $d$, the diameter of the circumcircle.

Completing this investigation should lead you to the following result.

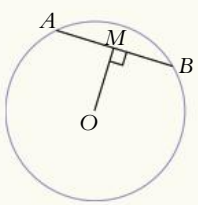



For a triangle $ABC$, the diameter $d$ of its circumcircle will be given by: $d = \dfrac{a}{\sin A}$

# 13:06 Tangent properties of circles

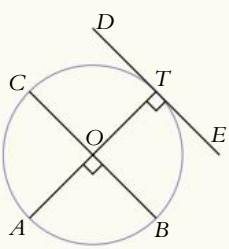
## PREP QUIZ 13:06

**1**



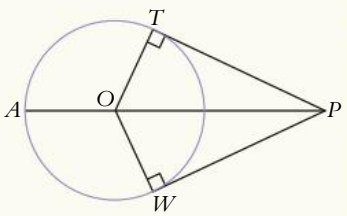

$AB = 12\,\text{m}$, $MB = \ldots$

**2**




Name the axis of symmetry.

**3**

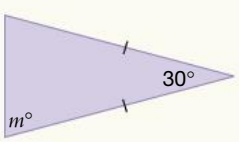



Name the axis of symmetry.

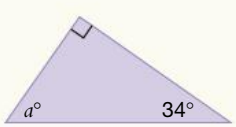
**4** Name the tangent in Question **2**.

**5** In Question **2**, what name is given to interval $OT$?

**6**

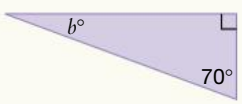



**7**



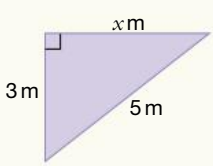

$a = \ldots$

**8**



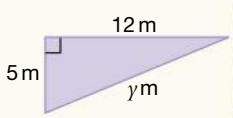

$b = \ldots$

**9**




$x = \ldots$

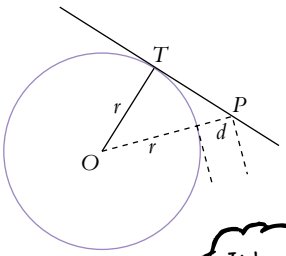
**10**

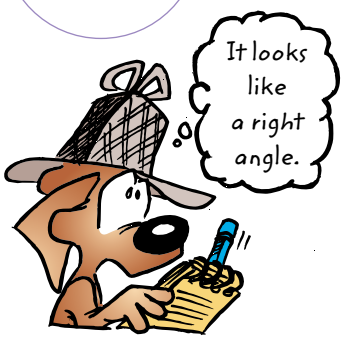



$y = \ldots$

A tangent is a straight line that touches the circle at only one point. This point is called the *point of contact*.

> The size of the angle between a tangent and the radius drawn to the point of contact is 90°.




Here, we aim to prove that $OT \perp TP$.
$TP$ is a tangent and $T$ is the point of contact.

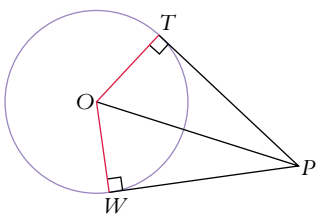
Construction: Join $O$ to any point $P$ on the tangent (other than $T$).

Proof: The shortest distance from a point to a line is the perpendicular distance.

Let $r$ be the radius of the circle.
$OT = r$

Every point other than $T$ is outside the circle.
$PO = r + d$, where $d$ is a positive quantity.
$\therefore\ OT < OP$
$\therefore$ $OT$ is the shortest distance to the line $TP$.
$\therefore\ OT \perp TP$
$\therefore\ \angle OTP = 90°$ QED

> From any external point, two equal tangents may be drawn to a circle. The line joining this point to the centre is an axis of symmetry.




$PT$ and $PW$ are tangents drawn to the circle from the point $P$.
Our aim is to prove that $PT = PW$ and that $OP$ is an axis of symmetry of the figure.

Construction: Draw in $OT$ and $OW$, the radii drawn to the points of contact.

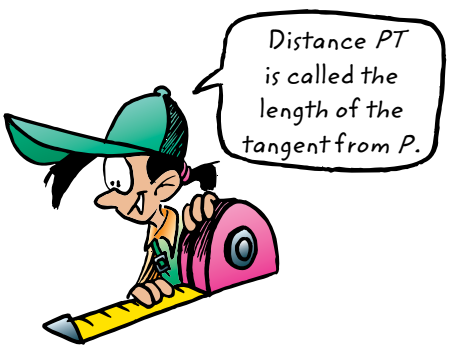
Proof: In $\Delta$s $OTP$ and $OWP$:

**1** $\angle OTP = \angle OWP = 90°$
(The angle between the tangent and radius is 90°.)

**2** $OP$ is common

**3** $OT = OW$ (radii of the circle)
$\therefore\ \Delta OTP \equiv \Delta OWP$ (RHS)
$\therefore\ PT = PW$ (matching sides of congruent $\Delta$s)

Also $OP$ is an axis of symmetry as $\Delta OTP$ is congruent to $\Delta OWP$.

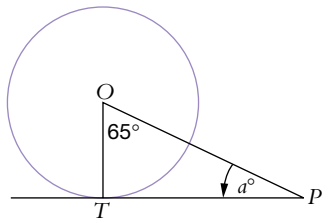



$\therefore$ The two tangents drawn from an external point are equal and the line joining this point to the centre is an axis of symmetry. QED

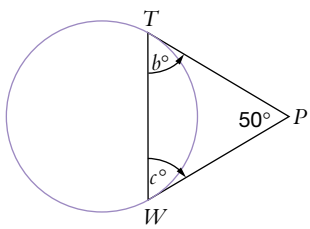
## WORKED EXAMPLES

Find the value of each pronumeral, giving reasons. In each case, $PT$ and $PW$ are tangents.

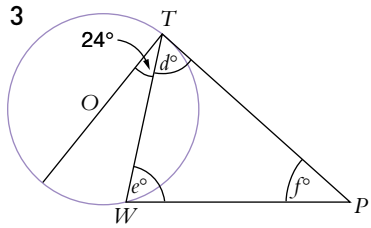
**1**




**2**




**3**




### Solutions

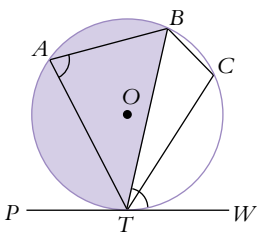
**1** $\angle OTP = 90°$
(radius $OT \perp$ tangent $TP$)
$a + 90 + 65 = 180$
$\therefore\ a = 25$
(angle sum of a $\Delta$)

**2** $PT = PW$
(equal tangents from $P$)
$\therefore\ b = c$ (isosceles $\Delta$)
$\therefore\ c + c + 50 = 180$
(angle sum of a $\Delta$)
$\therefore\ c = 65$ and $b = 65$

**3** $d + 24 = 90$
(radius $OT \perp$ tangent $TP$)
$\therefore\ d = 66$
$PT = PW$ (equal tangents)
$\therefore\ e = 66$ (isosceles $\Delta PTW$)
$\therefore\ f = 48$ (angle sum of a $\Delta$)

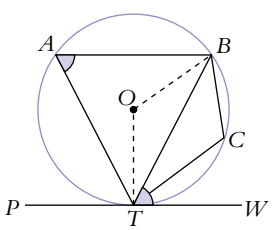
## The angle in the alternate segment

- $\angle BTW$ is the acute angle between the tangent $PW$ and the chord $BT$.
- The shaded segment of the circle is called the alternate segment to $\angle BTW$, while $\angle BAT$ is an angle in the alternate segment.
- $\angle BCT$ is an angle in the alternate segment to $\angle BTP$.




> An angle formed by a tangent to a circle with a chord drawn to the point of contact is equal to any angle in the alternate segment.

Data: The chord $BT$ meets the tangent $PW$ at the point of contact, $T$.
$O$ is the centre of the circle.
$\angle BAT$ is any angle in the segment alternate to $\angle BTW$.
$\angle BCT$ is any angle in the segment alternate to $\angle BTP$.




Aim: To prove: **1** that $\angle BTW = \angle BAT$ and
**2** that $\angle BTP = \angle BCT$.

Construction: Draw $OT$ and $OB$.

Proof: **1** Let $\angle BTW$ be $x°$.

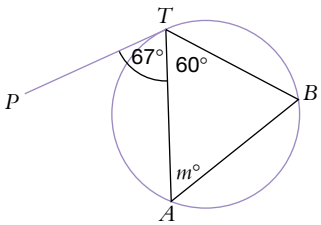
Now $\angle OTW = 90°$ (radius $OT \perp$ tangent $PW$)
$\therefore\ \angle OTB = 90° - x°$
$\therefore\ \angle OBT = 90° - x°$ ($\Delta TOB$ is isosceles, $OT$ and $OB$ are radii)
$\therefore\ \angle TOB = 180° - 2(90° - x°)$ (angle sum of $\Delta TOB$)
$= 2x°$
$\therefore\ \angle BAT = x°$ (angle at circumference is half the angle at the centre on the same arc)
$\therefore\ \angle BTW = \angle BAT$ QED

**2** $\angle BTP = 180° - x°$ ($\angle PTW$ is a straight angle)
but $\angle BCT = 180° - x°$ (supplementary to $\angle BAT$, as $ABCT$ is cyclic quad.)
$\therefore \angle BTP = \angle BCT$ QED

$\therefore$ The angle between the tangent and a chord drawn to the point of contact is equal to any angle in the alternate segment.

## WORKED EXAMPLES

Find the value of $m$.



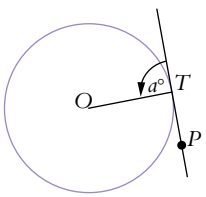

### Solution

$\angle ABT = 67°$ (angle in the alternate segment)
$m + 60 + 67 = 180$ (angle sum of $\Delta$)
$\therefore m = 53$

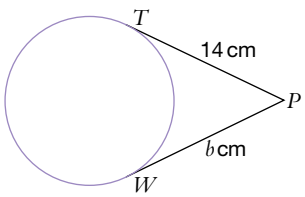
## Exercise 13:06

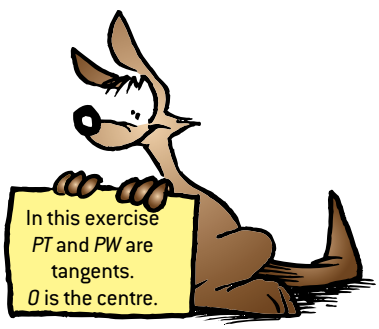
**1** Find the value of the pronumerals in each, giving reasons.

**a**

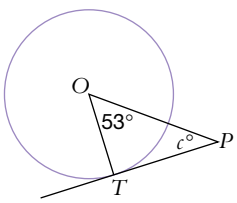



**b**

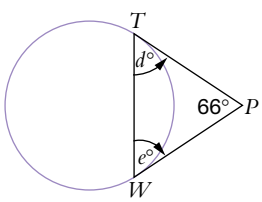


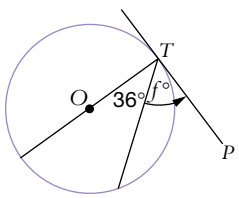

**c**



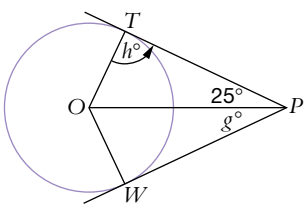

**d**



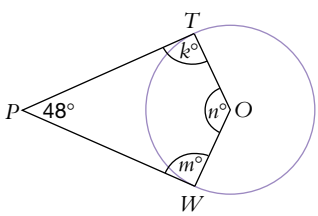

**e**



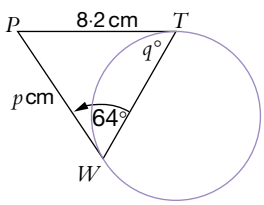

**f**




**g**

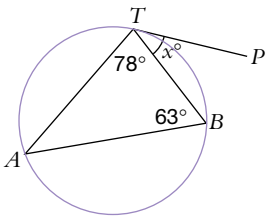



**h**

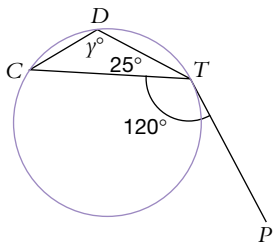


**2** Find the value of each pronumeral. $PT$ is a tangent in each diagram.

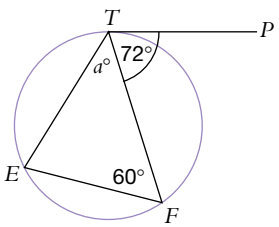
**a**



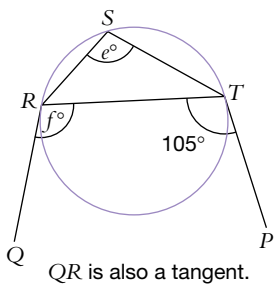

**b**




**c**



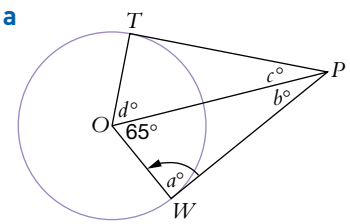

**d**



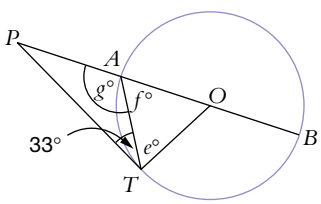

$QR$ is also a tangent.

**3** Find the value of the pronumerals in each.

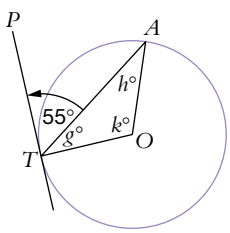
**a**



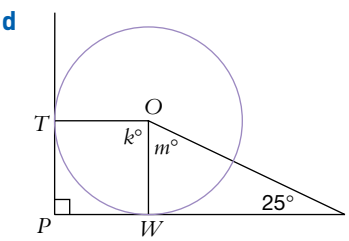

**b**



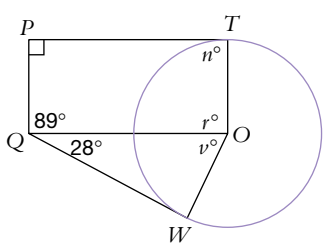

**c**



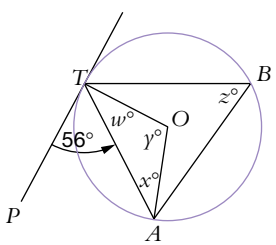

**d**



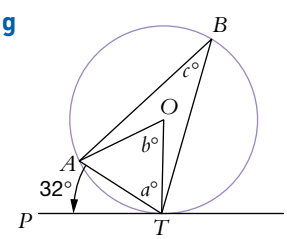

**e**



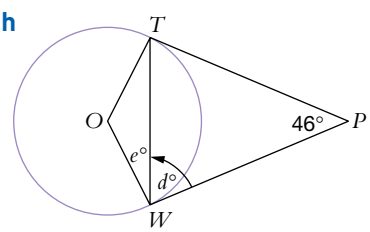

**f**



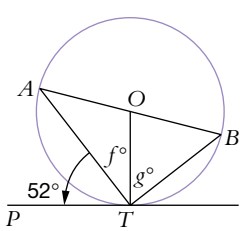

**g**




**h**

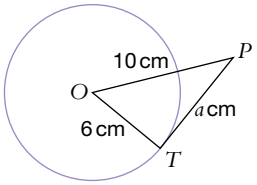



**i**

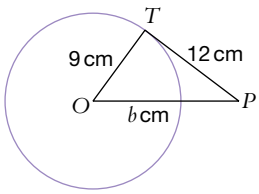


**4** Use Pythagoras' theorem to find the value of the pronumerals.

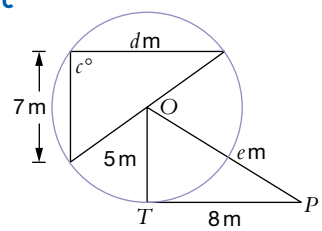
**a**



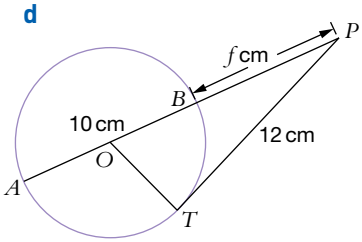

**b**



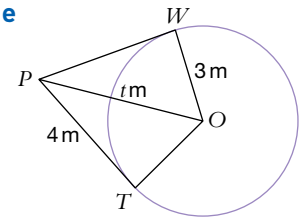
**c**




**d**




**e**

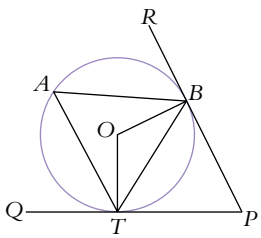



**5** $O$ is the centre of the circle. $BT$ is a chord that subtends $\angle BAT$ at the circumference and $\angle BOT$ at the centre. $PT$ and $PB$ are tangents to the circle. Prove that:

**a** $\angle BOT = 2\angle BTP$

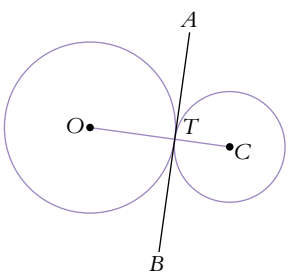
**b** $\angle ATQ + \angle RBA + \angle PBT = 180°$

**c** $\angle BPT = 180° - 2\angle BAT$

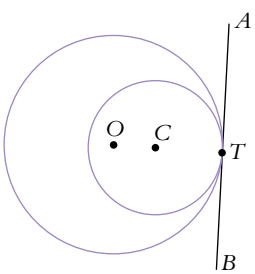



**6** From a point $T$ on a circle, chords of equal length are drawn to meet the circle at $A$ and $B$. Prove that the tangent at $T$ is parallel to the chord $AB$.

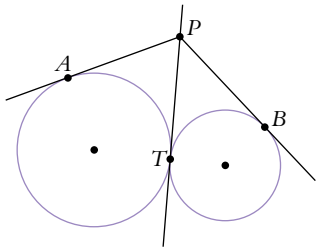
**7** **a** Two circles, with centres at $O$ and $C$, touch externally, thus having a common tangent at $T$. Prove that the interval $OC$ joining the centres is perpendicular to the common tangent.



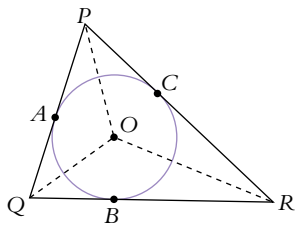

**b** Two circles, with centres at $O$ and $C$, touch internally, thus having a common tangent at $T$. Prove that the interval $OC$ produced is perpendicular to the common tangent.



**8** Two circles touch externally, having a common tangent at $T$. From a point $P$ on this tangent, a second tangent is drawn to each of the circles as shown, touching the circles at $A$ and $B$, respectively. Prove that $PA = PB$.

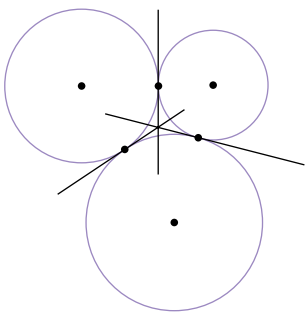



**9** Prove that the three angle bisectors of triangle $PQR$ are concurrent and that their point of intersection is the centre of a circle, which touches each side of the triangle. (This is called the incircle of the triangle.)




**10** Three different-sized circular discs touch each other. Prove that the three common tangents are concurrent.

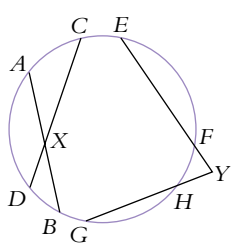
Concurrent lines intersect at a point.



# 13:07 Further circle properties

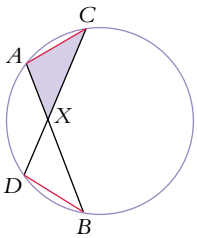
## Intersecting chords and secants

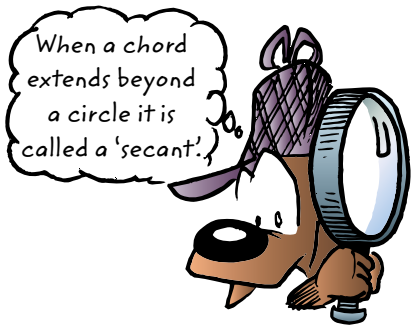
- $AB$ and $CD$ divide each other internally at $X$.
  $AX$ and $XB$ are called the intercepts of chord $AB$.
  $CX$ and $XD$ are called the intercepts of chord $CD$.
- $EF$ and $GH$ divide each other externally at $Y$.
  $EY$ and $YF$ are called the intercepts of secant $EY$.
  $GY$ and $YH$ are called the intercepts of secant $GY$.
- *Note:*
  (length $AX$) . (length $XB$) = (length $CX$) . (length $XD$)
  (length $EY$) . (length $YF$) = (length $GY$) . (length $YH$)

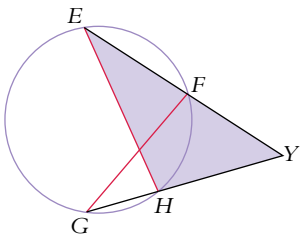



A dot can be used to show multiplication.

The products of intercepts of intersecting chords or secants are equal.




Data: Chords $AB$ and $CD$ meet at $X$.

Aim: To prove that $AX \cdot XB = CX \cdot XD$.

Construction: Draw $AC$ and $DB$.

Proof: In $\Delta$s $ACX$ and $DBX$:

**1** $\angle ACX = \angle DBX$ (angles subtended by the same arc $AD$)

**2** $\angle AXC = \angle DXB$ (vert. opp. angles)

$\therefore \Delta ACX \,|||\, \Delta DBX$ (equiangular)

$\therefore \dfrac{AX}{XD} = \dfrac{CX}{XB}$ (matching sides are in same ratio)

$\therefore AX \cdot XB = CX \cdot XD$ QED

Data: The secants $EY$ and $GY$ cut the circle at $F$ and $H$, respectively.

Aim: To prove that $EY \cdot YF = GY \cdot YH$.

Construction: Draw $FG$ and $EH$.

Proof: In $\Delta$s $EYH$ and $GYF$:

**1** $\angle EYH = \angle GYF$ (same angle)

**2** $\angle YEH = \angle YGF$ (angles subtended by the same arc $FH$)

$\therefore \Delta EYH \,|||\, \Delta GYF$ (equiangular)

$\therefore \dfrac{EY}{GY} = \dfrac{YH}{YF}$ (matching sides are in same ratio)

$\therefore EY \cdot YF = GY \cdot YH$ QED

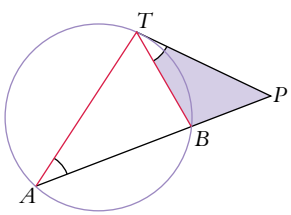
$\therefore$ The products of intercepts of intersecting chords or secants are equal.

A dot can be used to show multiplication.

The square of the length of a tangent is equal to the product of the intercepts of a secant drawn from an external point.

$$(PT)^2 = AP \cdot PB$$

Data: $PT$ is a tangent to the circle. $PA$ is a secant that cuts the circle at $A$ and $B$. (The chord $AB$ is divided externally at $P$.)




Aim: To prove that $(PT)^2 = AP \cdot PB$.

Construction: Draw $AT$ and $BT$.

Proof: In $\Delta$s $PTB$ and $PAT$:

**1** $\angle PTB = \angle PAT$ (angle in alternate segment)

**2** $\angle TPB = \angle APT$ (same angle)

$\therefore \Delta PTB \,|||\, \Delta PAT$ (equiangular)

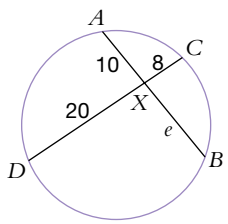
$\therefore \dfrac{PT}{AP} = \dfrac{PB}{PT}$ (matching sides of similar $\Delta$s are proportional)

$\therefore (PT)^2 = AP \cdot PB$ QED

## WORKED EXAMPLES

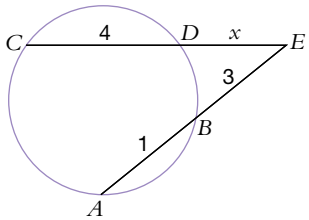
Find the value of each pronumeral, giving reasons.

**1**



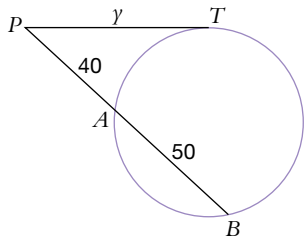

$AX = 10\,\text{mm}$, $CX = 8\,\text{mm}$, $DX = 20\,\text{mm}$, $XB = e\,\text{mm}$

**2**




$CD = 4\,\text{m}$, $AB = 1\,\text{m}$, $BE = 3\,\text{m}$, $DE = x\,\text{m}$

**3**




$PT = y\,\text{cm}$, $AP = 40\,\text{cm}$, $AB = 50\,\text{cm}$

### Solutions

**1** $AX \,.\, XB = CX \,.\, XD$
(products of intercepts of intersecting chords)
$10e = 8 \times 20$
$\therefore\ e = 16$

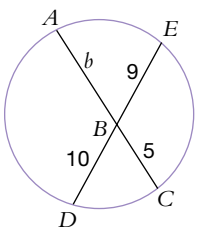
**2** $CE \,.\, ED = AE \,.\, EB$
(products of intercepts of intersecting secants)
$\therefore\ (4 + x) \,.\, x = (1 + 3) \,.\, 3$
$4x + x^2 = 4 \times 3$
$x^2 + 4x - 12 = 0$
$(x + 6)(x - 2) = 0$
$\therefore\ x = -6 \text{ or } 2$
$x = 2$
(a length must be positive)

**3** $(PT)^2 = BP \,.\, PA$
(square of the tangent equals the product of the intercepts)
$\therefore\ y^2 = (50 + 40) \,.\, 40$
$y^2 = 3600$
$\therefore\ y = \pm\sqrt{3600}$
$= \pm 60$
$y = 60$
(a length must be positive)

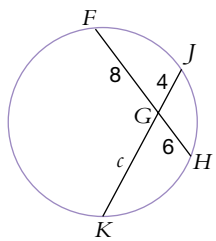
## Exercise 13:07

**1** Find the value of each pronumeral. All lengths are in centimetres and $PT$ is a tangent wherever it is used.

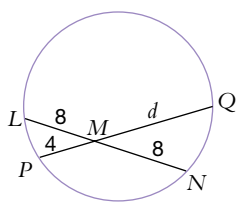
**a**



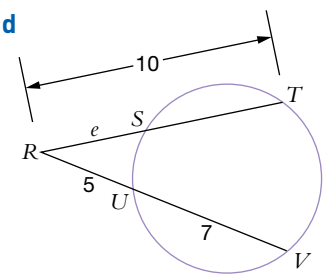

**b**



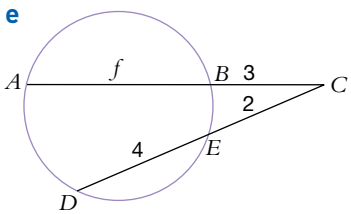

**c**



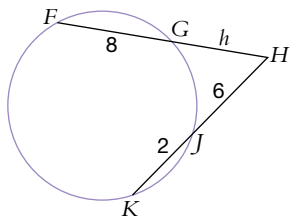

**d**



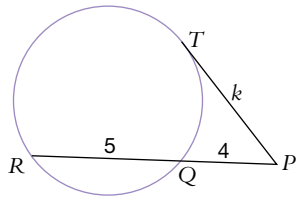

**e**



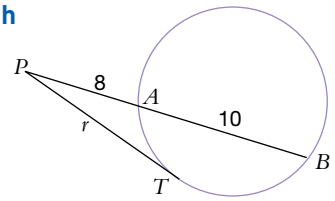

**f**



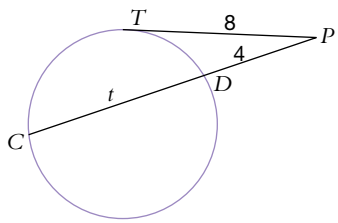
**g**



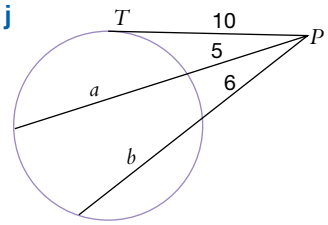

**h**



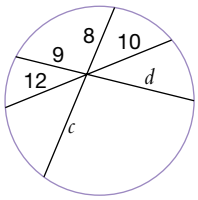

**i**



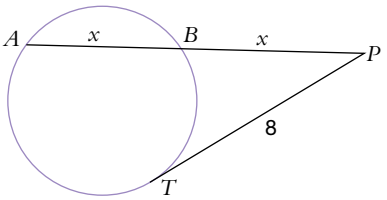

**j**



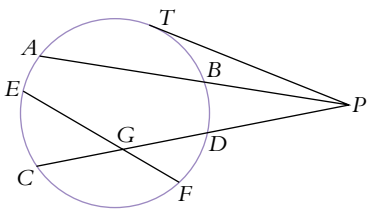

**k**




**l**




**2**




*PT* is a tangent

**a** Find *PT* if *AB* = 9 m and *BP* = 3 m.

**b** Find *AB* if *BP* = 10 cm and *PT* = 13 cm.

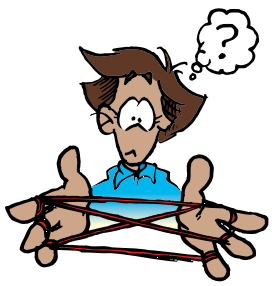
**c** Find *CD* if *DP* = 5 m, *AB* = 8 m, *BP* = 6 m.

**d** Find *EG* if *GF* = 20 m, *CG* = 30 m, *GD* = 25 m.

**e** Find *CD* if *CG* = 15 m, *EF* = 35 m, *EG* = 22 m.

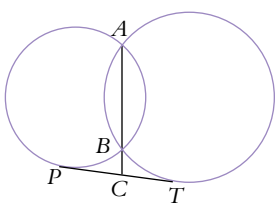
**f** Find *CD* if *TP* = 9 cm and *DP* = 5 cm.

**g** Find *PT* if *CD* = 8 cm and *DP* = 10 cm, correct to one decimal place.



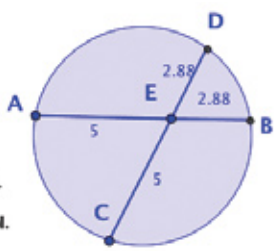
**3** *AB* is the diameter of a circle. *AB* bisects a chord *CD* at the point *E*. Find the length of *CE* if *AE* = 3 m and *BE* = 9 m.

**4** *PT* is a direct common tangent of the circles drawn. *AB* is a common chord that has been produced to meet the common tangent at *C*. Use the 'square of the tangent' result to prove that *CP* = *CT*.




## P GEOGEBRA ACTIVITY 13:07 INTERSECTING CHORDS AND SECANTS

Click and drag points to investigate intersecting chords and secants. You can then use a set of random exercises to test the knowledge you have learnt.




AE × BE = 5 × 2.88 = 14.4 to 1dec. pl.

CE × DE = 5 × 2.88 = 14.4 to 1 dec. pl.

**The products of the intercepts of intersecting chords are equal.**

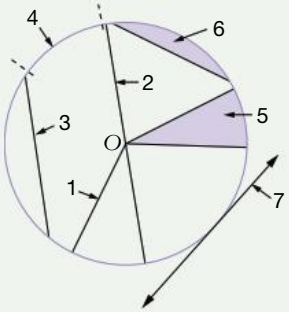
## FUN SPOT 13:07 HOW DO YOU MAKE A BUS STOP?

Answer each question and write the letter for that question in the box above the correct answer.

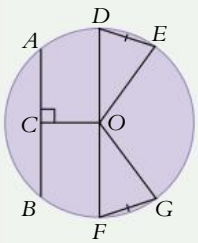
*From the circle below, give the number of the:*

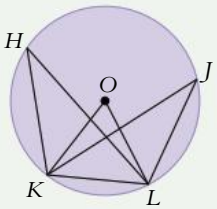
**A** sector
**B** radius
**D** tangent
**E** chord

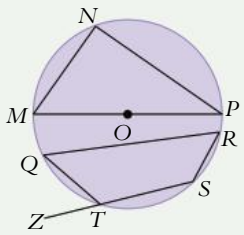
**A** diameter
**C** arc
**E** segment

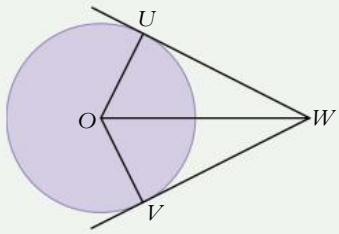


*In all these circles, O is the centre.*




*From the figures above, find what is equal to:*

**G** $AC$
**H** $\angle DOE$
**H** $\angle KHL$
**I** $2(\angle KJL)$
**I** $\angle UOW$
**N** $\angle VWO$
**N** $UW$
**N** $180° - \angle RQT$
**O** $\frac{1}{2}\angle MOP$
**S** $90° - \angle NMP$
**S** $\angle QTZ$
**T** What is the size of $\angle OUW$?

*If $\angle KHL = 40°$, find the size of:*

**T** $\angle KJL$
**W** $\angle KOL$
**W** $\angle OKL$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50° | $\angle KOL$ | 40° | $\angle KJL$ | 90° | 80° | $\angle MNP$ | 1 | 3 | $VW$ | 4 | $\angle FOG$ | 6 | $\angle NPM$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | $\angle RST$ | 7 | 2 | $\angle QRS$ | $\angle VOW$ | $CB$ | $\angle UWO$ |

# 13:08 Deductive exercises involving the circle

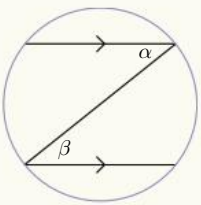
## PREP QUIZ 13:08

Use **ID Card 5** on page xxii to identify number:

**1** 13 **2** 14 **3** 15 **4** 16 **5** 18 **6** 19 **7** 20

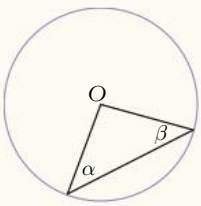
Give reasons why the fact under each figure is true.

**8**



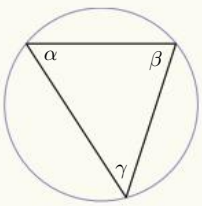

$\alpha = \beta$

**9**




$\alpha = \beta$

**10**



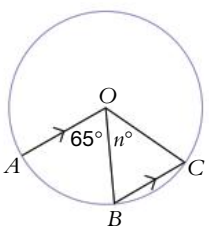

$\alpha + \beta + \gamma = 180°$

In this section, reasons must be given for any claim made in numerical exercises.

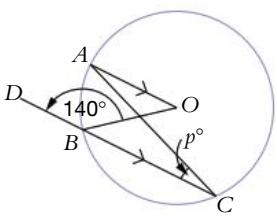
## WORKED EXAMPLES

Find the value of each pronumeral, giving reasons.

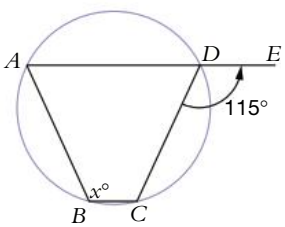
**1**




**2**




**3**




### Solutions

**1** $\angle OBC = 65°$ (alternate to $\angle AOB$, $AO \parallel BC$)

$\angle OCB = 65°$ ($\Delta OBC$ is isosceles)

$n + 130 = 180$ (angle sum of $\Delta OBC$)

$\therefore n = 50$

**2** $\angle AOB = 40°$ (cointerior to $\angle DBO$, $AO \parallel DC$)

$\angle ACB = 20°$ (half the angle at the centre)

$\therefore p = 20$

**3** $\angle ADC = 65°$ ($\angle ADE$ is a straight angle)

$\angle ABC = 115°$ (opposite angles of a cyclic quadrilateral are supplementary)

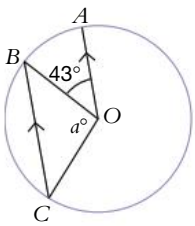
$\therefore x = 115$

# Exercise 13:08

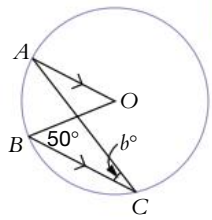
**1** Find the value of each pronumeral, giving reasons.

*O* is the centre. *PT* and *PW* are tangents.

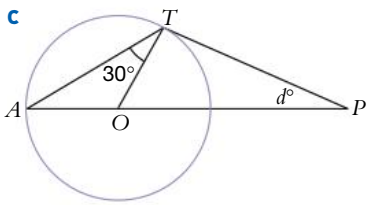
**a**



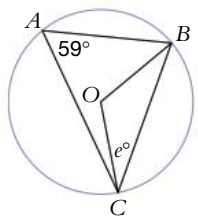

**b**



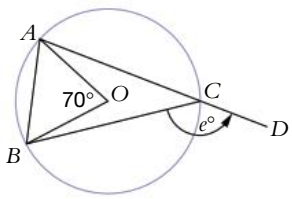

**c**



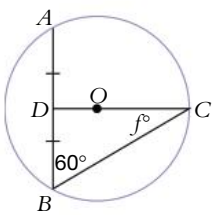

**d**



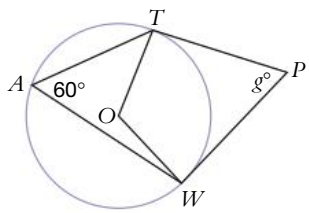

**e**



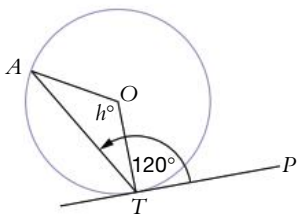

**f**



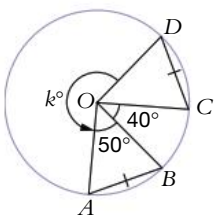

**g**



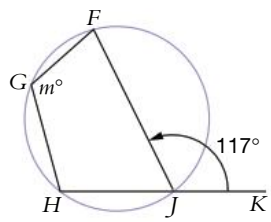

**h**



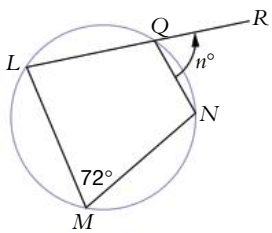

**i**



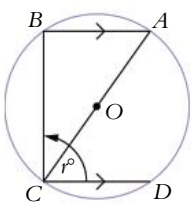

**j**



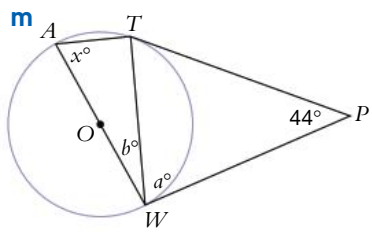

**k**



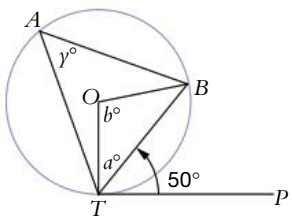

**l**



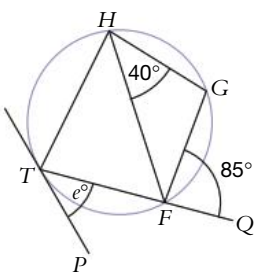

**m**




**n**

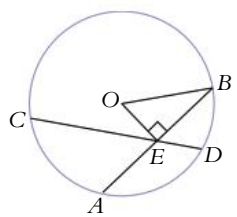



**2** **a**



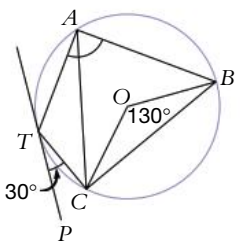

$\angle FHG = 40°$,
$\angle GFQ = 85°$,
$\angle PTF = e°$

**b**




$CE = 8$ cm, $ED = 2$ cm,
$OB = 5$ cm, $BE = x$ cm,
$OE = y$ cm

**c**




$\angle PTC = 30°$,
$\angle COB = 130°$,
$\angle BAT = x°$

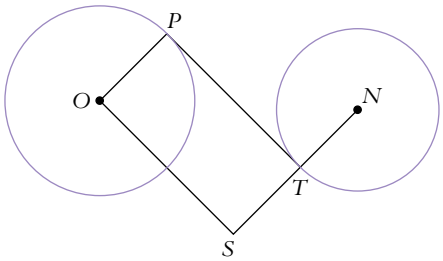
**3** $PT$ is a tangent to a circle, centre $O$, and the tangent touches the circle at $T$. $A$ is a point on the circle and $AP$ cuts the circle at $B$ such that $AB = BP$. Find the length of $AB$ if $PT$ is 8 cm.

**4** $PT$ is an indirect common tangent of the two circles that have centres $O$ and $N$. $OP = 6$ cm, $NT = 5$ cm and $ON = 15$ cm. $OS \parallel PT$.

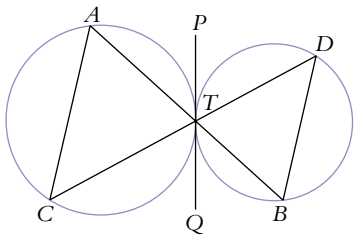
**a** Show that $\angle OSN = 90°$.

**b** Show that $OPTS$ is a rectangle.

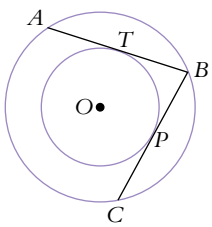
**c** Find the length of $PT$.



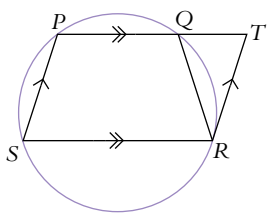

**5** $PQ$ is a common tangent. $AB$ and $CD$ intersect at the point where the tangent meets the circles. Prove that $AC \parallel DB$.



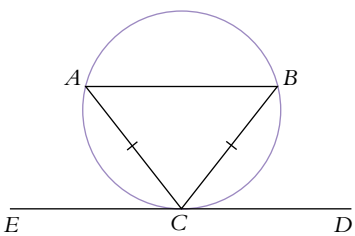

**6** The two circles are concentric. $AB$ and $CB$ are chords of the larger circle and tangents to the smaller circle. Prove that $AB = CB$.



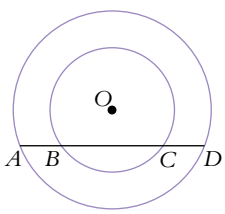

**7** $PQRS$ is a cyclic quadrilateral. Side $PQ$ has been produced to $T$ so that $PTRS$ is a parallelogram. Prove that $RQT$ is an isosceles triangle.




**8** $A$, $B$ and $C$ are three points on a circle where $AC = BC$. $ED$ is a tangent to the circle at $C$. Prove that $AB \parallel ED$.



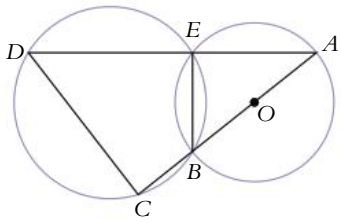

**9** In the diagram, $O$ is the centre of two concentric circles. $ABCD$ is a straight line. Prove that $AB = CD$.



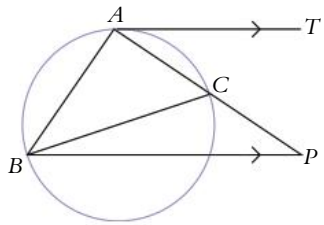
**10** $EB$ is the common chord of the intersecting circles. $AB$ is a diameter of the smaller circle that is produced to meet the larger circle at $C$. $DA$ passes through $E$.

**a** Prove that $\Delta AEB$ is similar to $\Delta ACD$.

**b** Hence or otherwise prove that $\angle ACD = 90°$.

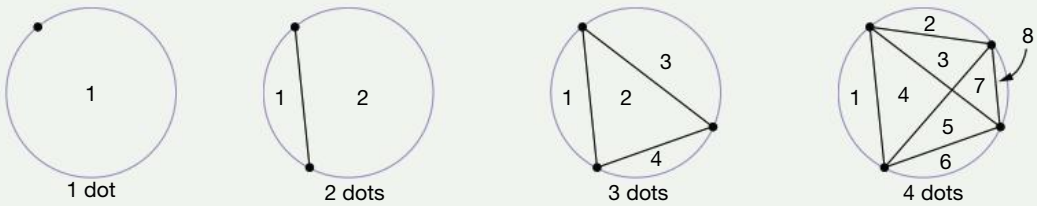



**11** $AT$ is a tangent and is parallel to $BP$.
$ACP$ is a straight line.
Prove that $\angle ABP = \angle ACB$.

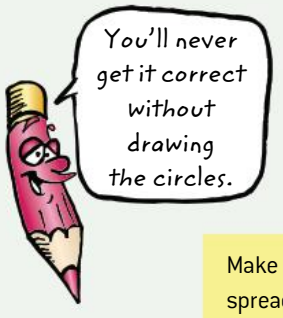



## FUN SPOT 13:08 HOW MANY SECTIONS?

Below are a series of circles. On them have been placed 1 dot, 2 dots, 3 dots and 4 dots. Chords have been drawn connecting every possible pair of dots. The number of chords and the number of divided sections of each circle were counted and recorded in the table.




| Dots | Chords | Sections |
|---|---|---|
| 1 | 0 | 1 |
| 2 | 1 | 2 |
| 3 | 3 | 4 |
| 4 | 6 | 8 |
| 5 | ? | ? |
| 6 | ? | ? |

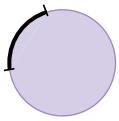



Make sure your dots are spread *unevenly* around the circle.

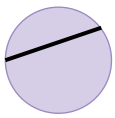
Can you find the next two lines of the table?

## MATHS TERMS 13

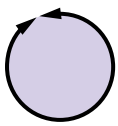
**arc**
- part of the circumference of a circle



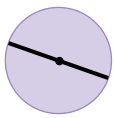
**chord**
- an interval joining two points on the circumference of a circle



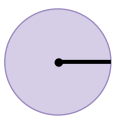
**circumference**
- the perimeter of a circle



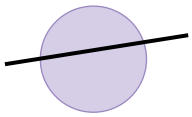
**diameter**
- a chord that passes through the centre of a circle
- the width of a circle



**radius (plural: radii)**
- an interval joining the centre of a circle to its circumference
- a radius is half the length of a diameter

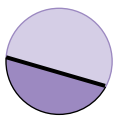


**secant**
- a line that intersects a circle in two places

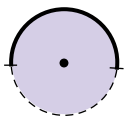


**sector**
- part of the area of a circle cut off by two radii

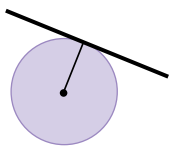
**segment**
- part of the area of a circle cut off by a chord



**semicircle**
- half a circle



**tangent**
- a line that touches a circle at one point
- a tangent and a radius are perpendicular at the point of contact



This medieval table hangs in the Great Hall in Winchester, England. It represents the Round Table of the legendary King Arthur.

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

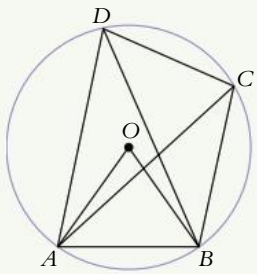
13:01

**1** From the figure shown, name:

**a** an angle subtended at the centre $O$

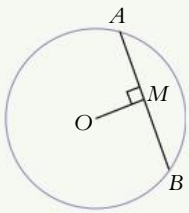
**b** an angle standing on the arc $AB$, subtended at the circumference

**c** an angle in the same segment as $\angle BAC$, standing on $BC$.



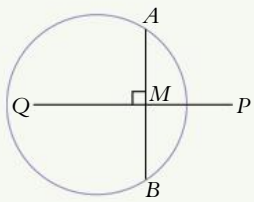

13:02

**2 a**



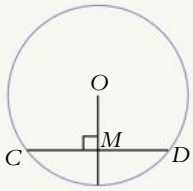

$AB = 9$ cm
Find the length of $AM$, giving reasons.

**b**




$AM = MB$
Give reasons why $PQ$ must pass through the centre of the circle.

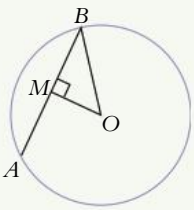
**c**




$CM = 18$ m
Find the length of $CD$, giving reasons.

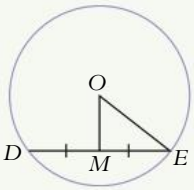
13:02

**3 a**



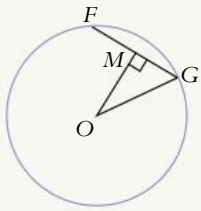

$BO = 26$ m,
$AB = 48$ m
Find the length of $OM$.

**b**




$OM = 9$ cm,
$ME = 12$ cm
Find the length of $OE$.

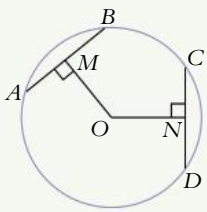
**c**




$OM = 40$ m,
$OG = 50$ m
Find the length of $FG$.

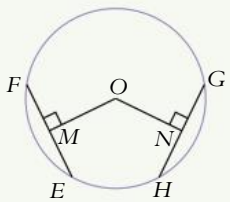
13:03

**4 a**



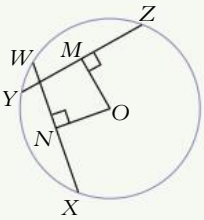

$AB = CD$,
$OM = 8$ m
Find the length of $ON$, giving reasons.

**b**




$OM = ON$,
$EF = 11$ km
Find the length of $GN$, giving reasons.

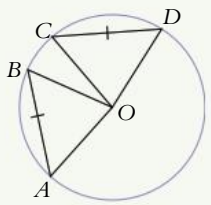
**c**




$OM = ON$,
$WX = 14$ m
Find the length of $YZ$, giving reasons.

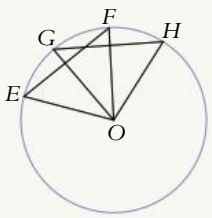
13:03

**5 a**



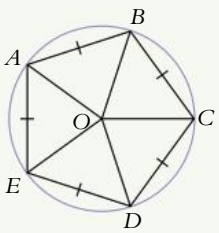

$AB = CD$, $\angle COD = 69°$
Find the size of $\angle AOB$, giving reasons.

**b**




$\angle GOH = \angle EOF$,
$GH = 1{\cdot}1$ m
Find the length of $EF$, giving reasons.

**c**

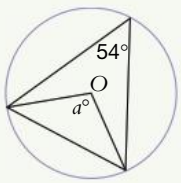



In this regular polygon, find the size of $\angle BOC$ and obtuse $\angle BOD$.

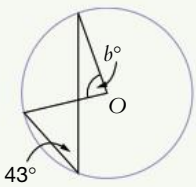
13:04

Find the value of the pronumerals in the following questions.

**6 a**



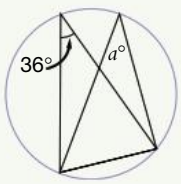

**b**



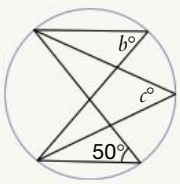

**c**

13:04

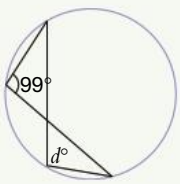
**7 a**




**b**

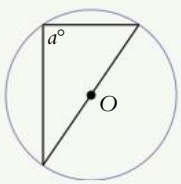



**c**

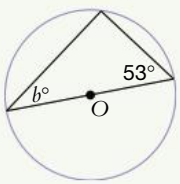



13:05

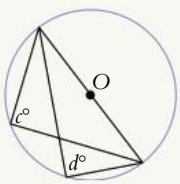
**8 a**




**b**

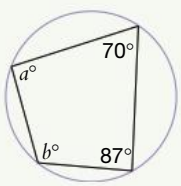



**c**

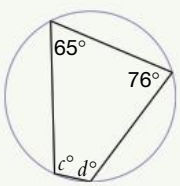



13:05

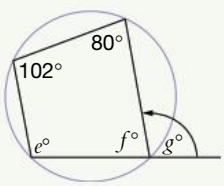
**9 a**




**b**



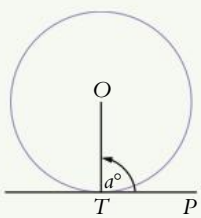

**c**



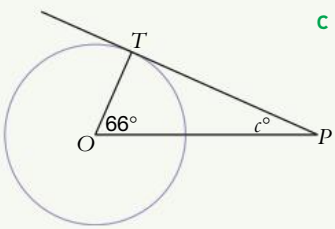

13:06

In Questions **10** to **13**, $PT$ and $PW$ are tangents.

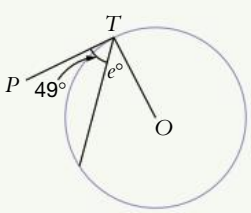
**10 a**



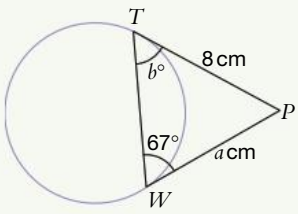

**b**



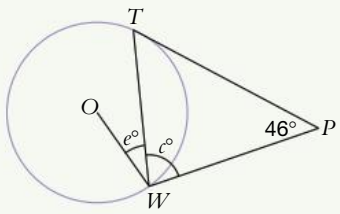

**c**



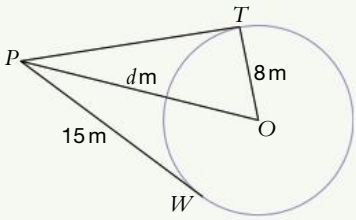
**11 a**




**b**

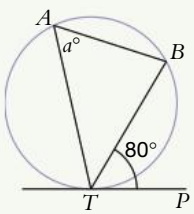



**c**

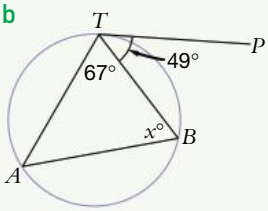



13:06

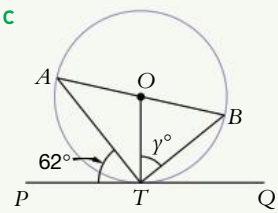
**12 a**




**b**

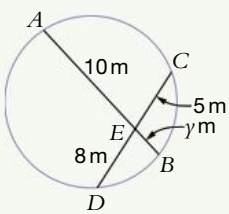



**c**

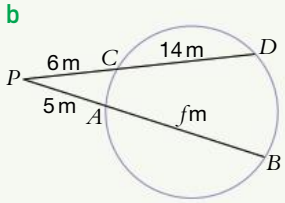



13:06

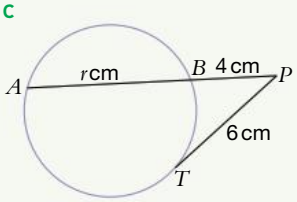
**13 a**




**b**




**c**




13:06

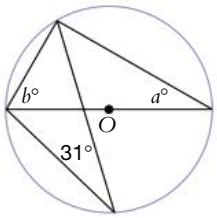
The circle in the sky seen throughout the world.

## ASSIGNMENT 13A Chapter review

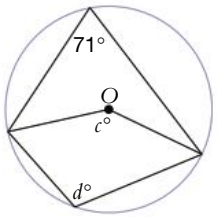
$O$ is the centre of each circle. $PT$ and $PW$ are tangents.

**1** Find the size of the pronumerals in each diagram.

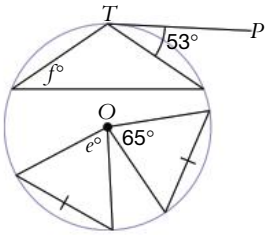
**a**



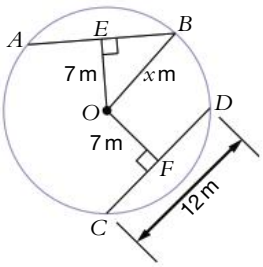

**b**



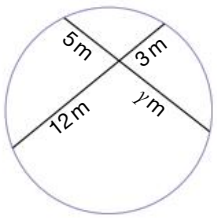

**c**



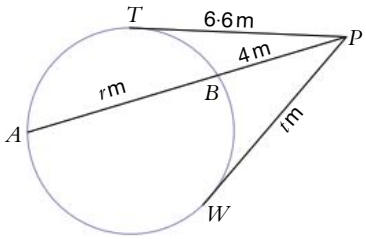

**d**



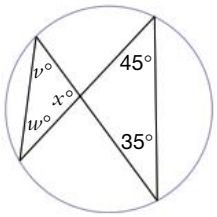

**e**



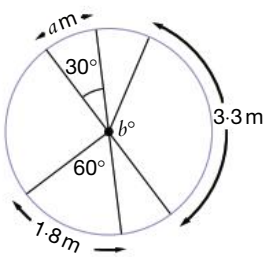

**f**



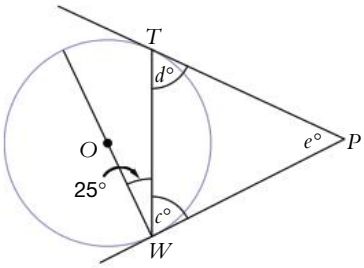

**g**



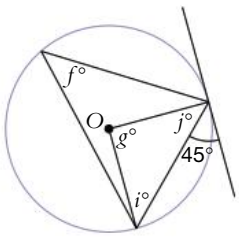

**h**



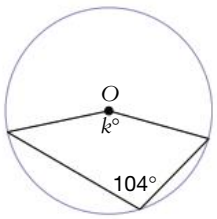

**i**



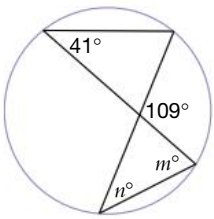

**j**



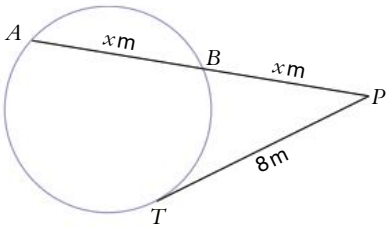

**k**



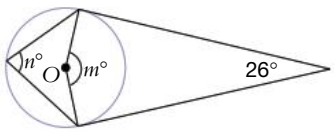

**l**




**m**



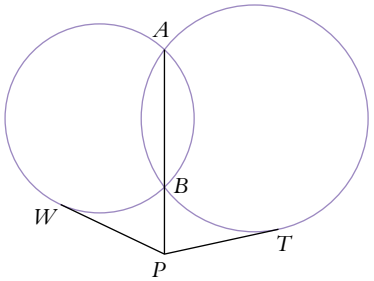

**n**



2 a $AB$ is the common chord of two circles and has been produced to $P$. From $P$, tangents $PT$ and $PW$ have been drawn to the circles.

i Prove that $WP = PT$.

ii If $BP = 14{\cdot}6$ cm and $PT = 19{\cdot}4$ cm, find the length of $AB$ correct to three significant figures.

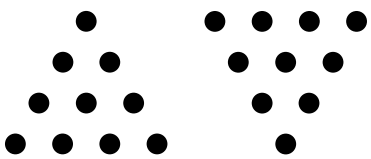



b Prove that the bisector of the angle between the tangents drawn to a circle from an external point passes through the centre.

c If $AB$, $AC$ are two tangents to a circle and $\angle BAC = 84°$, what are the sizes of the angles in the two segments into which $BC$ divides the circle?

d $P$ is a point within a circle of radius 13 cm and $XY$ is any chord drawn through $P$ so that $XP \cdot PY = 25$. Find the length of $OP$ if $O$ is the centre of the circle.

## ASSIGNMENT 13B Working mathematically

1 

Move three dots in the diagram on the left to obtain the diagram on the right.

2 A ladder hangs over the side of a ship. The rungs in the ladder are each 2·5 cm thick and are 18 cm apart. The fifth rung from the bottom of the ladder is just above the water level. If the tide is rising at a rate of 15·5 cm per hour, how many rungs will be under water in 3 hours?

3 Three circles are touching each other so that the distance between each pair of centres is 8 cm, 9 cm and 13 cm. What is the radius of each circle?

4 Four people are to be accommodated in two rooms. In how many ways can the people be arranged if there is at least one person in each room?

5 What is the smallest whole number that, when multiplied by 7, will give an answer consisting entirely of 8s?

6 a How many digits are needed to number the pages of a 50-page book?

b How many digits are needed to number the pages of a 500-page book?

c If a book had 5000 pages, how many digits would be needed?

d Can you determine an expression that will give the number of digits needed for a book with $n$ pages if $100 < n < 1000$?

# POLYNOMIALS

# 14

## Contents


**14:01** Polynomials
**14:02** Sum and difference of polynomials
**14:03** Multiplying and dividing polynomials by linear expressions
**14:04** Remainder and factor theorems
**14:05** Solving polynomial equations
**14:06** Sketching polynomials
GeoGebra activity 14:06 Sketching polynomials
Fun spot 14:06 How do you find a missing hairdresser?
**14:07** Sketching curves related to $y = P(x)$
GeoGebra activity 14:07 Sketching curves related to $y = P(x)$
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

Selections from ***Polynomials*** [Stage 5.3#]

- Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems (ACMNA266)
- Apply an understanding of polynomials to sketch a range of curves and describe the features of these curves from their equations (ACMNA268)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 14:01 Polynomials

You should be familiar with the terms monomial, binomial and trinomial, which are used to describe algebraic expressions with one, two and three terms, respectively.

- The name **polynomial** is used to describe an algebraic expression that is the sum of any number of terms, where each and every term is of the form $ax^n$ (and $n$ is a positive integer or zero).

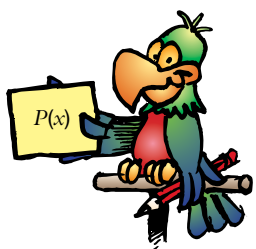
A polynomial $P(x)$ can be written in the form:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x^1 + a_0 x^0$

where $x$ is the variable

$a_n, a_{n-1}, \ldots a_2, a_1, a_0$ are the coefficients
$a_n x^n$ is the leading term
$n$ is the degree of the polynomial (only if $a_n \neq 0$) (note that $n$ is the highest power of $x$)
$a_n$ is the leading coefficient
$a_0$ is the constant term (since $x^0 = 1$)

- Note that expressions such as $x + \frac{1}{x}$, $x^2 + \sqrt{x}$ or $2^x + x$ are not polynomials in $x$, since each term is not a *positive integral* power of $x$ or a constant.
- The notation $P(x)$ is used for polynomials. $P(c)$ refers to the value of $P(x)$ at $x = c$.
- If the leading coefficient is 1, the polynomial is said to be *monic*.
- *Linear*, *quadratic* and *cubic* expressions, which have been met in earlier chapters, are polynomials of degree 1, 2 and 3 respectively.




## WORKED EXAMPLE 1

For each polynomial, state the degree, leading coefficient, constant term and number of terms.

**a** $3x^5 - 7x^4 + 2x^2 - 5$

**b** $7x - 4x^2 + x^3$

### Solutions

**a** degree = 5
leading coefficient = 3
constant term = −5
number of terms = 4

**b** rearranging in order: $x^3 - 4x^2 + 7x$
degree = 3
leading coefficient = 1 (monic)
constant term = 0
number of terms = 3

## WORKED EXAMPLE 2

$P(x)$ is a monic quadratic polynomial with a constant term of $-2$. If $P(2) = 8$, find the polynomial.

### Solution

$P(x)$ is quadratic (degree 2),
monic (leading coefficient = 1)
and has a constant term equal to $-2$.
$\therefore\ P(x) = x^2 + bx - 2$
Now, $P(2) = 2^2 + b \times 2 - 2$
$\therefore\ 8 = 2b + 2$
$\therefore\ b = 3$
$\therefore\ P(x) = x^2 + 3x - 2$

# Exercise 14:01

**1** Which expression (A, B or C) in each set is *not* a polynomial?

| | A | B | C |
|---|---|---|---|
| **a** | $x^2 + 7x - 3$ | $5x + 10$ | $3 - \frac{1}{x}$ |
| **b** | $x\sqrt{x} + 4x$ | $5x^2 - 7x^3$ | $7x$ |
| **c** | $5$ | $2^x + 1$ | $-2x^2$ |
| **d** | $x^2 + x^{-2}$ | $4x^3 - x + 3$ | $9 - x^2 + x^4$ |
| **e** | $9x - x^3 + x^5$ | $9 - x^4$ | $x^5 + 2x^{\frac{1}{2}}$ |

**2** Which of the following are monic polynomials?

**a** $2x^2 - x + 7$ **b** $x^2 - 3x - 1$ **c** $x^5 - x^3 - 1$
**d** $9x^3 - 4x^4$ **e** $x^5 + 2x^6$ **f** $6 - x - x^2$
**g** $5 + x^3 + x^6$ **h** $x^7 + 3x^4 + x$ **i** $4x^2 + 2x^3 + x^4$

**3** For each polynomial, state the degree, the leading coefficient and the constant term.

**a** $2x^3 + x^2 - x + 3$ **b** $x^5 - x^3 + x - 2$ **c** $9x^4 + x^2$
**d** $5 - 2x$ **e** $7x^6$ **f** $5 - 2x + x^2$
**g** $9 - x^2 + x^4 - x^6$ **h** $3$ **i** $x^3 - 2x^5$
**j** $\sqrt{3}x^2 + \sqrt{2}x + 1$ **k** $5x^3 + \frac{3x^4}{2}$ **l** $\frac{x^4}{3} + \frac{x^2}{2} + \frac{1}{5}$

**4** If $P(x) = x^2 + x - 3$, then $P(2)$ is found by substituting 2 for $x$. So the value of $P(2) = 3$. Similarly, find the value of $P(2)$ if:

**a** $P(x) = 3x^2 - x + 5$ **b** $P(x) = 5 - 3x + x^2$
**c** $P(x) = x^3 - x^2 + x - 2$ **d** $P(x) = \frac{x^4}{2} - \frac{x^2}{4} + 1$

**5** For each polynomial below, determine the values indicated.

| | | | |
|---|---|---|---|
| **a** $P(x) = 2x^2 - 3x + 4$ | **i** $P(1)$ | **ii** $P(-2)$ | **iii** $P(0)$ |
| **b** $P(x) = x^3 - 2x^2 + x - 3$ | **i** $P(0)$ | **ii** $P(10)$ | **iii** $P(-2)$ |
| **c** $P(x) = 6 - x^3 + x^6$ | **i** $P(2)$ | **ii** $P(5)$ | **iii** $P(-1)$ |

**6** Expand and simplify the following polynomials and state the degree, leading term and constant term for each.

**a** $(2x + 3)^2$ **b** $(x + 2)^2 - (x^2 + x + 1)$
**c** $(x + 3)^2 - (x - 3)^2$ **d** $2x(x^3 - x^2 + 1) + x^2(x^2 + 2x - 1)$
**e** $(3x - 6x^2)^2$ **f** $(x + 7)(2x - 3) - (x + 3)(2x - 7)$
**g** $(x + 1)(x + 2)(x + 3)$ **h** $(4x + 1)(3x - 1)^2$

**7** **a** $P(x)$ is a quadratic polynomial with two terms. It is known that $P(0) = 3$ and $P(1) = 5$. What is the polynomial?

**b** $P(x)$ is a cubic polynomial with a constant term of zero. If all its coefficients are equal and $P(2) = 28$, what is the polynomial?

**c** A monic cubic polynomial $P(x)$, has a constant term equal to 4. If the polynomial only has three terms and $P(2) = 14$, give the possible solutions for $P(x)$.

# 14:02 Sum and difference of polynomials

## PREP QUIZ 14:02

**A:** $3x^2 - x + 5$ **B:** $x^3 - 2x$ **C:** $6 - x^2 + x^4$

What is the degree of:

**1** A **2** B **3** C?

What is the leading coefficient of:

**4** A **5** B **6** C?

What is the constant term of:

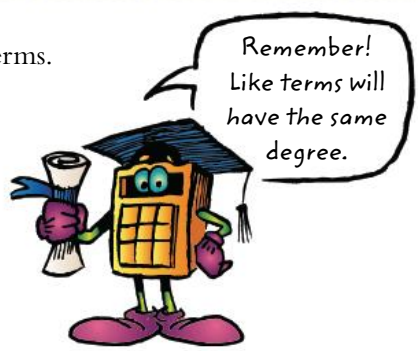
**7** A **8** B **9** C?

**10** Which polynomials above are monic?

When adding or subtracting polynomials, simply collect like terms.

If we have two polynomials $P(x)$ and $Q(x)$, then the sum $P(x) + Q(x)$ will also be a polynomial and the difference $P(x) - Q(x)$ will likewise be a polynomial.

If many terms are involved, the setting out in the worked examples on the following page may prove helpful.



## WORKED EXAMPLE 1

If $P(x) = 5x^4 + 2x^3 - x + 7$ and $Q(x) = 2x^4 - x^3 + 3x^2 - 1$, find:

**a** $P(x) + Q(x)$ **b** $P(x) - Q(x)$

### Solutions

**a**
$$\begin{array}{l} 5x^4 + 2x^3 \quad - x + 7 \\ \underline{2x^4 - \ x^3 + 3x^2 \quad - 1} \quad + \\ 7x^4 + \ x^3 + 3x^2 - x + 6 \end{array}$$

$\therefore P(x) + Q(x) = 7x^4 + x^3 + 3x^2 - x + 6$

**b**
$$\begin{array}{l} 5x^4 + 2x^3 \quad - x + 7 \\ \underline{2x^4 - \ x^3 + 3x^2 \quad - 1} \quad - \\ 3x^4 + 3x^3 - 3x^2 - x + 8 \end{array}$$

$\therefore P(x) - Q(x) = 3x^4 + 3x^3 - 3x^2 - x + 8$

## WORKED EXAMPLE 2

If $A(x) = x^3 - x^2 + x - 2$ and $B(x) = x^3 + x^2 - x + 1$, determine the degree and leading term of:

**a** $A(x) + B(x)$ **b** $A(x) - B(x)$

The process shown in Worked Example **1** can be done mentally.

### Solutions

**a** $A(x) + B(x)$
$= (x^3 - x^2 + x - 2) + (x^3 + x^2 - x + 1)$
$= 2x^3 - 1$
The degree of $A(x) + B(x)$ is 3 and the leading term is $2x^3$.

**b** $A(x) - B(x)$
$= (x^3 - x^2 + x - 2) - (x^3 + x^2 - x + 1)$
$= x^3 - x^2 + x - 2 - x^3 - x^2 + x - 1)$
$= -2x^2 + 2x - 3$
The degree of $A(x) - B(x)$ is 2 and the leading term is $-2x^2$.

## Exercise 14:02

**1** If $P(x) = x^3 + 2x^2 - 4x + 1$, $Q(x) = x^2 + 3x - 2$ and $R(x) = x^3 + 3x - 1$, determine each of the following.

**a** $P(x) + Q(x)$ **b** $P(x) + R(x)$ **c** $Q(x) + R(x)$
**d** $P(x) - Q(x)$ **e** $P(x) - R(x)$ **f** $Q(x) - R(x)$
**g** $R(x) - P(x)$ **h** $P(x) + Q(x) + R(x)$ **i** $P(x) + Q(x) - R(x)$
**j** $P(x) - Q(x) + R(x)$ **k** $P(x) - Q(x) - R(x)$ **l** $R(x) + Q(x) - P(x)$

**2** Simplify the following.

**a** $(5x^2 - 2x + 7) + (x^3 - 5x^2 - x + 3)$
**b** $(x^3 - x^2 + x + 1) + (x^3 + x^2 - x - 1)$
**c** $(2x^4 + x^2 - 1) + (x^4 - x^3 + x + 1)$
**d** $(x^4 - 2x^3 + 3x^2 - 4x + 5) + (5x^3 - x^2 + 7x - 3)$
**e** $(2x^3 + 3x^2 - 5x + 1) + (6 - 2x + 3x^2 - x^3)$
**f** $(4x^2 - x + 3) - (3x^2 + x + 1)$
**g** $(2x^3 + x^2 + 5x - 7) - (x^3 - 2x^2 + 5x + 4)$

**h** $(9x^4 + x^3 + 2x - 3) - (5x^4 + 7x^2 - 2x + 3)$
**i** $(5x^2 + 7x + 1) + (2x^2 + x - 3) + (x^2 - 10x + 7)$
**j** $(x^3 - x + 3) + (x^2 - 3x + 4) + (2x^3 - x^2 + 5)$
**k** $(x^4 + x^3 + x^2 - x - 1) + (2x^3 - x + 5) - (x^4 + 2x^2 - 7)$
**l** $(4x^5 + x^3 - 2x^2 + 7x) - (x^4 + 2x^3 - 7) + (2x^5 - x^4 + 3x^2 + 5)$

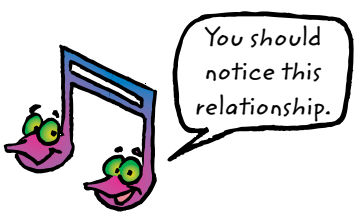
**3** For each of the following, state the degree of $A(x) + B(x)$ and its leading term.

| | | |
|---|---|---|
| **a** | $A(x) = x^3 + x^2 - x + 1$ | $B(x) = x^2 - 2x + 7$ |
| **b** | $A(x) = 2x^4 + 1$ | $B(x) = 2x^4 + 3x^2 - 7$ |
| **c** | $A(x) = 3x^3 - 2x^2 + x$ | $B(x) = 5x^4 - 3x^3 + 2x^2 + 7$ |
| **d** | $A(x) = 7x + 1$ | $B(x) = 3x^2 + 7x + 1$ |
| **e** | $A(x) = 2x^5 - x^3 + x$ | $B(x) = 4x^3 + 2x - 1$ |
| **f** | $A(x) = 7 + 3x - x^2$ | $B(x) = x^2 - 2x - 7$ |

What can be said about the degrees of $A(x)$, $B(x)$ and $A(x) + B(x)$?

**4** For each pair of polynomials $A(x)$ and $B(x)$ in Question **3**, determine the degree of $A(x) - B(x)$. What can be said about the degrees of $A(x)$, $B(x)$ and $A(x) - B(x)$?

**5** **a** $P(x) = x^3 + x^2 - x + 1$ and $Q(x) = x^2 + 2x + 3$.
Determine $A(x)$ and $B(x)$ if $A(x) = P(x) + Q(x)$ and $B(x) = P(x) - Q(x)$.
**b** Evaluate $A(2)$, $B(2)$, $P(2)$ and $Q(2)$.
**c** Evaluate $A(-1)$, $B(-1)$, $P(-1)$ and $Q(-1)$.




If $A(x) = P(x) \pm Q(x)$, then for any *x-value*, $a$, $A(a) = P(a) \pm Q(a)$.

# 14:03 Multiplying and dividing polynomials by linear expressions

A linear expression is of the form $ax + b$, where $a$ and $b$ are constants. It is a polynomial of degree 1.

## Multiplication

To multiply polynomials by a linear expression, we extend the procedure used in binomial expansions.

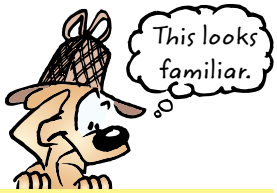
## WORKED EXAMPLES

Expand and simplify $(x + 5)(x^3 + 2x^2 - x + 1)$.

### Solution

$$\begin{aligned}(x + 5)(x^3 + 2x^2 - x + 1) &= x(x^3 + 2x^2 - x + 1) + 5(x^3 + 2x^2 - x + 1)\\ &= x^4 + 2x^3 - x^2 + x + 5x^3 + 10x^2 - 5x + 5\\ &= x^4 + 7x^3 + 9x^2 - 4x + 5\end{aligned}$$

## Division




When dividing a polynomial by another polynomial, a process is used that is similar to that known as 'long division' for integers.

This is shown to the right for the case 9736 ÷ 27.

9736 is the *dividend*
27 is the *divisor*
360 is the *quotient*
16 is the *remainder*

Note that:
Dividend = quotient × divisor + remainder
These terms are also used in polynomial division.

$$\begin{array}{r} 360 \\ 27\overline{)\,9736} \\ \underline{81\phantom{0}}\downarrow \\ 163 \\ \underline{162}\downarrow \\ 16 \\ \underline{0} \\ 16 \end{array}$$

$\therefore 9736 = 360 \times 27 + 16$

## WORKED EXAMPLES

Find the quotient and remainder when $2x^2 + 5x - 7$ is divided by $x + 2$.

### Solution

- At each step, we simply divide leading terms.
  $2x^2 \div x = 2x$ → $2x + 1$
- Then multiply $(x + 2)$ by $2x$ and subtract. → $2x^2 + 4x$
- Now, repeat the above procedure, this time dividing $(x - 7)$ by $(x + 2)$. → $x - 7$, $x + 2$, $-9$

$$\begin{array}{r} 2x + 1 \\ x + 2\overline{)\,2x^2 + 5x - 7} \\ \underline{2x^2 + 4x}\downarrow \\ x - 7 \\ \underline{x + 2} \\ -9 \end{array}$$

← Bring down next term.

So the quotient for this division is $2x + 1$ and the remainder is $-9$.

*Note:*

- The procedure is an iterative one: divide, multiply, subtract, divide, multiply, subtract … until the division process can no longer be carried out.
- The degree of the remainder must always be less than the degree of the divisor.
- If one polynomial is exactly divisible by another, the remainder will be zero. Thus, the divisor and quotient will be factors of the dividend.

## WORKED EXAMPLES

$$
\begin{array}{r}
x^2 + 10x + 49 \\
x - 5 \overline{)\, x^3 + 5x^2 - x + 7} \\
\underline{x^3 - 5x^2} \\
10x^2 - x \\
\underline{10x^2 - 50x} \\
49x + 7 \\
\underline{49x - 245} \\
252
\end{array}
$$

- Divide $x^3$ by $x$; write $x^2$ on top.
- Multiply $(x - 5)$ by $x^2$ and subtract.
- Divide $10x^2$ by $x$; write $10x$ on top.
- Multiply $(x - 5)$ by $10x$ and subtract.
- Divide $49x$ by $x$; write 49 on top.
- Multiply $(x - 5)$ by 49 and subtract.

The result of this division may be written in the following way.

$x^3 + 5x^2 - x + 7 = (x - 5)(x^2 + 10x + 49) + 252$

dividend = divisor × quotient + remainder

## Exercise 14:03

**1** Expand and simplify the following.

**a** $(x + 2)(x^2 + 3x + 1)$ **b** $(x + 4)(x^2 - 2x + 3)$ **c** $(x - 2)(x^2 + 5x - 4)$

**d** $(2x - 1)(x^2 - 2x + 5)$ **e** $(3x - 2)(x^3 - x + 1)$ **f** $(5 - x)(x^3 - 2x^2 + 1)$

**2** Check that the following are true using multiplication.

**a** $\dfrac{x^2 - 2x - 3}{x + 1} = x - 3$ **b** $\dfrac{x^3 - 3x + 2}{x + 2} = x^2 - 2x + 1$

**3** **a** If $P(x)$ is of degree 3, what is the degree of $(x - 3) \times P(x)$?

**b** If $P(x)$ is of degree 3, what is the degree of $Q(x)$ if $Q(x) = \dfrac{P(x)}{x - 3}$?

**4** **a** If $P(x) = x^3 + 2x^2 + x - 1$ and $Q(x) = x + 4$, find $R(x)$ if $R(x) = P(x) \cdot Q(x)$.

**b** Evaluate $P(1)$, $Q(1)$ and $R(1)$.

**c** Evaluate $P(-2)$, $Q(-2)$ and $R(-2)$.

**d** What relationship exists between $P(a)$, $Q(a)$ and $R(a)$?

A dot is sometimes used for *multiplication*.

**5** Complete the following divisions, expressing each result in the form dividend = divisor × quotient + remainder.

**a** $(x^2 + 2x - 3) \div (x + 1)$ **b** $(x^2 - 5x + 1) \div (x - 3)$

**c** $(3x^2 + x - 2) \div (x + 3)$ **d** $(2x^2 - 4x + 3) \div (x - 5)$

**e** $(x^3 + 2x^2 + x - 7) \div (x + 1)$ **f** $(3x^3 - 2x + 6) \div (x - 3)$

**g** $(2x^3 - x^2 + x - 1) \div (x - 2)$ **h** $(3x^3 + 2x - 5) \div (x - 1)$

**6** Determine the quotient and remainder for the following divisions.

**a** $(6x^2 - x + 2) \div (x - 3)$ **b** $(x^3 - 2x^2 + x + 1) \div (x + 2)$

**c** $(2x^3 - x^2 + 3x + 3) \div (2x + 1)$ **d** $(6x^3 - 5x^2 - 8x + 3) \div (2x - 3)$

**e** $(x^5 - x^4 + 2x^3 - x^2 + x + 3) \div (x + 1)$ **f** $(2x^6 + 2x^4 - 3x^2 + 4) \div (x - 1)$

**7** Show that the remainder is zero in each of the following, and hence write the dividend as the product of two factors.

**a** $(x^3 + 2x^2 + 3x + 2) \div (x + 1)$ **b** $(2x^3 - 3x^2 + 4x + 3) \div (2x + 1)$

**c** $(2x^3 - 3x^2 + 1) \div (x - 1)$ **d** $(2x^3 + x^2 + 1) \div (x + 1)$

# 14:04 Remainder and factor theorems

- In the last section, it was seen that if $P(x)$ is divided by $(x - a)$, the result can be expressed in the form:

$P(x) = (x - a)Q(x) + r$
where $Q(x)$ is the quotient and $r$ is the remainder.

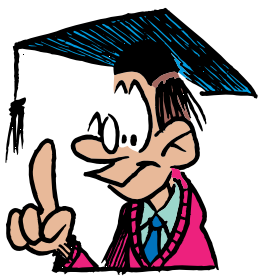
Now, if we let $x$ equal $a$ in this statement, it becomes:

$P(a) = (a - a)Q(a) + r$
$= 0 \times Q(a) + r$
$\therefore P(a) = r$

$Q(x)$ ← quotient
$x - a \overline{)\ P(x)}$ ← dividend
divisor
$r$ ← remainder
$P(x) = (x - a)\,Q(x) + r$

- If a given polynomial $P(x)$ is divided by $(x - a)$, then the remainder will be $P(a)$, which is the value of $P(x)$ when $a$ is substituted for $x$. This is called the **remainder theorem**.

> *Remainder theorem:* If a given polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.



- From the remainder theorem, it can be seen that if $r = 0$ then $P(x) = (x - a)\ Q(x)$. This means that $(x - a)$ is a factor of $P(x)$, and leads to the **factor theorem**.

> *Factor theorem:*
> $(x - a)$ divides $P(x)$ if and only if $P(a) = 0$.

*Note:* 'if and only if' means that the statement is also true in reverse, that is:
- If $(x - a)$ divides $P(x)$, then $P(a) = 0$.
- If $P(a) = 0$, then $(x - a)$ divides $P(x)$.

## WORKED EXAMPLE 1

Determine the remainder when $x^3 - 2x^2 + x - 1$ is divided by $x - 2$.
Is $x - 2$ a factor of $x^3 - 2x^2 + x - 1$?

### Solution

Divisor is $x - 2$, so remainder $r = P(2)$.

Let $P(x) = x^3 - 2x^2 + x - 1$
$\therefore P(2) = 2^3 - 2(2)^2 + 2 - 1$
$= 8 - 8 + 2 - 1$
$= 1$
$\therefore$ Remainder = 1
$x - 2$ is not a factor because the remainder is not zero.

## WORKED EXAMPLE 2

Show that $x - 3$ is a factor of $P(x) = x^3 - 2x^2 - 5x + 6$, and hence express $P(x)$ as a product of its factors.

### Solution

If $(x - 3)$ is a factor of $P(x)$, then $P(3)$ will equal zero.

Now $P(3) = (3)^3 - 2(3)^2 - 5(3) + 6$
$= 27 - 18 - 15 + 6$
$= 0$

$\therefore$ $(x - 3)$ must be a factor of $P(x)$.

If $P(x)$ is now divided by $x - 3$, this will enable the other factors to be found.

$\therefore$ $P(x) = (x - 3)(x^2 + x - 2)$

Further factorising $x^2 + x - 2$ gives:
$\therefore$ $P(x) = (x - 3)(x - 1)(x + 2)$

$$
\begin{array}{r}
x^2 + x - 2 \\
x - 3 \overline{)\, x^3 - 2x^2 - 5x + 6} \\
\underline{x^3 - 3x^2} \\
x^2 - 5x \\
\underline{x^2 - 3x} \\
-2x + 6 \\
\underline{-2x + 6} \\
0
\end{array}
$$

## WORKED EXAMPLE 3

If $f(x) = x^3 + ax + b$ is divisible by both $x + 2$ and $x - 3$, find the values of $a$ and $b$.

### Solution

If $f(x) = x^3 + ax + b$ is divisible by $x + 2$ and $x - 3$,

then $f(-2) = 0$ and $f(3) = 0$

$(-2)^3 + a(-2) + b = 0$ $(3)^3 + a(3) + b = 0$

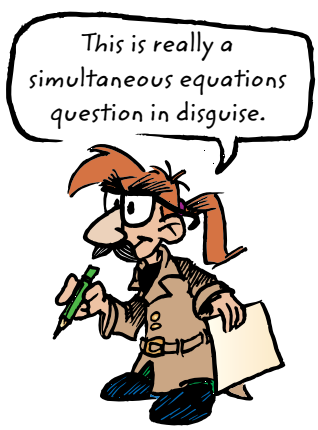
$-8 - 2a + b = 0$ $27 + 3a + b = 0$

$2a - b = -8$ $3a + b = -27$

Solving simultaneously:

$$
\begin{array}{rl}
2a - b = -8 & \\
3a + b = -27 & + \\
\hline
5a = -35 & \\
\therefore \quad a = -7 & \\
b = -6 &
\end{array}
$$

The values of $a$ and $b$ are $a = -7$, $b = -6$.
$\therefore$ $f(x) = x^3 - 7x - 6$




## Exercise 14:04

**1** Use the remainder theorem to find the remainder for the following divisions. Check your answer by carrying out the division.

**a** $(x^2 + 7x - 5) \div (x - 1)$
**b** $(x^2 - 3x - 10) \div (x + 2)$
**c** $(2x^2 + 3x + 7) \div (x + 1)$
**d** $(5x^2 + 4x - 10) \div (x - 2)$
**e** $(x^3 - 3x^2 + 7x - 5) \div (x - 3)$
**f** $(x^4 + 5x^2 - 6) \div (x - 1)$

**2** Find the remainder for each division when $P(x)$ is divided by $A(x)$.

**a** $P(x) = x^2 + 5x + 7$, $A(x) = x - 1$ **b** $P(x) = x^2 - 2x - 9$, $A(x) = x + 4$

**c** $P(x) = 2x^2 + x - 11$, $A(x) = x - 3$ **d** $P(x) = 3x^2 - 4x + 1$, $A(x) = x + 3$

**e** $P(x) = 2x^3 + 7x - 13$, $A(x) = x - 1$ **f** $P(x) = 5x^3 + x^2 - 4x$, $A(x) = x - 2$

**g** $P(x) = x^4 - x^2 + 1$, $A(x) = x - 2$ **h** $P(x) = x^5 - x^3 + 4x^2 + 7$, $A(x) = x + 2$

**3** Determine the remainder when $x^4 + 2x^3 - 5x^2 - 7x + 2$ is divided by:

**a** $(x + 1)$ **b** $(x - 1)$ **c** $(x - 3)$ **d** $(x + 3)$

**4** **a** If $x^3 - 4x^2 - 7x + k$ is divided by $(x - 4)$, the remainder is 2. Find the value of $k$.

**b** When $x^4 - 3kx^2 + 5k$ is divided by $(x + 1)$, the remainder is 5. Find the value of $k$.

**c** When $x^2 + x - 1$ is divided by $(x - k)$, the remainder is 5. Find all possible values of $k$.

**5** Find whether each polynomial $P(x)$ has the linear polynomial indicated as a factor.

**a** $P(x) = x^2 + 7x - 18; x + 9$ **b** $P(x) = 5x^2 - 9x - 2; x + 2$

**c** $P(x) = x^3 + x - 2; x + 1$ **d** $P(x) = x^3 - 2x^2 + 1; x - 1$

**e** $P(x) = 2x^3 + 3x^2 - 1; x + 1$ **f** $P(x) = x^3 - x^2 - 10x - 8; x + 2$

**6** Show that the first polynomial is a factor of $P(x)$ and then determine all the factors of $P(x)$.

**a** $x - 1, P(x) = x^3 + 4x^2 + x - 6$ **b** $x + 1, P(x) = x^3 - 6x^2 + 5x + 12$

**c** $x + 1, P(x) = x^3 + 4x^2 + 5x + 2$ **d** $x - 2, P(x) = x^3 - 5x^2 + 8x - 4$

**7** Determine the factors of the following polynomials.

**a** $x^3 - 2x^2 - 5x + 6$ **b** $x^3 - 6x^2 - 13x + 42$ **c** $x^3 - 4x^2 + x + 6$

**d** $x^3 - 21x + 20$ **e** $x^3 + 4x^2 - 15x - 18$ **f** $2 + 3x - 14x^2 - 15x^3$

**8** For what value of $k$ is:

**a** $x - 1$ a factor of $x^2 - 5x + k$ **b** $x + 2$ a factor of $2x^2 - x + k$

**c** $x + 1$ a factor of $x^2 + kx - 7$ **d** $x - 4$ a factor of $3x^2 - kx - 32$?

**9** If $(x + 1)$ and $(x + 2)$ are both factors of $x^3 + ax^2 + bx - 10$, find the values of $a$ and $b$.

**10** If $x^3 + px + q$ is divisible by both $x - 3$ and $x + 2$, find the values of $p$ and $q$.

# 14:05 Solving polynomial equations

- A polynomial $P(x)$ can take many values. The values of $x$ that make $P(x)$ zero are called, appropriately enough, **zeros**. These zeros must also be the solutions to the equation $P(x) = 0$. The solutions to equations are often called the roots of the equation.
- The factor theorem is the basis for solving equations of the form $P(x) = 0$.
- If a polynomial is of degree $n$, it can have at most $n$ factors,

i.e $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \underbrace{(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)}_{n \text{ factors}}$

This means that a polynomial of degree $n$ can have a maximum of $n$ zeros and that the equation $P(x) = 0$ can have a maximum of $n$ solutions (or roots).

The following example shows the steps involved in solving equations of this type.

## WORKED EXAMPLE

Solve the equation $x^3 - x^2 - 10x - 8 = 0$.

### Solution

**Step 1** Find a value of $x$ for which $P(x) = 0$

Let $P(x) = x^3 - x^2 - 10x - 8 = 0$

The values of $x$ to try are the factors of the constant term, $-8$, namely $\{1, -1, 2, -2, 4, -4, 8, -8\}$.

Starting with the smallest value, we use trial and error to find one that makes $P(x)$ equal zero.

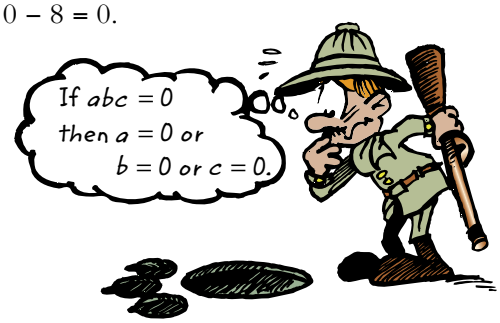
Now $P(1) = 1 - 1 - 10 - 8 \neq 0$
but $P(-1) = -1 - 1 + 10 - 8 = 0$.

**Step 2** Use the factor theorem.

Since $P(-1) = 0$, then $x + 1$ is a factor.

**Step 3** Divide $P(x)$ by the factor.

$$\begin{array}{r} x^2 - 2x - 8 \\ x + 1 \overline{)\, x^3 - x^2 - 10x - 8} \\ \underline{x^3 + x^2} \\ -2x^2 - 10x \\ \underline{-2x^2 - 2x} \\ -8x - 8 \\ \underline{-8x - 8} \\ 0 \end{array}$$




**Step 4** Write $P(x)$ as a product of its factors.

$$x^3 - x^2 - 10x - 8 = (x + 1)(x^2 + 2x - 8)$$
$$= (x + 1)(x + 2)(x - 4)$$

**Step 5** Write the solutions.

$x^3 - x^2 - 10x - 8 = 0$
$\therefore (x + 1)(x + 2)(x - 4) = 0$
$\therefore x = -1, -2$ or $4$

## Exercise 14:05

**1** The first step in solving the equation $x^3 - 2x^2 - 11x + 12 = 0$ is to find a value of $x$ that solves the equation. What values of $x$ should be tried?

**2** You have found that $x = 3$ is a solution to the equation $x^3 - 2x^2 - 11x + 12 = 0$. How do you use this to factorise $x^3 - 2x^2 - 11x + 12$?

**3** Factorise:

**a** $x^2 - x - 12$ **b** $2x^2 - 5x + 2$

**4** Solve:

**a** $(x - 3)(x - 2)(x + 1) = 0$ **b** $(x + 1)(x - 4)^2 = 0$

**c** $(2x - 1)(x + 4)(x - 1)(x + 2) = 0$ **d** $(x - 2)^2(x + 3)^2 = 0$

**5** Factorise completely and solve:

**a** $(x - 2)(x^2 - 2x - 8) = 0$ **b** $(p + 3)(2p^2 + 5p - 3) = 0$

**c** $(x - 1)(x^3 + 4x^2 + x - 6) = 0$ **d** $(p + 2)(2p^3 - p^2 - 13p - 6) = 0$

**6** These questions refer to the equation $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0$.

**a** Show that $x = -1$ is a solution.

**b** If $Q(x) = (2x^4 - 7x^3 - 2x^2 + 13x + 6) \div (x + 1)$, find $Q(x)$.

**c** Show that $x - 2$ is a factor of $Q(x)$.

**d** Write $2x^4 - 7x^3 - 2x^2 + 13x + 6$ as a product of its linear factors.

**e** Using your answer to **d**, write the roots of the equation $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0$.

**7** Find all real roots of these equations.

**a** $x^3 - 4x^2 + x + 6 = 0$ **b** $x^3 + 3x^2 + 7x + 10 = 0$

**c** $x^3 - x^2 - x + 1 = 0$ **d** $x^3 + 2x^2 - 9x - 18 = 0$

**e** $x^3 - 3x^2 + 3x - 1 = 0$ **f** $x^3 + 2x^2 + 4x + 21 = 0$

**8** **a** Solve the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$.

**b** Solve the equation $y^2(y - 3) = 6y - 8$.

**c** A box is to have a square base and a height that is 10 cm longer than the length of the base. If the volume of the box is to be 2000 $cm^3$, write a polynomial equation to represent this information and, by solving the equation, find the dimensions of the box.

# 14:06 Sketching polynomials

## PREP QUIZ 14:06

What is the degree of these polynomials?

**1** $2x - 3$ **2** $x^2 + 5x - 6$ **3** $4x^3 - 10$

**4** Expand $x(x - 1)(x + 2)$.

**5** What are the zeros of the polynomial $x(x - 1)(x + 2)$?

**6** Where will the curve $y = x(x - 1)(x + 2)$ cut the $x$-axis?

**7** What is the $y$-intercept of the curve $y = x(x - 1)(x + 2)$?

**8** Is $x - 1$ a factor of $x^3 + 2x^2 - 3$?

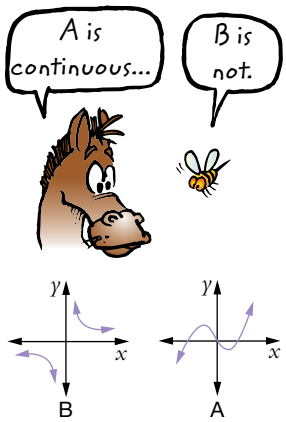
**9** If $x = 1\,000\,000$, which term of the polynomial $x^4 + 100x^3 + 1000x^2 + 1\,000\,000$ has the greatest value?

**10** For the polynomial $y = x^3 - 2x^2 + x - 6$, which term will have the biggest influence on the value of $y$ when $x$ is large?

**Polynomials**
Linear: degree 1
Quadratic: degree 2
Cubic: degree 3

- The definition of a polynomial in Section 14:01 reminds us that linear, quadratic and cubic expressions are in fact all polynomials. Hence, we already know how to sketch a considerable number of polynomials.
  The curves sketched in Exercises 7:04–7:05 and 7:09–7:12 are polynomials, and all of the techniques learnt there are applicable in this section of work.

- The $x$-and $y$-intercepts are important in sketching any curve.
  As $x$-intercepts occur when $y = 0$, the $x$-intercepts of $y = P(x)$ are the roots of the equation $P(x) = 0$. The $y$-intercept occurs when $x = 0$. So for $y = P(x)$, the $y$-intercept is $P(0)$.
  $P(0)$ is the constant term of the polynomial, so the $y$-intercept is the constant term of $P(x)$.
- Polynomials are examples of *continuous functions*.
  This means they have no gaps. Polynomials can have any $x$-value from negative infinity ($-\infty$) to positive infinity ($+\infty$), and every $x$-value has a corresponding $y$-value.
- Questions **9** and **10** in the Prep quiz illustrate that as $x$ becomes very large or very small (i.e. as $x$ approaches positive or negative infinity), the sign of $P(x)$ is the same as the sign of the leading term, $ax^n$.



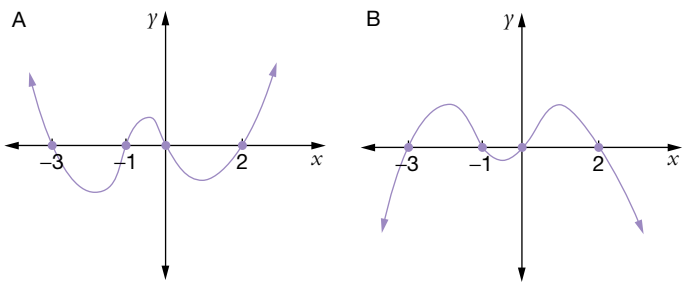

For the polynomial $y = P(x)$, as $x \to \pm\infty$, $P(x) \to ax^n$

- In this section we will consider only the graphs of polynomials that can be factorised.
- We will now see how we use these ideas in conjunction with what we have learnt in Chapter 7 to sketch the graphs of polynomials.

## WORKED EXAMPLE 1

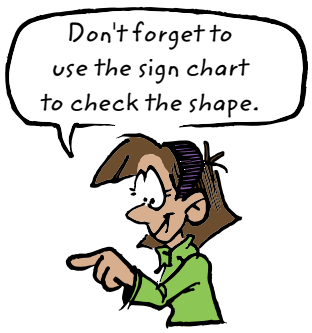
Sketch the graph of the polynomial $y = x(x + 1)(x - 2)(x + 3)$.

- When $y = 0$,
  $x(x + 1)(x - 2)(x + 3) = 0$
  $\therefore\ x = 0, -1, 2, -3$
  The curve has 4 $x$-intercepts.
- As the curve is continuous, there are only two possible shapes, i.e A or B.




Different factors produce different $x$-intercepts.

- If the polynomial were expanded it would be of degree 4, with a leading term of $x^4$. As $x \to \pm\infty$ it will become like $x^4$. Hence, sketch A is correct.




| $x$ | −4 | −3 | −2 | −1 | $-\frac{1}{2}$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | (+) | 0 | (−) | 0 | (+) | 0 | (−) | 0 | (+) |

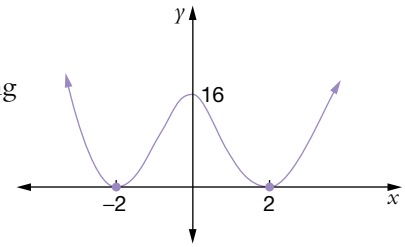
## WORKED EXAMPLE 2

Sketch the graph of the polynomial: $y = (x - 2)^2(x + 2)^2$

- If this were expanded, it would be of degree 4 with leading term $x^4$.
- It has only two $x$-intercepts, namely at $x = 2$ and $-2$. Each of these is said to be a double root.
- A sign analysis, the $y$-intercept and an analysis of the leading term as $x \to \pm\infty$ will confirm the shape of the curve.

| $x$ | $-3$ | $-2$ | $0$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $y$ | (+) | 0 | (+) | 0 | (+) |

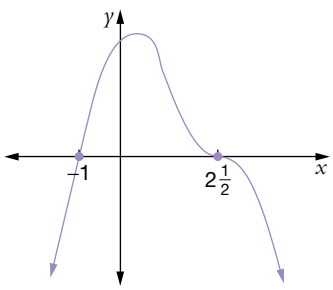
Don't forget to check the $y$-intercept. When $x = 0$, $y = 16$.




## WORKED EXAMPLE 3

Sketch the graph of the polynomial: $y = -(x + 1)(2x - 5)^3$

- If expanded it would be of degree 4 with leading term $-x^4$.
- It has only two $x$-intercepts, at $-1$ and $2\frac{1}{2}$. The polynomial has a single root at $x = -1$ and a triple root at $x = 2\frac{1}{2}$. The triple root has the effect of turning the curve over, as shown in the sketch.
- Sign analysis and the $y$-intercept will confirm the rest of the shape.

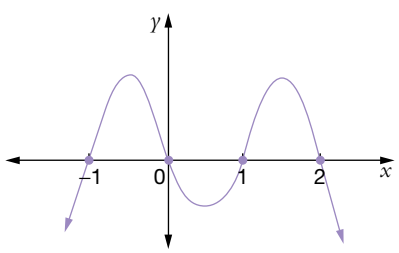



If $x = 3$,
$y = -(3 + 1)(6 - 5)^3$
$= -(4)(1)^3$
$= -4$

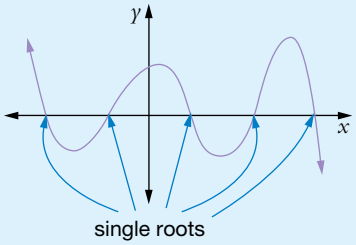
## WORKED EXAMPLE 4

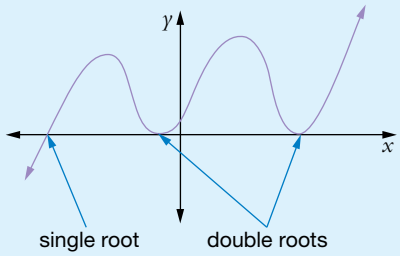
Sketch the graph of the polynomial: $y = -x^4 + 2x^3 + x^2 - 2x$

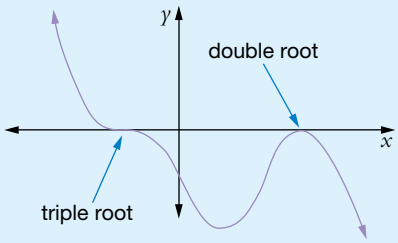
- First, we need to factorise $-x^4 + 2x^3 + x^2 - 2x$.
  $-x^4 + 2x^3 + x^2 - 2x = -x(x^3 - 2x^2 - x + 2)$
  Using the factor theorem on $(x^3 - 2x^2 - x + 2)$ shows that $x - 1$ is a factor.
  Dividing gives $y = -x(x - 1)(x^2 - x - 2)$
  $\therefore\ y = -x(x - 1)(x - 2)(x + 1)$
  $\therefore$ The polynomial has 4 single roots: 0, 1, 2 and $-1$.
- The leading term is $-x^4$, so $y$ behaves like $-x^4$ as $x \to \pm\infty$.
  As $x \to \pm\infty$, $y \to -\infty$.
  Hence, the sketch is as shown.



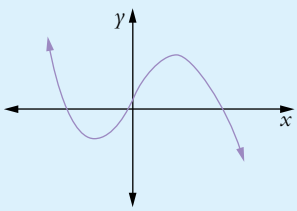
- When sketching $y = P(x)$, the nature of the roots will control the shape of the curve.

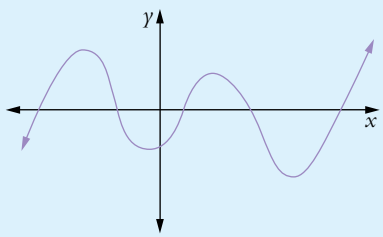


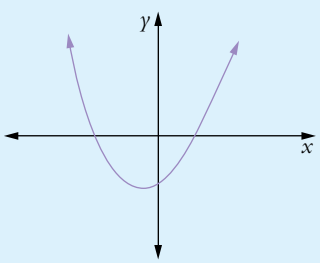



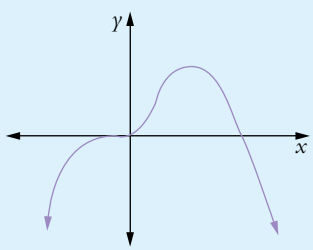

- If the degree of the polynomial is *odd*, the arrows on the ends of the curve will point in opposite directions.




- If the degree of the polynomial is *even*, the arrows on the ends of the curve will point in the same direction.




## Exercise 14:06

**1** The following polynomials are given in factored form. Determine their zeros and sketch their graphs, indicating clearly where they cut the $x$-axis.

**a** $y = (x + 1)(x - 3)$
**b** $y = 2x(x - 5)$
**c** $y = x(x + 1)(x - 1)$
**d** $y = (x - 2)(x - 1)(x + 1)$
**e** $y = (x + 4)(x + 1)(x - 2)$
**f** $y = x(x - 3)(x + 5)$
**g** $y = -x(x + 2)(x - 1)$
**h** $y = (3 - x)(x + 1)(x + 2)$
**i** $y = (x - 2)(x + 1)(1 - x)$
**j** $y = (2 - x)(x + 1)(x + 4)$
**k** $y = x(x + 2)(x + 1)(x - 3)$
**l** $y = (x + 3)(x + 1)(x - 1)(3 - x)$

**2** Each polynomial has a repeated root. Sketch the graph of each neatly.

**a** $y = (x - 1)^2$ **b** $y = -(x + 2)^2$ **c** $y = x(x + 2)^2$

**d** $y = (x + 1)(x - 2)^2$ **e** $y = (x + 3)^2(x - 1)$ **f** $y = -x(x - 1)^2$

**g** $y = x^2(5 - x)$ **h** $y = (x + 3)^3$ **i** $y = -(x + 3)(x - 5)^2$

**j** $y = (5 - x)^3$ **k** $y = x(x + 1)(x - 2)^2$ **l** $y = (x + 1)(x - 2)^3$

**3** Use the factor theorem to factorise the following and hence sketch their graphs, showing clearly the zeros of each polynomial.

**a** $y = x^3 + 4x^2 + x - 6$ **b** $y = x^3 + 6x^2 + 11x + 6$

**c** $y = x^3 - 6x^2 + 5x + 12$ **d** $y = x^3 - 5x^2 + 8x - 4$

**e** $y = x^3 + 8x^2 + 17x + 10$ **f** $y = x^3 - 3x^2 + 3x - 1$

**g** $y = -x^3 + 2x^2 + 5x - 6$ **h** $y = -x^3 + 4x^2 - x - 6$

**i** $y = x^4 - 17x^2 + 16$ **j** $y = x^4 - x^3 - 10x^2 - 8x$

**4** Write an equation that could represent each of the following and sketch the curve.

**a** $P(x)$ is of degree 3 and has a double root at 2 and a single root at $-1$. It has a negative $y$-intercept.

**b** The polynomial $y = P(x)$ has a double root at $-2$ and a triple root at 1. It is of degree 5 with a leading term of $-2x^5$.

**c** The polynomial is quartic (of degree 4). It has roots at $-3$, $-1$ and 4 and a positive $y$-intercept. The leading term is either $x^4$ or $-x^4$.

**d** The polynomial is quartic (of degree 4). It has roots at $-3$, $-1$ and 4 and a negative $y$-intercept. The leading term is either $x^4$ or $-x^4$.

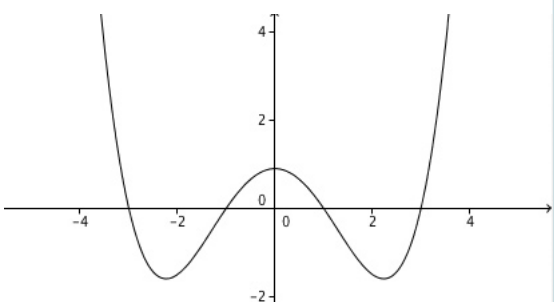
**5** Can a cubic equation have no roots? Give examples of cubic equations that have 1, 2 and 3 roots.

**6** A polynomial is monic and of degree 5. It has two double roots and a single root. Sketch the possible shapes of this polynomial.

## P GEOGEBRA ACTIVITY 14:06 SKETCHING POLYNOMIALS

In this activity you can study the graphs of different polynomials given in factored form and investigate the effect that repeated factors have on the shape. You can then test your skills by seeing if you can predict the equation of a randomly generated curve.

Another activity allows you to choose a polynomial that has an equation that is not in factored form. You will need to factorise it and sketch the curve in your workbook. You can then tick a checkbox to see the solution.




$y = 0.1\ (x - 3)\ (x - 1)\ (x + 1)\ (x + 3)$

## FUN SPOT 14:06 HOW DO YOU FIND A MISSING HAIRDRESSER?

Answer each question and write the letter for that part in the box that is above the correct answer.

*Consider the polynomial* $P(x) = x^3 - 2x^2 - x + 2$.

**E** What is the degree? **N** What is the leading coefficient?

**A** What is the constant term? **O** Evaluate $P(1)$.

**Y** What is the remainder when $P(x)$ is divided by $x + 2$?

*For the circle* $(x - 2)^2 + (y - 3)^2 = 16$, *find:*

**B** the radius **I** the coordinates of the centre.

*For the circle* $(x + 2)^2 + (y + 3)^2 = 25$, *find:*

**A** the radius **E** the coordinates of the centre.

**M** What is the $y$-intercept of the curve $y = x^3 + 8$?

**E** Find $a$ if the curve $y = ax^3 - 5$ passes through the point $(1, 10)$.

**T** Find $k$ if the curve $y = k(x + 1)(x - 1)(x - 2)^2$ passes through $(3, -16)$.

*What would be the equation of the curve that is obtained if* $y = x^3$ *is translated:*

**H** up 1 unit **O** down 1 unit **E** 1 unit to the right **R** 1 unit to the left?

*Solve the equations:*

**T** $(x - 1)(x + 2)(x - 3) = 0$ **U** $(x + 1)(x - 2)(x + 3) = 0$

**R** $x^3 - 2x^2 - x + 2 = 0$ **C** $x^3 + 2x^2 - x - 2 = 0$

| −12 | $y = x^3 - 1$ | −1, 2, −3 | −2, −1, 1 | 0 | −2 | 4 | −8 | $y = x^3 + 1$ | 3 | $y = (x - 1)^3$ | 1 | 1, −2, 3 | (2, 3) | $y = (x + 1)^3$ | 15 | 2 | −1, 1, 2 | (−2, −3) | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

# 14:07 Sketching curves related to $y = P(x)$

## Using $y = P(x)$ to sketch $y = P(x) + c$

In Chapter 7 we saw how $y = ax^n + d$ was obtained by moving the curve $y = ax^n$ vertically by $d$ units.

The tables below show that the same results hold for the curves $y = P(x) + c$ and $y = P(x)$. Clearly, $y = P(x) + c$ is obtained by moving $y = P(x)$ vertically $c$ units.

$y = P(x)$

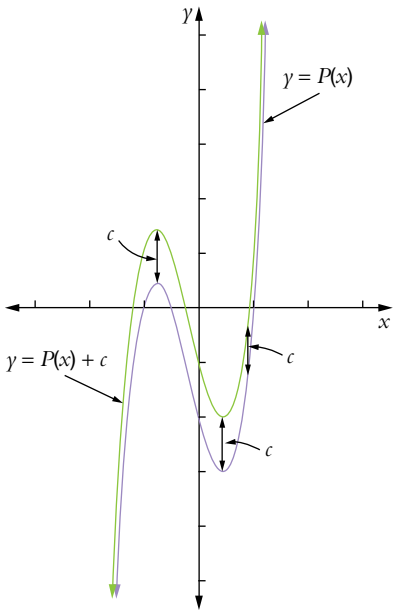
| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | $P(-2)$ | $P(-1)$ | $P(0)$ | $P(1)$ | $P(2)$ |

$y = P(x) + c$

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | $P(-2) + c$ | $P(-1) + c$ | $P(0) + c$ | $P(1) + c$ | $P(2) + c$ |

The tables show that if $c$ is added to each of the $y$-values in the table for $y = P(x)$, the values in the table for $y = P(x) + c$ are obtained.

If $y = P(x)$ is translated vertically by $c$ units, it produces the curve $y = P(x) + c$.




$y = P(x) + c$ is obtained by translating the curve $y = P(x)$ vertically by $c$ units.

- If $c$ is positive it moves up.
- If $c$ is negative it moves down.

## Using $y = P(x)$ to sketch $y = -P(x)$

$y = P(x)$

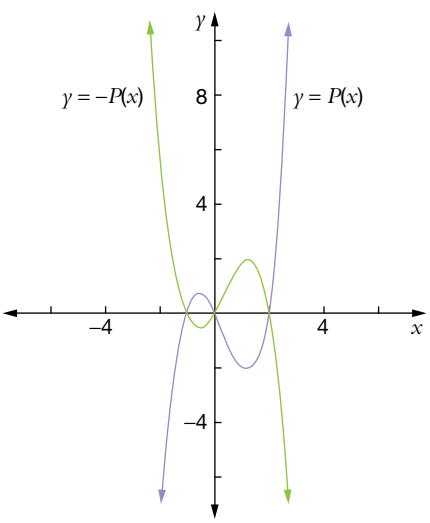
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $P(-2)$ | $P(-1)$ | $P(0)$ | $P(1)$ | $P(2)$ |

$y = -P(x)$

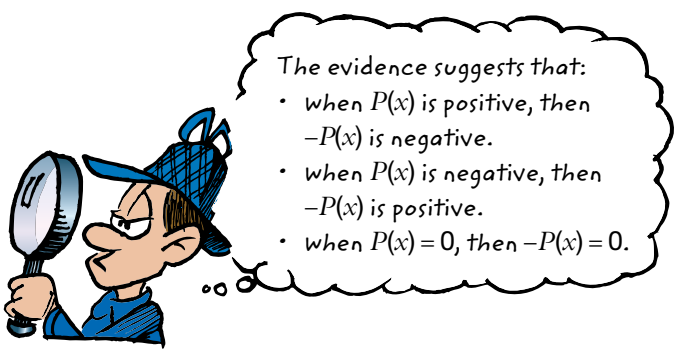
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $-P(-2)$ | $-P(-1)$ | $-P(0)$ | $-P(1)$ | $-P(2)$ |

The tables show that the curves have the same $x$-values but opposite $y$-values. Hence, to obtain the curve $y = -P(x)$ from $y = P(x)$, simply replace all the $y$-values in the table for $y = P(x)$ with their opposites.

Now, by keeping the $x$-values the same and changing the $y$-values to their opposites, we are in fact producing a curve that is the reflection of the first curve in the $x$-axis.




$y = -P(x)$ is obtained by reflecting the curve $y = P(x)$ in the $x$-axis.



## Using $y = P(x)$ to sketch $y = aP(x)$

$y = P(x)$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $P(-2)$ | $P(-1)$ | $P(0)$ | $P(1)$ | $P(2)$ |

$y = aP(x)$

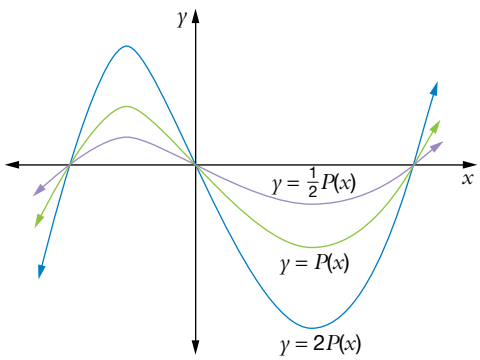
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $aP(-2)$ | $aP(-1)$ | $aP(0)$ | $aP(1)$ | $aP(2)$ |

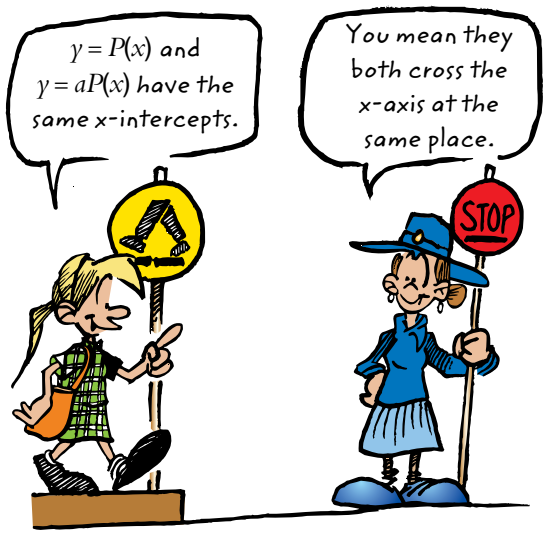
To obtain the values for $y = aP(x)$ from the values for $y = P(x)$, we simply multiply all the $y$-values by $a$.

The size and sign of $a$ are both important in determining the shape of $y = aP(x)$.

For positive $a$:

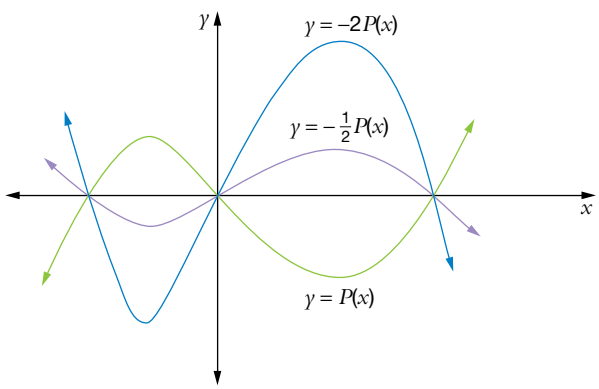
If $a > 1$, $y = aP(x)$ is obtained by stretching $y = P(x)$ vertically.
If $a < 1$, $y = aP(x)$ is obtained by squashing $y = P(x)$ vertically.
If $y = 0$ on $y = P(x)$, then $y = 0$ on $y = aP(x)$.




For negative $a$:

If $a < -1$, $y = aP(x)$ is obtained by reflecting $y = P(x)$ in the $x$-axis and then stretching it vertically.
If $-1 < a < 0$, (i.e. $a$ is a negative fraction), $y = aP(x)$ is obtained by reflecting $y = P(x)$ and then squashing it vertically.



## Using $y = P(x)$ to sketch $y = P(-x)$

The table for $y = P(x)$ is shown on the right.

Consider $y = P(-x)$.

As $x$ and $-x$ are opposites, swapping each $x$-value in Table 1 with its opposite ($-x$) (i.e. $-2$ with 2, $-1$ with 1, 1 with $-1$ and 2 with $-2$) will give you the values in Table 2.

Rearranging the values in Table 2, so that the $x$-values are in ascending order, produces Table 3, which is the table for $y = P(-x)$.

Hence, the points on $y = P(-x)$ can be obtained by moving each point on $y = P(x)$ horizontally from one side of the $y$-axis to an equal distance on the other side.

Geometrically, this means that $y = P(-x)$ can be obtained by reflecting $y = P(x)$ in the $y$-axis. The diagram below shows this procedure.

**Table 1** $y = P(x)$

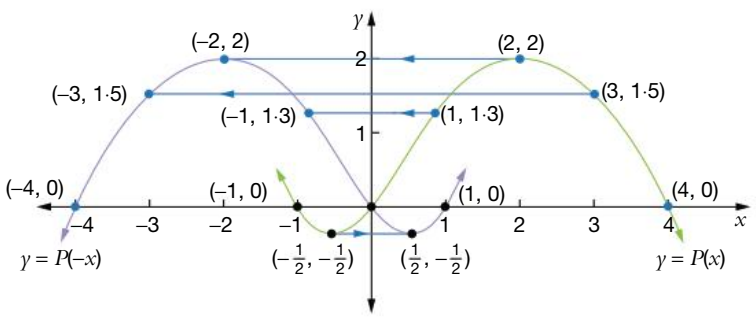
| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | $P(-2)$ | $P(-1)$ | $P(0)$ | $P(1)$ | $P(2)$ |

**Table 2** $y = P(-x)$

| $x$ | 2 | 1 | 0 | $-1$ | $-2$ |
|---|---|---|---|---|---|
| $y$ | $P(-2)$ | $P(-1)$ | $P(0)$ | $P(1)$ | $P(2)$ |

**Table 3** $y = P(-x)$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | $P(2)$ | $P(1)$ | $P(0)$ | $P(-1)$ | $P(-2)$ |




$y = P(-x)$ is obtained by reflecting the curve $y = P(x)$ in the $y$-axis.

## WORKED EXAMPLES

Use this graph of $y = P(x)$ to make sketch graphs of:

1 $y = P(x) + 1$

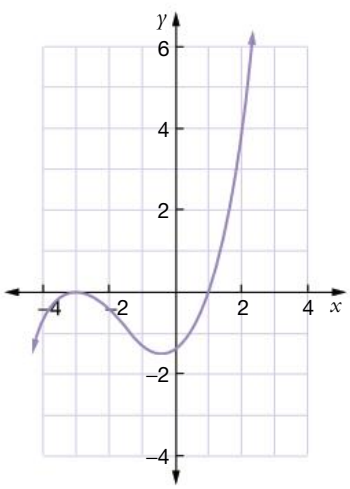
2 $y = P(x) - 2$

3 $y = \frac{1}{2}P(x)$

4 $y = -P(x)$

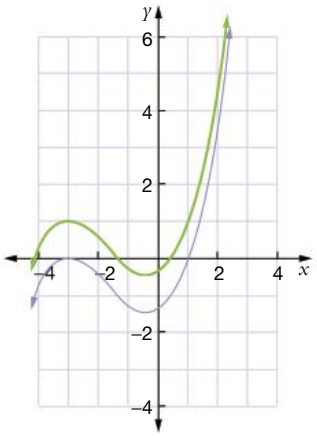
5 $y = P(-x)$

6 $y = 1 - \frac{1}{2}P(-x)$



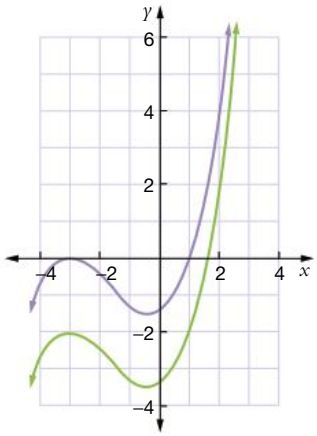
## Solutions

**1**



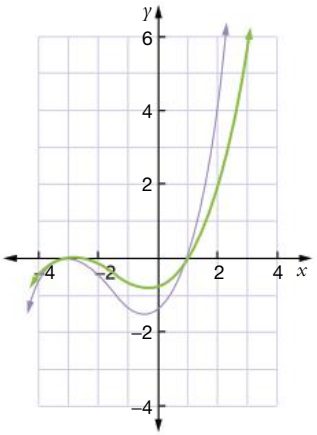

$y = P(x) + 1$ is obtained by moving each point on $y = P(x)$ up 1 unit.

**2**



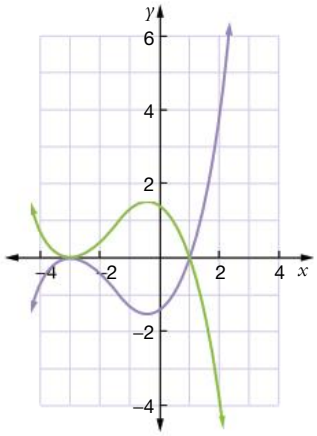

$y = P(x) - 2$ is obtained by moving each point on $y = P(x)$ down 2 units.

**3**



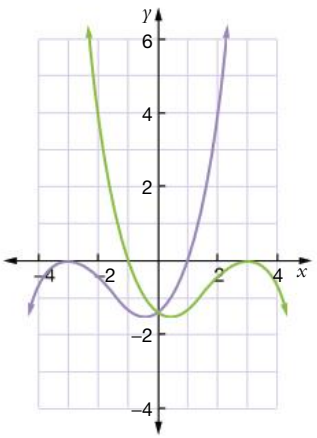

$y = \frac{1}{2}P(x)$ is obtained by halving the $y$-coordinate of each point on $y = P(x)$.

**4**



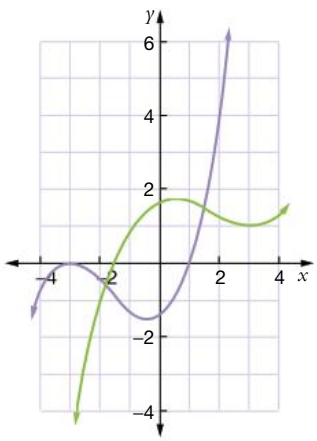

$y = -P(x)$ is the reflection of $y = P(x)$ in the $x$-axis.

**5**




$y = P(-x)$ is the reflection of $y = P(x)$ in the $y$-axis.

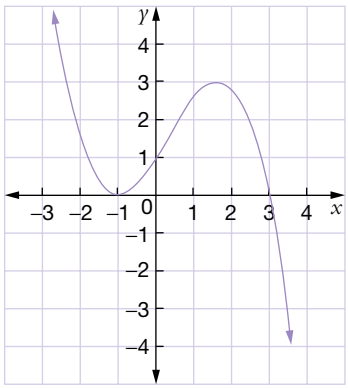
**6**




$y = 1 - \frac{1}{2}P(-x)$ is built from $y = P(x)$ in stages. $P(x)$ is reflected in the $y$-axis and then the $x$-axis to give $-P(-x)$. This is then halved and moved up 1 unit.

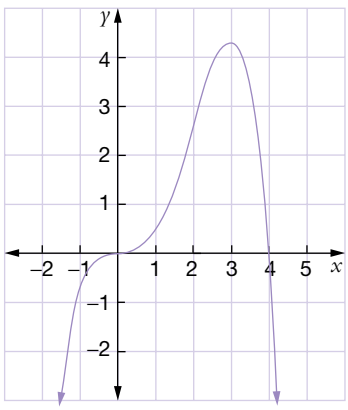
## Exercise 14:07

**1** Use the graph of $y = P(x)$ shown to sketch:



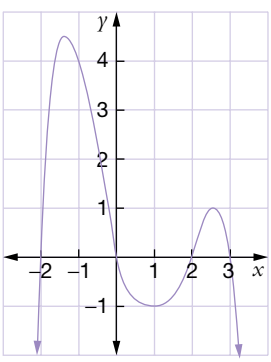

**a** $y = P(x) + 1$
**b** $y = P(x) - 2$
**c** $y = \frac{1}{2}P(x)$
**d** $y = 2P(x)$
**e** $y = -P(x)$
**f** $y = P(-x)$

**2** Use the graph of $y = H(x)$ shown to sketch:




**a** $y = H(x) + 1$
**b** $y = H(x) - 1$
**c** $y = \frac{1}{2}H(x)$
**d** $y = 2H(x)$
**e** $y = -H(x)$
**f** $y = H(-x)$

**3** Use the graph of $y = Q(x)$ shown to sketch:




**a** $y = Q(x) + 1$
**b** $y = Q(x) - 1$
**c** $y = \frac{1}{2}Q(x)$
**d** $y = 2Q(x)$
**e** $y = -Q(x)$
**f** $y = Q(-x)$

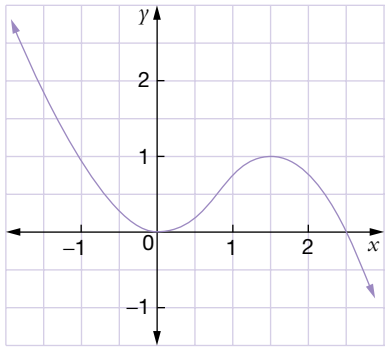
**4** Make a sketch of the polynomial $y = x(x - 1)(x + 2)$. Use this sketch to make sketches of:

**a** $y = \frac{1}{2}x(x - 1)(x + 2)$
**b** $y = 2x(x - 1)(x + 2)$
**c** $y = -\frac{1}{2}x(x - 1)(x + 2)$
**d** $y = -2x(x - 1)(x + 2)$

**5** The polynomial $y = P(x)$ is shown in the diagram.

**a** Sketch $y = -P(x)$ and then use this to produce a sketch of $y = 1 - P(x)$.

**b** Sketch $y = P(-x)$ and then use this to produce a sketch of $y = P(-x) + 1$.




**6** Using the sketch of $y = H(x)$ in Question **2**, make sketches of:

**a** $y = \frac{1}{2}H(x) - 1$ **b** $y = \frac{1}{2}H(-x)$ **c** $y = 2 - H(x)$

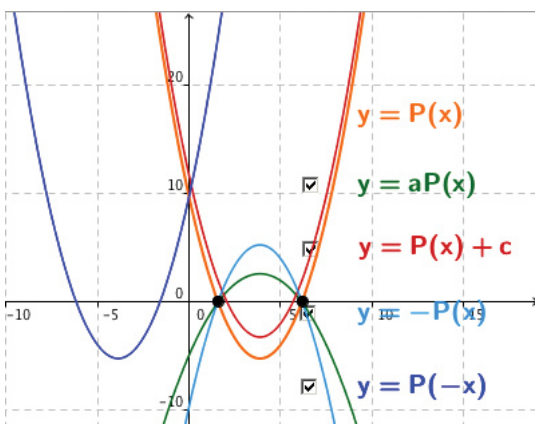
**7** If $P(x) = (x - 3)^2(x + 1)^3$, sketch:

**a** $y = P(x)$ **b** $y = P(-x)$ **c** $y = -P(x)$

**d** $y = 1 - P(x)$ **e** $y = P(-x) + 1$

## P GEOGEBRA ACTIVITY 14:07 SKETCHING CURVES RELATED TO $y = P(x)$

In this activity you can use sliders to create the polynomial $y = P(x)$. Use a checkbox to compare its graph with the graphs of $y = aP(x)$, $y = P(x) + c$, $y = P(-x)$ or $y = -P(x)$.



A time for reflection.

## MATHS TERMS 14

**constant term**

- the term in a polynomial that does not change in value as the variable changes,
  e.g. in $P(x) = x^2 + 6x - 2$, the constant term is $-2$

**continuous (curve)**

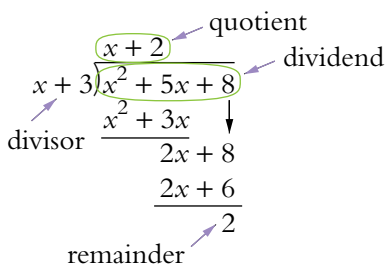
- a curve with no gaps
- no matter what $x$-value is chosen, a corresponding $y$-value can always be found

**degree**

- the highest power of any term in a polynomial

**dividing (a polynomial)**

- the divisor, dividend, quotient and remainder are names given to the different parts of the division process,
  e.g.




**factor (of a polynomial)**

- an expression that will divide into a polynomial and leave a remainder of zero

**factor theorem**

- a theorem that is used to find factors of a polynomial

**leading coefficient**

- the coefficient of the leading term

**leading term**

- the term that has the highest power of the variable

**monic**

- a polynomial that has a leading coefficient of 1

**polynomial**

- an expression that is the sum of any number of terms of the form $ax^n$ (where $n$ is a positive integer or zero),
  e.g. $P(x) = x^4 + 3x^2 - 6x + 8$
  $Q(x) = 2x^2 - x^3$

**remainder theorem**

- a theorem that is used to find the remainder when a polynomial is divided by $(x - a)$

**root**

- a solution of the polynomial equation $P(x) = 0$
- the degree of the polynomial gives the maximum number of roots that are possible. A polynomial of degree 3, for example, can have at most 3 roots
- roots can be single, double or triple roots,
  e.g. $y = (x - 1)(x - 2)(x - 3)$
  three single roots at $x = 1, 2, 3$
  $y = (x - 1)(x - 2)^2$
  single root at $x = 1$, double root at $x = 2$
  $y = (x - 1)^3$
  triple root at $x = 1$

**zero**

- a value of the variable that gives the polynomial a value of zero

# DIAGNOSTIC TEST 14 POLYNOMIALS

Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** What is the degree of each polynomial? 14:01

a $5x^2 + 7x + 3$ b $x^7 - x^5 + x^3 + x$ c $9 - 2x + x^2 - x^3$

**2** What is the leading term for each polynomial? 14:01

a $x^2 - 5x^3 + 3$ b $9x^4 + 7x^2 - 1$ c $5 + 3x + x^2$

**3** Which expression, A or B, is *not* a polynomial? 14:01

a A: $x^3 - x + 1$ B: $x^2 + x^{-1} + 3$

b A: $x^2 + \sqrt{x}$ B: $1 - 5x$

c A: $2 - 3x + x^3$ B: $x^2 + 2^x$

Questions **4** to **6** refer to the following polynomials: 14:02
$P(x) = x + 2$, $Q(x) = 2x^2 + 3x - 2$, $R(x) = 2x^3 + x^2 - 5x + 2$

**4** Determine the following.

a $Q(x) + R(x)$ b $R(x) - P(x)$ c $R(x) - Q(x)$

**5** Find the following products. 14:03

a $P(x) \cdot Q(x)$ b $P(x) \cdot R(x)$ c $(2x - 1) \cdot P(x) \cdot R(x)$

**6** Find: 14:03

a $Q(x) \div P(x)$ b $R(x) \div P(x)$ c $R(x) \div (x - 1)$

**7** Find the remainder for the following divisions. 14:04

a $(x^2 + 3x - 1) \div (x - 2)$
b $(x^3 + 2x^2 - 7) \div (x + 3)$
c $(2x^3 - x^2 + x - 4) \div (x - 5)$

**8** For the following, state whether or not $A(x)$ is a factor of $B(x)$. 14:04

a $A(x) = x - 3$, $B(x) = x^2 - 5x + 3$
b $A(x) = x + 1$, $B(x) = x^3 + x^2 - x - 1$
c $A(x) = x - 2$, $B(x) = x^4 - 2x^2 - x - 6$

**9** Solve the equations: 14:05

a $x(x - 1)(x + 2) = 0$
b $x^3 - 3x - 2 = 0$
c $x^3 - 4x^2 + x + 6 = 0$

**10** Make sketch graphs of the following. 14:06

a $y = -x(x + 1)(x - 2)$
b $y = (x + 1)^2(x - 1)(x + 3)$
c $y = (x - 1)(x + 2)^3$

14:07

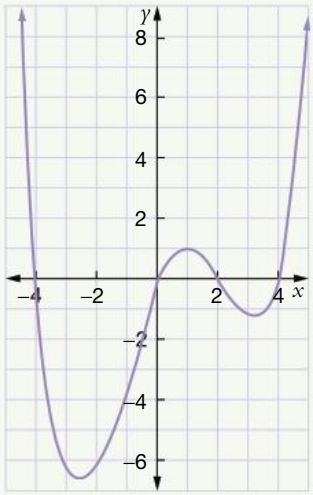
11 The graph represents the polynomial $y = P(x)$.
Use the graph to make sketches of:

a $y = -P(x)$

b $y = P(x) + 1$

c $y = P(-x)$

d $y = \frac{1}{2}P(x)$



Many mathematical curves can be seen in bridges.

## ASSIGNMENT 14A Chapter review

**1** $A(x) = 5x^2 - 3x^3 + 5x - 3$.

**a** State the degree, leading term and constant term of $A(x)$.

**b** Evaluate $A(-1)$.

**c** If $A(x)(ax + b)$ is monic and has a constant term of $-6$, what is the value of $a$ and $b$?

**d** What will the remainder be when $A(x)$ is divided by $(x + 2)$?

**2** Solve these polynomial equations.

**a** $2x(x - 3)(x + 4) = 0$

**b** $(x - 3)(2x + 3)(3x - 1) = 0$

**c** $x^3 - 4x^2 + 4x = 0$

**d** $x^4 - 8x^2 + 16 = 0$

**3** Sketch the graphs of the following polynomials.

**a** $y = (x - 1)(x + 1)(x - 4)$

**b** $y = (x + 1)^2(x - 4)$

**c** $y = -(x + 1)^2(x - 4)^2$

**4** A polynomial has zeros at 1, 2 and 4 and a $y$-intercept of 5. Make a possible sketch of the polynomial if it is of:

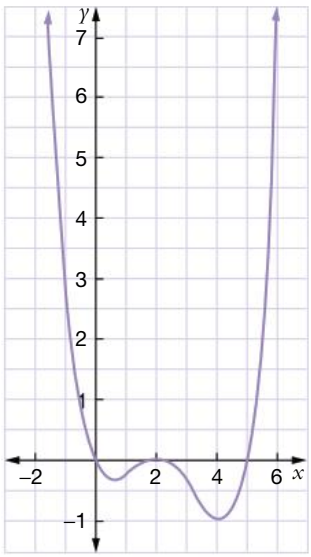
**a** degree 3

**b** degree 4

**5** The graph of $y = P(x)$ is shown. Use the graph to sketch:

**a** $y = P(x) + 1$ **b** $y = -0{\cdot}5P(x)$

**c** $y = P(-x)$




**6** **a** Give an equation of a polynomial of degree 4 that has:

**i** no zeros **ii** 1 zero

**iii** 2 zeros **iv** 3 zeros

**v** 4 zeros

**b** Sketch the polynomial $y = x(x - 2)^2$. How many zeros will each of the following polynomials have?

i $y = x(x - 2)^2 + 1$

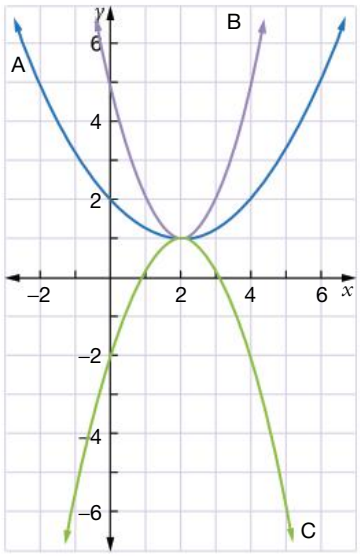
ii $y = x(x - 2)^2 - 1$

Rivers can look like polynomial curves winding their way to the sea.

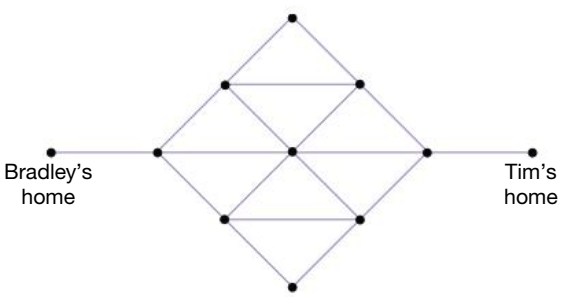
## ASSIGNMENT 14B Working mathematically

**1** What can be learned about the shape of a polynomial from its leading term?

**2** Find the equations of the parabolas A, B and C.

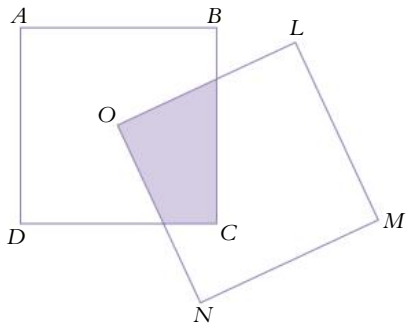



**3** Bradley must travel from his home to Tim's home regularly. He decides to travel a different route each day but, while travelling, he always chooses a way that will take him closer to Tim's home (see the diagram below).
How many days can he travel before he has to repeat an earlier route?




**4** A tank can be filled by a tap in 3 hours, and it can be emptied via a plug hole in 7 hours. If the tank is empty, how long will it take to fill if the tap is turned on and the plug hole is left open? (Assume that water drains out at a constant rate.)

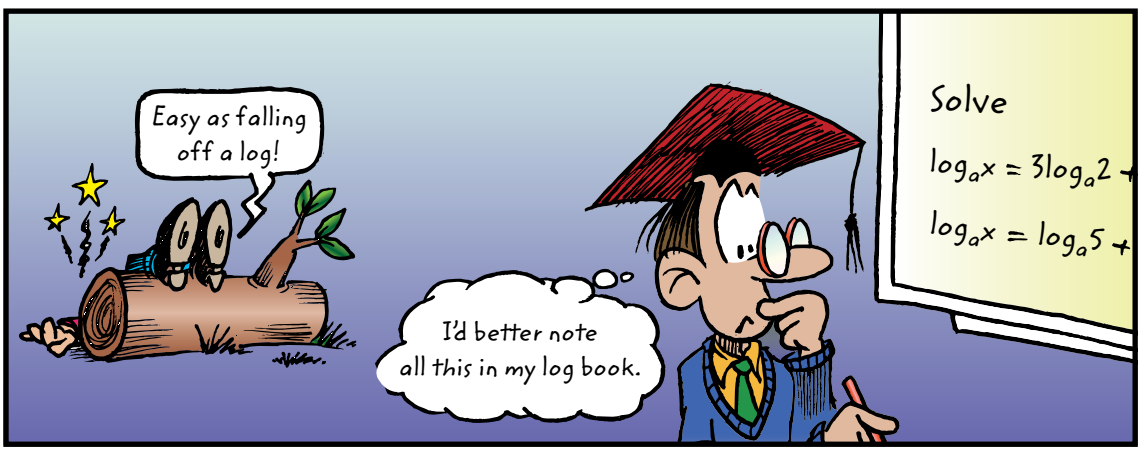
**5** Two identical squares are overlapped so that the vertex of one square is at the centre of the other. What fraction is the overlapping area of either square?




**6** If the line $ax + by + c = 0$ has a negative gradient, what does this say about $a$ and $b$?

# LOGARITHMS

15




## Contents


**15:01** Logarithms
Investigation 15:01 Logarithmic scales
**15:02** Logarithmic and exponential graphs
**15:03** Laws of logarithms
Investigation 15:03 Logarithmic scales and the history of calculating
**15:04** Simple exponential equations
Fun spot 15:04 What do you call lumberjack music?
Investigation 15:04 Solving harder exponential equations by 'guess and check'
Challenge 15:04 Further exponential equations
Maths terms, Diagnostic test, Assignments


## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

***Logarithms*** [Stage 5.3#]

- Use the definition of a logarithm to establish and apply the laws of logarithms (ACMNA265)
- Solve simple exponential equations (ACMNA270)

### Working Mathematically

- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 15:01 Logarithms

## PREP QUIZ 15:01

Evaluate:

**1** $8^2$ **2** $4^3$ **3** $2^6$

Solve the following:

**4** $64 = 8^x$ **5** $64 = 4^x$ **6** $64 = 2^x$

Write as powers of 2:

**7** 4 **8** 8 **9** $\frac{1}{2}$ **10** $\sqrt{2}$

In the Prep quiz, we saw that:

- the same *number* could be expressed as a *power* of a different number or base

  $64 = 8^2$ $64 = 4^3$ $64 = 2^6$
- different *numbers* could be expressed as *powers* of the same number or *base*.

  $4 = 2^2$ $8 = 2^3$ $\frac{1}{2} = 2^{-1}$

Numbers in index form are expressed as powers (indices) of another number (base).

Logarithms are indices. More specifically:

> The logarithm of a number to any base is the index when the number is expressed as a power of the base. If $y = a^x$ then $\log_a y = x$.

If $64 = 8^2$, then $\log_8 64 = 2$.
If $64 = 4^3$, then $\log_4 64 = 3$.
Also if $\log_2 64 = 6$, then $2^6 = 64$.

## WORKED EXAMPLE 1

Evaluate:

**a** $\log_4 16$ **b** $\log_2 1$ **c** $\log_9(\frac{1}{27})$

### Solution

**a** To evaluate an expression such as $\log_4 16$, we need to ask this question: '4 raised to what power will equal 16?'
If $4^2 = 16$, then $\log_4 16 = 2$.

**b** If $\log_2 1 = x$, then $2^x = 1$.
If $2^0 = 1$, then $\log_2 1 = 0$.

**c** If $\log_9(\frac{1}{27}) = x$, then
$9^x = \frac{1}{27}$
Solving: $3^{2x} = 3^{-3}$
$x = -\frac{3}{2}$
$\therefore \log_9(\frac{1}{27}) = -\frac{3}{2}$

## WORKED EXAMPLE 2

Solve the following logarithmic equations.

**a** $\log_{27} 3 = x$ **b** $\log_4 x = -2$ **c** $\log_x 8 = 1{\cdot}5$

### Solution

**a** If $\log_{27} 3 = x$
then $27^x = 3$
$3^{3x} = 3^1$
$3x = 1$
$\therefore x = \frac{1}{3}$

**b** If $\log_4 x = -2$
then $4^{-2} = x$
$\therefore x = \frac{1}{16}$

**c** If $\log_x 8 = 1{\cdot}5$
then $x^{1{\cdot}5} = 8$
$x^{\frac{3}{2}} = 2^3$
$x^{\frac{1}{2}} = 2$
$\therefore x = 4$

$$y = a^x \Leftrightarrow \log_a y = x$$

## Exercise 15:01

**1** Write each of these in logarithmic form.

**a** $8 = 2^3$ **b** $16 = 4^2$ **c** $7 = 7^1$ **d** $64 = 2^6$
**e** $3^4 = 81$ **f** $4^4 = 256$ **g** $2^5 = 32$ **h** $3^0 = 1$
**i** $\frac{1}{2} = 2^{-1}$ **j** $\frac{1}{9} = 3^{-2}$ **k** $\sqrt{5} = 5^{\frac{1}{2}}$ **l** $9 = 27^{\frac{2}{3}}$
**m** $25^{\frac{3}{2}} = 125$ **n** $36^{-\frac{1}{2}} = \frac{1}{6}$ **o** $16^{\frac{3}{4}} = 8$ **p** $8^{-\frac{5}{3}} = \frac{1}{32}$

**2** Rewrite each of these in index form.

**a** $\log_2 4 = 2$ **b** $\log_3 9 = 2$ **c** $\log_5 1 = 0$ **d** $\log_4 4 = 1$
**e** $\log_{10} 1000 = 3$ **f** $\log_3 27 = 3$ **g** $\log_2 16 = 4$ **h** $\log_4 16 = 2$
**i** $\log_7 243 = 3$ **j** $\log_5 625 = 4$ **k** $\log_2 128 = 7$ **l** $\log_6 6 = 1$
**m** $\log_2 \sqrt{2} = \frac{1}{2}$ **n** $\log_3 (\frac{1}{3}) = -1$ **o** $\log_2 (\frac{1}{4}) = -2$ **p** $\log_4 8 = \frac{3}{2}$

**3** Evaluate the following.

**a** $\log_3 9$ **b** $\log_2 8$ **c** $\log_5 25$ **d** $\log_7 1$
**e** $\log_3 3$ **f** $\log_2 32$ **g** $\log_4 64$ **h** $\log_8 64$
**i** $\log_6 216$ **j** $\log_{10} 1$ **k** $\log_{10} 10\,000$ **l** $\log_3 81$
**m** $\log_4 2$ **n** $\log_9 3$ **o** $\log_2 (\frac{1}{2})$ **p** $\log_4 8$

**4** Solve for $x$.

**a** $\log_2 16 = x$ **b** $\log_3 27 = x$ **c** $\log_5 625 = x$ **d** $\log_7 49 = x$
**e** $\log_4 1 = x$ **f** $\log_9 3 = x$ **g** $\log_6 6 = x$ **h** $\log_{10} \sqrt{10} = x$
**i** $\log_9 (\frac{1}{9}) = x$ **j** $\log_2 (\frac{1}{4}) = x$ **k** $\log_8 4 = x$ **l** $\log_{16} 64 = x$
**m** $\log_2 (0{\cdot}5) = x$ **n** $\log_5 (0{\cdot}04) = x$ **o** $\log_8 \sqrt{2} = x$ **p** $\log_{100} 1000 = x$

**5** Find $x$.

**a** $\log_2 x = 5$ **b** $\log_3 x = 3$ **c** $\log_2 x = 7$ **d** $\log_5 x = 0$
**e** $\log_7 x = 2$ **f** $\log_6 x = 3$ **g** $\log_3 x = -1$ **h** $\log_4 x = \frac{1}{2}$
**i** $\log_9 x = \frac{3}{2}$ **j** $\log_4 x = -2$ **k** $\log_2 x = -3$ **l** $\log_8 x = -\frac{1}{3}$
**m** $\log_{25} x = 1{\cdot}5$ **n** $\log_{10} x = -1$ **o** $\log_5 x = -\frac{1}{2}$ **p** $\log_{16} x = \frac{3}{4}$

**6** Solve the following.

**a** $\log_x 16 = 4$ **b** $\log_x 9 = 2$ **c** $\log_x 5 = 1$ **d** $\log_x 81 = 4$

**e** $\log_x 125 = 3$ **f** $\log_x 100 = 2$ **g** $\log_x 8 = 3$ **h** $\log_x 729 = 6$

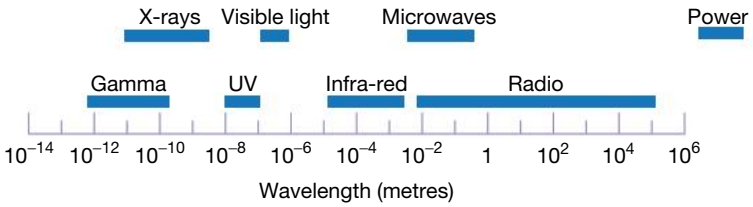
**i** $\log_x \sqrt{5} = \frac{1}{2}$ **j** $\log_x (\frac{1}{3}) = -1$ **k** $\log_x 2 = -1$ **l** $\log_x 8 = \frac{3}{2}$

**m** $\log_x 3 = \frac{1}{3}$ **n** $\log_x 4 = \frac{1}{2}$ **o** $\log_x 4 = \frac{1}{3}$ **p** $\log_x 27 = \frac{3}{4}$

## INVESTIGATION 15:01 LOGARITHMIC SCALES

In most measurement situations, the measuring scales are linear. In some situations, however, a linear scale would produce a large range of measurements. To overcome this, scales based on indices are used. These are called logarithmic scales.

### 1 Electromagnetic spectrum




The diagram above shows the variation in wavelength for different types of electromagnetic radiation. A logarithmic scale is used for wavelength.

**a** Copy the scale and mark approximately the position on the scale of the numbers:

**i** 1000 **ii** 10 **iii** 0·1 **iv** 0·000 005

**b** The scale is marked with equally spaced markings, but the difference in value between markings is not equal. What is the difference in value of:

**i** 1 and $10^2$ **ii** $10^4$ and $10^6$ **iii** $10^{-2}$ and 1?

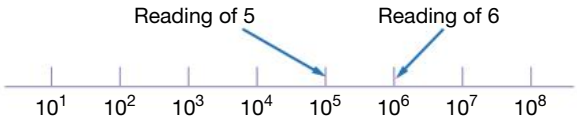
**c** Where would zero be in relation to the numbers on this scale?

**d** What is the range in wavelength of:

**i** radio waves **ii** visible light?

### 2 Richter scale

Another example of a logarithmic scale is the Richter scale, which measures the magnitude of earthquakes. In this question, we will assume that it is the same as the scale in the last question, in that every unit is ten times bigger than the unit before it. The scale is shown below.




**a** How many times bigger is:

**i** a reading of 6 than a reading of 5

**ii** a reading of 7 than a reading of 5?

**b** Where would a reading of 5·5 be on the scale?

**c** How many times bigger is a reading of 5·5 than a reading of 5?

### 3 pH scale

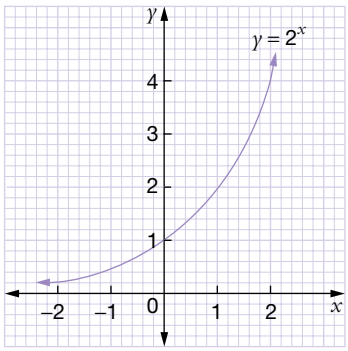
The concentration of an acid is measured by its pH. (See Assignment 7B Question **6**.)

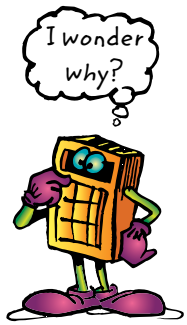
# 15:02 Logarithmic and exponential graphs

To draw the graph of an exponential function such as $y = 2^x$, we can plot the points in the following table of values.

| $x$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ | $1\frac{1}{2}$ | $2$ |
|---|---|---|---|---|---|---|---|
| $y$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $1{\cdot}4$ | $2$ | $2{\cdot}8$ | $4$ |

All exponential graphs cut the $y$-axis at 1.

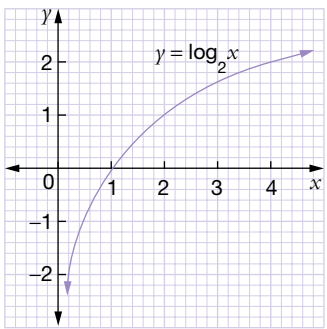



To draw the graph of a logarithmic function such as $y = \log_2 x$, we can use the fact that this is equivalent to $x = 2^y$. (The $x$- and $y$-values have changed places from $y = 2^x$).

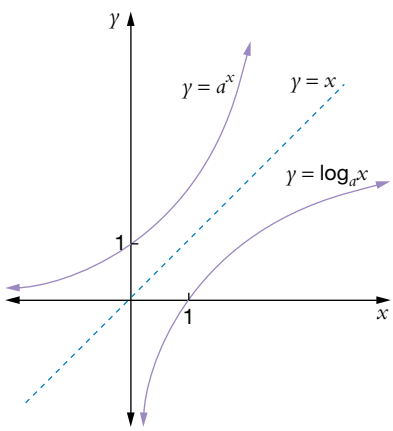
| $x$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $1{\cdot}4$ | $2$ | $2{\cdot}8$ | $4$ |
|---|---|---|---|---|---|---|---|
| $y$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ | $1\frac{1}{2}$ | $2$ |

A logarithmic graph always cuts the $x$-axis at 1.




Now we have seen that $y = \log_a x$ is the same as $x = a^y$. Interchanging the $x$ and $y$ in $x = a^y$ produces its inverse function, $y = a^x$.

$y = \log_a x$ and $y = a^x$ (for $a > 0$) are inverse functions and hence their graphs are reflections of each other in the line $y = x$.



## Exercise 15:02

**1** Copy and complete these tables of values and then draw their graphs on the same number plane. (Approximate to one decimal place when necessary.)

**a** $y = 3^x$

| $x$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ | $1\frac{1}{2}$ | $2$ |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** $y = \log_3 x$, $(x = 3^y)$

| $x$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| $y$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ | $1\frac{1}{2}$ | $2$ |

**2** **a** Using the fact that $y = \log_2 x$ and $x = 2^y$ are equivalent functions, complete the table of values below and use it to construct a graph of $y = \log_2 x$ (correct to two decimal places).

$y = \log_2 x$

| $x$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-3$ | $-2\frac{1}{2}$ | $-2$ | $-1\frac{1}{2}$ | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $1\frac{1}{2}$ | $2$ | $2\frac{1}{2}$ | $3$ |

**b** From your graph, estimate correct to one decimal place, the value of:

**i** $\log_2 3$ **ii** $\log_2 5$ **iii** $\log_2 1{\cdot}5$ **iv** $\log_2 0{\cdot}8$

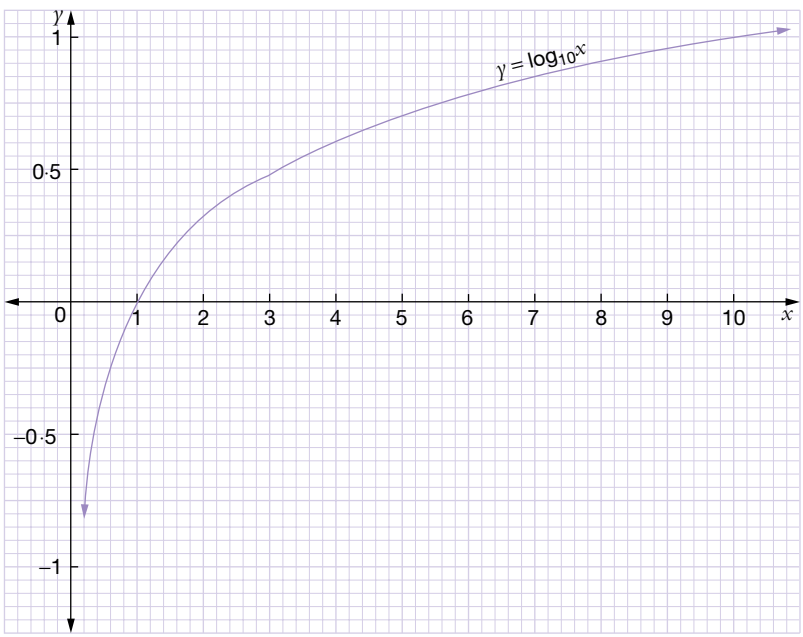
**3** The graphing of $y = \log_{10} x$ is aided by the use of the log key on your calculator.

**a** Complete the table of values below using your calculator (correct to two decimal places).

$y = \log_{10} x$

| $x$ | $0{\cdot}2$ | $0{\cdot}4$ | $0{\cdot}6$ | $0{\cdot}8$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | | | | | | |

**b** Using your table of values, draw a graph of $y = \log_{10} x$ like the one below. Use scales similar to those shown for each axis.




**c** Using your graph, find approximate solutions for the following equations.

**i** $\log_{10} x = 0{\cdot}7$ **ii** $\log_{10} x = 0{\cdot}25$ **iii** $\log_{10} x = -0{\cdot}5$

**4** On the same number plane, *sketch* the graphs of the following, showing their relative positions.

**a** $y = \log_2 x$ **b** $y = \log_3 x$ **c** $y = \log_5 x$

# 15:03 Laws of logarithms

## PREP QUIZ 15:03

Complete:

**1** If $8 = 2^3$ then $\log_2 8 = \ldots$

**2** If $m = 2^x$ then $\log_2 m = \ldots$

**3** If $m = a^x$ then $\log_a m = \ldots$

**4** If $n = a^y$ then $\log_a n = \ldots$

**5** If $p = a^{x+y}$ then $\log_a p = \ldots$

Simplify:

**6** $a^x \times a^y$ **7** $a^x \div a^y$ **8** $(a^x)^m$

**9** Write $\frac{1}{x}$ in index form. **10** Since $3^0 = 1$ then $\log_3 1 = \ldots$

Logarithms are indices, so the index laws can be used to deduce the logarithm laws.

$$\log_a xy = \log_a x + \log_a y$$

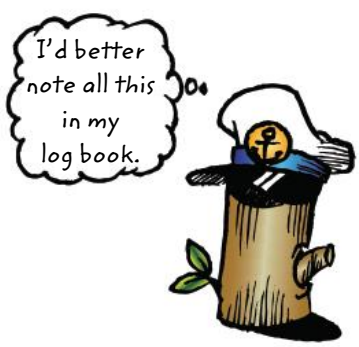
Proof: Let $x = a^m$ and $y = a^n$ $(\log_a x = m$ and $\log_a y = n)$

$$xy = a^m \times a^n$$
$$= a^{m+n}$$
$$\therefore \log_a xy = m + n$$
$$= \log_a x + \log_a n$$

$$\log \frac{x}{y} = \log_a x - \log_a y$$

Proof: Let $x = a^m$ and $y = a^n$ $(\log_a x = m$ and $\log_a y = n)$

$$\frac{x}{y} = a^m \div a^n$$
$$= a^{m-n}$$
$$\log_a \frac{x}{y} = m - n$$
$$= \log_a x - \log_a y$$




$$\log_a x^p = p \log_a x$$

Proof: Let $x = a^m$ $(\log_a x = m)$

$$x^p = (a^m)^p$$
$$= a^{mp}$$
$$\therefore \log_a x^p = mp$$
$$= p \log_a x$$

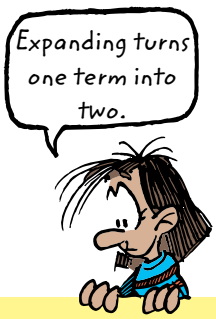
The following logarithms can be deduced from the four index equations:

$a^0 = 1$ $\quad a^1 = a$ $\quad a^x = a^x$ $\quad x^{-1} = \dfrac{1}{x}$

$\log_a 1 = 0$ $\quad \log_a a = 1$ $\quad \log_a a^x = x$ $\quad \log_a \dfrac{1}{x} = -\log_a x$

## WORKED EXAMPLE 1

The two main logarithm laws can be used to expand or contract expressions involving logarithms. Both processes are well illustrated in the following examples.




Contraction

$\log xy = \log x + \log y$

Expansion

Simplify:

**a** $\log_{10} 25 + \log_{10} 4$ **b** $\log_3 54 - \log_3 18$

**c** $5 \log_a a - \log_a a^4$ **d** $\dfrac{\log_a x^3}{\log_a \sqrt{x}}$

### Solutions

**a** $\log_{10} 25 + \log_{10} 4 = \log_{10} (25 \times 4)$
$= \log_{10} 100$
$= 2$

**b** $\log_3 54 - \log_3 18 = \log_3 (\frac{54}{18})$
$= \log_3 3$
$= 1$

**c** $5 \log_a a - \log_a a^4 = 5 \log_a a - 4 \log_a a$
$= \log_a a$
$= 1$

**d** $\dfrac{\log_a x^3}{\log_a \sqrt{x}} = \dfrac{\log_a x^3}{\log_a x^{\frac{1}{2}}}$
$= \dfrac{3 \log_a x}{\frac{1}{2} \log_a x}$
$= 6$

## WORKED EXAMPLE 2

If $\log_a 3 = 1{\cdot}4$ and $\log_a 4 = 1{\cdot}6$, evaluate:

**a** $\log_a 12$ **b** $\log_a (1\frac{1}{3})$ **c** $\log_a 9$ **d** $\log_a 2$

### Solutions

**a** $\log_a 12 = \log_a 3 + \log_a 4$
$= 1{\cdot}4 + 1{\cdot}6$
$= 3{\cdot}0$

**b** $\log_a (1\frac{1}{3}) = \log_a (\frac{4}{3})$
$= \log_a 4 - \log_a 3$
$= 1{\cdot}6 - 1{\cdot}4$
$= 0{\cdot}2$

**c** $\log_a 9 = \log_a 3^2$
$= 2 \log_a 3$
$= 2 \times 1{\cdot}4$
$= 2{\cdot}8$

**d** $\log_a 2 = \log_a 4^{\frac{1}{2}}$
$= \frac{1}{2} \log_a 4$
$= \frac{1}{2} \times 1{\cdot}6$
$= 0{\cdot}8$

## WORKED EXAMPLE 3

Solve:

**a** $\log_{10} 2x + \log_{10} 5 = 3$

**b** $\log_5 x - \log_5 (x - 1) = 1$

### Solutions

**a** $\log_{10} 2x + \log_{10} 5 = 3$

$\log_{10} (2x \times 5) = 3$

$\log_{10} 10x = 3$

$10x = 10^3$

$= 1000$

$\therefore\ x = 100$

**b** $\log_5 x - \log_5 (x - 1) = 1$

$\log_5 \dfrac{x}{x-1} = 1$

$\dfrac{x}{x-1} = 5$

$x = 5x - 5$

$4x = 5$

$\therefore\ x = \frac{5}{4}$

## WORKED EXAMPLE 4

Use the logarithm laws:

**a** to expand $\log_a \dfrac{x^2}{y}$

**b** to contract $\log_a x + \log_a y - 2\log_a z$.

### Solutions

**a** $\log_a \dfrac{x^2}{y} = \log_a x^2 - \log_a y$

$= 2\log_a x - \log_a y$

**b** $\log_a x + \log_a y - 2\log_a z = \log_a xy - \log_a z^2$

$= \log_a \dfrac{xy}{z^2}$

## Exercise 15:03

**1** Evaluate:

**a** $\log_6 9 + \log_6 4$ **b** $\log_5 20 - \log_5 4$ **c** $\log_2 48 - \log_2 3$

**d** $\log_{10} 2 + \log_{10} 5$ **e** $\log_a 4 + \log_a (\frac{1}{4})$ **f** $\log_5 1000 - \log_5 8$

**g** $\log_2 18 - 2\log_2 3$ **h** $2\log_{10} 5 + \log_{10} 40$ **i** $\log_3 24 - 3\log_3 2$

**j** $\log_{10} 125 - \log_{10} 4 + \log_{10} 32$ **k** $3\log_2 4 + \frac{1}{2}\log_2 81 - \log_2 18$

**2** If $\log_a 2 = 0{\cdot}301$ and $\log_a 3 = 0{\cdot}477$, evaluate:

**a** $\log_a 6$ **b** $\log_a 1{\cdot}5$ **c** $\log_a 9$ **d** $\log_a 0{\cdot}5$

**e** $\log_a \sqrt{3}$ **f** $\log_a 18$ **g** $\log_a 8$ **h** $\log_a 24$

**i** $\log_a 36$ **j** $\log_a (\frac{2}{3})$ **k** $\log_a (\frac{1}{4})$ **l** $\log_a \sqrt{12}$

**3** If $\log_x 3 = 1{\cdot}5$, $\log_x 5 = 1{\cdot}8$ and $\log_x 10 = 3{\cdot}0$, evaluate:

**a** $\log_x 15$ **b** $\log_x 2$ **c** $\log_x 9$ **d** $\log_x \sqrt{5}$

**e** $\log_x 150$ **f** $\log_x 250$ **g** $\log_x 45$ **h** $\log_x (\frac{1}{2})$

**i** $\log_x 1000$ **j** $\log_x 6$ **k** $\log_x \sqrt{50}$ **l** $\log_x 1{\cdot}5$

**4** Simplify:

**a** $\log_a(a^2)$ **b** $\log_a 5a - \log_a 5$ **c** $4\log_a a - \log_a(a^4)$

**d** $\dfrac{\log_a(x^5)}{\log_a x}$ **e** $\log_a \dfrac{x}{y} + \log_a \dfrac{y}{x}$ **f** $\dfrac{\log_a(x^3) - \log_a(x^2)}{\log_a \sqrt{x}}$

**5** Use the logarithm laws to expand the following expressions.

**a** $\log_a \dfrac{xy}{z}$ **b** $\log_a \dfrac{xa}{z^2}$ **c** $\log_a \dfrac{x}{(x+1)^2}$

**d** $\log_a x\sqrt{x+1}$ **e** $\log_y \sqrt{\dfrac{xy}{y+1}}$ **f** $\log_y \dfrac{x(a+b)}{y^2}$

**6** Use the logarithm laws to contract the following expressions.

**a** $\log_a x - \log_a y + 2\log_a z$ **b** $2\log_a x - \frac{1}{2}\log_a y + \log_a z$

**c** $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y$ **d** $\log_a(x+1) - \log_a x + \log_a(x+2)$

**e** $2\log_y x - \log_y z + \log_y y^3$ **f** $2\log_a x + \log_a y - 3$

**7** Find a relationship between $x$ and $y$ that does not involve logarithms.

**a** $\log_a x + \log_a y = \log_a(x+y)$ **b** $2\log_a x + \log_a 5 = \log_a y$

**c** $2\log_2 x + \log_2 y = 3$ **d** $\log_5 x = 3 + \log_5 y$

**e** $\log_a(1+x) - \log_a(1-x) = \log_a y$ **f** $\log_a y = \log_a 5 + 7\log_a x$

**8** Solve the following equations for $x$.

**a** $\log_a x = \log_a 5 + \log_a 2$ **b** $\log_a x = \log_a 10 - \log_a 5$

**c** $\log_a x = 3\log_a 2 + \log_a 4$ **d** $\log_a x = \frac{1}{2}\log_a 9 - \log_a 2$

**e** $\log_a 2 + \log_a x = \log_a 8$ **f** $\log_a 10 - \log_a x = \log_a 5$

**g** $\log_a x + \log_a 3 = \log_a(x+1)$ **h** $\log_a(4x) - \log_a 3 = \log_a(x+4)$

**9** Find values for $x$ if:

**a** $\log_{10} 2 + \log_{10} x = 2$ **b** $\log_2 x - \log_2 3 = 3$

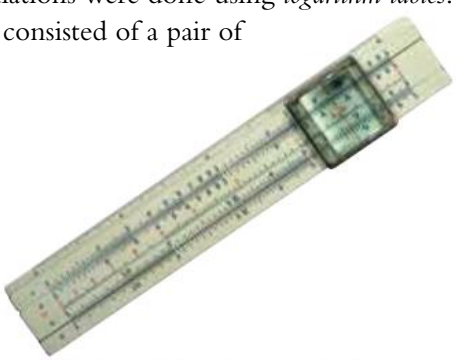
**c** $\log_{10} x = \dfrac{\log_{10} 8}{\log_{10} 2}$ **d** $\log_{10} x + 3 = 4\log_{10} x$

**e** $\log_{10} x - \log_{10}(x-1) = 1$ **f** $\log_{10} x = \log_{10} 5 - \log_{10} 2 - 1$

**g** $\log_2 x + \log_2(x+1) = 1$ **h** $\log_3 2x + \log_3(x-1) = 1$

## INVESTIGATION 15:03 LOGARITHMIC SCALES AND THE HISTORY OF CALCULATING

The use of electronic calculators to do mathematical calculations only became widespread in Australia during the late 1970s. Prior to then, many calculations were done using *logarithm tables*. Another common calculating device was a *slide-rule*. This consisted of a pair of logarithmic scales: one fixed and the other sliding.



1 Investigate the use of logarithms as a calculating tool. See if you can find a set of logarithm tables and find out how they were used.

2 Investigate the use of a slide-rule and try to construct one yourself using logarithmic graph paper.

# 15:04 Simple exponential equations

## PREP QUIZ 15:04

Write as powers of 2:

**1** 4 **2** 8 **3** $\frac{1}{2}$ **4** $\sqrt{2}$

**5** Write 125 as a power of 5.

**6** Write 243 as a power of 3.

**7** Write $\frac{1}{\sqrt{2}}$ as a power of 2.

**8** Is $4^{-\frac{3}{2}} = \frac{1}{8}$?

**9** If $x^3 = 4^3$, what is $x$?

**10** If $2^x = 2^5$, find $x$.

In an exponential equation the pronumeral is involved in the 'exponent' or 'power'.

e.g. $2^x = 16$ or $5^{x-1} = 125$

For simple equations of this type, both sides of the equation are written as powers of the same number or base and then a comparison of powers is used to find the solution.

## WORKED EXAMPLES

Find the pronumeral in each of the following.

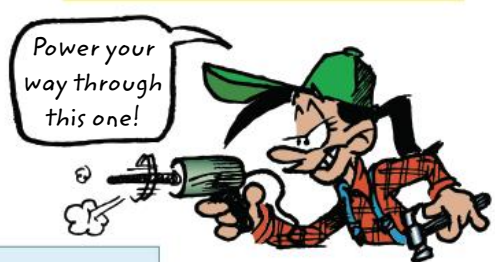
**1**
$2^x = 32$
$2^x = 2^5$
$\therefore\ x = 5$

**2**
$3^{2x-1} = 27$
$3^{2x-1} = 3^3$
$2x - 1 = 3$
$\therefore\ x = 2$

Each side of the equation must be written with the same base; then the indices must be equal.

**3**
$4^x = 8$
$(2^2)^x = 2^3$
$2^{2x} = 2^3$
$2x = 3$
$\therefore\ x = \frac{3}{2}$ or $1\frac{1}{2}$

**4**
$2^{x-1} = \frac{1}{4}$
$2^{x-1} = 2^{-2}$
$x - 1 = -2$
$\therefore\ x = -1$




To solve simple exponential equations, rewrite each side of the equation with the same base, and then equate the indices.

## Exercise 15:04

**1** Solve these equations.

**a** $2^x = 16$ **b** $2^x = 128$ **c** $3^x = 81$ **d** $3^x = 1$

**e** $4^x = 64$ **f** $5^x = 5$ **g** $6^x = 216$ **h** $2^x = 512$

**i** $3^x = 243$ **j** $4^x = 256$ **k** $10^x = 10\,000$ **l** $9^x = 729$
**m** $8^x = 4096$ **n** $11^x = 1331$ **o** $7^x = 16\,807$ **p** $6^x = 1296$

**2** Find $x$.

**a** $4^x = 2$ **b** $9^x = 3$ **c** $16^x = 2$ **d** $25^x = 5$
**e** $27^x = 3$ **f** $32^x = 2$ **g** $16^x = 4$ **h** $100^x = 10$
**i** $2^x = \frac{1}{2}$ **j** $3^x = \frac{1}{9}$ **k** $4^x = \frac{1}{4}$ **l** $2^x = \frac{1}{4}$
**m** $5^x = \frac{1}{25}$ **n** $3^x = \frac{1}{27}$ **o** $4^x = \frac{1}{2}$ **p** $9^x = \frac{1}{3}$

**3** Evaluate the pronumeral.

**a** $9^x = 27$ **b** $4^x = 32$ **c** $8^x = 4$ **d** $27^x = 81$
**e** $25^x = 125$ **f** $1000^x = 100$ **g** $128^x = 64$ **h** $32^x = 8$
**i** $8^x = \frac{1}{16}$ **j** $4^x = \frac{1}{8}$ **k** $27^x = \frac{1}{9}$ **l** $16^x = \frac{1}{8}$
**m** $(\frac{1}{4})^x = 8$ **n** $(\frac{1}{3})^x = 9$ **o** $(\frac{1}{9})^x = 27$ **p** $(0{\cdot}01)^x = 1000$

**4** Solve:

**a** $2^{x-2} = 8$ **b** $3^{1-x} = 27$ **c** $2^{2x} = 32$
**d** $4^{2x-1} = 64$ **e** $4^{x-1} = \sqrt{2}$ **f** $(\sqrt{3})^x = 81$
**g** $2^{2-x} = \sqrt{8}$ **h** $5^{3x-2} = 125$ **i** $8^{2-x} = 4$
**j** $9^{1+2x} = 243$ **k** $8^{3x} = 32$ **l** $100^{3x-1} = 100\,000$
**m** $4^{x+1} = \dfrac{1}{8\sqrt{2}}$ **n** $8^{2x-1} = 4\sqrt{8}$ **o** $3^{2-x} = \dfrac{1}{3\sqrt{3}}$
**p** $9^{3x-5} = \dfrac{\sqrt{3}}{27}$ **q** $2^{2-x} = 4^{x-1}$ **r** $9^{3x} = 27^{x+1}$
**s** $4^{2x-1} = (\frac{1}{8})^x$ **t** $(4\sqrt{2})^{x-1} = (2\sqrt{8})^{1-x}$

## FUN SPOT 15:04 — WHAT DO YOU CALL LUMBERJACK MUSIC?

Answer each question and write the letter for that question in the box above the correct answer.

*Solve for x:*

**T** $4^x = 64$ **M** $2^x = \frac{1}{4}$ **O** $8^x = 4$
**H** $\log_3 x = 2$ **E** $\log_x 16 = 4$ **S** $\log_5 625 = x$

*Simplify:*

**G** $\log_a a$ **Y** $\log_a (a^5)$ **R** $\log_a \dfrac{1}{a}$

*Evaluate:*

**H** $\log_4 24 - \log_4 3$
**G** $\log_{10} 2000 + \log_{10} 500$

*Solve for n:*

**R** $3^{n+5} = 9$ **L** $6^{2-n} = 36$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| 0 | $\frac{2}{3}$ | 6 | 1 | 2 | −3 | −1 | $\frac{3}{2}$ | 5 | 3 | 9 | −2 | 4 |

## INVESTIGATION 15:04 SOLVING HARDER EXPONENTIAL EQUATIONS BY 'GUESS AND CHECK'

Equations like $3^x = 5$ are more difficult than those in the previous exercise because the solution is obviously not an integer or simple fraction.

Later in this chapter, we will use logarithms to find the exact solution, but an approximate solution can be found by the 'guess and check' method.

### Example

Find an approximate solution for $3^x = 5$.
Now $3^1 = 3$ and $3^2 = 9$.

As 5 is between 3 and 9, $x$ must be between 1 and 2.

- $x = 1{\cdot}5$ gives $3^{1{\cdot}5} = 5{\cdot}196...$ (too big)
- $x = 1{\cdot}4$ gives $3^{1{\cdot}4} = 4{\cdot}655...$ (too small)
  $\therefore$ $x$ must be between $1{\cdot}4$ and $1{\cdot}5$.
- $x = 1{\cdot}45$ gives $3^{1{\cdot}45} = 4{\cdot}918...$ (too small)
  $\therefore$ $x$ must be between $1{\cdot}45$ and $1{\cdot}5$.

Repeating this procedure gives the solution:
$x = 1{\cdot}47$ (2 dec. pl.)

### Exercises

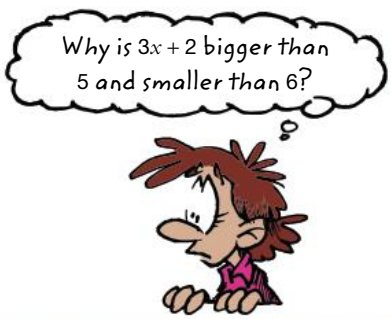
Use the 'guess and check' method to solve:

**1** $5^x = 500$ **2** $3^x = 1690$

**3** $7^{3x+2} = 19\,187$

Hint: If $3x + 2 > 5$ and $3x + 2 < 6$, then $x > 1$ and $x < 1\frac{1}{3}$.
$\therefore$ $x$ is between 1 and $1\frac{1}{3}$.

**4** $5^{2x-3} = 16\,755$




## CHALLENGE 15:04 FURTHER EXPONENTIAL EQUATIONS

In Exercise 15:04 the solution of exponential equations of the form $a^x = b$ was investigated.

In all cases, $b$ could be written as a power of $a$, and a comparison of the indices then allowed $x$ to be found.

If $b$ could not be written as a power of $a$, then we had to resort to the 'guess and check' method as outlined in Investigation 15:04.

Now, using the theory of logarithms, a quicker method can be found to solve these equations. The method outlined below makes use of the log key on your calculator. This key gives the logarithm of a number to the base 10.

## Examples

Solve the following:

**1** $5^x = 12$ **2** $3^{x+2} = 7$ **3** $2^x = 5^{x-1}$

## Solutions

The first step is to take logs of both sides, to base 10.

**1**
$$5^x = 12$$
$$\log_{10} 5^x = \log_{10} 12$$
$$x \log_{10} 5 = \log_{10} 12$$
$$x = \frac{\log_{10} 12}{\log_{10} 5}$$

Press: log 12 ) ÷ log 5 =

$= 1{\cdot}54$ (2 dec. pl.)

**2**
$$3^{x+2} = 7$$
$$\log_{10} 3^{x+2} = \log_{10} 7$$
$$(x + 2) \log_{10} 3 = \log_{10} 7$$
$$x + 2 = \frac{\log_{10} 7}{\log_{10} 3}$$
$$x + 2 = 1{\cdot}77 \text{ (2 dec. pl.)}$$
$$\therefore x = -0{\cdot}23$$

**3**
$$2^x = 5^{x-1}$$
$$\log_{10} 2^x = \log_{10} 5^{x-1}$$
$$x \log_{10} 2 = (x - 1) \log_{10} 5$$
$$= x \log_{10} 5 - \log_{10} 5$$
$$x \log_{10} 5 - x \log_{10} 2 = \log_{10} 5$$
$$x (\log_{10} 5 - \log_{10} 2) = \log_{10} 5$$
$$\therefore x = \frac{\log_{10} 5}{\log_{10} 5 - \log_{10} 2}$$
$$\therefore x = 1{\cdot}76 \text{ (2 dec. pl.)}$$

## Exercises

**1** Solve $x$ correct to three decimal places.

**a** $10^x = 700$ **b** $10^x = 41{\cdot}6$ **c** $10^x = 49\,168$

**d** $2^x = 7$ **e** $5^x = 100$ **f** $6^x = 2$

**2** Solve for $x$ correct to three decimal places.

**a** $10^{2x-3} = 1500$ **b** $5^{x+1} = 8$ **c** $2^{x-1} = 12$

**d** $3^{2x} = 15$ **e** $9^{2x-1} = 900$ **f** $5^{1-3x} = 27$

**3** Determine the value of $t$ correct to three decimal places.

**a** $1000 = 1{\cdot}8(10^{2t})$ **b** $1\,000\,000 = 100(2^{0{\cdot}8t})$ **c** $6(3^{2t-1}) + 3 = 27$

**4** Determine the value of $x$ correct to three decimal places.

**a** $3^x = 6^{x-1}$ **b** $2^x = 7^{x-3}$ **c** $3^{2x} = 7^x$

**d** $5^{x+1} = 8^{x-1}$ **e** $6^{x+2} = 10^{x-7}$ **f** $12^{2x} = 9^{x+7}$

## MATHS TERMS 15

**exponent**

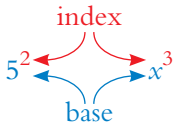
- another name for a power or index, e.g. for $2^3$, the number 3 is an exponent

**exponential equation**

- an equation where the unknown is part of an exponent, e.g. $2^x = 8$ or $3^{x-1} = 7$

**index (indices)**

- the raised symbol when a number is written as a power




**logarithm**

- an index
- if $y = a^x$, then $\log_a y = x$, e.g. if $32 = 2^5$ then $\log_2 32 = 5$

**power**

- repeated as a factor, e.g. 3 to the power of 5 $= 3^5$
  $= 3 \times 3 \times 3 \times 3 \times 3$

The nautilus shell is in the shape of a logarithmic spiral.

## DIAGNOSTIC TEST 15 LOGARITHMS

These questions reflect the important skills introduced in this chapter.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

**1** Rewrite each expression in the form $x = a^y$. 15:01

a $\log_2 8 = 3$ b $\log_3 9 = 2$ c $\log_4 2 = \frac{1}{2}$ d $\log_5 1 = 0$

**2** Rewrite each expression in the form $\log_a x = y$. 15:01

a $2^5 = 32$ b $4^3 = 64$ c $27^{\frac{1}{3}} = 3$ d $10^{-1} = 0{\cdot}1$

**3** Evaluate: 15:01

a $\log_3 3$ b $\log_5 25$ c $\log_{10} 1000$ d $\log_2 \frac{1}{8}$

**4** Find $x$. 15:03

a $\log_9 81 = x$ b $\log_x 4 = 2$ c $\log_2 x = 5$ d $\log_x 7 = 1$

**5** Simplify: 15:03

a $\log_a 7 + \log_a 3$ b $\log_a 15 - \log_a 5$

c $\log_a 6 + \log_a 10$ d $\log_a 20 - \log_a 4$

**6** Rewrite in the form $n\log_a 3$: 15:03

a $\log_a 3^4$ b $\log_a 9$ c $\log_a \sqrt{3}$ d $\log_a \frac{1}{3}$

**7** Use the logarithm laws to write in expanded form. 15:03

a $\log_a xy^2$ b $\log_a \sqrt{xy}$ c $\log_a \dfrac{x}{\sqrt{y}}$

**8** Use the logarithm laws to write in contracted form: 15:03

a $\log_a x - \log_a y - \log_a z$

b $2\log_a x + 3\log_a y + \frac{1}{2}\log_a z$

c $\log_a x + \frac{1}{2}\log_a (x - 1) - \frac{1}{2}\log_a (x + 1)$

**9** Solve these equations. 15:04

a $2^x = 128$ b $2^x = \frac{1}{4}$

c $10^{1-x} = 1000$ d $9^{2x-1} = 3$

# ASSIGNMENT 15A Chapter review

**1** Rewrite each expression in the form $x = a^y$.

**a** $\log_{10} 1000 = 3$

**b** $\log_2 0{\cdot}125 = -3$

**c** $\log_9 27 = \frac{3}{2}$

**2** Evaluate:

**a** $\log_2 128$ **b** $\log_{10} 0{\cdot}001$

**c** $\log_{16} 4$ **d** $\log_{25} 125$

**3** Solve:

**a** $5^x = 625$ **b** $9^x = 3^5$

**c** $4^x = 8$ **d** $3^{1-x} = \frac{1}{27}$

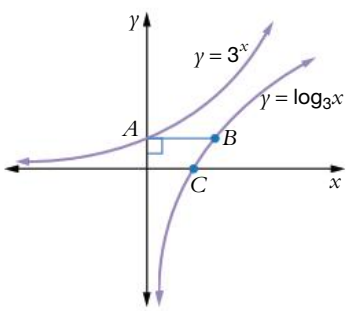
**4** If $\log_x 5 = 0{\cdot}56$, find:

**a** $\log_x 25$ **b** $\log_x \sqrt{5}$

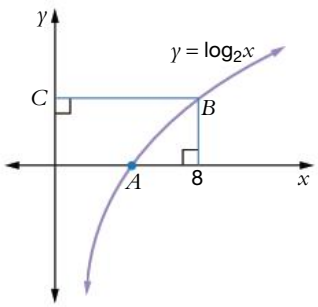
**c** $\log_x 5x$ **d** $\log_x 0{\cdot}2$

**5** For each graph give the coordinates of $A$, $B$ and $C$.

**a**




**b**




**6** Simplify and evaluate:

**a** $\log_3 54 - \log_3 2$

**b** $\log_6 18 + \log_6 2$

**c** $\log_5 150 - (\log_5 2 + \log_5 3)$

**d** $2\log_4 6 - 2\log_4 3$

**7** Simplify:

**a** $\log(x^2 - x - 2) - \log(x + 1)$

**b** $\log\sqrt{x^2 - 4x + 4} - \log(x - 2)$

**8** Find the value of $m$ if:

**a** $\log_a 4m - \log_a 3 = \log_a(m + 4)$

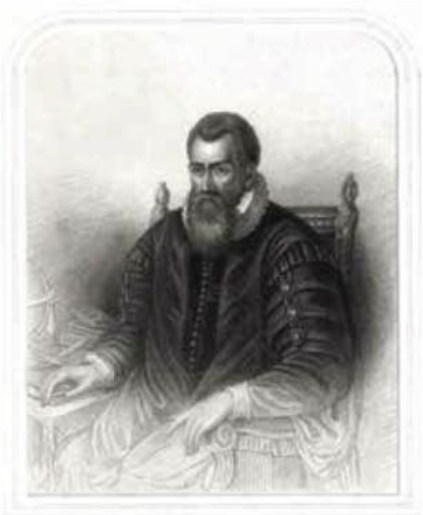
**b** $\log_a m + \log_a(m - 2) = \log_a 3$

**9** Find a relationship between $x$ and $y$, not involving logarithms.

**a** $\log x - \log y = \log(x + y)$

**b** $\log \dfrac{x^2}{y} = \log 2$

**c** $5\log x + 3\log y = \log 2$



John Napier was a mathematician who lived in the 16th century. His main achievement was the invention of logarithms, but he also invented a calculating device that was in use until the 20th century. The device was made of rods and the most expensive examples of these rods were made of horn or ivory. Hence, they became known as Napier's bones.

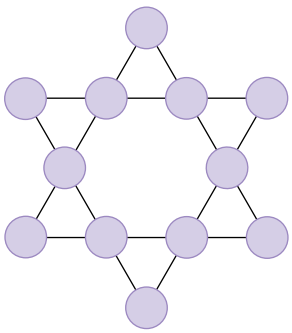
## ASSIGNMENT 15B Working mathematically

**1** **a** If 6 men can do a piece of work in 9 days, in what time will 18 men do it, working at the same rate?

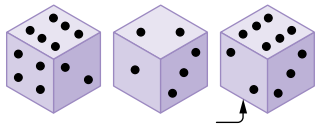
**b** If 14 men can do a piece of work in 12 days, how many men will be needed to do the work in 21 days, working at the same rate?

**2** A goods train completes a journey in $6\frac{1}{2}$ hours at an average speed of 42 km/h. What would be the average speed if the train completed the journey in $5\frac{1}{2}$ hours? (Answer correct to three significant figures.)

**3** A six-pointed star is formed by two intersecting equilateral triangles. Place the integers from 1 to 12 in the circles so that the sum of the four numbers along any side of a triangle, as well as the sum of the six integers forming the inner hexagon, is 26. (There is more than one solution to this problem.)

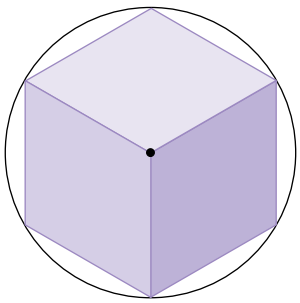


**4** Three views of the same 'unusual' dice are given. Determine the number of spots on the face on the bottom of the third cube.



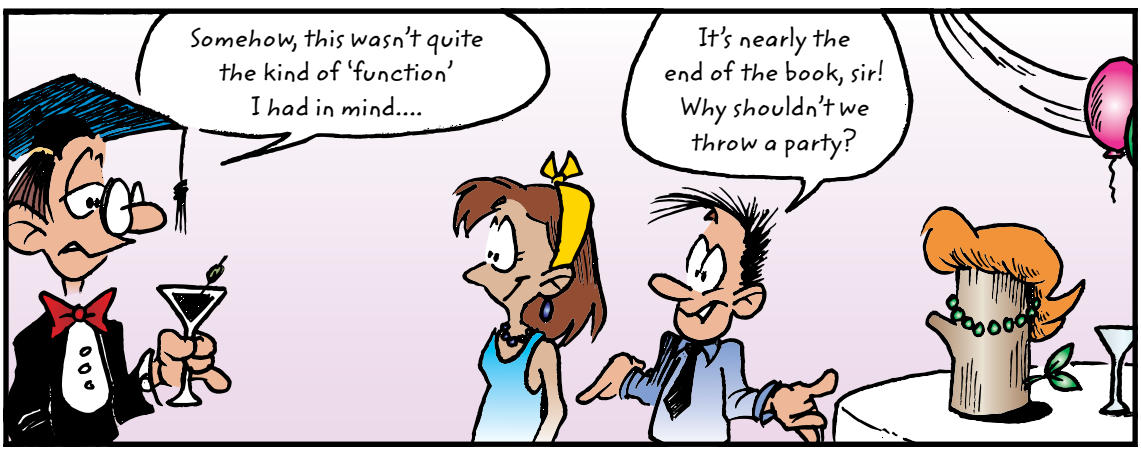
**5** A man and his granddaughter share the same birthday. For six consecutive birthdays the man's age is an exact multiple of his granddaughter's age. How old is each of them on the sixth of these birthdays? (Assume that the grandfather is less than 120 years old.)

**6** What is the edge length of the largest cube that can fit inside a sphere with a diameter of 100 cm?



# FUNCTIONS AND OTHER GRAPHS

# 16




## Contents

**16:01** Functions
**16:02** Inverse functions
Investigation 16:02 Quadratic functions and inverses
**16:03** The graphs of $y = f(x)$, $y = f(x) + k$ and $y = f(x - a)$
Fun spot 16:03 Where would you get a job playing a rubber trumpet?
Maths terms, Diagnostic test, Assignments

## Syllabus references (See pages x–xv for details.)

**Number and Algebra**

***Functions and Other Graphs*** [Stage 5.3#]

- Describe, interpret and sketch functions (NSW)

### Working Mathematically

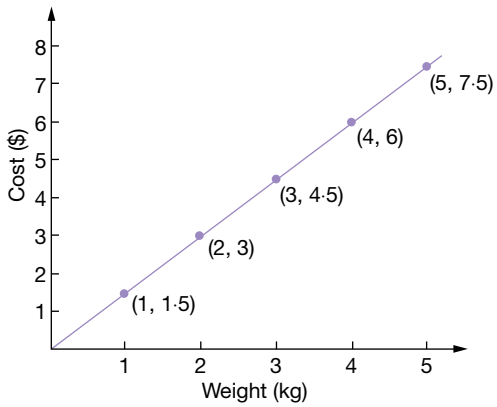
- Communicating
- Problem Solving
- Reasoning
- Understanding
- Fluency

# 16:01 Functions

- Graphs are used to show the relationship between variables.
- Graphs originate from ordered pairs, which in turn come from tables, as shown below.

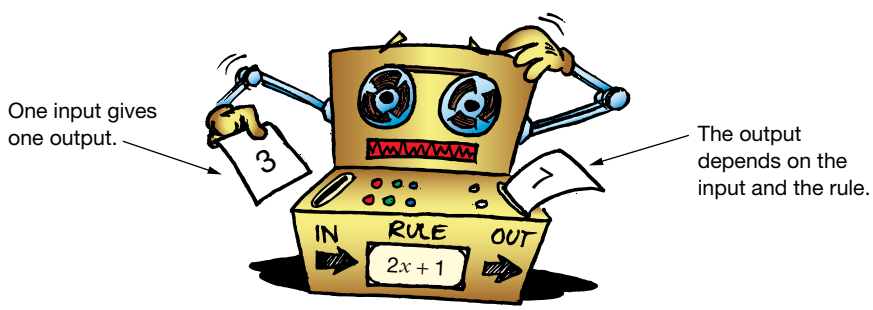
| Weight of butter (kg) | Cost ($) |
|---|---|
| 1 | $1.50 |
| 2 | $3 |
| 3 | $4.50 |
| 4 | $6 |
| 5 | $7.50 |

| Order pairs $(W, C)$ |
|---|
| (1, 1·5) |
| (2, 3) |
| (3, 4·5) |
| (4, 6) |
| (5, 7·5) |



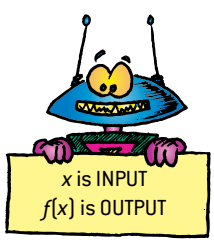

- Tables cannot include every possible ordered pair, so the relationship between variables is usually expressed as a rule or formula.
  In this case, $C = 1{\cdot}5W$.
  The value of $C$ depends on the value of $W$, so it is called the **dependent variable**.
  $W$ is called the **independent variable**.
- It should be clear that the rule $C = 1{\cdot}5W$ will produce an infinite set of ordered pairs $(W, C)$ and that every value of $W$ will produce only one value of $C$.
  When a set of ordered pairs has this property it is called a **function**.
- Functions can be thought of as being produced by a machine. The 'function machine' is given an input value, and using a rule produces a unique output value.
  The output value is dependent on both the input value and the rule being used.

> A function is a special rule or relationship that assigns a single output value to each input value.




## Function notation

- The relationship between the input value and the output value is represented by a statement such as $f(x) = 2x + 1$.
  This tells us that the function $f$ turns the input value $x$ into the output value $2x + 1$.
  Hence, $f(1) = 2 \times 1 + 1 = 3$ and $f(2) = 2 \times 2 + 1 = 5$



- Different letters can be used to represent different functions.

  e.g. $g(x) = x^2$ $\qquad$ $H(x) = (1 + x)^2$

  $g(3) = 3^2$ $\qquad$ $H(4) = (1 + 4)^2$

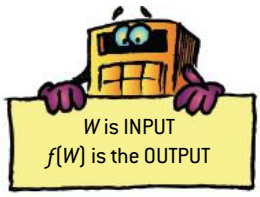
  $= 9$ $\qquad$ $= 25$

- The input value is usually represented by $x$, but other pronumerals can be used, as in the previous example concerning the cost of butter.

  e.g. $f(W) = 1{\cdot}5W$

  $f(3) = 1{\cdot}5 \times 3$

  $= 4{\cdot}5$




- Comparing $f(x) = 2x + 1$ and $y = 2x + 1$ suggests that $y = f(x)$ and, hence, there is an immediate connection with the number plane as $(x, y)$ becomes an (input value, output value) pair.

## Graphs of functions

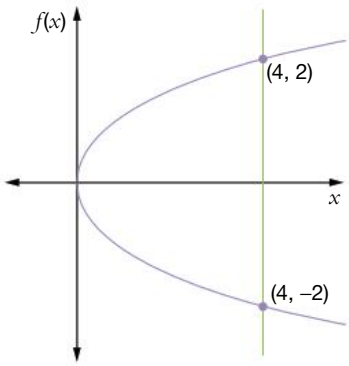
As stated earlier, a function is a set of ordered pairs, where each input value produces a unique output value.

This means that no two ordered pairs can have the same input value, or the same input value cannot produce different output values.

This gives rise to an easy test for deciding whether a graph is the graph of a function.

In this graph the vertical line shown cuts the graph at the points (4, 2) and (4, –2).

Because the graph has two ordered pairs with the same input value, it cannot represent the graph of a function.




This gives us the vertical line test.

> If a vertical line can be drawn to cut a graph in two or more places then the graph is not a function.

# Exercise 16:01

**1** In each of the following, state whether the set of ordered pairs represents a function.

**a** (1, 5), (2, 6), (3, 7), (4, 8)

**b** (8, 4), (7, 5), (6, 6), (5, 4)

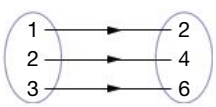
**c** (9, 3), (16, 4), (25, 5), (36, 6)

**d** (1, 3), (1, 4), (0, 3), (2, 4)

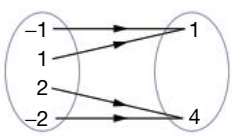
**e** (–1, 1), (1, 1), (2, 4), (–2, 4)

**2** In each of the following diagrams, an input value is joined to an output value by an arrow. State whether each diagram represents a function.

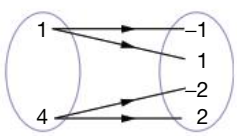
**a**




**b**




**c**



**3** If the arrows in the diagrams of Question **2** are reversed, state whether each diagram represents a function.

**4** If $f(x) = 2x - 5$, find:

**a** $f(0)$ **b** $f(2)$ **c** $f(-2)$

**5** If $H(x) = 2^x + 1$, find:

**a** $H(1)$ **b** $H(0)$ **c** $H(-1)$

**6** If $g(x) = \dfrac{x^2 + 1}{x}$, calculate:

**a** $g(1)$ **b** $g(2)$ **c** $g(\frac{3}{4})$

**7** It is known that $H(x) = 3x - 2$. Find $x$ if:

**a** $H(x) = 13$ **b** $H(x) = -14$

**8** Given that $f(x) = 2x - 5$, write an expression for:

**a** $f(2a)$ **b** $f(a + 1)$ **c** $f\left(\dfrac{1}{a}\right)$

**9** It is known that $C(W) = W^2 - 2W + 5$.

**a** Find $C(-1)$ **b** Find $C(1) - C(-1)$ **c** Find $C(W + 1)$

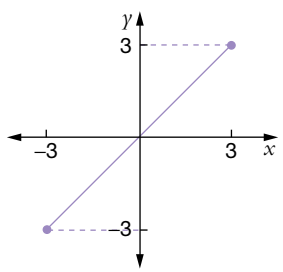
**10** If $F(p) = p^2$, find an expression for $\dfrac{F(p + h) - F(p)}{h}$.

**11** **a** If $f(x) = x^2$, show that $f(-x) = f(x)$.

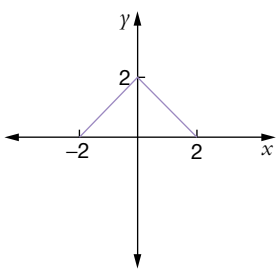
**b** If $f(x) = x^3$, show that $f(-x) = -f(x)$.

**12** Which of the following are graphs of functions?

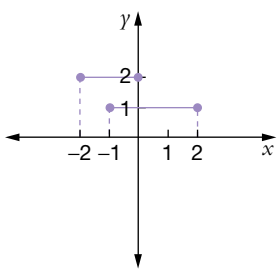
**a**



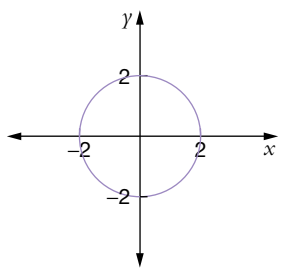

**b**



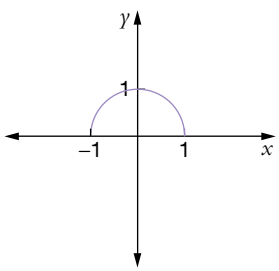

**c**



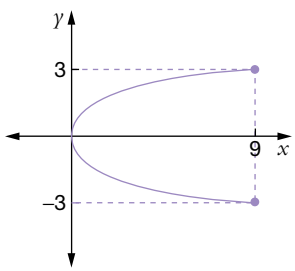

**d**



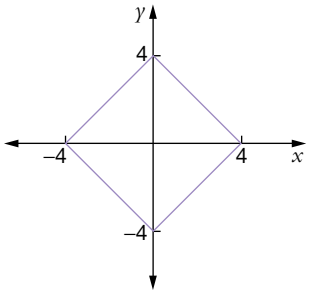

**e**



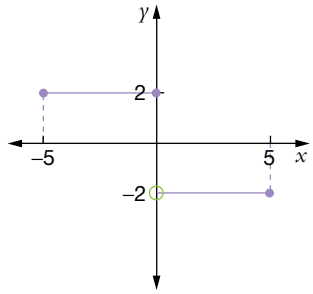

**f**



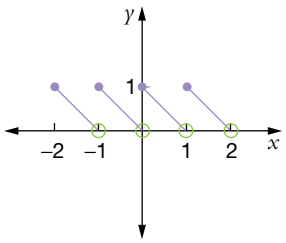
**g**




**h**




**i**



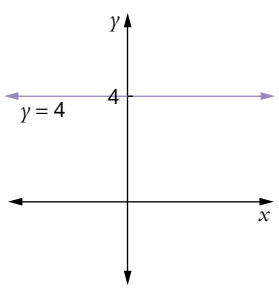
**13** Does a straight line graph always represent a function?

**14** Use this notation to describe the permissible $x$- and $y$-values for each of the graphs in Question **12**.

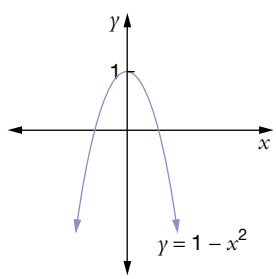
Mathematicians are often interested in the permissible $x$- and $y$-values that a function can take. For example, in Question **12a**, the possible $x$-values run from $-3$ to 3 inclusive. We write this as $\{-3 \leq x \leq 3\}$. Similarly, the $y$-values run from $-3$ to 3 inclusive. This is written as $\{-3 \leq y \leq 3\}$.

**15** Write the possible $y$-values for each of the following functions.

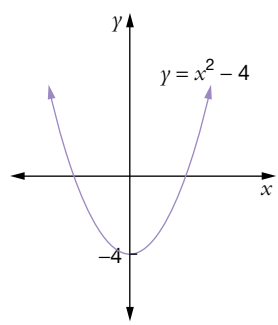
**a**




**b**

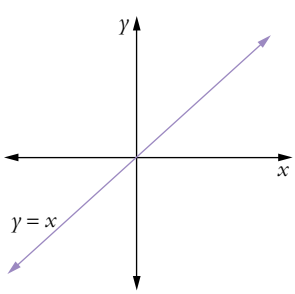



**c**

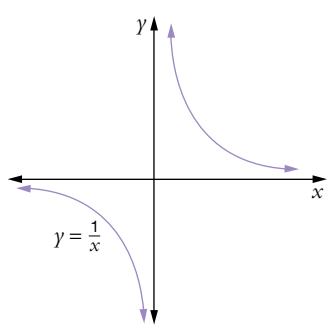



**16** Write the possible $x$-values for each of the following functions.

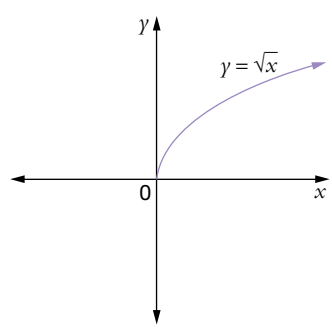
**a**




**b**




**c**



**17** By graphing each of the following or otherwise, write the permissible $x$- and $y$-values for each function.

**a** $y = 2x - 1$ **b** $y = x^2 + 1$ **c** $y = \dfrac{1}{x-1}$

**d** $y = 2^x$ **e** $y = (1 + x)^2$

**18** **a** The distance a stone falls when dropped from a cliff is a function of the time elapsed since it was dropped. The function is given by $f(t) = 4{\cdot}9t^2$, where $t$ is the time in seconds and $f(t)$ is the distance in metres. Explain in words what is meant by each of the following.

**i** $f(1{\cdot}5)$ **ii** $f(1{\cdot}6)$ **iii** $f(1{\cdot}6) - f(1{\cdot}5)$ **iv** $\dfrac{f(1{\cdot}6) - f(1{\cdot}5)}{1{\cdot}6 - 1{\cdot}5}$

**b** The period of a pendulum, $T$, is known to be a function of the length, $L$, where $T$ is measured in seconds and $L$ is measured in metres.

If $f(L) = 2\pi\sqrt{\dfrac{L}{g}}$ where $g = 9{\cdot}8$:

**i** find $f(9{\cdot}8)$

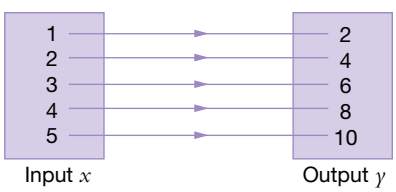
**ii** find the value of the period when $L = 4{\cdot}9$

**iii** what does $f(1{\cdot}5) - f(1)$ represent in terms of the pendulum's movement?

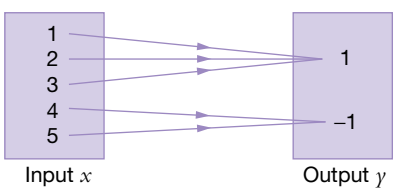
# 16:02 Inverse functions

In a function, an input value produces a unique output value. The following diagrams represent two different types of functions.

**One-to-one function**:
Each input produces one output and each output is produced from only one input.



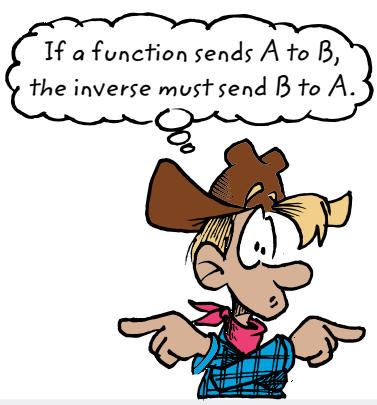

**Many-to-one function:**
Each output is produced from more than one input.



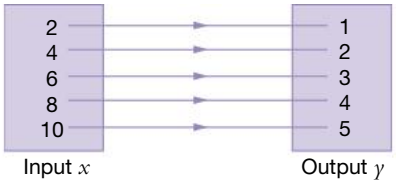

Given a certain function, $f$, we are now interested in finding a function that reverses $f$. If we can find such a function, it is called the *inverse function* and is denoted by $f^{-1}$.

To find the inverse function, it would seem that all we have to do is swap the input and output values or interchange the $x$ and $y$ in the original function.

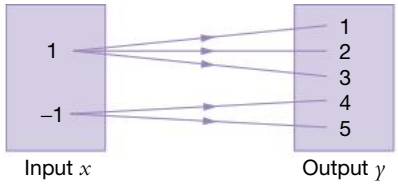
This has been done for the two functions above on the following page.



The reverse of the one-to-one function is also a function.




The reverse of the many-to-one function is **not** a function.




Reversing the process will only produce a function if the original function is one-to-one.

When the function is given in notation form rather than as a set of ordered pairs, the inverse function is produced by simply interchanging the $x$ and $y$ in the original equation and then making $y$ the subject. This is shown in the example below.

## WORKED EXAMPLE

Find the inverse function of the function $y = 3x + 1$.

### Solution

$y = 3x + 1$ where $f(x) = 3x + 1$

Interchange the $x$ and $y$ to obtain the inverse.

$x = 3y + 1$

$x - 1 = 3y$

$y = \dfrac{x-1}{3}$ where $f^{-1}(x) = \dfrac{x-1}{3}$

Hence, $y = \dfrac{x-1}{3}$ is the inverse function of $y = 3x + 1$.

If $y = f(x)$ is a one-to-one function, then the inverse function is found by:

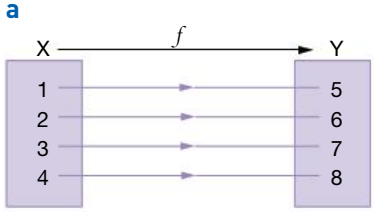
**1** interchanging $x$ and $y$ and then

**2** making $y$ the subject.

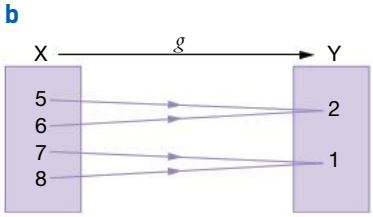
## Exercise 16:02

**1** Which of the following functions will have an inverse function?

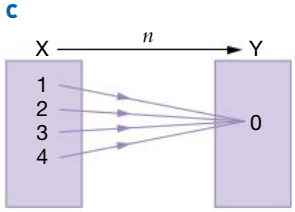
**a**




**b**




**c**



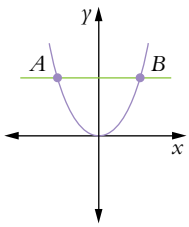
**2** **a** A function $H$ is represented by the ordered pairs $\{(0, 2), (2, 4), (3, 6), (4, 8)\}$. Does $H$ have an inverse function? If so, list the ordered pairs of the inverse function.

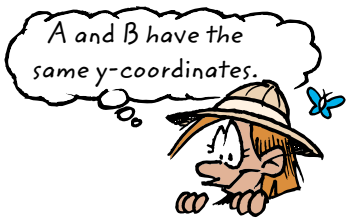
**b** A function $f$ is represented by the ordered pairs $\{(-1, 1), (1, 1), (0, 0), (-2, 2), (2, 2)\}$. Does $f$ have an inverse function? If so, list the ordered pairs of the inverse function.

**3** Find the inverse function of each of the following.

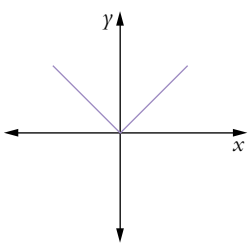
**a** $y = 2x$ **b** $y = 3x + 5$ **c** $y = \dfrac{x - 1}{3}$

**4** The function $y = x^2$ is an example of a many-to-one function as it has more than one point with the same $y$-value. As such it does not have an inverse function. Use the above to state whether the functions shown in the graphs below are one-to-one or many-to-one.

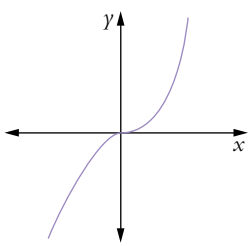


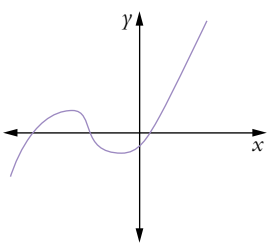

**a**



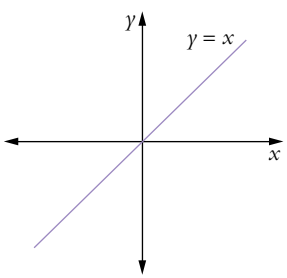

**b**



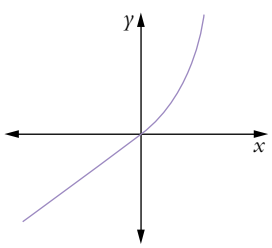

**c**



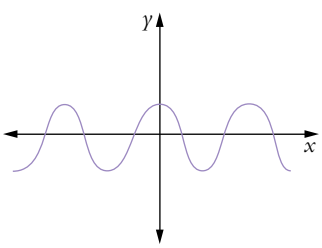

**d**




**e**




**f**




**5** Do all straight lines that are functions have inverses? Explain your answer.

**6** Can you think of a rule (similar to the vertical line test for functions) that could be used with the graph of the function to determine whether the function has an inverse or not?

**7** Find the inverse function for each of these linear functions.

**a** $f(x) = 4x$ **b** $f(x) = 1 - x$ **c** $f(x) = \dfrac{2 - x}{3}$

**8** **a** What is the inverse function of the line $y = x$?

**b** The inverse of the line $y = 2x$ is $y = \frac{1}{2}x$. Graph both lines and the line $y = x$ on the same number plane. What do you notice?

**c** On the same number plane graph $y = x + 1$, its inverse $y = x - 1$, and $y = x$. What do you notice?

**9** On the same number plane graph the lines $y = x$, and the line through the points $A(1, 1)$ and $B(3, 2)$.

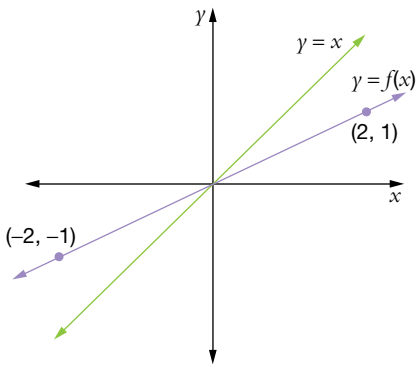
**a** Why do the points $(1, 1)$ and $(2, 3)$ lie on the inverse of line $AB$?

**b** Find the equation of the line $AB$ and hence find the equation of its inverse.

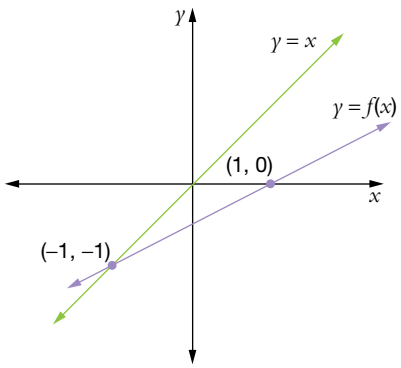
**c** Show that the points $(1, 1)$ and $(2, 3)$ satisfy the equation of the inverse.

**10** Copy the graphs below and add to each the graph of the inverse function $y = f^{-1}(x)$.

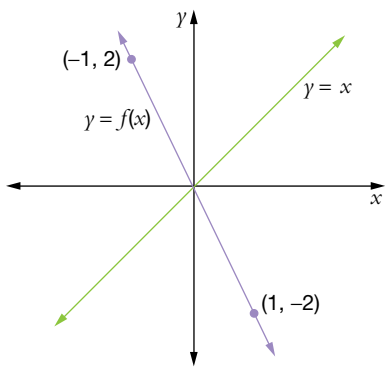
**a**




**b**




**c**




**11** The function $y = x^2$ does not have an inverse as it is a many-to-one function. If it is divided into two parts (each of which is one-to-one) it is then possible to find the inverse function.

**a** Sketch $y = x^2, x \geq 0$ (i.e. the part of $y = x^2$ that has zero or positive $x$-values).

**b** Use the fact that the inverse function is a reflection in the line $y = x$ to sketch the inverse function.

**c** Find the equation of the inverse function.

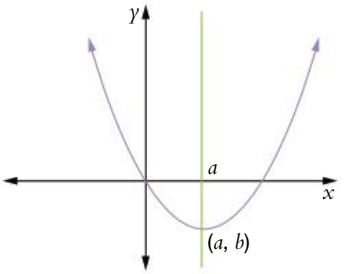
**d** Sketch the function $y = x^2, x \leq 0$ and its inverse. What is the equation of the inverse function?

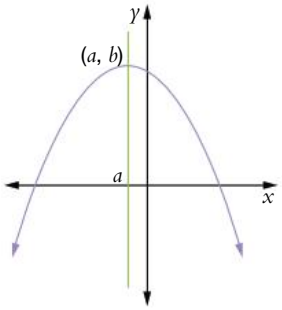
**12** Show on a graph the function $y = x^3$. Find the equation of its inverse and graph it on the same diagram.

## INVESTIGATION 16:02 QUADRATIC FUNCTIONS AND INVERSES

In Question **11** of Exercise 16:02, we considered $y = x^2$ and why it didn't have an inverse function. We now extend these ideas to any quadratic function.

**1** The two quadratic shapes we have met are given below.

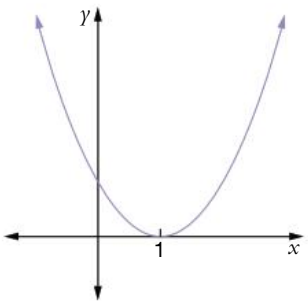



**a** Why don't the parabolas have inverse functions?

**b** How can the parabolas be divided to give two parts, each of which will have an inverse function?

**2** This graph shows $y = (x - 1)^2$.




**a** What are the coordinates of its vertex?

**b** Where must the parabola be divided to give two parts that will have inverse functions?

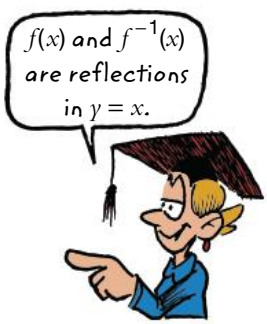
**c** Sketch the function $y = (x - 1)^2$, $x \geq 1$.

**d** Sketch the inverse of the function $y = (x - 1)^2$, $x \geq 1$.

**e** What is the equation of the inverse function?

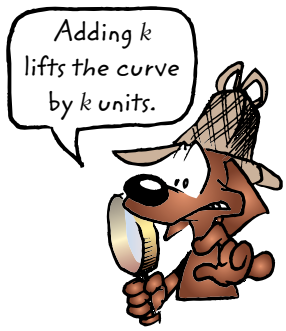
**3** **a** Sketch the function $y = 4 - x^2$, $x \geq 0$.

**b** Sketch the inverse of the function and find its equation.



# 16:03 The graphs of $y = f(x)$, $y = f(x) + k$ and $y = f(x - a)$

A relationship exists between these three curves that allows the graphs of $y = f(x) + k$ and $y = f(x - a)$ to be graphed given the shape of $y = f(x)$.




## Graphs of $y = f(x)$ and $y = f(x) + k$

| $x$ | 1 | 2 | 3 | $-1$ | $a$ |
|---|---|---|---|---|---|
| $f(x)$ | $f(1)$ | $f(2)$ | $f(3)$ | $f(-1)$ | $f(a)$ |
| $f(x) + k$ | $f(1) + k$ | $f(2) + k$ | $f(3) + k$ | $f(-1) + k$ | $f(a) + k$ |

The table shows that for the same $x$-values the $y$-values on the curve $y = f(x) + k$ are $k$ more (or less) than the $y$-values on the curve $y = f(x)$.

This means that the points on the curve $y = f(x) + k$ can be obtained by moving all the points on the curve $y = f(x)$ by $k$ units vertically.

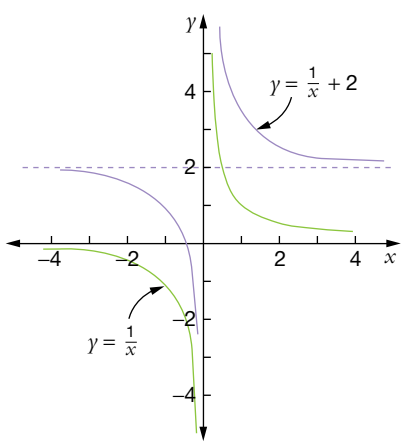
## WORKED EXAMPLE 1

Sketch the curve $y = \frac{1}{x}$ and use it to sketch the curves:

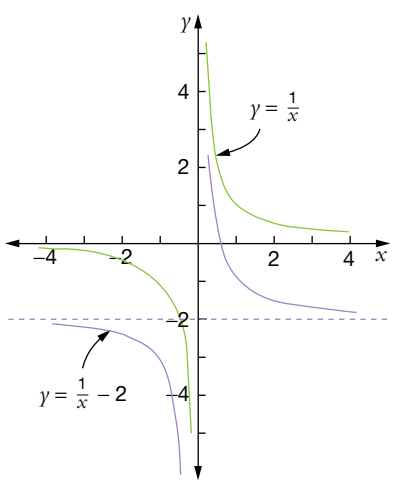
**a** $y = \frac{1}{x} + 2$ **b** $y = \frac{1}{x} - 2$

### Solutions

**a** To obtain $y = \frac{1}{x} + 2$, move $y = \frac{1}{x}$ up 2 units.




**b** To obtain $y = \frac{1}{x} - 2$, move $y = \frac{1}{x}$ down 2 units.



The curve $y = f(x) + k$ is obtained by moving the curve $y = f(x)$:

- up $k$ units if $k$ is positive
- down $k$ units if $k$ is negative.

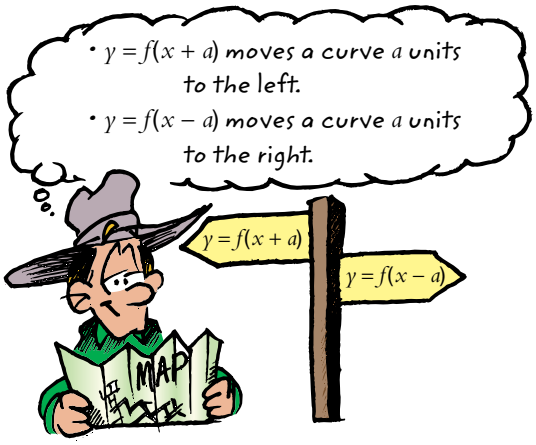
## Graphs of $y = f(x)$ and $y = f(x - a)$

| $x$ | 0 | 1 | 2 | −1 |
|---|---|---|---|---|
| $f(x)$ | $f(0)$ | $f(1)$ | $f(2)$ | $f(-1)$ |

| $x$ | $a$ | $a + 1$ | $a + 2$ | $a - 1$ |
|---|---|---|---|---|
| $f(x - a)$ | $f(0)$ | $f(1)$ | $f(2)$ | $f(-1)$ |

The tables show that when the $y$-values on $y = f(x)$ and $y = f(x - a)$ are equal, the $x$-values differ by $a$ units.

Hence, the points on the curve $y = f(x - a)$ can be obtained by moving all the points on the curve $y = f(x)$ $a$ units horizontally.

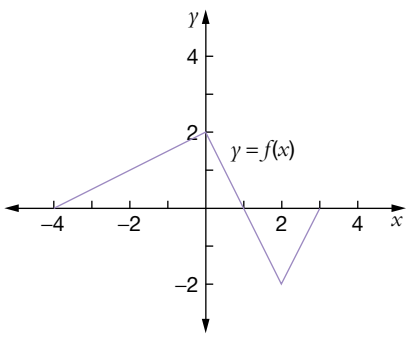



## WORKED EXAMPLE 2

Use the graph of $y = f(x)$ to sketch the following:

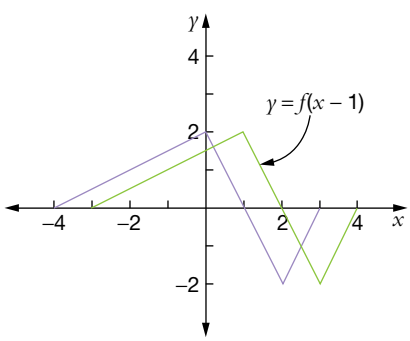
**a** $y = f(x - 1)$

**b** $y = f(x + 1)$

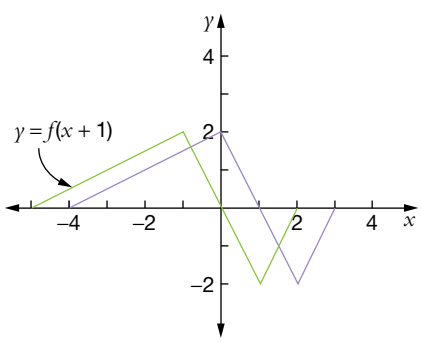



### Solution

**a** To obtain $y = f(x - 1)$, move $y = f(x)$ 1 unit to the right.




**b** To obtain $y = f(x + 1)$, move $y = f(x)$ 1 unit to the left.



The curve $y = f(x - a)$ is obtained by moving the curve $y = f(x)$:

- $a$ units to the right if $a$ is positive
- $a$ units to the left if $a$ is negative.

## Exercise 16:03

**1** State how the curve $y = x^2$ could be moved to produce each of the following curves.

**a** $y = (x + 1)^2$ **b** $y = x^2 + 1$ **c** $y = x^2 - 1$ **d** $y = (x - 1)^2$

**2** Give the equation of the curve that would result if the curve $y = 2^x$ was moved:

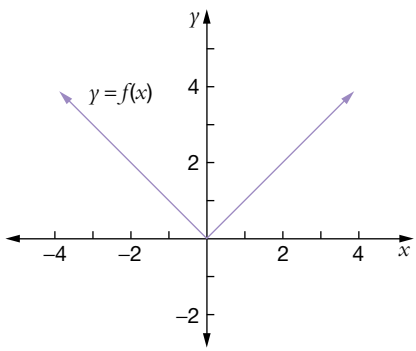
**a** 1 unit up **b** 1 unit down

**c** 1 unit to the right **d** 1 unit to the left

**3** Use the given graph of $y = f(x)$ to sketch the following functions.

**a** $y = f(x) + 1$ **b** $y = f(x) - 2$

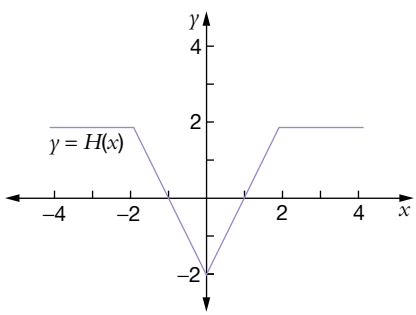
**c** $y = f(x - 1)$ **d** $y = f(x + 2)$




**4** Use the given graph of $y = H(x)$ to sketch the following functions.

**a** $y = H(x - 2)$ **b** $y = H(x) + 2$

**c** $y = H(x) - 1$ **d** $y = H(x + 1)$



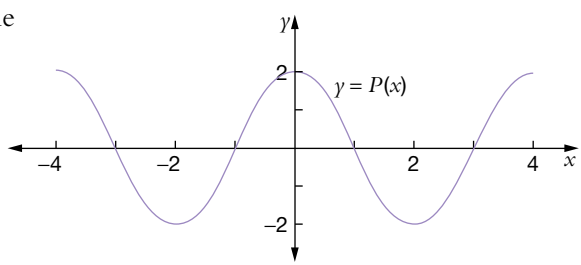

**5** Use the given graph of $y = P(x)$ to sketch the following functions.

**a** $y = P(x) + 2$

**b** $y = P(x + 1)$

**c** $y = P(x - 2)$

**d** $y = P(x) - 1$




**6** Sketch $y = F(x) + 1$ and $y = F(x + 1)$ if:

**a** $F(x) = x^3$ **b** $F(x) = x^2 - 1$ **c** $y = \frac{1}{x}$

## FUN SPOT 16:03

## WHERE WOULD YOU GET A JOB PLAYING A RUBBER TRUMPET?

Work out the answer to each part and write the letter for that part in the box that is above the correct answer.

If $f(x) = x^2$ and $g(x) = (x + 1)^2$, evaluate:

**N** $f(2)$ **T** $g(-2)$ **I** $f[g(2)]$

*State the permissible y-values for each of the functions:*

**C** $y = 4 - x^2$ **B** $y = x^2 + 1$

**A** $y = (x + 1)^2$ **I** $y = x^2 - 4$

*State the inverse function of:*

**N** $y = 2x - 1$ **D** $y = \frac{1}{2}x + 1$

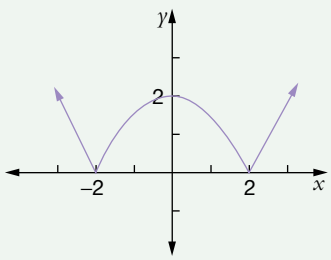
*State the equation of the resulting curve if the graph of $y = 2x$ is moved:*

**A** up 1 unit

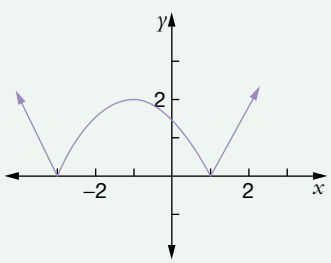
**L** left 1 unit.

The function $y = H(x)$ is shown on the right. Moving this curve produces the four curves below.

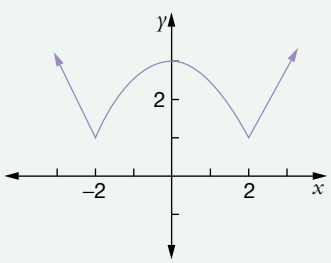
*What is the equation of each curve?*



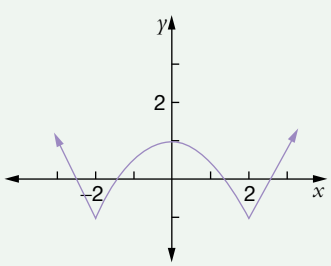

**A**



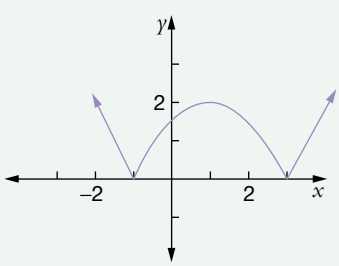

**S**




**E**




**N**




| $y \geq -4$ | 4 | $y \geq 0$ | $y = \frac{1}{2}x + \frac{1}{2}$ | $y = H(x) - 1$ | $y = 2x + 2$ | $y = 2x + 1$ | $y = H(x) + 1$ | 1 | 81 | $y \leq 4$ | $y \geq 1$ | $y = H(x + 1)$ | $y = H(x - 1)$ | $y = 2x - 2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | |

## MATHS TERMS 16

**dependent variable**

- a variable that depends on the value of another variable,
  e.g. for $y = x^2 + 1$
  $y$ is the dependent variable because its value depends on the value of $x$

**function**

- a special rule or relationship that assigns a single output value to each input value

**function notation**

e.g. $f(x) = x^2 + 1$ tells us that the function $f$ turns the input value $x$ into the output value $x^2 + 1$

**independent variable**

- a variable that does *not* depend on the value of any other variable,
  e.g. for $y = x^2 + 1$
  $x$ is the independent variable

**inverse function**

- denoted by $f^{-1}$
- an inverse function $f^{-1}$ reverses the function $f$
  e.g. if $f(x) = 2x + 1$,
  then $f^{-1}(x) = \dfrac{x - 1}{2}$

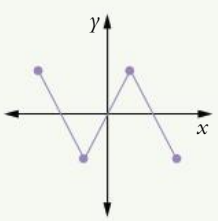
## DIAGNOSTIC TEST 16 FUNCTIONS AND OTHER GRAPHS

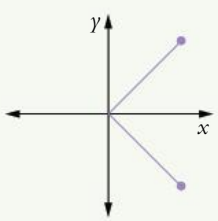
Each part of this test has similar items that test a certain type of question.
Errors made will indicate areas of weakness.
Each weakness should be treated by going back to the section listed.

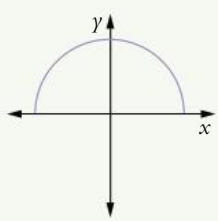
1 a If $g(x) = 3x - 1$, find $g(3)$.
b If $f(x) = x^2 + x$, find $f(-1)$.
c If $H(a) = 3a + a^2$, find $H(\frac{1}{2})$. 16:01

2 If $f(p) = 2p - 5$, write an expression for: 16:01
a $f(2p)$ b $f(a + 1)$ c $f(x^2)$

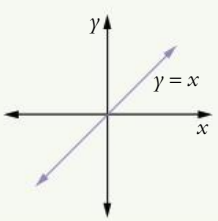
3 Which of the following represent graphs of functions? 16:01

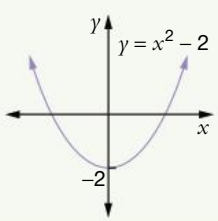
a 

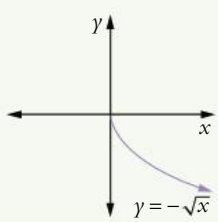

b 


c 


4 Write the possible set of $x$- and $y$-values for each of the following. 16:01

a 


b 


c 

5 a If $y = 2x - 1$, find the inverse function. 16:02

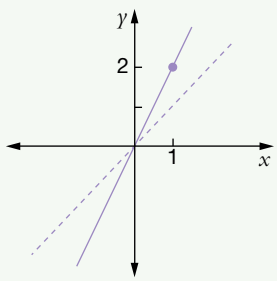
b If $f(x) = 3 - 2x$, find $f^{-1}(x)$.

c If $g(x) = \dfrac{3x + 1}{2}$, find $g^{-1}(x)$.

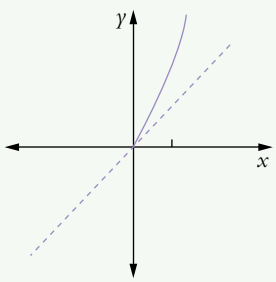
6 Which of the functions in Question 4 would have an inverse function? 16:02

7 Copy the following and add a sketch of the inverse function. 16:02

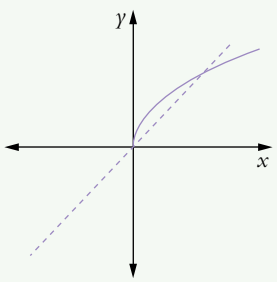
a




b




c



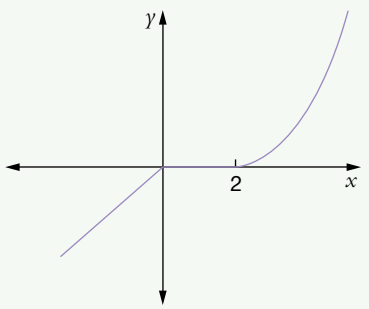

8 Use the sketch of $y = H(x)$ to sketch: 16:03

a $y = H(x) + 1$

b $y = H(x - 1)$

c $y = H(x + 1)$

d $y = H(x) - 1$



The curve formed by the chain is called a catenary.

# ASSIGNMENT 16A Chapter review

**1** A function is defined by $f(x) = 2x + 1$.

**a** Find $f(-2)$.

**b** If $f(a) = -5$, find $a$.

**c** Find $f^{-1}(x)$ and hence find $f^{-1}(2)$.

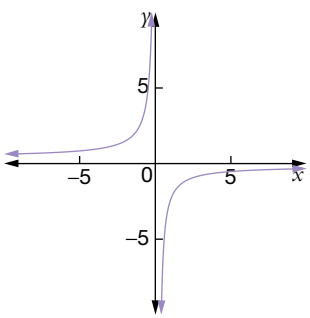
**2** Sketch the function $f(x) = x^2$ for the domain $-2 \leq x \leq 2$. For what values of $x$ will $f(x) = 2$? Does this function have an inverse?

**3** If $f(x) = x^2 + x + 1$, write an expression in its simplest form for:

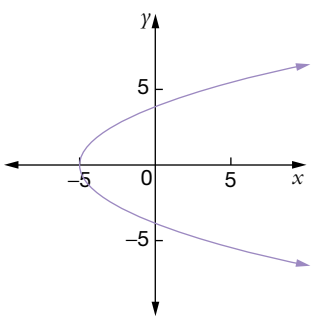
**a** $f(3x)$ **b** $f(-x)$ **c** $f(x - 1)$

**4** State whether each of the following graphs represents a function.

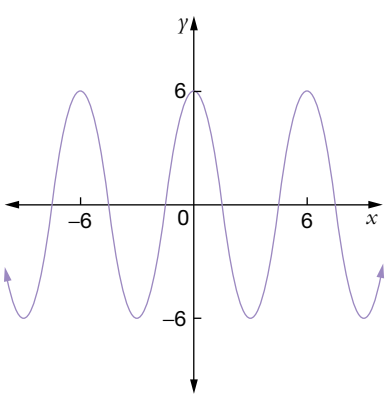
**a**




**b**




**c**



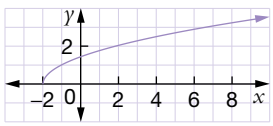

**5** For the graphs shown in Question **4**, determine all the possible values for $x$ and $y$.

**6** Find the inverse function $g^{-1}(x)$ for:

**a** $g(x) = 4x + 3$ **b** $g(x) = \dfrac{5 - 2x}{4}$

**c** $g(x) = (x + 1)^3$

**7** This is the graph of the function $y = \sqrt{x + 2}$.




**a** What are the possible $x$-values for this function?

**b** What are the possible $y$-values for this function?

**c** Determine the equation of the inverse function of $y = \sqrt{x + 2}$.

**d** What are the possible $x$-values for this inverse function?

**e** What are the possible $y$-values for this inverse function?

**f** Copy the diagram above and add the graph of the inverse function.

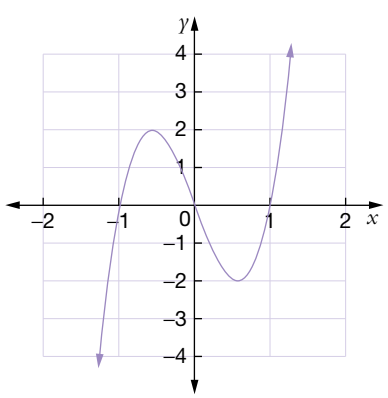
**8** **a** Find the inverse function for $f(x) = \dfrac{x + 1}{x - 1}$.

**b** Considering your answer to part **a**, what relationship exists between the graph of $y = f(x)$ and the line $y = x$?

**9** The graph of $y = G(x)$ is shown below. Sketch the graphs of:

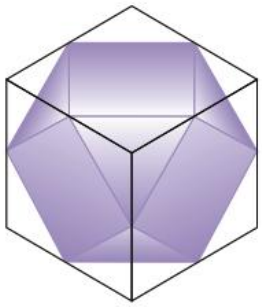
**a** $y = G(x) + 2$ **b** $y = G(x + 2)$

**c** $y = G(x - 2)$ **d** $y = G(x + 2) - 2$

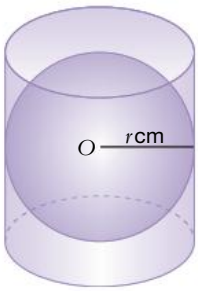


## ASSIGNMENT 16B Working mathematically

**1** A solid is formed from a cube by cutting off the corners in such a way that the vertices of the new solid will be at the midpoints of the edges of the original cube. How many edges will the new solid have?



**2** The diagram shows a sphere of radius $r$cm which fits inside a cylinder. The cylinder has the same diameter and height as the sphere. Calculate (in terms of $r$):




- **a** the volume of the sphere
- **b** the volume of the cylinder
- **c** the volume of the cylinder not occupied by the sphere
- **d** the volume of a cone with a radius of $r$cm and a height of $2r$cm.
- **e** What relationship can you see between the answers to parts **a**, **b** and **d**?

**3** John is given \$10 each month for a year. Anna is given 5 cents for the first month, 10 cents for the second, 20 cents for the third and so on until the twelfth month. Who is given the most in the year and by how much is the amount bigger?

**4** The table below gives the number of car occupants in various seating positions, and the number who died as a direct result of an impact to the side of the car. The figures exclude 38 occupants as there was insufficient information to determine their exact seating positions at the time of the crash.

- **a** How many occupants were in:
  - **i** the driver's seat
  - **ii** the front left side seat
  - **iii** a front seat
  - **iv** a passenger's side seat?
- **b** What percentage of drivers died?
- **c** What percentage of all the occupants who died were drivers?
- **d** For each of the six seating positions, calculate the number of deaths as a percentage of the total number of people in that seat. Can you infer anything about the relative safety of the various seating positions from this information? Give reasons for your answer.

| | Seating position | | | | | | All occupants | |
|---|---|---|---|---|---|---|---|---|
| | Left side (passenger's side) | | Centre seat | | Right side (driver's side) | | | |
| | Dead | Total | Dead | Total | Dead | Total | Dead | Total |
| Front seat | 114 | 187 | 0 | 1 | 199 | 316 | 313 | 504 |
| Rear seat | 29 | 63 | 2 | 19 | 20 | 44 | 51 | 126 |
| All occupants | 143 | 250 | 2 | 20 | 219 | 360 | 364 | 630 |

# Answers

# ID Cards

## ID Card 1 (Metric Units)

1 metres
2 decimetres
3 centimetres
4 millimetres
5 kilometres
6 square metres
7 square centimetres
8 square kilometres
9 hectares
10 cubic metres
11 cubic centimetres
12 seconds
13 minutes
14 hours
15 metres per second
16 kilometres per hour
17 grams
18 milligrams
19 kilograms
20 tonnes
21 litres
22 millilitres
23 kilolitres
24 degrees Celsius

## ID Card 2 (Symbols)

1 is equal to
2 is approximately equal to
3 is not equal to
4 is less than
5 is less than or equal to
6 is not less than
7 is greater than
8 is greater than or equal to
9 4 squared
10 4 cubed
11 the square root of 2
12 the cube root of 2
13 is perpendicular to
14 is parallel to
15 is congruent to
16 is similar to
17 per cent
18 therefore
19 for example
20 that is
21 pi
22 the sum of
23 the mean
24 probability of event $E$

## ID Card 3 (Language)

1 square
2 rectangle
3 parallelogram
4 rhombus
5 trapezium
6 regular pentagon
7 regular hexagon
8 regular octagon
9 kite
10 scalene triangle
11 isosceles triangle
12 equilateral triangle
13 circle
14 oval (or ellipse)
15 cube
16 rectangular prism
17 triangular prism
18 square pyramid
19 rectangular pyramid
20 triangular pyramid
21 cylinder
22 cone
23 sphere
24 hemisphere

## ID Card 4 (Language)

1 point $A$
2 interval $AB$
3 line $AB$
4 ray $AB$
5 collinear points
6 midpoint
7 number line
8 diagonals
9 acute-angled triangle
10 right-angled triangle
11 obtuse-angled triangle
12 vertices
13 $\Delta ABC$
14 hypotenuse
15 $180°$
16 $a° + b°$
17 $360°$
18 [$b$] $a° = b°$
19 $a° = 60°$
20 $3 \times 180° = 540°$
21 $AB$ is a diameter: $OC$ is a radius.
22 circumference
23 semicircle
24 $AB$ is a tangent. $CD$ is an arc. $EF$ is a chord.

## ID Card 5 (Language)

1 parallel lines
2 perpendicular lines
3 vertical, horizontal
4 concurrent lines
5 angle $ABC$ or $CBA$
6 *acute* angle
7 *right* angle
8 *obtuse* angle
9 straight angle
10 reflex angle
11 revolution
12 adjacent angles
13 complementary angles
14 supplementary angles
15 vertically opposite angles
16 $360°$
17 transversal
18 corresponding angles
19 alternate angles
20 co-interior angles
21 *bisecting* an interval
22 *bisecting* an angle
23 $\angle CAB = 60°$
24 $CD$ is *perpendicular* to $AB$

## ID Card 6 (Language)

1 Common Era, Anno Domini
2 Before Common Era, Before Christ
3 ante meridiem, before midday
4 post meridiem, after midday
5 hectare
6 regular shapes
7 net of a cube
8 cross-section
9 face
10 vertex
11 edge
12 axes of symmetry

**13** reflection (or flip) **14** translation (or slide) **15** rotation (or turn)
**16** tessellation **17** coordinates **18** tally
**19** picture graph **20** column graph **21** line graph
**22** sector (or pie) graph **23** bar graph **24** scatter diagram

# Chapter 1

## Exercise 1:01A

**1** **a** 2 **b** 1 **c** 0 **d** 14 **e** −3
**f** 17 **g** 12 **h** 30 **i** 70 **j** 20
**k** 300 **l** 25 **m** 19 **n** 11 **o** 18
**2** **a** 9 **b** 0 **c** 10 **d** 2 **e** $1\frac{1}{5}$
**f** 2 **g** 81 **h** 196 **i** 100

## Exercise 1:01B

**1** **a** $1\frac{3}{4}$ **b** $8\frac{1}{6}$ **c** $3\frac{3}{4}$ **d** $1\frac{3}{8}$
**2** **a** $\frac{11}{2}$ **b** $\frac{22}{7}$ **c** $\frac{35}{4}$ **d** $\frac{200}{3}$
**3** **a** $\frac{3}{5}$ **b** $\frac{7}{15}$ **c** $\frac{2}{3}$ **d** $\frac{5}{9}$
**4** **a** $\frac{18}{24}$ **b** $\frac{20}{50}$ **c** $\frac{8}{28}$ **d** $\frac{40}{120}$
**5** **a** $\frac{8}{15}$ **b** $\frac{1}{4}$ **c** $\frac{37}{40}$ **d** $\frac{9}{35}$
**6** **a** $9\frac{1}{10}$ **b** $2\frac{9}{20}$ **c** $10\frac{17}{20}$ **d** $3\frac{19}{40}$
**7** **a** $\frac{12}{35}$ **b** $\frac{27}{40}$ **c** $\frac{2}{15}$ **d** $\frac{7}{15}$
**8** **a** $4\frac{1}{2}$ **b** $4\frac{1}{2}$ **c** 20 **d** 15
**9** **a** $1\frac{7}{20}$ **b** $\frac{5}{8}$ **c** $\frac{2}{15}$ **d** $1\frac{5}{6}$

## Exercise 1:01C

**1** **a** 0·066, 0·6, 0·606, 0·66 **b** 0·153, 1·053, 1·53
**c** 0·017, 0·7, 0·77, 7 **d** 3·05, 3·4, 3·45, 3·5
**2** **a** 9·301 **b** 3·45 **c** 3·104 **d** 6·32
**e** 1·97 **f** 8·105 **g** 4·888 **h** 159·3
**3** **a** 0·036 **b** 0·006 **c** 0·585 **d** 0·0025
**4** **a** 31·4 **b** 500 **c** 0·03 **d** 38 000
**5** **a** 0·03 **b** 0·265 **c** 3·07 **d** 0·0025
**6** **a** $0{\cdot}4\dot{3}$ **b** $0{\cdot}8\dot{2}\dot{7}$ **c** $1{\cdot}\dot{5}$ **d** $0{\cdot}\dot{8}57\,142\dot{}$
**7** **a** 4·804 **b** 0·016 **c** 0·0009 **d** 0·000 65
**8** **a** 21 **b** 10·45 **c** 1500 **d** 2·8
**9** **a** $3\frac{17}{1000}$ **b** $\frac{1}{25}$ **c** $\frac{43}{50}$ **d** $16\frac{1}{200}$
**10** **a** 0·8 **b** 0·035 **c** 0·625 **d** $0{\cdot}\dot{7}\dot{2}$
**11** **a** $\frac{5}{9}$ **b** $\frac{257}{999}$ **c** $\frac{8}{11}$ **d** $\frac{214}{333}$
**12** **a** $\frac{5}{6}$ **b** $\frac{151}{165}$ **c** $\frac{98}{225}$ **d** $\frac{1489}{1665}$

## Exercise 1:01D

**1** **a** $\frac{27}{50}$ **b** $\frac{203}{100}$ or $2\frac{3}{100}$ **c** $\frac{49}{400}$ **d** $\frac{91}{1000}$
**2** **a** 55% **b** $44\frac{4}{9}$% **c** 125% **d** $66\frac{2}{3}$%
**3** **a** 0·16 **b** 0·086 **c** 0·03 **d** 0·1825
**4** **a** 47% **b** 6% **c** 37·5% **d** 130%
**5** **a** 144 m **b** 7·56 g **c** $2.72 **d** $86 360
**6** **a** $60 **b** 25 kg **c** $5 **d** 180 min or 3 h
**7** **a** 42·5% **b** 45% **c** 18·75%
**d** 12% (to nearest whole %)

## Exercise 1:01E

**1** **a** **i** 3 : 5 **ii** 1 : 10 **iii** 15 : 7 **iv** 3 : 1
**b** 14 : 1 **c** 2 : 3 **d** 15 : 4
**e** **i** 8 : 5 **ii** 41 : 130
**f** **i** $\frac{3}{5}$ : 1 **ii** $\frac{2}{7}$ : 1 **iii** $\frac{10}{3}$ : 1 **iv** $\frac{25}{4}$ : 1
**g** **i** $1 : \frac{5}{3}$ **ii** $1 : \frac{7}{2}$ **iii** $1 : \frac{3}{10}$ **iv** $1 : \frac{4}{25}$
**2** **a** $x = 50$ **b** 1098 million
**c** 2·5 people per $km^2$ **d** 4·2 million
**3** **a** Naomi gets 48, Luke gets 36
**b** 40°, 60°, 80°
**c** Tokyo, 12 million; Moscow, 10 million
**d** 36 males, 24 females

## Exercise 1:01F

**1** **a** 300 km/h **b** 8 m/mL
**c** 14·4 t/day **d** 2075 $cm^3$/kg
**2** **a** 6 miles per hour **b** $46.20
**c** $4\frac{2}{3}$ minutes/book **d** 10 $cm^3$/s

## Exercise 1:01G

**1** **a** 2 **b** 2 **c** 3 **d** 3 **e** 4
**f** 3 **g** 4 **h** 3 **i** 3 **j** 4
**k** 1 **l** 1 **m** 2 **n** 1 **o** 1
**p** 2 **q** 3 **r** 3 **s** 5 **t** 3
**u** 5 **v** 1 **w** 3 **x** 2
**y** It is ambiguous, 2 **z** It is ambiguous, 2
**2** **a** 2 **b** 3 **c** 1 **d** 1 **e** 2
**f** 2 **g** 3 **h** 5

## Exercise 1:01H

**1** **a** 4·6 **b** 0·8 **c** 3·2 **d** 0·1
**e** 15·2 **f** 8·1 **g** 1·0 **h** 121·6
**i** 0·1 **j** 47·4 **k** 0·4 **l** 2·8
**2** **a** 0·54 **b** 2·61 **c** 7·13 **d** 1·17
**e** 12·02 **f** 8·40 **g** 412·68 **h** 0·08
**i** 0·44 **j** 100·33 **k** 0·02 **l** 0·01
**3** **a** **i** 7 **ii** 7·3
**b** **i** 80 **ii** 85
**c** **i** 0·6 **ii** 0·63
**d** **i** 3 **ii** 2·6
**e** **i** 4 **ii** 4·2
**f** **i** 0·007 **ii** 0·0073
**g** **i** 0·08 **ii** 0·083
**h** **i** 3 **ii** 3·1
**i** **i** 0·009 **ii** 0·0093
**j** **i** 0·01 **ii** 0·0098

**k** i 8 ii 7·5
**l** i 0·04 ii 0·036

**4** **a** 2 **b** 14·6 **c** 2·2 **d** 0·9
**e** 4·1 **f** 7·37 **g** 0·724 **h** 6
**i** 31·69 **j** 0·007 **k** 0·8 **l** 0·0072

**5** **a** 5·6 **b** 0·2 **c** 0·44 **d** 15·4
**e** 8·33 **f** 413·8 **g** 72·0 **h** 3·067
**i** 10·0 **j** 4·800 **k** 0·08 **l** 0·004

## Exercise 1:01I

**1** Answers near these are acceptable.
**a** 74 **b** 120 **c** 31 **d** 7·2 **e** 110
**f** 60 **g** 18 **h** 5·8 **i** 7·7 **j** 1·8
**k** 0·4 **l** 52 **m** 15 **n** 21 **o** 310
**p** 1·1 **q** 7·0 **r** 17 **s** 59

## Exercise 1:01J

**1** **a** 12 000 L **b** 5000 kg **c** 3850 mg
**d** 8 000 000 m **e** 7 000 000 µm **f** 2 000 000 000 nm
**g** 6 000 000 000 B **h** 4000 GB **i** 3000 GB
**j** 25 000 µm **k** 5400 kB **l** 500 µs

**2** 400 GB of space has been used.

## Exercise 1:01K

**1** **a** 0·5 g **b** 0·05 m **c** 0·0005 L **d** 0·5 cm
**e** 0·05 t **f** 0·005 s **g** 500 L

**2** **a** 0·05 kg, smallest = 63·65 kg, largest = 63·75 kg
**b** from 11·25 cm up to 11·35 cm

## Exercise 1:02

**1** **a** $3a + 4b$ **b** $12ab$ **c** $k - m$
**d** $m - k$ **e** $\dfrac{x + y + z}{3}$ **f** $2(m + 5)$
**g** $(a - b)^2$ **h** $\sqrt{5m + 4n}$ **i** $n + 2$
**j** $m + (m + 1) + (m + 2) = 3m + 3$

**2** **a** 21 **b** 2 **c** 16 **d** −15
**e** −43 **f** 45 **g** −2 **h** 5
**i** $\frac{7}{2}$ or $3\frac{1}{2}$ **j** 3 **k** 1 **l** 3

**3** **a** $4a + 4b$ **b** $3ab$ **c** $2x^2 + 2x$
**d** $15xy$ **e** $18a^2b$ **f** $-10m^2n$
**g** $3a$ **h** 2 **i** $2a$
**j** $\frac{1}{3}$ **k** $\dfrac{3m}{2n}$ **l** $\dfrac{3y}{2x}$
**m** 21 **n** $10y$ **o** $5x$

**4** **a** $\dfrac{6a}{5}$ **b** $\dfrac{2x}{7}$ **c** $\dfrac{7}{y}$ **d** $\dfrac{7a}{12}$
**e** $\dfrac{7m}{15}$ **f** $\dfrac{1}{6n}$ **g** $\dfrac{ab}{12}$ **h** $\dfrac{2m^2}{15}$
**i** $3ay$ **j** $\frac{5}{2}$ **k** $2b$ **l** $xz$

**5** **a** $a + 3$ **b** $8m - 10$
**c** $5a + 15$ **d** $8n + 7$
**e** $5a - 19$ **f** $3 - 2x$
**g** $x^2 + 10x + 21$ **h** $y^2 - 5y + 4$
**i** $k^2 + 2k - 63$ **j** $2p^2 - 7p - 15$
**k** $18x^2 - 9x - 2$ **l** $6m^2 - 17m + 5$
**m** $m^2 - 49$ **n** $9a^2 - 16$
**o** $100 - 9q^2$ **p** $a^2 + 16a + 64$
**q** $4m^2 - 4m + 1$ **r** $16a^2 + 40a + 25$
**s** $x^2 - xy - 2y^2$ **t** $a^2 - 4b^2$
**u** $m^2 - 6mn + 9n^2$

**6** **a** $5(3a - 2)$ **b** $3m(m - 2)$
**c** $2n(2 + 3m)$ **d** $2m(3n - 2)$
**e** $5y(2y + 1)$ **f** $2a(3a - 1 + 2b)$
**g** $(x - 7)(x + 7)$ **h** $(10 - a)(10 + a)$
**i** $(4a - 3b)(4a + 3b)$ **j** $(x + 6)(x + 2)$
**k** $(x - 4)(x + 3)$ **l** $(x - 4)(x - 2)$
**m** $(a + 3)^2$ **n** $(y - 5)^2$
**o** $(1 - 2m)^2$ **p** $(2x + 1)(x + 3)$
**q** $(3m - 2)(m + 3)$ **r** $(3a - 4)(2a - 1)$
**s** $(2n + 3)^2$ **t** $(5x - 1)^2$
**u** $(3 - 4m)^2$ **v** $(a + x)(b - 4)$
**w** $(x - 2)(x + a)$ **x** $(2m - 1)(m + 3n)$

**7** **a** $2(x - 3)(x + 3)$ **b** $4(x + 3)(x - 2)$
**c** $3(a - b)(a - 2)$ **d** $2(2n - 1)^2$
**e** $9(1 - q)(1 + q)$ **f** $m^2(m - 1)(m + 1)$
**g** $(k^2 + 4)(k - 2)(k + 2)$ **h** $(y^2 + 1)(y + 1)$
**i** $(x + 1)(x - 1)^2$

**8** **a** $a + 4$ **b** 5 **c** $\dfrac{1}{a + 2}$
**d** $\dfrac{m}{m + 1}$ **e** $\dfrac{n - 3}{n + 3}$ **f** $\dfrac{x + 1}{2x + 3}$

**9** **a** $6x$ **b** $a + 1$ **c** $\frac{3}{2}$
**d** 5 **e** $\dfrac{a + 3}{a + 7}$ **f** $\dfrac{1}{3(n - 1)}$

**10** **a** $\dfrac{2a + 7}{(a + 3)(a + 4)}$ **b** $\dfrac{2x + 14}{(2x - 1)(4x + 3)}$
**c** $\dfrac{5x + 2}{x(x + 2)(x + 1)}$ **d** $\dfrac{2x + 5}{(x + 3)(x + 4)(x + 5)}$

**11** **a** $\dfrac{a}{(a - 1)(a + 1)}$ **b** $\dfrac{2x - 7}{3(x + 2)(x - 2)}$
**c** $\dfrac{5x - 4}{(x + 3)(x - 2)(x + 1)}$ **d** $\dfrac{3x - 12}{(x - 2)(x + 1)(x - 3)}$
**e** $\dfrac{2x^2 + x - 5}{(x - 3)(x + 3)(x - 2)}$ **f** $\dfrac{10n + 18}{(2n - 1)(n + 1)(n + 3)}$

## Exercise 1:03

**1** **a** $\frac{1}{3}$ **b** $\frac{1}{10}$ **c** $\frac{11}{30}$ **d** $\frac{7}{10}$

**2** **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **d** 1

**3** **a** $\frac{1}{3}$ **b** $\frac{7}{12}$ **c** $\frac{3}{4}$ **d** 0

**4** **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{13}$ **d** $\frac{12}{13}$ **e** $\frac{3}{13}$
**f** $\frac{1}{26}$ **g** $\frac{1}{4}$ **h** 0 **i** $\frac{3}{26}$ **j** $\frac{15}{52}$
**k** $\frac{1}{13}$ **l** $\frac{3}{4}$ **m** $\frac{11}{26}$ **n** $\frac{37}{52}$ **o** $\frac{4}{13}$
**p** $\frac{2}{13}$ **q** $\frac{23}{52}$

**5** **a** $\frac{10}{90}$ (or $\frac{1}{9}$), 11% **b** $\frac{53}{90}$, 59%
**c** $\frac{35}{90}$ (or $\frac{7}{18}$), 39% **d** $\frac{27}{90}$ (or $\frac{3}{10}$), 30%

**6** **a** $\frac{4}{85}$, 5% **b** $\frac{74}{85}$, 87%
**c** $\frac{40}{85}$ (or $\frac{8}{17}$), 47% **d** $\frac{33}{85}$, 39%

**7** **a**

Second choice (without replacement)

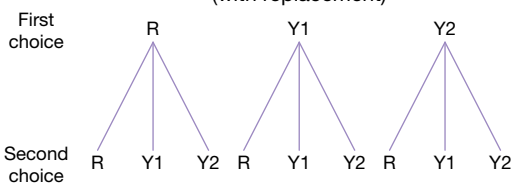
| First choice \ Second choice | R | G1 | G2 | Y1 | Y2 | Y3 |
|---|---|---|---|---|---|---|
| R | | RG | RG | RY | RY | RY |
| G1 | GR | | GG | GY | GY | GY |
| G2 | GR | GG | | GY | GY | GY |
| Y1 | YR | YG | YG | | YY | YY |
| Y2 | YR | YG | YG | YY | | YY |
| Y3 | YR | YG | YG | YY | YY | |

**b**

Second choice

| First choice \ Second choice | Red | Yellow 1 | Yellow 2 |
|---|---|---|---|
| Red | RR | RY | RY |
| Yellow 1 | YR | YY | YY |
| Yellow 2 | YR | YY | YY |

(with replacement)




## Exercise 1:04

**1** **a** 45·6 m **b** 24·4 cm **c** 39·3 m
**2** **a** 15·12 $m^2$ **b** 88·25 $cm^2$ **c** 23·04 $m^2$
**d** 28·08 $cm^2$ **e** 17·85 $cm^2$ **f** 11·52 $cm^2$
**3** **a** 63·9 $m^2$ **b** 35·7 $m^2$ **c** 23·3 $km^2$
**4** **a** 370·88 $cm^2$ **b** 648 $m^2$
**c** 333·55 $m^2$ (approx.)
**5** **a** **i** 77·41 $m^2$ **ii** 30·41 $m^2$ **iii** 107·82 $m^2$
**b** **i** 1658·76 $cm^2$ **ii** 760·27 $cm^2$
**iii** 2419·03 $cm^2$
**c** **i** 38·68 $m^2$ **ii** 73·49 $m^2$
**iii** 112·17 $m^2$
**6** **a** 461·04 $cm^3$ **b** 810 $m^3$
**c** 247·15 $m^3$ (approx.)
**7** **a** 85·15 $m^3$ (2 dec. pl.) **b** 9123·19 $cm^3$ (2 dec. pl.)
**c** 66·14 $m^3$ (2 dec. pl.)

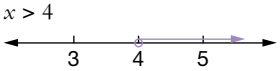
## Exercise 1:05

**1** **a** $a^3$ **b** $2^4$ **c** $n^5$ **d** $10^3$
**2** **a** $2^9$ **b** $a^5$ **c** $m^5$ **d** $10^8$
**e** $a^8$ **f** $y$ **g** $b^2$ **h** $10^3$
**i** $m^{12}$ **j** $a^6$ **k** $x^8$ **l** $10^{10}$
**m** 3 **n** 2 **o** 6 **p** $e^0$ or 1
**q** $30a^2$ **r** $2m^3$ **s** $30a^2$ **t** $16x^8$ or $4^2x^8$
**3** **a** $30a^5b^3$ **b** $56a^5b^3$ **c** $24a^4b^7$ **d** $10a^{10}b^3$
**e** $49x^6$ **f** $16m^8$ **g** $x^6y^9$ **h** $625x^4y^8$
**i** $6a^2$ **j** $5x^3$ **k** $3a$ **l** 8

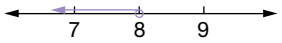
**4** **a** $\frac{1}{4}$ **b** $\frac{1}{10}$ **c** $\frac{1}{x}$ **d** $\frac{2}{a}$
**e** $\frac{1}{25}$ **f** $\frac{1}{8}$ **g** $\frac{1}{m^3}$ **h** $\frac{5}{x^2}$
**5** **a** $3^{-1}$ **b** $8^{-1}$ **c** $a^{-1}$ **d** $3x^{-1}$
**e** $2^{-4}$ **f** $10^{-6}$ **g** $y^{-4}$ **h** $5n^{-3}$
**6** **a** 3 **b** 6 **c** 2 **d** 3
**7** **a** $a^{\frac{1}{2}}$ **b** $y^{\frac{1}{3}}$ **c** $5m^{\frac{1}{2}}$ **d** $4x^{\frac{1}{2}}$
**8** **a** $x^2$ **b** $5m^5$ **c** $12n$ **d** $18y^{\frac{9}{2}}$
**e** $2x$ **f** $3x^2$ **g** $2a$ **h** 2
**i** $6x^2$
**9** **a** $1{\cdot}48 \times 10^8$ **b** $6{\cdot}8 \times 10^4$
**c** $1{\cdot}5 \times 10^{-4}$ **d** $1{\cdot}65 \times 10^{-6}$
**10** **a** 62 000 **b** 1 150 000
**c** 0·0074 **d** 0·000 069 1
**11** **a** $4{\cdot}8 \times 10^5$ B (2 sig. fig.) **b** $1{\cdot}4 \times 10^3$ B
**c** $3{\cdot}4 \times 10^{12}$ B **d** $8 \times 10^9$ B
**e** $8{\cdot}50 \times 10^6$ B **f** $4{\cdot}15 \times 10^9$ B
**12** **a** $5 \times 10^{-3}$ s **b** $4{\cdot}2 \times 10^{-6}$ s
**c** $7{\cdot}5 \times 10^{-9}$ s **d** $1{\cdot}25 \times 10^{-4}$ s
**e** $6{\cdot}5 \times 10^{-8}$ s **f** $8{\cdot}0 \times 10^{-2}$ s
**13** **a** $1{\cdot}05 \times 10^7$ L **b** $1{\cdot}02 \times 10^{11}$ L
**c** $8{\cdot}50 \times 10^{-1}$ g (2 sig. fig.) **d** $6{\cdot}7 \times 10^{-8}$ m

## Exercise 1:06

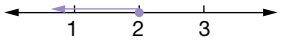
**1** **a** $a = 18$ **b** $m = 5$ **c** $x = 15$
**d** $y = -2$ **e** $p = 2\frac{1}{3}$ **f** $n = 12$
**g** $x = 2$ **h** $m = 2$ **i** $y = \frac{1}{5}$
**j** $k = \frac{2}{3}$ **k** $x = 2$ **l** $q = 3\frac{1}{2}$
**2** **a** $m = 5$ **b** $x = 4$ **c** $x = 7$
**d** $a = 8$ **e** $m = 4$ **f** $q = -1$
**g** $x = 2$ **h** $z = 1\frac{1}{2}$ **i** $m = -1\frac{1}{3}$
**3** **a** $a = 2$ **b** $x = 7$ **c** $x = 3$
**d** $a = -2$ **e** $x = -3$ **f** $m = 4\frac{1}{3}$
**g** $a = 4$ **h** $p = 8$ **i** $b = -18$
**j** $y = -1$ **k** $m = 7$ **l** $m = 5$
**m** $y = 2$ **n** $x = 2\frac{1}{2}$
**4** **a** $x = 4$ **b** $a = 9$ **c** $m = 6\frac{2}{3}$
**d** $n = 9$ **e** $x = 6$ **f** $p = -1$
**5** **a** $a = 6$ **b** $x = 15$ **c** $p = 6$
**d** $q = 36$ **e** $k = 24$ **f** $x = 60$
**g** $m = 6$ **h** $n = -1$ **i** $x = \frac{11}{13}$
**j** $x = \frac{55}{7}$ **k** $m = 33$ **l** $a = \frac{-19}{8}$
**6** **a** **i** $8n + 11 = 39$ **ii** $2n + 7 = 5$
**iii** $\frac{n+4}{10} = 7$ or $(n + 4) \div 10 = 7$
**b** **i** 11 **ii** 8
**iii** width = 34 m, length = 136 m
**7** **a** $x \leq 1$ **b** $x \geq 0$ **c** $x > 6$ **d** $x < -2$
**8** **a** $x > 4$



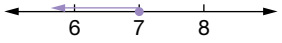
**b** $a < 8$



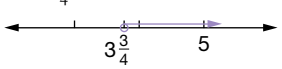

**c** $y \leq 2$



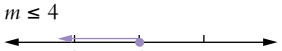

**d** $m \leq 7$



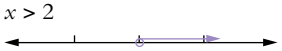

**e** $x > 3\frac{3}{4}$



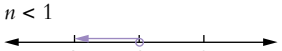

**f** $m \leq 4$



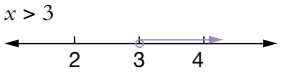

**g** $x > 2$



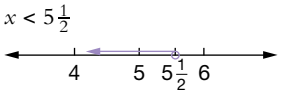

**h** $n < 1$




**i** $x > 3$




**j** $x < 5\frac{1}{2}$




**k** $a < 2$

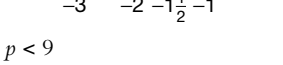
**l** $m < 0$

**m** $x < 10$

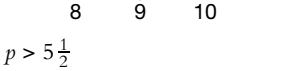
**n** $x \geq 8$

**o** $y \geq -1\frac{1}{2}$

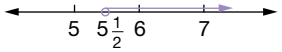
**p** $p < 9$



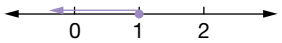

**q** $p > 5\frac{1}{2}$



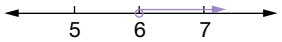

**r** $x \leq 1$



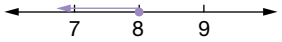

**s** $x > 6$




**t** $a \leq 8$




**u** $x \geq -18$




**9** **a** 92 **b** 5·81 **c** 15·7

**10** **a** 4 **b** 10 **c** 25 **d** $3{\cdot}08\dot{3}$ **e** $5{\cdot}2\dot{6}$

**11** **a** $y = \dfrac{ab - bx}{a}$ (or $y = \dfrac{b(a-x)}{a}$)

**b** $y = \pm\sqrt{\dfrac{x}{a}}$ **c** $y = \dfrac{B}{T^2}$

**d** $y = \dfrac{-1}{a-b}$ (or $y = \dfrac{1}{b-a}$)

## Exercise 1:07

**1** **a** \$211.60 **b** \$238.28
**c** \$350 **d** \$534.90

**2** **a** 701·36, 20·3% **b** \$5604
**c** \$65 746 **d** \$1179

**3** **a** The 15% discount is the best, by \$2.49.
**b** Jade's buy **c** Yet tea

**4** **a** **i** \$7.50 **ii** \$0.68 **iii** \$1.88
**b** **i** \$82.50 **ii** \$7.48 **iii** \$20.63
**c** **i** \$20 **ii** \$0.75 **iii** \$1.80

**5** **a** Buying on terms is a way of having an item while spreading the payment over a period of time. Interest is charged.
**b** He pays \$143.56. Interest is \$56.56.
**c** \$220 **d** \$545 700 **e** \$30.60

**6** **a** **i** $33\frac{1}{3}\%$ **ii** 25%
**b** **i** 25% **ii** $33\frac{1}{3}\%$

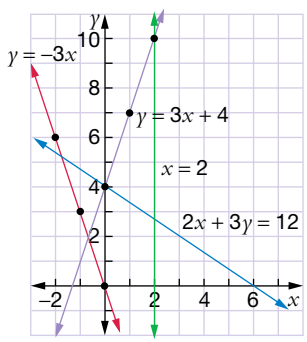
## Exercise 1:08

**1** **a** 1 **b** 7 **c** 0

**2** **a** (5, 8) **b** $(\frac{1}{2}, 1\frac{1}{2})$ **c** $(3\frac{1}{2}, 0)$

**3** **a** $\sqrt{20}$ units **b** $\sqrt{73}$ units **c** 10 units
**d** $\sqrt{53}$ units

**4**




**5** **a** gradient = 3, $y$-intercept = 5
**b** gradient = −1, $y$-intercept = −2
**c** gradient = −2, $y$-intercept = 5

**6** **a** $y = 3x + 4$ **b** $y = -\frac{2}{3}x + 4$
**c** $y = -3x$ **d** $x = 2$

**7** **a** $y = 5x - 2$ **b** $y = 4$
**c** $y = 2x + 5$ **d** $y = -x + 1$

**8** **a** $y = \frac{1}{2}x$ **b** $y = -x + 4$

**9** **a** C **b** B **c** A

**10** **a** B **b** C **c** A

## Exercise 1:09

**1** **a** $y = 15$ **b** $y = 1\frac{3}{4}$

**2** **a** $x = 5$ **b** $x = 6$

**3** **a** yes **b** no

**4** **a** $x = 4, y = 2$ **b** $x = -8, y = 2$
**c** $x = -4, y = -2$ **d** $x = 4, y = 2$
**e** $x = 0, y = -6$ **f** $x = 4, y = 2$

**5** **a** $x = 2, y = 8$ **b** $x = 7, y = 5$
**c** $x = 3, y = 1$ **d** $a = 1\frac{3}{7}, b = 2\frac{5}{7}$

**6** **a** $x = 5, y = 1\frac{2}{3}$ **b** $a = 2\frac{3}{5}, b = -\frac{1}{5}$
**c** $c = 8, d = -2$ **d** $x = -3, y = 5$

**7** 48

**8** \$20

## Exercise 1:10

**1** **a** Similar figures are figures that have the same shape but differet size. (Matching angles are equal and matching sides are in the same ratio.)
**b** C **c** E

**2** **a** true **b** true

**3** **a** yes
**b** no (as $\frac{40}{42}$ does not equal $\frac{30}{32}$)

**4** **a** $x = 9, y = 15$ **b** $a = 16, b = 20$
**c** $x = 32, y = 30$ **d** $x = 35, y = 81$

**5** **a** 14 cm by 7·5 cm **b** 1 : 400

## Exercise 1:11

**1** **a** $\frac{5}{12}$ **b** $\frac{15}{8}$ **c** $\frac{4}{3}$

**2** **a** $\frac{24}{25}$ **b** $\frac{7}{25}$ **c** $\frac{3}{5}$

**3** **a** 2·5 **b** 5·0 **c** 34·9

**4** **a** 5 **b** 6·9 **c** 6·5

**5** **a** 7·1 **b** 10·4 **c** 14·6

**6** **a** 40° **b** 51° **c** 40°

**7** **a** 36°52′ **b** 62°11′ **c** 53°8′

**8** 60·99 m

**9** 76·4 m

**10** 46·626 km

**11** **a** 6·16 cm **b** 29°7′

**12** **a** 3·93 cm **b** 4·84 cm

## Exercise 1:12

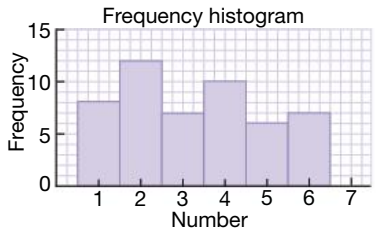
**1**

| Number | Tally | Frequency |
|---|---|---|
| 1 | 𝍸 \|\|\| | 8 |
| 2 | 𝍸 𝍸 \|\| | 12 |
| 3 | 𝍸 \|\| | 7 |
| 4 | 𝍸 𝍸 | 10 |
| 5 | 𝍸 \| | 6 |
| 6 | 𝍸 \|\| | 7 |

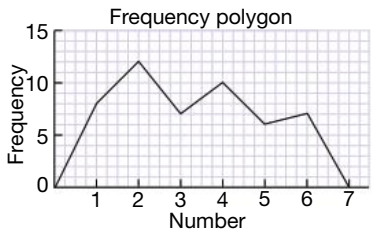
$\Sigma f = 50$

**a** 2 **b** 5 **c** 7 **d** 21 **e** 23

**2** **a**




**b**



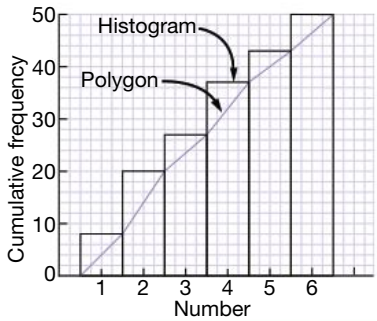

**3** **a** **i** 7 **ii** 5 **iii** $4\frac{1}{9}$ **iv** 4
**b** **i** 5 **ii** 2 **iii** 3·3 **iv** 3
**c**

| Number | Frequency | Cumulative Frequency |
|---|---|---|
| 1 | 8 | 8 |
| 2 | 12 | 20 |
| 3 | 7 | 27 |
| 4 | 10 | 37 |
| 5 | 6 | 43 |
| 6 | 7 | 50 |

$\Sigma f = 50$

**i** 37 **ii** 27 **iii** yes

**4**



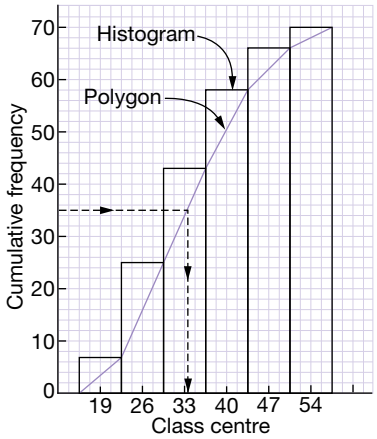

**5** **a**

| Class | Class centre | Tally | Frequency | Cumulative Frequency |
|---|---|---|---|---|
| 16–22 | 19 | 𝍸 \|\| | 7 | 7 |
| 23–29 | 26 | 𝍸 𝍸 𝍸 \|\|\| | 18 | 25 |
| 30–36 | 33 | 𝍸 𝍸 𝍸 \|\|\| | 18 | 43 |
| 37–43 | 40 | 𝍸 𝍸 𝍸 | 15 | 58 |
| 44–50 | 47 | 𝍸 \|\|\| | 8 | 66 |
| 51–57 | 54 | \|\|\|\| | 4 | 70 |

$\Sigma f = 70$

**b** 34·1 using table (34·6 using the individual numbers)

**c**




Median class = 30–36

**d** 23–29 and 30–36 cans.

**e** The survey was held over 70 days.

**6 a**

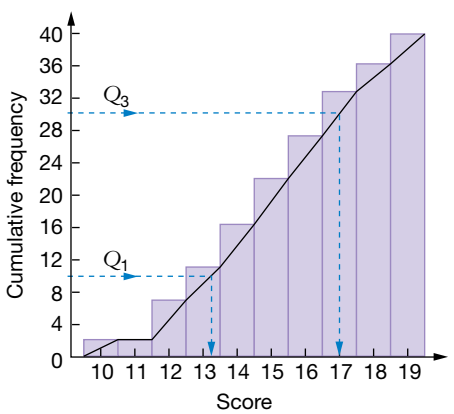
| Stem | Leaf |
|---|---|
| 0 | |
| 1 | 8 9 |
| 2 | 8 1 2 5 6 3 9 8 0 7 6 |
| 3 | 0 6 7 0 3 5 1 |
| 4 | 2 7 3 0 2 4 |
| 5 | 4 0 |

**b**

| Stem | Leaf |
|---|---|
| 0 | |
| 1 | 8 9 |
| 2 | 0 1 2 3 5 6 6 7 8 8 9 |
| 3 | 0 0 1 3 5 6 7 |
| 4 | 0 2 2 3 4 7 |
| 5 | 0 4 |

**7 a** 7·5

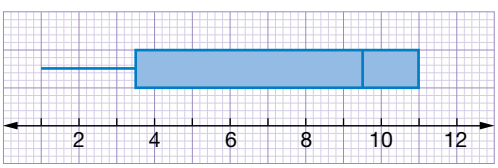
**b**




Interquartile range = $Q_3 - Q_1$
= 17 − 13
= 4

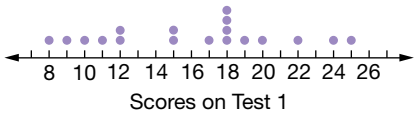
**c** 2·5 cm **d** 12 days

**8**




**9 a** 25 **b** range = 60, interquartile range ÷ 22

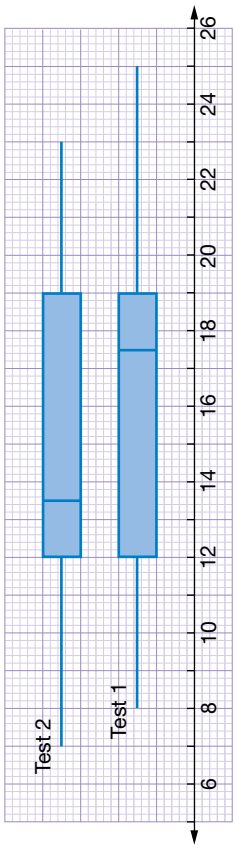
**10 a**




**b**

| Scores | | |
|---|---|---|
| Test 1 Leaf | Stem | Test 2 Leaf |
| 9 8 | 0 | 7 9 |
| 9 8 8 8 8 7 5 5 2 2 1 0 | 1 | 0 1 2 2 2 2 3 4 5 6 8 9 |
| 5 4 2 0 | 2 | 0 0 1 3 |

**c**




## Exercise 1:13

**1 a** 50 km/h **b** $8/kg **c** 24 L/h
**d** 25 g/cm$^3$

**2 a** 100 m/min **b** 2 L/s **c** 15 t/day
**d** 1 c/m$^2$

**3 a** direct **b** indirect **c** direct **d** indirect

**4 a** $k = 5$ **b** 85 **c** 34·8

**5 a** 40
**b** 13 (Number of people must be a whole number.)

**6 a** $D = 24T$ **b** $k = 24$ **c** 192 km **d** 9 h 30 min

**7 a** $k = 12$ **b** 264 **c** 16 h 40 min

**8 a** $t = \frac{48}{n}$ **b** $k = 48$ **c** 4 hours **d** 16

**9 a** $k = 120$ **b** 2 h 24 min **c** 60 km/h

**10 a** 750 Hz **b** 37·5 cm

## Exercise 1:14

**1** **a** 10 km **b** 20 km **c** 11:30 am
**d** Callum **e** 11:30 am **f** 50 km
**g** 30 km **h** 90 km

**2** **a** 3000 g **b** 4400 g **c** 900 g
**d** 500 g **e** 0–2 weeks

**3** **a** The maximum height is 24 m. The height of the roof is 18 m.
**b** after 4 seconds
**c** 18 m
**d** after 2 seconds (on the way up) and after 6 seconds (on the way down)
**e** 11 m

### ASSIGNMENT 1A

**1** **a**

| $t$ | 1 | 3 | 5 |
|---|---|---|---|
| $m$ | 3 | 7 | 11 |

, $m = 2t + 1$

**b**

| $t$ | 2 | 4 | 6 |
|---|---|---|---|
| $m$ | 5 | 10 | 15 |

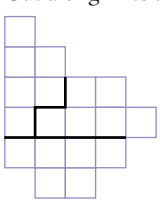
, $m = \frac{5t}{2}$

**c**

| $t$ | 1 | 3 | 6 |
|---|---|---|---|
| $m$ | 3 | 9 | 18 |

, $m = 3t$

**2** Cut along lines as shown.



**3** **a** 185 **b** 37
**4** 16 min 15 s
**5** 32
**6** **a** 11 290 000 **b** 485 400
**c** 11 300 (3 sig. fig.)
**d** Answers will vary.

# Chapter 2

## Prep quiz 2:01

**1** $x(x - 3)$ **2** $x(x + 7)$
**3** $(x + 1)(x + 2)$ **4** $(x - 6)(x + 2)$
**5** $(x + 5)(x - 4)$ **6** $(x - 7)(x - 1)$
**7** $x = 4$ **8** $x = 0$ **9** $x = 4$ **10** $x = -6$

## Exercise 2:01

**1** **a** $x = \pm 5$ **b** $x = \pm 9$ **c** $x = \pm 1$
**d** $x = \pm 10$ **e** $x = \pm 6$ **f** $x = \pm 8$
**g** $x = \pm 11$ **h** $x = \pm 20$

**2** **a** $x = \pm\sqrt{6}$ **b** $n = \pm\sqrt{3}$ **c** $y = \pm\sqrt{10}$
**d** $k = \pm\sqrt{15}$ **e** $x = \pm\sqrt{2}$ **f** $p = \pm\sqrt{\frac{3}{5}}$
**g** $q = \pm\sqrt{\frac{5}{4}}$ **h** $z = \pm\sqrt{\frac{5}{7}}$

**3** **a** $x = \pm 1{\cdot}414$ **b** $a = \pm 1{\cdot}871$
**c** $c = \pm 3{\cdot}464$ **d** $g = \pm 0{\cdot}612$

**4** **a** no real solutions
**b** square root of a negative number is not a real number

**5** **a** 0, 5 **b** 0, −7 **c** 0, −1
**d** 0, 2 **e** 0, −5 **f** 0, 7
**g** 2, 1 **h** 7, 3 **i** 5, 2
**j** −3, −4 **k** −3, −2 **l** −9, −5
**m** 6, −6 **n** −8, 7 **o** −1, 1
**p** $-1, \frac{1}{2}$ **q** $-\frac{2}{3}, 5$ **r** $0, \frac{1}{3}$
**s** $\frac{1}{4}, -\frac{1}{2}$ **t** $1\frac{1}{3}, \frac{1}{2}$ **u** $\frac{5}{6}, -\frac{3}{4}$
**v** $0, \frac{3}{5}$ **w** $-\frac{1}{9}, -\frac{2}{7}$ **x** $\frac{1}{5}, -\frac{1}{5}$

**6** **a** 0, −3 **b** 5, 0 **c** 0, −2
**d** 0, −2 **e** $0, \frac{1}{3}$ **f** 0, −2
**g** 2, −2 **h** 7, −7 **i** 6, −6
**j** 1, −1 **k** 10, −10 **l** 8, −8
**m** −1, −2 **n** 2, 3 **o** −5, −7
**p** 7, 3 **q** 8, 2 **r** 8, 3
**s** 4, −5 **t** 5, −7 **u** 9, −5
**v** 7, −8 **w** 7, 1 **x** 1, −10

**7** **a** $x = 2$ **b** one
**c** when the expression can be factorised as a 'perfect square'

**8** **a** $x = -2$ **b** $m = 3$ **c** $z = -1$
**d** $k = 4$ **e** $a = 5$ **f** $y = -10$

**9** **a** $\frac{1}{2}, -1$ **b** $-\frac{1}{3}, -2$ **c** $-\frac{2}{3}, -5$
**d** $4, 1\frac{1}{2}$ **e** $2\frac{1}{2}, -2$ **f** $7, -1\frac{1}{2}$
**g** $-\frac{1}{4}, -5$ **h** $5, -\frac{1}{4}$ **i** $5, \frac{1}{4}$
**j** $-\frac{1}{5}, -3$ **k** $1\frac{1}{2}, -8$ **l** $\frac{1}{7}, -7$
**m** $1\frac{1}{2}, -\frac{1}{2}$ **n** $\frac{1}{2}, -\frac{1}{3}$ **o** $-\frac{1}{3}, -\frac{2}{3}$
**p** $-\frac{2}{5}, -\frac{1}{2}$ **q** $\frac{1}{3}, \frac{1}{4}$ **r** $\frac{4}{5}, \frac{1}{2}$

**10** **a** 0, 3 **b** 0, 8 **c** 0, −5
**d** 4, 1 **e** 5, −3 **f** 2, 1
**g** 6, 3 **h** 9, −2 **i** 8, −4
**j** 1, −2 **k** 1, −3 **l** 5, 2
**m** 3, −6 **n** 4, −7 **o** 3, −5
**p** $\frac{1}{2}, -1$ **q** $3, -2\frac{1}{2}$ **r** $2, -\frac{3}{4}$
**s** $2\frac{1}{3}, 2$ **t** $3, \frac{2}{5}$ **u** $5, \frac{1}{2}$

## Exercise 2:02

**1** **a** 3 **b** 4 **c** 1 **d** 2
**e** $\frac{3}{2}$ **f** $\frac{7}{2}$ **g** $\frac{11}{2}$ **h** $\frac{1}{2}$
**i** $\frac{5}{4}$ **j** $\frac{1}{3}$

**2** **a** $2 \pm \sqrt{3}$ **b** $-1 \pm \sqrt{2}$ **c** $-5 \pm \sqrt{5}$
**d** $1 \pm \sqrt{10}$ **e** $3 \pm \sqrt{7}$ **f** $-2 \pm \sqrt{11}$
**g** $-3 \pm 2\sqrt{2}$ **h** $-10 \pm 2\sqrt{3}$ **i** $3 \pm 3\sqrt{2}$
**j** $-\frac{1}{2} \pm \sqrt{5}$ **k** $\frac{2}{3} \pm \sqrt{3}$ **l** $-1\frac{1}{2} \pm 2\sqrt{3}$
**m** $1 \pm \sqrt{\frac{5}{2}}$ **n** $-3 \pm \frac{3\sqrt{2}}{2}$ **o** $\frac{1}{3} \pm \frac{\sqrt{5}}{3}$

**3** **a** 0·41, −2·41 **b** 3·45, −1·45
**c** 5·46, −1·46 **d** 1·12, −7·12
**e** 5·65, 0·35 **f** −0·27, −3·73
**g** 0·48, −10·48 **h** 1·24, −3·24
**i** 12·08, −0·08 **j** −0·44, −4·56
**k** 0·41, −7·41 **l** 1·30, −2·30
**m** −0·35, −8·65 **n** 1·19, −4·19
**o** 10·52, 0·48 **p** 2·30, −1·30
**q** 0·56, −3·56 **r** 5·19, −0·19
**s** 2·22, −0·22 **t** 0·85, −2·35
**u** 3·87, 0·13 **v** 0·72, −1·39
**w** 1·27, −0·47 **x** 0·84, −0·59

## Exercise 2:03

**1** **a** −2, −3 **b** −2, −4 **c** −1, −9
**d** 5, −2 **e** 5, −3 **f** 2, −6
**g** 7, 2 **h** 6, 2 **i** 5, 1
**j** $-\frac{1}{3}$, −2 **k** $-\frac{1}{2}$, −5 **l** $-\frac{3}{4}$, −2
**m** 3, $-\frac{1}{2}$ **n** 2, $-\frac{1}{5}$ **o** 1, $\frac{2}{3}$
**p** $-\frac{1}{2}, -\frac{2}{3}$ **q** $\frac{1}{3}, -1\frac{1}{2}$ **r** $1\frac{1}{2}, \frac{1}{4}$

**2** **a** $\frac{-4 \pm \sqrt{8}}{2}$ **b** $\frac{-3 \pm \sqrt{5}}{2}$
**c** $\frac{-5 \pm \sqrt{13}}{2}$ **d** $\frac{-1 \pm \sqrt{5}}{2}$
**e** $\frac{-2 \pm \sqrt{12}}{2}$ **f** $\frac{-4 \pm \sqrt{20}}{2}$
**g** $\frac{2 \pm \sqrt{8}}{2}$ **h** $\frac{7 \pm \sqrt{41}}{2}$
**i** $\frac{6 \pm \sqrt{24}}{2}$ **j** $\frac{10 \pm \sqrt{136}}{2}$
**k** $\frac{8 \pm \sqrt{52}}{2}$ **l** no real solutions
**m** $\frac{-6 \pm \sqrt{28}}{4}$ **n** $\frac{-3 \pm \sqrt{17}}{4}$
**o** $\frac{7 \pm \sqrt{17}}{4}$ **p** $\frac{-10 \pm \sqrt{76}}{6}$
**q** $\frac{9 \pm \sqrt{57}}{6}$ **r** $\frac{-4 \pm \sqrt{56}}{10}$
**s** no real solutions **t** $\frac{3 \pm \sqrt{21}}{6}$
**u** $\frac{3 \pm \sqrt{41}}{8}$ **v** $\frac{-11 \pm \sqrt{161}}{4}$
**w** $\frac{9 \pm \sqrt{17}}{4}$ **x** $\frac{-2 \pm \sqrt{24}}{10}$

**3** **a** 3·73, 0·27 **b** 5·45, 0·55
**c** 0·58, −8·58 **d** −0·11, −8·89
**e** 1·45, −3·45 **f** 0·30, −3·30
**g** no solutions **h** 7·27, −0·27
**i** no solutions **j** 0·78, −1·28
**k** 2·85, −0·35 **l** −0·74, −2·26
**m** 3·19, 0·31 **n** 1·24, −0·64
**o** 0·80, −0·63

## Investigation 2:03

**1** 2 **2** 1 **3** 0 **4** 2
**5** 2 **6** 0 **7** 1 **8** 2
**9** 0 **10** 0 **11** 2 **12** 1

## Prep quiz 2:04

**1** $5x(x - 2)$ **2** $(x - 7)(x + 2)$
**3** $(x - 9)(x + 9)$ **4** $(x + 2)(x + 3)$
**5** 2, −7 **6** $1\frac{1}{2}, -\frac{1}{3}$ **7** 4, −4 **8** 0, 4
**9** 1, 2 **10** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## Exercise 2:04

**1** **a** −1, −6 **b** 6, 2
**c** 3, −8 **d** 2·62, 0·38
**e** 0·79, −3·79 **f** −0·59, −3·41
**g** 0, −8 **h** 10, 0
**i** 2, 0 **j** 9, −9
**k** 11, −11 **l** $1\frac{1}{2}, -1\frac{1}{2}$
**m** −0·29, −1·71 **n** 0·77, −0·43
**o** 2·28, 0·22 **p** −1, −2
**q** −2, −3 **r** 4, −1
**s** 9, −3 **t** 9, 4
**u** $4, -1\frac{1}{2}$ **v** 5, 5
**w** 9, 4 **x** 1·54, 0·26

**2** **a** −3, −4 **b** 5, 3 **c** 4, 2
**d** −2, −3 **e** 0, −7 **f** 3, −3
**g** $\frac{-2 \pm \sqrt{8}}{2}$ **h** $\frac{-4 \pm \sqrt{8}}{2}$ **i** $\frac{1 \pm \sqrt{5}}{2}$
**j** 1, −6 **k** 9, −2 **l** 8, −4
**m** 3, −1 **n** 0, −6 **o** $-5 \pm \sqrt{11}$
**p** $\frac{3}{2}, -\frac{5}{2}$ **q** $\frac{3 \pm \sqrt{7}}{5}$ **r** $\frac{7 \pm \sqrt{3}}{6}$
**s** 5, −3 **t** 7, −4 **u** 1, −2
**v** $\frac{5 \pm \sqrt{13}}{2}$ **w** $\frac{3 \pm \sqrt{21}}{2}$
**x** $\frac{-4 \pm \sqrt{24}}{4} = \frac{-2 \pm \sqrt{6}}{2}$

## Prep quiz 2:05

**1** 1, 2, 9, 10 **2** 1, 9 **3** 1, 9
**4** 9, 10 **5** $n + 1, n + 2$ **6** 12, 14
**7** $x + 2, x + 4$ **8** $x - 3$
**9** $a(a + 3)$ or $a^2 + 3a$ **10** $4a + 6$

## Exercise 2:05

**1** **a** 4, 5 **b** 9, 10 **c** 10, 12 **d** 7, 9
**2** **a** 9 **b** 11 **c** 8 **d** 0, 5
**e** 7, −6
**3** **a** 10 cm, 4 cm **b** 17 m, 15 m
**c** 7 cm **d** 17 cm
**4** **a** 2 s **b** 12

**c** i $2, -9$ ii $\frac{1}{2}, -6$ iii $\frac{-1 \pm \sqrt{85}}{6}$

**5** 15

**6** 7

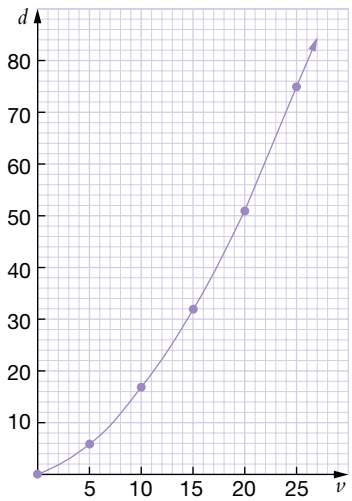
**7** 12 or 88

**8** **a** 0·0875

**b**

| $v$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $d$ | 0 | 6 | 17 | 32 | 51 | 75 |

**c** parabola

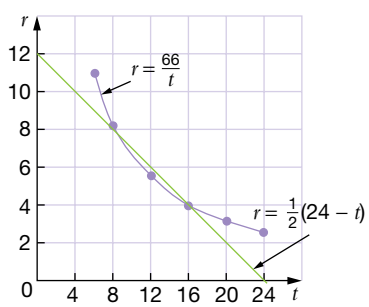



**d** approx. 17·2 m/s; by formula, 17·2927 m/s

**e** road and weather conditions, tyre and brake conditions on vehicles, physical conditions of drivers, other possible distractions, etc.

**9** **a** The rise must be no greater than 7·5 inches by formula 1 or $7.\dot{3}$ inches by formula 2.

**b**




**c** The points of intersection are: (8·5, 7·7) and (15·5, 4·3) By formula the solutions are:

$$\begin{cases} t \doteqdot 8{\cdot}536 \\ r \doteqdot 7{\cdot}732 \end{cases} \text{ and } \begin{cases} t \doteqdot 15{\cdot}464 \\ r \doteqdot 4{\cdot}268 \end{cases}$$

**d** $r = \frac{1}{2}(60 \cdot 960 - t)$ or $r = \frac{1}{2}(61 - t)$

$r = \dfrac{66 \times (2 \cdot 54)^2}{t}$ or $r = \dfrac{426}{t}$

$r$ and $t$ are in cm.

**e** Students' answers will vary.

## Investigation 2:05

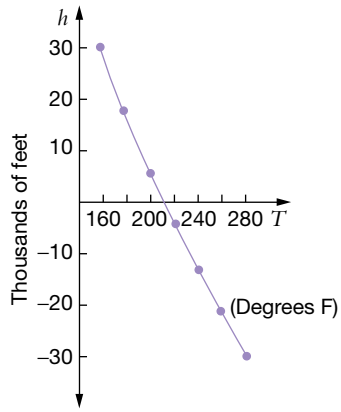
**1** $h = (212 - T)(520 + [212 - T])$
$= (212 - T)(732 - T)$

**2** **a** 6384 ft (or 1947 m)

**b** −18 316 ft (or −5586 m), −18 316 ft means 18 316 ft below sea level.

**3**

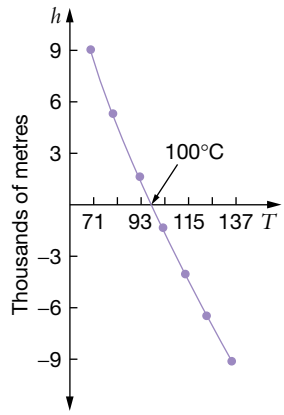
| $T$ in °F | 160 | 180 | 200 | 220 | 240 | 260 | 280 |
|---|---|---|---|---|---|---|---|
| $h$ in feet | 29 744 | 17 664 | 6384 | −4096 | −13 776 | −22 656 | −30 736 |




**4** **a** 210°F **b** 161°F **c** 215°F

**5** Check each using $T = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

**6**




**7** Students' answers will vary.

## Prep quiz 2:06

**1** $a = 0$ or 3 **2** $a = -\frac{1}{2}$ or 5 **3** $a = \frac{4}{3}$ or $-\frac{2}{5}$

**4** $a = -2$ or $-3$ **5** $a = -7$ **6** $a = 4$ or $-3$

**7** $a = -2$ or $-\frac{1}{3}$ **8** $a = \frac{3}{4}$ **9** $x = \pm 4$

**10** $x = \pm 2$

## Exercise 2:06

**1** **a** $x = \pm 1$ or $\pm 2$ **b** $a = \pm 1$ or $\pm 3$

**c** $n = \pm 2$ or $\pm 5$ **d** $k = \pm 2$

**e** $m = \pm 2$ or $\pm \sqrt{2}$ **f** $z = \pm 3$ or $\pm \sqrt{5}$

**2** **a** $x = \pm 1$ or $\pm \frac{1}{2}$ **b** $a = \pm 2$ or $\pm \frac{1}{3}$

**c** $n = \pm 3$ or $\pm \dfrac{1}{\sqrt{3}}$ **d** $k = \pm \frac{3}{2}$ or $\pm \dfrac{1}{\sqrt{2}}$

**3** **a** $y = 1$ or $2$ **b** $m = \pm 1$ or $\pm 2$
**c** $x = 2$ or $10$ **d** $n = \pm 1$ or $\pm 3$
**4** **a** $x = 0$ or $2$ **b** $x = 0$ or $1$
**c** $x = 2$ or $3$ **d** $x = 1$ or $2$
**5** **a** $x = \pm 1$ **b** $x = \pm\sqrt{2}$ or $\pm\sqrt{3}$
**c** $x = \pm\sqrt{6}$ **d** no real solutions
**e** $x = \pm\sqrt{2}$ or $\pm\sqrt{5}$ **f** $a = x = \pm\sqrt{5}$
**6** $x = -4, -1, 3, 6$
**7** $a = -6, -4, 0, 2$
**8** **a** $x = -3, -2, 4, 5$ **b** $x = 1 + \sqrt{6}, 1 - \sqrt{6}$

## Fun spot 2:06

If $x = y$, then $x - y = 0$; thus dividing by $x - y$ in the second last line is undefined.

## Diagnostic test 2

**1** **a** $x = \pm 7$ **b** $a = \pm 2{\cdot}646$
**c** $m = \pm 4$ **d** $y = \pm 1{\cdot}291$
**2** **a** $-7, 3$ **b** $0, 5$ **c** $\frac{1}{2}, -1$ **d** $-\frac{2}{3}, \frac{5}{4}$
**3** **a** $0, -5$ **b** $-2, -7$ **c** $-7, 7$ **d** $\frac{1}{2}, -3$
**4** **a** $9$ **b** $4$ **c** $\frac{9}{4}$ **d** $\frac{1}{4}$
**5** **a** $-1 \pm \sqrt{3}$ **b** $3 \pm 2\sqrt{2}$
**c** $\frac{3}{2} \pm \frac{\sqrt{29}}{2}$ **d** $\frac{5 \pm 3\sqrt{3}}{2}$
**6** **a** $\frac{-1 \pm \sqrt{13}}{2}$ **b** $\frac{5 \pm \sqrt{17}}{2}$
**c** $\frac{-2 \pm \sqrt{2}}{2}$ **d** $\frac{-1 \pm \sqrt{7}}{3}$
**7** **a** $6, -1$ **b** $3$
**c** $-4 \pm \sqrt{6}$ **d** $4, -2$
**8** **a** $x = \pm 2, \pm 4$ **b** $a = \pm 3, \pm\sqrt{2}$

### ASSIGNMENT 2A

**1** **a** $5, -6$ **b** $0, 7$ **c** $-4, 2$
**d** $\frac{3}{2}, -5$ **e** $4, -6$ **f** $-2, \frac{1}{3}$
**g** $7$ **h** $0, \frac{3}{2}$ **i** $10, -10$
**j** $7, -2$ **k** $7, -4$ **l** $\frac{-5 \pm \sqrt{21}}{2}$
**m** $3 \pm \sqrt{2}$ **n** $\frac{1}{2}, -\frac{1}{5}$ **o** $\pm\sqrt{20}$
**p** $0, -\frac{3}{5}$ **q** $-5$ **r** $-1 \pm \sqrt{5}$
**s** $1, -\frac{2}{3}$ **t** $\frac{-5 \pm \sqrt{17}}{4}$ **u** $3, -8$
**v** $-1, -2$ **w** $\frac{5 \pm \sqrt{17}}{2}$ **x** $2, -\frac{4}{3}$
**2** **a** $4, -8$ **b** $8, -5$ **c** $5 \pm \sqrt{21}$
**d** $\frac{-3 \pm \sqrt{15}}{2}$
**3** $3, 4, 5$
**4** **a** $5$ **b** $7$
**5** $40\,\text{m}$
**6** **a** $x = \pm 1, \pm 5$ **b** $n = \pm 3, \pm\sqrt{3}$
**c** $z = \pm 2{\cdot}602$ (3 dec. pl.) (i.e. $z = \pm\sqrt{\frac{5 + \sqrt{73}}{2}}$)

### ASSIGNMENT 2B

**1** 357
**2** 3
**3** **a** \$1
**b** the minimum monthly balance
**c** no
**4** \$14.40
**5** 8
**6** **a** **i** Oceania and Antartica
**ii** Americas
**b** **i** 14% **ii** 17%
**c** **i** 3800 **ii** 29 100

### ASSIGNMENT 2C

**1** **a** \$18.75 **b** 20% **c** \$87.50 **d** \$160
**2** **a** $4x - y$ **b** $2x$ **c** $\frac{19a}{15}$ **d** $\frac{3y}{2x}$
**3** **a** $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{1}{13}$ **d** $\frac{1}{52}$
**4** **a** $40{\cdot}04\,\text{cm}^2$ **b** $78{\cdot}1\,\text{cm}^2$ (1 dec. pl.)
**5** **a** 800 **b** 4 **c** $\frac{1}{16}$ **d** $\frac{1}{8}$ **e** 4
**6** **a** $9\sqrt{2}$ **b** $4\sqrt{2}$ **c** $-6$
**7** \$1448.40
**8** **a** $x = 9{\cdot}6$ **b** $x = 45$
**9** **a** 5 **b** 4 **c** 4 **d** $4\frac{7}{9}$
**e** 3·5 **f** 6 **g** 2·5

# Chapter 3

## Exercise 3:01A

**1** **a** $\angle HEB = 55°$ (adj. angles on straight line)
$\angle ABC = 55°$ (corresp. to $\angle HEB$, $DC \parallel GH$)
**b** $\angle EFB = 93°$ (vert. opp. $\angle HFG$)
$\angle ABC = 93°$ (corresp. to $\angle EFB$, $AD \parallel EG$)
**c** $\angle BDG = 106°$ (supp. adj. angles)
$\angle ABC = 106°$ (corresp. to $\angle BDG$, $HC \parallel FG$)
**2** **a** $\angle CBE = 39°$ (alt. to $\angle DEB$, $AC \parallel DF$)
$x = 141$ (adj. angles on a straight line)
**b** $\angle EBC = 88°$ (vert. opp. $\angle ABF$)
$x = 88$ (corresp. to $\angle EBC$, $EF \parallel GH$)
**c** $\angle EBC = 70°$ (corresp. to $\angle GCD$, $EF \parallel GH$)
$x = 70$ (vert. opp. $\angle EBC$)
**d** $\angle DAB = 92°$ (co-int. to $\angle CBA$, $AD \parallel BC$)
$x = 62$ (since $x° + 30° = 92°$)
**e** $\angle DCH = 125°$ (corresp. to $\angle CBF$, $EF \parallel GH$)
$x = 35$ (since $x° + 90° = 125°$)
**f** $\angle CBE = 48°$ (alt. to $\angle BED$, $AC \parallel DF$)
$x = 42$ (supp. adj. angles)

**3** **a** $\angle BCE = 82°$ (alt. to $\angle ABC$, $AB \parallel ED$)
$x = 98$ (co-int. to $\angle BCE$, $BC \parallel FG$)

**b** $\angle BCD = 77°$ (alt. to $\angle ABC$, $AB \parallel CD$)
$\angle CDE = 103°$ (co-int. to $\angle BCD$, $BC \parallel ED$)
$x = 103$ (alt. to $\angle CDE$, $EF \parallel CD$)

**c** $\angle BCF = 152°$ (co-int. to $\angle ABC$, $AB \parallel FC$)
$\angle DCF = 120°$ (co-int. to $\angle CDE$, $FC \parallel ED$)
$x° + 272° = 360°$ (angles at a point)
$x = 88$

**d** $\angle ACF = 45°$ (alt. to $\angle BAC$, $AB \parallel FC$)
$\angle DCF = 18°$ (alt. to $\angle CDE$, $FC \parallel DE$)
$x = 63$ ($\angle ACF + \angle DCF$)

**e** $\angle BCF = 100°$ (alt. to $\angle ABC$, $AB \parallel CF$)
$\angle DCF = 59°$ ($\angle BCF - \angle BCD$)
$x = 121$ (co-int. to $\angle DCF$, $DE \parallel CF$)

**f** Draw $CF$ right from $C \parallel AB$ and $DE$
$\angle ACF = 40°$ (co-int. to $\angle CAB$, $AB \parallel CF$)
$\angle DCF = 52°$ (co-int. to $\angle CDE$, $CF \parallel DE$)
$x = 92$ ($\angle ACF + \angle DCF$)

**g** $\angle ABD = 42°$ (co-int. to $\angle CDB$, $AB \parallel CD$)
$\angle ABF = 77°$ (co-int. to $\angle EFB$, $AB \parallel EF$)
$x = 35$ ($\angle ABF - \angle ABD$)

**h** Draw $EI$ up from $E \parallel AB$ and $CD$.
$\angle GEI = 73°$ (co-int. to $\angle CGE$, $CD \parallel EI$)
$\angle FEI = 106°$ (co-int. to $\angle AFE$, $AB \parallel EI$)
$x = 33$ ($\angle FEI - \angle GEI$)

**i** $\angle BDE = 76°$ (co-int. to $\angle DBC$, $AC \parallel DE$)
$\angle FDE = 38°$ ($\angle BDF = \angle FDE$, given)
$x = 142$ (co-int. to $\angle FDE$, $FG \parallel DE$)

## Exercise 3:01B

**1** **a** $\angle DCA = 40°$ (alt. to $\angle EDC$, $ED \parallel CA$)
$\angle ABC + 60° + 40° = 180°$ (angle sum of a $\Delta$)
$\angle ABC = 80°$

**b** $\angle DCB = 30°$ (alt. to $\angle EDC$, $ED \parallel BC$)
$\angle ABC + 30° + 100° = 180°$ (angle sum of a $\Delta$)
$\angle ABC = 50°$

**c** $\angle CDE + 40° + 75° = 180°$ (angle sum of a $\Delta$)
$\angle CDE = 65°$
$\angle ABC = 115°$ (co-int. to $\angle CDE$, $AB \parallel ED$)

**2** **a** $\angle CDB = 88°$ (suppl. to $\angle EDB$)
$\angle DBC = 72°$ (suppl. to $\angle DBA$)
$x° + 72° + 88° = 180°$ (angle sum of a $\Delta$)
$x = 20$

**b** $\angle ABD = 48°$ (vert. opp. to $\angle CBE$)
$x° + 75° + 48° = 180°$ (angle sum of a $\Delta$)
$x = 57$

**c** $\angle DBC = 111°$ (vert. opp. to $\angle ABE$)
$x° + 111° + 25° = 180°$ (angle sum of a $\Delta$)
$x = 44$

**3** **a** $\angle BCA = 80°$ (suppl. adj. angle)
$\angle BAC = 80°$ (base ang. of isos. $\Delta$)
$80° + 80° + \angle ABC = 180°$ (angle sum of $\Delta$)
$\angle ABC = 20°$

**b** $\angle CAB = 81°$ (alt. to $\angle ACD$, $AB \parallel DC$)
$\angle ABC = 81°$ (base angles of isos. $\Delta$)

**c** $\angle CBD = \angle CDB$ (base angles of isos. $\Delta$)
$68° + 2 \times \angle CBD = 180°$ (angle sum of a $\Delta$)
$\angle CBD = 56°$
$\angle ABC = 124°$ (suppl. adj. angles)

**4** **a** $\angle EBC = 60°$ (suppl. adj. angles)
$\angle ECB = 55°$ (suppl. adj. angles)
$\angle BEC = x°$ (vert. opp. angles)
$x° + 55° + 60° = 180°$ (angle sum of a $\Delta$)
$x = 65$

**b** $x = 110$ (alt. to $\angle EBC$, $DE \parallel AC$)

**c** $\angle DBC = 70°$ (corresp. to $\angle EAB$, $EA \parallel DB$)
$95° = x° + 70°$ (ext. angle of $\Delta DBC$)
$x = 25$

**d** $30° + 35° + \angle ABD = 180°$ (angle sum of a $\Delta$)
$\angle ABD = 115°$
$\angle DBC = 65°$ (suppl. adj. angles)
$x° = 50° + 65°$ (ext. angle of $\Delta BCD$)
$x = 115$

**e** $\angle BCE = 40°$ (suppl. adj. angles)
$96° = x° + 40°$ (ext. angle of $\Delta BCE$)
$x = 56$

**f** acute $\angle BAC = 80°$ (angle of revol'n)
acute $\angle ABC = 60°$ (angle of revol'n)
$x° + 60° + 80° = 180°$ (angle sum of a $\Delta$)
$x = 40$

**5** **a** $\angle BED = 58°$ (alt. to $\angle CBE$, $AC \parallel DE$)
$x° + 78° + 58° = 180°$ (angle sum of a $\Delta$)
$x = 44$

**b** $\angle BDA = x°$ (corresp. to $\angle CED$, $BD \parallel CE$)
$x° + 48° + 60° = 180°$ (angle sum of a $\Delta$)
$x = 72$

**c** $\angle CED = 55°$ ($DEF$ is a straight line)
$\angle BCE = x°$ (corresp. to $\angle ABF$, $BF \parallel CE$)
$x° = 46° + 55°$ (ext. angle of $\Delta CDE$)
$x = 101$

**d** $\angle ACD = x° + 42°$ (ext. angle of $\Delta ABC$)
$\angle CAD = x° + 42°$ (base angles of isos. $\Delta DAC$)
$122° = 2x° + 84°$ (ext. angle of $\Delta ACD$)
$x = 19$

**e** $\angle BCD = 125°$ (co-int. to $\angle ADC$, $BE \parallel AD$)
$\angle ACD = 55°$ (base angles of isos. $\Delta ACD$)
$\angle BCA = 70°$ ($\angle BCD - \angle ACD$)
$\angle ABC = 70°$ (base angles of isos. $\Delta ABC$)
$x° + 140° = 180°$ (angle sum of $\Delta ABC$)
$x = 40$

**f** $\Delta OAB$ and $\Delta OBC$ are isosceles (radii equal)
$\angle ABO = 60°$ (base angles of isos. $\Delta ABO$)
$\angle COB = 120°$ (ext. angle $\Delta ABO$)
$\angle OCB = x°$ (base angles of isos. $\Delta OBC$)
$2x° + 120° = 180°$ (angle sum of $\Delta OBC$)
$x = 30$

## Prep quiz 3:01C

| | | | |
|---|---|---|---|
| **1** 120 | **2** 60 | **3** 80 | **4** 105 |
| **5** 50 | **6** 80 | **7** 110 | **8** 70 |
| **9** 110 | **10** 360° | | |

## Exercise 3:01C

**1 a** $\angle CBE = 80°$ (vert. opp. $\angle FBA$)
$x = 80$ (opp. angles of a par'm)

**b** $\angle ABE = 78°$ (opp. angles of a par'm)
$x = 78$ (vert. opp. $\angle ABE$)

**c** $\angle CBE = 110°$ (vert. opp. $\angle ABF$)
$x° + 60° + 110° + 90° = 360°$ (angle sum of a quad.)
$x = 100$

**d** $\angle FBC = 80°$ (suppl. adj. angles)
$\angle CDF = 85°$ (suppl. adj. angles)
$x° + 85° + 80° + 80° = 360°$ (angle sum of a quad.)
$x = 115$

**e** $\angle EBC + 60° + 100° + 70° = 360°$ (angle sum of a quad.)
$\angle EBC = 130°$
$x = 50$ (suppl. adj. angles)

**f** Reflex $\angle ADC = 360° - 130° = 230°$
(angle sum at point $D$ is 360°)
$x° + 342° = 360°$ (angle sum of quad. $ABCD$)
$x = 18$

**2 a** $\angle CAB + 65° + 145° + 100° = 360°$ (angle sum of a quad.)
$\angle CAB = 50°$
$\angle CAB = \angle ABC$ (base angles of an isos. $\Delta$)
$\angle ABC = 50°$

**b** $\angle BAC + 70° + 150° + 85° = 360°$ (angle sum of a quad.)
$\angle BAC = 55°$
$\angle BCA = 55°$ (base angles of an isos. $\Delta$)
$\angle ABC + 55° + 55° = 180°$ (angle sum of a $\Delta$)
$\angle ABC = 70°$

**c** $\angle CAB + 95° + 80° + 115° = 360°$ (angle sum of a quad.)
$\angle CAB = 70°$
$\angle ABC = 70°$ (base angles of an isos. $\Delta$)

**d** $\angle EAD = \angle EDA$ (base angles of isos. $\Delta$)
$2\angle EAD + 100° = 180°$ (angle sum of a $\Delta$)
$\angle EAD = 40°$
$\angle BAD + \angle ADC + \angle DCB + \angle ABC = 360°$
(angle sum of a quad.)
$60° + 110° + 100° + \angle ABC = 360°$
$\angle ABC = 90°$

**e** $\angle BAC + 80° + 110° + 90° = 360°$ (angle sum of a quad.)
$\angle BAC = 80°$
$\angle ACB = 80°$ (base angles of isos. $\Delta$)
$\angle ABC + 80° + 80° = 180°$ (angle sum of a $\Delta$)
$\angle ABC = 20°$

**f** $\angle EAD = 80°$ (base angles of isos. $\Delta$)
$\angle BAC + 30° + 80° = 180°$ (angle sum of $\Delta$)
$\angle BAC = 70°$
$\angle ABC = 70°$ (base angle of isos. $\Delta$)

**3 a** $AE = EB$ (diag. of a rect. bisect each other)
$x = 30$ (base angle of isos. $\Delta$)

**b** $EB = EC$ (diag. of a rect. bisect each other and are equal)
$\angle ECB = x°$ (base angles of isos. $\Delta$)
$x° + x° + 40° = 180°$ (angle sum of a $\Delta$)
$x = 70$

**c** $DE = EC$ (diag. of a rect. bisect each other and are equal)
$x = 32$ (base angles of isos. $\Delta$)

**4 a** $\angle EAD = x°$ (base angles of isos. $\Delta$)
$\angle ADC = 2x°$ (ext. angle of $\Delta ADE$)
$2x° = 70°$ (opp. angles of par'm are equal)
$x = 35$

**b** $\angle GCF = 70°$ (corresp. angles, $AD \parallel BC$)
$\angle GFC = x°$ (corresp. angles, $BE \parallel AF$)
$\angle CGF = 30°$ (vert. opp. angles)
$x° + 30° + 70° = 180°$ (angle sum of a $\Delta$)
$x = 80$

**c** $\angle ADB = 40°$ (alt. angles, $AE \parallel BD$)
$\angle ADB = x°$ (alt. angles, $AD \parallel BC$)
$x = 40$

## Exercise 3:02

**1 a** In $\Delta$s $ABC$ and $DEF$:
(1) $AB = FE$ (data)
(2) $AC = FD$ (data)
(3) $\angle ABC = \angle FED$ (data)
$\therefore \Delta ABC \equiv \Delta DEF$ (RHS)

**b** In $\Delta$s $XYZ$ and $ACB$:
(1) $XY = AC$ (data)
(2) $XZ = AB$ (data)
(3) $YZ = CB$ (data)
$\therefore \Delta XYZ \equiv \Delta ACB$ (SSS)

**c** In $\Delta$s $ABC$ and $AED$:
(1) $BC = ED$ (data)
(2) $\angle BCA = \angle EDA$ (data)
(3) $CA = DA$ (data)
$\therefore \Delta ABC \equiv \Delta AED$ (SAS)

**d** In $\Delta$s $ABC$ and $DEF$:
(1) $\angle BCA = \angle EFD$ (data)
(2) $\angle BAC = \angle EDF$ (data)
(3) $CA = FD$ (data)
$\therefore \Delta ABC \equiv \Delta DEF$ (AAS)

**e** In Δs $LMN$ and $TRS$:
(1) $\angle NLM = \angle STR$ (data)
(2) $\angle NML = \angle SRT$ (data)
(3) $NM = SR$ (data)
$\therefore \Delta LMN \equiv \Delta TRS$ (AAS)

**2 a** In Δs $ABC$ and $DEC$:
(1) $BC = CE$ (given)
(2) $AC = DC$ (given)
(3) $\angle ACB = \angle DCE$ (vert. opp. ∠s)
$\therefore \Delta ABC \equiv \Delta DEC$ (SAS)

**b** In Δs $ABD$ and $BAC$:
(1) $BD = AC$ (given)
(2) $\angle ABC = \angle BAC$ (given)
(3) $AB$ is common
$\therefore \Delta ABD \equiv \Delta BAC$ (SAS)

**c** In Δs $ABD$ and $ACD$:
(1) $AB = AC$ (given)
(2) $AD$ is common
(3) $\angle ADB = \angle ADC$ (given)
$\therefore \Delta ABD \equiv \Delta ACD$ (RHS)

**d** In Δs $ABC$ and $ADC$:
(1) $AB = AD$ (given)
(2) $BC = DC$ (given)
(3) $AC$ is common
$\therefore \Delta ABC \equiv \Delta ADC$ (SSS)

**e** In Δs $ABD$ and $ACD$:
(1) $\angle ABD = \angle ACD$ (given)
(2) $\angle ADB = \angle ADC$ (given)
(3) $AD$ is common
$\therefore \Delta ABD \equiv \Delta ACD$ (AAS)

**f** In Δs $AOB$ and $COD$:
(1) $OA = OC$ (radii of circle)
(2) $OB = OD$ (radii of circle)
(3) $\angle AOB = \angle COD$ (given)
$\therefore \Delta AOB \equiv \Delta COD$ (SAS)

**g** In Δs $ABF$ and $CDE$:
(1) $\angle AFB = \angle CED$ (given)
(2) $AB = CD$ (sides of a square are equal)
(3) $\angle ABF = \angle EDC$ (angles of a square are equal)
$\therefore \Delta ABF \equiv \Delta CDE$ (AAS)

**3 a** In Δs $ABC$ and $ADC$:
(1) $AB = AD$ (given)
(2) $BC = DC$ (given)
(3) $AC$ is common
$\therefore \Delta ABC \equiv \Delta ADC$ (SSS)
$\therefore \angle BAC = \angle DAC$ (matching ∠s of cong't Δs)

**b** In Δs $ABC$ and $DBC$:
(1) $AC = CD$ (given)
(2) $AB = DB$ (given)
(3) $BC$ is common
$\therefore \Delta ABC \equiv \Delta DBC$ (SSS)
$\therefore \angle ABC = \angle CBD$ (matching ∠s of cong't Δs)

**c** In Δs $AXB$ and $DXC$:
(1) $AX = XD$ (given)
(2) $BX = XC$ (given)
(3) $\angle AXB = \angle CXD$ (vert. opp. ∠s)
$\therefore \Delta AXB \equiv \Delta DXC$ (SAS)
$\therefore \angle BAX = \angle CDX$ (matching ∠s of cong't Δs)
But $\angle ABX$ and $\angle DCX$ are alternate angles
$\therefore AB \parallel CD$ (equal alt. ∠s)

**d** In Δs $AOD$ and $BOD$:
(1) $AO = OB$ (radii of a circle)
(2) $OD$ is common
(3) $\angle ADO = \angle BDO$ (given)
$\therefore \Delta AOD \equiv \Delta BOD$ (RHS)
$\therefore AD = DB$ (matching sides of cong't Δs)

**e** In Δs $OAC$ and $OBC$:
(1) $OA = OB$ (radii of circle)
(2) $AC = CB$ (given)
(3) $OC$ is common
$\therefore \Delta OAC \equiv \Delta OBC$ (SSS)
$\therefore \angle OCA = \angle OCB$ (matching ∠s of cong't Δs)
Now $\angle OCA + \angle OCB = 180°$ (adj. supp. ∠s)
$\therefore 2 \times \angle OCA = 180°$
$\therefore \angle OCA = 90°$
$\therefore \angle OCA = \angle OCB = 90°$

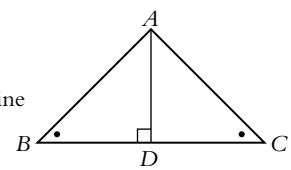
**f** In Δs $AOC$ and $BOD$:
(1) $OA = OB$ (radii of circle)
(2) $OC = OD$ (radii of circle)
(3) $\angle AOC = \angle BOD$ (vert. opp. angles)
$\therefore \Delta AOC \equiv \Delta BOD$ (SAS)
$AC = DB$ (matching sides of cong. Δs)
$\angle CAO = \angle DBO$ (corresp. angles of cong. Δs)
$\therefore AC \parallel DB$ (alt. angles $CAO$, $DBO$ are equal)

**4 a** In Δs $ABC$ and $DCB$:
(1) $\angle ACB = \angle DBC$ (given)
(2) $AC = DB$ (given)
(3) $BC$ is common
$\therefore \Delta ABC \equiv \Delta DCB$ (SAS)

**b** In Δs $DBC$ and $ECB$:
(1) $\angle BDC = \angle CEB$ (given)
(2) $\angle ABC = \angle ACB$ (given)
(3) $BC$ is common
$\therefore \Delta DBC \equiv \Delta EBC$ (AAS)

## Exercise 3:03

**1** $\angle B = \angle C$ (data)
Draw a perpendicular line from $A$ to $BC$, meeting $BC$ at $D$.




In Δs $ABD$ and $ACD$
(1) $\angle ABD = \angle ACD$ (data)
(2) $\angle ADB = \angle ADC = 90°$ (by construction)
(3) $AD$ is common
$\Delta ABD \equiv \Delta ACD$ (AAS)
$\therefore AB = AC$ (matching sides of cong't Δs)

**2 a** In Δs $ABD$ and $ACD$
(1) $AB = AC$ (data)
(2) $AD$ is common
(3) $\angle ADB = \angle ADC = 90°$ (by construction)
$\therefore \Delta ABD \equiv \Delta ACD$ (RHS)
$\therefore BD = CD$ (matching sides of cong't Δs)
$\therefore AD$ bisects $BC$

**b** In Δs $ABD$ and $ACD$
(1) $AB = AC$ (data)
(2) $AD$ is common
(3) $BD = CD$ (data)
$\therefore \Delta ABD \equiv \Delta ACD$ (SSS)
$\therefore \angle ADB = \angle ADC$ (matching ∠s of cong't Δs)
$\angle ADB + \angle ADC = 180°$ (adj. supp. ∠s)
$\therefore 2 \times \angle ADB = 180°$
$\angle ADB = 90°$
$\therefore AD \perp BC$

**3 a** $\angle B = \angle C$ ($AC = AB$)
$\angle A = \angle B$ ($AC = BC$)
$\therefore \angle A = \angle B = \angle C$

**b** $\angle A + \angle B + \angle C = 180°$ (∠ sum of Δ)
$\therefore 3 \times \angle A = 180°$ (using (a))
$\angle A = 60°$

**4 a** AAS
**b** matching sides of cong't Δs
**c** AAS
**d** matching sides of cong't Δs

**5 a** alt ∠s and || lines
**b** alt. ∠s and || lines
**c** yes
**d** In Δs $ABC$ and $CDA$:
(1) $\angle BAC = \angle DCA$ (alt. ∠s and || lines)
(2) $\angle BCA = \angle DAC$ (alt. ∠s and || lines)
(3) $AC$ is common
$\therefore \Delta ABC \equiv \Delta CDA$ (AAS)
$\therefore \angle ABC = \angle CDA$ (matching ∠s of cong't Δs)
$AB = DC$ (matching sides of cont'g Δs)
$BC = AD$ (matching sides of cong't Δs)

**6** In Δs $ABE$ and $CDE$:
(1) $AB = DC$ (opp. sides of a par'm)
(2) $\angle BAE = \angle DCE$ (alt. ∠s, $AB \parallel DC$)
(3) $\angle EBA = \angle EDC$ (alt. ∠s, $AB \parallel DC$)
$\therefore \Delta ABE \equiv \Delta CDE$ (AAS)
$\therefore AE = EC$ (matching sides of cong't Δs)
$\therefore EB = ED$ (matching sides of cong't Δs)

**7 a** SSS
**b** and **c** matching ∠s of cong't Δs
**d** SSS
**e** and **f** matching ∠s of cong't Δs

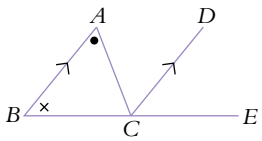
**8 a** Diagonals of a parallelogram bisect each other.
**b** In Δs $ABE$ and $CBE$:
(1) $AE = EC$ (diag. of par'm bisect each other)
(2) $BE$ is common
(3) $AB = BC$ (sides of a rhombus are equal)
$\therefore \Delta ABE \equiv \Delta CBE$ (SSS)
$\therefore \angle AEB = \angle BEC$ (matching ∠s of cong't Δs)
But $\angle AEB + \angle BEC = 180°$ (adj. supp. ∠s)
$\therefore 2 \times \angle AEB = 180°$
$\therefore \angle AEB = 90°$
$\therefore \angle AEB = \angle BEC = 90°$

**9 a** $\angle DAB = 90°$ (co-int. ∠s, $AB \parallel DC$)
$\angle ABC = 90°$ (co-int. ∠s, $AD \parallel BC$)
$\angle BCD = 90°$ (co-int. ∠s, $AB \parallel DC$)

**b** In Δs $ABD$ and $DCA$:
(1) $AB = DC$ (opp. sides of a par'm are equal)
(2) $AD$ is common
(3) $\angle ADC = \angle DAB$ (∠s of a rectangle)
$\therefore \Delta ABD \equiv \Delta DCA$ (SAS)
$\therefore AC = DB$ (matching sides of cong't Δs)

## Prep quiz 3:04

**1** and **2**




**3** $\angle DCE$ **4** $\angle ACD$
**5** $\angle ACD$ (or $\angle BCD$)
**6** $\angle ACD$ and $\angle DCE$ **7** $b$
**8** $180 - a$ **9** $c$ **10** $a = c$

## Exercise 3:04

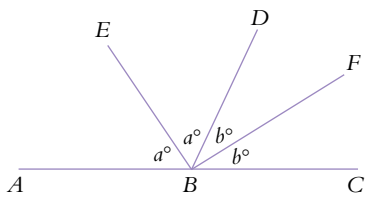
**1 a** $\angle AEF = x°$ (vert. opp. ∠s)
$\angle AEF = y°$ (corresp. ∠s, || lines)
$\therefore x = y$

**b** $\angle DBC = x°$ (vert. opp. ∠s)
$\angle BDE = 180° - x°$ (co-int. ∠s, $BC \parallel DE$)
$\angle BDE = 180° - y°$ (co-int. ∠s, $DB \parallel EC$)
$\therefore 180 - x = 180 - y$
$\therefore x = y$

**c** $\angle EFG = y°$ (equal ∠ of isos. Δ)
$\angle EFG = x°$ (alt. ∠, $AB \parallel CD$)
$\therefore x = y$

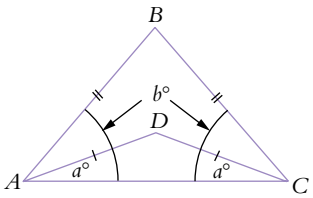
*Note:* Other proofs may exist for exercises in deductive reasoning.

**2**



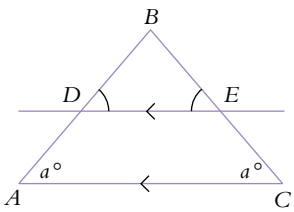

Let $\angle ABD$ be $2a°$ and $\angle DBC$ be $2b°$.
$\therefore \angle ABC = 2a° + 2b°$ ($\angle ABD + \angle DBC$)
$2(a + b)° = 180°$ ($\angle ABC$ is a straight angle)
$(a + b) = 90°$
Now $\angle EBD = a°$ ($BE$ bisects $\angle ABD$ given)
$\angle DBF = b°$ ($BF$ bisects $\angle DBC$; given)
$\therefore \angle EBF = (a + b)° = 90°$ ($\angle EBD + \angle DBF$)
$\therefore EB \perp BF$

**3**



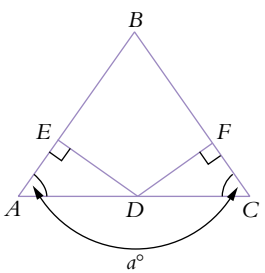

Let $\angle DAC$ and $\angle DCA$ be $a°$
(base angles of isos. $\Delta ADC$).
Let $\angle BAC$ and $\angle BCA$ be $b°$
(base angles of isos. $\Delta ABC$).
$\therefore \angle BAD = b° - a°$ ($\angle BAC - \angle DAC$)
$\angle BCD = b° - a°$ ($\angle BCA - \angle DCA$)
$\therefore \angle BAD = \angle BCD$

**4**



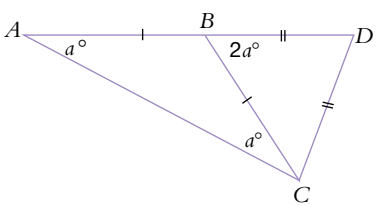

Let $\angle BAC$ and $\angle BCA$ be $a°$
(base angles of isos. $\Delta ABC$).
$\angle BDE = a°$ (corresp. to $\angle BAC$, $DE \parallel AC$)
$\angle BED = a°$ (corresp. to $\angle BCA$, $DE \parallel AC$)
$\therefore \angle BDE = \angle BED$ (ie base angles of $\Delta DBE$ are equal)
$\therefore \Delta DBE$ is isosceles

**5**



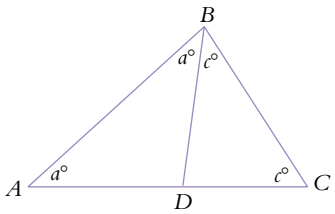

Let $\angle BAD$ and $\angle BCD$ be $a°$
(base angles of isos. $\Delta ABC$).
$\angle AED = \angle CFD = 90°$ ($DE \perp AB$, $DF \perp BC$)
$\angle EDA = 180° - a° - 90°$ (angle sum of $\Delta AED$)
$\angle FDC = 180° - a° - 90°$ (angle sum of $\Delta DFC$)
$\therefore \angle EDA = \angle FDC$

**6**



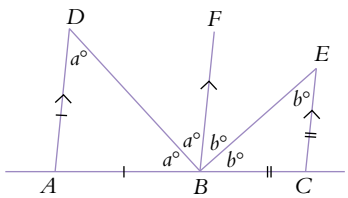

Let $\angle BAC$ and $\angle BCA$ be $a°$
(base angles of isos. $\Delta ABC$).
$\therefore \angle DBC = 2a°$ (ext. angle of $\Delta ABC$)
$\angle DCB = 2a°$ (base angles of isos. $\Delta BDC$)
$\therefore \angle BCD = 2 \times \angle BCA$

**7**



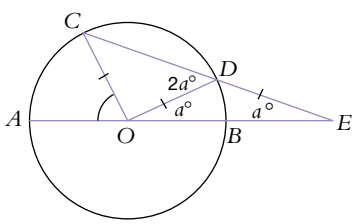

Let $\angle BAD$ be $a°$ and $\angle BCD$ be $c°$.
$\therefore \angle ABD = a°$ (base angles of isos. $\Delta ABD$)
$\angle DBC = c°$ (base angles of isos. $\Delta BCD$)
$2a° + 2c° = 180°$ (angle sum of $\Delta ABC$)
$a° + c° = 90°$ (dividing both sides by 2)
$\therefore \angle ABC = 90°$ ($\angle ABD + \angle DBC$)

**8**



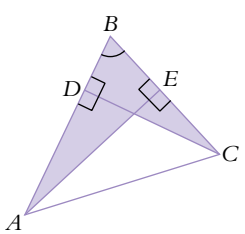

Construction: Draw $BF \parallel AD$ and $CE$.
Let $\angle ABD$ be $a°$ and $\angle CBE$ be $b°$.
$\angle ADB = a°$ (base angles of isos. $\Delta ADB$)
$\angle CEB = b°$ (base angles of isos. $\Delta BCE$)
$\angle FBD = a°$ (alt. to $\angle ADB$, $BF \parallel AD$)
$\angle FBE = b°$ (alt. to $\angle CEB$, $BF \parallel CE$)
$2a° + 2b° = 180°$ ($\angle ABC$ is a straight angle)
$\therefore a° + b° = 90°$ (dividing both sides by 2)
$\angle DBE = 90°$ ($\angle FBD + \angle FBE$)

**9**



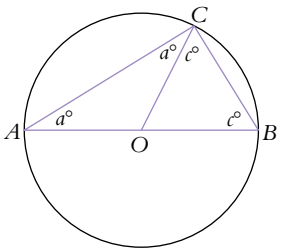

Let $\angle DOB$ and $\angle DEO$ be $a°$
(base angles of isos. $\Delta ODE$).
$\angle CDO = 2a°$ (ext. angle of $\Delta ODE$)
$\angle OCD = 2a°$ (base angles of isos. $\Delta OCD$, radii equal)
$\therefore \angle AOC = \angle DEO + \angle OCD$ (ext. angle of $\Delta OCE$)
$\therefore \angle AOC = a° + 2a° = 3a°$
$\therefore \angle AOC = 3 \times \angle DOB$

**10**



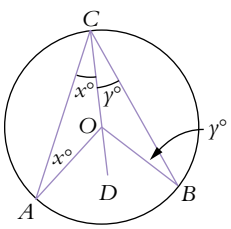

Let $\angle BAE = x°$, $\angle ABE = y°$ and $\angle BCD = z°$.
$x + y = 90$ (comp. $\angle$s in $\Delta ABE$)
$z + y = 90$ (comp. $\angle$s in $\Delta BCD$)
$\therefore x = z$
$\therefore \angle BAE = \angle BCD$

**11**




Construction: Draw interval $OC$
$OA = OC = OB$ (radii of circle)
Let $\angle OAC = a°$ and $\angle OBC = c°$.
$\angle ACO = a°$ (base angles of isos. $\Delta ACO$)
$\angle BCO = c°$ (base angles of isos. $\Delta BCO$)
$2a° + 2c° = 180°$ (angle sum of $\Delta ABC$)
$a° + c° = 90°$
$\therefore \angle ACB = 90°$ ($\angle ACO + \angle BCO$)

**12**




Construction: Join $CO$ and extend it to $D$.
$OA = OB = OC$ (radii of circle)
Let $\angle ACO = x°$ and $\angle BCO = y°$.
$\angle ACB = \angle ACO + \angle BCO$
$= x° + y°$
$\angle CAO = x°$ (base angles of isos. $\Delta OAC$)
$\angle AOD = 2x°$ (ext. angle of $\Delta OAC$)
Similarly,
$\angle CBO = y°$ and $\angle BOD = 2y°$
$\angle AOB = \angle AOD + \angle BOD$
$= 2x° + 2y°$
$= 2(x° + y°)$
$= 2 \times \angle ACB$

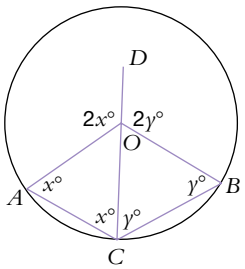
**13** $\angle BCE = \beta$ (alternate to $\angle ABC$, $AB \parallel CE$)
$\angle ECD = \alpha$ (corresponding to $\angle BAC$, $AB \parallel CE$)
$\angle BCA = \gamma$ (given)
$\therefore \beta + \alpha + \gamma = 180°$ ($\angle ACD$ is a straight angle)
$\therefore$ the angle sum of a triangle is 180°.

**14** Now $a + b + c = 180°$ (angle sum of $\Delta ABD$)
and $d + e + f = 180°$ (angle sum of $\Delta BCD$)
$\angle A + \angle B + \angle C + \angle D = b + (c + d) + e + (a + f)$
$= (a + b + c) + (d + e + f)$
$= 180° + 180°$
$= 360°$

**15** $\angle BCE = \beta$ (alt. $\angle$s, $AB \parallel CE$)
$\angle ECD = \alpha$ (corresp. $\angle$s, $AB \parallel CE$)
$\therefore \angle BCD = \angle BCE + \angle ECD$
$= \alpha + \beta$

**16** $x° + c° = 180°$ (adj. $\angle$s on a st. line)
$y° + b° = 180°$ (adj. $\angle$s on a st. line)
$z° + a° = 180°$ (adj. $\angle$s on a st. line)
$(x° + c°) + (y° + b°) + (z° + a°) = 540°$
$(x° + y° + z°) + (a° + b° + c°) = 540°$
But $x° + y° + z° = 180°$ ($\angle$ sum of a $\Delta$)
$\therefore 180° + (a° + b° + c°) = 540°$
$\therefore a° + b° + c° = 360°$

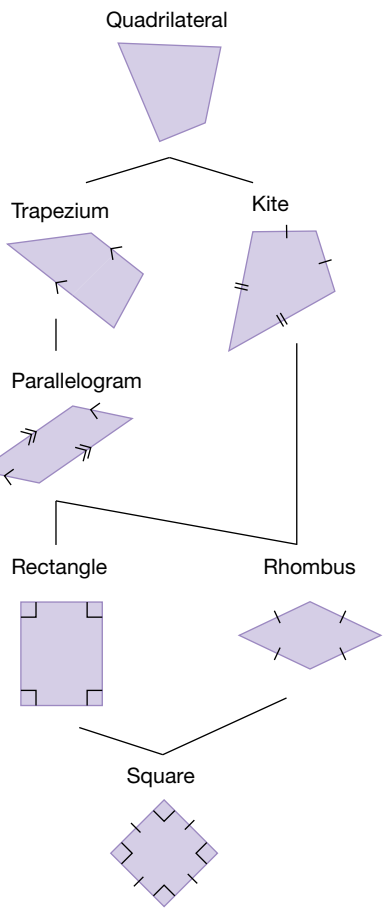
**17**




Produce $CO$ to $D$
$AO = BO = CO$ (radii of circle)
Let $\angle OCA = x°$ and $\angle OCB = y°$.
$\therefore \angle ACB = x° + y°$
$\angle OAC = x°$ (base $\angle$s of isos. $\Delta OAC$)
$\angle AOD = 2x°$ (ext. $\angle$ of $\Delta AOC$)
$\angle OBC = y°$ (base $\angle$s of isos. $\Delta OBC$)
$\angle BOD = 2y°$ (ext. $\angle$ of $\Delta BOC$)
Reflex $\angle AOB = \angle AOD + \angle BOD$
$= 2x° + 2y°$
$= 2(x° + y°)$
$= 2 \times \angle ACB$

## Exercise 3:05

**1**



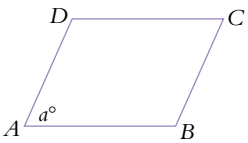
**2** **a** C **b** B **c** A **d** E
**e** F **f** G **g** H **h** D

**3** **a** true **b** false **c** true
**d** true **e** false **f** true
**g** true **h** false

**4** **a** A square is a parallelogram and it has two adjacent sides equal in length so it fits the definition of a rhombus. However, a rhombus is not necessarily a rectangle as it need not contain a right angle. This is part of the square's definition.

**b** An equilateral triangle has two sides equal so it satisfies the definition of a isosceles triangle. However, an isosceles triangle does not necessarily have three sides equal so it can't be called an equilateral triangle.

**5**



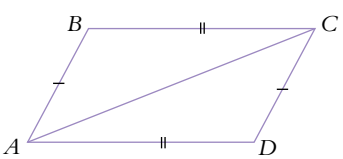

Let $\angle DAB = a°$
$\therefore \angle ADC = 180° - a°$ (co-int. $\angle$s, $AB \parallel DC$)
$\angle ABC = 180° - a°$ (co-int. $\angle$s, $AD \parallel BC$)
Now, $\angle DCB + \angle ABC = 180°$ (co-int. $\angle$s, $AB \parallel DC$)
$\angle DCB = 180° - \angle ABC$
$= 180° - (180° - a°)$
$= a°$
$\therefore \angle DAB = \angle DCB = a°$
and $\angle ABC = \angle ADC = 180° - a°$
$\therefore$ Opposite $\angle$s of par'm are equal.

**6** $\angle ADC = 90°$ (co-int. $\angle$s, $AB \parallel CD$)
$\angle ABC = 90°$ (co-int. $\angle$s, $AD \parallel BC$)
$\angle BCD = 90°$ (co-int. $\angle$s, $AB \parallel CD$)

**7** **a** Opposite sides of a parallelogram are equal.

**b** $AB = DC$ (opp. sides of a par'm)
$AD = BC$ (opp. sides of a par'm)
But $AB = AD$ (given)
$\therefore AB = AD = DC = BC$

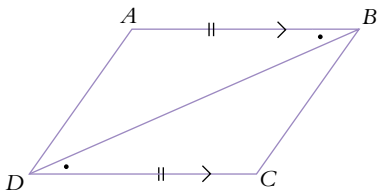
**8**




In $\Delta$s $ABC$ and $CDA$
(1) $AB = CD$ (given)
(2) $BC = DA$ (given)
(3) $AC$ is common
$\therefore \Delta ABC \equiv \Delta CDA$ (SSS)
$\angle BCA = \angle CAD$ (matching angles of cong. $\Delta$s)
$\therefore BC \parallel DA$ (alt. angles equal)
$\angle BAC = \angle DCA$ (matching angles of cong. $\Delta$s)
$\therefore AB \parallel CD$ (alt. angles equal)
$\therefore$ Opposite sides are parallel.
$\therefore ABCD$ is a parallelogram.

**9** **a** Vertically opposite $\angle$s
**b** SAS
**c** $\angle CBE$ (matching. $\angle$s of congruent $\Delta$s)
**d** $\angle ADE$ and $\angle CBE$ are equal alternate $\angle$s.
**e** SAS
**f** matching $\angle$s of congruent $\Delta$s
**g** Pairs of alternate angles are equal.

**10**

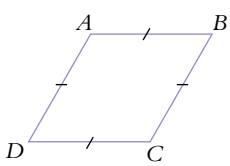



Data: $ABCD$ is a quadrilateral with $AB = DC$ and $AB \parallel DC$.
Aim: To prove $AD \parallel BC$.
Construction: Join $BD$.
Proof: In $\Delta$s $ABD$ and $CBD$
(1) $AB = DC$ (given)
(2) $BD$ is common.
(3) $\angle ABD = \angle CDB$ (alt. $\angle$s, $AB \parallel DC$)
$\therefore \Delta ABD \equiv \Delta CBD$ (SAS)
$\therefore \angle ADB = \angle CBD$ (matching $\angle$s of cong't $\Delta$s)
But $\angle ADB$ and $\angle CBD$ are also alternate $\angle$s
$\therefore AD \parallel BC$ (pair of alt. $\angle$s are equal)
$\therefore ABCD$ has both pairs of opposite sides parallel.
$\therefore ABCD$ is a parallelogram.

**11**

Proof: $ABCD$ is a parallelogram (opposite $\angle$s equal)
$4a° = 360°$ (angle sum of quad.)
$a = 90$
$\therefore ABCD$ is a parallelogram with a right angle.
$\therefore ABCD$ is a rectangle.

**12**




Proof: $ABCD$ is a parallelogram. (Both pairs of opposite sides are equal.)
$AB = BC$ (given)
$\therefore ABCD$ is a parallelogram with a pair of adjacent sides equal.
$\therefore ABCD$ is a rhombus. (See definition p. 73.)

**13** 1 quadrilateral 2 trapezium
3 parallelogram 4 rectangle
5 square 6 rhombus

**14**

| Properties | Parallelogram | Rhombus | Rectangle | Square |
|---|---|---|---|---|
| 1 Opposite sides parallel | yes | yes | yes | yes |
| 2 Opposite sides equal | yes | yes | yes | yes |
| 3 Opposite angles equal | yes | yes | yes | yes |
| 4 Diagonals bisect one another | yes | yes | yes | yes |
| 5 All sides equal | no | yes | no | yes |
| 6 All angles right angles | no | no | yes | yes |
| 7 Diagonals perpendicular | no | yes | no | yes |
| 8 Diagonals bisect angles through which they pass | no | yes | no | yes |
| 9 Diagonals are equal | no | no | yes | yes |

The numbers below refer to the table in Question **14**.

**15** **a** 1, 2, 3, 4, 5, 7, 8 **b** 1, 2, 3, 4, 6, 9
**c** 1, 2, 3, 4 **d** 1, 2, 3, 4

**16** **a** 5, 7, 8 **b** 6, 9 **c** 6, 9

**17** **a** yes **b** yes
**c** No, because it does not have all its sides equal.
**d** yes **e** yes
**f** No, because it does not have all its angles equal.

**18** **a** Parallelogram; both pairs of opposite sides are equal.
**b** Rectangle; diagonals are equal and bisect each other.
**c** Rhombus; all sides are equal.
**d** Rectangle; all angles are equal.
**e** Rhombus; diagonals bisect each other at right angles.
**f** Parallelogram; one pair of sides is both equal and parallel.

**19** Proof: $DC = AB$ (opp. sides of par'm $ABCD$)
$AB = FE$ (opp. sides of par'm $ABFE$)
$\therefore DC = FE$
Also $DC \parallel AB$ and $AB \parallel FE$
$\therefore DC \parallel FE$
$\therefore DC$ and $FE$ are parallel and equal.
$\therefore DCEF$ is a parallelogram.

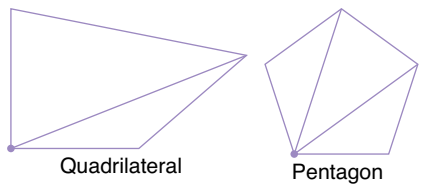
**20** Proof: $AK = KC$ (diag. of par'm bisect each other)
$EA = CG$ (given)
$\therefore AK + EA = KC + CG$
$EK = KG$
Similarly it can be shown that $KH = KF$.
$\therefore$ Diagonals $EG$ and $FH$ bisect each other.
$\therefore EFGH$ is a parallelogram.

## Investigation 3:05

**1** If a triangle has all angles equal then it has all its sides equal. (This converse is true.)
**2** If a quadrilateral has its diagonals equal then it is a square. (This converse is not true.)
**3** If a quadrilateral has its opposite angles equal then it is a parallelogram. (This converse is true.)
**4** If a quadrilateral has all its sides equal then it is a square. (This converse is not true.)
**5** If the square on the longest side of the triangle is equal to the sum of the squares on the two smaller sides then the triangle is right-angled. (This converse is true.)

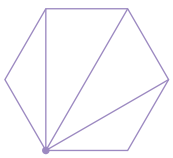
## Investigation 3:06A

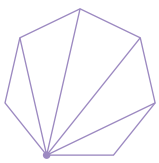
**1**

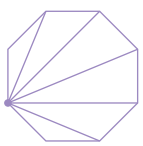


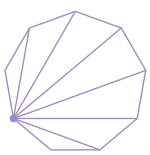




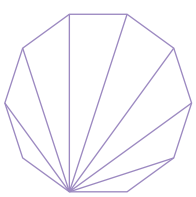


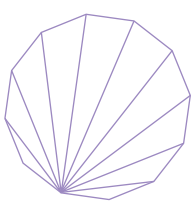




**2**




**3**

| No. of sides in polygon | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| No. of triangles | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 |

**4** The 'number of triangles' is two less than the 'number of sides'.
**5** $n - 2$
**6** $(n - 2) \times 180$
**7** $\dfrac{(n-2) \times 180}{n}$

## Investigation 3:06B

**1** 6
**2** $6 \times 180° = 1080°$
**3** 720°
**4** $1080° - 720° = 360°$
**5** (1) 5 (2) $5 \times 180° = 900°$ (3) 540°
(4) $900° - 540° = 360°$
**6** The exterior angles sum to 360°.

## Exercise 3:06

**1** **a** 720 **b** 1440 **c** 2340

**2** **a** **i** 5 **ii** 540°

**b** **i** 6 **ii** 720°

**c** **i** 7 **ii** 900°

**3** **a** 1080° **b** 1440° **c** 900°

**4** **a** 108° **b** 120° **c** 135° **d** 144°

**e** 140° **f** $158\frac{14}{17}°$

**5** **a** 3240° **b** 360° **c** 162° **d** 18°

**6** **a** 6 **b** 9 **c** 36 **d** 24

**7** **a** 240° **b** 228° **c** 315°

**8** **a** 80° **b** 110° **c** 120° **d** 135°

**e** 125° **f** 60°

## Investigation 3:06C

**1** 120°

**2** 3

**3** 360°

**4** A regular pentagon will not tessellate because the interior angles (108°) cannot add at a vertex to give 360°

**5** Square and equilateral triangle.

## Diagnostic test 3

**1** **a** $\angle FGC = 130°$ (corresp. $\angle$s, $AB \parallel CD$)
$x° + 130° = 180°$ (adj. supp. $\angle$s)
$x = 50$

**b** $\angle DBE = \angle EDB$ (base $\angle$s of isos. $\Delta$)
$2 \times \angle EDB + 40° = 180°$ ($\angle$ sum of $\Delta$)
$\therefore \angle EDB = 70°$
$\angle EDB = x°$ (alt. $\angle$s, $AC \parallel DE$)
$\therefore x = 70$

**c** $\angle LON + 90° + 70° + 85° = 360°$ ($\angle$ sum of quad.)
$\therefore \angle LON = 115°$
$\angle LOP = x°$ (base $\angle$s of isos. $\Delta$)
$x° + 115° = 180°$ (adj. supp. $\angle$s)
$x = 65$

**2** **a** In $\Delta$s $ABE$ and $DCE$,
(1) $AE = DE$ (given)
(2) $BE = CE$ (given)
(3) $\angle AEB = \angle DEC$ (vert. opp. $\angle$s)
$\therefore \Delta ABE \equiv \Delta DCE$ (SAS)

**b** In $\Delta$s $ACO$ and $BCO$,
(1) $AC = BC$ (given)
(2) $AO = BO$ (radii of circle)
(3) $OC$ is common
$\therefore \Delta ACO \equiv \Delta BCO$ (SSS)

**c** In $\Delta$s $ABC$ and $ADC$,
(1) $\angle ABC = \angle ADC$ (given)
(2) $\angle ACB = \angle ACD$ (given)
(3) $AC$ is common
$\therefore \Delta ABC \equiv \Delta ADC$ (AAS)

**3** **a** In $\Delta$s $ABE$ and $DCF$,
(1) $\angle AEB = \angle DFC$ (given)
(2) $AB = DC$ (given)
(3) $AE = FD$ (opp sides of rect $ADFE$)
$\therefore \Delta ABE \equiv \Delta DCF$ (RHS)
$\therefore \angle ABE = \angle DCF$
(matching $\angle$s of cong't $\Delta$s)

**b** In $\Delta$s $ABD$ and $ACD$,
(1) $AD$ is common
(2) $\angle ABD = \angle ACD$ (given)
(3) $\angle BAD = \angle CAD$ (given)
$\therefore \Delta ABD \equiv \Delta ACD$ (AAS)
$\therefore \angle ADB = \angle ADC$ (matching angles of cong't $\Delta$s)
$\angle ADB + \angle ADC = 180°$ (adj. supp. $\angle$s)
$2 \times \angle ADB = 180°$
$\angle ADB = 90°$
$\therefore AD \perp BC$

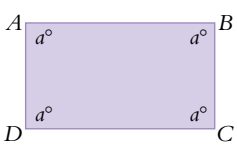
**c** In $\Delta$s $CAB$ and $CAD$,
(1) $AC$ is common
(2) $AB = AD$ (given)
(3) $BC = DC$ (given)
$\therefore \Delta CAB \equiv \Delta CAD$ (SSS)
$\therefore \angle CAB = \angle CAD$ (matching $\angle$s of cong't $\Delta$s)

**4** **a** $\angle DBA = x°$ (alt. $\angle$s, $AB \parallel CD$)
$\angle DBA = y°$ (alt. $\angle$s, $AC \parallel BD$)
$\therefore x = y$

**b** $\angle CBD = \angle CDB$ (base $\angle$s, of isos. $\Delta$)
$\therefore \angle ABD = 90° - \angle CBD$
$\angle ADB = 90° - \angle CDB$
$\therefore \angle ABD = \angle ADB$
$\therefore AB = AD$ (equal sides of isos. $\Delta$)

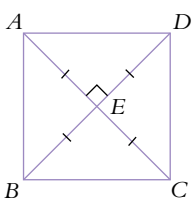
**c** Let $\angle ABC = \angle ACD = x°$
$\therefore \angle CAD = 90° - x°$ (comp. $\angle$s, $\Delta ADC$)
$\angle ATE = x°$ (comp. $\angle$s, $\Delta ATE$)
$\therefore \angle FTD = x°$ (vert. opp. $\angle$s)
$\therefore \angle ABC = \angle FTD$

**5** **a**




Proof: $ABCD$ is a parallelogram (opposite $\angle$s equal)
$4a° = 360°$ (angle sum of quad.)
$a = 90$
$\therefore ABCD$ is a parallelogram with a right angle.
$\therefore ABCD$ is a rectangle.

**b**




$ABCD$ is a rectangle (diagonals are equal and bisect each other)
$\Delta ADE \equiv \Delta CDE$ (SAS)
$\therefore AD = DC$ (matching sides of cong't $\Delta$s)
$\therefore ABCD$ is a square (rectangle with a pair of adjacent sides equal)

c One pair of sides is both equal and parallel.

**6** **a** 2340° **b** 135° **c** 18

## ASSIGNMENT 3A

**1** **a** $\angle BEG = 70°$ (comp. $\angle$s)
$x° + 70° = 180°$ (co-int. $\angle$s, $AB \parallel CD$)
$x = 110$

**b** $\angle EBA = 70°$ (base $\angle$s of isos. $\Delta EAB$)
$\angle DBC = 60°$ ($\angle$ of equil. $\Delta$)
$\therefore \angle EBD = 50°$ (adj. $\angle$s on a st. line)
$\angle BDE = 50°$ (base $\angle$s of isos. $\Delta EBD$)
$x + 50° + 50° = 180°$ ($\angle$ sum of $\Delta$)
$x = 80$

**c** $\angle BAC = x°$ (alt. $\angle$s, $DB \parallel AC$)
$\angle BCA = x°$ (base $\angle$s of isos. $\Delta$)
$2x° + 52° = 180°$ ($\angle$ sum of $\Delta$)
$x = 64$

**d** $\angle BCD + 74° + 138° + 96° = 360°$ ($\angle$ sum of quad.)
$\angle BCD = 52°$
$\angle BDC = x°$ (base $\angle$s of isos. $\Delta$)
$2x° + 52° = 180°$ ($\angle$ sum of $\Delta$)
$x = 64$

**2** **a** In $\Delta$s $AOB$ and $COD$,
(1) $AO = CO$ (radii of circle)
(2) $BO = DO$ (radii of circle)
(3) $\angle AOB = \angle COD$ (given)
$\therefore \Delta AOB \equiv \Delta COD$ (SAS)

**b** In $\Delta$s $ABD$ and $CBD$,
(1) $AB = CB$ (given)
(2) $AD = CD$ (given)
(3) $DB$ is common
$\therefore \Delta ABD \equiv \Delta CBD$ (SSS)
$\therefore \angle ADB = \angle CDB$
(matching $\angle$s of cong't $\Delta$s)

**3** In $\Delta$s $BED$ and $CFD$,
(1) $BD = CD$ ($D$ is midpoint of $BC$)
(2) $\angle BED = \angle CFD$ ($BE \perp AD$, $CF \perp AD$)
(3) $\angle BDE = \angle CDF$ (vert. opp. angles)
$\therefore \Delta BED \equiv \Delta CFD$ (AAS)
$BE = CF$ (matching sides of cong't $\Delta$s)

**4** **a** 3240°

**b** **i** 162° **ii** 18°

**5** $x = 126$

**6** $\angle CAB = x°$ (equal $\angle$s of isos. $\Delta$)
$\angle CAB = y°$ (alt. $\angle$s, $AB \parallel DC$)
$\therefore x = y$

**7** $AF = EC = 5$ (given)
$AF \parallel EC$ (Since $AB \parallel DC$)
$\therefore AFCE$ is a parallelogram
(One pair of sides is both equal and parallel).
$\therefore AE \parallel FC$

## ASSIGNMENT 3B

**1** One side of a triangle formed by the projection of one end of the board is 145·84.
3394·11 + 145·84 = 3550 mm (rounded up)

**2** The median and mode would not be useful as these are the same as the par of the hole in each case. Calculating the mean for each hole and the percentage of players who scored par or better gives a better indication of the degree of difficulty.

| Hole | Par | Mean score | % of players scoring par or better |
|---|---|---|---|
| 2 | 4 | 4·1 | 79% |
| 7 | 5 | 4·5 | 94% |
| 11 | 3 | 3·3 | 65% |
| 16 | 3 | 3·2 | 75% |

Both the mean and % of players scoring par or better indicate that the degree of difficulty of the holes are (1) Hole 11; (2) Hole 16; (3) Hole 2; (4) Hole 7

**3** green

**4** **a** 12 points **b** 8 points

## ASSIGNMENT 3C

**1** **a** $\frac{a^2}{6}$ **b** $\frac{5a}{6}$ **c** $\frac{2}{3}$ **d** $-\frac{a}{6}$

**2** **a** $\frac{2b}{3}$ **b** $\frac{b}{6}$ **c** $\frac{5n^4}{4m}$ **d** $\frac{1}{4ay^2}$

**3** **a** 7 500 000 B **b** 3100 µs **c** 5000 GB

**4** From 6·25 up to 6·35

**5** **a** 15·84 $m^2$ **b** 79·2 $m^3$

**6** **a** $x = 0$ or $-1\frac{2}{3}$ (or −1·67)

**b** $x = -7$ or 9 **c** $x = -0{\cdot}15$ or $-3{\cdot}35$

**d** $x = -2$ or 6 **e** $x = 4{\cdot}76$ or 9·24

**7** **a** $x = -1 + \sqrt{13}$ or $x = -1 - \sqrt{13}$

**b** $x = -\sqrt{7}, \sqrt{7}, -\sqrt{8}$ or $\sqrt{8}$

**8** Length of base = 8 cm

# Chapter 4

## Exercise 4:01

**1** A. Not B, as some people would have more than one ticket. Not C, as the person most suited is more likely to be chosen.

**2** Answers will vary.

**3** Answers will vary.

**4** **a** even chance **b** improbable **c** probable
**d** impossible **e** certain **f** even chance

**5** **a** $\frac{1}{8}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$
**d** $\frac{7}{8}$ **e** $\frac{3}{4}$ **f** 0

**6** **a** $\frac{1}{6}$ **b** $\frac{1}{6}$ **c** $\frac{5}{6}$
**d** $\frac{1}{3}$ **e** 0 **f** 1

**7** **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** 0

**d** $\frac{1}{13}$ **e** $\frac{12}{13}$ **f** $\frac{3}{13}$
**g** $\frac{1}{4}$ **h** $\frac{3}{4}$ **i** $\frac{2}{13}$

**8** **a** $\frac{4}{500}$ (or $\frac{1}{125}$) **b** $\frac{496}{500}$ (or $\frac{124}{125}$)

**9** **a** $\frac{16}{100}$ (or $\frac{4}{25}$) **b** $\frac{14}{100}$ (or $\frac{7}{50}$) **c** $\frac{43}{100}$
**d** $\frac{86}{100}$ (or $\frac{43}{50}$) **e** $\frac{30}{100}$ (or $\frac{3}{10}$) **f** $\frac{50}{100}$ (or $\frac{1}{2}$)

**10** **a** 50% **b** 38% **c** 1% **d** 20%
**e** 80% **f** 77% **g** 23%

**11** **a** $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** $\frac{3}{200}$ **d** $\frac{1}{200}$ **e** $\frac{3}{4}$
**f** $\frac{159}{200}$ **g** $\frac{197}{200}$ **h** $\frac{41}{200}$ **i** $\frac{1}{50}$

**12** **a** $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{99}$

**13** **a** $\frac{1}{8}$ **b** $\frac{5}{8}$ **c** $\frac{1}{4}$ **d** 0
**e** $\frac{2}{4}$ (or $\frac{1}{2}$) **f** $\frac{1}{2}$ **g** $\frac{1}{4}$ **h** $\frac{1}{3}$
**i** $\frac{1}{5}$ **j** $\frac{1}{4}$

**14** **a** Answers will vary. Knowing that it has not rained on Saturday gives us more information than the weather forecaster had when he gave his prediction on Tuesday. This will affect the probability that it will rain on Sunday. We say that these two events are dependent events.

**b** The chance that the next toss is a head, would be $\frac{1}{2}$ if the coin is a fair coin. However, the chance of getting 20 heads in a row is $\frac{1}{1048\,576}$ so one would have to expect that the coin either has two heads or is strongly biased towards heads.

**15** **a** $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$ **e** $\frac{1}{3}$

**16** **a** $\frac{48}{51}$ **b** 0·75 **c** 63% **d** 0·875

**e** **i** 99·5%

**ii** No, as \$135 is 1·35% of \$10 000 and this is clearly more than 0·5%. (There is 1 chance in 200 that I will die this year. 200 × \$135 = \$27 000. As I am insured for only \$10 000, it does not represent good value.)

## Exercise 4:02

**1** **a** $\frac{4}{9}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$ **d** $\frac{2}{9}$ **e** $\frac{4}{9}$

**2** **a** $\frac{2}{6}$ **b** 0 **c** $\frac{4}{6}$ **d** $\frac{2}{6}$ **e** $\frac{4}{6}$

**3** **a** $\frac{1}{9}$ **b** $\frac{3}{9}$ **c** $\frac{2}{9}$ **d** $\frac{1}{9}$ **e** $\frac{6}{9}$
**f** $\frac{1}{3}$ **g** 0 **h** $\frac{1}{3}$ **i** $\frac{2}{3}$

**4** **a** 0 **b** 0 **c** $\frac{2}{6}$ **d** $\frac{1}{6}$ **e** $\frac{4}{6}$
**f** $\frac{2}{6}$ **g** $\frac{2}{6}$ **h** 1 **i** 0 **j** 0
**k** $\frac{1}{2}$ **l** 1

**5** **a** $\frac{1}{200}$ **b** $\frac{1}{100}$ **c** $\frac{1}{10}$ **d** $\frac{1}{2}$ **e** 0

**6** If the result of the first step is known, then only the outcomes starting with the known result should be considered. This reduces the number of possible outcomes, and probably changes the number of ways that the event can occur.

## Investigation 4:02

**1** Answers will vary.

**2** **a** Here there is no replacement, so it is impossible to choose two greens (as we started with only 1 green counter).

**b** Answers will vary.

## Exercise 4:03

**1** **a** 8 **b** 2 (a head and a tail) **c** yes
**d** 3 **e** 3 **f** 1
**g** $\frac{1}{8}$ **h** $\frac{1}{8}$ **i** $\frac{3}{8}$
**j** $\frac{3}{8}$ **k** $\frac{7}{8}$ **l** $\frac{4}{8}$ (or $\frac{1}{2}$)

**2** **a** 2 **b** 3 **c** 2
**d** 12 **e** yes **f** 1
**g** 6 **h** 2 **i** 12
**j** 4 **k** 4 **l** $\frac{1}{12}$
**m** $\frac{6}{12}$ (or $\frac{1}{2}$) **n** $\frac{2}{12}$ (or $\frac{1}{6}$) **o** $\frac{12}{12}$ (or 1)
**p** $\frac{4}{12}$ (or $\frac{1}{3}$) **q** $\frac{4}{12}$ (or $\frac{1}{3}$)

**3** **a** 27 **b** 1 **c** 26
**d** 12 **e** $\frac{1}{27}$ **f** $\frac{1}{27}$
**g** $\frac{3}{27}$ **h** $\frac{3}{27}$ **i** $\frac{8}{27}$
**j** $\frac{7}{27}$

**4** **a** 6 **b** 0 **c** 6
**d** 2 **e** 0 **f** 0
**g** 1 **h** $\frac{2}{6}$ (or $\frac{1}{3}$) **i** $\frac{4}{6}$ (or $\frac{2}{3}$)

**5** **a** 60 **b** yes **c** 6
**d** 18 **e** 36 **f** 24
**g** There are only 2 yellow tickets, and there is no replacement.
**h** $\frac{6}{60}$ (or $\frac{1}{10}$) **i** $\frac{6}{60}$ (or $\frac{1}{10}$) **j** $\frac{36}{60}$ (or $\frac{3}{5}$)
**k** $\frac{24}{60}$ (or $\frac{2}{5}$) **l** $\frac{54}{60}$ (or $\frac{9}{10}$)

**6** **a** 210 **b** 24 **c** 72
**d** 6 **e** 204 **f** 90
**g** 108 **h** $\frac{24}{210}$ (or $\frac{4}{35}$) **i** $\frac{72}{210}$ (or $\frac{12}{35}$)
**j** $\frac{6}{210}$ (or $\frac{1}{35}$)

## Prep quiz 4:04

**1** 56 **2** 2 **3** 20 **4** 5 **5** 5
**6** 10 **7** 2 **8** 10 **9** 10 **10** 20

## Exercise 4:04

**1** **a** 10 **b** 30 **c** 20
**d** 12 **e** 8 **f** 24
**g** $\frac{1}{12}$ **h** $\frac{1}{8}$ **i** $\frac{1}{30}$
**j** $\frac{3}{30}$ (or $\frac{1}{10}$)

**2** **a** 60 **b** 40 **c** 120
**d** 48 **e** 8 **f** 125
**g** $\frac{2}{48}$ (or $\frac{1}{24}$) **h** $\frac{1}{60}$

**3** **a** 16 **b** 9 **c** 49
**d** $\frac{1}{16}$ **e** $\frac{8}{49}$ **f** $\frac{9}{9}$ (or 1)

**4** a 12 b 6 c 42
d $\frac{1}{12}$ e $\frac{8}{42}$ (or $\frac{4}{21}$) f $\frac{6}{6}$ (or 1)

**5** a 24 b 6 c 210
d $\frac{8}{210}$ (or $\frac{4}{105}$) e $\frac{6}{210}$ (or $\frac{3}{105}$) f $\frac{1}{24}$

**6** a $n(S) = 64$
b i $n(E_1) = 25$ ii $\frac{25}{64}$
c i $n(E_2) = 9$ ii $\frac{9}{64}$
d i $n(E_3) = 15$ ii $\frac{15}{64}$
e i $n(E_4) = 15$ ii $\frac{15}{64}$
f i $n(E_5) = 30$ ii $\frac{30}{64}$ (or $\frac{15}{32}$)

**7** a $n(S) = 56$
b i $n(E_1) = 20$ ii $\frac{20}{56}$ (or $\frac{5}{14}$)
c i $n(E_2) = 6$ ii $\frac{6}{56}$ (or $\frac{3}{28}$)
d i $n(E_3) = 15$ ii $\frac{15}{56}$
e i $n(E_4) = 15$ ii $\frac{15}{56}$
f i $n(E_5) = 30$ ii $\frac{30}{56}$ (or $\frac{15}{28}$)

**8** a $n(S) = 20$ b $\frac{1}{20}$
c $\frac{3}{20}$ d $\frac{4}{20}$ (or $\frac{1}{5}$)

**9** a $n(S) = 100$
b i $n(E_1) = 9$ ii $\frac{9}{100}$
c i $n(E_2) = 4$ ii $\frac{4}{100}$ (or $\frac{1}{25}$)
d i $n(E_3) = 10$ ii $\frac{10}{100}$ (or $\frac{1}{10}$)
e i $n(E_4) = 10$ ii $\frac{10}{100}$ (or $\frac{1}{10}$)
f i $n(E_5) = 20$ ii $\frac{20}{100}$ (or $\frac{1}{5}$)

**10** a $n(S) = 1320$
b i $n(E_1) = 24$ ii $\frac{24}{1320}$ (or $\frac{1}{55}$)
c i $n(E_2) = 120$ ii $\frac{120}{1320}$ (or $\frac{1}{11}$)
d i $n(E_3) = 12$ ii $\frac{12}{1320}$ (or $\frac{1}{110}$)

## Exercise 4:05

**1** a $\frac{1}{12}$ b $\frac{5}{16}$ c $\frac{1}{10}$
d $\frac{5}{48}$ e $\frac{1}{30}$ f $\frac{5}{40}$ (or $\frac{1}{8}$)
g $\frac{6}{48}$ (or $\frac{1}{8}$) h $\frac{3}{16}$ i $\frac{9}{40}$

**2** a $\frac{1}{4}$ b $\frac{1}{36}$ c $\frac{25}{64}$
d $\frac{1}{25}$

**3** a $\frac{16}{64}$ (or $\frac{1}{4}$) b $\frac{9}{64}$ c $\frac{12}{64}$ (or $\frac{3}{16}$)
d $\frac{12}{64}$ (or $\frac{3}{16}$) e $\frac{4}{64}$ (or $\frac{1}{16}$) f $\frac{4}{64}$ (or $\frac{1}{16}$)
g $\frac{3}{64}$ h $\frac{3}{64}$

**4** a $\frac{50}{99}$ b $\frac{40}{99}$ c $\frac{5}{99}$
d $\frac{4}{99}$

**5** a $\frac{1}{42}$ b $\frac{18}{42}$ (or $\frac{3}{7}$) c $\frac{2}{42}$ (or $\frac{1}{21}$)

**6** a $\frac{16}{100}$ (or $\frac{4}{25}$) b $\frac{24}{100}$ (or $\frac{6}{25}$) c $\frac{36}{100}$ (or $\frac{9}{25}$)
d $\frac{24}{100}$ (or $\frac{6}{25}$)

**7** a $\frac{8}{15}$ b $\frac{12}{25}$ c $\frac{2}{6}$ (or $\frac{1}{3}$)
d $\frac{3}{10}$

**8** a $\frac{9}{800}$ b $\frac{15}{800}$ (or $\frac{3}{160}$)

**9** a $\frac{36}{100}$ (or $\frac{9}{25}$) b $\frac{72}{150}$ (or $\frac{12}{25}$) c $\frac{8}{30}$ (or $\frac{4}{15}$)
d $\frac{18}{60}$ (or $\frac{3}{10}$)

**10** a $\frac{1}{8}$ b $\frac{125}{216}$ c $\frac{125}{512}$
d $\frac{27}{512}$ e $\frac{1}{125}$ f $\frac{5}{32}$
g $\frac{6}{96}$ (or $\frac{1}{16}$) h $\frac{20}{80}$ (or $\frac{1}{4}$)

**11** a $\frac{3}{20}$ b $\frac{3}{120}$ (or $\frac{1}{40}$) c $\frac{21}{160}$
d $\frac{21}{240}$ (or $\frac{7}{80}$) e $\frac{15}{120}$ (or $\frac{1}{8}$) f $\frac{1}{192}$
g $\frac{21}{960}$ (or $\frac{7}{320}$)

**12** a $\frac{18}{100}$ (or $\frac{9}{50}$)
b $\frac{11}{20}$ (The coin can be chosen from hat 1 or from hat 2.)

**13** a $\frac{1}{2}$
b i $\frac{3}{8}$ ii $\frac{5}{8}$

**14** Spending the day in both Sydney and London are dependent events. I cannot spend the day in both Sydney and London. If I spend the day in Sydney then the probability of my also spending the day in London is 0 not $\frac{2}{10}$, so the probability of spending the day in both Sydney and London = $\frac{7}{10} \times 0$ (or 0).

## Fun spot 4:05

**1** The contestant would improve their chance of choosing the car if they changed their choice.

**2** Yes. Originally the contestant had one chance in three of choosing the car. There were two chances in three that they had not chosen the car. Of the two doors not chosen, Monty always opened the door that did not contain the car, so the chance of the car being behind the other door is $\frac{2}{3}$. Changing choices at this point doubles the chance of choosing the car.

## Exercise 4:06

**1** a $\frac{1}{20}$ b $\frac{3}{20}$ c $\frac{6}{20}$

**2** a $\frac{1}{12}$ b $\frac{11}{12}$ c $\frac{1}{4}$

**3** a $\frac{1}{30}$ b $\frac{2}{30}$ (or $\frac{1}{15}$) c $\frac{28}{30}$ (or $\frac{14}{15}$)
d $\frac{1}{120}$ e $\frac{2}{120}$ (or $\frac{1}{60}$) f $\frac{118}{120}$ (or $\frac{59}{60}$)

**4** a $\frac{1}{380}$ b $\frac{90}{380}$ (or $\frac{9}{38}$) c $\frac{90}{380}$ (or $\frac{9}{38}$)
d $\frac{72}{380}$ (or $\frac{18}{95}$) e $\frac{110}{380}$ (or $\frac{11}{38}$) f $\frac{100}{380}$ (or $\frac{5}{19}$)
g 0 h $\frac{2}{380}$ (or $\frac{1}{190}$) i $\frac{504}{6840}$ (or $\frac{7}{95}$)
j $\frac{1}{6840}$ k $\frac{720}{6840}$ (or $\frac{2}{19}$) l $\frac{6120}{6840}$ (or $\frac{17}{19}$)

**5** a $\frac{2}{42}$ (or $\frac{1}{21}$) b $\frac{40}{42}$ (or $\frac{20}{21}$) c $\frac{20}{42}$ (or $\frac{10}{21}$)
d $\frac{20}{42}$ (or $\frac{10}{21}$) e $\frac{60}{210}$ (or $\frac{2}{7}$) f 0
g $\frac{40}{210}$ (or $\frac{4}{21}$) h $\frac{120}{210}$ (or $\frac{4}{7}$)

**6** a $\frac{12}{56}$ (or $\frac{3}{14}$) b $\frac{30}{56}$ (or $\frac{15}{28}$) c $\frac{2}{56}$ (or $\frac{1}{28}$)
d $\frac{54}{56}$ (or $\frac{27}{28}$) e $\frac{12}{56}$ (or $\frac{3}{14}$) f $\frac{24}{56}$ (or $\frac{3}{7}$)
g $\frac{12}{336}$ (or $\frac{1}{28}$) h $\frac{60}{336}$ (or $\frac{5}{28}$) i $\frac{120}{336}$ (or $\frac{5}{14}$)
j $\frac{216}{336}$ (or $\frac{9}{14}$)

**7** a $\frac{60}{336}$ (or $\frac{5}{28}$) b $\frac{90}{336}$ (or $\frac{15}{56}$)

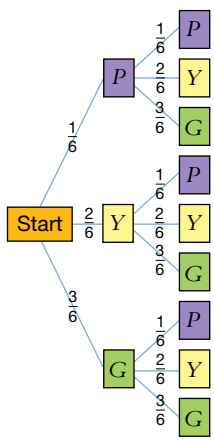
8 a $\frac{2}{20}$ (or $\frac{1}{10}$) b $\frac{8}{20}$ (or $\frac{2}{5}$) c $\frac{2}{20}$ (or $\frac{1}{10}$)
d $\frac{8}{20}$ (or $\frac{2}{5}$) e $\frac{10}{20}$ (or $\frac{1}{2}$) f $\frac{10}{20}$ (or $\frac{1}{2}$)
g $\frac{16}{20}$ (or $\frac{4}{5}$) h $\frac{4}{20}$ (or $\frac{1}{5}$)

9 a $\frac{2}{60}$ (or $\frac{1}{30}$) b $\frac{16}{60}$ (or $\frac{4}{15}$) c $\frac{2}{60}$ (or $\frac{1}{30}$)
d $\frac{8}{60}$ (or $\frac{2}{15}$) e $\frac{8}{60}$ (or $\frac{2}{15}$) f $\frac{4}{60}$ (or $\frac{1}{15}$)
g $\frac{10}{60}$ (or $\frac{1}{6}$) h $\frac{20}{60}$ (or $\frac{1}{3}$)

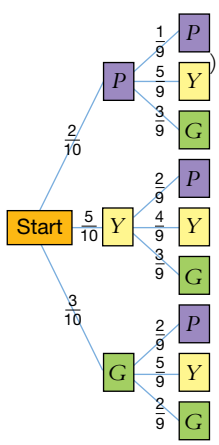
10 a $\frac{1}{144}$ b $\frac{25}{144}$ c $\frac{30}{144}$ (or $\frac{5}{24}$)
d $\frac{30}{144}$ (or $\frac{5}{24}$) e $\frac{60}{144}$ (or $\frac{5}{12}$) f $\frac{36}{144}$ (or $\frac{1}{4}$)
g $\frac{108}{144}$ (or $\frac{3}{4}$) h $\frac{5}{12}$ i $\frac{6}{12}$ (or $\frac{1}{2}$)
j $\frac{82}{144}$ (or $\frac{41}{72}$)

11 a $\frac{0}{12}$ (or 0) b $\frac{20}{132}$ (or $\frac{5}{33}$) c $\frac{30}{132}$ (or $\frac{5}{22}$)
d $\frac{30}{132}$ (or $\frac{5}{22}$) e $\frac{60}{132}$ (or $\frac{5}{11}$) f $\frac{30}{132}$ (or $\frac{5}{22}$)
g $\frac{102}{132}$ (or $\frac{17}{22}$) h $\frac{4}{11}$ i $\frac{5}{11}$
j $\frac{82}{132}$ (or $\frac{41}{66}$)

12 a $\frac{1}{36}$
b $\frac{9}{36}$ (or $\frac{1}{4}$)
c $\frac{6}{36}$ (or $\frac{1}{6}$)
d $\frac{6}{36}$ (or $\frac{1}{6}$)
e $\frac{12}{36}$ (or $\frac{1}{3}$)
f $\frac{16}{36}$ (or $\frac{4}{9}$)
g $\frac{20}{36}$ (or $\frac{5}{9}$)
h $\frac{3}{6}$ (or $\frac{1}{2}$)
i $\frac{2}{6}$ (or $\frac{1}{3}$)
j $\frac{22}{36}$ (or $\frac{11}{18}$)

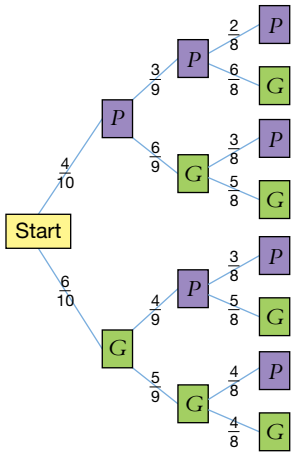



13 a $\frac{2}{90}$ (or $\frac{1}{45}$)
b $\frac{6}{90}$ (or $\frac{1}{15}$
c $\frac{15}{90}$ (or $\frac{1}{6}$)
d $\frac{15}{90}$ (or $\frac{1}{6}$)
e $\frac{30}{90}$ (or $\frac{1}{3}$)
f $\frac{20}{90}$ (or $\frac{2}{9}$)
g $\frac{70}{90}$ (or $\frac{7}{9}$)
h $\frac{2}{9}$
i $\frac{4}{9}$
j $\frac{31}{45}$




14 a $\frac{120}{504}$ (or $\frac{5}{21}$) b $\frac{384}{504}$ (or $\frac{16}{21}$) c $\frac{6}{504}$ (or $\frac{1}{84}$)
d $\frac{498}{504}$ (or $\frac{83}{84}$) e $\frac{126}{504}$ (or $\frac{7}{28}$) f $\frac{90}{504}$ (or $\frac{5}{28}$)
g $\frac{90}{504}$ (or $\frac{5}{28}$) h $\frac{90}{504}$ (or $\frac{5}{28}$) i $\frac{270}{504}$ (or $\frac{15}{28}$)
j $\frac{168}{504}$ (or $\frac{1}{3}$) k $\frac{336}{504}$ (or $\frac{2}{3}$) l $\frac{2}{56}$ (or $\frac{1}{28}$)
m $\frac{20}{56}$ (or $\frac{5}{14}$)

15




a $\frac{24}{720}$ (or $\frac{1}{30}$) b $\frac{696}{720}$ (or $\frac{29}{30}$) c $\frac{120}{720}$ (or $\frac{1}{6}$)
d $\frac{600}{720}$ (or $\frac{5}{6}$) e $\frac{576}{720}$ (or $\frac{4}{5}$) f $\frac{72}{720}$ (or $\frac{1}{10}$)
g $\frac{72}{720}$ (or $\frac{1}{10}$) h $\frac{72}{720}$ (or $\frac{1}{10}$) i $\frac{360}{720}$ (or $\frac{1}{2}$)
j $\frac{432}{720}$ (or $\frac{3}{5}$) k $\frac{288}{720}$ (or $\frac{2}{5}$) l $\frac{20}{72}$ (or $\frac{5}{18}$)
m $\frac{6}{72}$ (or $\frac{1}{12}$)

16 $\frac{7}{504}$ (or $\frac{1}{72}$)

## Diagnostic test 4

1 a i $\frac{2}{52}$ (or $\frac{1}{26}$) ii $\frac{50}{52}$ (or $\frac{25}{26}$)
b i $\frac{1}{3}$ ii $\frac{2}{3}$

2 B 50%

3 a $\frac{9}{36}$ (or $\frac{1}{4}$) b $\frac{9}{18}$ (or $\frac{1}{2}$) c $\frac{4}{36}$ (or $\frac{1}{9}$)

4 a $\frac{6}{30}$ (or $\frac{1}{5}$) b $\frac{6}{15}$ (or $\frac{2}{5}$) c $\frac{4}{30}$ (or $\frac{2}{15}$)

5 a $\frac{1}{6}$ b $\frac{0}{6}$ (or 0) c $\frac{6}{6}$ (or 1)

6 a 36 (6 × 6) b 8 (2 × 2 × 2) c 25 (5 × 5)
d 20 (5 × 4) e 125 (5 × 5 × 5)
f 60 (5 × 4 × 3)

7 a $\frac{12}{30}$ (or $\frac{2}{5}$) b $\frac{16}{30}$ (or $\frac{8}{15}$)

8 a $\frac{60}{336}$ (or $\frac{5}{28}$) b $\frac{90}{336}$ (or $\frac{15}{56}$)

9 a $\frac{9}{50}$ b $\frac{3}{24}$ (or $\frac{1}{8}$)

10 $\frac{1}{20}$

11 a i $\frac{6}{20}$ ii $\frac{12}{20}$ (or $\frac{3}{5}$)
b i $\frac{6}{60}$ (or $\frac{1}{10}$) ii $\frac{6}{60}$ (or $\frac{1}{10}$)

### ASSIGNMENT 4A

1 a $\frac{4}{13}$ b $\frac{3}{52}$ c $\frac{50}{52}$
d $\frac{49}{52}$

2 a 18% b 21% c 79%
d 48% e 69% f 41%

3 a $\frac{5}{100}$ (or 5%) b $\frac{5}{99}$ c $\frac{5}{98}$
d $\frac{4}{98}$

4 a $\frac{1}{16}$ b $\frac{1}{48}$ c $\frac{7}{20}$
d $\frac{2}{16}$ (or $\frac{1}{8}$) e $\frac{12}{80}$ (or $\frac{3}{20}$) f $\frac{18}{48}$ (or $\frac{3}{8}$)

## ASSIGNMENT 4B

**1** **a** 10 km **b** 20 km **c** 11:30 am
**d** Callum **e** 11:30 am

**2** **a** D **b** C **c** A **d** B

**3** **a** One side is longer than the sum of the other two. This cannot happen in any triangle.
**b** The angle sum to more than 180°.
**c** In a right-angled triangle the longest side must be opposite the right angle.

**4** 201

**5** 9

## ASSIGNMENT 4C

**1** **a** $\frac{x+y+z}{3}$
**b** $(m+n)^2$ or $m^2+2mn+n^2$
**c** $3m+6$

**2** **a** $(1, 6)$
**b** $\sqrt{40}$ units (or $2\sqrt{10}$ units)
**c** $\frac{1}{3}$
**d** $x-3y+17=0$
**e** $x-3y+31=0$

**3** **a** $y=-2$ **b** $x=7, y=5$
**c** $x=4+\sqrt{14}$ or $4-\sqrt{14}$

**4** **a** 104 **b** 20

**5** **a** 12·7 **b** 12·0 **c** 32·1

**6** **a** 22°37′ **b** 36°52′ **c** 72°33′

# Chapter 5

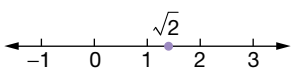
## Prep quiz 5:01

**1** 16 **2** 49 **3** 400 **4** 61
**5** 9 **6** 8 **7** 12 **8** 9
**9** 1·414 214 **10** 8·316 625

## Exercise 5:01

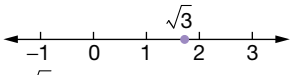
**1** **a** rational **b** irrational **c** rational
**d** rational **e** rational **f** irrational
**g** irrational **h** rational **i** rational
**j** rational **k** rational **l** rational
**m** rational **n** rational **o** rational
**p** irrational **q** rational **r** irrational
**s** rational **t** rational **u** rational
**v** rational **w** irrational **x** irrational
**y** rational **z** irrational

**2** **a**



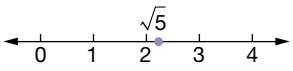

$\sqrt{2} \div 1{\cdot}4$

**b**



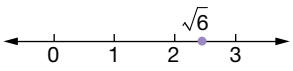

$\sqrt{3} \div 1{\cdot}7$

**c**



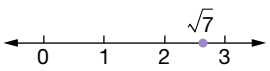

$\sqrt{5} \div 2{\cdot}2$

**d**



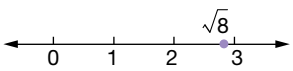

$\sqrt{6} \div 2{\cdot}4$

**e**



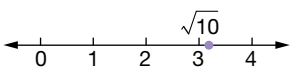

$\sqrt{7} \div 2{\cdot}6$

**f**



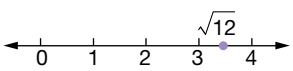

$\sqrt{8} \div 2{\cdot}8$

**g**



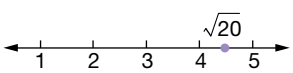

$\sqrt{10} \div 3{\cdot}2$

**h**



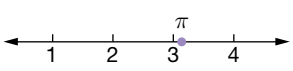

$\sqrt{12} \div 3{\cdot}5$

**i**




$\sqrt{20} \div 4{\cdot}5$

**j**




$\pi \div 3{\cdot}1$

**3** **a** 3, 4 **b** 4, 5 **c** 6, 7
**d** 8, 9 **e** 9, 10 **f** 11, 12
**g** 13, 14 **h** 15, 16 **i** 19, 20
**j** 30, 31

**4** **a** $\sqrt{3}, 2, \sqrt{5}$ **b** $\sqrt{8}, 3, \pi$
**c** $3, \sqrt{10}, \sqrt{12}$ **d** $\sqrt{40}, 6{\cdot}5, 7, \sqrt{50}$
**e** $\sqrt{2}, 2{\cdot}1, \pi, \sqrt{12}$ **f** $\sqrt{26}, \sqrt{30}, 5{\cdot}6, 6$
**g** $\sqrt{60}, 7{\cdot}9, \sqrt{65}, 8{\cdot}1$ **h** $\sqrt{98}, 10, \sqrt{102}, 10{\cdot}1$
**i** $\sqrt{9}, 3{\cdot}1, \pi, 3{\cdot}2$ **j** $4{\cdot}1, \sqrt{20}, 4{\cdot}5, \sqrt{21}$
**k** $\sqrt{390}, 20, \sqrt{420}, 21$ **l** $24, \sqrt{600}, \sqrt{610}, 25$

**5–7** Students' own diagrams.

## Prep quiz 5:02

**1** 4 **2** 3 **3** 6 **4** 5 **5** 7
**6** 12 **7** 12 **8** 2 **9** 2 **10** 16

## Exercise 5:02

**1** **a** $\sqrt{15}$ **b** $\sqrt{15}$ **c** $\sqrt{42}$ **d** $\sqrt{42}$
**e** $\sqrt{30}$ **f** $\sqrt{46}$ **g** $\sqrt{65}$ **h** $\sqrt{33}$
**i** $\sqrt{10}$ **j** $\sqrt{14}$ **k** $\sqrt{110}$ **l** $\sqrt{91}$
**m** $\sqrt{13}$ **n** $\sqrt{11}$ **o** $\sqrt{7}$ **p** $\sqrt{2}$
**q** $\sqrt{19}$ **r** $\sqrt{19}$ **s** $\sqrt{6}$ **t** $\sqrt{2}$

**2** **a** 16 **b** 9 **c** 1 **d** 100
**e** 5 **f** 8 **g** 15 **h** 73
**i** 8 **j** 45 **k** 12 **l** 75
**m** 147 **n** 28 **o** 891 **p** 180
**q** 1000 **r** 1620 **s** 1800 **t** 3375

**3** **a** $2\sqrt{2}$ **b** $2\sqrt{5}$ **c** $2\sqrt{3}$ **d** $5\sqrt{2}$
**e** $2\sqrt{6}$ **f** $4\sqrt{2}$ **g** $3\sqrt{5}$ **h** $3\sqrt{6}$
**i** $2\sqrt{7}$ **j** $3\sqrt{10}$ **k** $2\sqrt{14}$ **l** $3\sqrt{7}$
**m** $2\sqrt{11}$ **n** $2\sqrt{13}$ **o** $6\sqrt{3}$ **p** $2\sqrt{10}$
**q** $3\sqrt{11}$ **r** $2\sqrt{15}$ **s** $4\sqrt{6}$ **t** $2\sqrt{19}$
**u** $2\sqrt{17}$ **v** $3\sqrt{14}$ **w** $10\sqrt{2}$ **x** $9\sqrt{2}$

**4** **a** 4·2 **b** 5·2 **c** 2·8 **d** 3·5
**e** 5·6 **f** 6·9 **g** 7·1 **h** 12·7

**5** **a** $4\sqrt{3}$ **b** $6\sqrt{2}$ **c** $10\sqrt{2}$ **d** $12\sqrt{2}$
**e** $10\sqrt{5}$ **f** $10\sqrt{3}$ **g** $30\sqrt{3}$ **h** $6\sqrt{14}$
**i** $10\sqrt{5}$ **j** $12\sqrt{5}$ **k** $6\sqrt{6}$ **l** $6\sqrt{6}$
**m** $14\sqrt{14}$ **n** $18\sqrt{2}$ **o** $6\sqrt{11}$ **p** $15\sqrt{10}$
**q** $60\sqrt{2}$ **r** $35\sqrt{2}$ **s** $54\sqrt{3}$ **t** $10\sqrt{17}$

**6** **a** $\sqrt{12}$ **b** $\sqrt{18}$ **c** $\sqrt{20}$ **d** $\sqrt{54}$
**e** $\sqrt{32}$ **f** $\sqrt{75}$ **g** $\sqrt{63}$ **h** $\sqrt{50}$
**i** $\sqrt{72}$ **j** $\sqrt{150}$ **k** $\sqrt{90}$ **l** $\sqrt{112}$
**m** $\sqrt{252}$ **n** $\sqrt{250}$ **o** $\sqrt{98}$ **p** $\sqrt{200}$
**q** $\sqrt{243}$ **r** $\sqrt{256}$ **s** $\sqrt{441}$ **t** $\sqrt{432}$

## Prep quiz 5:03

**1** $2\sqrt{3}$ **2** $2\sqrt{5}$ **3** $4\sqrt{2}$ **4** $5\sqrt{2}$ **5** 5
**6** 3·6 **7** 2 **8** 3·5 **9** 11 **10** 7·8

## Exercise 5:03

**1** **a** $5\sqrt{2}$ **b** $11\sqrt{3}$ **c** $7\sqrt{6}$ **d** $3\sqrt{3}$
**e** $3\sqrt{5}$ **f** $\sqrt{2}$ **g** $5\sqrt{7}$ **h** $6\sqrt{3}$
**i** $8\sqrt{6}$ **j** $14\sqrt{5}$ **k** $9\sqrt{10}$ **l** $6\sqrt{3}$
**m** $3\sqrt{2}$ **n** $3\sqrt{3}$ **o** 0

**2** **a** $6\sqrt{5} + 3\sqrt{7}$ **b** $4\sqrt{7} + 2\sqrt{5}$
**c** $4\sqrt{3} + 8\sqrt{5}$ **d** $2\sqrt{3} + 5\sqrt{2}$
**e** $\sqrt{10} + 4\sqrt{7}$ **f** $2\sqrt{3} + 2\sqrt{5}$
**g** $8\sqrt{11} + 3\sqrt{7}$ **h** $\sqrt{2} + 12\sqrt{3}$
**i** $\sqrt{5} - \sqrt{2}$ **j** $2\sqrt{7} - 5\sqrt{5}$

**3** **a** $3\sqrt{2}$ **b** $4\sqrt{3}$ **c** $5\sqrt{2}$ **d** $6\sqrt{5}$
**e** $5\sqrt{3}$ **f** $5\sqrt{6}$ **g** $3\sqrt{2}$ **h** $\sqrt{5}$
**i** $\sqrt{2}$ **j** $7\sqrt{2}$ **k** $5\sqrt{5}$ **l** $2\sqrt{3}$
**m** $\sqrt{3}$ **n** $22\sqrt{2}$ **o** $23\sqrt{2}$ **p** $16\sqrt{7}$
**q** $8\sqrt{5}$ **r** $-2\sqrt{3}$

**4** **a** $4\sqrt{2}$ **b** $3\sqrt{5}$ **c** $6\sqrt{3}$
**d** $6\sqrt{7}$ **e** $3\sqrt{3} + 5\sqrt{2}$ **f** $8\sqrt{5} + 4\sqrt{2}$
**g** $18\sqrt{2} + 3\sqrt{3}$ **h** $21\sqrt{2} - 5\sqrt{3}$

## Exercise 5:04

**1** **a** $\sqrt{6}$ **b** $\sqrt{35}$ **c** $\sqrt{33}$ **d** $\sqrt{10}$
**e** $\sqrt{21}$ **f** $\sqrt{70}$ **g** 4 **h** 6
**i** 10 **j** $3\sqrt{2}$ **k** $5\sqrt{2}$ **l** $2\sqrt{5}$
**m** $2\sqrt{15}$ **n** $12\sqrt{10}$ **o** $14\sqrt{6}$ **p** 20
**q** 18 **r** 20 **s** $24\sqrt{5}$ **t** $8\sqrt{3}$
**u** $24\sqrt{5}$ **v** $6x$ **w** $4\sqrt{x}$ **x** $a^2x$

**2** **a** $\sqrt{5}$ **b** $\sqrt{3}$ **c** $\sqrt{2}$ **d** 3
**e** 2 **f** 3 **g** 5 **h** 6
**i** 2 **j** $4\sqrt{2}$ **k** $3\sqrt{5}$ **l** $\sqrt{7}$
**m** $6\sqrt{2}$ **n** $3\sqrt{2}$ **o** $2\sqrt{3}$ **p** $\sqrt{5}$
**q** $\sqrt{3}$ **r** $\sqrt{2}$ **s** $\sqrt{x}$ **t** $\sqrt{5}$
**u** $\sqrt{5}$

**3** **a** $3\sqrt{2}$ **b** $8\sqrt{3}$ **c** $\sqrt{2}$
**d** 1 **e** $\sqrt{6}$ **f** $6\sqrt{2}$
**g** $\sqrt{2}$ **h** 3 **i** $\frac{9}{4\sqrt{3}}$

**4** **a** $\sqrt{6} + 2$ **b** $5 + \sqrt{10}$ **c** $14 - \sqrt{14}$
**d** $5\sqrt{3} - 3$ **e** $2\sqrt{6} - \sqrt{2}$ **f** $10\sqrt{5} - 4\sqrt{10}$
**g** $4 + 2\sqrt{2}$ **h** $15 + 6\sqrt{5}$ **i** $4\sqrt{6} - 12$
**j** $3\sqrt{30} + 18$ **k** $14 - 2\sqrt{14}$ **l** $7\sqrt{3} - 9$
**m** $2\sqrt{6} + 8$ **n** $4\sqrt{10} - 40$ **o** $60 - 30\sqrt{3}$
**p** $a + \sqrt{a}$ **q** $2x + 3\sqrt{x}$ **r** $6y + 4\sqrt{xy}$

## Prep quiz 5:05

**1** $\sqrt{15}$ **2** 6 **3** 5
**4** 18 **5** $10\sqrt{2}$ **6** $8\sqrt{3}$
**7** 0 **8** $5\sqrt{7} - 20$
**9** $10 + \sqrt{30}$ **10** $6\sqrt{6} - 6$

## Exercise 5:05

**1** **a** $5 + 4\sqrt{2}$ **b** $4\sqrt{3} - 2$
**c** $17 - 7\sqrt{7}$ **d** $2 + \sqrt{6} + \sqrt{10} + \sqrt{15}$
**e** $\sqrt{35} - \sqrt{10} - \sqrt{14} + 2$
**f** $5\sqrt{2} + \sqrt{10} + \sqrt{30} + \sqrt{6}$
**g** $5\sqrt{3} + 5\sqrt{2} + \sqrt{6} + 2$
**h** $4 - 2\sqrt{5} - 2\sqrt{3} + \sqrt{15}$
**i** $5\sqrt{6} - 6 - 5\sqrt{2} + 2\sqrt{3}$
**j** $4 + \sqrt{2} + 2\sqrt{6} + \sqrt{3}$
**k** $13 + 3\sqrt{15}$ **l** $10 + 5\sqrt{6}$
**m** $7\sqrt{14} - 27$ **n** $9\sqrt{15} - 5$
**o** $26\sqrt{14} - 45$ **p** $-21\sqrt{21} - 40$
**q** $114 + 21\sqrt{70}$ **r** $77 + 39\sqrt{6}$
**s** $66 + 14\sqrt{6}$ **t** $42 - 27\sqrt{7}$
**u** $32\sqrt{3} - 1$ **v** $x + 5\sqrt{x} + 6$
**w** $2m + 3\sqrt{mn} + n$ **x** $6a + 5\sqrt{ab} - 6b$

**2** **a** $3 + 2\sqrt{2}$ **b** $28 - 10\sqrt{3}$
**c** $9 + 4\sqrt{5}$ **d** $5 + 2\sqrt{6}$
**e** $7 - 2\sqrt{10}$ **f** $13 + 2\sqrt{30}$
**g** $13 + 4\sqrt{3}$ **h** $34 - 24\sqrt{2}$
**i** $45 + 20\sqrt{5}$ **j** $15 + 4\sqrt{14}$
**k** $55 + 30\sqrt{2}$ **l** $52 - 6\sqrt{35}$
**m** $83 + 20\sqrt{6}$ **n** $167 - 28\sqrt{15}$

o $550 - 100\sqrt{30}$ p $x + 2\sqrt{xy} + y$

q $4m + 20\sqrt{m} + 25$ r $9p - 12\sqrt{pq} + 4q$

3 a 1 b 22 c −39 d 14

e −1 f 2 g 2 h 4

i −13 j 18 k −5 l 42

m 38 n −6 o $x - y$ p $4a - 9b$

## Prep quiz 5:06

1 5 2 10 3 6 4 10 5 12

6 1 7 1 8 23 9 10 10 23

## Exercise 5:06

1 a $\frac{\sqrt{2}}{2}$ b $\frac{\sqrt{5}}{5}$ c $\frac{2\sqrt{3}}{3}$

d $\frac{\sqrt{10}}{2}$ e $\frac{3\sqrt{2}}{2}$ f $2\sqrt{3}$

g $2\sqrt{5}$ h $\frac{2\sqrt{11}}{11}$ i $\frac{\sqrt{6}}{3}$

j $\frac{\sqrt{15}}{5}$ k $\frac{\sqrt{2}}{2}$ l $\frac{\sqrt{5}}{5}$

m $\frac{\sqrt{2}}{4}$ n $\frac{2\sqrt{3}}{15}$ o $\frac{7\sqrt{5}}{10}$

p $\frac{5\sqrt{3}}{3}$ q $\frac{\sqrt{2}}{2}$ r $\frac{\sqrt{10}}{10}$

s $\frac{\sqrt{6}}{3}$ t $\frac{\sqrt{35}}{3}$ u $\frac{2\sqrt{3}+3}{3}$

v $\frac{\sqrt{2}+\sqrt{10}}{2}$ w $\frac{7+\sqrt{21}}{14}$ x $\frac{2-\sqrt{2}}{10}$

2 a $\frac{1-\sqrt{2}}{-1} = \sqrt{2} - 1$ b $\frac{\sqrt{3}+1}{2}$

c $\frac{\sqrt{7}+\sqrt{5}}{2}$ d $\frac{\sqrt{10}-\sqrt{2}}{8}$

e $\frac{3\sqrt{3}-6}{-1} = -3\sqrt{3} + 6$ f $\frac{25+5\sqrt{2}}{23}$

g $\frac{10\sqrt{5}+10\sqrt{2}}{3}$ h $3\sqrt{7} + 3\sqrt{3}$

i $\frac{2\sqrt{3}-5}{-13}$ j $\frac{10+4\sqrt{2}}{17}$

k $\frac{3\sqrt{2}-2\sqrt{3}}{2}$ l $\frac{4\sqrt{3}+3\sqrt{2}}{30}$

m $\frac{27+10\sqrt{2}}{23}$ n $\frac{19+8\sqrt{3}}{13}$

o $4 - \sqrt{15}$ p $\frac{7-2\sqrt{6}}{5}$

3 a 0·894, $\frac{2\sqrt{5}}{5}$ b 1·13, $\frac{3\sqrt{7}}{7}$

c 0·612, $\frac{\sqrt{6}}{4}$ d 0·394, $\frac{\sqrt{35}}{15}$

e 2·44, $\frac{\sqrt{7}+\sqrt{5}}{2}$ f 0·551, $3 - \sqrt{6}$

g −11·9, $-6 - \sqrt{35}$ h 0·134, $\frac{19-4\sqrt{21}}{5}$

4 a $\frac{3\sqrt{2}+2\sqrt{3}}{6}$ b $\frac{2\sqrt{5}-5\sqrt{2}}{10}$

c $\frac{5\sqrt{6}+6\sqrt{5}}{30}$ d $\frac{\sqrt{10}-3\sqrt{5}}{5}$

e $\frac{13\sqrt{2}}{4}$ f $\frac{2\sqrt{3}-\sqrt{2}}{6}$

g $\frac{2\sqrt{2}+5\sqrt{10}}{10}$ h $\frac{3\sqrt{10}+2\sqrt{15}}{6}$

i $\frac{2\sqrt{10}-5\sqrt{6}}{10}$ j 4

k $\frac{2\sqrt{5}+\sqrt{3}-\sqrt{7}}{2}$ l $\frac{105+\sqrt{3}}{66}$

## Diagnostic test 5

1 a irrational b rational c irrational d rational

2 a 2·236 b 3·606 c 4·583 d 6·856

3 a $2\sqrt{5}$ b $3\sqrt{3}$ c $6\sqrt{2}$ d $10\sqrt{3}$

4 a $\sqrt{20}$ b $\sqrt{18}$ c $\sqrt{175}$ d $\sqrt{80}$

5 a $6\sqrt{3}$ b $5\sqrt{5}$ c $\sqrt{2}$ d $5\sqrt{3}$

6 a $\sqrt{30}$ b 6 c $2\sqrt{15}$ d 24

7 a $\sqrt{6}$ b 2 c 5 d $5\sqrt{2}$

8 a $11 + 5\sqrt{5}$ b $-5\sqrt{6}$ c $10 + 2\sqrt{21}$ d 22

9 a $\frac{3\sqrt{2}}{2}$ b $\sqrt{5}$ c $\frac{3+\sqrt{3}}{6}$ d $\frac{5\sqrt{2}-2}{10}$

10 a $\frac{5-\sqrt{2}}{23}$ b $3 + \sqrt{6}$ c $\frac{7-2\sqrt{10}}{3}$

d $\frac{31+5\sqrt{35}}{43}$

### ASSIGNMENT 5A

1 a 2 and 3 b 5 and 6 c 9 and 10

d 12 and 13 e 20 and 21 f 29 and 30

2 a $3\sqrt{2}$ b $3\sqrt{6}$ c $8\sqrt{2}$

d $10\sqrt{6}$ e $20\sqrt{10}$ f $45\sqrt{3}$

3 a $\sqrt{48}$ b $\sqrt{63}$ c $\sqrt{150}$

d $\sqrt{500}$

4 a $9\sqrt{5}$ b $3\sqrt{5}$ c 90

d 2

5 a $5\sqrt{5}$ b $9\sqrt{2}$ c $6\sqrt{2}$

d 0 e $14\sqrt{3}$ f $\sqrt{2}$

g $5\sqrt{2} + 16\sqrt{3}$ h $3\sqrt{7} - 12\sqrt{3} + 6\sqrt{2}$

6 a 20 b $6\sqrt{14}$ c $18\sqrt{2}$

d $24\sqrt{2}$ e $2\sqrt{5}$ f 1

g $3\sqrt{2}$ h $\frac{\sqrt{5}}{2}$

7 a $3 + 3\sqrt{5}$ b $5 + \sqrt{15} + \sqrt{10} + \sqrt{6}$

c $6 - 9\sqrt{2}$ d $9 - 2\sqrt{14}$

e $-28$ f $m + 2\sqrt{mn} + n$

g $m - n$

8 a $\frac{\sqrt{5}}{2}$ b $\frac{\sqrt{6}}{3}$

c $\frac{2\sqrt{3} + 3\sqrt{2}}{6}$ d $\frac{9\sqrt{5} - 10\sqrt{2}}{30}$

9 a $\frac{2\sqrt{5} + 2}{4}$ b $\frac{\sqrt{7} - \sqrt{2}}{5}$

c $9 - 4\sqrt{5}$ d $-5 - 2\sqrt{6}$

10 a $\frac{2\sqrt{3} - 3}{3}$ b $\frac{21 - 9\sqrt{5}}{2}$

c $\frac{84 - 36\sqrt{3} + 21\sqrt{7} - 9\sqrt{2}}{22}$

## ASSIGNMENT 5B

1 a 91 b 1392 c 18 893

2 a i 8 ii 12 iii 6 iv 1

b i 8 ii 24 iii 24 iv 8

3 a i 1, 4, 9, 16 ii 100

b 3, 7 (6 pointing up, 1 pointing down)

4 a 10 b 45 c $\frac{n}{2}(n+1)$

d i 14 ii 45

5 If $A$ = time Hawker drove, $B$ = time Saines drove
Then Hawker drove $A$ and rested for $\frac{1}{3}B$ hours and Saines drove $B$ and rested for $\frac{1}{2}A$ hours.

So $A + \frac{1}{3}B = B + \frac{1}{2}A$

i.e. $A = \frac{4}{3}B$

$\therefore$ Hawker drove for a longer time to go the same distance, so Saines was driving faster.

## ASSIGNMENT 5C

1 a 2·4 b 0·024 c 2 400 000

2 a $27x - 26$ b $6a^2 - a - 35$

c $16y^2 - 1$ d $9m^2 + 12m + 4$

e $4ab$

3 a $x = \frac{y + 5}{3}$

b $x = \frac{mn - my}{n}$ or $x = m - \frac{my}{n}$

c $x = \frac{a}{Y^2}$

4 a $y = 5x + 4$ b $y = 3x - 7$

c $y = -2x + 11$ d $y = -\frac{1}{2}x + 3$

5 a 16·2 b 55°57′

6 a 7 days and 4 hours b 8 men

7 a $x = \frac{1}{2}$ or $-\frac{2}{3}$ b $x = 8$ or $-3$

c $x = \frac{1}{2}$ d $x = 9$ or $-2$

e $\frac{-5 \pm \sqrt{17}}{2}$ f $x = \frac{3 \pm \sqrt{17}}{4}$

8 a In triangles $WYX$ and $YZW$
$WX = YZ$ (given)
$WZ = YX$ (given)
$WY = WY$ (common)
$\therefore \Delta WYX \equiv \Delta YZW$ (SSS)

b $\angle XWY = \angle ZYW$ (matching angles in congruent triangles)
$\therefore WX \parallel YZ$ (equal alternate angles)

9 a 12

b i $\frac{1}{3}$ ii $\frac{1}{4}$

10 a $\frac{1}{30}$ b $\frac{1}{6}$ c $\frac{4}{5}$

# Chapter 6

## Exercise 6:01

1 For the answers to this question, see the table on page 155.

2 a 0·5% p.a. for that month

b 2·65% p.a. c daily

d monthly e 0·007%

3 a 10 months

b i 4·75% ii 4·60% iii 4·87%

4 a 5, $3 per transaction

b i 4·25% p.a. ii 2·5% p.a. iii 4·75% p.a.

c $5000

5 a i 10 cents ii 1 cent

b $96 000

6 a 4·01% p.a.

b i $40.10 ii $3.34

c i 4 cents (only 0·01%p.a.)

ii $17.04 (4·01% p.a.)

## Prep quiz 6:02

1 $\frac{1}{12}$ 2 $\frac{5}{12}$ 3 $\frac{11}{12}$ 4 $\frac{1}{365}$

5 $\frac{7}{365}$ 6 $\frac{128}{365}$ 7 $58.50 8 $108.80

9 $15 000 10 $7.41

## Exercise 6:02

1 a $446.40 b $5850 c $7.02 d $246.33

e $157 500 f $247 500 g $140 h $12 000

i $15 174 j $1379

2 a $31.36 b $2052 c $15.70 d $3.23

e $139.20 f $362.10 g $22.77 h $210.70

3 a $66 b $273.60 c $6.05 d $113.40

e $20 160 f $0.81 g $43.61 h $6.72

i $2772.32 j $81.61

4 a $372 b $9.58 c $67.08 d $1283.33

e $1071 f $1133.99 g $1.41 h $4.72

i $107.80 j $12.60

5 a $16.32 b $161.61 c $6.75 d $727.10

e $67.40 f $267.12 g $23.75 h $1.27

i $126.55 j $21.74

## Prep quiz 6:03

**1** 0·18 **2** 0·05 **3** 0·054 **4** 0·055
**5** 0·0525 **6** 200 **7** 30 **8** 5
**9** 0·1 **10** 3

## Exercise 6:03

**1** **a** **i** $5878 **ii** $20 573
**b** $5080 **c** $1108.80 **d** $84
**e** $28 500 **f** $225
**g** **i** $2.36 **ii** $7.07 **iii** $68.34
**2** **a** 644 **b** $8{\cdot}4t$ **c** $1045.44 **d** $260.40
**3** **a** $3500 **b** $80 **c** $280 000 **d** $1000
**4** **a** $12 100 **b** $4255 **c** $32 310 **d** $15 755.83
**5** **a** $13 000 **b** $276 750 **c** $7550 **d** $30 900
**6** **a** 9% p.a. **b** $5600
**c** 6 **d** 14·25% p.a.

## Investigation 6:03

**1** **a** Money earned from typing in 2005 (Y9).
**b** Money earned from selling newspapers in 2003 (Y7).
**c** The category 'Babysitting'.
**d** The total amount earned washing cars from 2003 to 2006.
**e** The total income from these sources in 2004 (Y8).
**2** **a** B2 + C2 + D2 + E2
**b** C2 + C3 + C4 + C5 + C6 + C7 + C8
**3** $7026
**4** The sum of the rows and the sum of the columns are the same as they contain the same numbers.

## Prep quiz 6:04

**1** $11 000 **2** $12 100 **3** $13 310
**4** $14 641 **5** $16 105.10 **6** $101 000
**7** $102 010 **8** $103 030.10 **9** $104 060.40
**10** $2420

## Exercise 6:04

**1** **a** $2.04 **b** $21.76 **c** $41.82 **d** $70.70
**2** **a** $2420 **b** $7024.64 **c** $11 449
**d** $60 397.48 **e** $33 386.73
**3** **a** $A = \$501.76$, $I = \$101.76$
**b** $A = \$2741.50$; $I = \$841.50$
**c** $A = \$6356.34$, $I = \$1006.34$
**d** $A = \$125.97$, $I = \$25.97$
**e** $A = \$3321.27$, $I = \$447.27$
**f** $A = \$730\,340$, $I = \$80\,340$
**g** $A = \$122\,022.07$, $I = \$36\,322.07$
**4** $A = \$7320.50$, $I = \$2320.50$
**5** $20 776.96
**6** **a** $12.12 **b** $851.39
**c** $160.80 **d** $43.27
**7** 10% pa compound interest is better by $24.10
**8** 1166
**9** $207 360
**10** **a** $37 791.36 **b** $5.67 **c** $3.59

## Prep quiz 6:05

**1** 80% **2** 93% **3** 100% **4** 100%
**5** $280 **6** $280 **7** $744 **8** $744
**9** yes **10** yes

## Exercise 6:05

**1** **a** $19 926 **b** $213.90
**2** **a** $105 318.40 **b** $69 066.24 **c** $3066.62
**d** $7144.20 **e** 650 **f** 50 165
**3** **a** $30 681.60 **b** $12 533.76 **c** $1283.38
**d** $2655.80 **e** 190 **f** 6535
**4** **a** $800.15 **b** $1607.87
**5** **a** $32 931.50 **b** $38 587.50
**6** $1911.78
**7** 7% pa
**8** 3 years

## Prep quiz 6:06

**1** 0·09 **2** 0·065
**3** 0·1125 **4** 0·75% per month
**5** 0·67% per month **6** 0·54% per month
**7** 1·15 **8** 0·85
**9** $69 960·25 **10** $20 880·25

## Exercise 6:06

**1** **a** $7049.37 **b** $1380.97 **c** $58 361.78
**d** $657.02
**2** **a** $3049.37 **b** $520.97 **c** $41 001.78
**d** $180.52
**3** **a** $967.43 **b** $11 074.22 **c** $38.29
**4** **a** $326.68 **b** $7764.76 **c** $9946.39
**d** $300.52
**5** $552 904.08
**6** 11% p.a. compound interest; $24.54
**7** **a** $5095.57 **b** $1633.52 **c** $616.08
**8** **a** $7 788 322.36 **b** $2 748 960.63
**c** $2 921 255.08
**9** $4893.86
**10** $37 426.70
**11** **a** $1762.34 **b** $1790.85 **c** $1806.11
**d** $1816.70 **e** $1819.40
**f** $1821.94 However small we make the time period, the investment will not exceed $1822.12 in 5 years. Therefore there is a limit to which the investment will grow.
**12** **a** $38.07 **b** $1156.81 **c** $5513.57

## Investigation 6:06

**1** **a** amount = $45 044, int. = $25 044
**b** amount = $1 477 455, int. = $1 327 455
**c** amount = $11 953.50, int. = $4453.50
**2** **a** amount = $125 440
**b** amount = $126 970
**3** For **b**, interest is calculated more often. It accumulates 24 times instead of just twice as in part **a**.

## Diagnostic test 6

**1** **a** \$360 **b** \$1711.20 **c** \$102 000
**2** **a** \$279.50 **b** \$13 950 **c** \$1822.50
**3** **a** \$80 **b** \$65.10 **c** \$1023.75
**4** **a** \$3315 **b** 8% p.a. **c** \$100
**5** **a** \$12 986.07 **b** \$12 180.68 **c** \$17 892.57
**6** **a** \$960.40 **b** 512 **c** \$829.73

### ASSIGNMENT 6A

**1** **a** \$12.15 **b** \$12.20
**2** **a** \$1305 **b** \$1333.38
**c** \$2670; more interest of \$31.62
**3** **a** \$525 **b** \$3158.75 **c** \$73.33
**d** \$89.10
**4** \$2207.63
**5** \$2966.40
**6** \$82 978.56
**7** **a** **i** \$447 711.92 **ii** \$453 504.60
**iii** \$454 849.18
**b** \$7137.26
**8** **a** \$864 629.04 **b** \$550
**9** 5
**10** \$6702.71

### ASSIGNMENT 6B

**1** 59
**2** **a** 5 **b** 8 **c** 10
**3** 6 hours, because in 6 hours (half of 12) the hour hand will be pointing in the opposite direction to the present, and the minute hand will be in the same place.
**4** 4
**5** **a** **i** 1603 **ii** 7·6 **iii** 21 092 105
**b** 10th
**c** **i** Portugal **ii** Denmark

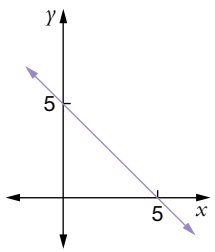
### ASSIGNMENT 6C

**1** **a** \$4.35 **b** \$5.20
**2** **a** $x = 4, y = -1$ **b** $a = -2, b = -4$
**3** **a** $x = \pm 8$ **b** $x = 9$ or $-2$
**c** $x = 4$ or $-3$ **d** $x = \frac{1}{3}$ or $-\frac{5}{2}$
**4** **a** $x = 3{\cdot}303$ or $-0{\cdot}303$
**b** $x = -0{\cdot}219$ or $-2{\cdot}281$
**c** no real solutions
**5** **a** $a = 115$ **b** $x = 50$ **c** $n = 25$
**6** $CE = BE$ (equal sides of isosceles $\Delta BEC$)
$AE = DE$ (given)
So $AC = DB$ ($CE + AE = BE + DE$)
Now in $\Delta ABC$ and $\Delta DCB$
$BC$ is common
$AC = DB$ (proved above)
$\angle ACB = \angle DBC$ (given)
$\therefore \Delta ABC \equiv \Delta DCB$ (SAS)
$\therefore AB = DC$ (corres. sides of cong. $\Delta$s)
**7** **a** 135° **b** 45°
**8** **a** $\frac{1}{2}$
**b** **i** $\frac{4}{9}$ **ii** $\frac{5}{9}$
**9** **a** $\frac{1}{8}$ **b** $\frac{3}{8}$
**10** **a** $\frac{\sqrt{6}}{4}$ **b** $\frac{4+\sqrt{10}}{2}$

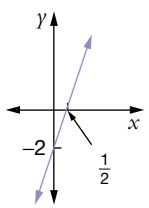
# Chapter 7

## Exercise 7:01

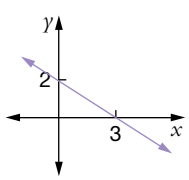
**1** **a** $\sqrt{45} = 3\sqrt{5}$ **b** $\frac{1}{2}$ **c** $(1, 3\frac{1}{2})$
**2** **a** $x$-int =5, $y$-int = 5



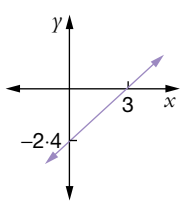

**b** $x$-int = $\frac{1}{2}$, $y$-int = $-2$




**c** $x$-int = 3, $y$-int = 2




**d** $x$-int = 3, $y$-int = $-2{\cdot}4$




**3** **a** $y = -x - 4$; $m = -1, b = -4$
**b** $y = \frac{1}{3}x - 3$; $m = \frac{1}{3}, b = -3$
**c** $y = -\frac{5}{2}x + 5$; $m = -\frac{5}{2}, b = 5$
**d** $y = \frac{4}{5}x + \frac{37}{5}$; $m = \frac{4}{5}, b = \frac{37}{5}$
**4** **a** $y = -2x + 6$ **b** $y = \frac{1}{2}x - \frac{3}{2}$
**c** $y = -\frac{2}{3}x + 6$ **d** $y = -\frac{4}{5}x + \frac{12}{5}$

e $y = \frac{4}{3}x - 4$

5 a $2x + y - 6 = 0$ b $x - 2y - 3 = 0$
c $2x + 3y - 18 = 0$ d $4x + 5y - 12 = 0$
e $4x - 3y - 12 = 0$

6 a $AB = \sqrt{61}$, $AC = \sqrt{61}$
b $AC$ is $y = \frac{6}{5}x + \frac{12}{5}$, $AB$ is $y = -\frac{6}{5}x + \frac{48}{5}$
c $D$ is $(0, \frac{12}{5})$, $E$ is $(0, \frac{48}{5})$

7 a $m = \frac{3}{5}$ b $y = \frac{3}{5}x - \frac{18}{5}$
c i $(6, 0)$ ii $(10, 2{\cdot}4)$ iii $(0, -3{\cdot}6)$

## Prep quiz 7:02

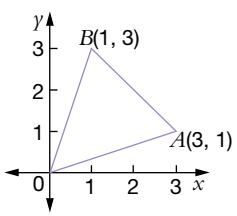
1 $\frac{1}{2}$ 2 yes 3 $\frac{1}{2}$
4 no 5 $m_1 = -\frac{1}{3}$ 6 $m_2 = 3$
7 $m_1 m_2 = -1$ 8 $m_3 = -2$ 9 $m_4 = \frac{1}{2}$
10 $m_3 m_4 = -1$

## Exercise 7:02

1 a yes b no c yes d yes
e yes f yes g yes h yes
2 a yes b yes c yes d yes
e no f yes g yes h yes
3 a $2x - y + 6 = 0$ and $y = 2x - 3$
b $x + y = 3$ and $y = -x + 8$
c $AB$ and $CD$
4 a $x + 2y = 4$, $y = -0{\cdot}5x + 5$
b $y = -1\frac{1}{2}x + 2$, $y = \frac{2}{3}x$
c $m$ of $AB$ is $-1$; $m$ of $BC$ is 1
$\therefore AB \perp BC$ (since $m_1 m_2 = -1$)
5 a $y = 5x + 3$ b $y = 3x - 1$
c $y = x + 3$ d $y = -x + 10$
6 a $y = 3x + 5$ b $y = x + 3$
c $y = 2x$ d $y = \frac{1}{2}x + 1{\cdot}5$
7 a $y = 5x - 7$ b $y = -3x + 3$
c $y = 3x + 5$ d $y = -3$
e $y = -3$
8 a $x + 2y - 8 = 0$ b $x - 3y - 1 = 0$
c $x + 3y - 15 = 0$ d $y = -3$
e $x = 3$
9 a $2x - 3y - 18 = 0$ b $3x - 4y + 11 = 0$
c $x = \frac{1}{2}$ d $a = \frac{29}{3}$
10 $A$ $(-1{\cdot}48, 1{\cdot}2)$; $B$ $(2, 1{\cdot}2)$; $C$ $(0, -2{\cdot}5)$; $D$ $(2, -1{\cdot}7)$; $E$ $(-1, 0)$

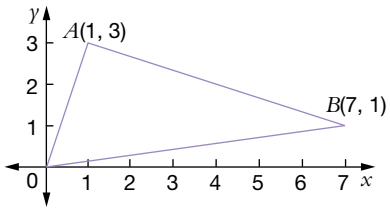
## Exercise 7:03

1 a



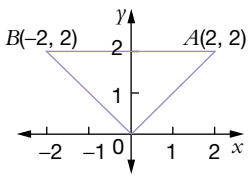

$OB = \sqrt{(3-0)^2 + (1-0)^2}$
$= \sqrt{10}$
$OA = \sqrt{(1-0)^2 + (3-0)^2}$
$= \sqrt{10}$
$\therefore \Delta OAB$ is isosceles (2 equal sides).

b



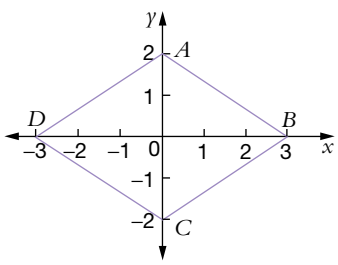

Slope of $OA = \frac{3-0}{1-0}$
$= 3$
Slope of $AB = \frac{3-1}{1-7}$
$= -\frac{1}{3}$
Now, slope of $OA \times$ slope of $AB = -1$
$\therefore OA \perp AB$
$\therefore \Delta OAB$ is right-angled at $A$.

c



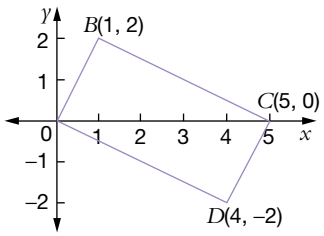

$OA = \sqrt{(2-0)^2 + (2-0)^2}$
$= \sqrt{8}$
$OB = \sqrt{(2-0)^2 + (-2-0)^2}$
$= \sqrt{8}$
$AB = 4$
Now, $OA = OB$
$\therefore \Delta OAB$ is isosceles.
Also, $4^2 = \left(\sqrt{8}\right)^2 + \left(\sqrt{8}\right)^2$
$\therefore AB^2 = OA^2 + OB^2$
$\therefore \Delta OAB$ is right-angled at $O$ (by Pythag. th'm)

2 a




$AO = OC = 2$ and
$OD = OB = 3$
Also, $AC \perp BD$
$\therefore$ Diagonals bisect each other at right angles.
$\therefore ABCD$ is a rhombus.
Note: This question could be done in other ways.

**b**




$OC = 5$

$BD = \sqrt{(2 - -2)^2 + (1 - 4)^2}$

$= \sqrt{4^2 + (-3)^2}$

$= 5$

$\therefore\ OC = BD$

Midpoint of $BD = \left(\frac{4+1}{2}, \frac{-2+2}{2}\right)$

$= (2\frac{1}{2}, 0)$

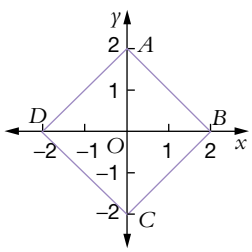
Midpoint of $OC = (2\frac{1}{2}, 0)$

$\therefore$ $BD$ and $OC$ bisect each other

$\therefore$ $OBCD$ is a rectangle

(equal diagonals that bisect each other)

**c**




$AB = \sqrt{2^2 + 2^2}$ (Pythag. th'm)

$= \sqrt{8}$

Similarly, $BC = DC = AD = \sqrt{8}$

$\therefore$ $ABCD$ is a parallelogram

(both pairs of opp. sides equal)

Now, slope of $AB = -1$

$\therefore$ slope of $AD = 1$

$\therefore$ $AB \perp AD$ (product of slopes is $-1$)

$\therefore$ $\angle DAB = 90°$

$\therefore$ $ABCD$ is a square (par'm with one right $\angle$)

**3 a** $(3, 2)$

**b** $OB = AC = \sqrt{13}$; The diagonals of a rectangle are equal in length.

**c** The midpoint of both diagonals is $(1\frac{1}{2}, 1)$. The diagonals bisect each other.

**4 a** Midpoint of $AB$ is $(3, 2)$; midpoint of $AC$ is $(2, -1)$.

**b** 3

**c** 3

**d** The line joining the midpoints of two sides of a triangle is parallel to the third side.

**5** Slope of $BD$ is 1; slope of $AC$ is $-1$; the diagonals are perpendicular.

**6 a** $(3, 2)$

**b** $\sqrt{13}$

**c** $\sqrt{13}$

**d** $E$ is equidistant from $A$, $B$ and $O$.

**7 a** $AB = BC = \sqrt{10}$ (by distance formula).
Hence $\Delta ABC$ is isosceles as it has two equal sides.

**b** $(0, 2)$

**c** 1

**d** slope of $AC = -1$; slope of $EB = 1$; $EB \perp AC$ since product of their slopes is $-1$.

**e** Find the lengths $AC$ and $BE$ and then evaluate $\frac{1}{2}AC \times BE$.

**8 a** Midpoint of $OA$ is $(\frac{1}{2}, 1)$; midpoint of $AB$ is $(2, 1)$.

**b** $1\frac{1}{2}$ units.

**c** $OB$ is 3 units in length. Hence $\frac{1}{2}OB = 1\frac{1}{2}$.

**9 a** $DT$ is $3x + 2y = 13$. $ET$ is $y = x - 4$

**b** $(4\frac{1}{5}, \frac{1}{5})$

**c** Perp. bisector of $BC$ is $y = -\frac{1}{4}x + \frac{5}{4}$
(or $x + 4y - 5 = 0$)
Now the coordinates of $T$ satisfy $BC$.
Hence $T$ lies on the perp. bisector.

**d** $\frac{\sqrt{442}}{5} \approx 4{\cdot}2$ units

**10 a** Median from $A$ is $y = -2x$; median from $B$ is $2x - 5y + 4 = 0$; median from $C$ is $x - y + 1 = 0$

**b** Point of intersection (of any two medians) is $(-\frac{1}{3}, \frac{2}{3})$.

**c** Substitute $(-\frac{1}{3}, \frac{2}{3})$ into the equation of the median not used in **b** and show that the coordinates satisfy the equation. This shows that the third median also passes through $(-\frac{1}{3}, \frac{2}{3})$.

## Exercise 7:04

**1 a**

| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

**b**

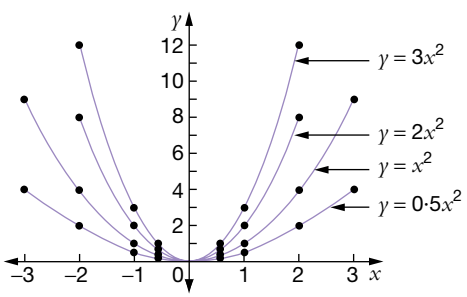
| $x$ | –2 | –1 | –0·5 | 0 | 0.5 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| $y$ | 8 | 2 | 0·5 | 0 | 0.5 | 2 | 8 |

**c**

| $x$ | –2 | –1 | –0·5 | 0 | 0·5 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| $y$ | 12 | 3 | 0·75 | 0 | 0·75 | 3 | 12 |

**d**

| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 4·5 | 2 | 0·5 | 0 | 0·5 | 2 | 4·5 |




The value of $a$ changes the curvature of the parabola.

**2 a** B **b** D **c** C **d** A

**3 a**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

**b**

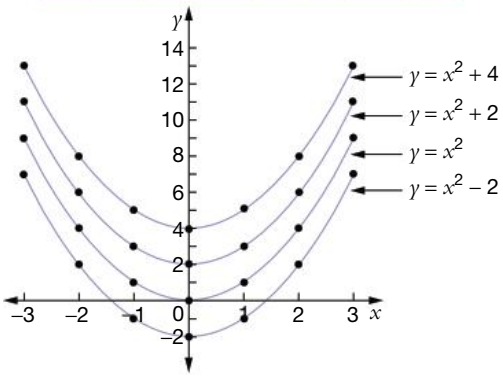
| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 11 | 6 | 3 | 2 | 3 | 6 | 11 |

**c**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 13 | 8 | 5 | 4 | 5 | 8 | 13 |

**d**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 7 | 2 | −1 | −2 | −1 | 2 | 7 |

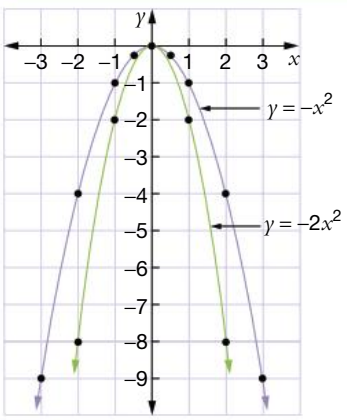



$y = x^2 + 2$ is the same curve as $y = x^2$ but raised two units. The value of $c$ determines where the vertex of the parabola cuts the $y$-axis.

**4 a** $y = -x^2$

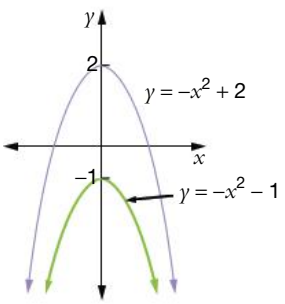
| $x$ | −3 | −2 | −1 | −0·5 | 0 | 0·5 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | −9 | −4 | −1 | −0·25 | 0 | −0·25 | −1 | −4 | −9 |

**b**




**c** Parabola is 'upside down' or 'concave down'.

**5**

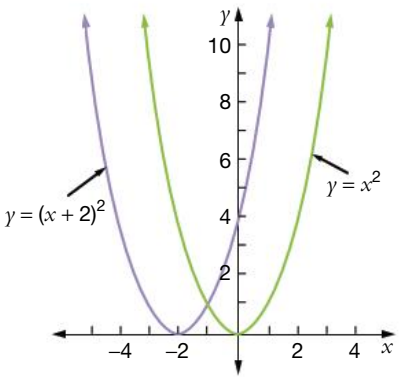


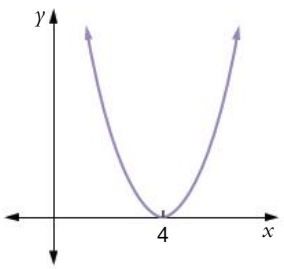

**6 a** E **b** C **c** B **d** A **e** D

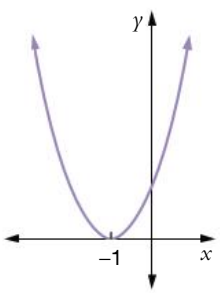
**7 a** The graphs are congruent. The graph of $y = (x - 2)^2$ can be produced by translating the graph of $y = x^2$ two units to the right.

**b** The parabola $y = (x + 2)^2$ is congruent to $y = x^2$. It can be produced by translating $y = x^2$ two units to the left.

**c**



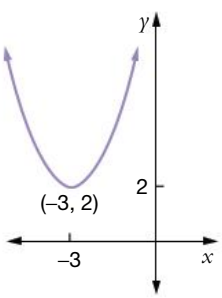

**d** i If $a$ is positive, the graph of $y = (x + a)^2$ is the graph of $y = x^2$ translated $a$ units to the left.

ii If $a$ is negative, the graph of $y = (x + a)^2$ is the graph of $y = x^2$ translated $a$ units to the right.

**8 a** It is the same graph translated up two units.

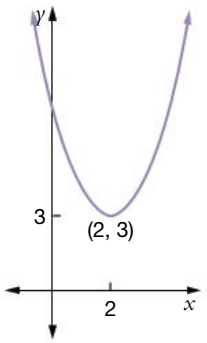
**b** It could be obtained by translating the graph of $y = (x - 3)^2$ down two units.

**c**

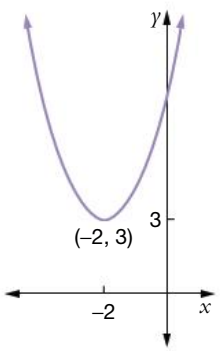


**d** The parabola $y = x^2$ has its vertex at $(0, 0)$. If the parabola is translated $p$ units sideways and $q$ units vertically so that its vertex is at the point $(p, q)$, then its equation is $y = (x - p)^2 + q$.

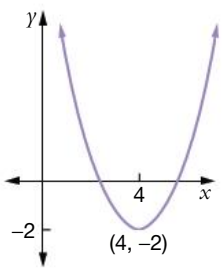
**e** i



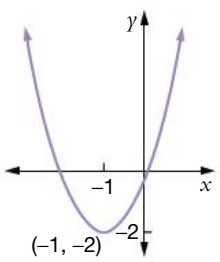

ii



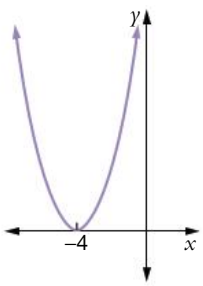

iii



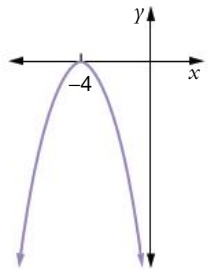

iv



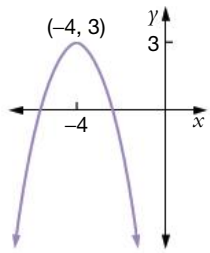

**9** **a**




**b**




**c**




**10** **a** $y = x^2 + 2$ **b** $y = x^2 - 2$
**c** $y = (x - 2)^2$ **d** $y = (x + 2)^2$
**e** $y = -x^2 + 4$ **f** $y = -x^2 - 2$
**g** $y = -x^2 - 2$ **h** $y = -(x - 2)^2$
**i** $y = (x + 2)^2 + 2$ **j** $y = -(x + 3)^2 - 2$

**11** **a** C **b** D **c** B **d** A
**e** F **f** E

**12** **a** E **b** D **c** A **d** B
**e** C **f** F

**13** **a** $y = -x^2 + 4$ **b** $y = (x + 3)^2$
**c** $y = -(x - 3)^2 + 3$ **d** $y = (x - 2)^2 - 3$
**e** $y = (x - 2)^2 + 1$ **f** $y = (x + 2)^2 - 1$

## Exercise 7:05

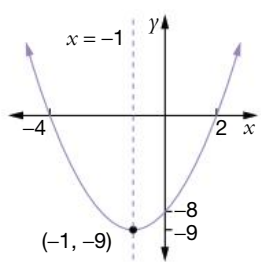
**1** **a** **i** 0 **ii** 0, 4 **iii** $x = 2$ **iv** $(2, -4)$
**b** **i** 3 **ii** $-3, 1$ **iii** $x = -1$ **iv** $(-1, 4)$
**c** **i** $-6$ **ii** $-1, 3$ **iii** $x = 1$ **iv** $(1, -8)$

**2** **a** 5 **b** $-8$ **c** $-6$

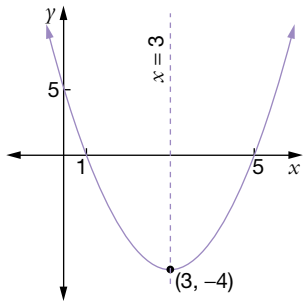
**3** **a** $-2, 4$ **b** $-4, \frac{2}{3}$ **c** $-\frac{7}{4}, 3$

**4** **a** $x = 4, (4, -1)$ **b** $x = -2, (-2, -48)$
**c** $x = -1, (-1, -\frac{9}{2})$ **d** $x = 3, (3, -2)$
**e** $x = 1\frac{1}{2}, (1\frac{1}{2}, 7\frac{1}{4})$ **f** $x = -1\frac{1}{2}, (-1\frac{1}{2}, 6\frac{1}{4})$

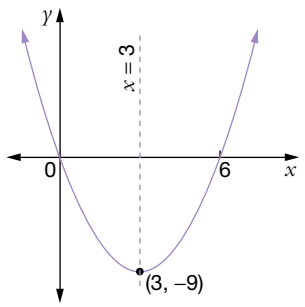
**5** **a** $(0, -8)$ **b** $(-4, 0), (2, 0)$
**c** $x = -1$ **d** $(-1, -9)$
**e**



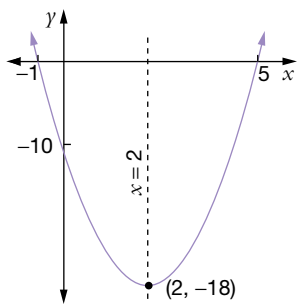
6 a



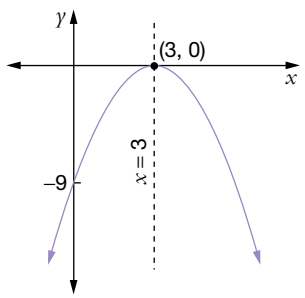

b



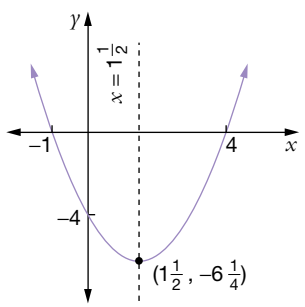

c



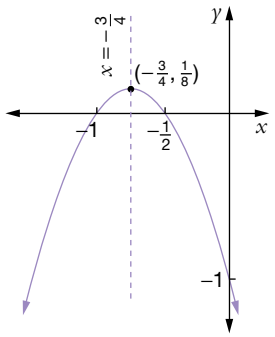

d




e

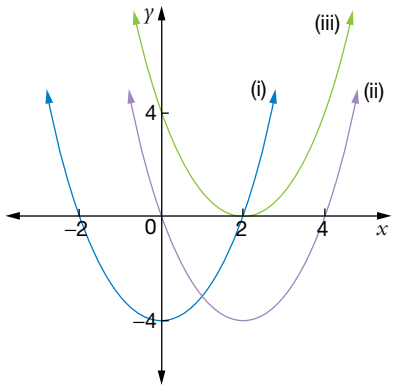



f

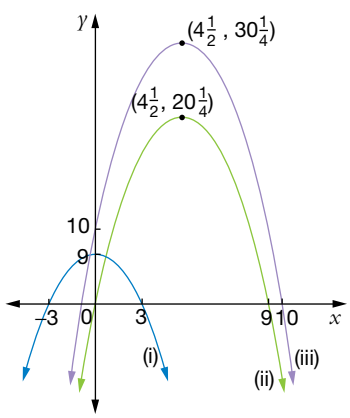



7 a D b B c F
d A e E f C

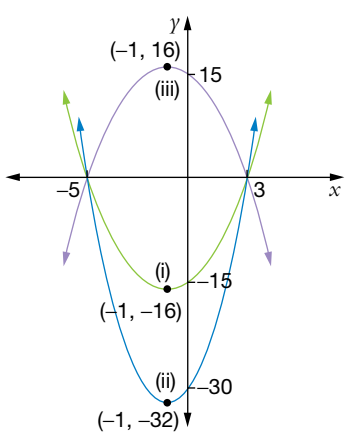
8 a



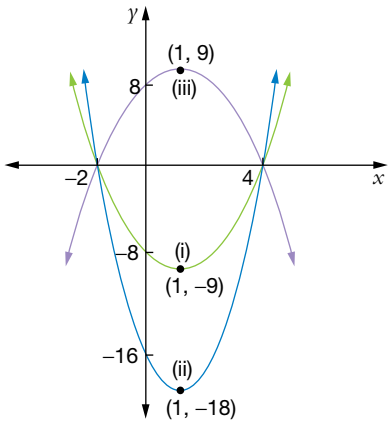

b



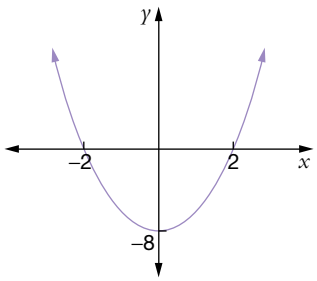

c



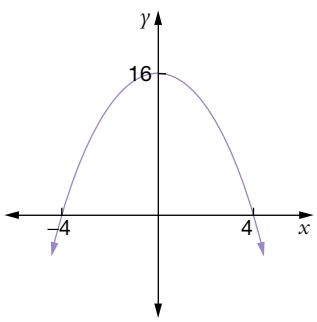
d



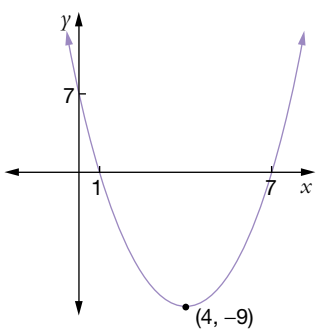

9 a



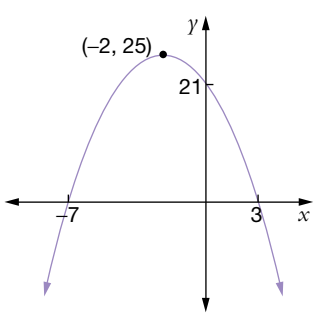

b



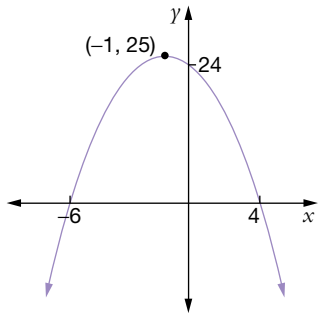

c



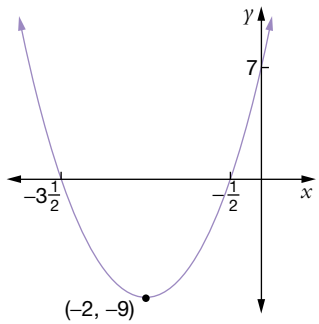

d



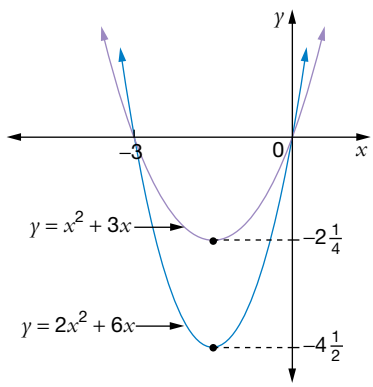

e




f

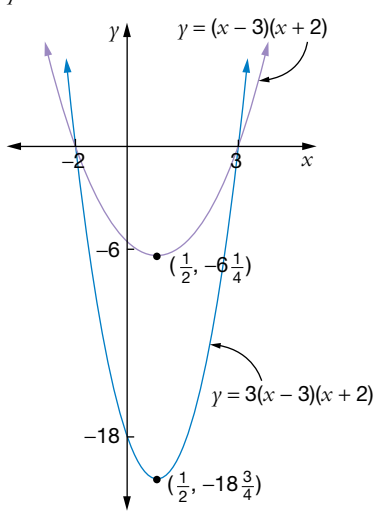



10 a




Same $x$- and $y$-intercepts and same concavity but they have different vertices. The $y$-coordinates of the vertices differ by a factor of 2, i.e. one $y$-coordinate is twice the other.

b



Same $x$-intercepts and concavity but the $y$-values of the vertices differ by a factor of 3, i.e. one is 3 times the other.

**11** **a** Substituting $(0, -5)$ into $y = ax^2 + bx + c$ gives $c = -5$

**b** $\frac{-b}{2a} = -1$

$\therefore\ b = 2a$

$\therefore\ y = ax^2 + 2ax - 5$

**c** $y = ax^2 + 2ax - 5$

$4 = a(1)^2 + 2a(1) - 5$

$4 = 3a - 5$

$a = 3$

**d** $y = 3x^2 + 6x - 5$

**12** **a** $y = \frac{1}{2}x^2 + 2$ **b** $y = -2x^2 + 8x - 8$

**c** $y = x^2 + 6x + 11$ **d** $y = \frac{1}{2}x^2 - x - 4$

**e** $y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$ **f** $y = 4x^2 - 16x + 8$

## Investigation 7:05

**1** $y = x^2 + 3x + 2$ **2** $y = x^2 - 10x + 16$

**3** $y = 2x^2 + x - 4$ **4** $y = 2x^2 - 3x + 5$

## Exercise 7:06

**1** **a** $(0, 1)$ **b** no

**c** $y = 5^x$ increases the fastest; $y = 2^x$ increases the slowest.

**d** for $x < 0$, it would be below the lowest curve ($y = 5^x$); for $x > 0$ it would be the steepest curve, (above $y = 5^x$)

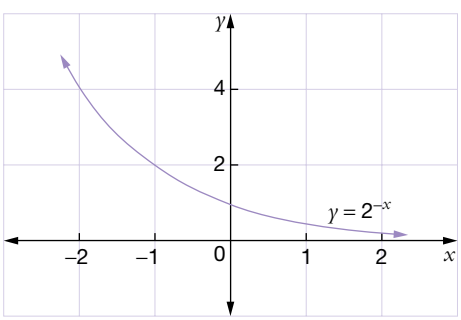
**e** for $x < 0$, it would be above the highest curve ($y = 2^x$); for $x > 0$ it would be the lowest curve, (below $y = 2^x$)

**2** **a** A is $y = 3^x$; B is $y = 6^x$; C is $y = 2^x$

**b** A is $y = 8^x$; B is $y = 4^x$; C is $y = 1{\cdot}9^x$

**3** **a**

| $x$ | −2 | −1.5 | −1 | −0.5 | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 4·0 | 2·8 | 2·0 | 1·4 | 1·0 | 0·71 | 0·50 | 0·35 | 0·25 |




**b** $y = 2^{-x}$ is a reflection of $y = 2^x$ in the $y$-axis.

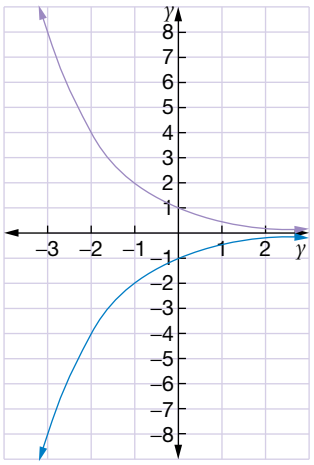
**4** **a** A **b** C **c** D **d** B

**5** **a** $y = 4^{-x}$ **b** $y = 6^{-x}$ **c** $y = 8^x$

**d** $y = 1{\cdot}2^{-x}$ **e** $y = 3{\cdot}6^x$

**6** **a** yes **b** yes

**c** yes




**7** **a** $y = -3^x$ **b** $y = -4^{-x}$ **c** $y = -1{\cdot}8^{-x}$

**d** $y = 1{\cdot}2^x$ **e** $y = 6^{-x}$

**8** **a** They are reflections of each other in the $y$-axis

**b** They are reflections of each other in the $x$-axis

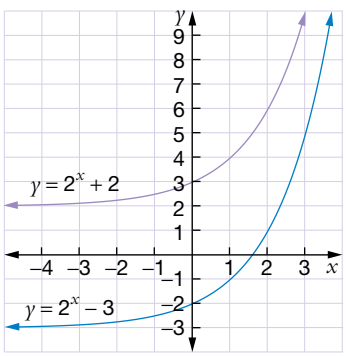
**9** **a** **i** vertical translation of 1 unit up

**ii** vertical translation of 1 unit down

**b** **i** $y = 2^x + 3$

**ii** $y = 2^x - 4$

**c**




**10** **a** $y = -3^{-x}$ **b** $y = 3^{-x}$ **c** $y = 3^x + 1$

**d** $y = 3^x - 1$ **e** $y = 3^{-x} - 1$ **f** $y = -3^x + 1$

**11** Curve A

**a** decreasing **b** concave down

**c** **i** they become very small (approach negative infinity)

**ii** they get closer and closer to 3

**d** 2

Curve B

**a** increasing **b** concave down

**c** **i** they get closer and closer to 5

**ii** they become very small (approach negative infinity)

**d** 4

Curve C

**a** increasing **b** concave up

**c** **i** they become very large (approach infinity)

**ii** they get closer and closer to 1

**d** 2

Curve D

**a** decreasing **b** concave up

**c** **i** they get closer and closer to −2

**ii** they become very large (approach infinity)

**d** −1

**12** **a** D, F, K **b** A, E, G, H, L

**c** B, C, D, F, I, J, K **d** A, E, G, L

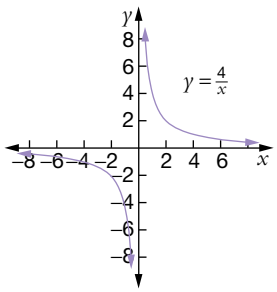
**e** H

**f** **i** A, B, D, G **ii** C, H, K

## Exercise 7:07

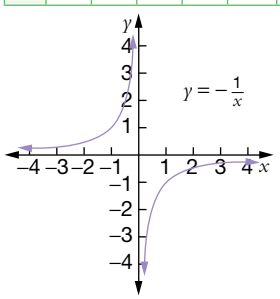
**1** **a**

| $x$ | −8 | −4 | −2 | −1 | −0·5 | 0·5 | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | −0·5 | −1 | −2 | −4 | −8 | 8 | 4 | 2 | 1 | 0·5 |




**b**

| $x$ | −4 | −2 | −1 | −0·5 | −0·25 | 0·25 | 0·5 | 1 | 2 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0·25 | 0·5 | 1 | 2 | 4 | −4 | −2 | −1 | −0·5 | −0·25 |




**2** **a** C **b** F **c** D **d** B

**e** E **f** A

**3** **a** If $k$ is positive the graph is in the 1st and 3rd quadrants. If $k$ is negative the graph is in the 2nd and 4th quadrants.

**b** As $k$ increases the hyperbola moves away from the origin.

**4** **a** Yes **b** −18 **c** 20

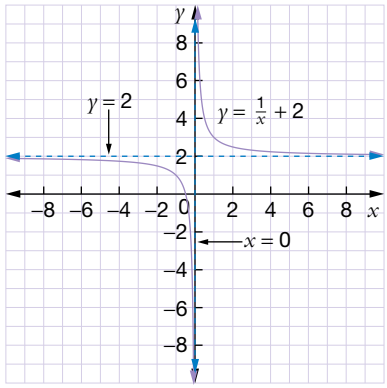
**5** **a** $(6, -2)$; $y = -\frac{12}{x}$ **b** $(4, 5)$; $y = \frac{20}{x}$

**6** **a** $y = \frac{1}{x} + 2$

| $x$ | −8 | −4 | −2 | −1 | −0·5 | 0·5 | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1·9 | 1·8 | 1·5 | 1 | 0 | 4 | 3 | 2·5 | 2·3 | 2·1 |

Asymptotes are $x = 0$ and $y = 2$

$x$-intercept $= -\frac{1}{2}$

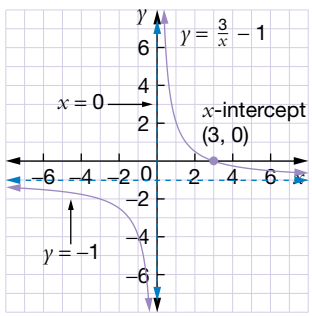



**b** $y = \frac{3}{x} - 1$

| $x$ | −6 | −3 | −2 | −1 | −0·5 | 0·5 | 1 | 2 | 3 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | −1·5 | −2 | −2·5 | −4 | −7 | 5 | 2 | 0·5 | 0 | −0·5 |

Asymptotes are $x = 0$ and $y = -1$

$x$-intercept $= 3$

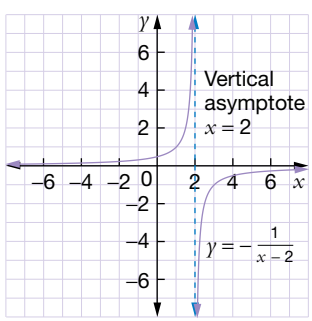



**c** $y = \frac{-1}{x - 2}$

| $x$ | −4 | −2 | 0 | 1 | 1·5 | 2·5 | 3 | 4 | 6 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0·2 | 0.3 | 0·5 | 1 | 2 | −2 | −1 | −0·5 | −0.3 | −0·2 |

Asymptotes are $x = 2$ and $y = 0$

$y$-intercept $= \frac{1}{2}$




**7** **A** $y = \frac{3}{x} - 2$ **B** $y = \frac{3}{x} + 1$

**C** $y = \frac{3}{x + 2}$ **D** $y = \frac{3}{x - 3}$

**8** **a** $y = \frac{1}{x} + 1$ **b** $y = \frac{1}{x} - 1$

**c** $y = \frac{1}{x-1}$ **d** $y = \frac{1}{x+1}$

**9** **a** move 3 units down

**b** move 2 units up

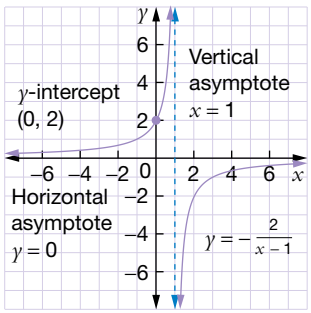
**c** move 3 units to the right

**d** move 2 units to the left

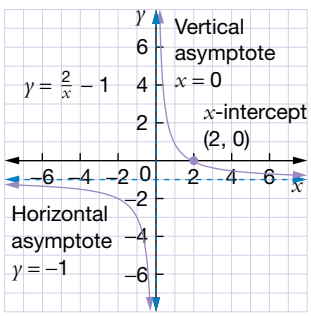
**10** **a** B **b** C **c** D **d** E

**e** F **f** A

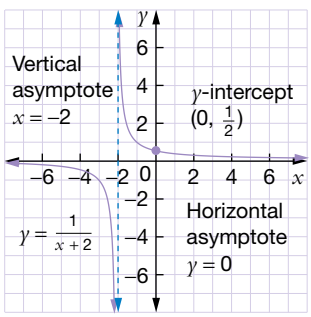
**11** **a**




**b**




**c**




**12** **a** $y = \frac{-2}{x} + 1$ **b** $y = \frac{1}{x} - 2$ **c** $y = \frac{2}{x-1}$

## Exercise 7:08

**1** **a** $(x-1)^2 + (y-1)^2 = 49$

**b** $(x-5)^2 + y^2 = 4$

**c** $(x+3)^2 + (y+5)^2 = 16$

**d** $(x-2)^2 + (y+5)^2 = 1$

**e** $x^2 + (y-2)^2 = \frac{1}{4}$

**f** $x^2 + y^2 = 9$

**2** **a** (2, 3); 8 **b** (−4, 1); 2 **c** (−3, −3); 3

**d** (6, 5); 10 **e** (0, −5); 4 **f** (3, 0); 1

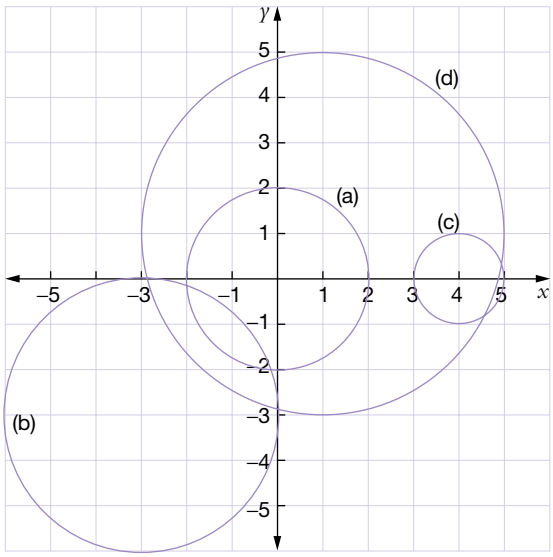
**g** (0, 0); 9 **h** (0, 0); 7 **i** (0, 0); $\sqrt{11}$

**j** (7, 8); $\sqrt{2}$

**3** **a** (5, −4); 3 **b** (−4, 7); 10 **c** (9, 10); 11

**d** $(4\frac{1}{2}, 0)$; $\sqrt{7}$

**4**




**5** **a** $x$-intercepts: $\pm\sqrt{5}$

$y$-intercepts: −1, 5

**b** $x$-intercepts: $2 \pm \sqrt{15}$

$y$-intercepts: $1 \pm \sqrt{12}$

**6** **a** $(x-3)^2 + (y-4)^2 = 25$

**b** $4x^2 + 4y^2 = 8$

**7** **a** $(x-2)^2 + y^2 = 4$

**b** $(x+2)^2 + y^2 = 4$

**c** $x^2 + (y-2)^2 = 4$

**d** $x^2 + (y+2)^2 = 4$

**8** $(x-4)^2 + (y+3)^2 = 4$

## Exercise 7:09

**1** **a** i B ii C iii A

**b** $y = 3x^3$

**c** $y = \frac{1}{2}x^3$

**d** By comparing the coefficients of $x^3$ you can tell which graph is the steepest. The larger the coefficient, the steeper the graph.

**2** **a** $y = 3x^3$ **b** $y = x^3$ **c** $y = 3x^3$

**3** **a** They are reflections of each other in the $x$-axis.

**b** The size of $a$ affects the steepness of the graph while the sign of $a$ indicates whether the graph is increasing or decreasing as you move from left to right.

**4** **a** increasing **b** decreasing **c** increasing

**d** increasing **e** decreasing **f** decreasing

**5** **a**

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $x^3$ | −8 | −1 | 0 | 1 | 8 |
| $x^3 + 2$ | −6 | 1 | 2 | 3 | 10 |
| $x^3 - 2$ | −10 | −3 | −2 | −1 | 6 |

**b** A is $y = x^3 + 2$
B is $y = x^3 - 2$

**c** $y = x^3 + 2$ is the curve $y = x^3$ translated up two units.

**d** $y = x^3 - 2$ is the curve $y = x^3$ translated down two units

**6** **a** Move the graph of $y = \frac{1}{2}x^3$ up one unit.

**b** Move the graph of $y = \frac{1}{2}x^3$ down one unit.

**c** Move the graph of $y = \frac{1}{2}x^3$ up two units.

**d** Move the graph of $y = \frac{1}{2}x^3$ down two units.

**e** Reflect the graph of $y = \frac{1}{2}x^3$ in the $x$-axis.

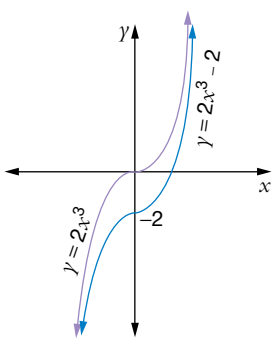
**f** Reflect the graph in the $x$-axis and then move it up one unit.

**7** **a** A $y = x^3 + 10$ B $y = x^3 - 10$

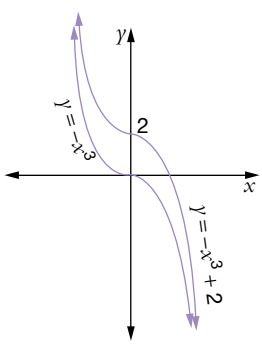
**b** A $y = -\frac{x^3}{3} + 6$ B $y = -\frac{x^3}{3} - 5$

**c** A $y = -\frac{1}{10}x^3 + 10$ B $y = -\frac{1}{10}x^3 - 15$

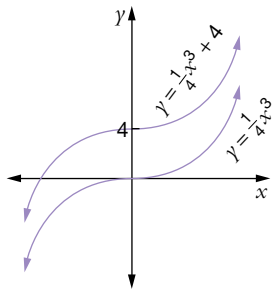
**8** **a**



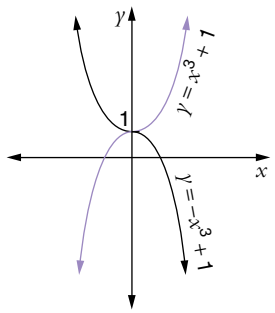

**b**



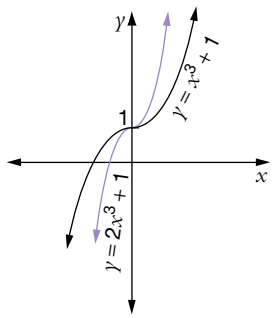

**c**




**9** **a**




**b**




**10** **a** B, H **b** D, E **c** C, F, I
**d** A, F, G **e** G **f** C, E

**11** **a** $y = \frac{5}{4}x^3 + 10$ **b** $y = -2x^3 - 4$

**12** The size of $a$ affects the steepness of the graph and the sign of $a$ indicates whether the graph is increasing or decreasing as it moves from left to right. The value and sign of $d$ affects how far and in which direction the graph is shifted vertically.

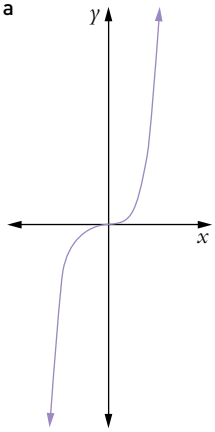
## Exercise 7:10

**1** **a** B **b** A **c** C **d** D **e** B
**f** A **g** C **h** A **i** D

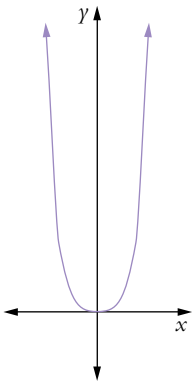
**2** **a** i $y = \frac{1}{2}x^4 - 4$ ii $y = \frac{1}{2}x^4 + 2$
**b** i $y = 2x^7 - 5$ ii $y = 2x^7 + 3$
**c** i $y = -5x^6 - 4$ ii $y = -5x^6 - 8$

**3** **a** i B ii A iii C
**b** i C ii B iii A
**c** i C ii B iii A

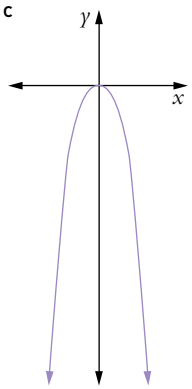
4 a



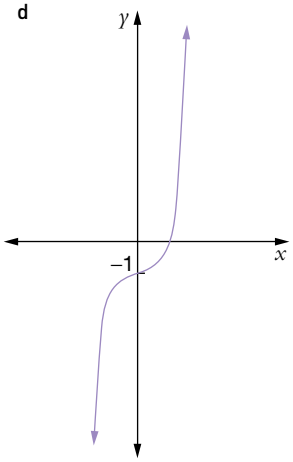

b



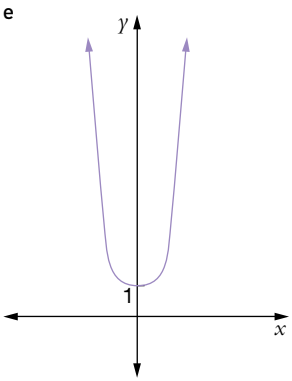

c



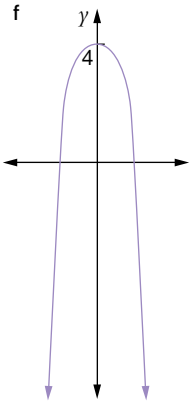

d



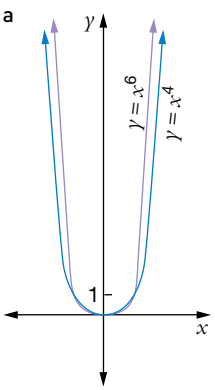

e



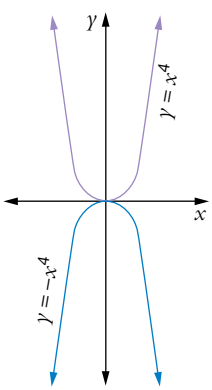

f



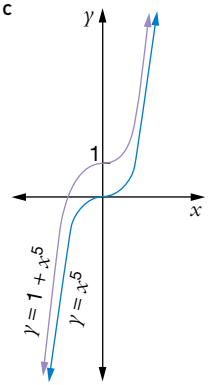

5 a



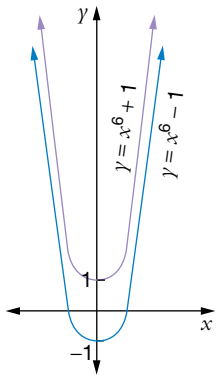

b



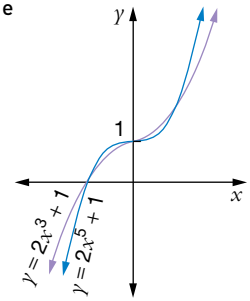

c



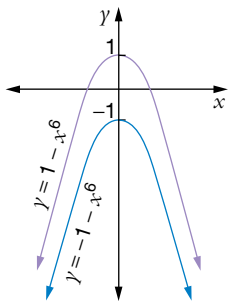

d




e




f

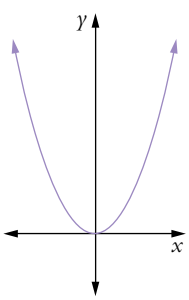



6 a The 4 tells us that the curve is shaped like a parabola, while the −2 tells us that it is upside down. The 1 tells us that the curve has been produced by moving the curve $y = -2x^4$ up one unit.

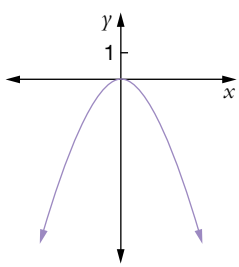
b The curve is similar in shape to $y = x^3$. It is steeper than this curve and it crosses the $y$-axis at $(0, -3)$.

c The curves are reflections of each other in the $x$-axis.

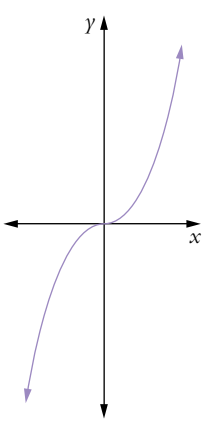
7 a



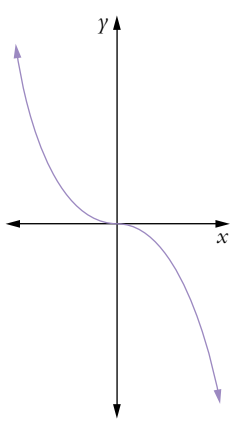

b

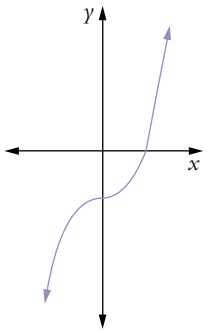


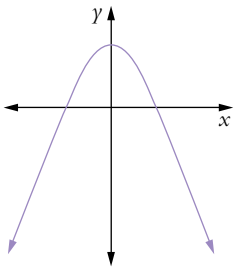

c

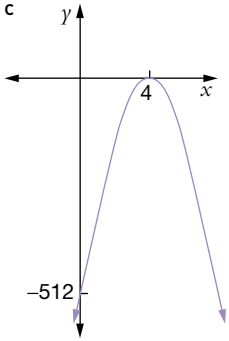


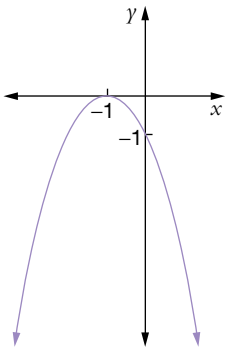

d



8 a 


b 


c 


d 

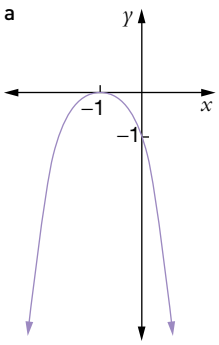

9 a $n$ is odd, $a$ is negative and $d$ is negative.
b $n$ is even, $a$ is negative and $d$ is positive.
c $n$ is even, $a$ is positive and $d$ is negative.

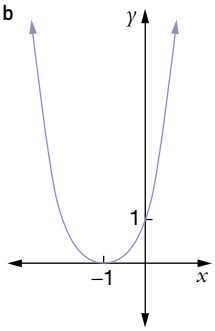
## Exercise 7:11

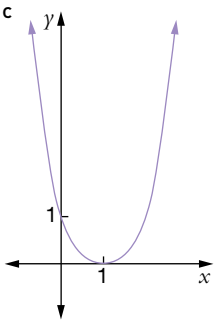
1 a A is $y = (x - 3)^5$, B is $y = (x + 4)^5$
b D

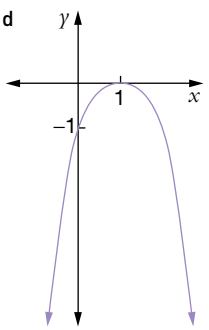
2 a $y = (x + 4)^4$ b $y = (x - 4)^4$
c $y = (x - 1)^4$ d $y = (x + \frac{1}{2})^4$

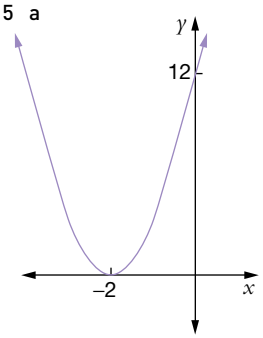
3 a 1 unit to the right b 3 units to the left
c $\frac{1}{2}$ a unit to the right

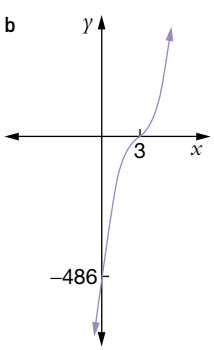
4 a 

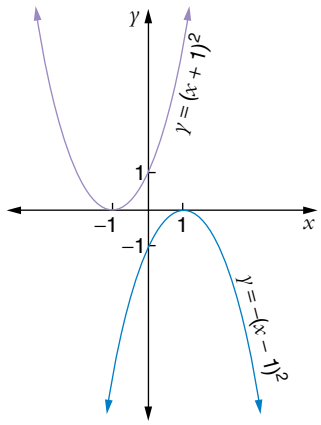

b 

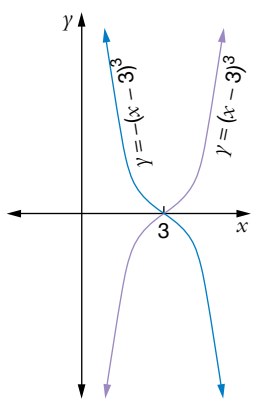

c 

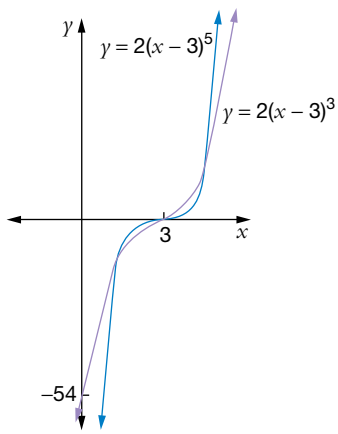

d 


5 a 

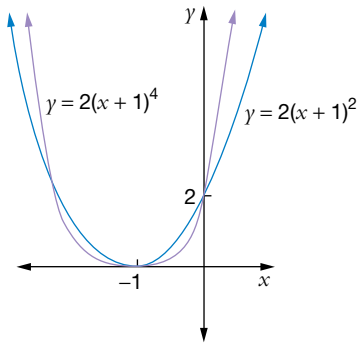

b 


6 a 


b 


c 

**d**




## Exercise 7:12

**1 a**

| $x$ | –5 | –4 | –3 | –2 | –1·5 | –1 | 0 |
|---|---|---|---|---|---|---|---|
| $y$ | – | 0 | + | 0 | – | 0 | + |

**b**

| $x$ | –3 | –2 | 0 | 2 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $y$ | + | 0 | – | 0 | + | 0 | – |

**c**

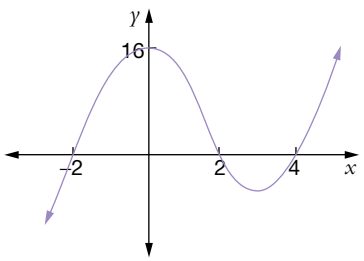
| $x$ | 0 | 1 | 2 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | – | 0 | – | 0 | + |

**d**

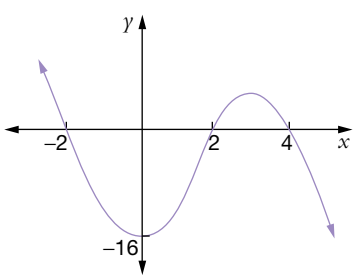
| $x$ | –2 | –1 | 0 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | + | 0 | – | 0 | – |

**2 a** 1, 2, 4 **b** 1, –2, 4 **c** 2, –5 **d** 0, –4, 2

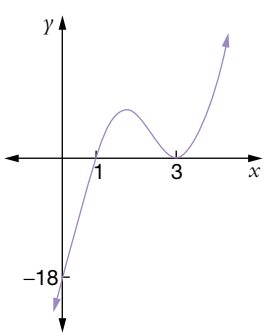
**3 a**



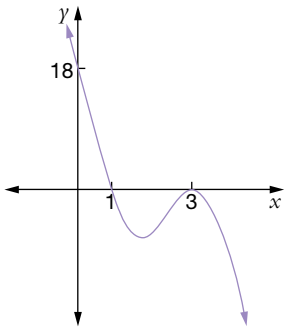

**b**




**c**




**d**




**4 a** 16; –16; –18; 18

**b** Let $x = 0$ to find the $y$-intercept. Ask yourself whether the graph could have this $y$-intercept.

**5 a** $y = -(x + 1)(x - 1)(x - 3)$

**b** $y = (x + 2)(x - 1)(x - 3)$

**c** $y = -(x + 4)(x + 2)(x - 2)$

**d** $y = -(x + 2)(x - 2)^2$

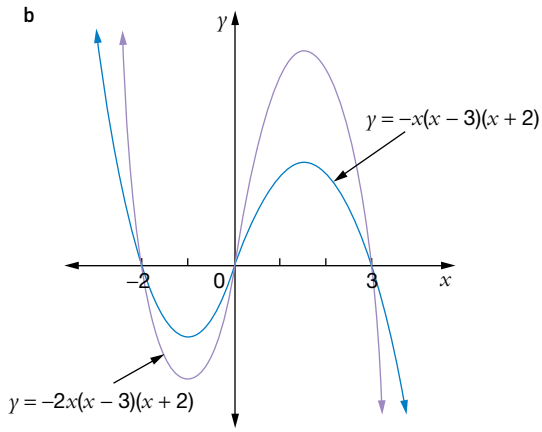
**e** $y = (x + 2)(x - 2)^2$

**f** $y = (x - 1)(x - 3)^2$

**6** You would need to know another point that the curve passes through.

**7 a** i B ii C iii A

**b**




**8 a** $y = (x - 1)(x + 1)(x - 4)$

**b** $y = (x + 4)(x + 2)(x - 1)$

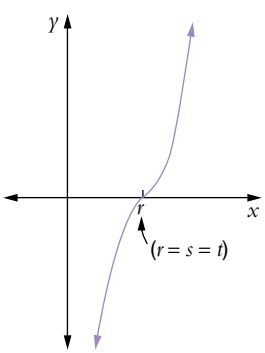
**c** $y = -x(x + 2)(x - 3)$

**d** $y = -x(x + 3)(x - 2)$

**9 a** $y = 2(x + 1)(x - 1)(x - 4)$

**b** $y = -2(x + 2)(x - 2)^2$
or $y = -\frac{2}{5}(x - 2)(x + 2)^2$

**10**



## Exercise 7:13

**1** **a** $(-1, 1), (2, 4)$ **b** $(-2, 4), (1, 1)$
**c** $(-6, -1), (2, 3)$ **d** $(-1, -3), (1, 3)$
**2** **a** $(-2, 4), (3, 9)$ **b** $(2, 4)$
**c** $(-1, 1), (2, 4)$ **d** $(-3, 9), (1, 1)$
**3** **a** $(-4, 3), (4, -3)$ **b** $(3, 4)$
**c** $(4{\cdot}7, -1{\cdot}8)$
**4** **a** A $(-0{\cdot}3, -0{\cdot}7)$, B $(2{\cdot}3, 4{\cdot}7)$
**b** A $(-0{\cdot}5, 0{\cdot}2)$, B $(1{\cdot}2, 1{\cdot}5)$
**5** and **6**
**a** $(0, 0), (2, 4)$
**b** $(-1, 1), (3, 9)$
**c** $(1, 1), (-1, -1)$
**d** $(1+\sqrt{2}, -1+\sqrt{2}), (1-\sqrt{2}, -1-\sqrt{2})$
**e** $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$
**f** $(0, -2), (1{\cdot}6, 1{\cdot}2)$
**7** **a** $(2, 3), (-1, 0)$
**b** $(2+\sqrt{2}, 2-\sqrt{2}), (2-\sqrt{2}, 2+\sqrt{2})$
**8** **a** A $(-0{\cdot}34, -0{\cdot}68)$, B $(2{\cdot}34, 4{\cdot}68)$
**b** A $(-0{\cdot}43, 0{\cdot}18)$, B $(1{\cdot}18, 1{\cdot}38)$
Answers to 2 d.p.

## Investigation 7:13

**1** **a** 1 **b** 2 **c** 3 **d** 4
**2** $(\sqrt{3}, 1), (-\sqrt{3}, 1), (2, 0), (-2, 0)$
**3** 4·25

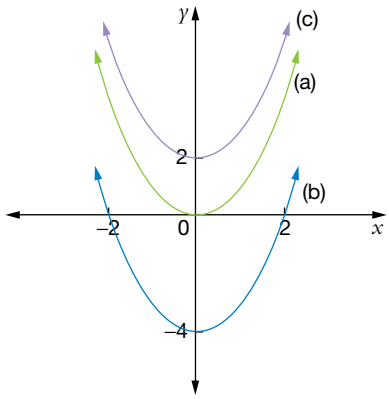
## Exercise 7:14

**1** **a** E, F, G **b** A, L **c** B, H, J
**d** C, I, O, P **e** D, N
**2** **a** E **b** D **c** B **d** P
**e** H **f** F **g** C **h** K
**i** A **j** N **k** G **l** P
**m** M **n** O **o** J **p** L
**3** **a** $y = (x-2)^2 - 2$ or $y = x^2 - 4x + 2$
**b** $y = x^4 - 2$
**c** $y = (x-2)^2 + 2$ or $y = x^2 - 4x + 6$
**d** $y = 3 - 2^{-x}$ **e** $y = 2(x-3)^3$
**f** $x^2 + (y-2)^2 = 4$ **g** $y = -x^3 - 4$
**h** $y = x^3 - 4$ **i** $y = 4 - 3^x$
**j** $(x+2)^2 + y^2 = 4$ **k** $y = x(x-1)(x+2)$
**l** $y = \dfrac{2}{x} - 3$ **m** $y = \dfrac{1}{x+2}$
**n** $y = \dfrac{1}{2-x}$ **o** $y = -\dfrac{4}{x} + 1$

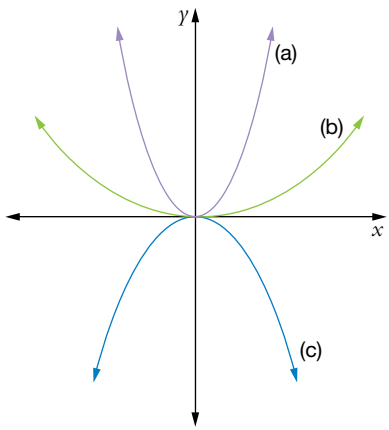
## Diagnostic test 7

**1** **a** $y = 4x + 2$
**b** $y = -3x + 10$
**c** $y = -\frac{3}{2}x + \frac{5}{2}$ or $3x + 2y - 5 = 0$
**d** $y = \frac{1}{2}x + 4\frac{1}{2}$ or $x - 2y + 9 = 0$

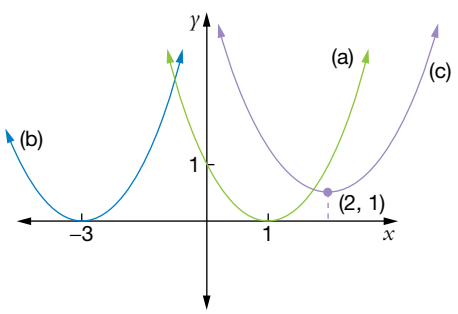
**2**



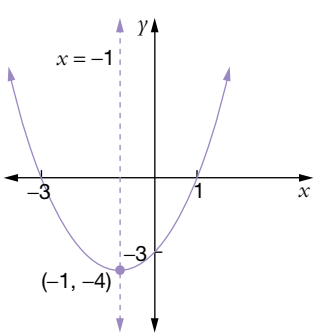

**3**



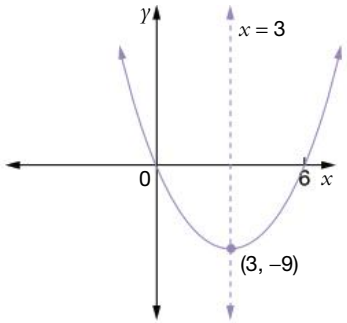

**4**



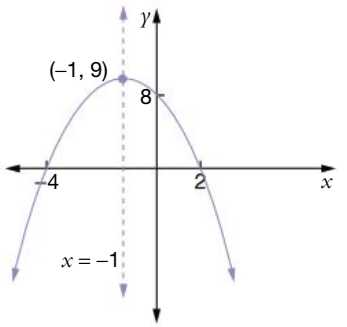

**5** **a** $(0, -3)$ **b** $(0, 0)$ **c** $(0, 8)$ **d** $(0, -5)$
**6** **a** $(1, 0), (-3, 0)$ **b** $(0, 0), (6, 0)$
**c** $(2, 0), (-4, 0)$ **d** $(\frac{1}{2}, 0)$ $(-\frac{5}{2}, 0)$
**7** **a** $x = -1$ **b** $x = 3$ **c** $x = -1$ **d** $x = -1$
**8** **a** $(-1, -4)$ **b** $(3, -9)$ **c** $(-1, 9)$ **d** $(-1, -9)$
**9** **a**



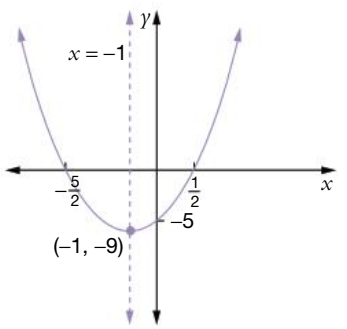
**b**



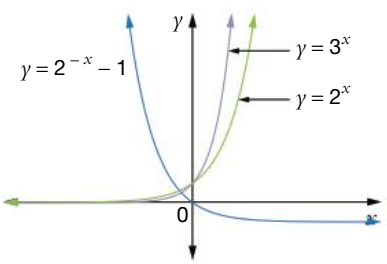

**c**



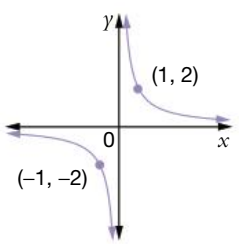

**d**



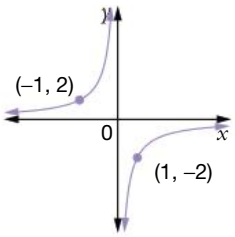

**10**



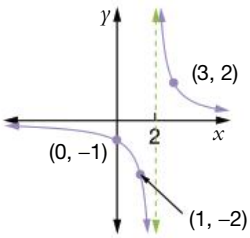

**11 a**




**b**



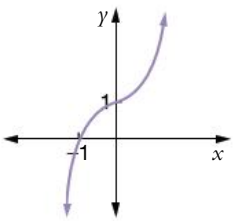

**c**



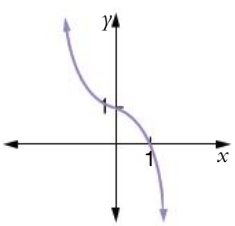

**12 a** $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 49$

**c** Centre is $(2, -1)$. radius = 4

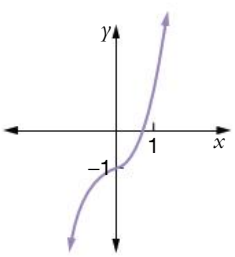
**13 a**



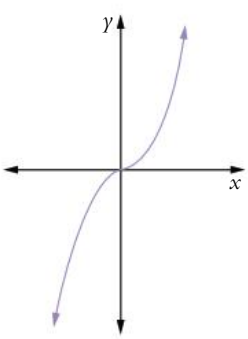

**b**



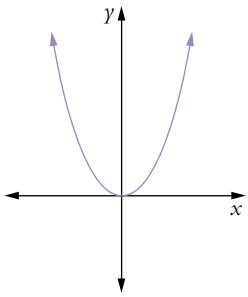

**c**



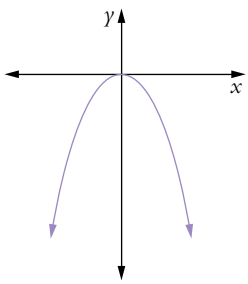

**14 a**



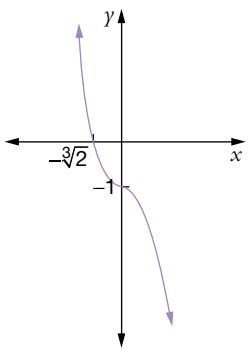
b



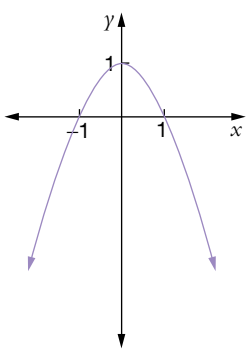

c



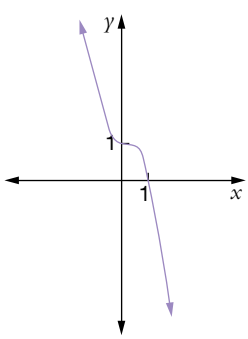

15 a



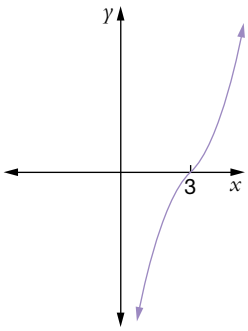

b



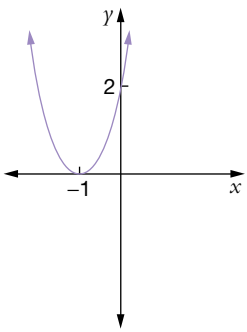

c



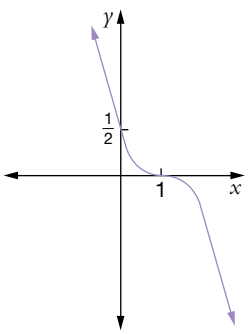

16 a



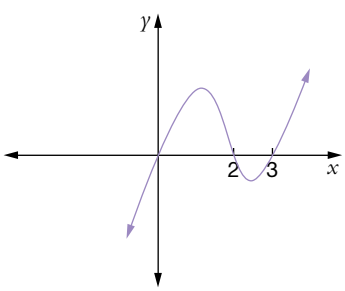

b



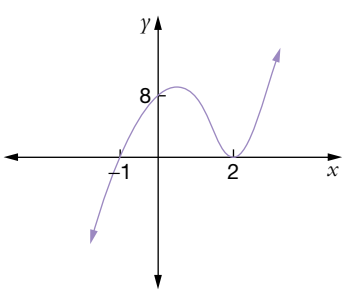

c



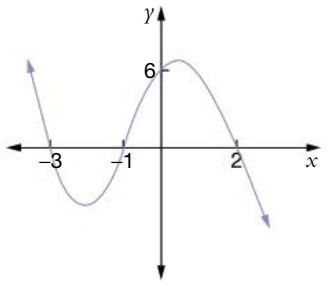

17 a




b



c




**18** **a** (2, 4) and (−4, −2)
**b** no points of intersection
**c** (1, 8) and (4, 2)

**19** **a** (3, 9) and (−4, 16) **b** (1, −2) and (2, −1)
**c** (1, 3) and (−3, −1)

**20** **a** E **b** C **c** B **d** A
**e** D **f** F

## ASSIGNMENT 7A

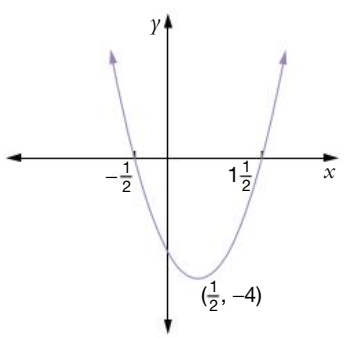
**1** **a** $y = 2x - 2$ **b** $y = -\frac{1}{2}x + 5\frac{1}{2}$

**2** $B(2\frac{1}{5}, 4\frac{2}{5})$

**3** Each graph is congruent to the graph of $y = x^2$. The graphs can be obtained by performing the following translations on $y = x^2$.
**a** moving it up five units
**b** moving it down five units
**c** moving it five units to the right
**d** moving it five units to the left
**e** moving it five units to the right and then five units up

**4** **a** **i** −3 **ii** $1\frac{1}{2}, -\frac{1}{2}$
**iii** $x = \frac{1}{2}$ **iv** $(\frac{1}{2}, -4)$

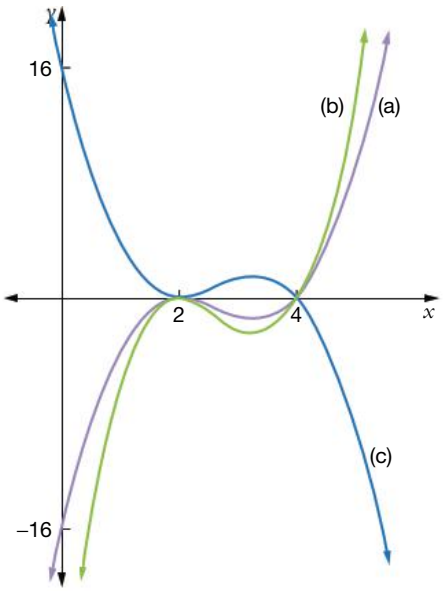
**b**



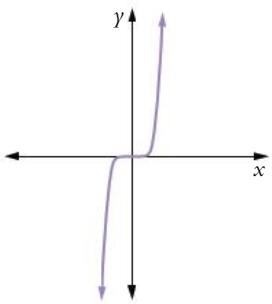

**5** **a** $y = \dfrac{4}{x} + 2$ **b** $y = \dfrac{4}{x - 2}$
**c** $y = \dfrac{4}{x + 2}$ **d** $y = -\dfrac{4}{x}$
**e** $y = -\dfrac{4}{x} + 2$

**6** **a** **i** $y = -x^5$ **ii** $y = -x^5$
**b** **i** $y = -x^5 + 2$ **ii** $y = -x^5 - 2$
**c** **i** $y = -\frac{1}{2}x^6$ **ii** $y = \frac{1}{2}x^6$
**d** **i** $y = 3(x - 2)^3$ **ii** $y = 3(x + 2)^3$

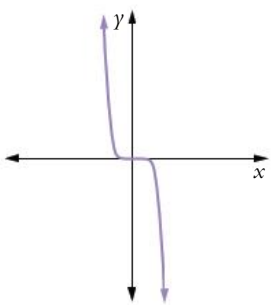
**7**



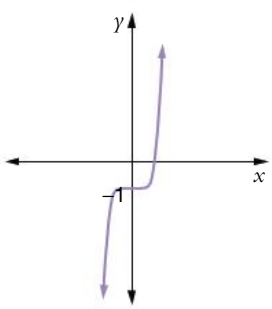

**8** **a**



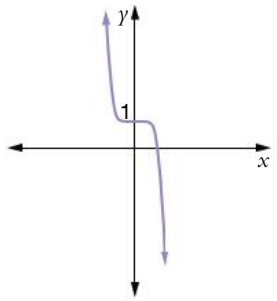

**b**



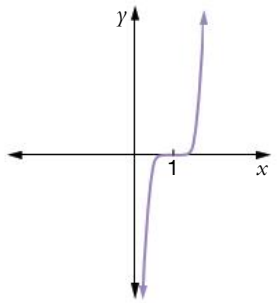

**c**



**d**




**e**




**9** **a** $(x + 3)^2 + y^2 = 9$ **b** $(x - 3)^2 + y^2 = 36$

**10** **a** $(-1, 1), (3, 9)$ **b** $(-3, -3), (1{\cdot}5, 6)$

## ASSIGNMENT 7B

**1** Sketch the curve by looking at the equation $y = ax^n + d$ in three stages. First, consider the curve $y = x^n$. If $n$ was even, it would mean that the curve was like a parabola. If it was odd, it would be like a cubic. Then look at $a$. The size of $a$ only affects the steepness of the curve and not the basic shape. The sign of $a$ would indicate whether the curve was the same shape as the curve $y = x^n$ or whether it was upside down. A positive value of $a$ would mean it was the same shape, while a negative value would mean it was upside down.
Finally, the value of $d$ would indicate how far the curve $y = ax^n$ had been translated to produce the curve $y = ax^n + d$. A positive value of $d$ would mean it had been translated $d$ units up, while a negative value would mean it had been translated $d$ units down.

**2** $y = -4x$ is a straight line with a negative slope. It passes through the origin.
$y = 4x^2$ is a parabola with its vertex at $(0, 0)$. It looks like $y = x^2$ but it is steeper.
$y = -4x^2 - 4$ is an upside-down parabola with its vertex at $(0, -4)$.
$y = (x - 4)^3$ is the curve that is produced by translating the curve $y = x^3$ four units to the right.
$y = 4^x$ is an increasing exponential curve. It has a $y$-intercept of $(0, 1)$.
$y = x^4$ is shaped like the parabola $y = x^2$.
$x^2 + y^2 = 4$ is a circle with its centre at $(0, 0)$ and a radius of two units.

**3** $50\,m^2$, 10 m by 5 m

**4** $D = \frac{C}{\pi} - 2d$

**5** $26 \times 26 \times 10 \times 10 \times 26 \times 26 = 45\,697\,600$

**6** **a** $0{\cdot}1 = 10^{-1}$, $0{\cdot}01 = 10^{-2}$, $0{\cdot}001 = 10^{-3}$

**b** The higher the pH, the lower the concentration of the acid. (To find the pH write the concentration as a power of 10. The pH is the negative of the power.)

**c** 5

**d** $10^{-4{\cdot}5}$ mole/L (or $0{\cdot}000\,031\,6$ mole/L to three significant figures)

**e** The acid with a pH of 2 is more concentrated. It is 100 times more concentrated.

## ASSIGNMENT 7C

**1** **a** $0, 6$ **b** $-10, -2$ **c** $-6, 16$ **d** $\frac{6 \pm \sqrt{180}}{2}$

**2** **a** **i** 174° **ii** 6°

**b** **i** 18 **ii** 18

**3** **a** $\frac{1}{216}$ **b** $\frac{125}{216}$ **c** $\frac{75}{216}$ **d** $\frac{91}{216}$

**4** **a** $7\sqrt{2}$ **b** $5\sqrt{3}$ **c** 45 **d** $\frac{\sqrt{2}}{2}$

**5** **a** 20 years **b** 15 years

**6** \$277 950

**7** **a** $x = 2, y = -2$ **b** $a = 2, b = 2$

**c** $m = \frac{38}{17}, n = \frac{12}{17}$

# Chapter 8

## Exercise 8:01

**1** **a** $544\,cm^2$ **b** $1721{\cdot}6\,cm^2$ **c** $348\,cm^2$

**2** **a** $x = 10{\cdot}82$ cm, $363{\cdot}84\,cm^2$

**b** $x = 8{\cdot}49$ cm, $622{\cdot}86\,cm^2$

**3** **a** $442\,cm^3$ **b** $226\,cm^3$ **c** $10\,179\,cm^3$

**4** **a** $370\,cm^2$ **b** $228\,cm^2$ **c** $3095\,cm^2$

**5** **a** $411\,cm^3$ **b** $47\,569\,cm^3$ **c** $519\,cm^3$

**6** **a** $368\,cm^2$ **b** $12\,528\,cm^2$ **c** $433\,cm^2$

**7** **a** $8{\cdot}6\,m^3$ **b** $354{\cdot}7\,m^3$ **c** $11{\cdot}1\,m^3$

**8** **a** $31{\cdot}3\,m^2$ **b** $354{\cdot}8\,m^2$ **c** $33{\cdot}2\,m^2$

## Exercise 8:02

**1** **a** $108\,cm^2$ **b** $1281{\cdot}48\,cm^2$

**2** **a** $156\,cm^2$ **b** $198{\cdot}75\,cm^2$

**c** $4\sqrt{17} + 2\sqrt{20} + 8 \doteqdot 33{\cdot}4\,cm^2$

**3** **a** **i** 4 cm **ii** $\sqrt{116}$ cm

**iii** $4\sqrt{116} = 43{\cdot}08\,cm^2$

**iv** $16\sqrt{116} + 64 = 236{\cdot}3\,cm^2$

**b** **i** 5 units **ii** $\sqrt{281}$ units

**iii** 6 units **iv** $\sqrt{292}$ units

**v** $6\sqrt{281}$ unit$^2$ **vi** $5\sqrt{292}$ unit$^2$

**vii** $12\sqrt{281} + 10\sqrt{292} + 120 = 492$ unit$^2$ (1 dec. pl.)

**4** **a** $360\,cm^2$ **b** $(100\sqrt{5}+100) \div 324\,cm^2$
**5** **a** $308{\cdot}1\,cm^2$ **b** $258{\cdot}8\,cm^2$ **c** $148{\cdot}4\,cm^2$
**6** **a** $106\,cm^2$ **b** $160{\cdot}13\,cm^2$
**7** **a** $104\,cm^2$ **b** $228\,cm^2$ **c** $62\,cm^2$
**8** $265\,cm^2$ (nearest $cm^2$)
**9** **a** $x = 40$ **b** $h = 10\sqrt{15}$

## Prep quiz 8:03

**1** $\pi s^2$ **2** $2\pi s$ **3** $\frac{r}{l}$
**4** $\pi rl$ **5** $\frac{1}{4}$ **6** $\frac{1}{3}$
**7** $\frac{1}{3}$ **8** $\frac{1}{2}$ **9** $\frac{r}{l}$
**10** 71·5 (1 dec. pl.)

## Exercise 8:03

**1** **a** $80\pi\,cm^2$ **b** $50\pi\,cm^2$ **c** $800\pi\,cm^2$
**d** $0{\cdot}48\pi\,m^2$
**2** **a** $180\pi\,cm^2$ **b** $75\pi\,cm^2$ **c** $1200\pi\,cm^2$
**d** $0{\cdot}64\pi\ m^2$
**3** **a** $144\pi\,cm^2$ **b** $5{\cdot}76\pi\,m^2$ **c** $200\pi\,cm^2$
**d** $0{\cdot}9\pi\,m^2$
**4** **a** $2092\,cm^2$ **b** $267{\cdot}0\,cm^2$ **c** $1013\,cm^2$
**d** $2{\cdot}730\,m^2$ **e** $23{\cdot}56\,m^2$
**5** **a** $1005{\cdot}3\,cm^2$ **b** $712{\cdot}6\,cm^2$ **c** $226{\cdot}5\,cm^2$
**6** **a** 15 cm **b** 5 cm **c** $10\sqrt{2} \div 14{\cdot}1$ cm
**7** **a** $\sqrt{1200} \div 34{\cdot}6$ cm **b** 23·2 cm

## Exercise 8:04

**1** **a** $100\pi\,cm^2, 314\,cm^2$ **b** $231{\cdot}04\pi\,cm^2, 726\,cm^2$
**c** $40{\cdot}96\pi\ m^2, 129\,m^2$ **d** $324\pi\,cm^2, 1020\,cm^2$
**e** $2{\cdot}56\pi\,m^2, 8{\cdot}04\,m^2$
**f** $6{\cdot}4 \times 10^7\pi\,km^2, 2{\cdot}01 \times 10^8\,km^2$
**2** **a** $432\pi\,cm^2$ **b** $108\pi\,cm^2$
**3** **a** $804{\cdot}25\,cm^2$ **b** $1182{\cdot}37\,m^2$ **c** $1140{\cdot}40\,m^2$
**4** $23{\cdot}12\,m^2$
**5** 4·0 cm
**6** **a** $333\,cm^2$ **b** $855\,cm^2$ **c** $103\,m^2$

## Investigation 8:05

Volume of a pyramid = $\frac{1}{3}$ of the volume of a prism.

## Exercise 8:05

**1** **a** $480\,cm^3$ **b** $193{\cdot}83\,cm^3$
**2** **a** $22{\cdot}18\,m^3$ **b** $234{\cdot}9\,cm^3$
**3** **a** $3{\cdot}6\,m^3$ **b** $31\,cm^3$
**4** **a** 60 cm **b** $10\frac{5}{12} \approx 10{\cdot}4$ cm
**c** area = $1600\,cm^2$, side length = 40 cm
**d** length = 20 cm, width = 10 cm
**5** **a** $300{\cdot}12\,cm^3$ **b** $324\,cm^3$
**6** **a** $0{\cdot}289\,m^3$ **b** $0{\cdot}608\,m^3$
**7** **a** $115{\cdot}7\,m^3$ **b** $960\,cm^3$ **c** $3500\,cm^3$
**d** $2110\,cm^3$ **e** $185{\cdot}625\,m^3$
**8** **a** $\frac{64\sqrt{193}}{3} \approx 296{\cdot}4\,cm^3$ **b** $144\sqrt{3}\ cm^3$
**9** $18\sqrt{2}\ cm^3$

## Exercise 8:06

**1** **a** $89{\cdot}22\,m^3$ **b** $17{\cdot}67\,m^3$ **c** $1707{\cdot}75\,cm^3$
**2** **a** $220\,cm^3$ **b** $636\,cm^3$ **c** $0{\cdot}7\,m^3$
**3** **a** $h = 5{\cdot}011$ m; $V = 32{\cdot}8\,m^3$
**b** $h = 15{\cdot}263$ cm; $V = 117\,cm^3$
**c** $h = 3{\cdot}175$ m; $V = 43{\cdot}1\,m^3$
**4** **a** $100\pi\,cm^3$ **b** $240\pi\,cm^3$
**5** **a** 9·5 cm **b** 8·5 cm
**c** $r = 7{\cdot}8$ cm, $h = 15{\cdot}6$ cm **d** $13\frac{1}{3}$ cm
**6** **a** $2080\,cm^3$ **b** $868\,m^3$ **c** $401\,cm^3$
**7** $600\,cm^3$

## Exercise 8:07

**1** **a** $36\pi\,cm^3$ **b** $\frac{32\pi}{3}\ m^3$
**c** $62{\cdot}208\pi\,m^3$ **d** $18{\cdot}432\pi\,m^3$
**2** **a** $7{\cdot}24\,m^3$ **b** $8780\,cm^3$
**c** $65{\cdot}5\,cm^3$ **d** $3{\cdot}62\,m^3$
**3** **a** $1570\,cm^3$ **b** $1690\,cm^3$ **c** $83{\cdot}8\,cm^3$
**4** $\frac{364\pi}{3}\ cm^3$
**5** **a** 7·8 cm **b** 11 cm **c** 0·63 m

## Exercise 8:08

**1** **a** $1{\cdot}1 \times 10^{21}\,m^3$ **b** $5{\cdot}9 \times 10^{21}$ t
**c** $3{\cdot}6 \times 10^8\,km^2$
**2** $33{\cdot}93\,m^3$
**3** $789\,m^2$
**4** $56\,cm^3$
**5** **a** $6000\,cm^2$ **b** $22\,000\,cm^3$
**6** **a** 44 kL **b** $9400
**7** **a** $6{\cdot}3\,m^3$ **b** 22·5 kL
**8** $0{\cdot}94\,m^3$
**9** $213{\cdot}1\,m^3$
**10** $80{\cdot}784\,m^3$
**11** **a** **i** 1·8 m **ii** $27{\cdot}57\,m^3$ (corr to 2 d.p.)
**b** $65{\cdot}9\,m^3$

## Diagnostic test 8

**1** **a** $384\,cm^2$ **b** $222{\cdot}78\,cm^2$ **c** $360\,cm^2$
**2** **a** $74{\cdot}3\,cm^2$ **b** $27{\cdot}1\,cm^2$ **c** $51{\cdot}3\,m^2$
**3** **a** $314{\cdot}16\,cm^2$ **b** $865{\cdot}70\,cm^2$ **c** $84{\cdot}82\,cm^2$
**4** **a** $44{\cdot}712\,m^3$ **b** $80{\cdot}136\,m^3$ **c** $180{\cdot}728\,cm^3$
**5** **a** $46{\cdot}0\,cm^3$ **b** $118{\cdot}1\,cm^3$ **c** $21{\cdot}8\,cm^3$
**6** **a** $523{\cdot}6\,cm^3$ **b** $333{\cdot}0\,cm^3$ **c** $883{\cdot}6\,cm^3$

## ASSIGNMENT 8A

**1** SA = 216 cm$^2$, vol. = 162 cm$^3$

**2** $200\pi$ cm$^2$ **3** 600 cm$^2$ **4** 2·879 m

**5** 476 cm$^3$ **6** 739 cm$^2$ **7** 12·3 m$^3$

**8** SA = $72\sqrt{3}$ cm$^2$; vol. = $72\sqrt{2}$ cm$^3$

## ASSIGNMENT 8B

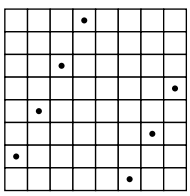
**1** \$417 605

**2** **a** $\frac{a\sqrt{3}}{3}$ **b** $\frac{a\sqrt{3}}{6}$

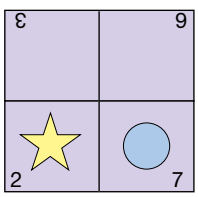
**3** $AX = \sqrt{12}$ units, vol. = 36 unit$^3$

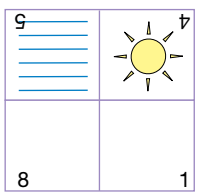
**4** 18, 42, 105

**5** 6 more queens can be placed. An example is shown to the left.



**6**




## ASSIGNMENT 8C

**1** **a** $y = 3x + 4$ **b** $y = \frac{1}{2}x - 3$

**2** **a** $y = -x - 2$, $y = -2x$, $y = x - 2$, $y = -2$, $x = 2$

**b** $y = x^2 - 2$, $y = -2x^2$, $y = 1 - 2x^2$

**c** $x^2 + y^2 = 1$

**d** $y = 2 - 2^x$, $y = 2^x$, $y = 2^{-x} - 2$

**3** **a** $1 + \sqrt{5}$ **b** 2 **c** $23 - 4\sqrt{15}$

**4** **a** $2\sqrt{3}$ **b** $2\sqrt{6}$ **c** $\frac{\sqrt{6}}{6}$ **d** $\frac{\sqrt{6}}{2}$

**5** **a** 97·0% **b** 3·0%

**6** **a** $\frac{-1+\sqrt{801}}{2} \approx 13{\cdot}7$, $\frac{-1-\sqrt{801}}{2} \approx -14{\cdot}7$

**b** 82, 83, 84

**7** $x = 105$, $a = 37{\cdot}5$

# Chapter 9

## Exercise 9:01

**1** **a** **i** 7 **ii** 4 **iii** 6 **iv** 5

**b** **i** 6 **ii** 14 **iii** 15·5 **iv** 15

**c** **i** 0·6 **ii** 1·7 and 1·9 **iii** 1·7125 **iv** 1·7

**d** **i** 4 **ii** 9 **iii** 8·2 **iv** 8

**e** **i** 20 **ii** 15 **iii** 14 **iv** 15

**2** The mode is Toyota with a frequency of 250.

**3** **a** mode = 5, mean = 6, median = 5

**b** mean = 6, median = 5·5

**4** **a** 12

**b** **i** 12·5 **ii** 11

**5** **a** 10 **b** 14 **c** 13·8 **d** 14

**6** **a** 50 **b** 30–<40 **c** 20–<30 **d** 27·25

**7** **a** **i**

| Score | Tally | Freq | $fx$ | $c.f.$ |
|---|---|---|---|---|
| 1 | \|\|\| | 3 | 3 | 3 |
| 2 | 𝍸 | 5 | 10 | 8 |
| 3 | 𝍸 \|\|\|\| | 9 | 27 | 17 |
| 4 | 𝍸 \|\| | 7 | 28 | 24 |
| 5 | \|\|\|\| | 4 | 20 | 28 |
| 6 | \|\| | 2 | 12 | 30 |
| | *Totals:* | 30 | 100 | |

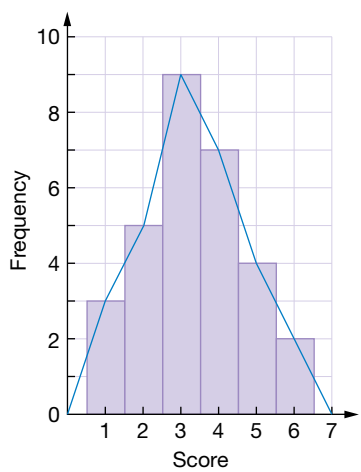
mean = $3\frac{1}{3}$ median = 3

**ii**

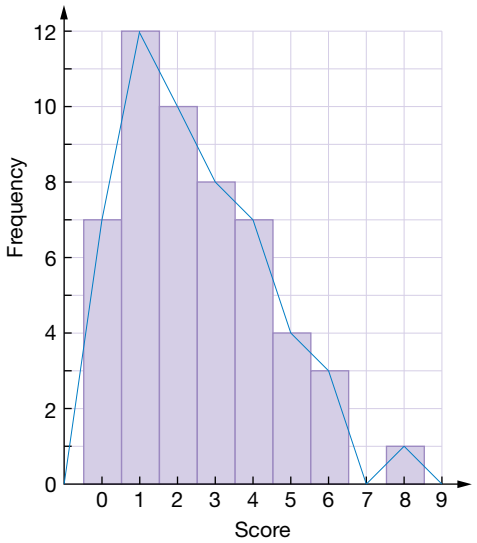
| Outcome | Tally | Freq | $fx$ | $c.f.$ |
|---|---|---|---|---|
| 0 | 𝍸 \|\| | 7 | 0 | 7 |
| 1 | 𝍸 𝍸 \|\| | 12 | 12 | 19 |
| 2 | 𝍸 𝍸 | 10 | 20 | 29 |
| 3 | 𝍸 \|\|\| | 8 | 24 | 37 |
| 4 | 𝍸 \|\| | 7 | 28 | 44 |
| 5 | \|\|\|\| | 4 | 20 | 48 |
| 6 | \|\|\| | 3 | 18 | 51 |
| 7 | | 0 | 0 | 51 |
| 8 | \| | 1 | 8 | 52 |
| | *Totals:* | 52 | 130 | |

mean = 2·5 median = 2

**b** **i**



ii

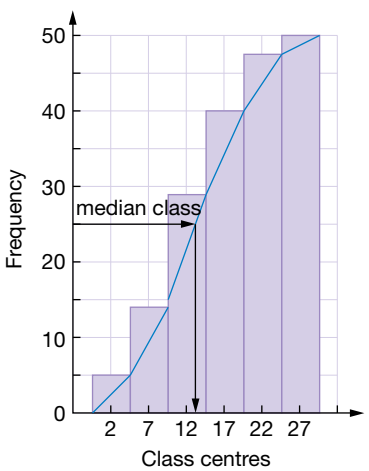



**8** **a**

| Class | *c.c.* | Tally | Freq | $f \times c.c.$ | *c.f.* |
|---|---|---|---|---|---|
| 0–4 | 2 | \|\|\|\| | 5 | 10 | 5 |
| 5–9 | 7 | 𝍸 \|\|\|\| | 9 | 63 | 14 |
| 10–14 | 12 | 𝍸 𝍸 𝍸 | 15 | 180 | 29 |
| 15–19 | 17 | 𝍸 𝍸 \| | 11 | 187 | 40 |
| 20–24 | 22 | 𝍸 \|\| | 7 | 154 | 47 |
| 25–29 | 27 | \|\|\| | 3 | 81 | 50 |
| | | *Totals:* | 50 | 675 | |

**b** 29 **c** 10–14 **d** 10–14 **e** 13·5

**f**




**9** **a** **i** median = 8, $Q_1 = 5$, $Q_3 = 9$, IQR = 4
**ii** median = 6·5, $Q_1 = 4·5$, $Q_3 = 8·5$, IQR = 4
**iii** median = 5, $Q_1 = 2·5$, $Q_3 = 6·5$, IQR = 4
**iv** median = 5, $Q_1 = 2·5$, $Q_3 = 7·5$, IQR = 5

**b** **i** (2, 5, 8, 9, 12)
**ii** (3, 4·5, 6·5, 8·5, 10)
**iii** (1, 2·5, 5, 6·5, 8)
**iv** (1, 2·5, 5, 7·5, 9)

**10** **a** **i** 40 **ii** 9 **iii** 5 **iv** 4
**v** 7 **vi** 3 **vii** (1, 4, 5, 7, 10)

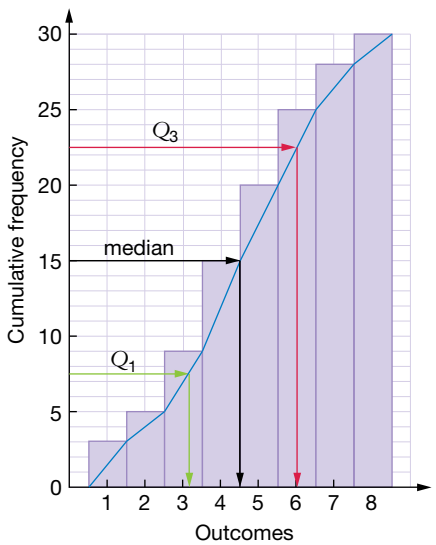
**b**

| Outcome | Frequency | *f. x.* | *c.f.* |
|---|---|---|---|
| 1 | 2 | 2 | 2 |
| 2 | 4 | 8 | 6 |
| 3 | 3 | 9 | 9 |
| 4 | 4 | 16 | 13 |
| 5 | 8 | 40 | 21 |
| 6 | 6 | 36 | 27 |
| 7 | 5 | 35 | 32 |
| 8 | 4 | 32 | 36 |
| 9 | 3 | 27 | 39 |
| 10 | 1 | 10 | 40 |
| *Totals:* | 40 | 215 | |

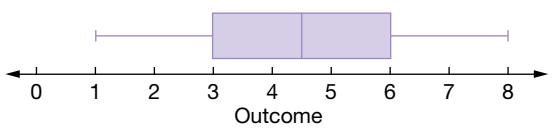
**c** **i** 5
**ii** 5·375

**11** **a** **i** 7 **ii** 4·5 **iii** 3 **iv** 6

**b**




**c**



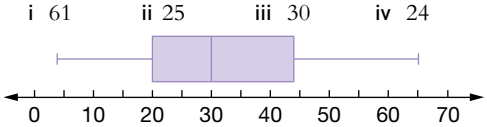

**12** **a** **i** (1, 2, 4, 7, 8)
**ii** (1, 4, 5·5, 7, 10)
**iii** (10, 12·5, 14, 16, 18)
**iv** (20, 28, 30, 33, 36)

**b** plot ii

**c** plot iv; it is negatively skewed

**13** **a** **i** 61 **ii** 25 **iii** 30 **iv** 24

**b**




**14** **a** Class A = 28, Class B = 24

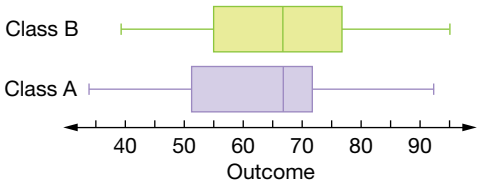
**b** A = 71, B = 67

**c** A = 67, B = 67

**d** A = 63, B = 67

**e** A: $Q_1 = 52$, $Q_3 = 71$; B: $Q_1 = 55{\cdot}5$, $Q_3 = 76{\cdot}5$

**f**




**g** Class A

**h** **i** Class A median = 68, so Class B would have lower median.

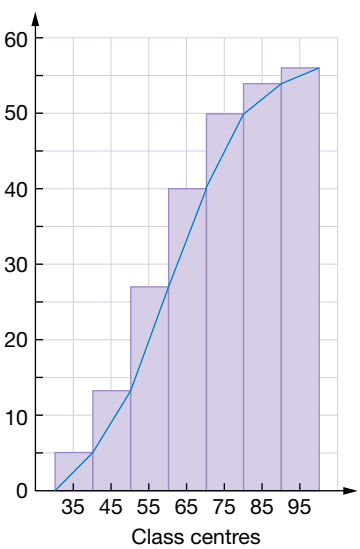
**ii** Class A mean = 67·375, so Class B would still have a slightly lower mean.

**15** **a**

| Income (× $1000) | *c.c.* | Freq | $f \times c.c.$ | *c.f.* |
|---|---|---|---|---|
| 30–<40 | 35 | 5 | 175 | 5 |
| 40–<50 | 45 | 8 | 360 | 13 |
| 50–<60 | 55 | 14 | 770 | 27 |
| 60–<70 | 65 | 13 | 845 | 40 |
| 70–<80 | 75 | 10 | 750 | 50 |
| 80–<90 | 85 | 4 | 340 | 54 |
| 90–<100 | 95 | 2 | 190 | 56 |
| | *Totals:* | 56 | 3430 | |

**b** 61·25

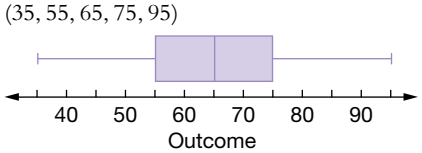
**c**




**d** median class = 60–<70,
first quartile class = 50–<60,
third quartile class = 70–<80

**e** (35, 55, 65, 75, 95)

**f**




**g** symmetrical

## Exercise 9:02

**1** $\overline{x} = 6$, $\sigma = 2$

**2** **a** **i** A **ii** C

**b** **i** C **ii** B

**c** **i** B **ii** A

**3** C, B, A, D

**4** **a** A **b** C **c** D **d** B

**5** **a** 5·4, 2·5 **b** 12·8, 3·3

**c** 5·3, 2·4 **d** 58·9, 19·3

**6** **a** Set A, 19·1; Set B, 5·4

**b** The first scores are different. The score of 20 in Set A is an outlier.

**c** yes

**7** **a** 2·95, 1·12 **b** 10·52, 1·16

**c** 33·4, 13·65 **d** 49·65, 1·48

**e** 29·2, 10·55

**8** **a** Set A: $\overline{x} = 65{\cdot}9$, $\sigma = 18{\cdot}8$
Set B: $\overline{x} = 68{\cdot}3$, $\sigma = 13{\cdot}7$

**b** The effect of the outlier will be reduced as the number of scores increases.

**9** **a** A **b** A **c** B

**10** **a** **i** George **ii** George **iii** Rick

**b** Rick's lower standard deviation suggests that he is consistently the better player.

**11** **a** B **b** B **c** A

**12** Maths: $\overline{x} = 69{\cdot}5$, $\sigma = 14{\cdot}2$, English: $\overline{x} = 59{\cdot}4$, $\sigma = 10{\cdot}4$. Tom's English mark is the better, relative to the rest of the class.

**13** **a** 70 **b** 62 **c** 63 **d** 18

**e** 40 **f** 48

**14** **a** Mean will be less, s.d. will be greater.

**b** Mean will be greater, s.d. will be less.

**c** Both mean and s.d. will be greater.

**15** **a** **i** Range = 4, $\overline{x} = 7$, $\sigma \div 1{\cdot}4$

**ii** Range = 4, $\overline{x} = 7$, $\sigma \div 1{\cdot}2$

**iii** Range = 4, $\overline{x} = 7$, $\sigma \div 1{\cdot}6$

**b** For all three graphs, the range and mean are the same. The standard deviation is smallest when scores tend to be clustered around the mean (as in case **ii**) and greatest when the scores tend to be further from the mean (as in case **iii**).

**c** The range does not change, the mean increases by 10 and the standard deviation does not change.

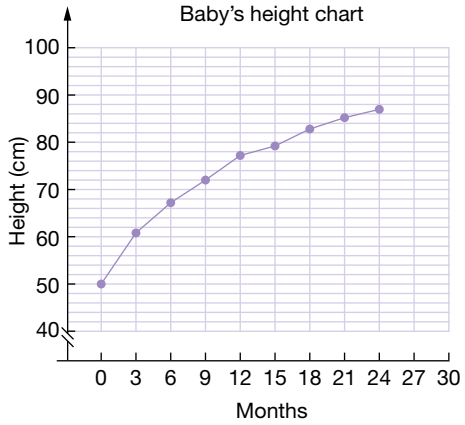
**d** The range, mean and standard deviation are all multiplied by 10.

## Exercise 9:03

**1** **a** 3·5 kg **b** 10·3 kg **c** 194%

**d** 12·6 kg, 22% **e** 13 kg **f** no

2 a




b Yes, the graph looks similar

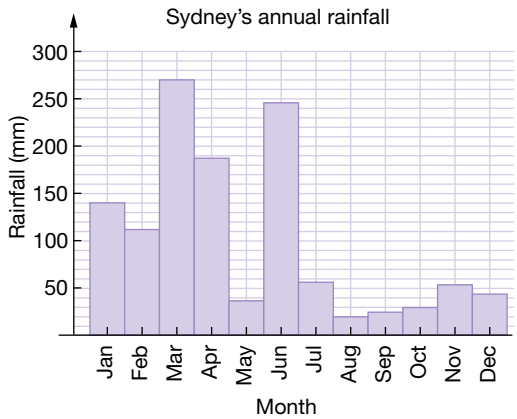
c yes d 78 kg

e no; growth spurts occur, particularly in teenage years.

3 a i Sept ii June

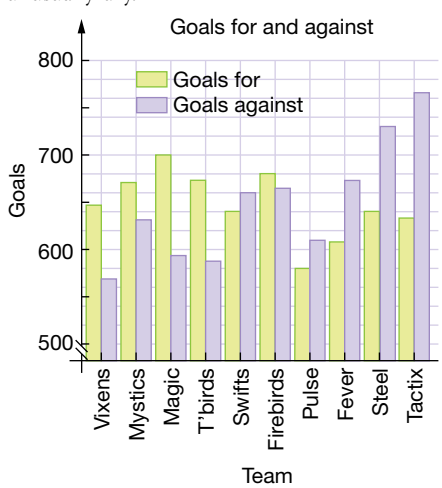
b first half

c




d Rainfall for 2012 is more uneven, however, the graph for average rainfall for over 150 years should be fairly even. For 2012 June is also a very wet month and, in general, the first half of the year is wetter. For 2012 the second half of the year seems unusually dry.

4 a




b Goals against seems to show a more steady pattern, so it would be the indicator of success. A coach might decide to pay more attention to defensive skills in training.

c Because both variables are involved in determining the result of any game and sometimes a team will perform much better or worse than they usually do. Winning a number of games by a narrow goal difference could be offset by a loss by a big goal difference.

5 a Denmark, United States, United Kingdom, France, Canada, Australia, New Zealand, Macau

b Macau, Australia, Canada, France, New Zealand, United Kingdom, Denmark, United States

c All eight.

d Life expectancy: Afghanistan, Zimbabwe, Chad, South Africa, Ethiopia, Kenya, Ghana, Bangladesh
Wealth: Ethiopia, Afghanistan, Nepal, Zimbabwe, Chad, Ghana, Bangladesh, Kenya

e Nepal

f South Africa

g By collecting similar data from more countries. By observing any change in life expectancy in countries where the per capita income has improved over time.

h Population density, availability of clean water and food, medical resources, political stability.

6 a 3·5; the green column is 3·5 times higher than the red column.

b WA

c Vic; 0·70

d Vic and Qld

e You might expect that the number of vehicles would have a greater influence on the number of fatalities but the rate of fatalities per population seems to show the more consistent pattern.

f No. Other factors such as distance travelled and quality of roads would have an influence.

7 a 2000

b 2010

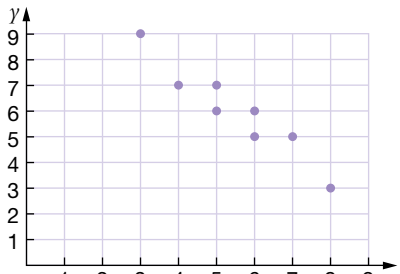
c The global financial crisis (GFC).

d Governments need to balance the amount of revenue they collect, largely through taxes and levies, with the amount they need to spend on services and infrastructure. The government has to decide which of the many areas that need funding have priority.

## Exercise 9:04

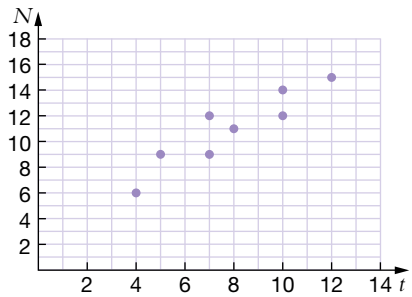
1 a yes, negative b yes, positive c no
d no e yes, negative f yes, positive

2 a



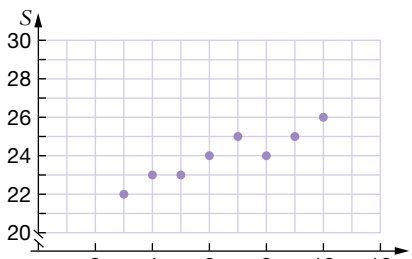

Negative linear relationship

b



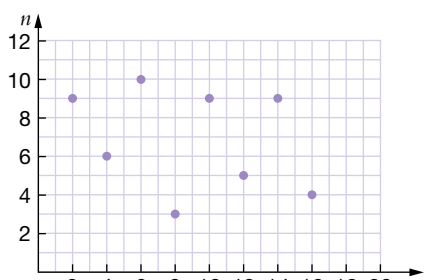

Positive linear relationship

c



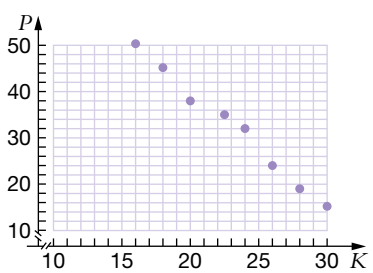

Positive linear relationship

d



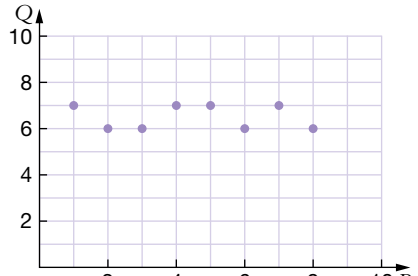

No relationship

e




Negative linear relationship

f




No relationship

3 a $y = \frac{1}{2}x + 2$ b $n = -\frac{2}{3}t + 7$

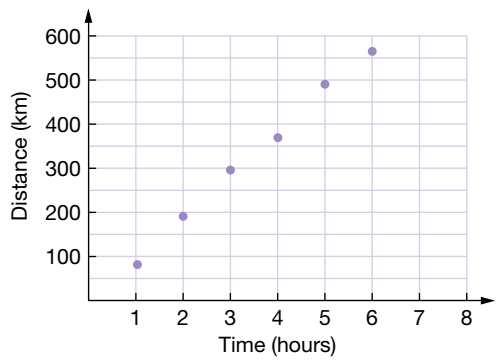
c $L = \frac{1}{10}t + 20$ d $T = -3P + 50$

4 a $y = x + 2$ b $D = -t + 13$

c $T = 2t + 10$ d $n = -2p + 60$

5 a $Y = 2X + 12$ b $Y = -3X + 80$

6 a

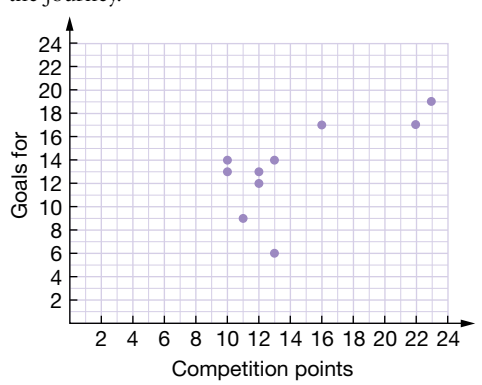



b positve linear relationship

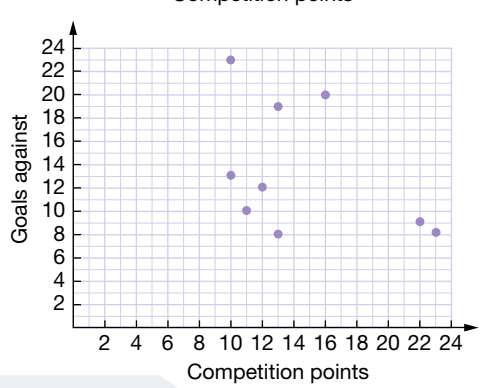
c 93·5 km/h

d The gradient represents the average speed of the journey.

7 a




b



c Neither has a strong relationship, but with the exception of one or two points, the 'goals for' scatter plot shows the stronger positive linear relationship.

8 a 23

b i Maths = 5, English = 7

ii Maths = 20, English = 19

iii Maths = 9, English = 13

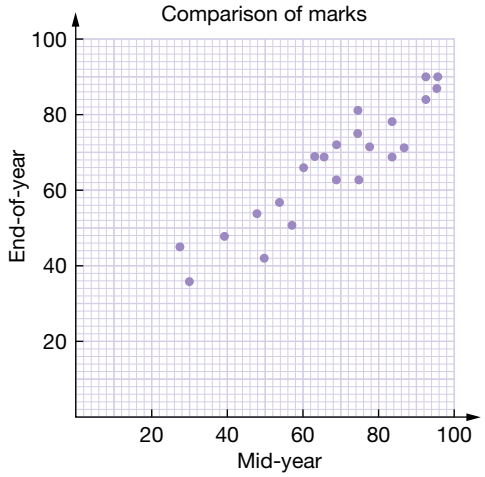
iv Maths = 18, English = 14

c 16 d 13 e 11 f 12

g a weak positive linear relationship

h $E = \frac{4}{5}M + 3$

9 a

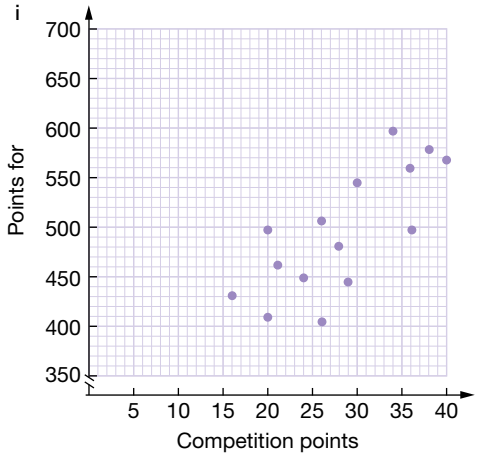



b From the pattern above, it is obvious that students tended to perform as well in both exams.

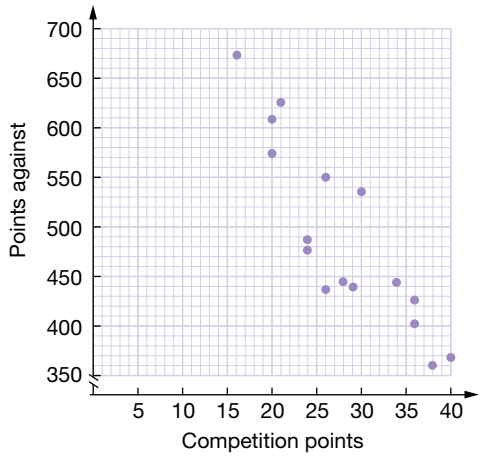
c equal 12th d 16th e 75

f 69 g $Y = 0{\cdot}7MY + 20$

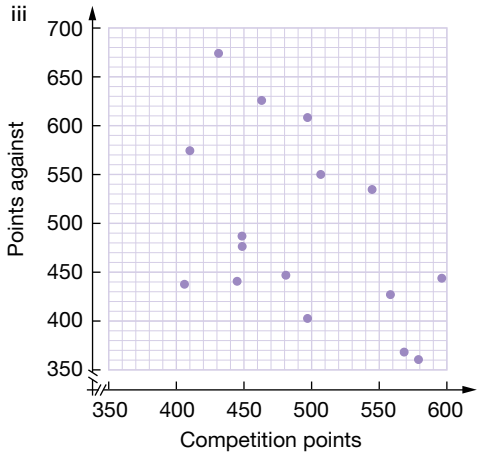
10 a i




ii




iii

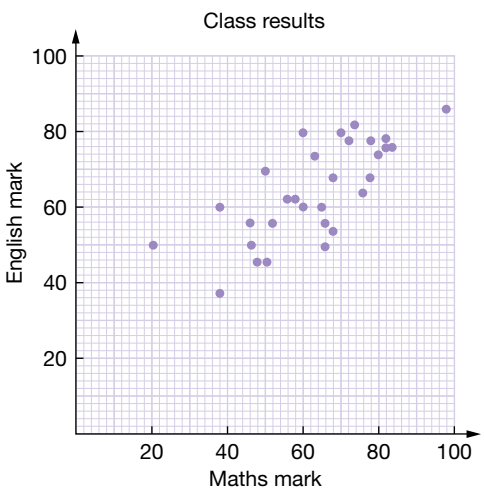



b Plot i

c Plot ii

d Competition points and Points against

11 a




b Maths: range = 78, median = 65·5
English: range = 48, median = 63

c A weak positive linear relationship

d $E = 0{\cdot}6M + 30$

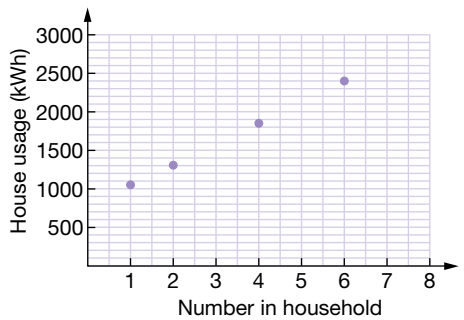
e about 70

## Exercise 9:05

1 a 29, assuming he batted at least once in each match.
b 130
c 52·34, average
d 2·94, economy rate
e 74·80, strike rate

2 a 100%
b The viewing audience, i.e. the total number of people watching TV when the surveys were done.
c i ABC, Seven, SBS
ii Nine, Ten
d More viewers attract more advertising investment
e Better programs, better personalities, more stations to provide diversity, etc.

3 a 1104·89 kilowatt hours (kWh)
b lower
c




d positive linear relationship
e about 2100 kWh
f i about 800 kWh
ii about 2900 kWh
iii about 6050 kWh
Although some services may be switched on, a household with no-one should use almost zero power. With a large number of people certain usage, such as lighting should not increase proportionately, so it is that less electricity per person may be used.

4 a 38 997, 0·17%
b Victoria, 14 312
c WA, 0·35%

5 a Avoca Beach
b No; if very few houses were sold in a particular month that were more extreme in their prices, the median might not be very representative.
c Avoca Beach
d Green Point and Kincumber are fairly symmetric and Avoca Beach is positively skewed.

6 a other
b 170
c After World War II, many were seeking re-settlement from Europe.
d Migrants from the UK were automatically considered as citizens as we were part of the British Commonwealth.
e 0
f 7

7 a 1982
b house prices are generally higher
c down
d Incomes rise and/or house prices come down.

8 a Both the number of licences and number of vehicles owned was increasing at a faster rate than the population. So the increase in fatalities was at an increasing rate against the population but not against the greater rate of increase of licences and vehicles.
b Fewer people owned cars.
c Random breath testing for alcohol was introduced in 1982.
d Compulsory seatbelts were introduced in 1971 for front seats and later for all passengers. Better roads, safer cars with equipment such as airbags, speed cameras, advertising campaigns, double demerit points in holiday times, school zones, decreased speed limits, etc.

9 a ACT; perhaps due to higher number of public servants.
b WA; perhaps due to the mining industry, which pays higher wages.
c positively
d 10% and 90% marks
e the median

10 a Home prices relative to the price in 1890 which has been given an index figure of 100.
b Around 1949, after World War II.
c No. Even if the home prices don't continue to rise, the graph cannot predict a 'crash'.
d Some reasons might be a fall in personal income making high prices unaffordable, an oversupply of homes or a fall in population.

## Exercise 9:06

1 a Z
b Y, as it gives a finishing time.

2 Student research
3 Student research
4 Student research
5 Student research

6 a Where you live may have more traffic, leading to more accidents or the area might be judged a higher risk for theft. Also whether the vehicle is sheltered in a garage or parked in the street may affect the premium.
b Whether the vehicle is for private or business use. Business use usually means the car is used more frequently, so the premium may be higher.
c Previous claims may cause you to be judged as a greater risk and result in a higher premium, particularly if you were at fault in an accident. Policies usually have a 'no-claim' bonus that reduces the premium.

d Obviously a more expensive car will result in a higher premium as the cost of repairs or replacement will be greater. Some cars may also be judged safer than others due to in-built safety features.

e Some features may be added to a vehicle that you want insured such as a sound system. Also varying the 'excess' paid before any claim is made will affect the premium.

7 a The data would suggest that males are more likely to be killed on the roads than females.

b It would be good to know how many people are in each category and how often they are on the roads. These figures obviously include passengers and perhaps pedestrians and not just drivers because of the 0–16 age group. Therefore, are there more male than female drivers? Do more males have jobs that involve driving?

8 Answers will vary. However, the length of time, the necessity of the activity and the reason for being in that place must be considered.

## Exercise 9:07

1 a It is very 'volatile', i.e. it fluctuates a lot.

b The 'spaces' represent each day, so the first section of the graph shows the index rising from the start of the day to the end of the day for Dec 03. So the 'points' on the graph indicate the end of one day and the beginning of the next.

c Scale not shown, but it is meant to show the value of the All Ordinaries index.

d The scale seems to start at 4510 and rise by 10 for each unit, so the top of the scale would be 4560.

e 4518; this would be the figure at the start of the graph, i.e. the beginning of Dec 03.

2 a yes b no c no

d As a visual means to attract attention. The bars indicate the order of the gains and losses but are not representative of the size of the gain or loss. What appears as a 'graph' is not a graph of the figures shown.

3 a Number of tests played

b No; the scale for the column graph

c 12

d The more times he batted would average out to his usual ability. Only batting a few times means there is a more likely chance of batting more significantly above usual ability. Of course it would be just as possible to score below average for only a few innings. The graph would seem to suggest that batting more has often resulted in a lower average, at least for Michael Hussey, but it has not always been the case.

4 a by 'obesity'

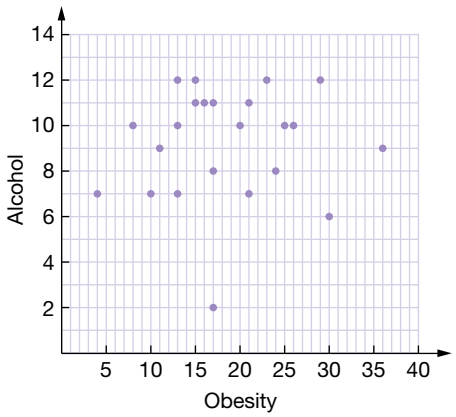
b by 'alcohol consumption'

c The percentage of the population with a body mass index of 30 or more and litres of alcohol consumed per person in a year.

d No: the variables are not measured against the same scale.

e yes

f




g no

h Neither variable truly represents the health or the wealth of a nation.

5 a i static; very little change

ii rising steadily to 2005 then falling steadily

b i 22·7°C

ii 22·9°C

c no

d It is impossible to know the pattern for either graph after 2011, so confidence would be very low to zero as the years increased.

6 a No, because of insufficient data.

b The answer would have been the same.

c increased

d By looking at the data for all years over an extended period. If data is only chosen for two years at random, it would be expected that one would be greater than the other.

7 a second table

b first table

c The area of each country, the mass of each country's vegetation, transport distances, energy needs

d China and India in particular have very large populations so their per capita ranking is very low. Arguing from this data would suggest their output of $CO_2$ is not significant. Also there are 12 countries ranked higher than Australia but some have very small populations that distorts the significance of the data.

## Diagnostic test 9

1 a i 8 ii 3 iii 6 iv 7

v 3 vi 9 vii 6

b i 6 ii 4 and 9 iii 6 iv 5·5

v 4 vi 8·5 vii 4·5

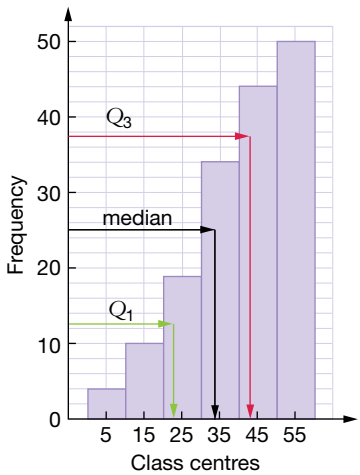
**2 a**

| Outcome | Tally | Freq | $f \times c.c.$ | $c.f.$ |
|---|---|---|---|---|
| 6 | 𝍸 \|\| | 7 | 42 | 7 |
| 7 | 𝍸 𝍸 \| | 11 | 77 | 18 |
| 8 | 𝍸 \|\|\| | 8 | 64 | 26 |
| 9 | 𝍸 | 5 | 45 | 31 |
| 10 | \|\|\|\| | 4 | 40 | 35 |
| 11 | \|\|\| | 3 | 33 | 38 |
| 12 | \|\| | 2 | 24 | 40 |
| | *Totals:* | 40 | 325 | |

**b** **i** 6 **ii** 7 **iii** 8·125
**iv** 8 **v** 7 **vi** 9

**3 a** **i** 30–<40 **ii** 30–<40 **iii** 32·8

**b**

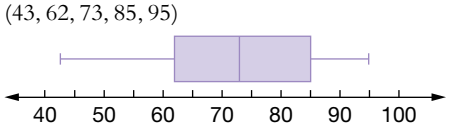



**c** $Q_1 = 23$, $Q_3 = 43$ ∴ IQR = 20

**4 a** **i** 52 **ii** 76 **iii** 73
**iv** 62 **v** 85 **vi** 23

**b** (43, 62, 73, 85, 95)

**c**

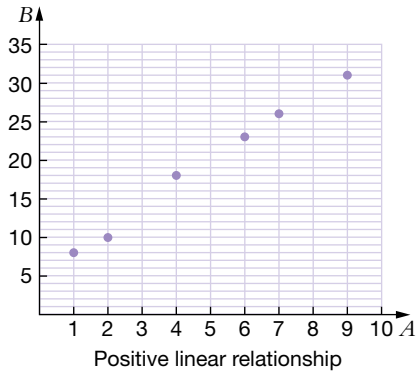



**5** Test A: s.d. = 2·00
Test B: s.d. = 4·15
Test C: s.d. = 4·52.
A mark of 15 is more impressive for Test C as it is 1·11 standard deviations above the mean (while it is only 1·00 for test A and 0·72 for Test B).

**6 a** time
**b** number of fatal crashes
**c** 14·00–16·00
**d** 02·00–04·00
**e** No. Need to consider the number of vehicles on the road at each time.

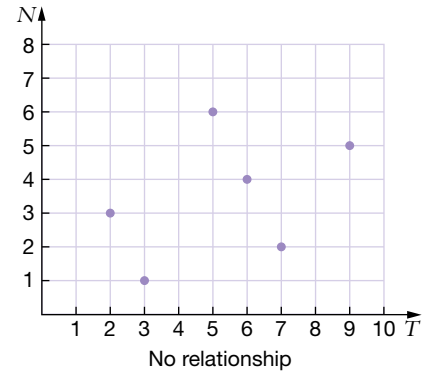
**7 a** yes, negative **b** no **c** yes, positive

**8 a**



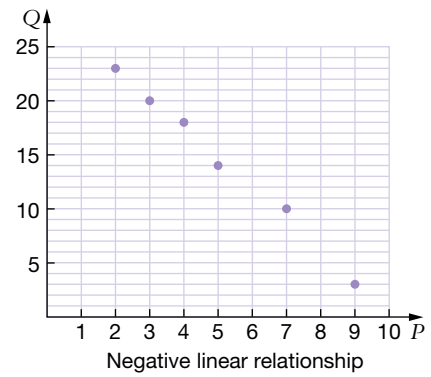
Positive linear relationship

**b**



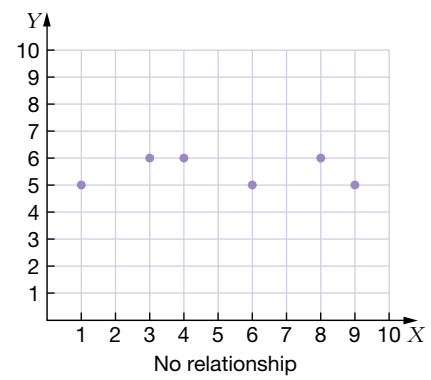
No relationship

**c**



Negative linear relationship

**d**



No relationship

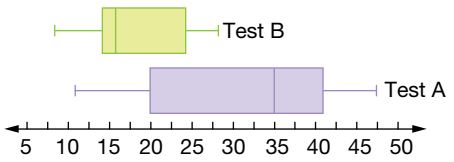
**9 a** $y = \frac{x}{2} + 3$ **b** $y = -2x + 14$

**10 a** 18 **b** 41
**c** **i** 37 **ii** 35 **iii** 20 **iv** 41
**d** (11, 20, 35, 41, 48)

**e** i 20 ii 16 iii 14 iv 24

**f** (8, 14, 16, 24, 28)

**g**



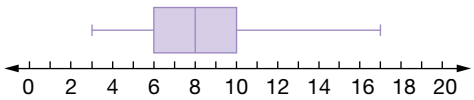

**h** i Test B ii Test A

## ASSIGNMENT 9A

**1 a** 27 **b** 7

**c** 8·22 (2 dec. pl.) **d** (3, 6, 8, 10, 17)

**e**




**f** slightly positively skewed

**2 a**

| Class | *c.c.* | Freq | $f \times c.c.$ | *c.f.* |
|---|---|---|---|---|
| 10–19 | 14·5 | 2 | 29 | 2 |
| 20–29 | 24·5 | 5 | 122·5 | 7 |
| 30–39 | 34·5 | 7 | 241·5 | 14 |
| 40–49 | 44·5 | 4 | 178 | 18 |
| 50–59 | 55·5 | 8 | 444 | 26 |
| 60–69 | 65·5 | 6 | 393 | 32 |
| 70–79 | 75·5 | 5 | 377·5 | 37 |
| 80–89 | 85·5 | 4 | 342 | 41 |
| 90–99 | 95·5 | 2 | 191 | 43 |
| | *Totals:* | 43 | 2318·5 | |

**b** 53·95 (2 dec. pl.)

**c** 53·92 (2 dec. pl.) slightly less than exact mean

**d** 35; no, as modal class is 50–59

**e** 56; yes, as median class is 50–59

**3** Without Sammy's score, the mean will be greater and the s.d. will be less.

**4 a** Nick: mean = 78, s.d. = 4·07
Dylan: mean = 78, s.d. = 5·06

**b** Nick is the more consistent golfer (because of his smaller s.d.)

**5 a** 15 hours

**b** 60 km/h

**c** At 90 km/h Liam would take 6 hours and 40 minutes.

**d** The relationship is not linear.

**6 a** i early mornings on Saturday and Sunday

ii Friday evenings followed closely by weekday afternoons

**b** i Many young people go out on Friday and Saturday evenings and are coming home in the early hours of Saturday and Sunday mornings. They could be tired or driving under the influence of alcohol and hence more likely to be involved in an accident.

ii Weekday afternoons are when people of this age group are driving home from work. It is a time of heavy traffic and their ability to react may be reduced because of fatigue or lack of concentration. This is exacerbated on Friday evenings when drivers are likely to be more tired and may be driving under the influence of alcohol having had a drink after work on Friday.

## ASSIGNMENT 9B

**1** 18

**2** 6

**3 a** 8 (100, 101, 104, 106, 109, 113, 116, 117)

**b** $685 = 3^2 + 26^2$ or $18^2 + 19^2$

**4** 26·3 cm

**5** 18·75 m

## ASSIGNMENT 9C

**1 a** 120 **b** 39·625 **c** 5 **d** 3·5

**2** Drink = \$3.50, Ice-cream = \$2.50

**3 a** 10 **b** (6, 1) **c** $-\frac{4}{3}$

**4 a** $y = 4x - 1$ **b** $y = -\frac{1}{4}x + 4$

**5 a** $x = \pm 10$ **b** $x = \frac{5}{3}$ or $-\frac{3}{2}$

**c** $x = 8$ or $-7$ **d** $x = \dfrac{1 \pm \sqrt{41}}{4}$

**6 a** $\frac{1}{36}$ **b** $\frac{1}{6}$ **c** $\frac{5}{6}$

**7 a** $5\sqrt{3}$ **b** $\sqrt{2}$ **c** 1

**d** $12 - 2\sqrt{35}$

**8 a** \$6480

**b** i \$8304.27 ii \$8567.15

**9 a** $y = 2x - 1$ **b** $x^2 + y^2 = 4$ **c** $y = 2^x$

**10 a** 725 $cm^2$ **b** 1123 $cm^3$

# Chapter 10

## Exercise 10:01

**1 a** In $\Delta$s $ABC$ and $DBE$

(1) $\angle CAB = \angle DBE$ (given)

(2) $\angle ABC = \angle BDE$ (vert. opp. $\angle$s)

$\therefore \Delta ABC \,|||\, \Delta DBE$ (2 pairs of equal $\angle$s)

$\therefore \dfrac{x}{4} = \dfrac{6}{5}$ (matching sides in the same ratio)

$\therefore x = 4{\cdot}8$

**b** In $\Delta$s $ABC$ and $DEC$

(1) $\angle ACB = \angle DCE$ (vert. opp. $\angle$s)

(2) $\dfrac{AC}{DC} = \dfrac{BC}{EC} = \dfrac{26}{16}$ (data)

$\therefore \Delta ABC \,|||\, \Delta DEC$

(1 pair of angles equal and sides about the angle in each triangle are in same ratio)

$\therefore \frac{x}{44} = \frac{16}{26}$ (ratio of matching sides)

$\therefore x = 27{\cdot}1$ (1 dec. pl.)

**c** In $\Delta$s $EFG$ and $JHG$

$\frac{JG}{EG} = \frac{JH}{EF} = \frac{GH}{GF} = \frac{7}{5}$

$\therefore \Delta EFG \,|||\, \Delta JHG$

(matching sides in same ratio)

$\therefore x = 83$ (matching $\angle$s of sim. $\Delta$s)

**d** In $\Delta$s $ABC$ and $AED$

$\frac{AD}{AC} = \frac{AE}{AB} = \frac{DE}{CB} = \frac{4}{3}$

$\therefore \Delta ABC \,|||\, \Delta AED$ (matching sides in same ratio)

$\therefore a = 53$ (matching $\angle$s of sim. $\Delta$s)

**e** In $\Delta$s $AEB$ and $DCB$

(1) $\angle AEB = \angle BCD$ (given)

(2) $\angle ABE = \angle DBC$ (vert. opp. $\angle$s)

$\therefore \Delta AEB \,|||\, \Delta DCB$ (2 pairs of equal $\angle$s)

$\therefore \frac{y}{21} = \frac{34}{24}$ (matching sides in the same ratio)

$\therefore y = 29{\cdot}75$

**2 a** Two angles of one are equal to two angles of the other.

$\frac{e}{3{\cdot}8} = \frac{f}{1{\cdot}9} = \frac{3{\cdot}8}{4{\cdot}6}$ (matching sides in same ratio)

$\therefore e = 3{\cdot}1, f = 1{\cdot}6$

**b** Two angles of one are equal to two angles of the other.

$\frac{x}{35} = \frac{30}{45}$ (matching sides in same ratio)

$x = 23\frac{1}{3}$

**c** Two pairs of sides are in the same ratio and the angles included by these sides are equal.

$\frac{x}{24} = \frac{66}{55}$ (matching sides in same ratio)

$x = 28{\cdot}8$

**d** Both triangles are right-angled and the ratio of hypotenuse to one side in one equals the ratio of hypotenuse to one side in the other triangle.

$h = 42$ (matching $\angle$s of sim. $\Delta$s)

**e** Two angles of one are equal to two angles of the other.

$a = 150$

$b = 282$

**f** Two angles of one are equal to two angles of the other.

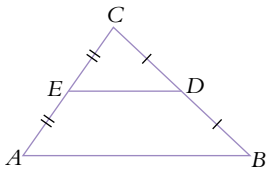
$\frac{x}{17} = \frac{50}{30}$ (matching sides in same ratio)

$\therefore x = 28\frac{1}{3}$

$\frac{30 + y}{25} = \frac{50}{30}$

$\therefore y = 11\frac{2}{3}$

**3**




In $\Delta$s $ECD$ and $ACB$

(1) $\angle ECD = \angle ACB$ (common)

(2) $\frac{EC}{AC} = \frac{DC}{BC} = \frac{1}{2}$ (given)

$\therefore \Delta ECD \,|||\, \Delta ACB$ (1 equal angle and sides about the angle are in the same ratio)

$\therefore \frac{ED}{AB} = \frac{1}{2}$ (matching sides in the same ratio)

$\therefore ED = \frac{1}{2}AB$

Also $\angle CED = \angle CAB$ (matching $\angle$s of sim. $\Delta$s)

But $\angle$s $CED$ and $CAB$ are corresp. $\angle$s

$\therefore ED \parallel AB$ (corresp. $\angle$s are equal)

**4 a** $x = 19$ (line through midpoint $\parallel$ to another side bisects third side)

**b** $x = 21$ (interval joining midpoints is $\frac{1}{2}$ third side)

**c** $x = 15$ (line through midpoint $\parallel$ to another side bisects third side)

$y = 24$ (interval joining midpoints is $\frac{1}{2}$ third side)

**5 a** **i** $\angle EBD = \angle ABC$ (common)

$\angle BED = \angle BAC$ (corresp. $\angle$s, $ED \parallel AC$)

$\therefore \Delta BED \,|||\, \Delta BAC$ (equiangular)

**ii** 4 : 9 **iii** $15\frac{5}{9}$ **iv** $19\frac{4}{9}$ **v** 4 : 5

**vi** From data and **v** we can see that $\frac{BE}{EA} = \frac{BD}{DC}$

**b** **i** 3 : 5 **ii** 3 : 5

**iii** $\angle EDH = \angle FDG$ (common)

$\frac{DE}{DF} = \frac{DH}{DG}$ (proved in **i** and **ii**)

$\therefore \Delta EDH \,|||\, \Delta FDG$ sides about an equal angle are in proportion)

**iv** and **v** Yes, matching $\angle$s of similar $\Delta$s.

**vi** Yes, since $\angle DEH$ and $\angle DFG$ are equal matching angles.

**6 a** $a = 18\frac{4}{7}$ **b** $b = 20\frac{4}{11}$ **c** $c = 25\frac{1}{5}$

**7 a** In $\Delta$s $ADE$ and $CBE$

(1) $\angle DAE = \angle BCE$ (given)

(2) $\angle AED = \angle CEB$ (vert. opp. $\angle$s)

$\therefore \Delta ADE \,|||\, \Delta CBE$ (equiangular)

$\therefore \frac{AE}{CE} = \frac{ED}{EB} = \frac{AD}{CB}$ (matching side ratios are equal)

$\therefore AE \times EB = CE \times ED$

**b** $\Delta$s $ABC$ and $DAC$ are isosceles with $\angle BAC = \angle ADC$

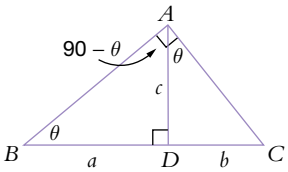
$\therefore$ All base $\angle$s are equal.

$\therefore \Delta ABC \,|||\, \Delta DAC$ (equiangular)

$\therefore \frac{a}{b} = \frac{b}{c}$ (matching side ratios are equal)

$\therefore ac = b^2$

**c**




$\angle ADC = \angle ADB = 90°$ ($AD \perp BC$)

Let $\angle ABD = \theta$

$\therefore \angle BAD = 90 - \theta$ (comp. $\angle$s)

$\therefore \angle DAC = \theta$ (comp. $\angle$s)

$\therefore \angle ABD = \angle DAC$

$\therefore \Delta ABD \,|||\, \Delta ACD$ (equiangular)

$\therefore \dfrac{AD}{DC} = \dfrac{BD}{AD}$

$\therefore \dfrac{c}{b} = \dfrac{a}{c}$

$\therefore ab = c^2$

## Prep quiz 10:02

**1** 5 : 3 **2** 25 : 9 **3** $a : b$ **4** $a^2 : b^2$ **5** $R : r$
**6** $R^2 : r^2$ **7** $k : 1$ **8** $k^2 : 1$ **9** 3 : 2 **10** 9 : 4

## Exercise 10:02

**1 a** 4 : 9 **b** 36 : 49 **c** 25 : 64
**d** 81 : 100 **e** 16 : 25 **f** 49 : 225

**2 a** $2{\cdot}6\,m^2$ **b** $270\,cm^2$
**c** $15{\cdot}625\,cm^2$ **d** $200\,m^2$
**e** $135\,cm^2$ **f** $78{\cdot}1\,m^2$

**3 a** 3 : 5, $x = 15$ **b** 3 : 4, $y = 24$
**c** 11 : 20, $x = 30$ **d** 4 : 5, $y = 2{\cdot}5$
**e** 2 : 3, $x = 15$

**4 a** Increased by a factor of 9.
**b** 4 : 5 **c** 22 500 : 1 **d** 4 : 7

**5** $0{\cdot}681\,m^2$ (3 dec. pl.)

**6 a** $418\,cm^2$ **b** 32 cm

**7** 40 kg

**8 a** 2 : 3 **b** 5 : 3 **c** 9 : 25
**d** 9 : 16 **e** 25 : 16

## Prep quiz 10:03

**1** $x : y$ **2** $x : y$ **3** $x^2 : y^2$ **4** $x^2 : y^2$
**5** $x^3 : y^3$ **6** $a : b$ **7** $a : b$ **8** $a^2 : b^2$
**9** $a^2 : b^2$ **10** $a^3 : b^3$

## Exercise 10:03

**1 a** **i** 4 : 9 **ii** 8 : 27
**b** **i** 16 : 25 **ii** 64 : 125
**c** **i** 25 : 36 **ii** 125 : 216
**d** **i** 49 : 64 **ii** 343 : 512
**e** **i** 9 : 25 **ii** 27 : 125
**f** **i** 81 : 100 **ii** 729 : 1000

**2 a** $1755\,m^2$ **b** $9056\,m^3$ **c** $19{\cdot}6875\,m^3$
**d** $140{\cdot}625\,cm^3$ **e** $306{\cdot}25\,m^2$ **f** $75\,m^2$
**g** $37{\cdot}5\,m^3$ **h** $183\,cm^2$ **i** $209\,cm^3$

**3 a** 5 : 7, 25 : 49 **b** 4 : 5, 16 : 25
**c** 15 : 16, 225 : 256

**4 a** $123{\cdot}48\,cm^2$ **b** 0·875 m
**c** $3{\cdot}23\,m^2$ (2 dec. pl.) **d** 17·1 m (1 dec. pl.)
**e** $198{\cdot}80\,m^2$ (2 dec. pl.)

**5 a** $8859{\cdot}375\,cm^3$ **b** 225 mL

**6 a** $6\,750\,000\,cm^3 = 6{\cdot}75\,m^3$
**b** 125 000 mL = 125 L

**7 a** 16 units **b** $1125\,cm^3$

**8** 250 glasses

**9 a** 81 **b** 729

**10 a** $\frac{1}{1024}$ **b** $\frac{1}{32\,768}$ **c** $2{\cdot}75\,cm^3$

## Diagnostic test 10

**1 a** 18 **b** 21·6 **c** 50

**2 a** $\angle ADE = \angle BAC$ (alt. $\angle$s, $BA \parallel DE$)
$\angle DAE = \angle ACB$ (alt. $\angle$s, $AE \parallel BC$)
$\therefore \Delta AED \,|||\, \Delta CBA$ (equiangular)

**b** $\dfrac{AB}{DB} = \dfrac{AD}{BC} = \dfrac{DB}{DC} = \dfrac{3}{2}$ (given)
$\therefore \Delta ABD \,|||\, \Delta BDC$ (matching sides in prop'n)

**c** $\angle DAC = \angle EAB$ (common)
$\dfrac{DA}{EA} = \dfrac{CA}{BA} = \dfrac{2}{1}$ (given)
$\therefore \Delta DAC \,|||\, \Delta EAB$ (sides about equal angles are in prop'n)

**3 a** **i** 100 : 49 **ii** $550\,m^2$
**b** **i** 25 : 64 **ii** $200\,m^2$
**c** **i** 9 : 10 **ii** 20 mm

**4 a** **i** 9 : 4 **ii** 27 : 8 **iii** $6750\,m^3$
**b** **i** 1 : 2 **ii** 1 : 8 **iii** $145{\cdot}6\,cm^3$
**c** $114{\cdot}8175\,m^3$

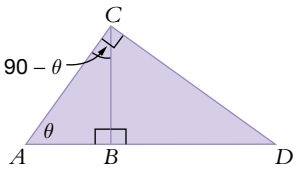
### ASSIGNMENT 10A

**1 a** $x = 9, y = 15$ **b** $a = 16, y = 20$
**c** $a = 32, y = 30$

**2 a** $x = 25{\cdot}2, y = 19{\cdot}6$ **b** $x = 21$
**c** $x = 5{\cdot}7, y = 6{\cdot}9$

**3** triangle C: 25·6 cm, 38·4 cm, 56·32 cm
triangle F: 10·8 cm, 16·2 cm, 23·76 cm

**4** $\angle ACB = 90 - \theta$ (comp. $\angle$s, $\Delta ABC$)
$\angle ADC = 90 - \theta$ (comp. $\angle$s, $\Delta ACD$)
$\therefore \angle BDC = 90 - \theta$ ($\angle ADC = \angle BDC$)
In $\Delta$s $ABC$ and $CBD$,
**1** $\angle ABC = \angle CBD$ (both right $\angle$s)
**2** $\angle ACB = \angle BDC$ (proved above)
$\therefore \Delta ABC \,|||\, \Delta CBD$ (equal angles)
$\therefore \dfrac{BD}{BC} = \dfrac{CD}{AC} = \dfrac{CB}{AB}$ (matching sides in same ratio)
$\therefore \dfrac{x}{4} = \dfrac{y}{5} = \dfrac{4}{3}$
$\therefore x = \frac{16}{3} = 5\frac{1}{3}, y = \frac{20}{3} = 6\frac{2}{3}$




5 a side of $A$ : side of $B$ = 10 : 7
b area of $A$ : area of $B$ = 100 : 49
c volume of $A$ : volume of $B$ = 1000 : 343

## ASSIGNMENT 10B

1 a 1 : 17 000 b 170 m c about 221 m
d about 136 m
2 $a = 108, b = 36, c = 36\ d = 72$
3 a $2\frac{2}{3}$ days b 8 men
4 a The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.
b The hypotenuse of a right-angled triangle must be the largest side.
5 The 40° and 20° angles cannot be correct. The angle that is marked 20° must be larger than the angle that is marked 40°.
6 83

## ASSIGNMENT 10C

1 a $x \leq -1\frac{1}{6}$ b $x < 3\frac{3}{4}$ c $x \neq \frac{1}{7}$
2 a $x$-intercepts = −1, 4; $y$-intercept = −4
b $x$-intercepts = −5, $1\frac{1}{2}$; $y$-intercept = −15
3 $a = 30, b = 30, c = 86$
4 a $12 - 10\sqrt{3}$ b −13 c $37 - 20\sqrt{3}$
5 a $\dfrac{\sqrt{3}+1}{2}$ b $\dfrac{3+\sqrt{3}}{2}$
c $\dfrac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}$
6 a $y = -2x^8 + 2$ b $y = -2x^8 - 2$
c $y = -2(x+2)^8$ d $y = -2(x-2)^8$
7 Vol. = 384 cm$^3$; S.A. = 384 cm$^2$
8 a $\dfrac{2000\pi}{3} \approx 2100\ \text{cm}^3$ b $\dfrac{14\,000\pi}{3} \approx 15\,000\ \text{cm}^3$
c $1300\pi \approx 4100\ \text{cm}^2$

# Chapter 11

## Exercise 11:01

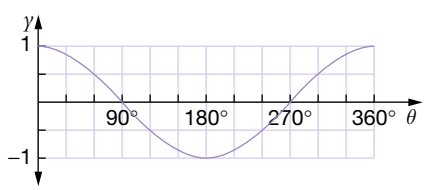
1 a 2nd b 3rd
c 4th d 1st
2 a positive b negative
c negative d positive
e negative f negative
g positive h positive
3 a 1st or 4th b 2nd or 3rd
c 1st or 2nd d 3rd or 4th
4 a 1st or 2nd b 3rd or 4th
c 1st or 4th d 2nd or 3rd
e 4th
5 a It goes from 0 to 1 b It goes from 1 to 0
c It goes from 0 to −1 d It goes from −1 to 0
6 a It goes from 1 to 0 b It goes from 0 to −1
c It goes from −1 to 0 d It goes from 0 to 1
7 a negative b 2nd and 4th
c 2nd and 4th d 1st and 3rd
8 a 1 and −1 b 1 and −1
c 1 and −1
9 a It goes from 0 up to very large numbers
b $\tan 90° = \dfrac{\sin 90°}{\cos 90°} = \dfrac{1}{0}$, but we cannot have a fraction with a denominator of zero.
10 a They are the same.
b They are the same.
c They are equal in size but opposite in sign.
d They are equal in size but opposite in sign.
11 a 30° and 150° b 50° and 130°
c 10° and 170° d 210° and 330°
e 230° and 310° f 190° and 350°
12 a 60° and 300° b 50° and 310°
c 30° and 330° d 120° and 240°
e 130° and 230° f 150° and 210°

## Investigation 11:01

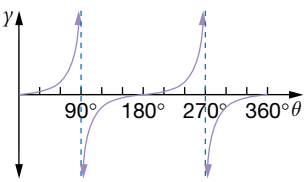
1 a

| $\theta$ | 300° | 660° | −60° |
|---|---|---|---|
| $\sin\theta$ | −0·866 | −0·866 | −0·866 |
| $\cos\theta$ | 0·5 | 0·5 | 0·5 |
| $\tan\theta$ | −1·732 | −1·732 | −1·732 |

b Yes
2 They have the same ratios because adding or subtracting 360° will not change the position of $P$ on the unit circle.
3 a no
b no
c i Translate the curve 360° to the right.
ii Translate the curve 360° to the left.
4

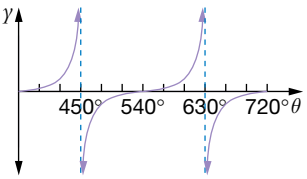



5 a Translate the curve 360° to the right.
b Translate the curve 360° to the left.
6 a



**b** The values of tan90° and tan 270° do not exist. As $\theta$ gets closer to these values, the value of tan $\theta$ becomes infinitely large.

**c**




**d** 180°

**7** **a** 360° **b** 360° **c** 180°

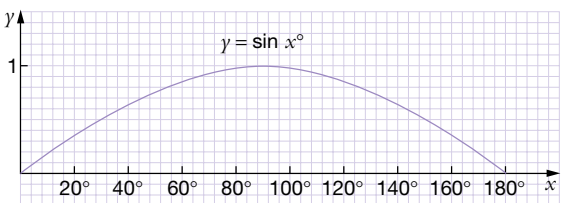
## Prep quiz 11:02

**1** 0·64 **2** 0·64 **3** 0·77 **4** −0·77
**5** yes **6** positive **7** positive **8** no
**9** $\theta = 30°$ or 150° **10** 60° or 300°

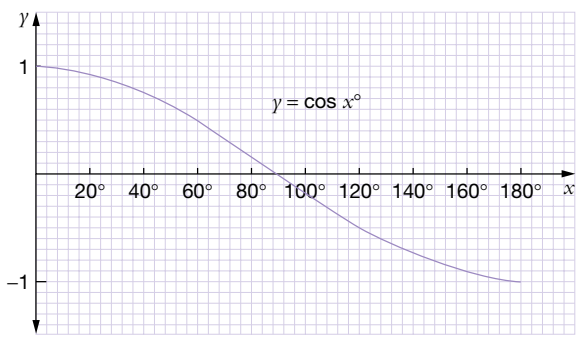
## Exercise 11:02

**1** **a** 0·216 **b** −0·770 **c** −0·570
**2** **a** sin 30° **b** −cos 80° **c** −tan 40°
**d** sin 55° **e** sin 39°28′ **f** −cos 12°29′
**g** −tan 78°57′ **h** −cos 87°10′
**3** **a** 44°16′ **b** 135°44′ **c** 83°4′
**d** 96°56′ **e** 120° **f** 135°
**g** 25°3′ **h** 110° **i** 62°0′
**j** 130° **k** 165°1′ **l** 52°59′
**m** 142°59′ **n** 139°28′ **o** 73°15′
**p** 175°34′
**4** **a** 30°, 150° **b** 46°53′, 133°7′
**c** 21°6′, 158°54′ **d** 26°56′, 153°4′
**e** 81°53′, 98°7′ **f** 21°30′, 158°30′
**g** 45°6′, 134°54′ **h** 3°15′, 176°45′
**5**

| $x°$ | 0 | 20° | 40° | 60° | 80° | 90° | 100° | 120° | 140° | 160° | 180° |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin x°$ | 0 | 0·34 | 0·64 | 0·87 | 0·98 | 1 | 0·98 | 0·87 | 0·64 | 0·34 | 0 |




**6**




**7** **a** 72° **b** 45° **c** 22° **d** 135°
**e** 162° **f** 30°
**8** **a** 54° **b** 121°

## Exercise 11:03

**1** **a** 22·6 **b** 5·2 **c** 4·7
**d** 25·3 (all answers correct to 1 dec. pl.)
**2** **a** $\frac{x}{\sin 55°} = \frac{9}{\sin 60°}$ $\therefore x = 8·5$

**b** $\frac{x}{\sin 36°} = \frac{12}{\sin 71°}$ $\therefore x = 7·5$

**c** $\frac{x}{\sin 120°} = \frac{6}{\sin 27°}$ $\therefore x = 11·4$

**3** **a** 5·3 **b** 5·8 **c** 7·7
**4** **a** **i** 46° **ii** 45°52′
**b** **i** 55° **ii** 54°44′
**c** **i** 12° **ii** 11°32′
**d** **i** 59° **ii** 59°26′
**e** **i** 39° **ii** 38°50′
**5** **a** $\frac{\sin\theta}{9} = \frac{\sin 63°}{10}$ $\therefore \theta = 53°$

**b** $\frac{\sin\theta}{3·6} = \frac{\sin 39°}{4·2}$ $\therefore \theta = 33°$

**c** $\frac{\sin\theta}{4·6} = \frac{\sin 66°}{8·9}$ $\therefore \theta = 28°$

**6** **a** 55°41′ **b** 63°16′ **c** 53°30′
**7** **a** 127° **b** 129° **c** 144°
**8** **a** 14·2 cm **b** 8·1 cm **c** 15°31′
**d** 18°33′
**9** **a** 35° **b** 77° **c** 8·4 cm
**10** **a** 66° **b** 74° **c** 28 km
**11** 37 m
**12** $P$ is 585 m from the plane; $Q$ is 864 m from the plane.

## Exercise 11:04

**1** **a** 59°, 121° **b** 12°, 168° **c** 41°, 139°
**d** 20°, 160°
**2** The obtuse angle is 139°. The acute angle is 41°.
**3** **a** In $\Delta BDC$, $\frac{\sin x°}{4} = \frac{\sin 45°}{2·9}$ $\therefore \sin x° = \frac{4\sin 45°}{2·9}$

In $\Delta ABC$, $\frac{\sin y°}{4} = \frac{\sin 45°}{2·9}$ $\therefore \sin y° = \frac{4\sin 45°}{2·9}$

**b** $x° = 103°$
$y° = 77°$
**4** **a** 64°9′ or 115°51′ **b** 24°59′ or 155°1′
**c** 37°9′ or 142°51′ **d** 31°45′ or 148°15′
**5** **a** 59°0′ or 121°0′ **b** 46°12′
**c** 73°15′ or 106°45′

## Exercise 11:05

**1** **a** 11·1 **b** 14·1 **c** 16·6 **d** 17·3
**2** **a** 53° **b** 127° **c** 53° **d** 117°
**3** **a** 4·3 **b** 10·0 **c** 6·6
**d** 8·6 **e** 7·2 **f** 13·1

**4** **a** 50°59′ **b** 62°58′ **c** 134°37′
**d** 31°55′ **e** 124°2′ **f** 47°17′
**5** **a** 9·1 cm **b** 2·8 m **c** 17·4 cm
**d** 73° **e** 46°33′
**6** 18 km
**7** 20°
**8** 21·9 cm
**9** 17·3 km
**10** 492 m
**11** 19·4 m
**12** $r$ = 731 m, $s$ = 510 m (nearest metre)

## Exercise 11:06

**1** **a** 18·0 $m^2$ **b** 84·0 $m^2$ **c** 38·7 $m^2$
**d** 24·3 $m^2$ **e** 22·9 $m^2$ **f** 50·9 $m^2$
**2** **a** 21 $cm^2$ **b** 69 $cm^2$
**3** **a** 26 $units^2$ **b** 120 $units^2$ **c** 4 $units^2$
**d** 1292 $units^2$ **e** 249 $units^2$ **f** 2033 $units^2$
**4** **a** 3064 $cm^2$ **b** 65·0 $cm^2$ **c** $\frac{325\sqrt{3}}{2}$ $cm^2$
**5** 17·2 cm

## Exercise 11:07

**1** **a** sine rule; 8·1 **b** cosine rule; 5·4
**c** sine rule; 8·2 **d** sine rule; 16·6
**e** cosine rule; 11·7
**2** **a** 25°13′ **b** 46°51′ **c** 23°52′
**d** 67°58′ **e** 48°15′ **f** 98°34′
**3** **a** 58°40′ **b** 7·7 **c** 1·1
**d** 6·9 **e** 13·2 **f** 60°0′
**4** 60°21′
**5** 1431 m
**6** 139 km (to nearest km)
**7** 20 238 $m^2$
**8** **a** 138·8 $cm^2$ **b** 7·7 cm, 19·8 cm
**9** S20°58′W
**10** 94°7′; 8·2 m

## Diagnostic test 11

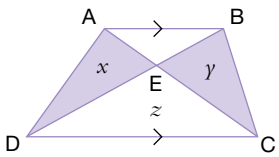
**1** **a** 150° **b** 130° **c** 100°
**2** **a** sin 40° **b** −cos 40° **c** −tan 40°
**3** **a** 128°41′ **b** 165°6′ **c** 91°2′
**4** **a** 11·2 **b** 5·7 **c** 15·4
**5** **a** 55° **b** 74° **c** 20°
**6** **a** 7·2 **b** 14·5 **c** 28·0
**7** **a** 75°31′ **b** 108°3′ **c** 119°11′
**8** **a** 26 $cm^2$ **b** 35 $cm^2$ **c** 4 $cm^2$

### ASSIGNMENT 11A

**1** **a** 150° **b** 120° **c** 135°
**2** **a** 11·9 cm (1 dec. pl.)
**b** 64°17′ (to the nearest minute)
**3** **a** $\theta$ = 43°, $\phi$ = 117°, $x$ = 15·6 cm
**b** $\theta$ = 137°, $\phi$ = 23°, $x$ = 6·9 cm
**4** **a** 85·6 $cm^2$ (1 dec. pl.) **b** 61·9 $cm^2$ (1 dec. pl.)
**5** **a** 83° (nearest degree) **b** 56°
**6** 11·5 km

### ASSIGNMENT 11B

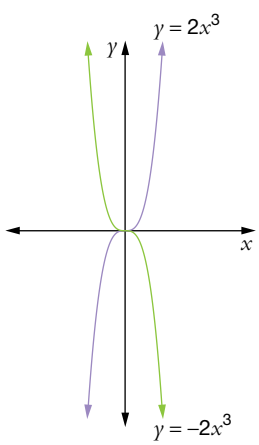
**1** $x$ = 32·5 mm, $y$ = 65 mm
**2** (1) $AD$ is bisected at $E$ by $BC$. $AB$ is parallel to $CD$.
(2) $AB$ is parallel to $CD$ and $BC$ bisects $AD$ at $E$.
(3) In $\Delta ABE$, $AE$ is produced to $D$ where $AE = ED$.
From $D$, a line is drawn parallel to $BA$.
This line meets $BE$ produced at $C$.
**3** area = 10 $unit^2$
**4** 400 $cm^2$
**5**




Proof: Let the area of $\Delta ADE$ = $x$ $unit^2$
Let the area of $\Delta BEC$ = $y$ $unit^2$
Let the area of $\Delta DEC$ = $z$ $unit^2$
Now, area of $\Delta ADC$ = $(x + z)$ $unit^2$
area of $\Delta BDC$ = $(y + z)$ $unit^2$
But area of $\Delta ADC$ = area of $\Delta BDC$
(Both $\Delta$s have same base and perp. ht.)
$\therefore\ x + z = y + z$
$\therefore\ x = y$
$\therefore$ area of $\Delta ADE$ = area $\Delta BEC$.
**6** 58·7 $unit^2$ (1 dec. pl.)

### ASSIGNMENT 11C

**1** **a** **C** **b** **E** **c** **B** **d** **D** **e** **A**
**2** $295
**3** **a** 9 : 1 **b** 25·2 $m^2$
**4** = 1·5$x$ − 5; 65·5
**5** **a** $15\sqrt{2}$ **b** 72 **c** 4 **d** $135\sqrt{5}$
**6** **a** $\frac{-5+\sqrt{17}}{2}, \frac{-5-\sqrt{17}}{2}$
**b** $\frac{3+\sqrt{89}}{8}, \frac{3-\sqrt{89}}{8}$
**7**



# Chapter 12

## Prep quiz 12:01

**1** $x = -3, 2$ **2** $x = \frac{1}{2}, -7$ **3** $x = 0, -4$
**4** $x = 2, -2$ **5** $x = 1, 2$ **6** $x = \frac{1}{2}, -4$
**7** $x = 1, 3$ **8** $x = -\frac{3}{2}, 1$ **9** $x = 2, y = 3$
**10** $x = 4, y = -1$

## Exercise 12:01

**1** **a** $x = 2, y = 8$
**b** $x = -1, y = 5$
**c** $x = 2, y = 4$
**d** $x = 3, y = 9$ or $x = -2, y = 4$
**e** $x = 2, y = 4$ or $x = -1, y = 1$
**f** $x = 1, y = 1$ or $x = -3, y = 9$

**2** Solutions same as for Question 1.

**3** **a** $x = -1, y = 1$ or $x = -3, y = 9$
**b** $x = 7, y = 49$ or $x = -7, y = 49$
**c** $x = 0, y = 0$ or $x = 3, y = 9$
**d** $x = 8, y = 64$ or $x = -7, y = 49$
**e** $x = 3, y = 9$ or $x = 7, y = 49$
**f** $x = -1, y = 1$ or $x = -2, y = 4$
**g** $x = 9, y = 86$ or $x = -5, y = 30$
**h** $x = 7, y = 56$ or $x = 1, y = 8$
**i** $x = 1, y = -9$ or $x = -10, y = 90$

**4** **a** $x = -1, y = 1$ or $x = 2, y = 4$
**b** $x = 1, y = 1$
**c** no solutions

**5** **a** $x = 1, y = 4$ or $x = -2, y = -2$
**b** **i** $x = 2, y = 2$ **ii** one
**iii** they only have one point in common

**6** **a** $x = 6, y = 36$ or $x = -1, y = 1$
**b** $x = 2, y = 5$ or $x = -1, y = 2$
**c** $x = 6, y = 33$ or $x = -2, y = 1$
**d** $x = 4, y = 12$ or $x = -1, y = 2$
**e** $x = 7, y = 28$ or $x = -2, y = 10$
**f** $x = 1, y = 4$ or $x = -7, y = 28$
**g** $x = 1, y = 4$ or $x = -1, y = -4$
**h** $x = -4, y = -2$ or $x = 1, y = 8$
**i** $x = 2, y = 3$ or $x = -6, y = -1$

**7** **a** $(2, 2)$ and $(-1, -1)$
**b** $(4, 6)$ and $(-5, 15)$
**c** $(2, 6)$
**d** $(4, 4)$ and $(-7, 15)$
**e** $(-3, 4)$
**f** no points of intersection
**g** $(5, 20)$ and $(-4, 2)$
**h** $(2, 0)$ and $(-7, 27)$
**i** $(3, 13)$ and $(-0{\cdot}5, -1)$

## Investigation 12:01

**Football scores**

Make $y$ the subject of the formula. $y = \frac{6x}{x-1}$. Since $x - 1$ is one less than $x$, $x - 1$ will not divide $x$ (when $x > 2$). Therefore answers that are positive integers can only be found when $x - 1$ divides 6 exactly.
$\therefore$ $x = 2, 3, 4$ or 7. The possible scores are 2·12·24, 3·9·27, 4·8·32 and 7·7·49.

**Number patterns**

**1** $-1, 0, 1$ or $2, 3, 4$

**2** The equation would be
$n + (n + 2)(n + 4) = n(n + 2 + n + 4)$.
This becomes $n^2 - n - 8 = 0$, which has solutions
$n = \frac{1 \pm \sqrt{33}}{2}$.
Since these answers are not integers, no solution that is an integer exists.

**3** No for 3, 4 and 5. Yes for 6. For integers differing by 6 the solution is: $-8, -2, 4$ or $9, 15, 21$.

## Prep quiz 12:02

**1** $x = 1$ **2** $x = \pm 1$ **3** $x = 1$
**4** $x = -1$ **5** no real solutions **6** $x = -1$
**7** 8 **8** no solution **9** 4
**10** $-4$

## Exercise 12:02

**1** **a** $x = 10$ **b** $a = -5$ **c** $p = 1$
**d** $k = -3$ **e** $y = \sqrt[3]{10}$ **f** $t = -\sqrt[3]{7}$
**g** $q = \sqrt[3]{2}$ **h** $n = -\sqrt[3]{20}$ **i** $z = -6$
**j** $x = 2$ **k** $z = -1$ **l** $p = \sqrt[3]{4}$

**2** **a** $x = 2{\cdot}71$ **b** $m = -2{\cdot}71$ **c** $y = 4{\cdot}64$
**d** $a = 1{\cdot}26$ **e** $x = 1{\cdot}5$ **f** $t = -2{\cdot}4$
**g** $z = 7$ **h** $n = -1{\cdot}87$ **i** $x = -12$

**3** **a** $x = 1$ **b** $m = -5$ **c** $t = 10$
**d** $z = -\frac{1}{2}$ **e** $x = \sqrt[3]{3}$ **f** $n = -\sqrt[3]{10}$
**g** $t = \sqrt[3]{20}$ **h** $z = -\sqrt[3]{\frac{1}{3}}$

**4** **a** $(-2, -8), (0, 0), (2, 8)$ **b** $x = -2, 0$ or 2
**c** three **d** $(-1, -1)$ and $(2, 8)$
**e** $x = -1$ and 2 **f** two **g** three

**5** **a** $x = 1$ **b** $x = 1$
**c** $x = -1, 0$ or 1 **d** $x = -2$ or 1

## Prep quiz 12:03

**1** $a = 27 - 15$ **2** $m = p - n$ **3** $x = \frac{35}{5}$
**4** $b = \frac{c}{a}$ **5** $x = 3$ **6** $n = -3$
**7** $m = 7\frac{1}{2}$ **8** $x = 13$ **9** $x = 81$
**10** $x = \pm 3$

## Exercise 12:03

**1** **a** $s = \frac{1-k}{2}$ **b** $S = \frac{a-b}{c}$ **c** $S = \frac{n+m}{a}$

**d** $S = \frac{y+z}{x}$ **e** $S = \frac{y-x}{a}$ **f** $S = \frac{g+q}{p}$

**g** $S = \frac{b}{a}$ **h** $S = \frac{bx}{a}$ **i** $S = \frac{uv}{t}$

**2** **a** $x = \frac{3a+b}{3}$ **b** $x = \frac{n-5m}{5}$ **c** $x = \frac{q-ap}{a}$

**d** $x = \frac{8-y}{4}$ **e** $x = \frac{tv-w}{t}$ **f** $x = \frac{m-12}{3}$

**g** $x = \frac{t-2a}{a}$ **h** $x = \frac{v-u}{u}$ **i** $x = \frac{h-km}{k}$

**3** **a** $x = \pm\sqrt{\frac{n}{m}}$ **b** $x = \pm\sqrt{\frac{a}{b}}$ **c** $x = \pm\sqrt{\frac{p}{5}}$

**d** $x = \pm\sqrt{a+b}$ **e** $x = \pm\sqrt{t-u}$ **f** $x = \pm\sqrt{v-w}$

**g** $x = \pm\sqrt{ay}$ **h** $x = \pm\sqrt{kz}$ **i** $x = \pm\sqrt{\frac{3m}{n}}$

**4** **a** $a = \frac{c^2}{b}$ **b** $a = \frac{t^2}{5}$ **c** $a = c^2 + b$

**d** $a = y^2 - 5$ **e** $a = b - r^2$ **f** $a = (m-n)^2$

**g** $a = (p+q)^2$ **h** $a = \frac{x^2}{y^2}$ **i** $a = \frac{(u-t)^2}{r^2}$

**5** **a** $x = \frac{b-a}{2}$ **b** $x = \frac{q}{a-p}$ **c** $x = \frac{a-b}{a-1}$

**d** $x = \frac{c+e}{b+d}$ **e** $x = \frac{15a}{8}$ **f** $x = \frac{4b}{4a-b^2}$

**g** $x = \frac{2a}{1-a}$ **h** $x = \frac{3a+2}{a-1}$

| | A | B | C |
|---|---|---|---|
| **6** **a** | $x = \frac{P}{y}$ | $q = p - r$ | $R = Q - P$ |
| **b** | $y = \frac{ab}{x}$ | $r = \frac{As}{3}$ | $m = \frac{2a}{n}$ |
| **c** | $x = \frac{a-y}{3}$ | $a = \frac{m-n}{t}$ | $a = \frac{v^2-u^2}{2s}$ |
| **d** | $b = \frac{x-5a}{5}$ | $n = \frac{4m}{3}$ | $t = \frac{P-Qr}{Q}$ |
| **e** | $n = \pm\sqrt{m-a}$ | $u = \pm\sqrt{v^2-t}$ | $x = \pm\sqrt{A-y^2}$ |
| **f** | $c = \frac{a^2}{b}$ | $x = \frac{Y^2}{a^2}$ | $a = x^2 - y$ |
| **g** | $Y = ax^2$ | $x = \frac{bR^2}{a}$ | $a = \frac{2V}{t^2}$ |
| **h** | $a = \frac{6y-3b}{2}$ | $m = \frac{15x+5n}{3}$ | $A = 2x - 3y$ |
| **i** | $b = m^2n - a$ | $k = \frac{h}{1-2h}$ | $m = na^2T^2$ |

**7** **a** $x \geq 4$ **b** $x \geq -3$ **c** $x \geq N$

**d** $x \leq 4$ **e** $x \leq 10$ **f** $x \leq Y$

**g** $x \geq \frac{-1}{3}$ **h** $x \geq \frac{2}{5}$ **i** $x \leq \frac{Q}{2}$

**j** 0

**8** **a** 3 **b** $-10$ **c** 1 **d** $q$ **e** $-s$

**f** $a$ **g** $2, -2$ **h** $\pm 4$ **i** $1, 2$

**9** **a** $t = 0, H = 10$

**b** $0 = 10 + 9t - t^2 \Rightarrow (10 - t)(t + 1) = 0 \Rightarrow t = 10$

**10** **a** $I = 0 \Rightarrow W = 350$ **b** $r \geq 0, r = \sqrt{\frac{S}{4\pi}}$

**c** **i** $B = \frac{A}{L}$ **ii** $L \geq 0$

**iii** $B$ will decrease

**d** **i** $l \geq 0$ **ii** $T$ will increase

**iii** $l = g\left(\frac{T}{2\pi}\right)^2 = g\frac{T^2}{4\pi^2}$

## Challenge 12:03

**a** 55, 89, 144 **b** 1·6180

**c** **i** 610 **ii** 6765 **iii** 832040

## Diagnostic test 12

**1** **a** $x = -2, y = 0$ or $x = 3, y = 5$

**b** $x = -1, y = -3$ or $x = 1, y = -3$

**c** $x = -3, y = 5$ or $x = 1, y = -3$

**2** **a** $x = 8, y = 64$ or $x = -1, y = 1$

**b** $x = 5, y = 15$ or $x = -4, y = 6$

**c** $x = 0, y = -5$ or $x = -3, y = -8$

**3** **a** $x = 3$ **b** $a = 2$ **c** $n = 2{\cdot}29$

**4** **a** $a = \frac{m-S}{n}$ **b** $a = \pm\sqrt{2P-b^2}$

**c** $a = p\left(\frac{K}{n}\right)^2$ **d** $a = \frac{Lm}{L-3}$

**5** **a** $x \not< 1$ **b** $x \not< -\frac{5}{2}$ **c** $x \neq 4$ **d** $x \neq \pm 3$

### ASSIGNMENT 12A

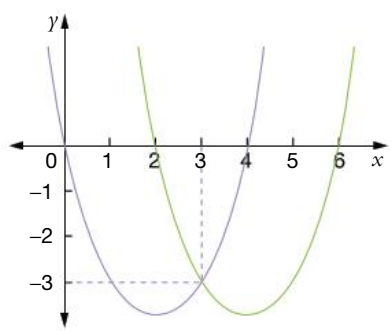
**1** **a** $x = -6, y = 0$ or $x = 1, y = 7$

**b** $x = \frac{1}{2}, y = \frac{9}{2}$ or $x = -3, y = -6$

**c** $x = \frac{1}{2}, y = \frac{5}{4}$ or $x = 3, y = 0$

**2** **a** $(3, -3)$

**b**




**3** **a** $x = 2, y = 2$ or $x = -1, y = -4$

**b** $x = 4, y = 1$ or $x = 2, y = 2$

**c** $x = 2, y = 2$

**4** **a** $x = 0{\cdot}375$ **b** $x = \pm 0{\cdot}612$ **c** $x = 0{\cdot}721$

**5** **a** $y = -64$ **b** no solutions **c** $y = -4$

**6** **a** $a = \dfrac{P + bc}{b}$ **b** $c = \dfrac{ab - P}{b}$

**c** $b = \dfrac{P}{a - c}$ **d** $a = \pm\sqrt{h^2 - b^2}$

**e** $p = \dfrac{q(Z + 1)}{Z - 1}$

**7** **a** $a \geq 4$ **b** $a \leq 4$ **c** $a \geq -4$ **d** $a \geq 0$

**8** **a** $n \neq 3$ **b** $n \neq -3$

**c** $n \neq 0$ **d** $n \neq 3$ or $-3$

**9** **a** $t = 0$ or 6

$\therefore$ ball returns to the ground after 6 seconds

**b** 90 metres **c** $0 \leq t \leq 6, 0 \leq h \leq 90$

**10** $Z = -2 \pm \sqrt{\dfrac{5}{X}}$

## ASSIGNMENT 12B

**1** 721 **2** 35·25 m **3** 12 cm

**4** 14 h 36 min, assuming a steady rate of climb and a slide back that takes only a second or two

**5** 66 **6** 11

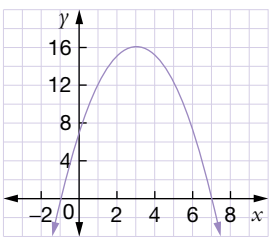
## ASSIGNMENT 12C

**1** **a** $x = 29$ **b** $x = -\frac{1}{3}$ or $\frac{5}{2}$

**c** $x = \pm 6$ or $\pm 1$

**2** **a** $2\sqrt{2}$ **b** $36\sqrt{2}$ **c** 30

**3**




**a** $(-1, 0), (7, 0)$ **b** $(0, 7)$

**c** $(3, 16)$ **d** max. value = 16

**4** Volume = 2111·2 cm$^3$, Surface Area = 1131·0 cm$^2$

**5** **a** mean = $67{\cdot}\dot{3}$, stand. dev. = 12·8

**b** 78 **c** It will be less.

**6** In $\Delta$s $ADE$ and $ACB$

$\angle A$ is common

$\angle ADE = \angle ACB$ (given)

$\therefore \Delta ADE \,|||\, \Delta ACB$ (2 pairs of corres. $\angle$s equal)

$\therefore x = 9{\cdot}35$

$\therefore y = 7{\cdot}18$ (2 dec. pl.)

**7** **a** 11·9 **b** 133°55′

# Chapter 13

## Prep quiz 13:01

**1** radius **2** diameter **3** chord

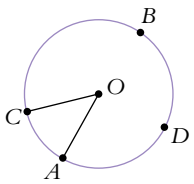
**4** arc **5** sector **6** segment

**7** tangent **8** circumference **9** semicircle

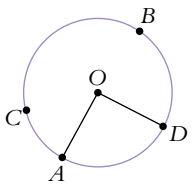
**10** 2

## Exercise 13:01

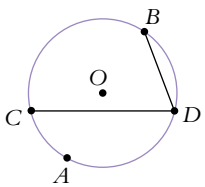
**1** **a**



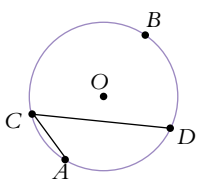

**b**




**c**




**d**




**2** **a** **i** $\angle AYX$ **ii** $\angle BXY$

**b** **i** $\angle XAY, \angle XBY$ **ii** $\angle AXB, \angle AYB$

**3** **a** $\angle ADE, \angle ABE$

**b** $\angle AOC, \angle ADC, \angle ABC$

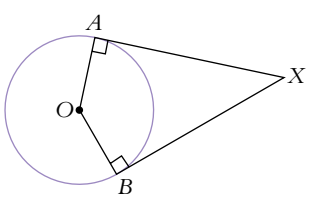
**c** 3 **d** $\angle AED$ **e** $AC$

**4** **a** **i** $\angle BDC, \angle BFC$ **ii** $\angle BAC, \angle BEC$

**b** **i** $\angle BAC, \angle BEC$ **ii** $\angle BDC, \angle BFC$

**5** **a** 90°

**b**




**6** See Student Book. Exercise 13:01 Question 6.

## Exercise 13:02

**1** **a** $MD = \frac{1}{2}CD$ (perp. from $O$ bisects chord)

$\therefore MD = 17{\cdot}5$

**b** $MF = EM$ (perp. from $O$ bisects chord)

$\therefore MF = 27$ cm

**c** $GH = 2GM$ (perp. from $O$ bisects chord)

$\therefore GH = 36$ m

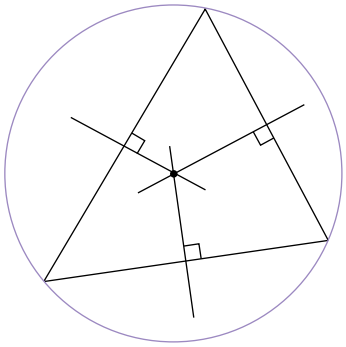
**d** $MH = 9$ cm, $JK = 18$ cm

**e** $\angle RMO = 90°$

**f** $TM = 9{\cdot}3$ cm, $MU = 9{\cdot}3$ cm

**2** **a** **i** Both $MP$ and $NP$ are perpendicular bisectors of chords.
**ii** A perpendicular bisector of a chord passes through the centre of the circle.
**iii** $P$ is the centre as both $MP$ and $NP$ pass through it.
**b** See Question **2a**.
**c** See Question **2a**.

**3**



**4** **a** 3 cm, 6 cm **b** 5·3 m, 10·6 m
**c** 5 mm, 10 mm **d** 10 cm, 16 cm
**e** 7 cm, 7·6 cm **f** 9 m, 4·4 m
**g** 5·3 cm **h** 6·4 cm

**5** **a** **i** 6·5 cm **ii** 6·7 cm **iii** 13·0 cm
**b** 12·2 cm **c** 9·2 cm

**6** **a** In $\Delta$s $POC$ and $QOC$
$PO = QO$ (equal radii)
$PC = QC$ (equal radii)
$OC$ is common.
$\therefore \Delta POC \equiv \Delta QOC$ (SSS)
**b** $\therefore \angle POC = \angle QOC$ (matching $\angle$s in cong. $\Delta$s)
**c** In $\Delta$s $PON$ and $QON$
$\angle POC = \angle QOC$ (proved in **b**)
$ON$ is common.
$PO = QO$ (equal radii)
$\Delta PON \equiv \Delta QON$ (SAS)
**d** $\therefore PN = QN$ (matching sides in cong. $\Delta$s)
$\therefore N$ bisects $PQ$.
$\angle PNO = \angle QNO$ (matching $\angle$s in cong. $\Delta$s)
Since $\angle PNO + \angle QNO = 180°$ (straight angle)
then $\angle PNO = \angle QNO = 90°$
$\therefore PQ \perp OC$

## Investigation 13:02

**1** B **2** G **3** H **4** J

## Exercise 13:03

**1** **a** $OM = 8$ m (equal chords are the same distance from the centre)
**b** $x = 8{\cdot}5$ (as in **a**)
**c** $ON = 3{\cdot}5$ cm (as in **a**)
**d** $CD = 11{\cdot}5$ m (chords equidistant from the centre are equal)
**e** $FG = 20$ m (as in **d**)
**f** $KL = 13{\cdot}8$ m (as in **d**)

**2** **a** $\angle COD = 55°$ (equal chords subtend equal angles at centre)
**b** $FE = 8$ m (as in **a**)
**c** $\angle JOK = 51°$ (as in **a**)
**d** $WX = 12$ m (perp. from $O$ bisects $WX$)
$\therefore ON = 5$ m (as in **a**)
**e** $QP = 14$ m (chords equidistant from the centre are equal.)
$\therefore QM = 7$ m (perp. from $O$ bisects $QP$)
**f** $\angle BOD = 125°$ (as in **a**)

**3** **a** 120° **b** 90° **c** 60°
**d** $51\frac{3}{7}°$ **e** 45° **f** 36°

**4** Your answers will look like **3a**, **3b**, **3c** and **3e**.

**5** Constructions should match those shown in question.

## Prep quiz 13:04

**1** 30 **2** 120 **3** 60 **4** 50 **5** 32
**6** 50 **7** 40 **8** 94 **9** 50 **10** 2

## Exercise 13:04

**1** **a** 44° **b** 25° **c** 34° **d** 39° **e** 39°
**f** 39° **g** 96 **h** 118 **i** 150

**2** **a** $a = 33, b = 33$ **b** $c = 62, d = 62$
**c** $e = 59, f = 59$ **d** $g = 39$
**e** $h = 58, i = 58$ **f** $k = 98$
**g** $m = 90, n = 90$ **h** $p = 180$
**i** $t = 90$

**3** **a** **i** 30° **ii** 60° **iii** 35° **iv** 70°
**v** 65° **vi** 130°
**b** **i** 30° **ii** 60° **iii** 35° **iv** 70°
**v** 65° **vi** 130° **vii** yes
**c** 120 **d** 125 **e** 204

## Exercise 13:05

**1** **a** $a = 90$ (angle in a semicircle)
**b** $b = 90$ (angle in a semicircle)
**c** $c = 90$ (angle in a semicircle)
**d** $\angle ABC = 90°$ (angle in a semicircle)
$\therefore d + 147 = 180$ (angle sum of $\Delta$)
$\therefore d = 33$
**e** $\angle EDF = 90°$ (angle in a semicircle)
$\therefore e + 160 = 180$ (angle sum of $\Delta$)
$\therefore e = 20$
**f** $\angle JHG = 90°$ (angle in a semicircle)
$\therefore f + 162 = 180$ (angle sum of $\Delta$)
$\therefore f = 18$
**g** $g + 76 = 180$ (cyclic quad.)
$\therefore g = 104$
**h** $h + 91 = 180$ (cyclic quad.)
$\therefore h = 89$
$i + 78 = 180$ (cyclic quad.)
$\therefore i = 102$
**i** $j + 82 = 180$ (cyclic quad.)
$\therefore j = 98$
$k + 80 = 180$ (cyclic quad.)
$\therefore k = 100$

**2** **a** 53 **b** 127
**c** $a = 53, b = 127$ **d** $c = 70, d = 110$
**e** $e = 75, f = 105$ **f** $g = 45, h = 135$
**g** $m = 50, n = 40$ **h** $m = 46, n = 44$

**3** **a** yes (opp. ∠s supplementary)
**b** no **c** yes (as in **a**)
**d** no **e** yes (as in **a**)
**f** no

**4** **a** $a = 110$ (cyclic quad.), $b = 70$ (adj. supp. angles)
**b** $c = 70$ (cyclic quad.), $d = 110$ (adj. supp. angles)
**c** $e = 85$ (cyclic quad.), $f = 95$ (adj. supp. angles)
**d** $g = 86$ (cyclic quad.), $h = 94$ (adj. supp. angles)
**e** $j = 60$ (cyclic quad.), $k = 120$ (adj. supp. angles)

## Prep quiz 13:06

**1** 6 m **2** $AT$ **3** $OP$ **4** $DT$ (or $DE$)
**5** radius **6** 75 **7** 56 **8** 20 **9** 4
**10** 13

## Exercise 13:06

**1** **a** $a = 90$ (radius $OT \perp$ tangent $TP$)
**b** $b = 14$ (equal tangents from $P$)
**c** $\angle OTP = 90°$ (radius $OT \perp$ tangent $TP$)
$\therefore c + 143 = 180$ (angle sum of Δ) $\therefore c = 37$
**d** $d = e$ ($TP = WP$, equal tangents from $P$)
$\therefore 2e + 66 = 180$ (angle sum of Δ) $\therefore e = d = 57$
**e** $f + 36 = 90$ (radius $OT \perp$ tangent $TP$) $\therefore f = 54$
**f** $h = 90$ (radius $OT \perp$ tangent $TP$),
$g = 25$ (corresp. to $\angle OPT$ about axis of symmetry $OP$)
**g** $k = 90$ (radius $OT \perp$ tangent $TP$),
$m = 90$ (radius $OW \perp$ tangent $PW$),
$n = 132$ (angle sum of a quadrilateral)
**h** $p = 8{\cdot}2$ (equal tangents from $P$),
$q = 64$ (Δ $PTW$ is isosceles)

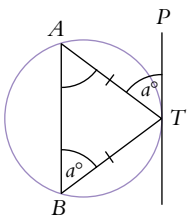
**2** **a** $x = 39$ **b** $y = 120$
**c** $a = 48$ **d** $e = 105, f = 105$

**3** **a** $a = 90, b = 25, c = 25, d = 65$
**b** $e = 57, f = 57, g = 123$
**c** $g = 35, h = 35, k = 110$
**d** $k = 90, m = 65$
**e** $n = 90, r = 91, v = 62$
**f** $w = 34, x = 34, y = 112, z = 56$
**g** $a = 58, b = 64, c = 32$
**h** $d = 67, e = 23$
**i** $f = 38, g = 52$

**4** **a** $a = 8$ **b** $b = 15$
**c** $c = 90, d = \sqrt{51}, e = \sqrt{89}$
**d** $f = 8$ **e** $t = 5$

**5** **a** Let $\angle BTP$ be $x°$.
$\therefore \angle BAT = x°$ (angle in the alternate segment)
$\angle BOT = 2x°$ (angle at centre is twice angle at circum. on same arc)
$\therefore \angle BOT = 2\angle BTP$
**b** $\angle ABT + \angle RBA + \angle PBT = 180°$
($\angle RBP$ is a straight angle)
But $\angle ATQ = \angle ABT$ (angle in the alternate segment)
$\therefore \angle ATQ + \angle RBA + \angle PBT = 180°$
**c** Let $\angle BAT$ be $a°$.
$\therefore \angle BTP = a°$ (angle in the alternate segment)
$\angle PBT = a°$ (angle in the alternate segment)
$\therefore \angle BPT + a° + a° = 180°$ (angle sum of a triangle)
$\therefore \angle BPT = 180° - 2a°$
$= 180° - 2\angle BAT$ QED

**6** Let $\angle PTA$ be $a°$
$\therefore \angle ABT = a°$ (angle in the alternate segment)
$\angle BAT = a°$ ($\angle BAT$ and $\angle ABT$ are base angles of isos. Δ$BAT$)
$\therefore \angle PTA = \angle BAT$ (both $a°$)
$\therefore AB \parallel PT$ (alternate angles are equal)

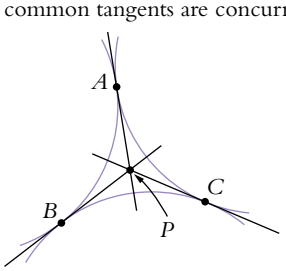



**7** **a** $\angle ATO = 90°$ (angle between tangent and radius)
$\angle ATC = 90°$ (angle between tangent and radius)
$\therefore \angle OTC = 180°$ ($\angle ATO + \angle ATC$)
$\therefore OC \perp AB$
**b** (first two lines the same as **a**)
$\therefore \angle ATO = \angle ATC$ (same angle)
$\therefore OC \perp AB$

**8** $PT = PA$ (equal tangents from $P$)
$PT = PB$ (equal tangents from $P$)
$\therefore PA = PB$

**9** $PA = PC$ (equal tangents from $P$)
$\angle APO = \angle CPO$ (given)
$\therefore PO$ is axis of symmetry of the circle passing through the centre $O$.
Similarly for $QO$ and $RO$.
$\therefore$ Angle bisectors are concurrent at the centre $O$.

**10** The tangents from $A$ and $B$ will meet at a point $P$ with $AP = BP$ (equal tangents from an external point). But from $P$, $AP$ will also equal $CP$ and $CP$ will equal $BP$. Since $P$ is the common external point, the three common tangents are concurrent.

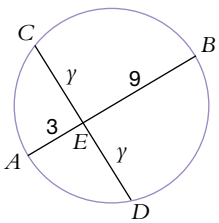


## Exercise 13:07

**1** **a** $b = 18$ **b** $c = 12$
**c** $d = 16$ **d** $e = 6$
**e** $f = 1$ **f** $h = 4$
**g** $k = 6$ **h** $r = 12$
**i** $t = 12$ **j** $a = 15, b = 10\frac{2}{3}$
**k** $c = 15, d = 13\frac{1}{3}$ **l** $x = \sqrt{32} = 4\sqrt{2}$

**2** **a** 6 m **b** 6·9 cm **c** 11·8 m
**d** 37·5 m **e** $34\frac{1}{15}$ m **f** 11·2 cm
**g** 13·4 cm

**3**




$\gamma^2 = 3 \times 9$
$\gamma = \sqrt{27}$ or $3\sqrt{3}$
$\therefore CE = 3\sqrt{3}$ m
(Note: Consider only positive solutions as lengths cannot be negative.)

**4** $PC^2 = AC.CB$ (square of the tangent equals the product of the intercepts)
$TC^2 = AC.CB$ (as for line 1)
$\therefore PC^2 = TC^2$
$\therefore PC = TC$ (lengths are positive)

## Prep quiz 13:08

**1** complementary angles
**2** supplementary angles
**3** vertically opposite angles
**4** 360°
**5** corresponding angles
**6** alternate angles
**7** co-interior angles
**8** alternate angles, lines parallel
**9** isosceles Δ since radii are equal
**10** angle sum of a Δ

## Exercise 13:08

**1** **a** $\angle OBC = 43°$ (alternate to $\angle AOB$, $AO \parallel BC$)
$\angle OCB = 43°$ ($\Delta OBC$ is isosceles)
$a + 86 = 180$ (angle sum of $\Delta OBC$)
$\therefore a = 94$

**b** $\angle AOB = 50°$ (alternate to $\angle OBC$, $AO \parallel BC$)
$\angle ACB = 25°$ (angle at circumference is half angle at centre)
$\therefore b = 25$

**c** $\angle TAO = 30°$ ($\Delta OTA$ is isosceles)
$\angle OTP = 90°$ (radius $OT \perp$ tangent $TP$)
$\therefore \angle ATP = 120°$
$d + 150 = 180$ (angle sum of $\Delta ATP$)
$\therefore d = 30$

**d** $\angle COB = 118°$ (angle at centre is twice angle at circumference)
$\angle CBO = e°$ ($\Delta OBC$ is isosceles)
$2e + 118 = 180$ (angle sum of $\Delta OBC$)
$\therefore e = 31$

**e** $\angle ACB = 35°$ (angle at circumference is half angle at centre)
$e + 35 = 180$ ($\angle ACD$ is a straight angle)
$\therefore e = 145$

**f** $\angle CDB = 90°$ (only the bisector of $AB$ which is perpendicular to $AB$ will pass through $O$)
$f + 150 = 180$ (angle sum of $\Delta BCD$)
$\therefore f = 30$

**g** $\angle TOW = 120°$ (angle at centre is twice angle at circumf.)
$\angle OTP = 90°$ (radius $OT \perp$ tangent $TP$)
$\angle OWP = 90°$ (radius $OW \perp$ tangent $WP$)
$g + 300 = 360$ (angle sum of quad. $WOTP$)
$\therefore g = 60$

**h** $\angle OTP = 90°$ (radius $OT \perp$ tangent $TP$)
$\therefore \angle OTA = 30°$ (120° − 90°)
$\therefore \angle OAT = 30°$ (isosceles $\Delta OAT$)
$h + 60 = 180$ (angle sum of $\Delta OAT$)
$\therefore h = 120$

**i** $\angle DOC = 50°$ (angles subtended by equal chords at the centre are equal)
$\therefore k + 140 = 360$ (angles at a point)
$\therefore k = 220$

**j** $\angle FJH = 63°$ (adjacent supplementary angle to $\angle FJK$)
$\angle FGH = 117°$ (cyclic quad. $FGHJ$)
$\therefore m = 117$

**k** $\angle NQL = 108°$ (cyclic quad. $LMNQ$)
$\angle NQR = 72°$ (adjacent supplementary angles)
$\therefore n = 72$

**l** $\angle ABC = 90°$ (angle in a semicircle)
$\therefore \angle BCD = 90°$ (co-interior to $\angle ABC$, $BA \parallel CD$)
$\therefore r = 90$

**m** $\angle PTW = \angle TWP$ (isosceles $\Delta TPW$ formed by equal tangents from $P$)
$\therefore 2a + 44 = 180$ (angle sum of $\Delta PTW$)
$\therefore a = 68$
$\angle AWP = 90°$ (radius $OW \perp$ tangent $PW$)
$\therefore b = 22$ (90° − 68° = 22°)
$\angle ATW = 90°$ (angle in a semicircle)
$\therefore x + 112 = 180$ (angle sum of $\Delta ATW$)
$\therefore x = 68$

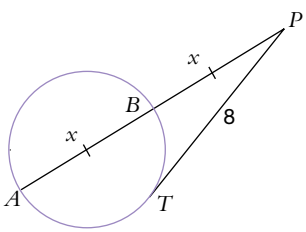
**n** $\angle OTP = 90°$ (radius $OT \perp$ tangent $TP$)
$\therefore a = 40$ $(90° - 50° = 40°)$
$\angle OBT = 40°$ ($\Delta OBT$ is isosceles)
$b + 80 = 180$ (angle sum of $\Delta OBT$)
$\therefore b = 100$
$\angle BAT = 50°$ (angle at circumference is half angle at centre)
$\therefore y = 50$

**2 a** $\angle THF = e°$ (angle in alternate segment)
$\angle THG = e° + 40°$
$\angle GFQ = \angle THG$ (ext. angle of cyclic quad.)
$\therefore 85 = e + 40$
$\therefore e = 45$

**b** $AE = x$ cm (perp. from centre bisects chord $AB$)
$\therefore x^2 = 8 \times 2$ (products of intersecting chords)
$x = 4$
In $\Delta OEB$, $OE^2 + EB^2 = OB^2$ (Pythagoras' theorem)
$\therefore 5^2 = y^2 + x^2$
$25 = y^2 + 16$
$\therefore y = 3$

**c** $\angle TAC = 30°$ (angle in alternate segment)
$\angle CAB = 65°$ (angle at circum. is half the angle subtended at centre on same arc)
$\angle BAT = \angle TAC + \angle CAB$
$\therefore x = 30 + 65$
$= 95$

**3** Let $AB$ and $BP$ be $x$ cm.
$\therefore 2x \times x = 64$
$\therefore x = \sqrt{32}$ (since $x > 0$)
$= 4\sqrt{2}$




**4 a** $\angle PTN = 90°$ (radius $NT \perp$ tangent $PT$)
$\angle OSN = 90°$ (corresp. to $\angle PTN$, $OS \parallel PT$)

**b** $\angle PTS = 90°$ ($\angle PTN = 90°$, $\angle NTS = 180°$)
$\angle OPT = 90°$ (radius $OP \perp$ tangent $PT$)
$\angle SOP = 90°$ (corresp. to $\angle OPT$, $OS \parallel PT$)
$\therefore OPTS$ is a rectangle (all angles 90°)

**c** $\Delta OSN$ is a right-angled $\Delta$.
$SN = 11$ cm, $ON = 15$ cm
$OS^2 + SN^2 = ON^2$
$OS^2 + 121 = 225$
$\therefore OS = \sqrt{104}$
$\therefore PT = \sqrt{104}$ ($PT$ and $OS$ are opposite sides of a rectangle)

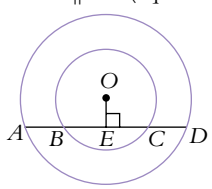
**5** Let $\angle CTQ = x°$
$\therefore \angle CAT = x°$ (angle in alternate segment)
$\angle PTD = x°$ (vert. opp. to $\angle CTQ$)
$\therefore \angle DBT = x°$ (angle in alternate segment)
$\therefore \angle CAT = \angle DBT$
$\therefore AC \parallel DB$ (alternate angles are equal)

**6** Let $BT$ be $x$ m.
$\therefore BP = x$ m (equal tangents)
Join $OT$ and $OP$.
$OT \perp AB$ (radius perp. to tangent)
$\therefore AT = x$ m (perp. from centre to chord bisects the chord)
$\therefore AB = 2x$ m
Similarly $CB = 2x$ m
$\therefore AB = CB$

**7** $\angle PSR = \angle PTR$ (opp. $\angle$s in parallelogram are equal)
$\angle PSR = \angle RQT$ (ext. angle of a cyclic quad.)
$\therefore \angle RQT = \angle QTR$
$\therefore \Delta RQT$ is isosceles.

**8** $\angle ECA = \angle ABC$ ($\angle$ in alt. seg't)
$\angle ABC = \angle BAC$ (base $\angle$s in isos. $\Delta$)
$\therefore \angle BAC = \angle ECA$
$\therefore AB \parallel ED$ (equal alt. $\angle$s)

**9**




Let $OE$ be perp. bisector of both chords.
Then $AE = DE$ and $BE = CE$
$\therefore AE - BE = DE - CE$
$\therefore AB = CD$

**10 a** In $\Delta$s $AEB$ and $ACD$
$\angle A$ is common.
$\angle AEB = \angle ACD$ (ext. angle of cyclic quad. equals interior opp. angle)
$\Delta AEB \parallel\!\parallel\!\parallel \Delta ACD$ (equiangular)

**b** But $\angle AEB = 90°$ (angle in a semicircle)
$\therefore \angle ACD = 90°$ (matching angles in similar triangles)

**11** $\angle TAC = \angle ABC$ ($\angle$ in alt. seg't)
$\angle TAC = \angle BPC$ (alt. $\angle$s, $AT \parallel BP$)
$\therefore$ In $\Delta$s $ABC$ and $ABP$
$\angle ABC = \angle BPC$ (shown above)
$\angle A$ is common.
$\therefore \angle ABP = \angle ACB$ (angle sum of $\Delta$)

## Fun spot 13:08

| Dots | Chords | Sections |
|---|---|---|
| 5 | 10 | 16 |
| 6 | 15 | 30 |

## Diagnostic test 13

**1** **a** $\angle AOB$ **b** $\angle ADB$ or $\angle ACB$ **c** $\angle BDC$

**2** **a** $AM = 4{\cdot}5$ cm (perp. from $O$ bisects $AB$)

**b** The perp. bisector of a chord passes through the centre (as it is an axis of symmetry).

**c** $CD = 36$ m (perp. from $O$ bisects $CD$)

**3** **a** $OM = 10$ m **b** $OE = 15$ cm **c** $FG = 60$ m

**4** **a** $ON = 8$ m (equal chords are the same distance from the centre)

**b** $GH = 11$ km (chords equidistant from the centre are equal) $\therefore$ $GN = 5{\cdot}5$ km (perp. from centre bisects the chord)

**c** $YZ = 14$ m (chords equidistant from the centre are equal)

**5** **a** $\angle AOB = 69°$ (angles subtended at centre by equal chords)

**b** $EF = 1{\cdot}1$ m (equal angles at the centre are subtended by equal chords at circumference)

**c** $\angle BOC = 72°$, $\angle BOD = 144°$

**6** **a** $a = 108$ **b** $b = 86$ **c** $c = 110$

**7** **a** $a = 36$ **b** $b = 50, c = 50$ **c** $d = 99$

**8** **a** $a = 90$ **b** $b = 37$ **c** $c = 90, d = 90$

**9** **a** $a = 93, b = 110$ **b** $c = 104, d = 115$

**c** $e = 100, f = 78, g = 102$

**10** **a** $a = 90$ **b** $c = 24$ **c** $e = 41$

**11** **a** $a = 8, b = 67$ **b** $c = 67, e = 23$ **c** $d = 17$

**12** **a** $a = 80$ **b** $x = 64$ **c** $y = 62$

**13** **a** $y = 4$ **b** $f = 19$ **c** $r = 5$

### ASSIGNMENT 13A

**1** **a** $a = 31, b = 59$ **b** $c = 142, d = 109$

**c** $e = 65, f = 53$ **d** $x = \sqrt{85}$

**e** $y = 7{\cdot}2$ **f** $r = 6{\cdot}89, t = 6{\cdot}6$

**g** $v = 45, w = 35, x = 100$

**h** $a = 0{\cdot}9, b = 110$

**i** $c = 65, d = 65, e = 50$

**j** $f = 45, g = 90, i = 45, j = 45$

**k** $k = 152$ **l** $n = 41, m = 68$

**m** $x = \sqrt{32} = 4\sqrt{2}$ **n** $m = 154, n = 77$

**2** **a** **i** $WP^2 = AP \,.\, PB$ (square of tangent = product of intercepts)

$PT^2 = AP \,.\, PB$ (same reason)

$\therefore$ $WP = PT$

**ii** $AB = 11{\cdot}2$ (to 3 sig. figs.)

**b** Join $WT$ to meet $PA$ at $B$.

In $\Delta$s $WPB$ and $TPB$

$BP$ is common

$\angle WPB = \angle TPB$ ($PB$ bisects $\angle WPT$)

$WP = PT$ (equal tangents)

$\therefore$ $\Delta WPB \equiv \Delta TPB$ (SAS)

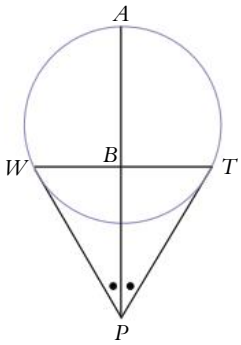
$\therefore$ $WP = BT$ (matching sides)

and $\angle WBP = \angle TBP$ (matching angles of congruent $\Delta$s)

$\therefore$ $\angle WBP = 90°$ ($\angle WBT = 180°$)

$\therefore$ $PA$ is the perp. bisector of chord $WT$.

$\therefore$ $PA$ passes through the centre of the circle.




**c** $\angle ABC = 48°$

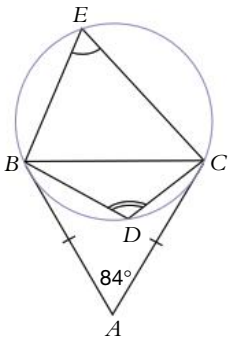
(isos. $\Delta ABC$)

$\angle BEC = 48°$

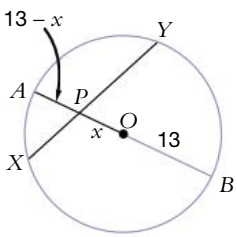
(angle in the alternate segment)

$\angle BDC = 132°$

(opp. angles of cyclic quad. are suppl.)




**d**




Let $OP$ be $x$ cm.

$\therefore$ $AP = (13 - x)$ cm

and $PB = (13 + x)$ cm

$AP \,.\, PB = XP \,.\, PY$

$(13 - x)(13 + x) = 25$

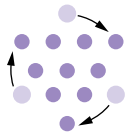
$169 - x^2 = 25$

$\therefore$ $x = 12$

$\therefore$ $OP$ is 12 cm

## ASSIGNMENT 13B

**1**



**2** 3 **3** 2 cm, 6 cm, 7 cm
**4** 10 **5** 126 984
**6** **a** 91 **b** 1392 **c** 18 893
**d** $(n - 99) \times 3 + 90 \times 2 + 9$

# Chapter 14

## Exercise 14:01

**1** **a** $C$ **b** $A$ **c** $B$ **d** $A$ **e** $C$
**2** b, c, g, h, i
**3** **a** 3, 2, 3 **b** 5, 1, −2 **c** 4, 9, 0
**d** 1, −2, 5 **e** 6, 7, 0 **f** 2, 1, 5
**g** 6, −1, 9 **h** 0, 3, 3 **i** 5, −2, 0
**j** $2, \sqrt{3}, 1$ **k** $4, \frac{3}{2}, 0$ **l** $4, \frac{1}{3}, \frac{1}{5}$
**4** **a** 15 **b** 3 **c** 4 **d** 8
**5** **a** **i** 3 **ii** 18 **iii** 4
**b** **i** −3 **ii** 807 **iii** −21
**c** **i** 62 **ii** 15 506 **iii** 8
**6** **a** $4x^2 + 12x + 9$; 2, $4x^2$, 9
**b** $3x + 3$; 1, $3x$, 3
**c** $12x$; 1, $12x$, 0
**d** $3x^4 - x^2 + 2x$; 4, $3x^4$, 0
**e** $9x^2 - 36x^3 + 36x^4$; 4, $36x^4$, 0
**f** $12x$; 1, $12x$, 0
**g** $x^3 + 6x^2 + 11x + 6$; 3, $x^3$, 6
**h** $36x^3 - 15x^2 - 2x + 1$; 3, $36x^3$, 1
**7** **a** $P(x) = 2x^2 + 3$
**b** $P(x) = 2x^3 + 2x^2 + 2x$
**c** $P(x) = x^3 + x + 4$ or $x^3 + \frac{1}{2}x^2 + 4$

## Prep quiz 14:02

**1** 2 **2** 3 **3** 4 **4** 3 **5** 1
**6** 1 **7** 5 **8** 0 **9** 6 **10** $B, C$

## Exercise 14:02

**1** **a** $x^3 + 3x^2 - x - 1$ **b** $2x^3 + 2x^2 - x$
**c** $x^3 + x^2 + 6x - 3$ **d** $x^3 + x^2 - 7x + 3$
**e** $2x^2 - 7x + 2$ **f** $-x^3 + x^2 - 1$
**g** $-2x^2 + 7x - 2$ **h** $2x^3 + 3x^2 + 2x - 2$
**i** $3x^2 - 4x$ **j** $2x^3 + x^2 - 4x + 2$
**k** $x^2 - 10x + 4$ **l** $-x^2 + 10x - 4$
**2** **a** $x^3 - 3x + 10$ **b** $2x^3$
**c** $3x^4 - x^3 + x^2 + x$ **d** $x^4 + 3x^3 + 2x^2 + 3x + 2$
**e** $x^3 + 6x^2 - 7x + 7$ **f** $x^2 - 2x + 2$
**g** $x^3 + 3x^2 - 11$ **h** $4x^4 + x^3 - 7x^2 + 4x - 6$
**i** $8x^2 - 2x + 5$ **j** $3x^3 - 4x + 12$
**k** $3x^3 - x^2 - 2x + 11$ **l** $6x^5 - 2x^4 - x^3 + x^2 + 7x + 12$
**3** **a** $3, x^3$ **b** $4, 4x^4$ **c** $4, 5x^4$
**d** $2, 3x^2$ **e** $5, 2x^5$ **f** $1, x$
**4** **a** 3 **b** 2 **c** 4
**d** 2 **e** 5 **f** 2
**5** **a** $A(x) = x^3 + 2x^2 + x + 4$, $B(x) = x^3 - 3x - 2$
**b** 22, 0, 11, 11 **c** 4, 0, 2, 2

## Exercise 14:03

**1** **a** $x^3 + 5x^2 + 7x + 2$
**b** $x^3 + 2x^2 - 5x + 12$
**c** $x^3 + 3x^2 - 14x + 8$
**d** $2x^3 - 5x^2 + 12x - 5$
**e** $3x^4 - 2x^3 - 3x^2 + 5x - 2$
**f** $-x^4 + 7x^3 - 10x^2 - x + 5$
**2** **a** true **b** true
**3** **a** 4 **b** 2
**4** **a** $x^4 + 6x^3 + 9x^2 + 3x - 4$
**b** $P(1) = 3$; $Q(1) = 5$; $R(1) = 15$
**c** $P(-2) = -3$; $Q(-2) = 2$; $R(-2) = -6$
**d** $R(a) = P(a) \cdot Q(a)$
**5** **a** $(x + 1)(x + 1) - 4$
**b** $(x - 3)(x - 2) - 5$
**c** $(x + 3)(3x - 8) + 22$
**d** $(x - 5)(2x + 6) + 33$
**e** $(x + 1)(x^2 + x) - 7$
**f** $(x - 3)(3x^2 + 9x + 25) + 81$
**g** $(x - 2)(2x^2 + 3x + 7) + 13$
**h** $(x - 1)(3x^2 + 3x + 5) + 0$
**6** **a** $6x + 17$; 53
**b** $x^2 - 4x + 9$; −17
**c** $x^2 - x + 2$; 1
**d** $3x^2 + 2x - 1$; 0
**e** $x^4 - 2x^3 + 4x^2 - 5x + 6$; −3
**f** $2x^5 + 2x^4 + 4x^3 + 4x^2 + x + 1$; 5
**7** **a** $(x + 1)(x^2 + x + 2)$
**b** $(2x + 1)(x^2 - 2x + 3)$
**c** $(x - 1)(2x^2 - x - 1)$
**d** $(x + 1)(2x^2 - x + 1)$

## Exercise 14:04

**1** **a** 3 **b** 0 **c** 6 **d** 18
**e** 16 **f** 0
**2** **a** 13 **b** 15 **c** 10 **d** 40
**e** −4 **f** 36 **g** 13 **h** −1
**3** **a** 3 **b** −7 **c** 71 **d** 5
**4** **a** 30 **b** 2 **c** 2 or −3
**5** **a** yes **b** no **c** no **d** yes
**e** yes **f** yes
**6** **a** $(x - 1)(x + 2)(x + 3)$
**b** $(x + 1)(x - 4)(x - 3)$
**c** $(x + 1)(x + 1)(x + 2)$
**d** $(x - 2)(x - 2)(x - 1)$
**7** **a** $(x - 1)(x - 3)(x + 2)$
**b** $(x - 2)(x - 7)(x + 3)$

**c** $(x + 1)(x - 2)(x - 3)$
**d** $(x - 1)(x + 5)(x - 4)$
**e** $(x + 1)(x + 6)(x - 3)$
**f** $(x + 1)(2 - 5x)(1 + 3x)$

**8** **a** 4 **b** −10 **c** −6 **d** 4

**9** $a = -2, b = -13$

**10** $p = -7, q = -6$

## Exercise 14:05

**1** the factors of 12, namely 1, 2, 3, 4, 6, 12 and their opposites (ie −1, −2, −3, −4, −6, −12)

**2** Divide $x^3 - 2x^2 - 11x + 12$ by $x - 3$.

**3** **a** $(x - 4)(x + 3)$ **b** $(2x - 1)(x - 2)$

**4** **a** $x = 3, 2, -1$ **b** $x = -1, 4$
**c** $x = \frac{1}{2}, -4, 1, -2$ **d** $x = 2, -3$

**5** **a** $(x - 2)(x - 4)(x + 2) = 0; x = 2, 4, -2$
**b** $(p + 3)(2p - 1)(p + 3) = 0; p = -3, \frac{1}{2}$
**c** $(x - 1)(x - 1)(x + 2)(x + 3) = 0; x = 1, -2, -3$
**d** $(p + 2)(p + 2)(p - 3)(2p + 1) = 0; p = -2, 3, -\frac{1}{2}$

**6** **a** Substitute $x = -1$ and show that it satisfies the equation.
**b** $2x^3 - 9x^2 + 7x + 6$
**c** Show that $Q(2) = 0$.
**d** $2x^4 - 7x^3 - 2x^2 + 13x + 6$
$= (x + 1)(x - 2)(2x + 1)(x - 3)$
**e** $-1, 2, -\frac{1}{2}, 3$

**7** **a** $-1, 2, 3$ **b** $-2$ **c** $-1, 1$
**d** $-2, -3, 3$ **e** 1 **f** $-3$

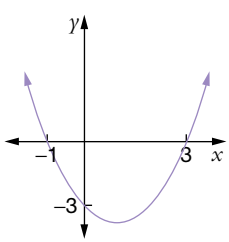
**8** **a** $-1, -2, 3, 4$ **b** $1, 4, -2$
**c** $x^2(x + 10) = 2000$; 10, 10, 20

## Prep quiz 14:06

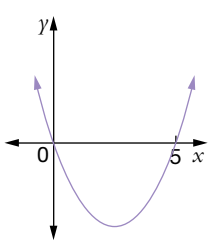
**1** 1 **2** 2 **3** 3
**4** $x^3 + x^2 - 2x$ **5** 0, 1, −2 **6** 0, 1, −2
**7** 0 **8** yes **9** $x^4$
**10** $x^3$

## Exercise 14:06

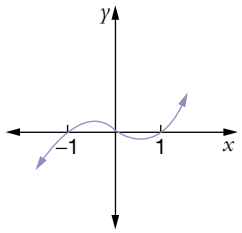
**1** **a**



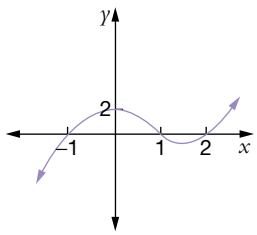

**b**



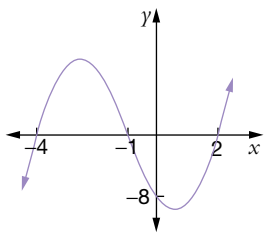

**c**



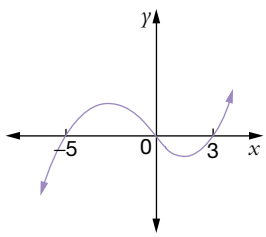

**d**



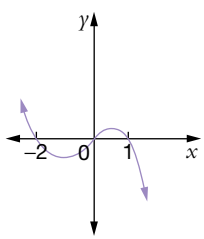

**e**



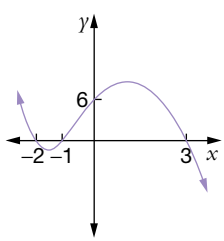

**f**



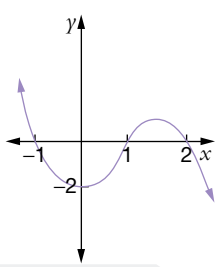

**g**



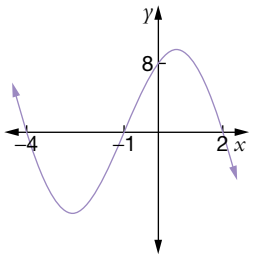

**h**



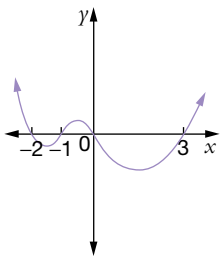

**i**



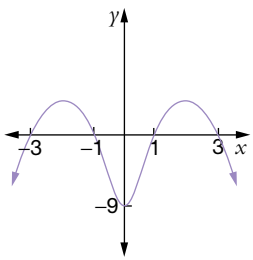
j



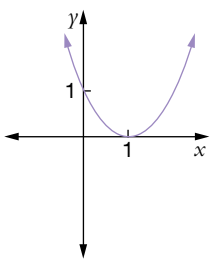

k



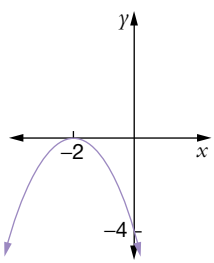

l



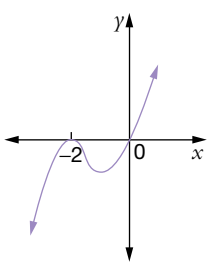

2 a



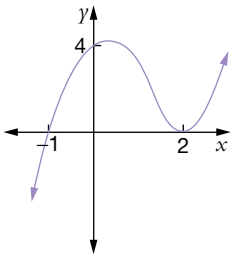

b



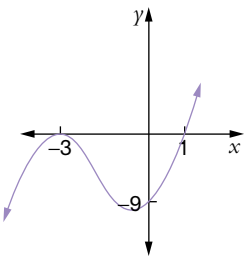

c



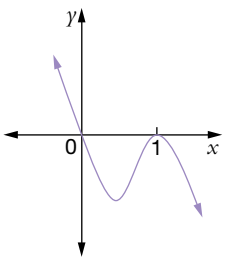

d



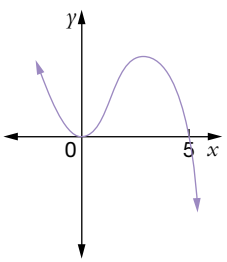

e



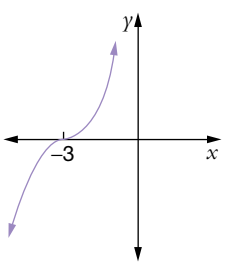

f



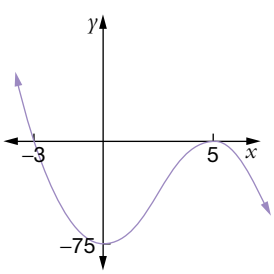

g



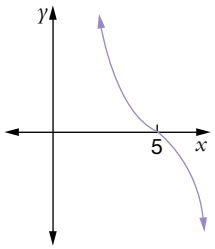

h



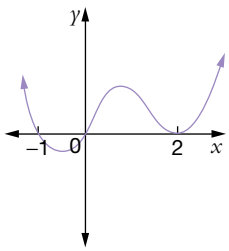

i



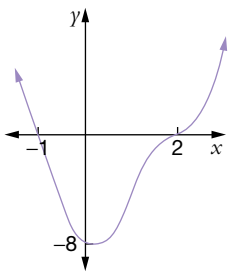
j



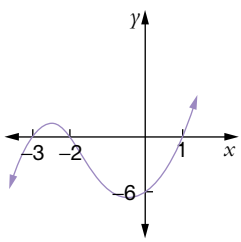

k



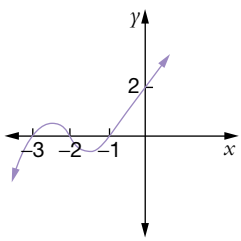

l



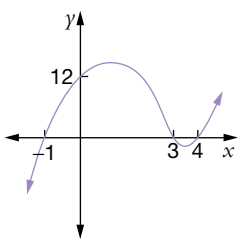

3 a



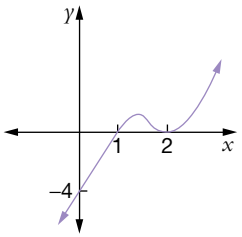

b



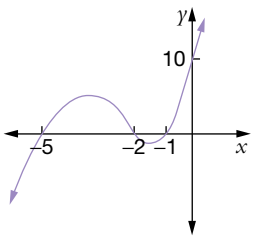

c



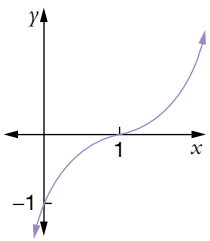

d



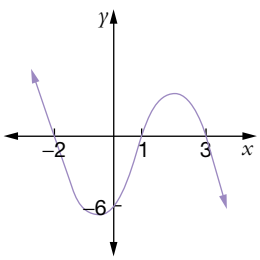

e



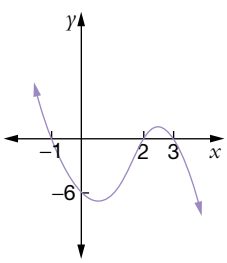

f



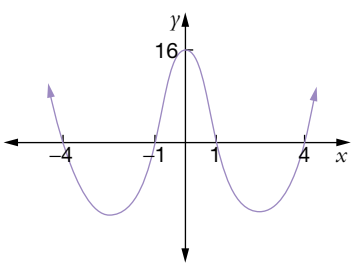

g



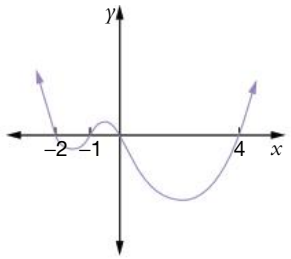

h




i



j




4 a $y = P(x)$ where $P(x) = -(x - 2)^2(x + 1)$

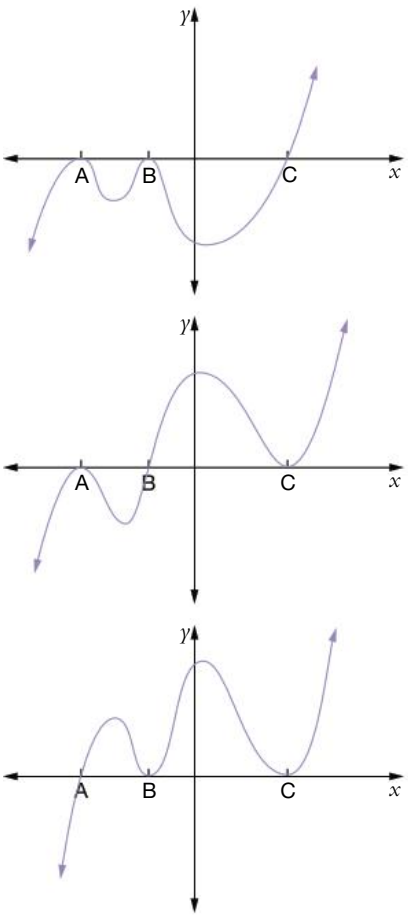
b $y = P(x)$ where $P(x) = -2(x + 2)^2(x - 1)^3$

c $y = (x + 3)(x + 1)(x - 4)^2$
or $-(x + 3)(x + 1)^2(x - 4)$
or $-(x + 3)^2(x + 1)(x - 4)$

d $y = -(x + 3)(x + 1)(x - 4)^2$
or $(x + 3)(x + 1)^2(x - 4)$
or $(x + 3)^2(x + 1)(x - 4)$

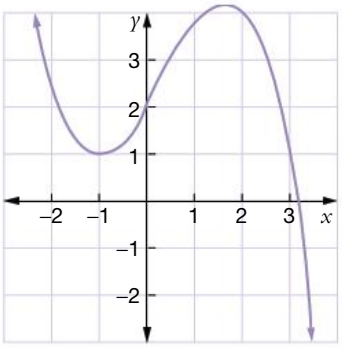
5 no; equations like: $(x - 1)^3 = 0$; $(x - 1)^2(x + 1) = 0$; $(x - 1)(x - 2)(x - 3) = 0$

6 There are three possible shapes. Each curve can be translated horizontally so that the roots occur in different positions.

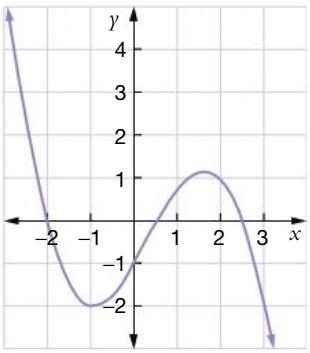



## Exercise 14:07

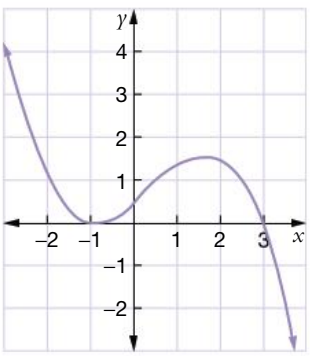
1 a



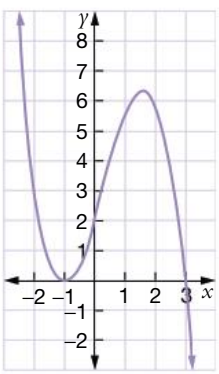

b



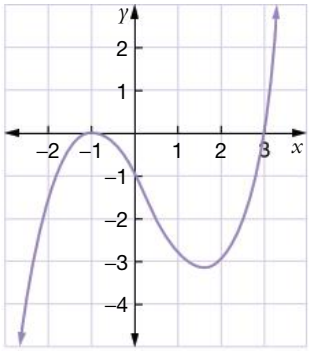

c



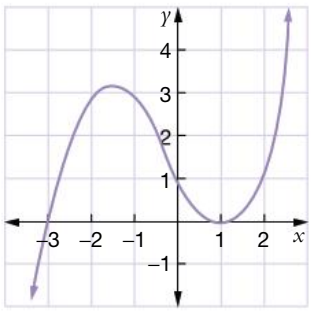

d



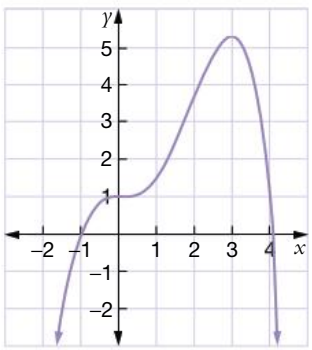
e



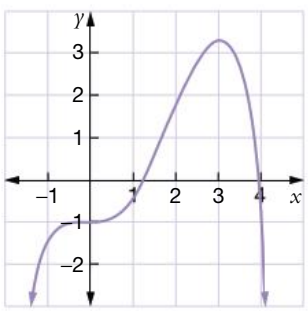

f



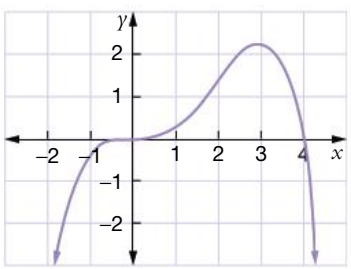

2 a



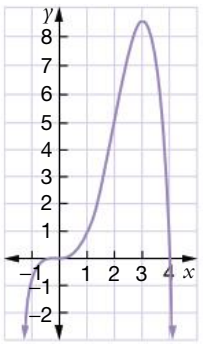

b



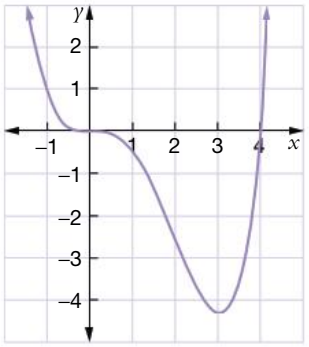

c



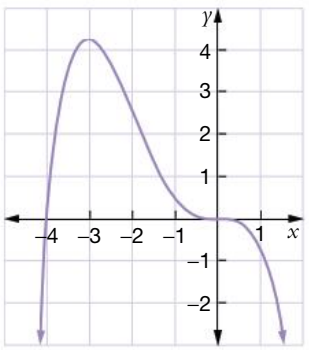

d



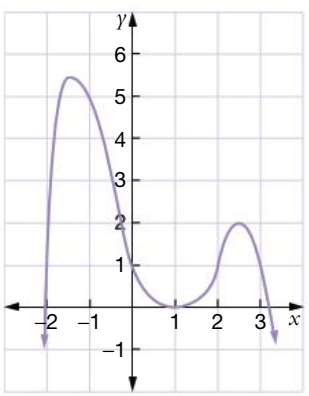

e



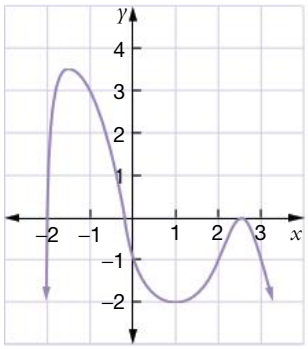

f



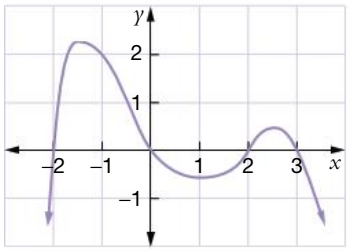

3 a



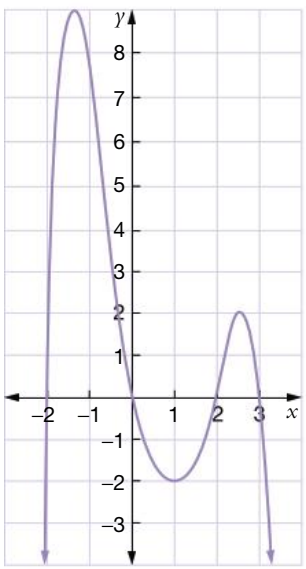
b


y
x
4
3
2
1
−1
−2
−2
−1
1
2
3

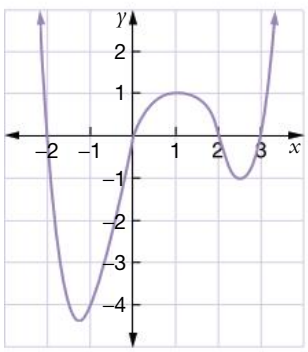
c


y
x
2
1
−1
−2
−1
1
2
3

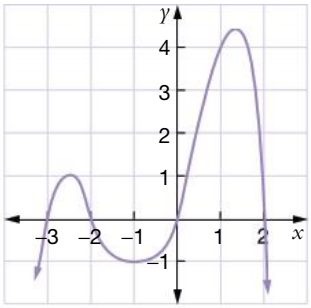
d


y
x
8
7
6
5
4
3
2
1
−1
−2
−3
−2
−1
1
2
3

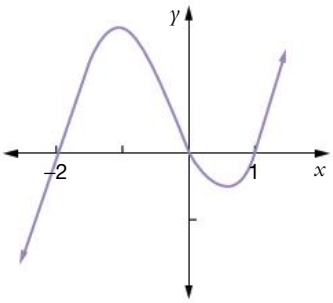
e


y
x
2
1
−1
−2
−3
−4
−2
−1
1
2
3

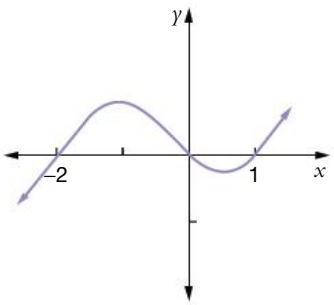
f


y
x
4
3
2
1
−1
−3
−2
−1
1
2

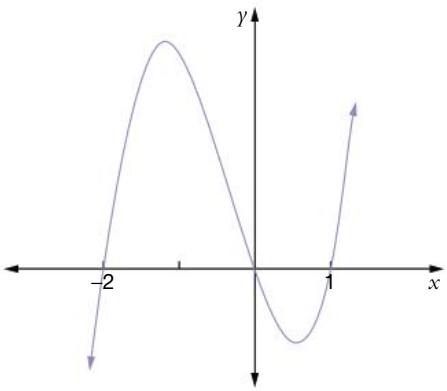
4


y
x
−2
1

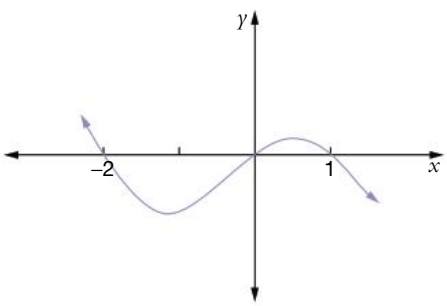
a


y
x
−2
1

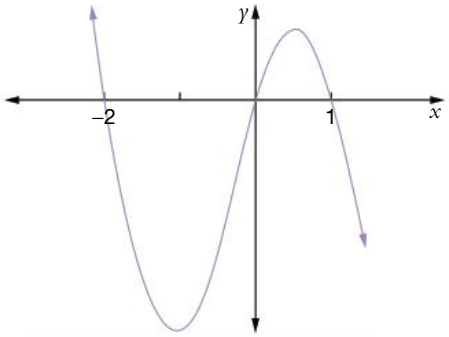
b


y
x
−2
1

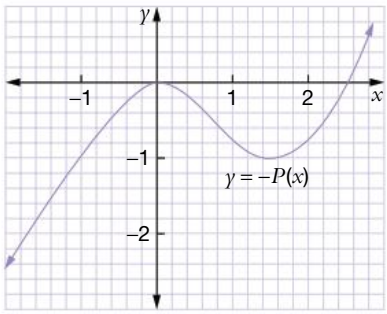
c

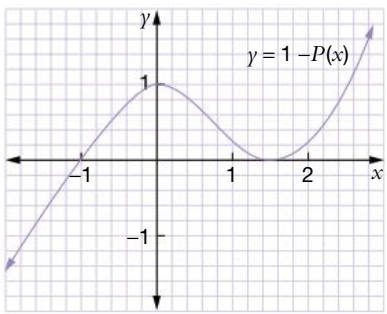

y
x
−2
1

d

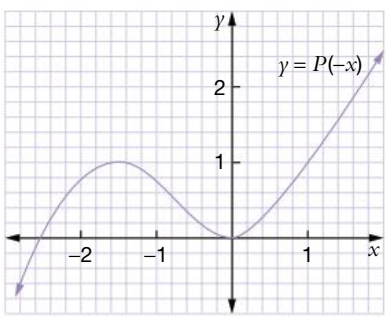


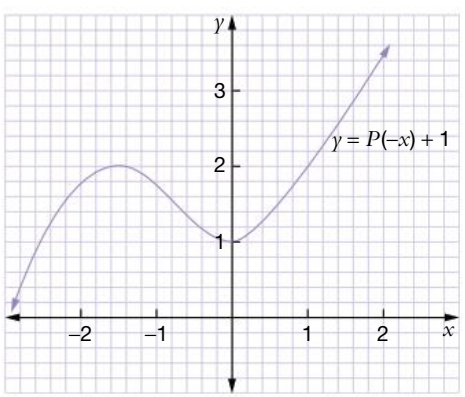

5 a



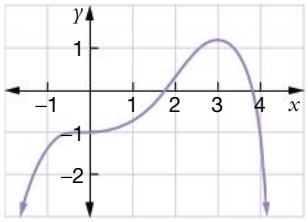




b

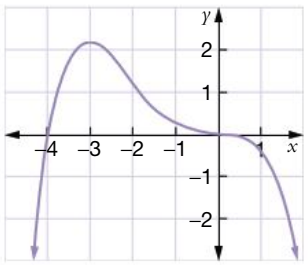


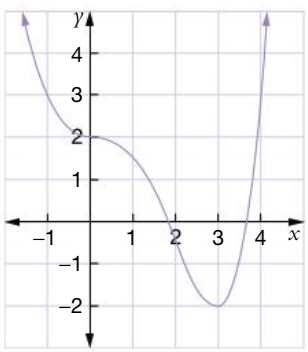

6 a



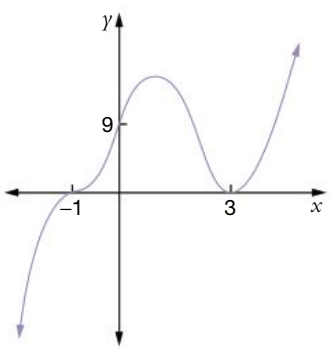

b



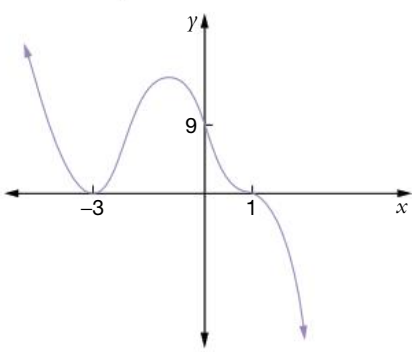

c



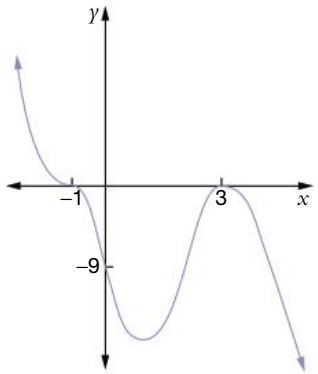

7 a



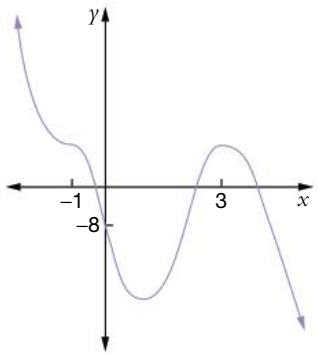

b



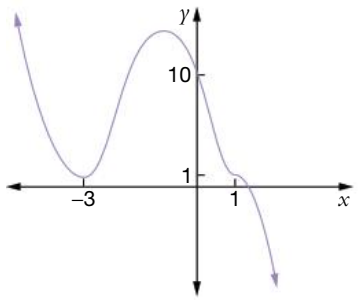
c




d




e




## Diagnostic test 14

**1** **a** 2 **b** 7 **c** 3

**2** **a** $-5x^3$ **b** $9x^4$ **c** $x^2$

**3** **a** $B$ **b** $A$ **c** $B$

**4** **a** $2x^3 + 3x^2 - 2x$

**b** $2x^3 + x^2 - 6x$

**c** $2x^3 - x^2 - 8x + 4$

**5** **a** $2x^3 + 7x^2 + 4x - 4$

**b** $2x^4 + 5x^3 - 3x^2 - 8x + 4$

**c** $4x^5 + 8x^4 - 11x^3 - 13x^2 + 16x - 4$

**6** **a** $2x - 1$

**b** $2x^2 - 3x + 1$

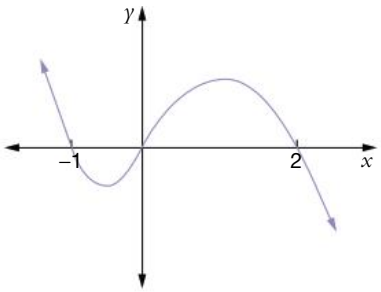
**c** $2x^2 + 3x - 2$

**7** **a** 9 **b** −16 **c** 226

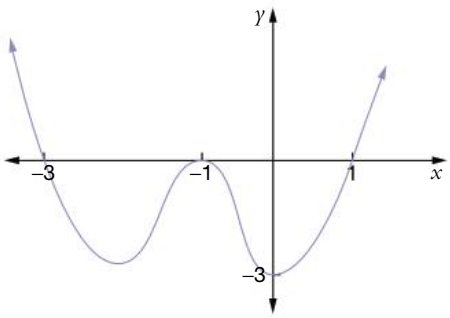
**8** **a** no **b** yes **c** yes

**9** **a** 0, 1, −2 **b** −1, 2 **c** −1, 2, 3

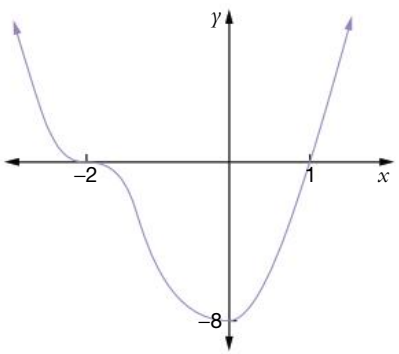
**10** **a**



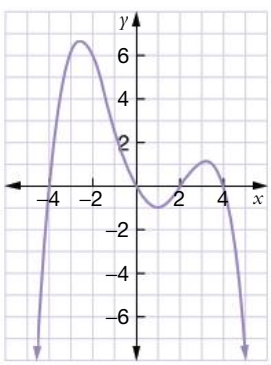

**b**



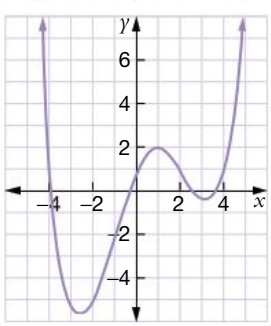

**c**



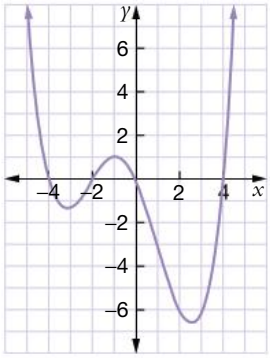

**11** **a**



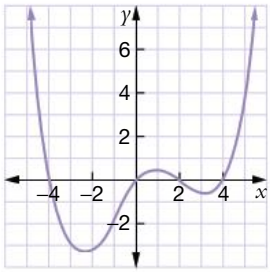

**b**



c




d




## ASSIGNMENT 14A

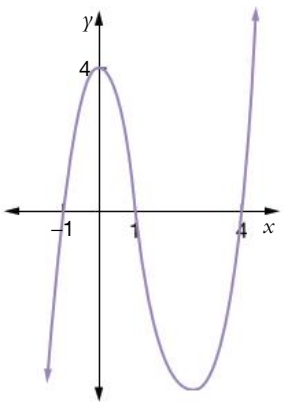
**1** **a** degree = 3, leading term = $-3x^3$, constant term = $-3$

**b** 0 **c** $a = -\frac{1}{3}, b = 2$ **d** 31

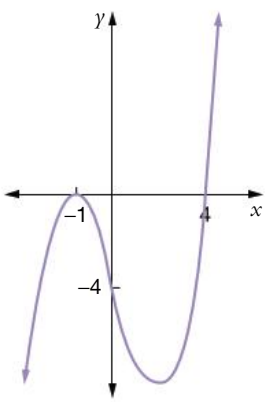
**2** **a** $x = 0, 3, -4$ **b** $x = 3, -\frac{3}{2}, \frac{1}{3}$

**c** $x = 0, 2$ **d** $x = -2, 2$

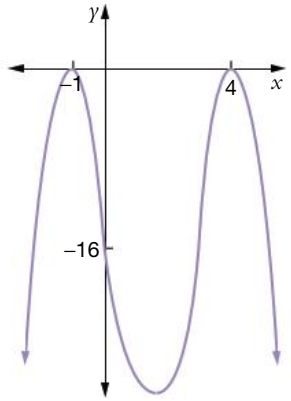
**3** **a**



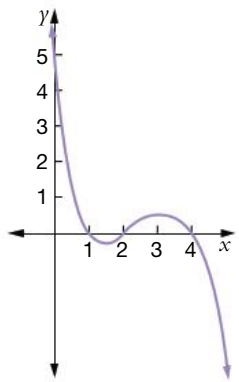

**b**



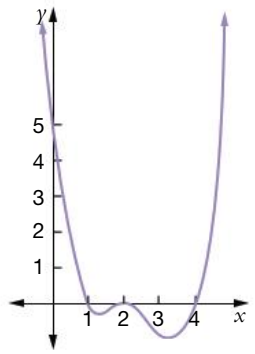

**c**



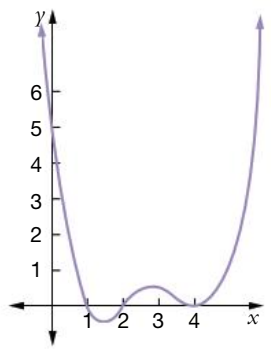

**4** **a**

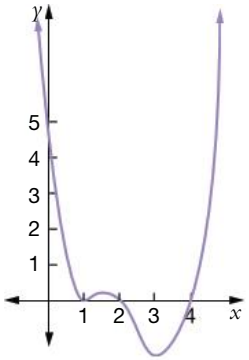



**b**

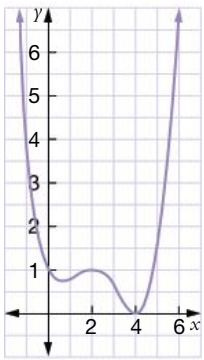



Or

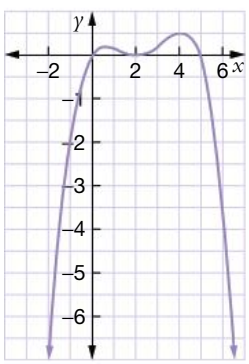




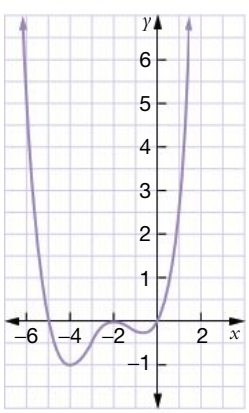

5 a




b




c




6 a *Note:* Many other answers are possible.

i $y = x^4 + 1$

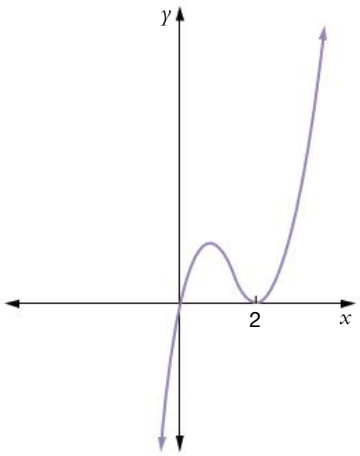
ii $y = x^4$

iii $y = x^2(x - 1)^2$

iv $y = x^2(x - 1)(x + 1)$

v $y = x(x - 1)(x - 2)(x - 3)$

b




i 1 ii 3

## ASSIGNMENT 14B

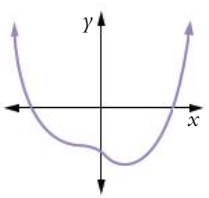
1 The leading term provides information on how the curve will behave as $x$ becomes very large or very small (ie as $x$ approaches positive or negative infinity). For these values of $x$, the sign of the polynomial $P(x)$ is the same as the sign of the leading term, $ax^n$.

If $n$ is even, then $ax^n$ is positive when $a$ is positive and negative when $a$ is negative. This means that the ends of the curve will point in the same direction as shown below.

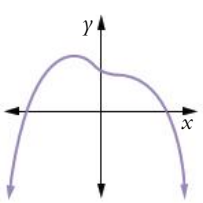
If $n$ is odd and $a$ is positive, then $ax^n$ will be positive when $x$ is positive and negative when $x$ is negative. When $a$ is negative, then $ax^n$ will be negative when $x$ is positive and positive when $x$ is negative.

This means that the ends of the curves will point in opposite directions as shown below.

$n$ is even

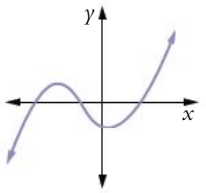



$a > 0$

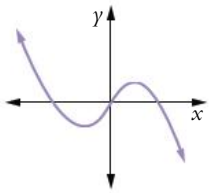



$a < 0$

$n$ is odd




$a > 0$




$a < 0$

**2** A is $y = \frac{1}{4}(x-2)^2 + 1$ or $y = \frac{1}{4}x^2 - x + 2$

B is $y = (x-2)^2 + 1$ or $y = x^2 - 4x + 5$

C is $y = -\frac{3}{4}(x-2)^2 + 1$ or $y = -\frac{3}{4}x^2 + 3x - 2$

**3** 11

**4** $5\frac{1}{4}$ h

**5** $\frac{1}{4}$ of the area of each square

**6** Rearranging $ax + by + c = 0$ into gradient–intercept form gives $y = -\frac{a}{b}x - \frac{c}{b}$. If the gradient is negative, then $-\frac{a}{b}$ is negative and hence $\frac{a}{b}$ is positive. This means that $a$ and $b$ would be either both positive or both negative.

# Chapter 15

## Prep quiz 15:01

**1** 64 **2** 64 **3** 64 **4** 2 **5** 3
**6** 6 **7** $2^2$ **8** $2^3$ **9** $2^{-1}$ **10** $2^{\frac{1}{2}}$

## Exercise 15:01

**1** **a** $\log_2 8 = 3$ **b** $\log_4 16 = 2$
**c** $\log_7 7 = 1$ **d** $\log_2 64 = 6$
**e** $\log_3 81 = 4$ **f** $\log_4 256 = 4$
**g** $\log_2 32 = 5$ **h** $\log_3 1 = 0$
**i** $\log_2(\frac{1}{2}) = -1$ **j** $\log_3(\frac{1}{9}) = -2$
**k** $\log_5 \sqrt{5} = \frac{1}{2}$ **l** $\log_{27} 9 = \frac{2}{3}$
**m** $\log_{25} 125 = \frac{3}{2}$ **n** $\log_{36}(\frac{1}{6}) = -\frac{1}{2}$
**o** $\log_{16} 8 = \frac{3}{4}$ **p** $\log_8(\frac{1}{32}) = -\frac{5}{3}$

**2** **a** $2^2 = 4$ **b** $3^2 = 9$ **c** $5^0 = 1$
**d** $4^1 = 4$ **e** $10^3 = 1000$ **f** $3^3 = 27$
**g** $2^4 = 16$ **h** $4^2 = 16$ **i** $7^3 = 243$
**j** $5^4 = 625$ **k** $2^7 = 128$ **l** $6^1 = 6$
**m** $2^{\frac{1}{2}} = \sqrt{2}$ **n** $3^{-1} = \frac{1}{3}$ **o** $2^{-2} = \frac{1}{4}$
**p** $4^{\frac{3}{2}} = 8$

**3** **a** 2 **b** 3 **c** 2 **d** 0
**e** 1 **f** 5 **g** 3 **h** 2
**i** 3 **j** 0 **k** 4 **l** 4
**m** $\frac{1}{2}$ **n** $\frac{1}{2}$ **o** −1 **p** $\frac{3}{2}$

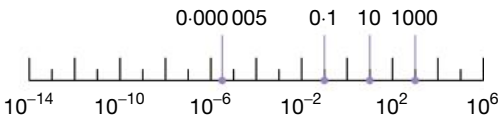
**4** **a** 4 **b** 3 **c** 4 **d** 2
**e** 0 **f** $\frac{1}{2}$ **g** 1 **h** $\frac{1}{2}$
**i** −1 **j** −2 **k** $\frac{2}{3}$ **l** $\frac{3}{2}$
**m** −1 **n** −2 **o** $\frac{1}{6}$ **p** $\frac{3}{2}$

**5** **a** 32 **b** 27 **c** 128 **d** 1
**e** 49 **f** 216 **g** $\frac{1}{3}$ **h** 2
**i** 27 **j** $\frac{1}{16}$ **k** $\frac{1}{8}$ **l** $\frac{1}{2}$
**m** 125 **n** $\frac{1}{10}$ **o** $\frac{1}{\sqrt{5}}$ **p** 8

**6** **a** 2 **b** 3 **c** 5 **d** 3
**e** 5 **f** 10 **g** 2 **h** 3
**i** 5 **j** 3 **k** $\frac{1}{2}$ **l** 4
**m** 27 **n** 16 **o** 64 **p** 81

## Investigation 15:01

**1** **a**




**b** **i** 99 **ii** 990 000 **iii** 0·99

**c** The zero is to the left of the numbers on the scale. (It could in fact never be shown on this axis.)

**d** **i** $10^{-2}$ m to $10^5$ m (0·01 m to 100 000 m)
**ii** $10^{-7}$ m to $10^{-6}$ m

**2** **a** **i** 10 times **ii** 100 times

**b** halfway between 5 and 6

**c** $10^{5\cdot5} \div 10^5 = 10^{\frac{1}{2}} \doteq 3\cdot16$

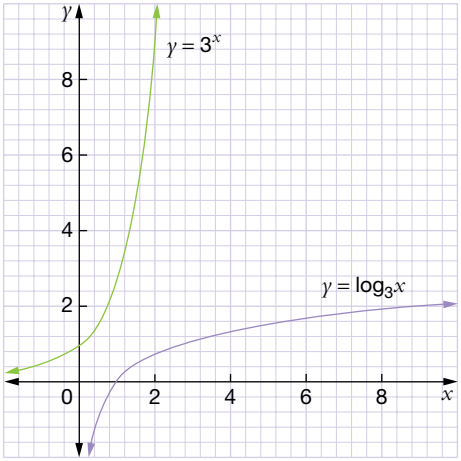
## Exercise 15:02

**1** **a** $y = 3^x$

| $x$ | −2 | −1 | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 |
|---|---|---|---|---|---|---|---|
| $y$ | 0·1 | 0·3 | 1 | 1·7 | 3 | 5·2 | 9 |

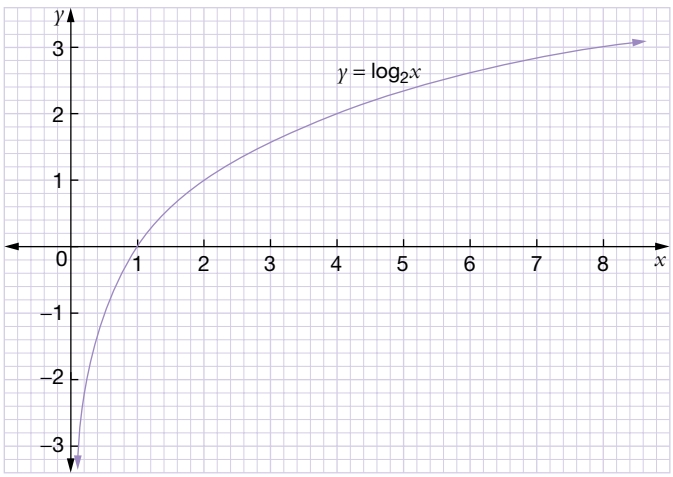
**b** $y = \log_3 x$

| $x$ | 0·1 | 0·3 | 1 | 1·7 | 3 | 5·2 | 9 |
|---|---|---|---|---|---|---|---|
| $y$ | −2 | −1 | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 |




**2** **a**

| $x$ | 0·13 | 0·18 | 0·25 | 0·35 | 0·5 | 0·71 | 1 | 1·4 | 2 | 2·8 | 4 | 5·7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | −3 | $-2\frac{1}{2}$ | −2 | $-1\frac{1}{2}$ | −1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 |




**b** **i** 1·6 **ii** 2·3 **iii** 0·6 **iv** −0·4

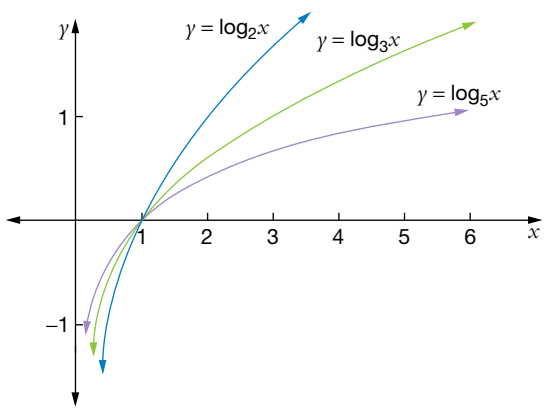
**3** **a**

| $x$ | 0·2 | 0·4 | 0·6 | 0·8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | −0·70 | −0·40 | −0·22 | −0·10 | 0 | 0·30 | 0·48 | 0·60 | 0·70 | 0·78 | 0·85 | 0·90 | 0·95 | 1 |

**b** same as the graph on page 491

**c** **i** 5·0 **ii** 1·8 **iii** 0·3

**4**




## Prep quiz 15:03

**1** 3 **2** $x$ **3** $x$
**4** $y$ **5** $x + y$ **6** $a^{x+y}$
**7** $a^{x-y}$ **8** $a^{mx}$ **9** $x^{-1}$
**10** 0

## Exercise 15:03

**1** **a** 2 **b** 1 **c** 4 **d** 1
**e** 0 **f** 3 **g** 1 **h** 3
**i** 1 **j** 3 **k** 5

**2** **a** 0·778 **b** 0·176 **c** 0·954
**d** −0·301 **e** 0·2385 **f** 1·255
**g** 0·903 **h** 1·380 **i** 1·556
**j** −0·176 **k** −0·602 **l** 0·5395

**3** **a** 3·3 **b** 1·2 **c** 3·0 **d** 0·9
**e** 6·3 **f** 6·6 **g** 4·8 **h** −1·2
**i** 9·0 **j** 2·7 **k** 2·4 **l** 0·3

**4** **a** 2 **b** 1 **c** 0 **d** 5
**e** 0 **f** 2

**5** **a** $\log_a x + \log_a y - \log_a z$
**b** $\log_a x + 1 - 2\log_a z$
**c** $\log_a x - 2\log_a (x + 1)$
**d** $\log_a x + \frac{1}{2}\log_a (x + 1)$
**e** $\frac{1}{2}\log_y x + \frac{1}{2} - \frac{1}{2}\log_y (y + 1)$
**f** $\log_y x + \log_y (a + b) - 2$

**6** **a** $\log_a \frac{xz^2}{y}$ **b** $\log_a \frac{x^2 z}{\sqrt{y}}$
**c** $\log_a \sqrt{xy}$ **d** $\log_a \frac{(x+1)(x+2)}{x}$
**e** $\log_y \frac{x^2 y^3}{z}$ **f** $\log_a \frac{x^2 y}{a^3}$

**7** **a** $xy = x + y$ **b** $5x^2 = y$ **c** $x^2 y = 8$
**d** $\frac{x}{y} = 125$ **e** $\frac{1+x}{1-x} = y$ **f** $y = 5x^7$

**8** **a** 10 **b** 2 **c** 32 **d** 1·5
**e** 4 **f** 2 **g** $\frac{1}{2}$ **h** 12

**9** **a** 50 **b** 24 **c** 1000 **d** 10
**e** $\frac{10}{9}$ **f** $\frac{1}{4}$ **g** 1, −2 **h** $\frac{1 \pm \sqrt{7}}{2}$

## Prep quiz 15:04

**1** $2^2$ **2** $2^3$ **3** $2^{-1}$ **4** $2^{\frac{1}{2}}$ **5** $5^3$
**6** $3^5$ **7** $2^{-\frac{1}{2}}$ **8** yes **9** 4 **10** 5

## Exercise 15:04

**1** **a** 4 **b** 7 **c** 4 **d** 0
**e** 3 **f** 1 **g** 3 **h** 9
**i** 5 **j** 4 **k** 4 **l** 3
**m** 4 **n** 3 **o** 5 **p** 4

**2** **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$ **d** $\frac{1}{2}$
**e** $\frac{1}{3}$ **f** $\frac{1}{5}$ **g** $\frac{1}{2}$ **h** $\frac{1}{2}$
**i** −1 **j** −2 **k** −1 **l** −2

m $-2$ n $-3$ o $-\frac{1}{2}$ p $-\frac{1}{2}$

**3** a $\frac{3}{2}$ b $\frac{5}{2}$ c $\frac{2}{3}$ d $\frac{4}{3}$

e $\frac{3}{2}$ f $\frac{2}{3}$ g $\frac{6}{7}$ h $\frac{3}{5}$

i $-\frac{4}{3}$ j $-\frac{3}{2}$ k $-\frac{2}{3}$ l $-\frac{3}{4}$

m $-\frac{3}{2}$ n $-2$ o $-\frac{3}{2}$ p $-\frac{3}{2}$

**4** a 5 b $-2$ c $\frac{5}{2}$ d 2

e $\frac{5}{4}$ f 8 g $\frac{1}{2}$ h $\frac{5}{3}$

i $\frac{4}{3}$ j $\frac{3}{4}$ k $\frac{5}{9}$ l $\frac{7}{6}$

m $-\frac{11}{4}$ n $\frac{13}{12}$ o $\frac{7}{2}$ p $\frac{5}{4}$

q $\frac{4}{3}$ r 1 s $\frac{2}{7}$ t 1

## Investigation 15:04

**1** 3·86 **2** 6·77 **3** 1·02 **4** 4·52

## Challenge 15:04

**1** a 2·845 b 1·619 c 4·692

d 2·807 e 2·861 f 0·387

**2** a 3·088 b 0·292 c 4·585

d 1·232 e 2·048 f $-0·349$

**3** a 1·372 b 16·610 c 1·131

**4** a 2·585 b 4·660 c 0

d 7·849 e 38·568 f 5·547

## Diagnostic test 15

**1** a $8 = 2^3$ b $9 = 3^2$ c $2 = 4^{\frac{1}{2}}$

d $1 = 5^0$

**2** a $\log_2 32 = 5$ b $\log_4 64 = 3$ c $\log_{27} 3 = \frac{1}{3}$

d $\log_{10} 0·1 = -1$

**3** a 1 b 2 c 3 d $-3$

**4** a 2 b 2 c 32 d 7

**5** a $\log_a 21$ b $\log_a 3$ c $\log_a 60$ d $\log_a 5$

**6** a $4\log_a 3$ b $2\log_a 3$ c $\frac{1}{2}\log_a 3$ d $-\log_a 3$

**7** a $\log_a x + 2\log_a y$ b $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y$

c $\log_a x - \frac{1}{2}\log_a y$

**8** a $\log_a \frac{x}{yz}$ b $\log_a x^2 y^3 \sqrt{z}$

c $\log_a x\sqrt{\frac{(x-1)}{(x+1)}}$

**9** a 7 b $-2$ c $-2$ d $\frac{3}{4}$

### ASSIGNMENT 15A

**1** a $1000 = 10^3$ b $0·125 = 2^{-3}$ c $27 = 9^{\frac{3}{2}}$

**2** a 7 b $-3$ c $\frac{1}{2}$ d $\frac{3}{2}$

**3** a 4 b $\frac{5}{2}$ c $\frac{3}{2}$ d 4

**4** a 1·12 b 0·28 c 1·56 d $-0·56$

**5** a $A(0, 1)$ $B(3, 1)$ $C(1, 0)$ b $A(1, 0)$, $B(8, 3)$, $C(0, 3)$

**6** a 3 b 2 c 2 d 1

**7** a $\log(x - 2)$ b 0

**8** a 12 b 3 or $-1$

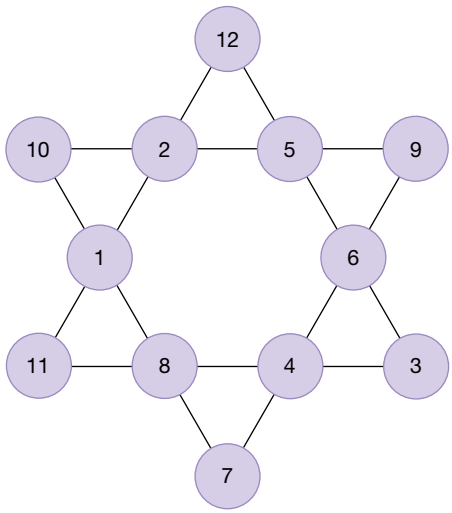
**9** a $\frac{x}{y} = x + y$ b $x^2 = 2y$ c $x^5 y^3 = 2$

### ASSIGNMENT 15B

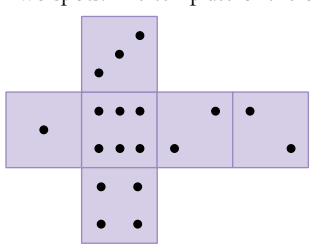
**1** a 3 days b 8 men

**2** 49·6 km/h

**3** This is one possible solution




**4** Two spots. The template of the cube looks like this:



**5** The granddaughter is 6 and the man is 66.

**6** $100 \div \sqrt{3} = 57·7$ cm (1 dec. pl.)

# Chapter 16

## Exercise 16:01

**1** a yes b yes c yes d no

e yes

**2** a yes b yes c no

**3** a yes b no c yes

**4** a $-5$ b $-1$ c $-9$

**5** a 3 b 2 c $1\frac{1}{2}$

**6** a 2 b $2\frac{1}{2}$ c $2\frac{1}{12}$

**7** a 5 b $-4$

**8** a $4a - 5$ b $2a - 3$ c $\frac{2}{a} - 5$

**9** a 8 b $-4$ c $W^2 + 4$

**10** $2p + h$

**11** **a** $f(-x) = (-x)^2$
$= x^2$
$= f(x)$

**b** $f(-x) = (-x)^3$
$= -x^3$
$= -f(x)$

**12** a, b, e, h, i

**13** No. If the line is vertical it is not a function.

**14** **a** $-3 \leq x \leq 3, -3 \leq y \leq 3$

**b** $-2 \leq x \leq 2, 0 \leq y \leq 2$

**c** $-2 \leq x \leq 2, y = 1$ or 2

**d** $-2 \leq x \leq 2, -2 \leq y \leq 2$

**e** $-1 \leq x \leq 1, 0 \leq y \leq 1$

**f** $0 \leq x \leq 9, -3 \leq y \leq 3$

**g** $-4 \leq x \leq 4, -4 \leq y \leq 4$

**h** $-5 \leq x \leq 5, y = -2$ or 2

**i** $-2 \leq x < 2, 0, 0 < y \leq 1$

**15** **a** $y = 4$ **b** $y \leq 1$ **c** $y \geq -4$

**16** **a** $x$ is any real number

**b** $x$ is any real number except 0

**c** $x \geq 0$

**17** **a** $x$ is any real number; $y$ is any real number

**b** $x$ is any real number; $y \geq 1$

**c** $x$ is any number except 1; $y$ is any number except 0

**d** $x$ is any real number; $y > 0$

**e** $x$ is any real number; $y \geq 0$

**18** **a** **i** the distance fallen in 1·5 seconds

**ii** the distance fallen in 1·6 seconds

**iii** the distance fallen from 1·5 seconds to 1·6 seconds

**iv** the average speed of the stone in the period from 1·5 to 1·6 seconds

**b** **i** $2\pi$ **ii** $\pi\sqrt{2}$

**iii** the increase in the period when the length is increased from 1 m to 1·5 m

## Exercise 16:02

**1** Only **a** will have an inverse.

**2** **a** yes: $\{(2, 0), (4, 2), (6, 3), (8, 4)\}$

**b** no

**3** **a** $y = \frac{1}{2}x$ **b** $y = \frac{1}{3}(x - 5)$

**c** $y = 3x + 1$

**4** **a**, **c**, **f** are many-to-one; **b**, **d**, **e** are one-to-one.

**5** Yes, because all straight lines (which are functions) are one-to-one.

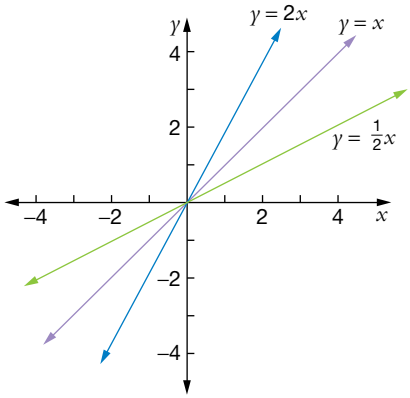
**6** If any horizontal line drawn through the graph of a function cuts the graph in more than one point, then the function does not have an inverse.

**7** **a** $f^{-1}(x) = \frac{1}{4}x$ **b** $f^{-1}(x) = 1 - x$

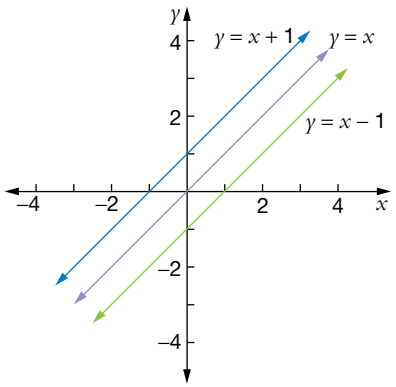
**c** $f^{-1}(x) = 2 - 3x$

**8** **a** $y = x$

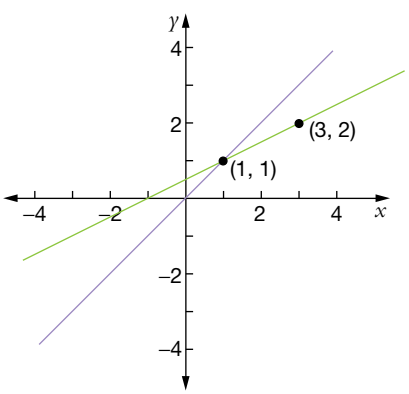
**b** The line $y = 2x$ and $y = \frac{1}{2}x$ are reflections of each other in the line $y = x$.




**c** $y = x + 1$ and $y = x - 1$ are reflections of each other in the line $y = x$.




**9**

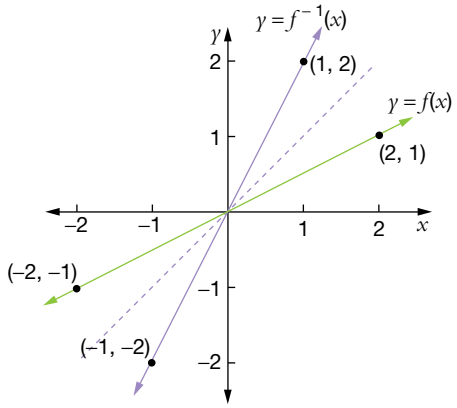



**a** Because they are the points obtained by reversing the $x$- and $y$-coordinates in the points (1, 1) and (3, 2).

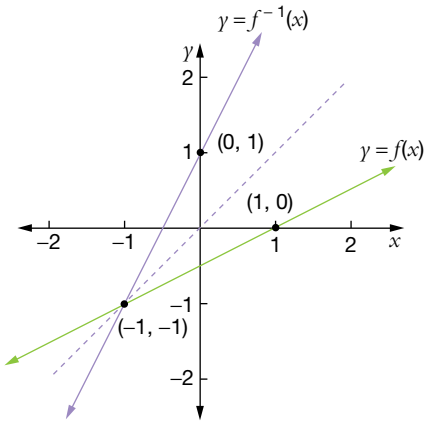
**b** $y = \frac{1}{2}(x + 1)$; $y = 2x - 1$

**c** For $y = 2x - 1$, $1 = 2 \times 1 - 1$ and $3 = 2 \times 2 - 1$. Hence (1, 1) and (2, 3) satisfy the equation $y = 2x - 1$.

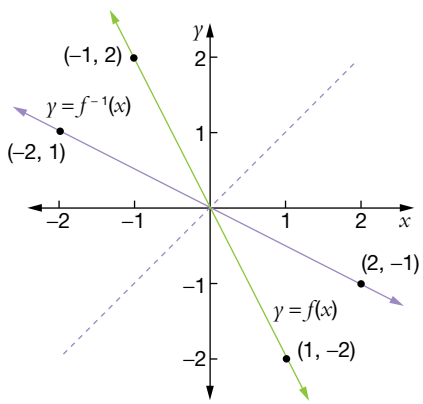
10 a



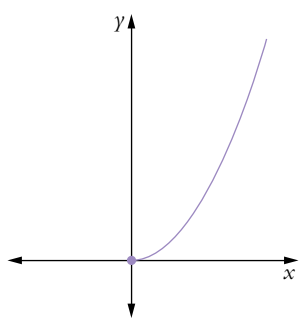

b



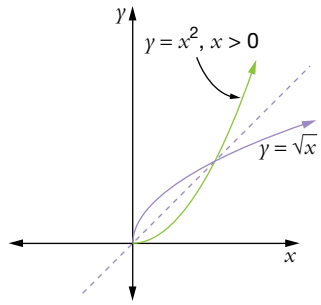

c




11 a

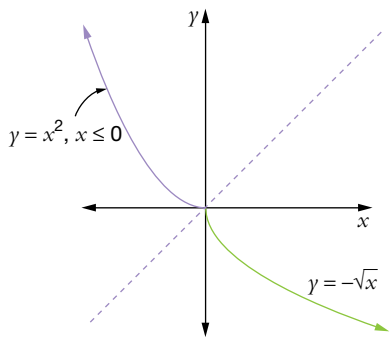


b



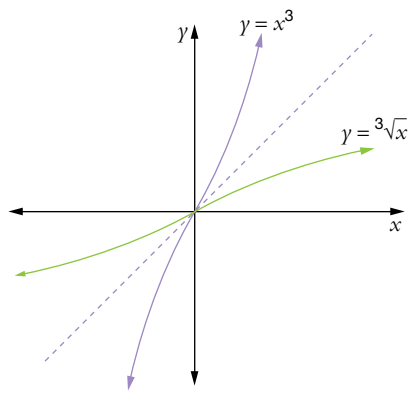

c $y = \sqrt{x}, y \geq 0$

d




inverse is $y = -\sqrt{x}$

12



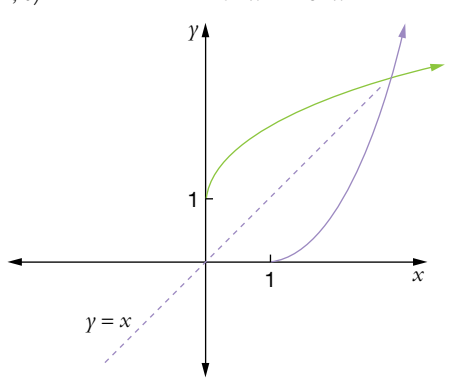

## Investigation 16:02

1 a The parabolas are not one-to-one functions.

b The parabola can be divided into two parts by taking that part where $x \geq a$ or the part where $x \leq a$.

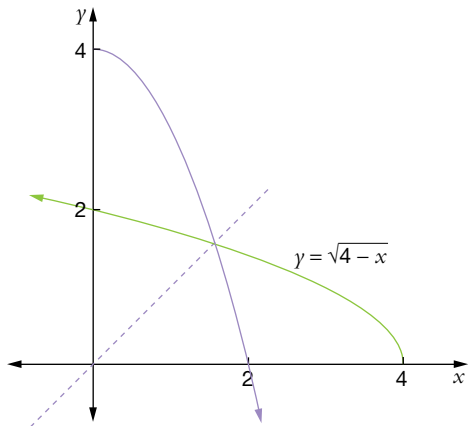
2 a $(1, 0)$ b $x \geq 1$ or $x \leq 1$

c, d



e $y = 1 + \sqrt{x}, y \geq 1$

3 a, b

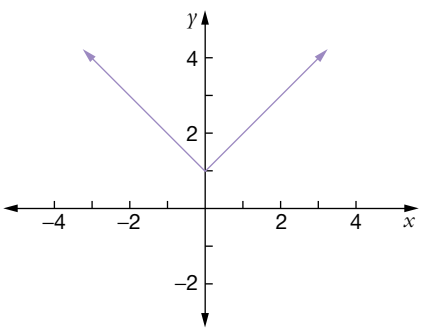



## Exercise 16:03

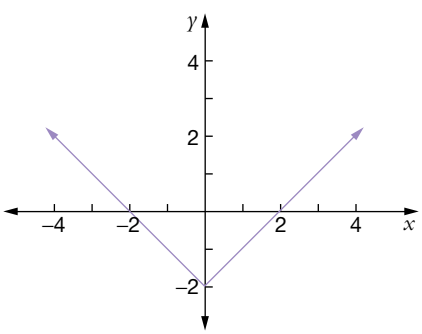
1 a 1 unit to the left
b 1 unit up
c 1 unit down
d 1 unit to the right

2 a $y = 1 + 2^x$
b $y = 2^x - 1$
c $y = 2^{x-1}$
d $y = 2^{x+1}$

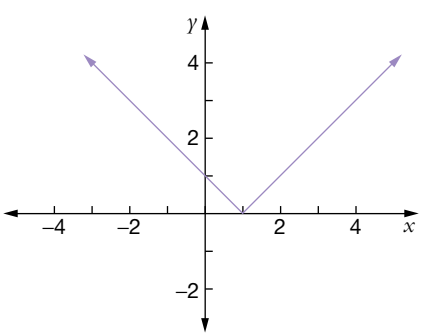
3 a



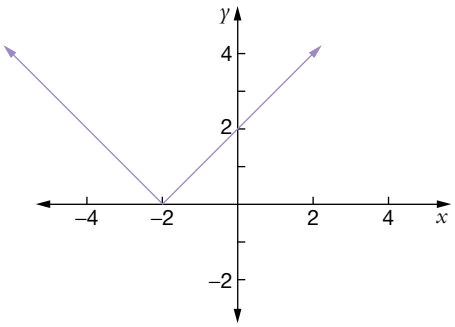

b



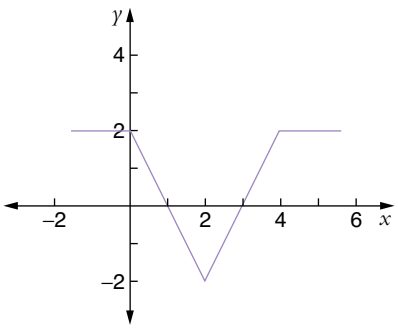

c



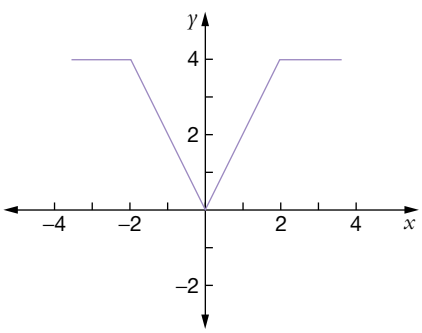

d



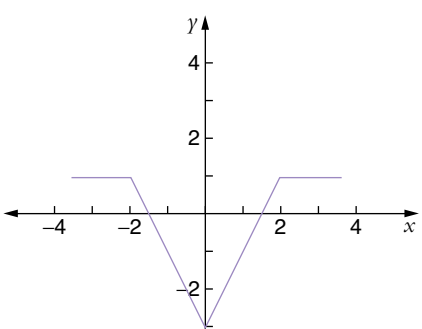

4 a



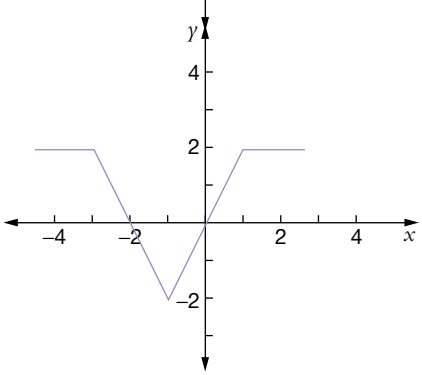

b



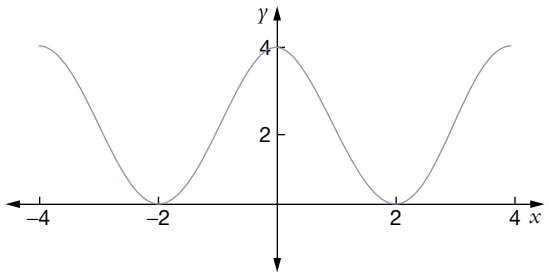

c



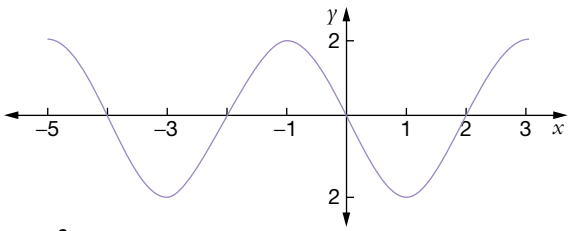

d



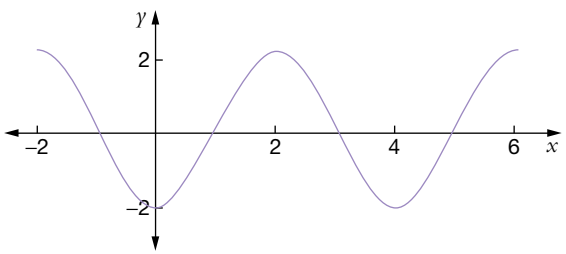
5 a



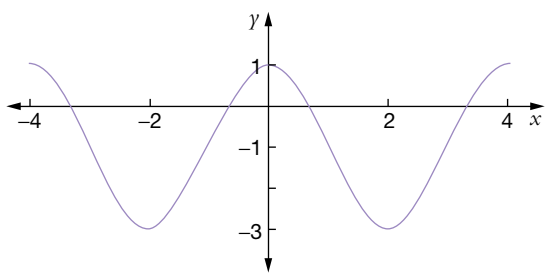

b



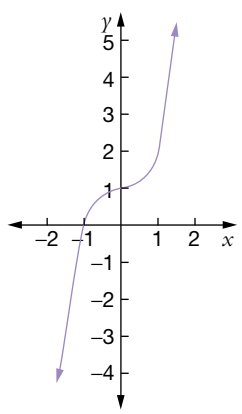

c



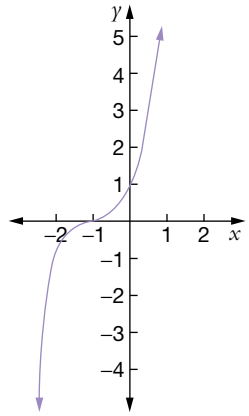

d




6 a

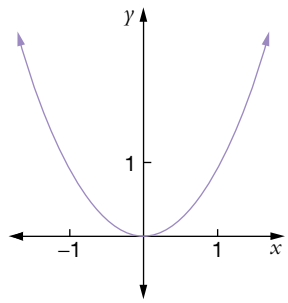



$y = F(x) + 1$

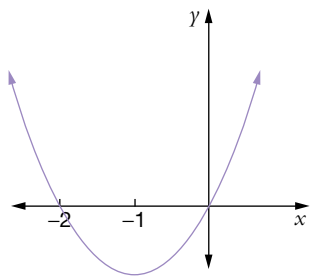



$y = F(x + 1)$

b

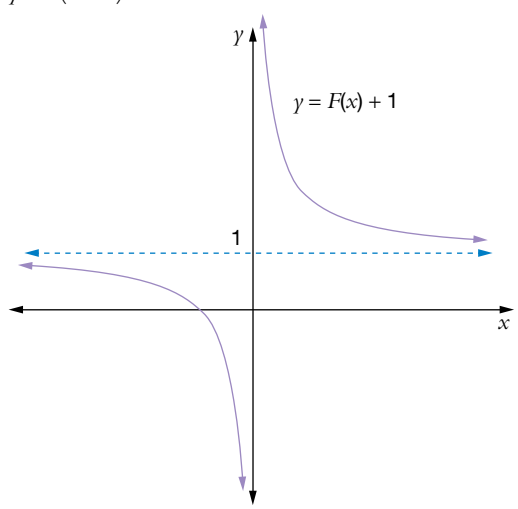


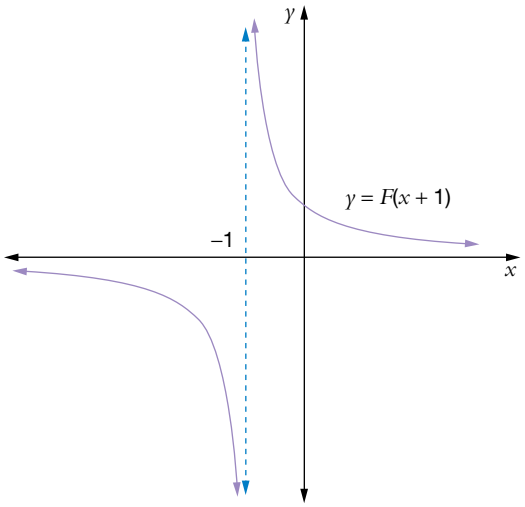

$y = F(x) + 1$




$y = F(x + 1)$

c






## Diagnostic test 16

**1** **a** 8 **b** 0 **c** $1\frac{3}{4}$

**2** **a** $4p - 5$ **b** $2a - 3$ **c** $2x^2 - 5$

**3** **a**, **c**

**4** **a** $x$ is any real number; $y$ is any real number

**b** $x$ is any real number; $y \geq -2$

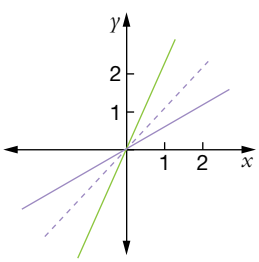
**c** $x \geq 0$; $y \leq 0$

**5** **a** $y = \frac{1}{2}(x + 1)$ **b** $f^{-1}(x) = \dfrac{3 - x}{2}$

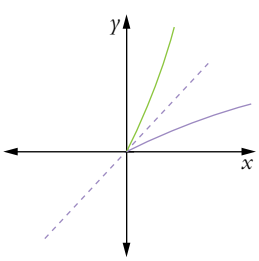
**c** $g^{-1}(x) = \dfrac{2x - 1}{3}$

**6** **a**, **c**

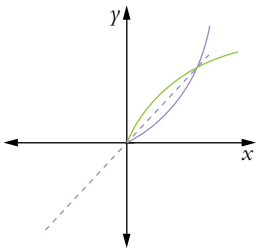
**7** **a**



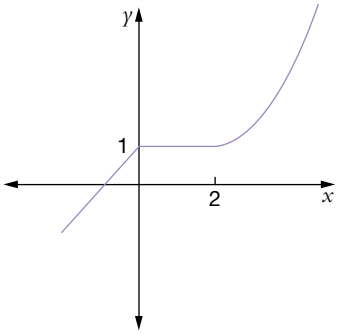

**b**



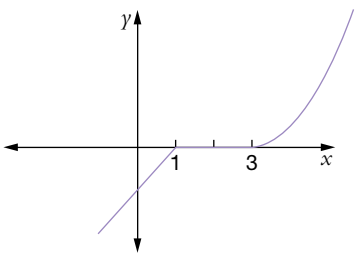

**c**



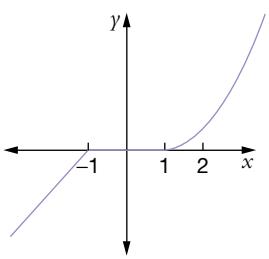

**8** **a**



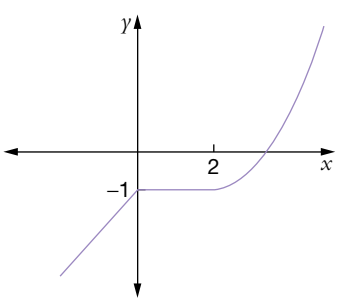

**b**




**c**




**d**

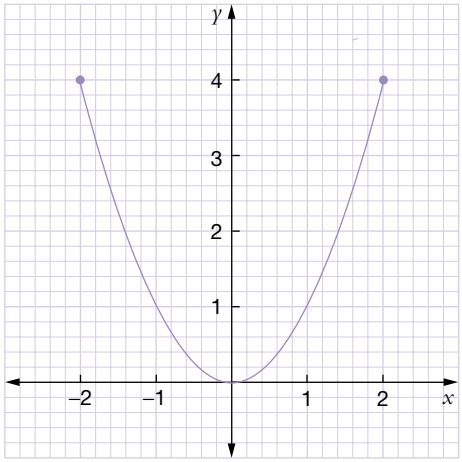



### ASSIGNMENT 16A

**1** **a** –3 **b** –3

**c** $f^{-1}(x) = \dfrac{x - 1}{2}$; $f^{-1}(2) = \frac{1}{2}$

**2** $x = \pm\sqrt{2}$; no




**3** **a** $9x^2 + 3x + 1$ **b** $x^2 - x + 1$

**c** $x^2 - x + 1$

**4** **a** yes **b** no **c** yes

**5** **a** $x$ is any real number, except $x = 0$; $y$ is any real number, except $y = 0$

**b** $x \geq -5$; $y$ is any real number

**c** $x$ is any real number; $-6 \leq y \leq 6$

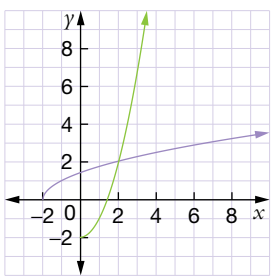
**6** **a** $g^{-1}(x) = \dfrac{x - 3}{4}$ **b** $g^{-1}(x) = \dfrac{5 - 4x}{2}$

**c** $g^{-1}(x) = \sqrt[3]{x} - 1$

**7** **a** $x \geq -2$ **b** $y \geq 0$ **c** $y = x^2 - 2$

**d** $x \geq 0$ **e** $y \geq -2$

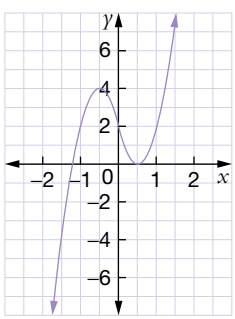
**f**



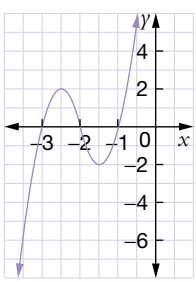

**8** **a** $f^{-1}(x) = \dfrac{x + 1}{x - 1}$

**b** It is symmetrical about the line $y = x$.

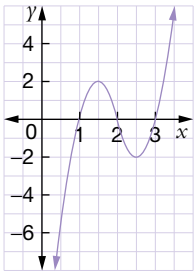
**9** **a**



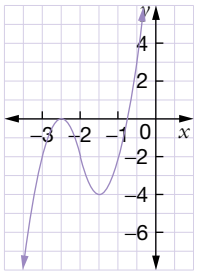

**b**




**c**




**d**




## ASSIGNMENT 16B

**1** 24

**2** **a** $\frac{4}{3}\pi r^3$ **b** $2\pi r^3$ **c** $\frac{2}{3}\pi r^3$ **d** $\frac{2}{3}\pi r^3$

**e** Volume of cylinder = volume of sphere + volume of cone

**3** Anna, $84.75

**4** **a** **i** 316 **ii** 187 **iii** 504 **iv** 250

**b** 63%

**c** 55%

**d** Front left 61%, Front centre 0%, Driver's seat (front right) 63%, Rear left 46%, Rear centre 11%, Rear right 45%.

There were too few occupants in the front centre to consider that spot. The safest spot seems to be the back centre, followed by the rear right and left. There isn't much difference between the driver's seat and the front left.