YEAR 11 ATAR COURSE REVISED EDITION

ACADEMIC TASK FORCE

REVISION SERIES

MATHEMATICS SPECIALIST

UNITS 1 & 2

OT LEE

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YEAR 11 ATAR COURSE
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SECOND EDITION

O. T. LEE

ACADEMIC GROUP
ACADEMIC TASK FORCE ACADEMIC ASSOCIATES THE EXAM EXPERTS
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About the Author
Dr O. T. Lee is an author of many books which are used extensively in WA schools. Dr Lee is an exceptional, insightful teacher with wide-ranging experience as a WACE marker.

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Mathematics Specialist Revision Series Units 1 & 2

Contents

Mathematics Specialist Revision Series
Units 1 & 2

- The Mathematics Specialist Revision Series Units 1 & 2 provides a comprehensive set of revision/review questions for the new year 11 Mathematics Specialist Units 1 & 2 course.

- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.

- This book exposes students to questions and problems at test/examination level.

- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.

- It is accompanied by a set of fully worked solutions with which students can measure their solutions. These solutions are often not the only solutions but they provide a model for students to work with. Students, interrogate your solutions to understand your errors and your successes. It may sometimes to possible to achieve a correct numerical answer with faulty reasoning!

- Do not memorise solutions. Understand the techniques and processes used in relation to the questions asked.

Notes

Combinatorics

- $^nC_r \equiv \binom{n}{r} \equiv \dfrac{n!}{r!\,(n-r)!}$

$$= \frac{\overbrace{n\times(n-1)\times(n-2)\times\ldots\ldots\times(n-r+1)}^{r\ \text{terms}}}{r!}$$

- $^nC_r = {}^nC_{n-r}$
- $^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$
- $^nC_0 = {}^nC_n = 1$
- $^nC_1 = {}^nC_{n-1} = n.$
- $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.
- $^nP_r = \dfrac{n!}{(n-r)!} = \underbrace{n\times(n-1)\times(n-2)\times\ldots\ldots(n-r+1)}_{r\ \text{terms}}$
- If Tasks A and B are mutually exclusive, then:
  - A or B can be completed in $n(A) + n(B)$ ways.
  - A & B can be completed in $n(A) \times n(B)$ ways.
- r objects can be chosen from n unlike objects:
  - without replacement in ^nC_r ways.
  - with replacement in n^r ways.
- r items from n items of type I and s items from m items of type II, can be chosen and then arranged in $^nC_r \times {}^mC_s \times (r + s)!$ ways.
- n unlike items can be arranged in a line in $n!$ ways.
- r objects out of n objects all different ($r \le n$) may be arranged in a line, with no object used more than once in ^nP_r or $^nC_r \times r!$ ways.
- n unlike objects can be arranged in a line with 2 specific objects apart from each other in $n! - (n-1)! \times 2!$ ways.
- n unlike items can be arranged in a line with r specific items adjacent to each other in $(n - r + 1)! \times r!$.

Inclusion-Exclusion Principle

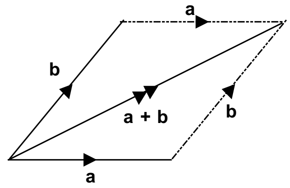
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

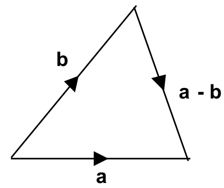
Pigeon Hole Principle

- If m items are placed in n containers, where $m > n$, at least one container will contain:
  - more than one item.
  - no more than $\textbf{Int}\left(\dfrac{m}{n}\right)$ items.
  - at least $\textbf{Int}\left(\dfrac{m}{n}\right) + 1$ items.

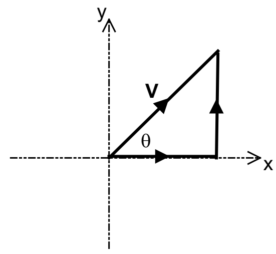
Vectors

- Addition of vectors   Subtraction of vectors





Vectors (resolving into components)



Component along

x – axis $v_x = v\cos\theta$

y – axis $v_y = v\sin\theta$

$v = v\cos\theta\ \mathrm{i} + v\sin\theta\ \mathrm{j}$

$= \langle v\cos\theta,\ v\sin\theta\rangle$

$= \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$

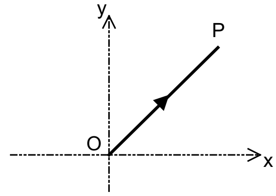
Vectors (magnitude and direction)

$\boldsymbol{u} = x\,\mathbf{i} + y\,\mathbf{j}$

- Magnitude of $\boldsymbol{u}$ $|\boldsymbol{u}| = \sqrt{(x^2 + y^2)}$
- Direction of $\boldsymbol{u}$ $= \tan^{-1}\dfrac{y}{x}$ with positive x-axis

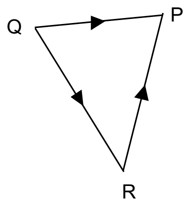
  (*Locate quadrant location first*)
- Unit vector in the direction of $\boldsymbol{u}$ is $\hat{\boldsymbol{u}} = \dfrac{1}{|\boldsymbol{u}|}\boldsymbol{u}$

Position Vectors



P is the point (x, y).

- The position vector of P with respect to the origin O is **OP** or $\overrightarrow{OP}$.
- **PQ = OQ – OP**



P, Q and R are points on the x-y plane (in space).

- **QP = QR + RP**

Parallel vectors

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel $\Leftrightarrow$ $\boldsymbol{u} = \lambda\,\boldsymbol{v}$.
- For $\boldsymbol{u} = \lambda\,\boldsymbol{v}$, if $\lambda > 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in the same direction.
- For $\boldsymbol{u} = \lambda\,\boldsymbol{v}$, if $\lambda < 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in the opposite direction.

Perpendicular vectors

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular $\Leftrightarrow$ then $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.

Relative Vectors

- The position vector or displacement vector of P relative to Q is given by:

$$_P\boldsymbol{r}_Q = \mathbf{OP} - \mathbf{OQ} = \boldsymbol{r}_P - \boldsymbol{r}_Q$$

- The velocity vector of P relative to Q is given by: $_P\boldsymbol{v}_Q = \boldsymbol{v}_P - \boldsymbol{v}_Q$

Scalar Product

Let $\boldsymbol{u} = a\,\mathbf{i} + b\,\mathbf{j}$ and $\boldsymbol{v} = x\,\mathbf{i} + y\,\mathbf{j}$

- scalar product $\boldsymbol{u} \bullet \quad = ax + by$
- Angle between $\boldsymbol{u}$ and $\boldsymbol{v}$: $\cos\theta = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

Projections

- The scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$

$$= |\boldsymbol{u}|\cos\theta = |\boldsymbol{u}| \times \frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \boldsymbol{u} \bullet \hat{\boldsymbol{v}}.$$

- The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$,

$$\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = (|\boldsymbol{u}|\cos\theta)\,\hat{\boldsymbol{v}} = (\boldsymbol{u} \bullet \hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}.$$

- The vector rejection of $\boldsymbol{u}$ onto $\boldsymbol{v}$,

$$\text{rej}_{\boldsymbol{v}}\,\boldsymbol{u} = \boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = \boldsymbol{u} - (\boldsymbol{u} \bullet \hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}$$

Work Done

- The work done by a force of $\boldsymbol{F}$ Newtons through a distance of s metres is given by: Scalar Projection of $\boldsymbol{F}$ along direction of motion × s Joules.
- The work done by a force $\boldsymbol{F}$ Newtons in moving a body over a displacement $\boldsymbol{d}$ metres is given by: Work done = $\boldsymbol{F} \bullet \boldsymbol{d}$ Joules.

Collision (interception) and closest approach

Displacement Method

- Position vector of A and B at time t:
  $\boldsymbol{r}_A = \boldsymbol{a} + t\,\boldsymbol{u}$ $\quad \boldsymbol{r}_B = \boldsymbol{b} + t\,\boldsymbol{v}$
- For collision (interception) at time t: $\boldsymbol{r}_A = \boldsymbol{r}_B$.
  Compare x components and solve for t, t_x.
  Compare y components and solve for t, t_y
- If $t_x = t_y$, $\Rightarrow$ collision (interception) occurs at t_x.
  If $t_x \neq t_y$ $\Rightarrow$ there is no collision (interception).
- Displacement vector between A and B at time t

$$\boldsymbol{d} = \boldsymbol{r}_A - \boldsymbol{r}_B$$

- Distance between A and B at time t, $s = |\boldsymbol{d}|$.
- If $s = 0$ has a real solution for t, then there is collision (interception).

Relative Vectors Method

- For collision (interception) at time t:

$$_B\boldsymbol{r}_A = t\ _A\boldsymbol{v}_B \text{ where } t > 0$$

  i.e. $\boldsymbol{b} - \boldsymbol{a} = t\,(\boldsymbol{u} - \boldsymbol{v})$
  Compare components and solve for t_x and t_y.
- If $t_x = t_y$, $\Rightarrow$ collision (interception) occurs at t_x.
  If $t_x \neq t_y$ $\Rightarrow$ there is no collision (interception).

Scalar product method

- At closest approach, $_B\boldsymbol{r}_A$ is perpendicular to $_B\boldsymbol{v}_A$

$$_B\boldsymbol{r}_A \bullet {}_B\boldsymbol{v}_A = 0$$

  Solve the resulting equation for t.
  This is the time of closest approach.
- Substitute value of t into $|_A\boldsymbol{r}_B|$ to obtain the distance of the closest approach.
- If $|_A\boldsymbol{r}_B| = 0$ on substituting value of t, then there is collision (interception).

Exact Values

θ°	θ rad	sin θ	cos θ	tan θ
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	$\rightarrow \infty$

General Solutions

Equation	General Solution $n \in \mathbb{Z}$
$\sin x = k$	$x = (-1)^n \sin^{-1} k + n\pi$ *OR* $x = \sin^{-1} k + 2n\pi$, or $-\sin^{-1} k + (2n+1)\pi$
$\cos x = k$	$x = 2n\pi \pm \cos^{-1} k$
$\tan x = k$	$x = \tan^{-1} k + n\pi$

Trigonometric Identities

- secant A = $\dfrac{1}{\cos A}$
- cosecant A = $\dfrac{1}{\sin A}$
- cotangent A = $\dfrac{1}{\tan A}$
- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta = 1 + \tan^2\theta$
- $\text{cosec}^2\theta = 1 + \cot^2\theta$
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Trigonometric Identities

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$
- $\sin 3A = 3 \sin A - 4 \sin^3 A$
- $\cos 3A = 4 \cos^3 A - 3 \cos A$
- $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$
- $\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]$
- $\sin A \pm \sin B = 2 \sin\left(\dfrac{A \pm B}{2}\right) \cos\left(\dfrac{A \mp B}{2}\right)$
- $\cos A + \cos B = 2 \cos\left(\dfrac{A + B}{2}\right) \cos\left(\dfrac{A - B}{2}\right)$
- $\cos A - \cos B = -2 \sin\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{A - B}{2}\right)$
- $a \sin x \pm b \cos x \equiv \sqrt{a^2 + b^2}\ \sin\left[x \pm tan^{-1}\left(\dfrac{b}{a}\right)\right]$
- $a \cos x \pm b \sin x \equiv \sqrt{a^2 + b^2}\ \cos\left[x \mp tan^{-1}\left(\dfrac{b}{a}\right)\right]$

Trigonometric Graphs

	$y = a \sin(bx + c) + d$ $y = a \cos(bx + c) + d$
Mean Line	$y = d$
Amplitude	$\lvert a \rvert$
Min./Max. y	Min: $d - \lvert a \rvert$, Max: $d + \lvert a \rvert$
Period	$360^\circ/b$ or $2\pi/b$
Phase shift	*Shifted c/b degrees/radians to the left*

	$y = a \tan(bx + c) + d$
Mean Line	$y = d$
Period	$180^\circ/b$ or π/b
Phase shift	*Shifted c/b degrees/radians to the left*

Matrices

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$
- In general, $\mathbf{AB} \neq \mathbf{BA}$.
- If $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then:
  - $\mathbf{A}^{-1} = \mathbf{B}$
  - $\mathbf{B}^{-1} = \mathbf{A}$
- $\mathbf{AB} = \mathbf{BA} = k\mathbf{I} \Rightarrow \mathbf{A}^{-1} = \frac{1}{k}\mathbf{B}$ and $\mathbf{B}^{-1} = \frac{1}{k}\mathbf{A}$.

- For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:
  - $\det(\mathbf{A}) = \lvert \mathbf{A} \rvert = ad - bc$
  - $\mathbf{A}^{-1}$, exists only if $\lvert \mathbf{A} \rvert \neq 0$ and is given by
    $\mathbf{A}^{-1} = \dfrac{1}{\lvert \mathbf{A} \rvert}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- If $\mathbf{A}^{-1}$ does not exist then $\mathbf{A}$ is **singular** or **non-invertible** [In which case $\lvert \mathbf{A} \rvert = 0$].
- $\mathbf{AX} = \mathbf{B} \Rightarrow \quad = \mathbf{A}^{-1}\mathbf{B}$ provided $\mathbf{A}^{-1}$ exists.

Linear Transformations

- linear transformation maps the point (0, 0) to the point (0, 0) and a set of parallel lines to another set (not necessarily the same) of parallel lines.
- Some transformation matrices:

Transformation	**Matrix**
Reflection about the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection about the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection about the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection about the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Reflection about $y = x \tan\theta$ $\theta \neq (2n + 1)\pi/2$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Rotation 90° clockwise about the origin	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation 90° anti-clockwise about the origin	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation 180° clockwise about the origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Rotation θ° anti-clockwise about the origin	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Dilation factor $k > 0$ along the x-axis	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Dilation factor $k > 0$ along the y-axis	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement factor $k > 0$ about the origin	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

- The absolute value of the determinant of a transformation matrix is defined as the scale factor for area, which is the ratio $\dfrac{\text{Area of Image}}{\text{Area of Object}}$.

Complex numbers

- $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.
- If $x + y\,i = a + b\,i$, then $x = a$, and $y = b$.
- $(x + y\,i) \pm (a + b\,i) = (x \pm a) + (y \pm b)\,i$
- $(x + y\,i)(a + b\,i) = (xa - yb) + (xb + ya)\,i$
- The **complex conjugate** of $z = x + y\,i$ is $\bar{z} = x - y\,i$. Note that, $z\bar{z} = x^2 + y^2$
- $\overline{u \pm v} = \bar{u} \pm \bar{v}$ and $\overline{u \times v} = \bar{u} \times \bar{v}$
- $\dfrac{1}{a+bi} = \dfrac{1}{a+bi} \times \dfrac{a-bi}{a-bi} = \dfrac{a-bi}{a^2+b^2}$
- $\dfrac{x+yi}{a+bi} = \dfrac{x+yi}{a+bi} \times \dfrac{a-bi}{a-bi} = \dfrac{(x+yi)(a-bi)}{a^2+b^2}$

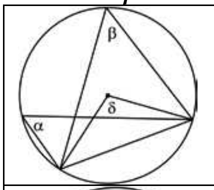
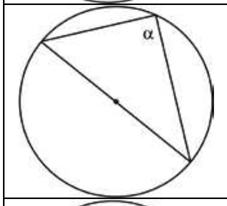
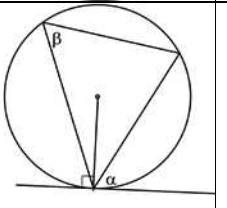
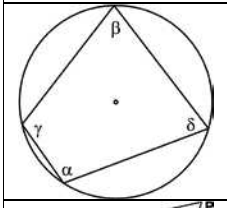
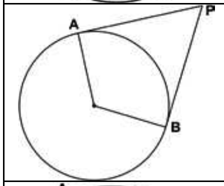
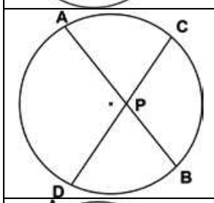
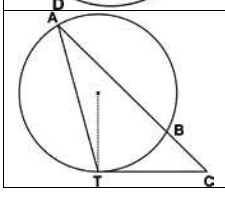
Similar Triangles

- AA Test
  Two angles of the first triangle are equal to two angles of the second triangle.
- SSS Test
  Ratios of corresponding side lengths are equal.
- SAS Test
  Ratios of the two corresponding side lengths are equal and included angles are equal.
- RHS Test
  Right Triangles with ratios of hypotenuse and one other corresponding side equal.

Congruent Triangles

- SSS Test
  All three corresponding sides of the two triangles have the same side length.
- SAS Test
  Two sides and the included angle of both triangles are the same.
- ASA Test
  Two angles and the included side of both triangles are equal.
- AAS Test
  Two angles and a corresponding non-included side of both triangles are equal.
- RHS Test
  Right Triangles with hypotenuse and one corresponding side equal.

Circle Properties

	$\alpha = \beta$ Angles in the same segment $\delta = 2\beta$ Angle at centre is twice angle at the circumference
	$\alpha = 90^\circ$ Angle in a semi-circle = 90°.
	Angle between radius and tangent = 90°. $\alpha = \beta$ Angle between a tangent and a chord is equal to the angle in the alternate segment.
	$\alpha + \beta = 180^\circ$ $\gamma + \delta = 180^\circ$ Opposite angles in a cyclic quadrilateral are supplementary.
	PA = PB Length of tangents to a circle from a common external point are the same.
	AP × PB = CP × PD
	AC × BC = TC^2

01 Combinatorics I

Calculator Free

1. [9 marks: 2, 2, 2, 3]

Find k if:

(a) $\dfrac{10!}{k!} = 720$

(b) $\dfrac{k!}{8!} = 990$

(c) ${}^{k}\mathrm{P}_{3} = 210$

(d) $\dfrac{10! - k!}{k!} = 30\,239$

Calculator Free

2. [5 marks: 3, 2]

(a) Prove that ${}^{n}\mathrm{P}_{r} = (n+1-r)\times {}^{n}\mathrm{P}_{r-1}$

(b) Simplify ${}^{100}\mathrm{P}_{20} - {}^{100}\mathrm{P}_{19}$.

Give your answer in the form $A \times {}^{100}\mathrm{P}_{19}$ where A is a constant.

Calculator Free

3. [4 marks: 1, 1, 1, 1]

The letters of the word RICHES are rearranged in a line. No letter is used more than once. Write mathematical expressions for:

(a) the total number of possible arrangements.

(b) the number of arrangements with the letters I and E are adjacent.

(c) the number of arrangements with the letters I and E not adjacent

(d) the number of arrangements where the vowels are adjacent and the letters C and H are adjacent.

---

4. [5 marks: 1, 2, 2]

The letters of the word HEXAGON are rearranged in a line. No letter is used more than once. Write mathematical expressions (do not evaluate) for:

(a) the total number of possible arrangements.

(b) the number of arrangements where all the consonants are adjacent to each other.

(c) the number of arrangements where all the consonants are apart.

Calculator Assumed

5. [9 marks: 1, 2, 3, 3]

Six digit numbers are formed using the digits 1, 2, 3, 4, 5, and 6. Each digit is used only once.

(a) How many six digit numbers are possible?

(b) How many six digit even numbers are possible?

(c) How many five digit even numbers greater than 40 000 are possible?

(d) How many even numbers greater than 40 000 are possible?

---

6. [5 marks]

How many 5 digit *odd* numbers greater than 30 000 but less than 80 000 can be formed using the digits of the 0 to 9 inclusive with no digit being used more than once.

Calculator Assumed

7. [4 marks: 2, 2]

Find the number of ways of *rearranging* the letters of the word EXPRESSIONS if:

(a) no other conditions apply.

(b) the arrangement must start with the letter X.

---

8. [9 marks: 2, 2, 2, 3]

Find the number of ways of *rearranging* the letters of the word REARRANGING if:

(a) no other conditions apply.

(b) the arrangement must start with the letter E.

(c) the arrangement must start with the letter A.

(d) the arrangement must start with a vowel.

Calculator Free

9. [9 marks: 2, 2, 2, 3]

Consider the letters in the word **ASSASSINATIONS.**
Write mathematical expressions for each of the following.

(a) The number of ways all the letters in this word can be arranged in a line.

(b) The number of ways all the letters in this word can be arranged in a line with all the S letters together.

(c) The number of ways all the letters in this word can be arranged in a line with all the A letters together?

(d) The number of ways all the letters in this word can be arranged in a line with all the A letters together <u>*or*</u> all the S letters together?

Calculator Assumed

10. [7 marks: 2, 2, 3]

Passwords consisting of between 8 and 12 characters inclusive are to be created using the letters of the alphabet (case sensitive) and the digits 0 to 9 inclusive.

(a) Write mathematical expressions for the number of possible passwords if:

(i) no character can be used more than once.

(ii) repetition of characters are permitted.

(b) A computer program is capable of checking 4 billion (1×10^9) passwords per second. How long will the computer take to check all the possible passwords in part (a) (ii)? Give your answer in years.

---

11. [4 marks: 2, 2]

The length of a password is determined by the number of characters in the password. Using the digits 0 to 9 inclusive and the case sensitive letters of the alphabet, find the minimum length of a password if the number of possible passwords is to exceed 1 trillion (1×10^{12}) if:

(a) repetition of characters is not permitted.

(b) repetition of characters is permitted.

02 Combinatorics II

Calculator Assumed

1. [4 marks: 1, 1, 2]

In a group of 40 students, there were 10 boys who were colour vision deficient (CVD) and 15 girls who were not CVD. There were as many boys who were not CVD as there were boys who were CVD. How many of these students:

(a) were boys?

(b) were either boys or colour vision deficient?

(c) were colour vision deficient?

---

2. [5 marks: 2, 2, 1]

In a survey of teachers teaching mathematics to year nines, 15 teachers had a mathematics degree, 55 men teachers did not have a mathematics degree and 10 women teachers had a mathematics degree. There were as many teachers who were either female or had a mathematics degree as there were men teachers.

(a) How many male teachers were there?

(b) How many female teachers were there?

(c) How many teachers were surveyed?

Calculator Assumed

3. [7 marks: 1, 1, 1, 1, 2, 1]

Flags of 10 different nations including that of Australia and New Zealand are to be flown from 10 flag poles set in a line. The flag poles are labelled poles 1 to 10. How many ways are there of assigning a flag to each of these poles if the:

(a) Australian flag must be flown from pole 1?

(b) Australian flag or New Zealand flag must be flown from pole 1?

(c) Australian flag and the New Zealand flag must be flown from poles 1 and 10 respectively?

(d) Australian flag must be flown from pole 1 and the New Zealand flag must not to be flown from pole 10.

(e) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10.

(f) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10 but not both at the same time.

Calculator Assumed

4. [6 marks: 2, 2, 2]

Twelve different coloured light bulbs, including a red bulb and a blue bulb are to be fitted into bulb sockets installed along a straight edge of a patio running East to West. The bulb socket at the extreme Eastern end is labelled E and the bulb socket at the extreme Western end is labelled W. Determine the number of arrangements with:

(a) the red light bulb not fitted into bulb sockets E or W.

(b) the red light bulb and the blue light bulb not fitted into bulb sockets E or W.

(c) the red light bulb or the blue light bulb not fitted into bulb sockets E or W.

---

5. [7 marks: 2, 2, 3]

Ten potted plants including four pots of roses of different shades of red and three pots of azaleas (each of a different colour) are to be arranged in a line along a footpath. How many arrangements will have:

(a) the potted roses adjacent to each other?

(b) the potted azaleas adjacent to each other?

(c) the roses adjacent to each other or the azaleas adjacent to each other?

Calculator Assumed

6. [4 marks]

In a group of 300 students:

- 140 students had holidayed in Bali
- 190 had holidayed in Singapore
- 160 had holidayed in Kuala Lumpur
- 80 had holidayed in both Bali and Kuala Lumpur
- 120 had holidayed in Kuala Lumpur and Singapore
- 90 had holidayed in Bali and Singapore
- there were as many students who had holidayed in all three places as those who had not holidayed in any of the three places

Show the use of the inclusion-exclusion principle to determine the number of students that had holidayed in all three places.

Calculator Assumed

7. [5 marks]

In a group of 35 students:

- 14 students were enrolled in Physics
- 16 were enrolled in Biology
- 13 students were enrolled in Chemistry
- 6 were enrolled in Physics and Chemistry
- 5 were enrolled in Physics and Biology
- 4 were not enrolled in any of these three subjects
- there were twice as many students who were enrolled in all these three subjects as those enrolled in Chemistry and Biology but not Physics.

Show the use of the inclusion-exclusion principle to determine the number of students enrolled in all three subjects.

Calculator Assumed

8. [5 marks]

In a group of 50 students:

- 27 students are enrolled in Accounting
- 7 students are enrolled in Accounting and Biology but not Chemistry
- 8 students are enrolled in Biology and Chemistry but not Accounting
- 4 students are enrolled in Accounting and Chemistry but not Biology
- 5 students were not enrolled in Accounting, Biology or Chemistry.
- there were as many students enrolled in Biology only as those enrolled in Chemistry only.

Show the use of the inclusion-exclusion principle to determine the number of students enrolled in Biology only?

Calculator Assumed

9. [8 marks: 1, 1, 1, 2, 1, 2]

Seven students including Amy, Brian and Catherine are to be arranged in a line. How many possible arrangements are there with:

(a) Amy or Brian or Catherine on the extreme left?

(b) Amy on the extreme left?

(c) Amy on the extreme left and Catherine on the extreme right?

(d) Amy on the extreme left or Catherine on the extreme right?

(e) Amy on the extreme left, Brian in the middle and Catherine on the extreme right?

(f) Amy on the extreme left or Brian in the middle or Catherine on the extreme right?

Calculator Assumed

10. [11 marks: 1, 1, 2, 3, 2, 2]

Ten different coloured balls, including a red, a blue and a green ball, are to be placed in ten different boxes labelled A to J. One ball is to be placed in each box. How many ways are there of placing the balls (one in each box) with:

(a) the red ball in box A?

(b) the red ball in box A and the blue ball in box B?

(c) the red ball in box A or the blue ball in box B?

(d) the red ball in box A or the blue ball in box B or the green ball in box C?

(e) the red ball in box A but the blue ball not in box B?

(f) the green ball in box C but the red ball not in box A and the blue ball not in box B?

Calculator Assumed

11. [8 marks: 1, 1, 1, 1, 2, 2]

Consider the set of integers between 1 000 and 9 999 inclusive.
How many integers in this set:

(a) are divisible by 2?

(b) are divisible by 3?

(c) are divisible by 2 and 3?

(d) are divisible by 2 and 6?

(e) are divisible by 2 or 3?

(f) are divisible by 2 or 3 but not both?

Calculator Assumed

12. [7 marks: 1, 1, 1, 2, 2]

Consider the set of integers between 500 and 5 000 inclusive.
How many integers in this set:

(a) are divisible by 5?

(b) are divisible by 10?

(c) are divisible by 5 and 10?

(d) are divisible by 5 or 10?

(e) are divisible by 5 or 10 and is an even number?

Calculator Assumed

13. [6 marks: 3, 3]

Consider the set of integers between 1 000 and 5 000 inclusive.

(a) Complete the following table listing the number of multiples of n within this set for the given values of n.

n	Number of multiples of n in this set.
2	
3	
5	
6	
10	
15	
30	

(b) Find the number of integers in this set that are multiples of 2, 3 or 5.

Calculator Assumed

14. [6 marks: 3, 3]

Consider the set of integers between 2500 and 10 000 inclusive.

(a) Complete the following table.

Multiples of	Number of multiples in this set.
2	
4	
5	
2 and 4	
2 and 5	
4 and 5	
2 and 4 and 5	

(b) Find the number of integers in this set that are multiples of 2, 4 or 5.

Calculator Assumed

15. [6 marks]

Consider the set of integers between 0 and 50 000 inclusive.
How many integers in this set are divisible by 2 *or* 3 *or* 5 *but not* 10?

Calculator Assumed

16. [10 marks: 2, 2, 2, 4]

Consider the set of integers between 100 000 and 500 000 inclusive.
How many integers in this set:

(a) are divisible by 2?

(b) are divisible by 3?

(c) are divisible by 2 and 3?

(d) are not divisible by 2 and not divisible by 3?

03 Combinatorics III

Calculator Free

1. [13 marks: 2, 2, 2, 4, 3]

(a) Find n and r if ${}^{n}C_{r} = \dfrac{n \times (n-1) \times 98}{3 \times 2 \times 1}$.

(b) Find n and r if ${}^{n}P_{r} = 20 \times 19 \times 18 \times 17$

(c) Find a and b if ${}^{30}C_{a} = {}^{3a}C_{b}$.

(d) Find all possible values of a and b if ${}^{12}C_{a} = {}^{12}C_{2a+b}$.

(e) Find a possible set of values for a and b if $10 \times {}^{9}P_{4} = 6 \times {}^{a}P_{b}$.

Calculator Free

2. [9 marks: 3, 3, 3]

(a) Find k if $\dfrac{(k+3)!}{k!} = 120$.

(b) (i) Show that ${}^{10}C_3 + {}^{10}C_4 = {}^{11}C_4$.

(ii) Hence, prove that ${}^{10}C_3 + {}^{10}C_4 + {}^{11}C_5 + {}^{12}C_6 + {}^{13}C_7 + {}^{14}C_8 = {}^{15}C_8$

Calculator Free

3. [3 marks]

(a) How many ways are there of choosing 2 students from a group of 100 students?

(b) How many ways are there of choosing 98 students from a group of 100 students?

---

4. [7 marks: 2, 2, 3]

A media folder has 10 video-clips. How many ways are there of choosing:

(a) seven of these clips?

(b) seven or eight of these clips?

(c) at least two of these clips?

Calculator Assumed

5. [4 marks: 2, 2]

Write a mathematical expression (do not evaluate) for the number of ways of dividing a group of 50 students into:

(a) one group with 30 students and the second group with 20 students?

(b) two groups each with 25 students?

---

6. [10 marks: 1, 3, 3, 3]

A committee of 9 people are to be selected from 10 Labor, 8 Liberal and 5 Green politicians. How many different ways can the committee be selected if:

(a) there are no restrictions

(b) all three political parties are equally represented

(c) the Liberal representatives are in the (absolute) majority

(d) a husband and wife pair, Alex and Alice, cannot be in the same committee.

Calculator Assumed

7. [10 marks: 1, 2, 3, 4] [TISC]

Wei has a collection of 20 stickers in her pink box and 25 stickers in her blue box. All these stickers are different from each other.

(a) In how many ways can Wei pick 3 stickers from her pink box?

(b) In how many ways can Wei pick 2 stickers from her blue box and arrange them in a line?

(c) In how many ways can Wei pick 3 stickers from her pink box and 2 stickers from her blue box and arrange them in a line if:

(i) there are no restrictions as to how the stickers are arranged?

(ii) all the stickers from the blue box must be together?

Calculator Free

8. [12 marks: 1, 2, 2, 2, 2, 3]

In how many ways can the letters of the word **COMBINE** be arranged in a straight line (no letter may be used more than once):

(a) using all seven letters?

(b) using all seven letters and starting with the letter **C**?

(c) using all seven letters and ending in **BONE?**

(d) using only 5 letters at a time?

(e) using all seven letters with the vowels in the first three places (from the left)?

(f) using only two vowels and two consonants?

Calculator Free

9. [12 marks: 1, 2, 2, 3, 4]

Consider the letters of the word CONQUEST. Write mathematical expressions for the number of ways the letters of this word can be *rearranged*:

(a) if the first letter must be a consonant.

(b) if the first three letters must be consonants.

(c) if one of the first three letters must be a consonant.

(d) if at least one of the first three letters must be a consonant.

(e) if the vowels are to be sandwiched between two consonants (that is, a vowel must be preceded by a consonant and this vowel must also be followed by a consonant).

Calculator Assumed

10. [14 marks: 2, 4, 4, 4]

Passwords are to be created using the the digits 0 to 9 inclusive and the 26 non-case sensitive letters of the English alphabet.

(a) **MARK** creates a password consisting of 10 characters and the passwood contains all the letters of his name. Write a mathematical expression for the number of possible passwords if no character is repeated.

(b) **FAREEHA** creates a password comprising 10 characters using all the letters in her name and three other distinct characters not found in her name. How many such passwords are possible?

(c) Write a mathematical expression for the number of possible passwords that comprises 3 digits *and* 7 letters with the three digits adding up to six with no character used more than once.

(d) How many 8 character passwords are possible if each password must include at least 3 digits *and* 3 letters?

Calculator Assumed

11. [5 marks]

In a school curriculum, six subjects are listed under List A, eight subjects are listed under list B and ten subjects are listed under List C.
A student is enrolled in six subjects. How many ways are there for this student to choose 2 List A subjects *or* 2 List B subjects?

---

12. [5 marks: 3, 2]

A password consisting of four characters is to be chosen from 10 digits, 26 upper case letters, 26 lower case letters and a set of 32 symbols.

(a) Complete the table below.

Composition of password	Number of different passwords
2 lower case letters	
2 symbols	
2 lower case letters and 2 symbols	

(b) Determine the number of possible passwords with either two lower case letters or two symbols.

Calculator Assumed

13. [10 marks: 2, 2, 3, 3]

Eight books are to be selected and arranged on a library display shelf. These eight books are to be selected from a collection of 8 adult novels, 4 non-fiction books and 5 illustrated children's books. How many of these arrangements will contain:

(a) four adult novels?

(b) three non-fiction books?

(c) exactly four adult novels and three non-fiction books?

(d) four adult novels or three non-fiction books?

Calculator Assumed

14. [6 marks]

Passwords consisting of 8 characters are to be formed using the digits 0 to 9 inclusive, the case sensitive letters of the English Latin (Roman) alphabet and the special symbols %, !, #, $, & and ?.

Write a mathematical expression for the number of possible passwords with no repeated characters with 4 digits *or* 4 letters *or* 4 special symbols.

04 Combinatorics IV

Calculator Free

1. [6 marks: 1, 2, 1, 1, 1]

A container has 10 different sized pairs of nuts and bolts with the nuts removed from the respective bolts.

(a) Five bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure one matching pair of nut and bolt?

(b) Six bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure two matching pairs of nut and bolt?

(c) What is the minimum number of items that need to be removed from this container to ensure one matching pair of nut and bolt?

(d) Fourteen items are randomly removed from the container.

(i) What is the minimum number of matching pairs of nuts and bolts?

(ii) What is the maximum possible number of matching pairs?

Calculator Free

2. [7 marks: 1, 2, 1, 1, 1, 1]

Twelve dog owners and their dogs (one dog per owner) meet at a dog park.

(a) Six dog owners are randomly chosen. What is the minimum number of dogs that need to be chosen to ensure a matching owner-dog pair?

(b) Seven dogs are randomly chosen. What is the minimum number of owners that need to be chosen to ensure three matching owner-dog pairs?

(c) A total of 15 owners and dogs were randomly selected.

(i) What is the minimum number of owner-dog pairs in this selection?

(ii) What is the maximum possible number of owner-dog pairs in this selection?

(d) If the owners came as couples, that is one dog per couple, what is the minimum number of persons and dogs that need to be chosen to ensure:

(i) a matching owner-dog pair?

(ii) more than two matching owner-dog pairs?

Calculator Free

3. [7 marks: 1 each]

A container has 5 red marbles, 6 green marbles and 9 yellow marbles. What is the minimum number of marbles that need to be drawn from this container to ensure:

(a) a marble of each colour?

(b) two marbles of each colour?

(c) two red marbles?

(d) two yellow marbles?

(e) two green marbles?

(f) two marbles of the same colour?

(g) three marbles of the same colour?

---

4. [7 marks: 1 each]

Dennis has 3 blue pens, 4 red pens and 5 black pens in his pencil case. What is the minimum number of pens that need to be drawn from the pencil case to ensure that:

(a) a red pen is drawn?

(b) a pen of each colour is drawn?

(c) a red and a blue pen is drawn?

Calculator Free

4. (d) two blue pens and two red pens are drawn?

   (e) two red pens and two black pens are drawn

   (f) two blue pens and two black pens are drawn?

   (g) two pens of the same colour are drawn?

---

5. [8 marks: 1, 2, 5]

   A box has four red pens, six blue pens and n black pens (where integer $n \geq 2$). Determine the minimum number of pens that need to be drawn from this box to ensure:

   (a) two pens of the same colour.

   (b) two black pens.

   (c) three red pens, two blue pens and one black pen.

Calculator Assumed

6. [4 marks: 2, 2]

There are 25 students in a class.

(a) Explain clearly why there must be at least 3 students that are born in a same month.

(b) Explain clearly why there must be at least one month which is a birth month shared by no more than 2 students.

Calculator Assumed

7. [6 marks: 1, 1, 2, 2]

A class has 30 students.

(a) How many students need to be chosen to ensure that there are:
 (i) two students who are born on the same day of the week?

 (ii) five students who are born on the same day of the week?

(b) There are at least x students who are born on the same day of the week. Find x. Justify your answer.

(c) There must be at least one day of the week which is the birth day of no more than y students. Find y. Justify your answer.

Calculator Assumed

8. [7 marks: 1, 1, 1, 2, 2]

There are 300 students in a primary school with names written using the English alphabet.

(a) How many students need to be chosen to ensure that there are:

(i) two students with family names that start with the same letter?

(ii) three students with family names that start with the same letter?

(iii) six students with family names that start with the same letter?

(b) There are at least x students with family names that start with the same letter. Find x. Justify your answer.

(c) There must be at least one letter that is the first letter of the family names of no more than y students. Find y. Justify your answer.

Calculator Assumed

9. [6 marks: 1, 2, 3]

There are 21 letters in the Italian alphabet. Italian names consist of a given-name followed by a surname. Consider a group of 100 students with names written using the Italian alphabet.

(a) How many of these students need to be chosen to ensure that there are two students with given-names that start with the same letter of the Italian alphabet?

(b) Explain why it is not possible to ensure that there are six students in this group with surnames that start with the same letter of the Italian alphabet?

(c) How many more students need to be chosen to ensure that there are at least two students with the same initials (initials comprise the first letter of the given name followed by the first letter of the surname)?

---

10. [4 marks: 1, 3]

Assume that humans with black/brown hair can have between 0 and 110 000 hairs on their head.

(a) How many persons need to be chosen to ensure that we have 3 persons with the same number of hair on their heads?

(b) A country has 1 billion black/brown haired people. There are at least x people in this country with the same number of hairs on their heads. Find x.

Calculator Assumed

11. [9 marks: 2, 2, 2, 3]

110 students sat an examination. Their performances in this examination were scored between 0 and 100 inclusive.

(a) Use the pigeon-hole principle to discuss the validity of each of the following statements. In each case you need to identify the "pigeons" and the "pigeon-holes".

(i) "At least two students scored 50 marks".

(ii) "At least two students scored the same mark".

(iii) "At least two students had marks that differ by 50".

(b) How many more students would be required to ensure that there are at least two pairs of students with marks that differ by the same amount?

Calculator Assumed

12. [4 marks: 2, 2]

n integers (all different) are to be chosen from the set of integers {1, 2, 3, …, 20}. It is required that there are always two numbers in the selection that sum to 21.

(a) Using the pigeon-hole principle, identify the "pigeon-holes" and the "pigeons" in this context.

(b) Determine the value of minimum value for n.

---

13. [4 marks]

In a bus with 20 passengers, use the pigeon-hole principle to explain why at least two passengers are friends with the same number of passengers.

05 Addition & Subtraction of Vectors (Using Trigonometry)

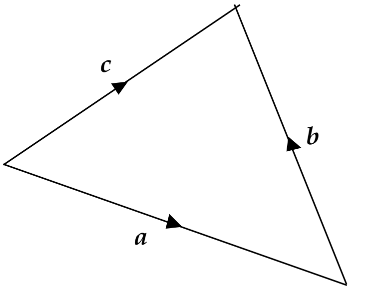
Calculator Free

1. [2 marks: 1, 1]

Vectors ***a***, ***b*** and ***c*** are as drawn in the accompanying diagram.

(a) Express ***c*** in terms of ***a*** and ***b***.

(b) Express ***a*** in terms of ***b*** and ***c***.



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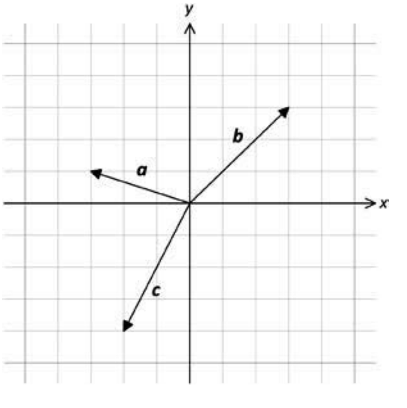
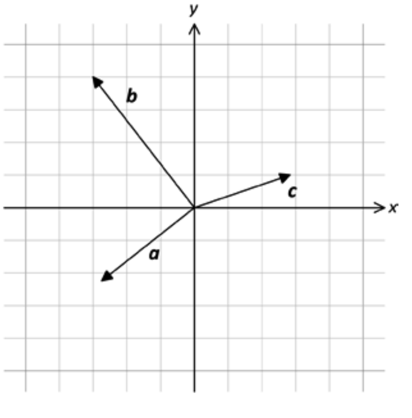
2. [4 marks: 2, 2]

The diagrams below shows the directed line segments representing the vectors ***a***, ***b*** and ***c***. Sketch on the provided axes,

(a) the directed line segment representing ***a*** + ***b*** + ***c***.



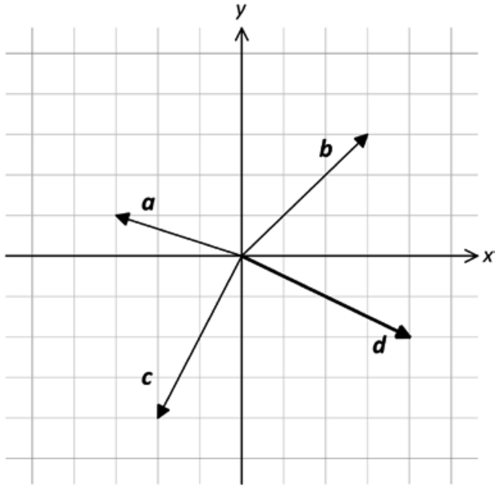
(b) Sketch on the same axes, the directed line segment representing ***a*** – ***b*** – ***c***.



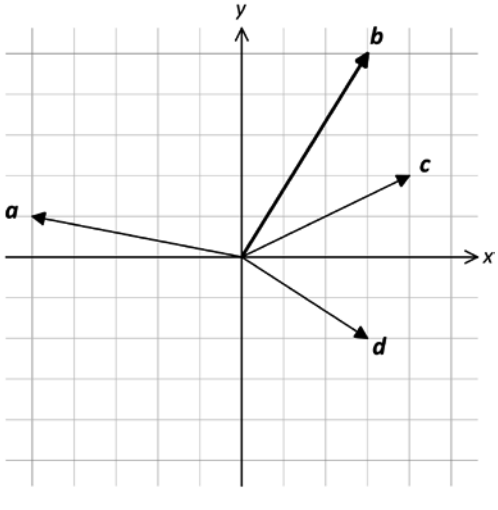
Calculator Free

3. [4 marks: 2, 2]

(a) The diagram below shows the directed line segments representing the vectors ***a***, ***b***, ***c*** and ***d***. Express ***d*** in terms of ***a*** and/or ***b*** and/or ***c***.



(b) The diagram below shows the directed line segments representing the vectors ***a***, ***b***, ***c*** and ***d***. Given that ***b*** = −***a*** + α***c*** + β***d***, determine the constants α and β.



Calculator Assumed

4. [6 marks: 2, 4]

An aircraft is flying with a speed of 400 kmh^{-1} along bearing 145°. The aircraft is buffeted by a strong wind of magnitude 80 kmh^{-1} blowing from bearing 240°.

(a) Draw a sketch to indicate the actual direction of the aircraft.

(b) Find the ground speed and direction of the aircraft.

Calculator Assumed

5. [6 marks: 3, 3]

A boy intends to swim across a river of width 20 metres to the opposite bank. The river flows at a steady rate of 1 kmh^{-1}. The boy can swim at a steady speed of 2 kmh^{-1}.

(a) In what direction should the boy be headed so that he ends up at the opposite bank directly opposite to where he started off ?

(b) Find the time taken for the swim in part (a).

Calculator Assumed

6. [5 marks]

   A current is flowing in the direction $N48^\circ E$ at $10\ \text{kmh}^{-1}$. With what speed and in what direction should a naval vessel be travelling to achieve a resultant speed of $40\ \text{kmh}^{-1}$ in the direction $N30^\circ W$.

Calculator Assumed

7. [9 marks: 3, 6]

A plane is to be flown from M to N. N is 3 000 km from M in the direction 210°.
A wind blows in the direction 310° at 30 km per hour.
The plane has a maximum speed of 800 km per hour.

(a) Draw a clearly labelled velocity diagram for the situation described above.

(b) Determine which direction the plane should be flown for it to arrive at N in minimum time. State the minimum flight time (to the nearest minute).

06 Components & Position Vectors I

Calculator Free

1. [8 marks: 2, 1, 1, 4]

Given that $\boldsymbol{a} = -2\,\mathbf{i} + 6\,\mathbf{j}$ and $\boldsymbol{b} = 5\,\mathbf{i} - 4\,\mathbf{j}$, find:

(a) $|\boldsymbol{a} + \boldsymbol{b}|$.

(b) the unit vector parallel to $\boldsymbol{a} + \boldsymbol{b}$.

(c) a vector that is parallel to $\boldsymbol{a} + \boldsymbol{b}$ but with a magnitude of 5.

(d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = 2\mathbf{i} + \mathbf{j}$ and $\boldsymbol{q} = -3\mathbf{i} + 2\mathbf{j}$.

Calculator Free

2. [6 marks]

OA = 3**i** + 10**j** , **OB** = 5**i** + *b***j** and **OC** = 9**i** + *c***j** .
Find *c* in terms of *b* if A, B and C are collinear.

---

3. [7 marks]

Vector *a***i** + (*a* + *b*)**j** has a magnitude of 5 and is parallel to vector 4**i** + 8**j** .
Find all possible values of *a* and *b*.

Calculator Free

4. [5 marks]

Vector $a\,\mathbf{i} + 10\mathbf{j}$ is of the same magnitude as $(b - 10)\,\mathbf{i} + (a - 2b)\,\mathbf{j}$ but acts in the opposite direction. Find the values of a and b.

5. [4 marks]

Vector $\begin{pmatrix} a \\ b \end{pmatrix}$ has magnitude 20 and is parallel to $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$.

Find the values of a and b.

6. [3 marks]

The point K divides the line segment AB internally in the ratio 4 : 1. Use a vector method to find the position vector of K if $\mathbf{OB} = -\,\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{AB} = 15\,\mathbf{i} - 5\,\mathbf{j}$.

Calculator Assumed

7. [4 marks]

The points A, C and K have position vectors < –1, 4 >, < 5, 10 > and < 2, –5 > respectively. The point C divides the line segment AB internally in the ratio 2 : 3. Use a vector method to determine **BK**.

---

8. [4 marks]

It is known that **OA** = a**i** + **j** and **OB** = 4**i** + b**j** .
K is a point such that AK: AB = 2 : 5 and **OK** = 4**i** – 3**j** . Find a and b .

Calculator Assumed

9. [5 marks]

Vector ***u*** has magnitude 100 kmh^{-1} and acts in the direction 040°.
Vector ***v*** has magnitude 150 kmh^{-1} and acts in the direction 280°.
Let ***i*** be the unit vector in the West-East direction
and ***j*** be the unit vector in the South-North direction.

Use vector components to find the magnitude and direction of ***u*** – 2***v***.

---

10. [5 marks]

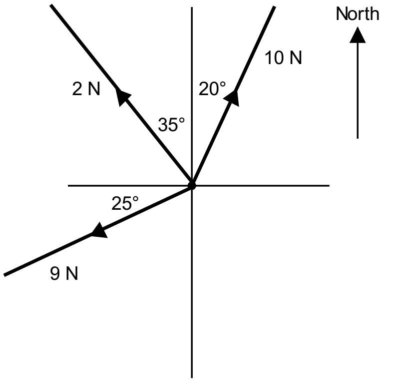
Given that ***u*** = < –4, 16 > and $|\boldsymbol{v}| = 100$, find ***v***
if ***u*** + ***v*** is to be in the same direction as the vector < 2, 2 >.

07 Components & Position Vectors II

Calculator Assumed

1. [6 marks: 3, 3]

   The diagram below shows the forces acting on a body. The forces are all on the same plane.



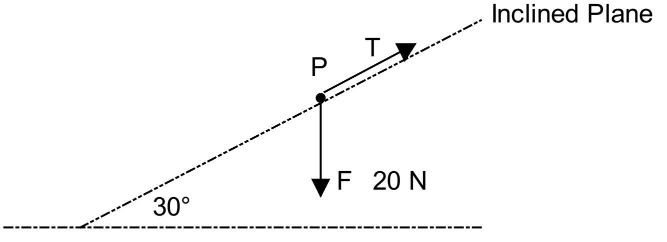
   (a) Find the magnitude of the resultant.

   (b) Find the magnitude and the direction of a single force that will keep this system in equilibrium.

Calculator Assumed

2. [5 marks: 2, 2, 1]

In the diagram below, a particle P is on a plane inclined at an angle of 30^{o} to the horizontal. A vertical force F of magnitude 20 N is acting on P as shown. Force T parallel to the inclined plane is applied to prevent P from slipping down the inclined plane.



(a) Find the magnitude of the component of F parallel to the inclined plane.

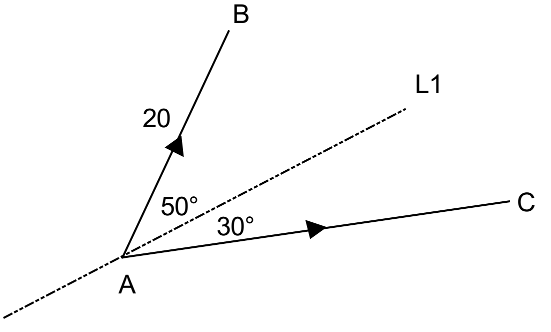
(b) Find the magnitude of the component of F perpendicular to the inclined plane.

(c) Find the magnitude of force T.

Calculator Assumed

3. [4 marks: 1, 1, 2]

In the diagram below, vector **AB** is of magnitude 20 units and is inclined at an angle of 50° to the line L1. Vector **AC** is inclined at angle of 30° to the line L1 as shown.



(a) Find the magnitude of the component of **AB** parallel to the line L1.

(b) Find the magnitude of the component of **AB** perpendicular to line L1.

(c) Find the magnitude of **AC** if the resultant of the vectors **AB** and **AC** is parallel to the line L1.

Calculator Assumed

4. [11 marks: 1, 1, 4, 2, 3]

A light plane can fly at 80 km per hour in still air. The pilot wishes to fly from O to a neighbouring airstrip Q, located 40 km from O in the direction 060°. A constant wind of 20 km per hour is blowing from the North. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Write in terms of **i** and **j** the position vector of Q relative to O.

(b) Write in terms of **i** and **j** the velocity vector of the wind.

(c) Find the velocity vector the pilot should set so that the plane flies directly to Q.

(d) Find the resultant speed of the plane.

(e) Find the difference in flying time (to the nearest minute) caused by the wind.

Calculator Assumed

5. [8 marks: 4, 2, 2]

A helicopter capable of flying at a speed of 100 km per hour in still air, takes off from O for a mining town located at A. The position vector of A relative to O is $200\mathbf{i} - 300\mathbf{j}$ km. Throughout the journey, the helicopter encounters a wind blowing with velocity $13\mathbf{i} + 5\mathbf{j}$ km per hour.

(a) Find the velocity vector the pilot should set so that the helicopter flies directly to A.

(b) Find the time taken for the journey.

(c) Find the velocity vector the pilot should set for a direct flight back to O. Assume that the wind blows with the same velocity throughout the flight back.

Calculator Assumed

6. [7 marks: 1, 6]

A drone is to fly from A to B where **AB** = < 20, –40 > m. A wind is blowing with a velocity of < 2, 3 > m/minute. The drone can maintain a speed of 50 m/minute. Let the velocity the drone should fly in order to reach B in minimum time be < x, y > m/minute.

(a) State in terms of x and y the resultant velocity of the drone.

(b) Calculate how long it will take the drone to reach B and the velocity < x, y >.

Calculator Assumed

7. [8 marks]

A drone is to be flown from P to Q with position vectors < 30, 70 > and < 50, 100 > respectively. A slight breeze is blowing with constant velocity < 1, 2 > ms^{-1}. Determine how long it will take the drone to be flown from P to Q at a speed of 2 ms^{-1}. State the direction the drone should be flown.

08 Components & Position Vectors III

Calculator Assumed

1. [6 marks: 1, 1, 1, 3]

A particle P, initially at 5**i** – 10**j** metres, moves with velocity 3**i** + 4**j** ms^{-1}.

(a) Find the position vector of P after 10 seconds.

(b) Find the distance travelled by P after 10 seconds.

(c) Find the position vector of P after *t* seconds.

(d) When is P at a point with position vector (65**i** + 70**j**) metres.

Calculator Assumed

2. [9 marks: 3, 3, 3]

The position vector of particles A and B, t hours after 12 noon, are
$\mathbf{r} = 12\mathbf{i} + 3\mathbf{j} + t\,(3\mathbf{i} + 4\mathbf{j})$ and $\mathbf{r} = -3\mathbf{i} - 5\mathbf{j} + t\,(2\mathbf{i} + 6\mathbf{j})$ metres respectively.

(a) Find in terms of t, the distance between A and B t hours after 12 noon.

(b) Find when A and B are 18 metres apart.

(c) Find when A is closest to B and find this distance.

Calculator Assumed

3. [6 marks: 1, 1, 2, 2]

Particle P starts moving from the point A with position vector $-2\mathbf{i} + 3\mathbf{j}$ metres with velocity $\mathbf{i} - 2\mathbf{j}$ metres per second. Particle Q starts moving from the point A at the same time with velocity $2\mathbf{i} + 3\mathbf{j}$ metres per second.

(a) Determine the position vector of P after t seconds.

(b) Determine the position vector of Q after t seconds.

(c) Find in terms of t, the distance between P and Q after t seconds.

(d) Use your answer in (c) to find when P and Q are 10 metres apart.

Calculator Assumed

4. [7 marks: 1, 2, 1, 3]

Particle P starts moving from the point with position vector 3**i** + 5**j** metres with velocity 2i – 3j metres per second.

(a) Determine the position vector of P after 3 seconds.

(b) Find the distance from P to the point with position vector –2**i** + **j** after 3 seconds.

(c) Determine the position vector of P after t seconds.

(d) Find when P is closest to the origin and state this distance.

Calculator Assumed

5. [9 marks: 2, 2, 2, 3]

A speed boat is moving at a constant velocity of 40 kmh^{-1} in the direction with bearing 060^o. Initially, the speed boat is $5\sqrt{2}$ km from a buoy and is in the direction with bearing 225^o from the buoy. Given that **i** and **j** are the unit vectors in the Easterly direction and Northerly direction respectively, find:

(a) the initial position vector of the speed boat with respect to the buoy in terms of **i** and **j**.

(b) the direction vector of the speed boat in terms of **i** and **j**.

(c) the position vector of the speed boat with respect to the buoy t hours later.

(d) the time when the speed boat is nearest to the buoy and the least distance between the speed boat and the buoy.

Calculator Assumed

6. [9 marks: 4, 2, 3]

Yacht A starts sailing from the point L with position vector 5**i** – 2**j** metres with velocity –**i** + 3**j** . Yacht B starts sailing 10 seconds later with velocity 2**i** + **j** metres per second, from the point M with position vector 4**i** – 3**j** metres per second.
t is time in seconds from the moment B starts sailing.

(a) Find in terms of t, the distance between A and B after t seconds.

(b) When will A and B be 40 metres apart?

(c) When will the two yachts be closest together? State this distance.

Calculator Assumed

7. [9 marks: 3, 3, 3]

At 0800 hours, object P is at the point with position vector $40\,\boldsymbol{i} - 70\boldsymbol{j}$ km and moving with constant velocity $-5\,\boldsymbol{i} + 2\,\boldsymbol{j}$ kmh^{-1}. At 0900 hours, object Q starts moving from a point with position vector $-64\,\boldsymbol{i} - 24\boldsymbol{j}$ km with constant velocity $4\,\boldsymbol{i} - 2\,\boldsymbol{j}$ km h^{-1}.

(a) Calculate the distance between P and Q at 1000 hours.

(b) Determine an expression for the distance between P and Q at time t hours after 1000 hours.

(c) If the directions of P and Q remain unchanged, use your answer in (b) to determine if P and Q will collide. If they do collide, provide the time and place of collision.

Calculator Assumed

8. [6 marks]

At 1400 hours, P is at the point with position vector $-5\boldsymbol{i} + 25\boldsymbol{j}$ metres and moving with constant velocity v metres per minute.

(a) Find v if P arrives at < 20, 75 > after 5 minutes.

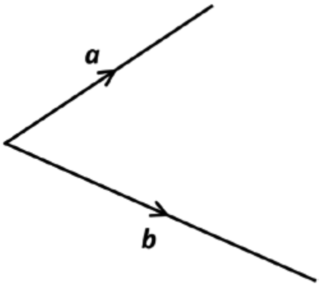
(b) Q which starts off from < 10, –20 > metres at 1400 hours travelling with a velocity of < – 4, 8 > metres per minute. P has a maximum speed of 12 metres per minute. Find $\boldsymbol{v}$ for P to intercept Q.

09 Relative Displacement

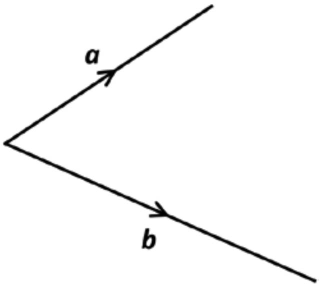
Calculator Free

1. [4 marks: 1, 1, 1, 1]

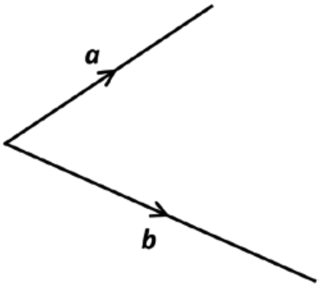
   (a) Indicate clearly in the diagram below $\boldsymbol{a} + \boldsymbol{b}$.



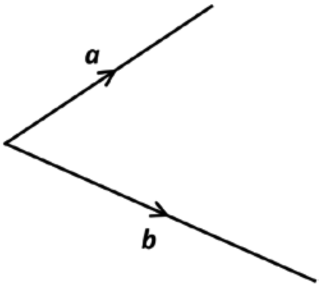
   (b) Indicate clearly in the diagram below $\boldsymbol{a} - \boldsymbol{b}$.



   (c) Indicate clearly in the diagram below the vector $\boldsymbol{b}$ relative to $\boldsymbol{a}$.



   (d) Indicate clearly in the diagram below the vector $\boldsymbol{a}$ relative to $\boldsymbol{b}$.



Calculator Free

2. [4 marks: 1, 1, 2]

The position vectors of the points A, B, and C with respect to the origin O are $\mathbf{i} + \mathbf{j}$, $2\mathbf{i} - \mathbf{j}$ and $-4\mathbf{i} + 5\mathbf{j}$ respectively.

(a) Find the position vector of A relative to B.

(b) Find the displacement of C relative to A.

(c) The position vector of the point D relative to C is $-2\mathbf{i} - 3\mathbf{j}$.
Find the position vector of D relative to O.

---

3. [5 marks: 2, 3]

The position vector the point A relative to the point B is < 2, 8 >.
The position vector of the point C relative to B is < –5, –2 >.

(a) Find the position vector of A relative to C.

(b) If in addition, the position vector of D relative to C is < 1, 5 > and the position vector of A is < –10, 2 >, find the position vector of D.

Calculator Assumed

4. [5 marks: 2, 1, 2]

The position vector of Peter relative to a flag pole is 20**i** + 40**j** metres. Relative to Peter, Joe has position vector 5**i** – 15**j** metres.

(a) Find the position vector of Joe relative to the flagpole.

(b) Hence, find the distance between Joe and the flag pole.

(c) The position vector of Kelly relative to Joe is a**i** + 20**j** metres.
Find the value of a if the distance between Kelly and Joe is 50 metres.

---

5. [4 marks]

Vectortown has position vector –20i + 10j km. The position vector of Trigtown relative to Vectortown is –40i – 15j km. The position vector of Easytown relative to Trigtown is 4i + 70j km. Find the position vector of Easytown.

10 Relative Velocity

Calculator Assumed

1. [7 marks: 1, 1, 3, 2]

Relative to an observer at O, A is moving with velocity 6**i** + 9**j** ms^{-1} and B is moving with velocity –3**i** + 4**j** ms^{-1} .

(a) Find the velocity of A relative to B

(b) What is the speed of A relative to B?

(c) The velocity of C relative to B is 4**i** – 5**j** ms^{-1}. Find the velocity of C relative to A.

(d) In what direction is C moving relative to A?

Calculator Assumed

2. [7 marks: 3, 4]

L is travelling along bearing 060° with a speed of 10 kmh^{-1}. M is travelling along bearing 210° with a speed of 5 kmh^{-1}.

(a) Draw a clearly labelled vector diagram indicating the velocity vector of L relative to M.

(b) Use trigonometry to find the speed and direction of L relative to M.

Calculator Assumed

3. [7 marks: 2, 2, 1, 2]

P is travelling along bearing 045° with a speed of 20 kmh^{-1}. Q is travelling along bearing 240° with a speed of 15 kmh^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Express the velocities of P and Q in terms of **i** and **j**.

(b) Find the velocity of Q relative to P.

(c) What is the speed of Q relative to P?

(d) What is the direction of Q relative to P?

11 Relative Vectors

Calculator Assumed

1. [4 marks: 2, 2]

James is running along bearing 050^o with a speed of 5 ms^{-1}. Wesley is running along bearing 300^o with speed 4 ms^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Find in component form the velocity of Wesley relative to James.

(b) Find how fast and in what direction is Wesley moving away from James.

Calculator Assumed

2. [6 marks: 2, 4]

A yacht is sailing on a bearing of 050° with speed 12 kmh^{-1}. Sarah on the yacht measures the wind as blowing with a speed of 10 kmh^{-1} from a bearing of 300°.

(a) Sketch a clearly labelled velocity vector diagram that shows the relationship between the velocity of the yacht, the true velocity of the wind and the velocity of wind relative to the yacht.

(b) Find the true speed and direction of the wind.

Calculator Assumed

3. [7 marks: 2, 2, 3]

A yacht Y is moving with velocity $2\mathbf{i} + 5\mathbf{j}$ kmh^{-1}. A sailor on board the yacht measures the wind as blowing with velocity $-3\mathbf{i} - 2\mathbf{j}$ kmh^{-1}.

(a) Find the velocity of the wind.

To a sailor on a second yacht Z, the wind appears to be blowing with velocity $2\mathbf{i} + 4\mathbf{j}$ kmh^{-1}.

(b) Find the velocity of the second yacht.

(c) How fast is yacht Z moving away from yacht Y and in what direction?

Calculator Assumed

4. [4 marks]

A ship is travelling with a speed of 20 knots along bearing 080°. Relative to the ship, the wind is blowing from 310° with a speed of 8 knots. Let **i** and **j** be unit vectors in the Easterly and Northerly directions respectively. By expressing the given velocities in component form, find the true speed and direction of the wind.

Calculator Assumed

5. [11 marks: 2, 2, 7]

May is running along 025° at 4 kmh^{-1}.
Relative to May, Fay is running with speed a kmh^{-1} along bearing 150°.

(a) Find in terms of a, the velocity of Fay, v_f.

Jane is running due East at 2 kmh^{-1}.
Relative to Jane, Fay is running with speed b kmh^{-1} along bearing 120°.

(b) Find in terms of b, the velocity of Fay, v_f.

(c) Use your answers in (a) & (b) to find a and b. Hence, find the true speed and direction with which Fay is running.

Calculator Assumed

6. [6 marks: 1, 1, 4]

At 0900 hours, P is located at $20\mathbf{i} - 10\mathbf{j}$ km and is travelling with a constant velocity of $-4\mathbf{i} + 6\mathbf{j}$ kmh^{-1}. At the same time, Q is located at $50\mathbf{i} + 40\mathbf{j}$ km and is travelling with a constant velocity of $-10\mathbf{i} - 4\mathbf{j}$ kmh^{-1} .

(a) Find the displacement of Q relative to P at 0900 hours.

(b) Find the velocity of P relative to Q at 0900 hours.

(c) Use your answers in (a) and (b) to determine when and where P will collide with Q.

Calculator Assumed

7. [11 marks: 2, 2, 5, 2]

When the clock struck one, relative to a clock tower, the position vector of a cat is $6\mathbf{i} + 10\mathbf{j}$ m. The cat is running with velocity $2\mathbf{i} + 2\mathbf{j}$ ms^{-1}. At the same time, relative to the same clock tower, the position vector of a dog is $-4\mathbf{i} - 6\mathbf{j}$. The dog can run at 5 ms^{-1}.

(a) Find the position vector of the cat relative to the dog.

Let the velocity vector of the dog for it to intercept the cat be $x\mathbf{i} + y\mathbf{j}$ ms^{-1} .

(b) Find in terms of x and y, the velocity of the dog relative to the cat.

(c) Use your answers in (a) and (b) to find x and y for the dog to intercept the cat. Assume that the cat continues running with the same speed and in the same direction.

(d) When and where does the interception take place.

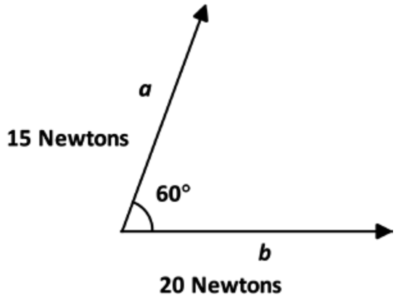
12 Scalar Product I

Calculator Free

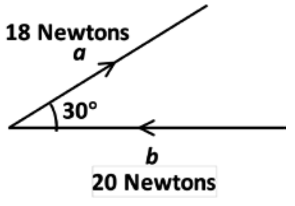
1. [4 marks: 2, 2]

Find the scalar product between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as given:

(a)



(b)



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2. [6 marks: 2, 2, 2]

Given that vector $\boldsymbol{u}$ has magnitude 10 ms^{-1} in the direction 030^o, $\boldsymbol{v}$ has magnitude 15 ms^{-1} in the direction 090^o and $\boldsymbol{w}$ has magnitude 5 ms^{-1} in the direction 180^o.
Find:

(a) $\boldsymbol{u} \cdot \boldsymbol{v}$

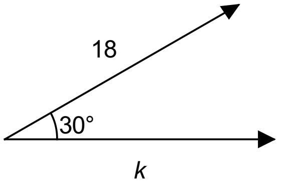
(b) $\boldsymbol{u} \cdot \boldsymbol{w}$

(c) the magnitude and direction of $(\boldsymbol{u} \cdot \boldsymbol{w})\, \boldsymbol{v}$.

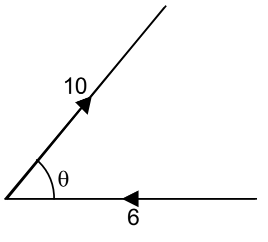
Calculator Free

3. [4 marks: 2, 2]

(a) The scalar product between the two vectors shown below is $180\sqrt{3}$. Find k.



(b) The scalar product between the two vectors shown below is –50. Find cos θ.



4. [4 marks]

Given that $|\boldsymbol{a}| = 20$ and $|\boldsymbol{b}| = 25$, determine with reasons the maximum and minimum value of $\boldsymbol{a} \cdot \boldsymbol{b}$.

5. [4 marks: 2, 1, 1]

Given that $|\boldsymbol{a}| = 8, |\boldsymbol{b}| = 5$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 20\sqrt{2}$:

(a) determine the acute angle between ***a*** and ***b.***

(b) determine the acute angle between 2***a*** and 3***b***

(c) ***a*** and –

Calculator Free

6. [6 marks: 3, 3]

Given that $|\boldsymbol{m}| = 10$ and $|\boldsymbol{n}| = 10$, find $\boldsymbol{n}$ in terms of $\boldsymbol{m}$ if:

(a) $\boldsymbol{m} \cdot \boldsymbol{n} = 100$

(b) $\boldsymbol{m} \cdot \boldsymbol{n} = -100$

7. [4 marks: 1, 1, 2]

Given that $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find:

(a) $(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c}$

(b) $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

(c) k if $\boldsymbol{a} \cdot (3\mathbf{i} + k\mathbf{j}) = 9$

Calculator Free

8. [3 marks: 1, 1, 1]

Expand and simplify:

(a) $(\boldsymbol{m} + \boldsymbol{n}) \cdot (\boldsymbol{m} + \boldsymbol{n})$

(b) $(\boldsymbol{c} + \boldsymbol{d}) \cdot (\boldsymbol{c} - \boldsymbol{d})$

(c) $(\boldsymbol{m} - 3\boldsymbol{n}) \cdot (2\boldsymbol{m} - \boldsymbol{n})$

---

9. [4 marks: 2, 2]

Given that $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular, prove that:

(a) $\boldsymbol{p} \cdot \boldsymbol{q} = 0$

(b) $(\boldsymbol{p} + \boldsymbol{q}) \cdot (\boldsymbol{p} + \boldsymbol{q}) = |\boldsymbol{p}|^2 + |\boldsymbol{q}|^2$

Calculator Free

10. [7 marks: 2, 2, 3]

Given that $|\boldsymbol{r}| = 10$ and $|\boldsymbol{s}| = 8$, find:
(a) $\boldsymbol{r} \cdot \boldsymbol{r}$

(b) $(\boldsymbol{r} + \boldsymbol{s}) \cdot (\boldsymbol{r} + \boldsymbol{s})$ if $\boldsymbol{r}$ and $\boldsymbol{s}$ are parallel and in the same direction.

(c) $|\boldsymbol{r} - \boldsymbol{s}|$ if $\boldsymbol{r}$ and $\boldsymbol{s}$ are perpendicular.

---

11. [7 marks: 2, 5]

The points A, B and C have position vectors $< -4, 3 >$, $< 2, 7 >$ and $< -1, 2 >$ respectively.
(a) Find the vectors **AB** and **CA**.

(b) Find a vector perpendicular to **AB** but with the same magnitude as **CA**.

Calculator Assumed

12. [7 marks: 2, 3, 2]

Given that $\boldsymbol{u} = \mathbf{i} + 3\mathbf{j}$, $\boldsymbol{v} = -6\mathbf{i} + 8\mathbf{j}$ and $\boldsymbol{w} = k\mathbf{i} - 2\mathbf{j}$, find:

(a) the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(b) k, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel in the opposite direction.

(c) k, if the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 120°.

Calculator Assumed

13. [6 marks: 3, 3]

Given that $|\boldsymbol{u}| = 10$ and $(\boldsymbol{u} + 2\boldsymbol{v}) \,.\, (\boldsymbol{u} + \boldsymbol{v}) = 408$.

(a) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

(b) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in opposite directions.

---

14. [6 marks: 3, 3]

Given that $\boldsymbol{u} = -\mathbf{i} + \mathbf{j}$, find in exact form, a unit vector $\hat{\boldsymbol{v}}$, if:

(a) $\boldsymbol{u}$ is perpendicular to $\hat{\boldsymbol{v}}$.

(b) the acute angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{v}}$ is 45°.

Calculator Assumed

15. [8 marks: 2, 3, 3]

Let $\boldsymbol{a} = \begin{pmatrix} -\sqrt{5} \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} t \\ 5 \end{pmatrix}$.

(a) Find the value of t if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

(b) Find value(s) of t if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

(c) Determine the value(s) of t if the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$.

---

16. [6 marks]

Given that $\boldsymbol{u} = 3\mathbf{i} + \mathbf{j}$ and $\boldsymbol{v} = x\mathbf{i} + y\mathbf{j}$, find x and y if $|\boldsymbol{v}| = \sqrt{10}$ and the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°.

Calculator Assumed

17. [7 marks: 3, 4]

The position vector the point P relative to the point Q is < –5, 4 >.
The position vector of the point R relative to Q is < 4, 5 >.
(a) Find the position vector of R relative to P.

(b) Determine the area of the triangle formed by the points P, Q and R.
Show clearly how you obtained your answer.

---

18. [6 marks: 2, 4]

The points P, Q and R have position vectors < –5, 10 >, < 2, 9 > and < –1, 6 > respectively.
(a) Find the **PQ** and **PR**.

(b) Determine the cosine of ∠QPR. Hence, find the area of ΔPQR.

13 Scalar Product II

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 3 and 4 respectively. The acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\dfrac{1}{3}$. Find in terms of the vectors $\boldsymbol{a}$ and/or $\boldsymbol{b}$:

(a) the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(b) the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(c) the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

(d) $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

(e) the magnitude of the $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

Calculator Free

2. [10 marks: 2, 2, 3, 3]

(a) The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 8 and 6 respectively. Find the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ if the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

(i) is 60°.

(ii) is 120°.

(b) Find θ, the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ if $|\boldsymbol{a}| = 8$ and the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ has a magnitude of $4\sqrt{3}$.

(c) The vector $\boldsymbol{b}$ has magnitude 10. The projection of vector $\boldsymbol{a}$ on $\boldsymbol{b}$ is $5\boldsymbol{b}$.
Find the magnitude of $\boldsymbol{a}$ if the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 60°.

Calculator Free

3. [4 marks: 2, 2]

Given ***a*** = < 5, 10 > and ***b*** = < 4, 3 >.

(a) Find the component of ***a*** that is parallel to ***b***.

(b) Find the component of ***a*** that is perpendicular to ***b***.

---

4. [6 marks: 2, 2, 2]

Given that ***a*** = < 2, 5 > + < 10, –4 >, find:

(a) the vector projection of ***a*** onto < 6, 15 >.

(b) the vector projection of ***a*** onto < –5, 2 >

(c) the vector projection of ***a*** onto < –5, 0 >

Calculator Free

5. [11 marks: 3, 2, 2, 4]

The acute angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°. The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is $< 4, -3 >$ and $|\boldsymbol{v}| = 10$.

(a) Find $\boldsymbol{v}$.

(b) Explain clearly why $\boldsymbol{u} = < 4, -3 > + \lambda < 3, 4 >$.

(c) Find $|\boldsymbol{u}|$.

(d) Find $\boldsymbol{u}$.

Calculator Assumed

6. [11 marks: 1, 3, 2, 3, 2]

Let $\boldsymbol{u} = < 2, 1 >$, $\boldsymbol{v} = < 1, -2 >$ and $\boldsymbol{w} = < 3, 9 >$.

(a) Show that $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

(b) Given that $\boldsymbol{w} = m\boldsymbol{u} + n\boldsymbol{v}$, find m and n.

(c) Find the vector projection of $\boldsymbol{w}$ onto $< -2, -1 >$.

(d) $\boldsymbol{w}$ is the vector projection of $\lambda < 2, 1 >$ onto $\boldsymbol{w}$. Find λ.

(e) If $\boldsymbol{w}$ is the vector projection of $\lambda < 2, 1 >$ onto $\boldsymbol{w}$, find the component of $\lambda < 2, 1 >$ that is perpendicular to $\boldsymbol{w}$.

Calculator Assumed

7 [7 marks: 3, 2, 2]

The position vector of the point A relative to the point B is < 3, 4 >.
The position vector of the point C relative to B is < –4, 5 >.

(a) Determine the vector projection of **BC** that is *parallel* to **BA**.

(b) Determine the vector projection of **BC** that is *perpendicular* to **BA**.

(c) Hence, or otherwise, determine the area of ΔABC.

14 Scalar Product III

Calculator Assumed

1. [6 marks: 2, 1, 3]

Given that vector $\boldsymbol{a}$ has magnitude 2 ms^{-1} in the direction 060°,
$\boldsymbol{b}$ has magnitude 4 ms^{-1} in the direction 045° :

(a) Find $\boldsymbol{a}$ and $\boldsymbol{b}$ in the form $x\,\boldsymbol{i} + y\,\boldsymbol{j}$.

(b) Find $\boldsymbol{a} \bullet$.

(c) Use your answers in (a) and/or (b) to find cos 15° in exact form.

Calculator Assumed

2. [4 marks: 2, 2]

(a) A force F has magnitude 2 000 N and acts at an angle of 010º with the horizontal. Calculate the work done by F in moving a object 10 m along the horizontal.

(b) Calculate the work done by the force $< 30, 50 >$ in moving an object through a displacement of $< 4, -1 >$ metres.

---

3. [5 marks: 3, 2]

A force F has magnitude 1 000 N and acts along bearing 030º.
Let $\boldsymbol{i}$ be the unit vector in the Easterly direction and $\boldsymbol{j}$ be the unit vector in the Northerly direction.

(a) Express F in the form $a\,\boldsymbol{i} + b\,\boldsymbol{j}$.

(b) Hence, or otherwise, calculate the work done when the force F moves an object through a displacement of $< 3, 4 >$ metres.

Calculator Assumed

4. [10 marks: 2, 2, 1, 2, 3]

A force $\boldsymbol{F}_1$ of $< 4, 4 >$ Newtons is applied to a body and causes the body to be displaced by $< 3, 1 >$ metres.

(a) Find the component of the applied force along the direction of motion.

(b) Find the component of the applied force perpendicular to the direction of motion.

(c) Determine the work done by the applied force.

(d) Another force $\boldsymbol{F}_2$ of $< x, y >$ Newtons is applied to the same body. But half as much work is required to cause the same displacement to the body.

(i) Find a possible pair of values for x and y.

(ii) Find x and y if $|F_2| = 2\sqrt{2}$ Newtons.

Calculator Assumed

5. [8 marks: 2, 1, 5]

Two forces < 1, 4 > Newtons and < 3, –2 > Newtons act on a single body P and causes the body P to be displaced by < 8, 4 >.

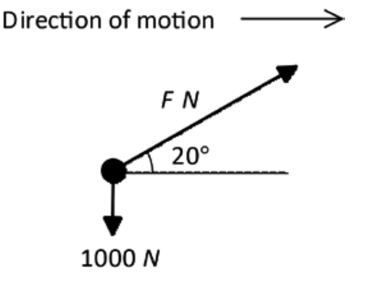
(a) Find the vector projection of the resultant force onto the displacement of P.

(b) Find the work done on P.

(c) A third force of < –2, 1 > Newtons is applied to P and causes it to move by 10 metres. The work done is 36 Joules. Find the displacement vector.

Calculator Assumed

6. [6 marks: 3, 2, 1]



An object of weight 1 000 Newtons is being pulled along a horizontal surface by a force of magnitude *F* Newtons inclined at an angle of 20° to the surface. The motion of the object is opposed by a horizontal force of magnitude 500 N.

(a) Find in terms of *F*, the magnitude of the pulling force perpendicular to the surface. Hence, find *F*.

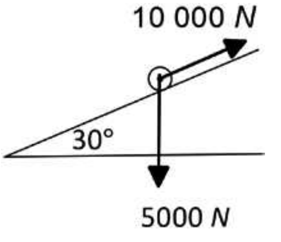
(b) Find the component of the pulling force in the direction of motion. Hence, find the magnitude of the resultant force in the direction of motion.

(c) Find the work done by the pulling force in moving the object 50 m in its direction of motion.

Calculator Assumed

7. [7 marks: 2, 1, 2, 2]

A body of weight 5 000 Newtons is pulled up along a plane inclined at an angle of 30° with the horizontal by a force of magnitude 10 000 N. There is a force of magnitude 500 N opposing the motion of the body.



(a) Find the component of the gravitational force on the body along the inclined plane. Hence, find the magnitude of the resultant force in the direction of the motion of the body.

(b) Find the work done by the resultant force when the body has moved 10 m along the inclined plane.

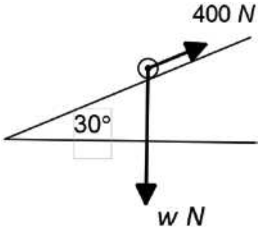
(c) Find the work done by the resultant force when the body has ascended a vertical distance of 10 m.

(d) The work done by the resultant force in moving the body up the inclined plane so that the change in its horizontal position is x metres is 280 kJ. Find x.

Calculator Assumed

8. [7 marks: 3, 4]

A cart of weight w N is at rest on a set of rail-tracks inclined at an angle of 30° with the horizontal. A force parallel to the inclined plane of magnitude 400 N just prevents the body from slipping down the rail-tracks.

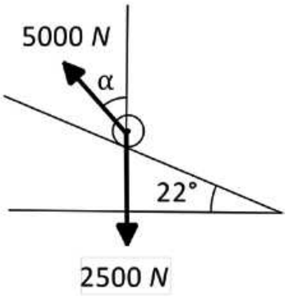


(a) Find the component of the gravitational force acting on the cart along the inclined rail-tracks. Hence, find the weight of the cart.

(b) A force of magnitude 1 000 N at an angle of 30° to the inclined rail-tracks acts on the cart and moves it a distance of 10 m along the tracks. Assume that the cart does not leave the rail-tracks. Find the work done by the resultant force along the inclined plane.

Calculator Assumed

9. [7 marks: 3, 4]



A body of weight 2500 N is pulled up an inclined plane with angle of inclination 22^o to the horizontal by a force F of magnitude 5000 N inclined at an angle of α^o to the vertical. The body remains in contact with the inclined plane. There is a force of 100 N opposing the motion of the body up the inclined plane.

(a) Calculate the value of α.

(b) Calculate the work done when the body is moved 2 m along the inclined plane by the force F.

15 Geometric Proofs using Vectors

Calculator Assumed

1. [6 marks: 2, 2, 2,]

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors. Find α and β if:

(a) $2\boldsymbol{a} + (\beta - 2)\,\boldsymbol{b} = (1 - \alpha)\,\boldsymbol{a}$

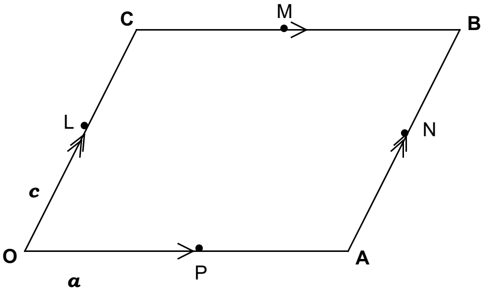
(b) $\alpha\,(3\boldsymbol{a} - 4\boldsymbol{b}) = 6\boldsymbol{a} + \beta\boldsymbol{b}$

(c) $\alpha\boldsymbol{a} + 5\boldsymbol{b}$ is parallel to $3\boldsymbol{a} + \beta\boldsymbol{b}$

---

2. [4 marks: 1, 1, 2]

OABC is a parallelogram. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. L, M, N and P are the midpoints of OC, CB, BA and AO respectively.



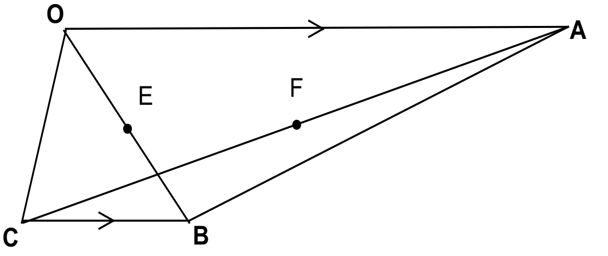
(a) Find **LM** in terms of **a** and **c** .

(b) Find **PN** in terms of **a** and **c**.

(c) Hence, use a vector method to show that LMNP is a parallelogram.

Calculator Assumed

3. [8 marks: 2, 4, 2]



OABC is a trapezium with **OA** = 3**CB**. **OA** = ***a*** and **OC** = ***c*** . E and F are midpoints of OB and CA respectively.

(a) Find **OE** in terms of ***a*** and ***c***.

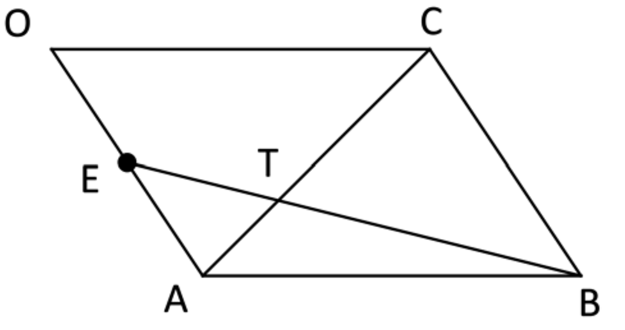
(b) Find **EF** in terms of ***a*** and ***c***.

(c) Prove that CEFB is a parallelogram.

Calculator Assumed

4. [7 marks: 3, 4]

OABC is a parallelogram with E as the midpoint of OA. The diagonal AC intersects the line segment BE at T such that AT = αAC and ET = βEB.
Let **OA** = ***a*** and **OC** = ***c***.



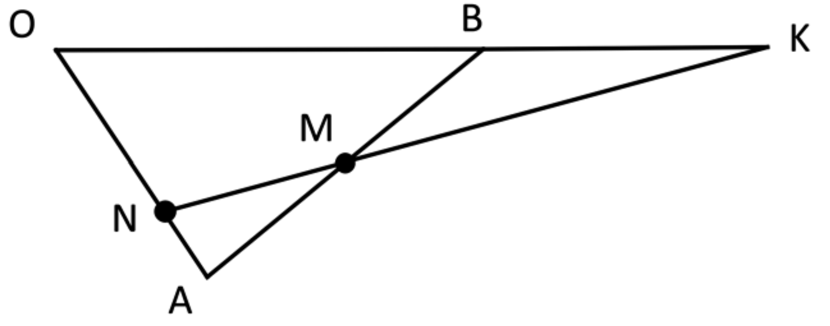
(a) Determine in terms of ***a*** and/or ***c***, the vectors **AT** and **ET**.

(b) Use **EA** = **ET** + **TA** to show that T divides the diagonal AC in the ratio 1 : 2.

Calculator Assumed

5. [8 marks: 4, 4]

In the diagram below, M is the midpoint of AB. N is a point on the line OA. OB extended meets NM extended at K. **ON** = $\frac{3}{4}$**OA**, **MK** = α**NK** and **BK** = β**OB**. Let **OA** = ***a*** and **OB** = ***b***.

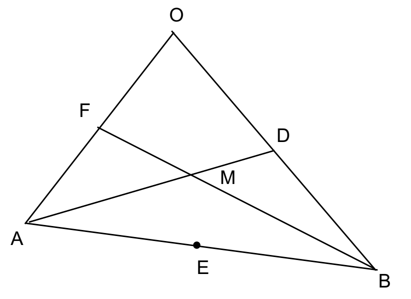


(a) Determine in terms of ***a*** and/or ***b***, the vectors **MB**, **OK** and **MK**.

(b) Use the vectors **BK**, **MB** and **MK** to determine the values of α and β.

Calculator Assumed

6. [14 marks: 2, 2, 5, 3, 2]



OAB is a triangle with **OA** = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. D, E and F are the midpoints of OB, AB, and OA respectively. **AM** = α**AD** and **MF** = β**BF**.

(a) Find **AD** and **BF** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Find **AM** and **MF** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Use your answers in (b) to find α and β.

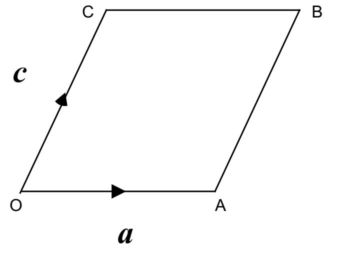
(d) Show that **OM** = μ**OE** giving the value of μ .

(e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.

Calculator Assumed

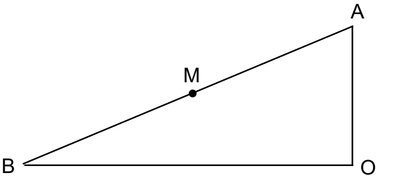
7. [4 marks]

OABC is a rhombus. **OA** = ***a*** and **OC** = ***c*** .
Use a vector method to show that the diagonals of a rhombus are perpendicular to each other.



8. [8 marks: 1, 3, 4]

OAB is a right angled triangle with $\angle$ AOB = 90°.
OA = ***a*** and **OB** = ***b***. M is the midpoint of AB.
(a) Explain why ***a*** **.** ***b*** = 0 .



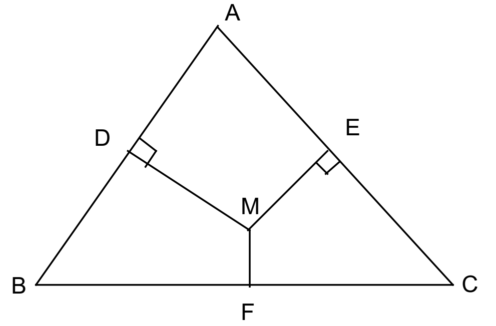
(b) Find $|\mathbf{BM}|^2$ in terms of a and b, where $|\boldsymbol{a}| = a$ and $|\boldsymbol{b}| = b$.

(c) Hence, prove that M is the centre of a circle passing through A, B and O.

Calculator Assumed

9. [10 marks: 1, 2, 2, 3, 2]

DM and EM are respectively the perpendicular bisectors of sides AB and AC of triangle ABC. F is midpoint of BC. Also, **AB** = $\boldsymbol{b}$, **AC** = $\boldsymbol{c}$ and **MD** = $\boldsymbol{d}$.



(a) Find **ME** in terms of $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

(b) Use your answer in (a) to show that $[\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c} = 0$

(c) Find **MF** in terms of $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

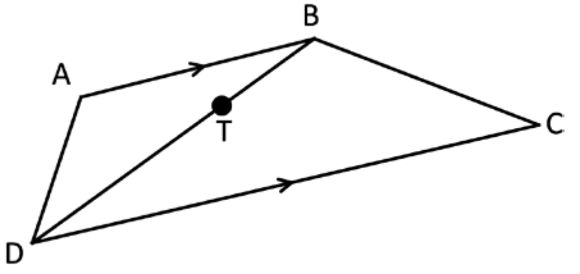
(d) Show that **MF . BC** = 0.

(e) State the significance of the result **MF . BC** = 0 .

Calculator Assumed

10. [10 marks: 1, 1, 2, 2, 3, 1]

In the quadrilateral ABCD shown below **AB** = ***u*** and **DA** = ***v***.
DC = 2AB and the point T divides the diagonal DB in the ratio 2:1.



(a) Determine in terms of ***u*** and/or ***v***:

(i) **DB**

(ii) **DT**

(iii) **CT**

(iv) **TA**

(b) Hence, or otherwise:

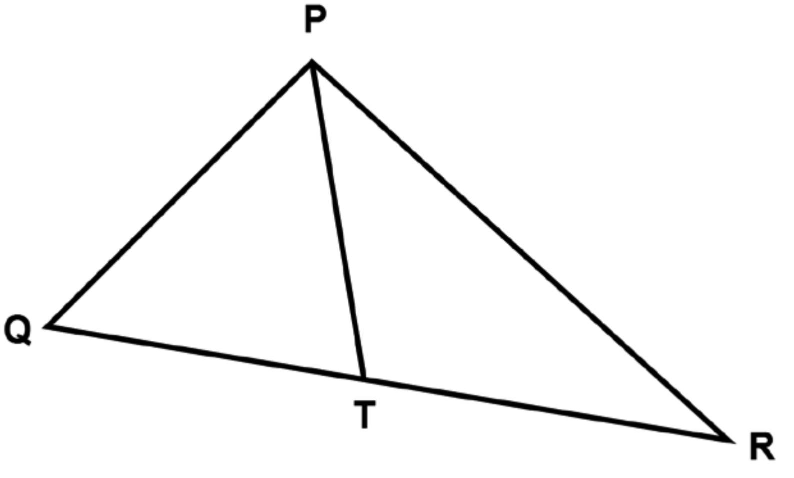
(i) prove that the points C, T and A are collinear.

(ii) determine the ratio T divides the line CA.

Calculator Assumed

11. [8 marks: 3, 5] [TISC]

In ΔPQR drawn below, the point T is the midpoint of QR. Let **PT** = ***a*** and **TR** = ***b***.



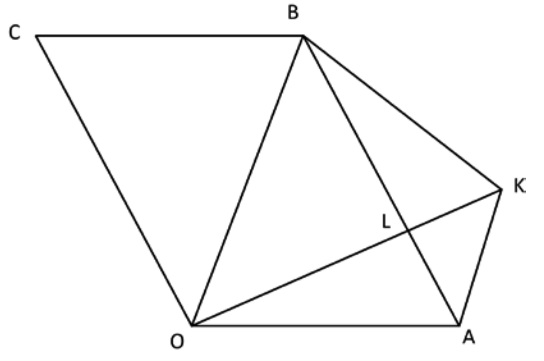
(a) Find **PR** and **PQ** in terms of ***a*** and ***b***.

(b) If T is equidistant to P and R, use a vector method to prove that ∠QPR = 90°.

Calculator Assumed

12. [8 marks: 1, 3, 4]

OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***. L is point on AB and OL is extended to K. **AL** = α **AB** and **LK** = $\frac{1}{2}$ **OL**.



(a) Express each of the following vectors in terms of ***a*** and or ***c***.

(i) **OB**

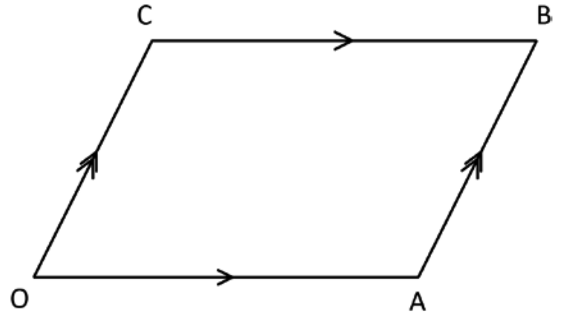
(ii) **LK**

(b) Use vectors to determine the value for α if OAKB is a trapezium.

Calculator Assumed

13. [9 marks: 3, 6]

OABC is a parallelogram with **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$.



(a) Prove that **OB** • **AC** = $|\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$.

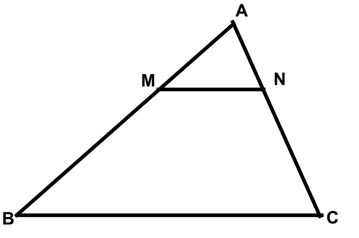
(b) Hence, prove that the diagonals of a parallelogram are perpendicular *if and only if* the parallelogram is a rhombus.

16 Geometric Proofs and Circle Properties

Calculator Free

1. [7 marks: 3, 2, 2]

   In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio 1 : 3.

   (a) Prove that ΔAMN and ΔABC are similar.

   (b) Hence, deduce that BC = 4MN.

   (c) Prove that MN is parallel to BC.

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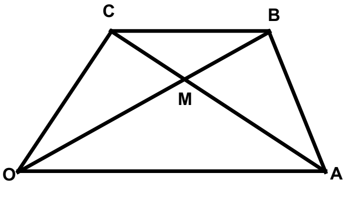
2. [5 marks: 3, 2]

   OABC is a trapezium with OA parallel to CB. The diagonals OB and AC intersect at M such that AM : MC = 3 : 1 .

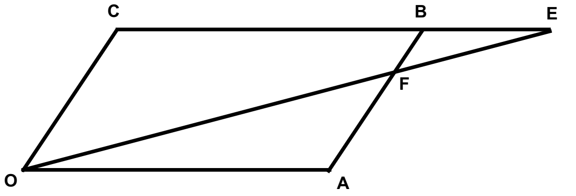
   

   (a) Prove that ΔMOA and ΔMBC are similar.

   (b) Hence deduce that OA = 3BC.

Calculator Free

3. [5 marks: 3, 2]



OABC is a parallelogram with OA parallel and congruent to CB. The point F divides AB in the ratio 2 : 1. OF extended meets the CB extended at E.

(a) Prove that ΔFOA and ΔFEB are similar.

(b) Hence, deduce that F divides OE in the ratio 2 : 1.

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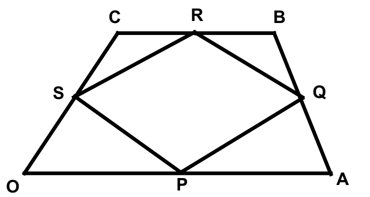
4. [5 marks]

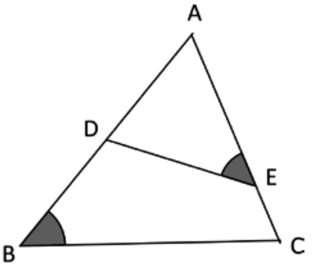


OABC is a trapezium with OA parallel to CB. P, Q, R and S are respectively the midpoints of OA, AB, BC and OC. Prove that the midpoints of the sides of a trapezium form a parallelogram, that is PQRS is a parallelogram.

Calculator Free

5. [6 marks]

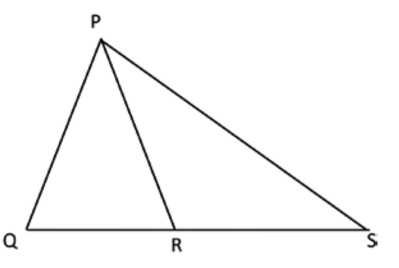
In the accompanying diagram, ∠ABC = ∠AED.
Also, AD = DB = 3 cm and AC = 5 cm.
Calculate with reasons, the length of EC.



6. [6 marks: 3, 3]

The diagram below shows the isosceles triangle PQR where PQ = 5 cm. The base QR is extended to S so that QS = PS = 8 cm.

(a) Prove that the triangles PQR and SQP are similar.

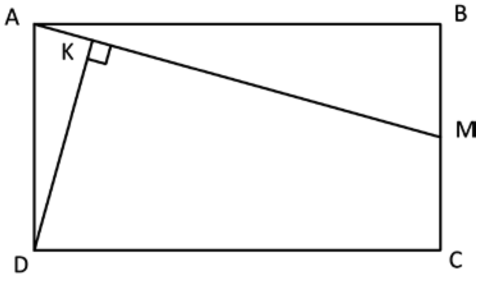


(b) Determine the length of RS. Show clearly how you obtained your answer.

Calculator Free

7. [9 marks: 4, 5]

In the diagram below, ABCD is a rectangle. M is a point on the side BC. The line segment DK is perpendicular to AM.



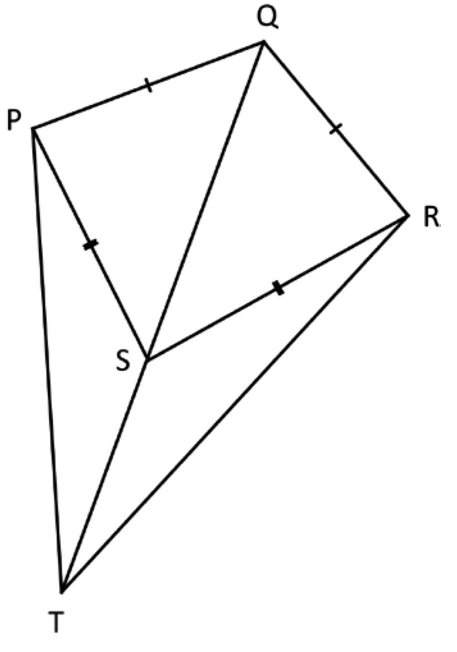
(a) Prove that triangles AKD and MBA are similar.

(b) Given that AK : MB = α, find in terms of α, the ratio of the area of ΔAKD to the area of ΔMBA.

Calculator Free

8. [8 marks]

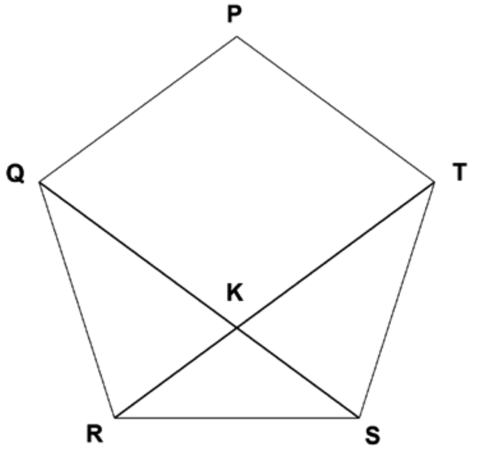
In the diagram shown below, PQRS is a quadrilateral with PQ = QR and SP = SR. The points Q, S and T are collinear. Prove that TP = TR.



Calculator Assumed

9. [9 marks: 2, 2, 5]

PQRST is a *regular* pentagon of side length 10 cm.



(a) Find the size of $\angle$ RST. Justify your answer.

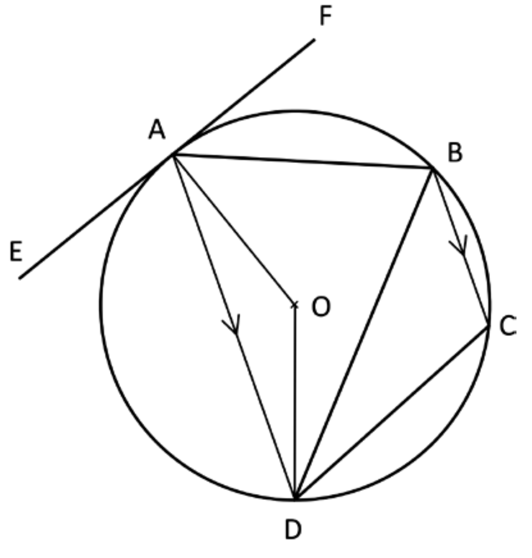
(b) Prove that $\angle STR = 36^\circ$.

(c) Find the length of KT. Show clearly how you obtained your answer.

Calculator Assumed

10. [6 marks: 4, 2]

ABCD is a cyclic quadrilateral with the sides AD parallel to BC. EAF is a tangent to the circle.



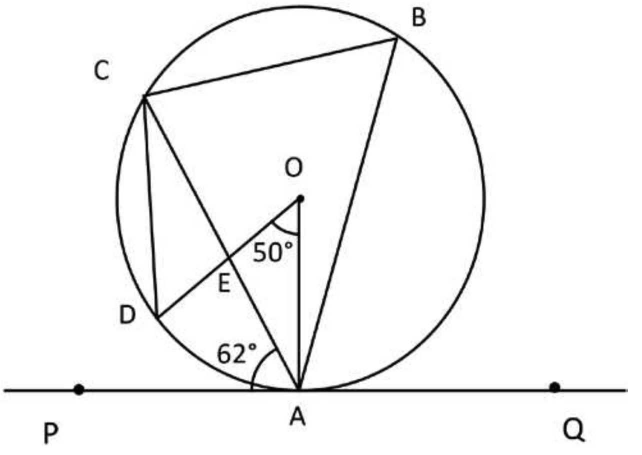
(a) Prove that $\angle DAE = \angle DBA$.
That is, prove that the angle between a chord and tangent is equal to the angle in the alternate segment.

(b) Hence, prove that $\angle FAB = \angle DBC$.

Calculator Assumed

11. [8 marks: 2, 3, 3]

The diagram below shows a circle centre O. PAQ is a tangent to the circle at A. B, C and D are points on the circumference of the circle. OD and AC intersect at E.



Determine with reasons, the size of each of the following angles.

(a) $\angle OAC$

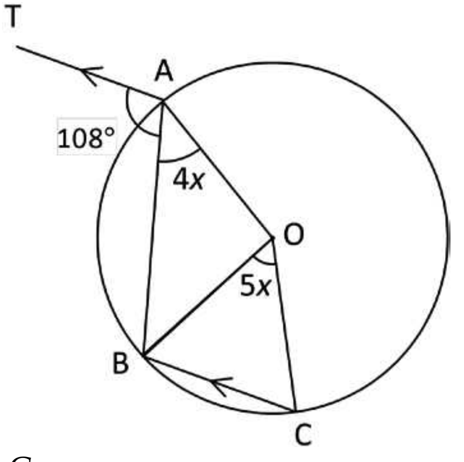
(b) $\angle ODC$

(c) $\angle ACB$

Calculator Assumed

12. [6 marks]

The points A, B and C lie on a circle centre O. The line AT is parallel to BC.
$\angle BAT = 108^\circ$, $\angle BAO = 4x$ and $\angle BOC = 5x$.

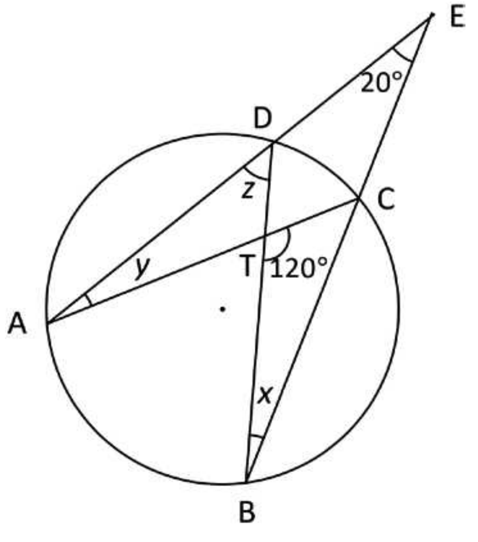


Find with reasons $\angle BAC$.

Calculator Assumed

13. [7 marks: 2, 5]

A, B, C and D are points on the circumference of a circle. AC meets BD at T. AD extended meets BC extended at E. ∠AEB = 20° and ∠BTC = 120°.



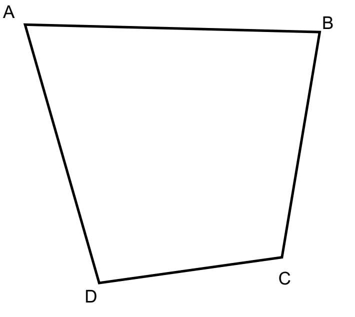
(a) State ∠ACE in terms of x. Justify your answer.

(b) Determine with reasons the values of x, y and z.

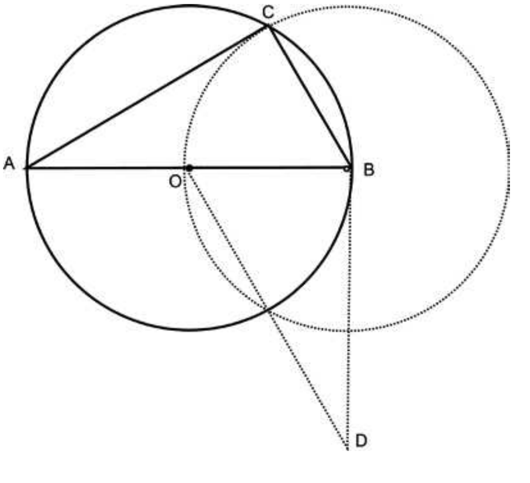
Calculator Assumed

14. [7 marks: 4, 3] [TISC]

(a) ABCD is a quadrilateral.
$\angle DAB + \angle DCB = 180^{o}$ and
$\angle ADC + \angle ABC = 180^{o}$.
Prove that there is a circle that passes through A, B, C and D.



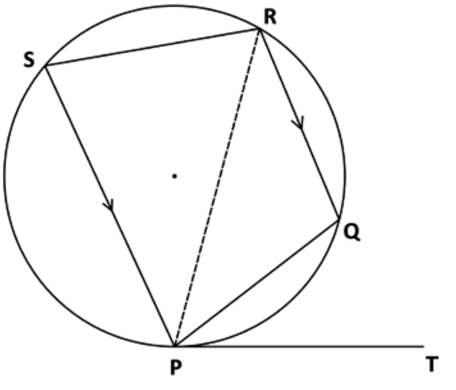
(b) AB is the diameter of the circle with centre O. B is the centre of another circle passing through O. The two circles intersect at C. BD is a tangent to the circle with centre O. If AC = BD, prove that $\angle BOD = \angle CBA$.



Calculator Assumed

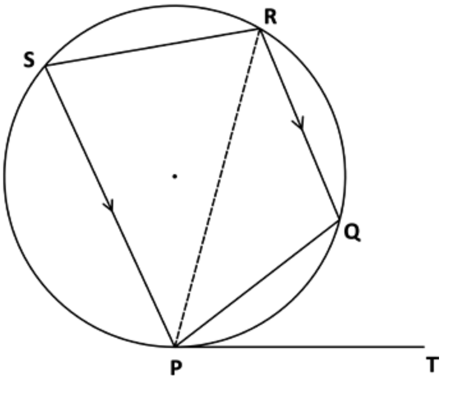
15. [7 marks: 3, 4] [TISC]

PQRS is a cyclic quadrilateral with RQ parallel to SP. PT is a tangent to the circle. The line PR bisects ∠SPQ.



(a) Prove that PQ bisects ∠TPR.

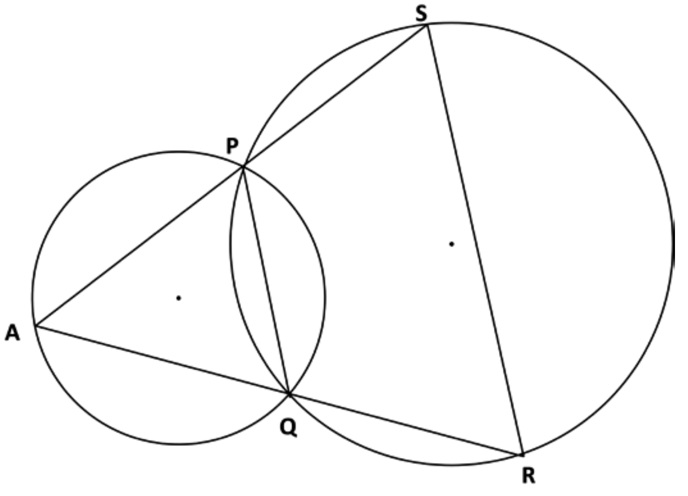
(b) Prove that PQ = SR.
[Hint: Prove that ΔRPQ is congruent to ΔQSR.]



Calculator Assumed

16. [7 marks: 3, 4]

In the diagram below, the two circles intersect at P and Q. S and P are points on the circumference of the larger circle. The points A, P and S are collinear. The points A, Q and R are collinear.



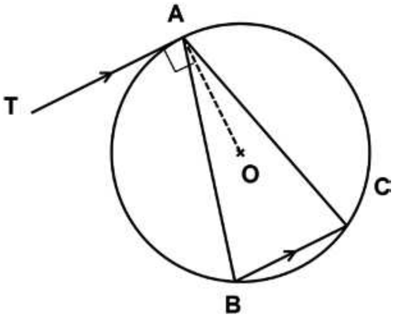
(a) Prove that ΔAPQ and ΔARS are similar.

(b) Given that AQ = 10 cm, AP = QR = 8 cm, find PS.

Calculator Assumed

17. [3 marks]

The points A, B and C lie on a circle centre O. TA is the tangent to the circle at A. If TA is parallel to BC, prove that ΔABC is isosceles.



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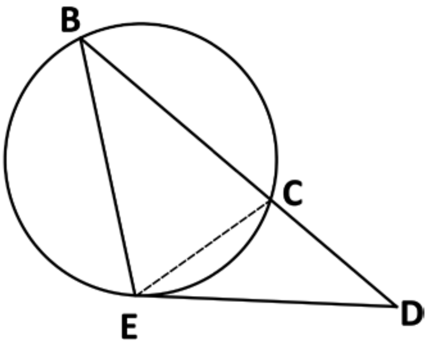
18. [5 marks: 3, 2]

The points B, C and E lie on the same circle. The chord BC extended meets the tangent to the circle at E at the point D.

(a) Prove that $ED^2 = CD \times BD$.

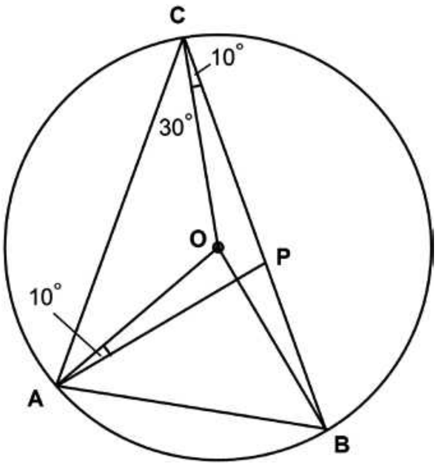


(b) BD is 20 cm long and the point C divides BD in the ratio 3: 2. Hence, find the length of DE.

Calculator Assumed

19. [6 marks: 4, 2]

(a) The points A, B and C lie on a circle centre O. $\angle ACO = 30^o$ and $\angle BCO = 10^o$. P is a point on the chord BC such that $\angle OAP = 10^o$. Find with reasons $\angle APB$.

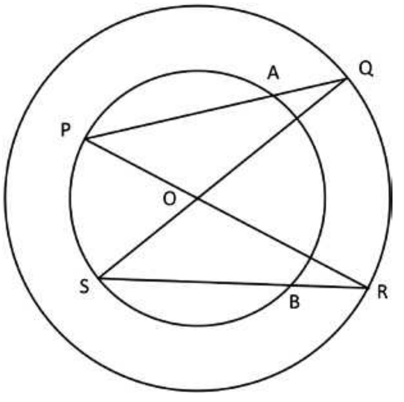


(b) If the points A, B and C lie on the circumference of a circle and O is a point inside the circle, prove or disprove the conjecture that if $\angle AOB = 2\angle ACB$, then O must be the centre of the circle.

Calculator Assumed

20. [8 marks: 3, 2, 3]

The diagram below shows two concentric circles centred at O.
S and P are points on the smaller circle while Q and R are points on the larger circle.
PQ intersects the inner circle at A and SR intersects the inner circle at B.



(a) Prove that triangles OPQ and OSR are congruent.

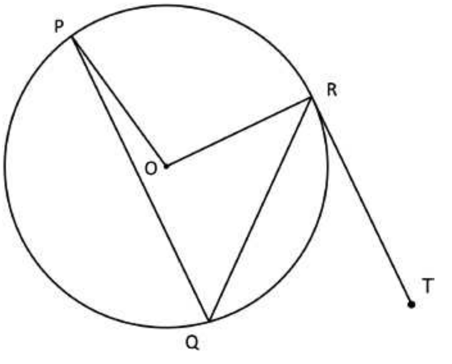
(b) Prove that the points P, Q, R and S are concyclic.

(c) Prove that AB is parallel to QR.

Calculator Assumed

21. [9 marks: 6, 3]

The diagram below shows a circle with centre at O. P, Q and R are points on the circle. RT is a tangent to the circle at R. Let the acute $\angle PQR = \theta$.



(a) Prove that obtuse $\angle POR = 2\theta$.

(b) If PO is parallel to RT, prove that $\angle PQR = 45^{\circ}$.

Calculator Assumed

22. [6 marks: 4, 2]

Consider the premise:

If ABCD is a square, then the diagonals AC and BD are perpendicular to each other.

(a) State the converse of this premise.
Determine with reasons if the converse of this premise is true or false.

(b) State the contrapositive of this statement.

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23. [5 marks]

Consider the premise: *For* ΔABC*, if* $AB = AC$*, then* $\angle ABC = \angle ACB$*.*
State the converse of this statement and prove that the converse of is true.

17 Trigonometric Equations I (Simple trigonometric ratios)

Calculator Free

1. [13 marks: 3, 3, 4, 3]

Solve for all values of θ (in degrees):

(a) $\sin\theta = \cos\theta$

(b) $(\cos\theta - 2)(2\cos\theta + 1) = 0$

(c) $2\sin^2\theta + 3\sin\theta - 2 = 0$

(d) $\sec^2\theta - 4\sec\theta + 4 = 0$

Calculator Free

2. [13 marks: 3, 3, 4, 3]

Given that $\cos 15^{\circ} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$, solve for θ in:

(a) $\cos(\theta + 5^{\circ}) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \leq \theta \leq 360^{\circ}$

(b) $\cos\theta = -\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right)$ for $0 \leq \theta \leq 360^{\circ}$

(c) $\sin\theta = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \leq \theta \leq 360^{\circ}$

(d) $\sec\theta = \sqrt{6} - \sqrt{2}$ for $0 \leq \theta \leq 360^{\circ}$

Calculator Free

3. [11 marks: 1, 3, 4, 3]

Given that $\sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4}$, find all solutions (in radians) to:

(a) $\sin x = \frac{\sqrt{5}-1}{4}$

(b) $\sin x = \frac{1-\sqrt{5}}{4}$

(c) $\cos x = \frac{\sqrt{5}-1}{4}$

(d) $\operatorname{cosec} x = 1 + \sqrt{5}$

Calculator Free

4. [12 marks: 1, 2, 2, 3, 4]

(a) The equation cos $(x) = a$ has general solution $x = 2n\pi \pm \dfrac{3\pi}{4}$ for $n \in \mathbb{Z}$.

(i) Determine the exact value of a.

(ii) Find all solutions in the domain $0 \le x \le 2\pi$.

(b) It is known that $\tan \alpha^{\circ} = \dfrac{\sqrt{7}}{3}$ where α is an acute angle.

For each of the following equations solve for all values of x° in terms of α°.

(i) $\tan x^{\circ} = \dfrac{-\sqrt{7}}{3}$

(ii) $\sin x^{\circ} = \dfrac{\sqrt{7}}{4}$

(iii) $\cos (2x^{\circ}) = \dfrac{3}{4}$

18 Trigonometric Identities I (Pythagorean)

Calculator Free

1. [12 marks: 3, 3, 3, 3]

(a) Prove that $\tan x = \dfrac{1}{\tan\left(\dfrac{\pi}{2} - x\right)}$. for $x \neq \dfrac{\pi}{2}$.

(b) Prove that $\cos^3 x \ \tan x + \sin^3 x = \sin x$.

(c) Prove that $\dfrac{1}{\sin x \tan x + \cos x} = \cos x$.

(d) Prove that $\dfrac{\sin^3 x - \cos^3 x}{1 + \sin x \ \cos x} = \sin x - \cos x$.

Calculator Free

2. [11 marks: 3, 4, 4]

(a) Prove $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$.

(b) Prove $(1 + \tan^2 P)(1 - \sin^2 P) = 1$

(c) Prove $\dfrac{1-2\cos^2 x}{\sin x+\cos x} = \sin x - \cos x$

Calculator Free

3. [10 marks: 4, 3, 3]

(a) Prove $\dfrac{1+\sin M}{\cos M} = \dfrac{\cos M}{1-\sin M}$

(b) Prove $\sec x \operatorname{cosec} x = \tan x + \cot x$

(c) Prove $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

Calculator Free

4. [8 marks: 2, 2, 4]

(a) Prove $\dfrac{1}{1+\cot x} = \dfrac{\tan x}{1+\tan x}$

(b) Prove $\dfrac{\sin x}{1+\cos x} = \dfrac{1}{\operatorname{cosec} x + \cot x}$

(c) Prove $\dfrac{\cos x}{1+\sin x} = \sec x - \tan x$

Calculator Free

5. [7 marks: 4, 3]

Prove each of the following:

(a) $\dfrac{\operatorname{cosec} x + 1}{\operatorname{cosec} x - 1} = \tan^2 x + 2 \tan x \sec x + \sec^2 x$

(b) $\dfrac{1}{\sec^2 x - 1} = \operatorname{cosec}^2 x - 1$

Calculator Free

6. [8 marks: 4, 4]

(a) Prove that $(\text{cosec}\ \theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

(b) Prove that $\dfrac{\cos^3\theta - \sin^3\theta}{\cos\theta - \sin\theta} + \dfrac{\cos^3\theta + \sin^3\theta}{\cos\theta + \sin\theta} = 2$

19 Trigonometric Identities II (Add/Sub Formulae)

Calculator Free

1. [7 marks: 1, 2, 1, 3]

For the acute angle θ^{o}, $\cos \theta^{o} = a$ where $0 < a < 1$, find in terms of a:

(a) $\sin(90^{o} - \theta^{o})$

(b) $\tan(\theta^{o})$

(c) $\cos(180^{o} + \theta^{o})$

(d) $\sin(\theta^{o} + 30^{o})$

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2. [10 marks: 3, 3, 4]

Use an appropriate trigonometric identity to find the exact value of :

(a) $\sin 75^{o}$

(b) $\cos 165^{o}$

Calculator Free

2. (c) $\tan \frac{7\pi}{12}$

3. [7 marks: 1, 1, 2, 3]

Given that $\sin A = \frac{4}{5}$ and $0 < A < \frac{\pi}{2}$, find the exact value of:

(a) $\cos A$

(b) $\tan A$

(c) $\sin (\frac{\pi}{2} + A)$

(d) $\cos (\frac{\pi}{4} - A)$

Calculator Assumed

4. [10 marks: 1, 1, 2, 2, 4]

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{4}$, where A and B are acute, use appropriate trigonometric identities (relationships) to find the exact value of:

(a) cos A

(b) sin B

(c) sin(A + B)

(d) cos (A – B)

(e) tan (A + B)

Calculator Assumed

5. [9 marks: 1, 1, 2, 2, 3]

Given that $\sin P = \frac{5}{13}$ and $\cos Q = -\frac{15}{17}$, where $\frac{\pi}{2} \le P \le \pi$ and $\frac{\pi}{2} \le Q \le \pi$,
use appropriate trigonometric identities to find the exact value of:

(a) $\cos P$

(b) $\sin Q$

(c) $\sin(P - Q)$

(d) $\cos(P + Q)$

(e) $\tan(P - Q)$

Calculator Assumed

6. [9 marks: 4, 5]

(a) Prove that $\dfrac{1}{\cot x - \cot 2x} = \sin 2x$.

(b) Prove that $\dfrac{\sin^2 A - \sin^2 B}{\sin^2(A+B)} = \dfrac{\tan A - \tan B}{\tan A + \tan B}$.

Calculator Assumed

7. [10 marks: 2, 3, 5]

(a) Prove that sin (–A) = – sin A

(b) Prove that $\dfrac{\sin(A-B)}{\sin A \sin B}$ = cot B – cot A

(c) Use your answers in parts (a) and (b) to rewrite $\dfrac{\sin 2x}{\sin x \sin 3x} + \dfrac{\sin 2x}{\sin 3x \sin 5x}$

in the form $\dfrac{\sin a}{\sin b \sin c}$, giving the values of a, b and c.

20 Trigonometric Identities III (Double angle)

Calculator Free

1. [11 marks: 1, 1, 3, 3, 3]

   Given that $\sin P = \frac{1}{4}$ and $\cos Q = \frac{2}{3}$, where $\frac{\pi}{2} \leq P \leq \pi$ and $\frac{3\pi}{2} \leq Q \leq 2\pi$, find the exact value of:

   (a) $\cos P$

   (b) $\sin Q$

   (c) $\cos (P + Q)$

   (d) $\tan 2Q$

   (e) $\sin \frac{Q}{2}$

Calculator Free

2. [9 marks: 1, 2, 3, 3]

For the acute angle θ^o, $\tan \theta^o = \frac{a}{b}$ where a and b are both positive numbers, find in terms of a and/or b:

(a) $\cos(\theta^o)$

(b) $\sin(-\theta^o)$

(c) $\tan(\theta^o + 45^o)$

(d) $\cos(2\theta^o)$

Calculator Free

3. [15 marks: 3, 3, 4, 5]

Prove each of the following identities:

(a) $\cos^4 x - \sin^4 x = \cos 2x$

(b) $\dfrac{\sin 2t}{1+\cos 2t} = \tan t$

(c) $\cos 6x = 4\cos^3 2x - 3\cos 2x$

(d) $\dfrac{1-\sin 2t}{\cos 2t} = \dfrac{1-\tan t}{1+\tan t}$

Calculator Free

4. [11 marks: 3, 4, 4]

   Prove each of the following:

   (a) $\dfrac{\cos x - \sin 2x}{\cos 2x + \sin x - 1} = \cot x$

   (b) $\cot \theta + \tan \theta = 2 \operatorname{cosec} 2\theta$

   (c) $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

Calculator Free

5. [9 marks: 3, 3, 3]

(a) Prove that $\sqrt{1-\cos x} = \sqrt{2}\,\sin\dfrac{x}{2}$.

(b) Prove that $\dfrac{\sin x}{1+\cos x} = \tan\dfrac{x}{2}$.

(c) Prove that $\tan^2\left(\dfrac{3x}{2}\right) = \dfrac{1-\cos 3x}{1+\cos 3x}$

Calculator Free

6. [13 marks: 4, 3, 6]

(a) Prove that $(a\cos x + b\sin x)^2 + (b\cos x - a\sin x)^2 = a^2 + b^2$.

(b) Prove that $\cos^4 2x - \sin^4 2x = \cos 4x$

(c) Prove that $\dfrac{1}{1+\tan x} - \dfrac{1}{1-\tan x} = -\tan 2x$.

Calculator Free

7. [11 marks: 3, 3, 2, 3]

(a) Prove that $\cos 3A = 4\cos^3 A - 3\cos A$.

(b) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

(c) Given that $\sin\theta = \dfrac{1}{4}$, where $0 < \theta < \dfrac{\pi}{2}$, find:

(i) $\sin 3\theta$

(ii) $\cos 3\theta$

21 Trigonometric Identities IV (Product to Sum and Sum to Product)

Calculator Free

1. [8 marks: 3, 3, 2]

(a) Use an appropriate compound angle formula to prove that

$$\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\frac{A}{2}\cos\frac{B}{2}.$$

(b) Use the result in (a) to prove that $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

(c) Use the result in (b) to evaluate $\sin 75^{\circ} + \sin 15^{\circ}$

Calculator Free

2. [9 marks: 3, 4, 2]

(a) Use an appropriate compound angle formula to prove that

$$\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right) = -2\sin\frac{A}{2}\sin\frac{B}{2}.$$

(b) Use the result in (a) to prove that $\cos P - \cos Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$.

(c) Use the result in (b) to evaluate $\cos 255^{o} - \cos 15^{o}$

Calculator Free

3. [11 marks: 2, 3, 2, 2, 2]

(a) Prove that $\cos P \cos Q = \frac{1}{2}[\cos(P + Q) + \cos(P - Q)]$.

(b) Use the result in (a) to prove that $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$.

(c) Use the results in (a) and/or (b) to evaluate $\cos^2 15^\circ$.

(d) Use the results in (a) and/or (b) to evaluate $\sin 15^\circ \cos 15^\circ$.

(e) Hence, evaluate $\cot 15^\circ$.

Calculator Free

4. [12 marks: 3, 4, 5]

(a) Show the use of an appropriate *product to sum formula* to simplify $\sin(15^{\circ}) \times \cos(75^{\circ})$.

(b) Show use of the formula $\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right)$ to simplify $\sin(15^{\circ}) - \sin(105^{\circ}) + \sin(135^{\circ})$

(c) Simplify $\tan(105^{\circ}) + \tan(165^{\circ})$

Calculator Free

5. [12 marks: 3, 4, 5]

(a) Prove that $\dfrac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} = \tan 3A \cot A$.

(b) Prove that $\sin 5\theta + 2\sin 3\theta + \sin\theta = 4\sin 3\theta\cos^2\theta$.

(c) Prove that $\dfrac{\sin P + \cos(2Q - P)}{\cos P - \sin(2Q - P)} = \cot\left(\dfrac{\pi}{4} - Q\right)$

Calculator Free

6. [7 marks: 1, 3, 3]

(a) Prove that $\sin A \cos A = \frac{1}{2} \sin 2A$

(b) Prove that $\sin 40^{o} \cos 40^{o} \cos 80^{o} = \frac{\sin 20^{o}}{4}$.

(c) Use your answer in (b) to prove that $\cos 20^{o} \cos 40^{o} \cos 80^{o} = \frac{1}{8}$.

Calculator Free

7. [9 marks: 2, 2, 5]

Let $a + b + c = 180^{\circ}$.

(a) Prove that $\sin \dfrac{c}{2} = \cos\left(\dfrac{a}{2} + \dfrac{b}{2}\right)$.

(b) Prove that $\cos\left(\dfrac{c}{2}\right) = \sin\left(\dfrac{a+b}{2}\right)$.

(c) Hence or otherwise prove that $\sin a + \sin b + \sin c = 4\left(\cos\dfrac{a}{2}\right)\left(\cos\dfrac{b}{2}\right)\left(\cos\dfrac{c}{2}\right)$.

22 Trigonometric Identities V (Auxiliary Angles)

Calculator Assumed

1. [8 marks: 4, 4]

(a) Given that $\cos x + \sqrt{3}\,\sin x = \mathrm{R}\sin(x + \alpha)$ where $\mathrm{R} > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, use the formula for sin (A + B) to find in exact form the value of R and the exact value of α.

(b) Hence, find the maximum value (in exact form) for $y = \sqrt{3}\,\cos x + 3\sin x$ and the smallest positive value of x at which this occurs.

Calculator Assumed

2. [10 marks: 4, 3, 3]

Compare 4 sin x + 7 cos x with the expansion of R sin(x + α) where $0 \le \alpha \le 90^{o}$.
Hence, find the exact value of R and the value of α to 2 decimal places

Hence, find:

(a) the maximum value (in exact form) for the expression 8 sin x + 14 cos x and the smallest positive value of x at which this occurs.

(b) the maximum value (in exact form) for the expression –4 sin x – 7 cos x and the smallest positive value of x at which this occurs.

Calculator Assumed

3. [10 marks: 4, 3, 3]

Compare $5 \cos x + 8 \sin x$ with the expansion of form $R \cos(x - \alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$. Hence, find the exact value of R and α to 4 decimal places.

Hence, find:

(a) the maximum value (in exact form) for the expression $\cos x + 1.6 \sin x$ and the values of x where $0 \le x \le 2\pi$ at which this occurs.

(b) the minimum value (in exact form) for the expression $10 \sin x + 16 \cos x$ and the values of x where $0 \le x \le 2\pi$ at which this occurs.

Calculator Assumed

4. [7 marks: 3, 1, 3]

Let $\theta = 5 - 3\cos(2t) + \sin(2t)$.

(a) Express θ in the form $5 - R\sin(2t + \alpha)$ where α (in degrees) is an acute angle.

(b) Find in exact form, the maximum value for θ.

(c) Find the smallest positive value for t when θ has a maximum value.

---

5. [7 marks: 3, 1, 3]

Let $\theta = 3\cos(\pi t) + 2\sin(\pi t) + 10$.

(a) Express θ in the form $R\sin(\pi t + \alpha) + 10$ where $0 < \alpha < \frac{\pi}{2}$.

(b) Find in exact form, the minimum value for θ.

(c) Find the smallest positive value for t when θ has a minimum value.

Calculator Assumed

6. [6 marks]

Use an appropriate trigonometric method to find the minimum value (in exact form) for $f(\theta) = 10 + 3\sin\theta + 5\cos\theta$ where $0 \le \theta \le 360^\circ$. Give also the smallest positive value for θ at which the minimum value of $f(\theta)$ occurs.

---

7. [5 marks]

Consider $A = a\sin\theta + b\cos\theta$.

A has a maximum value of 4 when $\theta = \frac{\pi}{3}$. Determine the value(s) of a and b.

23 Trigonometric Equations II

Calculator Free

1. [8 marks: 4, 4]

(a) Solve for x in $2\cos^2 x + 3\sin x = 0$ for $0 \le x \le 360^\circ$

(b) Solve for x in $\cos x - 3\sec x - 2 = 0$ for $0 \le x \le 2\pi$

---

2. [11 marks: 3, 4, 4]

(a) Solve for x in $\cos x + \sqrt{3}\sin x = 0$ where $0 \le x \le 360^\circ$:

(b) Solve for x in $\sin x - \cos 2x = 0$ where $-\pi < x \le \pi$.

Calculator Free

2. (c) Find all values of x (in degrees) in $\cos x + \sin 2x = 0$.

---

3. [8 marks: 4, 4]

(a) Solve for θ in $\cos 2\theta + \cos\theta + 1 = 0$ for $0 \le \theta \le 2\pi$.

(b) Find all solutions (in radians) to $\sin 2\theta - \sin\theta = 0$.

Calculator Free

4. [13 marks: 4, 4, 5]

(a) Find all solutions (in radians) for θ in $3\tan^2\theta + 5\sec\theta + 1 = 0$.

(b) Find all solutions (in degrees) for θ in $\tan\theta + \cot\theta - 2\sec\theta = 0$.

(c) Solve for all values of θ radians in $2\cos 2\theta + 2\sin^2\theta - 9\cos\theta - 5 = 0$.

Calculator Free

5. [17 marks: 3, 5, 4, 5]

(a) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 0$

(b) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 1$

(c) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos 2\theta = 1$.

(d) $\cos 4\theta - \cos^2 2\theta = 0$

Calculator Free

6. [16 marks: 5, 5, 6]

(a) Solve for all values of θ (in radians) in $2 \cot \theta = 3 \sec \theta$

(b) Solve for all values of θ (in radians) $\cos 3\theta + 2 \cos 2\theta + 3 \cos \theta = -2$.

(c) Solve for all values of θ in $1 + \sqrt{3} \tan \theta = \sqrt{3} \sec \theta$.

Calculator Free

7. [9 marks: 4, 5]

   (a) Solve for all values of θ in $\cos\theta + \cos 3\theta = 0$.

   (b) Solve $\cos\theta + \cos 3\theta + \cos 7\theta = 0$ for $0 \le \theta \le 180^{\circ}$.

Calculator Assumed

8. [10 marks: 1, 3, 6]

(a) Show that $\sin 2\theta + \sin 3\theta \equiv 2 \sin \frac{5\theta}{2} \cos \frac{\theta}{2}$.

(b) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta = 0$

(c) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta - \sin \theta = 0$.

Calculator Assumed

9. [10 marks: 4, 6]

(a) Prove that $\sin 2A + \sin 2B = 2 \sin (A + B) \cos (A - B)$.

(b) Use the result in (a) to solve for all values of θ in $\sin 6\theta + \sin 4\theta = 0$

Calculator Assumed

10. [11 marks: 4, 2, 5]

(a) Use the formula for tan 2A to show that $\tan\frac{\pi}{8} = -1 + \sqrt{2}$.

(b) Use your answer in (a) to find all solutions to $\sqrt{2}\cos\theta - \cos\theta - \sin\theta = 0$.

(c) Given that $\sin\frac{3\pi}{8} = \frac{1}{\sqrt{4-2\sqrt{2}}}$ and using the answer in (a),

solve for θ in $\sin\theta - (\sqrt{2}-1)\cos\theta = 1$ for $0 < \theta \le 2\pi$.

24 Trigonometric Graphs

Calculator Free

1. [6 marks]

Complete the following table.

Function	Period	Amplitude	Phase Shift
$y = 2\sin(2x^\circ)$			
$y = -4\cos(\frac{x}{2} + 30^\circ)$			
$v = 10\tan(3t + \pi)$			
$Q = 5\sin(\frac{\pi}{2} - t)$			
$y = \frac{\sqrt{2}}{2}\cos(\pi t) + 100$			
$T = 5 - \sin(\frac{\pi}{4} - \theta)$			

2. [5 marks]

Complete the table below.

Function	Minimum value of function	Maximum value of function
$y = 3\sin t$		
$y = 20\cos(\frac{2x}{3} - 45^\circ)$		
$v = 5\tan\theta$		
$M = 2\sin(\frac{\pi}{2} - 3t) + 4$		
$y = 5 - \cos(2\pi t)$		

Calculator Free

3. [6 marks: 3, 3]

A trigonometric function has equation $y = -4\sin(2x + 30^{\circ})$ for $0^{\circ} \leq x \leq 360^{\circ}$. Use an algebraic method to find:

(a) the maximum value for y and the corresponding value(s) for x.

(b) the minimum value for y and the corresponding value(s) for x.

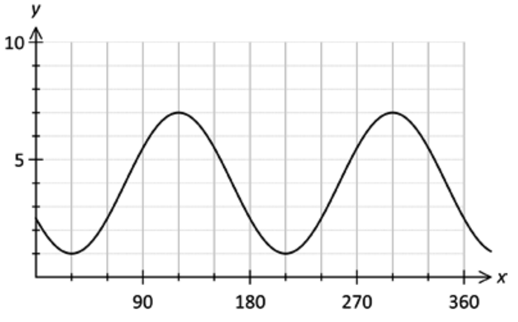
4. [5 marks]

A trigonometric function has equation $P = a\cos(bt + \frac{\pi}{4})$. Find the values of a (where $a > 0$) and b given that P has a maximum value of 4 when $t = \frac{\pi}{4}$.

Calculator Free

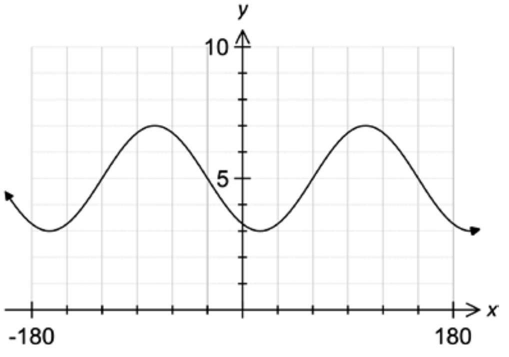
5. [4 marks]

The graph of $y = a + b \sin(cx + d)$ is shown in the accompanying diagram. Determine the values of a, b, c and d.



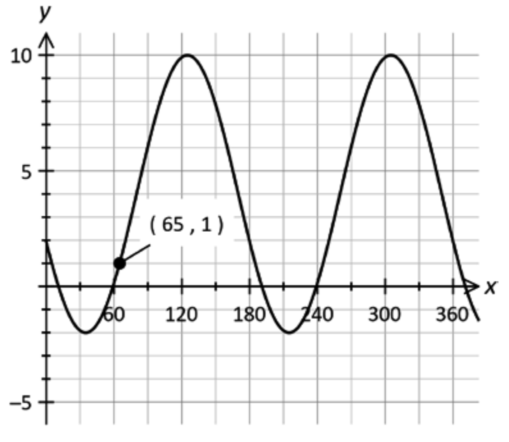
6. [4 marks]

The graph of $y = a + b \cos(cx + d)$ is shown in the accompanying diagram. Determine the values of a, b, c and d.



7. [4 marks]

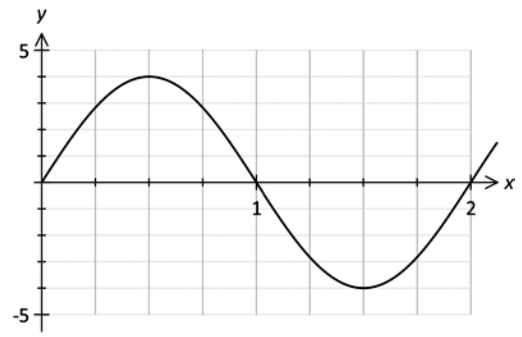
The accompanying diagram shows the graph of $y = a \sin(bx + c) + d$. Determine the values of the constants a, b, c and d where $0 \le c \le 90$.



Calculator Free

8. [3 marks]

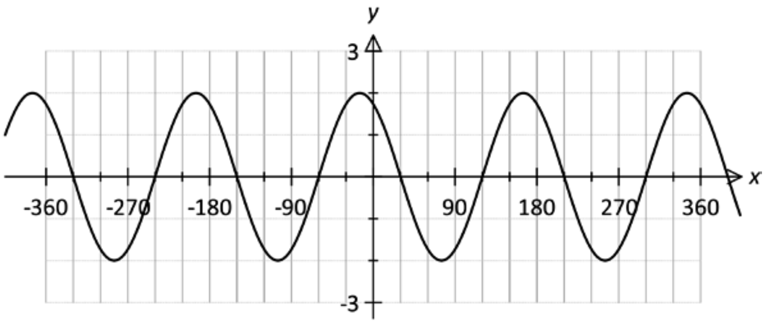
The accompanying diagram shows the graph of a trigonometric function. State the amplitude and period of the function. Hence, give the equation of this function.



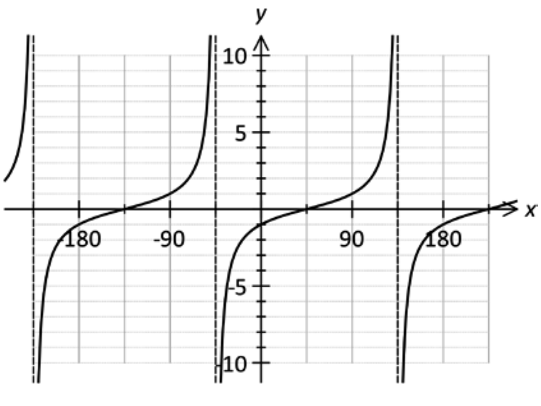
9. [7 marks: 4, 3]

Find the equation of the following trigonometric functions:

(a)



(b)



Calculator Free

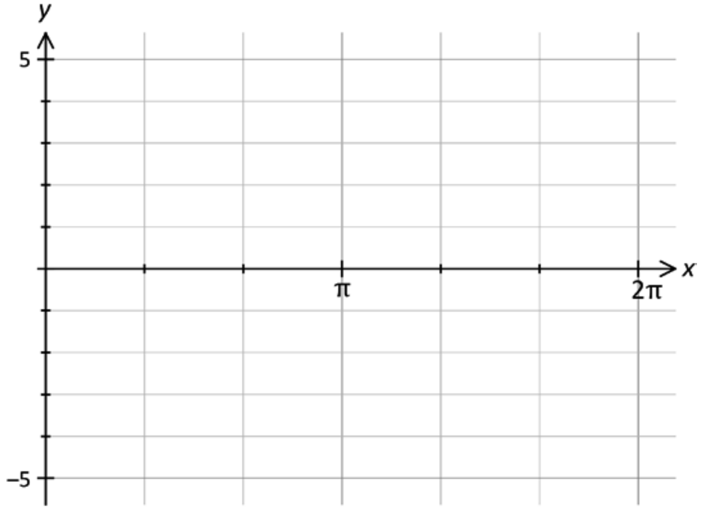
10. [9 marks: 4, 2, 3]

Consider $A = \sqrt{3}\sin\theta + \cos\theta$ where $0 \le \theta \le 2\pi$.

(a) Solve for values of θ for which $A = \sqrt{3}$.

(b) State the maximum value of A and the value(s) of θ at which this occurs.

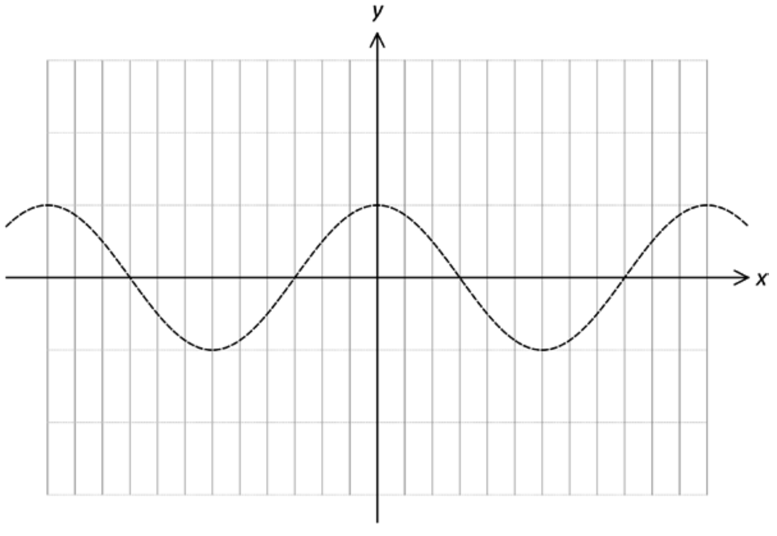
(c) In the axes provided below, sketch the graph of $A = \sqrt{3}\sin\theta + \cos\theta$ for $0 \le \theta \le 2\pi$.



Calculator Free

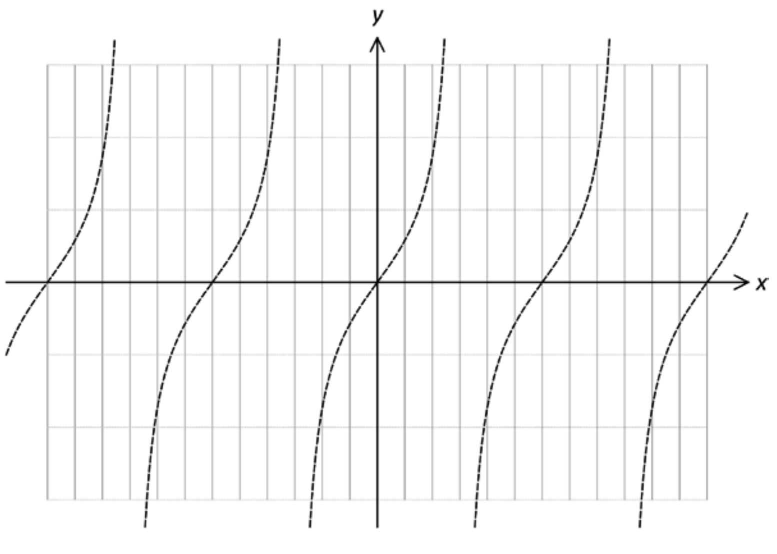
11. [3 marks]

The graph of $y = \cos(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \sec(ax)$.



12. [3 marks]

The graph of $y = \tan(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \cot(ax)$.

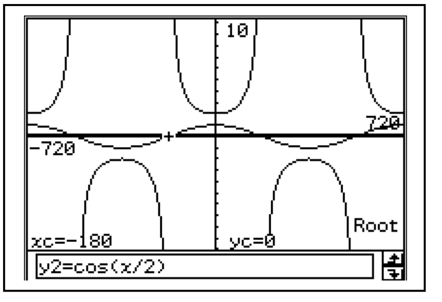


Calculator Assumed

13. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2\sec\left(\frac{x}{2}\right)$ for $-720^{\circ} < x < 720^{\circ}$.

(a) Determine the coordinates of the maximum turning point(s) of this curve.



(b) Determine the period of this curve.

(c) Determine the equations of the vertical asymptotes.

---

14. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2 + \text{cosec}\,(2x + 30^{\circ})$ for $-180^{\circ} < x < 180^{\circ}$.

(a) Determine the coordinates of the minimum turning point(s) of this curve.

(b) Determine the period of this curve.

(c) Determine the equations of the vertical asymptotes.

Calculator Assumed

15. [9 marks: 1, 1, 2, 2, 3]

The body temperature θ (Celsius) of a reptile in summer at time *t* hours after midnight is given by $\theta = 15 - 5\sin\left(\frac{\pi t}{12}\right)$.

(a) State the period for θ.

(b) What is the range of body temperature experienced by the reptile?

(c) Find the minimum body temperature of the reptile and state when this first occurs after midnight.

(d) Find the maximum body temperature of the reptile and state when this first occurs after midnight.

(e) Find for how many hours in a 24 hour day, the body temperature of the reptile is below 16^{o} Celsius. Give your answer to the nearest minute.

Calculator Assumed

16. [10 marks: 2, 3, 5]

The water depth, h metres, measured from the bottom of a harbour, t hours after 6 am is modelled by the equation $h = 12 - 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{4}\right)$ metres.

(a) Determine when the water depth is at its lowest in a 24-hour day (from 6 am). State the lowest depth.

(b) Repairs to the harbour can only be undertaken if the water depth is below 10 metres. What times of the day can this occur?

(c) The water depth is above k metres for 20% of a 24-hour day. Find k.

25 Matrix Algebra

Calculator Free

1. [6 marks: 1, 1, 4] [TISC]

Consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Given that matrix **X** can be added with matrix **A**, what is the size of matrix **X?**

(b) Given that **BY** = **YB**, what is the size of matrix **Y** ?

(c) Consider the matrices, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ k \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \end{pmatrix}$. Two different matrices are selected from the three given and then multiplied together. State all the possible products.

---

2. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & -1 \end{pmatrix}$ where a, b, x and y are constants.

(a) Using the matrices given, the product of two different matrices is a 2 × 1 matrix. Show how this product can be formed. State the resulting matrix.

Calculator Free

2. (b) Using the matrices given, the product of two different matrices is a 1 × 2 matrix. Show how this product can be formed. State the resulting matrix.

(c) Using the matrices given, the product of two non-square matrices is a square matrix. Show how this product can be formed. State the resulting matrix.

(d) Using the matrices given, the product of three different matrices is a 1 × 1 matrix. Show how this product can be formed. State the resulting matrix.

---

3. [8 marks: 1, 3, 4]

Let $\mathbf{A} = \begin{pmatrix} 0 & a \\ a & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} b & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix}$ where a, b and c are constants.

(a) Find **A** – **B**.

(b) Find the value of a if **A** – **B** is non-singular.

(c) Find if possible, the values of a and b if $(\mathbf{A} - \mathbf{B})^{-1} = \mathbf{C}$.

Calculator Free

4. [5 marks: 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} k & 1 \\ 8 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -8 & k \end{pmatrix}$

(a) Find the value(s) of k if **A** is a non-singular matrix.

(b) Find the value(s) of k if $\mathbf{A} \times \mathbf{B} = \mathbf{A} + \mathbf{B}$.

---

5. [6 marks: 2, 2, 2] [TISC]

Given that $\mathbf{A} = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}$.

(a) Find the relationship between a and b such that **A** is a singular matrix.

(b) Given that $b = 4$, find the value(s) of a for which **A** is non-singular.

(c) Explain clearly why **A** will always have an inverse if $b < 0$.

Calculator Free

6. [9 marks: 1, 3, 5]

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ b & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & -2 \\ 6 & 12 \end{pmatrix}$.

(a) Find **A** + **B**.

(b) Find the value of b if **B** is non-singular.

(c) Find if possible, the values of a and b if **A** × **B** = **C**.

---

7. [6 marks: 3, 3]

Let $\mathbf{A} = \begin{pmatrix} k^2 & 2 \\ k & 1 \end{pmatrix}$. Determine the value(s) of k if:

(a) $|\mathbf{A}| = 3$

(b) $\mathbf{A}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

Calculator Free

8. [4 marks: 2, 2] [TISC]

(a) If **A** is a non-singular square matrix, show that if $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{I}$ where **I** is the appropriate identity matrix.

(b) Find a 2 × 2 non-zero matrix **A,** where $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A} \neq \mathbf{I}$. (**I** is the 2 × 2 identity matrix.)

---

9. [6 marks: 3, 3]

(a) Given that **P** and **Q** are square matrices and $\mathbf{PQ} = \mathbf{P} + \mathbf{Q}$, show that $\mathbf{P} = \mathbf{Q}(\mathbf{Q} - \mathbf{I})^{-1}$, where **I** is the appropriate identity matrix.

(b) Given that **A** and **B** are 2 × 2 non-zero diagonal matrices, prove that **A** and **B** are commutative under multiplication.

Calculator Free

10. [9 marks: 2, 3, 4]

Given that the non-singular matrix **A**, where $\mathbf{A}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$

find each of the following. Justify each of your answers.

(a) $\mathbf{A}\begin{pmatrix}2\\-1\\3\end{pmatrix}$

(b) $\mathbf{A}^2\begin{pmatrix}1\\-1\\1\end{pmatrix}$

(c) $\mathbf{A}^{-1}\begin{pmatrix}1 & 3 & 4\\0 & 2 & 2\\2 & -1 & 1\end{pmatrix}$

Calculator Assumed

11. [5 marks]

Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with all non-zero elements, where $|\mathbf{M}| = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^2$, prove that $a + d = -1$.

---

12. [4 marks]

Let $\mathbf{X}$ be a $n \times 1$ matrix, $\mathbf{A}$ be a $n \times n$ matrix and λ be a real non-zero constant.
Given that $\mathbf{A\,X} = \lambda\mathbf{X}$, prove that $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

Calculator Assumed

13. [7 marks: 2, 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$.

(a) Show that $\mathbf{A}^2 = \mathbf{A} + k\mathbf{I}$ where k is a real constant.

(b) Use your result in (a) to find $\mathbf{A}^{-1}$ in the form $\alpha\mathbf{A} + \beta\mathbf{I}$.

(c) Find $\mathbf{A}^4$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

Calculator Assumed

14. [8 marks: 3, 1, 4]

(a) Given that **A**, and **B** are non-singular square matrices prove that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

(b) Hence, or otherwise, prove that $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

(c) Given that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$, find $(\mathbf{A}^2)^{-1}$ in the form $p\mathbf{A} + q\mathbf{I}$
where p and q are real constants and **I** is the identity matrix.

26 Systems of Equations

Calculator Free

1. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

(b) Find the product $\mathbf{A}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

(c) Consider the system of equations:

$$\begin{aligned} x + 2y &= 2 \\ -x + 3y &= -2 \end{aligned}$$

(i) Rewrite the given system of equations in the form **AX** = **B** where **X** is a column matrix and **A** and **B** are appropriate matrices.

(ii) Use a matrix method to solve for x and y.

Calculator Free

2. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

(b) Find the product $\begin{pmatrix} 3 & 4 \end{pmatrix} \times \mathbf{A}^{-1}$.

(c) Consider the system of equations:

$$3x + 5y = 3$$
$$x + 2y = 2$$

(i) Rewrite the given system of equations in the form **XA** = **B** where **X** is a row matrix and **A** and **B** are appropriate matrices.

(ii) Use a matrix method to solve for x and y.

Calculator Free

3. [4 marks]

Use a method involving the use of an inverse matrix to solve for x and y in:

$$2x + 3y = -1$$
$$3x - 2y = 18$$

4. [4 marks: 2, 2]

Consider the set of simultaneous equations:

$$x + y = 8$$
$$2x + 3y = 30$$

(a) Rewrite the system of equations using matrices in the form **XA** = **B**, where **X** = $\begin{pmatrix} x & y \end{pmatrix}$.

(b) Hence, use a method involving the use of an inverse matrix, solve for x and y.

Calculator Free

5. [4 marks]

Given that $\mathbf{A} \times \mathbf{B} = 4\mathbf{I}$, $\mathbf{B} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{A} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$,

where $\mathbf{I}$ is the 2×2 unit matrix, find x and y.

---

6. [5 marks: 1, 4]

Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

(a) Find product $\mathbf{A} \times \mathbf{B}$.

(b) Show how your answer in (a) can be used to solve for x and y in:
$3x + 5y = 1$ and $x + 2y = 2$.

Calculator Free

7. [5 marks]

Use a method involving the use of an inverse matrix to find the coordinates of the point of intersection between the lines $y = 2x + 6$ and $y = \dfrac{-3x}{2} - 1$:

8. [4 marks]

Given $\begin{pmatrix} -1 & 2 & 3 \\ -1 & 2 & 4 \\ 1 & -1 & -2 \end{pmatrix} \times \begin{pmatrix} 0 & -1 & 4 \\ -2 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, Use this result to solve:

$$-y + 4z = -12 \qquad -2x + y + 2z = -8 \qquad -x + y = -1$$

Calculator Free

9. [9 marks: 3, 6]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & -8 \\ -1 & -1 & 4 \\ 2 & 1 & -3 \end{pmatrix}$.

(a) Determine the values of p, q and r in $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & -8 \\ -1 & -1 & 4 \\ 2 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & r \\ 5 & q & -4 \\ p & 0 & 1 \end{pmatrix}$.

(b) Use your answer in (a) to solve for x, y and z in:

$$3x + 2y - 8z = -1$$
$$x + y - 4z = -1$$
$$2x + y - 3z = 1$$

27 Applications using Matrices

Calculator Assumed

1. [7 marks: 2, 3, 2]

A budget airline flies from Perth to Sydney and charges two different fares for its passengers; deluxe economy and economy. On a certain flight, there were: 200 fare paying passengers and one and a half times as many economy passengers as deluxe economy passengers
Let d: number of deluxe economy passengers on this flight
e: number of economy passengers on this flight

(a) Use the information given above to write down two equations involving d and e.

(b) Use a method involving the inverse of a matrix to find d and e.

(c) A deluxe economy ticket costs \$349, while an economy ticket costs 30% less. Use a matrix method to find the total amount in fares collected from this flight. You need to show clearly the matrices used, and the operation(s) used on these matrices.

Calculator Assumed

2. [5 marks: 1, 1, 3]

The table below shows the number of hours Jack worked last week at a fast food outlet.

Shift	Weekdays	Weekends
Morning (M)	16	4
Afternoon (A)	8	4
Night (N)	8	0

(a) Write a *row* matrix **A** describing the number of hours Jack worked on each shift on weekdays.

(b) Write a *column* matrix **B** describing the number of hours Jack worked on each shift on weekends.

The rates of pay are $15.00 per hour for weekday morning shifts, $12.00 per hour for weekday afternoon shifts and $20 per hour for weekday night shifts. Jack is paid twice as much per hour for weekend shifts as weekday shifts.

(c) Use matrices **A** and **B** and other matrices as required to find the total amount of money Jack earned last week.

Calculator Assumed

3. [7 marks: 2, 2, 3]

The number of different tickets available for a charity concert at a concert hall is given in the table below.

	Stalls	Gallery
Adults	500	300
Students/pensioners	150	50

The prices for these tickets are given in the table below.

	Stalls	Gallery
Adults	$150	$90
Students/pensioners	$120	$70

(a) Given that all the tickets were sold, use a matrix method to determine the revenue from:

(i) the adult members of audience

(ii) the stalls tickets.

(b) Given that 310 gallery tickets were sold realising a revenue of $26 900.
Use a matrix method to determine how many of the gallery tickets were sold to adults?

Calculator Assumed

4. [8 marks: 2, 3, 3]

In a city there are two companies, A and B, that supply gas to households. The table below shows the percentage of customers in each company that will remain with their original supplier and the percentage of customers that will switch to the competitor within two years.

<table>
<tr><td colspan="2" rowspan="2">%</td><td colspan="2">From</td></tr>
<tr><td>A</td><td>B</td></tr>
<tr><td rowspan="2">To</td><td>A</td><td>65</td><td>15</td></tr>
<tr><td>B</td><td>35</td><td>85</td></tr>
</table>

Initially, there were 750 000 and 250 000 customers with companies A and B respectively. Assume that the total number of customers remain unchanged.

(a) Use a matrix method to determine the number of customers with company A at the end of two years.

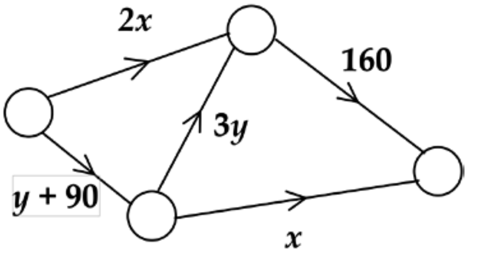
(b) Use a matrix method to determine the number of customers with company A at the end of four years.

(c) Using the table given, will company B ever have 750 000 customers? Justify your answer.

Calculator Assumed

5. [7 marks: 3, 4]

The diagram below shows the flow of fluid (in litres/minute) through a network of pipes. The numbers or letters indicate the flow rate through the pipe concerned. Assume that no fluid is lost in the process.



(a) Write down all equations involving x and y for the given network.

(b) Use a method involving the inverse of a matrix to solve for x and y.
Show clearly the matrices involved.

Calculator Assumed

6. [8 marks: 2, 1, 2, 3]

The table below shows the ratios of the different components for the different types of mortar for each twelve kg bag. For example, a 12 kg bag of retaining wall mortar has 2 kg of cement, 1 kg of lime and 9 kg of sand. There is an order for 10, 12 and 20 sixty kg bags of common mortar, retaining wall mortar and internal wall mortar.

	Cement	Lime	Sand
Common Mortar	1.5	1.5	9
Retaining Wall Mortar	2	1	9
Internal Wall Mortar	1	2	9

(a) Write a matrix **M** that describes the mass of the components in each sixty kg bag of common mortar, retaining wall mortar and internal wall mortar.

(b) Write a *row* matrix **B** that when multiplied with **M** will give the amount of cement, lime and sand required to fulfil this order.

(c) Use **M** and **B** to determine the amount of cement, lime and sand required to fulfil this order.

(d) Cement costs \$0.40 per kg, lime costs \$0.50 kg per kg and sand costs \$0.25 per kg. Use **M**, **B** and another appropriate matrix to determine the total cost for this order.

Calculator Assumed

7. [8 marks: 4, 4]

The table below shows the cost per share and dividend returns for shares in companies A, B and C.

	A	B	C
Cost per share	$2.40	$5.10	$1.20
Dividend per share	$0.20	$0.80	$0.10

(a) Ann had 4 000, 5 000 and 8 000 shares in companies A, B and C respectively. Show the use of matrices to calculate the total number of shares, total value of shares and the total amount of dividends received.

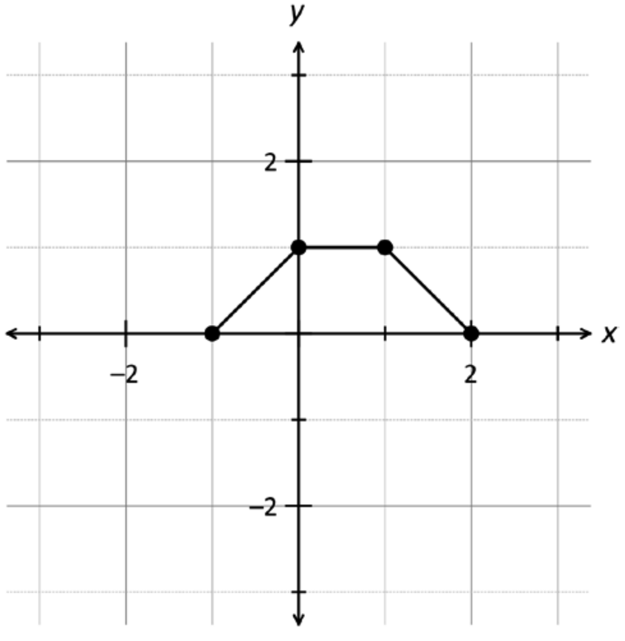
(b) Ben had a total of 25 000 shares worth a total of $82 800. The number of shares he held in companies A, B and C are a, b and c respectively. The total amount of dividends he received was $11 400. Show the use of a matrix equation to calculate the values of a, b and c.

28 Transformation Matrices

Calculator Free

1. [6 marks: 2, 2, 2]

The graph of $y = f(x)$ is given below. The transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is applied to this curve.



(a) Find the image of the points (–1, 0), (0, 1), (1, 1) and (2, 0).

(b) Sketch on the axes provided above, the graph of the resulting curve.

(c) The equation of the resulting curve is $y = a\,f(bx + c)$. Find a, b and c.

Calculator Free

2. [9 marks: 2, 3, 2, 2] [TISC]

Consider the curve with equation $y = f(x)$. The curve has a maximum point at A (–1, 3) and a minimum point at B (4, –7). The curve $y = f(x)$ is mapped onto the curve $y = g(x)$ by a transformation represented by the matrix $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe the effect the transformation represented by T has on the graph of $y = f(x)$.

(b) Find the coordinates of the images of the points A and B under **T**.

(c) Find the coordinates of the maximum and minimum points on the curve $y = g(x)$.

(d) Find the matrix that maps $y = g(x)$ back to $y = f(x)$.

Calculator Free

3. [6 marks: 1, 1, 2, 2] [TISC]

The transformation T is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe in words the transformation T.

(b) The transformation T is applied to the line with equation $y = x$.
Find the equation of the resulting line.

(c) The point A is mapped to the point with coordinates $(k, k + 1)$ under transformation T. Find the coordinates of the point A. Justify your answer.

(d) The transformation T is combined with the transformation represented by matrix **M**. All the entries in matrix **M** are positive. The effects of the combined transformation is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$.

Find the matrix **M**. Show clearly your reasoning.

Calculator Free

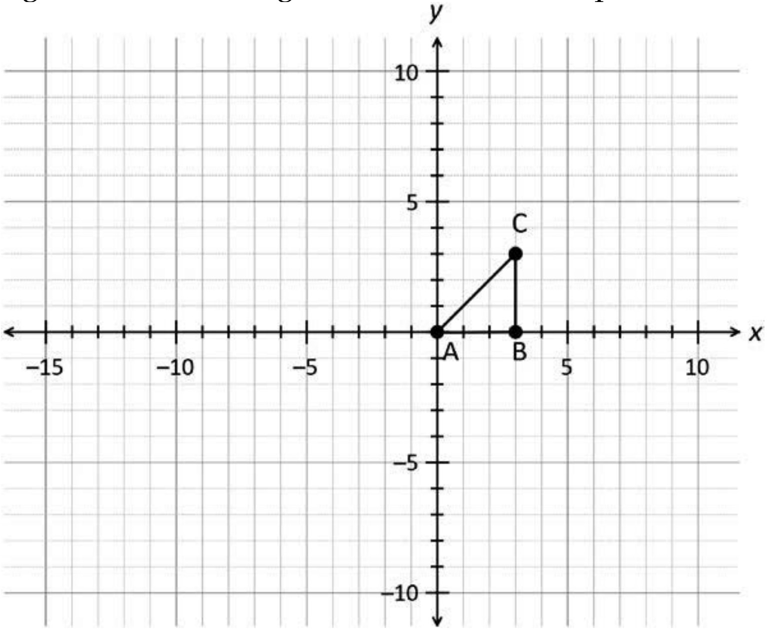
4. [7 marks: 3, 2, 1, 1] [TISC]

Triangle ABC with vertices A (0, 0), B (3, 0) and C (3, 3) is mapped to triangle A′B′C′ by the compound transformation represented by the matrix

$\mathbf{M} = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}$.

(a) Find the coordinates of the points A′, B′ and C′.

(b) Plot triangle ABC and triangle A′B′C′ on the axes provided below.



(c) The transformation applied is a combination of several simple transformations. What evidence is there to suggest that one of the simple transformations involved is :

(i) an enlargement?

(ii) a reflection ?

Calculator Free

5. [8 marks: 2, 3, 3]

The transformation **S** is represented by matrix $\begin{pmatrix} \cos 25^{o} & -\sin 25^{o} \\ \sin 25^{o} & \cos 25^{o} \end{pmatrix}$ and the transformation **T** is represented by matrix $\begin{pmatrix} \cos 35^{o} & -\sin 35^{o} \\ \sin 35^{o} & \cos 35^{o} \end{pmatrix}$.

(a) Describe in words, the transformation **S**.

(b) Determine the matrix that represents the combination of transformation **S** followed by transformation **T**. Give your answer in its simplest form using exact values.

(c) An object of area 10 units^2 is transformed by **T**.
Determine with reasons the area of the image.

Calculator Free

6. [11 marks: 4, 3, 4]

(a) The linear transformation **T** maps the point with coordinates (2, 1) to $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and the point with coordinates (2, –1) to $\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

(i) Determine the 2 × 2 matrix that can be used to represent **T**.

(ii) Describe in words the linear transformation **T**.

(b) ΔABC is mapped to $\Delta A'B'C'$ by the transformation represented by matrix $\begin{pmatrix} 0 & k \\ 3k & 0 \end{pmatrix}$.

Find the value of k if the area of ΔABC is three times the area of $\Delta A'B'C'$.

Calculator Assumed

7. [9 marks: 3, 2, 4] [TISC]

Consider two matrices $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The point A(1, *k*) is mapped to the point A'' using **T** followed by **S** as transformation matrices.

(a) Find the coordinates of A''.

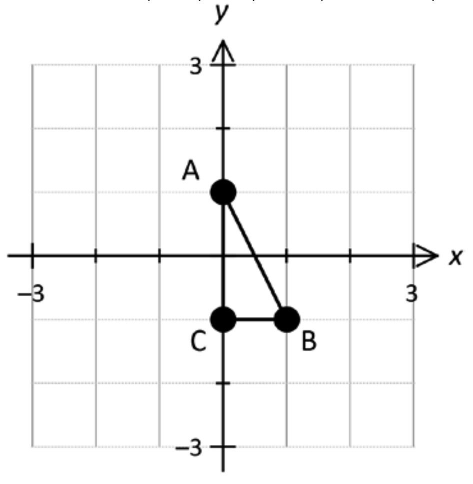
(b) Find a single transformation matrix that will map A'' back to A. Show how you obtained your answer.

(c) C is a circle of radius 1 with centre at (1, 1). C is transformed to circle C' by the transformation $\mathbf{T}$. Discuss the differences between the original circle C and its image C'. You need to comment on the coordinates of the centre, the radius and area of the two circles.

Calculator Assumed

8. [10 marks: 4, 6]

The triangle ABC has vertices A(0, 1), B(1, –1) and C(0, –1).



(a) ΔABC is mapped to $\Delta A'B'C'$ by a transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Draw and clearly label $\Delta A'B'C'$ on the diagram above.

(b) $\Delta A'B'C'$ is mapped to $\Delta A''B''C''$ by a transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The points A'', B'' and C'' have coordinates (2, 3), (–1, k) and (k, –3) respectively. Find matrix **M** and the value(s) of k.

Calculator Assumed

9. [9 marks: 3, 2, 4]

(a) Find the image of the line $y = x + 1$ under a transformation represented by $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$.

(b) Consider the transformation mapping T: $(x, y) \rightarrow (-x + 2, 2y - 1)$.

(i) Express the transformation mapping T as a matrix equation.

(ii) Identify the sequence of transformations that make up T.

Calculator Assumed

10. [9 marks: 3, 2, 4]

(a) Find the equation of the image of the line $y = 2x + 8$ under a transformation represented by $\begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix}$.

(b) Consider the transformation mapping **T**: $(x, y) \to (2y - 1, -4x + 3)$.

(i) Express the transformation mapping **T** as a matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ where **M** represents a linear transformation and a and b are constants.

(ii) Identify the sequence of transformations that make up **T**.

Calculator Assumed

11. [6 marks: 2, 4]

(a) Express the given transformation **T** as a matrix equation
T: $(x, y) \rightarrow (x - 3, 2y + 4)$. Identify the transformations involved.

(b) Write the transformation mapping for the point (x, y) under a 150° clockwise rotation about the origin followed by a translation of 2 units along the positive x-axis and -1 unit along the positive y-axis.

12. [5 marks]

Prove that a reflection about the line $y = \sqrt{3}\,x$, followed by a reflection about the x-axis is equivalent to a single rotation **R**. Describe this rotation.

Calculator Assumed

13. [8 marks: 2, 4, 2]

The reflection about the line $y = x \tan \alpha$ is represented by matrix **A**
The reflection about the line $y = x \tan \beta$ is represented by matrix **B**.

(a) Write the transformation matrices **A** and **B**.

(b) Find a single matrix that represents the combined transformations of
a reflection about the line $y = x \tan \alpha$ followed by
a reflection about the line $y = x \tan \beta$.
Simplify your answer.

(c) Hence, prove that the combined transformation in (b) is equivalent to an anti-clockwise rotation of $2(\alpha - \beta)$ about the origin.

29 Complex Numbers

Calculator Free

1. [12 marks: 1, 2, 2, 2, 2, 3]

Solve for $x \in \mathbb{C}$.

(a) $x^2 + 4 = 0$

(b) $x^2 + x + 4 = 0$

(c) $(x + 2)^2 + 9 = 0$

(d) $x^3 + 9x = 0$

(e) $(2z - 1)^2 + 16 = 0$

(f) $\dfrac{2}{2-z} = \dfrac{z}{2}$

Calculator Free

2. [5 marks: 1, 4]

Factorise using both real and complex factors where appropriate:

(a) $x^2 + 16$

(b) $x^2 + 2x + 5$

3. [6 marks: 3, 3]

(a) Factorize completely (including complex factors) $z^2 + 4z + 5$.

(b) Factorize completely (including complex factors) $z^3 + 5z^2 + 9z + 5$.

Calculator Free

4. [5 marks: 2, 3]

Express in the form $a + bi$:

(a) $(1 + i\sqrt{2})^2$

(b) $\dfrac{2+3i}{1-i}$

---

5. [5 marks]

Given that $(a + b\,i)^2 = -15 + 8\,i$, find a and b where a and b are real non-zero integers.

---

6. [5 marks]

Given that $(a + b\,i)^2 = 16 + 30\,i$, find a and b where a and b are real non-zero integers.

Calculator Free

7. [7 marks: 1, 3, 3]

Let $u = 1 + 3i$, $v = -1 + i$ and $w = -2i$.

(a) Find $u + v$.

(b) Find $\frac{v}{w}$.

(c) Find $v \times \bar{u}$.

8. [8 marks: 4, 4]

Find $w = a + bi$ where a and b are real non-zero integers if:

(a) $\bar{w} = \frac{5}{w}$,

(b) $w^2 = -3 - 4i$

Calculator Free

9. [9 marks: 3, 3, 3]

Let the complex numbers $z_1 = 2 + k\,i$ and $z_2 = -5 + 12\,i$, where k is a *real number*.
Determine all possible values of k if:

(a) $[\,\boldsymbol{Im}(z_1)\,]^2 = \boldsymbol{Im}(z_2)$

(b) $z_1{}^2 = z_2$

(c) $\dfrac{6z_1}{z_2} = -i.$

---

10. [8 marks: 2, 6]

Let the complex numbers $u = 3 + k\,i$ and $v = k + 2\,i$, where k is a *real number*.
Determine all possible values of k if:

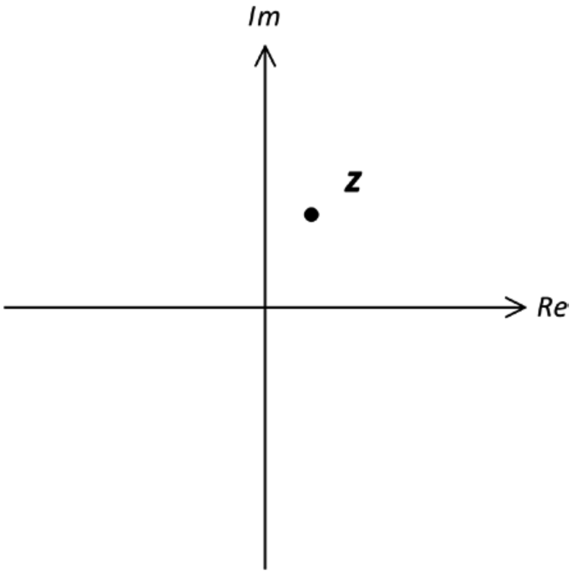
(a) $[\,\boldsymbol{Im}(u)\,]^2 = \boldsymbol{Re}(v)$

(b) $\dfrac{u}{v-1} = -1 - \dfrac{i}{2}$

Calculator Free

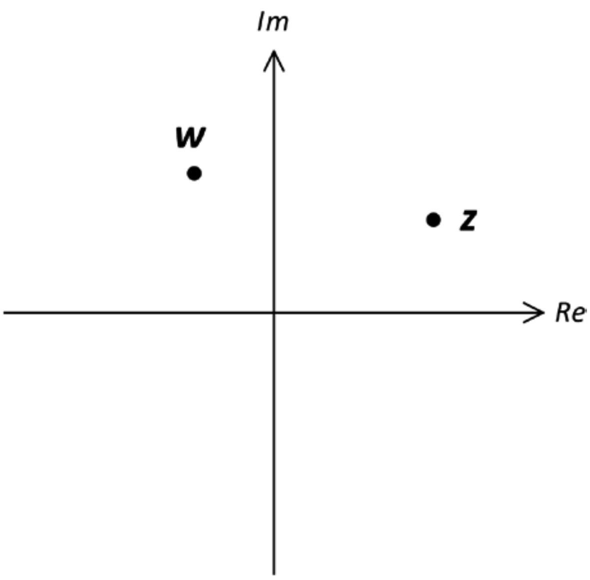
11. [3 marks]

The complex number z is plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $\overline{z}$, $z+\overline{z}$ and $z-\overline{z}$.



12. [4 marks]

The complex numbers w and z are plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $w+z$, $\overline{w+z}$, $w-z$ and $\overline{w-z}$.



30 Conjectures & Proofs

Calculator Assumed

1. [10 marks: 3, 3, 4]

(a) Prove that $15.\overline{15}$ is a rational number.

(b) Prove that $5.7\overline{35}$ is a rational number.

(c) Prove that $10 + 1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots$ is a rational number.

Calculator Free

2. [5 marks: 1, 1, 3]

(a) Given that $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

(i) $\mathbf{P} \times \mathbf{Q}$

(ii) $\mathbf{Q} \times \mathbf{P}$

(b) Prove or disprove the conjecture that if the matrices **P** and **Q** are such that $\mathbf{P} \times \mathbf{Q} = \mathbf{I}$, where I is the relevant identity matrix, then $\mathbf{P}^{-1} = \mathbf{Q}$.

Calculator Free

3. [9 marks: 3, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

(a) Consider the parallelograms ABCD and PQRS. If the diagonals AC = PR then ABCD and PQRS are congruent.

(b) Consider the parallelograms ABCD and PQRS. If AB = PQ and the diagonals AC = PR, ABCD and PQRS are congruent.

(c) If the diagonals of a quadrilateral are perpendicular then the quadrilateral must be a rhombus.

Calculator Assumed

4. [12 marks: 2, 2, 4, 4]

Provide a counter-example to disprove each of the following conjectures.
Show all attempts, successful and otherwise.

(a) $2^{2n} + 1$ is prime for integer $n \geq 1$.

(b) $n^2 - n + 5$ is prime for integer $n \geq 1$.

(c) $1^n + 5^n + 10^n + 18^n + 23^n + 27^n = 2^n + 3^n + 13^n + 15^n + 25^n + 26^n$ for integer $n \geq 1$.

(d) There are no integer solutions to $a^3 + b^3 = c^3 + d^3$ where $a \neq b \neq c \neq d$.

Calculator Free

5. [3 marks]

Prove that $2^n - 1$ is always odd for integer $n \geq 1$.

---

6. [3 marks]

Prove that the square of an odd number add 11 is a multiple of 4.

---

7. [5 marks: 2, 3]

(a) Prove that product of three consecutive integers is divisible by 3.

(b) Prove that product of three consecutive integers is divisible by 6.

Calculator Assumed

8. [3 marks]

Prove that $x^7 - x$ is divisible by 6 for integer $x \geq 1$.

9. [4 marks]

Prove that the product of any three consecutive even numbers must be a multiple of 24.

10. [3 marks]

Prove that $4n^3 - 4n$ is a multiple of 24 for integer $n \geq 1$.

Calculator Assumed

11. [7 marks: 4, 3]

(a) It is conjectured that a number is divisible by 4 if the sum of twice the tens digit and the ones digit is a multiple of 4. Prove that this conjecture is true for a four digit number.

(b) Consider the positive integers a and b. The arithmetic mean of these two integers is $M = \frac{a+b}{2}$ and the harmonic mean is $H = \frac{2ab}{a+b}$.
It is conjectured that $M \geq H$. Use the expansion of $(\sqrt{a} - \sqrt{b})^2$ to prove this conjecture.

Calculator Assumed

12. [7 marks: 2, 5]

(a) Provide a counter-example to disprove the conjecture that the cube of an even number greater than two less the number itself is divisible by 5.

(b) Prove that $x^5 + x^4 + x^3 + x^2$ is divisible by 4 for x as a whole number.

Calculator Assumed

13. [6 marks: 3, 1, 2]

(a) Prove that the sum of the cubes of two consecutive odd whole numbers is always a multiple of 4.

(b) Consider the conjecture:
The sum of the cubes of two consecutive odd whole numbers is always a multiple of 6.

(i) Disprove this conjecture.

(ii) Under what conditions will this conjecture be true?

Calculator Assumed

14. [4 marks]

Complete the table below.

Conjecture	
Negation of conjecture	
Contrapositive of conjecture	
Converse of conjecture	If x is an odd number then x^2 is an odd number.
Inverse of conjecture	

15. [5 marks: 2, 3]

Consider the conjecture:

If a is a factor of 20, then a is also factor of 40.

(a) Prove that the conjecture is true.

(b) State the inverse of the conjecture and determine with reasons whether it is true or false.

Calculator Assumed

16. [4 marks]

Use the method of contradiction to prove that for n as a *real* whole number $n(n + 1)$ is an even number.

17. [5 marks]

Use the method of contradiction to prove that $\sqrt{7}$ is an irrational number.
[You may assume that if p^2 is a multiple of q, then p is also a multiple of q for p as a whole number.]

Calculator Assumed

18. [7 marks: 5, 2]

Consider the statement: $\tan\theta = 0 \quad\Rightarrow\quad \sin\theta = 0.$

(a) Determine with reasons if the contrapositive of this statement is true or false.

(b) Hence or otherwise, determine with reasons if the conjecture is true of false.

---

19. [6 marks: 3, 3]

Consider the statement: $\cos\theta = \frac{1}{2} \quad\Rightarrow\quad \tan\theta = \sqrt{3}$

(a) Determine with reasons if the converse of this statement is true or false.

(b) Hence, or otherwise determine with reasons if the inverse is true or false.

Calculator Assumed

20. [8 marks: 1, 1, 1, 1, 4]

Consider the conjecture for whole number x :

If x is an odd number than x^2 is an odd number.

(a) State the converse of this conjecture.

(b) State the negation of this conjecture.

(c) State the inverse of this conjecture.

(d) State the contrapositive of this statement.

(e) Prove this conjecture by proving its contrapositive.

Calculator Assumed

21. [8 marks: 1, 1, 1, 1, 4]

Consider the conjecture for whole numbers p and q where $p < q$:

If p^2 is a multiple of q, then p is also a multiple of q.

(a) State the converse of this conjecture.

(b) State the negative of this conjecture.

(c) State the inverse of this conjecture.

(d) State the contrapositive of this statement.

(e) Prove this conjecture by proving its contrapositive.

Calculator Assumed

22. [9 marks: 5, 4]

(a) Negate the converse of the following conjecture:

If n is a multiple of 10, then n^2 is a multiple of 10.

Prove that the converse is true by proving that the negation of the converse is false.

(b) Use the result from (a) and the method of contradiction to prove that $\sqrt{10}$ is an irrational number.

Calculator Assumed

23. [7 marks: 5, 2]

(a) Use mathematical induction to prove that:

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{n}{5^n} = \frac{5}{16} - \frac{4n+5}{16(5^n)} \text{ for integer } n \geq 1.$$

(b) Discuss the sum as the number of terms increases indefinitely.

Calculator Assumed

24. [7 marks]

Use the method of mathematical induction to prove that for integer $n \geq 1$,

$$\frac{6}{5} + \left(\frac{6}{5}\right)^2 + \left(\frac{6}{5}\right)^3 + \left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right)^5 + \ldots + \left(\frac{6}{5}\right)^n = 6\times\left(\frac{6}{5}\right)^n - 6.$$

Calculator Assumed

25. [5 marks]

Prove inductively that $(1 + i)^{4n} = (-1)^n\, 2^{2n}$ for integer $n \geq 1$.

26. [5 marks]

Use mathematical induction to prove that $11^n + 4$ is divisible by 5 for integer $n \geq 1$.

Calculator Assumed

27. [6 marks]

Prove that $10^{n+1} + 3 \times 10^n + 5$ is a multiple of 9 for positive integer n.

---

28. [4 marks]

Given the non-singular commutative matrices **P** and **Q**, use mathematical induction to prove that for integer $n \geq 1$, $\mathbf{P}^n = \mathbf{Q}\,\mathbf{P}^n\,\mathbf{Q}^{-1}$.

Calculator Assumed

29. [5 marks] [TISC]

Use mathematical induction to prove that for integer $n \geq 1$:

$$\cos x + \cos 3x + \cos 5x + \ldots. + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}.$$

[Hint: Use the formula $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.]

Fully Worked Solutions

01 Combinatorics I

Calculator Free

1. [9 marks: 2, 2, 2, 3]

Find k if:

(a) $\frac{10!}{k!} = 720$

$$k! = \frac{10!}{720} \quad ✓$$
$$= \frac{10 \times 9 \times 8 \times 7!}{720} = 7!$$
Hence, $k = 7$ ✓

(b) $\frac{k!}{8!} = 990$

$k! = 990 \times 8!$ ✓
$= 11 \times 10 \times 9 \times 8! = 11!$
Hence, $k = 11$ ✓

(c) ${}^{k}P_3 = 210$

$\frac{k!}{(k-3)!} = 210$ ✓
$k(k-1)(k-2) = 210$
Hence, $k = 7$ ✓

(d) $\frac{10! - k!}{k!} = 30\,239$

$\frac{k!\{[10 \times 9 \times 8 \times \ldots \times (k+1)] - 1\}}{k!} = 30\,239$ ✓
$10 \times 9 \times 8 \times \ldots \times (k+1) = 30\,240$ ✓
$9 \times 8 \times \ldots \times (k+1) = 3\,024$
$8 \times \ldots \times (k+1) = 336$
$7 \times (k+1) = 42$
$k = 5$ ✓

Calculator Free

2. [5 marks: 3, 2]

(a) Prove that ${}^{n}P_r = (n+1-r) \times {}^{n}P_{r-1}$

RHS $= (n+1-r) \times {}^{n}P_{r-1}$ $= (n+1-r) \times \frac{n!}{(n-(r-1))!}$ ✓ $= (n+1-r) \times \frac{n!}{(n+1-r)!}$ ✓ $= (n+1-r) \times \frac{n!}{(n+1-r)(n-r)!}$ ✓ $= \frac{n!}{(n-r)!} = {}^{n}P_r$ = LHS	OR RHS $= (n+1-r) \times {}^{n}P_{r-1}$ $= (n+1-r) \times n \times (n-1) \times (n-2) \ldots \times (n-r+2)$ ✓ $= n \times (n-1) \times (n-2) \ldots \times (n-r+2) \times (n-r+1)$ ✓ $= {}^{n}P_r$ = LHS ✓

(b) Simplify ${}^{100}P_{20} - {}^{100}P_{19}$.

Give your answer in the form $A \times {}^{100}P_{19}$ where A is a constant.

${}^{100}P_{20} = (100 + 1 - 20)\ {}^{100}P_{19}$
$= 81\ {}^{100}P_{19}$ ✓
Hence:
${}^{100}P_{20} - {}^{100}P_{19} = 81\ {}^{100}P_{19} - {}^{100}P_{19}$
$= 80\ {}^{100}P_{19}$ ✓

Calculator Free

3. [4 marks: 1, 1, 1, 1]

The letters of the word RICHES are rearranged in a line. No letter is used more than once. Write mathematical expressions for:

(a) the total number of possible arrangements.

> $6!$ ✓

(b) the number of arrangements with the letters I and E are adjacent.

> $5! \times 2$ ✓

(c) the number of arrangements with the letters I and E not adjacent

> $6! - 5! \times 2$ ✓

(d) the number of arrangements where the vowels are adjacent and the letters C and H are adjacent.

> $4! \times 2! \times 2!$ ✓

---

4. [5 marks: 1, 2, 2]

The letters of the word HEXAGON are rearranged in a line. No letter is used more than once. Write mathematical expressions (do not evaluate) for:

(a) the total number of possible arrangements.

> $7!$ ✓

(b) the number of arrangements where all the consonants are adjacent to each other.

> $4! \times 4!$ ✓✓

(c) the number of arrangements where all the consonants are apart.

> $4! \times 3!$ ✓✓

Calculator Assumed

5. [9 marks: 1, 2, 3, 3]

Six digit numbers are formed using the digits 1, 2, 3, 4, 5, and 6. Each digit is used only once.

(a) How many six digit numbers are possible?

> $6! = 720$ ✓

(b) How many six digit even numbers are possible?

> $N = 5 \times 4 \times 3 \times 2 \times 1 \times 3 = 360$ ✓✓

(c) How many five digit even numbers greater than 40 000 are possible?

> If the first digit is odd: $N = 1 \times 4 \times 3 \times 2 \times 3 = 72$ ✓
> If the first digit is even: $N = 2 \times 4 \times 3 \times 2 \times 2 = 96$ ✓
> Hence, total = 72 + 96 = 168 ✓

(d) How many even numbers greater than 40 000 are possible?

> N = n(six digit even numbers > 40 000) + n(five digit even numbers > 40 000) ✓
> = 360 + 168 = 528 ✓✓

---

6. [5 marks]

How many 5 digit ***odd*** numbers greater than 30 000 but less than 80 000 can be formed using the digits of the 0 to 9 inclusive with no digit being used more than once.

> n(first digit is 4 or 6) ✓
> $= 2 \times 8 \times 7 \times 6 \times 5$
> = 3 360 ✓
>
> n(first digit is 3, 5 or 7) ✓
> $= 3 \times 8 \times 7 \times 6 \times 4$
> = 4 032 ✓
>
> Total = 7 392 ✓

Calculator Assumed

7. [4 marks: 2, 2]

Find the number of ways of *rearranging* the letters of the word EXPRESSIONS if:

(a) no other conditions apply.

$N = \dfrac{11!}{2!\,3!} = 3\,326\,400$ ✓✓

(b) the arrangement must start with the letter X.

$N = \dfrac{10!}{2!\,3!} = 302\,400$ ✓✓

---

8. [9 marks: 2, 2, 2, 3]

Find the number of ways of *rearranging* the letters of the word REARRANGING if:

(a) no other conditions apply.

$N = \dfrac{11!}{2!\ 2!\ 2!\ 3!} = 831\,600$ ✓✓

(b) the arrangement must start with the letter E.

$N = \dfrac{10!}{2!\ 2!\ 2!\ 3!} = 75\,600$ ✓✓

(c) the arrangement must start with the letter A.

$N = \dfrac{10!}{2!\ 2!\ 3!} = 151\,200$ ✓✓

(d) the arrangement must start with a vowel.

$N = n(\text{start with E}) + n(\text{start with I}) + n(\text{start with A})$ ✓✓
$= 75\,600 + 75\,600 + 151\,200$
$= 302\,400$ ✓

Calculator Free

9. [9 marks: 2, 2, 2, 3]

Consider the letters in the word **ASSASSINATIONS.**
Write mathematical expressions for each of the following.

(a) The number of ways all the letters in this word can be arranged in a line.

$N = \dfrac{14!}{(3!)(5!)(2!)(2!)}$ ✓✓

(b) The number of ways all the letters in this word can be arranged in a line with all the S letters together.

$N = \dfrac{10!}{(3!)(2!)(2!)}$ ✓✓

(c) The number of ways all the letters in this word can be arranged in a line with all the A letters together?

$N = \dfrac{12!}{(5!)(2!)(2!)}$ ✓✓

(d) The number of ways all the letters in this word can be arranged in a line with all the A letters together *or* all the S letters together?

$N(\text{A and S together}) = \dfrac{8!}{(2!)(2!)}$ ✓

Required number of ways

$= \dfrac{10!}{(3!)(2!)(2!)} + \dfrac{12!}{(5!)(2!)(2!)} - \dfrac{8!}{(2!)(2!)}$ ✓✓

Calculator Assumed

10. [7 marks: 2, 2, 3]

Passwords consisting of between 8 and 12 characters inclusive are to be created using the letters of the alphabet (case sensitive) and the digits 0 to 9 inclusive.

(a) Write mathematical expressions for the number of possible passwords if:

(i) no character can be used more than once.

$N = {}^{62}P_8 + {}^{62}P_9 + {}^{62}P_{10} + {}^{62}P_{11} + {}^{62}P_{12}$ ✓✓

(ii) repetition of characters are permitted.

$N = 62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12}$ ✓✓

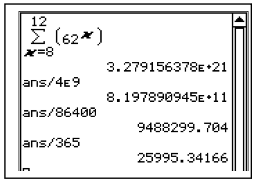
(b) A computer program is capable of checking 4 billion (1×10^9) passwords per second. How long will the computer take to check all the possible passwords in part (a) (ii)? Give your answer in years.

$N = 3.2792 \times 10^{21}$ ✓

$\text{Time} = \dfrac{3.2792 \times 10^{21}}{4 \times 10^9}$

$= 8.1979 \times 10^{11}$ seconds ✓

. 25 995.34 years. ✓

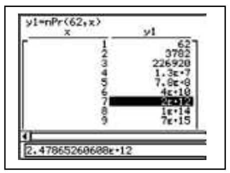


11. [4 marks: 2, 2]

The length of a password is determined by the number of characters in the password. Using the digits 0 to 9 inclusive and the case sensitive letters of the alphabet, find the minimum length of a password if the number of possible passwords is to exceed 1 trillion (1×10^{12}) if:

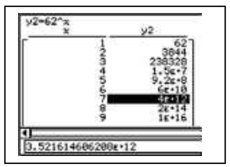
(a) repetition of characters is not permitted.

Use CAS table wizard:
${}^{62}P_6 = 4.4 \times 10^{10}$, ${}^{62}P_7 = 2.5 \times 10^{12}$ ✓
Hence, minimum length = 7 ✓



(b) repetition of characters is permitted.

Use CAS table wizard:
$62^6 = 5.7 \times 10^{10}$, $62^7 = 3.5 \times 10^{12}$ ✓
Hence, minimum length = 7 ✓



02 Combinatorics II

Calculator Assumed

1. [4 marks: 1, 1, 2]

In a group of 40 students, there were 10 boys who were colour vision deficient (CVD) and 15 girls who were not CVD. There were as many boys who were not CVD as there were boys who were CVD. How many of these students:

(a) were boys?

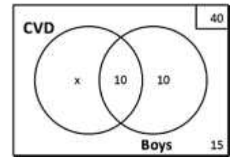
$n(\text{boys}) = n(\text{boys} \cap \text{CVD}) + n(\text{boys} \cap \overline{\text{CVD}})$
$= 10 + 10 = 20$ ✓

(b) were either boys or colour vision deficient?

$n(\text{boys} \cup \text{CVD}) = n(\text{total}) - n(\overline{\text{boys}} \cap \overline{\text{CVD}})$
$= 40 - 15 = 25$ ✓

(c) were colour vision deficient?

$n(\text{boys} \cup \text{CVD}) = n(\text{boys}) + n(\text{CVD}) - n(\text{boys} \cap \text{CVD})$
$25 = 20 + n(\text{CVD}) - 10$ ✓
$n(\text{CVD}) = 15$ ✓



(b) $15 + 10 + 10 + x = 40$
$x = 5$
$n(\text{boys} \cup \text{CVD})$
$= 10 + 10 + x = 25$

2. [5 marks: 2, 2, 1]

In a survey of teachers teaching mathematics to year nines, 15 teachers had a mathematics degree, 55 men teachers did not have a mathematics degree and 10 women teachers had a mathematics degree. There were as many teachers who were either female or had a mathematics degree as there were men teachers.

(a) How many male teachers were there?

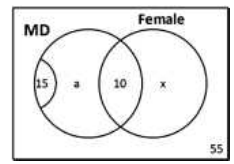
$n(\text{male}) = n(\text{male} \cap \text{MD}) + n(\text{male} \cap \text{no MD})$
$= (15 - 10) + 55 = 60$ ✓✓

(b) How many female teachers were there?

$n(\text{female} \cup \text{MD}) = n(\text{female}) + n(\text{MD}) - n(\text{female} \cap \text{MD})$
$60 = n(\text{female}) + 15 - 10$ ✓
$n(\text{female}) = 55$ ✓

(c) How many teachers were surveyed?

$n(\text{total}) = 55 + 60 = 115$ ✓



(a) $10 + a = 15$
$a = 5$
$n(\text{male}) = a + 55 = 60$
(b) $15 + x = 60$
$x = 45$
$n(\text{female}) = 10 + 45 = 55$

Calculator Assumed

3. [7 marks: 1, 1, 1, 1, 2, 1]

Flags of 10 different nations including that of Australia and New Zealand are to be flown from 10 flag poles set in a line. The flag poles are labelled poles 1 to 10. How many ways are there of assigning a flag to each of these poles if the:

(a) Australian flag must be flown from pole 1?

N = 1 × 9! = 362 880 ✓

(b) Australian flag or New Zealand flag must be flown from pole 1?

N = 2 × 9! = 725 760 ✓

(c) Australian flag and the New Zealand flag must be flown from poles 1 and 10 respectively?

N = 1 × 8! × 1 = 40 320 ✓

(d) Australian flag must be flown from pole 1 and the New Zealand flag must not to be flown from pole 10.

N = 1 × 8! × 8 = 322 560 ✓

(e) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10.

N = n(Aussie flag on pole 1) + n(NZ flag on pole 10) − n(Aussie on pole 1 ∩ NZ on pole 10)
= 362 880 + 362 880 − 40 320 = 685 440 ✓✓

(f) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10 but not both at the same time.

N = n(Aussie on pole 1 ∪ NZ on pole 10) − n(Aussie on pole 1 ∩ NZ on pole 10)
= 685 440 − 40 320 = 645 120 ✓

Calculator Assumed

4. [6 marks: 2, 2, 2]

Twelve different coloured light bulbs, including a red bulb and a blue bulb are to be fitted into bulb sockets installed along a straight edge of a patio running East to West. The bulb socket at the extreme Eastern end is labelled E and the bulb socket at the extreme Western end is labelled W. Determine the number of arrangements with:

(a) the red light bulb not fitted into bulb sockets E or W.

N = 11 × 10! × 10 = 399 168 000 ✓✓

(b) the red light bulb and the blue light bulb not fitted into bulb sockets E or W.

N = 10 × 10! × 9 = 326 592 000 ✓✓

(c) the red light bulb or the blue light bulb not fitted into bulb sockets E or W.

N = n(red not in E or W) + n(blue not in E or W) − n(red not in E or W ∩ blue not in E or W)
= 399 168 000 + 399 168 000 − 326 592 000 ✓
= 471 744 000 ✓

5. [7 marks: 2, 2, 3]

Ten potted plants including four pots of roses of different shades of red and three pots of azaleas (each of a different colour) are to be arranged in a line along a footpath. How many arrangements will have:

(a) the potted roses adjacent to each other?

N = 7! × 4! = 120 960 ✓✓

(b) the potted azaleas adjacent to each other?

N = 8! × 3! = 241 920 ✓✓

(c) the roses adjacent to each other or the azaleas adjacent to each other?

N = n(roses adjacent) + n(azaleas adjacent) − n(roses adjacent ∩ azaleas adjacent)
= 120 960 + 241 290 − 5! × 4! × 3! ✓✓
= 345 600 ✓

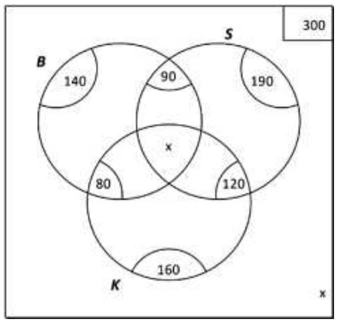
Calculator Assumed

6. [4 marks]

In a group of 300 students:
- 140 students had holidayed in Bali
- 190 had holidayed in Singapore
- 160 had holidayed in Kuala Lumpur
- 80 had holidayed in both Bali and Kuala Lumpur
- 120 had holidayed in Kuala Lumpur and Singapore
- 90 had holidayed in Bali and Singapore
- there were as many students who had holidayed in all three places as those who had not holidayed in any of the three places

Show the use of the inclusion-exclusion principle to determine the number of students that had holidayed in all three places.

Denote B, S and K as the sets of students who had holidayed in Bali, Singapore and Kuala Lumpur respectively.
Let $n(B \cap S \cap K) = n(\overline{B \cup S \cup K}) = x$.



Using the Venn Diagram as an aid:

$n(B \cup S \cup K) = n(B) + n(S) + n(K) - n(B \cap S) - n(B \cap K) - n(S \cap K) + n(B \cap S \cap K)$

$300 - x = 140 + 190 + 160 - 90 - 80 - 120 + x$ ✓ ✓ ✓

$x = 50$ ✓

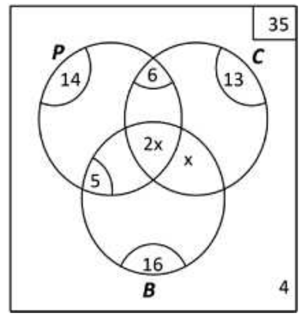
Calculator Assumed

7. [5 marks]

In a group of 35 students:
- 14 students were enrolled in Physics
- 16 were enrolled in Biology
- 13 students were enrolled in Chemistry
- 6 were enrolled in Physics and Chemistry
- 5 were enrolled in Physics and Biology
- 4 were not enrolled in any of these three subjects
- there were twice as many students who were enrolled in all these three subjects as those enrolled in Chemistry and Biology but not Physics.

Show the use of the inclusion-exclusion principle to determine the number of students enrolled in all three subjects.

Denote P, C and B as the sets of students enrolled in Physics, Chemistry and Biology respectively.
Let $n(P \cap C \cap B) = 2x$ and $n(\overline{P} \cap C \cap B) = x$.



Using the Venn Diagram as an aid:

$n(P \cup C \cup B) = n(P) + n(C) + n(B) - n(P \cap C) - n(P \cap B) - n(B \cap C) + n(P \cap C \cap B)$

$35 - 4 = 14 + 13 + 16 - 6 - 3x - 5 + 2x$ ✓ ✓ ✓

$x = 1$ ✓

Hence, $n(P \cap C \cap B) = 2$ ✓

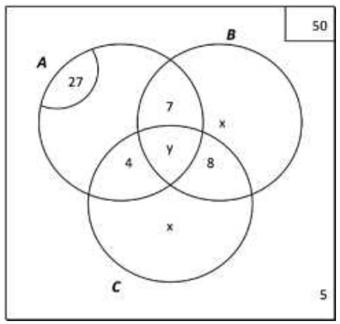
Calculator Assumed

8. [5 marks]

In a group of 50 students:
- 27 students are enrolled in Accounting
- 7 students are enrolled in Accounting and Biology but not Chemistry
- 8 students are enrolled in Biology and Chemistry but not Accounting
- 4 students are enrolled in Accounting and Chemistry but not Biology
- 5 students were not enrolled in Accounting, Biology or Chemistry.
- there were as many students enrolled in Biology only as those enrolled in Chemistry only.

Show the use of the inclusion-exclusion principle to determine the number of students enrolled in Biology only?

Denote A, B and C as the sets of students enrolled in Accounting, Biology and Chemistry respectively.
Let $n(B \text{ only}) = n(C \text{ only}) = x$ and $n(A \cap B \cap C) = y$.



Using the Venn Diagram as an aid:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$45 = 27 + (15 + x + y) + (12 + x + y) - (7 + y) - (4 + y) - (8 + y) + y$ ✓ ✓ ✓ ✓

$2x = 10$

$\Rightarrow x = 5$ ✓

Calculator Assumed

9. [8 marks: 1, 1, 1, 2, 1, 2]

Seven students including Amy, Brian and Catherine are to be arranged in a line. How many possible arrangements are there with:

(a) Amy or Brian or Catherine on the extreme left?

$N = 3 \times 6! = 2\,160$ ✓

(b) Amy on the extreme left?

$n(\text{A left}) = 1 \times 6! = 720$ ✓

(c) Amy on the extreme left and Catherine on the extreme right?

$n(\text{A left} \cap \text{C right}) = 1 \times 1 \times 5! = 120$ ✓

(d) Amy on the extreme left or Catherine on the extreme right?

$n(\text{A left} \cup \text{C right}) = n(\text{A left}) + n(\text{C right}) - n(\text{A left} \cap \text{C right})$
$= 720 + 720 - 120$ ✓
$= 1\,320$ ✓

(e) Amy on the extreme left, Brian in the middle and Catherine on the extreme right?

$n(\text{A left} \cap \text{B mid} \cap \text{C right}) = 1 \times 1 \times 1 \times 4! = 24$ ✓

(f) Amy on the extreme left or Brian in the middle or Catherine on the extreme right?

$n(\text{A left} \cup \text{B mid} \cup \text{C right}) = n(\text{A left}) + n(\text{B mid}) + n(\text{C right})$
$- n(\text{A left} \cap \text{B mid}) - n(\text{A left} \cap \text{C right}) - n(\text{B mid} \cap \text{C right})$
$+ n(\text{A left} \cap \text{B mid} \cap \text{C right})$
$= 720 \times 3 - 120 \times 3 + 24$ ✓
$= 1\,824$ ✓

Calculator Assumed

10. [11 marks: 1, 1, 2, 3, 2, 2]

Ten different coloured balls, including a red, a blue and a green ball, are to be placed in ten different boxes labelled A to J. One ball is to be placed in each box. How many ways are there of placing the balls (one in each box) with:

(a) the red ball in box A?

$n(\text{red in A}) = 1 \times 9! = 362\,880$ ✓

(b) the red ball in box A and the blue ball in box B?

$n(\text{red in A} \cap \text{blue in B}) = 1 \times 1 \times 8! = 40\,320$ ✓

(c) the red ball in box A or the blue ball in box B?

$$n(\text{red in A} \cup \text{blue in B}) = n(\text{red in A}) + n(\text{blue in B}) - n(\text{red in A} \cap \text{blue in B})$$
$$= 362\,880 + 362\,880 - 40\,320 \quad ✓$$
$$= 685\,440 \quad ✓$$

(d) the red ball in box A or the blue ball in box B or the green ball in box C?

$$n(\text{red in A} \cup \text{blue in B} \cup \text{green in C}) = n(\text{red in A}) + n(\text{blue in B}) + n(\text{green in C})$$
$$- n(\text{red in A} \cap \text{blue in B}) - n(\text{red in A} \cap \text{green in C})$$
$$- n(\text{blue in B} \cap \text{green in C})$$
$$+ n(\text{red in A} \cap \text{blue in B} \cap \text{green in C}))$$
$$= 3 \times 362\,880 - 3 \times 40\,320 + 1 \times 1 \times 1 \times 7! \quad ✓✓$$
$$= 972\,720 \quad ✓$$

(e) the red ball in box A but the blue ball not in box B?

$n(\text{red in A} \cap \overline{\text{blue in B}}) = 1 \times 8 \times 8! = 322\,560$ ✓✓

(f) the green ball in box C but the red ball not in box A and the blue ball not in box B?

$$n(\text{green in C} \cap \overline{\text{red in A}} \cap \overline{\text{blue in B}}) = n(\text{red in A} \cup \text{blue in B} \cup \text{green in C})$$
$$- n(\text{red in A} \cup \text{blue in B})$$
$$= 972\,720 - 685\,440 \quad ✓$$
$$= 287\,280 \quad ✓$$

Calculator Assumed

11. [8 marks: 1, 1, 1, 1, 2, 2]

Consider the set of integers between 1 000 and 9 999 inclusive.
How many integers in this set:

(a) are divisible by 2?

$1\,000 = 2 \times 500$
$9\,998 = 2 \times 4999$
Hence, $n(\text{divisible by 2}) = 4\,999 - 500 + 1 = 4\,500$ ✓

(b) are divisible by 3?

$1\,002 = 3 \times 334$
$9\,999 = 3 \times 3\,333$
Hence, $n(\text{divisible by 3}) = 3\,333 - 334 + 1 = 3\,000$ ✓

(c) are divisible by 2 and 3?

$1\,002 = 6 \times 167$
$9\,996 = 6 \times 1\,666$
Hence, $n(\text{divisible by 6}) = 1\,666 - 167 + 1 = 1\,500$ ✓

(d) are divisible by 2 and 6?

$$n(\text{divisible by 2 and 6}) = n(\text{divisible by 6})$$
$$= 1\,500 \quad ✓$$

(e) are divisible by 2 or 3?

$$n(\text{divisible by 2 or 3}) = n(\text{divisible by 2}) + n(\text{divisible by 3}) - n(\text{divisible by 2 and 3})$$
$$= 4\,500 + 3\,000 - 1\,500 = 6\,000 \quad ✓✓$$

(f) are divisible by 2 or 3 but not both?

$$n(\text{divisible by 2 or 3 but not both}) = n(\text{divisible by 2 or 3}) - n(\text{divisible by 2 and 3})$$
$$= 6\,000 - 1\,500 = 4\,500 \quad ✓✓$$

Calculator Assumed

12. [7 marks: 1, 1, 1, 2, 2]

Consider the set of integers between 500 and 5 000 inclusive.
How many integers in this set:

(a) are divisible by 5?

> 500 = 5 × 100
> 5 000 = 5 × 1 000
> Hence, n(divisible by 5) = 1 000 – 100 + 1 = 901 ✓

(b) are divisible by 10?

> 500 = 10 × 50
> 5 000 = 10 × 500
> Hence, n(divisible by 10) = 500 – 50 + 1 = 451 ✓

(c) are divisible by 5 and 10?

> LCM of 5 and 10 = 10
> Hence,
> n(divisible by 5 and 10) = n(divisible by 10)
> = 451 ✓

(d) are divisible by 5 or 10?

> n(divisible by 5 or 10) = n(divisible by 5) + n(divisible by 10) – n(divisible by 5 and 10)
> = 901+ 451 – 451 = 901 ✓✓
> OR
> All integers divisible by 10 are also divisible by 5.
> Hence, n(divisible by 5 or 10) = n(divisible by 5) = 901

(e) are divisible by 5 or 10 and is an even number?

> LCM of 2, 5 and 10 = 10
> Hence,
> n(divisible by 5 or 10 and is even) = n(divisible by 10) ✓
> = 451 ✓

Calculator Assumed

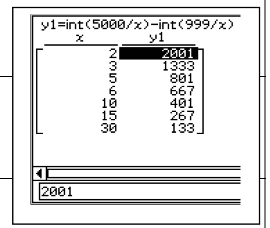
13. [6 marks: 3, 3]

Consider the set of integers between 1 000 and 5 000 inclusive.

(a) Complete the following table listing the number of multiples of n within this set for the given values of n.

> – 1 per error ✓✓✓

n	Number of multiples of n in this set.
2	$\textbf{Int}\left(\frac{5000}{2}\right) - \textbf{Int}\left(\frac{999}{2}\right) = 2\,001$
3	$\textbf{Int}\left(\frac{5000}{3}\right) - \textbf{Int}\left(\frac{999}{3}\right) = 1\,333$
5	$\textbf{Int}\left(\frac{5000}{5}\right) - \textbf{Int}\left(\frac{999}{5}\right) = 801$
6	$\textbf{Int}\left(\frac{5000}{6}\right) - \textbf{Int}\left(\frac{999}{6}\right) = 667$
10	$\textbf{Int}\left(\frac{5000}{10}\right) - \textbf{Int}\left(\frac{999}{10}\right) = 401$
15	$\textbf{Int}\left(\frac{5000}{15}\right) - \textbf{Int}\left(\frac{999}{15}\right) = 267$
30	$\textbf{Int}\left(\frac{5000}{30}\right) - \textbf{Int}\left(\frac{999}{30}\right) = 133$



(b) Find the number of integers in this set that are multiples of 2, 3 or 5.

> n(multiples of 2 or 3 or 5) = n(multiples of 2) + n(multiple of 3) + n(multiple of 5)
> – n(multiples of 2 and 3) – n(multiples of 2 and 5)
> – n(multiple of 3 and 5)
> + n(multiple of 2 and 3 and 5)
> = 2 001 + 1 333 + 801 – 667 – 401 –267 + 133 ✓✓
> = 2 933 ✓

Calculator Assumed

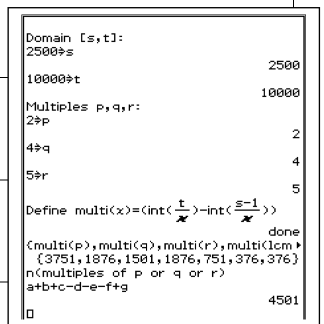
14. [6 marks: 3, 3]

Consider the set of integers between 2500 and 10 000 inclusive.

(a) Complete the following table.

– 1 per error ✓✓✓

Multiples of	Number of multiples in this set.
2	$\textbf{Int}\left(\frac{10\,000}{2}\right) - \textbf{Int}\left(\frac{2499}{2}\right) = 3\,751$
4	$\textbf{Int}\left(\frac{10\,000}{4}\right) - \textbf{Int}\left(\frac{2499}{4}\right) = 1\,876$
5	$\textbf{Int}\left(\frac{10\,000}{5}\right) - \textbf{Int}\left(\frac{2499}{5}\right) = 1\,501$
2 and 4	LCM of 2 and 4 = 4. Hence, 1 876.
2 and 5	LCM of 2 and 5 = 10. Hence, 751
4 and 5	LCM of 4 and 5 = 20 $\Rightarrow \textbf{Int}\left(\frac{10000}{20}\right) - \textbf{Int}\left(\frac{2499}{20}\right) = 376$
2 and 4 and 5	LCM of 2, 4 and 5 = 20. Hence, 376.



(b) Find the number of integers in this set that are multiples of 2, 4 or 5.

n(multiples of 2 or 4 or 10) = n(multiples of 2) + n(multiple of 4) + n(multiple of 5)
– n(multiples of 2 and 4) – n(multiples of 2 and 5)
– n(multiple of 4 and 5)
+ n(multiple of 2 and 4 and 5)
= 3 751 + 1 876 + 1 501 – 1 876 – 751 –376 + 376 ✓✓
= 4 501 ✓

Calculator Assumed

15. [6 marks]

Consider the set of integers between 0 and 50 000 inclusive.
How many integers in this set are divisible by 2 *or* 3 *or* 5 *but not* 10?

n(divisible by 2) = 25 000 + 1 = 25 001

n(divisible by 3) = 16 666

n(divisible by 5) = 10 000 + 1 = 10 001

n(divisible by $2 \cap 5$) = 5 000 + 1 = 5 001

n(divisible by $2 \cap 3$) = 8 333

n(divisible by $3 \cap 5$) = 3 333

n(divisible by $2 \cap 3 \cap 5$) = 1 666

n(divisible by $2 \cup 3 \cup 5$) = n(divisible by 2) + n(divisible by 3) + n(divisible by 5)
– n(divisible by $2 \cap 5$) – n(divisible by $2 \cap 3$) – n(divisible by $3 \cap 5$)
+ n(divisible by $2 \cap 3 \cap 5$) ✓

= 25 001 + 16 666 + 10 001 ✓
– 5 001 – 8 333 – 3 333 ✓
+ 1 666 ✓

= 36 667 ✓

Hence: number of integers in this set that are divisible by 2 *or* 3 *or* 5 *but not* 10
N = 36 667 – 5 001 = 31 666 ✓

Calculator Assumed

16. [10 marks: 2, 2, 2, 4]

Consider the set of integers between 100 000 and 500 000 inclusive.
How many integers in this set:

(a) are divisible by 2?

100 000 = 2 × 50 000
500 000 = 2 × 250 000
Hence, n(divisible by 2)
= 250 000 – 50 000 + 1 ✓
= 200 001 ✓

(b) are divisible by 3?

100 002 = 3 × 33 334
499 998 = 3 × 166 666
Hence, n(divisible by 3)
= 166 666 – 33 334 + 1 ✓
= 133 333 ✓

(c) are divisible by 2 and 3?

100 002 = 6 × 16 667
499 998 = 6 × 83 333
Hence, n(divisible by 6)
= 83 333 – 16 667 + 1 ✓
= 66 667 ✓

(d) are not divisible by 2 and not divisible by 3?

n(divisible by 2 or 3) = n(divisible by 2) + n(divisible by 3) – n(divisible by 2 and 3)
= 200 001 + 133 333 – 66 667 ✓
= 266 667 ✓
n(not divisible by 2 and not divisible by 3) = n(total) – n(divisible by 2 or 3)
= 400 001 – 266 667 ✓
= 133 334 ✓

03 Combinatorics III

Calculator Free

1. [13 marks: 2, 2, 2, 4, 3]

(a) Find n and r if ${}^{n}C_{r} = \dfrac{n \times (n-1) \times 98}{3 \times 2 \times 1}$.

$n = 100$ ✓
$r = 3$ or 97 ✓

(b) Find n and r if ${}^{n}P_{r} = 20 \times 19 \times 18 \times 17$

$n = 20$ ✓
$r = 4$ ✓

(c) Find a and b if ${}^{30}C_{a} = {}^{3a}C_{b}$.

$3a = 30 \Rightarrow a = 10$ ✓
$b = 10$ or 20 ✓

(d) Find all possible values of a and b if ${}^{12}C_{a} = {}^{12}C_{2a+b}$.

$a + 2a + b = 12$ ✓
$a = 4 - \dfrac{b}{3}$
As a and b must be non-negative integers:
$b = 0, a = 4$ or $b = 3, a = 3$
or $b = 6, a = 2$ or $b = 9, a = 1$
or $b = 12, a = 0$ ✓✓✓

(e) Find a possible set of values for a and b if $10 \times {}^{9}P_{4} = 6 \times {}^{a}P_{b}$.

$$10 \times \frac{9!}{5!} = 6 \times \frac{a!}{(a-b)!}$$
$$\frac{10 \times 9!}{6 \times 5!} = \frac{a!}{(a-b)!} \quad ✓$$
$$\frac{10!}{6!} = \frac{a!}{(a-b)!}$$
$$\Rightarrow \quad a = 10, b = 4 \quad ✓✓$$

Calculator Free

2. [9 marks: 3, 3, 3]

(a) Find k if $\frac{(k+3)!}{k!} = 120$.

$$\frac{(k+3)!}{k!} = \frac{(k+3)(k+2)(k+1)\,k!}{k!}$$
$$= (k+3)(k+2)(k+1) \quad ✓$$
$$\Rightarrow (k+3)(k+2)(k+1) = 120$$
$$= 6 \times 5 \times 4 \quad ✓$$
$$k = 3 \quad ✓$$

(b) (i) Show that ${}^{10}C_3 + {}^{10}C_4 = {}^{11}C_4$.

$${}^{10}C_3 + {}^{10}C_4 = \frac{10\times9\times8}{3\times2\times1} + \frac{10\times9\times8\times7}{4\times3\times2\times1} \quad ✓$$
$$= \frac{10\times9\times8}{3\times2\times1}\left(1+\frac{7}{4}\right) \quad ✓$$
$$= \frac{10\times9\times8}{3\times2\times1}\left(\frac{11}{4}\right)$$
$$= {}^{11}C_4 \quad ✓$$

(ii) Hence, prove that ${}^{10}C_3 + {}^{10}C_4 + {}^{11}C_5 + {}^{12}C_6 + {}^{13}C_7 + {}^{14}C_8 = {}^{15}C_8$

$$\text{LHS} = {}^{10}C_3 + {}^{10}C_4 + {}^{11}C_5 + {}^{12}C_6 + {}^{13}C_7 + {}^{14}C_8$$
$$= \left({}^{10}C_3 + {}^{10}C_4\right) + {}^{11}C_5 + {}^{12}C_6 + {}^{13}C_7 + {}^{14}C_8$$
$$= \left({}^{11}C_4 + {}^{11}C_5\right) + {}^{12}C_6 + {}^{13}C_7 + {}^{14}C_8 \quad ✓$$
$$= \left({}^{12}C_5 + {}^{12}C_6\right) + {}^{13}C_7 + {}^{14}C_8 \quad ✓$$
$$= \left({}^{13}C_6 + {}^{13}C_7\right) + {}^{14}C_8 \quad ✓$$
$$= {}^{14}C_7 + {}^{14}C_8$$
$$= {}^{15}C_8 = \text{RHS}$$

Calculator Free

3. [3 marks]

(a) How many ways are there of choosing 2 students from a group of 100 students?

$$N = {}^{100}C_2 = \frac{100 \times 99}{2 \times 1} = 4\,950 \quad ✓✓$$

(b) How many ways are there of choosing 98 students from a group of 100 students?

$$N = {}^{100}C_{98} = {}^{100}C_2 = 4\,950 \quad ✓$$

---

4. [7 marks: 2, 2, 3]

A media folder has 10 video-clips. How many ways are there of choosing:

(a) seven of these clips?

$$N = {}^{10}C_7 = {}^{10}C_3$$
$$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \quad ✓✓$$

(b) seven or eight of these clips?

$$N = {}^{10}C_7 + {}^{10}C_8 \quad ✓$$
$$= {}^{11}C_8$$
$$= \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 165 \quad ✓$$

(c) at least two of these clips?

$$N = {}^{10}C_2 + {}^{10}C_3 + \ldots + {}^{10}C_{10} \quad ✓$$
$$= 2^{10} - 1 - {}^{10}C_1 \quad ✓$$
$$= 1013 \quad ✓$$

Calculator Assumed

5. [4 marks: 2, 2]

Write a mathematical expression (do not evaluate) for the number of ways of dividing a group of 50 students into:

(a) one group with 30 students and the second group with 20 students?

$N = {}^{50}C_{30} \times {}^{20}C_{20}$ ✓ ✓

(b) two groups each with 25 students?

$N = \frac{1}{2}\left({}^{50}C_{25}\right)$ ✓ ✓ OR $N = {}^{49}C_{24}$ ✓✓

---

6. [10 marks: 1, 3, 3, 3]

A committee of 9 people are to be selected from 10 Labor, 8 Liberal and 5 Green politicians. How many different ways can the committee be selected if:

(a) there are no restrictions

No. of ways $= {}^{23}C_9$
$= 817\,190$ ✓

(b) all three political parties are equally represented

No. of ways $= {}^{10}C_3 \times {}^{8}C_3 \times {}^{5}C_3$ ✓✓
$= 67\,200$ ✓

(c) the Liberal representatives are in the (absolute) majority

No. of ways $= {}^{15}C_4 \times {}^{8}C_5 + {}^{15}C_3 \times {}^{8}C_6 + {}^{15}C_2 \times {}^{8}C_7 + {}^{15}C_1 \times {}^{8}C_8$ ✓✓
$= 90\,035$ ✓

(d) a husband and wife pair, Alex and Alice, cannot be in the same committee.

No. of ways $= {}^{23}C_9 - {}^{2}C_2 \times {}^{21}C_7$ ✓✓
$= 700\,910$ ✓

Calculator Assumed

7. [10 marks: 1, 2, 3, 4] [TISC]

Wei has a collection of 20 stickers in her pink box and 25 stickers in her blue box. All these stickers are different from each other.

(a) In how many ways can Wei pick 3 stickers from her pink box?

No. of ways $= {}^{20}C_3$
$= 1140$ ✓

(b) In how many ways can Wei pick 2 stickers from her blue box and arrange them in a line?

No. of ways $= {}^{25}C_2 \times 2!$ ✓
$= 600$ ✓

(c) In how many ways can Wei pick 3 stickers from her pink box and 2 stickers from her blue box and arrange them in a line if:

(i) there are no restrictions as to how the stickers are arranged?

No. of ways $= {}^{20}C_3 \times {}^{25}C_2 \times 5!$ ✓✓
$= 41\,040\,000$ ✓

(ii) all the stickers from the blue box must be together?

No. of ways $= {}^{20}C_3 \times {}^{25}C_2 \times 4! \times 2!$ ✓✓✓
$= 16\,416\,000$ ✓

Calculator Free

8. [12 marks: 1, 2, 2, 2, 2, 3]

In how many ways can the letters of the word **COMBINE** be arranged in a straight line (no letter may be used more than once):

(a) using all seven letters?

No. of ways = $7!$
$= 5040$ ✓

(b) using all seven letters and starting with the letter **C**?

No. of ways = $1 \times 6!$ ✓
$= 720$ ✓

(c) using all seven letters and ending in **BONE?**

No. of ways = $3! \times 1$ ✓
$= 6$ ✓

(d) using only 5 letters at a time?

No. of ways = ${}^{7}C_{5} \times 5!$ ✓
$= 2520$ ✓

(e) using all seven letters with the vowels in the first three places (from the left)?

No. of ways = $3! \times 4!$ ✓
$= 144$ ✓

(f) using only two vowels and two consonants?

No. of ways = ${}^{3}C_{2} \times {}^{4}C_{2} \times 4!$ ✓✓
$= 432$ ✓

Calculator Free

9. [12 marks: 1, 2, 2, 3, 4]

Consider the letters of the word CONQUEST. Write mathematical expressions for the number of ways the letters of this word can be *rearranged*:

(a) if the first letter must be a consonant.

$N = 5 \times 7!$ ✓

(b) if the first three letters must be consonants.

$N = {}^{5}C_{3} \times 3! \times 5!$ ✓✓

(c) if one of the first three letters must be a consonant.

$N = {}^{5}C_{1} \times {}^{3}C_{2} \times 3! \times 5!$ ✓✓

(d) if at least one of the first three letters must be a consonant.

$N = ({}^{5}C_{1} \times {}^{3}C_{2} \times 3! \times 5!) + ({}^{5}C_{2} \times {}^{3}C_{1} \times 3! \times 5!) + ({}^{5}C_{3} \times {}^{3}C_{0} \times 3! \times 5!)$ ✓ ✓ ✓

(e) if the vowels are to be sandwiched between two consonants (that is, a vowel must be preceded by a consonant and this vowel must also be followed by a consonant).

Let C: consonant and V: vowel.
The required arrangement must be of the form: CVCVCVCC or CCVCVCVC
or CVCCVCVC or CVCVCCVC ✓ ✓

Hence, $N = 5! \times 3! \times 4$ ✓✓

Calculator Assumed

10. [14 marks: 2, 4, 4, 4]

Passwords are to be created using the the digits 0 to 9 inclusive and the 26 non-case sensitive letters of the English alphabet.

(a) **MARK** creates a password consisting of 10 characters and the passwood contains all the letters of his name. Write a mathematical expression for the number of possible passwords if no character is repeated.

$N = {}^{4}C_{4} \times {}^{32}C_{6} \times 10!$ ✓ ✓

(b) **FAREEHA** creates a password comprising 10 characters using all the letters in her name and three other distinct characters not found in her name. How many such passwords are possible?

$N = {}^{31}C_{3} \times \frac{10!}{2!2!}$ ✓ ✓✓

$= 4\,077\,864\,000$ ✓

(c) Write a mathematical expression for the number of possible passwords that comprises 3 digits *and* 7 letters with the three digits adding up to six with no character used more than once.

Digits that sum to six:
{0, 1, 5} or {0, 2, 4} or {1, 2, 3} ✓

$N = \binom{3}{1} \times \binom{26}{7} \times 10!$ ✓ ✓ ✓

(d) How many 8 character passwords are possible if each password must include at least 3 digits *and* 3 letters?

$N = {}^{10}C_{3} \times {}^{26}C_{5} \times 8! + {}^{10}C_{4} \times {}^{26}C_{4} \times 8! + {}^{10}C_{5} \times {}^{26}C_{3} \times 8!$ ✓ ✓ ✓ ✓

$= 318\,269\,952\,000 + 126\,584\,640\,000 + 26\,417\,664\,000$

$= 471\,272\,256\,000$ ✓

Calculator Assumed

11. [5 marks]

In a school curriculum, six subjects are listed under List A, eight subjects are listed under list B and ten subjects are listed under List C.
A student is enrolled in six subjects. How many ways are there for this student to choose 2 List A subjects <u>*or*</u> 2 List B subjects?

$n(2A \cap 4 \text{ from B and C}) = \binom{6}{2}\binom{18}{4} = 45\,900$ ✓

$n(2B \cap 4 \text{ from A and C}) = \binom{8}{2}\binom{16}{4} = 50\,960$ ✓

$n(2A \cap 2B \cap 2C) = \binom{6}{2}\binom{8}{2}\binom{10}{2} = 18\,900$ ✓

$n(2A \cup 2B) = 45\,900 + 50\,960 - 18\,900$ ✓
$= 77\,960$ ✓

12. [5 marks: 3, 2]

A password consisting of four characters is to be chosen from 10 digits, 26 upper case letters, 26 lower case letters and a set of 32 symbols.

(a) Complete the table below.

Composition of password	Number of different passwords
2 lower case letters	${}^{26}C_2 \times {}^{68}C_2 \times 4! = 17\,768\,400$ ✓
2 symbols	${}^{32}C_2 \times {}^{62}C_2 \times 4! = 22\,510\,464$ ✓
2 lower case letters and 2 symbols	${}^{26}C_2 \times {}^{32}C_2 \times 4! = 3\,868\,800$ ✓

(b) Determine the number of possible passwords with either two lower case letters or two symbols.

$n(2 \text{ lower case} \cup 2 \text{ symbols}) = n(2 \text{ lower case}) + n(2 \text{ symbols}) - n(2 \text{ lower case} \cap 2 \text{ symbols})$ ✓
$= 17\,768\,400 + 22\,510\,464 - 3\,868\,800$
$= 36\,410\,064$ ✓

Calculator Assumed

13. [10 marks: 2, 2, 3, 3]

Eight books are to be selected and arranged on a library display shelf. These eight books are to be selected from a collection of 8 adult novels, 4 non-fiction books and 5 illustrated children's books. How many of these arrangements will contain:

(a) four adult novels?

No. of ways = ${}^8C_4\,{}^9C_4 \times 8!$ ✓
= 355 622 400 ✓

(b) three non-fiction books?

No. of ways = ${}^4C_3\,{}^{13}C_5 \times 8!$ ✓
= 207 567 360 ✓

(c) exactly four adult novels and three non-fiction books?

No. of ways = ${}^8C_4\,{}^4C_3\,{}^5C_1 \times 8!$ ✓✓
= 56 448 000 ✓

(d) four adult novels or three non-fiction books?

No. of ways = n(4 adult novels) + n(3 non-fiction) – n(4 adult novels ∩ 3 non-fiction)
= 355 622 400 + 207 567 360 – 56 448 000 ✓✓
= 506 741 760 ✓

Calculator Assumed

14. [6 marks]

Passwords consisting of 8 characters are to be formed using the digits 0 to 9 inclusive, the case sensitive letters of the English Latin (Roman) alphabet and the special symbols %, !, #, $, & and ?.

Write a mathematical expression for the number of possible passwords with no repeated characters with 4 digits *or* 4 letters *or* 4 special symbols.

Total = N(4 digits) + N(4 letters) + N(4 symbols)
– N(4 digits ∩ 4 letters) – N(4 digits ∩ 4 symbols) – N(4 letters ∩ 4 symbols)
+ N(4 digits ∩ 4 letters ∩ 4 symbols) ✓

$$= \binom{10}{4}\binom{58}{4}8! + \binom{52}{4}\binom{16}{4}8! + \binom{6}{4}\binom{62}{4}8!$$ ✓✓
$$- \binom{10}{4}\binom{52}{4}8! - \binom{10}{4}\binom{6}{4}8! - \binom{52}{4}\binom{6}{4}8!$$ ✓✓
$$+ 0$$ ✓

04 Combinatorics IV

Calculator Free

1. [6 marks: 1, 2, 1, 1, 1]

A container has 10 different sized pairs of nuts and bolts with the nuts removed from the respective bolts.

(a) Five bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure one matching pair of nut and bolt?

> Maximum number of incorrect nuts = 5
> Hence, minimum number of nuts = 5 + 1 = 6 ✓

(b) Six bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure two matching pairs of nut and bolt?

> Maximum number of incorrect nuts = 4
> Minimum number of nuts = 4 + 2 = 6 ✓✓

(c) What is the minimum number of items that need to be removed from this container to ensure one matching pair of nut and bolt?

> Minimum number of items = 10 + 1 = 11 ✓

(d) Fourteen items are randomly removed from the container.
(i) What is the minimum number of matching pairs of nuts and bolts?

> Minimum number of matching pairs = 14 – 10 = 4 ✓

(ii) What is the maximum possible number of matching pairs?

> Maximum possible number of matching pairs = $\frac{14}{2} = 7$ ✓

Calculator Free

2. [7 marks: 1, 2, 1, 1, 1, 1]

Twelve dog owners and their dogs (one dog per owner) meet at a dog park.

(a) Six dog owners are randomly chosen. What is the minimum number of dogs that need to be chosen to ensure a matching owner-dog pair?

> Maximum number of incorrect dogs = 6
> Hence, minimum number of dogs required = 6 + 1 = 7 ✓

(b) Seven dogs are randomly chosen. What is the minimum number of owners that need to be chosen to ensure three matching owner-dog pairs?

> Maximum number of incorrect owners = 5
> Hence, minimum number of owners required = 5 + 3 = 8 ✓✓

(c) A total of 15 owners and dogs were randomly selected.
(i) What is the minimum number of owner-dog pairs in this selection?

> Minimum number of owner-dog pairs = 15 – 12 = 3 ✓

(ii) What is the maximum possible number of owner-dog pairs in this selection?

> Maximum possible number of owner-dog pairs = $\frac{14}{2} = 7$. ✓

(d) If the owners came as couples, that is one dog per couple, what is the minimum number of persons and dogs that need to be chosen to ensure:
(i) a matching owner-dog pair?

> Minimum number of persons and dogs required = 24 + 1 = 25 ✓

(ii) more than two matching owner-dog pairs?

> Minimum number of persons and dogs required = 24 + 3 = 27 ✓

Calculator Free

3. [7 marks: 1 each]

A container has 5 red marbles, 6 green marbles and 9 yellow marbles. What is the minimum number of marbles that need to be drawn from this container to ensure:

(a) a marble of each colour?

> Minimum number of marbles = 9 + 6 + 1 = 16 ✓

(b) two marbles of each colour?

> Minimum number of marbles = 9 + 6 + 2 = 17 ✓

(c) two red marbles?

> Minimum number of marbles = 9 + 6 + 2 = 17 ✓

(d) two yellow marbles?

> Minimum number of marbles = 6 + 5 + 2 = 13 ✓

(e) two green marbles?

> Minimum number of marbles = 9 + 5 + 2 = 16 ✓

(f) two marbles of the same colour?

> Minimum number of marbles = 3 + 1 = 4 ✓

(g) three marbles of the same colour?

> Minimum number of marbles = 6 + 1 = 7 ✓

---

4. [7 marks: 1 each]

Dennis has 3 blue pens, 4 red pens and 5 black pens in his pencil case. What is the minimum number of pens that need to be drawn from the pencil case to ensure that:

(a) a red pen is drawn?

> Minimum number of pens = 5 + 3 + 1 = 9 ✓

(b) a pen of each colour is drawn?

> Minimum number of pens = 5 + 4 + 1 = 10 ✓

(c) a red and a blue pen is drawn?

> Minimum number of pens = 5 + 4 + 1 = 10 ✓

Calculator Free

4. (d) two blue pens and two red pens are drawn?

> Minimum number of pens = 5 + 4 + 2 = 11 ✓

(e) two red pens and two black pens are drawn

> Minimum number of pens = 3 + 5 + 2 = 10 ✓

(f) two blue pens and two black pens are drawn?

> Minimum number of pens = 4 + 5 + 2 = 11 ✓

(g) two pens of the same colour are drawn?

> Minimum number of marbles = 1 + 1 + 2 = 4 ✓

---

5. [8 marks: 1, 2, 5]

A box has four red pens, six blue pens and n black pens (where integer $n \geq 2$). Determine the minimum number of pens that need to be drawn from this box to ensure:

(a) two pens of the same colour.

> $N = 1 + 1 + 1 + 1 = 4$ ✓

(b) two black pens.

> $N = 4 + 6 + 2 = 12$ ✓ ✓

(c) three red pens, two blue pens and one black pen.

> To have 3 red pens, need $3 + 6 + n = n + 9$ pens ✓
> To have 2 blue pens, need $2 + 4 + n = n + 6$ pens ✓
> To have 1 black pen, need $1 + 4 + 6 = 11$ pens. ✓
> Since $n \geq 2$, $n + 9 \geq 11$. ✓
> Hence, minimum number $= n + 9$ pens ✓

Calculator Assumed

6. [4 marks: 2, 2]

There are 25 students in a class.

(a) Explain clearly why there must be at least 3 students that are born in a same month.

$25 = 2 \times 12 + 1$.
In a worst case scenario, we could have the following distribution.

Birth month	Jan	Feb	Mar	…	Oct	Nov	Dec
No. of students	2 + 1	2	2	2	2	2	2

Hence, there are at least 3 students that are born in the same month. ✓✓

OR
Assume that there are no more than 2 students who share a same birth month. If this is the case, then, we have a total of $2 \times 12 = 24$ students which is one less than 25. Hence, there must be at least one month which is the birth month of at least 3 students. ✓✓

(b) Explain clearly why there must be at least one month which is a birth month shared by no more than 2 students.

$25 = 2 \times 12 + 1$.
In a worst case scenario, we could have the following distribution.

Birth month	Jan	Feb	Mar	…	Oct	Nov	Dec
No. of students	2 + 1	2	2	2	2	2	2

Hence, there is at least one month which is the birth month of no more than 2 students. ✓✓

OR
Assume that there is no month which is the birth month shared by no more than 2 students. That is, there are at least 3 students that have birthdays in each month. If this is the case, then, we have a total of $3 \times 12 = 36$ students which is far more than the 25 students in the class.. Hence, there must be at least one month which is the birth month of no more than 2 students. ✓✓

Calculator Assumed

7. [6 marks: 1, 1, 2, 2]

A class has 30 students.

(a) How many students need to be chosen to ensure that there are:
(i) two students who are born on the same day of the week?

Minimum number of students = $7 + 1 = 8$ ✓

(ii) five students who are born on the same day of the week?

Minimum number of students = $4 \times 7 + 1 = 29$ ✓

(b) There are at least x students who are born on the same day of the week. Find x. Justify your answer.

$30 = 4 \times 7 + 2$
In a worst case scenario, we could have the following distribution of birthdays:

Born on:	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
No. of students	5	5	4	4	4	4	4

Hence, there are at least 5 students who are born on the same day of the week. That is $x = 5$. ✓✓

(c) There must be at least one day of the week which is the birth day of no more than y students. Find y. Justify your answer.

$30 = 4 \times 7 + 2$
In a worst case scenario, we could have the following distribution of birthdays:

Born on:	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
No. of students	5	5	4	4	4	4	4

Hence, there is at least one day of the week which is the birth day of no more than 4 students. That is $y = 4$. ✓✓

Calculator Assumed

8. [7 marks: 1, 1, 1, 2, 2]

There are 300 students in a primary school with names written using the English alphabet.

(a) How many students need to be chosen to ensure that there are:
(i) two students with family names that start with the same letter?

> Minimum number of students = 26 + 1 = 27 ✓

(ii) three students with family names that start with the same letter?

> Minimum number of students = 26 × 2 + 1 = 53 ✓

(iii) six students with family names that start with the same letter?

> Minimum number of students = 26 × 5 + 1 = 131 ✓

(b) There are at least x students with family names that start with the same letter. Find x. Justify your answer.

> 300 = 11 × 26 + 14.
> In a worst case scenario, we could have the following distribution.
> 300 = 14 letters with 12 students each + 12 letters with 11 students each.
> That is $x = 12$. ✓✓

(c) There must be at least one letter that is the first letter of the family names of no more than y students. Find y. Justify your answer.

> 300 = 11 × 26 + 14.
> In a worst case scenario, we could have the following distribution:
> 300 = 14 letters with 12 students each + 12 letters with 11 students each.
>
> Hence, there is at least one letter that is the first letter of family names of no more than 11 students. That is $y = 11$. ✓✓

Calculator Assumed

9. [6 marks: 1, 2, 3]

There are 21 letters in the Italian alphabet. Italian names consist of a given-name followed by a surname. Consider a group of 100 students with names written using the Italian alphabet.

(a) How many of these students need to be chosen to ensure that there are two students with given-names that start with the same letter of the Italian alphabet?

> Minimum number of students = 21 + 1 = 22 ✓

(b) Explain why it is not possible to ensure that there are six students in this group with surnames that start with the same letter of the Italian alphabet?

> Min. number of students required
> = 21 × 5 + 1 = 106 ✓
> But there are only 100 students which is less than the minimum number of students required. ✓

(c) How many more students need to be chosen to ensure that there are at least two students with the same initials (initials comprise the first letter of the given name followed by the first letter of the surname)?

> Number of possible initials = 21 × 21 = 441. ✓
> Hence, minimum number N = 441 + 1 = 442 ✓
> Therefore, need another 342 students. ✓

---

10. [4 marks: 1, 3]

Assume that humans with black/brown hair can have between 0 and 110 000 hairs on their head.

(a) How many persons need to be chosen to ensure that we have 3 persons with the same number of hair on their heads?

> $N = 110\,001 \times 2 + 1 = 220\,003$ ✓

(b) A country has 1 billion black/brown haired people. There are at least x people in this country with the same number of hairs on their heads. Find x.

> $N = \text{Ceiling}\left(\frac{1\,000\,000\,000}{110\,000}\right)$ ✓
> $= \text{Ceiling}(9\,090.909091)$ ✓
> $= 9\,091$ ✓

Calculator Assumed

11. [9 marks: 2, 2, 2, 3]

110 students sat an examination. Their performances in this examination were scored between 0 and 100 inclusive.

(a) Use the pigeon-hole principle to discuss the validity of each of the following statements. In each case you need to identify the "pigeons" and the "pigeon-holes".

(i) "At least two students scored 50 marks".

> There are 101 percentage marks (pigeon holes).
> It is not necessary for any one particular pigeon-hole to be filled. ✓
> Hence, statement is not valid. ✓

(ii) "At least two students scored the same mark".

> There are 101 possible percentage marks (pigeon holes).
> If one student (pigeon) was allocated a different mark each, there will be 9 students left over. Hence, at least one of the pigeon holes will need to cater to these 9 remaining students. ✓
> Hence, statement is valid. ✓

(iii) "At least two students had marks that differ by 50".

> There are 101 percentage marks (pigeon holes).
> It is possible for all students (pigeons) to score the same marks! ✓
> Hence, statement is not valid. ✓

(b) How many more students would be required to ensure that there are at least two pairs of students with marks that differ by the same amount?

> There are 101 possible marks differences (pigeon holes).
> Each mark difference needs to be allocated with a student pair (pigeons). ✓
> 203 students would be required to ensure that there are at least two student pairs with the same mark difference. ✓
> Therefore, we need another 93 students. ✓

Calculator Assumed

12. [4 marks: 2, 2]

n integers (all different) are to be chosen from the set of integers {1, 2, 3, …, 20}. It is required that there are always two numbers in the selection that sum to 21.

(a) Using the pigeon-hole principle, identify the "pigeon-holes" and the "pigeons" in this context.

> The pigeon-holes are the pairs of numbers in the selection that sum to 21. ✓
> The pigeons are the integers in the given set. ✓

(b) Determine the value of minimum value for n.

> There are 10 pairs of integers (pigeon holes) in the given set that sum to 21. ✓
> Hence, 11 integers (pigeons) need to be chosen to obtain one of these pairs. ✓
> Hence, $n = 11$.

---

13. [4 marks]

In a bus with 20 passengers, use the pigeon-hole principle to explain why at least two passengers are friends with the same number of passengers.

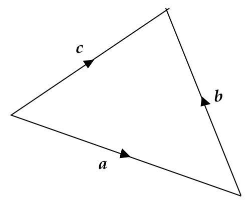
> Within the 20 passengers in this bus, it is not possible to have a passenger with no other friends in the same bus and a passenger who is friends with everyone in the same bus. ✓
> Hence, for any passenger it is possible for this passenger:
> - to be friends with no other passengers, or exactly two other passengers, …, or exactly eighteen other passengers.
> - to be friends with exactly one other passenger or exactly two other passengers, …, or exactly nineteen other passengers. ✓
>
> Hence, there are 0 to 18 or 1 to 19 possible friendships (pigeon holes).
> That is, there are 19 pigeon holes. ✓
>
> Since, there are 20 passengers (pigeons), with 19 pigeon holes, at least one pigeon hole will have two pigeons. ✓
> Hence, there is at least two passengers who are friend with the same number of passengers.

05 Addition & Subtraction of Vectors (Using Trigonometry)

Calculator Free

1. [2 marks: 1, 1]

Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are as drawn in the accompanying diagram.



(a) Express $\boldsymbol{c}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

> $c = a + b$ ✓

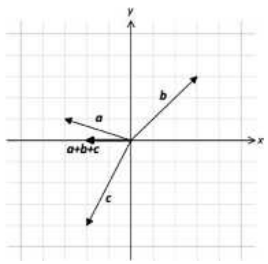
(b) Express $\boldsymbol{a}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

> $a = c - b$ ✓

2. [4 marks: 2, 2]

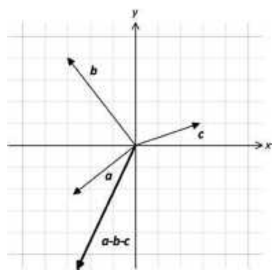
The diagrams below shows the directed line segments representing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Sketch on the provided axes,

(a) the directed line segment representing $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$.



> ✓ On x-axis
> ✓ Correct length.

(b) Sketch on the same axes, the directed line segment representing $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}$.

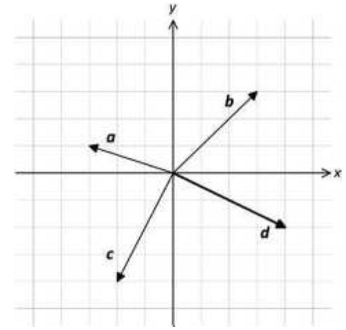


> ✓ Q3
> ✓ Correct .

Calculator Free

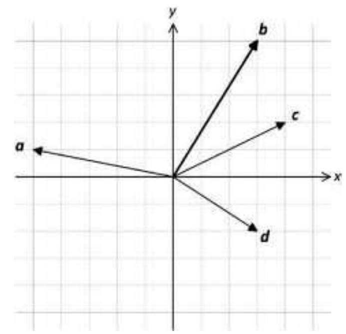
3. [4 marks: 2, 2]

(a) The diagram below shows the directed line segments representing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$. Express $\boldsymbol{d}$ in terms of $\boldsymbol{a}$ and/or $\boldsymbol{b}$ and/or $\boldsymbol{c}$.



> ✓ $c + b$ or $b - a$
> ✓ $c + b - a$

(b) The diagram below shows the directed line segments representing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$. Given that $\boldsymbol{b} = -\boldsymbol{a} + \alpha\boldsymbol{c} + \beta\boldsymbol{d}$, determine the constants α and β.



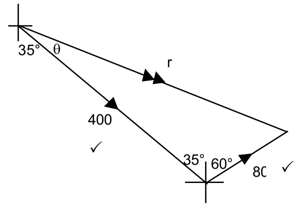
> ✓ $\alpha = 1$
> ✓ $\beta = -2$

Calculator Assumed

4. [6 marks: 2, 4]

An aircraft is flying with a speed of 400 kmh^{-1} along bearing 145°. The aircraft is buffeted by a strong wind of magnitude 80 kmh^{-1} blowing from bearing 240°.

(a) Draw a sketch to indicate the actual direction of the aircraft.



(b) Find the ground speed and direction of the aircraft.

$r^2 = 400^2 + 80^2 - 2(400(80)\cos 95$ ✓
$r = 414.7023$
Hence, ground speed of the aircraft is 414.7 kmh^{-1} ✓

$\frac{\sin\theta}{80} = \frac{\sin 95}{414.7023} \Rightarrow \theta = 11.08^o$ ✓

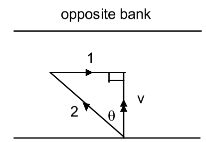
Hence, true direction of aircraft is bearing $145^o - 11.08^o$.= 133.9^o .✓

Calculator Assumed

5. [6 marks: 3, 3]

A boy intends to swim across a river of width 20 metres to the opposite bank. The river flows at a steady rate of 1 kmh^{-1}. The boy can swim at a steady speed of 2 kmh^{-1}.

(a) In what direction should the boy be headed so that he ends up at the opposite bank directly opposite to where he started off ?



$\sin\theta = ½ \Rightarrow \theta = 30^o$ ✓✓

Hence, the boy should be headed in a direction 60^o with the near-bank upstream. ✓

(b) Find the time taken for the swim in part (a).

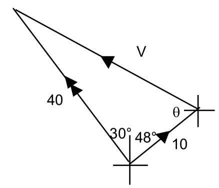
$v = \sqrt{(2^2 - 1^2)}$
$v = \sqrt{3}$ kmh^{-1} ✓

Hence, time taken $= \frac{0.020}{\sqrt{3}}$ ✓
= 0.01155 hour
= 41.6 seconds ✓

Calculator Assumed

6. [5 marks]

A current is flowing in the direction N48°E at 10 kmh^{-1}. With what speed and in what direction should a naval vessel be travelling to achieve a resultant speed of 40 kmh^{-1} in the direction N30°W.



$v^2 = 40^2 + 10^2 - 2(40(10)\cos 78$ ✓
$v = 39.1621$
Hence, the vessel should travel with a speed of 39.2 kmh^{-1} ✓

$\frac{\sin\theta}{40} = \frac{\sin 78}{39.1621} \Rightarrow \theta = 87.54^\circ$ ✓✓

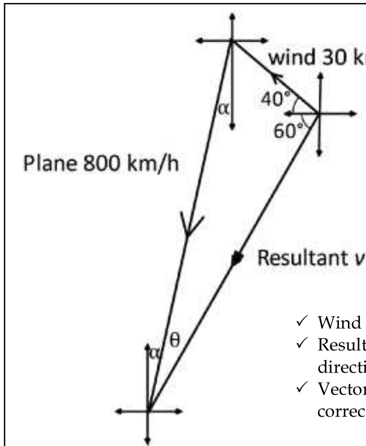
Hence, the vessel should travel in the direction $(180 + 48 + 87.5)^\circ$, that is 315.5°. ✓

Calculator Assumed

7. [9 marks: 3, 6]

A plane is to be flown from M to N. N is 3 000 km from M in the direction 210°.
A wind blows in the direction 310° at 30 km per hour.
The plane has a maximum speed of 800 km per hour.

(a) Draw a clearly labelled velocity diagram for the situation described above.



✓ Wind direction correct.
✓ Resultant in correct direction.
✓ Vector for plane correct.

(b) Determine which direction the plane should be flown for it to arrive at N in minimum time. State the minimum flight time (to the nearest minute).

Using the sine rule: $\frac{\sin\theta}{30} = \frac{\sin 100}{800}$ ✓

$\theta = 2.1164^\circ$ ✓

$\Rightarrow \alpha = (90 - 60 - 2.12) = 27.88^\circ$

Hence direction is 207.88° bearing. ✓

Using the cosine rule:

$800^2 = 30^2 + v^2 - 2 \times 30 \times v \times \cos 100$ ✓

$v = 794.2448$ (reject -804.6637) ✓

Hence, time of flight $= \frac{3000}{794.2448}$

$= 3.7772 = 3$ hours 47 minutes ✓

06 Components & Position Vectors I

Calculator Free

1. [8 marks: 2, 1, 1, 4]

Given that $\boldsymbol{a} = -2\,\mathbf{i} + 6\,\mathbf{j}$ and $\boldsymbol{b} = 5\,\mathbf{i} - 4\,\mathbf{j}$, find:

(a) $|\boldsymbol{a} + \boldsymbol{b}|$.

$$\begin{aligned} |\boldsymbol{a}+\boldsymbol{b}| &= |(-2\mathbf{i}+6\mathbf{j}) + (5\mathbf{i}-4\mathbf{j})| \quad ✓ \\ &= |3\mathbf{i}+2\mathbf{j}| \\ &= \sqrt{(3^2+2^2)} \\ &= \sqrt{13} \quad ✓ \end{aligned}$$

(b) the unit vector parallel to $\boldsymbol{a} + \boldsymbol{b}$.

Required unit vector = $\frac{1}{\sqrt{13}}(3\mathbf{i}+2\mathbf{j})$ ✓

(c) a vector that is parallel to $\boldsymbol{a} + \boldsymbol{b}$ but with a magnitude of 5.

$$\text{Required vector} = 5 \times \frac{1}{\sqrt{13}}(3\mathbf{i}+2\mathbf{j}) = \frac{5\sqrt{13}}{13}(3\mathbf{i}+2\mathbf{j}) \quad ✓$$

(d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = 2\mathbf{i} + \mathbf{j}$ and $\boldsymbol{q} = -3\mathbf{i} + 2\mathbf{j}$.

Let $\boldsymbol{a} = \mu\boldsymbol{p} + \lambda\boldsymbol{q}$

$(-2\mathbf{i}+6\mathbf{j}) = \mu(2\mathbf{i}+\mathbf{j}) + \lambda(-3\mathbf{i}+2\mathbf{j})$

$-2\mathbf{i}+6\mathbf{j} = (2\mu-3\lambda)\,\mathbf{i} + (\mu+2\lambda)\,\mathbf{j}$

Compare coefficients for **i** and **j** vectors:

$-2 = 2\mu - 3\lambda$ ✓

$6 = \mu + 2\lambda$ ✓

Solve simultaneously: $\mu = 2, \lambda = 2$

Hence, $\boldsymbol{a} = 2\boldsymbol{p} + 2\boldsymbol{q}$ ✓✓

Calculator Free

2. [6 marks]

$\mathbf{OA} = 3\mathbf{i} + 10\mathbf{j}$, $\mathbf{OB} = 5\mathbf{i} + b\mathbf{j}$ and $\mathbf{OC} = 9\mathbf{i} + c\mathbf{j}$.
Find c in terms of b if A, B and C are collinear.

$\mathbf{AB} = (5\mathbf{i}+b\mathbf{j}) - (3\mathbf{i}+10\mathbf{j}) = 2\mathbf{i} + (b-10)\,\mathbf{j}$ ✓

$\mathbf{AC} = (9\mathbf{i}+c\mathbf{j}) - (3\mathbf{i}+10\mathbf{j}) = 6\mathbf{i} + (c-10)\,\mathbf{j}$ ✓

For A, B and C to be collinear, $\mathbf{AB} = k\mathbf{AC}$.
Hence, $2\mathbf{i} + (b-10)\,\mathbf{j} = k[6\mathbf{i} + (c-10)\,\mathbf{j}]$ ✓

Compare **i** and **j** coefficients:

$2 = 6k \Rightarrow k = \frac{1}{3}$ ✓

$b - 10 = k(c-10)$ ✓

Hence, $b - 10 = \frac{1}{3}(c-10)$

$c = 3b - 20$ ✓

3. [7 marks]

Vector $a\mathbf{i} + (a+b)\mathbf{j}$ has a magnitude of 5 and is parallel to vector $4\mathbf{i} + 8\mathbf{j}$.
Find all possible values of a and b.

$a\mathbf{i} + (a+b)\mathbf{j} = \lambda(4\mathbf{i}+8\mathbf{j})$ ✓

Compare coefficients for **i** and **j** vectors:

$a = 4\lambda \Rightarrow \lambda = \frac{a}{4}$ ✓

$a + b = 8\lambda$ ✓

Hence, $a + b = 2a \Rightarrow a = b$ ✓

Magnitude of $a\mathbf{i} + (a+b)\mathbf{j}$ is 5.

Hence, $\sqrt{[a^2 + (a+b)^2]} = 5$ ✓

$\sqrt{(a^2 + 4a^2)} = 5$

$5a^2 = 25$

$a = b = \sqrt{5}$ ✓

$a = b = -\sqrt{5}$ ✓

Calculator Free

4. [5 marks]

Vector $a\,\mathbf{i} + 10\mathbf{j}$ is of the same magnitude as $(b-10)\,\mathbf{i} + (a-2b)\,\mathbf{j}$ but acts in the opposite direction. Find the values of a and b.

$a\,\mathbf{i} + 10\mathbf{j} = -[\,(b-10)\,\mathbf{i} + (a-2b)\,\mathbf{j}]$

$a\,\mathbf{i} + 10\mathbf{j} = -(b-10)\,\mathbf{i} - (a-2b)\,\mathbf{j}$ ✓

Compare **i** and **j** coefficients:

$a = -b + 10 \Rightarrow a + b = 10$ ✓

$10 = -a + 2b \Rightarrow -a + 2b = 10$ ✓

Solve simultaneously: $a = \frac{10}{3}$, $b = \frac{20}{3}$ ✓✓

5. [4 marks]

Vector $\begin{pmatrix} a \\ b \end{pmatrix}$ has magnitude 20 and is parallel to $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$.

Find the values of a and b.

$\left|\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}\right| = 2$ ✓

Hence, $\begin{pmatrix} a \\ b \end{pmatrix} = \pm 10\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$. ✓

$\Rightarrow a = b = \pm 10\sqrt{2}$ ✓✓

6. [3 marks]

The point K divides the line segment AB internally in the ratio 4 : 1. Use a vector method to find the position vector of K if $\mathbf{OB} = -\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{AB} = 15\,\mathbf{i} - 5\,\mathbf{j}$.

$\mathbf{OK} = \mathbf{OB} + \mathbf{BK}$

$= \mathbf{OB} + \frac{1}{5}\mathbf{BA}$ ✓

$= \langle -1, 2 \rangle + \frac{1}{5}\langle -15, 5 \rangle$ ✓

$= \langle -4, 3 \rangle$ ✓

Calculator Assumed

7. [4 marks]

The points A, C and K have position vectors $\langle -1, 4 \rangle$, $\langle 5, 10 \rangle$ and $\langle 2, -5 \rangle$ respectively. The point C divides the line segment AB internally in the ratio 2 : 3. Use a vector method to determine **BK**.

$3\mathbf{AC} = 2\mathbf{CB}$ ✓

$3\mathbf{OC} - 3\mathbf{OA} = 2\mathbf{OB} - 2\mathbf{OC}$

$5\begin{pmatrix} 5 \\ 10 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 4 \end{pmatrix} = 2\mathbf{OB}$ ✓

$\mathbf{OB} = \begin{pmatrix} 14 \\ 19 \end{pmatrix}$ ✓

$\mathbf{BK} = \mathbf{OK} - \mathbf{OB}$

$= \begin{pmatrix} 2 \\ -5 \end{pmatrix} - \begin{pmatrix} 14 \\ 19 \end{pmatrix}$

$= \begin{pmatrix} -12 \\ -24 \end{pmatrix}$ ✓

8. [4 marks]

It is known that $\mathbf{OA} = a\mathbf{i} + \mathbf{j}$ and $\mathbf{OB} = 4\mathbf{i} + b\mathbf{j}$.
K is a point such that AK: AB = 2 : 5 and $\mathbf{OK} = 4\mathbf{i} - 3\mathbf{j}$. Find a and b.

AK:AB = 2:5 $\Rightarrow$ $\mathbf{AK} = \frac{2}{5}\mathbf{AB}$ ✓

$\mathbf{AB} = \begin{pmatrix} 4 \\ b \end{pmatrix} - \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 4-a \\ b-1 \end{pmatrix}$

$\mathbf{AK} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 4-a \\ -4 \end{pmatrix}$

Hence $\begin{pmatrix} 4-a \\ -4 \end{pmatrix} = \frac{2}{5}\begin{pmatrix} 4-a \\ b-1 \end{pmatrix}$ ✓

Compare **i** and **j** coefficients:

$4 - a = \frac{2}{5}(4-a) \Rightarrow 4 - a = 0 \Rightarrow a = 4$ ✓

$-4 = \frac{2}{5}(b-1) \Rightarrow b = -9$ ✓

Calculator Assumed

9. [5 marks]

Vector $\boldsymbol{u}$ has magnitude 100 kmh^{-1} and acts in the direction 040°.
Vector $\boldsymbol{v}$ has magnitude 150 kmh^{-1} and acts in the direction 280°.
Let $\boldsymbol{i}$ be the unit vector in the West-East direction
and $\boldsymbol{j}$ be the unit vector in the South-North direction.

<u>Use vector components</u> to find the magnitude and direction of $\boldsymbol{u} - 2\boldsymbol{v}$.

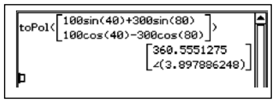
$\boldsymbol{u} = \begin{pmatrix} 100\sin 40 \\ 100\cos 40 \end{pmatrix}$ $\boldsymbol{v} = \begin{pmatrix} -150\sin 80 \\ 150\cos 80 \end{pmatrix}$ ✓✓

$\boldsymbol{u} - 2\boldsymbol{v} = \begin{pmatrix} 100\sin 40 + 300\sin 80 \\ 100\cos 40 - 300\cos 80 \end{pmatrix}$ ✓

$|\boldsymbol{u} - 2\boldsymbol{v}| = 360.6$ kmh^{-1} ✓

Direction 3.9° with the $\boldsymbol{i}$ vector. ✓

That is along bearing 086.1°.



10. [5 marks]

Given that $\boldsymbol{u} = \langle -4, 16 \rangle$ and $|\boldsymbol{v}| = 100$, find $\boldsymbol{v}$
if $\boldsymbol{u} + \boldsymbol{v}$ is to be in the same direction as the vector $\langle 2, 2 \rangle$.

$\begin{pmatrix} -4 \\ 16 \end{pmatrix} + \boldsymbol{v} = \lambda \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ ✓

$\boldsymbol{v} = \lambda \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 16 \end{pmatrix}$

$= \begin{pmatrix} 2\lambda + 4 \\ 2\lambda - 16 \end{pmatrix}$ ✓

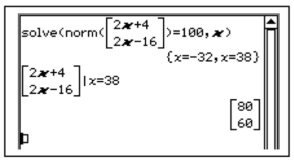
But $|\boldsymbol{v}| = 100$:

$\sqrt{(2\lambda+4)^2 + (2\lambda-16)^2} = 100$ ✓

$\lambda = -32, 38$ ✓

But $\lambda > 0$, hence, $\lambda = 38$

$\boldsymbol{v} = \begin{pmatrix} 80 \\ 60 \end{pmatrix}$ ✓

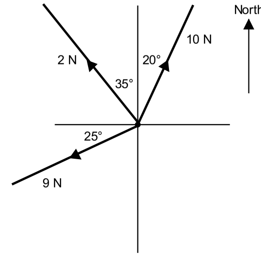


07 Components & Position Vectors II

Calculator Assumed

1. [6 marks: 3, 3]

The diagram below shows the forces acting on a body. The forces are all on the same plane.



(a) Find the magnitude of the resultant.

Vertical component of resultant
= 2 cos 35 + 10 cos 20 – 9 sin 25
= 7.2317 ✓

Horizontal component of resultant
= –2 sin 35 + 10 sin 20 – 9 cos 25
= –5.8837 ✓

Hence, magnitude of resultant
$= \sqrt{(7.2317^2 + 5.8837^2)}$
= 9.3228
= 9.32 N ✓

(b) Find the magnitude and the direction of a single force that will keep this system in equilibrium.

Resultant = – 5.8837 **i** + 7.2317 **j**

Hence, force required to maintain equilibrium
= –(– 5.8837 **i** + 7.2317 **j**)
= 5.8837 **i** – 7.2317 **j** ✓

Magnitude = 9.32 N ✓

$\tan\theta = \frac{7.2317}{5.8837}$

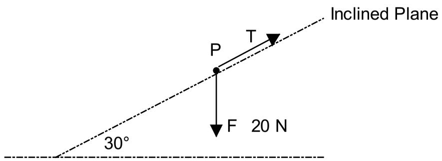
$\theta = 50.87°$

Hence, direction is 140.87°. ✓

Calculator Assumed

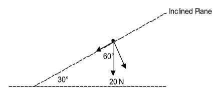
2. [5 marks: 2, 2, 1]

In the diagram below, a particle P is on a plane inclined at an angle of 30^o to the horizontal. A vertical force F of magnitude 20 N is acting on P as shown. Force T parallel to the inclined plane is applied to prevent P from slipping down the inclined plane.



(a) Find the magnitude of the component of F parallel to the inclined plane.

Magnitude of component of F parallel to the plane
= 20 cos 60 = 10 N ✓✓



(b) Find the magnitude of the component of F perpendicular to the inclined plane.

Magnitude of component of F perpendicular to the plane
= 20 sin 60 = 10√3 N ✓✓

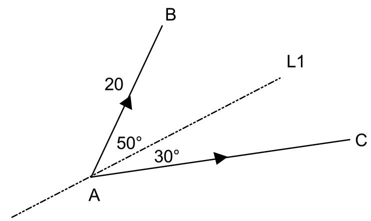
(c) Find the magnitude of force T.

Magnitude of T = magnitude of component of F parallel to the plane
= 10 N ✓

Calculator Assumed

3. [4 marks: 1, 1, 2]

In the diagram below, vector **AB** is of magnitude 20 units and is inclined at an angle of 50^o to the line L1. Vector **AC** is inclined at angle of 30^o to the line L1 as shown.



(a) Find the magnitude of the component of **AB** parallel to the line L1.

Magnitude of component of AB parallel to L1
= 20 cos 50
= 12.8557
= 12.86 units ✓

(b) Find the magnitude of the component of **AB** perpendicular to line L1.

Magnitude of component of F perpendicular to L1
= 20 sin 50
= 15.3208
= 15.32 units ✓

(c) Find the magnitude of **AC** if the resultant of the vectors **AB** and **AC** is parallel to the line L1.

Magnitude of Component of **AC** perpendicular to L1
= Magnitude of Component of **AB** perpendicular to L1

Hence, AC sin 30 = 15.3208
AC = 30.6416
= 30.64 units ✓

Hence, magnitude of **AC** is 30.64 units. ✓

Calculator Assumed

4. [11 marks: 1, 1, 4, 2, 3]

A light plane can fly at 80 km per hour in still air. The pilot wishes to fly from O to a neighbouring airstrip Q, located 40 km from O in the direction 060°. A constant wind of 20 km per hour is blowing from the North. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Write in terms of **i** and **j** the position vector of Q relative to O.

> $\mathbf{OQ} = 40\sin 60\,\mathbf{i} + 40\cos 60\,\mathbf{j}$
> $= 20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j}$ ✓

(b) Write in terms of **i** and **j** the velocity vector of the wind.

> $v = -20\,\mathbf{j}$ ✓

(c) Find the velocity vector the pilot should set so that the plane flies directly to Q.

> Let the velocity vector the pilot should set = v
> Resultant vector = $v + (-20\,\mathbf{j})$
>
> Hence: $v + (-20\,\mathbf{j}) = \lambda(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$ where $\lambda > 0$ ✓
> $v = (20\lambda\sqrt{3}\,\mathbf{i} + (20\lambda + 20)\,\mathbf{j}$
>
> Since speed of plane = 80, $|v| = 80$
> Hence, $(20\sqrt{3}\,\lambda)^2 + (20\lambda + 20)^2 = 6400$ ✓
> $4\lambda^2 + 2\lambda - 15 = 0$
> $\lambda = 1.7026$ (reject -2.20 as $\lambda > 0$) ✓
>
> Hence, the velocity vector the pilot should set is $58.98\,\mathbf{i} + 54.05\,\mathbf{j}$. ✓

(d) Find the resultant speed of the plane.

> Resultant velocity = $\lambda(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$
> $= 1.7026(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$ ✓
>
> Hence, resultant speed = $|1.7026(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})|$
> $= 68.104 \approx 68.1\ \text{kmh}^{-1}$ ✓

(e) Find the difference in flying time (to the nearest minute) caused by the wind.

> Travelling time in still air = $\frac{40}{80}$ = 0.5 hours = 30 minutes ✓
>
> Travelling time (with wind blowing) = $\frac{40}{68.104}$ = 0.5873 hours
> = 35.2 minutes ✓
> Hence, difference in travelling time = 35.2 – 30 = 9.5 = 5.2 minutes ✓

Calculator Assumed

5. [8 marks: 4, 2, 2]

A helicopter capable of flying at a speed of 100 km per hour in still air, takes off from O for a mining town located at A. The position vector of A relative to O is 200**i** – 300**j** km. Throughout the journey, the helicopter encounters a wind blowing with velocity 13**i** + 5**j** km per hour.

(a) Find the velocity vector the pilot should set so that the helicopter flies directly to A.

> Let the velocity vector be v.
>
> Hence, for $\lambda > 0$:
> $(13\,\mathbf{i} + 5\,\mathbf{j}) + v = \lambda(200\,\mathbf{i} - 300\,\mathbf{j})$ ✓
> $v = (200\lambda - 13)\,\mathbf{i} - (300\lambda + 5)\,\mathbf{j}$
>
> Since speed of plane $|v| = 100$,
> Hence, $(200\lambda - 13)^2 + (300\lambda + 5)^2 = 10\,000$ ✓
> $\lambda = 0.2832$ (reject -0.2663 as $\lambda > 0$) ✓
>
> Hence, the velocity vector the pilot should set is $43.645\,\mathbf{i} - 89.96\,\mathbf{j}$. ✓

(b) Find the time taken for the journey.

> Time taken = $\frac{1}{\lambda}$
> $= \frac{1}{0.2832}$ ✓
> = 3.53107 hours
> ≈ 3 hours 31.9 minutes ✓

(c) Find the velocity vector the pilot should set for a direct flight back to O. Assume that the wind blows with the same velocity throughout the flight back.

> For the return journey, $\lambda = -0.2663$ ✓
>
> Hence, the velocity vector the pilot should set is $-66.26\,\mathbf{i} + 74.89\,\mathbf{j}$. ✓

Calculator Assumed

6. [7 marks: 1, 6]

A drone is to fly from A to B where **AB** = < 20, –40 > m. A wind is blowing with a velocity of < 2, 3 > m/minute. The drone can maintain a speed of 50 m/minute. Let the velocity the drone should fly in order to reach B in minimum time be < x, y > m/minute.

(a) State in terms of x and y the resultant velocity of the drone.

Resultant velocity of drone = $\begin{pmatrix} x+2 \\ y+3 \end{pmatrix}$. ✓

(b) Calculate how long it will take the drone to reach B and the velocity < x, y >.

Let time of flight be t.

$$\begin{pmatrix} x+2 \\ y+3 \end{pmatrix} \times t = \begin{pmatrix} 20 \\ -40 \end{pmatrix} \quad ✓$$

$$x = \frac{20}{t} - 2 \quad ✓$$

$$y = \frac{-40}{t} - 3 \quad ✓$$

But $|< x, y >| = 50$.

Hence: $\left(\frac{20}{t} - 2\right)^2 + \left(\frac{-40}{t} - 3\right)^2 = 50^2$ ✓

$t = 0.9295$ ✓

Therefore: velocity of drone = $\begin{pmatrix} 19.5169 \\ -46.0339 \end{pmatrix}$ ✓

Calculator Assumed

7. [8 marks]

A drone is to be flown from P to Q with position vectors < 30, 70 > and < 50, 100 > respectively. A slight breeze is blowing with constant velocity < 1, 2 > ms^{-1}. Determine how long it will take the drone to be flown from P to Q at a speed of 2 ms^{-1}. State the direction the drone should be flown.

Let velocity of drone = $\begin{pmatrix} x \\ y \end{pmatrix}$.

Resultant velocity $v = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x+1 \\ y+2 \end{pmatrix}$. ✓

Displacement **PQ** = $\begin{pmatrix} 50-30 \\ 100-70 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$ ✓

Let time of flight be t.

$$t\begin{pmatrix} x+1 \\ y+2 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix} \quad ✓$$

$xt + t = 20$ (1)

$yt + 2t = 30$ (2) ✓

$x^2 + y^2 = 4$ (3)

Solve simultaneously:

```
xt+t=20
yt+2t=30
x^2+y^2=4 | t,x,y
{{x=-0.8679,y=-1.8019,t=151.4143},{x=1.3295,y=1.4942,t=8.5857}}
```

For $t = 8.59$ seconds: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.3295 \\ 1.4942 \end{pmatrix}$. ✓

Direction: along bearing 048.3°. ✓

or for $t = 151.41$ seconds:

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.8679 \\ -1.8019 \end{pmatrix}$. ✓

Direction: along bearing 244.3°. ✓

08 Components & Position Vectors III

Calculator Assumed

1. [6 marks: 1, 1, 1, 3]

A particle P, initially at 5**i** – 10**j** metres, moves with velocity 3**i** + 4**j** ms^{-1}.

(a) Find the position vector of P after 10 seconds.

> Position vector after 10 seconds
> **OP** = (5**i** – 10**j**) + 10(3**i** + 4**j**)
> = 35**i** + 30**j** ✓

(b) Find the distance travelled by P after 10 seconds.

> Distance travelled = |10(3**i** + 4**j**)|
> $= \sqrt{(30)^2 + (40)^2}$
> = 50 metres ✓

(c) Find the position vector of P after t seconds.

> **OP** = (5**i** – 10**j**) + t (3**i** + 4**j**)
> = (5 + 3t) **i** + (–10 + 4t) **j** ✓

(d) When is P at a point with position vector (65**i** + 70**j**) metres.

> (5 + 3t) **i** + (–10 + 4t) **j** = 65**i** + 70**j** ✓
>
> Compare coefficients of **i** and **j** vectors:
> $5 + 3t = 65 \Rightarrow t = 20$
> $-10 + 4t = 70 \Rightarrow t = 20$ ✓
>
> Hence P is at point with position vector (65**i** + 70**j**) at t = 20 seconds. ✓

Calculator Assumed

2. [9 marks: 3, 3, 3]

The position vector of particles A and B, t hours after 12 noon, are
r = 12**i** + 3**j** + t (3**i** + 4**j**) and **r** = –3**i** – 5**j** + t (2**i** + 6**j**) metres respectively.

(a) Find in terms of t, the distance between A and B t hours after 12 noon.

> **AB** = **OB** – **OA**
> = [(–3**i** – 5**j**) + t (2**i** + 6**j**)] – [(12**i** + 3**j**) + t (3**i** + 4**j**)]
> = (–15 – t) **i** + (–8 + 2t) **j** ✓✓
>
> Distance between A and B = |**AB**|
> $= \sqrt{(15 + t)^2 + (2t - 8)^2}$
> $= \sqrt{5t^2 - 2t + 289}$ metres ✓

(b) Find when A and B are 18 metres apart.

> When |**AB**| = 18, $5t^2 - 2t + 289 = 324$ ✓
> t = 2.8533 (reject –2.4533)
> = 2 hours 51 minutes ✓
> Hence, A and B are 18 metres apart at 2.51 pm. ✓

(c) Find when A is closest to B and find this distance.

> |**AB**| = $\sqrt{5t^2 - 2t + 289}$ metres
> |**AB**| is minimized when $t = -\frac{(-2)}{2(5)}$ ✓
> = 0.2 hours after 12 noon ✓
> = 12 minutes after 12 noon
> Minimum distance = $\sqrt{5(0.2)^2 - 2(0.2) + 289}$ = 16.99 m ✓
>
> OR
> ```
> fMin(√(5·t²-2·t+289),t)
>   {MinValue=16.99411663,t=0.2}
> ```
> ✓
> |AB| is minimized at 12.12 pm with ✓
> minimum distance 16.99 m. ✓

Calculator Assumed

3. [6 marks: 1, 1, 2, 2]

Particle P starts moving from the point A with position vector $-2\mathbf{i} + 3\mathbf{j}$ metres with velocity $\mathbf{i} - 2\mathbf{j}$ metres per second. Particle Q starts moving from the point A at the same time with velocity $2\mathbf{i} + 3\mathbf{j}$ metres per second.

(a) Determine the position vector of P after t seconds.

$\mathbf{OA} = (-2\mathbf{i} + 3\mathbf{j}) + t\,(\mathbf{i} - 2\mathbf{j})$
$= (-2 + t)\,\mathbf{i} + (3 - 2t)\,\mathbf{j}$ ✓

(b) Determine the position vector of Q after t seconds.

$\mathbf{OQ} = (-2\mathbf{i} + 3\mathbf{j}) + t\,(2\mathbf{i} + 3\mathbf{j})$
$= (-2 + 2t)\,\mathbf{i} + (3 + 3t)\,\mathbf{j}$ ✓

(c) Find in terms of t, the distance between P and Q after t seconds.

$\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$
$= [(-2 + 2t)\,\mathbf{i} + (3 + 3t)\,\mathbf{j}] - [(-2 + t)\,\mathbf{i} + (3 - 2t)\,\mathbf{j}]$
$= t\,\mathbf{i} + 5t\,\mathbf{j}$ ✓

Distance $|\mathbf{PQ}| = \sqrt{[t^2 + (5t)^2]}$
$= t\sqrt{26}$ metres ✓

(d) Use your answer in (c) to find when P and Q are 10 metres apart.

$t\sqrt{26} = 10$
$t = 1.96$ seconds ✓✓

Calculator Assumed

4. [7 marks: 1, 2, 1, 3]

Particle P starts moving from the point with position vector $3\mathbf{i} + 5\mathbf{j}$ metres with velocity $2\mathbf{i} - 3\mathbf{j}$ metres per second.

(a) Determine the position vector of P after 3 seconds.

$\mathbf{OP} = (3\mathbf{i} + 5\mathbf{j}) + 3 \times (2\mathbf{i} - 3\mathbf{j})$
$= 9\,\mathbf{i} - 4\,\mathbf{j}$ ✓

(b) Find the distance from P to the point with position vector $-2\mathbf{i} + \mathbf{j}$ after 3 seconds.

Let fixed point be B.

$\mathbf{PB} = \mathbf{OB} - \mathbf{OP}$
$= [-2\,\mathbf{i} + \mathbf{j}] - [9\,\mathbf{i} - 4\,\mathbf{j}]$
$= -11\,\mathbf{i} + 5\,\mathbf{j}$ ✓

Distance $|\mathbf{PB}| = \sqrt{[11^2 + 5^2]}$
$= 12.08$ metres ✓

(c) Determine the position vector of P after t seconds.

$\mathbf{OP} = (3\mathbf{i} + 5\mathbf{j}) + t\,(2\mathbf{i} - 3\mathbf{j})$
$= (3 + 2t)\,\mathbf{i} + (5 - 3t)\,\mathbf{j}$ ✓

(d) Find when P is closest to the origin and state this distance.

Distance $|\mathbf{OP}| = \sqrt{[(3 + 2t)^2 + (5 - 3t)^2]}$
$= \sqrt{(13t^2 - 18t + 34)}$ ✓

Distance is minimum when $t = -\dfrac{-18}{2(13)} = 0.69$ ✓

Hence, $|\mathbf{OP}| = 5.27$ metres ✓

OR

```
fMin(√((3+2t)²+(5-3t)²),t)
  {MinValue=5.269651864,t=0.6923076923}
```

Calculator Assumed

5. [9 marks: 2, 2, 2, 3]

A speed boat is moving at a constant velocity of 40 kmh^{-1} in the direction with bearing 060°. Initially, the speed boat is $5\sqrt{2}$ km from a buoy and is in the direction with bearing 225° from the buoy. Given that **i** and **j** are the unit vectors in the Easterly direction and Northerly direction respectively, find:

(a) the initial position vector of the speed boat with respect to the buoy in terms of **i** and **j**.

> $\mathbf{OP} = -5\sqrt{2}\sin 45\,\mathbf{i} - 5\sqrt{2}\cos 45\,\mathbf{j})$
> $= -5\,\mathbf{i} - 5\,\mathbf{j}$ ✓✓

(b) the direction vector of the speed boat in terms of **i** and **j**.

> $v = 40\cos 30\,\mathbf{i} + 40\sin 30\,\mathbf{j}$
> $= 20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j}$ ✓✓

(c) the position vector of the speed boat with respect to the buoy t hours later.

> $\boldsymbol{r} = (-5\,\mathbf{i} - 5\,\mathbf{j}) + t\,(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$
> $= (-5 + 20\sqrt{3}t)\,\mathbf{i} + (-5 + 20t)\,\mathbf{j}$ ✓✓

(d) the time when the speed boat is nearest to the buoy and the least distance between the speed boat and the buoy.

> Distance to buoy $= |\boldsymbol{r}|$
> $= \sqrt{[(-5 + 20\sqrt{3}t)^2 + (-5 + 20t)^2]}$ ✓
> $= \sqrt{[1600t^2 - (200\sqrt{3} + 200)\,t + 50)]}$
> Distance is minimum when $t = -\dfrac{-(200\sqrt{3} + 200)}{2(1600)} = 0.1708$ hours
> $= 10.2$ minutes ✓
> Hence, $|\boldsymbol{r}| = 1.8301$ km ✓
>
> Hence, speed boat is nearest the buoy after 10.2 minutes.
> Nearest distance = 1.83 km.
>
> OR
> fMin($\sqrt{(-5+20\sqrt{3}\,t)^2+(-5+20t)^2}$,t)
> {MinValue=1.830127019,t=0.1707531755}

Calculator Assumed

6. [9 marks: 4, 2, 3]

Yacht A starts sailing from the point L with position vector 5**i** – 2**j** metres with velocity –**i** + 3**j** . Yacht B starts sailing 10 seconds later with velocity 2**i** + **j** metres per second, from the point M with position vector 4**i** – 3**j** metres per second. t is time in seconds from the moment B starts sailing.

(a) Find in terms of t, the distance between A and B after t seconds.

> $\mathbf{OA} = (5\mathbf{i} - 2\mathbf{j}) + (t + 10)\,(-\mathbf{i} + 3\mathbf{j})$
> $= (-5 - t)\,\mathbf{i} + (28 + 3t)\,\mathbf{j}$ ✓
>
> $\mathbf{OB} = (4\mathbf{i} - 3\mathbf{j}) + t\,(2\mathbf{i} + \mathbf{j})$
> $= (4 + 2t)\,\mathbf{i} + (-3 + t)\,\mathbf{j}$ ✓
>
> $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$
> $= (9 + 3t)\,\mathbf{i} + (-31 - 2t)\,\mathbf{j}$ ✓
>
> Distance between A and B = $|\mathbf{AB}|$
> $= \sqrt{(13t^2 + 178t + 1042)}$ ✓

(b) When will A and B be 40 metres apart?

> $\sqrt{(13t^2 + 178t + 1042)} = 40$ ✓
> $t = 2.63$ (reject -16.3 as $t > 0$) ✓
> A and B will be 40 metres apart 2.6 seconds after B starts moving.

(c) When will the two yachts be closest together? State this distance.

> Use *f Min*($\sqrt{(13t^2 + 178t + 1042)}$, t, 0, ∞) ✓
>
> Hence, yachts will be closest together before B starts moving. ✓
> Closest distance = 32.3 metres. ✓
>
> fMin($\sqrt{13t^2+178t+1042}$,t,0,∞
> {MinValue=32.28002478,t=0}

Calculator Assumed

7. [9 marks: 3, 3, 3]

At 0800 hours, object P is at the point with position vector $40\,\boldsymbol{i} - 70\boldsymbol{j}$ km and moving with constant velocity $-5\,\boldsymbol{i} + 2\,\boldsymbol{j}$ kmh^{-1}. At 0900 hours, object Q starts moving from a point with position vector $-64\,\boldsymbol{i} - 24\boldsymbol{j}$ km with constant velocity $4\,\boldsymbol{i} - 2\,\boldsymbol{j}$ km h^{-1}.

(a) Calculate the distance between P and Q at 1000 hours.

At 1000 hours:

$\mathbf{OP} = \begin{pmatrix}40\\-70\end{pmatrix} + 2\begin{pmatrix}-5\\2\end{pmatrix} = \begin{pmatrix}30\\-66\end{pmatrix}$ ✓

$\mathbf{OQ} = \begin{pmatrix}-64\\-24\end{pmatrix} + \begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}-60\\-26\end{pmatrix}$

$\mathbf{PQ} = \begin{pmatrix}-60\\-26\end{pmatrix} - \begin{pmatrix}30\\-66\end{pmatrix} = \begin{pmatrix}-90\\40\end{pmatrix}$ ✓

$|\mathbf{PQ}| = 98.4886$ km ✓

(b) Determine an expression for the distance between P and Q at time t hours after 1000 hours.

t hours after 1000 hours:

$\mathbf{OP} = \begin{pmatrix}30\\-66\end{pmatrix} + t\begin{pmatrix}-5\\2\end{pmatrix}$ $\quad$ $\mathbf{OQ} = \begin{pmatrix}-60\\-26\end{pmatrix} + t\begin{pmatrix}4\\-2\end{pmatrix}$ ✓

$\mathbf{PQ} = \begin{pmatrix}30-5t\\-66+2t\end{pmatrix} - \begin{pmatrix}-60+4t\\-26-2t\end{pmatrix} = \begin{pmatrix}90-9t\\-40+4t\end{pmatrix}$ ✓

$|\mathbf{PQ}| = \sqrt{(90-9t)^2+(-40+4t)^2} = \sqrt{97}\,|x-10|$ ✓

(c) If the directions of P and Q remain unchanged, use your answer in (b) to determine if P and Q will collide. If they do collide, provide the time and place of collision.

Distance between P and Q $= \sqrt{97}\,|x-10|$

At collision, PQ = 0

$\sqrt{97}\,|x-10| = 0$ ✓

$t = 10$ ✓

Hence collision at 2000 hours.

Collide at $\left.\begin{pmatrix}30-5t\\-60+2t\end{pmatrix}\right|_{t=10} = \begin{pmatrix}-20\\-46\end{pmatrix}$. ✓

Calculator Assumed

8. [6 marks]

At 1400 hours, P is at the point with position vector $-5\boldsymbol{i} + 25\boldsymbol{j}$ metres and moving with constant velocity v metres per minute.

(a) Find v if P arrives at < 20, 75 > after 5 minutes.

$\begin{pmatrix}-5\\25\end{pmatrix} + 5v = \begin{pmatrix}20\\75\end{pmatrix}$ ✓

$v = \begin{pmatrix}5\\10\end{pmatrix}$ ✓

(b) Q which starts off from < 10, –20 > metres at 1400 hours travelling with a velocity of < – 4, 8 > metres per minute. P has a maximum speed of 12 metres per minute. Find $\boldsymbol{v}$ for P to intercept Q.

Let $v = \begin{pmatrix}x\\y\end{pmatrix}$,

$\mathbf{OP}(t) = \begin{pmatrix}-5+xt\\25+yt\end{pmatrix}$ ✓

$\mathbf{OQ}(t) = \begin{pmatrix}10-4t\\-20+8t\end{pmatrix}$ ✓

At interception, $\mathbf{OP}(t) = \mathbf{OQ}(t)$:

$-5 + xt = 10 - 4t$ (1)

$25 + yt = -20 + 8t$ (2)

Also $x^2 + y^2 = 144$ (3) ✓

Solve simultaneously:

$x = 2.57,\ y = -11.72,\ t = 2.28$ (reject $t < 0$)

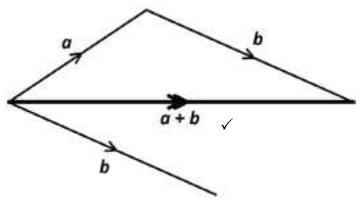
Hence: v = < 2.57, –11.72 > metres per minute. ✓

09 Relative Displacement

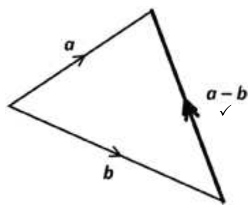
Calculator Free

1. [4 marks: 1, 1, 1, 1]

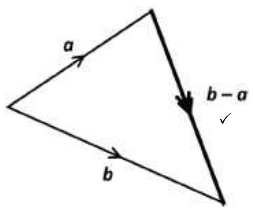
(a) Indicate clearly in the diagram below $\boldsymbol{a} + \boldsymbol{b}$.



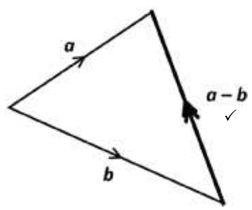
(b) Indicate clearly in the diagram below $\boldsymbol{a} - \boldsymbol{b}$.



(c) Indicate clearly in the diagram below the vector $\boldsymbol{b}$ relative to $\boldsymbol{a}$.



(d) Indicate clearly in the diagram below the vector $\boldsymbol{a}$ relative to $\boldsymbol{b}$.



Calculator Free

2. [4 marks: 1, 1, 2]

The position vectors of the points A, B, and C with respect to the origin O are $\mathbf{i} + \mathbf{j}$, $2\mathbf{i} - \mathbf{j}$ and $-4\mathbf{i} + 5\mathbf{j}$ respectively.

(a) Find the position vector of A relative to B.

$${}_A r_B = \mathbf{BA} = \mathbf{OA} - \mathbf{OB}$$
$$= (\mathbf{i} + \mathbf{j}) - (2\mathbf{i} - \mathbf{j})$$
$$= -\mathbf{i} + 2\mathbf{j} \quad ✓$$

(b) Find the displacement of C relative to A.

$$\mathbf{AC} = \mathbf{OC} - \mathbf{OA}$$
$$= (-4\mathbf{i} + 5\mathbf{j}) - (\mathbf{i} + \mathbf{j})$$
$$= -5\mathbf{i} + 4\mathbf{j} \quad ✓$$

(c) The position vector of the point D relative to C is $-2\mathbf{i} - 3\mathbf{j}$. Find the position vector of D relative to O.

Position vector of D relative to C, $\mathbf{CD} = -2\mathbf{i} - 3\mathbf{j}$

But $\mathbf{CD} = \mathbf{OD} - \mathbf{OC}$.

$\Rightarrow -2\mathbf{i} - 3\mathbf{j} = \mathbf{OD} - (-4\mathbf{i} + 5\mathbf{j})$ ✓

$\Rightarrow \mathbf{OD} = (-2\mathbf{i} - 3\mathbf{j}) + (-4\mathbf{i} + 5\mathbf{j})$

$= -6\mathbf{i} + 2\mathbf{j}$ ✓

---

3. [5 marks: 2, 3]

The position vector the point A relative to the point B is < 2, 8 >.
The position vector of the point C relative to B is < –5, –2 >.

(a) Find the position vector of A relative to C.

BA = < 2, 8 >   **BC** = < –5, – 2 >

CA = **CB** + **BA**

= < 5, 2 > + < 2, 8 > ✓

= < 7, 10 >. ✓

(b) If in addition, the position vector of D relative to C is < 1, 5 > and the position vector of A is < –10, 2 >, find the position vector of D.

CD = < 1, 5 >

OD = **OA** + **AB** + **BC** + **CD** ✓

= < –10, 2 > + < –2, –8 > + < –5, –2 > + < 1, 5 > ✓

= < –16, –3 >. ✓

Calculator Assumed

4. [5 marks: 2, 1, 2]

The position vector of Peter relative to a flag pole is 20**i** + 40**j** metres. Relative to Peter, Joe has position vector 5**i** – 15**j** metres.

(a) Find the position vector of Joe relative to the flagpole.

Let O: the flagpole, J : Joe and P: Peter

Position vector of Joe relative to Peter, **PJ** = 5**i** – 15**j**
But **PJ** = **OJ** – **OP** .
⇒ (5**i** – 15**j**) = **OJ** – (20**i** + 40**j**) ✓
⇒ **OJ** = (5**i** – 15**j**) + (20**i** + 40**j**)
= 25**i** + 25**j** ✓

(b) Hence, find the distance between Joe and the flag pole.

Distance = $\sqrt{(25^2 + 25^2)}$
= 35.4 m ✓

(c) The position vector of Kelly relative to Joe is a**i** + 20**j** metres. Find the value of a if the distance between Kelly and Joe is 50 metres.

JK = a**i** + 20**j**

Hence, $a^2 + 20^2 = 50^2$ ✓
$a = \pm 45.8$ m ✓

---

5. [4 marks]

Vectortown has position vector –20i + 10j km. The position vector of Trigtown relative to Vectortown is –40i – 15j km. The position vector of Easytown relative to Trigtown is 4i + 70j km. Find the position vector of Easytown.

Let V: Vectortown and T: Trigtown
OV = –20**i** + 10**j**
Position vector of T relative to V, **VT** = –40**i** – 15**j**
Position vector of E relative to T, **TE** = 4**i** + 70**j**

VT = **OT** – **OV** ⇒ **OT** = **VT** + **OV**
= (–40**i** – 15**j**) + (–20**i** + 10**j**)
= –60**i** – 5**j** ✓✓

Using **TE** = **OE** – **OT**
⇒ **OE** = **TE** + **OT**
= (4**i** + 70**j**) + (–60**i** – 5**j**)
= –56**i** + 65**j** ✓✓

10 Relative Velocity

Calculator Assumed

1. [7 marks: 1, 1, 3, 2]

Relative to an observer at O, A is moving with velocity 6**i** + 9**j** ms^{-1} and B is moving with velocity –3**i** + 4**j** ms^{-1} .

(a) Find the velocity of A relative to B

${}_A v_B = v_A - v_B$
= (6**i** + 9**j**) – (–3**i** + 4**j**)
= 9**i** + 5**j** ✓

(b) What is the speed of A relative to B?

Speed = $\sqrt{(9^2 + 5^2)}$
= 10.3 ms^{-1} . ✓

(c) The velocity of C relative to B is 4**i** – 5**j** ms^{-1}. Find the velocity of C relative to A.

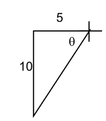
${}_C v_B = v_C - v_B \Rightarrow v_C = {}_C v_B + v_B$
= (4**i** – 5**j**) + (–3**i** + 4**j**)
= **i** – **j** ✓✓

${}_C v_A = v_C - v_A$
= (**i** – **j**) – (6**i** + 9**j**)
= –5**i** – 10**j** ✓

(d) In what direction is C moving relative to A?

${}_C v_A$ = –5**i** – 10**j**

In the triangle sketched,
tan θ = 2
⇒ θ = 63.4° ✓



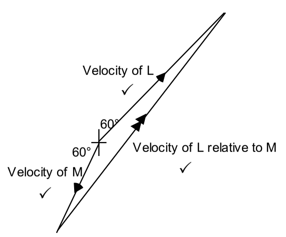
Hence, direction of C relative to A
is along bearing 270° – 63.4° = 206.6° . ✓

Calculator Assumed

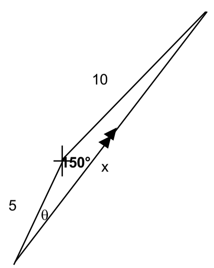
2. [7 marks: 3, 4]

L is travelling along bearing 060° with a speed of 10 kmh^{-1}. M is travelling along bearing 210° with a speed of 5 kmh^{-1}.

(a) Draw a clearly labelled vector diagram indicating the velocity vector of L relative to M.



(b) Use trigonometry to find the speed and direction of L relative to M.



In the diagram above:

$x^2 = 10^2 + 5^2 - 2(10)(5)\cos 150$

$x = 14.5466$ ✓

Also: $\frac{\sin\theta}{10} = \frac{\sin 150}{14.5466}$

$\Rightarrow \sin\theta = 0.3437$

$\theta = 20.1°$ ✓✓

Hence, relative to M, L is travelling at a speed of 14.5 kmh^{-1} along the bearing 050.1° . ✓

Calculator Assumed

3. [7 marks: 2, 2, 1, 2]

P is travelling along bearing 045° with a speed of 20 kmh^{-1}. Q is travelling along bearing 240° with a speed of 15 kmh^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Express the velocities of P and Q in terms of **i** and **j**.

$v_P = 20\cos 45\,\mathbf{i} + 20\sin 45\,\mathbf{j}$

$= 10\sqrt{2}\,\mathbf{i} + 10\sqrt{2}\,\mathbf{j}$ ✓

$v_Q = -15\cos 30\,\mathbf{i} - \quad \mathbf{j}$

$= -\frac{15\sqrt{3}}{2}\,\mathbf{i} - \frac{15}{2}\,\mathbf{j}$ ✓

(b) Find the velocity of Q relative to P.

${}_Qv_P = v_Q - v_P$

$= (-\frac{15\sqrt{3}}{2}\,\mathbf{i} - \frac{15}{2}\,\mathbf{j}) - (10\sqrt{2}\,\mathbf{i} + 10\sqrt{2}\,\mathbf{j})$

$= -27.1325\mathbf{i} - 21.6421\mathbf{j}$

$= -27.1\mathbf{i} - 21.6\mathbf{j}$ ✓✓

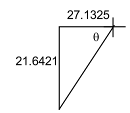
(c) What is the speed of Q relative to P?

Speed of Q relative to P $= \sqrt{[(-27.1325)^2 + (-21.6421)^2]}$

$= 34.7$ kmh^{-1} ✓

(d) What is the direction of Q relative to P?

${}_Qv_P = -27.1325\mathbf{i} - 21.6421\mathbf{j}$



In the triangle sketched,

$\tan\theta = \frac{21.6421}{27.1325}$

$\Rightarrow \theta = 38.6°$ ✓

Hence, direction of Q relative to P is along bearing $270° - 38.6° = 231.4°$. ✓

11 Relative Vectors

Calculator Assumed

1. [4 marks: 2, 2]

James is running along bearing 050° with a speed of 5 ms^{-1}. Wesley is running along bearing 300° with speed 4 ms^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

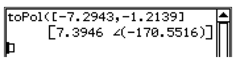
(a) Find in component form the velocity of Wesley relative to James.

Velocity of James v_J = 5 sin 50 **i** + 5 cos 50 **j**
Velocity of Wesley v_W = –4 sin 60 **i** + 4 cos 60 **j**
Hence, velocity of Wesley relative to James

$${}_Wv_J = v_W - v_J$$
$$= (-4 \sin 60\, \mathbf{i} + 4 \cos 60\, \mathbf{j}) - (5 \sin 50\, \mathbf{i} + 5 \cos 50\, \mathbf{j})$$
$$= -7.2943\mathbf{i} - 1.2139\mathbf{j}$$
$$= -7.3\mathbf{i} - 1.2\mathbf{j}$$ ✓✓

(b) Find how fast and in what direction is Wesley moving away from James.

${}_Wv_J = -7.3\mathbf{i} - 1.2\mathbf{j}$
= Polar[7.3946, –170.6°]
Wesley moving away from James at 7.4 ms^{-1} ✓
along 90 + 170.6 = 260.6°. ✓

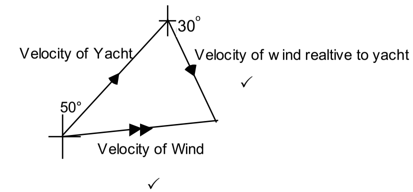


Calculator Assumed

2. [6 marks: 2, 4]

A yacht is sailing on a bearing of 050° with speed 12 kmh^{-1}. Sarah on the yacht measures the wind as blowing with a speed of 10 kmh^{-1} from a bearing of 300°.

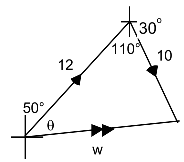
(a) Sketch a clearly labelled velocity vector diagram that shows the relationship between the velocity of the yacht, the true velocity of the wind and the velocity of wind relative to the yacht.



(b) Find the true speed and direction of the wind.

In the accompanying diagram:

$w^2 = 12^2 + 10^2 - 2(12)(10) \cos 110$
$w = 18.0578$ ✓



Also: $\frac{\sin\theta}{10} = \frac{\sin 110}{18.0578}$
$\Rightarrow \sin\theta = 0.5204$
$\theta = 31.36°$ ✓

Hence, wind is blowing with a speed of 18.1 kmh^{-1} along bearing (050 + 31.36)° = 081.4°. ✓✓

OR

${}_wv_y$ = < 10 cos 30, –10 sin 30 >, v_y = < 12 sin 50, 12 cos 50 >

$v_w = {}_wv_y + v_y$
= < 10 cos 30 + 12 sin 50, –10 sin 30 + 12 cos 50 > ✓
= < 17.8528, 2.7135 > ✓
= Polar[18.0578, 8.6423°]

Hence, wind is blowing with a speed of 18.1 kmh^{-1} along bearing (050 + 31.36)° = 081.4°. ✓✓

Calculator Assumed

3. [7 marks: 2, 2, 3]

A yacht Y is moving with velocity $2\mathbf{i} + 5\mathbf{j}$ kmh^{-1}. A sailor on board the yacht measures the wind as blowing with velocity $-3\mathbf{i} - 2\mathbf{j}$ kmh^{-1}.

(a) Find the velocity of the wind.

$v_y = \langle 2, 5 \rangle$

${}_w v_y = \langle -3, -2 \rangle \quad \Rightarrow \quad v_w - v_y = \langle -3, -2 \rangle$

Hence, $v_w = \langle 2, 5 \rangle + \langle -3, -2 \rangle$ ✓

$v_w = \langle -1, 3 \rangle$ ✓

To a sailor on a second yacht Z, the wind appears to be blowing with velocity $2\mathbf{i} + 4\mathbf{j}$ kmh^{-1}.

(b) Find the velocity of the second yacht.

$v_w = \langle -1, 3 \rangle$

${}_w v_z = \langle 2, 4 \rangle \quad \Rightarrow \quad v_w - v_z = \langle 2, 4 \rangle$

Hence, $v_z = \langle -1, 3 \rangle - \langle 2, 4 \rangle$ ✓

$v_z = \langle -3, -1 \rangle$ ✓

(c) How fast is yacht Z moving away from yacht Y and in what direction?

${}_z v_y = \langle -3, -1 \rangle - \langle 2, 5 \rangle$

$= \langle -5, -6 \rangle$ ✓

$= \text{Polar}\,[7.8102, -129.8056^\circ]$

Z is moving away with a speed of 7.8 kmh^{-1} ✓

along bearing 219.8°. ✓

```
toPol([-5,-6]
      [7.8102 ∠(-129.8056)]
```


Calculator Assumed

4. [4 marks]

A ship is travelling with a speed of 20 knots along bearing 080°. Relative to the ship, the wind is blowing from 310° with a speed of 8 knots. Let **i** and **j** be unit vectors in the Easterly and Northerly directions respectively. By expressing the given velocities in component form, find the true speed and direction of the wind.

Velocity of Ship $v_s = 20 \sin 80\, \mathbf{i} + 20 \cos 80\, \mathbf{j}$

Velocity of Wind relative to the ship ${}_W v_s = 8 \sin 50\, \mathbf{i} - 8 \cos 50\, \mathbf{j}$

${}_W v_s = \mathbf{v}_W - \mathbf{v}_s$

$\Rightarrow v_W = (8 \sin 50\, \mathbf{i} - 8 \cos 50\, \mathbf{j}) + (20 \sin 80\, \mathbf{i} + 20 \cos 80\, \mathbf{j})$

$= 25.8245\, \mathbf{i} - 1.6693\, \mathbf{j}$ ✓✓

$= \text{Polar}[25.8784, -3.698^\circ]$

Hence, true speed of the wind is 25.9 knots along bearing $(90 + 3.6984) = 093.7^\circ$. ✓✓

```
toPol([25.8245,-1.6693]
      [25.8784 ∠(-3.6985)]
```

Calculator Assumed

5. [11 marks: 2, 2, 7]

May is running along 025° at 4 kmh^{-1}.
Relative to May, Fay is running with speed a kmh^{-1} along bearing 150°.

(a) Find in terms of a, the velocity of Fay, v_f.

> ${}_f v_m = \langle a\sin 30, -a\cos 30\rangle$, $v_m = \langle 4\sin 25, 4\cos 25\rangle$
> $v_f = {}_f v_m + v_m$
> $= \langle a\sin 30 + 4\sin 25, -a\cos 30 + 4\cos 25\rangle$. ✓✓

Jane is running due East at 2 kmh^{-1}.
Relative to Jane, Fay is running with speed b kmh^{-1} along bearing 120°.

(b) Find in terms of b, the velocity of Fay, v_f.

> ${}_f v_j = \langle b\sin 60, -b\cos 60\rangle$, $v_j = \langle 2, 0\rangle$
> $v_f = {}_f v_j + v_j$
> $= \langle b\sin 60 + 2, -b\cos 60\rangle$. ✓✓

(c) Use your answers in (a) & (b) to find a and b. Hence, find the true speed and direction with which Fay is running.

> From (a), $v_f = \langle a\sin 30 + 4\sin 25, -a\cos 30 + 4\cos 25\rangle$. I
> From (b), $v_f = \langle b\sin 60 + 2, -b\cos 60\rangle$. II
>
> Compare I with II:
> $a\sin 30 + 4\sin 25 = b\sin 60 + 2$ III ✓
> $-a\cos 30 + 4\cos 25 = -b\cos 60$ IV ✓
>
> Solve III & IV simultaneously:
> $\Rightarrow a = 5.9696, b = 3.0891$ ✓✓
>
> Hence, $v_f = \langle 3.0891\sin 60 + 2, -3.0891\cos 60\rangle$
> $= \langle 4.6752, -1.5446\rangle$ ✓
> Hence, true speed for Fay = 4.9 kmh^{-1} ✓
> Direction = 90 + 18.3 = 108.3° ✓

```
⎧ a×sin(30)+4×sin(25)=b×sin(60)+2
⎩ -a×cos(30)+4×cos(25)=-b×cos(60) | a,b
                    {a=5.9696,b=3.0891}
[b×sin(60)+2,-b×cos(60)]|b=3.08911
                      [4.6752 -1.5446]
toPol(
                  [4.9238 ∠(-18.2819)]
```


Calculator Assumed

6. [6 marks: 1, 1, 4]

At 0900 hours, P is located at 20**i** – 10**j** km and is travelling with a constant velocity of –4**i** + 6**j** kmh^{-1}. At the same time, Q is located at 50**i** + 40**j** km and is travelling with a constant velocity of –10**i** – 4**j** kmh^{-1}.

(a) Find the displacement of Q relative to P at 0900 hours.

> ${}_Q r_P = r_Q - r_P$
> $= (50\mathbf{i} + 40\mathbf{j}) - (20\mathbf{i} - 10\mathbf{j})$
> $= 30\mathbf{i} + 50\mathbf{j}$ ✓

(b) Find the velocity of P relative to Q at 0900 hours.

> ${}_P v_Q = v_P - v_Q$
> $= (-4\mathbf{i} + 6\mathbf{j}) - (-10\mathbf{i} - 4\mathbf{j})$
> $= 6\mathbf{i} + 10\mathbf{j}$ ✓

(c) Use your answers in (a) and (b) to determine when and where P will collide with Q.

> For P to collide with Q, ${}_Q r_P = t\,{}_P v_Q$.
> Hence, $30\mathbf{i} + 50\mathbf{j} = t\,(6\mathbf{i} + 10\mathbf{j})$ ✓
> $\Rightarrow t = 5$ ✓
>
> P will collide with Q after 5 hours ($t = 5$), ie at 1400 hours. ✓
> The collision will take place at $(20\mathbf{i} - 10\mathbf{j}) + 5(-4\mathbf{i} + 6\mathbf{j}) = 20\mathbf{j}$. ✓

Calculator Assumed

7. [11 marks: 2, 2, 5, 2]

When the clock struck one, relative to a clock tower, the position vector of a cat is $6\mathbf{i} + 10\mathbf{j}$ m. The cat is running with velocity $2\mathbf{i} + 2\mathbf{j}$ ms^{-1}. At the same time, relative to the same clock tower, the position vector of a dog is $-4\mathbf{i} - 6\mathbf{j}$. The dog can run at 5 ms^{-1}.

(a) Find the position vector of the cat relative to the dog.

> ${}_{C}r_{D} = r_C - r_D$ [where C: cat, D: dog]
> $= (6\mathbf{i} + 10\mathbf{j}) - (-4\mathbf{i} - 6\mathbf{j}) = 10\mathbf{i} + 16\mathbf{j}$ ✓

Let the velocity vector of the dog for it to intercept the cat be $x\mathbf{i} + y\mathbf{j}$ ms^{-1}.

(b) Find in terms of x and y, the velocity of the dog relative to the cat.

> ${}_{D}v_{C} = v_D - v_C$
> $= (x\mathbf{i} + y\mathbf{j}) - (2\mathbf{i} - 2\mathbf{j}) = (x - \ )\mathbf{i} + (y - \ )\mathbf{j}$ ✓

(c) Use your answers in (a) and (b) to find x and y for the dog to intercept the cat. Assume that the cat continues running with the same speed and in the same direction.

> For the dog to intercept the cat, ${}_{C}r_{D} = t\ {}_{D}v_{C}$ where $t > 0$.
> Hence, $10\mathbf{i} + 16\mathbf{j} = t\,[(x - \ \ + (y - \ \ \mathbf{j}]$
> $\Rightarrow t(x-2) = 10$ I ✓
> $t(y-2) = 16$ II ✓
> But, speed of dog is 5.
> Hence, $x^2 + y^2 = 25$ III ✓
>
> Solve I, II and III simultaneously:
> $x = 3.1678$, $y = 3.8685$, $t = 8.5631$ ✓✓
>
> ```
> t(x-2)=10
> t(y-2)=16
> x^2+y^2=5^2 | x,y,t
> {{x=-2.0891,y=-4.5426,t=-2.4455},{x=3.1678,y=3.8685,t=8.5631}}
> ```

(d) When and where does the interception take place.

> Hence, interception will occur after 8.6 seconds at ✓
> $(6\mathbf{i} + 10\mathbf{j}) + 8.56\,(2\mathbf{i} + 2\mathbf{j}) = 23.12\mathbf{i} + 27.12\mathbf{j}$. ✓

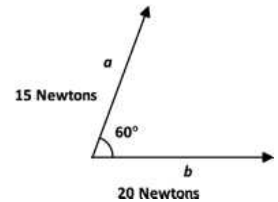
12 Scalar Product I

Calculator Free

1. [4 marks: 2, 2]

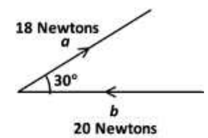
Find the scalar product between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as given:

(a)



> Scalar product = $15 \times 20 \times \cos 60$ ✓
> $= 150$ ✓

(b)



> Scalar product = $20 \times 18 \times \cos(180 - 30)$ ✓
> $= -180\sqrt{3}$ ✓

2. [6 marks: 2, 2, 2]

Given that vector $\boldsymbol{u}$ has magnitude 10 ms^{-1} in the direction 030°, $\boldsymbol{v}$ has magnitude 15 ms^{-1} in the direction 090° and $\boldsymbol{w}$ has magnitude 5 ms^{-1} in the direction 180°.
Find:

(a) $\boldsymbol{u} \cdot \boldsymbol{v}$

> $\boldsymbol{u} \cdot \boldsymbol{v} = 10 \times 15 \times \cos 60$
> $= 75$ ✓✓

(b) $\boldsymbol{u} \cdot \boldsymbol{w}$

> $\boldsymbol{u} \cdot \boldsymbol{w} = 10 \times 5 \times \cos 150$
> $= -25\sqrt{3}$ ✓✓

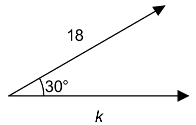
(c) the magnitude and direction of $(\boldsymbol{u} \cdot \boldsymbol{w})\,\boldsymbol{v}$.

> Magnitude = $|-25\sqrt{3}| \times 15 = 375\sqrt{3}$ ✓
> Direction = opposite to that of v = bearing 270° ✓

Calculator Free

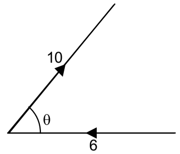
3. [4 marks: 2, 2]

(a) The scalar product between the two vectors shown below is $180\sqrt{3}$. Find k.



$18 \times k \times \cos 30 = 180\sqrt{3}$ ✓
$\Rightarrow k = 20$ ✓

(b) The scalar product between the two vectors shown below is –50. Find cos θ.



$10 \times 6 \times \cos(180 - \theta) = -50$ ✓
$-60\cos\theta = -50$
$\cos\theta = \frac{5}{6}$ ✓

---

4. [4 marks]

Given that $|\boldsymbol{a}| = 20$ and $|\boldsymbol{b}| = 25$, determine with reasons the maximum and minimum value of $\boldsymbol{a} \cdot \boldsymbol{b}$.

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$)
$= 20 \times 25 \times \cos\theta = 500\cos\theta$ ✓
But $-1 \le \cos\theta \le 1$ ✓
$\Rightarrow -500 \le \boldsymbol{a} \cdot \boldsymbol{b} \le 500$ ✓✓

---

5. [4 marks: 2, 1, 1]

Given that $|\boldsymbol{a}| = 8$, $|\boldsymbol{b}| = 5$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 20\sqrt{2}$:

(a) determine the acute angle between ***a*** and ***b*.**

$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{20\sqrt{2}}{8 \times 5}$ ✓
$= \frac{\sqrt{2}}{2} \Rightarrow \theta = 45^\circ$ ✓

(b) determine the acute angle between $2\boldsymbol{a}$ and $3\boldsymbol{b}$

Angle = 45° ✓

(c) ***a*** and –

Angle = 45° ✓

Calculator Free

6. [6 marks: 3, 3]

Given that $|\boldsymbol{m}| = 10$ and $|\boldsymbol{n}| = 10$, find ***n*** in terms of ***m*** if:

(a) $\boldsymbol{m} \cdot \boldsymbol{n} = 100$

Let the angle between ***m*** and ***n*** be θ.
$\Rightarrow \cos\theta = \frac{\boldsymbol{m} \cdot \boldsymbol{n}}{|\boldsymbol{m}||\boldsymbol{n}|} = \frac{100}{10 \times 10}$ ✓
$\Rightarrow \cos\theta = 1$
$\Rightarrow \theta = 0^\circ$ ✓
Since ***m*** and ***n*** have the same magnitude and are in the same direction, $\boldsymbol{n} = \boldsymbol{m}$. ✓

(b) $\boldsymbol{m} \cdot \boldsymbol{n} = -100$

Let the angle between ***m*** and ***n*** be θ.
$\Rightarrow \cos\theta = \frac{\boldsymbol{m} \cdot \boldsymbol{n}}{|\boldsymbol{m}||\boldsymbol{n}|} = \frac{-100}{10 \times 10}$ ✓
$\Rightarrow \cos\theta = -1$
$\Rightarrow \theta = 180^\circ$ ✓
Since ***m*** and ***n*** have the same magnitude and are in opposite directions, $\boldsymbol{n} = -\boldsymbol{m}$. ✓

---

7. [4 marks: 1, 1, 2]

Given that $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find:

(a) $(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c}$

$\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right] \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 10$ ✓

(b) $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

$\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right] \bullet \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
$= 20$ ✓

(c) k if $\boldsymbol{a} \cdot (3\mathbf{i} + k\mathbf{j}) = 9$

$(\mathbf{i} + 2\mathbf{j}) \cdot (3\mathbf{i} + k\mathbf{j}) = 9$ ✓
$\Rightarrow 3 + 2k = 9$
$k = 3$ ✓

Calculator Free

8. [3 marks: 1, 1, 1]

Expand and simplify:

(a) $(\boldsymbol{m}+\boldsymbol{n})\,.\,(\boldsymbol{m}+\boldsymbol{n})$

$$(m+n)\,.\,(m+n) = \boldsymbol{m}.\boldsymbol{m}+\boldsymbol{m}.\boldsymbol{n}+\boldsymbol{n}.\boldsymbol{m}+\boldsymbol{n}.\boldsymbol{n}$$
$$= |\boldsymbol{m}|^2 + 2\,\boldsymbol{m}.\boldsymbol{n} + |\boldsymbol{n}|^2 \quad ✓$$

(b) $(\boldsymbol{c}+\boldsymbol{d})\,.\,(\boldsymbol{c}-\boldsymbol{d})$

$$(c+d)\,.\,(c-d) = \boldsymbol{c}.\boldsymbol{c}-\boldsymbol{c}.\boldsymbol{d}+\boldsymbol{d}.\boldsymbol{c}-\boldsymbol{d}.\boldsymbol{d}$$
$$= |\boldsymbol{c}|^2 - |\boldsymbol{d}|^2 \quad ✓$$

(c) $(\boldsymbol{m}-3\boldsymbol{n})\,.\,(2\boldsymbol{m}-\boldsymbol{n})$

$$(m-3n)\,.\,(2m-n) = \boldsymbol{m}.\boldsymbol{2m}-\boldsymbol{m}.\boldsymbol{n}-3\boldsymbol{n}.\boldsymbol{2m}+3\boldsymbol{n}.\boldsymbol{n}$$
$$= 2|\boldsymbol{m}|^2 - 7\,\boldsymbol{m}.\boldsymbol{n} + 3|\boldsymbol{n}|^2 \quad ✓$$

---

9. [4 marks: 2, 2]

Given that $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular, prove that:

(a) $\boldsymbol{p}\,.\,\boldsymbol{q} = 0$

$$\boldsymbol{p}.\boldsymbol{q} = |\boldsymbol{p}|\,|\boldsymbol{q}|\cos 90$$
$$= 0 \quad ✓✓$$

(b) $(\boldsymbol{p}+\boldsymbol{q})\,.\,(\boldsymbol{p}+\boldsymbol{q}) = |\boldsymbol{p}|^2 + |\boldsymbol{q}|^2$

$$(p+q)\,.\,(p+q) = \boldsymbol{p}.\boldsymbol{p}+\boldsymbol{p}.\boldsymbol{q}+\boldsymbol{q}.\boldsymbol{p}+\boldsymbol{q}.\boldsymbol{q}$$
$$= |\boldsymbol{p}|^2 + 2\,\boldsymbol{p}.\boldsymbol{q} + |\boldsymbol{q}|^2 \quad ✓$$
$$= |\boldsymbol{p}|^2 + |\boldsymbol{q}|^2 \quad \text{since } \boldsymbol{p}.\boldsymbol{q} = 0 \quad ✓$$

Calculator Free

10. [7 marks: 2, 2, 3]

Given that $|\boldsymbol{r}| = 10$ and $|\boldsymbol{s}| = 8$, find:

(a) $\boldsymbol{r}\,.\,\boldsymbol{r}$

$$\boldsymbol{r}.\boldsymbol{r} = |\boldsymbol{r}|\,|\boldsymbol{r}|\cos 0$$
$$= 10\times 10 \times \cos 0 \quad ✓$$
$$= 100 \quad ✓$$

(b) $(\boldsymbol{r}+\boldsymbol{s})\,.\,(\boldsymbol{r}+\boldsymbol{s})$ if $\boldsymbol{r}$ and $\boldsymbol{s}$ are parallel and in the same direction.

$$(r+s)\,.\,(r+s) = |\boldsymbol{r}|^2 + 2\,\boldsymbol{r}.\boldsymbol{s} + |\boldsymbol{s}|^2$$
$$= 10^2 + 2[\,|\boldsymbol{r}|\,|\boldsymbol{s}|\cos 0] + 8^2 \quad ✓$$
$$= 100 + (2\times 10\times 8) + 64$$
$$= 324 \quad ✓$$

(c) $|\boldsymbol{r}-\boldsymbol{s}|$ if $\boldsymbol{r}$ and $\boldsymbol{s}$ are perpendicular.

$$|\boldsymbol{r}-\boldsymbol{s}|^2 = (r-s)\,.\,(r-s)$$
$$= |\boldsymbol{r}|^2 - 2\,\boldsymbol{r}.\boldsymbol{s} + |\boldsymbol{s}|^2 \quad ✓$$
$$= 10^2 - 2\times 0 + 8^2 \quad ✓ \qquad [\,\boldsymbol{r} \text{ and } \boldsymbol{s} \text{ perpendicular} \Rightarrow \boldsymbol{r}.\boldsymbol{s} = 0]$$
$$= 164$$

$$|\boldsymbol{r}-\boldsymbol{s}| = 2\sqrt{41} \quad ✓$$

---

11. [7 marks: 2, 5]

The points A, B and C have position vectors < –4, 3 >, < 2, 7 > and < –1, 2 > respectively.

(a) Find the vectors **AB** and **CA**.

AB = < 2, 7 > – < –4, 3 > = < 6, 4 > ✓
CA = < –4, 3 > – < –1, 2 > = < –3, 1 > ✓

(b) Find a vector perpendicular to **AB** but with the same magnitude as **CA**.

$|\mathbf{CA}| = \sqrt{10}$ ✓
A vector perpendicular to **AB** = < –4, 6 > ✓
$|\mathbf{AB}| = \sqrt{42} = 2\sqrt{13}$ ✓

$$\text{Required vector} = \frac{\sqrt{10}}{2\sqrt{13}} < -4, 6 > \quad ✓$$
$$= \frac{\sqrt{10}}{\sqrt{13}} < -2, 3 > \quad ✓$$

Calculator Assumed

12. [7 marks: 2, 3, 2]

Given that $\boldsymbol{u} = \mathbf{i} + 3\mathbf{j}$, $\boldsymbol{v} = -6\mathbf{i} + 8\mathbf{j}$ and $\boldsymbol{w} = k\mathbf{i} - 2\mathbf{j}$, find:

(a) the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}-6\\8\end{pmatrix}}{\sqrt{10}\times 10} \quad ✓ \quad [\theta \text{ is the angle between } \boldsymbol{u} \text{ and } \boldsymbol{v}]$$

$$= \frac{18}{10\sqrt{10}} \quad \Rightarrow \quad \theta = 55.30^\circ \quad ✓$$

```
angle([1 3],[-6 8])
                55.3048
```

(b) k, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel in the opposite direction.

$\Rightarrow \ \boldsymbol{v} = -\lambda\boldsymbol{w}$ where λ is a positive constant

$-6\mathbf{i} + 8\mathbf{j} = -\lambda(k\mathbf{i} - 2\mathbf{j})$ ✓

Compare **i** and **j** coefficients:

$-6 = -\lambda k$

$8 = 2\lambda \ \Rightarrow \ \lambda = 4$ ✓

Hence, $k = \frac{3}{2}$ ✓

(c) k, if the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 120°.

$$\cos 120 = \frac{\begin{pmatrix}-6\\8\end{pmatrix}\begin{pmatrix}k\\-2\end{pmatrix}}{10\times\sqrt{(k^2+4)}} \quad ✓$$

$$\Rightarrow \ -\frac{1}{2} = \frac{\begin{pmatrix}-6\\8\end{pmatrix}\begin{pmatrix}k\\-2\end{pmatrix}}{10\times\sqrt{(k^2+4)}}$$

$-6k - 16 = -5\sqrt{(k^2 + 4)}$

$\Rightarrow \ k = -0.8543$ ✓

```
solve(-6x-16=-5√(x²+4) ,x)
              {x=-0.8543}
```

OR

Use CAS:

solve(angle(< –6, 8 >, < k, –2 >) = 120, k) ✓

$\Rightarrow \ k = -0.8543$ ✓

```
solve(angle([-6 8],[k -2])=120,k)
          {k=-0.8543145111}
```


Calculator Assumed

13. [6 marks: 3, 3]

Given that $|\boldsymbol{u}| = 10$ and $(\boldsymbol{u} + 2\boldsymbol{v}) . (\boldsymbol{u} + \boldsymbol{v}) = 408$.

(a) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

$(\boldsymbol{u} + 2\boldsymbol{v}) . (\boldsymbol{u} + \boldsymbol{v}) = |\boldsymbol{u}|^2 + 3\,\boldsymbol{u} . \boldsymbol{v} + 2|\boldsymbol{v}|^2$ ✓

$\Rightarrow \ 408 = 100 + (3 \times 0) + 2|\boldsymbol{v}|^2$ ✓

$\Rightarrow \ 2|\boldsymbol{v}|^2 = 308$

$|\boldsymbol{v}| = \sqrt{154}$ ✓

(b) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in opposite directions.

$(\boldsymbol{u} + 2\boldsymbol{v}) . (\boldsymbol{u} + \boldsymbol{v}) = |\boldsymbol{u}|^2 + 3\,\boldsymbol{u} . \boldsymbol{v} + 2|\boldsymbol{v}|^2$

$\Rightarrow \ 408 = 100 + (3 \times 10 \times |\boldsymbol{v}| \times \cos 180) + 2|\boldsymbol{v}|^2$ ✓

$\Rightarrow \ 2|\boldsymbol{v}|^2 - 30|\boldsymbol{v}| - 308 = 0$ ✓

$|\boldsymbol{v}| = 22$ or -7 (reject as $|\boldsymbol{v}| > 0$)

Hence, $|\boldsymbol{v}| = 22$. ✓

14. [6 marks: 3, 3]

Given that $\boldsymbol{u} = -\mathbf{i} + \mathbf{j}$, find in exact form, a unit vector $\hat{v}$, if:

(a) $\boldsymbol{u}$ is perpendicular to $\hat{v}$.

A vector that is perpendicular to $\boldsymbol{u}$ is $\mathbf{i} + \mathbf{j}$. ✓

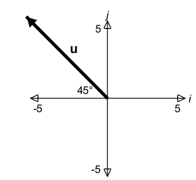
$|\mathbf{i} + \mathbf{j}| = \sqrt{2}$

Hence, $\hat{v} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$= \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$ ✓✓

[other equivalent answers possible]

(b) the acute angle between $\boldsymbol{u}$ and $\hat{v}$ is 45°.



From sketch that $\hat{v} = -\mathbf{i}$ or $\mathbf{j}$. ✓✓✓

Calculator Assumed

15. [8 marks: 2, 3, 3]

Let $\boldsymbol{a} = \begin{pmatrix} -\sqrt{5} \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} t \\ 5 \end{pmatrix}$.

(a) Find the value of t if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

$\begin{pmatrix} -\sqrt{5} \\ 1 \end{pmatrix} \bullet \begin{pmatrix} t \\ 5 \end{pmatrix} = 0$ ✓

$-\sqrt{5}\ t + 5 = 0 \Rightarrow t = \sqrt{5}$ ✓

(b) Find value(s) of t if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

$\begin{pmatrix} t \\ 5 \end{pmatrix} = \lambda \begin{pmatrix} -\sqrt{5} \\ 1 \end{pmatrix}$ ✓

$\lambda = \pm 5$ ✓

$t = \pm 5\sqrt{5}$ ✓

(c) Determine the value(s) of t if the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$.

$$\frac{\begin{pmatrix} -\sqrt{5} \\ 1 \end{pmatrix} \bullet \begin{pmatrix} t \\ 5 \end{pmatrix}}{\left|\begin{pmatrix} -\sqrt{5} \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} t \\ 5 \end{pmatrix}\right|} = \frac{1}{\sqrt{6}} \quad ✓$$

$$\frac{-t\sqrt{5}+5}{\sqrt{6}\sqrt{25+t^2}} = \frac{1}{\sqrt{6}} \quad ✓$$

$t = 0$ ✓

16. [6 marks]

Given that $\boldsymbol{u} = 3\mathbf{i} + \mathbf{j}$ and $\boldsymbol{v} = x\mathbf{i} + y\mathbf{j}$, find x and y if $|\boldsymbol{v}| = \sqrt{10}$ and the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°.

$$\cos 60 = \frac{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}{\sqrt{10} \times \sqrt{10}} \quad ✓✓$$

$\frac{1}{2} = \frac{3x+y}{10}$ where $3x + y > 0$

$\Rightarrow 3x + y = 5$ I ✓

$|\boldsymbol{v}| = \sqrt{10} \Rightarrow x^2 + y^2 = 10$ II ✓

Solve I & II simultaneously:

Hence, $x = 0.6340$, $y = 3.0981$ ✓

or $x = 2.3660$, $y = -2.0981$ ✓

```
{3x+y=5
{x²+y²=10 | x,y
{{x=0.6340,y=3.0981},{x=2.3660,y=-2.0981}}
```


Calculator Assumed

17. [7 marks: 3, 4]

The position vector the point P relative to the point Q is < –5, 4 >.
The position vector of the point R relative to Q is < 4, 5 >.

(a) Find the position vector of R relative to P.

QP = < –5, 4 >   **QR** = < 4, 5 > ✓

PR = **PQ** + **QR**

= < 5, –4 > + < 4, 5 > ✓

= < 9, 1 >. ✓

(b) Determine the area of the triangle formed by the points P, Q and R. Show clearly how you obtained your answer.

QP • **QR** = < –5, 4 > • < 4, 5 > ✓

= 0

Hence $\angle PQR = 90^\circ$. ✓

Hence, area of $\Delta PQR = \frac{1}{2} \times \sqrt{41} \times \sqrt{41}$ ✓

$= \frac{41}{2}$ ✓

18. [6 marks: 2, 4]

The points P, Q and R have position vectors < –5, 10 >, < 2, 9 > and < –1, 6 > respectively.

(a) Find the **PQ** and **PR**.

PQ = < 2, 9 > – < –5, 10 > = < 7, –1 > ✓

PR = < –1, 6 > – < –5, 10 > = < 4, –4 > ✓

(b) Determine the cosine of $\angle QPR$. Hence, find the area of ΔPQR.

$\cos \angle QPR = \frac{<7,-1>\bullet<4,-4>}{\sqrt{50} \times 4\sqrt{2}}$ ✓

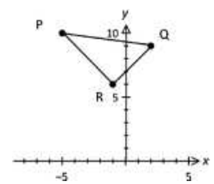
$= \frac{32}{40} = \frac{4}{5}$

$\cos \angle QPR = \frac{4}{5} \Rightarrow \sin \angle QPR = \frac{3}{5}$ ✓

Area $= \frac{1}{2} \times PQ \times PR \times \sin \angle QPR$

Hence, area $= \frac{1}{2} \times \sqrt{50} \times 4\sqrt{2} \times \frac{3}{5}$ ✓

$= 12$ ✓

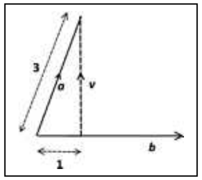


13 Scalar Product II

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 3 and 4 respectively. The acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\frac{1}{3}$. Find in terms of the vectors $\boldsymbol{a}$ and/or $\boldsymbol{b}$:



(a) the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

Scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b} = 3 \times \frac{1}{3} = 1$. ✓

(b) the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

$\text{proj}_{\boldsymbol{b}}\,\boldsymbol{a} = 1 \times \hat{\boldsymbol{b}}$ ✓

$= 1 \times \frac{1}{4}\boldsymbol{b} = \frac{1}{4}\boldsymbol{b}$ ✓

(c) the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

$\text{proj}_{\boldsymbol{a}}\,\boldsymbol{b} = 4 \times \frac{1}{3} \times \hat{\boldsymbol{a}}$ ✓

$= \frac{4}{3} \times \frac{1}{3}\boldsymbol{a} = \frac{4}{9}\boldsymbol{a}$ ✓

(d) $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

$\boldsymbol{v} + \text{proj}_{\boldsymbol{b}}\,\boldsymbol{a} = \boldsymbol{a}$

$\boldsymbol{v} = \boldsymbol{a} - \frac{1}{4}\boldsymbol{b}$ ✓✓

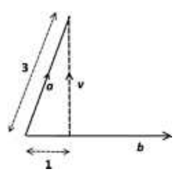
(e) the magnitude of the $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

Using Pythagoras Theorem:

$|\boldsymbol{v}|^2 = 3^2 - |\text{proj}_{\boldsymbol{b}}\,\boldsymbol{a}|^2$ ✓

$= 9 - 1$

$|\boldsymbol{v}| = 2\sqrt{2}$ ✓



Calculator Free

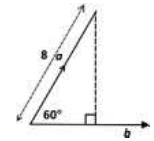
2. [10 marks: 2, 2, 3, 3]

(a) The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 8 and 6 respectively. Find the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ if the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

(i) is 60°.

$\text{proj}_{\boldsymbol{b}}\,\boldsymbol{a} = 8\cos 60 \times \hat{\boldsymbol{b}}$ ✓

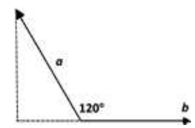
$= 4 \times \frac{1}{6}\boldsymbol{b} = \frac{2}{3}\boldsymbol{b}$ ✓



(ii) is 120°.

$\text{proj}_{\boldsymbol{b}}\,\boldsymbol{a} = 8\cos 120 \times \hat{\boldsymbol{b}}$ ✓

$= -4 \times \frac{1}{6}\boldsymbol{b} = -\frac{2}{3}\boldsymbol{b}$ ✓



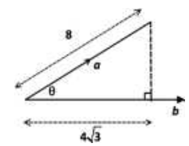
(b) Find θ, the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ if $|\boldsymbol{a}| = 8$ and the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ has a magnitude of $4\sqrt{3}$.

$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\boldsymbol{a}\cdot\hat{\boldsymbol{b}}}{|\boldsymbol{a}|}$

But $\boldsymbol{a}\cdot\hat{\boldsymbol{b}} = 4\sqrt{3}$

Hence, $\cos\theta = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$ ✓✓

$\theta = 30^{\circ}$ ✓



(c) The vector $\boldsymbol{b}$ has magnitude 10. The projection of vector $\boldsymbol{a}$ on $\boldsymbol{b}$ is $5\boldsymbol{b}$. Find the magnitude of $\boldsymbol{a}$ if the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 60°.

$|\text{proj}_{\boldsymbol{b}}\,\boldsymbol{a}| = |5\boldsymbol{b}|$

$= 5 \times 10 = 50$ ✓

But $|\text{proj}_{\boldsymbol{b}}\,\boldsymbol{a}| = |\boldsymbol{a}|\cos 60.$

$\Rightarrow |\boldsymbol{a}|\cos 60 = 50$ ✓

$|\boldsymbol{a}| = 100.$ ✓

Calculator Free

3. [4 marks: 2, 2]

Given $\boldsymbol{a} = <5, 10>$ and $\boldsymbol{b} = <4, 3>$.

(a) Find the component of $\boldsymbol{a}$ that is parallel to $\boldsymbol{b}$.

$\text{proj}_b\, \boldsymbol{a} = (\boldsymbol{a} \bullet \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$

$= [<5, 10> \bullet \frac{1}{5}<4, 3>]\frac{1}{5}<4, 3>$ ✓

$= 10 \times \frac{1}{5}<4, 3>$

$= <8, 6>$ ✓

(b) Find the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

Let required vector be $\boldsymbol{v}$.

Hence, $\boldsymbol{v} + \text{proj}_b\, \boldsymbol{a} = \boldsymbol{a}$

$\boldsymbol{v} = <5, 10> - <8, 6>$ ✓

$= <-3, 4>$ ✓

---

4. [6 marks: 2, 2, 2]

Given that $\boldsymbol{a} = <2, 5> + <10, -4>$, find:

(a) the vector projection of $\boldsymbol{a}$ onto $<6, 15>$.

$<2, 5>$ and $<10, -4>$ are components of $\boldsymbol{a}$ parallel and perpendicular to $<6, 15>$ respectively. ✓

Hence, vector projection of $\boldsymbol{a}$ onto $<6, 15> = <2, 5>$. ✓

(b) the vector projection of $\boldsymbol{a}$ onto $<-5, 2>$

$<10, -4>$ and $<2, 5>$ are components of $\boldsymbol{a}$ parallel and perpendicular to $<-5, 2>$ respectively. ✓

Hence, vector projection of $\boldsymbol{a}$ onto $<-5, 2> = <10, -4>$. ✓

(c) the vector projection of $\boldsymbol{a}$ onto $<-5, 0>$

$\boldsymbol{a} = <12, -1> = <12, 0> + <0, -1>$.

$<12, 0>$ and $<0, -1>$ are components of $\boldsymbol{a}$ parallel and perpendicular to $<-5, 0>$. ✓

Hence, vector projection of $\boldsymbol{a}$ onto $<-5, 0>$ is $<12, 0>$. ✓

Calculator Free

5. [11 marks: 3, 2, 2, 4]

The acute angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60^{o}. The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is $<4, -3>$ and $|\boldsymbol{v}| = 10$.

(a) Find $\boldsymbol{v}$.

$\text{proj}_v\, \boldsymbol{u} = <4, -3>$

Hence, $\boldsymbol{v} = \mu <4, -3>$. ✓

But $|\boldsymbol{v}| = 10$.

$|\mu <4, -3>| = 10$ ✓

$\mu = 2$.

Hence, $\boldsymbol{v} = <8, -6>$. ✓

(b) Explain clearly why $\boldsymbol{u} = <4, -3> + \lambda <3, 4>$.

$<4, -3>$ is the vector component of $\boldsymbol{u}$ parallel to $<8, -6>$. ✓

$\lambda <3, 4>$ is perpendicular to $<8, -6>$ as $\lambda <3, 4> \bullet <8, 6> = 0$.

Hence, $\lambda <3, 4>$ is the vector component of $\boldsymbol{u}$ perpendicular to $<8, -6>$. ✓

Hence, $\boldsymbol{u} = <4, -3> + \lambda <3, 4>$.

(c) Find $|\boldsymbol{u}|$.

$\text{proj}_v\, \boldsymbol{u} = <4, -3>$

Hence, $|\boldsymbol{u}|\cos 60 = |<4, -3>|$ ✓

$|\boldsymbol{u}| = 10$ ✓

(d) Find $\boldsymbol{u}$.

$\boldsymbol{u} = <4, -3> + \lambda <3, 4>$.

$\Rightarrow |<4 + 3\lambda, -3 + 4\lambda>| = 10$ ✓

$(4 + 3\lambda)^2 + (-3 + 4\lambda)^2 = 100$ ✓

$16 + 24\lambda + 9\lambda^2 + 9 - 24\lambda + 16\lambda^2 = 100$

$25\lambda^2 = 75$

$\lambda = \pm\sqrt{3}$. ✓

Hence, $\boldsymbol{u} = <4, 3> \pm \sqrt{3}\,<3, 4>$. ✓

Calculator Assumed

6. [11 marks: 1, 3, 2, 3, 2]

Let $\boldsymbol{u} = \langle 2, 1 \rangle$, $\boldsymbol{v} = \langle 1, -2 \rangle$ and $\boldsymbol{w} = \langle 3, 9 \rangle$.

(a) Show that $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

> $\langle 2, 1 \rangle \bullet \langle 1, -2 \rangle = 0$. ✓
> Hence, $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

(b) Given that $\boldsymbol{w} = m\boldsymbol{u} + n\boldsymbol{v}$, find m and n.

> x-components: $2m + n = 3$ ✓
> y-components: $m - 2n = 9$ ✓
> $m = 3, n = -3$ ✓
>
> $\begin{cases} 2x+y=3 \\ x-2y=9 \end{cases} \Big|_{x,y}$ $\{x=3, y=-3\}$

(c) Find the vector projection of $\boldsymbol{w}$ onto $\langle -2, -1 \rangle$.

> From (b), $3\boldsymbol{u}$ and $-3\boldsymbol{v}$ are components of $\boldsymbol{w}$
> parallel and perpendicular to $\langle 2, 1 \rangle$ respectively. ✓
> Since $\langle -2, -1 \rangle$ is parallel to $\boldsymbol{u}$,
> vector projection of $\boldsymbol{w}$ onto $\langle -2, -1 \rangle$ is $3\boldsymbol{u} = \langle 6, 3 \rangle$. ✓

(d) $\boldsymbol{w}$ is the vector projection of $\lambda \langle 2, 1 \rangle$ onto $\boldsymbol{w}$. Find λ.

> $\hat{w} = \frac{1}{\sqrt{90}} \langle 3, 9 \rangle$.
> $\Rightarrow \lambda \langle 2, 1 \rangle \bullet \frac{1}{\sqrt{90}} \langle 3, 9 \rangle = \sqrt{90}$ ✓✓
> $15\lambda = 90$
> $\lambda = 6$ ✓

(e) If $\boldsymbol{w}$ is the vector projection of $\lambda \langle 2, 1 \rangle$ onto $\boldsymbol{w}$, find the component of $\lambda \langle 2, 1 \rangle$ that is perpendicular to $\boldsymbol{w}$.

> Let required vector be $\boldsymbol{a}$.
> Hence, $\boldsymbol{a} + \boldsymbol{w} = \lambda \langle 2, 1 \rangle = 6 \langle 2, 1 \rangle$ ✓
> $\boldsymbol{a} = \langle 12, 6 \rangle - \langle 3, 9 \rangle$
> $= \langle 9, -3 \rangle$. ✓

Calculator Assumed

7 [7 marks: 3, 2, 2]

The position vector of the point A relative to the point B is $\langle 3, 4 \rangle$.
The position vector of the point C relative to B is $\langle -4, 5 \rangle$.

(a) Determine the vector projection of **BC** that is *parallel* to **BA**.

> Unit vector parallel to **BA** $= \frac{1}{5} \langle 3, 4 \rangle$ ✓
> scalar projection of **BC** onto **BA** $= \begin{pmatrix} -4 \\ 5 \end{pmatrix} \bullet \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ✓
> $= \frac{8}{5}$
> Vector projection of **BC** onto **BA** $= \frac{8}{5} \times \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$
> $= \frac{8}{25}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ✓

(b) Determine the vector projection of **BC** that is *perpendicular* to **BA**.

> $\frac{8}{25}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + v = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ ✓
> $v = \frac{1}{25}\begin{pmatrix} -124 \\ 93 \end{pmatrix} = \frac{31}{25}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ ✓

(c) Hence, or otherwise, determine the area of ΔABC.

> Area of $\Delta ABC = \frac{1}{2} \times |\mathbf{BA}| \times height$ ✓
> $= \frac{1}{2} \times \left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| \times \left|\frac{31}{25}\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right|$
> $= \frac{1}{2} \times 5 \times \frac{31}{25} \times 5 = \frac{31}{2}$ ✓

14 Scalar Product III

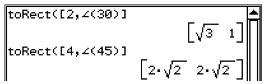
Calculator Assumed

1. [6 marks: 2, 1, 3]

Given that vector $\boldsymbol{a}$ has magnitude 2 ms^{-1} in the direction 060^o,
$\boldsymbol{b}$ has magnitude 4 ms^{-1} in the direction 045^o :

(a) Find $\boldsymbol{a}$ and $\boldsymbol{b}$ in the form $x\,\boldsymbol{i} + y\,\boldsymbol{j}$.

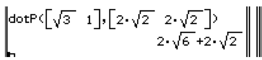
$\boldsymbol{a}$ = < 2 cos 30^o, 2 sin 30^o >
= < $\sqrt{3}$, 1 > ✓
$\boldsymbol{b}$ = < 4 cos 45^o, 4 sin 45^o >
= < $2\sqrt{2}$, $2\sqrt{2}$ > ✓



(b) Find $\boldsymbol{a} \bullet \boldsymbol{b}$.

$\boldsymbol{a} \bullet$ = < $\sqrt{3}$, 1 > • < $2\sqrt{2}$, $2\sqrt{2}$ >

= $2(\sqrt{2} + \sqrt{6})$ ✓



(c) Use your answers in (a) and/or (b) to find cos 15^o in exact form.

Angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 15^o. ✓

Hence, $\cos 15^o = \dfrac{<\sqrt{3},1>\bullet<2\sqrt{2},2\sqrt{2}>}{|<\sqrt{3},1>||<2\sqrt{2},2\sqrt{2}>|}$ ✓

$= \dfrac{2(\sqrt{2}+\sqrt{6})}{2\times 4}$

$= \dfrac{\sqrt{2}+\sqrt{6}}{4}$ ✓

Calculator Assumed

2. [4 marks: 2, 2]

(a) A force F has magnitude 2 000 N and acts at an angle of 010^o with the horizontal. Calculate the work done by F in moving a object 10 m along the horizontal.

Component of F parallel to horizontal = 2 000 cos 10 ✓
Work done = 2 000 cos 10 × 10
= 19 696.15 J. ✓

(b) Calculate the work done by the force < 30, 50 > in moving an object through a displacement of < 4, –1 > metres.

Work done = $\begin{pmatrix}30\\50\end{pmatrix} \bullet \begin{pmatrix}4\\-1\end{pmatrix}$ ✓
= 120 – 50
= 70 J ✓

3. [5 marks: 3, 2]

A force F has magnitude 1 000 N and acts along bearing 030^o.
Let $\boldsymbol{i}$ be the unit vector in the Easterly direction and $\boldsymbol{j}$ be the unit vector in the Northerly direction.

(a) Express F in the form $a\,\boldsymbol{i} + b\,\boldsymbol{j}$.

F = < 1000 cos 60, 1000 sin 60 > ✓✓
= < 500, 500$\sqrt{3}$ > N. ✓

(b) Hence, or otherwise, calculate the work done when the force F moves an object through a displacement of < 3, 4 > metres.

Work done = $\begin{pmatrix}500\\500\sqrt{3}\end{pmatrix} \bullet \begin{pmatrix}3\\4\end{pmatrix}$ ✓
= 1500 + 2000$\sqrt{3}$
= 4964.1 J ✓

Calculator Assumed

4. [10 marks: 2, 2, 1, 2, 3]

A force $\boldsymbol{F}_1$ of $<4, 4>$ Newtons is applied to a body and causes the body to be displaced by $<3, 1>$ metres.

(a) Find the component of the applied force along the direction of motion.

> Component along direction of motion
> $= \left| <4,4> \bullet \frac{1}{\sqrt{10}} <3,1> \right| \frac{1}{\sqrt{10}} <3,1>$ ✓
> $= \frac{8}{5} <3,1>$ ✓

(b) Find the component of the applied force perpendicular to the direction of motion.

> Component perpendicular to direction of motion
> $= <4,4> - \frac{8}{5} <3,1>$ ✓
> $= \frac{4}{5} <-1,3>$ ✓

(c) Determine the work done by the applied force.

> Work done $= <4,4> \bullet <3,1>$
> $= 16$ Joules. ✓

(d) Another force $\boldsymbol{F}_2$ of $<x, y>$ Newtons is applied to the same body. But half as much work is required to cause the same displacement to the body.

(i) Find a possible pair of values for x and y.

> Work done: $<x,y> \bullet <3,1> = 8$
> $3x + y = 8$ ✓
> Possible $\boldsymbol{F}_2$: $<0, 8>$ ✓

(ii) Find x and y if $|F_2| = 2\sqrt{2}$ Newtons.

> $3x + y = 8$ I
> $|F_2| = 2\sqrt{2} \Rightarrow x^2 + y^2 = 8$ II ✓
> Solve I & II: $x = 2, y = 2$ ✓
> or $x = \frac{14}{5}, y = -\frac{2}{5}$ ✓

Calculator Assumed

5. [8 marks: 2, 1, 5]

Two forces $<1, 4>$ Newtons and $<3, -2>$ Newtons act on a single body P and causes the body P to be displaced by $<8, 4>$.

(a) Find the vector projection of the resultant force onto the displacement of P.

> Resultant force $= <1,4> + <3,-2>$
> $= <4, 2>$. ✓
> Since resultant force is parallel to displacement vector,
> projection of resultant force onto displacement vector $= <4, 2>$. ✓

(b) Find the work done on P.

> Work done $= <4,2> \bullet <8,4>$
> $= 40$ Joules ✓

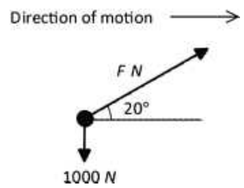
(c) A third force of $<-2, 1>$ Newtons is applied to P and causes it to move by 10 metres. The work done is 36 Joules. Find the displacement vector.

> Resultant force $= <4,2> + <-2,1> = <2,3>$. ✓
> Let displacement vector be $<x, y>$.
> Then, $x^2 + y^2 = 100$ I ✓
> Also $<2,3> \bullet <x,y> = 36$
> $2x + 3y = 36$ II ✓
> Solve I & II $x = 6, y = 8$
> or $x = \frac{66}{13}, y = \frac{112}{13}$
> Hence, displacement vector is $<6, 8>$ or $< \frac{66}{13}, \frac{112}{13} >$ metres. ✓✓

Calculator Assumed

6. [6 marks: 3, 2, 1]

An object of weight 1 000 Newtons is being pulled along a horizontal surface by a force of magnitude F Newtons inclined at an angle of 20^o to the surface. The motion of the object is opposed by a horizontal force of magnitude 500 N.



(a) Find in terms of F, the magnitude of the pulling force perpendicular to the surface. Hence, find F.

> Magnitude of pulling force perpendicular to surface
> $= F \sin 20^o$ N ✓
>
> Since, there is no motion vertical to the surface,
> $F \sin 20^o = 1\,000$ ✓
> $F = 2\,923.8044 \approx 2\,923.8$ N ✓

(b) Find the component of the pulling force in the direction of motion. Hence, find the magnitude of the resultant force in the direction of motion.

> Magnitude of pulling force parallel to surface
> $= F \cos 20^o = 2\,747.4774$ ✓
>
> Magnitude of resultant force in direction of motion
> $= 2\,747.4774 - 500$
> $= 2\,247.4774 \approx 2\,247.5$ N ✓

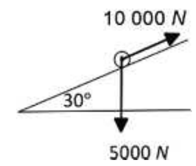
(c) Find the work done by the pulling force in moving the object 50 m in its direction of motion.

> Work done $= 2\,747.4774 \times 50$
> $\approx 137\,373.9$ J or 137.4 kJ ✓

Calculator Assumed

7. [7 marks: 2, 1, 2, 2]

A body of weight 5 000 Newtons is pulled up along a plane inclined at an angle of 30° with the horizontal by a force of magnitude 10 000 N. There is a force of magnitude 500 N opposing the motion of the body.



(a) Find the component of the gravitational force on the body along the inclined plane. Hence, find the magnitude of the resultant force in the direction of the motion of the body.

> Component of gravitational force along inclined plane
> $= 5\,000 \sin 30^o = 2\,500$ N ✓
>
> Magnitude of resultant force in direction of motion
> $= 10\,000 - 2\,500 - 500 = 7\,000$ N ✓

(b) Find the work done by the resultant force when the body has moved 10 m along the inclined plane.

> Work done $= 7\,000 \times 10$ J = 70 kJ ✓

(c) Find the work done by the resultant force when the body has ascended a vertical distance of 10 m.

> Distance moved along the inclined plane
> $= \dfrac{10}{\sin 30} = 20$ m ✓
> Hence, work done $= 7\,000 \times 20$ J = 140 kJ ✓

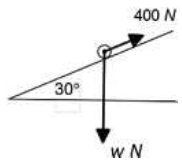
(d) The work done by the resultant force in moving the body up the inclined plane so that the change in its horizontal position is x metres is 280 kJ. Find x.

> Distance moved along plane $= \dfrac{280\,000}{7\,000} = 40$ m. ✓
> Hence, $x = 40 \cos 30 = 20\sqrt{3}$ m. ✓

Calculator Assumed

8. [7 marks: 3, 4]

A cart of weight w N is at rest on a set of rail-tracks inclined at an angle of 30° with the horizontal. A force parallel to the inclined plane of magnitude 400 N just prevents the body from slipping down the rail-tracks.



(a) Find the component of the gravitational force acting on the cart along the inclined rail-tracks. Hence, find the weight of the cart.

Component of gravitational force along plane
$= w \sin 30^\circ$ ✓
Since cart is at rest on the plane:
$w \sin 30^\circ = 400$ ✓
$w = 800$ N ✓

(b) A force of magnitude 1 000 N at an angle of 30° to the inclined rail-tracks acts on the cart and moves it a distance of 10 m along the tracks. Assume that the cart does not leave the rail-tracks. Find the work done by the resultant force along the inclined plane.

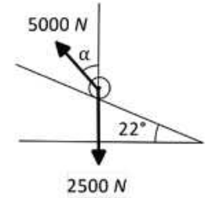
Component of force along track
$= 1\,000 \cos 30^\circ = 500\sqrt{3}$ ✓

Magnitude of resultant force along direction of motion
$= 500\sqrt{3} - 800 \sin 30^\circ$ ✓
$= 500\sqrt{3} - 400$ N ✓
Hence, work done $= (500\sqrt{3} - 400) \times 10$ J
$= 4.66$ kJ ✓

Calculator Assumed

9. [7 marks: 3, 4]

A body of weight 2500 N is pulled up an inclined plane with angle of inclination 22° to the horizontal by a force F of magnitude 5000 N inclined at an angle of α° to the vertical. The body remains in contact with the inclined plane. There is a force of 100 N opposing the motion of the body up the inclined plane.



(a) Calculate the value of α.

Component of F along the vertical $= 5\,000 \cos \alpha$ ✓
Hence: $5\,000 \cos \alpha = 2500$ ✓
$\alpha = 60^\circ$ ✓

(b) Calculate the work done when the body is moved 2 m along the inclined plane by the force F.

Angle of F with the inclined plane $= 90 - 22 - 60 = 8^\circ$ ✓
Component of F parallel to incline $= 5\,000 \cos 8$ ✓
Resultant force up the incline $= 5\,000 \cos 8 - 2500 \sin 22 - 100$ ✓
$= 3914.8239$
Work done $= 3914.8239 \times 2$
$= 7829.6$ J. ✓

15 Geometric Proofs using Vectors

Calculator Assumed

1. [6 marks: 2, 2, 2,]

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors. Find α and β if:

(a) $2\boldsymbol{a} + (\beta - 2)\,\boldsymbol{b} = (1 - \alpha)\,\boldsymbol{a}$

$1 - \alpha = 2 \Rightarrow \alpha = -1$ ✓
$\beta - 2 = 0 \Rightarrow \beta = 2$ ✓

(b) $\alpha\,(3\boldsymbol{a} - 4\boldsymbol{b}) = 6\boldsymbol{a} + \beta\boldsymbol{b}$

$3\alpha = 6 \Rightarrow \alpha = 2$ ✓
$\beta = -4\alpha \Rightarrow \beta = -8$ ✓

(c) $\alpha\boldsymbol{a} + 5\boldsymbol{b}$ is parallel to $3\boldsymbol{a} + \beta\boldsymbol{b}$

$\alpha\boldsymbol{a} + 5\boldsymbol{b} = \lambda(3\boldsymbol{a} + \beta\boldsymbol{b})$
$\alpha = 3\lambda \Rightarrow \lambda = \alpha/3$
$\lambda\beta = 5 \Rightarrow \alpha\beta = 15$ or $\alpha = 15/\beta$ for $\beta \neq 0, \alpha \neq 0$ ✓✓

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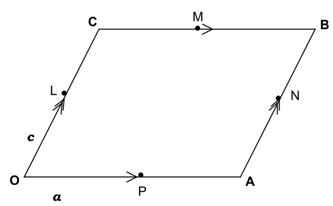
2. [4 marks: 1, 1, 2]



OABC is a parallelogram. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. L, M, N and P are the midpoints of OC, CB, BA and AO respectively.

(a) Find **LM** in terms of **a** and **c** .

LM = LC + CM
$= \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{a}$ ✓

(b) Find **PN** in terms of **a** and **c**.

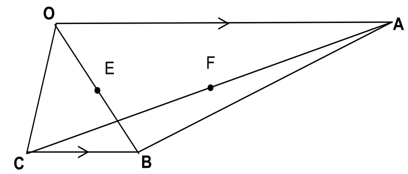
PN = PA + AN
$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ ✓

(c) Hence, use a vector method to show that LMNP is a parallelogram.

Clearly **LM = PN**.
Hence, LMNP has a pair of opposite sides that are parallel and congruent. ✓
Hence, LMNP is a parallelogram. ✓

Calculator Assumed

3. [8 marks: 2, 4, 2]



OABC is a trapezium with **OA** = 3**CB**. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. E and F are midpoints of OB and CA respectively.

(a) Find **OE** in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

$\mathbf{OE} = \frac{1}{2}\mathbf{OB} = \frac{1}{2}(\mathbf{OC} + \mathbf{CB})$ ✓
$= \frac{1}{2}(\boldsymbol{c} + \frac{1}{3}\boldsymbol{a})$. ✓

(b) Find **EF** in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

$\mathbf{AF} = \frac{1}{2}\mathbf{AC} = \frac{1}{2}(\mathbf{AO} + \mathbf{OC})$ ✓
$= \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c})$. ✓
$\mathbf{EF} = \mathbf{EO} + \mathbf{OA} + \mathbf{AF}$
$= -\frac{1}{2}(\boldsymbol{c} + \frac{1}{3}\boldsymbol{a}) + \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c})$ ✓
$= \frac{1}{3}\boldsymbol{a}$ ✓

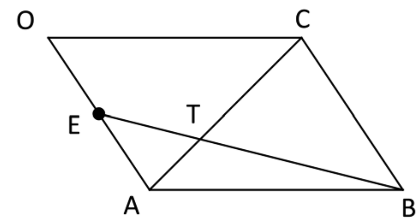
(c) Prove that CEFB is a parallelogram.

Clearly **EF = CB**.
Hence, CEFB has a pair of opposite sides that are parallel and congruent. ✓
Hence, CEFB is a parallelogram. ✓

Calculator Assumed

4. [7 marks: 3, 4]

OABC is a parallelogram with E as the midpoint of OA. The diagonal AC intersects the line segment BE at T such that AT = αAC and ET = βEB.
Let **OA** = ***a*** and **OC** = ***c***.



(a) Determine in terms of ***a*** and/or ***c***, the vectors **AT** and **ET**.

$$\mathbf{AT} = \alpha\mathbf{AC}$$
$$= \alpha(-\boldsymbol{a} + \boldsymbol{c}) \quad ✓$$
$$\mathbf{ET} = \beta\mathbf{EB}$$
$$= \beta(\mathbf{EA} + \mathbf{AB}) \quad ✓$$
$$= \beta(\tfrac{1}{2}\boldsymbol{a} + \boldsymbol{c}) \quad ✓$$

(b) Use **EA = ET + TA** to show that T divides the diagonal AC in the ratio 1 : 2.

$$\mathbf{EA} = \mathbf{ET} + \mathbf{TA}$$
$$\frac{1}{2}\boldsymbol{a} = \beta(\frac{1}{2}\boldsymbol{a} + \boldsymbol{c}) + \alpha(-\boldsymbol{c} + \boldsymbol{a}) \quad ✓$$
$$(\frac{1}{2} - \frac{1}{2}\beta - \alpha)\,\boldsymbol{a} = (\beta - \alpha)\,\boldsymbol{c} \quad ✓$$

Hence: $\alpha = \beta$

and $\frac{1}{2} - \frac{1}{2}\beta - \alpha = 0$ ✓

$\alpha = \frac{1}{3}$. ✓

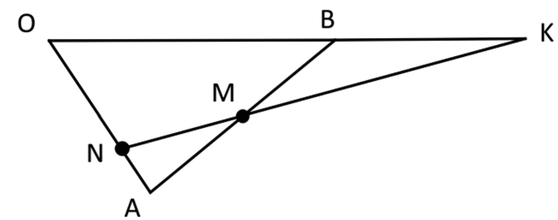
Therefore, T divides AC in the ratio 1 : 2.

Calculator Assumed

5. [8 marks: 4, 4]

In the diagram below, M is the midpoint of AB. N is a point on the line OA. OB extended meets NM extended at K. $\mathbf{ON} = \frac{3}{4}\mathbf{OA}$, **MK** = α**NK** and **BK** = β**OB**.

Let **OA** = ***a*** and **OB** = ***b***.



(a) Determine in terms of ***a*** and/or ***b***, the vectors **MB**, **OK** and **MK**.

$$\mathbf{MB} = \frac{1}{2}\mathbf{AB}$$
$$= \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b}) \quad ✓$$
$$\mathbf{OK} = \mathbf{OB} + \mathbf{BK}$$
$$= (1 + \beta)\boldsymbol{b} \quad ✓$$
$$\mathbf{MK} = \alpha\mathbf{NK}$$
$$= \alpha(\mathbf{NO} + \mathbf{OK}) \quad ✓$$
$$= \alpha(\frac{-3}{4}\boldsymbol{a} + (1 + \beta)\boldsymbol{b}) \quad ✓$$

(b) Use the vectors **BK**, **MB** and **MK** to determine the values of α and β.

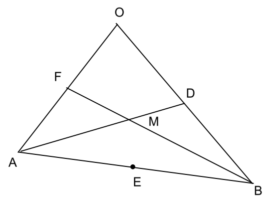
$$\mathbf{MK} = \mathbf{MB} + \mathbf{BK} \quad ✓$$
$$\alpha(\frac{-3}{4}\boldsymbol{a} + (1 + \beta)\boldsymbol{b}) = \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b}) + \beta\boldsymbol{b}$$
$$\frac{-3\alpha}{4}\boldsymbol{a} + \alpha(1 + \beta)\boldsymbol{b} = \frac{-1}{2}\boldsymbol{a} + \left(\frac{1}{2} + \beta\right)\boldsymbol{b} \quad ✓$$

Hence:

$$\frac{-3\alpha}{4} = \frac{-1}{2} \quad \Rightarrow \alpha = \frac{2}{3} \quad ✓$$
$$\alpha(1 + \beta) = \left(\frac{1}{2} + \beta\right) \quad \Rightarrow \beta = \frac{1}{2} \quad ✓$$

Calculator Assumed

6. [14 marks: 2, 2, 5, 3, 2]



OAB is a triangle with **OA** = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. D, E and F are the midpoints of OB, AB, and OA respectively. **AM** = α**AD** and **MF** = β**BF**.

(a) Find **AD** and **BF** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

> **AD** = $-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$ ✓
> **BF** = $\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}$ ✓

(b) Find **AM** and **MF** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

> **AM** = $\alpha(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}) = -\alpha\boldsymbol{a} + \frac{\alpha}{2}\boldsymbol{b}$ ✓
> **MF** = $\beta(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}) = \frac{\beta}{2}\boldsymbol{a} - \beta\boldsymbol{b}$ ✓

(c) Use your answers in (b) to find α and β.

> **AF** = **AM** + **MF** ✓
> $-\frac{1}{2}\boldsymbol{a} = (-\alpha\boldsymbol{a} + \frac{\alpha}{2}\boldsymbol{b}) + (\frac{\beta}{2}\boldsymbol{a} - \beta\boldsymbol{b}) = (-\alpha + \frac{\beta}{2})\boldsymbol{a} + (\frac{\alpha}{2} - \beta)\boldsymbol{b}$ ✓
> Compare coefficients for $\boldsymbol{b}$ and $\boldsymbol{a}$ vectors:
> $\frac{\alpha}{2} - \beta = 0 \Rightarrow \beta = \frac{\alpha}{2}$ ✓
> $-\alpha + \frac{\beta}{2} = -\frac{1}{2} \Rightarrow -\alpha + \frac{1}{4}\alpha = -\frac{1}{2} \Rightarrow \alpha = \frac{2}{3},\ \beta = \frac{1}{3}$ ✓✓

(d) Show that **OM** = μ**OE** giving the value of μ.

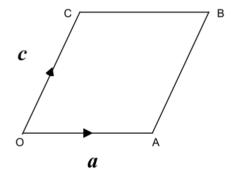
> **OM** = **OA** + **AM**
> $= \boldsymbol{a} + (-\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}) = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$ ✓
> **OE** = **OA** + **AE**
> $= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ ✓
> Hence, **OM** = $\frac{2}{3}$**OE** ✓

(e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.

> AD and BF are medians of ΔOAB. AD and BF meet at M, such that AM = $\frac{2}{3}$AD and BM = $\frac{2}{3}$BF.
> Since OM = $\frac{2}{3}$OE, and OE is also a median, all three medians ✓
> meet at M, two-thirds down from the respective vertices. ✓

Calculator Assumed

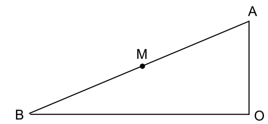
7. [4 marks]



OABC is a rhombus. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$.
Use a vector method to show that the diagonals of a rhombus are perpendicular to each other.

> Diagonal **AC** = $-\boldsymbol{a} + \boldsymbol{c}$
> Diagonal **OB** = $\boldsymbol{a} + \boldsymbol{c}$
>
> **AC** • **OB** = $(-\boldsymbol{a} + \boldsymbol{c}) \bullet (\boldsymbol{a} + \boldsymbol{c})$ ✓
> $= -\boldsymbol{a} \bullet \boldsymbol{a} - \boldsymbol{a} \bullet \boldsymbol{c} + \boldsymbol{c} \bullet \boldsymbol{a} + \boldsymbol{c} \bullet \boldsymbol{c}$
> $= -|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$ ✓
> $= 0$ since for a rhombus $|\boldsymbol{a}| = |\boldsymbol{c}|$. ✓✓

8. [8 marks: 1, 3, 4]



OAB is a right angled triangle with ∠ AOB = 90°.
OA = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. M is the midpoint of AB.

(a) Explain why $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

> OA is perpendicular to OB.
> Hence, **OA** . **OB** = 0.
> Therefore, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. ✓

(b) Find $|\mathbf{BM}|^2$ in terms of a and b, where $|\boldsymbol{a}| = a$ and $|\boldsymbol{b}| = b$.

> **BM** = $\frac{1}{2}$**BA** = $\frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$ ✓
> $|\mathbf{BM}|^2 = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b}) \cdot \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$ ✓
> $= \frac{1}{4}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,\boldsymbol{a} \cdot \boldsymbol{b})$
> $= \frac{1}{4}(a^2 + b^2)$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ✓

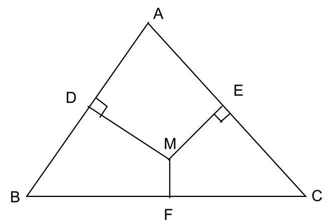
(c) Hence, prove that M is the centre of a circle passing through A, B and O.

> **OM** = $\boldsymbol{b}$ + **BM** = $\boldsymbol{b} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$
> $= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ ✓
> $|\mathbf{OM}|^2 = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \cdot \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$
> $= \frac{1}{4}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a} \cdot \boldsymbol{b})$
> $= \frac{1}{4}(a^2 + b^2)$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ✓
> Hence, $|\mathbf{OM}| = |\mathbf{BM}| = |\mathbf{MA}|$. ✓
> Therefore, M is equidistant from O, A and B.
> M is then the centre of a circle passing through A, B and O. ✓

Calculator Assumed

9. [10 marks: 1, 2, 2, 3, 2]

DM and EM are respectively the perpendicular bisectors of sides AB and AC of triangle ABC. F is midpoint of BC. Also, **AB** = $\boldsymbol{b}$, **AC** = $\boldsymbol{c}$ and **MD** = $\boldsymbol{d}$.



(a) Find **ME** in terms of $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

> **ME** = **MD** + **DA** + **AE**
> $= \boldsymbol{d} + \frac{1}{2}(-\boldsymbol{b}) + \frac{1}{2}\boldsymbol{c}$
> $= \boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})$ ✓

(b) Use your answer in (a) to show that $[\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c} = 0$

> ME is perpendicular to AC. ✓
> Hence, **ME** . **AC** = 0 .
> Therefore, $[\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c} = 0$ ✓

(c) Find **MF** in terms of $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

> **MF** = **MD** + **DB** + **BF**
> $= \boldsymbol{d} + \frac{1}{2}(\boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})$ ✓
> $= \boldsymbol{d} + \frac{1}{2}\boldsymbol{c}$ ✓

(d) Show that **MF . BC** = 0.

> **MF . BC** $= (\boldsymbol{d} + \frac{1}{2}\boldsymbol{c}) \cdot (\boldsymbol{c} - \boldsymbol{b})$ ✓
> $= \boldsymbol{d} \cdot \boldsymbol{c} - \boldsymbol{d} \cdot \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{c}$
> $= \boldsymbol{d} \cdot \boldsymbol{c} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{c}$ since $\boldsymbol{d} \cdot \boldsymbol{b} = 0$ ✓
> $= [(\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c}$ ✓
> $= 0$

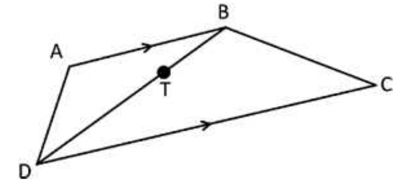
(e) State the significance of the result **MF . BC** = 0 .

> **MF . BC** = 0 ⇒ **MF** is perpendicular to **BC.**
> But F is the midpoint of BC.
> Hence, **MF** is the perpendicular bisector of **BC.** ✓
>
> MD, ME are also perpendicular bisectors.
> Hence, the perpendicular bisectors of the sides of a triangle meet at the same point (in this case, M). ✓

Calculator Assumed

10. [10 marks: 1, 1, 2, 2, 3, 1]

In the quadrilateral ABCD shown below **AB** = $\boldsymbol{u}$ and **DA** = $\boldsymbol{v}$.
DC = 2AB and the point T divides the diagonal DB in the ratio 2:1.



(a) Determine in terms of $\boldsymbol{u}$ and/or $\boldsymbol{v}$:

(i) **DB**

> **DB** = $\boldsymbol{u} + \boldsymbol{v}$ ✓

(ii) **DT**

> **DT** = $\frac{2}{3}(\boldsymbol{u} + \boldsymbol{v})$ ✓

(iii) **CT**

> **CT** = **CD** + **DT** ✓
> $= -2\boldsymbol{u} + \frac{2}{3}(\boldsymbol{u} + \boldsymbol{v})$
> $= \frac{-4}{3}\boldsymbol{u} + \frac{2}{3}\boldsymbol{v}$ ✓

(iv) **TA**

> **TA** = **TD** + **DA** ✓
> $= -\frac{2}{3}(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{v}$
> $= \frac{-2}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v}$ ✓

(b) Hence, or otherwise:

(i) prove that the points C, T and A are collinear.

> **TA** $= \frac{-2}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v}$ and **CT** $= \frac{-4}{3}\boldsymbol{u} + \frac{2}{3}\boldsymbol{v}$
> Hence, **CT** = 2**TA** ✓
> ⇒ CT is parallel to TA. ✓
> ⇒ With C as a point in common, C, T and A are collinear. ✓

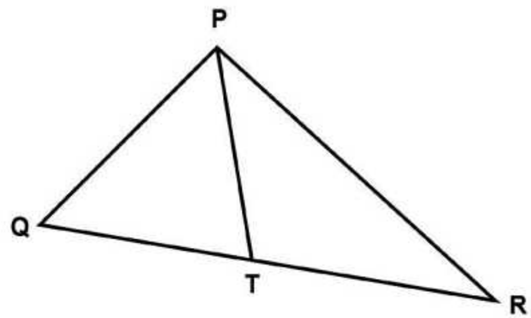
(ii) determine the ratio T divides the line CA.

> Since **CT** = 2**TA**
> ⇒ T divides the line CA in the ratio 2:1. ✓

Calculator Assumed

11. [8 marks: 3, 5] [TISC]

In ΔPQR drawn below, the point T is the midpoint of QR. Let **PT** = $\boldsymbol{a}$ and **TR** = $\boldsymbol{b}$.



(a) Find **PR** and **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

PR = PT + TR
$= \boldsymbol{a} + \boldsymbol{b}$ ✓
PQ = PT + TQ
But **TQ** = −**TR** = $-\boldsymbol{b}$ ✓
Hence, **PQ** = $\boldsymbol{a} - \boldsymbol{b}$ ✓

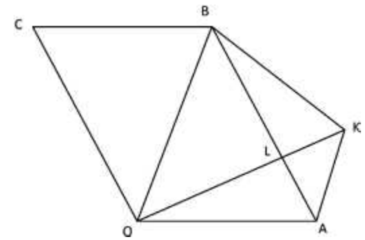
(b) If T is equidistant to P and R, use a vector method to prove that ∠QPR = 90°.

Since PT = TR $\Rightarrow$ $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$ ✓
PR . PQ = $(\boldsymbol{a}+\boldsymbol{b}) . (\boldsymbol{a}-\boldsymbol{b})$ ✓
$= \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b}$
$= \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b}$
$= \lvert\boldsymbol{a}\rvert^2 - \lvert\boldsymbol{b}\rvert^2$ ✓
$= 0$ ✓
Hence, PR is perpendicular to PQ and ∠QPR = 90°. ✓

Calculator Assumed

12. [8 marks: 1, 3, 4]

OABC is a parallelogram with **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. L is point on AB and OL is extended to K. **AL** = α **AB** and **LK** = $\frac{1}{2}$ **OL**.



(a) Express each of the following vectors in terms of $\boldsymbol{a}$ and or $\boldsymbol{c}$.

(i) **OB**

OB = $\boldsymbol{a} + \boldsymbol{c}$ ✓

(ii) **LK**

OL = OA + AL ✓
= **OA** + α **AB**
$= \boldsymbol{a} + \alpha\,\boldsymbol{c}$ ✓
LK = $\frac{1}{2}$ **OL** = $\frac{1}{2}(\boldsymbol{a} + \alpha\,\boldsymbol{c})$ ✓

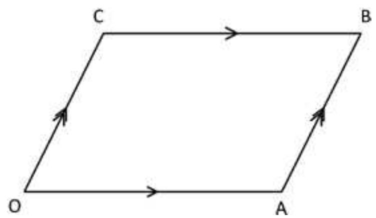
(b) Use vectors to determine the value for α if OAKB is a trapezium.

OB = $\boldsymbol{a} + \boldsymbol{c}$
AK = AL + LK
$= \alpha\,\boldsymbol{c} + \frac{1}{2}(\boldsymbol{a} + \alpha\,\boldsymbol{c})$ ✓
OB parallel to **AK**:
$\Rightarrow$ **AK** = λ **OB** ✓
$\alpha\,\boldsymbol{c} + \frac{1}{2}(\boldsymbol{a} + \alpha\,\boldsymbol{c}) = \lambda(\boldsymbol{a} + \boldsymbol{c})$ ✓
$\frac{1}{2}\boldsymbol{a} + \frac{3}{2}\alpha\,\boldsymbol{c} = \lambda\,\boldsymbol{a} + \lambda\,\boldsymbol{c}$
$\lambda = \frac{1}{2} \Rightarrow \alpha = \frac{1}{3}$ ✓

Calculator Assumed

13. [9 marks: 3, 6]

OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***.



(a) Prove that **OB** • **AC** = $|c|^2 - |a|^2$.

> **OB** = $a + c$
> **AC** = $-a + c$ ✓
> **OB** • **AC** = $(c + a)\bullet(c - a)$ ✓
> $= c\bullet c - a\bullet c + a\bullet c - a\bullet a$ ✓
> $= |c|^2 - |a|^2$

(b) Hence, prove that the diagonals of a parallelogram are perpendicular *if and only if* the parallelogram is a rhombus.

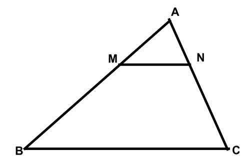
> When diagonals are perpendicular, **OB** • **AC** = 0. ✓
> Hence: $|c|^2 - |a|^2 = 0$ ✓
> $\Rightarrow |a| = |c|$
> $\Rightarrow$ Parallelogram is a rhombus. ✓
>
> If parallelogram is a rhombus, $|a| = |c|$ ✓
> Hence: **OB** • **AC** = $|c|^2 - |a|^2$
> $= 0$ ✓
> Hence, diagonals are perpendicular. ✓

16 Geometric Proofs and Circle Properties

Calculator Free

1. [7 marks: 3, 2, 2]

In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio 1 : 3.



(a) Prove that ΔAMN and ΔABC are similar.

> $\frac{AM}{AB} = \frac{AN}{AC} = \frac{1}{4}$ (given) ✓
> ∠MAN = ∠BAC (common) ✓
> Hence, ΔAMN and ΔABC are similar (SAS). ✓

(b) Hence, deduce that BC = 4MN.

> Since ΔAMN and ΔABC are similar, $\frac{AM}{AB} = \frac{MN}{BC}$. ✓
> But $\frac{AM}{AB} = \frac{1}{4}$. $\Rightarrow \frac{MN}{BC} = \frac{1}{4}$. $\Rightarrow$ BC = 4MN. ✓

(c) Prove that MN is parallel to BC.

> Since ΔAMN and ΔABC are similar, ∠AMN = ∠ABC. ✓
> Hence, MN must be parallel to BC. These angles being corresponding angles. ✓

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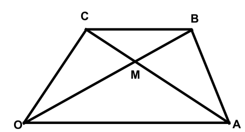
2. [5 marks: 3, 2]

OABC is a trapezium with OA parallel to CB.
The diagonals OB and AC intersect at M such that AM : MC = 3 : 1 .



(a) Prove that ΔMOA and ΔMBC are similar.

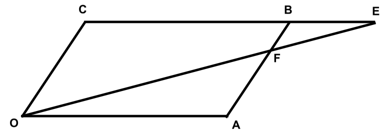
> ∠OMA = ∠BMC (vertically opposite) ✓
> ∠AOM = ∠CBM (alternate angles, OA parallel to CB) ✓
> Hence ΔMOA and ΔMBC are similar (AA). ✓

(b) Hence deduce that OA = 3BC.

> Since ΔMOA and ΔMBC are similar, $\frac{AM}{CM} = \frac{OA}{BC}$. ✓
> But $\frac{AM}{CM} = \frac{3}{1}$. $\Rightarrow \frac{OA}{BC} = \frac{3}{1}$. $\Rightarrow$ OA = 3BC. ✓

Calculator Free

3. [5 marks: 3, 2]



OABC is a parallelogram with OA parallel and congruent to CB. The point F divides AB in the ratio 2 : 1. OF extended meets the CB extended at E.

(a) Prove that ΔFOA and ΔFEB are similar.

∠OFA = ∠EFB (vertically opposite) ✓
∠AOF = ∠BEF (alternate angles, OA parallel to BE) ✓
Hence ΔFOA and ΔFEB are similar (AA). ✓

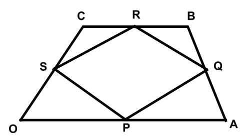
(b) Hence, deduce that F divides OE in the ratio 2 : 1.

Since ΔFOA and ΔFEB are similar, $\frac{FA}{FB} = \frac{FO}{FE}$. ✓

But $\frac{FA}{FB} = \frac{2}{1}$. ⇒ $\frac{FO}{FE} = \frac{2}{1}$. ✓

⇒ F divides OE in the ratio 2 : 1.

4. [5 marks]



OABC is a trapezium with OA parallel to CB. P, Q, R and S are respectively the midpoints of OA, AB, BC and OC. Prove that the midpoints of the sides of a trapezium form a parallelogram, that is PQRS is a parallelogram.

$\frac{CR}{CB} = \frac{CS}{CO} = \frac{1}{2}$ [Given: R and S midpoints of CB and CO respectively]
∠SCR = ∠OCB [Common]
Hence, ΔCSR and COB are similar (SAS). ✓
Therefore SR is parallel to OB and SR = $\frac{1}{2}$OB. ✓

$\frac{AQ}{AB} = \frac{AP}{AO} = \frac{1}{2}$ [Given: Q and P midpoints of AB and AO respectively]
∠QAP = ∠BAO [Common]
Hence, ΔAPQ and AOB are similar (SAS). ✓
Therefore PQ is parallel to OB and PQ = $\frac{1}{2}$OB. ✓

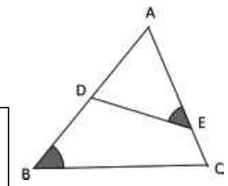
Hence, SR is congruent and parallel to PQ. [SR = PQ = $\frac{1}{2}$OB]

Therefore, PQRS has a pair of parallel congruent sides, which makes it a parallelogram. ✓

Calculator Free

5. [6 marks]

In the accompanying diagram, ∠ABC = ∠AED.
Also, AD = DB = 3 cm and AC = 5 cm.
Calculate with reasons, the length of EC.



∠ABC = ∠AED Given.
∠BAC = ∠DAE Common. ✓

Hence, $\Delta\frac{ABC}{AED}$ are similar. AA ✓

Corresponding sides of similar triangles have similar ratios:

$\Rightarrow \frac{AC}{AD} = \frac{AB}{AE}$ ✓

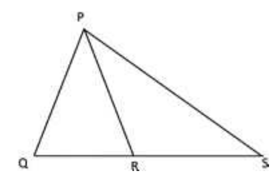
$\frac{5}{3} = \frac{6}{AE}$ ✓

$AE = \frac{18}{5} = 3.6$ ✓

Hence, EC = 5 – 3.6 = 1.4 cm ✓

6. [6 marks: 3, 3]

The diagram below shows the isosceles triangle PQR where PQ = 5 cm. The base QR is extended to S so that QS = PS = 8 cm.



(a) Prove that the triangles PQR and SQP are similar.

∠PQR = ∠SQP Common. ✓

$\frac{PQ}{SQ} = \frac{PR}{SP} = \frac{5}{8}$ Given lengths. ✓

Hence $\Delta\frac{PQR}{SQP}$ are similar. (SAS) ✓

(b) Determine the length of RS. Show clearly how you obtained your answer.

$\frac{QR}{QP} = \frac{PR}{SP} = \frac{5}{8}$ ✓

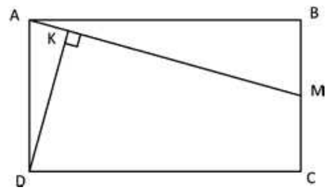
$\frac{QR}{5} = \frac{5}{8} \Rightarrow QR = \frac{25}{8}$ ✓

Therefore: $RS = 8 - \frac{25}{8} = \frac{39}{8}$ ✓

Calculator Free

7. [9 marks: 4, 5]

In the diagram below, ABCD is a rectangle. M is a point on the side BC. The line segment DK is perpendicular to AM.



(a) Prove that triangles AKD and MBA are similar.

$\angle AKD = \angle MBA = 90^o$	Given.	✓
Let $\angle KAD = \theta$.		
Then $\angle BAM = 90 - \theta$	$\angle KAD$ & $\angle BAM$ are complementary.	✓
$\Rightarrow \angle BMA = 180 - 90 - (90 - \theta) = \theta$	Sum of angles in a Δ is 180^o.	
Hence $\angle KAD = \angle BMA$		✓
Therefore Δ^{AKD}_{MBA} are similar.	AA	✓

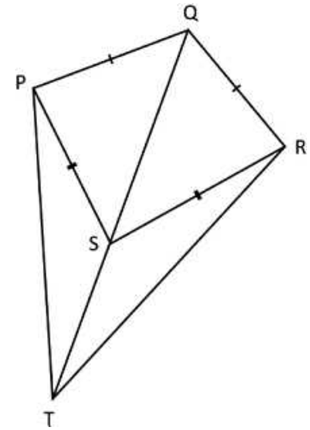
(b) Given that AK : MB = α, find in terms of α, the ratio of the area of ΔAKD to the area of ΔMBA.

$\frac{AK}{MB} = \frac{KD}{BA} = \alpha$	
Hence $AK = \alpha MB$ and $KD = \alpha AB$	✓
Area of $\Delta MBA = \frac{1}{2} MB \times AB$	✓
Area of $\Delta AKD = \frac{1}{2} AK \times KD$	✓
$= \frac{1}{2} \alpha MB \times \alpha AB$	
$= \alpha^2 \left(\frac{1}{2} MB \times AB\right)$	✓
Hence: $\frac{\text{Area of } \Delta AKD}{\text{Area of } \Delta MBA} = \alpha^2$.	✓

Calculator Free

8. [8 marks]

In the diagram shown below, PQRS is a quadrilateral with PQ = QR and SP = SR. The points Q, S and T are collinear. Prove that TP = TR.

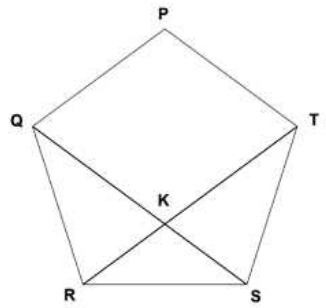


For ΔQPS and ΔQRS:		
QP = QR	Given.	
SP = SR	Given.	
QS is common.		
Hence Δ^{QPS}_{QRS} are congruent.	SSS	✓✓✓
For ΔQPT and ΔQRT:		
QP = QR	Given.	
QT is common.		
$\angle PQT = \angle RQT$	Corresponding angles Δ^{QPS}_{QRS} are congruent.	
Hence Δ^{QPT}_{QRT} are congruent.	SAS	✓✓✓
Therefore: TP = TR	Corresponding sides Δ^{QPT}_{QRT} are congruent.	✓✓

Calculator Assumed

9. [9 marks: 2, 2, 5]

PQRST is a *regular* pentagon of side length 10 cm.



(a) Find the size of ∠ RST. Justify your answer.

Since PQRST is a pentagon, the sum of all its interior angles adds up to 540°. ✓
As PQRST is a regular pentagon, all five interior angles are congruent.
Hence, interior angle ∠ RST = $\frac{540}{5}$ = 108°. ✓

(b) Prove that ∠ STR = 36°.

SR = ST — sides of a regular pentagon
Hence, Δ STR is isosceles. ✓
Hence, ∠ STR = ∠ SRT = $\frac{180-108}{2}$ = 36°. ✓

(c) Find the length of KT. Show clearly how you obtained your answer.

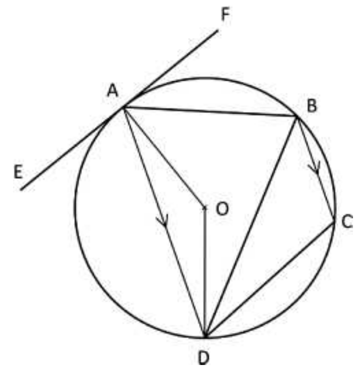
QR = TS	Sides of a regular pentagon
RS is common.	
∠ QRS = ∠ TSR	Interior angle of a regular pentagon
Hence, Δ QRS, TSR are congruent.	SAS ✓✓

⇒ ∠ QSR = ∠ TRS = 36°.
∠ TSK = ∠ TSR – ∠ QSR = 108 – 36 = 72°. ✓
In Δ TKS, ∠TKS = 180 – 36 – 72 = 72°. Sum angles in a triangle.
Therefore Δ TKS is isosceles. ✓
⇒ TK = TS = 10 cm. ✓

Calculator Assumed

10. [6 marks: 4, 2]

ABCD is a cyclic quadrilateral with the sides AD parallel to BC. EAF is a tangent to the circle.



(a) Prove that ∠DAE = ∠DBA.
That is, prove that the angle between a chord and tangent is equal to the angle in the alternate segment.

Let ∠DAE = θ.	
Then ∠OAD = 90 – θ.	Radius OA perpendicular to tangent AE ✓
∠OAD = ∠ODA = 90 – θ	Base angles of isosceles triangle OAD, OA = OD. ✓
∠DOA = 180 – 2(90 – θ)	
= 2θ	Sum of angles in a triangle is 180°. ✓
Hence, ∠DBA = θ	Angle at centre, twice angle at circumference. ✓

(b) Hence, prove that ∠FAB = ∠DBC.

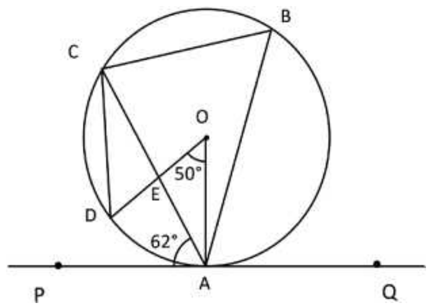
∠FAB = ∠ADB	Angle between chord and tangent equals angle in the alternate segment. ✓
∠ADB = ∠DBC	Alternate angles, AB parallel to BC. ✓

Hence, ∠FAB = ∠DBC.

Calculator Assumed

11. [8 marks: 2, 3, 3]

The diagram below shows a circle centre O. PAQ is a tangent to the circle at A. B, C and D are points on the circumference of the circle. OD and AC intersect at E.



Determine with reasons, the size of each of the following angles.

(a) ∠OAC

∠OAC = 90 – 62 = 28°	✓
∠OAP = 90°; radius OA perpendicular to tangent PAQ.	✓

(b) ∠ODC

∠CEO = 50 + 28 = 78	Exterior angle of a triangle is the sum of the 2 interior angles.	✓
∠DCE = 25	Angle on circumference half angle at centre.	✓
∠ODC = 78 – 25 = 53°	Exterior angle of a triangle is the sum of the 2 interior angles.	✓

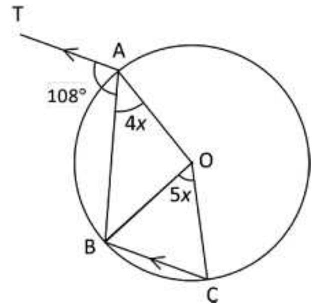
(c) ∠ACB

∠BAC = ∠ODC = 53	Angles in the same segment.	✓
∠ABC = ∠CAP = 62	Angle between chord and tangent = interior opposite angle.	✓
∠ACB = 180 – 53 – 62 = 65°	Sum of angles in a triangle = 180°.	✓

Calculator Assumed

12. [6 marks]

The points A, B and C lie on a circle centre O. The line AT is parallel to BC. ∠BAT = 108°, ∠BAO = $4x$ and ∠BOC = $5x$.



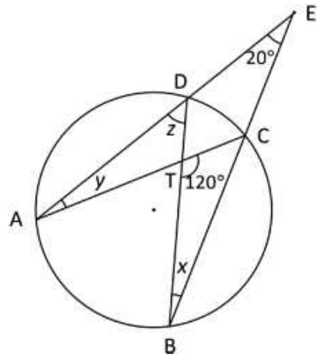
Find with reasons ∠BAC.

∠OBA = ∠OAB = $4x$	Base angles of isosceles ΔOAB	✓
∠OBC = $\frac{1}{2}(180-5x)$	Base angle of isosceles ΔOBC	✓
Hence, ∠ABC = $4x + \frac{1}{2}(180-5x)$ $= \frac{3x}{2} + 90$		✓
But ∠ABC = ∠BAT = 108	Alternate angles CB parallel to AT	✓
Hence, $\frac{3x}{2} + 90 = 108$ $x = 12°$		
∠BAC = $\frac{1}{2}$ ∠BOC	Angle at centre twice angle at circumference.	✓
$= \frac{1}{2} \times 5 \times 12 = 30°$		✓

Calculator Assumed

13. [7 marks: 2, 5]

A, B, C and D are points on the circumference of a circle. AC meets BD at T. AD extended meets BC extended at E. $\angle AEB = 20^\circ$ and $\angle BTC = 120^\circ$.



(a) State $\angle ACE$ in terms of x. Justify your answer.

$\angle ACE = 120 + x$ ✓
External angle of triangle
is the sum of the two internally opposite angles. ✓

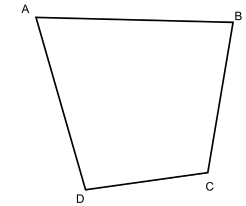
(b) Determine with reasons the values of x, y and z.

$\angle EAC = \angle DBE \Rightarrow x = y$	Angles in the same segment.	✓
Hence, in ΔACE:		
$x + 120 + x + 20 = 180$	Sum of angles in a Δ is 180°.	✓
$x = 20^\circ$		
$y = 20^\circ$		✓
$\angle ADB = \angle AEB + \angle EBD$	External angle of triangle is the sum of the two internally opposite angles.	✓
Hence: $z = x + 20 = 40^\circ$		✓

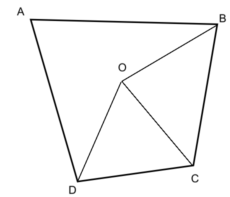
Calculator Assumed

14. [7 marks: 4, 3] [TISC]

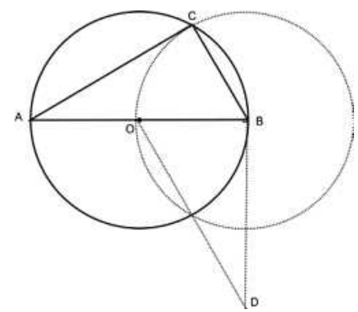
(a) ABCD is a quadrilateral.
$\angle DAB + \angle DCB = 180^\circ$ and
$\angle ADC + \angle ABC = 180^\circ$.
Prove that there is a circle that passes through A, B, C and D.



Let O be the centre of the circle passing through B, C and D.
Let $\angle DCB = x^\circ$.
Then, reflex $\angle DOB = 2x^\circ$. ✓
[Angle at centre = 2 × angle at circumference]
$\Rightarrow$ obtuse $\angle DOB = 360^\circ - 2x^\circ = 2 \times (180^\circ - x^\circ)$. ✓
But given that $\angle DAB = 180^\circ - x^\circ$.
Hence, obtuse $\angle DOB = 2 \times \angle DAB$. ✓
[Angle at centre = 2 × angle at circumference]
Therefore, A could lie on the same circle as B, C and D. ✓



(b) AB is the diameter of the circle with centre O. B is the centre of another circle passing through O. The two circles intersect at C. BD is a tangent to the circle with centre O. If AC = BD, prove that $\angle BOD = \angle CBA$.

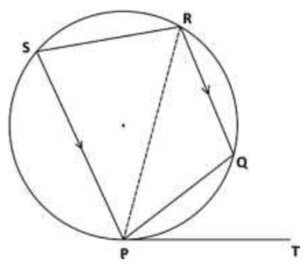


AC = BD	Given		
BC = BO	Radius circle centre B.		✓
$\angle ACB = 90^\circ$	Angle in a semi-circle.		
$\angle OBD = 90^\circ$	BO is a tangent to the circle with radius OB		
$\Rightarrow \angle ACB = \angle OBD$			✓
Hence, ΔBCA and ΔOBD are congruent.		SAS or RHS	✓
Hence, $\angle BOD = \angle CBA$.			

Calculator Assumed

15. [7 marks: 3, 4] [TISC]

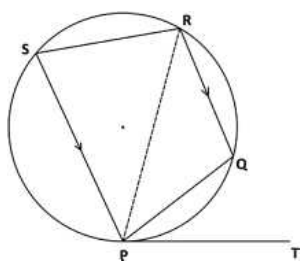
PQRS is a cyclic quadrilateral with RQ parallel to SP. PT is a tangent to the circle. The line PR bisects ∠SPQ.



(a) Prove that PQ bisects ∠TPR.

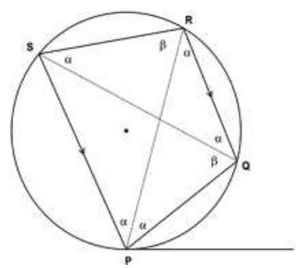
Let ∠SPR = ∠RPQ = α	Given PR bisects ∠SPQ	
But ∠SPR = ∠PRQ = α	Alternate angles: RQ parallel to SP.	✓
Also ∠TPQ = ∠PRQ = α	Angle between tangent and chord = angle in the opposite segment	✓
Hence ∠TPQ = ∠QPR = α		✓
Therefore, PQ bisects ∠TPR.		

(b) Prove that PQ = SR.
[Hint: Prove that ΔRPQ is congruent to ΔQSR.]



In ΔRPQ and ΔQSR, RQ is common.

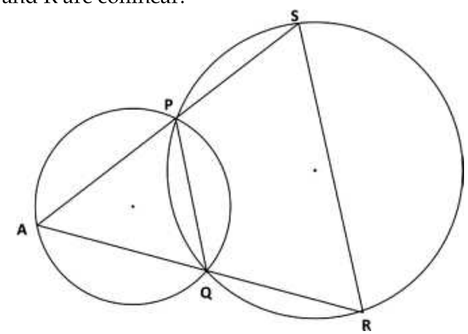
Let ∠SPR = ∠RPQ = α	Given PR bisects ∠SPQ	
∠SPR = ∠PRQ = α	Alternate angles: PQ parallel to SP.	
∠SPR = ∠SQR = α	Angle in the same segment (chord SR)	✓
Hence, ∠SQR = ∠PRQ = α		✓
∠SRP = ∠ SQP = β	Angle in the same segment (chord SP)	✓
Hence ∠SRQ = ∠PQR = α + β		
Hence, $\Delta\frac{RPQ}{QSR}$ are congruent	(ASA)	✓
⇒ PQ = SR		



Calculator Assumed

16. [7 marks: 3, 4]

In the diagram below, the two circles intersect at P and Q. S and P are points on the circumference of the larger circle. The points A, P and S are collinear. The points A, Q and R are collinear.



(a) Prove that ΔAPQ and ΔARS are similar.

Let ∠APQ = x^o.		
Hence, ∠SPQ = 180 – x^o.	Supplementary to ∠PAB.	✓
Then, ∠SRQ = x^o.	Opposite angles of a cyclic quadrilateral are supplementary, PQRS is a cyclic quadrilateral.	✓
Hence, ∠APQ = ∠SRQ		
∠PAQ = ∠RAS	Common.	
Hence, ΔAPQ and ΔARS are similar.	AA	✓

(b) Given that AQ = 10 cm, AP = QR = 8 cm, find PS.

ΔAPQ and ΔARS are similar ⇒ $\frac{AS}{AQ} = \frac{AR}{AP}$ ✓

$\frac{AS}{10} = \frac{18}{8}$ ✓

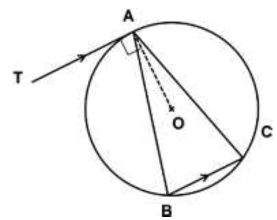
$AS = 10 \times \frac{18}{8} = \frac{45}{2}$ ✓

Therefore $PS = \frac{45}{2} - 8 = \frac{29}{2}$ cm ✓

Calculator Assumed

17. [3 marks]

The points A, B and C lie on a circle centre O. TA is the tangent to the circle at A. If TA is parallel to BC, prove that ΔABC is isosceles.

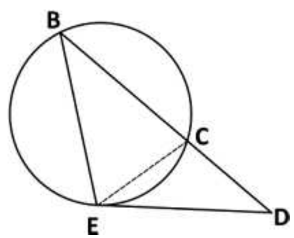


Let ∠BAT = α.		
⇒ ∠ABC = α	Alternate angles: TA parallel to BC	✓
∠ACB = ∠BAT = α	Angle in the alternate segment: TA is a tangent to the circle.	✓
Hence, ∠ACB = ∠ABC = α		✓
Therefore ΔABC is isosceles.		

18. [5 marks: 3, 2]

The points B, C and E lie on the same circle. The chord BC extended meets the tangent to the circle at E at the point D.

(a) Prove that $ED^2 = CD \times BD$.



Let ∠CDE = α and ∠CED = β.		
∠CBE = ∠CED = β	Alternate segment theorem.	
∠CDE = ∠BDE = α	Common	
Hence $\Delta\frac{CED}{EBD}$ are similar (AA).		✓✓
Therefore:	$\frac{CD}{ED} = \frac{ED}{BD}$	✓
	$ED^2 = CD \times BD$	

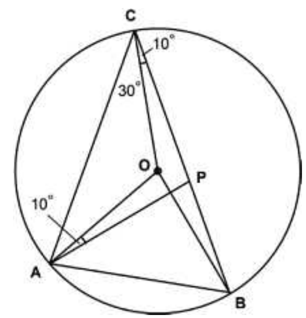
(b) BD is 20 cm long and the point C divides BD in the ratio 3: 2. Hence, find the length of DE.

BC = 12 cm and CD = 8 cm		
Hence	$ED^2 = 8 \times 20$	✓
	$ED = 4\sqrt{10}$ cm	✓

Calculator Assumed

19. [6 marks: 4, 2]

(a) The points A, B and C lie on a circle centre O. ∠ACO = 30° and ∠BCO = 10°. P is a point on the chord BC such that ∠OAP = 10°. Find with reasons ∠APB.



ΔAOC is isosceles as OA = OC =radius of circle.		
⇒ ∠OAC = ∠OCA = 30°	Base angles of isosceles ΔBOC	✓✓
∠APB is external to ΔAPC.		
Hence: ∠APB = ∠PAC + ∠PCA		✓
= 10° + 30° + 30° + 10°		
= 80°		✓

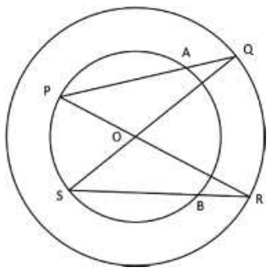
(b) If the points A, B and C lie on the circumference of a circle and O is a point inside the circle, prove or disprove the conjecture that if ∠AOB = 2∠ACB, then O must be the centre of the circle.

Conjecture is False.	✓
Counter-example: See part (a). ∠APB = 2∠ACB but P is not the centre of the circle.	✓

Calculator Assumed

20. [8 marks: 3, 2, 3]

The diagram below shows two concentric circles centred at O.
S and P are points on the smaller circle while Q and R are points on the larger circle.
PQ intersects the inner circle at A and SR intersects the inner circle at B.



(a) Prove that triangles OPQ and OSR are congruent.

∠POQ = ∠SOR	Vertically Opposite.	✓
OP = OS	Radius of smaller circle.	
OQ = OR	Radius of larger circle.	✓
Hence $\Delta\frac{OPQ}{OSR}$ are congruent.	SAS	✓

(b) Prove that the points P, Q, R and S are concyclic.

∠PQS = ∠PRS	Corresponding angles of congruent triangles.	✓
These angles are subtended by the same arc PS.		✓
Hence, P, Q, R and S are concyclic.		

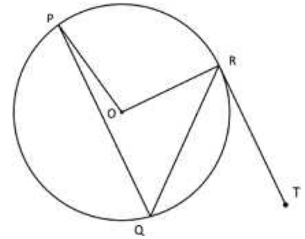
(c) Prove that AB is parallel to QR.

Let ∠PQR = x.		
Then ∠PSR = 180 – x	PQRS is a cyclic quadrilateral and opposite angles of cyclic quadrilateral are supplementary.	✓
⇒ ∠PAB = x	PABS is a cyclic quadrilateral and opposite angles of cyclic quadrilateral are supplementary.	✓
Hence, ∠PQR = ∠PAB		
Since PAQ is a transversal, this establishes ∠PQR and ∠PAB as corresponding angles of two parallel lines.		✓
⇒ AB is parallel to QR.		

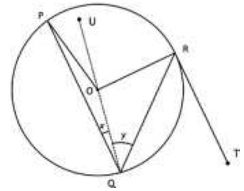
Calculator Assumed

21. [9 marks: 6, 3]

The diagram below shows a circle with centre at O. P, Q and R are points on the circle. RT is a tangent to the circle at R. Let the acute ∠PQR = θ.



(a) Prove that obtuse ∠POR = 2θ.



Construct line QOU to form two isosceles triangles.		✓
Let ∠OQP = x and ∠OQR = y. θ = $x + y$		
ΔOPQ is isosceles as OP = OQ = radius.		
⇒ ∠OPQ = ∠OQP = x	Base angles of isosceles are equal.	
⇒ ∠POU = 2x	External angle is sum of interior opposite angles.	✓✓
ΔORQ is isosceles as OR = OQ = radius.		
⇒ ∠OQR = ∠ORQ = y	Base angles of isosceles are equal.	
⇒ ∠POU = 2y	External angle is sum of interior opposite angles.	✓✓
Hence, ∠POR = 2x + 2y = 2θ		✓

(b) If PO is parallel to RT, prove that ∠PQR = 45°.

∠ORT = 90°	RT is a tangent to the circle and angle between tangent and radius is 90°.	✓
⇒ ∠ROP = 90°	Alternate angles PO parallel to RT.	✓
⇒ ∠PQR = 45°	Angle at circumference half angle at centre.	✓

Calculator Assumed

22. [6 marks: 4, 2]

Consider the premise:
If ABCD is a square, then the diagonals AC and BD are perpendicular to each other.

(a) State the converse of this premise.
Determine with reasons if the converse of this premise is true or false.

> Converse:
> If AC and BD are perpendicular, then ABCD is a square. ✓✓
>
> Converse is false. ✓
> ABCD could also be a kite or a rhombus
> as diagonals of kites and rhombuses are perpendicular. ✓

(b) State the contrapositive of this statement.

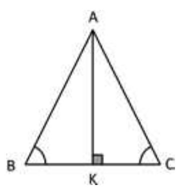
> Contrapositive:
> If AC and BD are not perpendicular, then ABCD is not a square. ✓✓

---

23. [5 marks]

Consider the premise: *For ΔABC, if AB = AC, then ∠ABC = ∠ACB.*
State the converse of this statement and prove that the converse of is true.

> Converse: For ΔABC, if ∠ABC = ∠ACB, then AB = AC. ✓



> Construct AK such that AK is perpendicular to BC. ✓
>
> ∠ABK = ∠ACK — Given.
> ∠AKB = ∠AKC — As constructed.
> AK is common.
>
> Hence $\Delta\begin{smallmatrix}ABK\\ACK\end{smallmatrix}$ are congruent. — AAS ✓✓
>
> ⇒ AB = AC. — Corresponding sides of two congruent triangles. ✓

17 Trigonometric Equations I (Simple trigonometric ratios)

Calculator Free

1. [13 marks: 3, 3, 4, 3]

Solve for all values of θ (in degrees):

(a) $\sin\theta = \cos\theta$

> $\sin\theta = \cos\theta \Rightarrow \tan\theta = 1$ ✓
> $\tan^{-1}\theta = 45^\circ$. ✓
> Hence, $\theta = 45^\circ + 180^\circ n \quad n \in \mathbb{Z}$ ✓

(b) $(\cos\theta - 2)(2\cos\theta + 1) = 0$

> $\Rightarrow \cos\theta = 2$ or $\cos\theta = -\frac{1}{2}$ ✓
> $\cos\theta = 2$ gives no solution.
> Hence, $\cos\theta = -\frac{1}{2}$
> $\cos^{-1}\theta = 120^\circ$ ✓
> $\Rightarrow \theta = 360^\circ n \pm 120^\circ \quad n \in \mathbb{Z}$ ✓

(c) $2\sin^2\theta + 3\sin\theta - 2 = 0$

> Factorise, $(2\sin\theta - 1)(\sin\theta + 2) = 0$ ✓
> $\sin\theta = \frac{1}{2}$ or $\sin\theta = -2$ ✓
> $\sin\theta = -2$ gives no solution.
> Hence, $\sin\theta = \frac{1}{2}$
> $\sin^{-1}\theta = 30^\circ$ ✓
> $\Rightarrow \theta = (-1)^n \times 30^\circ + 180^\circ n \quad n \in \mathbb{Z}$ ✓

(d) $\sec^2\theta - 4\sec\theta + 4 = 0$

> $(\sec\theta - 2)^2 = 0$ ✓
> $\sec\theta = 2$
> $\cos\theta = \frac{1}{2}$ ✓
> $\theta = 360^\circ n \pm 60^\circ \quad n \in \mathbb{Z}$ ✓

Calculator Free

2. [13 marks: 3, 3, 4, 3]

Given that $\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$, solve for θ in:

(a) $\cos(\theta + 5^\circ) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \le \theta \le 360^\circ$

> Reference angle for $\theta + 5^\circ = 15^\circ$. ✓
> $\theta + 5^\circ$ is in Quadrant 1 and Quadrant 4.
> Hence, $\theta + 5^\circ = 15^\circ, 345^\circ$
> $\theta = 10^\circ, 340^\circ$ ✓✓

(b) $\cos\theta = -\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right)$ for $0 \le \theta \le 360^\circ$

> Reference angle for $\theta = 15^\circ$. ✓
> θ is in Quadrant 2 and Quadrant 3.
> Hence, $\theta = 180^\circ - 15^\circ, 180^\circ + 15^\circ$
> $\theta = 165^\circ, 195^\circ$ ✓✓

(c) $\sin\theta = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \le \theta \le 360^\circ$

> $\sin\theta = \cos(90^\circ - \theta)$ ✓
> Hence: $\cos(90^\circ - \theta) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$
> $90^\circ - \theta$ is in Quadrant 1 and Quadrant 4.
> $90^\circ - \theta = 15^\circ, 345^\circ$ ✓
> $\theta = 75^\circ, -255^\circ$ ✓
> $\theta = 75^\circ, 105^\circ$ ✓

(d) $\sec\theta = \sqrt{6} - \sqrt{2}$ for $0 \le \theta \le 360^\circ$

> $\dfrac{1}{\cos\theta} = \sqrt{6} - \sqrt{2}$ ✓
> $\Rightarrow \cos\theta = \dfrac{1}{\sqrt{6}-\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ ✓
> $\theta = 15^\circ, 345^\circ$ ✓

Calculator Free

3. [11 marks: 1, 3, 4, 3]

Given that $\sin\left(\dfrac{\pi}{10}\right) = \dfrac{\sqrt{5}-1}{4}$, find all solutions (in radians) to:

(a) $\sin x = \dfrac{\sqrt{5}-1}{4}$

> $x = (-1)^n \times \left(\dfrac{\pi}{10}\right) + n\pi \quad n \in \mathbb{Z}$ ✓

(b) $\sin x = \dfrac{1-\sqrt{5}}{4}$

> $\sin x = -\left(\dfrac{\sqrt{5}-1}{4}\right)$ ✓
> $\sin^{-1} x = -\dfrac{\pi}{10}$ ✓
> $x = (-1)^n \times \left(-\dfrac{\pi}{10}\right) + n\pi \quad n \in \mathbb{Z}$ ✓

(c) $\cos x = \dfrac{\sqrt{5}-1}{4}$

> $\cos x = \sin\left(\dfrac{\pi}{2} - x\right)$
> Hence: $\sin\left(\dfrac{\pi}{2} - x\right) = \dfrac{\sqrt{5}-1}{4}$ ✓
> $\sin^{-1}\left(\dfrac{\pi}{2} - x\right) = \dfrac{\pi}{10}$ ✓
> $\left(\dfrac{\pi}{2} - x\right) = (-1)^n \times \left(\dfrac{\pi}{10}\right) + n\pi$ ✓
> $x = \dfrac{\pi}{2} - [(-1)^n \times \left(\dfrac{\pi}{10}\right) + n\pi] \quad n \in \mathbb{Z}$ ✓

(d) $\operatorname{cosec} x = 1 + \sqrt{5}$

> $\dfrac{1}{\sin x} = 1 + \sqrt{5}$ ✓
> $\Rightarrow \sin x = \dfrac{1}{1+\sqrt{5}} = \dfrac{\sqrt{5}-1}{4}$ ✓
> $x = (-1)^n \times \left(\dfrac{\pi}{10}\right) + n\pi \quad n \in \mathbb{Z}$ ✓

Calculator Free

4. [12 marks: 1, 2, 2, 3, 4]

(a) The equation cos $(x) = a$ has general solution $x = 2n\pi \pm \frac{3\pi}{4}$ for $n \in \mathbb{Z}$.

(i) Determine the exact value of a.

$$a = \cos\frac{3\pi}{4} = \frac{-\sqrt{2}}{2} \quad ✓$$

(ii) Find all solutions in the domain $0 \le x \le 2\pi$.

$n = 0$	$x = \frac{3\pi}{4}$	✓
$n = 1$	$x = 2\pi - \frac{3\pi}{4} = \frac{5\pi}{4}$	✓

(b) It is known that $\tan\alpha^{o} = \frac{\sqrt{7}}{3}$ where α is an acute angle.

For each of the following equations solve for all values of x^{o} in terms of α^{o}.

(i) $\tan x^{o} = \frac{-\sqrt{7}}{3}$

Reference angle for $x = \alpha$ ✓
Hence: $x = 180n - \alpha \quad n \in \mathbb{Z}$ ✓

(ii) $\sin x^{o} = \frac{\sqrt{7}}{4}$

$\tan\alpha^{o} = \frac{\sqrt{7}}{3} \Rightarrow \sin\alpha^{o} = \frac{\sqrt{7}}{4}$ ✓
Reference angle for $x = \alpha$ ✓
Hence: $x = (-1)^{n}\alpha + 180n \quad n \in \mathbb{Z}$ ✓

(iii) $\cos(2x^{o}) = \frac{3}{4}$

$\tan\alpha^{o} = \frac{\sqrt{7}}{3} \Rightarrow \cos\alpha^{o} = \frac{3}{4}$ ✓
Reference angle for $2x = \alpha$ ✓
Hence: $2x = 360n \pm \alpha$ ✓
$x = 180n \pm \frac{\alpha}{2} \quad n \in \mathbb{Z}$ ✓

18 Trigonometric Identities I (Pythagorean)

Calculator Free

1. [12 marks: 3, 3, 3, 3]

(a) Prove that $\tan x = \frac{1}{\tan\left(\frac{\pi}{2} - x\right)}$. for $x \ne \frac{\pi}{2}$.

$$\text{RHS} = \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)} \quad ✓$$
$$= \frac{\sin x}{\cos x} \quad ✓✓$$
$$= \tan x = \text{LHS}$$

(b) Prove that $\cos^3 x \tan x + \sin^3 x = \sin x$.

$$\text{LHS} = (\cos^3 x)\times\left(\frac{\sin x}{\cos x}\right) + \sin^3 x \quad ✓$$
$$= \cos^2 x \sin x + \sin^3 x$$
$$= \sin x(\cos^2 x + \sin^2 x) \quad ✓$$
$$= \sin x \quad ✓$$
$$= \text{LHS}$$

(c) Prove that $\frac{1}{\sin x \tan x + \cos x} = \cos x$.

$$\text{LHS} = \frac{1}{\sin x\left[\frac{\sin x}{\cos x}\right] + \cos x} \quad ✓$$
$$= \frac{1}{\frac{\sin^2 x}{\cos x} + \cos x} = \frac{1}{\frac{\sin^2 x + \cos^2 x}{\cos x}} \quad ✓$$
$$= \cos x \quad ✓$$
$$= \text{RHS}$$

(d) Prove that $\frac{\sin^3 x - \cos^3 x}{1 + \sin x \cos x} = \sin x - \cos x$.

$$\text{LHS} = \frac{(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)}{1 + \sin x \cos x} \quad ✓$$
$$= \frac{(\sin x - \cos x)(1 + \sin x \cos x)}{1 + \sin x \cos x} \quad ✓$$
$$= \sin x - \cos x \quad ✓$$
$$= \text{RHS}$$

Calculator Free

2. [11 marks: 3, 4, 4]

(a) Prove $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$.

$$\begin{aligned}
\text{LHS} &= \frac{(1-\sin\theta)+(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \quad ✓\\
&= \frac{2}{1-\sin^2\theta} \quad ✓\\
&= \frac{2}{\cos^2\theta} = 2\sec^2\theta \quad ✓\\
&= \text{RHS}
\end{aligned}$$

(b) Prove $(1 + \tan^2 P)(1 - \sin^2 P) = 1$

$$\begin{aligned}
\text{LHS} &= \left[1+\frac{\sin^2 P}{\cos^2 P}\right](1-\sin^2 P) \quad ✓\\
&= \left[\frac{\cos^2 P+\sin^2 P}{\cos^2 P}\right](\cos^2 P) \quad ✓✓\\
&= 1 \quad ✓\\
&= \text{RHS}
\end{aligned}$$

(c) Prove $\dfrac{1-2\cos^2 x}{\sin x+\cos x} = \sin x - \cos x$

$$\begin{aligned}
\text{LHS} &= \frac{(\cos^2 x+\sin^2 x)-2\cos^2 x}{\sin x+\cos x} \quad ✓\\
&= \frac{\sin^2 x-\cos^2 x}{\sin x+\cos x} \quad ✓\\
&= \frac{(\sin x+\cos x)(\sin x-\cos x)}{\sin x+\cos x} \quad ✓\\
&= \sin x-\cos x \quad ✓\\
&= \text{RHS}
\end{aligned}$$

Calculator Free

3. [10 marks: 4, 3, 3]

(a) Prove $\dfrac{1+\sin M}{\cos M} = \dfrac{\cos M}{1-\sin M}$

$$\begin{aligned}
\text{LHS} &= \left[\frac{1+\sin M}{\cos M}\right]\times\left[\frac{\cos M}{\cos M}\right] \quad ✓\\
&= \left[\frac{(1+\sin M)\cos M}{\cos^2 M}\right]\\
&= \left[\frac{(1+\sin M)\cos M}{1-\sin^2 M}\right] \quad ✓\\
&= \left[\frac{(1+\sin M)\cos M}{(1+\sin M)(1-\sin M)}\right] \quad ✓\\
&= \left[\frac{\cos M}{(1-\sin M)}\right] = \text{RHS} \quad ✓
\end{aligned}$$

(b) Prove $\sec x \operatorname{cosec} x = \tan x + \cot x$

$$\begin{aligned}
\text{RHS} &= \frac{\sin x}{\cos x}+\frac{\cos x}{\sin x} \quad ✓\\
&= \frac{\sin^2 x+\cos^2 x}{\cos x\,\sin x} \quad ✓\\
&= \frac{1}{\cos x\,\sin x} = \sec x\operatorname{cosec} x \quad ✓\\
&\equiv \text{LHS}
\end{aligned}$$

(c) Prove $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

$$\begin{aligned}
\text{LHS} &= (\cos^2 x-\sin^2 x)(\cos^2 x+\sin^2 x) \quad ✓\\
&= [(1-\sin^2 x)-\sin^2 x]\times 1 \quad ✓\\
&= 1-2\sin^2 x \quad ✓\\
&= \text{RHS}
\end{aligned}$$

Calculator Free

4. [8 marks: 2, 2, 4]

(a) Prove $\dfrac{1}{1+\cot x} = \dfrac{\tan x}{1+\tan x}$

$$\text{LHS} = \frac{1}{1+\frac{1}{\tan x}} \quad ✓$$
$$= \frac{1}{\left[\frac{\tan x+1}{\tan x}\right]} \quad ✓$$
$$= \frac{\tan x}{1+\tan x} \equiv \text{RHS}$$

(b) Prove $\dfrac{\sin x}{1+\cos x} = \dfrac{1}{\operatorname{cosec} x+\cot x}$

$$\text{RHS} = \frac{1}{\frac{1}{\sin x}+\frac{\cos x}{\sin x}} \quad ✓$$
$$= \frac{1}{\left[\frac{1+\cos x}{\sin x}\right]} \quad ✓$$
$$= \frac{\sin x}{1+\cos x} \equiv \text{LHS}$$

(c) Prove $\dfrac{\cos x}{1+\sin x} = \sec x - \tan x$

$$\text{RHS} = \frac{1}{\cos x} - \frac{\sin x}{\cos x} \quad ✓$$
$$= \frac{1-\sin x}{\cos x} \quad ✓$$
$$= \frac{\cos x(1-\sin x)}{\cos^2 x} \quad ✓$$
$$= \frac{\cos x(1-\sin x)}{1-\sin^2 x}$$
$$= \frac{\cos x(1-\sin x)}{(1-\sin x)(1+\sin x)} \quad ✓$$
$$= \frac{\cos x}{(1+\sin x)}$$
$$\equiv \text{LHS}$$

Calculator Free

5. [7 marks: 4, 3]

Prove each of the following:

(a) $\dfrac{\operatorname{cosec} x+1}{\operatorname{cosec} x-1} = \tan^2 x + 2\tan x \sec x + \sec^2 x$

$$\text{LHS} = \frac{\operatorname{cosec} x+1}{\operatorname{cosec} x-1}$$
$$= \frac{\left[\frac{1}{\sin x}+1\right]}{\left[\frac{1}{\sin x}-1\right]} \quad ✓$$
$$= \frac{1+\sin x}{1-\sin x}$$
$$= \frac{1+\sin x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} \quad ✓$$
$$= \frac{\sin^2 x+2\sin x+1}{1-\sin^2 x} \quad ✓$$
$$= \frac{\sin^2 x+2\sin x+1}{\cos^2 x} \quad ✓$$
$$= \tan^2 x + 2\tan x \sec x + \sec^2 x$$
$$= \text{RHS}$$

(b) $\dfrac{1}{\sec^2 x-1} = \operatorname{cosec}^2 x - 1$

$$\text{LHS} \equiv \frac{1}{\sec^2 x-1}$$
$$\equiv \frac{1}{1+\tan^2 x-1} \quad ✓$$
$$\equiv \frac{1}{\tan^2 x}$$
$$\equiv \cot^2 x \quad ✓$$
$$\equiv \operatorname{cosec}^2 x - 1 \quad ✓$$
$$\equiv \text{RHS}$$

Calculator Free

6. [8 marks: 4, 4]

(a) Prove that $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

$$\begin{aligned}\text{LHS} &= (\text{cosec}\,\theta - \cot\theta)^2\\ &= \left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)^2 \quad ✓\\ &= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2\\ &= \frac{(1-\cos\theta)^2}{\sin^2\theta} \quad ✓\\ &= \frac{(1-\cos\theta)^2}{1-\cos^2\theta} \quad ✓\\ &= \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} \quad ✓\\ &= \frac{1-\cos\theta}{1+\cos\theta} = \text{RHS}\end{aligned}$$

(b) Prove that $\dfrac{\cos^3\theta-\sin^3\theta}{\cos\theta-\sin\theta} + \dfrac{\cos^3\theta+\sin^3\theta}{\cos\theta+\sin\theta} = 2$

$$\begin{aligned}\text{LHS} &= \frac{\cos^3\theta-\sin^3\theta}{\cos\theta-\sin\theta}+\frac{\cos^3\theta+\sin^3\theta}{\cos\theta+\sin\theta}\\ &= \frac{(\cos^3\theta-\sin^3\theta)(\cos\theta+\sin\theta)+(\cos^3\theta+\sin^3\theta)(\cos\theta-\sin\theta)}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)} \quad ✓\\ &= \frac{\cos^4\theta+\cos^3\theta\sin\theta-\sin^3\theta\cos\theta-\sin^4\theta+\cos^4\theta-\cos^3\theta\sin\theta+\sin^3\theta\cos\theta-\sin^4\theta}{\cos^2\theta-\sin^2\theta} \quad ✓\\ &= \frac{2(\cos^4\theta-\sin^4\theta)}{\cos^2\theta-\sin^2\theta} \quad ✓\\ &= \frac{2(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}{\cos^2\theta-\sin^2\theta} \quad ✓\\ &= 2 = \text{RHS}\end{aligned}$$

OR

$$\begin{aligned}\text{LHS} &= \frac{(\cos\theta-\sin\theta)(\cos^2\theta+\cos\theta\sin\theta+\sin^2\theta)}{\cos\theta-\sin\theta}+\frac{(\cos\theta+\sin\theta)(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta)}{\cos\theta+\sin\theta} \quad ✓✓\\ &= \cos^2\theta+\cos\theta\sin\theta+\sin^2\theta + \cos^2\theta-\cos\theta\sin\theta+\sin^2\theta \quad ✓\\ &= 2(\cos^2\theta+\sin^2\theta) \quad ✓\\ &= 2 = \text{RHS}\end{aligned}$$

19 Trigonometric Identities II (Add/Sub Formulae)

Calculator Free

1. [7 marks: 1, 2, 1, 3]

For the acute angle θ^o, $\cos\theta^o = a$ where $0 < a < 1$, find in terms of a:

(a) $\sin(90^o - \theta^o)$

$\sin(90^o - \theta^o) = \cos\theta^o = a$ ✓

(b) $\tan(\theta^o)$

$\tan(\theta^o) = \dfrac{\sqrt{1-a^2}}{a}$ ✓✓

(c) $\cos(180^o + \theta^o)$

$\cos(180^o + \theta^o) = -a$ ✓

(d) $\sin(\theta^o + 30^o)$

$$\begin{aligned}\sin(\theta^o + 30^o) &= \sin\theta^o\cos 30^o + \cos\theta^o \sin 30^o \quad ✓\\ &= \sqrt{1-a^2}\times\frac{\sqrt{3}}{2} + a\times\frac{1}{2} \quad ✓✓\end{aligned}$$

2. [10 marks: 3, 3, 4]

Use an appropriate trigonometric identity to find the exact value of :

(a) $\sin 75^o$

$$\begin{aligned}\sin 75^o &= \sin(30^o+45^o) = \sin 30^o\cos 45^o + \cos 30^o \sin 45^o \quad ✓\\ &= \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} \quad ✓\\ &= \frac{\sqrt{2}(1+\sqrt{3})}{4} \quad ✓\end{aligned}$$

(b) $\cos 165^o$

$$\begin{aligned}\cos 165^o &= \cos(120^o+45^o) = \cos 120^o\cos 45^o - \sin 120^o \sin 45^o \quad ✓\\ &= -\frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} \quad ✓\\ &= -\frac{\sqrt{2}(1+\sqrt{3})}{4} \quad ✓\end{aligned}$$

Calculator Free

2\. (c) $\tan \frac{7\pi}{12}$

$$\tan \frac{7\pi}{12} = \tan\left[\frac{\pi}{3} + \frac{\pi}{4}\right] = \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}} \quad \checkmark$$
$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} \quad \checkmark$$
$$= \frac{(\sqrt{3}+1)^2}{-2} \quad \checkmark$$
$$= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3} \quad \checkmark$$

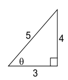
3\. [7 marks: 1, 1, 2, 3]

Given that $\sin A = \frac{4}{5}$ and $0 < A < \frac{\pi}{2}$, find the exact value of:

(a) cos A

A is an acute angle with $\sin A = \frac{4}{5}$.

From the triangle sketched, $\cos A = \frac{3}{5}$ ✓



(b) tan A

$\tan A = \frac{4}{3}$ ✓

(c) $\sin(\frac{\pi}{2} + A)$

$$\sin(\frac{\pi}{2} + A) = \sin\frac{\pi}{2}\cos A + \cos\frac{\pi}{2}\sin A \quad \checkmark$$
$$= \cos A = \frac{3}{5} \quad \checkmark$$

(d) $\cos(\frac{\pi}{4} - A)$

$$\cos(\frac{\pi}{4} - A) = \cos\frac{\pi}{4}\cos A + \sin\frac{\pi}{4}\sin A \quad \checkmark$$
$$= \frac{\sqrt{2}}{2} \times \frac{3}{5} + \frac{\sqrt{2}}{2} \times \frac{4}{5} \quad \checkmark$$
$$= \frac{\sqrt{2}}{2}(\frac{3}{5} + \frac{4}{5}) = \frac{7\sqrt{2}}{10} \quad \checkmark$$

Calculator Assumed

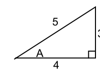
4\. [10 marks: 1, 1, 2, 2, 4]

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{4}$, where A and B are acute, use appropriate trigonometric identities (relationships) to find the exact value of:

(a) cos A

From the triangle sketched, with $\sin A = \frac{3}{5}$,

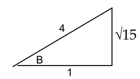
$\Rightarrow \cos A = \frac{4}{5}$ ✓



(b) sin B

From the triangle sketched, with $\cos B = \frac{1}{4}$,

$\Rightarrow \sin B = \frac{\sqrt{15}}{4}$ ✓



(c) sin(A + B)

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
$$= \frac{3}{5} \times \frac{1}{4} + \frac{4}{5} \times \frac{\sqrt{15}}{4} \quad \checkmark$$
$$= \frac{3 + 4\sqrt{15}}{20} \quad \checkmark$$

(d) cos (A – B)

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$
$$= \frac{4}{5} \times \frac{1}{4} + \frac{3}{5} \times \frac{\sqrt{15}}{4} \quad \checkmark$$
$$= \frac{4 + 3\sqrt{15}}{20} \quad \checkmark$$

(e) tan (A + B)

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$= \frac{\frac{3}{4} + \sqrt{15}}{1 - \frac{3}{4}\sqrt{15}} = \frac{3 + 4\sqrt{15}}{4 - 3\sqrt{15}} \quad \checkmark\checkmark$$
$$= \frac{3 + 4\sqrt{15}}{4 - 3\sqrt{15}} \times \frac{4 + 3\sqrt{15}}{4 + 3\sqrt{15}}$$
$$= \frac{192 + 25\sqrt{15}}{-119} \quad \checkmark\checkmark$$

Calculator Assumed

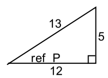
5. [9 marks: 1, 1, 2, 2, 3]

Given that $\sin P = \frac{5}{13}$ and $\cos Q = -\frac{15}{17}$, where $\frac{\pi}{2} \le P \le \pi$ and $\frac{\pi}{2} \le Q \le \pi$, use appropriate trigonometric identities to find the exact value of:

(a) cos P

P is an obtuse angle. From the triangle sketched,

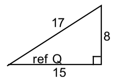
with sin (reference angle for P) = $\frac{5}{13}$, $\cos P = -\frac{12}{13}$ ✓



(b) sin Q

Q is an obtuse angle. From the triangle sketched,

with cos (reference angle for Q) = $\frac{15}{17}$, $\sin Q = \frac{8}{17}$ ✓



(c) sin(P – Q)

$\sin(P - Q) = \sin P \cos Q - \cos P \sin Q$

$= \frac{5}{13} \times (-\frac{15}{17}) - (-\frac{12}{13}) \times \frac{8}{17}$ ✓

$= \frac{21}{221}$ ✓

(d) cos (P + Q)

$\cos(P + Q) = \cos P \cos Q - \sin P \sin Q$

$= (-\frac{12}{13}) \times (-\frac{15}{17}) - \frac{5}{13} \times \frac{8}{17}$ ✓

$= \frac{140}{221}$ ✓

(e) tan (P – Q)

$\tan(P - Q) = \frac{\tan P - \tan Q}{1 + \tan P \tan Q}$

$= \frac{\left[-\frac{5}{12}\right] - \left[-\frac{8}{15}\right]}{1 + \left[-\frac{5}{12}\right] \times \left[-\frac{8}{15}\right]}$ ✓✓

$= \frac{\frac{7}{60}}{\frac{11}{9}} = \frac{21}{220}$ ✓

Calculator Assumed

6. [9 marks: 4, 5]

(a) Prove that $\frac{1}{\cot x - \cot 2x} = \sin 2x$.

$\text{LHS} = \frac{1}{\cot x - \cot 2x}$

$= \frac{1}{\left(\frac{\cos x}{\sin x}\right) - \left(\frac{\cos 2x}{\sin 2x}\right)}$ ✓

$= \frac{1}{\left(\frac{\sin 2x \cos x - \cos 2x \sin x}{\sin x \sin 2x}\right)}$ ✓

$= \frac{1}{\left(\frac{\sin(2x - x)}{\sin x \sin 2x}\right)}$ ✓

$= \frac{1}{\left(\frac{\sin x}{\sin x \sin 2x}\right)}$ ✓

$= \sin 2x = \text{RHS}$

(b) Prove that $\frac{\sin^2 A - \sin^2 B}{\sin^2(A + B)} = \frac{\tan A - \tan B}{\tan A + \tan B}$.

$\text{RHS} = \frac{\tan A - \tan B}{\tan A + \tan B}$

$= \frac{\left(\frac{\sin A}{\cos A}\right) - \left(\frac{\sin B}{\cos B}\right)}{\left(\frac{\sin A}{\cos A}\right) + \left(\frac{\sin B}{\cos B}\right)}$ ✓

$= \frac{\sin A \cos B - \cos A \sin B}{\sin A \cos B + \cos A \sin B}$ ✓

$= \frac{\sin A \cos B - \cos A \sin B}{\sin A \cos B + \cos A \sin B} \times \frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B + \cos A \sin B}$ ✓

$= \frac{(\sin A \cos B)^2 - (\cos A \sin B)^2}{(\sin(A + B))^2}$

$= \frac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{(\sin(A + B))^2}$ ✓

$= \frac{\sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B}{(\sin(A + B))^2}$ ✓

$= \frac{\sin^2 A - \sin^2 B}{\sin^2(A + B)} = \text{RHS}$

Calculator Assumed

7. [10 marks: 2, 3, 5]

(a) Prove that sin (–A) = – sin A

$$\sin(0 - A) = \sin 0 \cos A - \cos 0 \sin A \quad ✓$$
$$= -\sin A \quad ✓$$

(b) Prove that $\dfrac{\sin(A-B)}{\sin A \sin B} = \cot B - \cot A$

$$\text{LHS} \equiv \frac{\sin A \cos B + \cos A \sin B}{\sin A \sin B} \quad ✓$$
$$\equiv \frac{\cos B}{\sin B} - \frac{\cos A}{\sin A} \quad ✓$$
$$\equiv \frac{1}{\tan B} - \frac{1}{\tan A} \quad ✓$$
$$\equiv \cot B - \cot A \equiv \text{RHS}$$

(c) Use your answers in parts (a) and (b) to rewrite $\dfrac{\sin 2x}{\sin x \sin 3x} + \dfrac{\sin 2x}{\sin 3x \sin 5x}$ in the form $\dfrac{\sin a}{\sin b \sin c}$, giving the values of a, b and c.

$$\frac{\sin 2x}{\sin x \sin 3x} + \frac{\sin 2x}{\sin 3x \sin 5x} = \frac{-\sin(x-3x)}{\sin x \sin 3x} + \frac{-\sin(3x-5x)}{\sin 3x \sin 5x} \quad ✓✓$$
$$= -(\cot 3x - \cot x) - (\cot 5x - \cot 3x) \quad ✓✓$$
$$= \cot x - \cot 5x$$
$$= \frac{\sin 4x}{\sin x \sin 5x} \quad ✓$$

20 Trigonometric Identities III (Double angle)

Calculator Free

1. [11 marks: 1, 1, 3, 3, 3]

Given that $\sin P = \dfrac{1}{4}$ and $\cos Q = \dfrac{2}{3}$, where $\dfrac{\pi}{2} \le P \le \pi$ and $\dfrac{3\pi}{2} \le Q \le 2\pi$, find the exact value of:

(a) cos P

$$\cos P = -\frac{\sqrt{15}}{4} \quad ✓$$

(b) sin Q

$$\sin Q = -\frac{\sqrt{5}}{3} \quad ✓$$

(c) cos (P + Q)

$$\cos(P+Q) = \cos P \cos Q - \sin P \sin Q \quad ✓$$
$$= -\frac{\sqrt{15}}{4} \times \frac{2}{3} - \frac{1}{4} \times -\frac{\sqrt{5}}{3} \quad ✓$$
$$= \frac{\sqrt{5}(1-2\sqrt{3})}{12} \quad ✓$$

(d) tan 2Q

$$\tan 2Q = \frac{2\tan Q}{1-\tan^2 Q} \quad ✓$$
$$= \frac{2\left(-\frac{\sqrt{5}}{2}\right)}{1-\left(-\frac{\sqrt{5}}{2}\right)^2} \quad ✓$$
$$= 4\sqrt{5} \quad ✓$$

(e) $\sin \dfrac{Q}{2}$

$$\sin^2 \frac{Q}{2} = \frac{1}{2}(1 - \cos Q)$$
$$= \frac{1}{2}\left(1 - \frac{2}{3}\right) \quad ✓$$
$$\sin \frac{Q}{2} = \pm\frac{\sqrt{6}}{6} \quad ✓$$
$$\text{But } \frac{3\pi}{4} \le \frac{Q}{2} \le \pi, \Rightarrow \sin\frac{Q}{2} = \frac{\sqrt{6}}{6} \quad ✓$$

Calculator Free

2. [9 marks: 1, 2, 3, 3]

For the acute angle θ^o, $\tan\theta^o = \frac{a}{b}$ where a and b are both positive numbers, find in terms of a and/or b:

(a) $\cos(\theta^o)$

$$\cos\theta = \frac{b}{\sqrt{a^2+b^2}} \quad ✓$$

(b) $\sin(-\theta^o)$

$$\sin(-\theta) = \frac{-a}{\sqrt{a^2+b^2}} \quad ✓✓$$

(c) $\tan(\theta^o + 45^o)$

$$\tan(\theta+45) = \frac{\tan\theta+\tan 45}{1-\tan\theta\tan 45} \quad ✓$$
$$= \frac{\frac{a}{b}+1}{1-\frac{a}{b}} \quad ✓$$
$$= \frac{b+a}{b-a} \quad ✓$$

(d) $\cos(2\theta^o)$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad ✓$$
$$= \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 - \left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 \quad ✓$$
$$= \frac{b^2-a^2}{b^2+a^2} \quad ✓$$

Calculator Free

3. [15 marks: 3, 3, 4, 5]

Prove each of the following identities:

(a) $\cos^4 x - \sin^4 x = \cos 2x$

$$\text{LHS} \equiv (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) \quad ✓$$
$$\equiv (\cos 2x) \times 1 \quad ✓✓$$
$$\equiv \cos 2x \equiv \text{RHS}$$

(b) $\frac{\sin 2t}{1+\cos 2t} = \tan t$

$$\text{LHS} \equiv \frac{2\sin t\cos t}{1+(2\cos^2 t-1)} \quad ✓$$
$$\equiv \frac{2\sin t\cos t}{2\cos^2 t} \quad ✓$$
$$\equiv \frac{\sin t}{\cos t} \quad ✓$$
$$\equiv \tan t$$
$$\equiv \text{RHS}$$

(c) $\cos 6x = 4\cos^3 2x - 3\cos 2x$

$$\text{LHS} \equiv \cos(2x+4x)$$
$$\equiv \cos 2x\cos 4x - \sin 2x\sin 4x \quad ✓$$
$$\equiv \cos 2x(2\cos^2 2x - 1) - \sin 2x(2\sin 2x\cos 2x) \quad ✓✓$$
$$\equiv 2\cos^3 2x - \cos 2x - 2\sin^2 2x\cos 2x$$
$$\equiv 2\cos^3 2x - \cos 2x - 2(1-\cos^2 2x)\cos 2x \quad ✓$$
$$\equiv 4\cos^3 2x - 3\cos 2x$$
$$\equiv \text{RHS}$$

(d) $\frac{1-\sin 2t}{\cos 2t} = \frac{1-\tan t}{1+\tan t}$

$$\text{RHS} \equiv \frac{1-\frac{\sin t}{\cos t}}{1+\frac{\sin t}{\cos t}} \equiv \frac{\cos t-\sin t}{\cos t+\sin t} \quad ✓✓$$
$$\equiv \frac{\cos t-\sin t}{\cos t+\sin t}\times\frac{\cos t-\sin t}{\cos t-\sin t} \equiv \frac{\cos^2 t+\sin^2 t-2\sin t\cos t}{\cos^2 t-\sin^2 t} \quad ✓✓✓$$
$$\equiv \frac{1-\sin 2t}{\cos 2t} \equiv \text{LHS}$$

Calculator Free

4. [11 marks: 3, 4, 4]

Prove each of the following:

(a) $\dfrac{\cos x - \sin 2x}{\cos 2x + \sin x - 1} = \cot x$

$$\begin{aligned}
\text{LHS} &\equiv \frac{\cos x - 2\sin x\cos x}{(1-2\sin^2 x) + \sin x - 1} && \checkmark\\
&\equiv \frac{\cos x(1-2\sin x)}{\sin x(1-2\sin x)} && \checkmark\checkmark\\
&\equiv \frac{1}{\tan x}\\
&\equiv \cot x \equiv \text{RHS}
\end{aligned}$$

(b) $\cot\theta + \tan\theta = 2\operatorname{cosec} 2\theta$

$$\begin{aligned}
\text{LHS} &\equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} && \checkmark\\
&\equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} && \checkmark\\
&\equiv \frac{1}{\sin\theta\cos\theta} && \checkmark\\
&\equiv \frac{2}{\sin 2\theta} && \checkmark\\
&\equiv 2\operatorname{cosec} 2\theta \equiv \text{RHS}
\end{aligned}$$

(c) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$

$$\begin{aligned}
\text{RHS} &\equiv \frac{\frac{\cos^2 x}{\sin^2 x} - 1}{2\times\frac{\cos x}{\sin x}} && \checkmark\\
&\equiv \frac{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}{2\times\frac{\cos x}{\sin x}} && \checkmark\\
&\equiv \frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} && \checkmark\\
&\equiv \frac{\cos 2x}{\sin 2x} && \checkmark\\
&\equiv \cot 2x \equiv \text{LHS}
\end{aligned}$$

Calculator Free

5. [9 marks: 3, 3, 3]

(a) Prove that $\sqrt{1-\cos x} = \sqrt{2}\sin\dfrac{x}{2}$.

$$\begin{aligned}
\text{LHS} &\equiv \sqrt{1-\cos[2\times\tfrac{x}{2}]} && \checkmark\\
&\equiv \sqrt{1-(1-2\sin^2\tfrac{x}{2})} && \checkmark\\
&\equiv \sqrt{2}\sin\tfrac{x}{2} \equiv \text{RHS} && \checkmark
\end{aligned}$$

(b) Prove that $\dfrac{\sin x}{1+\cos x} = \tan\dfrac{x}{2}$.

$$\begin{aligned}
\text{LHS} &\equiv \frac{\sin[2\times\frac{x}{2}]}{1+\cos[2\times\frac{x}{2}]}\\
&\equiv \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+[2\cos^2\frac{x}{2}-1]} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} && \checkmark\checkmark\\
&\equiv \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\\
&\equiv \tan\tfrac{x}{2} \equiv \text{RHS} && \checkmark
\end{aligned}$$

(c) Prove that $\tan^2\left(\dfrac{3x}{2}\right) = \dfrac{1-\cos 3x}{1+\cos 3x}$

$$\begin{aligned}
\text{RHS} &\equiv \frac{1-\cos 3x}{1+\cos 3x}\\
&\equiv \frac{1-\left[1-2\sin^2\left(\frac{3x}{2}\right)\right]}{1+\left[2\cos^2\left(\frac{3x}{2}\right)-1\right]} && \checkmark\checkmark\\
&\equiv \frac{2\sin^2\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)} && \checkmark\\
&\equiv \tan^2\left(\frac{3x}{2}\right)\\
&= \text{RHS}
\end{aligned}$$

Calculator Free

6. [13 marks: 4, 3, 6]

(a) Prove that $(a\cos x + b\sin x)^2 + (b\cos x - a\sin x)^2 = a^2 + b^2$.

$$\text{LHS} \equiv (a\cos x + b\sin x)^2 + (b\cos x - a\sin x)^2$$
$$\equiv a^2\cos^2 x + 2ab\sin x\cos x + b^2\sin^2 x + b^2\cos^2 x - 2ab\sin x\cos x + a^2\sin^2 x \quad ✓✓$$
$$\equiv a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x) \quad ✓✓$$
$$\equiv a^2 + b^2 = \text{RHS}$$

(b) Prove that $\cos^4 2x - \sin^4 2x = \cos 4x$

$$\text{LHS} \equiv \cos^4 2x - \sin^4 2x$$
$$\equiv (\cos^2 2x - \sin^2 2x)(\cos^2 2x + \sin^2 2x) \quad ✓$$
$$\equiv \cos^2 2x - (1 - \cos^2 2x) \quad ✓$$
$$\equiv 2\cos^2 2x - 1 \quad ✓$$
$$\equiv \cos 4x \equiv \text{RHS}$$

(c) Prove that $\dfrac{1}{1+\tan x} - \dfrac{1}{1-\tan x} = -\tan 2x$.

$$\text{LHS} \equiv \frac{1}{1+\tan x} - \frac{1}{1-\tan x}$$
$$\equiv \frac{1}{\left(1+\frac{\sin x}{\cos x}\right)} - \frac{1}{\left(1-\frac{\sin x}{\cos x}\right)} \quad ✓$$
$$\equiv \frac{1}{\left(\frac{\cos x + \sin x}{\cos x}\right)} - \frac{1}{\left(\frac{\cos x - \sin x}{\cos x}\right)} \quad ✓$$
$$\equiv \frac{\cos x}{\cos x + \sin x} - \frac{\cos x}{\cos x - \sin x} \quad ✓$$
$$\equiv \frac{\cos x(\cos x - \sin x) - \cos x(\cos x + \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}$$
$$\equiv \frac{\cos^2 x - \cos x\sin x - \cos^2 x - \cos x\sin x}{(\cos^2 x - \sin^2 x)} \quad ✓$$
$$\equiv \frac{-2\sin x\cos x}{\cos^2 x - \sin^2 x} \quad ✓$$
$$\equiv \frac{-\sin 2x}{\cos 2x}$$
$$\equiv -\tan 2x \equiv \text{RHS} \quad ✓$$

Calculator Free

7. [11 marks: 3, 3, 2, 3]

(a) Prove that $\cos 3A = 4\cos^3 A - 3\cos A$.

$$\text{LHS} \equiv \cos 3A$$
$$\equiv \cos(2A + A)$$
$$\equiv \cos 2A\cos A - \sin 2A\sin A \quad ✓$$
$$\equiv (2\cos^2 A - 1)\cos A - (2\sin A\cos A)\sin A \quad ✓$$
$$\equiv 2\cos^3 A - \cos A - 2\cos A\sin^2 A$$
$$\equiv 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A) \quad ✓$$
$$\equiv 4\cos^3 A - 3\cos A \equiv \text{RHS}$$

(b) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

$$\sin 3A \equiv \sin(2A + A)$$
$$\equiv \sin 2A\cos A + \cos 2A\sin A \quad ✓$$
$$\equiv 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A \quad ✓$$
$$\equiv 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A \quad ✓$$
$$\equiv 3\sin A - 4\sin^3 A \equiv \text{RHS}$$

(c) Given that $\sin\theta = \dfrac{1}{4}$, where $0 < \theta < \dfrac{\pi}{2}$, find:

(i) $\sin 3\theta$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$= 3 \times \frac{1}{4} - 4 \times \left(\frac{1}{4}\right)^3 \quad ✓$$
$$= \frac{3}{4} - \frac{1}{16} = \frac{11}{16} \quad ✓$$

(ii) $\cos 3\theta$

$$\cos\theta = \frac{\sqrt{15}}{4} \quad ✓$$
$$\cos 3\theta = 4 \times \left(\frac{\sqrt{15}}{4}\right)^3 - 3 \times \frac{\sqrt{15}}{4} \quad ✓$$
$$= \frac{3\sqrt{15}}{16} \quad ✓$$

21 Trigonometric Identities IV (Product to Sum and Sum to Product)

Calculator Free

1. [8 marks: 3, 3, 2]

(a) Use an appropriate compound angle formula to prove that

$$\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\frac{A}{2}\cos\frac{B}{2}.$$

$$\begin{aligned}
\text{LHS} &\equiv \sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right)\\
&\equiv \sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\\
&\quad + \sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2} \quad ✓✓\\
&\equiv 2\sin\frac{A}{2}\cos\frac{B}{2} \equiv \text{RHS} \quad ✓
\end{aligned}$$

(b) Use the result in (a) to prove that $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

Let $P + Q = A$ I
$P - Q = B$ II

I + II $P = \frac{A+B}{2}$ III ✓

I − II $Q = \frac{A-B}{2}$ IV ✓

From (a): $\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\frac{A}{2}\cos\frac{B}{2}$

Substitute I, II, III & IV: ✓

Hence: $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

(c) Use the result in (b) to evaluate $\sin 75^o + \sin 15^o$

$$\begin{aligned}
\sin 75^o + \sin 15^o &= 2\sin\left(\frac{75^o+15^o}{2}\right)\cos\left(\frac{75^o-15^o}{2}\right) \quad ✓\\
&= 2\sin 45^o\cos 30^o\\
&= 2\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2} \quad ✓
\end{aligned}$$

Calculator Free

2. [9 marks: 3, 4, 2]

(a) Use an appropriate compound angle formula to prove that

$$\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right) = -2\sin\frac{A}{2}\sin\frac{B}{2}.$$

$$\begin{aligned}
\text{LHS} &\equiv \cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right)\\
&\equiv \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\\
&\quad - \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2} \quad ✓✓\\
&\equiv -2\sin\frac{A}{2}\sin\frac{B}{2} \equiv \text{RHS} \quad ✓
\end{aligned}$$

(b) Use the result in (a) to prove that $\cos P - \cos Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$.

Let $P + Q = A$ I
$P - Q = B$ II

I + II $P = \frac{A+B}{2}$ III ✓

I − II $Q = \frac{A-B}{2}$ IV ✓

From (a): $\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right) = -2\sin\frac{A}{2}\sin\frac{B}{2}$

Substitute I, II, III & IV:

Hence: $\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$ V ✓

But $-\sin\left(\frac{P-Q}{2}\right) = \sin\left[-\left(\frac{P-Q}{2}\right)\right]$
$= \sin\left(\frac{Q-P}{2}\right)$ ✓

Hence, V becomes: $\cos P - \cos Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$

(c) Use the result in (b) to evaluate $\cos 255^o - \cos 15^o$

$$\begin{aligned}
\cos 255^o - \cos 15^o &= 2\sin\left(\frac{255^o+15^o}{2}\right)\sin\left(\frac{15^o-255^o}{2}\right) \quad ✓\\
&= 2\sin 135^o\sin(-120^o)\\
&= 2\times\frac{\sqrt{2}}{2}\times\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{6}}{2} \quad ✓
\end{aligned}$$

Calculator Free

3. [11 marks: 2, 3, 2, 2, 2]

(a) Prove that $\cos P \cos Q = \frac{1}{2}[\cos(P+Q) + \cos(P-Q)]$.

$$\begin{aligned} \text{RHS} &\equiv \frac{1}{2}[\cos(P+Q) + \cos(P-Q)] \\ &\equiv \frac{1}{2}[\cos P\cos Q - \sin P\sin Q + \cos P\cos Q + \sin P\sin Q] \quad ✓ \\ &\equiv \frac{1}{2} \times 2\cos P\cos Q \\ &\equiv \cos P\cos Q \equiv \text{LHS} \quad ✓ \end{aligned}$$

(b) Use the result in (a) to prove that $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

Let $P = 90^o - A$ and $Q = B$. ✓
Hence identity in (a) becomes:

$$\cos(90^o - A)\cos B = \frac{1}{2}[\cos(90^o - A + B) + \cos(90^o - A - B)]$$

$$\cos(90^o - A)\cos B = \frac{1}{2}[\cos(90^o - (A-B)) + \cos(90^o - (A+B))] \quad \text{I} \quad ✓$$

Since $\cos(90^o - A) = \sin A$, I becomes:

$$\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)] \quad ✓$$

(c) Use the results in (a) and/or (b) to evaluate $\cos^2 15^o$.

$$\begin{aligned} \cos 15^o \cos 15^o &= \frac{1}{2}[\cos(15^o + 15^o) + \cos(15^o - 15^o)] \quad ✓ \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right) \quad ✓ \end{aligned}$$

(d) Use the results in (a) and/or (b) to evaluate $\sin 15^o \cos 15^o$.

$$\begin{aligned} \sin 15^o \cos 15^o &= \frac{1}{2}[\sin(15^o + 15^o) + \sin(15^o - 15^o)] \quad ✓ \\ &= \frac{1}{4} \quad ✓ \end{aligned}$$

(e) Hence, evaluate $\cot 15^o$.

$$\begin{aligned} \cot 15^o &= \frac{\cos 15^o \cos 15^o}{\sin 15^o \cos 15^o} \quad ✓ \\ &= \frac{\frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)}{\frac{1}{4}} = 2 + \sqrt{3} \quad ✓ \end{aligned}$$

Calculator Free

4. [12 marks: 3, 4, 5]

(a) Show the use of an appropriate *product to sum formula* to simplify $\sin(15^o) \times \cos(75^o)$.

$$\begin{aligned} \sin(15^o) \times \cos(75^o) &= \frac{1}{2}[\sin(15+75) + \sin(15-75)] \quad ✓ \\ &= \frac{1}{2}[\sin(90^o) + \sin(-60^o)] \quad ✓ \\ &= \frac{2-\sqrt{3}}{4} \quad ✓ \end{aligned}$$

(b) Show use of the formula $\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right)$ to simplify $\sin(15^o) - \sin(105^o) + \sin(135^o)$

$$\begin{aligned} &\sin(15^o) - \sin(105^o) + \sin(135^o) \\ &= 2\left[\sin\left(\frac{15-105}{2}\right)\cos\left(\frac{15+105}{2}\right)\right] + \sin(135) \quad ✓ \\ &= 2\sin(-45)\cos(60) + \sin(135) \quad ✓ \\ &= \sin(-45) + \sin(135) \\ &= 2\left[\sin\left(\frac{-45+135}{2}\right)\cos\left(\frac{-45-135}{2}\right)\right] \quad ✓ \\ &= 2\sin(45)\cos(-90) = 0 \quad ✓ \end{aligned}$$

(c) Simplify $\tan(105^o) + \tan(165^o)$

$$\begin{aligned} &\tan(105^o) + \tan(165^o) \\ &= \tan(60+45) + \tan(120+45) \quad ✓ \\ &= \frac{\tan 60 + \tan 45}{1 - \tan 60 \tan 45} + \frac{\tan 120 + \tan 45}{1 - \tan 120 \tan 45} \quad ✓ \\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}} + \frac{-\sqrt{3}+1}{1+\sqrt{3}} \quad ✓✓ \\ &= \frac{(\sqrt{3}+1)^2 + (-\sqrt{3}+1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = -4 \quad ✓ \end{aligned}$$

Calculator Free

5. [12 marks: 3, 4, 5]

(a) Prove that $\dfrac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} = \tan 3A \cot A$.

$$\begin{aligned} \text{LHS} &= \frac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} \\ &= \frac{2\sin\left(\frac{4A+2A}{2}\right)\cos\left(\frac{4A-2A}{2}\right)}{2\sin\left(\frac{4A-2A}{2}\right)\cos\left(\frac{4A+2A}{2}\right)} \quad ✓✓ \\ &= \frac{2\sin 3A\cos A}{2\sin A\cos 3A} \quad ✓ \\ &= \tan 3A \cot A = \text{RHS} \end{aligned}$$

(b) Prove that $\sin 5\theta + 2\sin 3\theta + \sin\theta = 4\sin 3\theta\cos^2\theta$.

$$\begin{aligned} \text{LHS} &\equiv \sin 5\theta + 2\sin 3\theta + \sin\theta \\ &\equiv (\sin 5\theta + \sin\theta) + 2\sin 3\theta \quad ✓ \\ &\equiv 2\sin\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) + 2\sin 3\theta \quad ✓ \\ &\equiv 2\sin 3\theta\cos 2\theta + 2\sin 3\theta \\ &\equiv 2\sin 3\theta(\cos 2\theta + 1) \quad ✓ \\ &\equiv 2\sin 3\theta(2\cos^2\theta - 1 + 1) \quad ✓ \\ &\equiv 4\sin 3\theta\cos^2\theta \equiv \text{RHS} \end{aligned}$$

(c) Prove that $\dfrac{\sin P + \cos(2Q-P)}{\cos P - \sin(2Q-P)} = \cot\left(\dfrac{\pi}{4} - Q\right)$

$$\begin{aligned} \text{LHS} &\equiv \frac{\sin P + \cos(2Q-P)}{\cos P - \sin(2Q-P)} \\ &\equiv \frac{\cos\left(\frac{\pi}{2}-P\right) + \cos(2Q-P)}{\sin\left(\frac{\pi}{2}-P\right) - \sin(2Q-P)} \quad ✓✓ \\ &\equiv \frac{2\cos\left(\frac{\frac{\pi}{2}-P+(2Q-P)}{2}\right)\cos\left(\frac{\frac{\pi}{2}-P-(2Q-P)}{2}\right)}{2\sin\left(\frac{\frac{\pi}{2}-P-(2Q-P)}{2}\right)\cos\left(\frac{\frac{\pi}{2}-P+(2Q-P)}{2}\right)} \quad ✓✓ \\ &\equiv \frac{\cos\left(\frac{\pi}{4}-Q\right)}{\sin\left(\frac{\pi}{4}-Q\right)} \quad ✓ \\ &\equiv \cot\left(\frac{\pi}{4}-Q\right) \equiv \text{RHS} \end{aligned}$$

Calculator Free

6. [7 marks: 1, 3, 3]

(a) Prove that $\sin A\cos A = \dfrac{1}{2}\sin 2A$

$$\begin{aligned} \text{RHS} &\equiv \frac{1}{2}\sin 2A \\ &\equiv \frac{1}{2}(2\sin A\cos A) \quad ✓ \\ &\equiv \sin A\cos A \equiv \text{LHS} \end{aligned}$$

(b) Prove that $\sin 40^o\cos 40^o\cos 80^o = \dfrac{\sin 20^o}{4}$.

$$\begin{aligned} \text{LHS} &\equiv \sin 40^o\cos 40^o\cos 80^o \\ &\equiv \frac{1}{2}\sin(2\times 40^o)\cos 80^o \equiv \frac{1}{2}\sin(80^o)\cos 80^o \quad ✓ \\ &\equiv \frac{1}{2}\times\left(\frac{1}{2}\sin(2\times 80^o)\right) \equiv \frac{1}{4}\sin 160^o \quad ✓ \\ &\equiv \frac{1}{4}\sin(180^o - 160^o) \quad ✓ \\ &\equiv \frac{1}{4}\sin 20^o \equiv \text{RHS} \end{aligned}$$

(c) Use your answer in (b) to prove that $\cos 20^o\cos 40^o\cos 80^o = \dfrac{1}{8}$.

From (b):

$\sin 40^o\cos 40^o\cos 80^o = \frac{1}{4}\sin 20^o$ I

But $\sin 40^o = 2\sin 20^o\cos 20^o$. ✓

Hence, I becomes:

$(2\sin 20^o\cos 20^o)\times\cos 40^o\cos 80^o = \frac{1}{4}\sin 20^o$ ✓

$\Rightarrow \cos 20^o\cos 40^o\cos 80^o = \frac{1}{8}$ ✓

Calculator Free

7. [9 marks: 2, 2, 5]

Let $a + b + c = 180^{\circ}$.

(a) Prove that $\sin\frac{c}{2} = \cos\left(\frac{a}{2}+\frac{b}{2}\right)$.

$$\text{LHS} = \sin\frac{c}{2}$$
$$= \sin\left(\frac{180-(a+b)}{2}\right) = \sin\left(90-\frac{a+b}{2}\right) \quad ✓$$
$$= \cos\left(\frac{a}{2}+\frac{b}{2}\right) = \text{RHS} \quad ✓$$

(b) Prove that $\cos\left(\frac{c}{2}\right) = \sin\left(\frac{a+b}{2}\right)$.

$$\text{LHS} = \cos\frac{c}{2}$$
$$= \cos\left(\frac{180-(a+b)}{2}\right) = \cos\left(90-\frac{a+b}{2}\right) \quad ✓$$
$$= \sin\left(\frac{a}{2}+\frac{b}{2}\right) = \text{RHS} \quad ✓$$

(c) Hence or otherwise prove that $\sin a + \sin b + \sin c = 4\left(\cos\frac{a}{2}\right)\left(\cos\frac{b}{2}\right)\left(\cos\frac{c}{2}\right)$.

$$\text{LHS} = \sin a + \sin b + \sin c$$
$$= 2\sin\left(\frac{a+b}{2}\right)\times\cos\left(\frac{a-b}{2}\right) + 2\left(\sin\frac{c}{2}\right)\left(\cos\frac{c}{2}\right) \quad ✓✓$$
$$= 2\cos\left(\frac{c}{2}\right)\times\cos\left(\frac{a}{2}-\frac{b}{2}\right) + 2\left(\sin\frac{c}{2}\right)\left(\cos\frac{c}{2}\right) \quad ✓$$
$$= 2\cos\left(\frac{c}{2}\right)\left[\cos\left(\frac{a}{2}-\frac{b}{2}\right) + \left(\sin\frac{c}{2}\right)\right]$$
$$= 2\cos\left(\frac{c}{2}\right)\left[\cos\left(\frac{a}{2}-\frac{b}{2}\right) + \cos\left(\frac{a}{2}+\frac{b}{2}\right)\right] \quad ✓$$
$$= 2\cos\left(\frac{c}{2}\right)\left[2\cos\left(\frac{a}{2}+\frac{b}{2}+\overline{\frac{a}{2}-\frac{b}{2}}\right)\cos\left(\frac{a}{2}+\frac{b}{2}-\overline{\frac{a}{2}-\frac{b}{2}}\right)\right] \quad ✓$$
$$= 4\left(\cos\frac{a}{2}\right)\left(\cos\frac{b}{2}\right)\left(\cos\frac{c}{2}\right) = \text{RHS}$$

22 Trigonometric Identities V (Auxiliary Angles)

Calculator Assumed

1. [8 marks: 4, 4]

(a) Given that $\cos x + \sqrt{3}\sin x = \text{R}\sin(x+\alpha)$ where $\text{R} > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, use the formula for sin (A + B) to find in exact form the value of R and the exact value of α.

$\text{R}\sin(x+\alpha) = \text{R}\cos\alpha\sin x + \text{R}\sin\alpha\cos x$

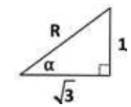
Hence: $\text{R}\cos\alpha = \sqrt{3}$

$\text{R}\sin\alpha = 1$ ✓

$\Rightarrow \text{R} = 2$ ✓

$\tan\alpha = \frac{1}{\sqrt{3}}$ ✓

$\alpha = \frac{\pi}{6}$ ✓



(b) Hence, find the maximum value (in exact form) for $y = \sqrt{3}\cos x + 3\sin x$ and the smallest positive value of x at which this occurs.

$y = \sqrt{3}\cos x + 3\sin x$

$= \sqrt{3}(\cos x + \sqrt{3}\sin x)$

$= \sqrt{3}\times 2\sin(x + \frac{\pi}{6})$ ✓

Hence, maximum value for $y = 2\sqrt{3}$ ✓

when $\sin(x+\frac{\pi}{6}) = 1$ ✓

$x + \frac{\pi}{6} = \frac{\pi}{2}$

$x = \frac{\pi}{3}$ ✓

Calculator Assumed

2. [10 marks: 4, 3, 3]

Compare $4 \sin x + 7 \cos x$ with the expansion of $R \sin(x + \alpha)$ where $0 \le \alpha \le 90^\circ$. Hence, find the exact value of R and the value of α to 2 decimal places

Let $4 \sin x + 7 \cos x \equiv R \sin (x + \alpha)$
$\Rightarrow 4 \sin x + 7 \cos x \equiv R \sin x \cos \alpha + R \cos x \sin \alpha$

Compare coefficients for sin x and cos x:
$R \cos \alpha = 4$ (I)
$R \sin \alpha = 7$ (II) ✓
(II) ÷ (I) $\Rightarrow$ $\tan \alpha = \frac{7}{4}$ ✓
$\alpha = 60.26^\circ$ ✓

$(I)^2 + (II)^2 \Rightarrow$ $R^2 \cos^2\alpha + R^2 \sin^2\alpha = 4^2 + 7^2$
$R^2 (\cos^2\alpha + \sin^2\alpha) = 65$
$R = \sqrt{65}$ ✓

Hence, find:
(a) the maximum value (in exact form) for the expression $8 \sin x + 14 \cos x$ and the smallest positive value of x at which this occurs.

Since, $4 \sin x + 7 \cos x \equiv \sqrt{65} \sin (x + 60.26^\circ)$,
$8 \sin x + 14 \cos x \equiv 2\sqrt{65} \sin (x + 60.26^\circ)$
Maximum value for expression = $2\sqrt{65}$. ✓

This occurs when $\sin (x + 60.26^\circ) = 1$. ✓
$\Rightarrow x + 60.26^\circ = 90^\circ$
$x = 29.7^\circ$ ✓

(b) the maximum value (in exact form) for the expression $-4 \sin x - 7 \cos x$ and the smallest positive value of x at which this occurs.

Since, $4 \sin x + 7 \cos x \equiv \sqrt{65} \sin (x + 60.26^\circ)$,
$-4 \sin x - 7 \cos x \equiv -\sqrt{65} \sin (x + 60.26^\circ)$
Maximum value for expression = $\sqrt{65}$. ✓

This occurs when $\sin (x + 60.26^\circ) = -1$. ✓
$\Rightarrow x + 60.26^\circ = 270^\circ$
$x = 209.7^\circ$ ✓

Calculator Assumed

3. [10 marks: 4, 3, 3]

Compare $5 \cos x + 8 \sin x$ with the expansion of form $R \cos(x - \alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$. Hence, find the exact value of R and α to 4 decimal places.

Let $5 \cos x + 8 \sin x \equiv R \cos (x - \alpha)$
$\Rightarrow 5 \cos x + 8 \sin x \equiv R \cos x \cos \alpha + R \sin x \sin \alpha$

Compare coefficients for cos x and sin x:
$R \cos \alpha = 5$ (I)
$R \sin \alpha = 8$ (II) ✓
(II) ÷ (I) $\Rightarrow$ $\tan \alpha = \frac{8}{5}$ ✓
$\alpha = 1.0122$ ✓

$(I)^2 + (II)^2 \Rightarrow$ $R^2 \cos^2\alpha + R^2 \sin^2\alpha = 5^2 + 8^2$
$R^2 (\cos^2\alpha + \sin^2\alpha) = 89$
$R = \sqrt{89}$ ✓
Hence, $5 \cos x + 8 \sin x \equiv \sqrt{89} \cos (x - 1.0122)$

Hence, find:
(a) the maximum value (in exact form) for the expression $\cos x + 1.6 \sin x$ and the values of x where $0 \le x \le 2\pi$ at which this occurs.

Since, $5 \cos x + 8 \sin x \equiv \sqrt{89} \cos (x - 1.0122)$,
$\cos x + 1.6 \sin x \equiv \frac{\sqrt{89}}{5} \cos (x - 1.0122)$
Maximum value for expression = $\frac{\sqrt{89}}{5}$. ✓

This occurs when $\cos (x - 1.0122) = 1$. ✓
$\Rightarrow x - 1.0122 = 0$
$x = 1.01$ radians ✓

(b) the minimum value (in exact form) for the expression $10 \sin x + 16 \cos x$ and the values of x where $0 \le x \le 2\pi$ at which this occurs.

Since, $5 \cos x + 8 \sin x \equiv \sqrt{89} \cos (x - 1.0122)$, ✓
$10\cos x + 16 \sin x \equiv 2\sqrt{89} \cos (x - 1.0122)$

Mminimum value for expression = $-2\sqrt{89}$.

This occurs when $\cos (x - 1.0122) = -1$. ✓
$\Rightarrow x - 1.0122 = \pi$
$x = 4.15$ radians ✓

Calculator Assumed

4. [7 marks: 3, 1, 3]

Let $\theta = 5 - 3\cos(2t) + \sin(2t)$.

(a) Express θ in the form $5 - R\sin(2t + \alpha)$ where α (in degrees) is an acute angle.

$\theta = 5 + [\sin(2t) - 3\cos(2t)]$ ✓
$= 5 + \sqrt{10}\sin[2t - \tan^{-1} 3]$ ✓
$= 5 + \sqrt{10}\sin(2t + 71.57^o)$ ✓

(b) Find in exact form, the maximum value for θ.

Maximum $= 5 + \sqrt{10}$ ✓

(c) Find the smallest positive value for t when θ has a maximum value.

$\sin(2t + 18.43^o) = 1$ ✓
$(2t + 18.43^o) = 90$ ✓
$t = 9.2^o$ ✓

5. [7 marks: 3, 1, 3]

Let $\theta = 3\cos(\pi t) + 2\sin(\pi t) + 10$.

(a) Express θ in the form $R\sin(\pi t + \alpha) + 10$ where $0 < \alpha < \frac{\pi}{2}$.

$\theta = \sqrt{2^2+3^2}\ \sin[\pi t + \tan^{-1}\left(\frac{3}{2}\right)] + 10$ ✓✓
$= \sqrt{13}\sin(\pi t + 0.98279) + 10$ ✓

(b) Find in exact form, the minimum value for θ.

Maximum $= 10 - \sqrt{13}$ ✓

(c) Find the smallest positive value for t when θ has a minimum value.

$\sin(\pi t + 0.98279) = -1$ ✓
$(\pi t + 0.98279) = \frac{3\pi}{2}$ ✓
$t = 1.1872$ ✓

Calculator Assumed

6. [6 marks]

Use an appropriate trigonometric method to find the minimum value (in exact form) for $f(\theta) = 10 + 3\sin\theta + 5\cos\theta$ where $0 \le \theta \le 360^o$. Give also the smallest positive value for θ at which the minimum value of $f(\theta)$ occurs.

$f(\theta) = 10 + 3\sin\theta + 5\cos\theta$
$= 10 + \sqrt{34}\sin(\theta + 59.04^o)$ ✓✓✓

Minimum value for $f(\theta) = 10 - \sqrt{34}$. ✓

This occurs when $\sin(\theta + 59.04^o) = -1$. ✓
$\Rightarrow \theta + 59.04^o = 270^o$
$\theta = 210.96^o$ ✓

7. [5 marks]

Consider $A = a\sin\theta + b\cos\theta$.

A has a maximum value of 4 when $\theta = \frac{\pi}{3}$. Determine the value(s) of a and b.

Let $a\sin\theta + b\cos\theta = R\cos(\theta - \alpha)$
$R^2 = a^2 + b^2$ ✓
$\tan\alpha = \frac{a}{b}$ ✓
Max value for $A = |R|$ when $\theta = \alpha$
But Max for $A = 4$ when $\theta = \frac{\pi}{3}$.
$\Rightarrow a^2 + b^2 = 16$ (1) ✓
$\frac{a}{b} = \tan\frac{\pi}{3}$
$\Rightarrow a = \sqrt{3}\,b$ (2) ✓
Solve (1) & (2);
$b = 2, a = 2\sqrt{3}$ ✓
Reject $b = -2, a = -2\sqrt{3}$

23 Trigonometric Equations II

Calculator Free

1. [8 marks: 4, 4]

(a) Solve for x in $2\cos^2 x + 3\sin x = 0$ for $0 \le x \le 360^\circ$

$2(1-\sin^2 x) + 3\sin x = 0$ ✓
$2\sin^2 x - 3\sin x - 2 = 0$
$(2\sin x + 1)(\sin x - 2) = 0$ ✓
$\Rightarrow \sin x = -\frac{1}{2}$ or 2 (reject)
For $\sin x = -\frac{1}{2}$: $x = 180^\circ + 30^\circ, 360^\circ - 30^\circ$
$= 210^\circ, 330^\circ$ ✓✓

(b) Solve for x in $\cos x - 3\sec x - 2 = 0$ for $0 \le x \le 2\pi$

Rewrite as $\cos x - \frac{3}{\cos x} - 2 = 0$
Multiply both sides of equation with $\cos x$.
$\Rightarrow \cos^2 x - 2\cos x - 3 = 0$ ✓
$(\cos x + 1)(\cos x - 3) = 0$ ✓
$\Rightarrow \cos x = -1$ or 3 (reject) ✓
$\cos x = -1 \Rightarrow x = \pi$ radians ✓

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2. [11 marks: 3, 4, 4]

(a) Solve for x in $\cos x + \sqrt{3}\sin x = 0$ where $0 \le x \le 360^\circ$:

$\sqrt{3}\sin x = -\cos x$
$\tan x = -\frac{1}{\sqrt{3}}$ ✓
$\Rightarrow x = 150^\circ, 330^\circ$ ✓✓

(b) Solve for x in $\sin x - \cos 2x = 0$ where $-\pi < x \le \pi$.

$\sin x - (1 - 2\sin^2 x) = 0$ ✓
$2\sin^2 x + \sin x - 1 = 0$ ✓
$(2\sin x - 1)(\sin x + 1) = 0$
$\Rightarrow \sin x = \frac{1}{2}$ or -1
$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ ✓
$\sin x = -1 \Rightarrow x = -\frac{\pi}{2}$ ✓

Calculator Free

2. (c) Find all values of x (in degrees) in $\cos x + \sin 2x = 0$.

$\cos x + 2\sin x\cos x = 0$ ✓
$\cos x(1 + 2\sin x) = 0$ ✓
$\Rightarrow \cos x = 0$ or $\sin x = -\frac{1}{2}$
$\cos x = 0 \Rightarrow x = 360^\circ n \pm 90^\circ \quad n \in \mathbb{Z}$ ✓
$\sin x = -\frac{1}{2} \quad x = -30^\circ \times (-1)^n + 180^\circ n \quad n \in \mathbb{Z}$ ✓

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3. [8 marks: 4, 4]

(a) Solve for θ in $\cos 2\theta + \cos\theta + 1 = 0$ for $0 \le \theta \le 2\pi$.

$(2\cos^2\theta - 1) + \cos\theta + 1 = 0$ ✓
$2\cos^2\theta + \cos\theta = 0$
$\cos\theta(2\cos\theta + 1) = 0$ ✓
$\Rightarrow \cos\theta = 0$ or $-\frac{1}{2}$
$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ radians ✓
$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ radians ✓

(b) Find all solutions (in radians) to $\sin 2\theta - \sin\theta = 0$.

Rewrite as: $2\sin\theta\cos\theta - \sin\theta = 0$. ✓
$\sin\theta(2\cos\theta - 1) = 0$ ✓
$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$
$\sin\theta = 0 \Rightarrow \theta = n\pi \quad n \in \mathbb{Z}$ ✓
$\cos\theta = \frac{1}{2} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3} \quad n \in \mathbb{Z}$ ✓

OR

$2\sin\left(\frac{2\theta - \theta}{2}\right)\cos\left(\frac{2\theta + \theta}{2}\right) = 0$ ✓✓
$\sin\frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = n\pi$
$\theta = 2n\pi \quad n \in \mathbb{Z}$ ✓
$\cos\frac{3\theta}{2} = 0 \Rightarrow \frac{3\theta}{2} = \frac{(2n+1)\pi}{2}$
$\theta = \frac{(2n+1)\pi}{3} \quad n \in \mathbb{Z}$ ✓

Calculator Free

4. [13 marks: 4, 4, 5]

(a) Find all solutions (in radians) for θ in $3\tan^2\theta + 5\sec\theta + 1 = 0$.

$3(\sec^2\theta - 1) + 5\sec\theta + 1 = 0$ ✓
$3\sec^2\theta + 5\sec\theta - 2 = 0$
$(3\sec\theta - 1)(\sec\theta + 2) = 0$ ✓
$\Rightarrow \sec\theta = \frac{1}{3}$ or -2 ✓
$\sec\theta = \frac{1}{3} \Rightarrow \cos\theta = 3$ has no solution.
$\sec\theta = -2 \Rightarrow \cos\theta = -\frac{1}{2}$
$\theta = 2n\pi \pm \frac{2\pi}{3} \quad n \in \mathbb{Z}$ ✓

(b) Find all solutions (in degrees) for θ in $\tan\theta + \cot\theta - 2\sec\theta = 0$.

Rewrite as $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} - \frac{2}{\cos\theta} = 0.$ ✓
$\frac{\sin^2\theta + \cos^2\theta - 2\sin\theta}{\sin\theta\cos\theta} = 0$
$\Rightarrow \sin^2\theta + \cos^2\theta - 2\sin\theta = 0$ ✓
$1 - 2\sin\theta = 0$
$\Rightarrow \sin\theta = \frac{1}{2}$ ✓
Hence: $\theta = (-1)^n \times 30^o + 180^o n \quad n \in \mathbb{Z}$ ✓

(c) Solve for all values of θ radians in $2\cos 2\theta + 2\sin^2\theta - 9\cos\theta - 5 = 0$.

$2(2\cos^2\theta - 1) + 2(1 - \cos^2\theta) - 9\cos\theta - 5 = 0$ ✓✓
$2\cos^2\theta - 9\cos\theta - 5 = 0$ ✓
$(2\cos\theta + 1)(\cos\theta - 5) = 0$
$\cos\theta = -\frac{1}{2}$ or 5 (reject) ✓
$= 2n\pi \pm \left(\frac{2\pi}{3}\right) \quad n \in \mathbb{Z}$ ✓

Calculator Free

5. [17 marks: 3, 5, 4, 5]

(a) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 0$

Rewrite as: $\tan\theta = -\frac{1}{\sqrt{3}}$ ✓
$\theta = n\pi + \left(-\frac{\pi}{6}\right) \quad n \in \mathbb{Z}$ ✓✓

(b) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 1$

Hence: $\sqrt{3}\sin\theta + \cos\theta \equiv 2\sin(\theta + \frac{\pi}{6})$ ✓✓
$\Rightarrow 2\sin(\theta + \frac{\pi}{6}) = 1$
$\sin(\theta + \frac{\pi}{6}) = \frac{1}{2}$ ✓
$\theta + \frac{\pi}{6} = (-1)^n \times \frac{\pi}{6} \pm n\pi$ ✓
$\theta = (-1)^n \times \frac{\pi}{6} \pm n\pi - \frac{\pi}{6} \quad n \in \mathbb{Z}$ ✓

(c) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos 2\theta = 1$.

Rewrite: $\sqrt{3}\sin\theta + (1 - 2\sin^2\theta) = 1$ ✓
$\sin\theta(\sqrt{3} - 2\sin\theta) = 0$ ✓
$\sin\theta = 0$ or $\frac{\sqrt{3}}{2}$
$\sin\theta = 0 \Rightarrow \theta = n\pi \quad n \in \mathbb{Z}$ ✓
$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = (-1)^n \times \frac{\pi}{3} \pm n\pi \quad n \in \mathbb{Z}$ ✓

(d) $\cos 4\theta - \cos^2 2\theta = 0$

$(2\cos^2 2\theta - 1) - \cos^2 2\theta = 0$ ✓
$\cos^2 2\theta = 1$
$\cos 2\theta = 1 \Rightarrow 2\theta = 2n\pi$ ✓
$\theta = n\pi$ for $n \in \mathbb{Z}$ ✓
$\cos 2\theta = -1 \Rightarrow 2\theta = (2n + 1)\pi$ ✓
$\theta = \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$ ✓

Calculator Free

6. [16 marks: 5, 5, 6]

(a) Solve for all values of θ (in radians) in $2\cot\theta = 3\sec\theta$

$\dfrac{2\cos\theta}{\sin\theta} = \dfrac{3}{\cos\theta}$ ✓

$2\cos^2\theta - 3\sin\theta = 0$ ✓

$2(1-\sin^2\theta) - 3\sin\theta = 0$

$2\sin^2\theta + 3\sin\theta - 2 = 0$ ✓

$(2\sin\theta - 1)(\sin\theta + 2) = 0$

$\sin\theta = \dfrac{1}{2}$ or -2 (reject) ✓

$\theta = \dfrac{(-1)^n\pi}{6} + n\pi$ for $n \in \mathbb{Z}$ ✓

(b) Solve for all values of θ (in radians) $\cos 3\theta + 2\cos 2\theta + 3\cos\theta = -2$.

$(4\cos^3\theta - 3\cos\theta) + 2(2\cos^2\theta - 1) + 3\cos\theta = -2$ ✓✓

$4\cos^3\theta + 4\cos^2\theta = 0$

$4\cos^2\theta(\cos\theta + 1) = 0$ ✓

$\Rightarrow \cos\theta = 0$ or $\cos\theta = -1$

$\cos\theta = 0 \Rightarrow \theta = \dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$ ✓

$\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$ for $n \in \mathbb{Z}$ ✓

(c) Solve for all values of θ in $1 + \sqrt{3}\tan\theta = \sqrt{3}\sec\theta$.

Rewrite: $1 + \dfrac{\sqrt{3}\sin\theta}{\cos\theta} = \dfrac{\sqrt{3}}{\cos\theta}$ where $\cos\theta \neq 0$.

$\cos\theta + \sqrt{3}\sin\theta = \sqrt{3}$ where $\cos\theta \neq 0$ ✓

Rewrite: $\cos\theta + \sqrt{3}\sin\theta \equiv 2\cos(\theta - \frac{\pi}{3})$ ✓✓

Hence: $2\cos(\theta - \frac{\pi}{3}) = \sqrt{3}$

$\cos(\theta - \frac{\pi}{3}) = \dfrac{\sqrt{3}}{2}$ ✓

$(\theta - \frac{\pi}{3}) = 2n\pi \pm \dfrac{\pi}{6}$

$\theta = 2n\pi + \dfrac{\pi}{6}$ or $2n\pi + \dfrac{\pi}{2}$

But $\cos\theta \neq 0$, hence, reject $\theta = 2n\pi + \dfrac{\pi}{2}$. ✓

Therefore: $\theta = 2n\pi + \dfrac{\pi}{6}$ $n \in \mathbb{Z}$ ✓

Calculator Free

7. [9 marks: 4, 5]

(a) Solve for all values of θ in $\cos\theta + \cos 3\theta = 0$.

$2\cos\left(\dfrac{3\theta+\theta}{2}\right)\cos\left(\dfrac{3\theta-\theta}{2}\right) = 0$ ✓

$\cos 2\theta = 0$ or $\cos\theta = 0$ ✓

For $\cos 2\theta = 0 \Rightarrow 2\theta = \dfrac{(2n+1)\pi}{2} \Rightarrow \theta = \dfrac{(2n+1)\pi}{4}$ $n \in \mathbb{Z}$ ✓

For $\cos\theta = 0 \Rightarrow \theta = \dfrac{(2n+1)\pi}{2}$ $n \in \mathbb{Z}$ ✓

OR

$\cos 3\theta = -\cos\theta$

$\cos 3\theta = \cos(\pi - \theta)$ ✓

$\Rightarrow 3\theta = 2n\pi \pm (\pi - \theta)$ ✓

$\theta = \dfrac{(2n+1)\pi}{4}$ or $\dfrac{(2n-1)\pi}{2}$ $n \in \mathbb{Z}$ ✓✓

(b) Solve $\cos\theta + \cos 3\theta + \cos 7\theta = 0$ for $0 \le \theta \le 180^\circ$.

Rewrite as: $\cos 7\theta + \cos\theta + \cos 3\theta = 0$

$2\cos\left(\dfrac{7\theta+\theta}{2}\right)\cos\left(\dfrac{7\theta-\theta}{2}\right) + \cos 3\theta = 0$ ✓

$2\cos 4\theta\cos 3\theta + \cos 3\theta = 0$

$\cos 3\theta(2\cos 4\theta + 1) = 0$ ✓

$\cos 3\theta = 0$ or $\cos 4\theta = -\dfrac{1}{2}$ ✓

For $\cos 3\theta = 0 \Rightarrow 3\theta = 90^\circ, 270^\circ, 450^\circ$

$\theta = 30^\circ, 90^\circ, 150^\circ$ ✓

For $\cos 4\theta = -\dfrac{1}{2} \Rightarrow 4\theta = 120^\circ, 240^\circ, 480^\circ$

$\theta = 30^\circ, 60^\circ, 120^\circ$ ✓

Calculator Assumed

8. [10 marks: 1, 3, 6]

(a) Show that $\sin 2\theta + \sin 3\theta \equiv 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}$.

$$\text{LHS} \equiv \sin 2\theta + \sin 3\theta$$
$$\equiv 2\sin\left(\frac{3\theta+2\theta}{2}\right)\cos\left(\frac{3\theta-2\theta}{2}\right) \quad ✓$$
$$\equiv 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2} \equiv \text{RHS}$$

(b) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta = 0$

$$2\sin\left(\frac{3\theta+2\theta}{2}\right)\cos\left(\frac{3\theta-2\theta}{2}\right) = 0$$
$$\sin\frac{5\theta}{2} = 0 \text{ or } \cos\frac{\theta}{2} = 0 \quad ✓$$

For $\sin\frac{5\theta}{2} = 0 \Rightarrow \frac{5\theta}{2} = n\pi \Rightarrow \theta = \frac{2n\pi}{5} \quad n \in \mathbb{Z}$ ✓

For $\cos\frac{\theta}{2} = 0 \Rightarrow \theta = (2n+1)\pi \quad n \in \mathbb{Z}$ ✓

(c) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta - \sin\theta = 0$.

Rewrite: $2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\theta = 0$

But $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.

Hence: $2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 0$ ✓

$2\cos\left(\frac{\theta}{2}\right)\left[\sin\left(\frac{5\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right] = 0$ ✓

$2\cos\left(\frac{\theta}{2}\right) \times 2\sin\left(\frac{\frac{5\theta}{2}-\frac{\theta}{2}}{2}\right)\cos\left(\frac{\frac{5\theta}{2}+\frac{\theta}{2}}{2}\right) = 0$

$4\cos\left(\frac{\theta}{2}\right)\sin\theta\cos\left(\frac{3\theta}{2}\right) = 0$ ✓

$\sin\theta = 0 \Rightarrow \theta = 2n\pi \quad n \in \mathbb{Z}$ ✓

$\cos\left(\frac{\theta}{2}\right) = 0 \Rightarrow \theta = (2n+1)\pi \quad n \in \mathbb{Z}$ ✓

$\cos\left(\frac{3\theta}{2}\right) = 0 \Rightarrow \frac{3\theta}{2} = \frac{(2n+1)\pi}{2}$

$\theta = \frac{(2n+1)\pi}{3} \quad n \in \mathbb{Z}$ ✓

Calculator Assumed

9. [10 marks: 4, 6]

(a) Prove that $\sin 2A + \sin 2B = 2\sin(A+B)\cos(A-B)$.

RHS = $2\sin(A+B)\cos(A-B)$

$= 2(\sin A\cos B + \cos A\sin B)(\cos A\cos B + \sin A\sin B)$ ✓

$= 2(\sin A\cos A\cos^2 B + \sin B\cos B\sin^2 A$
$\quad + \sin B\cos B\cos^2 A + \sin A\cos A\sin^2 B)$ ✓

$= \sin 2A\cos^2 B + \sin 2B\sin^2 A + \sin 2B\cos^2 A + \sin 2A\sin^2 B$ ✓

$= \sin 2A(\cos^2 B + \sin^2 B) + \sin 2B(\sin^2 A + \cos^2 A)$ ✓

$= \sin 2A + \sin 2B \equiv$ RHS

(b) Use the result in (a) to solve for all values of θ in $\sin 6\theta + \sin 4\theta = 0$

Let $A = 3\theta$ and $B = 2\theta$.

From (a):
$\sin 6\theta + \sin 4\theta = 2\sin(3\theta + 2\theta)\cos(3\theta - \theta)$
$= 2\sin 5\theta\cos\theta$ ✓

Hence: $2\sin 5\theta\cos\theta = 0$ ✓

$\sin 5\theta = 0$ ✓
$\Rightarrow 5\theta = n\pi$
$\theta = \frac{n\pi}{5} \quad n \in \mathbb{Z}$ ✓

$\cos\theta = 0$ ✓
$\theta = \frac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}$ ✓

Calculator Assumed

10. [11 marks: 4, 2, 5]

(a) Use the formula for tan 2A to show that $\tan\frac{\pi}{8} = -1 + \sqrt{2}$.

$$\tan\left(2\times\frac{\pi}{8}\right) = \frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} \quad ✓$$
$$1-\tan^2\left(\frac{\pi}{8}\right) = 2\tan\left(\frac{\pi}{8}\right)$$
$$\tan^2\left(\frac{\pi}{8}\right) + 2\tan\left(\frac{\pi}{8}\right) - 1 = 0 \quad ✓$$
$$\tan\frac{\pi}{8} = \frac{-2\pm\sqrt{4+4}}{2} \quad ✓$$
$$= -1 \pm \sqrt{2}$$
But $\tan\frac{\pi}{8} > 0$, $\Rightarrow \tan\frac{\pi}{8} = -1+\sqrt{2}$ ✓

(b) Use your answer in (a) to find all solutions to $\sqrt{2}\cos\theta - \cos\theta - \sin\theta = 0$.

Rewrite: $(\sqrt{2}-1)\cos\theta - \sin\theta = 0$
$$\tan\theta = \sqrt{2}-1 \quad ✓$$
$$\theta = n\pi + \frac{\pi}{8} \quad ✓$$

(c) Given that $\sin\frac{3\pi}{8} = \frac{1}{\sqrt{4-2\sqrt{2}}}$ and using the answer in (a),

solve for θ in $\sin\theta - (\sqrt{2}-1)\cos\theta = 1$ for $0 < \theta \le 2\pi$.

Rewrite: $\sin\theta - (\sqrt{2}-1)\cos\theta \equiv R\sin(\theta-\alpha)$
$$R^2 = (\sqrt{2}-1)^2 + 1^2$$
$$R = \sqrt{4-2\sqrt{2}}$$
$$\alpha = \tan^{-1}(\sqrt{2}-1) = \frac{\pi}{8}$$
Hence: $\sqrt{4-2\sqrt{2}}\sin(\theta - \frac{\pi}{8}) = 1$ ✓✓
$$\sin(\theta-\frac{\pi}{8}) = \frac{1}{\sqrt{4-2\sqrt{2}}} \quad ✓$$
Hence: $\theta - \frac{\pi}{8} = \frac{3\pi}{8}$ or $\frac{5\pi}{8}$ ✓
$$\theta = \frac{\pi}{2}, \frac{3\pi}{4} \quad ✓$$

24 Trigonometric Graphs

Calculator Free

1. [6 marks]

Complete the following table.

Function	Period	Amplitude	Phase Shift
$y = 2\sin(2x^o)$	180^o	2	0
$y = -4\cos(\frac{x}{2} + 30^o)$	720^o	4	-60^o
$v = 10\tan(3t+\pi)$	$\frac{\pi}{3}$	n/a	$-\frac{\pi}{3}$
$Q = 5\sin(\frac{\pi}{2} - t)$	2π	5	$\frac{\pi}{2}$
$y = \frac{\sqrt{2}}{2}\cos(\pi t) + 100$	2	$\frac{\sqrt{2}}{2}$	0
$T = 5 - \sin(\frac{\pi}{4} - \theta)$	2π	1	$\frac{\pi}{4}$

[–1 mark per error]

2. [5 marks]

Complete the table below.

Function	Minimum value of function	Maximum value of function
$y = 3\sin t$	–3	3
$y = 20\cos(\frac{2x}{3} - 45^o)$	–20	20
$v = 5\tan\theta$	n/a	n/a
$M = 2\sin(\frac{\pi}{2} - 3t) + 4$	–2 + 4 = 2	2 + 4 = 6
$y = 5 - \cos(2\pi t)$	5 – 1 = 4	5 – (–1) = 6

[–1 mark per error]

Calculator Free

3. [6 marks: 3, 3]

A trigonometric function has equation $y = -4\sin(2x + 30^{\circ})$ for $0^{\circ} \le x \le 360^{\circ}$. Use an algebraic method to find:

(a) the maximum value for y and the corresponding value(s) for x.

> Maximum value for $y = -4 \times -1 = 4$. ✓
>
> This occurs when $\sin(2x + 30^{\circ}) = -1$ ✓
>
> $\Rightarrow \quad 2x + 30^{\circ} = 270^{\circ}$
>
> $x = 120^{\circ}$ ✓

(b) the minimum value for y and the corresponding value(s) for x.

> Minimum value for $y = -4 \times 1 = -4$. ✓
>
> This occurs when $\sin(2x + 30^{\circ}) = 1$ ✓
>
> $\Rightarrow \quad 2x + 30^{\circ} = 90^{\circ}$
>
> $x = 30^{\circ}$ ✓

4. [5 marks]

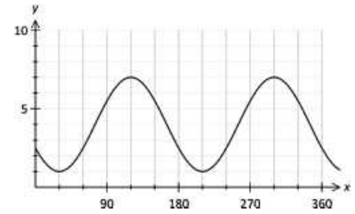
A trigonometric function has equation $P = a\cos(bt + \frac{\pi}{4})$. Find the values of a (where $a > 0$) and b given that P has a maximum value of 4 when $t = \frac{\pi}{4}$.

> $a = 4$ ✓
>
> when $\cos(b \times \frac{\pi}{4} + \frac{\pi}{4}) = 1$. ✓
>
> Hence, $\frac{b\pi}{4} + \frac{\pi}{4} = 2n\pi \quad n \in \mathbb{Z}$ ✓
>
> $\frac{b\pi}{4} = 2n\pi - \frac{\pi}{4}$
>
> $b = 8n - 1 \quad n \in \mathbb{Z}$ ✓✓

Calculator Free

5. [4 marks]

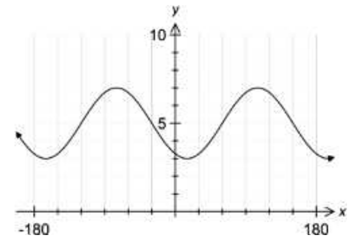
The graph of $y = a + b\sin(cx + d)$ is shown in the accompanying diagram. Determine the values of a, b, c and d.



> $a = 4 \quad b = -3$ ✓✓
>
> $c = 2 \quad d = 30^{\circ}$ ✓✓

6. [4 marks]

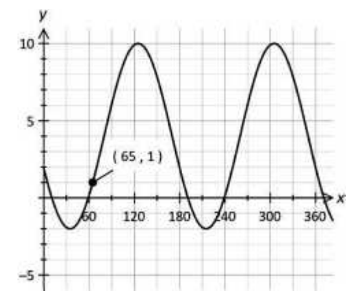
The graph of $y = a + b\cos(cx + d)$ is shown in the accompanying diagram. Determine the values of a, b, c and d.



> $a = 5 \quad b = -2$ ✓✓
>
> $c = 2 \quad d = -30^{\circ}$ ✓✓

7. [4 marks]

The accompanying diagram shows the graph of $y = a\sin(bx + c) + d$. Determine the values of the constants a, b, c and d where $0 \le c \le 90$.

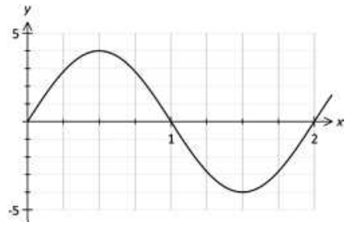


> $a = -6 \quad b = 2$ ✓✓
>
> $c = 20^{\circ} \quad d = 4$ ✓✓

Calculator Free

8. [3 marks]

The accompanying diagram shows the graph of a trigonometric function. State the amplitude and period of the function. Hence, give the equation of this function.



Amplitude = 4 ✓

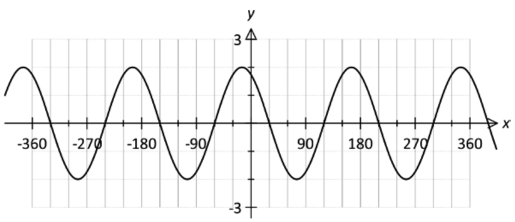
Period = 2 ✓

$f(x) = 4\sin \pi x$ ✓

9. [7 marks: 4, 3]

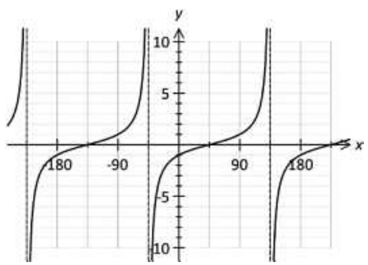
Find the equation of the following trigonometric functions:

(a)



Equation is $f(x) = 2\cos(2x + 30^\circ)$ ✓✓✓✓

(b)



Equation is $y = \tan(x - 45^\circ)$ ✓✓✓

Calculator Free

10. [9 marks: 4, 2, 3]

Consider $A = \sqrt{3}\sin\theta + \cos\theta$ where $0 \le \theta \le 2\pi$.

(a) Solve for values of θ for which $A = \sqrt{3}$.

$\sqrt{3}\sin\theta + \cos\theta = \sqrt{3}$

$\Rightarrow 2\sin\left(\theta + \frac{\pi}{6}\right) = \sqrt{3}$ ✓✓

$\sin\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\theta + \frac{\pi}{6} = \frac{\pi}{3}, \frac{2\pi}{3}$ ✓

$\theta = \frac{\pi}{6}, \frac{\pi}{2}$ ✓

(b) State the maximum value of A and the value(s) of θ at which this occurs.

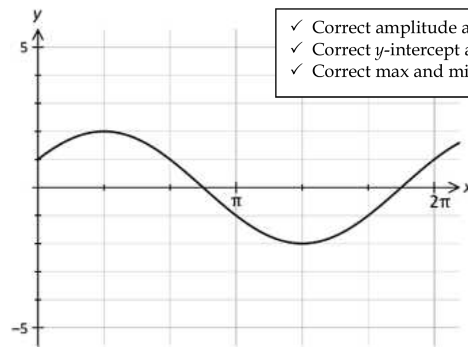
$A = 2\sin\left(\theta + \frac{\pi}{6}\right)$

Hence: Max for $A = 2$ ✓

when $\theta + \frac{\pi}{6} = \frac{\pi}{2}$

$\theta = \frac{\pi}{3}$ ✓

(c) In the axes provided below, sketch the graph of $A = \sqrt{3}\sin\theta + \cos\theta$ for $0 \le \theta \le 2\pi$.

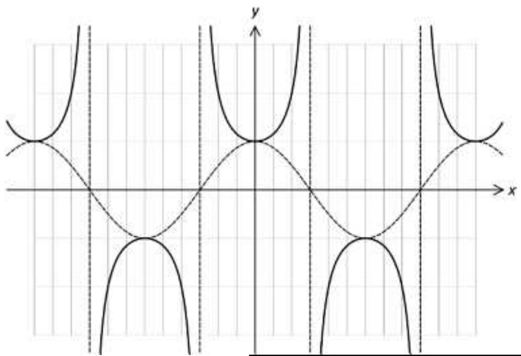


✓ Correct amplitude and period
✓ Correct y-intercept and roots
✓ Correct max and min.

Calculator Free

11. [3 marks]

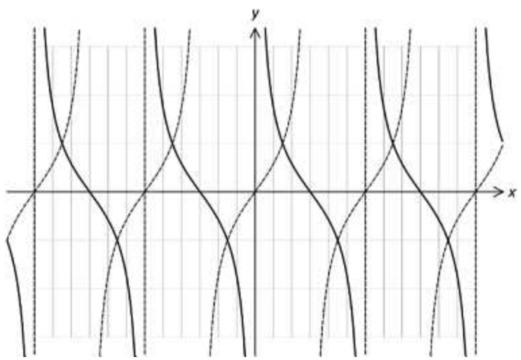
The graph of $y = \cos(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \sec(ax)$.



Correct asymptotes.	✓
Correct minimum points.	✓
Correct shape/symmetry.	✓

12. [3 marks]

The graph of $y = \tan(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \cot(ax)$.



Correct asymptotes.	✓
Correct roots.	✓
Correct points of intersection between curves.	✓

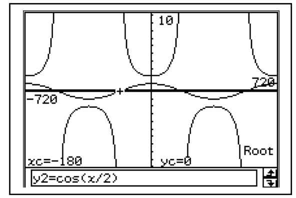
Calculator Assumed

13. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2\sec\left(\frac{x}{2}\right)$ for $-720^o < x < 720^o$.

(a) Determine the coordinates of the maximum turning point(s) of this curve.

$(-360^o, -2)$ & $(360^o, -2)$ ✓✓



(b) Determine the period of this curve.

Period = 720^o ✓

(c) Determine the equations of the vertical asymptotes.

$x = \pm 540^o, \pm 180^o$ ✓✓

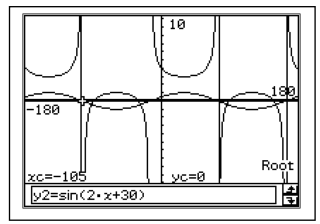
[Asymptotes coincide with the roots of $y = \cos\left(\frac{x}{2}\right)$.]

14. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2 + \text{cosec}\,(2x + 30^o)$ for $-180^o < x < 180^o$.

(a) Determine the coordinates of the minimum turning point(s) of this curve.

$(-150^o, 3)$ & $(30^o, 3)$ ✓✓



(b) Determine the period of this curve.

Period = 180^o ✓

(c) Determine the equations of the vertical asymptotes.

$x = -105^o, -15^o, 75^o, 165^o$ ✓✓

[Asymptotes coincide with the roots of $y = \sin(2x + 30^o)$.]

Calculator Assumed

15. [9 marks: 1, 1, 2, 2, 3]

The body temperature θ (Celsius) of a reptile in summer at time t hours after midnight is given by $\theta = 15 - 5\sin\left(\frac{\pi t}{12}\right)$.

(a) State the period for θ.

Period = $\frac{2\pi}{\pi/12}$ = 24 hours ✓

(b) What is the range of body temperature experienced by the reptile?

10^o Celsius to 20^o Celsius . Hence, Range = 10^o . ✓

(c) Find the minimum body temperature of the reptile and state when this first occurs after midnight.

Minimum body temperature is 10^o Celsius . ✓
This occurs when $t = 6$ i.e. at 6.00am. ✓

(d) Find the maximum body temperature of the reptile and state when this first occurs after midnight.

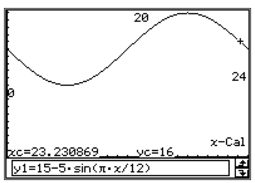
Maximum temperature is 20^o Celsius . ✓
This occurs when $t = 18$ i.e. at 6.00 pm. ✓

(e) Find for how many hours in a 24 hour day, the body temperature of the reptile is below 16^o Celsius. Give your answer to the nearest minute.

From graph: $\theta \geq 16 \Rightarrow 12.7691 \leq t \leq 23.2309$. ✓

That is, $\theta \geq 16$ for 10.4618 hours. ✓

Hence, $\theta < 16$ for 13.5382 hours, i.e. for 13 hours and 32 minutes. ✓



Calculator Assumed

16. [10 marks: 2, 3, 5]

The water depth, h metres, measured from the bottom of a harbour, t hours after 6 am is modelled by the equation $h = 12 - 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{4}\right)$ metres.

(a) Determine when the water depth is at its lowest in a 24-hour day (from 6 am). State the lowest depth.

Lowest depth = 8 metres ✓
at $t = 1.5, 13.5$ i.e. at 7.30 am and 7.30 pm. ✓

(b) Repairs to the harbour can only be undertaken if the water depth is below 10 metres. What times of the day can this occur?

From graph: $h < 10$ for $0 < t < 3.5$, $11.5 < t < 15.5$, $23.5 < t < 24$.

That is: between 6.00 am and 9.30 am, ✓
between 5.30 pm and 9.30 pm ✓
between 5.30 am and 6.00 am of the next day. ✓

(c) The water depth is above k metres for 20% of a 24-hour day. Find k.

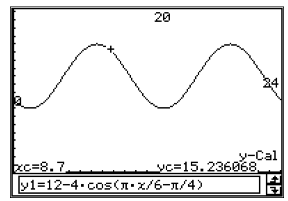
$h > k$ for $24 \times 0.2 = 4.8$ hours. ✓

The water depth changes with a period of 12 hours.
There are two cycles within a 24-hour day. ✓
Hence, within one cycle, the water depth is above k metres for 2.4 hours. ✓

That is, $h > k$ for 1.2 hours before the maximum depth and for 1.2 hours after the maximum depth. ✓

Maximum depth occurs at $t = 7.5$
When $t = 7.5 + 1.2 = 8.7$, $h = 15.24$.

Hence, $k = 15.24$ metres. ✓



25 Matrix Algebra

Calculator Free

1. [6 marks: 1, 1, 4] [TISC]

Consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Given that matrix **X** can be added with matrix **A**, what is the size of matrix **X?**

> dimension = 3 × 2 ✓

(b) Given that **BY** = **YB**, what is the size of matrix **Y** ?

> dimension = 2 × 2 ✓

(c) Consider the matrices, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ k \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \end{pmatrix}$. Two different matrices are selected from the three given and then multiplied together. State all the possible products.

> $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} -1 \\ 1+k \end{pmatrix}$ ✓
>
> $\begin{pmatrix} 1 \\ k \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -k & k \end{pmatrix}$ ✓
>
> $\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} -1+k \end{pmatrix}$ ✓
>
> $\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}$ ✓

---

2. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & -1 \end{pmatrix}$ where a, b, x and y are constants.

(a) Using the matrices given, the product of two different matrices is a 2 × 1 matrix. Show how this product can be formed. State the resulting matrix.

> $\mathbf{A} \times \mathbf{C} = \begin{pmatrix} a \\ 2b \end{pmatrix}$ or $\mathbf{B} \times \mathbf{C} = \begin{pmatrix} 2x+1 \\ y \end{pmatrix}$ ✓

Calculator Free

2. (b) Using the matrices given, the product of two different matrices is a 1 × 2 matrix. Show how this product can be formed. State the resulting matrix.

> $\mathbf{D} \times \mathbf{A} = \begin{pmatrix} 2a & -b \end{pmatrix}$ or $\mathbf{D} \times \mathbf{B} = \begin{pmatrix} -y+2 & 2x \end{pmatrix}$ ✓

(c) Using the matrices given, the product of two non-square matrices is a square matrix. Show how this product can be formed. State the resulting matrix.

> $\mathbf{C} \times \mathbf{D} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ ✓✓

(d) Using the matrices given, the product of three different matrices is a 1 × 1 matrix. Show how this product can be formed. State the resulting matrix.

> $\mathbf{D} \times \mathbf{A} \times \mathbf{C} = \begin{pmatrix} 2a-2b \end{pmatrix}$ or $\mathbf{D} \times \mathbf{B} \times \mathbf{C} = \begin{pmatrix} 4x-y+2 \end{pmatrix}$ ✓✓

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3. [8 marks: 1, 3, 4]

Let $\mathbf{A} = \begin{pmatrix} 0 & a \\ a & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} b & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix}$ where a, b and c are constants.

(a) Find **A – B**.

> $\mathbf{A} - \mathbf{B} = \begin{pmatrix} -b & a-1 \\ a & 0 \end{pmatrix}$ ✓

(b) Find the value of a if **A – B** is non-singular.

> $\det(\mathbf{A}-\mathbf{B}) = -a^2 + a = a(1-a)$ ✓
>
> **A** – $\Rightarrow \det(\mathbf{A}-\mathbf{B}) \neq 0$ ✓
>
> Hence, $a \neq 0 \quad a \neq 1$ ✓

(c) Find if possible, the values of a and b if $(\mathbf{A}-\mathbf{B})^{-1} = \mathbf{C}$.

> $\begin{pmatrix} -b & a-1 \\ a & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓
>
> $\begin{pmatrix} -2a+2 & -2b-4a+4 \\ 0 & 2a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓
>
> Comparing elements: $a = \frac{1}{2}$ ✓
>
> $-2b - 2 + 4 = 0$
>
> $b = 1$ ✓

> $\begin{pmatrix} -b & a-1 \\ a & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix}$ ✓
>
> $\frac{1}{-a^2+a}\begin{pmatrix} 0 & -a+1 \\ -a & -b \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix}$ ✓
>
> $\frac{-a}{-a^2+a} = -2 \Rightarrow a = \frac{1}{2}$ ✓
>
> $\frac{-b}{-a^2+a} = -4 \Rightarrow b = 1$ ✓

Calculator Free

4. [5 marks: 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} k & 1 \\ 8 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -8 & k \end{pmatrix}$

(a) Find the value(s) of k if **A** is a non-singular matrix.

> $|\mathbf{A}| = 2k - 8 \neq 0$ ✓
> $k \neq 4$ ✓

(b) Find the value(s) of k if $\mathbf{A} \times \mathbf{B} = \mathbf{A} + \mathbf{B}$.

> $\begin{pmatrix} 2k-8 & 0 \\ 0 & 2k-8 \end{pmatrix} = \begin{pmatrix} k+2 & 0 \\ 0 & k+2 \end{pmatrix}$ ✓✓
> $\Rightarrow 2k - 8 = k + 2$
> $k = 10$ ✓

5. [6 marks: 2, 2, 2] [TISC]

Given that $\mathbf{A} = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}$.

(a) Find the relationship between a and b such that **A** is a singular matrix.

> As **A** is a singular matrix, $|\mathbf{A}| = 0$. ✓
> $|\mathbf{A}| = a^2 - b = 0$ ✓

(b) Given that $b = 4$, find the value(s) of a for which **A** is non-singular.

> Since A is non-singular, $|\mathbf{A}| \neq 0$.
> Hence, $a^2 - 4 \neq 0$ ✓
> $\Rightarrow$ $a \neq \pm 2$ ✓ (must give both values)

(c) Explain clearly why **A** will always have an inverse if $b < 0$.

> If $b < 0$, then $a^2 - b > 0 \ \forall a$. ✓
> Hence, $|\mathbf{A}| \neq 0$. ✓
> $\Rightarrow$ **A** will always have an inverse.

Calculator Free

6. [9 marks: 1, 3, 5]

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ b & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & -2 \\ 6 & 12 \end{pmatrix}$.

(a) Find **A** + **B**.

> $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 2+a \\ b & 7 \end{pmatrix}$. ✓

(b) Find the value of b if **B** is non-singular.

> $\det(\mathbf{B}) = 4 - 2b$ ✓
> **B** non-singular $\Rightarrow \det(\mathbf{B}) \neq 0$ ✓
> Hence, $b \neq 2$ ✓

(c) Find if possible, the values of a and b if $\mathbf{A} \times \mathbf{B} = \mathbf{C}$.

> $\begin{pmatrix} 1 & a \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ b & 4 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 6 & 12 \end{pmatrix}$
> $\begin{pmatrix} 1+ab & 2+4a \\ 3b & 12 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 6 & 12 \end{pmatrix}$ ✓
> Comparing elements: $a = -1$ ✓
> $b = 2$ ✓
> $1 + ab = -2$
> $ab = -3$ ✓
> But $ab = -2 \neq -3$. Hence, no solution. ✓

7. [6 marks: 3, 3]

Let $\mathbf{A} = \begin{pmatrix} k^2 & 2 \\ k & 1 \end{pmatrix}$. Determine the value(s) of k if:

(a) $|\mathbf{A}| = 3$

> $|\mathbf{A}| = 3 \Rightarrow k^2 - 2k = 3$ ✓
> $(k-3)(k+1) = 0$
> $k = -1$, or 3 ✓✓

(b) $\mathbf{A}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

> $\begin{pmatrix} k^2 & 2 \\ k & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓
> $\begin{pmatrix} -k^2+2 & 2k^2-2 \\ -k+1 & 2k-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓
> Comparing elements: $k = 1$ ✓

Calculator Free

8. [4 marks: 2, 2] [TISC]

(a) If **A** is a non-singular square matrix, show that if $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{I}$ where **I** is the appropriate identity matrix.

$$\mathbf{A}^2 = \mathbf{A}$$
$$\mathbf{A}^2\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} \quad ✓$$
$$\mathbf{A} = \mathbf{I} \quad ✓$$

(b) Find a 2 × 2 non-zero matrix **A**, where $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A} \neq \mathbf{I}$. (**I** is the 2 × 2 identity matrix.)

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ✓✓$$

---

9. [6 marks: 3, 3]

(a) Given that **P** and **Q** are square matrices and $\mathbf{PQ} = \mathbf{P} + \mathbf{Q}$, show that $\mathbf{P} = \mathbf{Q}(\mathbf{Q} - \mathbf{I})^{-1}$, where **I** is the appropriate identity matrix.

$$\mathbf{PQ} - \mathbf{P} = \mathbf{Q} \quad ✓$$
$$\mathbf{P}(\mathbf{Q} - \mathbf{I}) = \mathbf{Q} \quad ✓$$
$$\mathbf{P} = \mathbf{Q}(\mathbf{Q} - \mathbf{I})^{-1} \quad ✓$$

(b) Given that **A** and **B** are 2 × 2 non-zero diagonal matrices, prove that **A** and **B** are commutative under multiplication.

Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$

$$\mathbf{AB} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & by \end{pmatrix} \quad ✓$$

$$\mathbf{BA} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & by \end{pmatrix} \quad ✓$$

Hence, **AB** = **BA**.
That is, **A** and **B** are commutative under multiplication. ✓

Calculator Free

10. [9 marks: 2, 3, 4]

Given that the non-singular matrix **A**, where $\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

find each of the following. Justify each of your answers.

(a) $\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \mathbf{A}\left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$
✓ ✓

(b) $\mathbf{A}^2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\mathbf{A}\left[\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right] = \mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$
✓ ✓ ✓

(c) $\mathbf{A}^{-1}\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \Rightarrow \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad ✓$$

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad ✓$$

$$\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad ✓$$

$$\mathbf{A}^{-1}\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix} \quad ✓$$

Calculator Assumed

11. [5 marks]

Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with all non-zero elements, where $|\mathbf{M}| = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^2$, prove that $a + d = -1$.

$\mathbf{M}^{-1} = \mathbf{M}^2 \Rightarrow \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$

$|\mathbf{M}| = 1 \Rightarrow ad - bc = 1$

Hence $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$ ✓✓✓

$\Rightarrow a^2 + bc = d$ I $\quad ab + bd = -b$ II

$ac + cd = -c$ III $\quad d^2 + bc = a$ IV

From II: $b(a + d) = -b \Rightarrow a + d = -1$ ✓

From III: $c(a + d) = -c \Rightarrow a + d = -1$ ✓

Hence: $a + d = -1$

---

12. [4 marks]

Let **X** be a $n \times 1$ matrix, **A** be a $n \times n$ matrix and λ be a real non-zero constant. Given that $\mathbf{A}\,\mathbf{X} = \lambda\mathbf{X}$, prove that $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$\mathbf{A}\,\mathbf{X} = \lambda\mathbf{X} \Rightarrow \mathbf{AX} - \lambda\mathbf{X} = \mathbf{0}$

$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{0}$ ✓

$|(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X}| = |\mathbf{0}|$ ✓

$|(\mathbf{A} - \lambda\mathbf{I})| \times |\mathbf{X}| = 0$ ✓

But $|\mathbf{X}|$ is not defined.

$\Rightarrow |(\mathbf{A} - \lambda\mathbf{I})| = 0$ ✓

Calculator Assumed

13. [7 marks: 2, 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$.

(a) Show that $\mathbf{A}^2 = \mathbf{A} + k\mathbf{I}$ where k is a real constant.

$\mathbf{A}^2 = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}^2 = \begin{pmatrix} 9 & 1 \\ 5 & 6 \end{pmatrix}$

$\begin{pmatrix} 9 & 1 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix} + k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$k = 7$ ✓

(b) Use your result in (a) to find $\mathbf{A}^{-1}$ in the form $\alpha\mathbf{A} + \beta\mathbf{I}$.

$\mathbf{A}^2 = \mathbf{A} + 7\mathbf{I}$

$\mathbf{A}^2 - \mathbf{A} = 7\mathbf{I}$

$\mathbf{A}\,(\mathbf{A} - \mathbf{I}) = 7\mathbf{I}$ ✓

$\mathbf{A}^{-1} = \frac{1}{7}\mathbf{A} - \frac{1}{7}\mathbf{I}.$ ✓

(c) Find $\mathbf{A}^4$ in terms of **A** and **I**.

$\mathbf{A}^2 = \mathbf{A} + 7\mathbf{I}$

$\mathbf{A}^4 = [\mathbf{A} + 7\mathbf{I}]^2$ ✓

$= \mathbf{A}^2 + 14\mathbf{A} + 49\mathbf{I}$

$= \mathbf{A} + 7\mathbf{I} + 14\mathbf{A} + 49\mathbf{I}$ ✓

$= 15\mathbf{A} + 56\mathbf{I}$ ✓

Calculator Assumed

14. [8 marks: 3, 1, 4]

(a) Given that **A**, and **B** are non-singular square matrices prove that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

Let $(\mathbf{AB})^{-1} = \mathbf{P}$.

$\Rightarrow$ $(\mathbf{AB}) \times \mathbf{P} = \mathbf{I}$

$\mathbf{A}^{-1}\mathbf{AB} \times \mathbf{P} = \mathbf{A}^{-1}$ ✓

$\mathbf{B} \times \mathbf{P} = \mathbf{A}^{-1}$ ✓

$\mathbf{B}^{-1}\mathbf{B} \times \mathbf{P} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$\mathbf{P} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ ✓

(b) Hence, or otherwise, prove that $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

$(\mathbf{AA})^{-1} = \mathbf{A}^{-1}\mathbf{A}^{-1}$ ✓

$(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

(c) Given that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$, find $(\mathbf{A}^2)^{-1}$ in the form $p\mathbf{A} + q\mathbf{I}$ where p and q are real constants and **I** is the identity matrix.

$\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}(2\mathbf{A} + \mathbf{I})$ ✓

$\mathbf{A} = 2\mathbf{I} + \mathbf{A}^{-1}$

$\mathbf{A}^{-1} = \mathbf{A} - 2\mathbf{I}$ ✓

From (b) $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$

$= (\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 2\mathbf{I})$ ✓

$= \mathbf{A}^2 - 4\mathbf{A} + 4\mathbf{I}$

$= (2\mathbf{A} + \mathbf{I}) - 4\mathbf{A} + 4\mathbf{I}$

$= -2\mathbf{A} + 5\mathbf{I}$ ✓

26 Systems of Equations

Calculator Free

1. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1} = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$ ✓

(b) Find the product $\mathbf{A}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ✓

(c) Consider the system of equations:

$$x + 2y = 2$$
$$-x + 3y = -2$$

(i) Rewrite the given system of equations in the form **AX** = **B** where **X** is a column matrix and **A** and **B** are appropriate matrices.

Rewrite system as: $x + 2y = 2$, $x - 3y = 2$ ✓

Hence: $\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ ✓

(ii) Use a matrix method to solve for x and y.

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ ✓

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Hence, $x = 2$, $y = 0$ ✓

Calculator Free

2. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

$$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} \quad ✓$$

(b) Find the product $\begin{pmatrix} 3 & 4 \end{pmatrix} \times \mathbf{A}^{-1}$.

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 4 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 3 \end{pmatrix} \quad ✓$$

(c) Consider the system of equations:

$$3x + 5y = 3$$
$$x + 2y = 2$$

(i) Rewrite the given system of equations in the form **XA** = **B** where **X** is a row matrix and **A** and **B** are appropriate matrices.

Rewrite system as: $3x + 5y = 3$, $2x + 4y = 4$ ✓

Hence: $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \end{pmatrix}$ ✓

(ii) Use a matrix method to solve for x and y.

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 3 & 4 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}^{-1} \quad ✓$$
$$= \begin{pmatrix} -4 & 3 \end{pmatrix}$$

Hence: $x = -4, y = 3$ ✓

Calculator Free

3. [4 marks]

Use a method involving the use of an inverse matrix to solve for x and y in:

$$2x + 3y = -1$$
$$3x - 2y = 18$$

Matrix Equation:

$$\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 18 \end{pmatrix} \quad ✓$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 18 \end{pmatrix}$$
$$= \frac{1}{13}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 18 \end{pmatrix} \quad ✓$$
$$= \frac{1}{-13}\begin{pmatrix} -52 \\ 39 \end{pmatrix} \quad ✓$$
$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$
$$x = 4, y = -3 \quad ✓$$

4. [4 marks: 2, 2]

Consider the set of simultaneous equations:

$$x + y = 8$$
$$2x + 3y = 30$$

(a) Rewrite the system of equations using matrices in the form **XA** = **B**, where $\mathbf{X} = \begin{pmatrix} x & y \end{pmatrix}$.

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 30 \end{pmatrix} \quad ✓✓$$

(b) Hence, use a method involving the use of an inverse matrix, solve for x and y.

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 8 & 30 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} \quad ✓$$
$$= \begin{pmatrix} -6 & 14 \end{pmatrix}$$
$$x = -6, y = 14 \quad ✓$$

Calculator Free

5. [4 marks]

Given that $\mathbf{A}\times\mathbf{B} = 4\mathbf{I}$, $\mathbf{B}\times\begin{pmatrix}-2\\5\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{A}\times\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2\\5\end{pmatrix}$,

where **I** is the 2 × 2 unit matrix, find x and y.

$$\begin{pmatrix}x\\y\end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix}-2\\5\end{pmatrix} \quad ✓$$

But $\mathbf{A}\times\mathbf{B} = 4\mathrm{I} \Rightarrow \mathbf{A}^{-1} = \frac{1}{4}\times\mathbf{B}$ ✓

Hence,

$$\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{4}\times\mathbf{B}\times\begin{pmatrix}-2\\5\end{pmatrix}$$

$$= \frac{1}{4}\times\begin{pmatrix}1\\2\end{pmatrix}. \quad ✓$$

$$x = \frac{1}{4},\ y = \frac{1}{2} \quad ✓$$

6. [5 marks: 1, 4]

Let $\mathbf{A} = \begin{pmatrix}1 & 4\\-1 & 3\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}3 & 5\\1 & 2\end{pmatrix}$.

(a) Find product $\mathbf{A}\times\mathbf{B}$.

$$\mathrm{A}\times\mathrm{B} = \begin{pmatrix}1 & 4\\-1 & 3\end{pmatrix}\times\begin{pmatrix}3 & 5\\1 & 2\end{pmatrix} = \begin{pmatrix}7 & 13\\0 & 1\end{pmatrix} \quad ✓$$

(b) Show how your answer in (a) can be used to solve for x and y in:
$3x + 5y = 1$ and $x + 2y = 2$.

Rewrite as: $\begin{pmatrix}3 & 5\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

Pre-multiply both sides with $\begin{pmatrix}1 & 4\\-1 & 3\end{pmatrix}$:

$$\begin{pmatrix}1 & 4\\-1 & 3\end{pmatrix}\begin{pmatrix}3 & 5\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 4\\-1 & 3\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} \quad ✓$$

$$\begin{pmatrix}7 & 13\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}9\\5\end{pmatrix} \quad ✓$$

Hence: $y = 5$ ✓

$7x + 13(5) = 9 \Rightarrow x = -8$ ✓

Calculator Free

7. [5 marks]

Use a method involving the use of an inverse matrix to find the coordinates of the point of intersection between the lines $y = 2x + 6$ and $y = \frac{-3x}{2} - 1$:

Rewrite equations as:
$2x - y = -6$
$3x + 2y = -2$ ✓

Matrix equation:

$$\begin{pmatrix}2 & -1\\3 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-6\\-2\end{pmatrix} \quad ✓$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & -1\\3 & 2\end{pmatrix}^{-1}\begin{pmatrix}-6\\-2\end{pmatrix} \quad ✓$$

$$= \frac{1}{7}\begin{pmatrix}2 & 1\\-3 & 2\end{pmatrix}\begin{pmatrix}-6\\-2\end{pmatrix} \quad ✓$$

$$= \begin{pmatrix}-2\\2\end{pmatrix}$$

Hence: Point of intersection (–2, 2). ✓

8. [4 marks]

Given $\begin{pmatrix}-1 & 2 & 3\\-1 & 2 & 4\\1 & -1 & -2\end{pmatrix}\times\begin{pmatrix}0 & -1 & 4\\-2 & 1 & 2\\1 & -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0\\0 & -2 & 0\\0 & 0 & 2\end{pmatrix}$, Use this result to solve:

$$-y + 4z = -12 \qquad -2x + y + 2z = -8 \qquad -x + y = -1$$

Matrix Equation:

$$\begin{pmatrix}0 & -1 & 4\\-2 & 1 & 2\\1 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-12\\-8\\1\end{pmatrix} \quad ✓$$

$$\begin{pmatrix}-1 & 2 & 3\\-1 & 2 & 4\\1 & -1 & -2\end{pmatrix}\times\begin{pmatrix}0 & -1 & 4\\-2 & 1 & 2\\1 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1 & 2 & 3\\-1 & 2 & 4\\1 & -1 & -2\end{pmatrix}\begin{pmatrix}-12\\-8\\1\end{pmatrix} \quad ✓$$

$$\begin{pmatrix}-1 & 0 & 0\\0 & -2 & 0\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\0\\-6\end{pmatrix} \quad ✓$$

$$x = 1,\ y = 0,\ z = -3 \quad ✓$$

Calculator Free

9. [9 marks: 3, 6]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & -8 \\ -1 & -1 & 4 \\ 2 & 1 & -3 \end{pmatrix}$.

(a) Determine the values of p, q and r in $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & -8 \\ -1 & -1 & 4 \\ 2 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & r \\ 5 & q & -4 \\ p & 0 & 1 \end{pmatrix}$.

> $p = 1 \quad q = 2 \quad r = 0$ ✓✓✓

(b) Use your answer in (a) to solve for x, y and z in:

$$3x + 2y - 8z = -1$$
$$x + y - 4z = -1$$
$$2x + y - 3z = 1$$

> Rewrite system as a matrix equation:
>
> $\begin{pmatrix} 3 & 2 & -8 \\ -1 & -1 & 4 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ✓
>
> $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & -8 \\ -1 & -1 & 4 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ✓
>
> $\begin{pmatrix} 1 & 0 & 0 \\ 5 & 2 & -4 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ ✓
>
> $\Rightarrow x = 1$ ✓
>
> $x + z = 2 \Rightarrow z = 1$ ✓
>
> $5x + 2y - 4z = 5 \Rightarrow y = 2$ ✓

27 Applications using Matrices

Calculator Assumed

1. [7 marks: 2, 3, 2]

A budget airline flies from Perth to Sydney and charges two different fares for its passengers; deluxe economy and economy. On a certain flight, there were: 200 fare paying passengers and one and a half times as many economy passengers as deluxe economy passengers

Let d: number of deluxe economy passengers on this flight
e: number of economy passengers on this flight

(a) Use the information given above to write down two equations involving d and e.

> $d + e = 200$ ✓
>
> $e - 1.5d = 0$ ✓

(b) Use a method involving the inverse of a matrix to find d and e.

> $\begin{pmatrix} 1 & 1 \\ -1.5 & 1 \end{pmatrix}\begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 200 \\ 0 \end{pmatrix}$ ✓
>
> $\begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1.5 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 200 \\ 0 \end{pmatrix}$ ✓
>
> $= \begin{pmatrix} 80 \\ 120 \end{pmatrix}$
>
> Hence: $d = 80$, $e = 120$ ✓

(c) A deluxe economy ticket costs \$349, while an economy ticket costs 30% less. Use a matrix method to find the total amount in fares collected from this flight. You need to show clearly the matrices used, and the operation(s) used on these matrices.

> $\begin{pmatrix} 349 & 349 \times 0.7 \end{pmatrix}\begin{pmatrix} 80 \\ 120 \end{pmatrix} = (57\,236)$ ✓
>
> Hence, total fare collected = \$57 236 ✓

Calculator Assumed

2. [5 marks: 1, 1, 3]

The table below shows the number of hours Jack worked last week at a fast food outlet.

Shift	Weekdays	Weekends
Morning (M)	16	4
Afternoon (A)	8	4
Night (N)	8	0

(a) Write a *row* matrix **A** describing the number of hours Jack worked on each shift on weekdays.

$$\mathbf{A} = \text{Weekday} \begin{pmatrix} \overset{M}{16} & \overset{A}{8} & \overset{N}{8} \end{pmatrix} \quad ✓$$

(b) Write a *column* matrix **B** describing the number of hours Jack worked on each shift on weekends.

Weekend

$$\mathbf{B} = \begin{matrix} M \\ A \\ N \end{matrix} \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad ✓$$

The rates of pay are $15.00 per hour for weekday morning shifts, $12.00 per hour for weekday afternoon shifts and $20 per hour for weekday night shifts. Jack is paid twice as much per hour for weekend shifts as weekday shifts.

(c) Use matrices **A** and **B** and other matrices as required to find the total amount of money Jack earned last week.

$$\begin{pmatrix} 16 & 8 & 8 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 20 \end{pmatrix} + \begin{pmatrix} 30 & 24 & 40 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad ✓✓$$

$$= (496) + (216)$$
$$= (712)$$

Amount of money earned = $712. ✓

Calculator Assumed

3. [7 marks: 2, 2, 3]

The number of different tickets available for a charity concert at a concert hall is given in the table below.

	Stalls	Gallery
Adults	500	300
Students/pensioners	150	50

The prices for these tickets are given in the table below.

	Stalls	Gallery
Adults	$150	$90
Students/pensioners	$120	$70

(a) Given that all the tickets were sold, use a matrix method to determine the revenue from:

(i) the adult members of audience

$$\begin{pmatrix} 150 & 90 \end{pmatrix} \begin{pmatrix} 500 \\ 300 \end{pmatrix} = (102\,000) \quad ✓$$

Hence, $102 000 . ✓
(No marks if matrices are not used)

(ii) the stalls tickets.

$$\begin{pmatrix} 150 & 120 \end{pmatrix} \begin{pmatrix} 500 \\ 150 \end{pmatrix} = (93\,000) \quad ✓$$

Hence, $93 000 . ✓
(No marks if matrices are not used)

(b) Given that 310 gallery tickets were sold realising a revenue of $26 900. Use a matrix method to determine how many of the gallery tickets were sold to adults?

Let a: number of gallery tickets sold to adults
p: number of gallery tickets sold to pensioners/students.

Then:
$$a + p = 310$$
$$90a + 70p = 26\,900$$

Hence:
$$\begin{pmatrix} 1 & 1 \\ 90 & 70 \end{pmatrix} \begin{pmatrix} a \\ p \end{pmatrix} = \begin{pmatrix} 310 \\ 26\,900 \end{pmatrix} \quad ✓$$

$$\begin{pmatrix} a \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 90 & 70 \end{pmatrix}^{-1} \begin{pmatrix} 310 \\ 26\,900 \end{pmatrix} \quad ✓$$

$$= \begin{pmatrix} 260 \\ 50 \end{pmatrix}$$

Hence, 260 tickets were sold to adults. ✓

Calculator Assumed

4. [8 marks: 2, 3, 3]

In a city there are two companies, A and B, that supply gas to households. The table below shows the percentage of customers in each company that will remain with their original supplier and the percentage of customers that will switch to the competitor within two years.

%		From	
		A	B
To	A	65	15
	B	35	85

Initially, there were 750 000 and 250 000 customers with companies A and B respectively. Assume that the total number of customers remain unchanged.

(a) Use a matrix method to determine the number of customers with company A at the end of two years.

$$\begin{pmatrix}0.65 & 0.15\end{pmatrix}\begin{pmatrix}750\,000\\250\,000\end{pmatrix} = \begin{pmatrix}525\,000\end{pmatrix} \quad ✓$$

Hence, 525 000 customers with A after 2 years. ✓

(b) Use a matrix method to determine the number of customers with company A at the end of four years.

After two years, there will be 525 000 customers with A and 475 000 customers with B. ✓

Hence, after 4 years: $\begin{pmatrix}0.65 & 0.15\end{pmatrix}\begin{pmatrix}525\,000\\475\,000\end{pmatrix} = \begin{pmatrix}412\,500\end{pmatrix}$ ✓

That is, there will be 412 500 customers with A. ✓

(c) Using the table given, will company B ever have 750 000 customers? Justify your answer.

Distribution of customers after n lots of 2 years is given by:

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0.65 & 0.15\\0.35 & 0.85\end{pmatrix}^n\begin{pmatrix}750\,000\\250\,000\end{pmatrix} \quad ✓$$

For $n = 30$: $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}300\,000\\700\,000\end{pmatrix}$

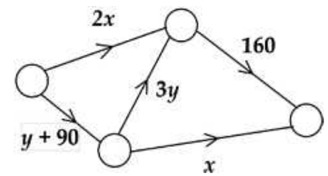
For $n = 31$: $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}300\,000\\700\,000\end{pmatrix}$

Hence, no, as the maximum number of customers for B is 700 000. ✓✓

Calculator Assumed

5. [7 marks: 3, 4]

The diagram below shows the flow of fluid (in litres/minute) through a network of pipes. The numbers or letters indicate the flow rate through the pipe concerned. Assume that no fluid is lost in the process.



(a) Write down all equations involving x and y for the given network.

$2x + 3y = 160$ (1) ✓

$y + 90 = x + 3y$

$\Rightarrow x + 2y = 90$ (2) ✓

$2x + y + 90 = 160 + x$

$x + y = 70$ (3) ✓

(b) Use a method involving the inverse of a matrix to solve for x and y. Show clearly the matrices involved.

Using (1) and (2):

$$\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}160\\90\end{pmatrix} \quad ✓$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}160\\90\end{pmatrix} \quad ✓$$

$$= \begin{pmatrix}50\\20\end{pmatrix}$$

Hence: $x = 50, y = 20$ ✓

Check: Clearly $x = 50, y = 20$ satisfies (3). ✓

Calculator Assumed

6. [8 marks: 2, 1, 2, 3]

The table below shows the ratios of the different components for the different types of mortar for each twelve kg bag. For example, a 12 kg bag of retaining wall mortar has 2 kg of cement, 1 kg of lime and 9 kg of sand. There is an order for 10, 12 and 20 sixty kg bags of common mortar, retaining wall mortar and internal wall mortar.

	Cement	Lime	Sand
Common Mortar	1.5	1.5	9
Retaining Wall Mortar	2	1	9
Internal Wall Mortar	1	2	9

(a) Write a matrix **M** that describes the mass of the components in each sixty kg bag of common mortar, retaining wall mortar and internal wall mortar.

$$\mathbf{M} = 5\begin{pmatrix} 1.5 & 1.5 & 9 \\ 2 & 1 & 9 \\ 1 & 2 & 9 \end{pmatrix} \quad ✓✓$$

(b) Write a *row* matrix **B** that when multiplied with **M** will give the amount of cement, lime and sand required to fulfil this order.

$$\mathbf{B} = \begin{pmatrix} 10 & 12 & 20 \end{pmatrix} \quad ✓$$

(c) Use **M** and **B** to determine the amount of cement, lime and sand required to fulfil this order.

$$\begin{pmatrix} 10 & 12 & 20 \end{pmatrix} \times 5\begin{pmatrix} 1.5 & 1.5 & 9 \\ 2 & 1 & 9 \\ 1 & 2 & 9 \end{pmatrix} = \begin{pmatrix} 295 & 335 & 1890 \end{pmatrix} \quad ✓$$

Hence: 295 kg of cement, 335 kg of lime and 1890 kg of sand. ✓

(d) Cement costs \$0.40 per kg, lime costs \$0.50 kg per kg and sand costs \$0.25 per kg. Use **M**, **B** and another appropriate matrix to determine the total cost for this order.

Let the unit cost matrix be $\mathbf{C} = \begin{pmatrix} 0.4 \\ 0.5 \\ 0.25 \end{pmatrix}$

$$\begin{pmatrix} 10 & 12 & 20 \end{pmatrix} \times 5\begin{pmatrix} 1.5 & 1.5 & 9 \\ 2 & 1 & 9 \\ 1 & 2 & 9 \end{pmatrix} \times \begin{pmatrix} 0.4 \\ 0.5 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 758 \end{pmatrix} \quad ✓✓$$

Hence, cost = \$758. ✓

Calculator Assumed

7. [8 marks: 4, 4]

The table below shows the cost per share and dividend returns for shares in companies A, B and C.

	A	B	C
Cost per share	\$2.40	\$5.10	\$1.20
Dividend per share	\$0.20	\$0.80	\$0.10

(a) Ann had 4 000, 5 000 and 8 000 shares in companies A, B and C respectively. Show the use of matrices to calculate the total number of shares, total value of shares and the total amount of dividends received.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2.4 & 5.1 & 1.2 \\ 0.2 & 0.8 & 0.1 \end{pmatrix}\begin{pmatrix} 4\,000 \\ 5\,000 \\ 8\,000 \end{pmatrix} = \begin{pmatrix} 17\,000 \\ 44\,700 \\ 5\,600 \end{pmatrix} \quad ✓✓$$

Hence: Total number of shares = 17 000 ✓
Total value of shares = \$44 700
Total dividends received = \$5 600. ✓

(b) Ben had a total of 25 000 shares worth a total of \$82 800. The number of shares he held in companies A, B and C are a, b and c respectively. The total amount of dividends he received was \$11 400. Show the use of a matrix equation to calculate the values of a, b and c.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2.4 & 5.1 & 1.2 \\ 0.2 & 0.8 & 0.1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 25\,000 \\ 82\,800 \\ 11\,400 \end{pmatrix} \quad ✓$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2.4 & 5.1 & 1.2 \\ 0.2 & 0.8 & 0.1 \end{pmatrix}^{-1}\begin{pmatrix} 25\,000 \\ 82\,800 \\ 11\,400 \end{pmatrix} \quad ✓$$

$$= \begin{pmatrix} 5\,000 \\ 12\,000 \\ 8\,000 \end{pmatrix} \quad ✓$$

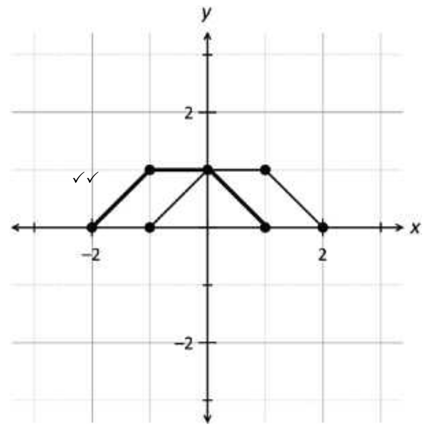
Hence: $a = 5\,000$, $b = 12\,000$, $c = 8\,000$. ✓

28 Transformation Matrices

Calculator Free

1. [6 marks: 2, 2, 2]

The graph of $y = f(x)$ is given below. The transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is applied to this curve.



(a) Find the image of the points (–1, 0), (0, 1), (1, 1) and (2, 0).

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$ ✓

Hence, images are respectively (1, 0), (0, 1), (–1, 1) and (–2, 0). ✓

(b) Sketch on the axes provided above, the graph of the resulting curve.

(c) The equation of the resulting curve is $y = a\,f(bx + c)$. Find a, b and c.

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection about the y-axis. ✓

Hence $y = f(x)$ is transformed to $y = f(-x)$.
$\Rightarrow a = 1, b = -1, c = 0$. ✓

Calculator Free

2. [9 marks: 2, 3, 2, 2] [TISC]

Consider the curve with equation $y = f(x)$. The curve has a maximum point at A (–1, 3) and a minimum point at B (4, –7). The curve $y = f(x)$ is mapped onto the curve $y = g(x)$ by a transformation represented by the matrix $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe the effect the transformation represented by T has on the graph of $y = f(x)$.

The transformation reflects ✓ the graph of $y = f(x)$ about the x-axis. ✓

(b) Find the coordinates of the images of the points A and B under **T**.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 3 & -7 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -3 & 7 \end{pmatrix}$$ ✓

Hence, image of A is (–1, –3). ✓
image of B is (4, 7). ✓

(c) Find the coordinates of the maximum and minimum points on the curve $y = g(x)$.

Maximum point is (4, 7). ✓
Minimum point is (–1, –3) ✓

(d) Find the matrix that maps $y = g(x)$ back to $y = f(x)$.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection about the x-axis.

Hence, reverse transformation has matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ✓✓

Calculator Free

3. [6 marks: 1, 1, 2, 2] [TISC]

The transformation T is represented by the matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

(a) Describe in words the transformation T.

A reflection about the x-axis. ✓

(b) The transformation T is applied to the line with equation $y = x$. Find the equation of the resulting line.

$y = -x$ ✓

(c) The point A is mapped to the point with coordinates $(k, k + 1)$ under transformation T. Find the coordinates of the point A. Justify your answer.

Since T is a reflection about the x-axis, ✓
the coordinates of A is $(k, -k - 1)$. ✓

(d) The transformation T is combined with the transformation represented by matrix **M**. All the entries in matrix **M** are positive. The effects of the combined transformation is represented by the matrix $\begin{pmatrix}1 & 0\\-2 & -1\end{pmatrix}$.

Find the matrix **M**. Show clearly your reasoning.

$\mathbf{M}\times\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\-2 & -1\end{pmatrix}. \Rightarrow \mathbf{M} = \begin{pmatrix}1 & 0\\-2 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{-1}$

$= \begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}.$

But all entries in **M** are positive. Hence $\mathbf{M} \neq \begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}$. ✓

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\times\mathbf{M} = \begin{pmatrix}1 & 0\\-2 & -1\end{pmatrix}. \Rightarrow \mathbf{M} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}$

$= \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}.$ ✓

Calculator Free

4. [7 marks: 3, 2, 1, 1] [TISC]

Triangle ABC with vertices A (0, 0), B (3, 0) and C (3, 3) is mapped to triangle A′B′C′ by the compound transformation represented by the matrix

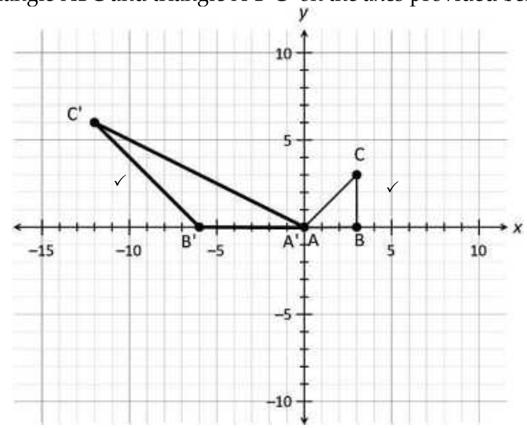
$\mathbf{M} = \begin{pmatrix}-2 & -2\\0 & 2\end{pmatrix}.$

(a) Find the coordinates of the points A′, B′ and C′.

$\begin{pmatrix}-2 & -2\\0 & 2\end{pmatrix}\begin{pmatrix}0 & 3 & 3\\0 & 0 & 3\end{pmatrix} = \begin{pmatrix}0 & -6 & -12\\0 & 0 & 6\end{pmatrix}$

Hence, A′(0, 0) , B′(–6, 0) and C′(–12, 6) ✓✓✓

(b) Plot triangle ABC and triangle A′B′C′ on the axes provided below.



(c) The transformation applied is a combination of several simple transformations. What evidence is there to suggest that one of the simple transformations involved is :

(i) an enlargement?

The size of the image is larger than the size of the object ✓

(ii) a reflection ?

The vertices of the object is read in an anticlockwise manner while the vertices of the image is read in a clockwise manner. ✓
Or
The x-coordinates of the each vertex have their signs reversed whereas the signs of the y-coordinates remain unchanged

Calculator Free

5. [8 marks: 2, 3, 3]

The transformation **S** is represented by matrix $\begin{pmatrix}\cos 25^\circ & -\sin 25^\circ \\ \sin 25^\circ & \cos 25^\circ\end{pmatrix}$ and the transformation **T** is represented by matrix $\begin{pmatrix}\cos 35^\circ & -\sin 35^\circ \\ \sin 35^\circ & \cos 35^\circ\end{pmatrix}$.

(a) Describe in words, the transformation **S**.

> T: Rotation 25° ✓
> anti-clockwise about origin. ✓

(b) Determine the matrix that represents the combination of transformation **S** followed by transformation **T**. Give your answer in its simplest form using exact values.

> $\begin{pmatrix}\cos 35^\circ & -\sin 35^\circ \\ \sin 35^\circ & \cos 35^\circ\end{pmatrix}\begin{pmatrix}\cos 25^\circ & -\sin 25^\circ \\ \sin 25^\circ & \cos 25^\circ\end{pmatrix}$ ✓
>
> $= \begin{pmatrix}\cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ\end{pmatrix}$ ✓
>
> $= \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$ ✓

(c) An object of area 10 units^2 is transformed by **T**.
Determine with reasons the area of the image.

> $\left|\begin{pmatrix}\cos 35^\circ & -\sin 35^\circ \\ \sin 35^\circ & \cos 35^\circ\end{pmatrix}\right| = \cos^2 35^\circ + \sin^2 35^\circ$ ✓
>
> $= 1.$
>
> Hence, scale factor for area = 1. ✓
>
> Therefore area of image = 10 units^2. ✓

Calculator Free

6. [11 marks: 4, 3, 4]

(a) The linear transformation **T** maps the point with coordinates (2, 1) to $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and the point with coordinates (2, –1) to $\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

(i) Determine the 2 × 2 matrix that can be used to represent **T**.

> $\mathbf{M}\begin{pmatrix}2 & 2 \\ 1 & -1\end{pmatrix} = \begin{pmatrix}\frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2}\end{pmatrix}$ ✓
>
> $\mathbf{M} = \frac{\sqrt{2}}{2}\begin{pmatrix}3 & 1 \\ 1 & 3\end{pmatrix}\begin{pmatrix}2 & 2 \\ 1 & -1\end{pmatrix}^{-1}$ ✓
>
> $\mathbf{M} = \frac{\sqrt{2}}{2}\begin{pmatrix}3 & 1 \\ 1 & 3\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & 2 \\ 1 & -2\end{pmatrix}$ ✓
>
> $= \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}$
>
> $= \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2}\end{pmatrix}$ ✓

(ii) Describe in words the linear transformation **T**.

> $\mathbf{M} = \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2}\end{pmatrix} = \begin{pmatrix}\cos 45^\circ & \sin 45^\circ \\ \sin 45^\circ & -\cos 45^\circ\end{pmatrix}$ ✓
>
> Reflection about the line ✓
>
> with equation $y = x\tan 22.5^\circ$ ✓

(b) ΔABC is mapped to $\Delta A'B'C'$ by the transformation represented by matrix $\begin{pmatrix}0 & k \\ 3k & 0\end{pmatrix}$.

Find the value of k if the area of ΔABC is three times the area of $\Delta A'B'C'$.

> $\det\begin{pmatrix}0 & k \\ 3k & 0\end{pmatrix} = -3k^2$ ✓
>
> Scale factor for area = $\frac{\text{Area of Image}}{\text{Area of Object}} = \frac{1}{3}$ ✓
>
> Hence: $3k^2 = \frac{1}{3}$ ✓
>
> $k = \pm\frac{1}{3}$ ✓

Calculator Assumed

7. [9 marks: 3, 2, 4] [TISC]

Consider two matrices $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The point A(1, k) is mapped to the point A″ using **T** followed by **S** as transformation matrices.

(a) Find the coordinates of A″.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} \quad ✓✓$$
$$= \begin{pmatrix} 2 \\ -2k \end{pmatrix}$$

Hence, coordinates of A´´ are (2, –2k). ✓

(b) Find a single transformation matrix that will map A″ back to A. Show how you obtained your answer.

Hence, reverse mapping has matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
✓ ✓

(c) C is a circle of radius 1 with centre at (1, 1). C is transformed to circle C′ by the transformation **T**. Discuss the differences between the original circle C and its image C′. You need to comment on the coordinates of the centre, the radius and area of the two circles.

Centre is now at (2, 2). ✓

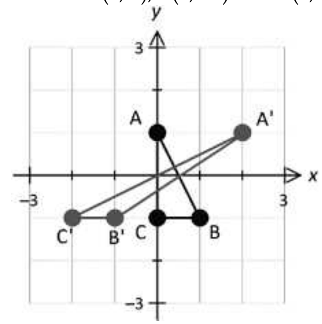
Radius is now 2 × 1 = 2 units ✓

Area is now _four_ times larger ie 4π square units. ✓✓

Calculator Assumed

8. [10 marks: 4, 6]

The triangle ABC has vertices A(0, 1), B(1, –1) and C(0, –1).



(a) ΔABC is mapped to ΔA′B′C′ by a transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Draw and clearly label ΔA′B′C′ on the diagram above.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix} \quad ✓$$

Hence A′ (2, 1), B′ (–1, –1) and C′ (–2, –1). ✓✓✓ Marked on diagram.

(b) ΔA′B′C′ is mapped to ΔA″B″C″ by a transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The points A″, B″ and C″ have coordinates (2, 3), (–1, k) and (k, –3) respectively. Find matrix **M** and the value(s) of k.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & -1 & -2 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & k \\ 3 & k & -3 \end{pmatrix}$$

$2a + b = 2$ (1)
$-a - b = -1$ (2) ✓ Method
$\Rightarrow a = 1, b = 0$ ✓

But $-2a - b = k \Rightarrow k = -2$ ✓

$2c + d = 3$ (3)
$-c - d = k = -2$ (4) ✓ Method
$\Rightarrow c = 1, d = 1$ ✓

Check: $-2c - d = -3 \Rightarrow -2(1) - 1 = -3$ True. ✓

Hence, $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $k = -2$.

Calculator Assumed

9. [9 marks: 3, 2, 4]

(a) Find the image of the line $y = x + 1$ under a transformation represented by $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$.

> Position vector of point on line = $\begin{pmatrix} t \\ t+1 \end{pmatrix}$
>
> Hence, $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} t \\ t+1 \end{pmatrix} = \begin{pmatrix} 2t \\ 3(t+1) \end{pmatrix}$. ✓
>
> $x = 2t$
>
> $y = 3t + 1$ ✓
>
> $\Rightarrow\ y = \dfrac{3x}{2} + 1$ ✓

(b) Consider the transformation mapping T: $(x, y) \to (-x + 2, 2y - 1)$.

(i) Express the transformation mapping T as a matrix equation.

> **T:** $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ 2y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ✓
>
> $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ✓

(ii) Identify the sequence of transformations that make up T.

> 1. Reflection about the y-axis. ✓
> 2. Dilation along the y-axis factor 2. ✓
> 3. Translation –3 units in direction of positive x-axis. ✓
> 4. Translation 4 units in the direction of the positive y-axis. ✓

Calculator Assumed

10. [9 marks: 3, 2, 4]

(a) Find the equation of the image of the line $y = 2x + 8$ under a transformation represented by $\begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix}$.

> Position vector of point on line = $\begin{pmatrix} t \\ 2t+8 \end{pmatrix}$
>
> Hence, $\begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} t \\ 2t+8 \end{pmatrix} = \begin{pmatrix} -4t-16 \\ 4t \end{pmatrix}$ ✓
>
> Therefore: $x = -4t - 16$
>
> $y = 4t$ ✓
>
> Image as equation: $x = -y - 16$ ✓

(b) Consider the transformation mapping **T**: $(x, y) \to (2y - 1, -4x + 3)$.

(i) Express the transformation mapping **T** as a matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ where **M** represents a linear transformation and a and b are constants.

> **T:** $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y \\ -4x \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ ✓
>
> $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ ✓

(ii) Identify the sequence of transformations that make up **T**.

	OR
$\begin{pmatrix} 0 & 2 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
1. Dilation along the x-axis factor 4.	1. Dilation along the x-axis factor 2.
2. Dilation along the y-axis factor 2. ✓	2. Dilation along the y-axis factor 4. ✓
3. Rotation 90° clockwise about O. ✓	3. Rotation 90° clockwise about O. ✓
4. Translation –1 units in direction of positive x-axis ✓	4. Translation –1 units in direction of positive x-axis ✓
5. Translation 3 units in the direction of the positive y-axis. ✓	5. Translation 3 units in the direction of the positive y-axis. ✓

Calculator Assumed

11. [6 marks: 2, 4]

(a) Express the given transformation **T** as a matrix equation
T: $(x, y) \to (x-3, 2y+4)$. Identify the transformations involved.

$$\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ 2y \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad ✓$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad ✓$$

(b) Write the transformation mapping for the point (x, y) under a 150° clockwise rotation about the origin followed by a translation of 2 units along the positive x-axis and -1 unit along the positive y-axis.

$$\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos(-150) & -\sin(-150) \\ \sin(-150) & \cos(-150) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad ✓✓$$

$$\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{-\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad ✓$$

$$\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{-x\sqrt{3}}{2} + \frac{y}{2} + 2 \\ -\frac{x}{2} - \frac{\sqrt{3}}{2} - 1 \end{pmatrix} \quad ✓$$

---

12. [5 marks]

Prove that a reflection about the line $y = \sqrt{3}\,x$, followed by a reflection about the x-axis is equivalent to a single rotation **R**. Describe this rotation.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos 120 & \sin 120 \\ \sin 120 & -\cos 120 \end{pmatrix} \quad ✓✓$$

$$= \begin{pmatrix} \cos 120 & \sin 120 \\ -\sin 120 & \cos 120 \end{pmatrix} \quad ✓$$

$$= \begin{pmatrix} \cos(-120) & -\sin(-120) \\ \sin(-120) & \cos(-120) \end{pmatrix} \quad ✓$$

R is a 120° clockwise rotation about O. ✓

Calculator Assumed

13. [8 marks: 2, 4, 2]

The reflection about the line $y = x\tan\alpha$ is represented by matrix **A**
The reflection about the line $y = x\tan\beta$ is represented by matrix **B**.

(a) Write the transformation matrices **A** and **B**.

$$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \quad ✓$$

$$\mathbf{B} = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \quad ✓$$

(b) Find a single matrix that represents the combined transformations of
a reflection about the line $y = x\tan\alpha$ followed by
a reflection about the line $y = x\tan\beta$.
Simplify your answer.

$$\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \quad ✓$$

$$= \begin{pmatrix} \cos 2\beta\cos 2\alpha + \sin 2\beta\sin 2\alpha & \cos 2\beta\sin 2\alpha - \sin 2\beta\cos 2\alpha \\ \sin 2\beta\cos 2\alpha - \cos 2\beta\sin 2\alpha & \cos 2\beta\cos 2\alpha + \sin 2\beta\sin 2\alpha \end{pmatrix} \quad ✓✓$$

$$= \begin{pmatrix} \cos(2\alpha-2\beta) & \sin(2\alpha-2\beta) \\ \sin(2\alpha-2\beta) & -\cos(2\alpha-2\beta) \end{pmatrix} \quad ✓$$

(c) Hence, prove that the combined transformation in (b) is equivalent to an anti-clockwise rotation of $2(\alpha - \beta)$ about the origin.

Anti-clockwise rotation of $2(\alpha - \beta)$ about the origin

$$= \begin{pmatrix} \cos(2\alpha-2\beta) & \sin(2\alpha-2\beta) \\ \sin(2\alpha-2\beta) & -\cos(2\alpha-2\beta) \end{pmatrix} \quad ✓$$

Hence, proved. ✓

29 Complex Numbers

Calculator Free

1. [12 marks: 1, 2, 2, 2, 2, 3]

Solve for $x \in \mathbb{C}$.

(a) $x^2 + 4 = 0$

$x^2 = -4 \Rightarrow x = \pm 2i$ ✓

(b) $x^2 + x + 4 = 0$

$x = \dfrac{-1 \pm \sqrt{1^2 - 4\times1\times4}}{2}$ ✓

$= \dfrac{-1 \pm i\sqrt{15}}{2}$ ✓

(c) $(x+2)^2 + 9 = 0$

$(x + 2) = \pm 3i$ ✓

$x = -2 \pm 3i$ ✓

(d) $x^3 + 9x = 0$

$x(x^2 + 9) = 0$

$x = 0, \pm 3i$ ✓✓

(e) $(2z - 1)^2 + 16 = 0$

$2z - 1 = \pm 4i$ ✓

$z = \frac{1}{2} \pm 2i$ ✓

(f) $\dfrac{2}{2-z} = \dfrac{z}{2}$

$z^2 - 2z + 4 = 0$ ✓

$z = 1 \pm i\sqrt{3}$ ✓✓

Calculator Free

2. [5 marks: 1, 4]

Factorise using both real and complex factors where appropriate:

(a) $x^2 + 16$

$x^2 + 16 \equiv (x - 4i)(x + 4i)$ ✓

(b) $x^2 + 2x + 5$

Zeroes are: $x = \dfrac{-2 \pm \sqrt{2^2 - 4\times1\times5}}{2}$ ✓

$= -1 \pm 2i$ ✓

Hence: $x^2 + 2x - 1 \equiv [x - (-1 + 2i)][x - (-1 - 2i)]$

$\equiv (x + 1 - 2i)(x + 1 + 2i)$ ✓✓

---

3. [6 marks: 3, 3]

(a) Factorize completely (including complex factors) $z^2 + 4z + 5$.

zeros are $z = \dfrac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$ ✓

$= -2 \pm i$

Hence, $z^2 + 4z + 5 = [z - (-2 + i)][z - (-2 - i)]$ ✓✓

$= (z + 2 - i)(z + 2 + i)$

(b) Factorize completely (including complex factors) $z^3 + 5z^2 + 9z + 5$.

When $z = -1$, $z^3 + 5z^2 + 9z + 5 = 0$

Hence, $(z + 1)$ is a factor. ✓

$z^3 + 5z^2 + 9z + 5 = (z + 1)(z^2 + 4z + 5)$ ✓

$= (z + 1)(z + 2 - i)(z + 2 + i)$ ✓

Calculator Free

4. [5 marks: 2, 3]

Express in the form $a + bi$:

(a) $(1 + i\sqrt{2})^2$

$(1 + i\sqrt{2})^2 = 1 + 2\sqrt{2}\,i + (i\sqrt{2})^2$
$= -1 + 2\sqrt{2}\,i$ ✓✓

(b) $\dfrac{2+3i}{1-i}$

$\dfrac{2+3i}{1-i} = \dfrac{(2+3i)(1+i)}{(1-i)(1+i)}$ ✓
$= \dfrac{-1}{2} + \dfrac{5i}{2}$ ✓✓

---

5. [5 marks]

Given that $(a + b\,i)^2 = -15 + 8\,i$, find a and b where a and b are real non-zero integers.

$a^2 - b^2 + 2abi = -15 + 8i$ ✓

Hence: $a^2 - b^2 = -15$ ✓
$ab = 4$ ✓

By inspection: $a = 1,\ b = 4$ ✓
$a = -1, b = -4$ ✓

---

6. [5 marks]

Given that $(a + b\,i)^2 = 16 + 30\,i$, find a and b where a and b are real non-zero integers.

$a^2 - b^2 + 2abi = 16 + 30i$ ✓

Hence: $a^2 - b^2 = 16$ ✓
$ab = 15$ ✓

Therefore $a = 5, b = 3$ ✓
$a = -5, b = -3$ ✓

Calculator Free

7. [7 marks: 1, 3, 3]

Let $u = 1 + 3\,i$, $v = -1 + i$ and $w = -2\,i$.

(a) Find $u + v$.

$u + v = 4\,i$ ✓

(b) Find $\dfrac{v}{w}$.

$\dfrac{v}{w} = \dfrac{-1 + i}{-2i}$
$= \dfrac{-1 + i}{-2i} \times \dfrac{i}{i}$ ✓
$= \dfrac{-1 - i}{2}$ ✓✓

(c) Find $v \times \bar{u}$.

$(-1 + i) \times (1 - 3\,i)$ ✓
$= -1 + 3\,i + i + 3$
$= 2 + 4\,i$ ✓✓

---

8. [8 marks: 4, 4]

Find $w = a + b\,i$ where a and b are real non-zero integers if:

(a) $\bar{w} = \dfrac{5}{w}$,

$w = a + b\,i \Rightarrow \bar{w} = a - b\,i$ ✓
Hence $(a + b\,i) \times (a - b\,i) = 5$
$\Rightarrow a^2 + b^2 = 5$ ✓
Since, a and b are real non-zero integers, $a = \pm 2$, $b = \pm 1$ ✓
or $b = \pm 2$, $a = \pm 1$ ✓

(b) $w^2 = -3 - 4\,i$

$(a + b\,i)^2 \equiv a^2 - b^2 + 2ab\,i = -3 - 4\,i$
Compare Re parts: $a^2 - b^2 = -3$ ✓
Compare Im parts: $ab = -2$ ✓
Since, a and b are real non-zero integers, $a = 1, b = -2$ ✓
$a = -1, b = 2$ ✓

Calculator Free

9. [9 marks: 3, 3, 3]

Let the complex numbers $z_1 = 2 + k\,i$ and $z_2 = -5 + 12\,i$, where k is a *real number*.
Determine all possible values of k if:

(a) $[\,\boldsymbol{Im}(z_1)\,]^2 = \boldsymbol{Im}(z_2)$

$k^2 = 12$ ✓
$\Rightarrow\ k = \pm 2\sqrt{3}$ ✓✓

(b) $z_1^{\ 2} = z_2$

$(2 + k\,i)^2 = -5 + 12\,i$
$(4 - k^2) + 4\,k\,i = -5 + 12\,i$ ✓
Equate Real parts: $4 - k^2 = -5 \Rightarrow k = \pm 3$ ✓
Equate Im parts $4k = 12 \Rightarrow k = 3$
Hence, $k = 3$. ✓

(c) $\dfrac{6z_1}{z_2} = -i$.

$6\,z_1 = -i \times z_2$
$6\,(2 + k\,i) = -i \times (-5 + 12\,i)$ ✓
$12 + 6\,k\,i = 12 + 5\,i$ ✓
$\Rightarrow\ k = 5/6$ ✓

10. [8 marks: 2, 6]

Let the complex numbers $u = 3 + k\,i$ and $v = k + 2\,i$, where k is a *real number*.
Determine all possible values of k if:

(a) $[\,\boldsymbol{Im}(u)\,]^2 = \boldsymbol{Re}(v)$

$k^2 = k$
$k = 0, 1$ ✓✓

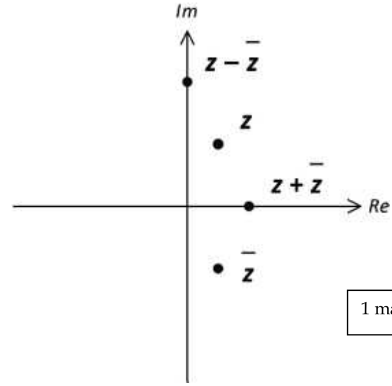
(b) $\dfrac{u}{v-1} = -1 - \dfrac{i}{2}$

$(3 + ki) = (k + 2i - 1)\left(-1 - \frac{i}{2}\right)$
$3 + ki = (k - 1 + 2i)\left(-1 - \frac{i}{2}\right)$ ✓
Real Parts: $3 = -(k-1) + 1 \Rightarrow k = -1$ ✓✓
Im Parts: $k = -2 - \frac{k-1}{2} \Rightarrow k = -1$ ✓✓
Hence, $k = -1$. ✓

Calculator Free

11. [3 marks]

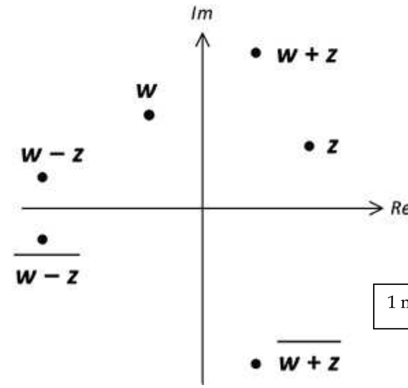
The complex number z is plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $\overline{z}$, $z + \overline{z}$ and $z - \overline{z}$.



1 mark each ✓✓✓

12. [4 marks]

The complex numbers w and z are plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $w + z$, $\overline{w+z}$, $w - z$ and $\overline{w-z}$.



1 mark each ✓✓✓✓

30 Conjectures & Proofs

Calculator Assumed

1. [10 marks: 3, 3, 4]

(a) Prove that $15.\overline{15}$ is a rational number.

Let	$n = 15.\overline{15}$	I	✓
	$100n = 1515.\overline{15}$	II	✓
II – I	$99n = 1500$		
Hence:	$n = \frac{1500}{99} = \frac{500}{33}$.		✓
Therefore, $15.\overline{15}$ is a rational number.			

(b) Prove that $5.7\overline{35}$ is a rational number.

Let	$n = 5.7\overline{35}$	I	
	$10n = 57.\overline{35}$	II	✓
	$1000n = 5735.\overline{35}$	III	✓
III – II	$990n = 5678$		
Hence:	$n = \frac{5678}{990} = \frac{2839}{495}$		✓
Therefore, $5.7\overline{35}$ is a rational number.			

(c) Prove that $10 + 1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots$ is a rational number.

The terms of the series,

$$10, 1, \frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}, \ldots\ldots$$

form an infinite geometric sequence

with first term $a = 10$ and common ratio $r = \frac{1}{10}$. ✓✓

Since $|r| < 1$, the infinite sequence has a finite sum. ✓

Sum of infinite sequence $= \frac{a}{1-r}$

$$= \frac{10}{1-\frac{1}{10}}$$

$$= \frac{100}{9}.$$ ✓

Hence, series sum is rational.

Calculator Free

2. [5 marks: 1, 1, 3]

(a) Given that $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

(i) **P × Q**

$$\mathbf{P} \times \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

(ii) **Q × P**

$$\mathbf{Q} \times \mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✓

(b) Prove or disprove the conjecture that if the matrices **P** and **Q** are such that **P × Q = I**, where I is the relevant identity matrix, then $\mathbf{P}^{-1} = \mathbf{Q}$.

Conjecture is false. ✓

Counter-example:

From (a) (i):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

But $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ does not have an inverse as it is not a square matrix. ✓

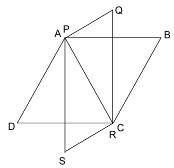
Calculator Free

3. [9 marks: 3, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

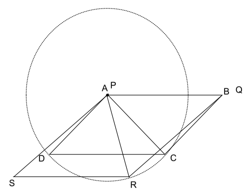
(a) Consider the parallelograms ABCD and PQRS. If the diagonals AC = PR then ABCD and PQRS are congruent.

As shown in the accompanying diagram, AC = PR but ABCD and PQRS are not congruent. ✓✓✓



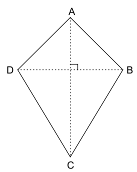
(b) Consider the parallelograms ABCD and PQRS. If AB = PQ and the diagonals AC = PR, ABCD and PQRS are congruent.

As shown in the accompanying diagram, AB = PQ and diagonals AC = PR, but ABCD and PQRS are not congruent. ✓✓✓



(c) If the diagonals of a quadrilateral are perpendicular then the quadrilateral must be a rhombus.

As shown in the accompanying diagram, the diagonals AC and BD are perpendicular but ABCD is not a rhombus. ✓✓✓



Calculator Assumed

4. [12 marks: 2, 2, 4, 4]

Provide a counter-example to disprove each of the following conjectures.
Show all attempts, successful and otherwise.

(a) $2^{2n} + 1$ is prime for integer $n \geq 1$.

$n = 3$, $2^{2n} + 1 = 65$
$= 13 \times 5$ which is not prime. ✓✓

(b) $n^2 - n + 5$ is prime for integer $n \geq 1$.

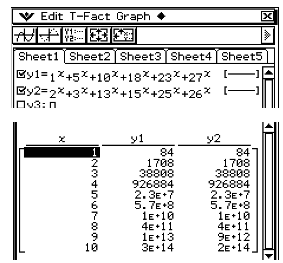
$n = 5$, $n^2 - n + 5 = 25$
$= 5 \times 5$ which is not prime. ✓✓

(c) $1^n + 5^n + 10^n + 18^n + 23^n + 27^n = 2^n + 3^n + 13^n + 15^n + 25^n + 26^n$ for integer $n \geq 1$.

$n = 9$. ✓✓

LHS $\approx 1 \times 10^{13}$ ✓
RHS $\approx 9 \times 10^{12}$ ✓

Hence, LHS ≠ RHS



(d) There are no integer solutions to $a^3 + b^3 = c^3 + d^3$ where $a \neq b \neq c \neq d$.

For $a = 1$, $b = -1$, $c = 2$, $d = -2$: ✓✓✓✓
LHS = 0
RHS = 0.

Calculator Free

5. [3 marks]

Prove that $2^n - 1$ is always odd for integer $n \geq 1$.

> 2^n for $n \geq 1$ will always be a multiple of 2
> and hence is even for $n \geq 1$. ✓✓
> Hence, $2^n - 1$ will always be odd. ✓

6. [3 marks]

Prove that the square of an odd number add 11 is a multiple of 4.

> $(2k + 1)^2 + 11 = 4k^2 + 4k + 12$ ✓✓
> $= 4(k^2 + k + 3)$ ✓
> which is a multiple of 4.

7. [5 marks: 2, 3]

(a) Prove that product of three consecutive integers is divisible by 3.

> In a run of three consecutive integers, one
> of the integers must be a multiple of 3. ✓
> Hence, the product must be a multiple of 3. ✓

(b) Prove that product of three consecutive integers is divisible by 6.

> In a run of three consecutive integers, one
> of the integers must be a multiple of 3 ✓
> and at least one of the integers must be even. ✓
> Hence, the product must be a multiple of 3 & 2.
> That is, the product must be multiple of 6 ✓

Calculator Assumed

8. [3 marks]

Prove that $x^7 - x$ is divisible by 6 for integer $x \geq 1$.

> $x^7 - x = (x - 1)\, x\, (x + 1)\, (x^2 + x + 1)(x^2 - x + 1)$ ✓
> But $(x - 1)\, x\, (x + 1)$ is a product of 3 consecutive integers.
> As one of the three integers must be a multiple of 3,
> and at least one must be even,
> the product must be a multiple of 6. ✓
> Hence, $(x - 1)\, x\, (x + 1) = 6k$.
> $\Rightarrow\ x^7 - x = 6k\,(x^2 + x + 1)(x^2 - x + 1)$ ✓
> = multiple of 6.

9. [4 marks]

Prove that the product of any three consecutive even numbers must be a multiple of 24.

> $(2n) \times (2n + 2) \times (2n + 4) = 8 \times n(n + 1)(n + 2)$ ✓
> But $n(n + 1)(n + 2)$ = product of 3 consecutive integers
> = multiple of 3 as one of the three integers must be a multiple of 3
> $= 3k$ ✓
> Hence:
> $(2n) \times (2n + 2) \times (2n + 4) = 8 \times n(n + 1)(n + 2)$
> $= 8 \times 3k$ ✓
> $= 24k$ ✓
> = multiple of 24.

10. [3 marks]

Prove that $4n^3 - 4n$ is a multiple of 24 for integer $n \geq 1$.

> $4n^3 - 4n = 4n(n^2 - 1)$
> $= 4n(n - 1)(n + 1)$ ✓
>
> $(n - 1)n(n + 1)$ is the product of 3 consecutive integers,
> one of which must be a multiple of 3 and one of which
> must be even.
> Hence, $(n - 1)n(n + 1)$ must be a multiple of 6. ✓
> $\Rightarrow\ 4n(n - 1)(n + 1)$ = multiple of 4×6, ✓
>
> Hence, $4n^3 - 4n$ is a multiple of 24.

Calculator Assumed

11. [7 marks: 4, 3]

(a) It is conjectured that a number is divisible by 4 if the sum of twice the tens digit and the ones digit is a multiple of 4. Prove that this conjecture is true for a four digit number.

> Let the number be *abcd*.
> Value of number $V = 1000a + 100b + 10c + d$ ✓
>
> Required sum in conjecture $= 2c + d$. ✓
> If $2c + d$ is a multiple of 4: $2c + d = 4k$
> $d = 4k - 2c$ ✓
>
> Hence, $V = 1000a + 100b + 10c + 4k - 2c$
> $= 1000a + 100b + 8c + 4k$
> $= 4(250a + 25b + 2c + 1)$ ✓
> $\Rightarrow$ V is a multiple of 4, and *abcd* is a multiple of 4.

(b) Consider the positive integers a and b. The arithmetic mean of these two integers is $M = \dfrac{a+b}{2}$ and the harmonic mean is $H = \dfrac{2ab}{a+b}$.

It is conjectured that $M \geq H$. Use the expansion of $(\sqrt{a} - \sqrt{b})^2$ to prove this conjecture.

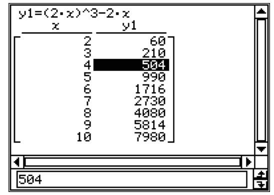
> $(\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$
> Clearly $(\sqrt{a} - \sqrt{b})^2 \geq 0$ ✓
> Hence, $a + b - 2\sqrt{ab} \geq 0$
> $a + b \geq 2\sqrt{ab}$ ✓
> $(a+b)^2 \geq 4ab$
> Hence: $\dfrac{a+b}{2} \geq \dfrac{2ab}{a+b}$ ✓
> That is $M \geq H$.

Calculator Assumed

12. [7 marks: 2, 5]

(a) Provide a counter-example to disprove the conjecture that the cube of an even number greater than two less the number itself is divisible by 5.

> Number $= (2n)^3 - 2n$ for $n \geq 2$
>
> Counter example: $n = 4$ Number $= 8^3 - 8 = 504$ ✓✓
> which is not divisible by 5.



(b) Prove that $x^5 + x^4 + x^3 + x^2$ is divisible by 4 for x as a whole number.

> Expression $= x^5 + x^4 + x^3 + x^2 = x^2(x^2 + 1)(x + 1)$ ✓
>
> If x is an even number: $x = 2n$ ✓
> Expression $= (2n)^2\,[(2n)^2 + 1](2n + 1)$
> $= 4n^2(4n^2 + 1)(2n + 1)$ ✓
> which is divisible by 4.
>
> If x is an odd number: $x = 2n + 1$ ✓
> Expression $= (2n + 1)^2\,[(2n + 1)^2 + 1](2n + 1 + 1)$
> $= (2n + 1)^2\,[4n^2 + 4n + 1 + 1](2n + 1 + 1)$
> $= (2n + 1)^2\,[4n^2 + 4n + 2](2n + 2)$
> $= 4\,(2n + 1)^2\,(2n^2 + 2n + 1)(n + 1)$ ✓
> which is divisible by 4.
>
> Hence, expression is always divisible by 4.

Calculator Assumed

13. [6 marks: 3, 1, 2]

(a) Prove that the sum of the cubes of two consecutive odd whole numbers is always a multiple of 4.

> Let the two consecutive odd numbers be
> $2n + 1$ and $2n + 3$. ✓
> Required sum $= (2n+1)^3 + (2n+3)^3$
> $= 4(n+1)(4n^2 + 8n + 7)$ ✓
> Clearly required sum is a multiple of 4. ✓

(b) Consider the conjecture:
The sum of the cubes of two consecutive odd whole numbers is always a multiple of 6.
(i) Disprove this conjecture.

> Counter example: (3, 5)
> $3^3 + 5^3 = 152$ which is not a multiple of 6. ✓

(ii) Under what conditions will this conjecture be true?

> Sum is a multiple of 6 for
> (5, 7), (11, 13), (17, 19) etc. ✓
> That is sum is a multiple of 6 if the pair is of the form $(6k - 1, 6k + 1)$ for $k \in \mathbb{N}$. ✓

y1=((2·x+1)^3+(2·x+3)^3)/6

x	y1
1	25.333
2	78
3	178.67
4	343.33
5	588
6	928.67
7	1381.3
8	1962
9	2686.7
10	3571.3
11	4632

Calculator Assumed

14. [4 marks]

Complete the table below.

> 1 mark each ✓✓✓✓

Conjecture	If x^2 is an odd number then x is an odd number.
Negation of conjecture	If x^2 is an odd number then x is not an odd number.
Contrapositive of conjecture	If x is not an odd number then x^2 is not an odd number.
Converse of conjecture	If x is an odd number then x^2 is an odd number.
Inverse of conjecture	If x^2 is not an odd number then x is not an odd number.

---

15. [5 marks: 2, 3]

Consider the conjecture:
If a is a factor of 20, then a is also factor of 40.

(a) Prove that the conjecture is true.

> a is a factor of 20
> $\Rightarrow \exists\ t$ such that $at = 20$ where t is a whole number. ✓
>
> Since $20 \times 2 = 40 \Rightarrow at \times 2 = 40$
> $a(2t) = 40$ ✓
> Hence, a is also a of 40.

(b) State the inverse of the conjecture and determine with reasons whether it is true or false.

> Inverse:
> If a is not a factor of 20 $\Rightarrow$ a is not a factor of 40. ✓
>
> Inverse is false. ✓
> Counter-example:
> 40 is not a factor of 20 but is a factor of 40. ✓

Calculator Assumed

16. [4 marks]

Use the method of contradiction to prove that for n as a *real* whole number $n(n + 1)$ is an even number.

> Assume that for n as a real whole number $n(n + 1)$ is an odd number. ✓
>
> $\Rightarrow$ $n(n + 1) = 2k + 1$ for some integer k. ✓
>
> If n is an even number, $n = 2a$ for integer a.
> $\Rightarrow$ $n(n + 1) \equiv 2a(2a + 1)$ is an even number. ✓
>
> Which contradicts $n(n + 1)$ as an odd number. ✓
>
> Hence, $n(n + 1)$ must be even.

17. [5 marks]

Use the method of contradiction to prove that $\sqrt{7}$ is an irrational number.
[You may assume that if p^2 is a multiple of q, then p is also a multiple of q for p as a whole number.]

> Assume that $\sqrt{7}$ is rational. ✓
>
> $\Rightarrow$ $\sqrt{7} = \frac{a}{b}$ where a and b are integers with no common factors. ✓
> $\Rightarrow$ $a = \sqrt{7}\, b$
> $a^2 = 7b$
> $\Rightarrow$ a is a multiple of 7. ✓
> $\Rightarrow$ $a = 7n$ for some integer n.
>
> Hence, $7b = 49n^2$
> $\Rightarrow$ $b = 7n^2$
>
> This implies that a and b are both multiples of 7. ✓
> This contradicts the initial condition that a and b have no factors in common.
> Hence, $\sqrt{7}$ cannot be rational. ✓

Calculator Assumed

18. [7 marks: 5, 2]

Consider the statement: $\tan\theta = 0 \quad \Rightarrow \quad \sin\theta = 0.$

(a) Determine with reasons if the contrapositive of this statement is true or false.

> Contrapositive:
> $\sin\theta \neq 0 \;\Rightarrow\; \tan\theta \neq 0.$ ✓
>
> Consider the negation of the contrapositive:
> $\sin\theta \neq 0 \;\Rightarrow\; \tan\theta = 0.$ ✓
>
> Since $\sin\theta \neq 0$, $\sin\theta = k$ $\quad k \neq 0, -1 \le k \le 1$, ✓
> $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{k}{\sqrt{1-k^2}}$
> But $\tan\theta = 0 \;\Rightarrow\; \frac{k}{\sqrt{1-k^2}} = 0$
> $\Rightarrow\; k = 0$
> which contradicts $k \neq 0$. ✓
>
> Hence, the negation of the contrapositive is false and contrapositive must be true. ✓

(b) Hence or otherwise, determine with reasons if the conjecture is true of false.

> Since the contrapositive of the conjecture is true, ✓
> the conjecture is true. ✓

19. [6 marks: 3, 3]

Consider the statement: $\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$

(a) Determine with reasons if the converse of this statement is true or false.

> Converse: $\tan\theta = \sqrt{3} \;\Rightarrow\; \cos\theta = \frac{1}{2}$. ✓
> Converse is false. ✓
> Counter-example: $\tan 240^\circ = \sqrt{3}$ but $\cos 240^\circ = -\frac{1}{2} \neq \frac{1}{2}$. ✓

(b) Hence, or otherwise determine with reasons if the inverse is true or false.

> Inverse: $\cos\theta \neq \frac{1}{2} \;\Rightarrow\; \tan\theta \neq \sqrt{3}$. ✓
> Inverse is false as the converse has been shown to be false. ✓✓

Calculator Assumed

20. [8 marks: 1, 1, 1, 1, 4]

Consider the conjecture for whole number x :
If x is an odd number than x^2 is an odd number.

(a) State the converse of this conjecture.

If x^2 is an odd number then x is an odd number. ✓

(b) State the negation of this conjecture.

If x is an odd number then x^2 is <u>*not*</u> an odd number. ✓

(c) State the inverse of this conjecture.

If x is <u>*not*</u> an odd number then x^2 is <u>*not*</u> an odd number. ✓

(d) State the contrapositive of this statement.

If x^2 is <u>*not*</u> an odd number then x is <u>*not*</u> an odd number. ✓

(e) Prove this conjecture by proving its contrapositive.

Consider the negation of the contrapositive.
If x^2 is <u>*not*</u> an odd number $\Rightarrow$ x is an odd number. ✓
Since x^2 is even $\Rightarrow$ $x^2 = 2n$.
x is an odd number $\Rightarrow$ $x = 2m + 1$ ✓
$x^2 = (2m+1)^2$
$= 4m^2 + 4m + 1$ ✓
But x^2 is even $\Rightarrow$ 1 is divisible by 2
which is a contradiction. ✓
Hence negation of contrapositive is false.
Therefore contrapositive is true.
Which means that the conjecture must be true.

Calculator Assumed

21. [8 marks: 1, 1, 1, 1, 4]

Consider the conjecture for whole numbers p and q where $p < q$:
If p^2 is a multiple of q, then p is also a multiple of q.

(a) State the converse of this conjecture.

If p is a multiple of q, then p^2 is a multiple of q. ✓

(b) State the negative of this conjecture.

If p^2 is a multiple of q, then p is <u>*not*</u> a multiple of q. ✓

(c) State the inverse of this conjecture.

If p^2 is not a multiple of q, then p is not a multiple of q. ✓

(d) State the contrapositive of this statement.

If p is a not multiple of q, then p^2 is not a multiple of q. ✓

(e) Prove this conjecture by proving its contrapositive.

Let $p = nq + m$ where $1 \le m < q$ and $n \in \mathbb{N}$ ✓
$p^2 = n^2q^2 + 2nqm + m^2$ ✓
$= q(n^2q + 2mn) + m^2$. ✓
Since $1 \le m < q \Rightarrow 1 \le m^2 < q^2$
Hence, m^2 is not a multiple of q. ✓
Therefore p^2 is not a multiple of q.

Calculator Assumed

22. [9 marks: 5, 4]

(a) Negate the converse of the following conjecture:

If n is a multiple of 10, then n^2 is a multiple of 10.

Prove that the converse is true by proving that the negation of the converse is false.

> Converse of conjecture:
>
> If n^2 is a multiple of 10, then n is a multiple of 10. ✓
>
> Assume that the negation of the converse is true:
> Given that n^2 is a multiple of 10, then n is <u>not</u> a multiple of 10. ✓
>
> As n is not a multiple of 10, $n = 10a + b$
> for some integer a and integer b where $1 \le b \le 9$. ✓
>
> Hence, $n^2 = (10a + b)^2$
> $= 100a^2 + 20ab + b^2$
> $= 10(10a^2 + 2ab) + b^2$ I ✓
>
> As n^2 is a multiple of 10, b^2 in statement I must also be a multiple of 10.
> Which is a contradiction as for $1 \le b \le 9$, b^2 is not a multiple of 10. ✓
>
> Therefore the negation of the converse cannot be true.
> Hence, if n^2 is a multiple of 10, then n must be a multiple of 10.

(b) Use the result from (a) and the method of contradiction to prove that $\sqrt{10}$ is an irrational number.

> Assume that $\sqrt{10}$ is rational.
>
> $\Rightarrow \sqrt{10} = \frac{a}{b}$, where a and b are integers with no factors in common. ✓
>
> $\Rightarrow a^2 = 10b^2$
>
> $\Rightarrow a^2$ and hence a are multiples of 10 (from (a)). ✓
>
> Hence, $a = 10k$ for some integer k.
>
> $\Rightarrow (10k)^2 = 10b^2$
>
> $\Rightarrow b^2 = 10k^2$
>
> $\Rightarrow b^2$ and hence b are multiples of 10 (from (a)). ✓
>
> Hence, a and b are both multiples of 10;
> which contradicts the initial premise that a and b have no factors in common. ✓
> Hence, $\sqrt{10}$ must be irrational.

Calculator Assumed

23. [7 marks: 5, 2]

(a) Use mathematical induction to prove that:

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{n}{5^n} = \frac{5}{16} - \frac{4n+5}{16(5^n)} \quad \text{for integer } n \ge 1.$$

> For $n = 1$: LHS $= \frac{1}{5}$, RHS $= \frac{5}{16} - \frac{9}{80} = \frac{1}{5}$. ✓
> Hence, conjecture is true for $n = 1$.
>
> Assume that the conjecture is true for $n = k$.
> That is: $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{k}{5^k} = \frac{5}{16} - \frac{4k+5}{16(5^k)}$.
>
> For $n = k + 1$: LHS $= [\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{k}{5^k}] + \frac{k+1}{5^{k+1}}$
>
> $= \frac{5}{16} - \frac{4k+5}{16(5^k)} + \frac{k+1}{5^{k+1}}$
>
> $= \frac{5}{16} + \frac{-5(4k+5)+16(k+1)}{16(5^{k+1})}$ ✓
>
> $= \frac{5}{16} + \frac{-4k-9}{16(5^{k+1})}$
>
> $= \frac{5}{16} - \frac{4(k+1)+5}{16(5^{k+1})}$. ✓✓
>
> RHS $= \frac{5}{16} - \frac{4(k+1)+5}{16(5^{k+1})}$.
>
> Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓
> Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.
>
> Hence, conjecture is true for all integers $n \ge 1$.

(b) Discuss the sum as the number of terms increases indefinitely.

> As $n \to \infty$, sum $= \frac{5}{16} - \frac{4n+5}{16(5^n)} \to \frac{5}{16}$. ✓✓

Calculator Assumed

24. [7 marks]

Use the method of mathematical induction to prove that for integer $n \geq 1$,

$$\frac{6}{5} + \left(\frac{6}{5}\right)^2 + \left(\frac{6}{5}\right)^3 + \left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right)^5 + \ldots + \left(\frac{6}{5}\right)^n = 6\times\left(\frac{6}{5}\right)^n - 6.$$

For $n = 1$: LHS = $\frac{6}{5}$

RHS = $\frac{6}{5}$

Hence, conjecture is true for $n = 1$. ✓

Assume that conjecture is true for $n = k$, $k \in \mathbb{N}$

That is:

$$\frac{6}{5} + + \left(\frac{6}{5}\right)^2 + \left(\frac{6}{5}\right)^3 + \left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right)^5 + \ldots + \left(\frac{6}{5}\right)^n = 6\times\left(\frac{6}{5}\right)^n - 6 \quad ✓$$

For $n = k + 1$:

$$\text{LHS} = \frac{6}{5} + \left(\frac{6}{5}\right)^2 + \left(\frac{6}{5}\right)^3 + \left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right)^5 + \ldots + \left(\frac{6}{5}\right)^k + \left(\frac{6}{5}\right)^{k+1} \quad ✓$$

$$= 6\times\left(\frac{6}{5}\right)^k - 6 + \left(\frac{6}{5}\right)^{k+1} \quad ✓$$

$$= 6\times\left(\frac{6}{5}\right)^k + \frac{6}{5}\left(\frac{6}{5}\right)^k - 6$$

$$= \left(\frac{36}{5}\right)\left(\frac{6}{5}\right)^k - 6$$

$$= 6\times\left(\frac{6}{5}\right)^{k+1} - 6 \quad ✓$$

$$\text{RHS} = 6\times\left(\frac{6}{5}\right)^{k+1} - 6 \quad ✓$$

Hence, if conjecture is true for $n = k$,
it must necessarily be true for $n = k + 1$.

But conjecture is true for $n = 1$,
hence, it must be true for $n = 1 + 1 = 2$,
which means that it must also be true for $n = 3$, and ….. . ✓

Hence, conjecture is true for all integer $n \geq 1$.

Calculator Assumed

25. [5 marks]

Prove inductively that $(1 + i)^{4n} = (-1)^n\, 2^{2n}$ for integer $n \geq 1$.

For $n = 1$: LHS = $(1 + i)^4 = -4$, RHS = $(-1)^1 2^2 = -4$. ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is: $(1 + i)^{4k} = (-1)^k\, 2^{2k}$
For $n = k + 1$: LHS = $(1 + i)^{4(k+1)}$

$= (1 + i)^{4k} \times (1 + i)^4$ ✓

$= (-1)^k\, 2^{2k} \times -4$ ✓

$= (-1)^{k+1}\, 2^{2(k+1)}$ ✓

RHS = $(-1)^{k+1}\, 2^{2(k+1)}$

Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \geq 1$.

26. [5 marks]

Use mathematical induction to prove that $11^n + 4$ is divisible by 5 for integer $n \geq 1$.

For $n = 1$: $11^n + 4 = 15$ which is divisible by 5 ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is: $11^k + 4 = 5m$ for integer $m \geq 1$.
For $n = k + 1$: Expression = $11^{k+1} + 4$

$= 11(11^k) + 4$ ✓

$= 11(5m - 4) + 4$ ✓

$= 55m - 40$

$= 5(11m - 8)$ which is divisible by 5. ✓

Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \geq 1$.

Calculator Assumed

27. [6 marks]

Prove that $10^{n+1} + 3 \times 10^n + 5$ is a multiple of 9 for positive integer n.

For $n = 1$, Expression $= 10^2 + 3 \times 10 + 5$
$= 135 = 9 \times 15$
Hence, conjecture is true for $n = 1$. ✓

Assume that the conjecture is true for $n = k$.
That is, $10^{k+1} + 3 \times 10^k + 5 = 9x$ for some integer x I ✓

For $n = k + 1$:
Expression $= 10^{k+2} + 3 \times 10^{k+1} + 5$ ✓
$= 10(10^{k+1}) + 30(10^k) + 5$

But from I, $10^{k+1} = 9x - 5 - 3(10^k)$ ✓

Hence, Expression $= 10[9x - 5 - 3(10^k)] + 30(10^k) + 5$
$= 90x - 45 = 9(10x - 5)$ which is a multiple of 9. ✓

Hence, if it is assumed true for $n = x$, it will be true for $n = x + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \geq 1$.

---

28. [4 marks]

Given the non-singular commutative matrices **P** and **Q**, use mathematical induction to prove that for integer $n \geq 1$, $\mathbf{P}^n = \mathbf{Q}\,\mathbf{P}^n\,\mathbf{Q}^{-1}$.

For $n = 1$: LHS $= \mathbf{P}$ RHS $= \mathbf{Q}\,\mathbf{P}\,\mathbf{Q}^{-1} = \mathbf{P}\,\mathbf{Q}\,\mathbf{Q}^{-1} = \mathbf{P}$. ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is: $\mathbf{P}^k = \mathbf{Q}\,\mathbf{P}^k\,\mathbf{Q}^{-1}$.
For $n = k + 1$: RHS $= \mathbf{Q}\,\mathbf{P}^{k+1}\,\mathbf{Q}^{-1}$
$= \mathbf{Q}\,\mathbf{P}\,\mathbf{P}^k\,\mathbf{Q}^{-1}$ ✓
$= \mathbf{P}\,(\mathbf{Q}\,\mathbf{P}^k\,\mathbf{Q}^{-1})$ ✓
$= \mathbf{P}\,\mathbf{P}^k = \mathbf{P}^{k+1}$
LHS $= \mathbf{P}^{k+1}$

Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.
Hence, conjecture is true for all integers $n \geq 1$.

Calculator Assumed

29. [5 marks] [TISC]

Use mathematical induction to prove that for integer $n \geq 1$:

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}.$$

[Hint: Use the formula $2\cos A \sin B = \sin(A + B) - \sin(A - B)$.]

For $n = 1$: LHS $= \cos x$
RHS $= \dfrac{\sin 2x}{2\sin x}$
$= \dfrac{2\sin x\cos x}{2\sin x}$
$= \cos x$. ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is:
$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \dfrac{\sin 2kx}{2\sin x}$. ✓

For $n = k + 1$:
LHS $= \cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$
$= \dfrac{\sin 2kx}{2\sin x} + \cos(2k+1)$
$= \dfrac{\sin 2kx + 2\times[\cos(2k+1)x]\sin x}{2\sin x}$ ✓
$= \dfrac{\sin 2kx + \{[\sin(2kx+x+x)] - [\sin(2kx+x-x)]\}}{2\sin x}$ ✓
$= \dfrac{\sin 2(k+1)x}{2\sin x}$.
RHS $= \dfrac{\sin 2(k+1)x}{2\sin x}$.

Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \geq 1$.

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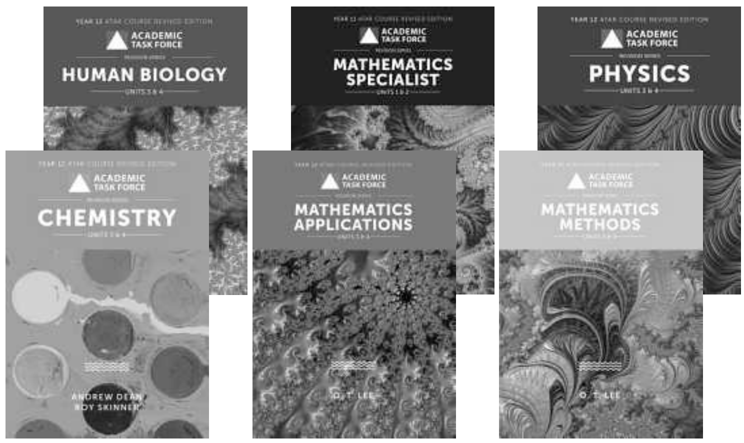

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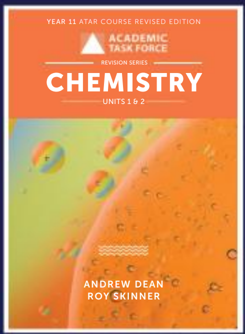

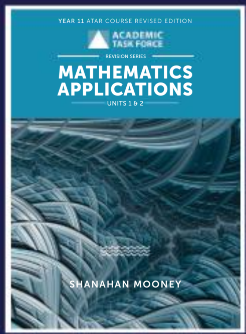

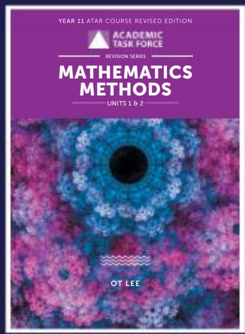

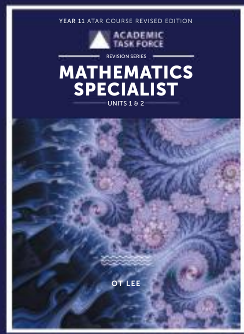

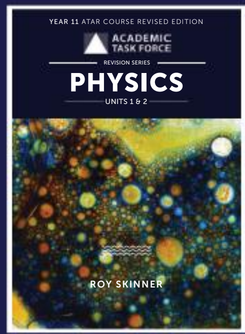

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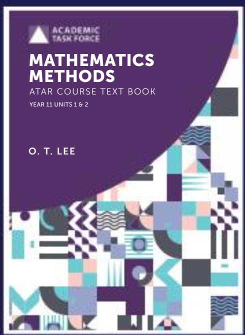

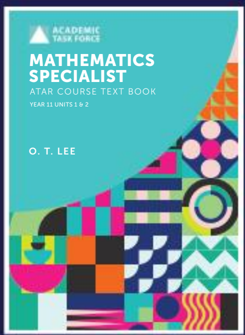

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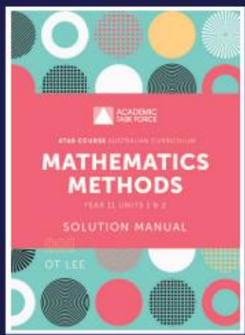

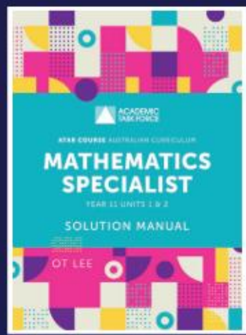

- This book provides a comprehensive set of revision/review questions for the Year 11 Mathematics Specialist ATAR Course, Units 1 & 2.
- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.
- This book exposes students to questions and problems at test/examination level.
- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.
- Questions are accompanied by a set of fully worked solutions with which students can measure their answers.
- Use this book alongside your school textbook and lesson notes to review and improve your understanding of each concept.
- The worked examples demonstrate, step by step, important concepts and calculation methods.
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