PEARSON

SPECIALIST MATHEMATICS

QUEENSLAND

STUDENT BOOK

11

UNITS 1 & 2



QCE 2019 SYLLABUS

PEARSON

SPECIALIST MATHEMATICS

QUEENSLAND

STUDENT BOOK



UNITS 1 & 2

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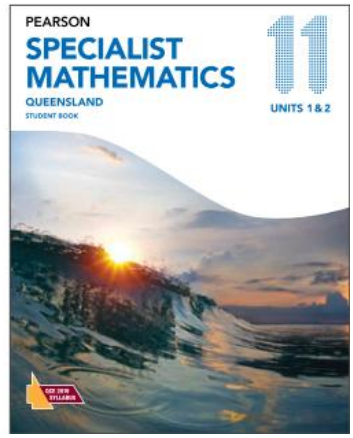
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Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non CAS)
- CASIO fx-CG50AU

Throughout the student book are Technology worked examples strategically placed within the theory. These are suitable for both the TI-Nspire CX (non CAS) and CASIO fx-CG50AU. The examples clearly demonstrate how the technology can be used effectively and efficiently for the content covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

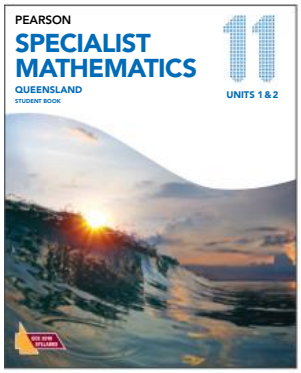
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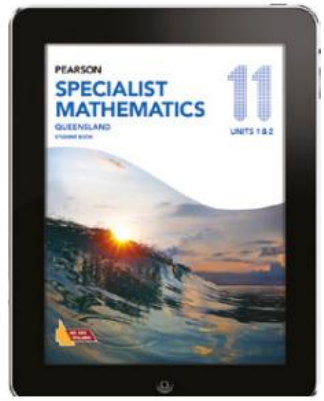
Specialist Mathematics 11
Student book

Student book

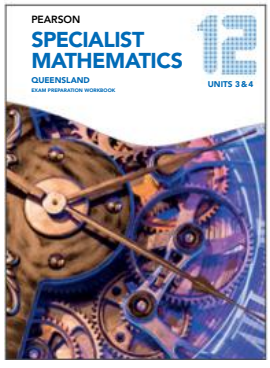
The student book has been authored by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next-generation eBook. This is an electronic textbook that students can access on any device, online or offline, and is linked to features, interactives and visual media that will help consolidate their understanding of concepts and ideas, as well as other useful content developed specifically for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



Specialist Mathematics 11
eBook



Specialist Mathematics 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Specialist Mathematics 11 Queensland Units 1 & 2

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teacher support.

Explore further

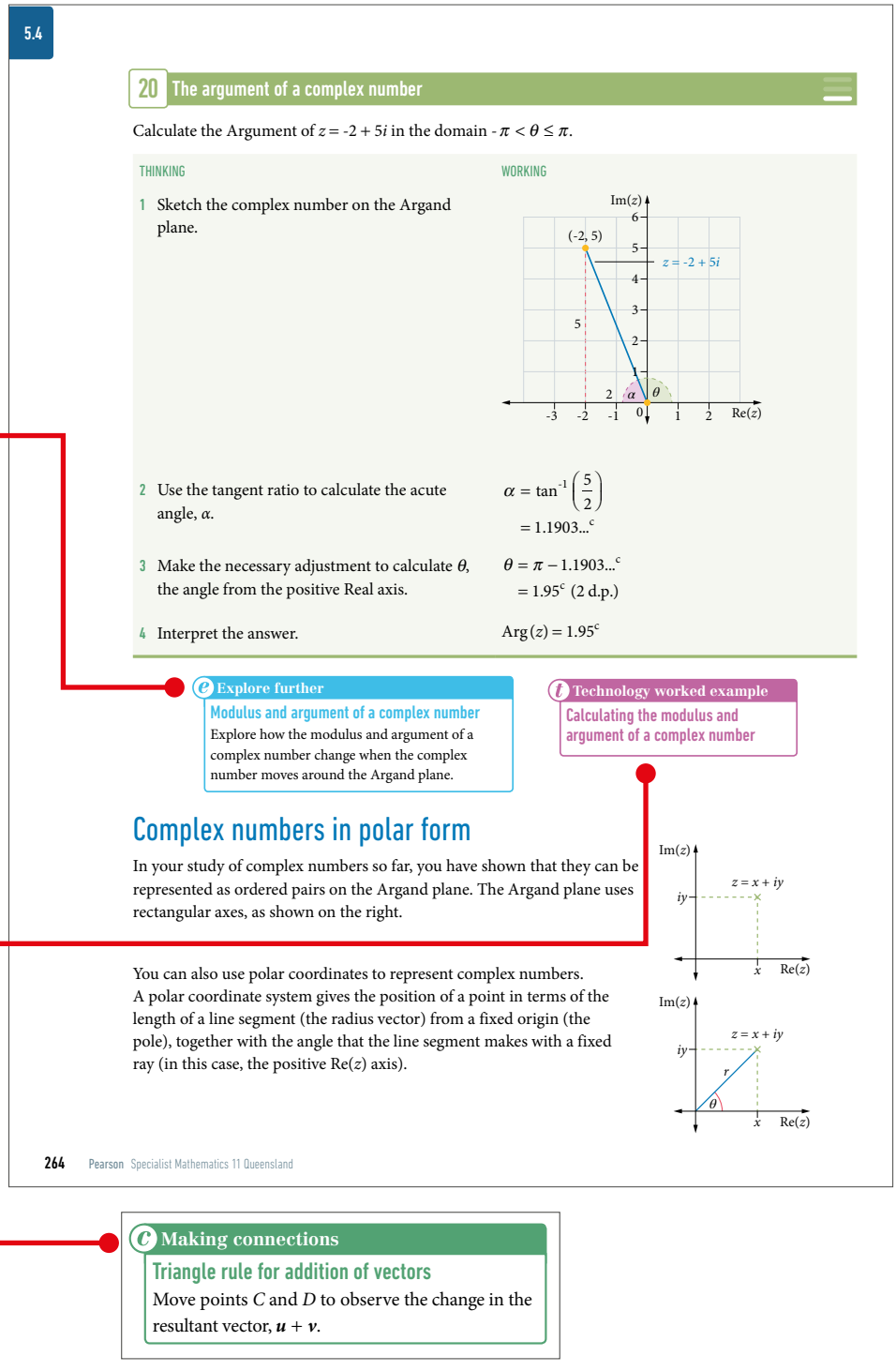
This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Technology worked examples

These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Making connections

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.



Key information

Key information and rules are highlighted throughout the chapter.

Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

Meeting the needs of the QCE syllabus

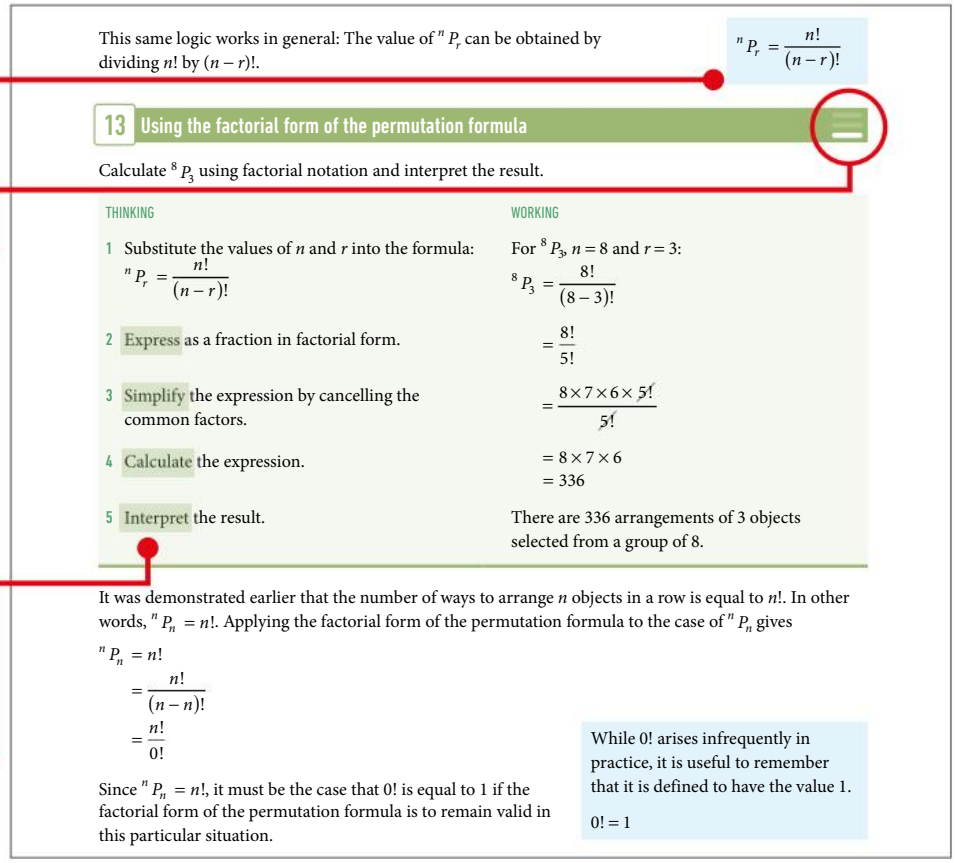
The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

Tech-free questions

These questions are designed to provide students with the opportunity to practice algebraic manipulations to prepare them for technology-free examination papers.

Worked solutions

Fully worked solutions are provided for every question in the student textbook and can be accessed from the accompanying eBook.



This same logic works in general: The value of ${}^nP_r$ can be obtained by dividing $n!$ by $(n-r)!$.

$${}^nP_r = \frac{n!}{(n-r)!}$$

13 Using the factorial form of the permutation formula

Calculate ${}^8P_3$ using factorial notation and interpret the result.

THINKING	WORKING
1 Substitute the values of n and r into the formula: ${}^nP_r = \frac{n!}{(n-r)!}$	For ${}^8P_3$, $n = 8$ and $r = 3$: ${}^8P_3 = \frac{8!}{(8-3)!}$
2 Express as a fraction in factorial form.	$= \frac{8!}{5!}$
3 Simplify the expression by cancelling the common factors.	$= \frac{8 \times 7 \times 6 \times 5!}{5!}$
4 Calculate the expression.	$= 8 \times 7 \times 6$ $= 336$
5 Interpret the result.	There are 336 arrangements of 3 objects selected from a group of 8.

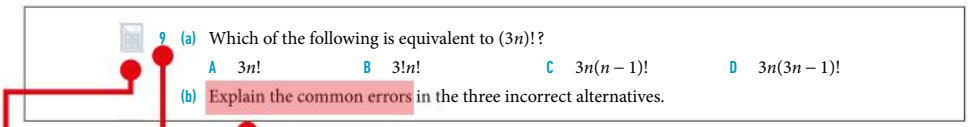
It was demonstrated earlier that the number of ways to arrange n objects in a row is equal to $n!$. In other words, ${}^nP_n = n!$. Applying the factorial form of the permutation formula to the case of ${}^nP_n$ gives

$${}^nP_n = n!$$
$$= \frac{n!}{(n-n)!}$$
$$= \frac{n!}{0!}$$

Since ${}^nP_n = n!$, it must be the case that $0!$ is equal to 1 if the factorial form of the permutation formula is to remain valid in this particular situation.

While 0! arises infrequently in practice, it is useful to remember that it is defined to have the value 1.

$0! = 1$



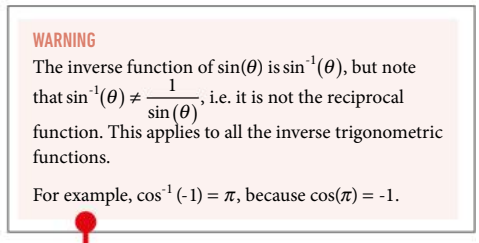
9 (a) Which of the following is equivalent to $(3n)!$?

A $3n!$ B $3!n!$ C $3n(n-1)!$ D $3n(3n-1)!$

(b) Explain the common errors in the three incorrect alternatives.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.



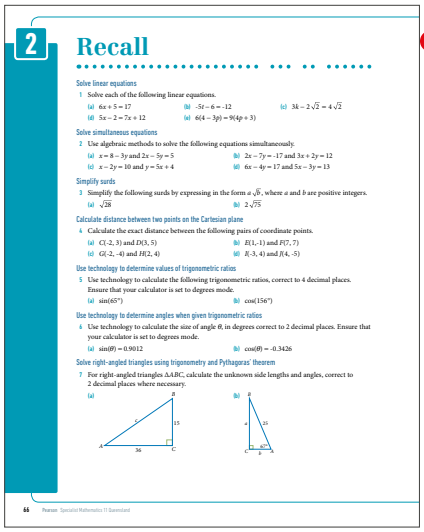
WARNING

The inverse function of $\sin(\theta)$ is $\sin^{-1}(\theta)$, but note that $\sin^{-1}(\theta) \neq \frac{1}{\sin(\theta)}$, i.e. it is not the reciprocal function. This applies to all the inverse trigonometric functions.

For example, $\cos^{-1}(-1) = \pi$, because $\cos(\pi) = -1$.

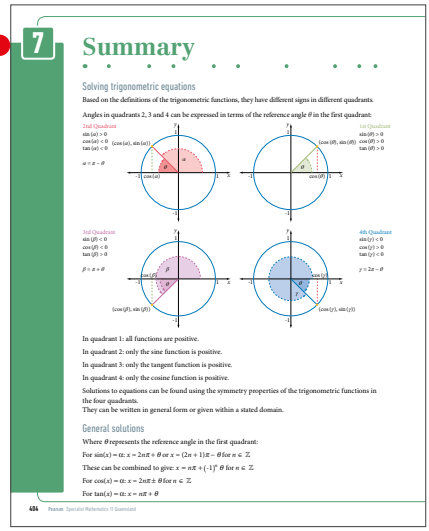
Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.


2
Recall
Solve linear equations
1 Solve each of the following linear equations.
(a) $6x + 5 = 17$ (b) $-5t - 6 = -12$ (c) $3k - 2\sqrt{2} = 4\sqrt{2}$
(d) $5x - 2 = 7x + 12$ (e) $6(4 - 3p) = 9(4p + 3)$
Solve simultaneous equations
2 Use algebraic methods to solve the following equations simultaneously.
(a) $x = 8 - 3y$ and $2x - 5y = 5$ (b) $2x - 7y = -17$ and $3x + 2y = 12$
(c) $x - 2y = 10$ and $y = 5x + 4$ (d) $6x - 4y = 17$ and $5x - 3y = 13$
Simplify surds
3 Simplify the following surds by expressing in the form $a\sqrt{b}$, where a and b are positive integers.
(a) $\sqrt{28}$ (b) $2\sqrt{75}$
Calculate distance between two points on the Cartesian plane
4 Calculate the exact distance between the following pairs of coordinate points.
(a) $C(-2, 3)$ and $D(3, 5)$ (b) $E(1, -1)$ and $F(7, 7)$
(c) $G(-2, -4)$ and $H(2, 4)$ (d) $I(-3, 4)$ and $J(4, -5)$
Use technology to determine values of trigonometric ratios
5 Use technology to calculate the following trigonometric ratios, correct to 4 decimal places. Ensure that your calculator is set to degrees mode.
(a) $\sin(65°)$ (b) $\cos(156°)$
Use technology to determine angles when given trigonometric ratios
6 Use technology to calculate the size of angle θ, in degrees correct to 2 decimal places. Ensure that your calculator is set to degrees mode.
(a) $\sin(\theta) = 0.9012$ (b) $\cos(\theta) = -0.3426$
Solve right-angled triangles using trigonometry and Pythagoras' theorem
7 For right-angled triangles ΔABC, calculate the unknown side lengths and angles, correct to 2 decimal places where necessary.
(a) (b)
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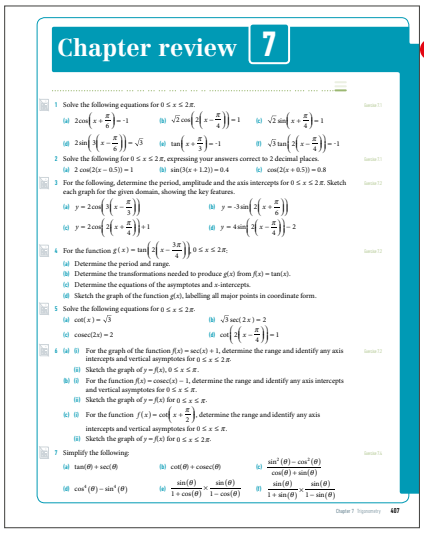
Recall

Each chapter begins with a review of assumed knowledge for the chapter.


7
Summary
Solving trigonometric equations
Based on the definitions of the trigonometric functions, they have different signs in different quadrants.
Angles in quadrants 2, 3 and 4 can be expressed in terms of the reference angle θ in the first quadrant:
2nd Quadrant
$\sin(\alpha) > 0$
$\cos(\alpha) < 0$
$\tan(\alpha) < 0$
$\alpha = \pi - \theta$
1st Quadrant
$\sin(\theta) > 0$
$\cos(\theta) > 0$
$\tan(\theta) > 0$
3rd Quadrant
$\sin(\beta) < 0$
$\cos(\beta) < 0$
$\tan(\beta) > 0$
$\beta = \pi + \theta$
4th Quadrant
$\sin(\gamma) < 0$
$\cos(\gamma) > 0$
$\tan(\gamma) < 0$
$\gamma = 2\pi - \theta$
In quadrant 1: all functions are positive.
In quadrant 2: only the sine function is positive.
In quadrant 3: only the tangent function is positive.
In quadrant 4: only the cosine function is positive.
Solutions to equations can be found using the symmetry properties of the trigonometric functions in the four quadrants.
They can be written in general form or given within a stated domain.
General solutions
Where θ represents the reference angle in the first quadrant:
For $\sin(x) = \alpha$: $x = 2n\pi + \theta$ or $x = (2n + 1)\pi - \theta$ for $n \in \mathbb{Z}$
These can be combined to give: $x = n\pi + (-1)^n \theta$ for $n \in \mathbb{Z}$
For $\cos(x) = \alpha$: $x = 2n\pi \pm \theta$ for $n \in \mathbb{Z}$
For $\tan(x) = \alpha$: $x = n\pi + \theta$
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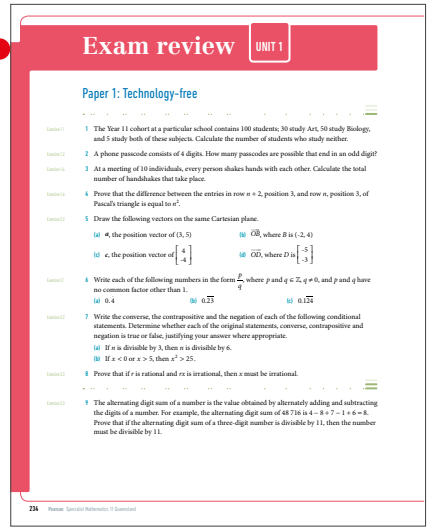
Summary

At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.


Chapter review 7
1 Solve the following equations for $0 \le x \le 2\pi$.
(a) $2\cos\left(x + \frac{\pi}{6}\right) = -1$ (b) $\sqrt{2}\cos\left(2\left(x - \frac{\pi}{4}\right)\right) = 1$ (c) $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$
(d) $2\sin\left(3\left(x + \frac{\pi}{6}\right)\right) = -\sqrt{3}$ (e) $\tan\left(x + \frac{\pi}{3}\right) = -1$ (f) $\sqrt{3}\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1$
2 Solve the following for $0 \le x \le 2\pi$, expressing your answers correct to 2 decimal places.
(a) $2\cos(2(x - 0.5)) = 1$ (b) $\sin(3(x + 1.2)) = 0.4$ (c) $\cos(2(x + 0.5)) = 0.8$
3 For the following, determine the period, amplitude and the axis intercepts for $0 \le x \le 2\pi$. Sketch each graph for the given domain, showing the key features.
(a) $y = 2\cos\left(3\left(x - \frac{\pi}{3}\right)\right)$ (b) $y = -3\sin\left(2\left(x + \frac{\pi}{6}\right)\right)$
(c) $y = 2\cos\left(2\left(x + \frac{\pi}{4}\right)\right) + 1$ (d) $y = 4\sin\left(3\left(x - \frac{\pi}{4}\right)\right) - 2$
4 For the function $g(x) = \tan\left(2\left(x - \frac{3\pi}{4}\right)\right)$, $0 \le x \le 2\pi$.
(a) Determine the period and range.
(b) Determine the transformations needed to produce $g(x)$ from $f(x) = \tan(x)$.
(c) Determine the equations of the asymptotes and x-intercepts.
(d) Sketch the graph of the function $g(x)$, labelling all major points in coordinate form.
5 Solve the following equations for $0 \le x \le 2\pi$.
(a) $\cot(x) = \sqrt{3}$ (b) $\sqrt{3}\sec(2x) = 2$
(c) $\operatorname{cosec}(2x) = 2$ (d) $\cot\left(2\left(x - \frac{\pi}{4}\right)\right) = 1$
6 (a) (i) For the graph of the function $f(x) = \sec(x) + 1$, determine the range and identify any axis intercepts and vertical asymptotes for $0 \le x \le 2\pi$.
(ii) Sketch the graph of $y = f(x)$, $0 \le x \le \pi$.
(b) (i) For the function $f(x) = \operatorname{cosec}(x) - 1$, determine the range and identify any axis intercepts and vertical asymptotes for $0 \le x \le \pi$.
(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 2\pi$.
(c) (i) For the function $f(x) = \cot\left(x + \frac{\pi}{2}\right)$, determine the range and identify any axis intercepts and vertical asymptotes for $0 \le x \le \pi$.
(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 2\pi$.
7 Simplify the following.
(a) $\tan(\theta) + \sec(\theta)$ (b) $\cot(\theta) + \operatorname{cosec}(\theta)$ (c) $\frac{\sin^2(\theta) - \cos^2(\theta)}{\cos(\theta) + \sin(\theta)}$
(d) $\cos^4(\theta) - \sin^4(\theta)$ (e) $\frac{\sin(\theta)}{1 + \cos(\theta)} \times \frac{\sin(\theta)}{1 - \cos(\theta)}$ (f) $\frac{\sin(\theta)}{1 + \sin(\theta)} \div \frac{\sin(\theta)}{1 - \sin(\theta)}$
Chapter 7 Trigonometry 407

Chapter review

Every chapter review follows the QCAA examination proportions for level of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.


Exam review UNIT 1
Paper 1: Technology-free
1 The Year 11 cohort at a particular school contains 100 students; 30 study Art, 50 study Biology, and 5 study both of these subjects. Calculate the number of students who study neither.
2 A phone passcode consists of 4 digits. How many passcodes are possible that end in an odd digit?
3 At a meeting of 10 individuals, every person shakes hands with each other. Calculate the total number of handshakes that take place.
4 Prove that the difference between the entries in row $n + 2$, position 3, and row n, position 3, of Pascal's triangle is equal to n^2.
5 Draw the following vectors on the same Cartesian plane.
(a) $\mathbf{a}$, the position vector of $(3, 5)$ (b) $\overrightarrow{OB}$, where B is $(-2, 4)$
(c) $\mathbf{c}$, the position vector of $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$ (d) $\overrightarrow{OD}$, where D is $\begin{bmatrix} -5 \\ -3 \end{bmatrix}$
6 Write each of the following numbers in the form $\frac{p}{q}$, where p and $q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.
(a) 0.4 (b) $0.\dot{2}\dot{3}$ (c) $0.1\dot{2}\dot{4}$
7 Write the converse, the contrapositive and the negation of each of the following conditional statements. Determine whether each of the original statements, converse, contrapositive and negation is true or false, justifying your answer where appropriate.
(a) If n is divisible by 3, then n is divisible by 6.
(b) If $x < 0$ or $x > 5$, then $x^2 > 25$.
8 Prove that if r is rational and rx is irrational, then x must be irrational.
9 The alternating digit sum of a number is the value obtained by alternately adding and subtracting the digits of a number. For example, the alternating digit sum of 48 716 is $4 - 8 + 7 - 1 + 6 = 8$. Prove that if the alternating digit sum of a three-digit number is divisible by 11, then the number must be divisible by 11.
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Exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each Unit.





1 Combinatorics

Recall 4

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1.3 Permutations 23

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1.6 Pascal's triangle 46

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1 Recall

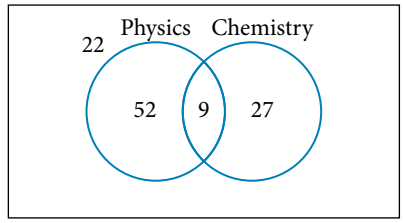
Determine the elements in sets satisfying given restrictions

1 Consider the set of whole numbers from 1 to 10 inclusive. Determine the numbers that are included in the following sets.

(a) even and greater than 5 (b) even or greater than 5 (c) neither even nor greater than 5

Interpret simple Venn diagrams

2 The Venn diagram shows the number of students who study Chemistry/Physics at a particular school.



(a) Determine the number of students who study Physics.
(b) Determine the number of students who study both Physics and Chemistry.
(c) Determine the number of students who study Physics or Chemistry.
(d) Determine the number of students who study Physics but not Chemistry.
(e) Determine the number of students who study neither Physics nor Chemistry.
(f) Determine the number of students who study Physics or Chemistry, but not both.

Create a list of all the possible outcomes for a scenario

3 Determine all possible 2-digit numbers that can be formed using the digits 1, 2, 3 and 4 under the following circumstances.

(a) Each digit is used only once. (b) Digits can be repeated.

Calculate routine probabilities

4 Determine the following probabilities.

(a) a vowel being obtained when a letter from the alphabet is selected at random
(b) a blue marble being drawn from a bag containing 5 red, 6 blue and 9 green marbles
(c) a number greater than 4 being obtained when a die is rolled.

Simplify numerical expressions involving fractions

5 Simplify and solve the following.

(a) $\dfrac{20 \times 19}{2}$ (b) $\dfrac{100 \times 99 \times 98 \times 97 \times 96}{98 \times 97 \times 96}$ (c) $\dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 4 \times 3 \times 2 \times 1}$

Substitute values and solve expressions

6 Determine the value of N by substituting the value(s) indicated.

(a) $N = n(n-1)(n-2)$, $n = 6$ (b) $N = \dfrac{n(n-1)(n-2)}{(n-1)(n-2)}$, $n = 10$
(c) $N = 10 \times \dfrac{n+1-k}{k}$, $n = 5$ and $k = 2$

Simplify algebraic fractions

7 Simplify the following expressions.

(a) $\dfrac{abc}{bc}$ (b) $\dfrac{n(n-1)(n-2)(n-3)}{2 \times 1 \times (n-2)(n-3)}$ (c) $\dfrac{n(n-1)}{2} + n$

Solve quadratic equations

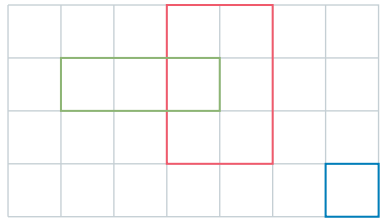
8 Solve the following equations.

(a) $x^2 - 4x - 12 = 0$ (b) $2x^2 + 3x = 4$

1.1 Counting techniques

Combinatorics is a branch of Mathematics concerned with patterns and structures that can be counted. It has many applications to other areas of mathematics, such as algebra and geometry. It has been used to solve many problems in fields such as computer science, biology, chemistry and physics.

While counting is something that you have been doing since you were very young, there are many situations in which specialised techniques are required. For example, suppose you wanted to count the total number of rectangles that appear in the diagram on the right. (Several such rectangles are shown in colour). Simply counting the rectangles one by one would be next to impossible.



In this chapter, a variety of principles and techniques are explained that will enable you to count almost anything.

Sets and the inclusion–exclusion principle

A *set* is an unordered collection of distinct objects. Sets form the basis of nearly all of pure mathematics. While the idea of a set is very simple, there is a significant amount of terminology and notation that it is necessary to understand.

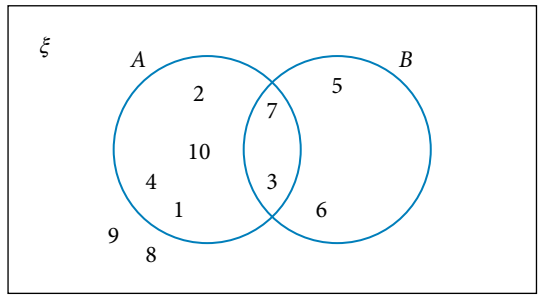
The objects in a set are referred to as the *elements* of the set. Uppercase letters are usually used to represent sets, and the elements are written inside braces. For example, if the symbol E were to be used to represent the set of positive even numbers less than 10, then you could write $E = \{2, 4, 6, 8\}$. The order in which the elements in the braces are listed does not matter, and there are no repeated elements.

The symbol ξ (alternatively ε, U or S) is used to represent the *universal set*. The universal set contains all possible objects relevant to a particular situation.

The symbol $\varnothing$ is used to denote the *empty set* (also called the *null set*). The empty set doesn't contain any elements.

Venn diagrams, named after their creator, John Venn (1883–1923), are used to represent sets in a visual form. They consist of a number of circles (one circle for each set) that are usually overlapping. Elements of the sets are categorised into the different regions of the diagram to show which elements are shared and which elements are unique to one or to none of the sets.

The Venn diagram shows the following set scenario: $\xi = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 2, 3, 4, 7, 10\}$ and $B = \{3, 5, 6, 7\}$.

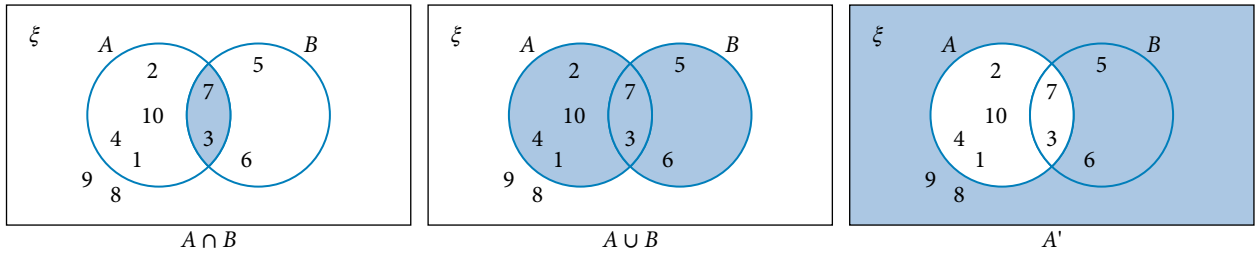


The notation $A \cap B$ is used to denote the *intersection* of sets A and B; this is the set containing all elements that are in both A and B. For the sets A and B described previously, $A \cap B = \{3, 7\}$.

The notation $A \cup B$ is used to denote the *union* of sets A and B; this is the set containing all elements that are in either A or B (or both). For the sets A and B described, $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 10\}$.

The notation A' is used to denote the *complement* of set A; this is the set containing all elements of the universal set that are not elements of A. For the sets A and ξ described, $A' = \{5, 6, 8, 9\}$.

The shading on the diagrams below shows the relevant regions of the Venn diagram previously described that correspond to $A \cap B$, $A \cup B$, and A'.



$A \cap B$ $A \cup B$ A'

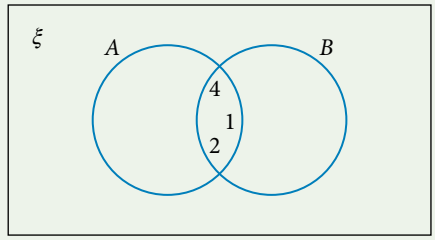
1 Working with Venn diagrams for two sets

Consider the sets, $A = \{0, 1, 2, 3, 4\}$, $B = \{1, 2, 4, 5, 6, 8\}$, and $\xi = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

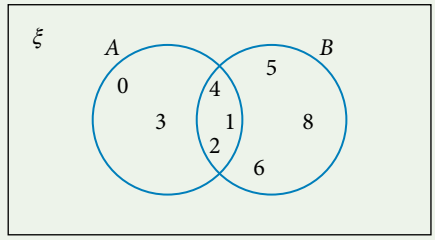
(a) Represent the sets on a Venn diagram.

THINKING

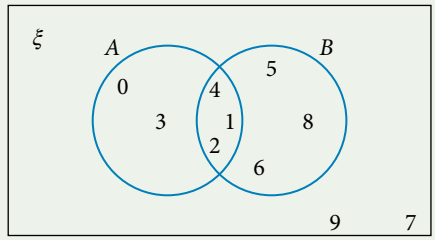
1 Draw two intersecting circles. Then focus on the elements in the sets. Start by adding to the diagram the elements that are common to both sets.



2 Add the elements that occur in exactly one set.



3 Add the elements that are not in either of the two sets, but are in the universal set.



(b) With the aid of the Venn diagram, determine the elements in the following sets.

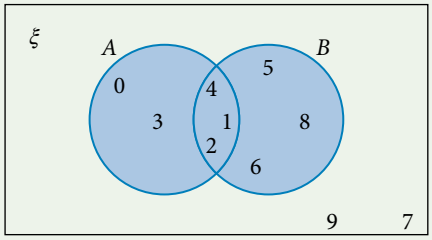
(i) $A \cup B$ **(ii)** $A \cap B$

(iii) A' **(iv)** B'

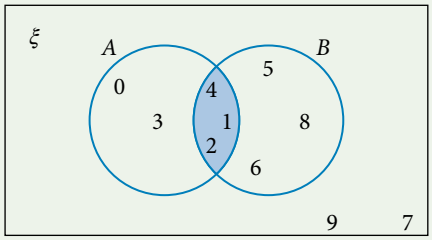
(v) $A' \cup B$ **(vi)** $A \cup B'$

(vii) $(A \cup B)'$ **(viii)** $(A \cap B)'$

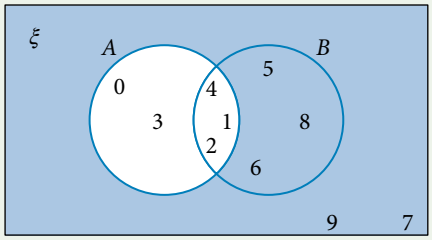
In each case, identify the relevant region of the Venn diagram (these have been shaded below) and then determine the elements.



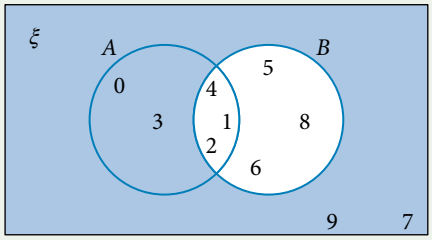
(i) $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8\}$



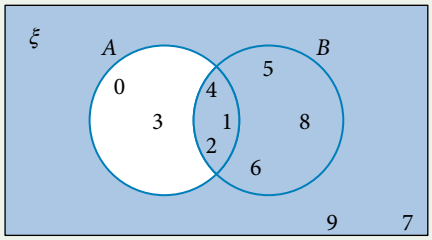
(ii) $A \cap B = \{1, 2, 4\}$



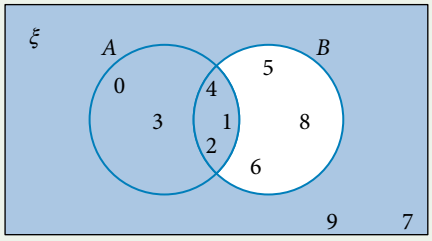
(iii) $A' = \{5, 6, 7, 8, 9\}$



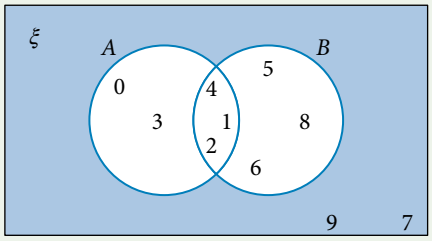
(iv) $B' = \{0, 3, 7, 9\}$



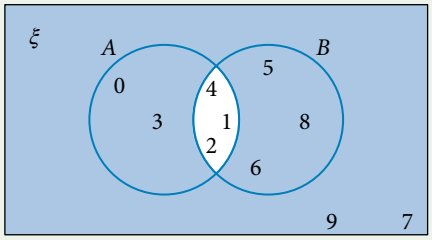
(v) $A' \cup B = \{1, 2, 4, 5, 6, 7, 8, 9\}$



(vi) $A \cup B' = \{0, 1, 2, 3, 4, 7, 9\}$



(vii) $(A \cup B)' = \{7, 9\}$



(viii) $(A \cap B)' = \{0, 3, 5, 6, 7, 8, 9\}$

2 Working with Venn diagrams for three sets

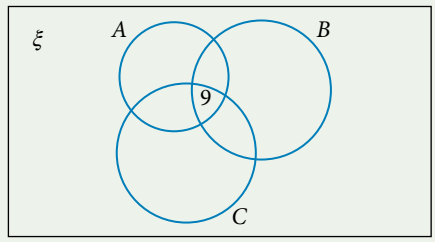
Consider the sets $A = \{1, 8, 9, 11, 12, 15, 16\}$, $B = \{4, 5, 6, 9, 10, 11, 14\}$, $C = \{2, 3, 4, 7, 8, 9, 13, 15\}$ and $\xi = \{1, 2, 3, ..., 18\}$.

(a) Represent the sets on a Venn diagram.

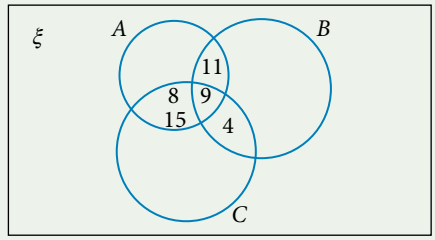
THINKING

1 Draw three intersecting circles. Then focus on the elements in the sets. Start by adding to the diagram the elements that are common to all three sets.

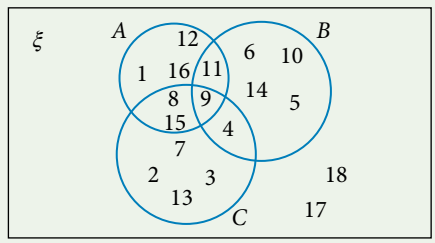
WORKING



2 Add the elements that are common to two sets, but not the third.



3 Add the elements that occur in exactly one set. Finally, add the elements that are not in any of the three sets, but are in the universal set.



(b) With the aid of the Venn diagram, determine the elements in the following sets.

(i) A'	**(ii)** B'	**(iii)** $A \cup B$
(iv) $B \cap C$	**(v)** $A \cap B \cap C$	**(vi)** $A' \cup B$
(vii) $B' \cap C$	**(viii)** $(A \cup B \cup C)'$	**(ix)** $A \cap B \cap C'$

In each case, identify the relevant region of the Venn diagram, and then determine the elements.

(i) $A' = \{2, 3, 4, 5, 6, 7, 10, 13, 14, 17, 18\}$

(ii) $B' = \{1, 2, 3, 7, 8, 12, 13, 15, 16, 17, 18\}$

(iii) $A \cup B = \{1, 4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16\}$

(iv) $B \cap C = \{4, 9\}$

(v) $A \cap B \cap C = \{9\}$

(vi) $A' \cup B = \{2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 17, 18\}$

(vii) $B' \cap C = \{2, 3, 7, 8, 13, 15\}$

(viii) $(A \cup B \cup C)' = \{17, 18\}$

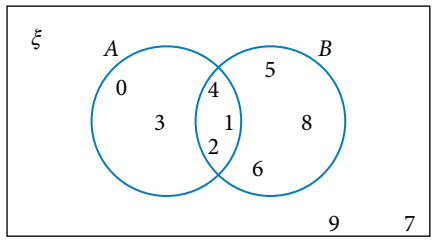
(ix) $A \cap B \cap C' = \{11\}$

The inclusion–exclusion principle for two sets

Of particular relevance to this chapter are techniques that can be used to determine the number of elements in particular sets. The notation $n(A)$ or $|A|$ is used to denote the number of elements in the set A.

One important technique for determining the number of elements in a particular set is the *inclusion–exclusion principle*. This principle states that if A and B are sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

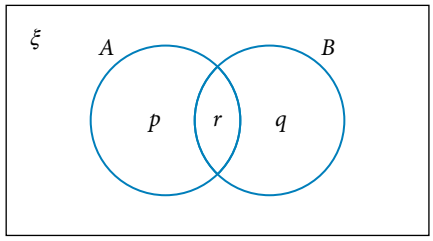
Consider the Venn diagram from Worked Example 1 (see right).



In this situation, $n(A) = 5$ and $n(B) = 6$. To obtain the number of elements in $A \cup B$, you could start by adding together $n(A)$ and $n(B)$ to obtain 11, however, this would have counted the three elements in $A \cap B$ twice. To correct this, you must subtract 3 (the value of $n(A \cap B)$) from the total to obtain 8. You can verify from the diagram that the value of $n(A \cup B)$—the number of elements in the set $A \cup B$—is, in fact, 8 where
$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8\}$.

A formal proof of the result is as follows.

Let p, q, r represent the *number of elements* in the regions of the Venn diagram shown. For example, the variable p represents the number of elements that belong to A but do not belong to B.



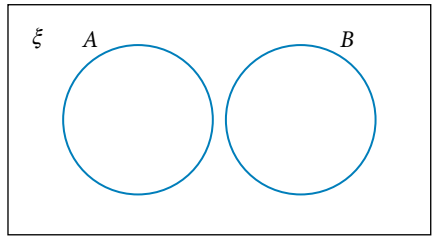
From the diagram: $n(A \cup B) = p + q + r$.

$$\begin{aligned} n(A) + n(B) - n(A \cap B) &= (p + r) + (q + r) - r \\ &= p + q + r \end{aligned}$$

It follows that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

If two sets do not contain any common elements, then $(A \cap B) = \varnothing$, and the sets are said to be disjoint or mutually exclusive. On a Venn diagram, this situation would contain two non-overlapping circles, as shown.

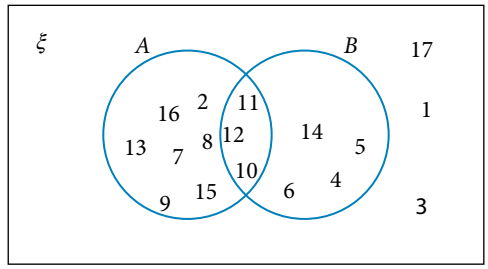
The inclusion–exclusion principle for two sets, A and B: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ which can also be written as $|A \cup B| = |A| + |B| - |A \cap B|$



Note that if sets A and B are disjoint, then $n(A \cup B) = n(A) + n(B)$ since $n(A \cap B) = 0$ in this case. This result is known as the *addition principle.*

3 Exploring the inclusion–exclusion principle for two sets

The Venn diagram for $\xi = \{1, 2, 3, \ldots, 15, 16, 17\}$, $A = \{2, 7, 8, 9, 10, 11, 12, 13, 15, 16\}$ and $B = \{4, 5, 6, 10, 11, 12, 14\}$ is shown to the right.



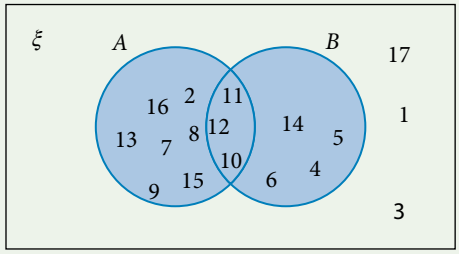
(a) Determine the following.

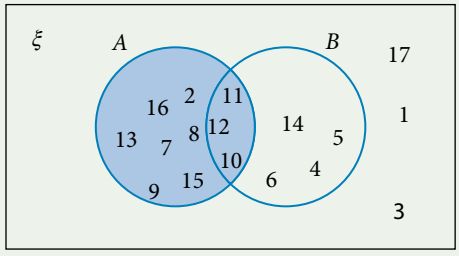
(i) $n(A \cup B)$ **(ii)** $n(A)$

(iii) $n(B)$ **(iv)** $n(A \cap B)$

THINKING

In each case, identify the relevant regions on the Venn diagram (these have been shaded) and then determine the number of elements in each region.

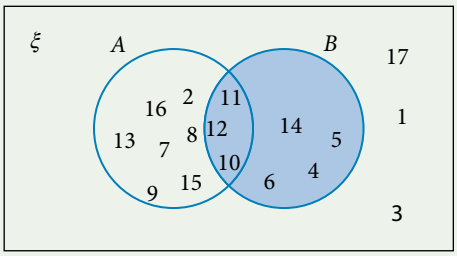




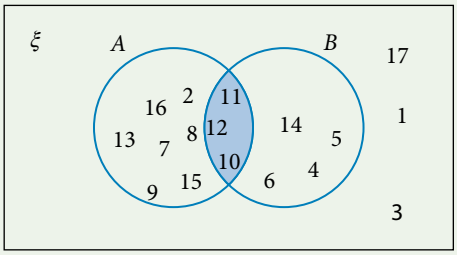
WORKING

(i) $n(A \cup B) = 14$

(ii) $n(A) = 10$



(iii) $n(B) = 7$



(iv) $n(A \cap B) = 3$

(b) Evaluate the reasonableness of your solution by verifying the inclusion–exclusion principle for these sets.

1 Recall the inclusion–exclusion principle.

2 Demonstrate that the left-hand side (LHS) and the right-hand side (RHS) of the equation are equal, given the values obtained for the sets under consideration.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\text{LHS} = n(A \cup B) = 14$

$\text{RHS} = n(A) + n(B) - n(A \cap B)$

$= 10 + 7 - 3$

$= 14$

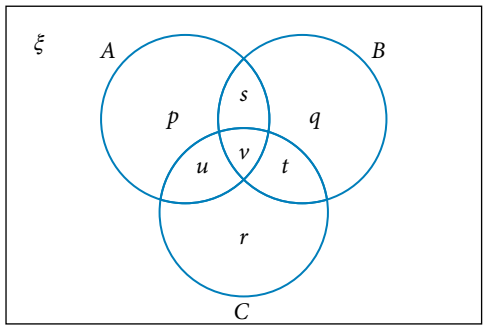
$\text{LHS} = \text{RHS}$

The inclusion–exclusion principle for three sets

The inclusion–exclusion principle can be extended to describe the relationship between three sets:
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

This result can be proven in a similar way to the two-set version of the principle.

Let p, q, r, s, t, u, v represent the number of elements in the regions of the Venn diagram shown. The variable p represents the number of elements that belong to A only, that is, the elements belong to A but do not belong to B or C.



From the diagram on the right:
$n(A \cup B \cup C) = p + q + r + s + t + u + v.$

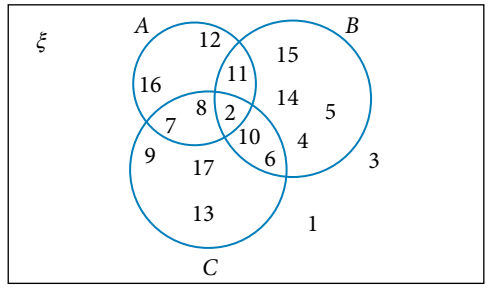
Similarly:
$n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
$= (p + s + u + v) + (q + s + t + v) + (r + t + u + v) - (s + v) - (t + v) - (u + v) + v$
$= p + q + r + s + t + u + v$

It follows that:
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

The inclusion–exclusion principle for three sets is
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

4 Exploring the inclusion–exclusion principle for three sets

The elements in three sets are shown on the Venn diagram for $\xi = \{1, 2, 3, \ldots, 17\}$.



(a) Determine number of elements in the following sets.

(i) $n(A \cup B \cup C)$
(ii) $n(A)$
(iii) $n(B)$
(iv) $n(C)$
(v) $n(A \cap B)$
(vi) $n(B \cap C)$
(vii) $n(A \cap C)$
(viii) $n(A \cap B \cap C)$

THINKING	WORKING
(i) For $n(A \cup B \cup C)$ count the elements in the three circles.	$n(A \cup B \cup C) = 15$
(ii) For $n(A)$ count the elements in the A circle.	$n(A) = 6$
(iii) For $n(B)$ count the elements in the B circle.	$n(B) = 8$
(iv) For $n(C)$ count the elements in the C circle.	$n(C) = 8$
(v) For $n(A \cap B)$ count the elements common to A and B.	$n(A \cap B) = 2$
(vi) For $n(B \cap C)$ count the elements common to B and C.	$n(B \cap C) = 3$
(vii) For $n(A \cap C)$ count the elements common to A and C.	$n(A \cap C) = 3$
(viii) For $n(A \cap B \cap C)$ count the elements common to the three circles.	$n(A \cap B \cap C) = 1$

(b) Verify the inclusion–exclusion principle for these sets.

THINKING	WORKING
1 Recall the inclusion–exclusion principle.	$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
2 Demonstrate that both sides of the equation are equal, given the values obtained for the sets under consideration.	LHS $= n(A \cup B \cup C) = 15$ RHS $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$ $= 6 + 8 + 8 - 2 - 3 - 3 + 1$ $= 15$ LHS = RHS, therefore the inclusion–exclusion principle holds true as required.

Using the inclusion–exclusion principle to solve worded problems

The inclusion–exclusion principle can be used to solve worded problems based on real-life situations, as the following examples illustrate.

5 Applying the inclusion–exclusion principle (two-set situation)

Fifty people who played tennis and/or basketball were surveyed. It was found that thirty people played tennis and forty people played basketball.

Of the people surveyed, how many people played both tennis and basketball?

THINKING	WORKING
1 Define variables for the sets involved and identify key values.	Let T be the set of people who play tennis, and let B be the set of people who play basketball. $n(T \cup B) = 50$ $n(T) = 30$ $n(B) = 40$
2 Recall the inclusion–exclusion principle and substitute known values.	$n(T \cup B) = n(T) + n(B) - n(T \cap B)$ $50 = 30 + 40 - n(T \cap B)$
3 Solve for the unknown and interpret the result.	$n(T \cap B) = 30 + 40 - 50$ $= 20$ Twenty people played both tennis and basketball.

When it is not clear that the inclusion–exclusion principle can be applied directly to solve a problem, it may be beneficial to draw a Venn diagram, using the information in the question to label various regions with the number of elements represented by these regions. The following example illustrates this approach.

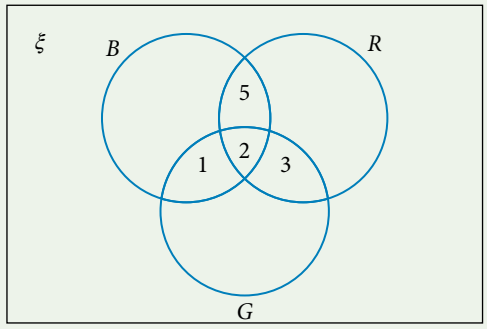
6 Applying the inclusion–exclusion principle (three-set situation)

A store contains a collection of jumpers that are all decorated with stripes of different colours; some with two different coloured stripes and some with three different coloured stripes. Ten jumpers have blue stripes, sixteen jumpers have red stripes and thirteen jumpers have green stripes.

One jumper has blue and green stripes but no red stripes, five have blue and red stripes but no green stripes, while three have red and green stripes but no blue stripes.

Two jumpers have stripes of all three colours.

How many jumpers are there in total?

THINKING	WORKING
1 Define variables for the sets involved.	Let B represent the set of jumpers with blue stripes. Let R represent the set of jumpers with red stripes. Let G represent the set of jumpers with green stripes.
2 Draw a Venn diagram and fill in each region with the number of elements belonging to that particular set.	
3 Recall the inclusion–exclusion principle and substitute known values.	$n(R \cup B \cup G) = n(R) + n(B) + n(G)$ $- n(R \cap B) - n(B \cap G)$ $- n(R \cap G) + n(R \cap B \cap G)$ $= 16 + 10 + 13 - 7 - 5 - 3 + 2$ $= 26$
4 Interpret the result.	There are 26 jumpers in total.

EXERCISE 1.1 Counting techniques

Worked Example 1

1 Consider the sets $A = \{1, 2, 4, 5, 8\}$, $B = \{1, 2, 3, 6, 8\}$ and $\xi = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) Represent these sets on a Venn diagram.

(b) With the aid of the Venn diagram, determine the elements in the following sets.

(i) $A \cup B$ (ii) $A \cap B$ (iii) A'

(iv) B' (v) $A' \cup B$ (vi) $A \cup B'$

(vii) $(A \cup B)'$ (viii) $(A \cap B)'$

2 Consider the sets $A = \{5, 6, 7, 9, 12, 13, 18\}$, $B = \{2, 3, 4, 9, 10, 14, 15, 18\}$, $C = \{1, 3, 4, 5, 9, 13, 15\}$ and $\xi = \{1, 2, 3, \ldots, 18\}$. 2

(a) Represent these sets on a Venn diagram.

(b) With the aid of the Venn diagram, determine the elements in the following sets.

(i) $B \cap C$ **(ii)** $A \cup B$ **(iii)** $A \cap B \cap C$

(iv) $B' \cap C$ **(v)** $A' \cup B$ **(vi)** $A \cup B \cup C$

(vii) $A \cap B \cap C'$ **(viii)** $(A \cup B \cup C)'$

3 Consider the sets $A = \{2, 3, 6, 7, 10, 13, 14\}$, $B = \{3, 4, 5, 6, 10, 11, 14, 15\}$ and $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. 3

(a) Represent these sets on a Venn diagram.

(b) With the aid of the Venn diagram, determine the following.

(i) $n(A \cup B)$ **(ii)** $n(A)$

(iii) $n(B)$ **(iv)** $n(A \cap B)$

(c) Verify the inclusion–exclusion principle for these sets.

4 Consider the sets $A = \{2, 3, 4, 5, 7, 10, 11\}$, $B = \{1, 3, 4, 5, 7, 14, 15\}$ and $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. Verify the inclusion–exclusion principle for these sets.

5 Consider the sets $A = \{2, 4, 6, 8, 11\}$, $B = \{1, 2, 3, 5, 6, 7, 8, 10, 13\}$, $C = \{2, 4, 6, 10, 12\}$ and $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. 4

(a) Represent these sets on a Venn diagram.

(b) Determine number of elements in the following sets.

(i) $n(A \cup B \cup C)$ **(ii)** $n(A)$ **(iii)** $n(B)$

(iv) $n(C)$ **(v)** $n(A \cap B)$ **(vi)** $n(B \cap C)$

(vii) $n(A \cap C)$ **(viii)** $n(A \cap B \cap C)$

(c) Verify the inclusion–exclusion principle for these sets.

6 Consider the sets $A = \{1, 2, 3, 5\}$, $B = \{5, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$ and $\xi = \{1, 2, 3, \ldots, 11\}$. Verify the inclusion–exclusion principle for these sets.

7 When thirty artists were surveyed it was found that twenty people were painters and fifteen people were sculptors. Determine the number of artists who were both painters and sculptures. 5

8 When fifteen students were asked whether they like lettuce and tomato, ten responded that they like lettuce, twelve responded that they like tomato, and two responded that they did not like either.

(a) How many students like both lettuce and tomato?

(b) How many students like lettuce but not tomato?

9 At a particular school, 420 students play a musical instrument, 330 students play a sport, 60 play both a musical instrument and a sport and 20 students play neither.

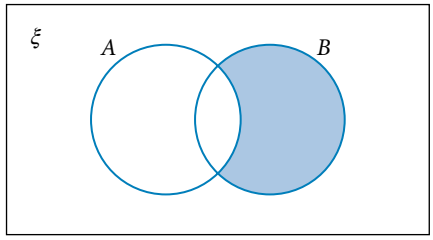
How many students do at least one activity?

10 **(a)** If A and B are sets, which of the following sets is equivalent to $(A \cap B)'$?

A $A \cap B$ **B** $A' \cap B'$ **C** $A \cup B$ **D** $A' \cup B'$

(b) What are the common errors that could lead to selecting the alternative answers?

11 Consider the Venn diagram pictured. Which of the following sets is represented by the shaded region?



A A' **B** B **C** $A' \cap B$ **D** $A \cap B'$

Worked Example 6

12 Nineteen families owned dogs, cats or both dogs and cats. Three families owned dogs only while nine families owned cats only. How many families owned both dogs and cats?

13 Thirty people were asked what pets they kept. Fifteen had dogs, twelve had cats and six had fish. Four had dogs and cats only, three had dogs and fish only, and one had a cat and fish only. If one person had all three types of pets, how many people had none of these pets?

14 A careers advisor helped 30 students choose their subjects for the following year. Sixteen students said they wanted to study French, 16 wanted to study Chinese and 11 wanted to study Latin. Five said they wanted to study both French and Latin, and of these, 3 wanted to study Chinese as well. Five wanted to study only Latin, and 8 wanted only to study Chinese. How many students wanted to study French only?

15 If $n(A \cap B) = 2$, $n(\xi) = 19$, $n(A) = 6$, and $n(A \cup B)' = 5$, calculate $n(B)$.

16 Let A and B be sets such that $n(A \cup B) = 39$, $n(A \cap B) = 5$, and $n(A) = 3 \times n(B)$. By making use of the inclusion–exclusion principle, determine $n(A' \cap B)$.

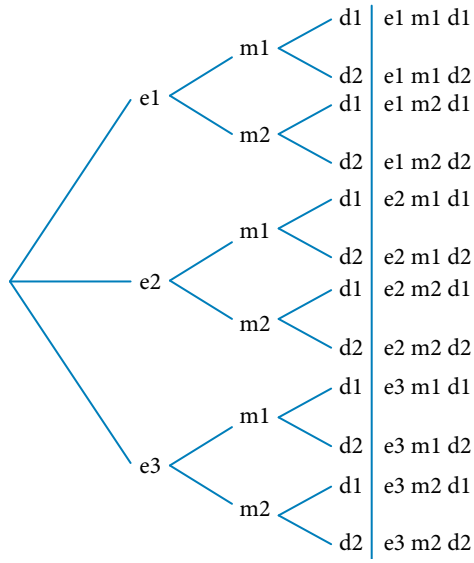
17 Determine the number of positive integers less than 6000 that are divisible by either 2 or 3. Hint: A number is divisible by both 2 and 3 precisely when it is divisible by 6. Evaluate the reasonableness of your result.

18 A group of 82 senior students all study at least one of Biology, Chemistry and Physics. The number of students who study all three subjects is equal to the number who study Biology and Chemistry but not Physics. The number of students who study all three subjects is also equal to the number who study Chemistry and Physics but not Biology. Finally, there are twice as many students who study Biology and Physics but not Chemistry, as those who study all three subjects. If the numbers of students studying Biology, Physics and Chemistry are 41, 35, and 30 respectively, determine the number of students who study both Biology and Physics.

The multiplication and addition principles 1.2

The multiplication principle

Suppose that the menu at a particular restaurant contains 3 entrées (for convenience, let the entrées be e1, e2, and e3), 2 main courses (m1 and m2), and 2 desserts (d1 and d2). If you were interested in determining the number of different three-course meals that are possible, you could construct a diagram similar to the one on the right.



From the diagram, it's clear that for each choice of entrée, there are 2 ways to choose the main course. Therefore the number of possible combinations of an entrée and a main course is $3 \times 2 = 6$ (see middle column in the diagram.) Furthermore, for each of these combinations, there are 2 desserts to choose from. Hence, there are $6 \times 2 = 12$ different three-course meals that are possible. In summary, the number of different three-course meals can be obtained by multiplying together the number of ways to choose each course $3 \times 2 \times 2 = 12$.

3 entrées	2 main courses	2 desserts
e1	m1	d1
e2	m2	d2
e3		

In the restaurant example, the task was choosing a three-course meal. The first step was choosing an entrée, the second step was choosing a main course, and the third step was choosing a dessert. Note that in this situation, the number of ways each step could be performed did not depend on the particular choices made in any previous step. For example, the number of ways to choose a main course is 2, no matter whether e1, e2, or e3 was chosen in the first step. This may sound obvious, but you should always ensure that this is the case when attempting to use the multiplication principle.

Suppose a task can be broken up into a certain number of steps in which the number of ways each step can be performed is not dependent on the choices made in previous steps. Then the number of ways to perform the entire task is equal to the *product* of the number of ways each step can be performed. This is the multiplication principle.

7 Using the multiplication principle

Sam has 3 different styles of hats, 2 jumpers, 5 shirts and 3 pairs of jeans. How many different outfits consisting of a hat, a jumper, a shirt and a pair of jeans could Sam wear?

THINKING	WORKING
1 Identify the number of choices for each stage in the task.	Number of choices of each item: hat: 3 jumper: 2 shirt: 5 jeans: 3
2 Use the multiplication principle to calculate the answer.	Number of outfits $= 3 \times 2 \times 5 \times 3 = 90$

In many situations, there are restrictions placed on the possible choices. The multiplication principle can still be used in these situations, however, it is often necessary to take care in deciding which step should be performed first.

8 Using the multiplication principle with restrictions

Consider the numbers 5, 6, 7 and 8.

(a) How many three-digit numbers can be formed if there are no restrictions?

THINKING	WORKING
1 Identify the number of choices for each stage in the task.	Number of choices of each digit:

1st digit	2nd digit	3rd digit
4 options	4 options	4 options

2 Use the multiplication principle to calculate the answer.	Number of three-digit numbers: $4 \times 4 \times 4 = 64$

(b) How many three-digit numbers can be formed if the digits of the three-digit number are all different?

1 Identify the number of possible choices for each digit as each digit is selected.	Number of choices of each digit:

1st digit	2nd digit	3rd digit
4 options	3 options	2 options

2 Use the multiplication principle to calculate the answer.	Number of three-digit numbers with the given restriction: $4 \times 3 \times 2 = 24$

(c) How many three-digit numbers can be formed if the digits of the three-digit number are all different, and the final digit is greater than 6?

1 Identify the possible values of the restricted digit, hence determine the number of possible choices.	The final digit is greater than 6, i.e. {7, 8}. There are two possible options for the final digit.

1st digit	2nd digit	3rd digit
		2 options

2 Determine the number of possible choices remaining for the other digits.	

1st digit	2nd digit	3rd digit
2 options	3 options	2 options

3 Use the multiplication principle to calculate and interpret the answer.	The number of three-digit numbers possible with a final digit greater than 6: $2 \times 3 \times 2 = 12$

Part (c) of the previous example illustrates an important subtlety of applying the multiplication principle:

> When applying the multiplication principle, it is important to remember that the number of choices in any step cannot depend on the actual choices made in previous steps. To avoid a scenario in which the number of choices in a particular step *does* depend on the choices made in previous steps, make the first step of the task the one containing the restriction.

The addition principle

Consider the restaurant scenario described at the beginning of this section. Suppose this time that you are interested in the total number of possible two-course meals (suppose that a two-course meal consists of an entrée and main course, or a main course and a dessert.)

In this case, the objects being counted (two-course meals) can be classified into two different types.

Entrée	Main
3 options	2 options

Main	Dessert
2 options	2 options

There are 6 different options to have a two-course meal consisting of an entrée and main meal.

There are 4 different options to have a two-course meal consisting of a main meal and a dessert.

Adding together the number of entrée–main and main–dessert options results in the total number of two-course meal options: $6 + 4 = 10$. In total, there are 10 different ways to order a two-course meal.

This is actually an example of using the addition principle, mentioned briefly in section 1.1 for the case of two disjoint sets. The addition principle can be generalised to any number of mutually exclusive sets.

> Suppose a collection of objects can be classified into a number of different types, such that every object belongs to exactly one type. Then the total number of objects is equal to the *sum* of the number of objects of each type. This is the addition principle.

The condition of every object belonging to exactly one type corresponds to the need for the types to be disjoint (not containing any common elements). If there were objects belonging to more than one type, then the addition principle would not apply, and the more general inclusion–exclusion principle would be needed.

The addition principle is very powerful in solving more complex counting problems. Typically, it is used in conjunction with other counting techniques, such as the multiplication principle.

9 Using the addition principle in conjunction with the multiplication principle

How many positive integers less than 500 with no repeated digits can be formed where each digit is one of 4, 5, 6 or 7?

THINKING	WORKING
1 Identify the number of digits needed for the value to exist.	A value less than 500 contains either one, two or three digits.

2 Determine the number of possible one-digit values.	A one-digit value will be less than 500. 1st digit: 4 options There are 4 different one-digit numbers less than 500.
3 Determine the number of possible two-digit values.	The digits must be different, however, all two-digit values will be less than 500. 1st digit: 4 options; 2nd digit: 3 options There are $4 \times 3 = 12$ different two-digit numbers less than 500 (with no repeated digits).
4 Determine the number of possible three-digit values.	In order for the number to be less than 500, the first digit must be a 4. 1st digit: 1 option; 2nd digit: ; 3rd digit: Then select different digits for the other digits. 1st digit: 1 option; 2nd digit: 3 options; 3rd digit: 2 options There are $3 \times 2 = 6$ different three-digit numbers less than 500 (with no repeated digits).
5 Use the addition principle to calculate and interpret the total number of integers.	There are $4 + 12 + 6 = 22$ integer values less than 500 containing no repeated digits and created from the digits {4, 5, 6, 7}.

To help you determine when to use the *multiplication principle* and when to use the *addition principle*, make a list of all the objects that you are counting. If you can classify the objects into a number of types, then the addition principle can be used to add up the number of objects of each type. When determining the number of objects of a *particular* type, the multiplication principle can be used if the process of forming a single object of that type (such as a two-digit number less than 500), can be broken up into steps (choosing the first digit, then the second digit).

Explore further

Problem solving with counting methods

Make a list of all the objects that you are counting.
Classify the objects into a number of types.
Use the addition principle to add up the number of objects of each type.

EXERCISE 1.2 The multiplication and addition principles

Worked Example 7

1 In Mark's wardrobe there are 5 different shirts, 2 different pairs of pants and 3 different pairs of shoes. Determine the number of different outfits that are possible consisting of a shirt, a pair of pants and a pair of shoes.

2 Sandwiches can be made using white bread, multigrain bread or rye bread with margarine or butter spreads and fillings of either meat, pickle or egg. Determine the number of different sandwiches that can be made.

3 License plates in a particular country consist of a single letter (from A to Z) followed by three digits (from 0 to 9). Determine the total number of possible license plates.

8

4 Using the numbers 2, 3, 4, 5 and 6, how many three-digit positive integers can be formed under the following conditions?

(a) There are no restrictions.

(b) The digits of the three-digit number are all different.

(c) The digits of the three-digit number are all different, and the three-digit number is even.

5 From a class of 18 students, a music captain, sport captain and service captain must be chosen to form the leadership team of the class. (An individual cannot hold more than one of these three roles.) How many different leadership teams are possible under the following conditions?

(a) There are no restrictions.

(b) The sport captain must be either Emma or Greg.

(c) A particular student, Pari, cannot be chosen as either the sport captain or service captain.

6 How many four-letter sequences are possible if each letter is either A, B, C, D, E or F under the following conditions?

(a) There are no other restrictions.

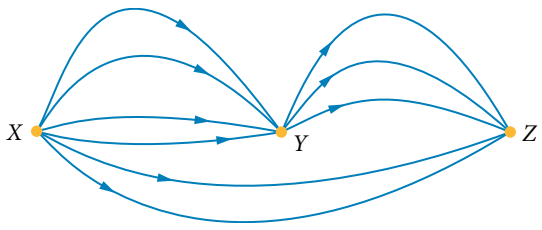
(b) The sequence begins and ends with the letter A, but no other letter occurs more than once, assuming there are only 2 A's in the sequence.

7 Calculate the number of ways that the collection of letters A, B, C, D, E and F could be arranged in a row.

9

8 How many positive integers less than 4000 with no repeated digits can be formed where each digit is either 2, 3, 4, 5 or 6?

9 Consider the diagram.



(a) How many distinct paths are there from X to Z?

A $4+3+2$ B $4\times3\times2$

C $4\times3+2$ D $4+3\times2$

(b) What are the common errors in the three incorrect alternatives?

10 A restaurant offers a choice of 7 pizzas, 6 pasta dishes, 5 cakes, 4 tarts and 3 sundaes. A two-course meal consists of either a pizza or pasta dish, and a dessert. A three-course meal consists of both a pizza and pasta dish, and a dessert. What is the total number of possible meals?

A $(7 \times 6) + (5 \times 4 \times 3)$ **B** $(7 + 6) \times (5 + 4 + 3)$

C $7 \times 6 \times 5 \times 4 \times 3 + (7 + 6) \times 4 \times 3$ **D** $(7 + 6) \times (5 + 4 + 3) + 7 \times 6 \times (5 + 4 + 3)$

11 From a collection of 10 different fonts, three different ones must be chosen to be used on a webpage: one for the heading, one for the standard text and one for quotations. Determine the number of different ways the fonts can be chosen and allocated, given that one of the ten fonts, MT Heading, is only appropriate for the heading.

12 How many different four-digit numbers are possible if each digit of the number is either 0, 1, 2, 3, 4 or 5, and numbers starting with 3 cannot be even?

13 A committee consisting of a president, a treasurer, and a secretary is to be formed from four people: Ann, Bob, Cal and Dan. Suppose that, for various reasons, Ann cannot be president and either Cal or Dan must be secretary. A student attempting to determine the number of possible committee combinations reasons there are 3 ways to choose the president, 3 ways to choose the secretary, and 2 ways to choose the treasurer, so the number of possibilities is equal to $3 \times 3 \times 2 = 18$. Explain why this student's reasoning is invalid, and determine the correct number of possible committees.

14 Lucinda goes to the local pizzeria for dinner with a group of friends. The menu offers 4 types of soup, 5 main meals, 6 desserts and 3 different types of drinks. Lucinda will order 2 different courses and a drink. How many dinner options does she have?

15 Simon has a pack of 9 cards, numbered from 1 to 9. Three cards are chosen and placed in a line, with the proviso that an even-numbered card cannot be to the right of an odd-numbered card. How many ways can this be done?

16 Andrew has a locker at school that is currently locked with a combination lock. The combination is a sequence of three numbers, from 1 to 20, that must be entered in order. Andrew can only remember that the second number is a multiple of 6 (and is not equal to 6) and that the third number is greater than the second number. If it takes Andrew 12.3 seconds to test any particular combination, determine the longest possible time it will take him (in minutes) for him to unlock his locker.

17 From a class of secondary school students, a captain and vice-captain must be chosen. Three of the students are not eligible to be the captain, but they may be vice-captain. Given that there are 255 possible ways for students in this class to be assigned to these roles, determine the number of students in the class. Evaluate the reasonableness of your solution.

Permutations

1.3

Factorial notation

The solutions to many counting problems involve expressions such as $6 \times 5 \times 4 \times 3 \times 2 \times 1$, in which all positive integers less than or equal to a particular integer must be multiplied together. The notation 6! can be used to represent this expression in a more compact form. The symbol '!' is known as the factorial operator; it can be thought of as shorthand for the product of all positive integers less than or equal to the integer preceding it. In words, 6! would be expressed as 'six factorial'.

The first few factorials of positive integers are given below.

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

If n is a positive integer:
$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

One particularly useful property of any factorial is that it is easy to express it in terms of the factorial of any positive integer below it. For example, 6! can be expressed in any of the following forms:

$6 \times 5!$ (since 5! is simply shorthand for $5 \times 4 \times 3 \times 2 \times 1$)

$6 \times 5 \times 4!$ (since 4! is simply shorthand for $4 \times 3 \times 2 \times 1$)

$6 \times 5 \times 4 \times 3!$ (since 3! is simply shorthand for $3 \times 2 \times 1$)

$6 \times 5 \times 4 \times 3 \times 2!$ (since 2! is simply shorthand for 2×1)

$6 \times 5 \times 4 \times 3 \times 2 \times 1!$ (since 1! is equal to 1)

While most calculators have a factorial button or command, it is nonetheless important to become proficient at simplifying and evaluating expressions involving factorials in order to solve certain counting problems.

10 Evaluating factorial expressions

Calculate the following.

(a) 5!

THINKING	WORKING
5! is shorthand for $5 \times 4 \times 3 \times 2 \times 1$. Write this in expanded form, then multiply the numbers.	$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

(b) $\frac{7!}{5!}$

METHOD 1

Express 7! and 5! as products and cancel common factors. Then calculate the result.

$$\begin{aligned}\frac{7!}{5!} &= \frac{7\times6\times\cancel{5\times4\times3\times2\times1}}{\cancel{5\times4\times3\times2\times1}}\\ &= 7\times6\\ &= 42\end{aligned}$$

METHOD 2

Write the product of the larger number in terms of the factorial of the smaller number and then calculate the result.

$$\begin{aligned}\frac{7!}{5!} &= \frac{7\times6\times\cancel{5!}}{\cancel{5!}}\\ &= 7\times6\\ &= 42\end{aligned}$$

(c) $\frac{40!}{43!}$

METHOD 1

Express 40! and 43! as products and cancel common factors. Then calculate the result.

$$\begin{aligned}\frac{40!}{43!} &= \frac{\cancel{40\times39\times38\times\ldots\times3\times2\times1}}{43\times42\times41\times\cancel{40\times39\times38\times\ldots\times3\times2\times1}}\\ &= \frac{1}{43\times42\times41}\\ &= \frac{1}{74\,046}\end{aligned}$$

METHOD 2

Write the product of the larger number in terms of the factorial of the smaller number and then calculate the result.

$$\begin{aligned}\frac{40!}{43!} &= \frac{\cancel{40!}}{43\times42\times41\times\cancel{40!}}\\ &= \frac{1}{43\times42\times41}\\ &= \frac{1}{74\,046}\end{aligned}$$

11 Simplifying algebraic expressions involving factorials

Simplify $\frac{n!}{(n-2)!}$

METHOD 1

THINKING

Write the expressions for $n!$ and $(n-2)!$ as products. Then cancel common factors.

WORKING

$$\begin{aligned}\frac{n!}{(n-2)!} &= \frac{n(n-1)\cancel{(n-2)(n-3)\times\ldots\times3\times2\times1}}{\cancel{(n-2)(n-3)\times\ldots\times3\times2\times1}}\\ &= n(n-1)\end{aligned}$$

METHOD 2

Write the product of the larger number in terms of the factorial of the smaller, recalling that $(n-1)$ is the number 1 less than n.

$$\frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$$
$$= n(n-1)$$

Permutation notation

Consider the number of ways that the collection of letters A, B, C, D, E and F could be arranged in a row. Some possible arrangements are ABCDEF, BACDEF, FACDEB, and so on. Each of these arrangements is known as a *permutation*.

A permutation is an *ordered* arrangement of objects.

The multiplication principle can be used to determine the number of different permutations possible in the letter example: There are 6 ways to choose the first letter, 5 ways to choose the second, and so on. Hence the total number of permutations is equal to $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$. This reasoning works whenever you need to determine the number of ways to arrange any number of objects in a row.

The number of ways to arrange n different objects in a row is equal to $n!$

Now suppose you wish to determine the number of ways that 4 of the letters from the collection A, B, C, D, E and F could be arranged in a row. In this case, some possible arrangements are ABCD, ABDC, DCAB, FEBA, FCAB, and so on. Each of these arrangements is known as a *4-permutation*.

An r-permutation of a collection of objects is an *ordered* arrangement of r of the n objects, where r is a number less than n.

The multiplication principle can once again be used to determine the number of different 4-permutations possible in the letter example:

There are 6 ways to choose the first letter, 5 ways to choose the second, 4 ways to choose the third, and 3 ways to choose the fourth. Hence the total number of 4-permutations is equal to $6 \times 5 \times 4 \times 3$.

The notation ${}^{6}P_{4}$ represents the number of 4-permutations of a collection of 6 objects; in other words, ${}^{6}P_{4}$ represents the number of ways to select 4 objects from a collection of 6 objects, and arrange them in order. Thus ${}^{6}P_{4} = 6 \times 5 \times 4 \times 3$.

As another example, ${}^{12}P_{3}$ represents the number of ways to select 3 objects from a set of 12 objects and arrange them in order. To determine this number, you can use the same logic as in the letter example: there are 12 ways to choose the first object, 11 ways to choose the second, and 10 ways to choose the third. Thus ${}^{12}P_{3} = 12 \times 11 \times 10$.

Notice that for ${}^{12}P_{3}$, the top number (12) represents the number of ways to choose the first item in the arrangement, while the bottom number (3) represents the number of items in the arrangement, and hence, how many numbers must be multiplied together.

The notation ${}^{n}P_{r}$ represents the number of ways to select r objects from a collection of n objects and arrange them in order.

$${}^{n}P_{r} = \underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}_{r \text{ terms}}$$

${}^6P_6$ represents the number of ways to select and arrange all 6 objects from a group of 6 objects. Thus, ${}^6P_6 = 6!$.

Explore further

Permutations

On a spreadsheet use the function PERMUT(number, number_chosen).

12 Evaluating ${}^nP_r$

(a) Calculate ${}^6P_2$ and verify your answer by listing all permutations of two letters from the letters A, B, C, D, E and F.

THINKING

1 Recall the rule ${}^nP_r$, where n represents the total number of objects and r represents the number of objects to be selected and arranged.

2 List the permutations in a systematic way.

WORKING

$${}^6P_2 = 6 \times 5 = 30$$

	A	B	C	D	E	F
A		AB	AC	AD	AE	AF
B	BA		BC	BD	BE	BF
C	CA	CB		CD	CE	CF
D	DA	DB	DC		DE	DF
E	EA	EB	EC	ED		EF
F	FA	FB	FC	FD	FE	

There are 30 different ways to arrange 6 objects selected two at a time. This verifies the earlier result.

(b) Calculate ${}^{10}P_3$.

Recall the rule ${}^nP_r$, where n represents the total number of objects and r represents the number of objects to be selected and arranged.

$${}^{10}P_3 = 10 \times 9 \times 8 = 720$$

The factorial form of the permutation formula

Consider ${}^{10}P_3$. This was calculated in the previous example as follows: ${}^{10}P_3 = 10 \times 9 \times 8$.

Notice that the exact same expression can be obtained by dividing 10! by (10 – 3)! as demonstrated below:

$$\frac{10!}{(10-3)!} = \frac{10!}{7!}$$
$$= \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}}$$
$$= 10 \times 9 \times 8$$

This same logic works in general: The value of ${}^{n}P_{r}$ can be obtained by dividing $n!$ by $(n-r)!$.

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

13 Using the factorial form of the permutation formula

Calculate ${}^{8}P_{3}$ using factorial notation and interpret the result.

THINKING	WORKING
1 Substitute the values of n and r into the formula: ${}^{n}P_{r} = \frac{n!}{(n-r)!}$	For ${}^{8}P_{3}$, $n = 8$ and $r = 3$: ${}^{8}P_{3} = \frac{8!}{(8-3)!}$
2 Express as a fraction in factorial form.	$= \frac{8!}{5!}$
3 Simplify the expression by cancelling the common factors.	$= \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}}$
4 Calculate the expression.	$= 8 \times 7 \times 6$ $= 336$
5 Interpret the result.	There are 336 arrangements of 3 objects selected from a group of 8.

It was demonstrated earlier that the number of ways to arrange n objects in a row is equal to $n!$. In other words, ${}^{n}P_{n} = n!$. Applying the factorial form of the permutation formula to the case of ${}^{n}P_{n}$ gives

$${}^{n}P_{n} = n!$$
$$= \frac{n!}{(n-n)!}$$
$$= \frac{n!}{0!}$$

Since ${}^{n}P_{n} = n!$, it must be the case that 0! is equal to 1 if the factorial form of the permutation formula is to remain valid in this particular situation.

While 0! arises infrequently in practice, it is useful to remember that it is defined to have the value 1.

$0! = 1$

The factorial form of the permutation formula may seem less useful than the form presented initially. Indeed, it is more intuitive to remember (and understand that), for example, ${}^{8}P_{3}$ represents the number of ways to select 3 objects from a set of 8 objects and arrange them in order, and thus, using the multiplication rule, it is equal to $8 \times 7 \times 6$. That said, there are situations where the factorial form of the formula is useful. For example, expressing ${}^{100}P_{20}$ as $\frac{100!}{80!}$ is simpler than $100 \times 99 \times 98 \times \ldots \times 82 \times 81$. The factorial form of the permutation formula also provides the basis for another important formula that will be introduced in the next section.

t Technology worked example
Calculating permutations

Note: Most calculators have an ${}^{n}P_{r}$ function or command that allows you to input values of n and r to obtain the value of ${}^{n}P_{r}$.

Permutations with restrictions

Problems involving permutations with restrictions will typically be solved using the multiplication principle and addition principles in the same manner as similar problems in Exercise 1.2. It is also helpful to remember that the number of ways to arrange n objects in a row is $n!$

14 Permutations with restrictions

Calculate the number of different arrangements of the letters in the word INTEGRAL under the following conditions.

(a) There are no restrictions.

THINKING	WORKING
1 Determine the expression, where the number of ways to arrange n (distinguishable) objects in a row is $n!$	Number of arrangements: $8! = 40\,320$
2 Interpret the answer.	There are 40 320 arrangements of the letters INTEGRAL.

(b) The G and the R must be next to each other.

1 Identify the options for the selections if the letters are to be positioned next to each other.	The order of the letters may be GR or RG, in which case there are 2 possible scenarios for the arrangements. The arrangement comes from one of the sets: {I, N, T, E, GR, A, L} {I, N, T, E, RG, A, L}
2 Use the multiplication principle to calculate the number of possible arrangements in each scenario.	The are 7 distinguishable objects to arrange in each set. There are $7! = 5040$ arrangements of the letters in the word INTEGRAL containing the object GR and another $7! = 5040$ arrangements with the object RG.
3 Use the addition principle to calculate and interpret the result.	$7! + 7! = 2 \times 7!$ $= 10\,080$ There are 10 080 different arrangements of the letters in the word INTEGRAL with the letters G and R next to each other.

(c) The G and R are not next to each other.

1 Identify known information.	In total, there are 40 320 arrangements of the letters in the word INTEGRAL, 10 080 of which have the letters G and R together.

2 Recall the rule for complementary events where $n(A') = n(\xi) - n(A)$

Calculate the number of arrangements.

$40\,320 - 10\,080 = 30\,240$

3 Interpret the result.

There are 30 240 arrangements of the letters in the word INTEGRAL where the letters G and R are not together.

(d) Whenever the first letter is a vowel, the final letter must not be a vowel.

1 Identify the options for the arrangements.

Option 1

The arrangement consists of a vowel as the first letter, a consonant as the last and the other letters arranged between them.

Option 2

The arrangement consists of a consonant as the first letter and then the rest of the letters arranged after it.

2 Use the multiplication principle to determine and interpret the number of arrangements possible for the first option.

Option 1

The vowels are {I, E, A}.

Therefore there are 3 different ways to select the first letter.

The consonants are {N, T, G, R, L}.

Therefore there are 5 different ways to select the last letter.

First letter (vowel)	Last letter (consonant)	Remaining letters
3	5	$6 \times 5 \times 4 \times 3 \times 2 \times 1$

The number of arrangements with a vowel as the first letter and a consonant as the last:

$3 \times 5 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 3 \times 5 \times 6!$

$= 10\,800$

3 Use the multiplication principle to determine and interpret the number of arrangements possible for the second option.

Option 2

First letter (consonant)	Remaining letters
5	$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

The number of arrangements with a consonant as the first letter:

$5 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 5 \times 7!$

$= 25\,200$

4 Use the addition principle to calculate and interpret the result.

There are $10\,800 + 25\,200 = 36\,000$ possible arrangements of the letters of the word INTEGRAL, given the restriction that when the first letter is a vowel the final letter must not be.

Permutations with indistinguishable objects

Consider the problem of determining the number of different arrangements of the letters in the word VECTORS. The answer is 7! because the seven letters are all different. But what about if you wanted to determine the number of different arrangements of the letters in the word WEEKEND. This problem is actually much more difficult as there is a letter that occurs more than once. The letter 'E' occurs three times and they are indistinguishable from each other.

In order to solve this problem, it helps to firstly imagine that each 'E' is distinguishable from the others, for example, E_1, E_2 and E_3. In this case, the problem would be equivalent to the VECTORS one, with an answer of 7!. The problem with this is that you would be counting many arrangements more than once. For example, the arrangements:

$WE_1E_2KE_3ND$, $WE_3E_1KE_2ND$, $WE_2E_3KE_1ND$, $WE_1E_3KE_2ND$, $WE_3E_2KE_1ND$, $WE_2E_1KE_3ND$

would all be counted as different arrangements, but you want them to be considered the same arrangement since each 'E' is actually indistinguishable from the others. To help understand what to do, imagine making a giant list of every single arrangement. You can break this list up into groups as shown to the right.

$E_1WKE_2E_3ND$
$E_1WKE_3E_2ND$
$E_2WKE_3E_1ND$
$E_2WKE_1E_3ND$
$E_3WKE_1E_2ND$
$E_3WKE_2E_1ND$
} $EWKEEND$

$WE_1KE_2E_3ND$
$WE_1KE_3E_2ND$
$WE_2KE_3E_1ND$
$WE_2KE_1E_3ND$
$WE_3KE_1E_2ND$
$WE_3KE_2E_1ND$
} $WEKEEND$

$DE_1KE_2WE_3N$
$DE_1KE_3WE_2N$
$DE_2KE_3WE_1N$
$DE_2KE_1WE_3N$
$DE_3KE_1WE_2N$
$DE_3KE_2WE_1N$
} $DEKEWEN$
⋮

Notice that all the arrangements within any particular group correspond to our problem where each E is indistinguishable from the others (shown to the right of the group).

The number of arrangements in each group is 3! since this is the number of ways that E_1, E_2 and E_3 can be arranged. Since there are 7! arrangements in total, the number of groups, and hence the number of arrangements in our problem is $\frac{7!}{3!} = 840$.

Now suppose the word under consideration has multiple letters that are repeated. For example, consider the number of different arrangements of the letters in the word REFEREE. This word also has 7 letters in total, four of the letter E and two of the letter R. Using a similar method to the case of WEEKEND, it can be shown that the number of arrangements is obtained by starting with 7! and dividing by *both* 4! and 2!. That is, the number of arrangements is equal to $\frac{7!}{4!2!} = 105$.

Suppose that in a collection of n objects, there is one or more objects that are indistinguishable copies. If the numbers of copies of each of these are a, b, c, d and so on, then the number of arrangements of the objects is equal to $\frac{n!}{a!b!c!d!\ldots}$

In summary, to determine the total number of arrangements of a collection of objects, where there is one or more objects that are indistinguishable copies, divide the factorial of the total number of objects by the factorials of the frequencies of each of the different types of objects. In symbols: $\frac{n!}{a!b!c!d!\ldots}$

15 Permutations with indistinguishable objects

Calculate the number of different arrangements of the letters in the word PASSATA under the following conditions.

(a) There are no restrictions.

THINKING	WORKING
1 Identify the total number of objects and any indistinguishable objects.	PASSATA contains 7 letters with three indistinguishable A's and two indistinguishable S's.
2 Recall the permutations formula and calculate.	number of permutations $= \frac{n!}{a!b!c!\ldots}$ $= \frac{7!}{3!2!}$ $= 420$
3 Interpret the result.	There are 420 different arrangements of the letters in the word PASSATA.

(b) The three A's must all be together.

1 Identify the total number of objects and any indistinguishable objects.	PASSATA contains 7 letters but if the three A's are together, PSST(AAA) contains only 5 different 'objects' with two indistinguishable S's.
2 Recall the permutations formula and calculate.	number of permutations $= \frac{n!}{a!b!c!\ldots}$ $= \frac{5!}{2!}$ $= 60$
3 Interpret the result.	There are 60 different arrangements of the letters in the word PASSATA if all the A's are together.

(c) The first and last letters must be A.

1 Identify an appropriate arrangement and recall the permutations formula.	number of permutations $= \frac{n!}{a!b!c!\ldots}$ The first and last letters must be an A, which leaves 5 letters, PSSTA, with two indistinguishable S's.

First letter	Remaining letters	Last letter
1	$\frac{5!}{2!}$	1

2 Use the multiplication principle to determine the number of arrangements.	$1 \times \frac{5!}{2!} \times 1 = 60$
3 Interpret the result.	There are 60 different arrangements of the letters in the word PASSATA that start and end with an A.

EXERCISE 1.3 Permutations

Worked Example

10

1 Calculate the following.

(a) $5!$ (b) $4!$ (c) $\dfrac{5!}{4!}$

(d) $\dfrac{10!}{7!}$ (e) $\dfrac{100!}{98!}$ (f) $3\times\dfrac{6!}{4!}$

11

2 Simplify the following expressions.

(a) $\dfrac{n!}{(n-3)!}$ (b) $\dfrac{n!}{(n-1)!}$ (c) $\dfrac{2(n+1)!}{n!}$

(d) $\dfrac{3(n-1)!}{(n-2)!}$ (e) $\dfrac{(n+2)!}{5n!}$ (f) $\dfrac{n(n-1)!}{n!}$

(g) $\dfrac{(n+2)!}{(n-1)!}$ (h) $\dfrac{(n-1)!}{(n+1)!}$ (i) $\dfrac{(n-5)!}{(n-7)!}$

12

3 (a) (i) Calculate ${}^5P_2$.

(ii) Verify your answer to (i) by listing all permutations of two letters from the letters A, B, C, D and E.

(b) (i) Calculate ${}^3P_3$.

(ii) Verify your answer to part (i) by listing all permutations of the letters A, B and C.

4 Express each of the following using ${}^nP_r$ notation, and calculate.

(a) the number of ways to select 3 people from a group of 7 people and arrange them in a row

(b) the number of ways to arrange 5 people in a row

(c) the number of different arrangements of the letters A, B, C and D

(d) the number of different three-letter sequences that can be formed using letters A, B, C and D if letters cannot be repeated

(e) the number of different permutations containing 2 objects from a set of 6 objects

(f) from a set of 5 individuals, the number of ways to select 3 people to form a committee consisting of a president, a secretary and a treasurer

5 Write a simplified expression for each of the following.

(a) ${}^nP_1$ (b) ${}^nP_2$ (c) ${}^nP_3$

13

6 Calculate the following using factorial notation.

(a) ${}^6P_2$ (b) ${}^7P_3$ (c) ${}^5P_5$

7 Express each of the following as a fraction involving factorials.

(a) ${}^nP_{n-10}$ (b) ${}^{n+r}P_r$ (c) ${}^{2n}P_n$ (d) ${}^{5n+1}P_{2n+7}$

8 Calculate the following.

(a) ${}^{10}P_4$ (b) ${}^{12}P_5$

9 (a) Which of the following is equivalent to $(3n)!$?

A $3n!$ B $3!n!$ C $3n(n-1)!$ D $3n(3n-1)!$

(b) Explain the common errors in the three incorrect alternatives.

10 Let n and r be positive integers with $n > r$. The expression ${}^{n+1}P_{r+1}$ is equivalent to:

A $\frac{(n+1)!}{(r+1)!}$ B $\frac{(n+1)!}{(n-r)!}$ C $\frac{(n+1)!}{(n-r+2)!}$ D $\frac{n!}{(n-r)!}$

Worked Example 14

11 How many ways can the letters of the word GREAT be arranged under the following conditions?

(a) There are no restrictions.

(b) The two vowels must be next to each other in any order.

(c) The two vowels must not be next to each other.

(d) Whenever the first letter is G or R, the last letter is not A.

12 Seven individuals need to be seated in a row along one side of a long table. How many seating arrangements are possible if two of the individuals are not to sit together?

13 How many ways can the letters in the word PENCIL be arranged under the following conditions?

(a) There is no restriction.

(b) The arrangement starts with a vowel.

(c) The vowels are positioned at the start and at the end of the arrangement.

(d) The arrangement starts with a C or a P and ends with a vowel.

14 Four girls and three boys are to sit on a straight bench for the school's scuba diving club photograph. How many ways can the students be arranged if the three boys are not to sit together on the bench?

15 How many ways can eight different books be arranged on a bookshelf if two particular books must occupy the first and last positions, in order to hold the books in place, and two of the remaining books are not to be placed together?

Worked Example 15

16 How many ways can the letters of the word STRESSED be arranged under the following conditions?

(a) There are no restrictions.

(b) The two E's must be next to each other.

(c) The first letter must be S and the last letter must be T.

17 A party decoration kit has 12 coloured flags, which can be attached to a rope in different orders. The kit contains 3 identical red flags, 4 identical blue flags, 3 identical green flags and 2 identical yellow flags. What is the number of permutations you can make with this kit?

18 A binary string is an ordered sequences of 0's and 1's (for example, 0100101). Determine the number of binary strings there are consisting of five 0's and three 1's.

19 A necklace has 5 red beads, 3 blue beads, 2 green beads and 4 yellow beads, which can be threaded in a straight line in different orders. Calculate the number of ways the coloured beads can be arranged under the following conditions.

(a) There are no restrictions.

(b) The first and last beads must be green.

(c) The red beads are kept together.

20 Suppose that 5 apples, 4 bananas, and 3 oranges are to be distributed to 12 different people so that each person receives one piece of fruit. Calculate the number of different ways there are to distribute the fruit under the following conditions.

(a) There are no restrictions.

(b) A particular individual, Mark, is to receive an apple.

21 Solve the following equation for n, where n is a non-negative integer.

$$\frac{(n+1)!}{(n-3)!} = \frac{18n!}{(n-2)!}$$

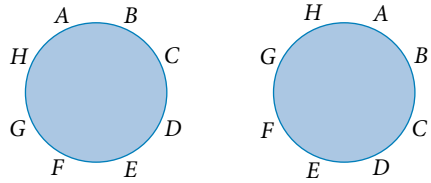
22 Suppose that n married individuals are required to stand next to their partners in a long row for a photo. Determine an expression in terms of n for the number of ways this can be done.

23 A photograph of a diving team is to be taken with the team members in two rows. The team has four girls and two boys. The back row has four places while the front row has the coach in the middle with one diver to the left and one diver to the right. How many ways can the divers be arranged in the photograph if the boys and girls are to alternate in the back row?

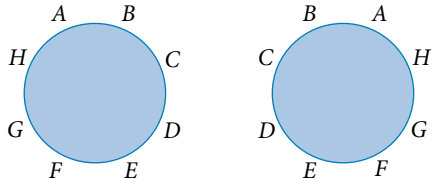
24 A group of four adults, three girls and two boys are to be seated on one side of a long table. How many seating arrangements are possible if the adults are seated together but the boys must not be seated together?

25 Suppose that 8 individuals are to be seated around a circular table.

(a) How many different seating arrangements are possible if two arrangements are considered to be the same if every individual has the same person on their right and the same person on their left? So, for example, the arrangements shown on the right would be considered the same. Ensure you clearly explain your reasoning.



(b) How many different seating arrangements are possible if two arrangements are considered to be the same if every individual is seated next to the same two people? So, for example, the arrangements shown on the right would be considered the same. Note that these arrangements would be considered different in the part **(a)** scenario since, for example, F is to the left of E in the first arrangement, but to the right of E in the second. Ensure you clearly explain your reasoning.



Combinations

1.4

Consider the problem of determining the number of *unordered* selections of three letters from the letters A, B, C, D and E. An unordered selection is known as a *combination*.

A combination is an *unordered* selection of objects.

One way to approach the problem described would be to make a list of all ordered arrangements of three letters from A, B, C, D and E, (the number of these is ${}^5P_3 = 60$), but group these arrangements in such a way that all arrangements within a particular group correspond to the same combination of 3 letters. This has been done below.

Below is the list of all possible ordered arrangements of 3 letters from the letters A, B, C, D and E.

Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7	Group 8	Group 9	Group 10
ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	ECB	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	EBC	EDB	EDC

Since each group contains all possible arrangements of three letters, the number of arrangements in each group is $3! = 6$.

Further, since each group corresponds to a single combination, the number of combinations of 3 letters from the letters A, B, C, D and E is equal to the total number of ordered arrangements, divided by 3!. That is, the number of combinations is equal to $\frac{{}^5P_3}{3!} = \frac{60}{6} = 10$.

The notation ${}^nC_r$ is used to denote the number of combinations of r objects from a collection of n objects; in other words, the number of ways to select or choose r objects from a collection of n objects without arranging them in any particular order.

The example described previously illustrates that ${}^5C_3 = \frac{{}^5P_3}{3!}$. In general, ${}^nC_r = \frac{{}^nP_r}{r!}$. Furthermore,

$$\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n!}{(n-r)!} \div r! \\ &= \frac{n!}{(n-r)!} \times \frac{1}{r!} \\ &= \frac{n!}{r!(n-r)!} \end{aligned}$$

The number of ways to select or choose r objects from a collection of n objects without arranging them in any particular order is denoted by ${}^nC_r$ or alternatively $\binom{n}{r}$.

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Explore further

Combinations

Use the spreadsheet function COMBIN(number, number_chosen) to determine combination formulas.

Making connections

Permutations vs combinations

This activity illustrates the difference between permutations and combinations.

16 Evaluating the combination formula and verifying the result

(a) Calculate ${}^6C_2$.

THINKING	WORKING
1 Substitute into the formula ${}^nC_r = \frac{n!}{r!(n-r)}$	${}^6C_2 = \frac{6!}{2!(6-2)!}$ $= \frac{6!}{2!4!}$
2 Simplify and calculate.	$= \frac{6 \times 5 \times \cancel{4!}}{2!\cancel{4!}}$ $= \frac{6 \times 5}{2}$ $= 15$
3 Interpret the answer.	There are 15 different ways to select 2 objects from a group of 6 objects.

(b) Evaluate the reasonableness of your solution to part **(a)** by determining all combinations of two letters from the letters A, B, C, D, E and F.

1 List all combinations systematically, remembering that order is not relevant (so BD is the same combination as DB for instance).	AB, AC, AD, AE, AF BC, BD, BE, BF CD, CE, CF DE, DF EF There are 15 different ways to select a group of two letters from six. This verifies the result from part **(a)**.

Problems involving counting unordered selections can involve restrictions similar to those encountered in previous sections.

Technology worked example

Calculating combinations

17 Combinations with restrictions

A netball team of 7 players is to be selected from a squad of 12 players. How many different teams are possible with the following restrictions?

(a) There are no restrictions.

THINKING	WORKING
Identify the total number of objects available, and the number to be selected.	The number of possible teams is ${}^{12}C_7 = 792$.

(b) The coach has already decided that a certain player is definitely in the team.

If a certain player is already in the team, there are now only 6 players to select from the remaining 11 players.	The number of possible teams is ${}^{11}C_6 = 462$.

(c) Two of the players play only the position of goal keeper and therefore will not be in a team together.

1 Recall the rule for complementary events where $n(A') = n(\xi) - n(A)$ Identify the total number of teams possible.	In total, there are 792 possible teams of 7 players selected from 12 players. $n(\text{both}') = 792 - n(\text{both})$
2 Determine $n(A)$.	If both players are on the team, then there are 10 players remaining to fill 5 positions. $n(\text{both}) = {}^{10}C_5$ $= 252$
3 Calculate $n(A')$ and interpret the result.	There are 792 possible ways to select a team of 7 players from 12, and 252 of those teams contain both goal keepers. $792 - 252 = 540$ different teams with either one or none of the goal keepers.

18 Applications of combinations

A mixed softball team of eight players is selected from a group of seven girls and five boys. What is the total number of possible teams that can be selected with the following restrictions?

(a) There are no restrictions.

THINKING	WORKING
Identify the total number of objects available, and the number to be selected.	The number of possible teams is ${}^{12}C_8 = 495$.

(b) The team must contain five girls and three boys.

1 Identify the number of ways of selecting the girls and the boys separately.	The number of ways to choose five girls from seven is ${}^7C_5 = 21$. The number of ways to choose three boys from five is ${}^5C_3 = 10$.
2 Use the multiplication principle.	Therefore, the total number of possible teams containing five girls and three boys: ${}^7C_5 \times {}^5C_3$ $= 21 \times 10$ $= 210$

(c) The team must contain five girls and three boys including a particular girl and a particular boy.

1 Identify the reduced sample size.	Remove one girl and one boy from the group and select players for the team from the remaining players.
2 Identify the number of ways of selecting the girls and the boys separately.	The number of ways to choose four girls from six is ${}^6C_4 = 15$. The number of ways to choose two boys from four is ${}^4C_2 = 6$.
3 Use the multiplication principle.	Therefore, the total number of possible teams with the given restriction: ${}^6C_4 \times {}^4C_2$ $= 15 \times 6$ $= 90$

EXERCISE 1.4 Combinations

Worked Example 16

1 **(a)** Calculate ${}^5C_2$ and verify your answer by listing all relevant combinations of the letters A, B, C, D and E.

(b) Calculate ${}^4C_3$ and verify your answer by listing all relevant combinations of the letters A, B, C and D.

(c) Calculate ${}^6C_1$ and verify your answer by listing all relevant combinations of the letters A, B, C, D, E and F.

(d) Calculate ${}^7C_7$ and verify your answer by listing all relevant combinations of the letters A, B, C, D, E, F and G.

2 Express each of the following using ${}^nC_r$ notation and calculate.

(a) The number of ways to choose three letters from A, B, C, D, E and F.

(b) The number of different teams of five players, where each is chosen from a squad of seven players.

3 Express each of the following using $\binom{n}{r}$ notation and calculate.

(a) The number of combinations of 3 objects from a collection of 10 objects.

(b) The number of ways to select 7 single-digit positive integers.

4 Write a simplified expression for each of the following.

(a) ${}^{n}C_{0}$ **(b)** ${}^{n}C_{1}$ **(c)** ${}^{n}C_{2}$

(d) ${}^{n}C_{3}$ **(e)** ${}^{n}C_{n-2}$ **(f)** ${}^{n+1}C_{n-1}$

5 Express each of the following as a fraction involving factorials.

(a) ${}^{n+1}C_{r+1}$ **(b)** ${}^{2\,n}C_{n}$ **(c)** ${}^{3n+1}C_{n+4}$

6 Calculate the following using technology.

(a) ${}^{13}C_{5}$ **(b)** ${}^{20}C_{10}$ **(c)** $\binom{22}{9}$ **(d)** $\binom{40}{9}$

Worked Example 17

7 A cross-country team of 5 runners is to be formed from a group of 11 runners. How many different teams are possible with the following restrictions?

(a) There are no restrictions.

(b) Two particular runners from the squad have already qualified for the team.

(c) Two particular runners cannot be on the team together.

8 A chess team of six players is to be chosen from ten students for an interstate competition. How many different chess teams are possible with the following restrictions?

(a) There are no restrictions.

(b) A particular student must be on the team.

(c) Three team members are from the same family, so either all go to the competition or none of them do.

9 A bookshelf contains 18 different books. A selection of 5 books is made such that one of two particular books (Book A or Book B) is chosen, but not both. In how many ways could such a selection be made?

10 **(a)** Which of the following is equal to ${}^{n}C_{0}$?

A 0 **B** 1 **C** n **D** $n-1$

(b) What misconception could lead to selecting ${}^{n}C_{0} = 0$?

11 **(a)** Which of the following is *not* equal to ${}^{n}C_{3}$?

A the number of ways to choose a captain, vice-captain and sport captain from a class of n students

B the number of subsets of three numbers chosen from the set $\{0, 1, 2, \ldots, n-1\}$

C the number of three-letter combinations of a set of n letters

D the number of ways to choose three school prefects from a class of n students

(b) Explain the common error that could lead to selecting the third incorrect alternative.

Worked Example

18

12 A mixed football team of nine players is selected from a group of five girls and seven boys. How many different teams can be selected with the following restrictions?

(a) There are no restrictions.

(b) The team contains four girls and five boys.

(c) The team consists of three girls and six boys where a particular girl and a particular boy must be on the team.

13 A mixed netball squad of 10 must be chosen from a group of 7 men and 8 women. How many squads can be chosen with the following restrictions?

(a) There are no restrictions.

(b) The squad contains 6 women.

(c) The squad contains the same number of men and women.

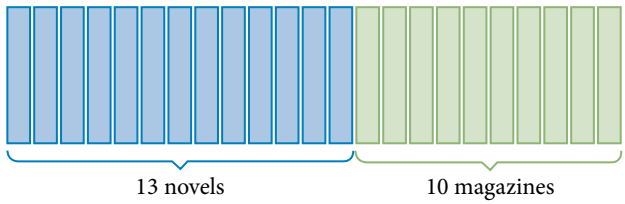
14 A school representative council with 5 members is to be chosen from a group of 10 students and 7 teachers. How many different formations of the council are possible with the following restrictions?

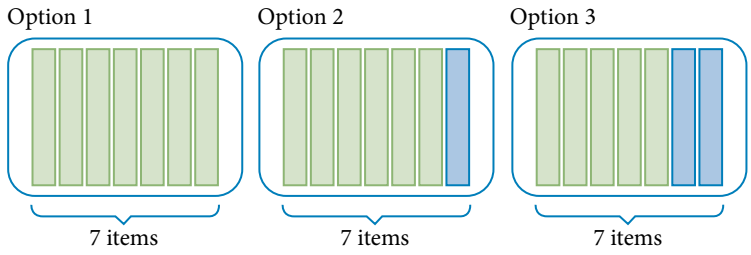
(a) There are no restrictions.

(b) The council must contain 3 students and 2 teachers.

15 A collection of dots is drawn in a circular configuration and straight lines are drawn between every possible pair of dots. If the number of triangles in the drawing is exactly five times the number of lines, how many dots are there?

16 A book display in the school library is to made using novels and magazines. There are thirteen novels and ten magazines, which can be used in the display. If only seven items can be included in the display, how many ways can they be chosen if the display must include at least five magazines?





17 A soccer team of 11 players, one of whom is chosen as the goalkeeper, is to be chosen from a squad of 15 players. Assuming that two possible teams consisting of the same 11 players are considered different if a different player is the goalkeeper in each, determine the number of possible teams that could be chosen if Amy and Brett refuse to be on the team together.

18 A committee of 10 people is to be formed from a group of 11 women and a certain number of men. If the total number of possible committees containing at most 2 men is 11 561, determine the number of men in the initial group. Evaluate the reasonableness of your solution.

1.5 Applications to probability

If all possible outcomes in a situation are equally likely, then the probability of an event, $P(\text{event})$, can be obtained by dividing the number of desirable outcomes (that is, the number of outcomes in which the event occurs) by the total number of outcomes.

$$P(\text{event}) = \frac{\text{number of desirable outcomes}}{\text{total number of outcomes}}$$

The counting techniques discussed in previous sections are often useful for determining both the total number of outcomes and the number of desirable outcomes in many situations.

19 Calculating probabilities

A bag contains 10 marbles numbered 1 to 10. Four marbles are drawn from the box simultaneously. Determine the probability that the four marbles drawn all had numbers less than 6.

THINKING	WORKING
1 First, determine the total number of possible outcomes; that is, the number of ways four marbles can be selected.	The total number of ways that four marbles can be drawn from the box is ${}^{10}C_4 = 210$.
2 Calculate the number of desirable outcomes; that is, the number satisfying the given restriction.	Five of the marbles have numbers less than 6. Hence, the number of selections of four marbles from the 5 marbles is ${}^{5}C_4 = 5$.
3 Calculate the required probability by dividing the number of successful outcomes by the total number of outcomes.	The required probability is $\frac{5}{210} = \frac{1}{42}$.

Games involving standard playing cards are a common context in which probability-related questions arise. If you are not familiar with the makeup of a standard deck of playing cards, the diagram below illustrates this.



Note how there are 52 cards in total, arranged into 13 'denominations' and 4 'suits'.

Typically, situations involving cards involve combinations since there is nothing important about the order in which the cards are dealt. Only the identity of the cards matters.

Note that the rule for complementary events for probability may be useful.

For example, suppose A represents the event that it rains tomorrow. If the probability of it not raining tomorrow is $P(A') = 0.2$, then the probability of it raining tomorrow can be shown as $P(A) = 1 - 0.2 = 0.8$.

The rule for complementary events is used in the second part of the following example.

If A represents some event, and A' represents the complement of this event (that is, the event that A does not occur), then $P(A) = 1 - P(A')$. This is the rule for complementary events.

20 Calculating probabilities involving cards

Three cards are dealt from a standard deck. Calculate the following probabilities.

(a) Two of the cards are hearts and the remaining card is a club.

THINKING	WORKING
1 Determine the number of possible outcomes.	There are ${}^{52}C_3 = 22100$ different ways to select 3 cards from a pack of 52.
2 Use the multiplication principle to determine the number of desirable outcomes.	There are ${}^{13}C_2 = 78$ different ways to select 2 hearts from 13. There are ${}^{13}C_1 = 13$ different ways to select 1 club from 13. ${}^{13}C_2 \times {}^{13}C_1 = 78 \times 13$ $= 1014$ There are 1014 different ways to select 3 cards where 2 of them are hearts and 1 is a club.
3 Calculate the probability of obtaining the desired outcome by dividing the number of desired outcomes by the total number of outcomes.	$P(2\heartsuit + 1\clubsuit) = \frac{1014}{22100}$ $= \frac{39}{850}$

(b) At least one of the cards is an ace.

1 Identify known information.	In total, there are 22 100 different ways to select 3 cards from a pack of 52.
2 'At least one' can be determined by calculating the probability of obtaining 'none' and subtracting this from 1. Recall the rule for complementary events: $n(A) = n(\xi) - n(A')$ Calculate the number of arrangements.	In order to select no aces, there are $52 - 4 = 48$ possible cards that could be drawn. There are ${}^{48}C_3 = 17296$ different ways to select 3 cards and be guaranteed that none of the hands contains an ace and, therefore, $22100 - 17296 = 4804$ ways to select a hand with one or more aces.

3 Calculate the probability of obtaining the desired outcome by dividing the number of desired outcomes by the total number of outcomes.

$$P(\geq 1 \text{ aces}) = \frac{4804}{22100} = \frac{1201}{5525}$$

It can be difficult to determine when to use the combination formula, and when to use the permutation formula.

The combination formula is used in situations involving a selection of objects in which there is no order to the objects, and no special status given to any of the chosen objects. For example, when choosing a committee of three members from a group of five candidates, only the identity of the chosen members is relevant.

The permutation formula is used in situations in which there is a specific order to the objects chosen, or there is special status given to each chosen object. For example, when choosing a president, secretary and treasurer from a group of five candidates, it's not just the identity of the chosen members that is relevant here, but also the roles assigned to them.

EXERCISE 1.5 Applications to probability

Worked Example 19

1 A bag contains 12 marbles numbered 1 to 12. Three marbles are drawn from the box simultaneously. Calculate the probability that the three marbles drawn all had numbers greater than or equal to 7.

2 Seven individuals chosen at random from a group of ten individuals are seated at random along one side of a long table. Calculate the probability that a particular individual is chosen to sit at the table, and is placed in the middle seat.

3 A committee of three members is chosen from a group of six men and four women. If the members are chosen at random, calculate the probability that all members are women.

4 If the letters A, B, C, D, E and F are randomly arranged, calculate the probability that the letters A and B will be next to each other.

5 Four girls and three boys are seated randomly in a row. Calculate the following probabilities.

(a) The girls occupy the first three places.

(b) They are arranged to alternate girl, boy, girl, etc.

6 A team of 8 players is to be chosen from a squad of 12 players. If the members of the team were chosen at random, calculate the probability that three friends would either all be chosen or all not be chosen.

7 The letters EEEHRRVW are mixed up and re-arranged in a random fashion. What is the probability that the resulting arrangement will spell 'WHEREVER'?

8 Four cards are randomly chosen from a standard deck. What is the probability that exactly two of the cards are red?

A $\frac{{}^{26}C_2}{{}^{52}C_4}$ **B** $\frac{{}^{26}P_2}{{}^{52}P_4}$ **C** $\frac{\left({}^{26}P_2\right)^2}{{}^{52}P_4}$ **D** $\frac{\left({}^{26}C_2\right)^2}{{}^{52}C_4}$

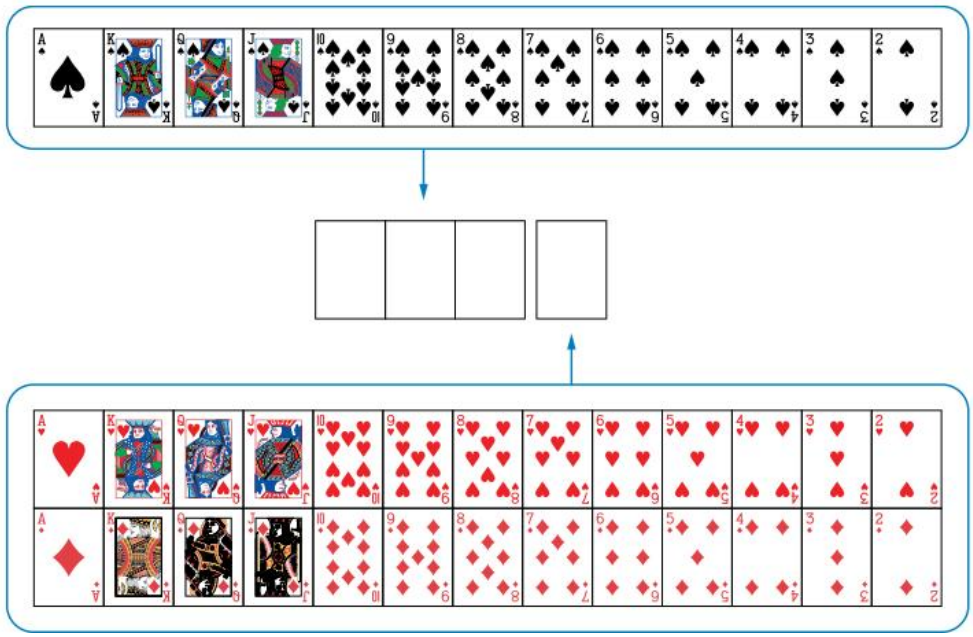
9 If license plates consist of three letters followed by three numbers, what is the probability that a randomly selected license plate contains no repeated letters or numbers?

Worked Example 20

A $\frac{{}^{26}C_3 \times {}^{10}C_3}{26^3 \times 10^3}$ **B** $\frac{{}^{26}P_3 \times {}^{10}P_3}{26^3 \times 10^3}$ **C** $\frac{{}^{26}C_3 \times {}^{10}C_3}{26 \times 25 \times 24 \times 10 \times 9 \times 8}$ **D** $\frac{{}^{36}P_6}{36^6}$

10 Four cards are dealt from a standard deck. Calculate the following probabilities.

(a) Three of the cards are spades and the remaining card is red.



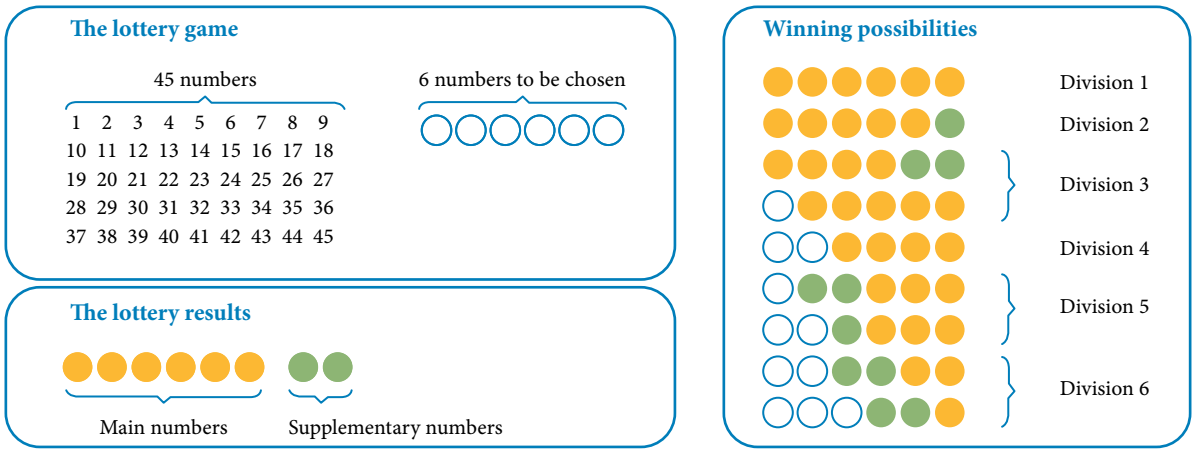
(b) At least one of the cards is either a jack, queen or king.

11 In the card game of Poker, each player is dealt a hand of five cards. A player is said to have a 'four of a kind' hand if their hand contains four cards of the same denomination. Calculate the probability of obtaining a 'four of a kind' hand. Express your answer correct to 7 decimal places. Evaluate the reasonableness of your solution.

12 The digits 3, 4, 5, 6 and 7 are randomly arranged in a row, and the number formed is noted. Calculate the probability that the number is greater than 75 000.

13 A coin is tossed ten times. Calculate the probability of obtaining exactly three tails.

14 To enter in a lottery game, a player makes a selection of six numbers from the numbers 1, 2, …, 45. Initially, six numbers (called the 'main' numbers) are randomly chosen, followed by two extra numbers (called the 'supplementary' numbers). A player wins the first division prize if they chose the exact six main numbers that are drawn. A player wins the second division prize if they chose five of the main numbers and one of the supplementary numbers.



Calculate the following probabilities.

(a) A player wins the first division prize.

(b) A player wins the second division prize.

15 Given a class of 20 students, calculate the probability that there is at least one pair of students who share a birthday. Specify any assumptions made.

16 Recall that in the card game of Poker, each player is dealt a hand of five cards. A player is said to have a 'three of a kind' hand if their hand consists of three cards of the same denomination, and two cards of denominations different from the aforementioned three cards, and different from each other. Calculate the probability of obtaining a 'three of a kind' hand. Express your answer correct to 5 decimal places.

17 Consider the numbers 1, 2, 3, …, n. All combinations of five of these numbers are written, and one combination is chosen at random. If the probability that this chosen combination does not contain the number 7 is 0.875, determine the value of n.

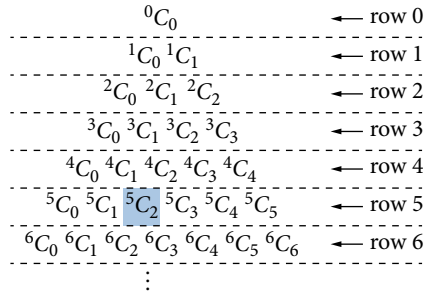
1.6 Pascal's triangle

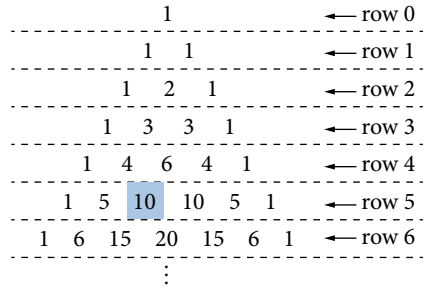
Consider infinitely many rows of ${}^nC_r$ expressions following the pattern pictured to the right.

Note that if you start numbering the rows at 0, and think of the very first expression from the left in a particular row having position 0, the next having position 1, and so on, then the expression in row n, position r is equal to ${}^nC_r$. For example, the expression in row 5, position 2 (highlighted) is ${}^5C_2$.

When the expressions in this diagram are calculated, the triangle of values to the right is obtained. This is known as Pascal's triangle, named after the famous French mathematician Blaise Pascal (1623–62).

Remember that in Pascal's triangle, the number in row n, position r (where numbering starts from 0) is equal to ${}^nC_r$.





The first and last number in each row of Pascal's triangle are both 1. All other numbers are equal to the sum of the two numbers directly above.

The fact that the first few rows of Pascal's triangle can be constructed so readily makes it a useful way to determine the value of ${}^nC_r$ for small values of n and r, without the need for the formula or technology.

Making connections

Pascal's triangle from the top down

See a simple animation that shows the construction of Pascal's triangle from the top down, one entry at a time.

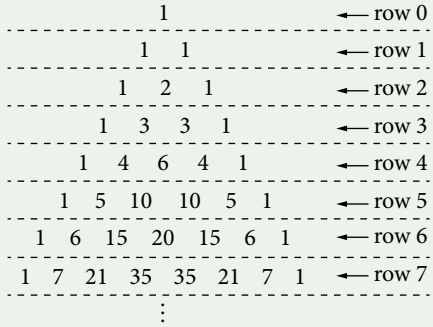
21 Using Pascal's triangle to calculate ${}^nC_r$

Draw Pascal's triangle up to row 7 and use it to write down the values of both ${}^5C_2$ and ${}^7C_5$.

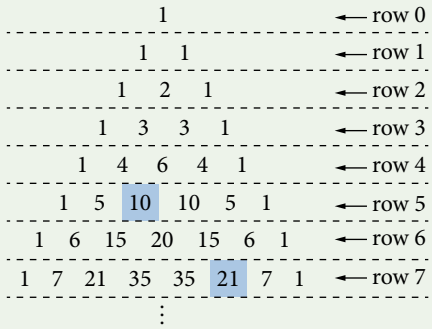
THINKING

1 Draw Pascal's Triangle up to row 7. Start with row 0 (which is just the number 1). For every subsequent row, the first and last numbers are 1, while all other numbers are obtained by adding the two numbers directly above.

WORKING



2 Determine the appropriate values in the triangle. Remember that ${}^nC_r$ is equal to the number in row n, position r. In this case, you want the values of ${}^5C_2$ and ${}^7C_5$.



3 Determine the values of ${}^5C_2$ and ${}^7C_5$.

${}^5C_2 = 10$, ${}^7C_5 = 21$

Conversely, the fact that ${}^nC_r$ can be calculated readily with technology enables you to obtain the value of any entry in Pascal's triangle, without actually having to draw the triangle of values.

Pascal's formula

In order to explain why any value in Pascal's triangle is equal to the sum of the numbers directly above, consider a particular case: In row 5, entry 2 is the number 10. This entry corresponds to the expression ${}^5C_2$.

It is equal to the sum of the 4 and the 6 in the row above. These numbers correspond to the expressions ${}^4C_1$ and ${}^4C_2$. A counting argument can be used to explain why it should be expected that ${}^5C_2 = {}^4C_1 + {}^4C_2$.

Consider the problem of determining the number of possible 2-person selections from a group of 5 individuals. There are ${}^5C_2 = 10$ possible ways this can occur and it can be demonstrated by selecting groups of 2 from the 5 objects A, B, C, D and E.

AB	BC	CD	DE
AC	BD	CE	
AD	BE		
AE			

However, another way to determine the total number of selections would be to select a particular individual from the group of 5, for example, A, and determine the number of selections containing this individual and 1 partnering object from B, C, D and E. ${}^4C_1 = 4$ represents the 4 selections including A, that is, AB, AC, AD and AE.

Then determine the number of selections that do not include the individual A (i.e. select pairs from B, C, D and E). ${}^4C_2 = 6$ represents the 6 selections that do not include A, that is BC, BD, BE, CD, CE and DE.

Thus ${}^5C_2 = {}^4C_1 + {}^4C_2$.

This reasoning works in general, leading to the result, known as Pascal's formula.

Pascal's formula: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

The following example illustrates how Pascal's formula can be verified algebraically for a given value of r.

22 Prove Pascal's rule for a given value of r

Prove algebraically that Pascal's formula holds when $r = 2$.

THINKING	WORKING
1 Determine the result to be proved by substituting $r = 2$ into Pascal's formula.	To prove: ${}^{n}C_2 = {}^{n-1}C_1 + {}^{n-1}C_2$
2 Demonstrate that the left-hand side is equal to the right-hand side.	LHS $= {}^{n}C_2$ $= \frac{n!}{2!(n-2)!}$ $= \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}$ $= \frac{n(n-1)}{2}$
	RHS $= {}^{n-1}C_1 + {}^{n-1}C_2$ $= \frac{(n-1)!}{1!(n-2)!} + \frac{(n-1)!}{2!(n-3)!}$ $= \frac{(n-1)\cancel{(n-2)!}}{1!\cancel{(n-2)!}} + \frac{(n-1)(n-2)\cancel{(n-3)!}}{2!\cancel{(n-3)!}}$ $= \frac{n-1}{1} + \frac{(n-1)(n-2)}{2}$ $= \frac{2(n-1)}{2} + \frac{(n-1)(n-2)}{2}$ $= \frac{(n-1)(2+(n-2))}{2}$ $= \frac{(n-1)(n)}{2}$ $= \frac{n(n-1)}{2}$ $=$ LHS
3 Interpret the result.	Given that ${}^{n}C_2 = {}^{n}C_1 + {}^{n-1}C_2$, then Pascal's formula ${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ holds true for $r = 2$.

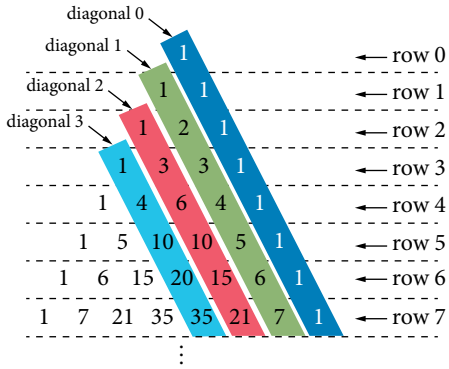
Further interesting properties of Pascal's triangle

There are many interesting patterns and properties of Pascal's triangle, besides Pascal's formula. For instance, the diagram to the right shows that for the first few rows, the sum of the entries is equal to 2 raised to the power of the row number. This result is true in general.

							Total
			1				$1\ (= 2^0)$
		1		1			$2\ (= 2^1)$
	1		2		1		$4\ (= 2^2)$
	1	3		3		1	$8\ (= 2^3)$
1		4	6		4	1	$16\ (= 2^4)$
1	5	10		10	5	1	$32\ (= 2^5)$
1	6	15	20		15	6	1 $64\ (= 2^6)$
1	7	21	35	35	21	7	1 $128\ (= 2^7)$
			⋮				

The sum of the numbers in row n of Pascal's triangle is equal to 2^n.

Many of the interesting properties of Pascal's triangle involve the diagonals. The diagonals are numbered as indicated in the diagram to the right.

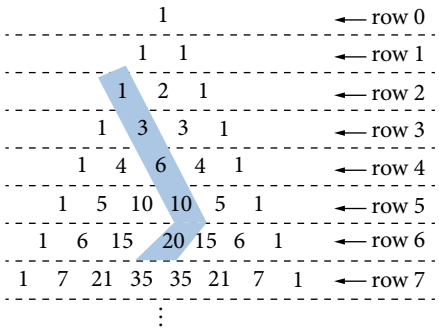


Explore further

Using a spreadsheet to generate a Pascal's triangle

Generate Pascal's triangle by copying a simple formula.

In the diagram of Pascal's triangle below, a collection of values in the shape of a hockey stick has been highlighted. Notice that the single number after the 'turn' of the hockey stick (20) is equal to the sum of the numbers before the turn (1 + 3 + 6 + 10). This pattern always holds and is known as the hockey-stick property.



The hockey-stick property: If you add the numbers going down any diagonal, starting with 1 and stopping at any point, then the sum will be equal to the number in the next row, slightly to the left of the last number added.

The hockey-stick property can be used to establish formulas for sums of numbers that follow particular patterns, as the following example illustrates.

23 Establishing number formulas from Pascal's Triangle

Notice that the positive integers 1, 2, 3, … are given by diagonal 1 of Pascal's triangle.

(a) Determine the row and position of each of the first 5 positive integers in this diagonal.

THINKING	WORKING
Observe the patterns in the triangle.	The number 1 is in row 1, position 0. The number 2 is in row 2, position 1. The number 3 is in row 3, position 2. The number 4 is in row 4, position 3. The number 5 is in row 5, position 4.

(b) Use the hockey-stick property of Pascal's triangle to determine the row and position of the number that would give the sum of the first 5 positive integers.

Using the hockey-stick property, the sum of the numbers described is equal to the number in the next row (row 6), in the position slightly to the left of the last number added. The last number added was in position 4. In the next row, the number just to the left will also be position 4 (since this row starts slightly to the left of the previous row).

The sum of the values described previously is the entry in row 6, position 4.

(c) Determine a simplified formula for the sum first n positive integers.

1 It's already established that the sum of the first 5 integers is equal to the entry in row 6, position 4.

Note: The row number is one more than 5, while the position number is one less. Use this to generalise the result to the sum of the first n positive integers.

Extending the result established to the sum of the first n positive integers, this sum must be equal to the entry in row $n+1$, position $n-1$.

2 Determine a simplified formula for this value.

This entry is equal to

$$
\begin{aligned}
{}^{n+1}C_{n-1} &= \frac{(n+1)!}{(n-1)!2!} \\
&= \frac{(n+1)n\,\cancel{(n-1)!}}{\cancel{(n-1)!}\,2!} \\
&= \frac{(n+1)n}{2}
\end{aligned}
$$

EXERCISE 1.6 Pascal's triangle

Worked Example 21

1 Draw rows 0 to 7 of Pascal's triangle and use them to determine the values of the following.

(a) ${}^5C_2$ **(b)** ${}^3C_0$ **(c)** ${}^7C_4$ **(d)** ${}^3C_1$

2 Determine the values of the following numbers in Pascal's triangle.

(a) row 18, position 0 **(b)** row 10, position 3 **(c)** row 9, position 7

3 Given that row 15 of Pascal's triangle contains all odd numbers, determine the number of odd numbers in row 16. Explain your reasoning.

4 Recall that the sum of the numbers in row n of Pascal's triangle is 2^n. Use this result to determine ${}^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + \ldots + {}^{15}C_{15}$.

5 Recall that the sum of the first n positive integers is equal to the entry in row $n+1$, position $n-1$ of Pascal's triangle. Use this result to calculate the sum of the first 240 positive integers.

Worked Example 22

6 Recall Pascal's formula: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$. Prove algebraically that this formula holds when $r = 3$.

7 The third smallest integer in a particular row of Pascal's triangle is 120. What is the row number?

A 17 **B** 16 **C** 10 **D** 11

8 The middle number in a particular row of Pascal's triangle is 48 620. What is the row number?

A 18 **B** 19 **C** 8 **D** 9

9 Notice that the square of any number in diagonal 1 of Pascal's triangle is the sum of the number immediately to its left and the number in the row below slightly to the left. For example, the square of the number in row 5, position 4 (25) is equal to the sum of the numbers in row 5, position 3 (10), and row 6, position 4 (15). Prove that this property must be true in general.

23

10 If a collection of dots is arranged into the shape of an equilateral triangle, as shown below, then the number of dots used is known as a triangular number. The first few triangular numbers are 1, 3, 6, 10, … Notice that the triangular numbers are given by diagonal 2 of Pascal's triangle.



(a) Determine the row and position in Pascal's triangle of the following.

(i) 1st triangular number **(ii)** 2nd triangular number

(iii) 3rd triangular number **(iv)** 4th triangular number

(v) 5th triangular number **(vi)** 6th triangular number

(b) Use the hockey-stick property of Pascal's triangle to determine the row and position of the number that would give the sum of the first 6 triangular numbers.

(c) Determine a simplified formula for the first n triangular numbers.

11 The numbers that form diagonal 3 of Pascal's triangle as known as the tetrahedral numbers. Use Pascal's triangle to determine a simplified formula for the sum of the first n tetrahedral numbers.

12 Prove that the sum of the numbers in row $n+1$ of Pascal's triangle is exactly double the sum of the numbers in row n. Hint: Start by letting the numbers in row n of Pascal's triangle be, in order, 1, a_1, a_2, a_3, …, a_{n-1}, 1.

13 Prove Pascal's formula algebraically: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

1.7 The Pigeon-hole principle

The ceiling of a number

If x is a real number, then the notation $\lceil x \rceil$, is used to denote the smallest integer greater than or equal to x; this is often referred to as the ceiling of x. For example, $\lceil 5.3 \rceil = 6$ since 6 is the smallest integer greater than or equal to 5.3. It's clear that to determine the ceiling of a number that is not an integer, you simply round up to the next integer. The ceiling of an integer is simply itself. For example, $\lceil 5 \rceil = 5$ since 5 is the smallest integer that is greater than or equal to 5.

The notation $\lceil x \rceil$ represents the ceiling of x; that is, the smallest integer greater than or equal to x.

24 Evaluating the ceiling function

Determine the value of each of the following.

(a) $\lceil 7.23 \rceil$

THINKING	WORKING
$\lceil x \rceil$ represents the smallest integer greater than or equal to x.	$\lceil 7.23 \rceil = 8$

(b) $\lceil 3 \rceil$

$\lceil x \rceil$ represents the smallest integer greater than or equal to x.	$\lceil 3 \rceil = 3$

(c) $\left\lceil \frac{21}{10} \right\rceil$

$\lceil x \rceil$ represents the smallest integer greater than or equal to x.	$\left\lceil \frac{21}{10} \right\rceil = \lceil 2.1 \rceil = 3$

The Pigeon-hole principle

Suppose that in a bag of 16 marbles, every marble is coloured red, green or blue, but you do not know the number of marbles of each colour in the bag. Also suppose that you are interested in knowing the greatest number of marbles that you can safely guarantee will have the same colour.

One way to think about this would be to imagine a red box, a green box and a blue box, and imagine distributing the 16 marbles in these boxes according to their colour.

Notice that if you divide the number of marbles by the number of boxes, you obtain $\frac{16}{3} = 5.\dot{3}$. This number essentially tells you the number of marbles that would be in each box if each box contained the

same number of marbles. The fact that this number is slightly greater than 5 means that you couldn't possibly end up with each box containing 5 marbles. At least one of the boxes must contain at least 6 marbles, as shown to the right.



Thus, you can safely guarantee that at least $\left\lceil \frac{16}{3} \right\rceil = \lceil 5.\dot{3} \rceil = 6$ marbles will have the same colour.

In general, if n objects are to be placed in k boxes, then it is certain that at least $\left\lceil \frac{n}{k} \right\rceil$ objects. This result is known as the Pigeon-hole principle. (The name arises from imagining the boxes as pigeon-holes, and the objects as pigeons that fill the pigeon-holes.)

The Pigeon-hole principle: if n objects are to be placed in k boxes, then at least one box will contain at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

This principle was first used by Dirichlet (1805–59). Although it is extremely simple, it can be used in many situations to establish results that otherwise are extremely difficult to prove. For example, it can be used to prove that if 5 points are placed on a sphere, there is a hemisphere that will contain 4 of the points.

In order to solve problems using the Pigeon-hole principle, the challenge is to establish what the boxes are, and what the objects are. Generally, you want the boxes to represent some property of the objects that you are interested in, so that the objects may be assigned to them based on this property (such as marble colour in the example described previously).

In what follows, five examples are presented illustrating the most common scenarios in which the Pigeon-hole principle may be applied.

In the first scenario, the number of objects is one more than the number of boxes. This is the simplest case of applying the Pigeon-hole principle, and one that is intuitively almost obvious: for example, if 13 objects are to be placed in 12 boxes, it is clear that at least $\left\lceil \frac{13}{12} \right\rceil = 2$ objects.

25 The Pigeon-hole principle: the simplest case

Six netball players were given a number of attempts to score a goal in a training session. Each player scored either 5, 6, 7, 8 or 9 goals. Prove that at least two students scored the same number of goals.

THINKING	WORKING
1 Identify boxes and objects.	Think of the 5 possible numbers of goals scored (4, 5, 6, 7, 8) as the boxes, and the 6 players as the objects to be assigned to boxes based on their number of goals scored.
2 Use the Pigeon-hole principle to prove the result.	Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{6}{5} \right\rceil = 2$ objects. This means there must be at least 2 players who scored the same number of goals.

26 The Pigeon-hole principle: a standard problem

There are 25 students in a class. Each student achieved an A, B or C on a test. Prove that there are at least 9 students who received the same grade.

THINKING	WORKING
1 Identify boxes and objects	Think of the 3 letter grades as boxes, and the 25 students as objects to be assigned to these boxes based on their grade.
2 Use the Pigeon-hole principle to prove the result.	Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{25}{3} \right\rceil = 9$ objects. This means there must be at least 9 students who received the same grade.

One of the most powerful ways to implement the Pigeon-hole principle is break up a set of objects into groups, such that all objects within the same group share an important property. These groups then can be thought of as the boxes.

27 The Pigeon-hole principle: breaking up the objects into groups

Twenty-six different integers are chosen from the integer set {1, 2, 3, 4, ..., 49, 50}. Prove that at least two integers must be consecutive.

THINKING	WORKING
1 Identify boxes and objects. The goal is to prove at least two integers are consecutive, so one idea is to break up the 50 integers into groups of consecutive integers.	The positive integers from 1 to 50 can be broken up into the following 25 groups: [1, 2], [3, 4], [5, 6],…, [49, 50]. (Notice that each group contains a pair of consecutive integers.) Think of these 25 groups as boxes, and the 26 chosen integers as objects assigned to boxes based on their value (for example, if the number 5 was chosen, it would be placed in the third box listed since it is the one whose label includes 5).
2 Use the Pigeon-hole principle to prove the result.	Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{26}{25} \right\rceil = 2$ objects. Since each box contains a pair of consecutive integers, the result follows that at least two integers must be consecutive.

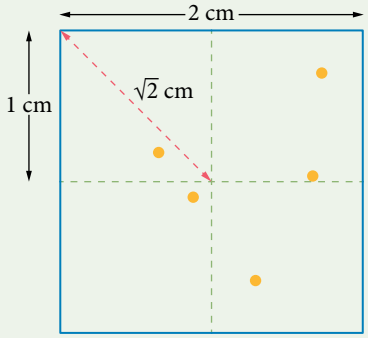
28 The Pigeon-hole principle: a divisibility context

Prove that for six different integers it is possible to determine two whose difference is divisible by 5.

THINKING	WORKING
1 Note that the difference between two numbers is divisible by 5 if the two numbers have the same remainder when divided by 5. (For example, 7 and 22 both have a remainder of 2 when divided by 5; $7 = 5 \times 1 + 2$ and $22 = 5 \times 4 + 2$. When determining the difference between these two numbers, the remainders cancel and the result is a multiple of 5.	The difference between two numbers is divisible by 5 if the two numbers have the same remainder when divided by 5.
2 Identify boxes and objects.	Think of the possible remainders as boxes (there are 5 of them: 0, 1, 2, 3, 4), and the six chosen integers as objects to be assigned to the boxes based on their remainders upon division by 5.
3 Use the Pigeon-hole principle to prove the result.	Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{6}{5} \right\rceil = 2$ objects. This means that at least two of the integers have the same remainder upon division by 5, and hence their difference must be divisible by 5.

29 The Pigeon-hole principle: a geometric context

Five points are placed inside a square of side length 2 cm. Prove that at least two points are within $\sqrt{2}$ cm of each other.

THINKING	WORKING
1 Draw the square and identify the maximum distance points can be from one another. The key insight needed in such geometric questions is to think of a way to divide the square into smaller sections.	In this case, divide the square into four smaller squares of side length 1 cm. 

The maximum distance between points in the larger square, from corner to corner:

$l^2 = 2^2 + 2^2$

$= 8$

$\therefore l = \sqrt{8}$

$= \sqrt{4} \times \sqrt{2}$

$= 2\sqrt{2}$

Therefore the maximum distance within a smaller square (from corner to corner) is $\sqrt{2}$ cm.

2 Identify boxes and objects.

Think of the 4 smaller squares as boxes, and the dots as objects to be placed in the boxes.

3 Use the Pigeon-hole principle to prove the result.

Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{5}{4} \right\rceil = 2$ objects.
This means that at least two points are within the same small square.

Since the maximum distance between any two points inside (or on the edge of) a square of side length 1 cm is $\sqrt{2}$ cm (using Pythagoras' theorem), the result follows that least two points are within $\sqrt{2}$ of each other.

EXERCISE 1.7 The Pigeon-hole principle

Worked Example

24

1 Determine the value of each of the following.

(a) $\lceil 5.12 \rceil$ (b) $\lceil 7 \rceil$ (c) $\left\lceil \frac{6}{5} \right\rceil$

(d) $\left\lceil \frac{13}{4} \right\rceil$ (e) $\left\lceil \frac{42}{5} \right\rceil$

25

2 Eight students were given a number of hits at a hockey goal from the penalty spot. Each student scored either 0, 1, 2, 3, 4, 5 or 6 goals. Prove that at least two students scored the same number of goals.

3 Eleven telephone numbers are randomly chosen from a speed-dialling computer program. Prove that at least two telephone numbers will have the same last digit.

26

4 There are 25 students in a class. Each student achieved an A, B, C or D on a test. Prove that there are at least seven students who received the same grade.

5 There are 27 students in a room.

(a) Prove that there must be at least two students whose first names start with the same letter.

(b) Prove that there must be at least three students who were born in the same month.

(c) Prove that there must be at least four students born on the same day of the week.

6 In a class of 22 students, each student made no more than 9 mistakes on a test. Prove that at least three students made the same number of mistakes on the test.

7 What is the minimum number of two-digit numbers that would need to be selected to ensure that there are at least three that have the same first digit?

A 19 **B** 27 **C** 31 **D** 30

8 Let r be a positive integer such that whenever 50 pigeons are placed in r pigeon holes, it is always the case that at least one pigeon hole contains at least 3 pigeons. Which of the following is a possible value for r?

A 50 **B** 40 **C** 30 **D** 20

Worked Example 27

9 Six positive, even integers less than 21 are chosen. Prove that at least one pair will have a sum of 22.

Worked Example 28

10 Consider four different integers. Prove that for four different integers, it is possible to determine two of them whose difference is divisible by three.

11 Prove that if you have 100 whole numbers, you can choose 15 of them so that the difference of any two is divisible by 7.

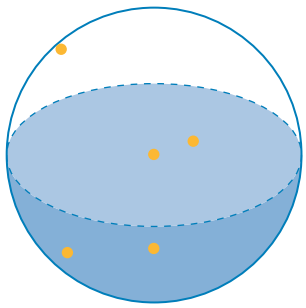
Worked Example 29

12 Four points are placed into a rectangle with side lengths 3 cm and 1 cm. If no point can coincide with any other point, prove that at least two points are within $\sqrt{2}$ cm of each other.

13 Thirteen points are placed inside a regular hexagon of side length 1 cm. Prove that at least three points are all within 1 cm of each other. Hint: Think about how a regular hexagon may be broken up.

14 If 168 three-digit numbers are selected, prove that it is possible to determine that eight of them have the same digit sum.

15 Suppose that 5 points lie on a sphere. Prove that there exists a hemisphere (half a sphere including boundary) that contains 4 of the points.



16 **(a)** Prove that if one selects any 11 integers from a set of 20 consecutive integers, then at least two of the chosen integers must differ by 10.

(b) Use the result from part **(a)**, together with the Pigeon-hole principle, to prove that however you select 51 integers of the integers from 1 to 100, there will be at least two that differ by 10.

1 Summary

The inclusion–exclusion principle

The inclusion–exclusion principle for two sets states that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

For three sets, it states:
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

The multiplication principle

Suppose a task can be broken up into a certain number of steps in which the number of ways each step can be performed is not dependent on the choices made in previous steps. Then the number of ways to perform the entire task is equal to the product of the number of ways each step can be performed.

The addition principle

Suppose a collection of objects can be classified into a number of different types, such that every object belongs to exactly one type. Then the total number of objects is equal to the sum of the number of objects of each type.

Factorial notation

If n is a positive integer, then $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

The value of $0!$ is defined to be 1.

Permutations

A permutation is an **ordered** arrangement of objects.

The number of ways to arrange n objects in a row is equal to $n!$.

The notation ${}^{n}P_{r}$ represents the number of ways to select r objects from a collection of n objects and arrange them in order.

$${}^{n}P_{r} = \underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}_{r \text{ terms}} = \frac{n!}{(n-r)!}$$

If, in a collection of n objects, there is one or more type of object that contains several indistinguishable copies, and the numbers of copies of each of these are a, b, c, d, and so on, then the number of arrangements of the objects is equal to $\frac{n!}{a!b!c!d!\cdots}$.

Combinations

A combination is an **unordered** selection of objects.

The number of ways to select or choose r objects from a collection of n objects without arranging them in any particular order is denoted by ${}^{n}C_{r}$ or alternatively $\binom{n}{r}$.

$${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!(n-r)!}$$

Pascal's triangle

Pascal's triangle is an infinite list of numbers arranged in rows in a triangular pattern. The number in row n, position r (where numbering starts from 0) is equal to ${}^{n}C_{r}$. The first and last number in each row of Pascal's triangle are both 1. All other numbers are equal to the sum of the two numbers directly above.

Two useful properties of Pascal's triangle are as follows:

- The sum of the numbers in row n of Pascal's triangle is equal to 2^{n}.
- If one adds the numbers going down any diagonal, starting with 1, and stopping at any point, then the sum will be equal to the number in the next row, slightly to the left of the last number added; this result is known as the hockey-stick property.

The Pigeon-hole principle

The notation $\lceil x \rceil$ represents the ceiling of x; that is, the smallest integer greater than or equal to x.

The Pigeon-hole principle states that if n objects are to be placed in k boxes, then at least one box will contain at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

Chapter review 1

Exercise 1.1

1 In a group of people, every one of whom plays at least one of tennis and cricket, 10 play tennis, 22 play cricket, and 5 play both. How many people are in the group?

Exercise 1.1

2 Consider the sets $A = \{1, 2, 3, 5\}$, $B = \{5, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$, and $\xi = \{1, 2, 3, ..., 11\}$. Verify the inclusion–exclusion principle for these sets.

Exercise 1.1

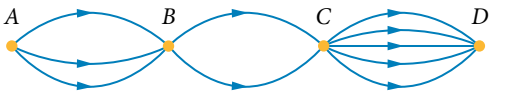
3 In a group of 30 students studying at least one of Accounting and Economics, 19 students study Accounting and 22 students study Economics. How many students study Accounting only?

A 8 **B** 9 **C** 10 **D** 11

Exercise 1.2

4 Examine the diagram and determine how many different ways there are to get from A to D.

A 10 **B** 20 **C** 30 **D** 40



Exercise 1.3

5 Calculate the following.

(a) 6! **(b)** $\frac{4!}{3!}$ **(c)** $\frac{10!}{8!}$

Exercise 1.3

6 Simplify the following expressions.

(a) $\frac{(n+1)!}{(n-1)!}$ **(b)** $\frac{3n!}{(n-2)!}$

Exercise 1.3

7 Express each of the following as a fraction involving factorials.

(a) ${}^{n}P_{n-5}$ **(b)** ${}^{n+r-1}P_{r-1}$

Exercise 1.4

8 Express each of the following using ${}^{n}C_{r}$ notation and calculate.

(a) the number of ways to choose four letters from A, B, C, D, E, F

(b) the number of different teams of 4 players, where each is chosen from a squad of 5 players

Exercise 1.4

9 Write a simplified expression for each of the following.

(a) ${}^{n}C_{n}$ **(b)** ${}^{n+1}C_{n-2}$

Exercise 1.5

10 If the letters A, B, C, D, and E are randomly arranged, calculate the probability that the letters A, B and C will not be all together in a group.

Exercise 1.6

11 Determine a simplified expression for the third number in diagonal n of Pascal's triangle.

Exercise 1.7

12 Forty-six students sat a test and received a mark out of 10; half marks were possible. Prove that there was a group of at least three students who received the same mark.

Exercise 1.2

13 License plates in a particular country consist of either 3 or 4 numbers, following by 2 letters from A to Z. Determine the number of different license plates.

14 In a group of 10 campers, someone needs to carry the tents, another person needs to carry the food, and yet another must carry the climbing equipment. How many ways can the campers be assigned to these roles under the following conditions? Exercise 1.2

(a) There are no restrictions.

(b) Mark and Sally are the navigators and therefore do not carry anything.

(c) Jaden demands that if he carries the tent equipment, then his friend Alex must carry the food.

15 How many ways can the letters in the word OECOLOGY be arranged to make a word under the following conditions? Exercise 1.3

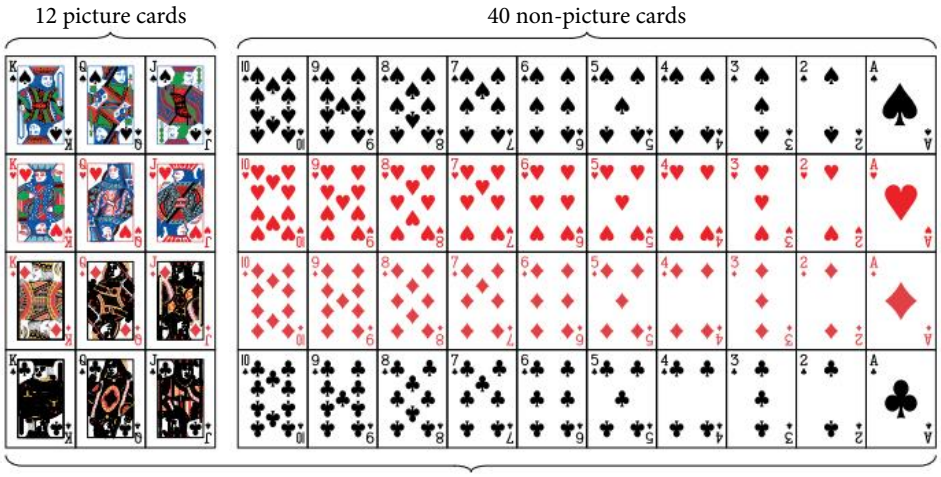
(a) There are no restrictions.

(b) The three Os are together in the word.

(c) The vowels and consonants alternate.

16 A bag contains 10 marbles numbered 1 to 10. Three marbles are drawn from the bag simultaneously. Determine the probability that the three marbles drawn all had numbers less than 5. Exercise 1.5

17 Three cards are selected from a standard deck. Calculate the probability that the three cards are all picture cards, that is, either jacks, queens or kings. Exercise 1.5



18 Determine the value in row 17, position 5 of Pascal's triangle. Exercise 1.6

19 Determine the value of ${}^{19}C_1 + {}^{19}C_2 + \ldots + {}^{19}C_{19}$ Exercise 1.6

20 There are 4 schools in a particular town where there are 813 school-age children. Prove that at least one of the schools contains at least 204 students. Exercise 1.7

21 Fifty people were asked what pets they kept: 20 had cats, 20 had dogs, and 17 had fish. But 4 had cats and dogs only, 3 had dogs and fish only, and 5 had cats and fish only. If 19 people had no pets, how many people had all three types of pet (cats, dogs and fish)? Exercise 1.1

Exercise 1.2 **22** How many positive integers between 70 and 5000 with no repeated digits can be formed where each digit is either 4, 5, 6, 7 or 8?

Exercise 1.6 **23** Recall Pascal's formula: ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$. Prove algebraically that this formula holds when $n = 5$.

Exercise 1.7 **24** An equilateral triangle of side length 2 cm is drawn. Prove that if 9 points are placed on the edge or inside of this triangle, at least 3 must all be within 1 cm of each other.

Exercise 1.4 **25** A sport team of 6 players is to be formed from a group of 10 players. How many different teams are possible if two particular players refuse to be on the team together?

Exercise 1.4 **26** A mixed basketball squad of twelve is to be chosen from a group of nine woman and eight men. How many different squads are possible under the following conditions?

(a) There are no restrictions.

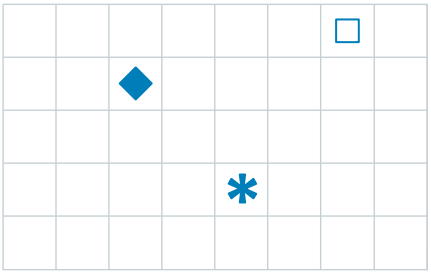
(b) The squad contains eight women and four men.

(c) The squad must contain at least eight women.

Exercise 1.5 **27** A team of 5 players is to be chosen from a squad of 10 players. If the team were chosen at random, calculate the probability that either Adam or Beth will be chosen, but not both.

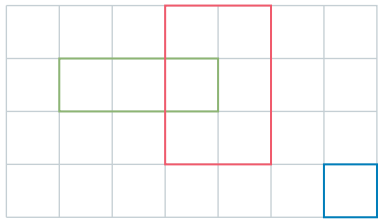
Exercise 1.1 **28** Let A, B and C be sets with $n(A) + n(B) + n(C) = 130$. The number of elements that belong to exactly two of these sets is double the number that belong to all three. The number of elements that belong to exactly one of these sets is triple the number that belong to exactly two. Determine $n(A \cap B \cap C)$.

Exercise 1.2 **29** Three different symbols are to be placed on a grid of squares consisting of 5 rows and 8 columns such that no two symbols appear in the same row or column. One possible way this can be done is shown below. In how many ways can this be done?



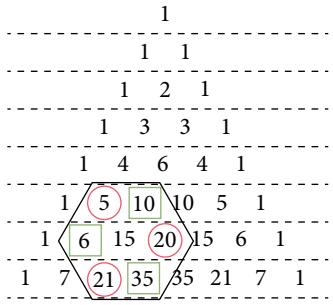
Exercise 1.3 **30** A passcode to enter a building requires people to enter a sequence of four different numbers, each from 1 to n, in a particular order. Jim has forgotten his passcode, but remembers that one of the numbers in his code is 13. He has calculated that there are 13 800 possible passcodes that contain the number 13 in them. Determine the value of n.

31 Determine the number of rectangles (including squares) that can be drawn in the grid below. Three possible shapes have been drawn already. Exercise 1.4



32 In the card game of Poker, each player is dealt a hand of five cards. A player is said to have a 'straight' hand if their hand contains five cards of sequential denomination (such as 3, 4, 5, 6, 7 or 9, 10, J, Q, K). If all the cards happen to be of the same suit, then the hand is called a 'straight-flush'. Calculate the probability of obtaining either a straight or a straight-flush in Poker. Assume for the purpose of this question that the ace represents the highest denomination, so for example, J, Q, K, A, 2 is not considered a straight; nor is A, 2, 3, 4, 5. Exercise 1.5

33 Choose any interior number from Pascal's triangle; its 6 surrounding numbers form a hexagon. Draw alternating squares and circles around these numbers (as shown below), and calculate the product of the numbers in the circles, and the product of the numbers in the squares, excluding the number in the centre. You should discover that both products are equal. Prove this will always happen, no matter which interior number you start with. Exercise 1.6



34 Consider the set of five-digit numbers where every digit is from 1 to 9, and no digit is repeated. Prove that if 379 numbers are selected, at least 4 will have the exact same digits. Exercise 1.7





2

Vectors in the plane

Recall 66
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2 Recall

Solve linear equations

1 Solve each of the following linear equations.

(a) $6x + 5 = 17$ (b) $-5t - 6 = -12$ (c) $3k - 2\sqrt{2} = 4\sqrt{2}$

(d) $5x - 2 = 7x + 12$ (e) $6(4 - 3p) = 9(4p + 3)$

Solve simultaneous equations

2 Use algebraic methods to solve the following equations simultaneously.

(a) $x = 8 - 3y$ and $2x - 5y = 5$ (b) $2x - 7y = -17$ and $3x + 2y = 12$

(c) $x - 2y = 10$ and $y = 5x + 4$ (d) $6x - 4y = 17$ and $5x - 3y = 13$

Simplify surds

3 Simplify the following surds by expressing in the form $a\sqrt{b}$, where a and b are positive integers.

(a) $\sqrt{28}$ (b) $2\sqrt{75}$

Calculate distance between two points on the Cartesian plane

4 Calculate the exact distance between the following pairs of coordinate points.

(a) $C(-2, 3)$ and $D(3, 5)$ (b) $E(1,-1)$ and $F(7, 7)$

(c) $G(-2, -4)$ and $H(2, 4)$ (d) $I(-3, 4)$ and $J(4, -5)$

Use technology to determine values of trigonometric ratios

5 Use technology to calculate the following trigonometric ratios, correct to 4 decimal places. Ensure that your calculator is set to degrees mode.

(a) $\sin(65°)$ (b) $\cos(156°)$

Use technology to determine angles when given trigonometric ratios

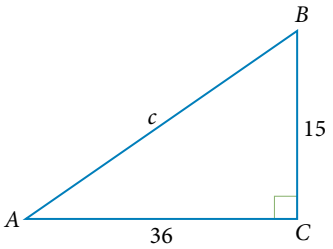
6 Use technology to calculate the size of angle θ, in degrees correct to 2 decimal places. Ensure that your calculator is set to degrees mode.

(a) $\sin(\theta) = 0.9012$ (b) $\cos(\theta) = -0.3426$

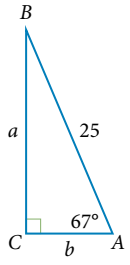
Solve right-angled triangles using trigonometry and Pythagoras' theorem

7 For right-angled triangles ΔABC, calculate the unknown side lengths and angles, correct to 2 decimal places where necessary.

(a)



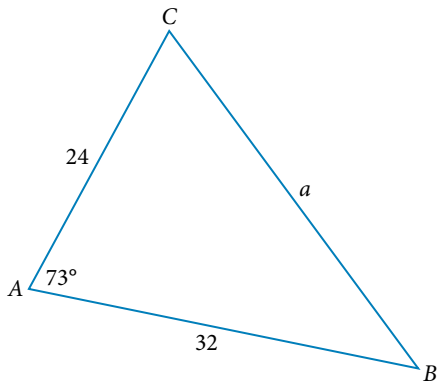
(b)



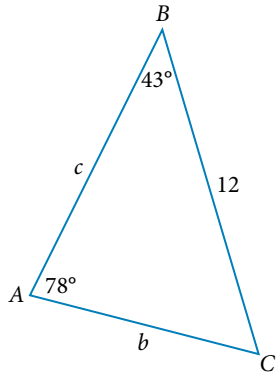
Solve non-right-angled triangles using the sine and cosine rules

8 For triangles ΔABC, calculate the unknown side lengths and angles, correct to 2 decimal places where necessary.

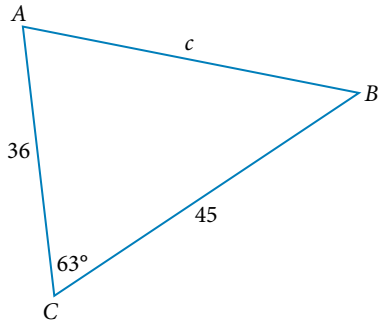
(a)



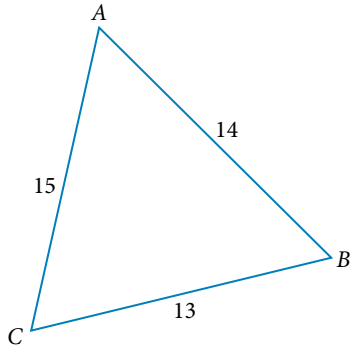
(b)



(c)

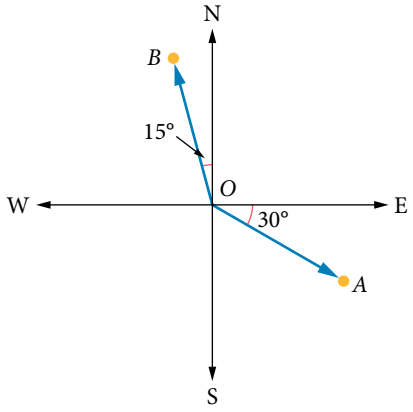


(d)



Work with conventional and true bearings

9 (a) Give the conventional bearing of the points A and B from the origin, O.



(b) Give the true bearing of the points A and B from the origin, O.

(c) Hence, calculate the true bearing of the origin, O, from points A and B.

2.1 Introduction to vectors

Scalar quantities and vector quantities

A scalar quantity is one that has magnitude (size) only. Examples of scalar quantities include mass, length, speed and time.

A vector quantity is one that has magnitude and direction. Examples of vector quantities include:

- displacement (change in position of an object)
- velocity (rate of change of displacement with respect to time)
- acceleration (rate of change of velocity with respect to time)
- force (a push or pull that alters the state of motion of an object)
- momentum (the product of an object's mass and velocity).

Vectors are important in mathematics as they allow you to solve problems in two or even three dimensions. In this chapter you will be working only with vectors in the two-dimensional plane, however, in future chapters you will extend your knowledge to three-dimensional space.

A vector is a quantity possessing both magnitude and direction.

1 Scalar quantities and vector quantities

Determine whether the following quantities are scalar quantities or vector quantities.

(a) 6.5 km

THINKING	WORKING
You are given only a measure of distance; no direction is indicated.	6.5 km is a scalar quantity.

(b) 24 ms^{-1} south-west

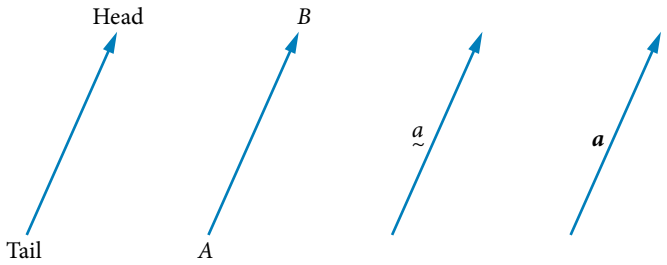
You are given the speed (24 ms^{-1}) and direction (south-west).	24 ms^{-1} south-west is a vector quantity.

(c) -10°C

Although this temperature has a negative value, this is simply an indication of its relative magnitude; it does not imply a direction.	-10°C is a scalar quantity.

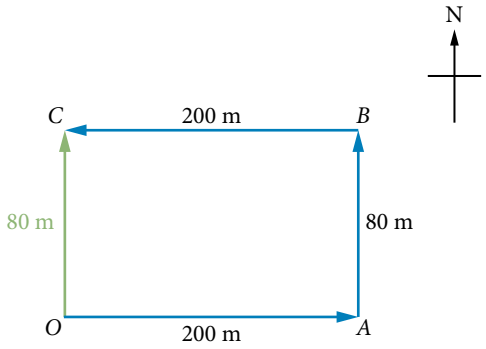
Vectors and vector notation

Because vector quantities have a magnitude and direction, a convenient way to represent a vector is by using a directed line segment, which is a straight line with an arrowhead. The length of the line segment indicates the magnitude of the vector, and the arrow indicates its direction. By convention, the beginning of a vector is called its tail, and the end (where the arrowhead is located) is called its head.



Often the vector will be labelled using a bold italic letter, such as $\boldsymbol{a}$. However, this notation can be impractical for handwriting solutions, so often the tilde notation $\underset{\sim}{a}$ (as seen in the worked solutions for this chapter) is used. This notation is often used as it is useful for distinguishing vectors from variables.

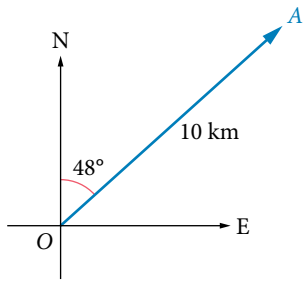
By representing vectors using arrows, it is possible to set up and analyse complex scenarios. For example, if Mark walks from point O to point C via points A and B, as shown in the diagram on the right, he walks a total *distance* of 480 m. However, Mark ends his journey only 80 m directly north of his starting point at O. Therefore, Mark's *displacement* from O (the change in his position from the starting point) is 80 m north.



Magnitude of a vector

The magnitude of a vector is a measure of its size or length. Magnitude is a scalar quantity. For example, the diagram on the right shows a vector of magnitude 10 km in a direction N48°E.

The magnitude of vector $\overrightarrow{AB}$ is denoted $\left|\overrightarrow{AB}\right|$ and the magnitude of vector $\boldsymbol{a}$ is denoted $|\boldsymbol{a}|$.



Equality of vectors

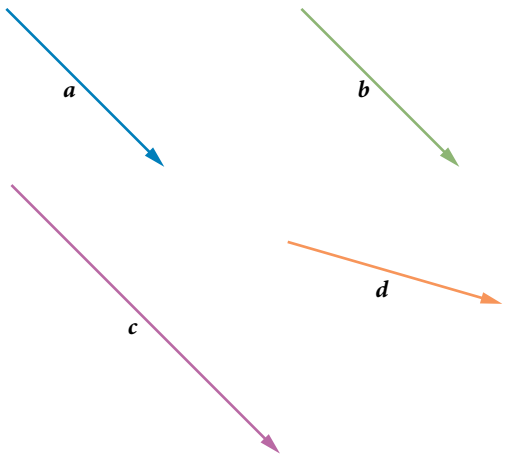
Two vectors are equal if, and only if, they have the same magnitude and the same direction, regardless of their positions.

For example, in the diagram to the right:

$\boldsymbol{a} = \boldsymbol{b}$ (Vectors have the same magnitude and direction.)

$\boldsymbol{a} \neq \boldsymbol{c}$ (Vectors have the same direction but different magnitudes.)

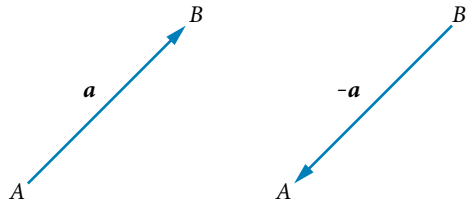
$\boldsymbol{a} \neq \boldsymbol{d}$ (Vectors have the same magnitude but different directions.)



Negative of a vector

If $\boldsymbol{a}$ is the vector $\overrightarrow{AB}$, then $-\boldsymbol{a}$ is the vector $\overrightarrow{BA}$.

This means that the negative of a vector has the same magnitude as the original, but the direction is reversed.



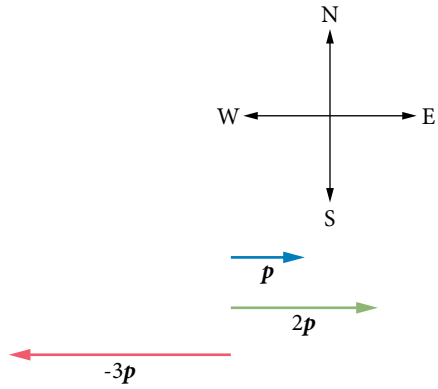
Multiplying a vector

Multiplying a vector by a scalar (a number), k, where $k \in \mathbb{R} \setminus \{0\}$, results in a vector with k times the magnitude that is parallel to the original vector.

The zero vector, denoted $\mathbf{0}$, is a vector of zero magnitude and undefined direction. For example, $\boldsymbol{a} + -\boldsymbol{a} = \mathbf{0}$.

If $k > 0$, then $k\,.\,\boldsymbol{a}$ has the same direction as $\boldsymbol{a}$ and has k times the magnitude.
If $k = 0$, then $k\,.\,\boldsymbol{a} = \mathbf{0}$.
If $k < 0$ then $k\,.\,\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has k times the magnitude.

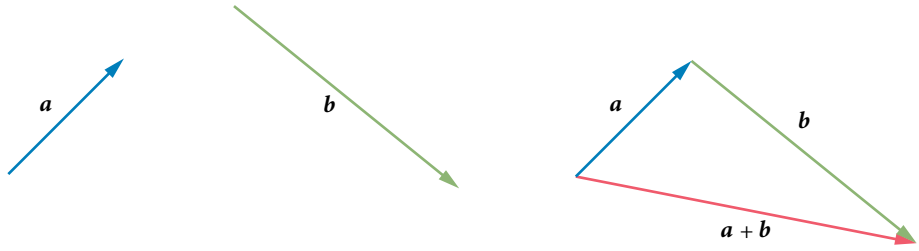
For example, consider vector $\boldsymbol{p}$ in the diagram on the right. Suppose that $\boldsymbol{p}$ has a magnitude of 5 and a direction of east. This would mean that vector $2\boldsymbol{p}$ would have a magnitude of 10 and a direction of east, while vector $-3\boldsymbol{p}$ would have a magnitude of 15 and a direction of west.



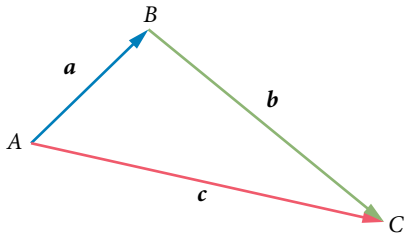
Addition of vectors

The triangle rule for the addition of vectors

To add two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the vectors are placed with the head of $\boldsymbol{a}$ at the tail of $\boldsymbol{b}$. The vector sum, also known as the resultant vector, is a single vector joining the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$. This vector addition forms a triangle, hence the *triangle rule*, and is sometimes called the head-to-tail rule for vector addition.



Also, consider ΔABC below. In this triangle, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, or $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$.



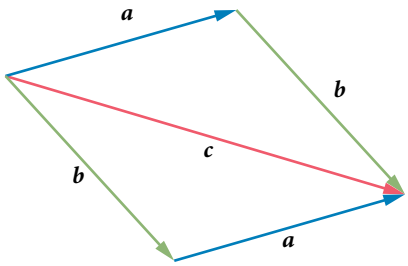
C Making connections

Triangle rule for addition of vectors

Move points C and D to observe the change in the resultant vector, $\boldsymbol{u} + \boldsymbol{v}$.

The parallelogram rule for the addition of vectors

If two vectors to be added have the same initial point (see vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ on the right), they can be added by drawing a parallelogram rather than moving one vector to the head of the other vector. The diagonal of the parallelogram, drawn from the initial point to the final point, represents the vector sum.



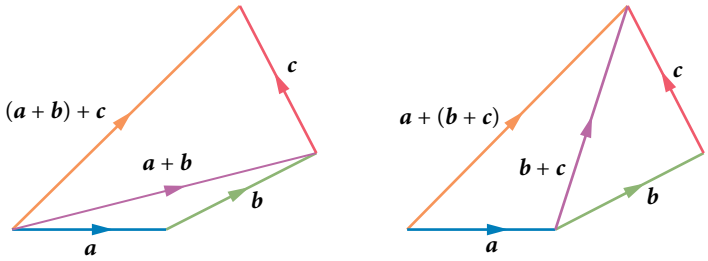
In the diagram to the right, note that a parallelogram is formed from vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{c}$ is the resultant vector. This illustrates that $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{c}$, which means that vector addition is *commutative*.

C Making connections

Parallelogram rule for addition of vectors

Move points C and D to observe the change in the resultant vector, $\boldsymbol{u} + \boldsymbol{v}$.

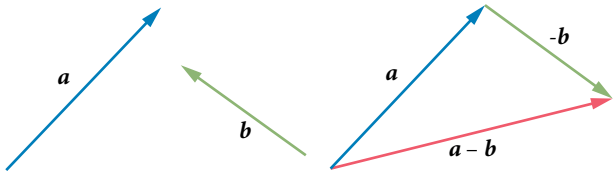
Vector addition is also *associative*. Therefore, if three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are added, then $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$.



$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$

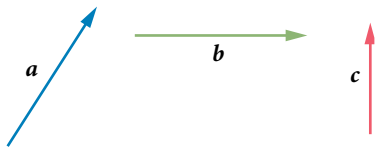
Subtraction of vectors

Subtraction of a vector is defined as the addition of a negative vector. Therefore, $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$.



2 Adding and subtracting vectors

If three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are given, as shown in the diagram to the right, construct the following vectors.



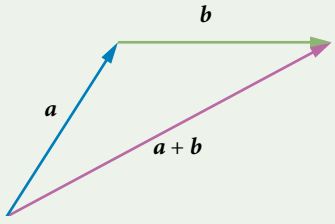
(a) $\boldsymbol{a} + \boldsymbol{b}$

THINKING

To add vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, place the head of $\boldsymbol{a}$ at the tail of $\boldsymbol{b}$.

The resultant vector or vector sum is a vector joining the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$.

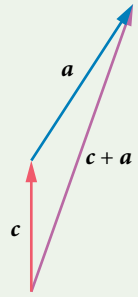
WORKING



(b) $\boldsymbol{c} + \boldsymbol{a}$

To add vectors $\boldsymbol{c}$ and $\boldsymbol{a}$, place the head of $\boldsymbol{c}$ at the tail of $\boldsymbol{a}$.

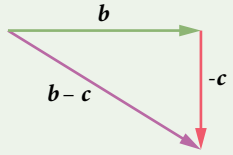
The resultant vector or vector sum is a vector joining the tail of $\boldsymbol{c}$ to the head of $\boldsymbol{a}$.



(c) $\boldsymbol{b} - \boldsymbol{c}$

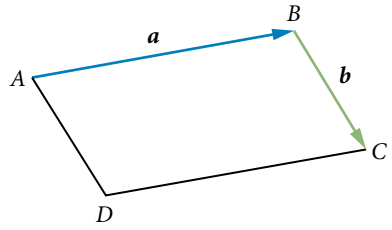
To subtract vector $\boldsymbol{c}$ from vector $\boldsymbol{b}$, place the tail of $-\boldsymbol{c}$ at the head of $\boldsymbol{b}$.

The resultant vector or vector sum is a vector joining the tail of $\boldsymbol{b}$ to the head of $-\boldsymbol{c}$.



3 Expressing vectors in a parallelogram in terms of *a* and *b*.

$ABCD$ is a parallelogram (right). If $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$, express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



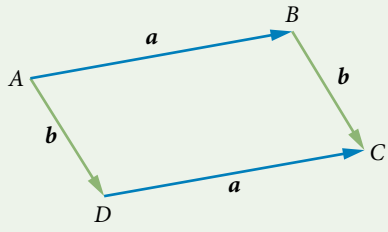
(a) $\overrightarrow{DA}$

THINKING

1 Draw a labelled diagram.

A parallelogram has sides equal in magnitude and direction, so the other sides can also be labelled.

WORKING



2 $\overrightarrow{DA}$ has the same magnitude as $\boldsymbol{b}$, in the opposite direction.

$\overrightarrow{DA} = -\boldsymbol{b}$

(b) $\overrightarrow{AC}$

Identify the vectors joining the vertices A and C. Determine the vector addition from $\overrightarrow{AC}$ expressing the answer in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= \boldsymbol{a} + \boldsymbol{b}$

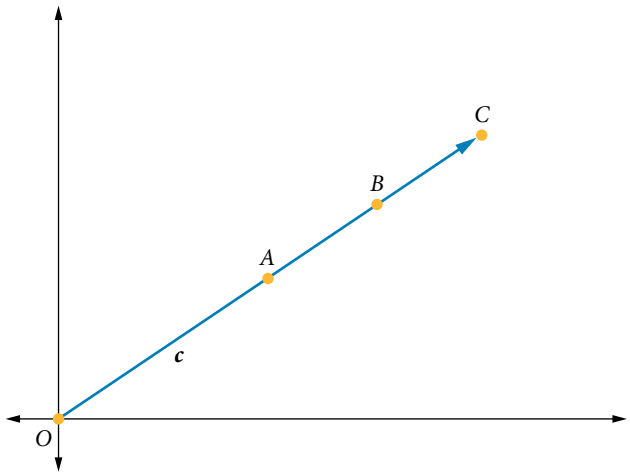
(c) $\overrightarrow{BD}$

Identify the vectors joining the vertices B and D. Determine the vector addition from $\overrightarrow{BD}$ expressing the answer in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$ (or $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$)
$\overrightarrow{BD} = -\boldsymbol{a} + \boldsymbol{b}$
$= \boldsymbol{b} - \boldsymbol{a}$

4 Midpoints of a vector

Consider points O, A, B and C, which are collinear as shown in the diagram to the right. Let vector $\overrightarrow{OC} = \boldsymbol{c}$. Let A be the midpoint of OC and let B be the midpoint of AC. Express the following vectors in terms of $\boldsymbol{c}$.



(a) $\overrightarrow{OA}$

THINKING

Compare the relationship between the given length and direction to the known vector from $\overrightarrow{OC}$.

Then express this relationship in terms of the given vector $\boldsymbol{c}$.

WORKING

$\overrightarrow{OA} = \frac{1}{2}\overrightarrow{OC}$
$= \frac{1}{2}\boldsymbol{c}$

(b) $\overrightarrow{BC}$

The midpoint bisects the vector. Use this information to establish a relationship between the vectors.

$\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AC}$
$= \frac{1}{2} \times \frac{1}{2}\boldsymbol{c}$
$= \frac{1}{4}\boldsymbol{c}$

(c) $\overrightarrow{BA}$

The midpoint bisects the vector. Use this information to establish a relationship between the vectors. In this case, the vector will be in the opposite direction, so it is negative.

$\overrightarrow{BA} = -\frac{1}{2}\overrightarrow{AC}$
$= -\frac{1}{2} \times \frac{1}{2}\boldsymbol{c}$
$= -\frac{1}{4}\boldsymbol{c}$

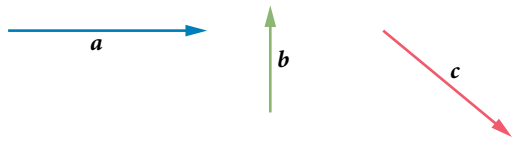
EXERCISE 2.1 Introduction to vectors

Worked Example

1 Determine which of the following are *scalar* quantities, and which are *vector* quantities.

(a) 10 km/h south-west (b) 50 N towards the ground (c) 15 km

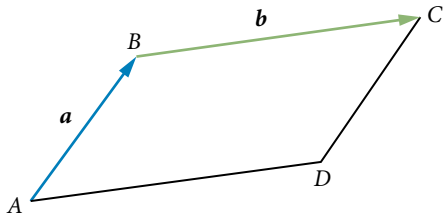
2 If three vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, are given, construct the following vectors.



(a) $\boldsymbol{a}+\boldsymbol{c}$ (b) $\boldsymbol{c}+\boldsymbol{b}$

(c) $\boldsymbol{a}-\boldsymbol{b}$ (d) $\boldsymbol{c}-\boldsymbol{a}$

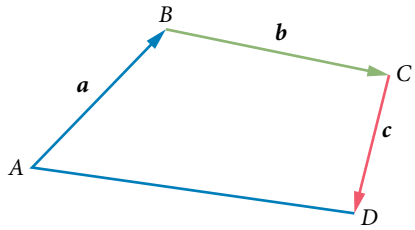
3 $ABCD$ is a parallelogram. If $\overrightarrow{AB}=\boldsymbol{a}$ and $\overrightarrow{BC}=\boldsymbol{b}$, express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



(a) $\overrightarrow{CD}$ (b) $\overrightarrow{AD}$

(c) $\overrightarrow{CA}$ (d) $\overrightarrow{DB}$

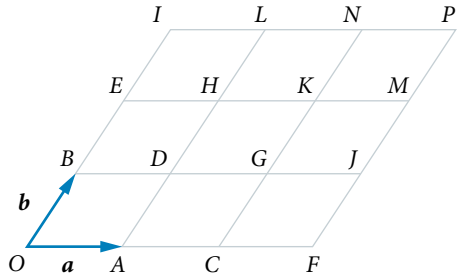
4 $ABCD$ is a quadrilateral. If $\overrightarrow{AB}=\boldsymbol{a}$, $\overrightarrow{BC}=\boldsymbol{b}$ and $\overrightarrow{CD}=\boldsymbol{c}$, express each of the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



(a) $\overrightarrow{AC}$ (b) $\overrightarrow{AD}$

(c) $\overrightarrow{DA}$ (d) $\overrightarrow{DB}$

5 In the diagram on the right, line segments $\overline{OA}$, $\overline{AC}$, $\overline{CF}$, etc. are the same length. Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



(a) $\overrightarrow{ON}$

(b) $\overrightarrow{CP}$

(c) $\overrightarrow{PB}$

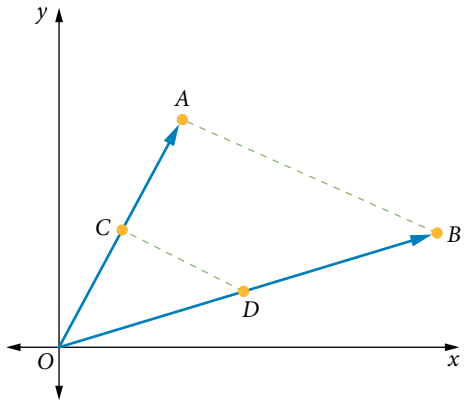
6 Demonstrate graphically that $2(\boldsymbol{a}+\boldsymbol{b})=2\boldsymbol{a}+2\boldsymbol{b}$.

7 (a) Given the vectors $\overrightarrow{AB}=\boldsymbol{a}$ and $\overrightarrow{AC}=\boldsymbol{c}$, what is the representation of the vector $\overrightarrow{BC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$?

A $\boldsymbol{b}+\boldsymbol{c}$ B $\boldsymbol{b}-\boldsymbol{a}$ C $\boldsymbol{c}+\boldsymbol{a}$ D $\boldsymbol{c}-\boldsymbol{a}$

(b) Explain why the alternatives are incorrect and what the common errors are.

8 Examine the diagram and consider C and D as the midpoints of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.



(a) Identify the false statement.

A $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

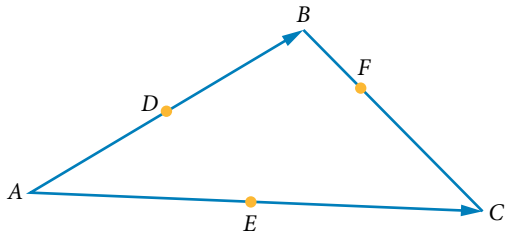
B $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{AB}$

C OCB forms a triangle.

D $2\overrightarrow{AB} = \overrightarrow{CD}$

(b) Explain the reasoning behind your choice.

9 ΔABC is a triangle with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{c}$. D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. F is a point on $\overline{BC}$ such that $\overline{FC} = 2 \times \overline{BF}$.



(a) Express the vectors $\overrightarrow{BC}$ and $\overrightarrow{DE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

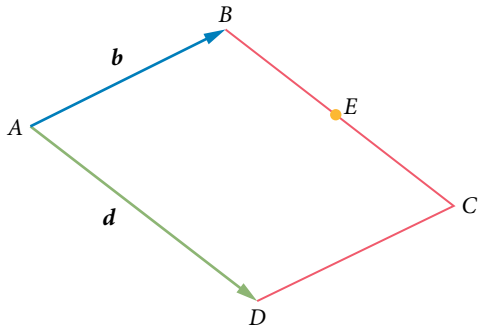
(b) Express $\overrightarrow{DE}$ in terms of $\overrightarrow{BC}$.

(c) What geometric property of a triangle does the answer to part **(b)** demonstrate?

(d) Express the vectors $\overrightarrow{BF}$ and $\overrightarrow{FC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(e) Demonstrate that $\overrightarrow{AF} = \frac{1}{3}(2\boldsymbol{a} + \boldsymbol{c})$.

10 $ABCD$ is a parallelogram in which $\overrightarrow{AB} = \boldsymbol{b}$ and $\overrightarrow{AD} = \boldsymbol{d}$ and E is the midpoint of $\overline{BC}$.



(a) Express $\overrightarrow{AC}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$.

(b) Express $\overrightarrow{AE}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$.

(c) Express $\overrightarrow{DE}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$.

Suppose F is a point on $\overrightarrow{DE}$ and $\overrightarrow{DF} = \frac{2}{3}\overrightarrow{DE}$.

(d) Express $\overrightarrow{DF}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$.

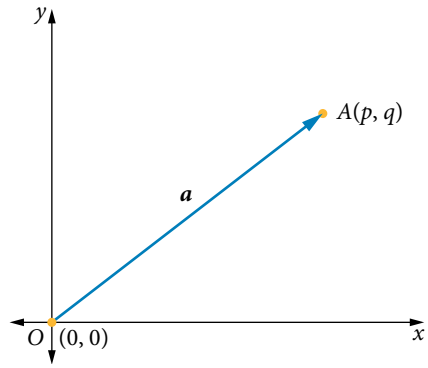
(e) Express $\overrightarrow{AF}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$, and hence demonstrate that F lies on $\overrightarrow{AC}$.

(f) Calculate the ratio $\overline{AF} : \overline{FC}$.

2.2 Vectors in two dimensions

Position vectors

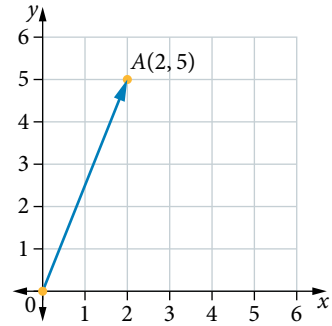
Position vectors can be drawn on the Cartesian plane and represented in different ways, but they always begin at O, the origin. Consider the vector connecting the origin, O, with point A, whose coordinates are (p,q).



The vector $\overrightarrow{OA}$ represents the position vector of point $A(p,q)$ relative to the origin. Therefore it can be said that $\overrightarrow{OA} = (p,q)$. Or you can give this vector a different label, for example, $\overrightarrow{OA} = \boldsymbol{a}$.

A position vector is any vector drawn with its tail at O, the origin.

A position vector can be represented by a *coordinate pair*, (p,q). Starting from the tail at the origin, this vector represents a directed line segment with the head of the arrow p units in the positive x-direction and q units in the positive y-direction from the origin, O.



A position vector can also be represented as a *column matrix*. A matrix is a rectangular array of numbers, and a column matrix consists of a single column only. The matrix $\begin{bmatrix} p \\ q \end{bmatrix}$ represents the position vector (p,q) that is a directed line segment p units in the positive x-direction and q units in the positive y-direction.

For example, if A has the coordinates (2, 5), then the position vector $\overrightarrow{OA}$ can be represented by the coordinate pair (2, 5), or the column matrix $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

5 Drawing position vectors on the Cartesian plane

Draw the following position vectors on the Cartesian plane.

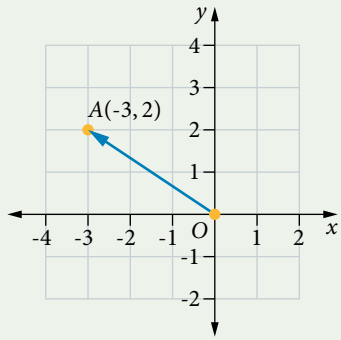
(a) $\overrightarrow{OA}$, where A is (-3, 2).

THINKING

1 Identify the coordinates of the position vector and plot them on the Cartesian plane.

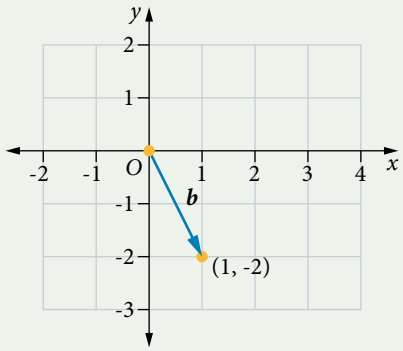
2 Draw the position vector (a line with an arrow to represent magnitude and direction) with the tail at the origin, O, and the head at the given coordinates.

WORKING



(b) $\boldsymbol{b}$, the position vector (1, -2)

1. Identify the coordinates of the position vector and plot them on the Cartesian plane.
2. Draw the position vector with the tail at the origin (0,0) and the head at the given coordinates.

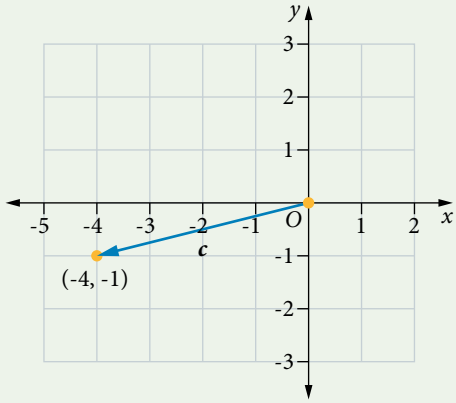


(c) $\boldsymbol{c}$, the position vector $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$

1. Recognise the coordinates of the position vector from the column matrix.
2. Plot the coordinates on the Cartesian plane.

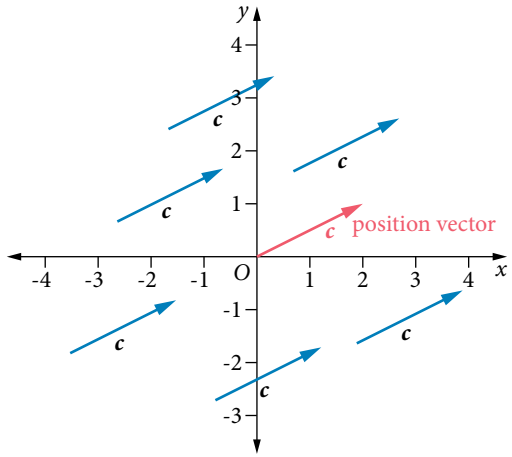
   Draw the position vector with the tail at the origin, O, and the head at the given coordinates.

The column matrix $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$ corresponds to the coordinate pair (-4,-1).



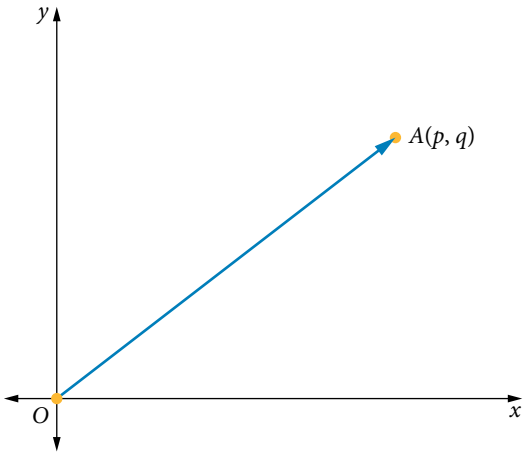
Location of vectors on the Cartesian plane

Remember that vectors that are equal do not have to start at the same point; they just have to have the same magnitude and direction. Consider the vector $\boldsymbol{c}$ represented by the ordered pair (2, 1). To draw this as a position vector, start at the origin and draw a vector 2 units to the right and 1 unit up. However, $\boldsymbol{c}$ could also be placed anywhere on the Cartesian plane, as long as its head is 2 units to the right and 1 unit up from its tail. Only the vector starting at the origin is called a position vector.

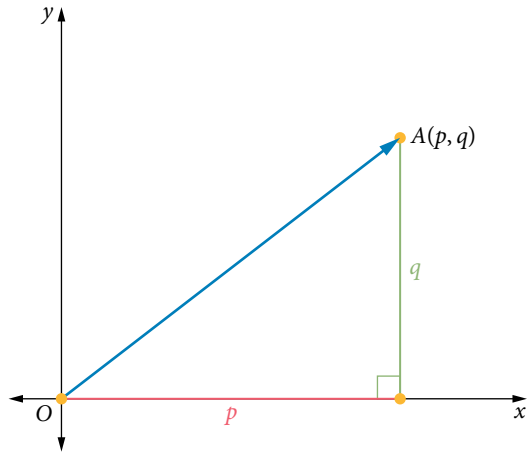


Magnitude of a position vector

Consider again the position vector connecting the origin, O, to point $A(p,q)$.



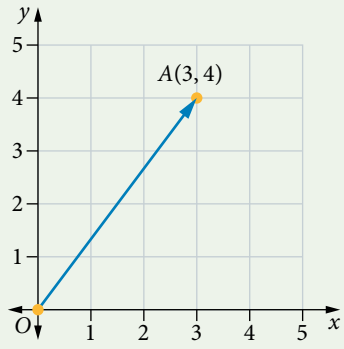
By constructing a right-angled triangle, the *magnitude* of position vector $\overrightarrow{OA} = (p, q)$ or $\overrightarrow{OA} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be calculated using Pythagoras' theorem. Therefore, $\left|\overrightarrow{OA}\right| = \sqrt{p^2 + q^2}$.



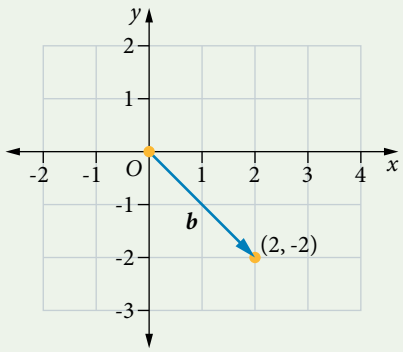
6 Magnitude of a vector

Calculate the magnitude of the following vectors:

(a) $\overrightarrow{OA}$, where A is (3, 4).

THINKING	WORKING
1 On the Cartesian plane, draw the position vector with the tail at the origin, O, to the given coordinates.	
2 Calculate the magnitude of the position vector using Pythagoras' theorem.	$\left\|\overrightarrow{AB}\right\| = \sqrt{3^2 + 4^2}$ $= \sqrt{25}$ $= 5$ units
3 Interpret the answer.	The length of the position vector (3,4) is 5 units.

(b) $\boldsymbol{b}$, the position vector $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

1 Recognise the coordinates of the position vector from the column matrix.	The column matrix $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ corresponds to the coordinate pair (2,-2).
2 On the Cartesian plane, draw a vector with the tail at the origin, O, to the give coordinates.	
3 Calculate the magnitude of the position vector using Pythagoras' theorem.	$\lvert\boldsymbol{b}\rvert = \sqrt{2^2 + (-2)^2}$ $= \sqrt{8}$ $= \sqrt{4} \times \sqrt{2}$ $= 2\sqrt{2}$ units
4 Interpret the answer.	The length of the position vector $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is $2\sqrt{2}$ units.

Technology worked example
Calculate the magnitude of a vector

Multiplying a vector by a scalar

Recall that multiplying a vector by a scalar k, where k is a real number, results in a vector with k times the magnitude that is parallel to the original vector. When working with vectors in coordinate pair or column matrix form, you simply multiply each coordinate in the vector by the scalar k.

If $\boldsymbol{a} = (p, q)$ then $k\boldsymbol{a} = (kp, kq)$, where $k \in \mathbb{R} \setminus \{0\}$

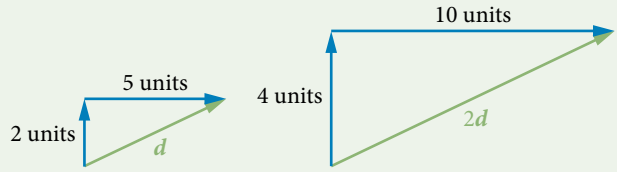
If $\boldsymbol{a} = \begin{bmatrix} p \\ q \end{bmatrix}$ then $k\boldsymbol{a} = \begin{bmatrix} kp \\ kq \end{bmatrix}$, where $k \in \mathbb{R} \setminus \{0\}$

7 Multiplying a vector by a scalar

(a) If vector $\boldsymbol{d}$ is represented by the ordered pair (2, 5), specify the ordered pair for $2\boldsymbol{d}$.

THINKING	WORKING
1 Identify the ordered pair and recall the rule to multiply a vector $\boldsymbol{a} = (p, q)$ by a scalar k: $k\boldsymbol{a} = (kp, kq)$.	If $\boldsymbol{d} = (2, 5)$, then $2\boldsymbol{d} = (2 \times 2, 2 \times 5)$ $= (4, 10)$

2 Verify your answer with a diagram.



$2\boldsymbol{d}$ is a parallel vector that is twice as long as $\boldsymbol{d}$ pointing in the same direction.

3 Interpret the answer.

The ordered pair that represents $2\boldsymbol{d}$ is $(4, 10)$.

(b) If the vector $\boldsymbol{e}$ is represented by the column matrix $\begin{bmatrix} 4 \\ -6 \end{bmatrix}$, specify a column matrix for $-3\boldsymbol{e}$.

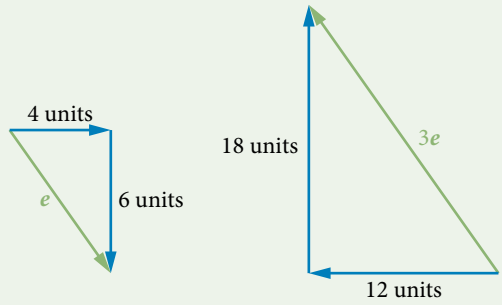
1 Identify the ordered pair and recall the rule to multiply a vector $\boldsymbol{a} = \begin{bmatrix} p \\ q \end{bmatrix}$ by a scalar k:

$k\boldsymbol{a} = \begin{bmatrix} kp \\ kq \end{bmatrix}$.

If $\boldsymbol{e} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$, then

$$-3\boldsymbol{e} = \begin{bmatrix} -3 \times 4 \\ -3 \times -6 \end{bmatrix} = \begin{bmatrix} -12 \\ 18 \end{bmatrix}$$

2 Verify your answer with a diagram.



$-3\boldsymbol{e}$ is a parallel vector 3 times the length of $\boldsymbol{e}$ and pointing in the opposite direction.

3 Interpret the answer.

The column matrix that represents $-3\boldsymbol{e}$ is $\begin{bmatrix} -12 \\ 18 \end{bmatrix}$.

(c) Vector $\boldsymbol{a}$ is represented by (-2, 3) and vector $\boldsymbol{b}$ is represented by (8, -9). Determine whether $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

1 If $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$, then there exists a real number, k such that $k\boldsymbol{a} = \boldsymbol{b}$.

Assume that $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$.

$k\boldsymbol{a} = \boldsymbol{b}$

$k(-2, 3) = (8, -9)$

Test the x-value:

$-2k = 8$

$k = -4$

Test the y-value:

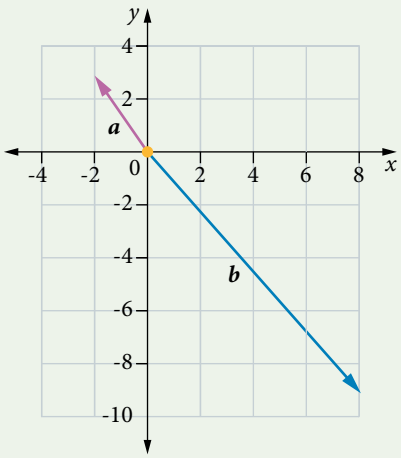
$3k = -9$

$k = -3$

2 Check to see if there is a single value of k that satisfies this equation.

3 Verify your answer graphically.

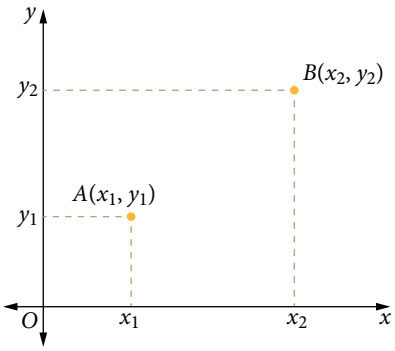
Values of k are different, therefore $\boldsymbol{a}$ is not parallel to $\boldsymbol{b}$.



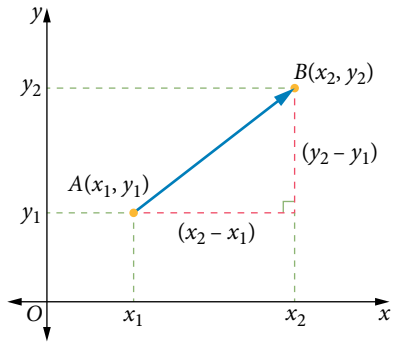
Relative position vectors

So far you have considered position vectors, which represent the position of one point in relation to the origin. A relative position vector represents a point's position in relation to another point.

Consider two points $A(x_1,\ y_1)$ and $B(x_2, y_2)$ as shown on the Cartesian plane below:

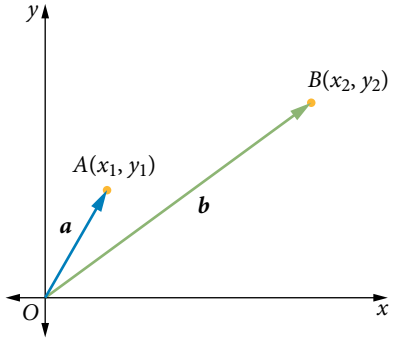


Vector $\overrightarrow{AB}$ can be drawn to connect these points. This vector will have a horizontal component of $(x_2 - x_1)$ and a vertical component of $(y_2 - y_1)$.



If A is the point (x_1, y_1) and B is the point (x_2, y_2), then:

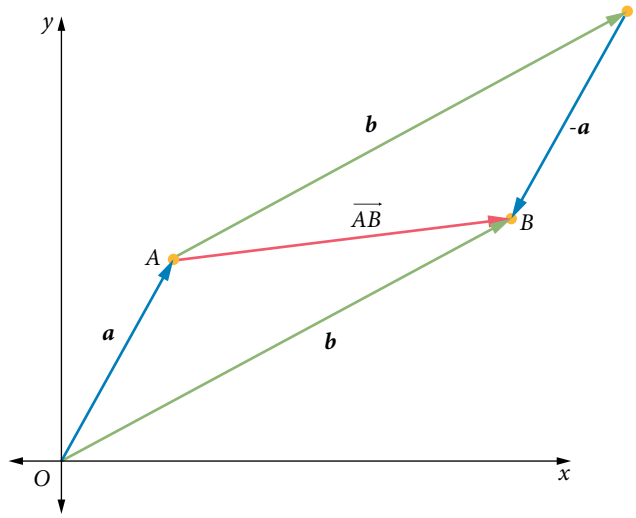
- $\overrightarrow{AB} = \left(x_2 - x_1, y_2 - y_1\right)$ (in coordinate pair form)
- $\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$ (in column matrix form)



This can also be demonstrated using the parallelogram rule of addition. Consider again the points $A(x_1, y_1)$ and $B(x_2, y_2)$, which can be represented by the position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

To determine vector $\overrightarrow{AB}$, start at point A. If you trace backwards along vector $\boldsymbol{a}$ to the origin and then forwards along $\boldsymbol{b}$, you arrive at point B. This corresponds to adding vectors $-\overrightarrow{OA}$ and $\overrightarrow{OB}$.

If A is defined by the position vector $\boldsymbol{a}$ and B is defined by the position vector $\boldsymbol{b}$ then $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.



8 Relative position vectors

Let A be the point (1, -4) and B be the point (8, 2).

(a) Determine $\overrightarrow{AB}$ in column matrix form.

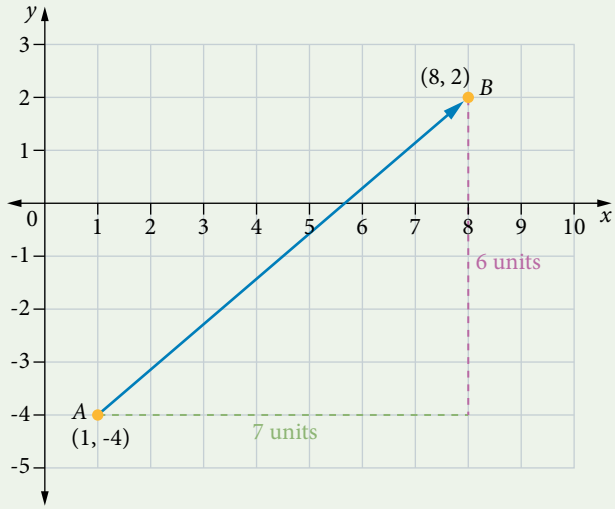
THINKING

1 Calculate the change in the horizontal component and the change in the vertical component between A and B.

WORKING

$$\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$
$$= \begin{bmatrix} 8 - 1 \\ 2 - (-4) \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

2 Evaluate the reasonableness of your solution by drawing a graph.



(b) Calculate the length of $\overrightarrow{AB}$, correct to 2 decimal places.

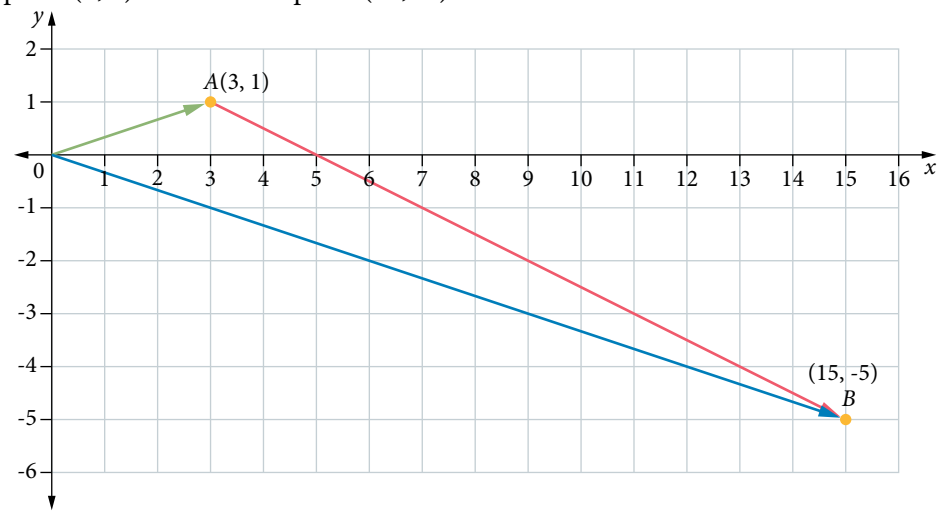
As you have already found $\overrightarrow{AB}$, you can apply Pythagoras' theorem to calculate $\left|\overrightarrow{AB}\right|$.

$$\overrightarrow{AB} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{7^2 + 6^2}$$
$$= \sqrt{85}$$
$$= 9.22 \text{ (2 d.p.)}$$

9 Sections of vectors

Let A be the point (3, 1) and B be the point (15, -5).



(a) Calculate the position vector $\overrightarrow{OM}$, where M is the midpoint of $\overrightarrow{AB}$.

THINKING

1 Calculate the vector $\overrightarrow{AB}$.

WORKING

$$\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$
$$= \begin{bmatrix} 15 - 3 \\ -5 - 1 \end{bmatrix}$$
$$= \begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

Point B is positioned 12 units to the right and 6 units down from point A.

2 The midpoint splits $\overrightarrow{AB}$ in half.

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$$
$$= \frac{1}{2}\begin{bmatrix} 12 \\ -6 \end{bmatrix}$$
$$= \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

The midpoint, M, of the vector joining A and B is positioned 6 units to the right and 3 units down from point A.

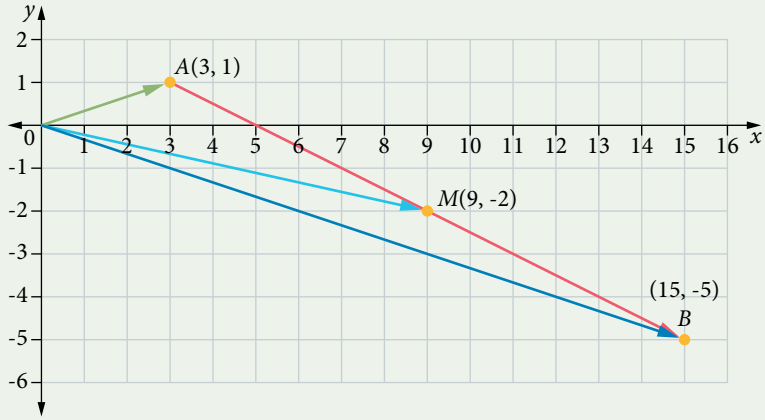
3 Use vector addition to calculate the position vector.

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$
$$= \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

4 Answer the question.

The position vector joining the origin, O, and the midpoint, M, of the vector joining points A and B, is given by $\overrightarrow{OM} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$ or $\overrightarrow{OM} = (9, -2)$.

5 Evaluate the reasonableness of your solution by drawing a graph.



(b) Calculate the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, where P and Q are the points of trisection of $\overrightarrow{AB}$.

1 You know $\overrightarrow{AB}$ from the previous question.

$$\overrightarrow{AB} = \begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

2 The points of trisection split $\overrightarrow{AB}$ into thirds.

$$\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$$
$$= \frac{1}{3}\begin{bmatrix} 12 \\ -6 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$
$$\overrightarrow{AQ} = \frac{2}{3}\overrightarrow{AB}$$
$$= \frac{2}{3}\begin{bmatrix} 12 \\ -6 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

The point P, $\frac{1}{3}$ of the vector $\overrightarrow{AB}$, is 4 units to the right and 2 units down from point A.

The point Q, $\frac{2}{3}$ of the vector $\overrightarrow{AB}$, is 8 units to the right and 4 units down from point A.

3 Use vector addition to calculate the position vector.

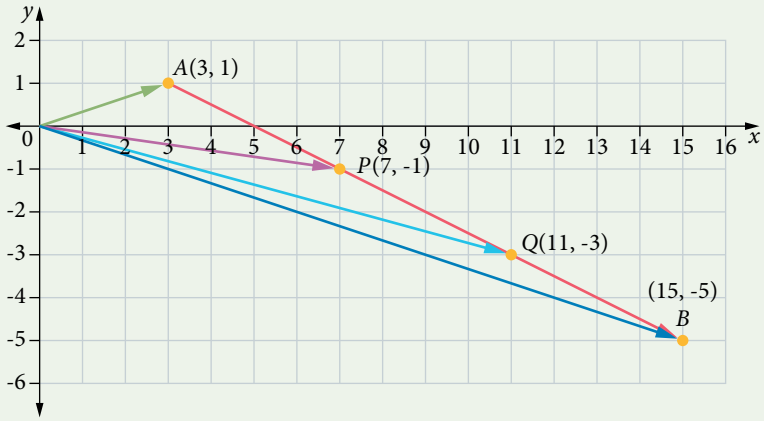
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$
$$= \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

4 Answer the question.

$$\overrightarrow{OP} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \text{ and } \overrightarrow{OQ} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

5 Evaluate the reasonableness of your solution by drawing a graph.



C Making connections

Vector between two points

Move points A and B to explore the relationship between the vector $\overrightarrow{AB}$ and the position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

EXERCISE 2.2 Vectors in two dimensions

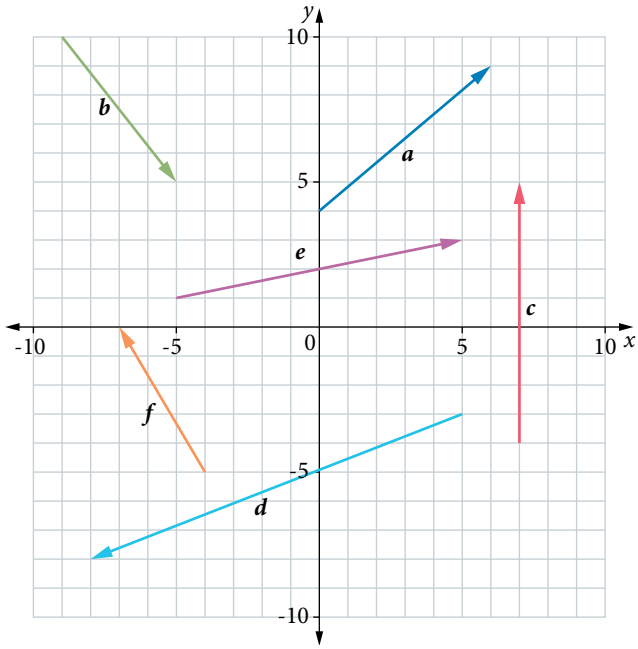
Worked Example 5

1 Draw the following position vectors on the Cartesian plane.

(a) $\overrightarrow{OA}$, where A is (4, 7)

(b) $\boldsymbol{b}$, the position vector (-1, -5)

2 Represent each of the vectors $\boldsymbol{a}$ to $\boldsymbol{f}$ in the plane shown as an ordered pair.

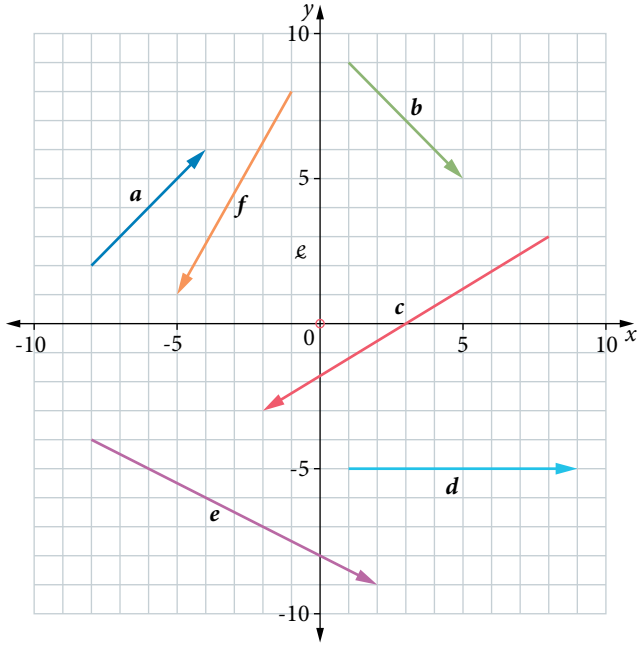


Worked Example

8

6

3 (a) Represent each of the vectors $\boldsymbol{a}$ to $\boldsymbol{f}$ in the plane shown as a column matrix.



(b) Determine the magnitude of each vector shown in part **(a)**.

7

4 If vector $\boldsymbol{a}$ is represented by the ordered pair (2, -6), specify an ordered pair to represent each of the following vectors.

(a) $3\boldsymbol{a}$ **(b)** $\frac{1}{2}\boldsymbol{a}$ **(c)** $-\boldsymbol{a}$ **(d)** $0.4\boldsymbol{a}$

5 (a) Given $\overrightarrow{OA}$ and $\overrightarrow{OB}$, what is the correct way to calculate $\overrightarrow{BA}$?

A $\overrightarrow{OA} + \overrightarrow{OB}$ **B** $\overrightarrow{BO} + \overrightarrow{AO}$ **C** $\overrightarrow{OB} + \overrightarrow{OA}$ **D** $-\overrightarrow{OB} + \overrightarrow{OA}$

(b) Explain why the alternatives are incorrect.

6 If $\overrightarrow{OA} = (p, q)$ and $\overrightarrow{AB} = (s, t)$, what is $\overrightarrow{OB}$ equal to?

A (ps, qt) **B** $(p+q, s+t)$ **C** $(p+s, q+t)$ **D** $(q+t, p+s)$

7 **(a)** If the vector $\boldsymbol{c} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$ has an initial point (2, 1), determine the coordinates of its end point.

(b) If the vector $\boldsymbol{d} = \begin{bmatrix} -7 \\ 13 \end{bmatrix}$ has a terminal point (12, -2), determine the coordinates of its initial point.

8 The points A, B and C have coordinates (-2, -3), (2, 3) and (8, -1) respectively.

(a) Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ and express them in column matrix form.

(b) Calculate $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{BC}\right|$ and $\left|\overrightarrow{AC}\right|$.

(c) Use Pythagoras' theorem to prove that ΔABC is a right-angled triangle.

(d) Determine the coordinates of a point D such that $ABCD$ forms a square.

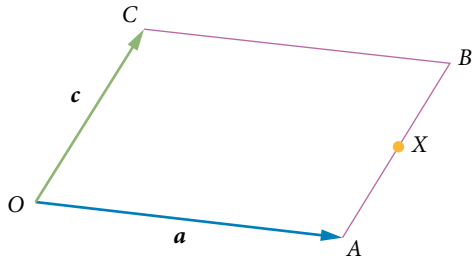
(e) Determine the coordinates of the point of intersection of the diagonals of the square $ABCD$.

9 Point A has coordinates (1, 4) and point B has coordinates (5, q). Calculate the possible value(s) for q such that $\left|\overrightarrow{AB}\right| = \sqrt{20}$.

10 Thomas and Ruth begin their day at base camp and each embark on a different journey. Thomas finishes the day 4 km north and 7 km east of base camp, while Ruth ends up 10 km due west of base camp. Determine the distance between the two hikers at the end of the day.

Worked Example 9

11 $OABC$ is a parallelogram with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$. X is the midpoint of $\overline{AB}$ as shown.



(a) Express the vectors $\overrightarrow{OB}$ and $\overrightarrow{OX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

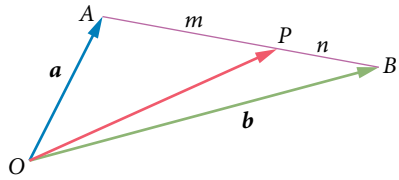
(b) Express the vector $\overrightarrow{CX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(c) If Y is a point on $\overrightarrow{CX}$ such that $\overrightarrow{CY} = \frac{2}{3}\overrightarrow{CX}$, express $\overrightarrow{CY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(d) Determine $\overrightarrow{OY}$ and hence prove that Y lies on $\overrightarrow{OB}$.

(e) Calculate the ratio $\overline{OY} : \overline{YB}$.

12 **(a)** The position vectors of points A and B are (-3, 6) and (2, 9) respectively. Determine the position vector of the midpoint of $\overline{AB}$.



(b) Let the position vectors of points A and B be $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Determine the position vector of the midpoint of $\overline{AB}$.

(c) The position vectors of points D and E are (1, 7) and (5, 3) respectively. Determine the position vectors of the points of trisection of $\overline{DE}$.

(d) Let the position vectors of points D and E be $\boldsymbol{d}$ and $\boldsymbol{e}$ respectively. Determine the position vectors of the points of trisection of $\overline{DE}$.

(e) Let the position vectors of points A and B be $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Let P divide $\overline{AB}$ in the ratio m:n so that $\frac{\overline{AP}}{\overline{AB}} = \frac{m}{m+n}$. Demonstrate that the position vector of point P is $\frac{n}{m+n}\boldsymbol{a} + \frac{m}{m+n}\boldsymbol{b}$.

2.3 Vectors in component form

Unit vectors

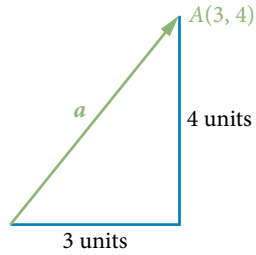
A *unit vector* is a vector with a magnitude of 1 unit. To obtain a unit vector from a given vector, divide that vector by its own magnitude.

If vector $\boldsymbol{a}$ has a magnitude of $|\boldsymbol{a}|$, then a unit vector in the direction of $\boldsymbol{a}$, denoted by $\hat{\boldsymbol{a}}$, can be found by dividing vector $\boldsymbol{a}$ by its own magnitude $|\boldsymbol{a}|$. That is, $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$.

The unit vector in the direction of $\boldsymbol{a}$ is denoted $\hat{\boldsymbol{a}}$, where $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ and $|\hat{\boldsymbol{a}}| = 1$.

Consider the position vector $\boldsymbol{a} = (3, 4)$:

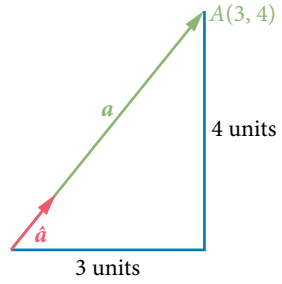
$$\begin{aligned}|\boldsymbol{a}| &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \text{ units}\end{aligned}$$



A unit vector $\hat{\boldsymbol{a}}$ represents the vector in the direction of $\boldsymbol{a}$ with a magnitude of 1.

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

Verify that the magnitude or length of the vector is 1 unit using Pythagoras' theorem.



$$\begin{aligned}|\hat{\boldsymbol{a}}| &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\ &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\ &= \sqrt{\frac{25}{25}} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

Vectors in Cartesian form

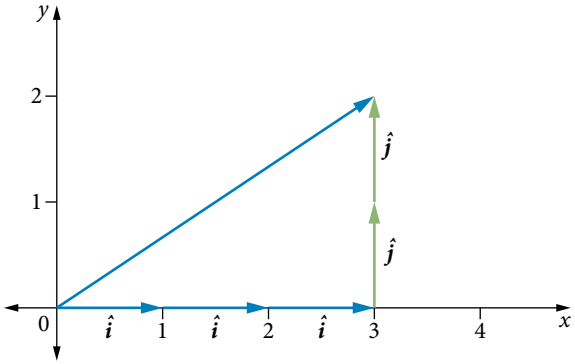
Recall that a vector can be represented as an ordered pair (x, y) or as a column matrix, $\begin{bmatrix} x \\ y \end{bmatrix}$ where the first value represents the distance parallel to the x-axis and the second value the distance parallel to the y-axis.

Using this information, it is often convenient to express a vector as a sum of unit vectors parallel to the axes of the Cartesian plane. Consider a vector of 1 unit magnitude in the positive x direction—let this be equal to vector $\hat{\boldsymbol{i}}$. An equivalent vector of 1 unit magnitude in the positive y direction is defined as vector $\hat{\boldsymbol{j}}$.

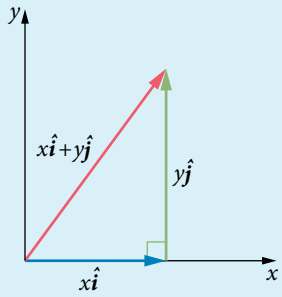
Vector $\hat{\boldsymbol{i}}$ is defined as a vector of 1 unit magnitude in the positive x direction.

Vector $\hat{\boldsymbol{j}}$ is defined as a vector of 1 unit magnitude in the positive y direction.

For example, consider the vector that could be represented in ordered pair form as (3, 2) or as column matrix $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Using unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, this vector could be constructed as follows:



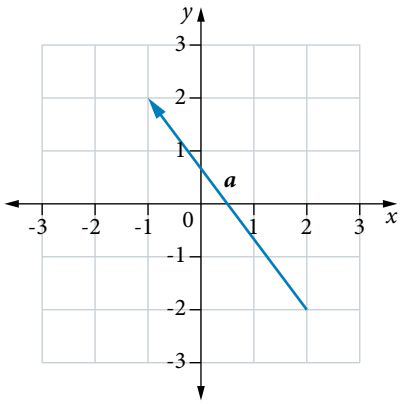
Vector $\boldsymbol{a}$, represented as an ordered pair by (x,y) or as a column matrix as $\begin{bmatrix} x \\ y \end{bmatrix}$, can be expressed as $\boldsymbol{a} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. This is called the Cartesian form (or component form) of the vector.



Using vector addition, the vector (3, 2) could be expressed as $3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$. This notation is known as the Cartesian form (or component form) of a vector.

10 Expressing a vector in Cartesian form

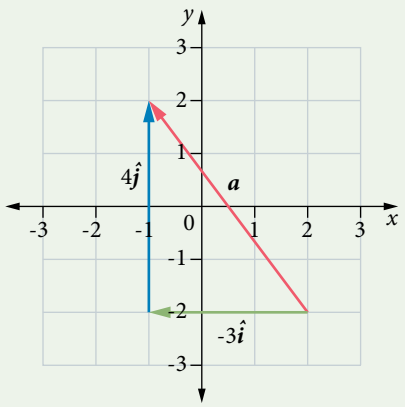
Express the vector $\boldsymbol{a}$ in Cartesian form.



THINKING

1 From the tail of the vector, draw another vector across and then up (or down) to meet the head of the original vector. Label these vectors using $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ components.

WORKING



2 Express the original vector in terms of the components $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

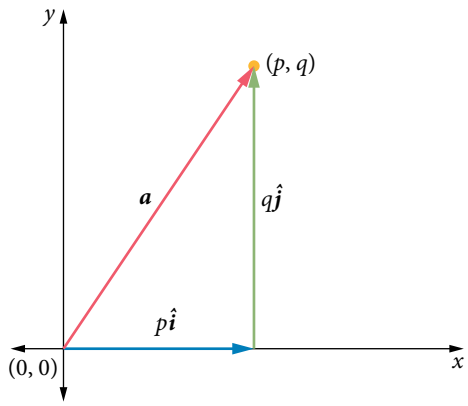
$\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

When determining the components of a vector, start at the tail end. It does not matter whether you move parallel to the x-axis first (to find the $\hat{\boldsymbol{i}}$ component) or parallel to the y-axis first (to find the $\hat{\boldsymbol{j}}$ component), however, by convention, the $\hat{\boldsymbol{i}}$ component is given first when expressing a vector in component form.

Magnitude of a vector in Cartesian form

To calculate the magnitude of a vector given in Cartesian form, use Pythagoras' theorem.

If $\boldsymbol{a} = p\hat{\boldsymbol{i}} + q\hat{\boldsymbol{j}}$, then $|\boldsymbol{a}| = \sqrt{p^2 + q^2}$.

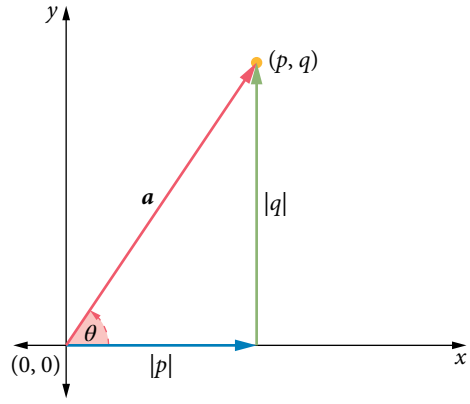


Direction of a vector in Cartesian form

Using trigonometry, it is possible to calculate the angle that a vector forms with the positive direction of the x-axis. Conventionally, the direction of a vector is given as an angle rotated anticlockwise from the positive x-axis. (If the angle is negative, this indicates a clockwise rotation from the positive x-axis).

Consider the vector $\boldsymbol{a} = p\hat{\boldsymbol{i}} + q\hat{\boldsymbol{j}}$ that forms an angle θ with the positive x-axis. This acute angle is given by $\theta = \tan^{-1}\left(\dfrac{|q|}{|p|}\right)$.

Note that if vector $\boldsymbol{a}$ lies in the second, third or fourth quadrant (that is, if either of the vertical or horizontal components is negative), then you will need to make the necessary adjustments to measure the direction of the vector anticlockwise from the positive x-axis.



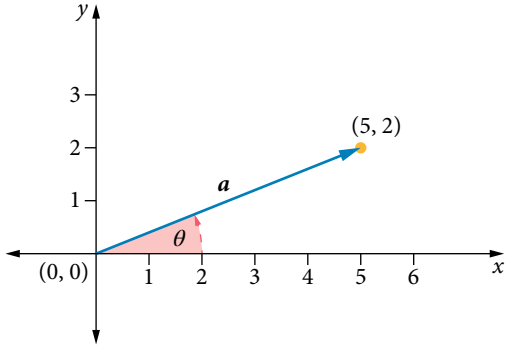
Vectors in polar form

An alternative to expressing a vector in terms of its horizontal and vertical components is to use polar form. When expressing a vector in polar form, the magnitude and direction (from the positive x-axis) of the vector are stated directly.

If vector $\boldsymbol{a}$ has a magnitude of $|\boldsymbol{a}|$ and forms an angle of θ with the positive x-axis, where $0° \leq \theta < 360°$, then the vector can be written in polar form as $\boldsymbol{a} = (|\boldsymbol{a}|, \theta°)$. To avoid confusing polar and Cartesian form, the angle symbol can be added to give $\boldsymbol{a} = (|\boldsymbol{a}|, \angle\theta°)$.

11 Expressing the polar form of a vector from a diagram

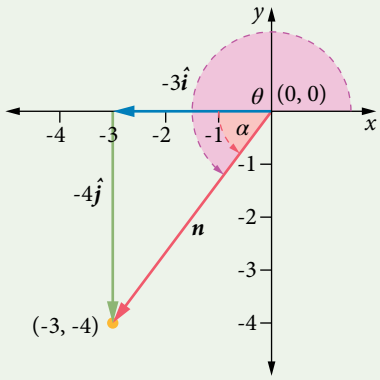
Calculate the magnitude and direction of the vector shown below, hence express the vector in polar form.



THINKING	WORKING
1 Express the vector in Cartesian form.	$\boldsymbol{a} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$
2 Use Pythagoras' theorem to calculate the magnitude of the vector.	$\lvert\boldsymbol{a}\rvert = \sqrt{p^2 + q^2}$ $= \sqrt{5^2 + 2^2}$ $= \sqrt{29}$ units.
3 Use trigonometry to calculate the acute angle that the vector forms with the x-axis. This angle is in the first quadrant, so the answer represents the direction of the vector.	$\theta = \tan^{-1}\left(\frac{\lvert 2\rvert}{\lvert 5\rvert}\right)$ $= 21.80°$ (2 d.p.)
4 Express the vector in polar form, $\boldsymbol{a} = (\lvert\boldsymbol{a}\rvert, \angle\theta°)$.	$\boldsymbol{a} = \left(\sqrt{29}, 21.80°\right)$

12 Expressing the polar form of a vector from the Cartesian form

Calculate the magnitude and direction of the vector $\boldsymbol{n} = -3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, hence express the vector in polar form.

THINKING	WORKING
1 Draw a diagram to help visualise where this vector lies on the Cartesian plane.	
2 Use Pythagoras' theorem to calculate the magnitude of the vector.	$\lvert n\rvert = \sqrt{p^2 + q^2}$ $= \sqrt{(-3)^2 + (-4)^2}$ $= 5$ units
3 Use trigonometry to calculate the acute angle that the vector forms with the x-axis.	$\alpha = \tan^{-1}\left(\frac{\lvert -4\rvert}{\lvert -3\rvert}\right)$ $= 53.1301...°$
4 As this vector is in the third quadrant, add 180° to obtain the direction of the vector from the positive x-axis in an anticlockwise direction.	$\angle\theta° = 53.1301...° + 180°$ $= 233.13°$ (2 d.p.)
5 Express the vector in polar form, $\boldsymbol{a} = (\lvert\boldsymbol{a}\rvert, \angle\theta°)$.	$n = (5, 233.13°)$

Resolving vectors into Cartesian form

If you are given a vector in polar form, then you can use trigonometry to express that vector in Cartesian (component) form.

Consider vector $\boldsymbol{a}$ which has a magnitude of $|\boldsymbol{a}|$ and forms an angle θ with the x-axis. This vector can be resolved into component form (see right).

Using trigonometric rules to determine the values of x:

$$\cos(\theta) = \frac{\text{A}}{\text{H}}$$
$$\cos(\theta) = \frac{x}{|\boldsymbol{a}|}$$
$$x = |\boldsymbol{a}| \times \cos(\theta)$$

And to determine y:

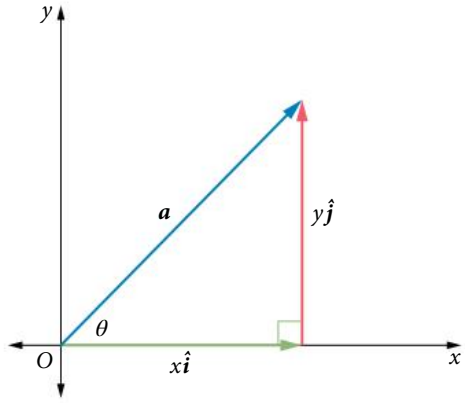
$$\sin(\theta) = \frac{\text{O}}{\text{H}}$$
$$\sin(\theta) = \frac{y}{|\boldsymbol{a}|}$$
$$y = |\boldsymbol{a}| \times \sin(\theta)$$

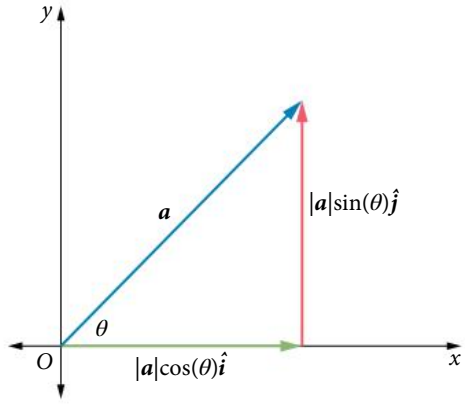
Therefore, $\boldsymbol{a} = (|\boldsymbol{a}|, \theta)$ expressed in Cartesian form is:

$\boldsymbol{a} = |\boldsymbol{a}|\cos(\theta)\hat{\boldsymbol{i}} + |\boldsymbol{a}|\sin(\theta)\hat{\boldsymbol{j}}$ (see right).

Note that if vector $\boldsymbol{a}$ lies in the second, third or fourth quadrant, then it is often easier to determine the angle θ to the nearest x-axis before calculating the components.

Technology worked example
Calculating the magnitude and direction of a vector





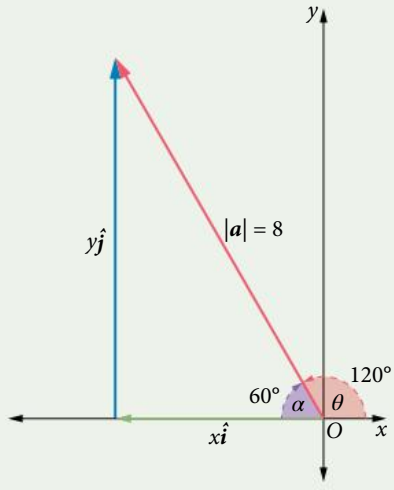
13 Resolving vectors into Cartesian form

Resolve the vector $\boldsymbol{a}$ into Cartesian form, given that $\boldsymbol{a} = (8, 120°)$.

THINKING

1 Draw a diagram to help visualise where this vector lies on the Cartesian plane. You should also indicate the acute angle to the nearest x-axis.

WORKING



2 Calculate horizontal and vertical components and check the reasonableness of your results.

Horizontal: $x = |\boldsymbol{a}|\cos(\theta)$
$= 8\cos(120°)$
$= -4$
Vertical: $y = |\boldsymbol{a}|\sin(\theta)$
$= 8\sin(120°)$
$= 8 \times \frac{\sqrt{3}}{2}$
$= 4\sqrt{3}$
$= 6.93$ (2 d.p.)
As expected, the horizontal component is negative and the vertical component is positive.

3 Express the vector in component form. Check to see whether any components are negative.

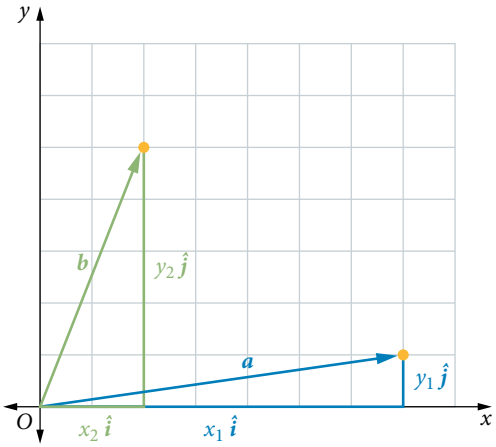
$\boldsymbol{a} = -4\hat{\boldsymbol{i}} + 6.93\hat{\boldsymbol{j}}$ (2 d.p.)

Addition and subtraction of vectors in Cartesian form

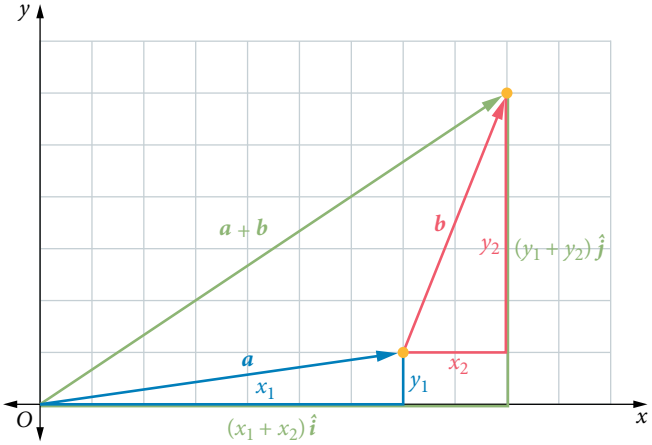
Addition and subtraction of vectors is performed by adding or subtracting the $\hat{\boldsymbol{i}}$ components and the $\hat{\boldsymbol{j}}$ components.

Technology worked example
Resolving a vector into Cartesian form

Consider $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$.



And $\boldsymbol{a} + \boldsymbol{b} = \left(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}\right) + \left(x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}\right)$
$= x_1\hat{\boldsymbol{i}} + x_2\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + y_2\hat{\boldsymbol{j}}$
$= \left(x_1 + x_2\right)\hat{\boldsymbol{i}} + \left(y_1 + y_2\right)\hat{\boldsymbol{j}}$



Similarly,

$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b} &= \left(x_1\hat{\boldsymbol{i}}+y_1\hat{\boldsymbol{j}}\right)-\left(x_2\hat{\boldsymbol{i}}+y_2\hat{\boldsymbol{j}}\right)\\ &= \left(x_1-x_2\right)\hat{\boldsymbol{i}}+\left(y_1-y_2\right)\hat{\boldsymbol{j}}\end{aligned}$$

If $\boldsymbol{a}=x_1\hat{\boldsymbol{i}}+y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b}=x_2\hat{\boldsymbol{i}}+y_2\hat{\boldsymbol{j}}$, then

$$\boldsymbol{a}+\boldsymbol{b}=\left(x_1+x_2\right)\hat{\boldsymbol{i}}+\left(y_1+y_2\right)\hat{\boldsymbol{j}}$$
$$\boldsymbol{a}-\boldsymbol{b}=\left(x_1-x_2\right)\hat{\boldsymbol{i}}+\left(y_1-y_2\right)\hat{\boldsymbol{j}}$$

Scalar multiplication of vectors in Cartesian form

When multiplying a vector by a scalar, simply multiply each component by that scalar.

$$\begin{aligned}k\boldsymbol{a} &= k\left(x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}\right)\\ &= kx\hat{\boldsymbol{i}}+ky\hat{\boldsymbol{j}}\end{aligned}$$

If $\boldsymbol{a}=x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}$, then $k\boldsymbol{a}=kx\hat{\boldsymbol{i}}+ky\hat{\boldsymbol{j}}$

14 Addition, subtraction and scalar multiplication of vectors in Cartesian form

If $\boldsymbol{a}=\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}$ and $\boldsymbol{b}=-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}$, determine the following.

(a) $\boldsymbol{a}+\boldsymbol{b}$

THINKING	WORKING
1 Write the sum as the sum of the vectors in component form.	$\boldsymbol{a}+\boldsymbol{b}=\left(\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}\right)+\left(-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)$
2 Group the coefficients of the components together and simplify the expression.	$=(1-3)\hat{\boldsymbol{i}}+(-5+2)\hat{\boldsymbol{j}}$ $=-2\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}$

(b) $\boldsymbol{b}-\boldsymbol{a}$

1 Write the difference as the sum of the vectors in component form.	$\boldsymbol{b}-\boldsymbol{a}=\left(-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)-\left(\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}\right)$
2 Group the coefficients of the components together and simplify the expression.	$=(-3-1)\hat{\boldsymbol{i}}+\left(2-(-5)\right)\hat{\boldsymbol{j}}$ $=-4\hat{\boldsymbol{i}}+7\hat{\boldsymbol{j}}$

(c) $-4\boldsymbol{a}+7\boldsymbol{b}$

1 Write the difference as the sum of the vectors in component form.	$-4\boldsymbol{a}+7\boldsymbol{b}=-4\left(\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}\right)+7\left(-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)$
2 Multiply each vector by the given scalar quantity.	$=-4\times\hat{\boldsymbol{i}}-4\times-5\hat{\boldsymbol{j}}+7\times-3\hat{\boldsymbol{i}}+7\times2\hat{\boldsymbol{j}}$ $=-4\hat{\boldsymbol{i}}+20\hat{\boldsymbol{j}}-21\hat{\boldsymbol{i}}+14\hat{\boldsymbol{j}}$
3 Group the coefficients of the components together and simplify the expression.	$=(-4-21)\hat{\boldsymbol{i}}+(20+14)\hat{\boldsymbol{j}}$ $=-25\hat{\boldsymbol{i}}+34\hat{\boldsymbol{j}}$

Equality of vectors in component form

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$, then $\boldsymbol{a} = \boldsymbol{b}$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

15 Equality of vectors in component form

Calculate the values of m and n if $7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} = (3m+1)\hat{\boldsymbol{i}} + (4n-9)\hat{\boldsymbol{j}}$.

THINKING	WORKING
1 Equate coefficients of the vector components.	$\hat{\boldsymbol{i}}$ components: $7 = 3m + 1$ $\hat{\boldsymbol{j}}$ components: $-5 = 4n - 9$
2 Solve the equations to calculate the unknown values.	$7 = 3m + 1$ and $-5 = 4n - 9$ $3m = 6$ $\quad$ $4n = 4$ $m = 2$ $\quad$ $n = 1$
3 Verify your answer by substituting the values of m and n into the vector equation given.	For $m = 2$ and $n = 1$: $(3m+1)\hat{\boldsymbol{i}} + (4n-9)\hat{\boldsymbol{j}}$ $= (3(2)+1)\hat{\boldsymbol{i}} + (4(1)-9)\hat{\boldsymbol{j}}$ $= (6+1)\hat{\boldsymbol{i}} + (4-9)\hat{\boldsymbol{j}}$ $= 7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$
4 State your answer.	For $m = 2$ and $n = 1$, $7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} = (3m+1)\hat{\boldsymbol{i}} + (4n-9)\hat{\boldsymbol{j}}$

Parallel vectors

Two vectors are parallel if they are scalar multiples of each other.

For example, if $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$, then $\boldsymbol{b} = 4\left(\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$.

Therefore, $\boldsymbol{b} = 4\boldsymbol{a}$, so $\boldsymbol{b}$ is parallel to $\boldsymbol{a}$.

If $\boldsymbol{b} = k\boldsymbol{a}$, where $k \in \mathbb{R}$, then $\boldsymbol{b}$ is parallel to $\boldsymbol{a}$.

Unit vectors in component form

Recall that a unit vector has a magnitude of 1 and $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$.

Therefore, if you are given a vector in component form such that $\boldsymbol{a} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, you can calculate the unit vector in the direction of $\boldsymbol{a}$ as follows:

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{\sqrt{x^2 + y^2}}$$
$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right)$$

You will recall from your work with surds that it is often desirable to rationalise the denominator of a fraction by multiplying by a root or conjugate. To express the unit vector with a rational denominator, proceed as follows:

$$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{x^2+y^2}}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right) \times \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$\hat{\boldsymbol{a}} = \frac{\sqrt{x^2+y^2}}{x^2+y^2}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right)$$

If $\boldsymbol{a} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$,

then $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{x^2+y^2}}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right) = \frac{\sqrt{x^2+y^2}}{x^2+y^2}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right)$

Unit vectors are useful for indicating a particular direction. If you wish to determine a vector $\boldsymbol{b}$ with a given magnitude and in the direction of another vector, $\boldsymbol{a}$, calculate $\hat{\boldsymbol{a}}$ and multiply it by the given magnitude.

16 Unit vectors in component form

If $\boldsymbol{c} = 3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$

(a) Calculate $\hat{\boldsymbol{c}}$.

THINKING	WORKING
1 Calculate the magnitude of the vector.	$\lvert\boldsymbol{c}\rvert = \sqrt{3^2+(-6)^2}$ $= \sqrt{45}$ $= 3\sqrt{5}$
2 Determine the unit vector by dividing the vector by its magnitude and express the unit vector with a rational denominator	$\boldsymbol{c} = \frac{3}{3\sqrt{5}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ $= \frac{1}{\sqrt{5}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

(b) Determine $\boldsymbol{d}$, in the direction of $\boldsymbol{c}$ with a magnitude of 5.

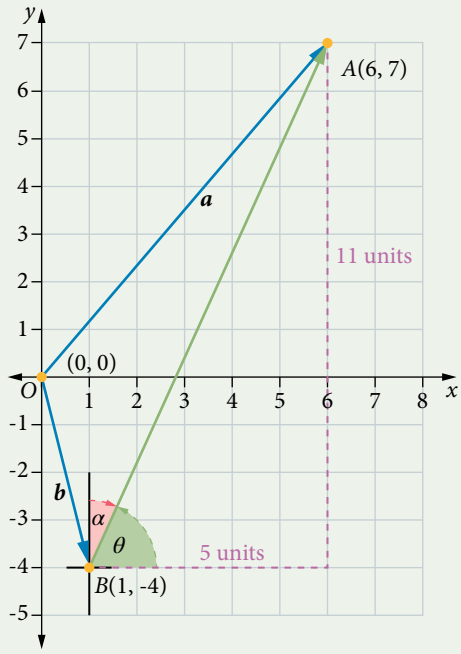
1 Identify the scalar multiplication.	$\boldsymbol{d} = 5\hat{\boldsymbol{c}}$
2 Multiply the unit vector by the given scalar quantity.	$\boldsymbol{d} = \frac{5}{\sqrt{5}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ $= \sqrt{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

Relative position vectors in component form

Recall that if point A is defined by the position vector $\boldsymbol{a}$ and point B is defined by the position vector $\boldsymbol{b}$ then $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$. This information can be applied to vectors in component form, and utilised to solve a variety of different problems.

17 Relative position vectors in component form

Appleton is 7 km north and 6 km east of Riverdale, while Burrowdale is 4 km south and 1 km east of Riverdale. Calculate the direct distance between Burrowdale and Appleton, as well as the true bearing of Appleton from Burrowdale stated to the nearest degree.

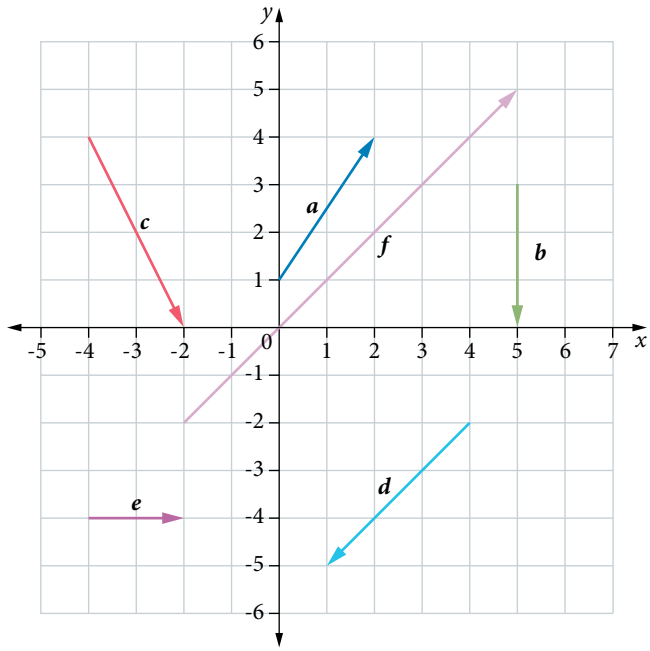
THINKING	WORKING
1 Choose a fixed point as the origin, and create position vectors for each town.	Let Riverdale be the origin, (0, 0). Therefore, Appleton will have the position vector $\boldsymbol{a} = 6\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and Burrowdale will have the position vector $\boldsymbol{b} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$.
2 Create a graphical representation of the scenario.	
3 Determine the relative position vector.	$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b}$ $= \left(6\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}\right) - \left(\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right)$ $= 5\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}$
4 Calculate the magnitude of the relative position vector.	$\left\lvert\overrightarrow{BA}\right\rvert = \left\lvert 5\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}\right\rvert$ $= \sqrt{5^2 + 11^2}$ $= \sqrt{146}$ ≈ 12.083 km
5 Calculate the direction of the relative position vector (to the positive x-axis), and use this to determine the bearing.	$\theta = \tan^{-1}\left(\frac{\lvert 11\rvert}{\lvert 5\rvert}\right)$ $= 65.556...°$ $\alpha = 90 - 65.556...°$ $= 24.44°$ (2 d.p.)
6 Interpret the solution.	The distance between Burrowdale and Appleton is 12.08 km (2 d.p.), and the bearing of Appleton from Burrowdale is 024°T.

EXERCISE 2.3 Vectors in component form

Worked Example

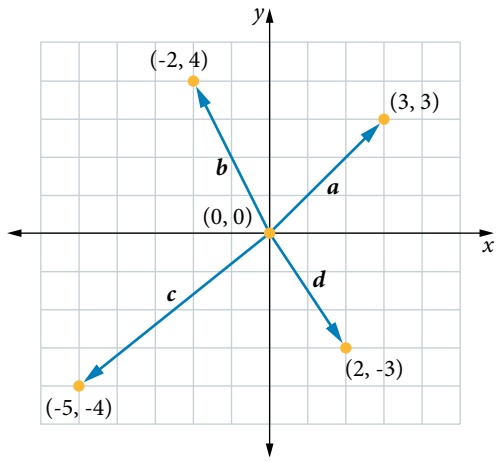
10

1 Express in Cartesian (component) form each vector (***a*** to ***f***) shown.



11

2 Calculate the magnitude and direction of the four vectors shown, and hence express the vectors in polar form.



12

3 Express the following vectors in polar form.

(a) $\boldsymbol{a} = 10\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ (b) $\boldsymbol{b} = 8\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ (c) $\boldsymbol{c} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ (d) $\boldsymbol{d} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

13

4 Resolve the following vectors into component form. Give answers correct to 2 decimal places.

(a) $\boldsymbol{a} = (15, 35°)$ (b) $\boldsymbol{b} = (23, 121°)$ (c) $\boldsymbol{c} = (10, 315°)$ (d) $\boldsymbol{d} = (36, 185°)$

14

5 If $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, express the following in component form.

(a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{b} - \boldsymbol{a}$ (c) $2\boldsymbol{a} + 7\boldsymbol{b}$ (d) $3\boldsymbol{a} - 5\boldsymbol{b}$

15

6 Calculate the values of m and n if $-6\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} = (4m - 2)\hat{\boldsymbol{i}} + (5 - 2n)\hat{\boldsymbol{j}}$.

7 (a) Which of the following is a unit vector?

A $\boldsymbol{a} = 0.5\hat{\boldsymbol{i}} + 0.5\hat{\boldsymbol{j}}$ B $\boldsymbol{b} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ C $\boldsymbol{c} = \hat{\boldsymbol{i}}$ D $\boldsymbol{d} = \dfrac{|\hat{\boldsymbol{i}}|}{|\hat{\boldsymbol{j}}|}$

(b) Explain why the alternatives are incorrect and what the common errors are.

8 If $\boldsymbol{a}$ and $\boldsymbol{b}$ represent vectors on the plane, which of these is correct?

A $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}| + |\boldsymbol{b}|$ B $|\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$ C $|\boldsymbol{a}+\boldsymbol{b}| \geq |\boldsymbol{a}| + |\boldsymbol{b}|$ D $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{b}| - |\boldsymbol{a}|$

9 Determine the vector equation $\boldsymbol{a}$ with a horizontal component of -28 that is parallel to $\boldsymbol{b} = 14\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$.

10 Determine the values of the unknown pronumerals in each of the following vector equations.

(a) $5\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} = 3a\hat{\boldsymbol{i}} + 2b\hat{\boldsymbol{j}}$ (b) $(x+2y)\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} = -3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$

Worked Example 16

11 Determine the unit vector, $\hat{\boldsymbol{a}}$, for each of the following vectors.

(a) $\boldsymbol{a} = 8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$ (c) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$ (d) $\boldsymbol{a} = -\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$

12 Let $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}}$.

(a) Determine $\hat{\boldsymbol{b}}$. (b) Determine vector $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$ with a magnitude of 15.

13 Calculate the values of the unknown pronumerals in each of the following vector equations:

(a) $(2g+1)\hat{\boldsymbol{i}} - (3h-1)\hat{\boldsymbol{j}} = 3\left(h\hat{\boldsymbol{i}} + 2g\hat{\boldsymbol{j}}\right)$ (b) $(3p-18)\hat{\boldsymbol{i}} + (3q-8)\hat{\boldsymbol{j}} = 2p\left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right) - q\left(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)$

14 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, determine $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = 2\boldsymbol{a} - 3\boldsymbol{b}$.

15 If $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, determine the vector $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$ where $|\boldsymbol{c}| = |\boldsymbol{a}|$.

16 The position vectors of points C and D are $\overrightarrow{OC} = -8\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ and $\overrightarrow{OD} = 13\hat{\boldsymbol{i}} - 23\hat{\boldsymbol{j}}$ respectively.

(a) Calculate $\overrightarrow{DC}$. (b) Calculate $\left|\overrightarrow{DC}\right|$.

17 In parallelogram $ABCD$, $\overrightarrow{AB} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\overrightarrow{AD} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$. Calculate the following.

(a) $\overrightarrow{CD}$ (b) $\overrightarrow{CA}$ (c) $\overrightarrow{DB}$

(d) $\left|\overrightarrow{AB}\right|$ (e) $\left|\overrightarrow{CA}\right|$ (f) $\left|\overrightarrow{DB}\right|$

18 Point A can be represented by the position vector $\boldsymbol{a} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and point B can be represented by the position vector $\boldsymbol{b} = -6\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$. 17

(a) Calculate $\left|\overrightarrow{AB}\right|$ (b) Calculate the direction of $\overrightarrow{AB}$.

19 Let $\boldsymbol{a} = 3p\hat{\boldsymbol{i}} + 4p\hat{\boldsymbol{j}}$, $p > 0$ and $|\boldsymbol{a}| = 2$.

(a) Determine the exact value of p. (b) Hence, calculate $\hat{\boldsymbol{a}}$.

(c) Determine the vector $\boldsymbol{b}$ which is parallel to $\hat{\boldsymbol{a}}$ and $|\boldsymbol{b}| = 10$.

(d) If $\boldsymbol{c} = 7q\hat{\boldsymbol{i}} + 24q\hat{\boldsymbol{j}}$, $q > 0$ and $|\boldsymbol{c}| = 4$, calculate the exact value of q.

(e) Hence, calculate $\hat{\boldsymbol{c}}$. (f) Determine the vector $\boldsymbol{d}$ in the direction of $\hat{\boldsymbol{c}}$ where $|\boldsymbol{d}| = 50$.

(g) Determine the vector with magnitude 10 that is parallel to the vector $\boldsymbol{b} + \boldsymbol{d}$.

20 Calculate the values of the unknown pronumerals in each of the following vector equations.

(a) $\left(x^2 + 5x\right)\hat{\boldsymbol{i}} + \left(y^3 - 1\right)\hat{\boldsymbol{j}} = -6\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ (b) $\left(x^2 - 4x\right)\hat{\boldsymbol{i}} + \left(y^4 + 9\right)\hat{\boldsymbol{j}} = 2\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}$

21 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.

(a) Calculate $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$. (b) Hence, calculate the exact values of x and y, such that $2x\hat{\boldsymbol{a}} + 3y\hat{\boldsymbol{b}} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

22 Let $\boldsymbol{c} = 4\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}}$ and $\boldsymbol{d} = 8\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.

(a) Calculate $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{d}}$. (b) Hence, calculate the exact values of v and w, such that $2v\hat{\boldsymbol{c}} + 3w\hat{\boldsymbol{d}} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

2.4 Scalar product of vectors

The scalar product

The *scalar product* (also called the dot product) is a way of multiplying two vectors. The result of the multiplication is a scalar quantity (magnitude but no direction). The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $\boldsymbol{a} \cdot \boldsymbol{b}$ (and read as '$\boldsymbol{a}$ dot $\boldsymbol{b}$').

Consider the vectors where $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 6$ as shown:



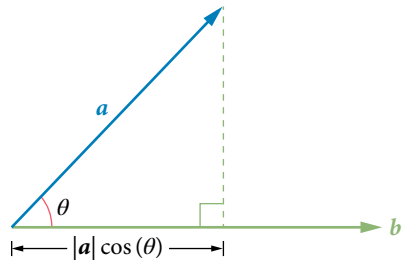
If you were to calculate the dot product $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| = 5 \times 6 = 30$, which represents the product of the magnitude of the vectors (in the same direction), the answer is 30.

The following diagram shows $\boldsymbol{a}$ and $\boldsymbol{b}$ if they are vectors in opposite directions.



The product of the magnitude of the vectors (in the same direction) is -30 because the vectors, although aligned, are in opposite directions.

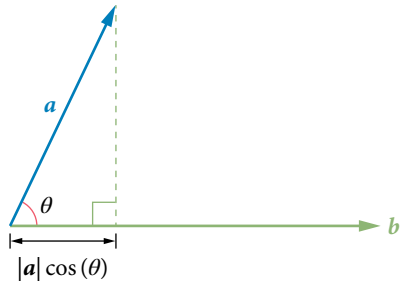
Consider then, that the vectors are placed at any given angle θ apart:



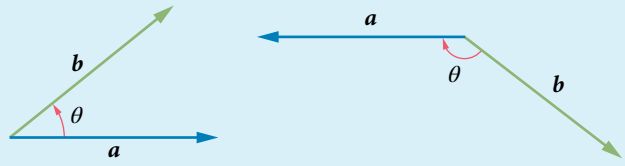
To calculate the dot product, the value of the product of the magnitude of the vectors (in the same direction) is given by: $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cos(\theta) \times |\boldsymbol{b}|$

$= |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta)$

As $\theta \to 90°$ the product of the magnitude of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ decreases, $\boldsymbol{a} \cdot \boldsymbol{b} \to 0$.



If θ is the angle between the positive directions of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar product is defined to be $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta)$.

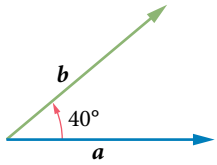


Note: You always use the acute or obtuse angle, not the reflex angle.

18 Calculating the scalar product of two vectors in the plane

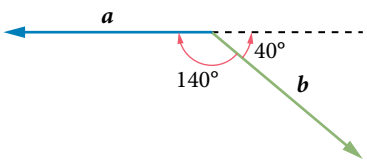
If $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 6$, calculate the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$, correct to 2 decimal places, in each of the following.

(a)



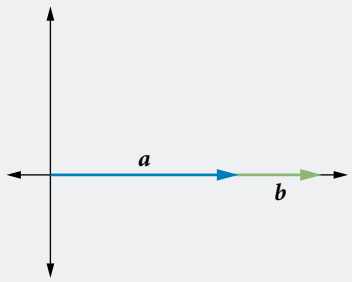
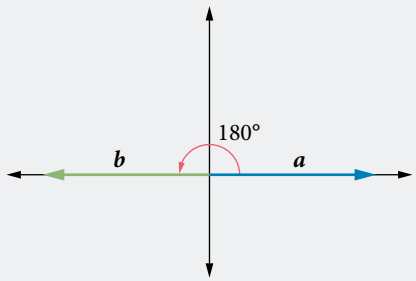
THINKING	WORKING
If θ is the angle between the positive directions of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar product is defined to be $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$.	$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$ $= 5 \times 6 \times \cos(40°)$ $= 22.98$

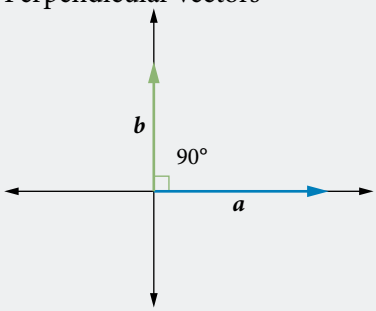
(b)



If θ is the angle between the positive directions of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar product is defined to be $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$.	$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$ $= 5 \times 6 \times \cos(140°)$ $= -22.98$

Special cases of the scalar product

Special Case	Explanation
Parallel vectors (in the same direction) 	$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$ In this case, $\theta = 0°$ and $\cos(0°) = 1$. Therefore, $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(0°)$ $= \|\boldsymbol{a}\|\|\boldsymbol{b}\| \times 1$ $= \|\boldsymbol{a}\|\|\boldsymbol{b}\|$
Parallel vectors (in the opposite direction) 	$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$ In this case, $\theta = 180°$ so $\cos(180°) = -1$. Therefore, $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(180°)$ $= \|\boldsymbol{a}\|\|\boldsymbol{b}\| \times -1$ $= -\|\boldsymbol{a}\|\|\boldsymbol{b}\|$

Special Case	Explanation
Perpendicular vectors 	$\boldsymbol{a}\cdot\boldsymbol{b}=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos(\theta)$ In this case, $\theta = 90°$ so $\cos(90°)=0$. Therefore, $\boldsymbol{a}\cdot\boldsymbol{b}=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos(90°)$ $=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\times 0$ $=0$

This final case is particularly important. If the dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to zero, then either $|\boldsymbol{a}|=0$, or $|\boldsymbol{b}|=0$, or $\theta = 90°$.

For two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, if $\boldsymbol{a}\cdot\boldsymbol{b}=0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

Scalar product for vectors in component form

Recall that unit vector $\hat{\boldsymbol{i}}$ is defined as a vector of magnitude 1 in the direction of the positive x-axis and unit vector $\hat{\boldsymbol{j}}$ is defined as a vector of magnitude 1 in the direction of the positive y-axis. Because these unit vectors are mutually perpendicular, this means that $\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}=\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{i}}=0$. Also, using the property of parallel vectors, you know that $\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{i}}=\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}}=1$.

The dot product is a commutative operation: $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$.

The dot product is also distributive: $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$.

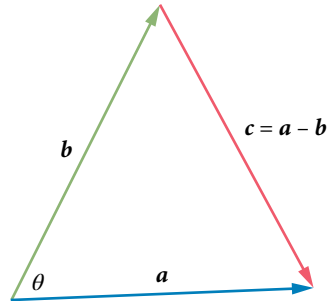
Therefore, consider the two vectors in component form, $\boldsymbol{a}=x_1\hat{\boldsymbol{i}}+y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b}=x_2\hat{\boldsymbol{i}}+y_2\hat{\boldsymbol{j}}$. Using the above information, the dot product $\boldsymbol{a}\cdot\boldsymbol{b}$ can be calculated as follows:

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}&=\left(x_1\hat{\boldsymbol{i}}+y_1\hat{\boldsymbol{j}}\right)\cdot\left(x_2\hat{\boldsymbol{i}}+y_2\hat{\boldsymbol{j}}\right)\\&=(x_1x_2)\left(\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{i}}\right)+(x_1y_2)\left(\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}\right)+(y_1x_2)\left(\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{i}}\right)+(y_1y_2)\left(\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}}\right)\\&=x_1x_2+y_1y_2\end{aligned}$$

If $\boldsymbol{a}=x_1\hat{\boldsymbol{i}}+y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b}=x_2\hat{\boldsymbol{i}}+y_2\hat{\boldsymbol{j}}$, then $\boldsymbol{a}\cdot\boldsymbol{b}=x_1x_2+y_1y_2$.

Relationship to the cosine rule

Consider the triangle formed by vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, where $\boldsymbol{c}=\boldsymbol{a}-\boldsymbol{b}$. Calculating the dot product of vector $\boldsymbol{c}$ with itself:



$$\begin{aligned}\boldsymbol{c}\cdot\boldsymbol{c}&=(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})\\\boldsymbol{c}\cdot\boldsymbol{c}&=(\boldsymbol{a}\cdot\boldsymbol{a})-(\boldsymbol{a}\cdot\boldsymbol{b})-(\boldsymbol{b}\cdot\boldsymbol{a})+(\boldsymbol{b}\cdot\boldsymbol{b})\\\boldsymbol{c}\cdot\boldsymbol{c}&=(\boldsymbol{a}\cdot\boldsymbol{a})-2(\boldsymbol{a}\cdot\boldsymbol{b})+(\boldsymbol{b}\cdot\boldsymbol{b})\\|\boldsymbol{c}||\boldsymbol{c}|\cos(0°)&=|\boldsymbol{a}||\boldsymbol{a}|\cos(0°)-2\left(|\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)\right)+|\boldsymbol{b}||\boldsymbol{b}|\cos(0°)\\|\boldsymbol{c}|^2&=|\boldsymbol{a}|^2-2|\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)+|\boldsymbol{b}|^2\\|\boldsymbol{c}|^2&=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-2|\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)\end{aligned}$$

Thus, the dot product can be used to verify the cosine rule.

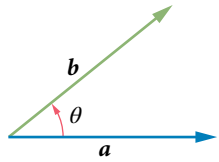
19 Scalar product for vectors in component form

Calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ given $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.

THINKING	WORKING
1 Create a scalar product.	$\boldsymbol{a} \cdot \boldsymbol{b} = \left(2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right) \cdot \left(-3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$
2 Recall the scalar product rule for vectors in component form.	$\boldsymbol{a} \cdot \boldsymbol{b} = x_1 x_2 + y_1 y_2$
3 Multiply the coefficients of the like components and add.	$= 2 \times (-3) + 5 \times 4$ $= 14$

Calculating the angle between two vectors

The scalar product can be used to calculate the angle between two vectors. Let θ be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in the diagram.



By the definition of the dot product, $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta)$.

$$\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

When working in Cartesian form, you also know that if $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = x_1 x_2 + y_1 y_2$.

Therefore,

$$|\boldsymbol{a}||\boldsymbol{b}| \cos(\theta) = x_1 x_2 + y_1 y_2$$

$$\cos(\theta) = \frac{x_1 x_2 + y_1 y_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

If θ is the angle between vectors $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$, then $\cos(\theta) = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{x_1 x_2 + y_1 y_2}{|\boldsymbol{a}||\boldsymbol{b}|}$.

20 Calculating the angle between two vectors

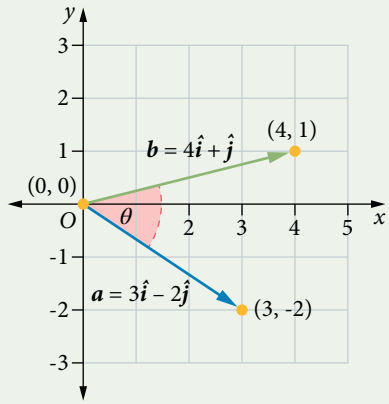
Calculate the angle, in degrees correct to 2 decimal places, between vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

THINKING	WORKING
1 Calculate the scalar product of the vectors, $\boldsymbol{a} \cdot \boldsymbol{b}$.	$\boldsymbol{a} \cdot \boldsymbol{b} = \left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right) \cdot \left(4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)$ $= 3 \times 4 + (-2) \times 1$ $= 10$
2 Calculate the magnitude of each of the vectors.	$\|\boldsymbol{a}\| = \sqrt{3^2 + (-2)^2}$ $= \sqrt{13}$ $\|\boldsymbol{b}\| = \sqrt{4^2 + 1^2}$ $= \sqrt{17}$

3 Substitute into the rule $\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$ and simplify.

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$$
$$= \cos^{-1}\left(\frac{10}{\sqrt{13}\sqrt{17}}\right)$$
$$= 47.73° \text{ (2 d.p.)}$$

4 Evaluate the reasonableness of your solution graphically.



The angle between vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ is approximately 47.73°.

Explore further

The angle between two vectors

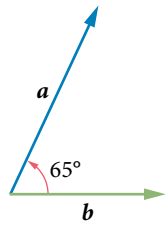
Create dynamic vectors and calculate the angles between them.

EXERCISE 2.4 Scalar product of vectors

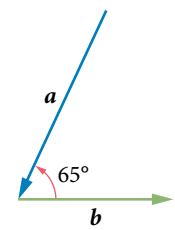
Worked Example 18

1 If $|\boldsymbol{a}| = 8$ and $|\boldsymbol{b}| = 7$, calculate the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$, correct to 2 decimal places, in each of the following.

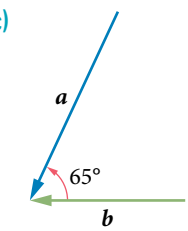
(a)



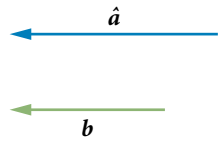
(b)



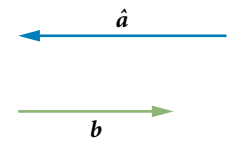
(c)



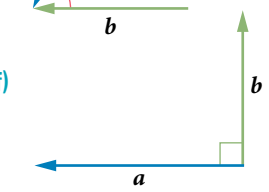
(d)



(e)



(f)



19

2 Calculate the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$, given:

(a) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

(b) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

(c) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$

(d) $\boldsymbol{a} = -4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 7\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$

(e) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -7\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

(f) $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

3 If $\boldsymbol{a} \cdot \boldsymbol{b} = -|\boldsymbol{b}|^2$, then which of these is true?

A Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal in magnitude and opposite in direction.

B Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal in magnitude and in the same direction.

C Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal in magnitude and perpendicular to one another.

D Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ must be unit vectors.

4 (a) Given $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$, identify the correct statement.

A Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

B Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

C Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ must have the same magnitude.

D Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ must be equal.

(b) Explain the common errors made in selecting the incorrect options.

5 Demonstrate that the vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 7\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ are perpendicular.

6 If the vectors $\boldsymbol{e} = 7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{f} = x\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ are perpendicular, calculate the value of x.

7 If $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$, what is the value of $\boldsymbol{a} \cdot \boldsymbol{b}$?

20

8 Calculate the angle, correct to the nearest degree, between each of the following pairs of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

(a) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

(b) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$

(c) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

(d) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

9 Determine the vector $\boldsymbol{d}$ that is perpendicular to $\boldsymbol{c} = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and has a magnitude of 10.

10 If vectors $\boldsymbol{a} = 2p\hat{\boldsymbol{i}} - q\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = q\hat{\boldsymbol{i}} + 5q\hat{\boldsymbol{j}}$ are perpendicular, then express p in terms of q.

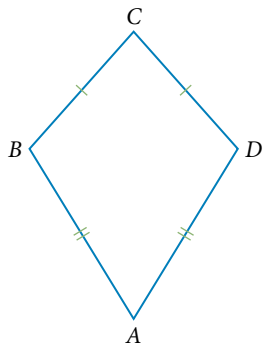
11 Point A is represented by position vector $\boldsymbol{a} = -2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ and point C is represented by position vector $\boldsymbol{c} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$. Point B lies on the curve $y = x^2$. Using vector methods, determine the coordinates of B such that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.

12 The points A, B and C have position vectors $\overrightarrow{OA} = -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, $\overrightarrow{OB} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OC} = 8\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$. Using vector techniques, demonstrate that these points form a right-angled, isosceles triangle.

13 $ABCD$ is a kite as shown in the diagram. A kite has two pairs of adjacent sides that are equal in length, that is, $\overline{AB} = \overline{AD}$ and $\overline{CB} = \overline{CD}$.

(a) Demonstrate that the quadrilateral $ABCD$ whose vertices have position vectors $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\overrightarrow{OB} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, $\overrightarrow{OC} = 8\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and $\overrightarrow{OD} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ respectively is a kite.

(b) Demonstrate that the diagonals of this kite cross at right angles.

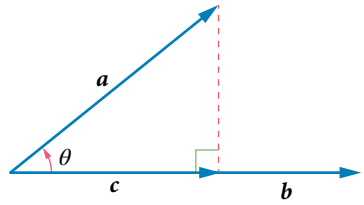


2.5 Projections of vectors

A vector can be resolved into a sum of two vectors that are perpendicular to each other. It is usual to resolve the vector into two perpendicular components with one in a specified direction (i.e. parallel to a given vector) and the other perpendicular to the specified direction.

Scalar projection

In the diagram, vector $\boldsymbol{c}$ is the vector projection of vector $\boldsymbol{a}$ in the direction of vector $\boldsymbol{b}$.



The magnitude of $\boldsymbol{c}$, $|\boldsymbol{c}|$, is called the scalar projection of vector $\boldsymbol{a}$ onto vector $\boldsymbol{b}$. (You may also see this referred to as the scalar resolute of vector $\boldsymbol{a}$ onto vector $\boldsymbol{b}$).

Using trigonometry, $\cos(\theta) = \dfrac{|\boldsymbol{c}|}{|\boldsymbol{a}|}$.

Using the definition of the dot product, $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$ and so

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\frac{|\boldsymbol{c}|}{|\boldsymbol{a}|}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{b}||\boldsymbol{c}|$$

$$\therefore |\boldsymbol{c}| = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$$

Using the definition of the unit vector,

$$\frac{\boldsymbol{b}}{|\boldsymbol{b}|} = \hat{\boldsymbol{b}}$$

$$\therefore |\boldsymbol{c}| = \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$$

The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\boldsymbol{a} \cdot \hat{\boldsymbol{b}}$, where $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

21 Calculate the scalar projection

Consider the vectors $\boldsymbol{a} = 5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.

(a) Calculate the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

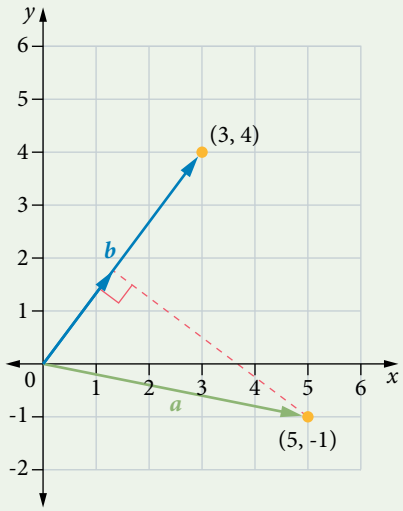
THINKING	WORKING
1 Calculate the unit vector $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{\|\boldsymbol{b}\|}$ by first calculating the magnitude of vector $\boldsymbol{b}$.	$\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ $\|\boldsymbol{b}\| = \sqrt{3^2 + 4^2}$ $= \sqrt{25}$ $= 5$ $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{\|\boldsymbol{b}\|} = \dfrac{1}{5}\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$

2 Calculate the scalar projection $\boldsymbol{a} \cdot \hat{\boldsymbol{b}}$.

$$\begin{aligned}\boldsymbol{a} \cdot \hat{\boldsymbol{b}} &= \left(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right) \cdot \frac{1}{5}\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right) \\ &= \frac{15}{5} - \frac{4}{5} \\ &= \frac{11}{5}\end{aligned}$$

3 Interpret the answer.

The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\frac{11}{5} = 2.2$.



(b) Calculate the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

1 Calculate $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ by first calculating $|\boldsymbol{a}|$.

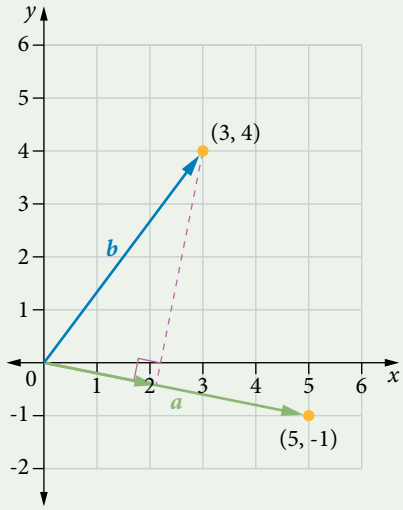
$$\begin{aligned}\boldsymbol{a} &= 5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} \\ |\boldsymbol{a}| &= \sqrt{5^2 + (-1)^2} \\ &= \sqrt{26} \\ \hat{\boldsymbol{a}} &= \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{1}{\sqrt{26}}\left(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)\end{aligned}$$

2 Calculate the scalar projection $\boldsymbol{b} \cdot \hat{\boldsymbol{a}}$.

$$\begin{aligned}\boldsymbol{b} \cdot \hat{\boldsymbol{a}} &= \left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right) \cdot \frac{1}{\sqrt{26}}\left(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right) \\ &= \frac{15}{\sqrt{26}} - \frac{4}{\sqrt{26}} \\ &= \frac{11}{\sqrt{26}} \\ &= \frac{11\sqrt{26}}{26}\end{aligned}$$

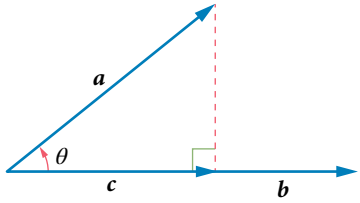
3 Interpret the answer.

The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\frac{11}{\sqrt{26}} = 2.16$ (2 d.p.).



Vector projection

Vector $\boldsymbol{c}$ in the diagram is the *vector projection* of vector $\boldsymbol{a}$ onto $\boldsymbol{b}$. This vector will have the same magnitude as the scalar projection, and will be parallel to vector $\boldsymbol{b}$. (You may also see this referred to as the *vector resolute* of vector $\boldsymbol{a}$ onto vector $\boldsymbol{b}$.) Hence,



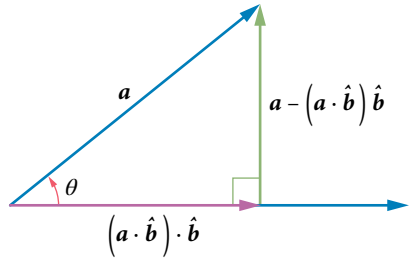
$$\boldsymbol{c} = |\boldsymbol{c}|\hat{\boldsymbol{b}}$$
$$= \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$$

The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is

$$\boldsymbol{a} - \boldsymbol{c} = \boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$$

This is also known as the vector rejection of $\boldsymbol{a}$ from $\boldsymbol{b}$.

In general, the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ does not equal the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.



The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is $\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

t Technology worked example

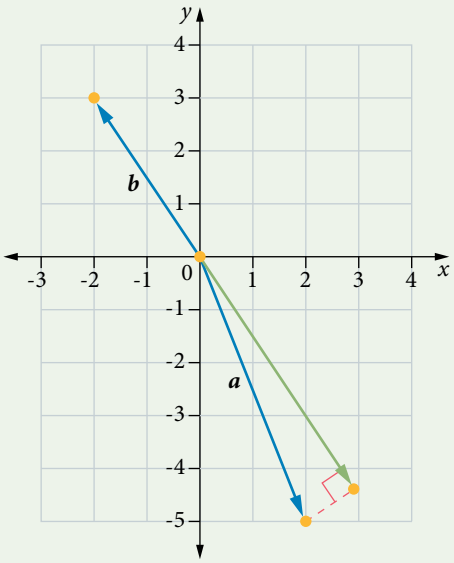
Calculating vector and scalar projections

22 Calculating vector projections

Consider the vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.

(a) Calculate the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

THINKING	WORKING
1 Calculate $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{\lvert\boldsymbol{b}\rvert}$ by first calculating $\lvert\boldsymbol{b}\rvert$.	$\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ $\lvert\boldsymbol{b}\rvert = \sqrt{(-2)^2 + 3^2}$ $= \sqrt{13}$ $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{\lvert\boldsymbol{b}\rvert} = \dfrac{1}{\sqrt{13}}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$
2 Calculate the scalar projection $\boldsymbol{a} \cdot \hat{\boldsymbol{b}}$.	$\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \left(2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}\right) \cdot \dfrac{1}{\sqrt{13}}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$ $= \dfrac{-4}{\sqrt{13}} + \dfrac{-15}{\sqrt{13}}$ $= \dfrac{-19}{\sqrt{13}}$
3 Calculate the vector projection $\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.	$\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} = \dfrac{-19}{\sqrt{13}} \times \dfrac{1}{\sqrt{13}}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$ $= \dfrac{-19}{13}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$
4 Answer the question.	The vector projection of $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ onto $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ is $\dfrac{-19}{13}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$.



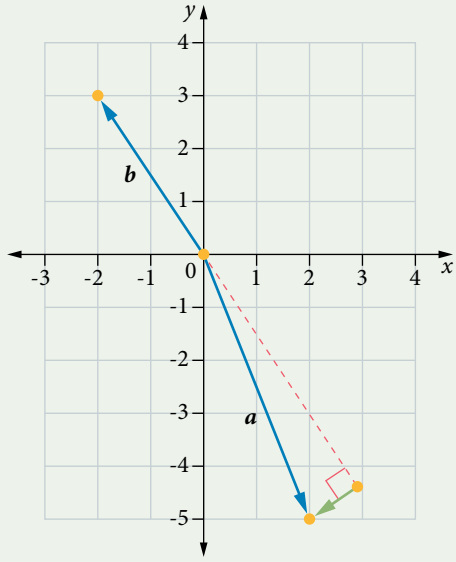
(b) Calculate the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

1 Calculate $\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

$$\begin{aligned}\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} &= 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - \frac{-19}{13}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right) \\ &= 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - \frac{38}{13}\hat{\boldsymbol{i}} + \frac{57}{13}\hat{\boldsymbol{j}} \\ &= -\frac{12}{13}\hat{\boldsymbol{i}} - \frac{8}{13}\hat{\boldsymbol{j}} \\ &= -\frac{4}{13}\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)\end{aligned}$$

2 Answer the question.

The vector projection of $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ perpendicular to $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ is $-\frac{4}{13}\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)$.



(c) Verify that the parallel and perpendicular vector projections of $\boldsymbol{a}$ onto $\boldsymbol{b}$ are perpendicular.

1 Define the vector projections in the form $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

Let

$$\boldsymbol{c} = \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} \qquad \boldsymbol{d} = \boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$$

$$= -\frac{19}{13}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right) \quad \text{and} \quad = -\frac{4}{13}\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)$$

$$= \frac{38}{13}\hat{\boldsymbol{i}} - \frac{57}{13}\hat{\boldsymbol{j}} \qquad = -\frac{12}{13}\hat{\boldsymbol{i}} - \frac{8}{13}\hat{\boldsymbol{j}}$$

2 Recall the formula to calculate the scalar product of the projections $\boldsymbol{a} \cdot \boldsymbol{b} = x_1x_2 + y_1y_2$.

$$\begin{aligned}\boldsymbol{c} \cdot \boldsymbol{d} &= \frac{38}{13} \times -\frac{12}{13} + \left(-\frac{57}{13} \times -\frac{8}{13}\right) \\ &= -\frac{456}{169} + \frac{456}{169} \\ &= 0\end{aligned}$$

3 Interpret the answer.

Given that the scalar product of the parallel and perpendicular vector projections is 0, the determined projections are in fact perpendicular, that is:

$$\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} \cdot \left[\boldsymbol{a} - \left(\boldsymbol{a} - \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}\right] = 0$$

EXERCISE 2.5 Projections of vectors

Worked Example 21

1 For each of the following pairs of vectors, calculate the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(a) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ (c) $\boldsymbol{a} = 8\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$

(d) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ (e) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ (f) $\boldsymbol{a} = -5\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

2 Consider $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ are unit vectors, and $|\boldsymbol{c}|$ is the scalar projection of $\hat{\boldsymbol{a}}$ onto $\hat{\boldsymbol{b}}$.

(a) Which of the following statements is true?

A $|\boldsymbol{c}| = 1$ B $-1 < |\boldsymbol{c}| < 1$ C $-1 \leq |\boldsymbol{c}| \leq 1$ D $|\boldsymbol{c}| \geq 1$

(b) Explain why the other statements are incorrect and what the common errors are.

22

3 For each of the following pairs of vectors, calculate:

(i) the vector projections of $\boldsymbol{a}$ onto $\boldsymbol{b}$

(ii) the vector projections of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

(a) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ (c) $\boldsymbol{a} = 8\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$

(d) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ (e) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ (f) $\boldsymbol{a} = -5\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

4 Vector $\boldsymbol{a} = k\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ and vector $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$. Determine k such that the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b} = 10$.

5 A road is constructed through the jungle such that its direction is given by the vector $\boldsymbol{r} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, where the vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ represent the direction of east and north respectively. At a certain point along this road, an explorer ventures off the road and into the jungle. She eventually sets up camp at a point defined by the position vector $\boldsymbol{e} = 6\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}$, $\boldsymbol{d}$ is the vector through the jungle to camp. (All measurements are in kilometres.)

(a) Construct a diagram to represent the vectors.

(b) Calculate the vector projection of $\boldsymbol{e}$ perpendicular to $\boldsymbol{r}$.

(c) Calculate the *minimum distance* from the road to the explorer's camp, and the *bearing* on which the explorer must travel from her camp to reach the road along this shortest path.

(d) Identify any assumptions you have made and state any effects these assumptions would have on your answer.

6 $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are unit vectors in the Cartesian plane.

(a) Demonstrate that $\boldsymbol{a} = \cos(\alpha)\hat{\boldsymbol{i}} + \sin(\alpha)\hat{\boldsymbol{j}}$.

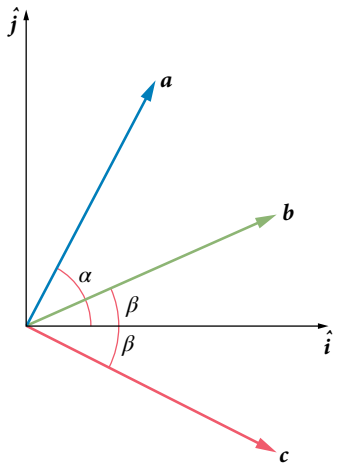
(b) Derive similar expressions for $\boldsymbol{b}$ and $\boldsymbol{c}$.

(c) Calculate $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \cdot \boldsymbol{c}$.

(d) Hence, deduce the compound angle formulas:

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$



2.6 Vectors in geometric proofs

Many geometric theorems can be solved using vector methods. Remember that, in a proof, showing all of your working is very important.

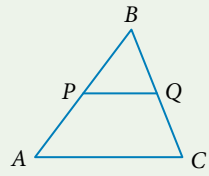
23 Collinear points

Demonstrate that the points $A(1, 5)$, $B(2, 8)$ and $C(4, 14)$ are collinear.

THINKING	WORKING
1 Define two vectors starting from the same point.	$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ $= \left(2\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}\right) - \left(\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$ $= \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$ $= \left(4\hat{\boldsymbol{i}} + 14\hat{\boldsymbol{j}}\right) - \left(\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$ $= 3\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$
2 Determine whether the vectors are parallel.	If $\overrightarrow{AC} = k.\overrightarrow{AB}$, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$. $\overrightarrow{AC} = 3\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$ $= 3\left(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$ $= 3\overrightarrow{AB}$
3 Interpret the result.	Since $\overrightarrow{AC} = 3\overrightarrow{AB}$ and both vectors include point A, points $A(1, 5)$, $B(2, 8)$ and $C(4, 14)$ are collinear.

24 Vector proof involving sides of a triangle

Use vector methods to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

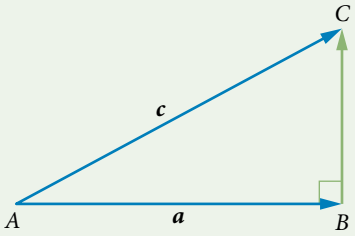
THINKING	WORKING
1 Draw any triangle and label the vertices.	Consider ΔABC. Label P and Q, the midpoints of sides $\overline{AB}$ and $\overline{BC}$ respectively. 

2 Define relevant vectors.	Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$.
3 Define other vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.	Since P is the midpoint of $\overline{AB}$ and Q is the midpoint of $\overline{BC}$, then $\overrightarrow{PB} = \frac{1}{2}\boldsymbol{a}$ and $\overrightarrow{BQ} = \frac{1}{2}\boldsymbol{b}$.
4 Express the required vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.	$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ $= \boldsymbol{a} + \boldsymbol{b}$ $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$ $= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$
5 Express one vector in terms of the other.	$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$
6 Interpret the result.	Thus, $\overline{PQ}$ is parallel to $\overline{AC}$ and half the length of $\overline{AC}$.

Pythagoras' theorem is used extensively throughout mathematics, particularly in measurement and trigonometry. You have used it throughout this vectors topic to calculate the magnitude of a vector. Pythagoras' theorem can be proved using vector methods.

25 Vector proof involving sides of a right-angled triangle

Use vector methods to prove Pythagoras' theorem; that is, in any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares on the other two sides.

THINKING	WORKING
1 Draw a right-angled triangle and label the vertices.	Consider ΔABC. 
2 Define relevant vectors.	Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{c}$
3 Define other vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.	$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$ $= -\boldsymbol{a} + \boldsymbol{c}$ $= \boldsymbol{c} - \boldsymbol{a}$
4 Use the scalar product of vectors to calculate $\left\lvert\overrightarrow{AC}\right\rvert^2$.	$\left\lvert\overrightarrow{AC}\right\rvert^2 = \overrightarrow{AC} \cdot \overrightarrow{AC}$ $= \boldsymbol{c} \cdot \boldsymbol{c}$ $= \lvert\boldsymbol{c}\rvert^2$

5 Use the property of perpendicular vectors to simplify the expression. That is, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular, then $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$.

That is, $\boldsymbol{a} \cdot (\boldsymbol{c} - \boldsymbol{a}) = 0$.

$$\boldsymbol{a} \cdot (\boldsymbol{c} - \boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{a} = 0$$

Thus, $\boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{a}$

6 Use the property of parallel vectors to simplify the expression further. That is, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are like parallel vectors, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$.

$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{a} &= |\boldsymbol{a}||\boldsymbol{a}| \\ &= |\boldsymbol{a}|^2\end{aligned}$$

so $\left|\overrightarrow{AB}\right|^2 = |\boldsymbol{a}|^2$

and

$$\begin{aligned}(\boldsymbol{c} - \boldsymbol{a}) \cdot (\boldsymbol{c} - \boldsymbol{a}) &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{c} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{a} \\ &= \boldsymbol{c} \cdot \boldsymbol{c} - 2\boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{a}\end{aligned}$$

Since $\boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{a}$, then

$$\begin{aligned}(\boldsymbol{c} - \boldsymbol{a}) \cdot (\boldsymbol{c} - \boldsymbol{a}) &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{c} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{a} \\ &= \boldsymbol{c} \cdot \boldsymbol{c} - 2\boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{a} \\ &= \boldsymbol{c} \cdot \boldsymbol{c} - 2\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{a} \\ &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{a} \\ &= |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2\end{aligned}$$

Thus, $\left|\overrightarrow{BC}\right|^2 = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$

7 Verify Pythagoras' theorem.

Thus,

$$\left|\overrightarrow{BC}\right|^2 = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$$

$$\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2$$

26 Vector proof involving angle in a semicircle

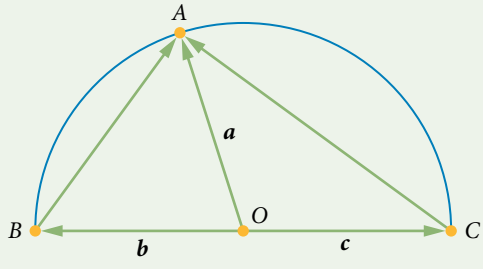
Use vector methods to prove that the angle inscribed in a semicircle is a right angle.

THINKING

1 Draw a semicircle and label the relevant points.

WORKING

Consider ΔABC.



2 Define relevant vectors.	Let $\overrightarrow{OA} = \boldsymbol{a}$, let $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$
3 Determine conditions for proof, using the definition of the dot product.	$\angle BAC$ will be a right angle if $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are perpendicular, in other words, if $\overrightarrow{BA} \cdot \overrightarrow{CA} = 0$
4 Determine relationships between vectors.	Because $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ all represent the radius of a circle, it follows that $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert = \lvert\boldsymbol{c}\rvert$. Also, because they lie on the same diameter, $\boldsymbol{c} = -\boldsymbol{b}$. (They are in opposite directions.)
5 Define vectors $\overrightarrow{BA}$ and $\overrightarrow{CA}$ in terms of the other defined vectors.	$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b}$ and $\overrightarrow{CA} = \boldsymbol{a} - \boldsymbol{c}$
6 Calculate the dot product of $\overrightarrow{BA}$ and $\overrightarrow{CA}$.	$\overrightarrow{BA} \cdot \overrightarrow{CA} = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{c})$ $= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c}$ Because $\boldsymbol{a} \cdot \boldsymbol{a} = \lvert\boldsymbol{a}\rvert$ and $\boldsymbol{c} = -\boldsymbol{b}$, then $\overrightarrow{BA} \cdot \overrightarrow{CA} = \lvert\boldsymbol{a}\rvert^2 + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{a} - \lvert\boldsymbol{b}\rvert^2$ $= \lvert\boldsymbol{a}\rvert^2 + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} - \lvert\boldsymbol{b}\rvert^2$ $= \lvert\boldsymbol{a}\rvert^2 - \lvert\boldsymbol{b}\rvert^2$ And because $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert = \lvert\boldsymbol{c}\rvert$, then $\overrightarrow{BA} \cdot \overrightarrow{CA} = \lvert\boldsymbol{a}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$ $= 0$
7 Interpret the result.	Because $\overrightarrow{BA} \cdot \overrightarrow{CA} = 0$, then $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are perpendicular.

EXERCISE 2.6 Vectors in geometric proofs

Worked Example 23

1 Using a vector method, demonstrate that $\boldsymbol{p} = a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}$ and $\boldsymbol{q} = -b\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}}$ are perpendicular.

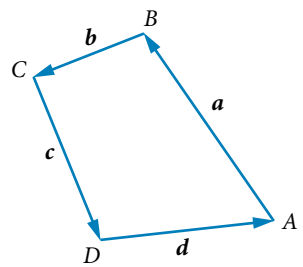
2 Prove that the points $A(-2, -3)$, $B(0, 1)$ and $C(3, 7)$ are collinear.

3 Consider the quadrilateral $ABCD$ as shown in the diagram. Let $\boldsymbol{a} = \overrightarrow{AB}$, $\boldsymbol{b} = \overrightarrow{BC}$, $\boldsymbol{c} = \overrightarrow{CD}$ and $\boldsymbol{d} = \overrightarrow{DA}$.

Which one of the following statements is correct?

A $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{c} - \boldsymbol{d}$ B $\boldsymbol{a} - \boldsymbol{c} = \boldsymbol{b} - \boldsymbol{d}$

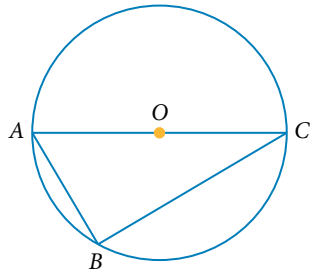
C $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d}$ D $\boldsymbol{a} + \boldsymbol{c} = -\boldsymbol{b} - \boldsymbol{d}$



Worked Example

26

4 Consider the circle with centre O and radius $\overrightarrow{OA} = \boldsymbol{a}$. B is a point on the circle and the diameter is $\overrightarrow{AC}$. Let $\overrightarrow{BC} = \boldsymbol{b}$.



(a) Which one of the following statements must be true?

A $\boldsymbol{a} = \frac{1}{2}\boldsymbol{b}$ B $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{b}$

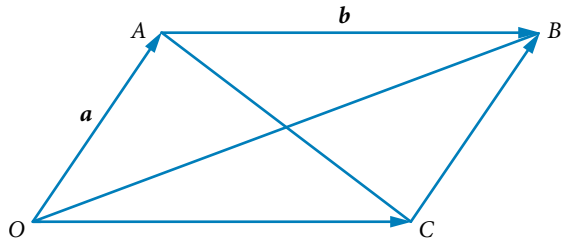
C $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{a}$ D $2\boldsymbol{a} \cdot \boldsymbol{b} = -\boldsymbol{b} \cdot \boldsymbol{b}$

(b) Explain why the first of the incorrect options is not correct.

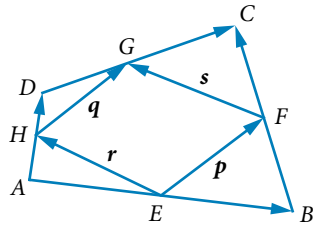
25

5 Prove that the diagonals of a rectangle are equal in length.

6 Consider the parallelogram $OABC$ where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$ as shown. Prove that the diagonals of a parallelogram meet at right angles if and only if it is a rhombus.



7 Let $ABCD$ be the vertices of a quadrilateral and E, F, G and H be the midpoints as shown. Prove that the quadrilateral with vertices $EFGH$ is a parallelogram



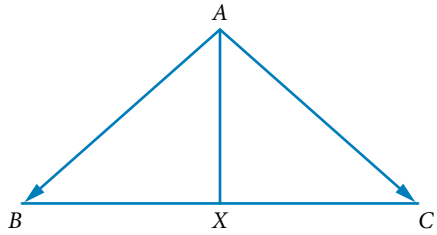
24

8 Use vector methods to prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from all vertices.

9 $ABCD$ is a rectangle. Prove that the diagonals of a rectangle bisect each other.

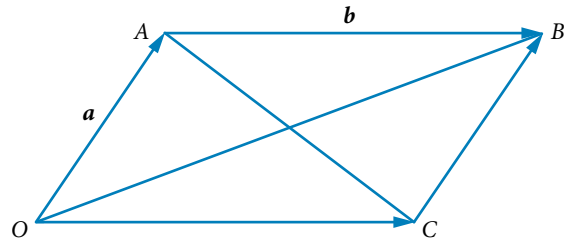
10 Apollonius' theorem relates the length of a median of a triangle to the lengths of its sides. In any triangle, the sum of the squares on any two sides is equal to twice the square of half the third side together with twice the square of the median bisecting the third side.

In the triangle ABC at right, $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{b}$. Apollonius' theorem would state that $\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 2\left(\left|\overrightarrow{AX}\right|^2 + \left|\overrightarrow{BX}\right|^2\right)$. Prove Apollonius' theorem for this triangle.



11 Consider the parallelogram $OABC$ where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$.

Prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

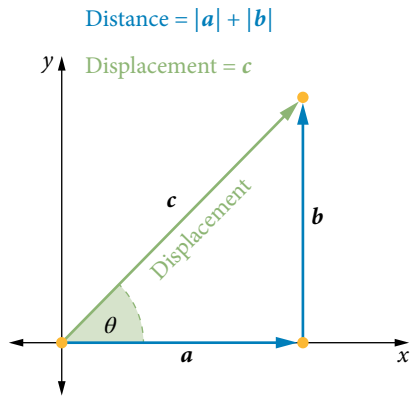


Displacement and velocity problems

2.7

Distance and displacement

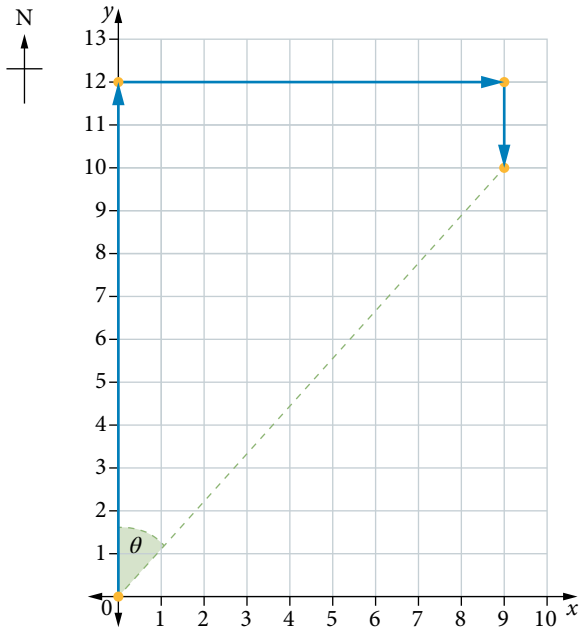
You will recall from the start of this chapter that distance and displacement are related but separate concepts. *Displacement* is the change in an object's position from its starting point to its finishing point. It is a vector quantity and consists of a magnitude and direction. *Distance*, on the other hand, is a scalar quantity—this is a measure of how far an object has travelled, and does not take direction into account.



The SI unit for displacement and distance is the metre (m), and other common units are the millimetre (mm) and the centimetre (cm) for relatively small displacement and distances, and the kilometre (km) for relatively large measurements. Displacement calculations also include a direction, which is often given as a compass direction or a true or three-figure bearing for real-world, two-dimensional problems.

27 Distance and displacement

Harrison embarks on a journey with three separate parts. First he travels 12 km north, then he travels 9 km east, and finally he travels 2 km south. Use a vector technique to calculate Harrison's total distance and displacement from his starting point.



THINKING	WORKING
1 Represent each leg of the journey using vectors.	First leg: $\boldsymbol{a} = 12\hat{\boldsymbol{j}}$ Second leg: $\boldsymbol{b} = 9\hat{\boldsymbol{i}}$ Third leg: $\boldsymbol{c} = -2\hat{\boldsymbol{j}}$
2 To calculate the total distance travelled, add up the distances of each leg of the journey.	Distance = 12 + 9 + 2 = 23 km

3 To calculate the displacement, start at the origin and add the vectors together.	$\text{Displacement} = 12\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ $= 9\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}$
4 Calculate the magnitude and direction of the resultant displacement vector.	$\text{Magnitude} = \sqrt{9^2 + 10^2}$ $= \sqrt{181}$ $\approx 13.45\,\text{km}$ $\theta = \tan^{-1}\left(\frac{9}{10}\right)$ $\approx 42°$
5 Interpret the answer.	The total distance that Harrison travelled is 23 km. Harrison's displacement from his starting point is 13.45 km at 042°T (as a true bearing).

Speed and velocity

Velocity is the rate at which the displacement of an object changes with respect to time. Because displacement is a vector quantity, velocity is also a vector quantity—it has magnitude and direction. The magnitude of velocity is the scalar quantity *speed*. The SI unit for velocity and speed is the metre per second (m/s or m s^{-1}), or the derived unit kilometres per hour (km/h or km h^{-1}).

The average velocity of an object over the course of a journey is defined by:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time taken}} = \frac{\text{displacement}}{\text{time}}$$

On the other hand, the average speed over the course of a journey is defined by:

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Note: It is often necessary to convert from m s^{-1} to km h^{-1}, or vice versa. Because 1 km = 1000 m and 1 h = 3600 seconds, then $1\frac{\text{km}}{\text{h}} = \frac{1000\,\text{m}}{3600\,\text{s}}$ or $3.6\,\text{km h}^{-1} = 1\,\text{m s}^{-1}$.

So, to convert from m s^{-1} to km h^{-1}, multiply by 3.6.

28 Speed and velocity

If each of the three legs of Harrison's journey from the previous worked example took a total of 1 hour, calculate Harrison's average speed and his average velocity for the whole journey. Express your answer in m s^{-1}.

THINKING	WORKING
1 Calculate the average speed.	$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$ $= \frac{23\,\text{km}}{3\,\text{h}}$ $= 7.\dot{6}\,\text{km h}^{-1}$ $7.\dot{6} \div 3.6 = 2.13\,\text{m s}^{-1}$ (2 d.p.)

2 Calculate the velocity as a vector.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$
$$= \frac{\left(9\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}\right)}{3}$$
$$= \left(3\hat{\boldsymbol{i}} + \frac{10}{3}\hat{\boldsymbol{j}}\right) \text{ km h}^{-1}$$

3 Calculate the magnitude and direction of the velocity vector.

$$\text{magnitude} = \sqrt{3^2 + \left(\frac{10}{3}\right)^2}$$
$$= \sqrt{\left(\frac{181}{9}\right)}$$
$$= 4.4845... \text{ km h}^{-1}$$

$4.4845... \div 3.6 = 1.25 \text{ m s}^{-1}$ (2 d.p.)

The direction will be the same as the displacement vector, so the true bearing is 042°T.

4 Interpret the answer.

Harrison's average speed for the journey was 2.13 m s^{-1} (2 d.p.). His average velocity for the journey was 1.25 m s^{-1} (2 d.p.), at bearing 42°T.

Displacement is the change in position of an object relative to its starting point. It is a vector quantity, described by both magnitude and direction.

Distance is how far an object has travelled in total. It is a scalar quantity so has magnitude only.

Velocity is the rate at which the displacement of an object changes with respect to time. It is a vector quantity described by both magnitude and direction.

Speed is the magnitude of an object's velocity. It is a scalar quantity so has magnitude only.

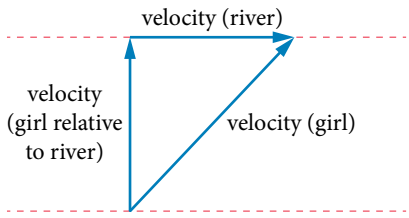
Making connections

Relative velocity interactive

Move the sliders and observe position, distance and displacement.

Relative velocity

Vectors are very useful for analysing situations involving multiple velocities. Consider the case in which a girl is attempting to swim north across a river with an easterly current. Because of the motion of the river, the girl's actual velocity will be in a direction east of north, as represented in the diagram.



In the vector diagram, you will note that the following vector sum applies:

$$\text{velocity(girl)} = \text{velocity(girl relative to river)} + \text{velocity(river)}$$

A more general equation for relative velocity gives the relationship:

$$\boldsymbol{v}_a = \boldsymbol{v}_{a \text{ rel } b} + \boldsymbol{v}_b \text{ or alternatively } \boldsymbol{v}_{a \text{ rel } b} = \boldsymbol{v}_a - \boldsymbol{v}_b$$

where $\boldsymbol{v}_a$ is the velocity of A, $\boldsymbol{v}_b$ is the velocity of B, and $\boldsymbol{v}_{a \text{ rel } b}$ is the velocity of A relative to B.

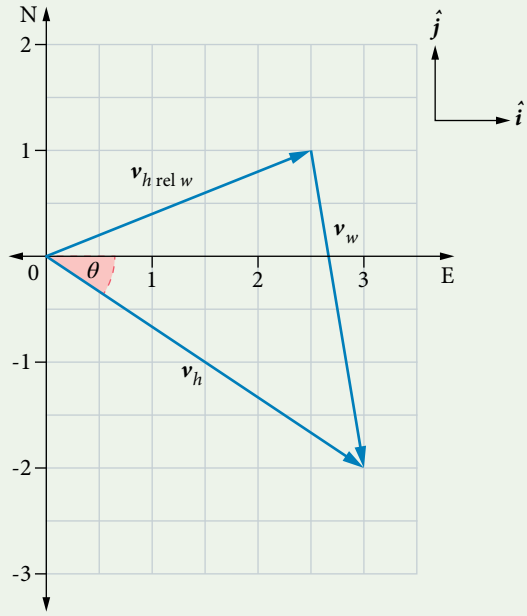
Explore further

Relative velocity

Construct dynamic relative velocity vectors using graphing technology.

29 Relative velocity

Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the east and north directions respectively. Hamish is swimming in the ocean and his velocity relative to the water, $\boldsymbol{v}_{h\,\text{rel}\,w}$ m s^{-1}, is given by the vector $\boldsymbol{v}_{h\,\text{rel}\,w} = 2.5\hat{\boldsymbol{i}} + 1.0\hat{\boldsymbol{j}}$. The ocean's current has a velocity $\boldsymbol{v}_w$ m s^{-1}, where $\boldsymbol{v}_w = 0.5\hat{\boldsymbol{i}} - 3.0\hat{\boldsymbol{j}}$. Calculate the magnitude and direction as a true bearing of Hamish's actual velocity, $\boldsymbol{v}_h$ m s^{-1}, correct to 1 decimal place.

THINKING	WORKING
1 Draw a vector diagram to illustrate the situation.	
2 Calculate the resultant velocity, $\boldsymbol{v}_h$ by adding vectors.	$\boldsymbol{v}_h = \boldsymbol{v}_{h\,\text{rel}\,w} + \boldsymbol{v}_w$ $= 2.5\hat{\boldsymbol{i}} + 1.0\hat{\boldsymbol{j}} + 0.5\hat{\boldsymbol{i}} - 3.0\hat{\boldsymbol{j}}$ $= 3.0\hat{\boldsymbol{i}} - 2.0\hat{\boldsymbol{j}}$
3 Determine the magnitude and direction of the resultant vector. *Note*: Because θ is an acute angle in the diagram, use positive values for side lengths.	$\lvert\boldsymbol{v}_h\rvert = \sqrt{(3.0)^2 + (-2.0)^2}$ $= \sqrt{13.0}$ $= 3.6$ m s^{-1} (1 d.p.) $\theta = \tan^{-1}\left(\frac{2.0}{3.0}\right)$ $= 33.7°$ (1 d.p.) The direction is $90 + 33.7 = 123.7°$ T
4 Interpret the answer.	Hamish is swimming at 3.6 m s^{-1} in a direction of 123.7° T.

Technology worked example
Resultant and relative velocity vectors

EXERCISE 2.7 Displacement and velocity problems

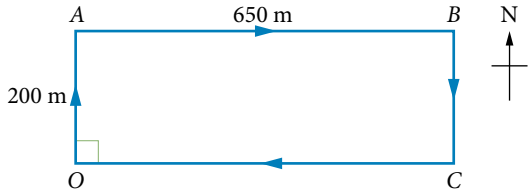
Worked Example

1 Devin begins his day at home. At 8:00 am he travels 5.6 km in a northerly direction to school and stays there until 3:00 pm, at which time he drives 9.4 km to the shops and then 9 km home, arriving back at his house at 4:00 pm. 28

(a) Calculate Devin's average speed between 8:00 am and 4:00 pm.

(b) Calculate Devin's average velocity between 8:00 am and 4:00 pm.

2 Louise is going for a walk around the block. She starts at O and moves to point A and then to point B, where she stops to have a rest. 27



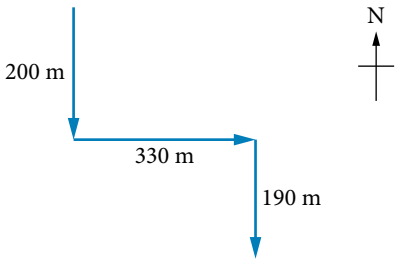
(a) Calculate the distance Louise travelled from O to B.

(b) Calculate Louise's displacement from O to B, correct to 1 decimal place.

(c) If Louise takes 12 minutes to reach point B, calculate her average speed, in metres per second, correct to 1 decimal place.

(d) Calculate Louise's average velocity from O to B, in metres per second, correct to 1 decimal place.

3 Kudzie walks 200 m south, then 330 m east, and finally 190 m south. Calculate Kudzie's displacement, correct to 1 decimal place, for his entire walk. Give your answer to 1 decimal place and express the direction as a true bearing.



4 A mountain-climbing expedition establishes its base camp and two intermediate camps at positions A and B. Camp A is 8400 m west of and 1800 m above base camp. Camp B is 5900 m west of Camp A and 850 m higher than Camp A. Calculate the displacement of Camp B from the base camp. (Give answers correct to 1 decimal place.)

5 Ronin is running at 8 m s^{-1} north, and Charlie is chasing after him at 6 m s^{-1} north. What is Charlie's velocity relative to Ronin?

6 If a river flows at 5 km h^{-1} and William rows at 10 km h^{-1}, in what direction should William row to go straight across the river?

A downstream at angle of 60° to the bank
B upstream at angle of 30° to the bank
C upstream at angle of 60° to the bank
D not possible

7 If a river flows at 10 km h^{-1} and William rows at 5 km h^{-1}, in what direction should he row to go straight across the river?

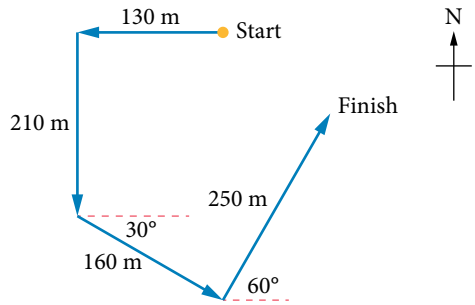
A downstream at angle of 60° to the bank
B upstream at angle of 30° to the bank
C upstream at angle of 60° to the bank
D not possible

8 (a) Karen, who is standing still on the ground, is watching an aircraft flying in a tail wind and travelling at a velocity of 100 m s^{-1} north, relative to the air. If the velocity of the wind is 10 m s^{-1} north, calculate the actual velocity of the aircraft relative to Karen.

(b) Later, when the wind has changed direction to 10 m s^{-1} south, Karen observes a second aircraft flying with a velocity of 125 m s^{-1} north, relative to the air. Calculate the actual velocity of the aircraft relative to Karen.

(c) The next day, when the wind has changed to 25 m s^{-1} west, Karen observes another aircraft flying with a velocity of 100 m s^{-1} north, relative to the air. Calculate the actual velocity of the aircraft relative to Karen.

9 Mark is taking his dog for a walk along the path in the local park, as shown in the diagram. Calculate the magnitude and direction of Mark's resultant displacement, correct to the nearest whole number.



10 While exploring a recently discovered cave system, Eylece starts at the entrance of the cave and makes the following movements: 85 m north, 190 m east, 250 m N45°E, and 100 m south.

(a) Calculate Eylece's final displacement from the cave entrance.

(b) If the time taken for each leg of the journey was 12 min, 14 min, 14 min and 5 min respectively, calculate Eylece's average velocity (in km h^{-1}).

11 Sia and Mitchell are riding in a boat whose speed relative to the water is 3 m s^{-1}. The boat points upstream at an angle of 35° with the bank on a river flowing at 0.75 m s^{-1}.

(a) If the river is 50 m wide, calculate (correct to 2 decimal places) the time it takes for their boat to reach the opposite shore.

Worked Example

(b) Calculate the distance (correct to 2 decimal places) between the points at which Sia and Mitchell embark and disembark from the boat.

29

12 Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the east and north directions respectively. Ben is swimming in the ocean and his velocity relative to the water, $\boldsymbol{v}_{b\,\text{rel}\,w}$ m s^{-1}, is given by the vector $\boldsymbol{v}_{b\,\text{rel}\,w} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$. The ocean's current has a velocity $\boldsymbol{v}_w$ m s^{-1}, where $\boldsymbol{v}_w = -0.5\hat{\boldsymbol{i}} + 2.5\hat{\boldsymbol{j}}$. Calculate the magnitude and direction of Ben's actual velocity, $\boldsymbol{v}_b$ m s^{-1}, correct to 1 decimal place.

13 When a car travels at 60 km h^{-1}, the rain appears to fall towards the car at an angle of 45° to the horizontal. When the car travels at 60 km h^{-1} in the opposite direction, the rain appears to fall towards the car at an angle of 55° to the horizontal. Calculate the actual angle at which the rain is falling, relative to the horizontal ground. State any assumptions you make and comment on the effect of these assumptions on your answer.

2.8 Problems involving forces

Forces exist everywhere and are fundamental to the structure of the universe and to the nature of matter. A force can be thought of as a push or a pull acting on a body or object. It is convenient to consider that the forces acting on an object act through a single point at which the object's mass is concentrated.

The action of a force affects an object or body by changing its shape, speed or the direction of its motion. The amount of force acting is measured using the SI unit the newton (N). A force of 1 newton will accelerate an object of mass 1 kilogram at a rate of 1 metre per second per second; i.e. $1\,\text{N} = 1\,\text{kg m s}^{-2}$.

Every object near the Earth's surface is subject to a force called gravity and this is known as the *weight* of the object. Weight is a force that acts vertically downwards on an object towards the centre of the Earth. At the surface of the Earth, the weight force due to gravity of a body of mass 1 kg is approximately 9.8 N. For example, a person whose mass is 80 kg has a weight of $80g$ N, where $g = 9.8\,\text{kg m s}^{-2}$. Thus, a person with a mass of 80 kg weighs $80g = 80 \times 9.8 = 784$ N.

Force is a vector quantity so it needs both magnitude and direction to fully describe it. If more than one force acts on an object, the *resultant force* or *net force*, R, acts as if only one force is actually acting on the object.

The resultant or net force, R, is the vector sum of all the forces acting on an object.

If forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, ... act on a particle, then $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \ldots$

Assumptions of force calculations

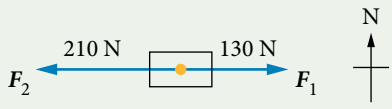
When considering real-world problems involving forces, there are many subtle complexities in play. Accuracy can be increased by developing sophisticated mathematical models to account for variables such as aerodynamics and material composition. For the purposes of this unit, models will be simplified by making the following assumptions (unless otherwise specified):

- All forces act through a *single point* on the object under consideration.
- The material of the object is *rigid and inextensible*. In other words, assume that the object cannot have its shape altered by the applied forces.
- Objects are *smooth* and do not exert a friction force.
- Other forces such as air resistance are negligible when velocities are small.

30 Calculating resultant forces

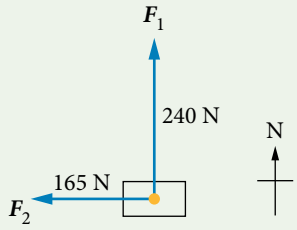
The following pairs of separate forces are acting on an object. Calculate the resultant force acting on the object each time.

(a) $\boldsymbol{F}_1 = 130\,\text{N}$ east, $\boldsymbol{F}_2 = 210\,\text{N}$ west

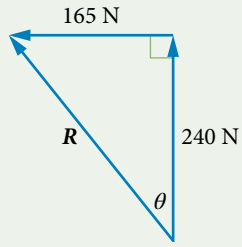
THINKING	WORKING
1 Draw a diagram to show the forces acting on the object.	
2 Write a vector equation.	$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$
3 Calculate the resultant force.	$\boldsymbol{R} = 130\,\text{N east} + 210\,\text{N west} = 80\,\text{N west}$

(b) $\boldsymbol{F}_1 = 240\,\text{N}$ north, $\boldsymbol{F}_2 = 165\,\text{N}$ west

1 Draw a diagram to show the forces acting on the object.



2 Draw a vector diagram to add the forces.



3 Calculate the magnitude of the resultant force using Pythagoras' theorem.

$$|\boldsymbol{R}| = \sqrt{240^2 + 165^2}$$
$$= 291.25\,\text{N}$$

4 Calculate the direction of the resultant force.

$$\tan(\theta) = \frac{165}{240}$$
$$\theta = \tan^{-1}\left(\frac{165}{240}\right)$$
$$= 34.51°$$

5 Calculate the resultant force.

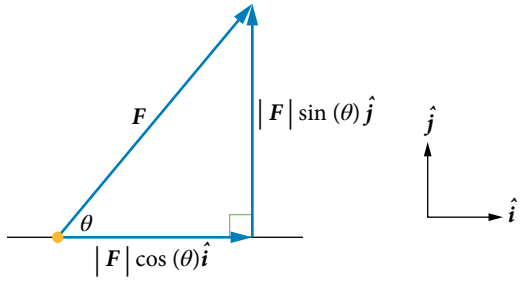
The resultant force is 291.25 N in the direction N34.51°W.

Calculating resultant forces using vector components

If the forces acting on an object are in the same plane, then each of the forces and the resultant force can be expressed in component form as a sum of two perpendicular vectors, usually $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ components.

If a force $\boldsymbol{F}$ acts at an angle of θ to the horizontal, then $\boldsymbol{F}$ can be written as the sum of two components, one horizontal ($\hat{\boldsymbol{i}}$) and the other vertical ($\hat{\boldsymbol{j}}$). The force $\boldsymbol{F}$ is resolved into two perpendicular components, that is, $\boldsymbol{F} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

From the diagram, $\boldsymbol{F} = |\boldsymbol{F}|\cos(\theta)\hat{\boldsymbol{i}} + |\boldsymbol{F}|\sin(\theta)\hat{\boldsymbol{j}}$.



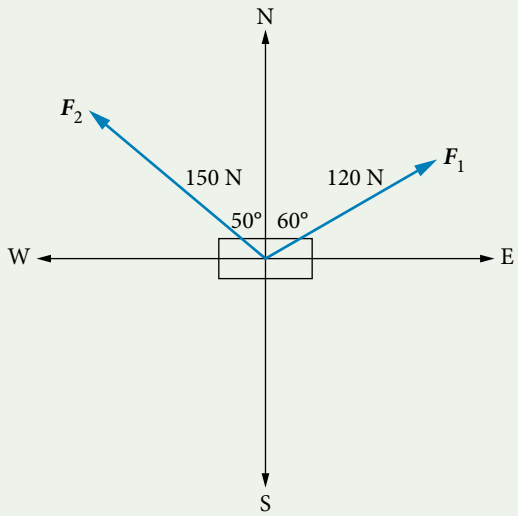
31 Calculating resultant forces using vector components

Two forces are acting on an object: $\boldsymbol{F}_1 = 120\,\text{N}$ acting at N60°E, and $\boldsymbol{F}_2 = 150\,\text{N}$ acting at N50°W. Calculate the resultant force $\boldsymbol{R}$ acting on the object.

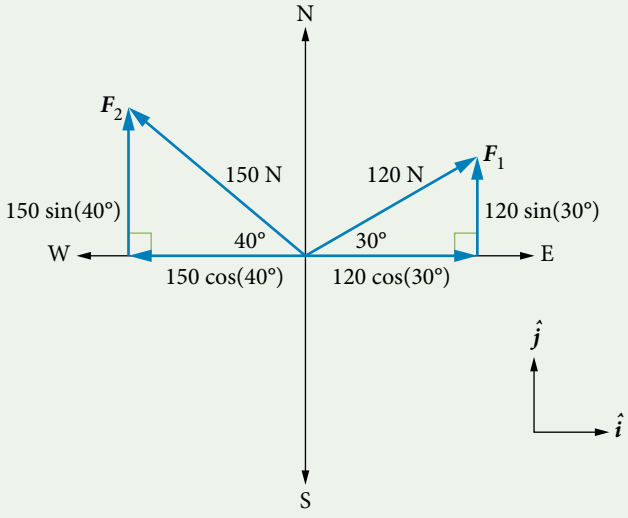
THINKING

1 Draw a diagram to show the forces acting on the object.

WORKING



2 Resolve each force into horizontal ($\hat{\boldsymbol{i}}$) and vertical ($\hat{\boldsymbol{j}}$) components.



3 Calculate the horizontal ($\hat{\boldsymbol{i}}$) and vertical ($\hat{\boldsymbol{j}}$) components of each force.

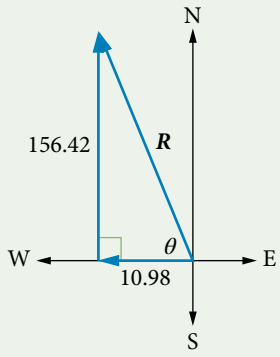
$$\begin{aligned}\boldsymbol{F}_1 &= 120\cos(30°)\hat{\boldsymbol{i}} + 120\sin(30°)\hat{\boldsymbol{j}}\\ &= 103.92\hat{\boldsymbol{i}} + 60.00\hat{\boldsymbol{j}}\end{aligned}$$

$$\begin{aligned}\boldsymbol{F}_2 &= -150\cos(40°)\hat{\boldsymbol{i}} + 150\sin(40°)\hat{\boldsymbol{j}}\\ &= -114.91\hat{\boldsymbol{i}} + 96.42\hat{\boldsymbol{j}}\end{aligned}$$

4 Add the forces by adding horizontal ($\hat{\boldsymbol{i}}$) and vertical ($\hat{\boldsymbol{j}}$) components.

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{F}_1 + \boldsymbol{F}_2\\ &= 103.92\hat{\boldsymbol{i}} + 60.00\hat{\boldsymbol{j}} - 114.91\hat{\boldsymbol{i}} + 96.42\hat{\boldsymbol{j}}\\ &= (103.92 - 114.91)\hat{\boldsymbol{i}} + (60.00 + 96.42)\hat{\boldsymbol{j}}\\ &= -10.98\hat{\boldsymbol{i}} + 156.42\hat{\boldsymbol{j}}\end{aligned}$$

5 Draw a diagram of the resultant vector.



6 Calculate the magnitude of the resultant vector using Pythagoras' theorem and the angle at which the resultant force is acting.

$$|\boldsymbol{R}| = \sqrt{(-10.98)^2 + 156.42^2}$$
$$= 156.80\,\text{N}$$
$$\theta = \tan^{-1}\left(\frac{156.42}{10.98}\right)$$
$$= 86°$$
$$\therefore \text{Bearing is } (90 - 86) = \text{N4°W}$$

7 Describe the resultant force.

$\boldsymbol{R} = 156.80\,\text{N}$ acting in the direction N4°W.

This method can be applied to situations in which more than two forces are acting on an object.

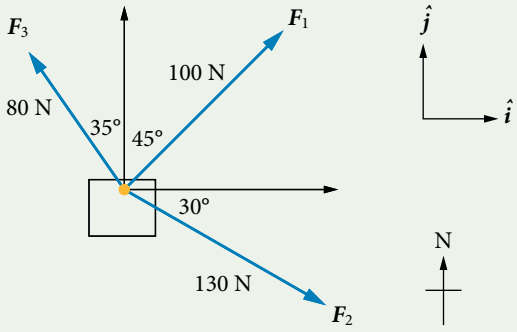
32 Calculating the resultant of three forces using vector components

Three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on an object: $\boldsymbol{F}_1 = 100\,\text{N}$ at 045°T, $\boldsymbol{F}_2 = 130\,\text{N}$ at 120°T and $\boldsymbol{F}_3 = 80\,\text{N}$ at 325°T. Calculate the magnitude and direction of the resultant force, $\boldsymbol{R}$.

THINKING

1 Draw a force diagram for the problem.

WORKING



2 Determine each force in component form.

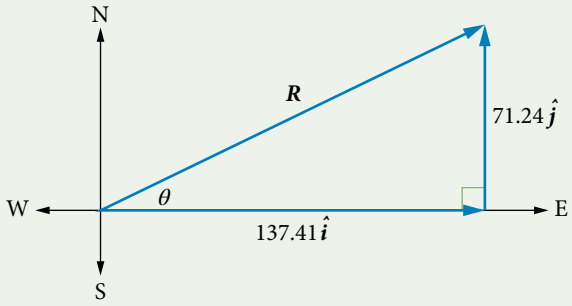
$$\boldsymbol{F}_1 = 100\cos(45°)\hat{\boldsymbol{i}} + 100\sin(45°)\hat{\boldsymbol{j}}$$
$$\approx 70.71\hat{\boldsymbol{i}} + 70.71\hat{\boldsymbol{j}}$$

$$\boldsymbol{F}_2 = 130\cos(30°)\hat{\boldsymbol{i}} - 130\sin(30°)\hat{\boldsymbol{j}}$$
$$\approx 112.58\hat{\boldsymbol{i}} - 65.00\hat{\boldsymbol{j}}$$
$$\boldsymbol{F}_3 = -80\cos(55°)\hat{\boldsymbol{i}} + 80\sin(55°)\hat{\boldsymbol{j}}$$
$$\approx -45.89\hat{\boldsymbol{i}} + 65.53\hat{\boldsymbol{j}}$$

3 Determine the resultant force, $\boldsymbol{R}$, by adding the components together.

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$$
$$= (70.71 + 112.58 - 45.89)\hat{\boldsymbol{i}} + (70.71 - 65.00 + 65.53)\hat{\boldsymbol{j}}$$
$$= 137.40\ldots\hat{\boldsymbol{i}} + 71.24\ldots\hat{\boldsymbol{j}}$$

4 Calculate the magnitude and direction of the resultant vector.



$$|\boldsymbol{R}| = \sqrt{137.40\ldots^2 + 71.24\ldots^2}$$
$$= 154.78\,\text{N}$$

$$\tan(\theta) = \frac{71.24}{137.41}$$
$$\theta = 27.41°$$

$\therefore$ The acting angle is $(90° - 27.41°) = 62.59°$.

5 Describe the resultant force in terms of its magnitude and direction.

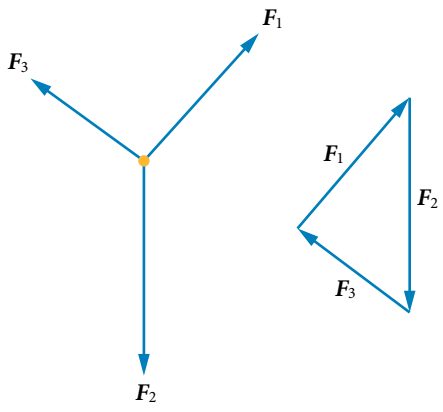
The resultant force is 154.78 N acting at a rounded angle of 063°T.

Objects in equilibrium

If the resultant force acting on an object is zero, the object's motion will remain unchanged. That is, it will remain at rest or continue with constant velocity. This is known as Newton's first law of motion and the object is said to be in *equilibrium*.

Newton's first law of motion states that an object at rest stays at rest and an object in motion stays in motion with the same velocity unless acted upon by an unbalanced force.

If the forces acting on an object are in equilibrium, then the resultant force or net force acting on the object is zero. For example, if the three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ that act on a particle are in equilibrium, then $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$. The forces can be displayed using a triangle of forces (see right).

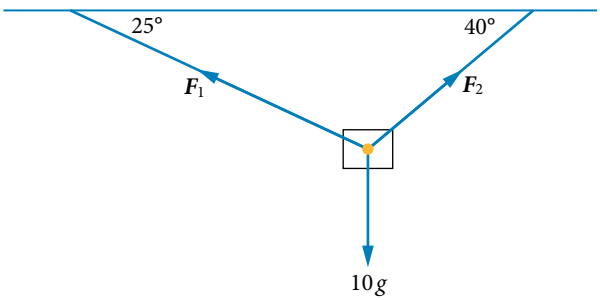


The magnitude of the forces acting and the angles between the forces can be found using trigonometric ratios (if the triangle contains a right angle) or by using the sine rule.

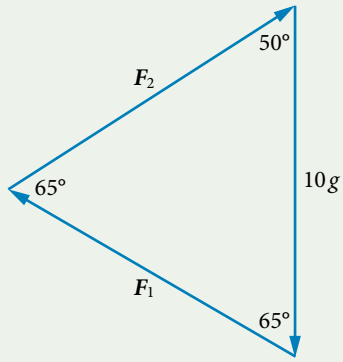
Another method involves the use of vectors in component form—if all forces are in equilibrium, then the resultant vector $\boldsymbol{R}$ will have horizontal ($\hat{\boldsymbol{i}}$) and vertical ($\hat{\boldsymbol{j}}$) components of zero.

33 Forces in equilibrium

A particle of mass 10 kg is suspended by two strings attached to two points in the same horizontal plane. The two strings make angles of 25° and 40° respectively with the horizontal.



(a) Calculate the magnitude of the tension in each string, in newtons correct to 2 decimal places, using a vector triangle method.

THINKING	WORKING
1 Represent the forces as a triangle. Use geometric properties to determine the angles within the triangle.	
2 Use the sine rule $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ to determine the magnitudes of the tension forces.	$\frac{\lvert F_1 \rvert}{\sin(50°)} = \frac{\lvert F_2 \rvert}{\sin(65°)} = \frac{10g}{\sin(65°)}$
3 Calculate the magnitude of the first force.	$\frac{\lvert F_1 \rvert}{\sin(50°)} = \frac{10g}{\sin(65°)}$ $\lvert F_1 \rvert = \frac{98}{\sin(75°)} \times \sin(50°)$ $= 82.83\,\text{N}$ (2 d.p.)
4 Calculate the magnitude of the second force.	$\frac{\lvert F_2 \rvert}{\sin(65°)} = \frac{10g}{\sin(65°)}$ $\lvert F_2 \rvert = \frac{98}{\sin(65°)} \times \sin(65°)$ $= 98\,\text{N}$
5 Interpret the answer.	The tension in string 1 is 82.83 N. The tension in string 2 is 98 N.

(b) Calculate the magnitude of the tension in each string, in newtons, correct to 2 decimal places, using a component vector method.

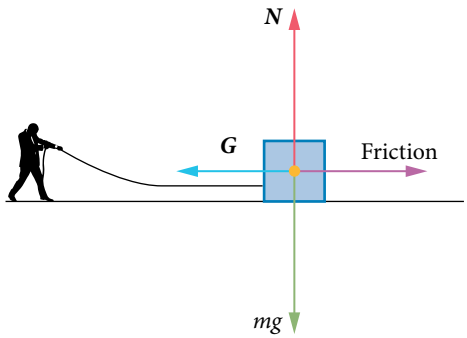
1 Express each force in component form. (Let $\boldsymbol{F}_3$ be the weight force due to gravity.)	$\boldsymbol{F}_1 = -\lvert F_1 \rvert \cos(25°)\hat{\boldsymbol{i}} + \lvert F_1 \rvert \sin(25°)\hat{\boldsymbol{j}}$ $\boldsymbol{F}_2 = \lvert F_2 \rvert \cos(40°)\hat{\boldsymbol{i}} + \lvert F_2 \rvert \sin(40°)\hat{\boldsymbol{j}}$ $\boldsymbol{F}_3 = -10g\hat{\boldsymbol{j}}$
2 As the forces are in equilibrium, the sum of the horizontal and vertical components will equal zero.	Horizontal ($\hat{\boldsymbol{i}}$) components: $-\lvert F_1 \rvert \cos(25°) + \lvert F_2 \rvert \cos(40°) = 0$ [1] Vertical ($\hat{\boldsymbol{j}}$) components: $\lvert F_1 \rvert \sin(25°) + \lvert F_2 \rvert \sin(40°) - 10g = 0$ [2]

3 Solve the equations simultaneously to determine one of the forces.	Rearranging [1] gives: $\lvert \boldsymbol{F}_1 \rvert = \dfrac{\lvert \boldsymbol{F}_2 \rvert \cos(40°)}{\cos(25°)}$ Substituting into [2] gives: $\dfrac{\lvert \boldsymbol{F}_2 \rvert \cos(40°)}{\cos(25°)} \sin(25°) + \lvert \boldsymbol{F}_2 \rvert \sin(40°) - 10g = 0$ $1\lvert \boldsymbol{F}_2 \rvert = 98$ $\lvert \boldsymbol{F}_2 \rvert = 98\,\text{N}$
4 Substitute to determine the other force.	$-\lvert \boldsymbol{F}_1 \rvert \cos(25°) + \lvert \boldsymbol{F}_2 \rvert \cos(40°) = 0$ $-\lvert \boldsymbol{F}_1 \rvert \cos(25°) = -98\cos(40°)$ $\lvert \boldsymbol{F}_1 \rvert = 82.83\,\text{N}$ (2 d.p.)
5 Interpret the answer.	The tension in string 1 is 82.83 N, the tension in string 2 is 98 N.

Friction

Friction is a force that resists the sideways movement of an object that is in contact with a surface. You will already have an intuitive understanding about how friction works, as our everyday lives depend on it. When you stand, walk or drive, friction helps to control your motion.

The force of friction always acts in the *opposite* direction to actual or potential motion. Consider a person who is attempting to pull a heavy object across a rough surface as shown at right.



Now in this scenario, gravity is pulling the object downwards to the floor with a weight force of *mg*. To counter the weight force, the floor exerts a *normal reaction force*, ***N***, that is, a force that is equal in magnitude to the weight force, but opposite in direction. Therefore, the vertical forces are balanced. As the person pulls on the rope, she exerts a force of ***G*** in the direction she is pulling. If the object remains stationary on the floor, it is because a friction force balances the force ***G*** in the opposite direction.

If the person is able to exert a greater force than the maximum friction force, then the forces will no longer be in equilibrium and the object will begin to move. It is important to note that the friction force will only ever be large enough to balance the pushing or pulling force—it can never be greater in magnitude.

The maximum friction force depends on two factors—the size of ***N*** (the normal reaction force) and a scalar quantity known as the *coefficient of friction* (μ). The coefficient of friction is a measure of the 'roughness' of the contact surfaces and has a theoretical range of zero to greater than one.

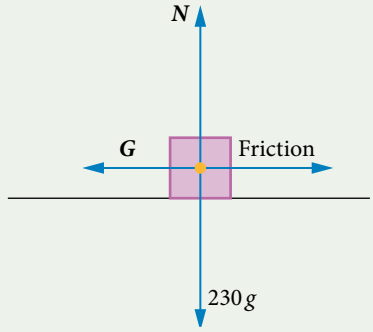
Note that in this topic we will only consider *static friction*, that is, the friction that applies before an object begins to move. Once an object begins to move across a surface, it typically encounters a relatively smaller friction force.

Friction ($\boldsymbol{F}$) is the product of the normal reaction force ($\boldsymbol{N}$) and the coefficient of friction (μ).

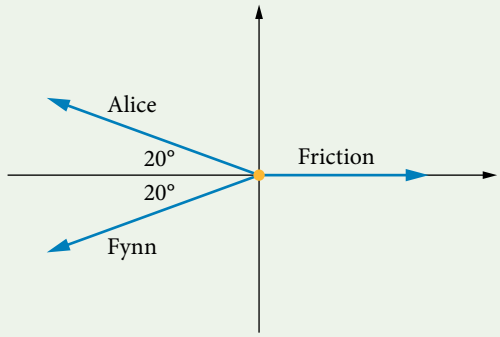
$\boldsymbol{F} = \mu \boldsymbol{N}$

34 Friction

(a) Michael attempts to push a table weighing 230 kg across a wooden floor with coefficient of friction $\mu = 0.4$. What force must he exert in order to move the table?

THINKING	WORKING
1 Draw a diagram representing the forces as vectors.	
2 Calculate normal reaction and friction forces.	$\boldsymbol{N} = 230\,\text{g}$ $= 2254\,\text{N}$ $\boldsymbol{F} = \mu\boldsymbol{N}$ $= 0.4 \times 2254\,\text{N}$ $= 901.6\,\text{N}$
3 Answer the question.	Michael must exert a force greater than 901.6 N in order to move the table.

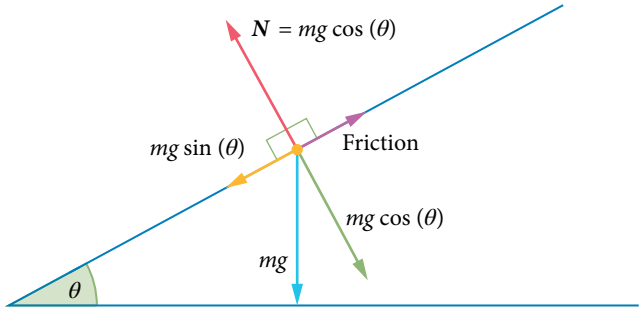
(b) Alice and Fynn attempt to pull a 150 kg sled across a rough surface by each pulling on a rope. When they each exert a pulling force of 400 N at an angle of 20° to the direction of the sled's motion, the sled just begins to move. Calculate μ, the coefficient of friction of the rough surface (correct to 2 decimal places).

1 Draw a top-down diagram to show the force exerted by Alice and Fynn.	
2 Calculate the combined horizontal components of the forces in the diagram. Note that the vertical components will cancel out.	Combined force $= 2 \times 400\cos(20°)$ $= 751.7541...\ \text{N}$
3 Calculate the friction force, hence the coefficient of friction.	Since the sled has just started to move, the maximum friction force has been reached. $\boldsymbol{F} = \mu\boldsymbol{N}$ $751.7541... = \mu \times 150g$ $\mu = 0.51$ (2 d.p.)

Inclined plane problems

Consider the forces acting on a plane that is inclined at angle θ to the horizontal.

When calculating forces on an inclined plane, it is often easier to change the frame of reference so that the $\hat{\boldsymbol{i}}$ vector is parallel to the plane and the $\hat{\boldsymbol{j}}$ vector is perpendicular to the plane. Therefore, the component of the weight force that acts down the plane will be $mg\sin(\theta)$ and the normal reaction force will be $\boldsymbol{N} = mg\cos(\theta)$. Note that, as θ increases, the force that pulls the object down the plane will increase, and it will require an increased friction force to hold the object in equilibrium.



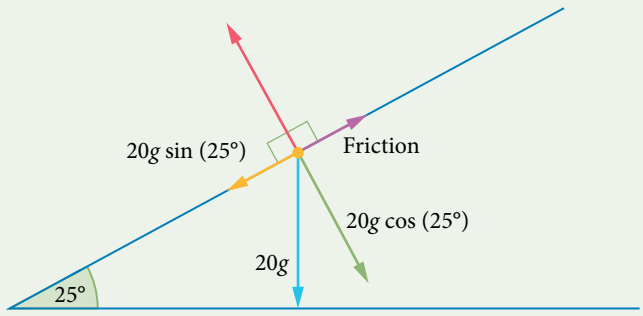
Explore further

Forces on an inclined plane

Investigate forces on an inclined plane.

35 Inclined plane problems

An object of mass 20 kg rests on a rough plane inclined 25° to the horizontal. The coefficient of friction is $\mu = 0.3$. Determine whether the object will begin to slide down the plane.

THINKING	WORKING
1 Draw a diagram representing the forces as vectors.	
2 Calculate normal reaction force and the force, $\boldsymbol{G}$ down the plane.	$\boldsymbol{N} = 20g \times \cos(25°)$ $= 177.6363...$ N $\boldsymbol{G} = 20g \times \sin(25°)$ $= 82.83$ N (2 d.p.)
3 Calculate the maximum friction force.	$\boldsymbol{F} = \mu\boldsymbol{N}$ $= 0.3 \times 177.6363...$ $= 53.29$ N (2 d.p.)
4 Compare the force down the plane with the maximum friction force.	The force down the plane is 82.83 N (2 d.p.), which is greater than the maximum friction force of 53.29 N (2 d.p.). Therefore, the object will slide down the plane.

EXERCISE 2.8 Problems involving forces

Worked Example

30

1 The following separate forces are acting on objects. Calculate the resultant force acting on each object.

(a) $\boldsymbol{F}_1 = 150\,\text{N}$ north, $\boldsymbol{F}_2 = 120\,\text{N}$ south

(b) $\boldsymbol{F}_1 = 67\,\text{N}$ east, $\boldsymbol{F}_2 = 83\,\text{N}$ west

(c) $\boldsymbol{F}_1 = 320\,\text{N}$ east, $\boldsymbol{F}_2 = 210\,\text{N}$ west, $\boldsymbol{F}_3 = 140\,\text{N}$ east

(d) $\boldsymbol{F}_1 = 64\,\text{N}$ south, $\boldsymbol{F}_2 = 56\,\text{N}$ north, $\boldsymbol{F}_3 = 48\,\text{N}$ south

2 The following separate forces are acting on objects. Calculate the resultant force, correct to 2 decimal places, acting on each object.

(a) $\boldsymbol{F}_1 = 160\,\text{N}$ north, $\boldsymbol{F}_2 = 120\,\text{N}$ west

(b) $\boldsymbol{F}_1 = 80\,\text{N}$ east, $\boldsymbol{F}_2 = 90\,\text{N}$ south

(c) $\boldsymbol{F}_1 = 125\,\text{N}$ west, $\boldsymbol{F}_2 = 85\,\text{N}$ south

(d) $\boldsymbol{F}_1 = 340\,\text{N}$ north, $\boldsymbol{F}_2 = 110\,\text{N}$ east

3 Two tug boats are applying simultaneous forces to a container ship as it attempts to berth at the port. One tug boat is applying a force of 3200 N in an easterly direction and the second tug boat is applying a 2300 N force directly south. Calculate the resultant force being applied to the container ship, giving the direction as a conventional bearing.

31

4 Calculate the resultant force $\boldsymbol{R}$ acting on each of the following objects that have two forces acting on them. Give answers correct to 2 decimal places.

(a) $\boldsymbol{F}_1 = 100\,\text{N}$ acting at N40°E and $\boldsymbol{F}_2 = 90\,\text{N}$ acting at N35°W. (Express the direction as a conventional bearing.)

(b) $\boldsymbol{F}_1 = 200\,\text{N}$ acting at N39°W and $\boldsymbol{F}_2 = 160\,\text{N}$ acting at S40°W. (Express the direction as a conventional bearing.)

(c) $\boldsymbol{F}_1 = 15\,\text{N}$ acting at 152°T and $\boldsymbol{F}_2 = 23\,\text{N}$ acting at 065°T. (Express the direction as a true bearing.)

(d) $\boldsymbol{F}_1 = 2050\,\text{N}$ acting at 037°T and $\boldsymbol{F}_2 = 1560\,\text{N}$ acting at 226°T. (Express the direction as a true bearing.)

5 Three forces act simultaneously in the same plane on an object, creating a resultant force of $\boldsymbol{F} = 360\,\text{N}$ due north. If the first force is $\boldsymbol{F}_1 = 440\,\text{N}$ north and the second force is $\boldsymbol{F}_2 = 560\,\text{N}$ south, calculate the magnitude and direction of the third force, $\boldsymbol{F}_3$.

6 Amanda and Curly are attempting to move a broken-down car. Amanda applies a force of 250 N west and Curly applies a force of 300 N north. What is the direction of their resultant force? Give your answer as a bearing.

34

7 Gaurav attempts to push an object weighing 160 kg across a wooden floor with coefficient of friction $\mu = 0.3$. What force must he exert in order to move the table?

8 An object sits on a rough surface, and a force $\boldsymbol{G}$ is applied to the object in a parallel direction to the surface. Which of the following would result in a *reduced* friction force?

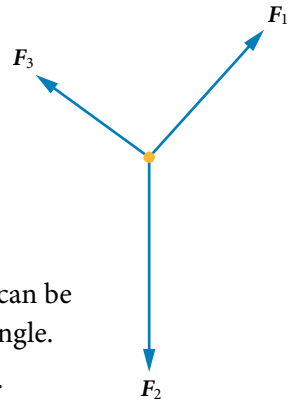
A an increase in the object's weight

B an increase in μ, the coefficient of friction

C an increase in force $\boldsymbol{G}$

D a decrease in force $\boldsymbol{G}$

9 In the diagram, the three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on the particle, which remains in equilibrium.



(a) Which of the following statements is false?

A The object's motion will remain unchanged.

B $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$

C The object experiences no acceleration.

D The magnitude of the forces acting and the angles between the forces can be found using trigonometric ratios only if the triangle contains a right angle.

(b) Identify the key concepts required to choose the correct answer in part (a).

10 Kelly and Sam are moving a piano weighing 850 kg across a rough surface. When Kelly pushes with a force of 450 N parallel to the floor and Sam pulls in the same direction with a force of 500 N, the piano just starts to move. Calculate the coefficient of friction of the floor, correct to 2 decimal places.

Worked Example 32

11 Three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on an object: $\boldsymbol{F}_1 = 200$ N at 057°T, $\boldsymbol{F}_2 = 220$ N at 170°T and $\boldsymbol{F}_3 = 150$ N at 245°T.

(a) Resolve each of the forces in horizontal $\hat{\boldsymbol{i}}$ components and vertical $\hat{\boldsymbol{j}}$ components. Give answers correct to 2 decimal places.

(b) Determine the resultant force, $\boldsymbol{R}$. Give answers correct to 2 decimal places.

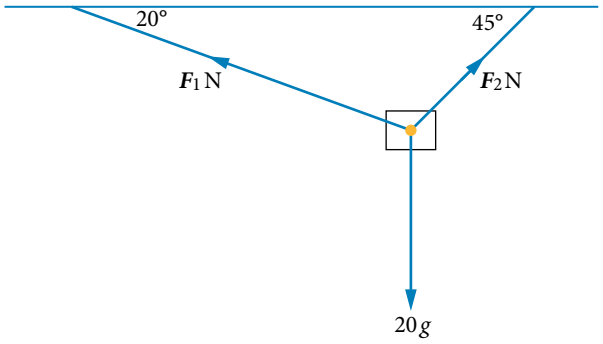
12 Three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on an object: $\boldsymbol{F}_1 = 250$ N at N45°E, $\boldsymbol{F}_2 = 270$ N at S35°W and $\boldsymbol{F}_3 = 350$ N at S 25° E.

(a) Resolve each of the forces in horizontal $\hat{\boldsymbol{i}}$ components and vertical $\hat{\boldsymbol{j}}$ components. Give answers correct to 2 decimal places.

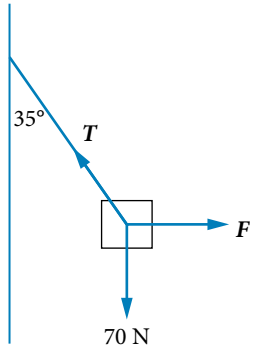
(b) Determine the resultant force $\boldsymbol{R}$. Give answers correct to 2 decimal places.

13 A particle of mass 20 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 20° and 45° respectively to the horizontal, calculate the magnitude of the tension in each string, in newtons correct to 2 decimal places.

33



14 An object that has a vertically downwards force of 70 N is suspended by a string of negligible mass. The object is pulled to the right by a horizontal force so that the string makes an angle of 35° with the vertical as shown in the diagram. Calculate the magnitudes of the horizontal force, $\boldsymbol{F}$ N, and the tension in the string, $\boldsymbol{T}$ N. Give answers correct to 2 decimal places.



15 An object that has a vertically downwards force of 250 N is suspended by a string of negligible mass. The object is pulled to the right by a horizontal force so that the string makes an angle of 22° with the vertical.

(a) Draw the force diagram for the problem. Let the horizontal force be $\boldsymbol{F}$ N and the tension in the string be $\boldsymbol{T}$ N.

(b) Calculate the magnitudes of the horizontal force, $\boldsymbol{F}$ N, and the tension in the string, $\boldsymbol{T}$ N. Give answers correct to 2 decimal places.

16 Two tug boats are towing a large boat, of mass 15 000 kg, in an easterly direction in the channel towards the dock. The first tug boat is pulling with a force of $\boldsymbol{F}_1 = 8000$ N at an angle of 28° north of the forward motion and the second tug boat is pulling with a force of $\boldsymbol{F}_2 = 8250$ N at an angle θ south of the forward motion. If there is a resistive force of 4200 N opposing the eastern motion of the large boat, calculate the easterly force acting on the large boat, to the nearest newton.

17 Two tug boats are towing a large boat, of mass M kg, back to shore. The first tug boat is pulling with a force of $\boldsymbol{F}_1$ N at an angle of 27° north of the forward motion and the second tug boat is pulling with a force of $\boldsymbol{F}_2$ N at an angle 30° south of the forward motion. If the large boat is being pulled with constant velocity, and there is a resistive force of 5500 N to the motion, calculate the magnitudes of the two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ to the nearest newton.

18 A section of a new bridge is being moved into position by four cranes, which will move it horizontally into position. The chains on the four cranes that are connected to the bridge section exert forces that are acting on the bridge simultaneously and in the same plane. The four forces are $\boldsymbol{F}_1 = 2050$ N acting at 037°T, $\boldsymbol{F}_2 = 1560$ N acting at 130°T, $\boldsymbol{F}_3 = 1650$ N acting at 237°T and $\boldsymbol{F}_4 = 1930$ N acting at 316°T.

(a) Calculate the magnitude and direction of the resultant force applied to the bridge section. Give answers correct to the nearest whole number.

(b) Due to a wind shift, an adjustment is needed. The first force is changed to $\boldsymbol{F}_1 = 2150$ N at 037°T and the second force is changed to $\boldsymbol{F}_1 = 1560$ N at 135°T. Calculate the magnitude and direction of the new resultant force applied to the bridge section. Give answers correct to 2 decimal places.

19 Brooke is attempting to pull a cart weighing 20 kg by applying a force along a handle that is elevated at 30° to the horizontal. The coefficient of friction of the surface is $\mu = 0.3$. What force must Brooke pull with in order to move the cart?

Worked Example 35

20 A crate of weight 100 kg rests on a plane that is inclined θ to the horizontal. The surface of the plane is rough and has a coefficient of friction of $\mu = 0.6$. At what angle must the plane be inclined so that the crate begins to slide down the plane? (Give your answer to 1 decimal place.)

Summary

2

Vector notation

A scalar quantity is one that is completely specified by its magnitude.

A vector quantity is one that is completely described by its magnitude and direction.

The vector from point A to point B is denoted $\overrightarrow{AB}$ or $\boldsymbol{a}$. Its magnitude is denoted $\left|\overrightarrow{AB}\right|$ or $|\boldsymbol{a}|$ respectively.

Two vectors are equal if and only if they have the same magnitude and the same direction.

The negative of vector $\overrightarrow{AB} = \boldsymbol{a}$ is a vector of equal magnitude but with opposite direction; that is $-\overrightarrow{AB} = \overrightarrow{BA} = -\boldsymbol{a}$.

A vector of zero magnitude, denoted O, is represented as a point and has neutral direction.

Multiplying a vector by a scalar k results in a vector with k times the magnitude and parallel to the original vector.

Vector addition and subtraction

The triangle rule for the addition of two vectors: To add two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the vectors are placed head to tail. The resultant vector is the vector joining the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$ to form a triangle. Therefore, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$.

The parallelogram law for addition: If two vectors to be added have the same initial point, they can be added by drawing a parallelogram. The diagonal of the parallelogram, drawn from the initial point to the final point, represents the vector sum.

To subtract $\boldsymbol{b}$ from $\boldsymbol{a}$, add $-\boldsymbol{b}$ to $\boldsymbol{a}$; that is $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$.

Position vectors

A position vector is drawn with its tail at O, the origin. Vectors drawn on the Cartesian plane can be represented as ordered pairs; for example, the position vector of point $A(3, 2)$ is $\overrightarrow{OA}$.

A relative position vector represents a point's position in relation to another point. The position vector of B relative to (as seen from) A is $\overrightarrow{AB}$, where $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

If A is defined by the position vector $\boldsymbol{a}$ and B is defined by the position vector $\boldsymbol{b}$ then $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.

Vectors in Cartesian (component) form

A vector can be represented as an ordered pair (x, y) or a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where the first value represents the distance parallel to the x-axis and the second value the distance parallel to the y-axis.

This information can also be defined using the vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, where $\hat{\boldsymbol{i}}$ is a vector of one unit magnitude in the positive x-direction and $\hat{\boldsymbol{j}}$ is a vector of one unit magnitude in the positive y-direction.

$\boldsymbol{a} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ is called the component form of vector $\boldsymbol{a}$ and can also be represented in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}$.

For the vector $\boldsymbol{a} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$.

Vectors in polar form

If vector $\boldsymbol{a}$ has a magnitude of $|\boldsymbol{a}|$ and forms an angle of θ with the positive x-axis, where $0° \leq \theta < 360°$, then the vector can be written in polar form as $\boldsymbol{a} = (|\boldsymbol{a}|, \theta°)$. To avoid confusing polar and Cartesian form, the angle symbol can be added to give $\boldsymbol{a} = (|\boldsymbol{a}|, \angle\theta°)$.

Unit vectors

A unit vector is a vector that has a magnitude of 1. The vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors.

A unit vector in the direction of $\boldsymbol{a}$, denoted $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$. If $\boldsymbol{a} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, then $\hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{x^2 + y^2}}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right)$

For vectors in the Cartesian plane, if $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$, then

- $\boldsymbol{a} + \boldsymbol{b} = (x_1 + x_2)\hat{\boldsymbol{i}} + (y_1 + y_2)\hat{\boldsymbol{j}}$
- $\boldsymbol{a} - \boldsymbol{b} = (x_1 - x_2)\hat{\boldsymbol{i}} + (y_1 - y_2)\hat{\boldsymbol{j}}$
- $k\boldsymbol{a} = kx_1\hat{\boldsymbol{i}} + ky_1\hat{\boldsymbol{j}}$
- $\boldsymbol{a} = \boldsymbol{b}$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

In matrix notation, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}$ and $k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix}$.

The scalar product of vectors

The scalar product or dot product is a way of multiplying two vectors. If θ is the angle between the positive directions of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar product is $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$ (if vectors are in the same direction) or $\boldsymbol{a} \cdot \boldsymbol{b} = -|\boldsymbol{a}||\boldsymbol{b}|$ (if vectors are in opposite directions). Therefore, for the unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} = 1$.

If $\boldsymbol{a} = \boldsymbol{b}$, then $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.

If, for two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

Unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are mutually perpendicular, therefore $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = 0$.

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = x_1x_2 + y_1y_2$.

The scalar product can be used to calculate the angle between two vectors: $\theta = \cos^{-1}\left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$

Scalar and vector projections

The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\boldsymbol{a} \cdot \hat{\boldsymbol{b}}$, where $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

A vector can be resolved into a sum of two vectors that are perpendicular to each other. It is usual to resolve the vector into two perpendicular components with one in a specified direction (i.e. parallel to a given vector):

The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is $\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$

Problems involving displacement and velocity

The vector quantity *displacement* is the change in position of an object relative to its starting point. The scalar quantity *distance* is how far an object has travelled in total.

The vector quantity *velocity* is the rate at which the displacement of an object changes with respect to time. The scalar quantity *speed* is the magnitude of the object's velocity.

The average velocity of an object between two positions is defined by:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time taken}} = \frac{\text{displacement}}{\text{time taken}}$$

The average speed between two points is defined by: $\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$

Problems involving forces

A force can be thought of as a push or a pull acting on a body or object and it can be considered that the forces acting on an object act through a single point at which the object's mass is concentrated. Force is a vector quantity so it needs both magnitude and direction to fully describe it.

If more than one force acts on an object, the resultant force or net force, acts as if only one force is actually acting on the object. The resultant or net force, $\boldsymbol{R}$, is the vector sum of all the forces acting on an object. If forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, ... act on a particle, then $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + ...$

If the forces acting on an object are in the same plane, then each of the forces and the resultant force can be expressed in component form as a sum of two perpendicular vectors, usually $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ components. If a force $\boldsymbol{F}$ acts at an angle of θ to the horizontal, then $\boldsymbol{F}$ can be written as the sum of a horizontal ($\hat{\boldsymbol{i}}$) and a vertical ($\hat{\boldsymbol{j}}$) component; i.e. $\boldsymbol{F} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ or $\boldsymbol{F} = |\boldsymbol{F}|\cos(\theta)\hat{\boldsymbol{i}} + |\boldsymbol{F}|\sin(\theta)\hat{\boldsymbol{j}}$.

If the resultant force acting on an object is zero, the object's motion will remain unchanged. This is known as Newton's first law of motion and the object is said to be in equilibrium. For example, if the three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ that act on a particle are in equilibrium, then $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$.

The magnitude of the forces acting and the angles between the forces can be found using trigonometric ratios (if the triangle contains a right angle) or by using the sine rule.

If more than one force acts on an object, then the resultant force can be determined from adding the forces. If the object is in equilibrium, then the sum of the $\hat{\boldsymbol{i}}$ components and the sum of the $\hat{\boldsymbol{j}}$ components are each zero.

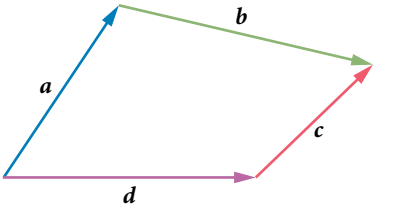
Friction ($\boldsymbol{F}$) always acts in the opposite direction to the actual or potential motion of an object across a rough surface. It is the product of the normal reaction force ($\boldsymbol{N}$) and the coefficient of friction (μ): $\boldsymbol{F} = \mu\boldsymbol{N}$.

When calculating forces acting on an object resting on a plane inclined at angle θ to the horizontal, then the component of the weight force that acts down the plane will be $mg\sin(\theta)$ and the normal reaction force will be $\boldsymbol{N} = mg\cos(\theta)$.

Chapter review 2

Exercise 2.1

1 Four vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are shown in the diagram. Which one of the following statements is true?

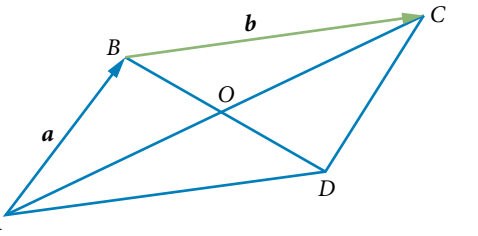


A $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{d}$ B $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}+\boldsymbol{d}$

C $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}=\boldsymbol{0}$ D $\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}+\boldsymbol{d}=\boldsymbol{0}$

Exercise 2.1

2 In the parallelogram $ABCD$ shown, the point of intersection of the diagonals is O. Express vector $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



Exercise 2.2

3 The vector $\boldsymbol{a}$ is represented by the ordered pair (-2, 3). Calculate $-3\boldsymbol{a}$.

Exercise 2.2

4 Determine the column matrix vector that runs from the point (-3, 1) to the point (3, -2)

Exercise 2.3

5 Which of the following vectors is *not* parallel to $\boldsymbol{f} = -6\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$?

A $\boldsymbol{a} = -18\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ B $\boldsymbol{b} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

C $\boldsymbol{c} = 4\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$ D $\boldsymbol{d} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

Exercise 2.3

6 If $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$, determine the two vectors that are double the magnitude of $\boldsymbol{a}$ and parallel to $\boldsymbol{a}$.

Exercise 2.3

7 If $\boldsymbol{a} = -4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, calculate $2\boldsymbol{a} - \boldsymbol{b}$.

Exercise 2.3

8 Express the vector $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ in polar form.

Exercise 2.3

9 If the vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - x\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ are perpendicular, determine the value of x.

Exercise 2.7

10 A car is moving at a speed of 100 km h^{-1} and a truck is moving at 90 km h^{-1} in the opposite direction. Calculate the relative velocity of the car with respect to the truck.

Exercise 2.7

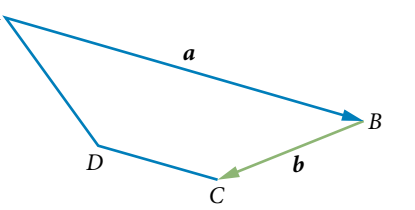
11 On his way home from training, Mitch walks along three streets after leaving the recreation reserve. He walks 150 m north, 300 m east and 250 m north. What is the magnitude of Mitch's resultant displacement?

Exercise 2.8

12 Three forces act simultaneously in the same plane on an object. The resultant force is $\boldsymbol{F} = 520$ N due east. If two of the forces are $\boldsymbol{F}_1 = 350$ N east and $\boldsymbol{F}_2 = 270$ N west, calculate the third force.

Exercise 2.1

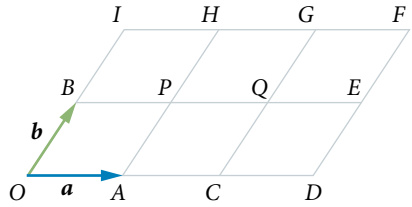
13 $ABCD$ is a trapezium with $\overline{AB}$ parallel to, and three times the length of, $\overline{DC}$. If $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$, express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



(a) $\overrightarrow{CD}$ (b) $\overrightarrow{AC}$

(c) $\overrightarrow{AD}$ (d) $\overrightarrow{DB}$

14 The line segments $\overline{OA}$, $\overline{OB}$, $\overline{AC}$, etc. are the same length. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. Exercise 2.1



(a) $\overrightarrow{OP}$ (b) $\overrightarrow{OG}$
(c) $\overrightarrow{OQ}$ (d) $\overrightarrow{CE}$
(e) $\overrightarrow{AB}$ (f) $\overrightarrow{DI}$
(g) $\overrightarrow{FQ}$ (h) $\overrightarrow{DE} + \overrightarrow{EO}$

15 Given vectors $\boldsymbol{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$, express each of the following vectors in column matrix form. Exercise 2.2

(a) $4\boldsymbol{c} - 3\boldsymbol{d}$ (b) $7\boldsymbol{d} - 8\boldsymbol{c}$

16 If $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, express each of the following vectors in component form. Exercise 2.3

(a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $3\boldsymbol{b} - 2\boldsymbol{a}$ (c) $-2\boldsymbol{a} - 7\boldsymbol{b}$

17 If $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, calculate the magnitude of each of the following vectors. Exercise 2.3

(a) $|\boldsymbol{a} + \boldsymbol{b}|$ (b) $|-2\boldsymbol{a} - 7\boldsymbol{b}|$

18 The position vectors of points A and B are $\overrightarrow{OA} = 15\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = -6\hat{\boldsymbol{i}} - 19\hat{\boldsymbol{j}}$, respectively. Exercise 2.3

(a) Express the vector $\overrightarrow{AB}$ in component form. (b) Calculate $\left|\overrightarrow{AB}\right|$.

19 Calculate the magnitude of each of the following vectors. Exercise 2.3

(a) $\boldsymbol{a} = 10\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}}$ (b) $\boldsymbol{b} = -6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$

20 Resolve the following vectors into Cartesian (component) form. Give answers correct to 2 decimal places. Exercise 2.3

(a) $\boldsymbol{a} = (16, 48°)$ (b) $\boldsymbol{b} = (24, 148°)$

21 Calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ given the following pairs of vectors. Exercise 2.4

(a) $\boldsymbol{a} = -4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

22 Calculate the scalar product and hence prove that the vectors $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 10\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ are perpendicular. Exercise 2.4

23 For each of the following pairs of vectors, determine the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$. Exercise 2.5

(a) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = -5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$

24 The following separate forces are acting on objects. Calculate the resultant force acting on each object, correct to 2 decimal places. Exercise 2.8

(a) $\boldsymbol{F}_1 = 250\,\text{N}$ north, $\boldsymbol{F}_2 = 450\,\text{N}$ south, $\boldsymbol{F}_3 = 125\,\text{N}$ north

(b) $\boldsymbol{F}_1 = 100\,\text{N}$ east, $\boldsymbol{F}_2 = 100\,\text{N}$ south

25 Which one of the following vectors is parallel to the vector $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and has a magnitude of $2\sqrt{58}$? Exercise 2.3

A $-6\hat{\boldsymbol{i}} + 14\hat{\boldsymbol{j}}$ B $-24\hat{\boldsymbol{i}} + 28\hat{\boldsymbol{j}}$ C $-\frac{3}{2}\hat{\boldsymbol{i}} + \frac{7}{2}\hat{\boldsymbol{j}}$ D $3\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$

Exercise 2.5 **26** The vector projection of $\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ onto $\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ is $-\frac{7}{13}\left(\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$. What is the vector projection of $\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ perpendicular to $\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$?

Exercise 2.3 **27** Calculate the exact values of the unknown pronumerals in each of the following vector equations.

(a) $\left(2a - 3b\right)\hat{\boldsymbol{i}} - 2b\hat{\boldsymbol{j}} = 5\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$ **(b)** $\left(2f + 5\right)\hat{\boldsymbol{i}} + \left(8 - 7g\right)\hat{\boldsymbol{j}} = f\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right) + 2g\left(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$

Exercise 2.3 **28** Consider the vector $\boldsymbol{a} = -9\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$.

(a) Calculate $\hat{\boldsymbol{a}}$.

(b) Determine the vector $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ with a magnitude of 5.

Exercise 2.4 **29** Calculate the angle between the vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

Exercise 2.4 **30** Calculate the angle between the vectors $\boldsymbol{a} = -5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

Exercise 2.5 **31** If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, calculate:

(a) the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ **(b)** the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

Exercise 2.7 **32** Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the east and north directions respectively. Lawton and Ted finish school on the last day of term and decide to take a road trip across the desert in Ted's new car. Their trip involves the following movements: 43 km at 337°T, 65 km at 270°T and 22 km at 93°T. Ted's car breaks down after the last leg of the trip. Calculate the distance and direction of Lawton and Ted from the school, giving your answers to 1 decimal place.

Exercise 2.8 **33** A particle of mass 50 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings both make angles of 30° to the horizontal, calculate the magnitude of the tension in each string, in newtons, given that $\sin\left(30°\right) = \frac{1}{2}$.

Exercise 2.8 **34** A particle of mass 25 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 20° and 45° respectively to the horizontal, use the component form to calculate the magnitude of the tension in each string, in newtons, correct to 1 decimal place.

Exercise 2.8 **35** Nick attempts to pull an object of weight 130 kg across a floor that has a coefficient of friction of 0.7. Calculate the force that Nick must exert in order to make the object move. State any assumptions you make and comment on the effect of your assumptions.

Exercise 2.3 **36** Calculate the exact values of the unknown pronumerals in the following vector equation:
$\left(a^2 - 9a\right)\hat{\boldsymbol{i}} + \left(2b^3 + 1\right)\hat{\boldsymbol{j}} = 10\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$

Exercise 2.5 **37** Consider two vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

(a) Calculate the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$

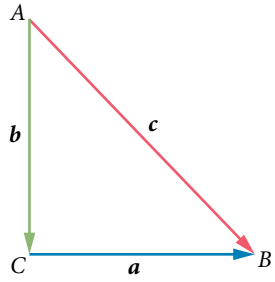
(b) Calculate the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(c) Calculate the vector projection of $\boldsymbol{a}$ perpendicular to the direction of $\boldsymbol{b}$.

(d) Hence, express the vector $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ in terms of projections onto and perpendicular to $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

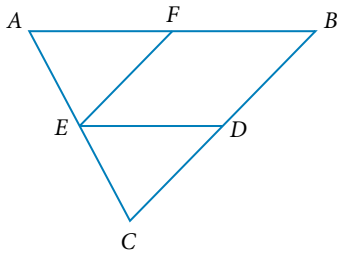
38 Use vector methods to prove that, if for ΔABC, $|\boldsymbol{c}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$, $\angle ACB$ is a right angle.

Exercise 2.6



39 $BDEF$ is a parallelogram contained within a triangle ABC, as shown in the diagram. Let $\overrightarrow{AF} = \boldsymbol{a}$ and $\overrightarrow{BD} = \boldsymbol{b}$ and F is the midpoint of $\overline{AB}$.

Exercise 2.6



(a) Express the vector $\overrightarrow{AE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Use vector methods to prove that $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$

40 Freddie rows his boat in a direction perpendicular to the river's banks. The river is 200 m wide. The boat's speed in this direction is 2 m s^{-1}. The river current speed is 1 m s^{-1}. Give answers correct to 2 decimal places, where necessary.

Exercise 2.7

(a) Calculate the velocity of the boat relative to the bank. State any assumptions you make, and comment on the effect of these assumptions on your answer.

(b) At what distance downstream from point A, on the opposite bank but directly opposite starting point O, will Freddie get to the other bank? State any assumptions you make, and comment on the effect of these assumptions on your answer.

(c) Calculate Freddie's final displacement from starting point O.

(d) Rohanne is going to cross the river in another rowboat. She can row at the speed of 2 m s^{-1} relative to the river. At what direction should Rohanne head her rowboat to get to the other bank at point A directly opposite the starting point O, and how long will it take her? Give the answer correct to 2 decimal places.

41 Lachie's car of mass 850 kg is broken down on a hill inclined at 20° to the horizontal. The coefficient of friction of the road surface is 0.3. Calculate how hard Lachie must push against his car to stop it from rolling down the hill.

Exercise 2.8





3

Introduction to proof

Recall 144

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3

Recall

Identify numbers with certain properties

1 Identify the numbers from 0, 1, 2, 3, …, 10 with the following properties.

(a) even (b) odd (c) prime

(d) divisible by 4 (e) factor of 4 (f) perfect square

Express a given integer as a product of prime powers

2 Express each of the following numbers as a product of prime powers.

(a) 8 (b) 12 (c) 90

(d) 63 (e) 50

Simplify surds

3 Simplify each of the following.

(a) $\sqrt{50}$ (b) $2\sqrt{28}-3\sqrt{63}+8\sqrt{7}$ (c) $(5-2\sqrt{2})(5+2\sqrt{2})$

Rationalise denominators of fractions involving surds

4 Express each of the following fractions with integer denominators.

(a) $\dfrac{5}{\sqrt{6}}$ (b) $\dfrac{2}{3-2\sqrt{5}}$ (c) $\dfrac{4+\sqrt{3}}{4-\sqrt{3}}$

Simplify algebraic expressions involving surds

5 Simplify each of the following. Assume x and y are positive real numbers.

(a) $\sqrt{x^6}$ (b) $\sqrt{y^7}$ (c) $\left(\sqrt{x}+\sqrt{y}\right)^2$

Simplify algebraic expressions involving indices

6 Simplify each of the following.

(a) $3^{n+1}\times 3^{n-1}$ (b) $81\times 3^{n-5}$ (c) $\dfrac{3^{n+5}}{81}$

Solve simple inequalities

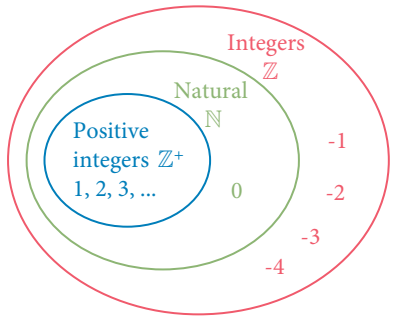
7 Solve the following inequalities.

(a) $3x+5\leq 17$ (b) $-2x+1>7$

3.1 Rational and irrational numbers

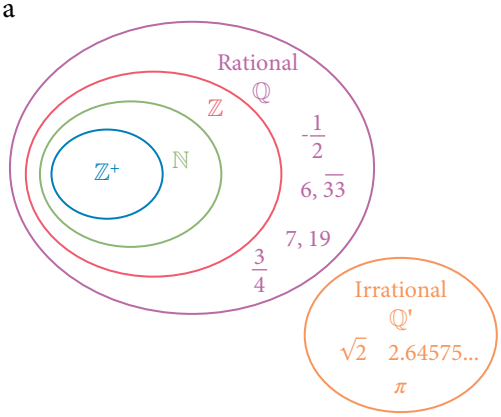
Early civilisations used numbers exclusively to count things; there was no need for negative numbers, fractional numbers or even the number zero. These other types of numbers were introduced gradually over many thousands of years with contributions from many cultures. Today, much of mathematics is based on a foundation of numbers, and it is therefore important to understand the different types of numbers that mathematicians work with. This chapter focuses on real numbers; that is, numbers that can be placed on a number line. (In Chapter 5, you will be introduced to the concept of imaginary numbers, which cannot be placed on a number line.)

The most fundamental type of number is a *positive integer*. The symbol $\mathbb{N}$ (or less commonly, *N* or *J*) is used to denote the set of non-negative integers, {0, 1, 2, 3...}; non-negative integers are also known as *natural numbers*. Note that some sources do not include 0 in the set of natural numbers, and use the terms 'natural number' and 'positive integer' interchangeably.



The set of integers, $\mathbb{Z}$ (or less commonly, *Z*), includes all *whole numbers* (both positive and negative), and zero (which is neither positive nor negative). The symbol $\mathbb{Z}^+$ is used to denote just the set of positive integers, {1, 2, 3...}, while the symbol $\mathbb{Z}^-$ is used to denote just the set of negative integers, {-1, -2, -3...}.

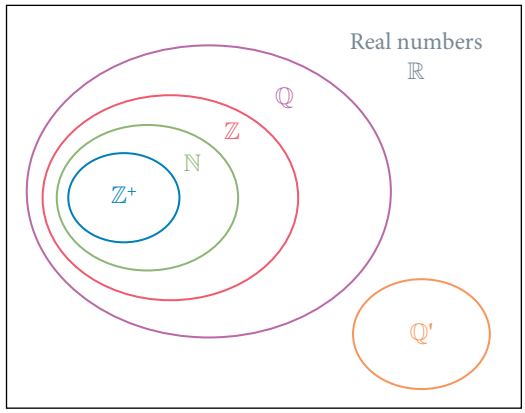
A *rational number* is any number that can be expressed in the form $\frac{p}{q}$ where p and q are both integers, and $q \neq 0$. Any number with a decimal representation that either terminates or repeats in a regular pattern (recurs) is rational. The symbol $\mathbb{Q}$ (or less commonly, *Q*) is used to denote the set of all rational numbers. Note that all integers are rational (since an integer, n, can be expressed as $\frac{n}{1}$).

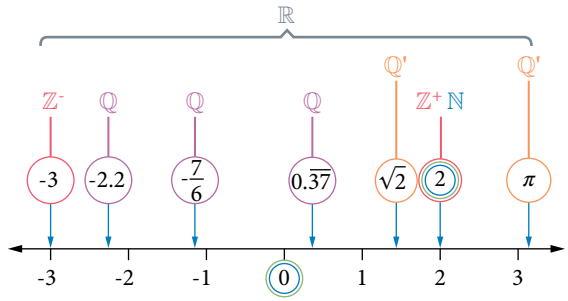


Numbers that are not rational, but can still be placed on a number line, are said to be *irrational*. The symbol $\mathbb{Q}'$ (or less commonly, *Q* or *I*) is used to denote the set of irrational numbers. The number π is known to be irrational, however, it is often approximated as $\frac{22}{7}$. The decimal representations of irrational numbers go on forever in no predictable pattern; thus, irrational number cannot be written exactly using decimals.

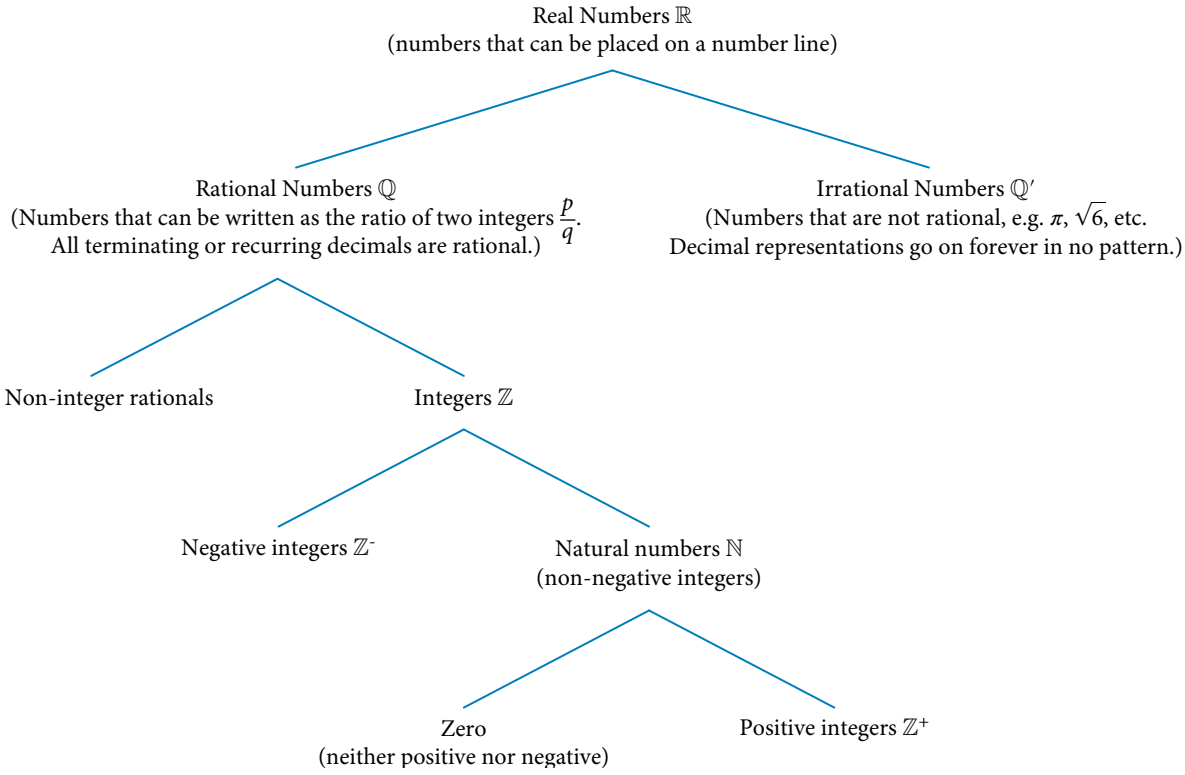
As well as the number π, any number that can only be expressed exactly using a radical symbol $\left(\sqrt{\ }\right)$ is *irrational*. For example, $\sqrt{2}$ is irrational. Note that $\sqrt{9}$ is not irrational because it is possible to express this number without the radical symbol $\left(\sqrt{9} = 3\right)$.

The sets of rational and irrational numbers together comprise the set of real numbers, denoted by the symbol $\mathbb{R}$ (or less commonly, R). As stated previously, the set of real numbers can be thought of as the set of all numbers that can be placed on a number line; see diagram below.





The following diagram summarises the symbols and relationships between the different sets of numbers described.



1 Classifying numbers

For each of the following numbers, determine which sets from $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the number is an element of.

(a) -7

THINKING	WORKING
1 Consider whether or not the number belongs to each of the sets listed.	-7 is a negative integer. Therefore it is also an integer, a rational number, and a real number.
2 Answer the question.	Hence, -7 is an element of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}^-$.

(b) $4.\overline{23}$

1 Consider whether or not the number belongs to each of the sets listed.	$4.\overline{23}$ is a recurring decimal. Therefore, it is a non-integer rational number.
2 Answer the question.	Hence, $4.\overline{23}$ is an element of $\mathbb{R}$ and $\mathbb{Q}$.

(c) $6\sqrt{10}$

1 Consider whether or not the number belongs to each of the sets listed. If a number cannot be expressed without a radical symbol, it is irrational. Further, the product (or sum, or difference) of an irrational number and a rational number is irrational.	The number is part of the Real number system, $\mathbb{R}$, but cannot be written without a radical. It is therefore classified as an Irrational number, $\mathbb{Q}'$.
2 Answer the question.	Hence, $6\sqrt{10}$ is an element of $\mathbb{R}$ and $\mathbb{Q}'$.

(d) $\sqrt{\frac{25}{49}}$

1 Consider whether or not the number belongs to each of the sets listed. If a number cannot be expressed without a radical symbol, it is irrational.	$\sqrt{\frac{25}{49}} = \sqrt{\left(\frac{5}{7}\right)^2} = \frac{5}{7}$ Thus, this number is rational.
2 Answer the question.	Hence, $\sqrt{\frac{25}{49}}$ is an element of $\mathbb{R}$ and $\mathbb{Q}$.

(e) $\sqrt{-100}$

1 Consider whether or not the number belongs to each of the sets listed.	Square roots of negative numbers are not real numbers (and hence, do not belong to any of the other sets listed).
2 Answer the question.	$\sqrt{-100}$ does not belong to any of the sets listed.

As previously stated, all rational numbers can be expressed as either terminating or recurring decimals. The following example illustrates how to express a rational number in decimal form without the aid of a calculator.

2 Expressing rational numbers as decimals

Express each of the following rational numbers as a terminating or recurring decimal.

(a) $\frac{3}{25}$

THINKING	WORKING
If the denominator of a fraction divides into 10, 100, 1000, etc. then the number can readily be written as a terminating decimal by first rewriting the fraction with a denominator that is a power of 10.	$\frac{3}{25} = \frac{12}{100}$ $= 0.12$

(b) $\frac{3}{80}$

If it is not obvious how to express the fraction as a decimal, use short division. It may also be helpful to first divide the top and bottom of the fraction by 10 to obtain a single-digit denominator.

$\frac{3}{80} = \frac{0.3}{8}$

$$8\overline{)0.\,3\;0^{6}0^{4}0}\quad 0.\,0\;3\;7\;5$$

$\therefore \frac{3}{80} = 0.0375$

(c) $\frac{1}{6}$

If it is not obvious how to express the fraction as a decimal, use short division. When the same remainder (or patterns of remainders) appear, the resulting digits must recur.

$$6\overline{)1.\,^{4}0\,^{4}0\,^{4}0\ldots}\quad 0.\;\;1\;\;6\;\;6\ldots$$

$\therefore \frac{1}{6} = 0.1\dot{6}$

You may notice from experience that a simplified fraction can be expressed as a terminating decimal if the denominator has no prime factors other than 2 or 5.

If the denominator of a fraction contains no prime factors other than 2 or 5, then this number must divide into 10 or 100 or 1000 etc. meaning that the fraction can be expressed with a power of 10 denominator, and thus be expressed as a terminating decimal.

A simplified fraction can be expressed as a terminating decimal if the denominator has no prime factors other than 2 or 5; otherwise, it can only be expressed as a recurring decimal.

3 Expressing rational numbers as decimals

For each of the following rational numbers, determine whether it must be recurring or terminating, justifying your answer.

(a) $\frac{7}{16}$

THINKING	WORKING
1 Ensure the fraction is in its simplest form, and consider the prime factors of the denominator.	The only prime factor of 16 is 2, since $16 = 2^4$. $\frac{7}{16}$ can be expressed as a terminating decimal, because the denominator contains no prime factors other than 2 or 5.
2 Evaluate the reasonableness of your answer by converting the fraction to a decimal.	$$16\overline{)7.0^{6}0^{12}0^{8}0}\ \ 0.4\ 3\ 7\ 5$$

(b) $\frac{2}{15}$

1 Ensure the fraction is in its simplest form, and consider the prime factors of the denominator.	The prime factors of 15 are 3 and 5, since $15 = 5 \times 3$. $\frac{2}{15}$ can only be expressed as a recurring decimal, because the denominator contains a prime factor other than 2 or 5, namely 3.
2 Evaluate the reasonableness of your answer by converting the fraction to a decimal.	$$15\overline{)2.0^{5}0^{5}0^{5}0\ldots}\ \ 0.1\ 3\ 3\ 3\ldots$$

While terminating decimals can readily be converted to simplified fractions, converting a recurring decimal to a simplified fraction is not quite so obvious. The following examples illustrate one technique that can be used to accomplish this.

4 Expressing recurring decimals as fractions

Write each of the following numbers in the form $\frac{p}{q}$ where p and $q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.

(a) $0.\dot{2}$

THINKING	WORKING
1 Assign the number to a variable.	Let $x = 0.2222...$
2 Multiply by a suitable power of 10 to obtain a number with the same repeated digits following the decimal point.	$10x = 2.2222...$
3 Subtracting these two expressions should result in a rational number.	$10x - x = 2.2222... - 0.2222...$ $9x = 2$ $x = \frac{2}{9}$

(b) $0.1\overline{23}$

1 Assign the number to a variable, then multiply by a suitable power of 10 so that the recurring digits occur immediately after the decimal point.	Let $x = 0.123\,2323...$ $10x = 1.232\,323....$
2 Multiply by a suitable power of 10 to obtain a number with the same repeated digits following the decimal point.	$1000x = 123.2323...$
3 Subtracting these two expressions should result in a rational number.	$1000x - 10x = 123.\cancel{2323...} - 1.\cancel{2323...}$ $990x = 122$ $x = \frac{122}{990}$ $= \frac{61}{495}$

EXERCISE 3.1 Rational and irrational numbers

Worked Example

1

1 For each of the following numbers, determine which sets from $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the number is an element of.

(a) 17 **(b)** -3 **(c)** 0 **(d)** $\frac{3}{5}$

(e) $\frac{24}{6}$ **(f)** $-4\frac{1}{3}$ **(g)** $6.\dot{3}$ **(h)** $-0.1\overline{23}$

(i) $\sqrt{16}$ **(j)** $2\sqrt{6}$ **(k)** $\sqrt{\frac{1}{4}}$ **(l)** $\sqrt[3]{6}$

(m) $\sqrt[3]{0.001}$ **(n)** π **(o)** $\sqrt{-9}$ **(p)** $\sqrt[3]{0}$

2 Express each of the following rational numbers as a terminating or recurring decimal. 2

(a) $\frac{4}{10}$ (b) $\frac{213}{100}$ (c) $\frac{3}{40}$ (d) $\frac{7}{12}$

(e) $\frac{2}{11}$ (f) $\frac{7}{11}$ (g) $\frac{11}{30}$ (h) $\frac{34}{9}$

3 For each of the following rational numbers, without calculating the decimal representation, determine whether it can be written as a terminating decimal, or only as a recurring decimal, justifying your answer. 3

(a) $\frac{5}{8}$ (b) $\frac{11}{12}$ (c) $\frac{3}{160}$ (d) $\frac{6}{70}$

4 Express each of the following numbers in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1. 4

(a) 1.8 (b) 0.14 (c) 4.111 (d) $0.\dot{3}$

(e) $0.\dot{4}$ (f) $0.\dot{1}\dot{4}$ (g) $0.\overline{125}$ (h) $0.2\dot{3}$

(i) $0.5\overline{4}$ (j) $0.3\dot{1}$ (k) $0.34\dot{7}$ (l) $0.1\overline{28}$

5 Which of the following numbers is equal to $0.1\overline{21}$?

A $0.\overline{12}$ B $0.\overline{21}$ C $0.1\dot{2}$ D $0.\overline{121}$

6 (a) Which of the following numbers is equal to $0.4\dot{9}$?

A 0.499999 B $0.\overline{49}$ C 0.5 D $0.\overline{499}$

(b) Explain the misconceptions or errors that lead to selecting the incorrect alternatives.

7 Demonstrate that $0.\dot{a}$, where $a \in \mathbb{Z}$, is equal to $\frac{a}{9}$.

8 Demonstrate that $0.\overline{ab}$, where $a, b \in \mathbb{Z}$, can be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$.

9 Demonstrate that $0.a\dot{b}$, where $a, b \in \mathbb{Z}$, can be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$.

10 Let $E = \sqrt{11+\sqrt{x}} - \sqrt{11-\sqrt{x}}$. By considering E^2, deduce which positive integer values of x make E an integer.

3.2 The language and logic of proof

Mathematical knowledge is unlike the knowledge of other scientific disciplines. Instead of consisting of theories based on observation and evidence that can possibly be falsified, it consists of *theorems*: significant mathematical statements that have been proven to be absolutely true (think of Pythagoras' theorem, for instance). Scientific theories are similar to mathematical *conjectures*. A conjecture is a statement that mathematicians have reason to believe may be true, but which has not been proven definitively. One of the most famous mathematical conjectures is the *Goldbach conjecture*, named after the eighteenth-century mathematician Christian Goldbach:
'Every even integer greater than 2 can be expressed as the sum of two prime numbers.'

The Goldbach conjecture (see right) seems likely to be true; in fact, it has been shown to be true for every integer up to 4×10^{18}. However, it is, at present, unproven—there may, in fact, be a very large even integer that cannot be expressed as the sum of two prime numbers.

$$\begin{aligned} 4 &= 2+2 \\ 6 &= 3+3 \\ 8 &= 3+5 \\ 10 &= 3+7 = 5+5 \\ 12 &= 5+7 \\ 14 &= 3+11 = 7+7 \\ &\vdots \\ 4 \times 10^{18} &= \cdots \end{aligned}$$

The idea of mathematical proof is extremely powerful. It enables mathematics to be a rich and robust system of knowledge that will never be falsified; further, any result that has been proven can safely be used to help establish further useful results, adding to this system of knowledge.

In order to prove mathematical statements, it is important to use clear, unambiguous language and valid logic. The focus of this section is the language and logic used to construct, combine and evaluate the truth or falsehood of mathematical statements about numbers.

Note that the word 'statement' will be used to not only refer to an assertion that is true or false (such as 'The number 7 is prime' or 'All multiples of 10 are also multiples of 5'), but also to refer to an assertion involving one or more variables that becomes true or false whenever values are substituted for the variable (such as 'n is a multiple of 5' or '$x^2 < 20$').

Negating statements

Statement	Example
The negation of a mathematical statement is the statement that is true precisely when the original statement is false, and vice versa.	The negation of the statement $x > 0$ is $x \leq 0$. Notice that, for any number that is substituted in place of x, if the statement $x > 0$ is true, then the negation $x \leq 0$ will be false, and vice-versa.
The negation of a statement can be obtained by preceding the statement with the phrase: 'It is not the case that…'.	If n represents an integer, then the negation of the statement 'n is an even number' is 'It is not the case that n is an even number'. This is equivalent to saying that n is an odd number.
A negation of a statement can be written using a symbol representation. In this case the symbol '$\neg$' represents the word '*not*'.	If P represents any statement, then the negation of P can be written as $\neg P$ or simply '*not P*'.

The negation of statements involving the words 'and' or 'or' can sometimes cause confusion.	Consider negating the statement 'Either $x = 5$ or $x = 7$'. If it is not the case that x is equal to 5 or 7 then it must be the case that $x \neq 5$ *and* $x \neq 7$. As another example, consider the negation of the statement: $x > 0$ and $x < 10$. If it is not the case that x is between 0 and 10, then either $x \leq 0$ or $x \geq 10$.

The negation of '*P* and *Q*' is '*not P* or *not Q*'. The negation of '*P* or *Q*' is '*not P* and *not Q*'.

The preceding rules about negating 'and' and 'or' statements are known as De Morgan's laws.

5 Negating statements using De Morgan's laws

Negate the following statements.

(a) n is divisible by 2 or n is divisible by 3

THINKING	WORKING
Think about preceding the statement with the phrase, 'it is not the case that…', and then use De Morgan's laws.	n is not divisible by 2 and n is not divisible by 3. Equivalently, n is divisible by neither 2 nor 3.

(b) $x > 0$ and $x \leq 5$

Think about preceding the statement with the phrase, 'It is not the case that…', and then use De Morgan's laws.	$x \leq 0$ or $x > 5$

Statements involving quantifiers

Statement	Example
Often, mathematical statements involving variables are true for certain values of the variable and false for others.	The statement $(x - 2)(x - 5) = 0$ is true if x is equal to either 2 or 5, but false otherwise. But consider the following statement, where x represents a real number: $x^2 \geq 0$. This is true for every possible real number that can be substituted in place of x. Therefore, you could say: 'For all real numbers x, $x^2 \geq 0$'.
Statements that assert that some property is true for all possible values of a variable are so common that a special symbol is often used to enable such statements to be written in a more compact way. The symbol $\forall$ is known as the universal quantifier and is used as shorthand for the phrase 'for all'.	The statement could be written as follows: '$\forall$ real numbers x, $x^2 \geq 0$'. This can be condensed further by making use of the notation $x \in \mathbb{R}$ to signify that the variable x is an element of the set of real numbers. So, you could write the statement as follows: '$\forall x \in \mathbb{R}$, $x^2 \geq 0$'.

Sometimes you want to consider at least one possible value of the variable that makes it true and not all the values, as was mentioned before. For such statements, the symbol ∃ (known as the existential quantifier), can be used as shorthand for the phrase 'there exists'.	Consider the statement: '∃ an integer, n such that $n^2 + 2n$ is a prime number'. This can also be condensed further as follows: '$\exists n \in \mathbb{Z}$ such that $n^2 + 2n$ is a prime number'.

The symbol ∀ (known as the *universal quantifier*) is used to mean 'for all'.

The symbol ∃ (known as the *existential quantifier*) is used to mean 'there exists'.

6 Translating statements involving quantifier symbols

Translate the following statements into everyday language. Also determine whether or not each statement is true, justifying your answer.

(a) $\exists x \in \mathbb{R}$ such that $x^2 = \sqrt{x}$.

THINKING	WORKING
1 Determine the meaning of the statement.	There is at least one real number whose square is equal to its square root.
2 In order to prove a 'there exists' statement is true, you need to be able to provide one example.	$(1)^2 = \sqrt{1}$
3 Interpret the answer.	Given that $(1)^2 = \sqrt{1}$, the statement is true. There exists at least one real number for which $x^2 = \sqrt{x}$.

(b) ∀ integers n, the number $5n$ is even.

1 Determine the meaning of the statement.	The product of any integer n and the number 5 results in an even number.
2 In order to prove a 'for all' statement is false, you need to provide a single counter-example. That is, an example demonstrating falsehood.	Let $n = 3$. $3 \times 5 = 15$
3 Interpret the answer.	For $n = 3$, $5n = 15$ which is not an even number, so the statement is false. There may be integers that, when multiplied by 5, result in an even number, however, this does not apply in every case ($\forall n$).

Providing a single example is always sufficient to prove that a 'there exists' statement is true.

Providing a single counter-example is always sufficient to prove that a 'for all' statement is false.

Examples and counter examples

In part (a) of Worked Example 6, it was claimed that the statement '$\exists x \in \mathbb{R}$ such that $x^2 = \sqrt{x}$' is true, and to justify this claim, a single example was provided of a real number whose square was equal to its square root (namely, $x = 1$). Providing a single example is always sufficient to prove that a 'there exists' statement is true.

In part (b) of Worked Example 6, it was claimed that that the statement '$\forall$ integers n, the number $5n$ is even' is false, and to justify this claim, a single example was provided of an integer (namely, $n = 3$) which, when multiplied by 5, gives a number that is not even (namely, 15). An example such as this, that demonstrates the falsehood of a statement, is known as a counter-example. Providing a single counter-example is always sufficient to prove that a 'for all' statement is false.

7 Writing statements with quantifier symbols

Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, determine whether or not each statement is true, justifying your answer.

(a) The square-root of any positive integer is less than or equal to the integer.

THINKING	WORKING
1 Identify the quantifier.	'Of any' specifies that for all positive integers. So the quantifier is $\forall$.
2 Create a mathematical statement.	For all positive integers n, the square root of n is less than or equal to n. $\forall n \in \mathbb{Z}^+, \sqrt{n} \leq n$
3 Determine whether the $\forall$ 'for all' statement can be proved to be false using a counter-example.	For $n = 1$ $\sqrt{1} \leq 1$ For $n = 2$ $\sqrt{2} \leq 2$ This could continue for $n = 1, 2, 3, 4$ and 5. There is not an obvious choice for a counter-example and the true statements cannot be calculated for infinite positive integers. It is assumed to be true, but has not been proved to be so.

(b) There is at least one real number that, when squared, results in a smaller number.

1 Identify the quantifier.	'There is at least one' specifies the existence of a value. So the quantifier is $\exists$.
2 Create a mathematical statement.	$\exists n \in \mathbb{R}$ such that $n^2 < n$.

3 Determine whether the ∃ 'there exists' statement can be proved to be correct by calculating an example.	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ $\therefore \left(\frac{1}{2}\right)^2 < \frac{1}{2}$ This statement is true as shown by the existence of a single example: $n = \frac{1}{2}$.

Note that the symbols ∀ and ∃ may be used together in a single statement, however, the order in which they appear is important. As an example, consider the following two statements:

- ∀ integers n, ∃ an integer m such that $n + m$ is a multiple of 5.
- ∃ an integer n, such that ∀ integers m, $n + m$ is a multiple of 5.

The first statement is true as it says that for every integer, you can find another integer to add to it to give a sum that is a multiple of 5.

The second statement is false as it says there is a special integer that has the property that when you add any other integer to it, you always obtain a multiple of 5.

When the symbols ∀ and ∃ appear together in the same statement, the order in which they appear is important.

Negating statements involving quantifiers

Consider the negation of the statement '∀ real numbers x, $x^2 \geq 0$'. If it is not the case that the square of every real number is greater than or equal to zero, then it must mean that there is at least one real number whose square is less than zero. Thus, the negation is '∃ a real number x such that $x^2 < 0$'. Notice how the negation of a 'for all' statement is a 'there exists' statement. The reverse is also true. For example, the negation of the statement: '∃ a real number x such that $3x = x$' is '∀ real numbers x, $3x \neq x$'.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

8 Negating statements involving quantifiers

Determine the negation of each of the following statements. Also determine whether the original statement or the negation is true, justifying your answer.

(a) ∀ integers n, $2n$ is even

THINKING	WORKING
1 Interpret the meaning of the statement.	For all integers, n, multiplying any integer by 2 will result in an even number.
2 Write a negating statement. The negation of a 'for all' statement is a 'there exists' statement.	There exists an integer, n such that $2 \times n$ is not even (odd). $\exists n \in \mathbb{Z}$ such that $2n$ is not even.
3 Determine whether the original or the negation is true. Justify your answer.	The original statement is true. By definition, an even number is any integer value that is a multiple of 2.

(b) $\exists$ a real number x, such that $x^2 = -1$.

1 Interpret the meaning of the statement.	There exists a real number x, where $x^2 = -1$.
2 Write a negating statement. The negation of a 'there exists' statement is a 'for all' statement.	For all real numbers x, $x^2 \neq -1$. $\forall x \in \mathbb{R}, x^2 \neq -1$
3 Determine whether the original or the negation is true. Justify your answer.	The negation is true. The square of any real number is greater than or equal to 0 and therefore is not equal to -1.

(c) $\exists$ an integer n, such that n is even and n is prime.

1 Interpret the meaning of the statement.	There exists a real number n, where n is both even and prime.
2 Write a negating statement. The negation of a 'there exists' statement is a 'for all' statement.	For all real numbers n, n is neither even nor prime. $\forall n \in \mathbb{Z}$, n is odd or not prime.
3 Determine whether the original or the negation is true. Justify your answer.	The original statement is true. This is proven by the single example where $n = 2$. Two is the only even and prime number. Every other even number is a multiple of 2 and therefore not prime.

Conditional statements

Consider the following statement: if n is a multiple of 10, then n is an even number. This is an example of a conditional statement. A conditional statement (also known as an 'if–then' statement, or an 'implication') is one that asserts that *if* some condition holds, *then* it must be the case that some property is true. Conditional statements are so common in mathematics that there is a variety of ways to express them. The given statement could be represented in any of the following ways:

- If n is a multiple of 10, then n is an even number.
- n is an even number if n is a multiple of 10.
- n being a multiple of 10 is a *sufficient* condition to conclude that n is even.
- n being even is *necessary* if n is a multiple of 10.
- n is a multiple of 10 *implies that* n is an even number.

Finally, the implication symbol, $\Rightarrow$, is used to mean 'implies that'. Thus, the statement could be also written as follows: n is a multiple of 10 $\Rightarrow$ n is an even number.

Each of the following means the same as $P \Rightarrow Q$:

- If P, then Q.
- Q if P.
- P is a *sufficient* condition to conclude that Q.
- Q is *necessary* if P.
- P *implies that* Q.

9 Using the implication symbol

Rewrite the following conditional statements using the implication symbol, $\Rightarrow$.

(a) If n ends in a zero, where $n \in \mathbb{Z}$, then n is even.

THINKING	WORKING
If P, then Q can be written as $P \Rightarrow Q$.	If n ends in a zero $\Rightarrow$ n is even.

(b) $n > 3$ is a sufficient condition to conclude that n is positive.

'P is a sufficient condition to conclude Q' means the same as 'If P, then Q'.	If $n > 3$, then n is positive. $n > 3 \Rightarrow n$ is positive.

(c) $n > 3$ is necessary if n is greater than 4.

'P is a necessary condition if Q' means the same as 'If Q, then P'.	If $n > 4$ then $n > 3$. $n > 4 \Rightarrow n > 3$

The converse, contrapositive and negation of a conditional statement

The converse

The *converse* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol. For example, consider the conditional statement previously introduced:

- original: n is a multiple of 10 $\Rightarrow$ n is an even number
- converse: n is an even number $\Rightarrow$ n is a multiple of 10

Notice that the converse is not saying the same thing as the original. The original statement is claiming that if a number is a multiple of 10, then it must be even (which is true). But the converse is claiming that if a number is even, then it must be a multiple of 10 (which is definitely not true.)

For a statement of the form $P \Rightarrow Q$ that involves some variable, the converse is the statement $Q \Rightarrow P$.

The contrapositive

The *contrapositive* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol, and also negating both statements. Again, consider the conditional statement previously introduced:

- original: n is a multiple of 10 $\Rightarrow$ n is an even number
- contrapositive: n is not an even number $\Rightarrow$ n is not multiple of 10

As a classic illustrative real-life example, the contrapositive of the statement, 'If an animal is a poodle, then it is a dog' is 'If an animal is not a dog, then it is not a poodle'. Notice, again, how the original and the contrapositive statements are essentially saying the same thing.

For $P \Rightarrow Q$, the contrapositive is in the form *not* $Q \Rightarrow$ *not* P.

The negation of a conditional statement

Now consider the *negation* of the statement 'n is a multiple of 10 $\Rightarrow$ n is an even number'. This statement is essentially saying that for every integer that is a multiple of 10, this integer must also be even. If this were not the case, it would mean that there must exist some integer that is a multiple of 10 but is not even. Using the real-life example from earlier, the negation of 'If an animal is a poodle, then it is a dog' would be 'There exists some animal which is a poodle, but not a dog'.

The negation of a conditional statement of the form $P \Rightarrow Q$ that involves some variable is 'There exists some value of the variable for which P is true, but Q is false'.

WARNING

Be aware that the negation of a conditional statement is different from both the converse and the contrapositive.

10 Writing the converse, contrapositive and negatiown of a conditional statement

Consider the following conditional statement: 'If n is a perfect square, then n is divisible by 3'.

(a) Write the statement in the form $P \Rightarrow Q$ and determine whether the statement is true or false.

THINKING	WORKING
1 Write the conditional statement in the form $P \Rightarrow Q$.	$\forall n$, n is a perfect square $\Rightarrow$ n is divisible by 3.
2 A 'for all' statement can be proven false by providing a counter-example.	The perfect squares: $\{1, 4, 9, 16, 25\ldots\}$ There exists a value of n for which n is a perfect square but is not divisible by 3. A counter-example: $n = 16$ is a perfect square but is not divisible by 3.
3 Interpret the answer.	The conditional statement, 'If n is a perfect square, then n is divisible by 3', is false.

(b) Write the converse to the conditional statement and determine whether the converse statement is true or false.

1 For $P \Rightarrow Q$, the converse is in the form $Q \Rightarrow P$.	$\forall n$, n is divisible by 3 $\Rightarrow$ n is a perfect square.
2 A 'for all' statement can be proven false by providing a counter-example.	The multiples of 3: $\{3, 6, 9, 12, 15\ldots\}$ There exists a value of n for which n is divisible by 3 but is not a perfect square. A counter-example: 12 is divisible by 3 but is not a perfect square.
3 Interpret the answer.	The converse statement, 'If n is divisible by 3, then n is a perfect square', is false.

(c) Write the contrapositive to the conditional statement and determine whether the contrapositive statement is true or false.

1 For $P \Rightarrow Q$, the contrapositive is in the form *not* $Q \Rightarrow$ *not* P.	$\forall n$, n is not divisible by 3 $\Rightarrow$ n is not a perfect square.

2 A 'for all' statement can be proven false by providing a counter-example. Alternatively, recall that the contrapositive of a conditional statement is true whenever the original statement is true.	Not divisible by 3: {1, 2, 4, 5, 7, 8…} There exists a value of n for which n is not divisible by 3, but is a perfect square. A counter example: 4 is not divisible by 3 but it is a perfect square.
3 Interpret the answer.	The contrapositive statement, 'If n is not divisible by 3, then n is not a perfect square', is false.

(d) Write the negation to the conditional statement and determine whether the negation statement is true or false.

1 For $P \Rightarrow Q$, the negation is in the form 'There exists some value of the variable for which P is true, but Q is false'.	$\exists n$, n is a perfect square but n is not divisible by 3.
2 A 'there exists' statement can be proven true by providing one example.	Perfect squares: {1, 4, 9, 16, 25…} There exists a value of n for which n is a perfect square, but not divisible by 3. An example: 4 is a perfect square but is not divisible by 3.
3 Interpret the answer.	The negation statement, 'There exists some value of n, where n is a perfect square not divisible by 3', is true.

Logical equivalent statements

Recall that the conditional statement: 'n is a multiple of 10 $\Rightarrow$ n is an even number' is true, however, its converse 'n is an even number $\Rightarrow$ n is a multiple of 10' is not. Sometimes, however, a conditional statement and its converse are both true. As an example, notice that if $x = 5$ then $2x = 10$ and, conversely, if $2x = 10$, then $x = 5$. This means that for the two statements, $x = 5$ and $2x = 10$, whenever one of these is true, the other must also be true. Such statements are said to be *logically equivalent*.

Two statements are logically equivalent if whenever one is true, the other must also be true.

There is a variety of ways to represent the fact that $x = 5$ and $2x = 10$ are logically equivalent:

- $x = 5$ is *necessary and sufficient* for $2x = 10$
- $x = 5$ *if and only if* $2x = 10$
- $x = 5 \Rightarrow 2x = 10$ and $2x = 10 \Rightarrow x = 5$

Finally, the symbol $\Leftrightarrow$ is often used to denote logical equivalence. Thus, $x = 5 \Leftrightarrow 2x = 10$ (or equivalently, $2x = 10 \Leftrightarrow x = 5$).

Each of the following can be used to express the fact that P and Q are logically equivalent:

- P is *necessary and sufficient* for Q.
- P *if and only if* Q.
- $P \Rightarrow Q$ and $Q \Rightarrow P$
- $P \Leftrightarrow Q$

11 Using the logical equivalence symbol

Rewrite the following statement using the logical equivalence symbol, $\Leftrightarrow$: 'For n to be divisible by 5, it is both necessary and sufficient that n end in either 0 or 5'.

THINKING	WORKING
'P is a necessary and sufficient condition for Q' means that P and Q are logically equivalent.	n is divisible by 5 $\Leftrightarrow$ n ends in 0 or 5.

EXERCISE 3.2 The language and logic of proof

Worked Example

1 Determine the negation of each of the following statements. 5

(a) P and Q are both even.
(b) $x > 5$ or $x < -5$
(c) x is divisible by either 7 or 8.
(d) $x = 0$ or $y = 0$

2 Translate the following statements into everyday language. Also determine whether or not the statement is true, justifying your answer where appropriate. 6

(a) $\forall$ integers n, the number $2n + 3$ is odd.
(b) $\exists$ a real number x such that $\frac{1}{x} = x$.
(c) $\forall$ real numbers x, $x^2 > 0$.
(d) $\exists x \in \mathbb{R}$ such that $x^2 = -1$.
(e) $\forall n \in \mathbb{Z}$, the number $n(n + 1)$ is divisible by 3.
(f) $\forall$ real numbers x and y, $x - y > 0$.
(g) $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.
(h) $\exists$ a real number x, such that $\forall$ real numbers y, $xy = y$.

3 Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, determine whether or not the statement is true, justifying your answer where appropriate. 7

(a) The square of any integer is greater than the integer.
(b) There is a real number that, when multiplied by 5 gives an answer of 0.
(c) The sum of any two consecutive integers is odd.
(d) There is a real number equal to its square.
(e) The sum of the squares of any two real numbers is less than the product of the numbers.
(f) There is a special real number with the property that whenever another real number is divided by it, this other real number is obtained.
(g) Every integer is divisible by at least one integer.

4 Determine the negation of each of the following statements. Also determine whether the original statement or the negation is true, justifying your answer where appropriate. 8

(a) $\forall$ real numbers x, $x^2 > 0$.
(b) $\exists$ a real number x such that $x^2 = x$.
(c) $\forall$ positive integers n, $10n > n$.
(d) $\forall$ real numbers x, x is either positive or negative.
(e) $\exists$ an integer n such that $n \neq 0$ and $n^2 < 1$.
(f) $\forall$ integers n, either $(-1)^n = 1$ or $(-1)^n = -1$.

9

5 Rewrite the following statements using the implication symbol, $\Rightarrow$.

(a) If $x > 3$, then $x^2 > 9$.

(b) If n is divisible by 9, then n is divisible by 3.

(c) $n > 5$ implies that $n > 4$.

(d) $7P$ is positive if $P > 3$.

(e) Q is even is necessary if $2Q$ is a perfect square.

(f) m is a multiple of 6 is a sufficient condition to conclude that m is divisible by 3.

(g) It is necessary that $x^2 > 2$ if $x < -2$.

(h) n is even and greater than 2 is a sufficient condition to conclude that n is not prime.

10

6 Write the converse, the contrapositive and the negation of each of the following conditional statements. Determine whether each of the original statements, converse, contrapositive and negation is true or false, justifying your answer where appropriate.

(a) If n is divisible by 20, then n is divisible by 5.

(b) If n is divisible by 3, then n^2 is divisible by 3.

(c) If $x > 7$, then $10x > 70$.

(d) If $xy = 0$, then either $x = 0$ or $y = 0$.

(e) If n is divisible by 5, then the final digit of n is 5.

(f) If $x = 4$ and $y = 4$, then $xy = 16$.

(g) If n is divisible by 24, then n is even and n is divisible by 3.

11

7 Rewrite the following statements using the logical equivalence symbol, $\Leftrightarrow$.

(a) n is even if and only if n^2 is even.

(b) $x + y = 0$ if and only if $x = -y$.

(c) n being even and divisible by 3 is necessary and sufficient for n to be divisible by 6.

8 For each of the following statements, provide three examples that are consistent with the statement, and then a single counter-example to prove that the statement is actually false.

(a) If a positive integer is divisible by 7, then it is not divisible by 3.

(b) $\forall x \in \mathbb{R}, x^2 \geq x$

(c) If P is a prime number, then $2^P - 1$ is prime.

(d) If a positive integer is divisible by both 10 and 6, then it is also divisible by 60.

(e) Suppose that x and y are real numbers. If $x > 3$, then $x^2 - 2y > 5$.

(f) Suppose that x and y are positive real numbers. Then $xy > x + y$.

9 (a) Which of the following statements is true?

A $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 6$.

B $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, xy = 6$.

C $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, x + y = 6$.

D $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 6$.

(b) Why are the alternatives incorrect?

10 What is the negation of the statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 6$?

A $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y \neq 6$.

B $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, x + y \neq 6$.

C $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y \neq 6$.

D $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y \neq 6$.

11 Write the negation of the following statement, where x represents a real number:
$x > 0$ and $x < 10 \Rightarrow x \geq 0$ and $x \leq 10$. Also, determine whether the original statement or the negation is true.

12 Consider the following conjecture: Start with any positive integer. If the integer is even, halve it. If the integer is odd, triple it and add one. Repeat this process. Eventually, the integer 1 will be obtained. This is known as the '$3x + 1$' conjecture. It is yet to be proved, but it has been shown to be true for all integers up to roughly 10^{14}. Verify this conjecture for the following positive integers:

(a) 6 **(b)** 13 **(c)** 7

Methods of proof

3.3

Aside from examples and counter-examples (which can be used to prove the truth of a 'there exists' statement or the falsehood of a 'for all' statement), a mathematical proof typically consists of a sequence of statements with each statement following directly from either definitions, previous steps or other established results. In this section, several common strategies for constructing proofs are illustrated.

As many of the statements proved are concerned with even and odd numbers, and divisibility more generally, it is necessary to be familiar with the following definitions:

An integer n is said to be even if n can be expressed in the form $n = 2k$, for $k \in \mathbb{Z}$.

For $k = 5$, $n = 2 \times 5 = 10$

Therefore, the number 10 is said to be even, because it is the product of 2 and an integer, 5.

Similarly, an integer n is said to be odd if n can be expressed in the form $n = 2k + 1$, for $k \in \mathbb{Z}$.

For $k = 5$, $n = 2 \times 5 + 1 = 10 + 1 = 11$

Therefore, the number 11 is said to odd, because it is one more than the product of 2 and an integer, 5.

Finally, an integer is said to be divisible by an integer m, if n can be expressed in the form $n = mk$ for $k \in \mathbb{Z}$.

For $m = 3$, $n = 3 \times k = 3k$

Therefore, consider the set of numbers $k = \{1, 2, 3, 4\ldots\}$ then $n = \{3, 6, 9, 12\ldots\}$, where n represents the set of integers divisible by 3.

Direct proof

The most straightforward way to prove a statement is to use a direct proof. A direct proof typically starts by introducing any relevant variables, and assuming any given condition holds, and then establishes the desired result via a logical sequence of valid statements. Note that if the statement to be proved has the form 'If P, then Q', then you assume that P is true, and then proceed to prove that Q must be true.

12 Direct proof

Use a direct proof to prove that if a number is odd, then its square is also odd.

THINKING	WORKING
1 Define a numerical variable, and assume the information given—in this case, that the number is odd.	Let p be an odd integer.
2 Use the fact that an odd number can always be expressed in the form $2k + 1$.	Then $p = 2k + 1$ for some integer k.
3 Aim to demonstrate that the statement is true—in this case, that p^2 is odd. Thus, obtain an expression for p^2 and demonstrate that it can be expressed in the form 2(integer) + 1.	$p^2 = (2k + 1)^2$ $= 4k^2 + 4k + 1$ $= 2(2k^2 + 2k) + 1$ As $2k^2 + 2k$ must be an integer, it follows that p^2 is odd.

Proof by contraposition

Recall that the contrapositive of the statement $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. As the contrapositive is logically equivalent to the original statement, $P \Rightarrow Q$, the original statement can be proved indirectly by proving $\neg Q \Rightarrow \neg P$; that is, by assuming that Q is false, and then proceeding to prove that P must be false.

13 Proof by contraposition

Use a contrapositive proof to prove that if $5n + 3$ is odd, then n is even.

THINKING	WORKING
1 Determine the contrapositive statement.	The contrapositive statement is: if n is not even, then $5n + 3$ is not odd. This means, if n is odd, then $5n + 3$ is even.
2 Introduce appropriate variables and proceed to prove the contrapositive statement.	Let n be an odd integer. Then $n = 2k + 1$ for some integer k. $5n + 3 = 5(2k + 1) + 3$ $= 10k + 5 + 3$ $= 10k + 8$ $= 2(5k + 4)$ As $5k + 4$ must be an integer, it follows that $5n + 3$ is even if n is odd.
3 Write your conclusion.	Because n is odd, then $5n + 3$ is even. Therefore, by the contrapositive statement, if $5n + 3$ is odd, then n is even.

Proof by contradiction

Another form of indirect proof, but one that is not restricted to proving conditional statements, is 'proof by contradiction'. The basic idea of such a proof is to assume that the statement needing to be proved is *false*, and then demonstrate that this assumption leads to an absurd and impossible result. This then must mean that the initial assumption—that the statement was false—cannot be true, meaning that the original statement must be true!

Two proofs by contradiction are illustrated in the following example. The first is a famous proof that $\sqrt{2}$ is irrational. The second is a proof of the statement from the previous worked example, only using contradiction instead of contraposition.

14 Proof by contradiction

Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{2}$ is irrational.

THINKING	WORKING
1 Assume the negation of the statement to be proved.	Suppose, for a contradiction, that $\sqrt{2}$ is rational.
2 Recall that a rational number can be expressed in the form $\frac{p}{q}$ for integers p and q, with $q \neq 0$, and with p and q having no common factor other than 1.	Then $\sqrt{2} = \frac{p}{q}$ for integers p and q, with $q \neq 0$, and with p and q having no common factor other than 1.
3 Using a series of valid steps, arrive at a contradiction.	$2 = \frac{p^2}{q^2}$ $p^2 = 2q^2$, hence p^2 is divisible by 2. If p^2 is divisible by 2, then p is divisible by 2. Therefore $p = 2m$ for some integer m. $4m^2 = 2q^2$ $2m^2 = q^2$ Hence q^2 is also divisible by 2, so q is divisible by 2. Since p and q are both divisible by 2 then there is a contradiction since p and q have no common factors other than 1.
4 Interpret your result.	It follows that $\sqrt{2}$ is an irrational number.

(b) If $5n + 3$ is odd, then n is even.

1 Assume the negation of the statement to be proved.	Suppose, for a contradiction, that $5n + 3$ is odd, but n is also *odd*.
2 Using a series of valid steps, arrive at a contradiction.	Then $n = 2k + 1$ for some integer k. $5n + 3 = 5(2k + 1) + 3$ $= 10k + 5 + 3$ $= 10k + 8$ $= 2(5k + 4)$ As $5k + 4$ must be an integer, it follows that $5n + 3$ is even if n is odd. But this is a contradiction, as the number $5n + 3$ is odd, and a number cannot be both even and odd.
3 Interpret your result.	Hence, the number n cannot be odd, and so must be even.

Note that the proof in part **(a)** of the previous example used the fact that if the square of an integer is divisible by 2, then the original integer must also be divisible by 2. This fact is true not just for the number 2 but for any integers with no perfect square factors other than 1.

For example, if k^2 is even then k is also even:

$$\begin{aligned} k^2 &= 36 \\ k &= \pm\sqrt{36} \\ &= \pm 6 \end{aligned}$$

where k is even because it can be expressed in the form $k = 2m$, $-6 = 2 \times -3$ or $6 = 2 \times 3$.

This stems from the fact that, for an even number, $k = 2m$, where $m \in \mathbb{Z}$ then:

$$\begin{aligned} k^2 &= (2m)^2 \\ &= 4m^2 \\ &= 2\left(2m^2\right) \end{aligned}$$

You may use this fact when modifying the previous proof to prove the irrationality of other surds in the subsequent exercise.

You may have noticed that the contradiction proof in part **(b)** looks very similar to the contraposition proof from Worked Example 13. The actual logic used is almost identical—the difference is that there is no actual contradiction obtained in the contrapositive proof since it was never assumed that $5n + 3$ was odd. Note also that a contradiction proof does not aim for a *particular* contradiction—any contradiction is sufficient. In a situation such as this, when a contradiction proof is very similar to a contrapositive proof, the contrapositive proof is considered more efficient and elegant.

Proving logical equivalences

The simplest method to prove a statement of the form $P \Leftrightarrow Q$ is to separately prove both $P \Rightarrow Q$ and $Q \Rightarrow P$, as demonstrated in the following example.

15 Proving logical equivalences

Let n be a positive integer. Prove that $n + 9$ is odd if and only if $n - 8$ is even.

THINKING	WORKING
1 First, suppose that $n + 9$ is odd. Then $n + 9 = 2k + 1$ for some integer k.	$n - 8 = (n + 9) - 17$ $= (2k + 1) - 17$ $= 2k - 16$ $= 2(k - 8)$ Thus, $n - 8$ is even if $n + 9$ is odd.
2 Conversely, suppose that $n - 8$ is even. Then $n - 8 = 2k$ for some integer k.	$n + 9 = (n - 8) + 17$ $= 2k + 17$ $= 2k + 16 + 1$ $= 2(k + 8) + 1$ Thus, $n + 9$ is odd if $n - 8$ is even.

The final example presented is a more complex proof of a well-known divisibility result.

16 Proving divisibility results

Prove that a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

THINKING	WORKING
1 Define the variables for the digits and write an expression for the number.	Let a, b, c be the digits, in order, of a three-digit number, N. The actual number is therefore $N = 100a + 10b + c$.
2 Since the statement involves 'if and only if', it is necessary to split it into two conditional proofs. First suppose that the number is divisible by 3 and demonstrate that this implies that the sum of the digits is divisible by 3.	First suppose that N is divisible by 3. Thus, $100a + 10b + c = 3k$ for some integer k. Then $99a + 9b + a + b + c = 3k$ $a + b + c = 3k - 99a - 9b$ $a + b + c = 3(k - 33a - 3b)$ Thus, the sum of the digits is divisible by 3.
3 Finally, demonstrate the converse of the previous step to complete the proof.	Conversely, suppose the sum of the digits is divisible by 3. Then $a + b + c = 3k$ for some integer k. Then $N = 100a + 10b + c$ $= 99a + 9b + a + b + c$ $= 99a + 9b + 3k$ $= 3(33a + 3b + k)$ Thus, N is divisible by 3.

EXERCISE 3.3 Methods of proof

Worked Example

12

1 Use a direct proof to prove each of the following statements.

(a) The sum of any two odd integers is even.

(b) The sum of an odd integer and an even integer is always odd.

(c) The product of two odd integers is odd.

(d) The sum of two consecutive odd numbers is divisible by 4.

(e) The sum of the squares of five consecutive integers is divisible by 5.

(f) The product of two rational numbers is rational.

(g) The sum of two rational numbers is rational.

(h) If n is odd, then n^2 is odd.

(i) If n is divisible by 7, then n^2 is divisible by 7.

(j) If $m + n$ and $n + p$ are even, where m, n, p are integers, then $m + p$ is even.

13

2 Use a contrapositive proof to prove each of the following statements.

(a) Let n be an integer. If $3n + 2$ is even, then n is even.

(b) If a and b are integers and ab is even, then at least one of a and b is even.

(c) Let n be an integer. If $n^3 + 5$ is odd, then n is even.

(d) If x is irrational then $\sqrt{x}$ is irrational.

(e) If x is irrational, then $\frac{1}{x}$ is irrational.

14

3 Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{3}$ is irrational.

(b) $\sqrt{5}$ is irrational.

(c) The sum of a rational and an irrational number is irrational.

(d) The product of a rational and an irrational number is irrational.

(e) There are no integers a and b such that $18a + 6b = 1$.

15

4 Prove each of the following logical equivalences.

(a) Let n be a positive integer. $n + 9$ is even if and only if $n + 6$ is odd.

(b) Let n be a positive integer. $n - 3$ is odd if and only if $n + 2$ is even.

(c) Let n be a positive integer. n is even if and only if $13n + 4$ is even.

(d) Let n be a positive integer. n is odd if and only if $7n + 6$ is odd.

(e) Let n be a positive integer. n is even if and only if n^2 is even.

5 Consider the following statement: 'If two integers have an even product, then at least one of the two integers must be even.'

(a) To prove this statement by contraposition, it would be necessary to:

A suppose that at least one of the two integers is even, and then show that the product must be even.

B suppose that at least one of the integers is odd, and then show that the product must be odd.

C suppose that both integers are odd, and then show that the product must be odd.

D suppose that the two integers have an even product and that both integers are odd, and then show that a contradiction arises.

(b) Explain the misconceptions or errors that lead to selecting the incorrect alternatives.

6 Prove that the number $1 - 5\sqrt{2}$ is irrational.

7 Let a, b, c be positive real numbers such that $ab = c$. Prove that $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$.

Worked Example 16

8 Prove that if $a^2 - 2a + 7$ is even, then a is odd.

9 Prove that a four-digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

10 (a) Prove that for all real numbers a and b, $a^2 + b^2 \geq 2ab$.

(b) Using the result from part (a) or otherwise, prove that for all positive real numbers a and b, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

(c) Using the result from part (a) or otherwise, prove that for all positive real numbers a and b, $\frac{a+b}{2} \geq \sqrt{ab}$.

11 Prove that every odd integer can be expressed as the difference between two perfect squares.

12 Prove that if a and $b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

13 Let k be a positive integer. Prove that if $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+3} + 3^{3k+3}$ is also divisible by 5.

14 It is known that $\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square. Use this result to help prove that $\sqrt{6} + \sqrt{10}$ is irrational.

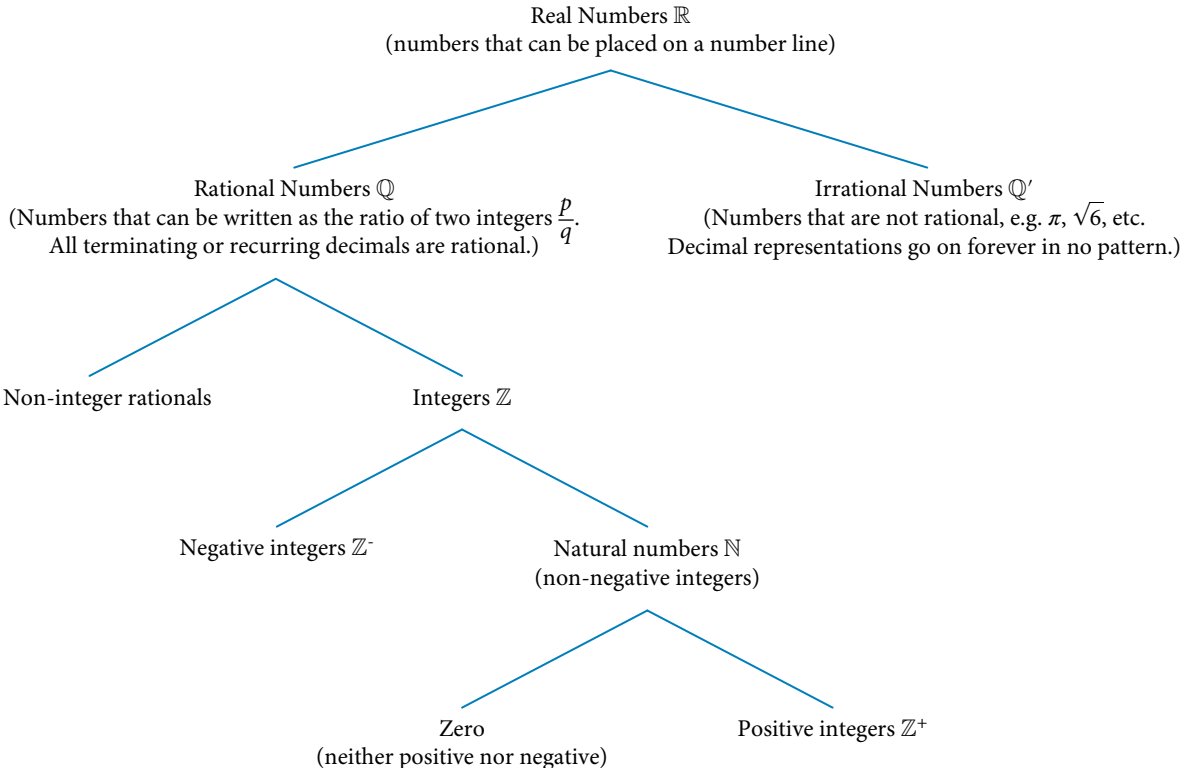
15 Use a proof by contradiction to demonstrate that there is no rational solution to the equation $x^3 + x + 1 = 0$.
Hint: Start by supposing, for a contradiction, that $r = \frac{p}{q}$ is a rational solution to the equation, where p and q are integers with no common factor other than 1 and with $q \neq 0$. Then consider what would happen if both p and q were odd, or if one of them was even and the other odd.

3

Summary

Real numbers

The diagram below summarises the different types of real numbers.



A simplified fraction can be expressed as a terminating decimal if the denominator has no prime factors other than 2 or 5; otherwise, it can only be expressed as a recurring decimal.

When converting a recurring decimal to a simplified fraction, multiply by the power of 10 equal to the number of digits that are recurring.

Negating 'and' and 'or' statements

The negation of '*P* and *Q*' is '*not P* or *not Q*'.

The negation of '*P* or *Q*' is '*not P* and *not Q*'.

Quantifiers

The symbol $\forall$ (known as the *universal quantifier*) is used to mean 'for all'.

The symbol $\exists$ (known as the *existential quantifier*) is used to mean 'there exists'.

Providing a single example is always sufficient to prove that a 'there exists' statement is true.

Providing a single counter-example is always sufficient to prove that a 'for all' statement is false.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

Conditional statement

Each of the following means the same as $P \Rightarrow Q$:

- If P, then Q.
- Q if P.
- P is a *sufficient* condition to conclude that Q.
- Q is *necessary* if P.
- P *implies that* Q.

For a statement of the form $P \Rightarrow Q$ that involves some variable:

- The converse is the statement $Q \Rightarrow P$.
- The contrapositive is the statement *not* $Q \Rightarrow$ *not* P.
- The negation is the statement 'there exists some value of the variable for which P is true, but Q is false'.

Logically equivalent statements

Two statements are logically equivalent if whenever one is true, the other must also be true.

Each of the following can be used to express the fact that P and Q are logically equivalent:

- P is *necessary and sufficient* for Q.
- P *if and only if* Q.
- $P \Rightarrow Q$ and $Q \Rightarrow P$.
- $P \Leftrightarrow Q$.

Methods of proof

A *direct proof* starts by assuming any given conditions hold, and then proceeds directly to the desired result.

A *contrapositive proof* is a direct proof of the contrapositive of a conditional statement. As this involves proving a statement that is not the same, but is logically equivalent to the original, it is known as an indirect proof.

A *proof by contradiction* involves assuming that the statement needing to be proved is *false* and then demonstrating that this assumption leads to an absurd and impossible result. This then must mean that the assumption was false, meaning that the initial statement must be true.

The simplest method to prove a statement of the form $P \Leftrightarrow Q$ is to separately prove both $P \Rightarrow Q$ and $Q \Rightarrow P$.

Chapter review 3

Exercise 3.1 **1** For each of the following numbers, determine which sets from $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the number is an element of.

(a) $\frac{6}{11}$ **(b)** $-5.\overline{83}$ **(c)** -3 **(d)** 0 **(e)** $\sqrt{10}$

Exercise 3.1 **2** Express each of the following rational numbers as a terminating or recurring decimal.

(a) $\frac{17}{100}$ **(b)** $\frac{11}{40}$ **(c)** $\frac{5}{12}$

Exercise 3.1 **3** Write each of the following numbers in the form $\frac{p}{q}$ where p and $q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.

(a) 0.38 **(b)** $0.\dot{5}$ **(c)** $0.2\overline{31}$

Exercise 3.1 **4** Which one of the following numbers can be written as a terminating decimal?

A $\frac{47}{150}$ **B** $\frac{2}{900}$ **C** $\frac{47}{320}$ **D** $\frac{55}{140}$

Exercise 3.2 **5** Determine the negation of each of the following statements.

(a) n is divisible by either 5 or 6. **(b)** $0 \leq x \leq 10$ **(c)** p is prime and $p \neq 2$.

Exercise 3.2 **6** Translate the following statements into everyday language. Also determine whether or not the statement is true, justifying your answer where appropriate.

(a) $\forall$ integers n, the number $10n$ is divisible by 4.

(b) $\exists$ a real number x such that $x^2 = 2$.

(c) $\forall$ real numbers x, $\exists$ a real number y such that $xy = 1$.

Exercise 3.2 **7** Determine the negation of each of the following statements. Also determine whether the original statement or the negation is true, justifying your answer where appropriate.

(a) $\exists$ a real number x such that $x^2 < 0$.

(b) $\forall$ integers n, $5n + 3$ is odd.

(c) $\forall$ integers n, $\exists$ an integer m such that $2m = n$.

Exercise 3.2 **8** Rewrite the following statements using the implication symbol, $\Rightarrow$.

(a) $3x$ is negative if $x < -10$.

(b) n is a multiple of 5 is necessary if n ends in a zero.

(c) n is a multiple of 4 is a sufficient condition to conclude that n is even.

(d) $x > 3$ is a sufficient condition to conclude that $x^2 > 9$.

9 Write the converse, the contrapositive and the negation of each of the following conditional statements. Determine whether each of the original statements, converse, contrapositive and negation is true or false, justifying your answer where appropriate. Exercise 3.2

(a) If n is divisible by 4, then n is divisible by 12.

(b) If $xy = 1$, then either $x = 1$ or $y = 1$.

(c) If $x > 0$ and $x < 3$, then $x^2 < 9$.

10 Which of the following statements is true? Exercise 3.2

A $x = 3 \Leftrightarrow x^2 = 9$ **B** $x < 3 \Leftrightarrow x < 4$ **C** $x = 3 \Leftrightarrow x^3 = 27$ **D** $x > 1 \Leftrightarrow 2x > 1$

11 Use a direct proof to prove that the sum of the squares of any two odd integers is even. Exercise 3.3

12 Use a contrapositive proof to prove that if $n^3 - n$ is not divisible by 4, then n must be even. Exercise 3.3

13 Use a proof by contradiction to prove that $\sqrt{7}$ is irrational. Exercise 3.3

14 Use a proof by contradiction to prove that for all integers a and b, $12a - 9b \neq 20$. Exercise 3.3

15 Prove that n is odd if and only if n^3 is odd. Exercise 3.3

16 Demonstrate that $0.a\overline{3b}$, where a and $b \in \mathbb{Z}$, can be expressed in the form $\frac{p}{q}$ where p and $q \in \mathbb{Z}$, $q \neq 0$. Exercise 3.1

17 Prove that if a, b, c and d are consecutive integers with $a < b < c < d$, then $-a^2 + b^2 - c^2 + d = a + b + c + d$. Exercise 3.3

18 Prove that $\sqrt[3]{2}$ is irrational. Exercise 3.3

19 Prove that if a and $b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$. Exercise 3.3

20 Prove that if $6^k + 4$ is divisible by 5, then $6^{k+1} + 4$ is divisible by 10. Exercise 3.3

21 Prove that for all positive real numbers x and y, $\frac{1}{7} + \frac{x}{y} \neq \frac{1+x}{7+y}$. Exercise 3.3

22 Prove that a four-digit number is divisible by 7 if and only if the number obtained by removing the final digit and subtracting it twice from the original number is divisible by 7. Exercise 3.3

23 Prove that if a, b, c are odd integers, then the equation $ax^2 + bx + c$ cannot possibly have a rational solution. Exercise 3.3

24 Solve the following: $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$. Hint: Consider modifying the method used to convert a recurring decimal to a fraction. Exercise 3.1

25 Prove that if $P = \dfrac{\left(\frac{1}{3}\right)^n + 1}{\left(\frac{1}{3}\right)^n + 3}$, then $\dfrac{1}{4 - 3P} = \dfrac{\left(\frac{1}{3}\right)^{n+1} + 1}{\left(\frac{1}{3}\right)^{n+1} + 3}$. Exercise 3.3





4

Geometric properties and proofs

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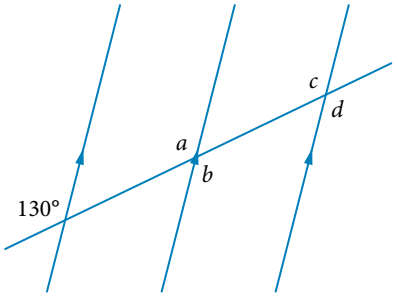
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Recall

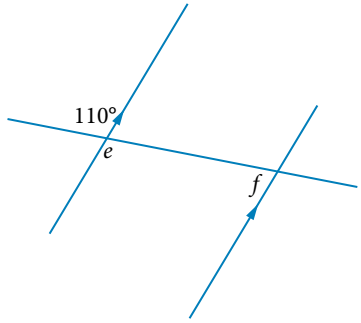
Calculate angles associated with parallel lines

1 Calculate the values of the pronumerals.

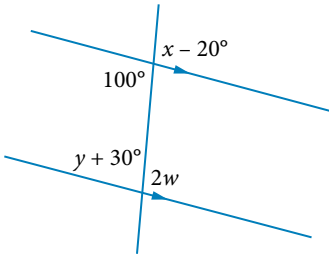
(a)



(b)



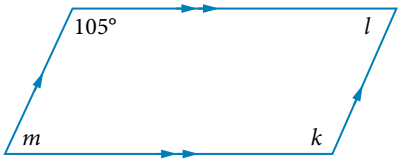
(c)



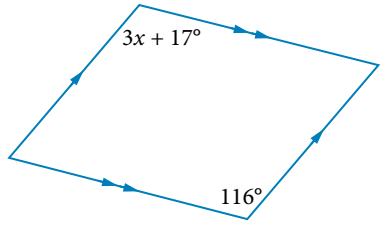
Calculate angles in parallelograms

2 Calculate the values of the pronumerals.

(a)



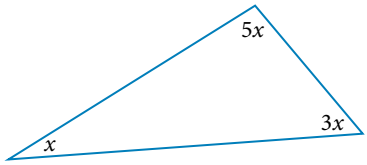
(b)



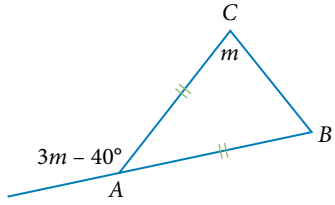
Calculate angles associated with triangles

3 Calculate the value of the pronumeral in each of the following diagrams.

(a)



(b)



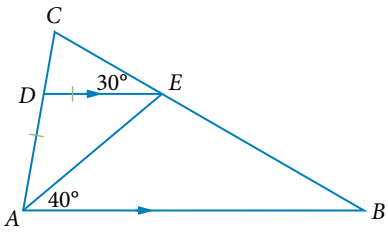
Calculate angles in a polygon

4 (a) Calculate the sum of the angles in a quadrilateral.

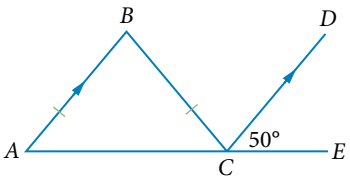
(b) Calculate the magnitude of each interior angle of a regular hexagon.

Deduce angles from given information

5 (a) In triangle ABC, DE is parallel to AB and $DA = DE$. If $\angle DEC = 30°$ and $\angle EAB = 40°$, calculate $\angle ACB$.

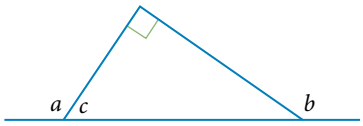


(b) Triangle ABC is isosceles and CD is parallel to AB. If $\angle DCE = 50°$, calculate $\angle BCD$.



Deduce relationships between angles

6 Demonstrate that $a + b = 270°$.



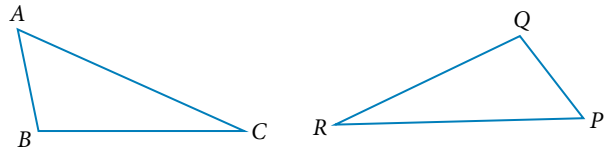
4.1 Review of congruence and similarity

Geometry is a deductive system, which starts with a small number of simple, assumed statements and then derives more complex statements from them.

In this chapter, you will first review some concepts from earlier years, and as the chapter progresses, you will have the opportunity to apply these concepts in different situations.

Congruence

Consider the triangles shown. They both appear to have the same size and shape but a different orientation. Is it correct to say that these triangles are identical? The strict mathematical answer is to say that they are not identical since they occupy different positions in space and have different orientations. The correct mathematical term is to say they are congruent.



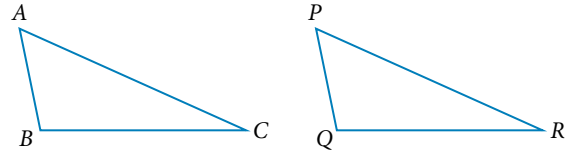
Congruent shapes have the same shape and size but may have a different orientation.

Two triangles are said to be congruent if and only if one triangle can be transformed into the other by a combination of translations, rotations and reflections.

The expression 'if and only if' is sometimes abbreviated to 'iff'. It means that the statement in which 'iff' is found applies both ways. As a simple example, the statement 'I will play golf iff it is not raining' means both of the following:

- I will play golf if it is not raining.
- If it is not raining I will play golf.

In this section, you will deal exclusively with triangles. If two triangles are congruent, the magnitude of the sides and angles of one will correspond exactly to the magnitude of the sides and angles of the other. When making statements based on congruence, it is important that the vertices of both triangles are written in the corresponding order. The symbol used in this text to express congruence is $\cong$ although other symbols such as $\equiv$ are sometimes seen.



For the triangles illustrated, the statement of congruence would be written as $\Delta ABC \cong \Delta PQR$.

If the two triangles, ΔABC and ΔPQR, are congruent, then corresponding sides are equal in length and corresponding angles are equal in magnitude. For these triangles, the corresponding sides are:

$|AB| = |PQ|$
$|BC| = |QR|$
$|AC| = |PR|$

Note that the length of a line segment from point A to point B is written with the modulus sign as $|AB|$.

The corresponding angles are:

$\angle BAC = \angle QPR$
$\angle ABC = \angle PQR$
$\angle BCA = \angle QRP$

In the case of the internal angles in triangles, these are sometimes abbreviated to $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$. This is appropriate when it is clear to which angle the single letter refers. In more complex diagrams this is not always possible.

If $\Delta ABC \cong \Delta PQR$ then corresponding sides are equal in length and corresponding angles are equal in magnitude.

Much of the early work on geometry was conducted by Greek mathematicians such as Thales, Euclid and Pythagoras. From their work, several tests for congruence of triangles were developed. In these tests it is not necessary to check all six of the equalities already outlined. Rather, a smaller subset is tested, and if they prove to be correct, then the rest are implied.

Testing for congruence of triangles

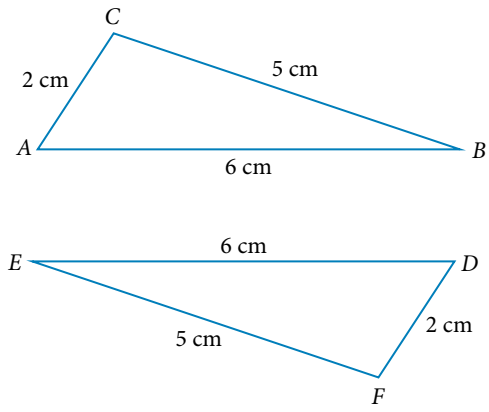
There are four tests for congruence that are commonly used. Each will be looked at in turn in the work that follows.

Side, Side, Side (SSS)

Two triangles are congruent if the three side lengths in one triangle are equal to the lengths of the corresponding three sides in the second triangle.

$\Delta ABC \cong \Delta DEF$ since

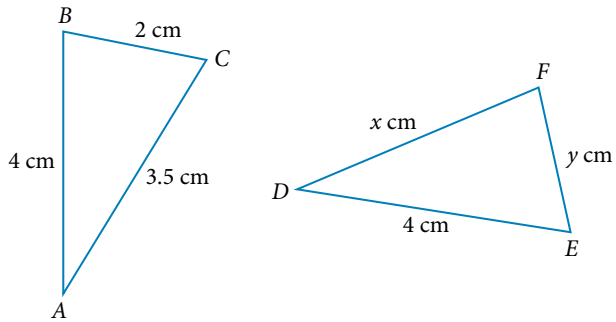
$|AB| = |DE|$
$|BC| = |EF|$
$|AC| = |DF|$



To make things easier to check, put the corresponding vertices of the triangles in the same order when making statements about congruence. In the triangles shown, for instance, the shortest side connects the first and third letters in the sequence in each case and the longest side connects the first two letters in the sequence.

1 Using SSS to determine the length of an unknown side of a triangle

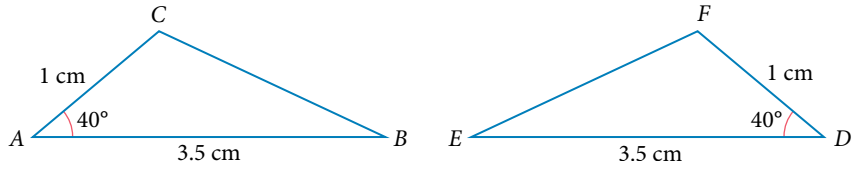
In this diagram, $\Delta ABC \cong \Delta DEF$.
Determine the values of x and y.



THINKING	WORKING
1 Name the triangles being considered, and match the three side lengths in each triangle based on $\Delta ABC \cong \Delta DEF$.	In ΔABC and ΔDEF $\lvert DE \rvert = \lvert AB \rvert = 4\,\text{cm}$ $\lvert EF \rvert = \lvert BC \rvert = 2\,\text{cm}$ $\lvert DF \rvert = \lvert AC \rvert = 3.5\,\text{cm}$
2 Determine the values for the unknowns based on these equalities.	$\lvert DF \rvert = x = 3.5\,\text{cm}$ $\lvert EF \rvert = y = 2\,\text{cm}$

Side, Angle, Side (SAS)

Two triangles are congruent if two pairs of corresponding side lengths are equal and their included angle is equal. It is vital that the angle is the included angle. This is the angle formed where the two named sides intersect. In the following diagram, $\Delta ABC \cong \Delta DEF$ based on SAS.

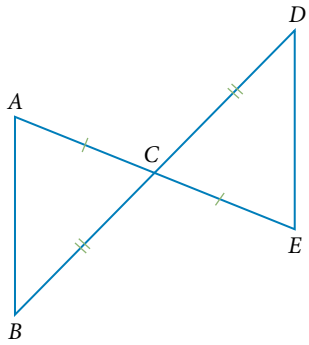


These two triangles are congruent as they both have sides of length 1 cm and 3.5 cm and an included angle (between the two known sides) with magnitude 40°.

2 Checking congruence using SAS

Consider the following diagram.

Determine whether ΔABC and ΔEDC are congruent.



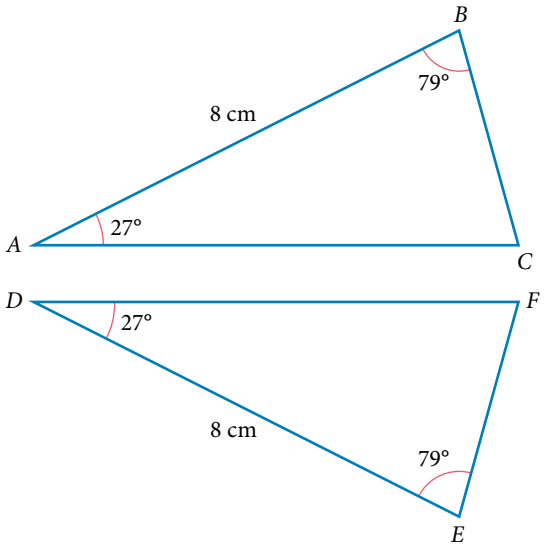
THINKING	WORKING
1 Name the triangles being considered, and determine any equalities you can see involving side lengths in the two triangles.	In ΔABC and ΔEDC $\lvert BC \rvert = \lvert DC \rvert$ $\lvert AC \rvert = \lvert EC \rvert$
2 Determine any equality you can see involving angles in the two triangles. Give a reason for this equality, if necessary.	Vertically opposite angles on straight lines are equal, therefore $\angle ACB = \angle ECD$. This is the included angle for the two pairs of equal sides.
3 Check if this satisfies the requirements for congruence based on SAS.	By SAS: $\Delta ACB \cong \Delta ECD$

Angle, Side, Angle (ASA)

Two triangles are congruent if two pairs of angles are equal and one pair of corresponding sides has the same magnitude. It is not important if the side is the included side, so this could just as easily be abbreviated as AAS or even SAA. This is because, if two pairs of angles are equal then the third pair must also be equal since the sum of the internal angles in every triangle is 180°.

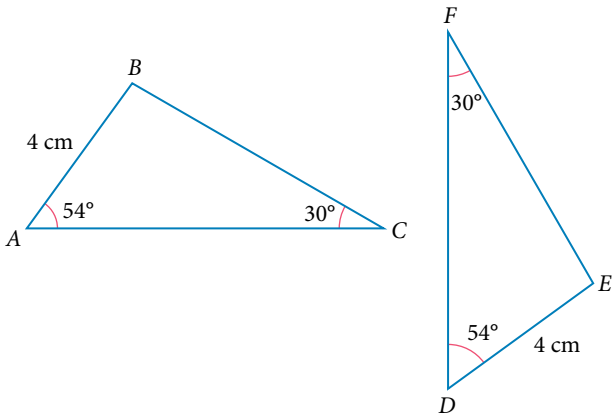
In the diagram to the right, $\Delta ABC \cong \Delta DEF$ based on ASA.
$\angle BAC = \angle EDF$
$\angle ABC = \angle DEF$
$|AB| = |DE|$

It is clear the sides stated are corresponding sides because both sides connect two angles that are the same size. However, the diagram could just as easily have demonstrated that $|BC| = |EF|$ or $|AC| = |DF|$.



3 Checking for congruence using ASA

Consider the following diagram.
Prove $\Delta ABC \cong \Delta DEF$.



THINKING	WORKING
1 Name the triangles being considered, and determine any equalities you can see involving angles in the two triangles.	In ΔABC and ΔDEF: $\angle BAC = \angle EDF$ $\angle ACB = \angle DFE$
2 Determine any corresponding sides in the two triangles. Include a comment as to how you know the sides are corresponding.	$\|AB\| = \|DE\|$ These sides are both the shortest sides in the triangles, so corresponding.
3 Check if this satisfies the requirements for congruence based on ASA.	$\Delta ABC \cong \Delta DEF$ (ASA)

Right angle, Hypotenuse, Side (RHS)

This final test is used exclusively with right-angled triangles. It really does not need to be covered separately as it is a special case of SSS. This is because Pythagoras' theorem could be used to demonstrate that the third side is also equal for these triangles. However, it is a useful shortcut for right-angled triangles so it is included.

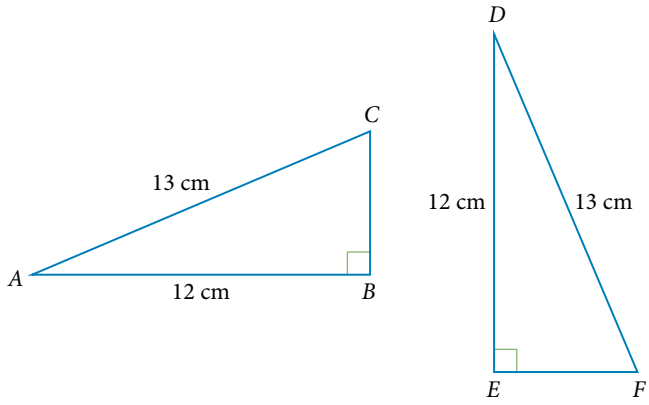
Two right-angled triangles are congruent if the magnitudes of their hypotenuses are equal and another set of corresponding sides are equal in magnitude. The right angle is the third piece of information required.

In the diagram to the right, $\Delta ABC \cong \Delta DEF$ based on RHS.

The hypotenuses are equal: $|AC| = |DF|$.
The pair of sides that are the second longest are equal: $|AB| = |DE|$.

The triangles are both right-angled, $\angle ABC = \angle DEF$.

Thus $\Delta ABC \cong \Delta DEF$ (RHS).



Verify $|BC| = |EF|$ using Pythagoras' theorem:

$13^2 = 12^2 + x^2$

$169 = 144 + x^2$

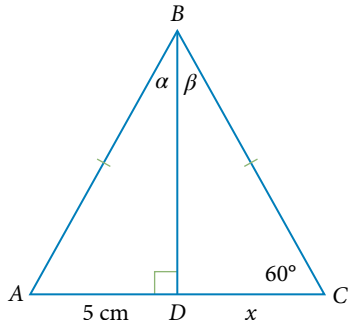
$25 = x^2$

$\therefore x = 5$

$|BC| = |EF| = 5$ cm for each triangle, so SSS could have been used to verify congruence.

4 Using the RHS congruency test to determine unknown sides and angles

Determine the value of the unknown side and angles in the diagram.



THINKING	WORKING
1 Name the triangles being considered and check if the triangles are congruent. Write the known information.	ΔABD is a right-angled triangle. ΔCBD is a right-angled triangle. $\|AB\| = \|CB\|$ (given) $\|BD\| = \|BD\|$ (common) $\therefore \Delta ADB \cong \Delta CDB$ (RHS)

2 Make a statement of equality involving the unknown side length.	$x = \|CD\|$ Since corresponding sides in congruent triangles are equal: $\|CD\| = \|AD\|$ $\|CD\| = 5$ $x = 5$
3 Make a statement of equality involving the unknown angles.	$\alpha + \beta = 60°$ (ΔABC is an equilateral triangle.) $\alpha = \beta$ $(\Delta ADB \cong \Delta CDB)$ Therefore $\alpha = 30°$ $\beta = 30°$
4 Answer the questions.	$x = 5$ cm $\alpha = 30°$ $\beta = 30°$

The four tests for congruence of triangles

Test for congruence		Description
SSS	Side, Side, Side	Three pairs of corresponding sides are equal.
SAS	Side, Angle, Side	Two pairs of corresponding sides are equal in magnitude and the included angles are also equal.
ASA	Angle, Side, Angle	Two pairs of angles are equal and one pair of corresponding sides are equal in magnitude.
RHS	Right Angle, Hypotenuse, Side	In a right-angled triangle, corresponding hypotenuses and another pair of corresponding sides are of equal magnitude.

Similarity

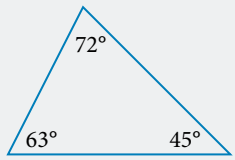
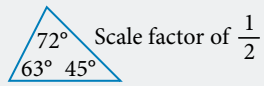
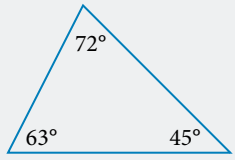
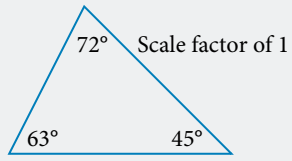
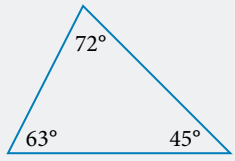
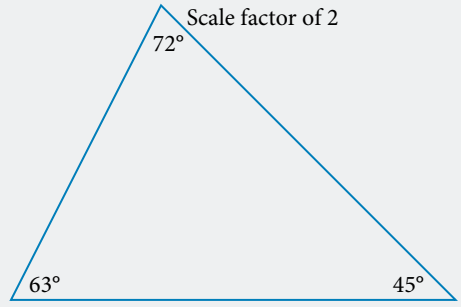
Consider the way in which a photocopier works. It copies an image or text and produces an exact copy, an enlargement or a reduction of the original. An enlargement or reduction involves increasing or decreasing all dimensions by the same scale factor. Typically, the term 'enlargement' is used to describe increasing all of the dimensions by the same scale factor, that is a scale factor greater than 1. A reduction describes the decrease of all dimensions by the same scale factor, that is a scale factor between 0 and 1. However, the angles in the shape are unaltered by the dilation of the dimensions. It can be said that the original and the copy are similar figures.

The previous section was about congruent triangles, which is a special case of similar triangles. It occurs when the scale factor is 1.

Consider the triangles in the table. All the triangles are similar.

The markings indicate that all three corresponding angles match in each of the triangles. However, in relation to the 'original' triangle, the top one has undergone dilation by a factor of $\frac{1}{2}$, the middle one dilation by a factor of 1 (congruence) and the bottom one dilation by a factor of 2.

Similar triangles are a special case of similar figures. To prove that triangles are similar it is only necessary to prove that their corresponding angles are equal. This is not a sufficient proof for similar figures in general. For other figures, it is necessary to also demonstrate that corresponding sides are in the same ratio. There are four tests for similarity of triangles and these tests are like the tests for congruence.

Original figure	Figure after transformation
	
	
	

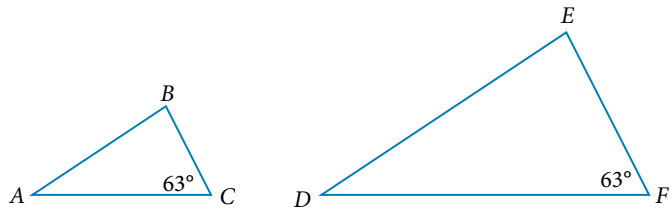
In the two triangles on the right, if ΔABC is similar to ΔDEF ($\Delta ABC \sim \Delta DEF$), then the following four conditions must all hold true:

$\angle CAB = \angle FDE$

$\angle ABC = \angle DEF$

$\angle ACB = \angle DFE$

$$\frac{|AC|}{|DF|} = \frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$$

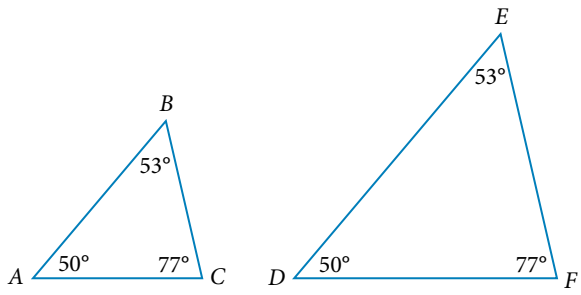


Tests for similarity

Angle, Angle, Angle (AAA)

Two triangles are similar if the three pairs of corresponding angles are equal. This could just as easily be referred to as AA since, if two pairs of corresponding angles are the same, then the third pair must be also.

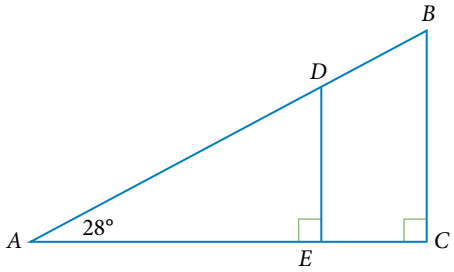
In the diagram, $\Delta ABC \sim \Delta DEF$.



5 Using AAA to test for similarity

Consider the diagram.

Demonstrate that $\Delta ABC \sim \Delta ADE$.

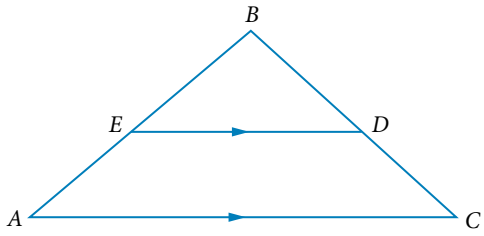


THINKING	WORKING
1 Name the triangles being considered. You need to demonstrate that the corresponding angles are equal. It is sufficient to demonstrate that two pairs of angles are corresponding, but the calculation could be shown for all three pairs of angles for completeness. Write equalities based on the information provided in the diagram.	In ΔABC and ΔADE $\angle BAC = \angle DAE = 28°$ (common) $\angle ACB = \angle AED = 90°$ (given)
2 Consider the third angle in each triangle.	$\angle ABC = \angle ADE$ (Third angle adds to 180° in each triangle.)
3 Write a conclusion about the two triangles.	$\Delta ABC \sim \Delta ADE$ (AAA)

6 Using parallel lines with similar figures

Consider the diagram.

Demonstrate that $\Delta BED \sim \Delta BAC$



THINKING	WORKING
1 Name the triangles being considered, and write equalities based on the information provided in the diagram.	In ΔBED and ΔBAC $\angle EBD = \angle ABC$ (common) $\angle BED = \angle BAC$ (corresponding angles since lines are parallel) $\angle BDE = \angle BCA$
2 Write a conclusion about the two triangles.	$\Delta BED \sim \Delta BAC$ (AAA)

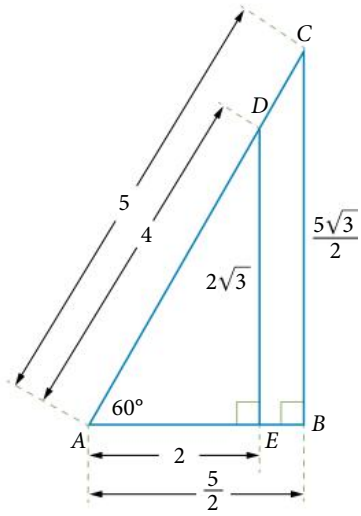
Side, Side, Side (SSS)

With many pairs of similar triangles, it is relatively easy to see that the three pairs of corresponding angles are equal, but it is not as obvious that the corresponding sides are in the same ratio.

Two triangles are similar if the ratios of the side lengths of all three pairs of corresponding sides are equal.

That common ratio is often referred to as the scale factor.

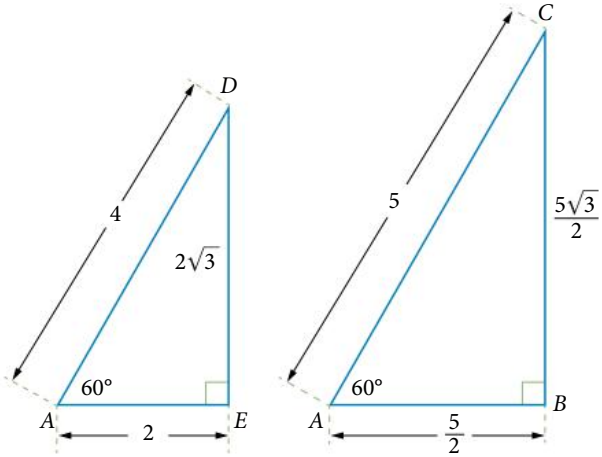
There is a way to demonstrate that the side lengths are in the same ratio. Consider the diagram on the right, showing similar triangles.



It can be demonstrated that $\Delta ABC \sim \Delta AED$ (by SSS) since

$$\frac{|AB|}{|AE|} = \frac{|AC|}{|AD|} = \frac{|BC|}{|ED|} = \frac{5}{4}$$

However, looking at pairs of sides within each triangle and the corresponding pairs of sides in the other triangle you can see that:



In ΔABC

$$\frac{|AB|}{|AC|} = \frac{\frac{5}{2}}{5} = \frac{1}{2} \qquad \frac{|AC|}{|BC|} = \frac{5}{\frac{5\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \qquad \frac{|AB|}{|BC|} = \frac{\frac{5}{2}}{\frac{5\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

In ΔAED

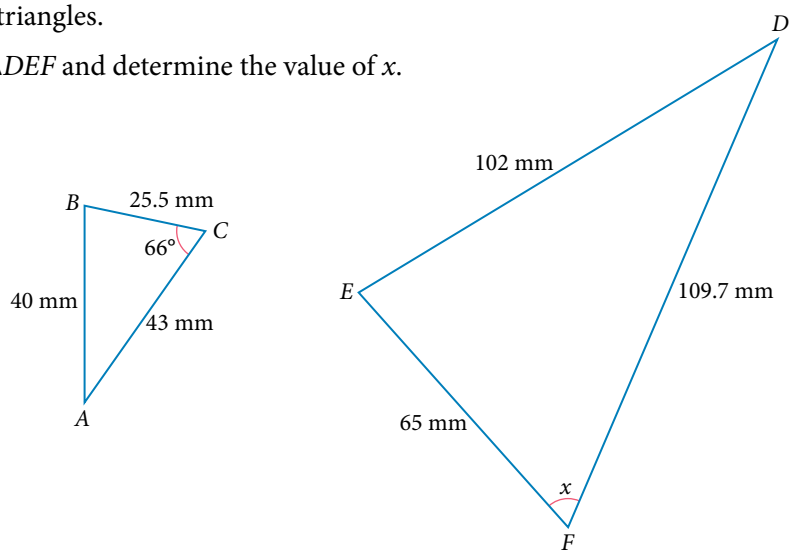
$$\frac{|AE|}{|AD|} = \frac{2}{4} = \frac{1}{2} \qquad \frac{|AD|}{|ED|} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}} \qquad \frac{|AE|}{|ED|} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

In each case, the ratio between the side lengths of the corresponding pairs is the same. The scale factor can be found by looking at the ratio between any two corresponding sides, for example AC and AD, so the calculation is $\frac{5}{4} = 1.25$ as seen previously.

7 Using the SSS test for similarity

Consider the pair of triangles.

Prove that $\Delta ABC \sim \Delta DEF$ and determine the value of x.



THINKING	WORKING
1 Name the triangles being considered. Use the SSS test to check for similarity of ΔABC and wΔDEF.	For ΔABC and ΔDEF: $\frac{\lvert AB\rvert}{\lvert DE\rvert} = \frac{40}{102}, \frac{\lvert BC\rvert}{\lvert EF\rvert} = \frac{25.5}{65}, \frac{\lvert AC\rvert}{\lvert DF\rvert} = \frac{43}{109.7}$ $\approx 0.39 \quad \approx 0.39 \quad \approx 0.39$
2 Write a conclusion based on these calculations.	$\therefore \Delta ABC \sim \Delta DEF$ (SSS)
3 The given angle corresponds to the unknown angle. Use this to calculate its value.	$\angle ACB = \angle DFE$ (corresponding angles) $66 = x$ (in similar triangles) $\therefore x = 66°$

To prove that two triangles are similar it is necessary to demonstrate one of the following conditions:

All three pairs of corresponding angles are equal.	AAA
The ratio (scale factor) between all three pairs of corresponding sides is the same.	SSS
Two pairs of corresponding side lengths are in the same ratio and the angles between them are equal.	SAS
The hypotenuse and one other side of a right-angled triangle are in the same ratio to the hypotenuse and corresponding side of another right-angled triangle.	RHS

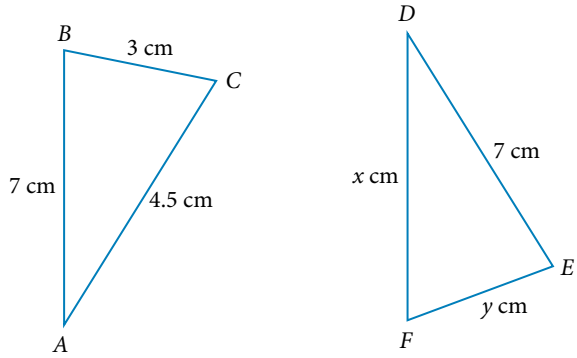
When two triangles are similar, it is written as $\Delta ABC \sim \Delta DEF$, for example.

EXERCISE 4.1 Review of congruence and similarity

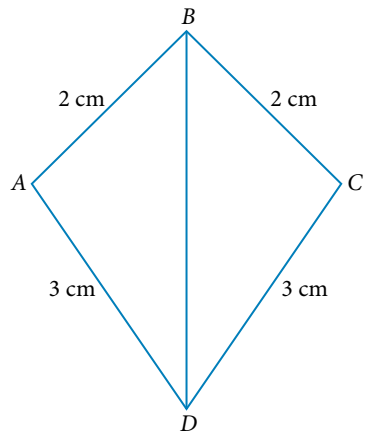
Worked Example

1

1 In the diagram, $\Delta ABC \cong \Delta DEF$. Determine the values of x and y.

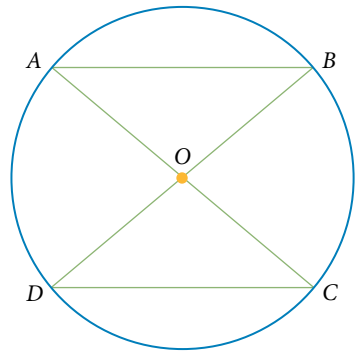


2 For the diagram, demonstrate that $\Delta ABD \cong \Delta CBD$.

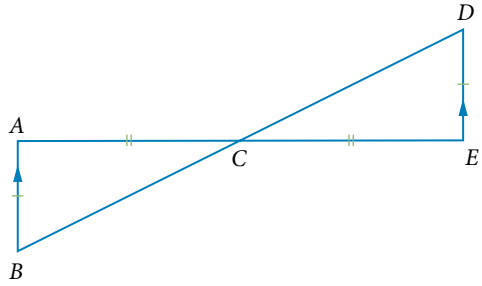


2

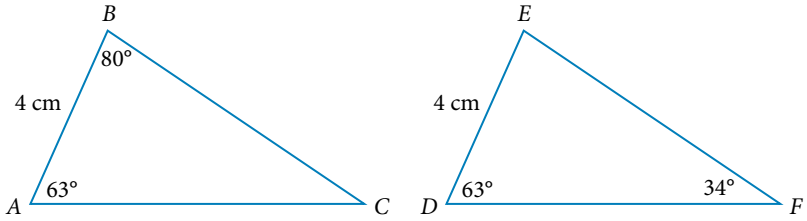
3 In the diagram, O is the centre of the circle. Demonstrate that $\Delta ABO \cong \Delta DCO$.



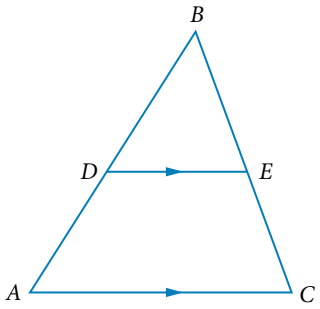
4 In the diagram $|AC| = |CE|$, $|AB| = |DE|$ and AB is parallel to DE. Demonstrate that $\Delta ABC \cong \Delta EDC$.



5 For the diagram, check if $\Delta ABC \cong \Delta DEF$. 3



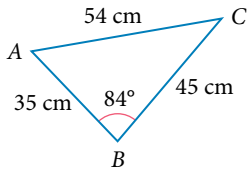
6 For the diagram, demonstrate that $\Delta ABC \sim \Delta DBE$. 5

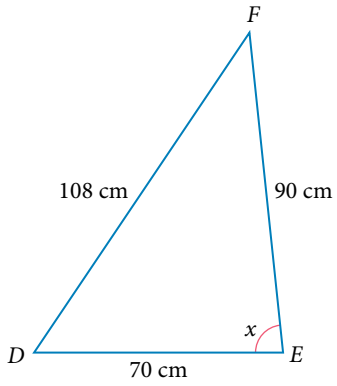


7 Consider the pair of triangles. 7

(a) Demonstrate that the triangles are similar.

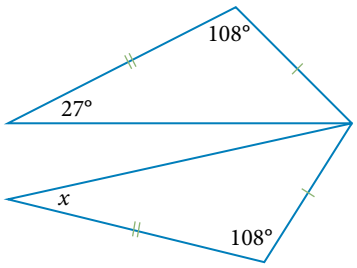
(b) Determine the value of x.



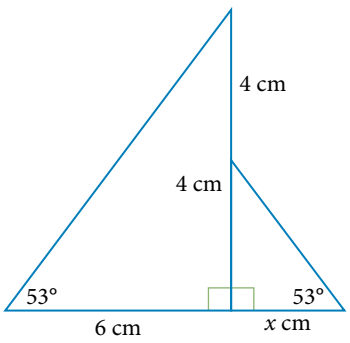


8 Determine the value of the pronumeral in each of the following diagrams. 4

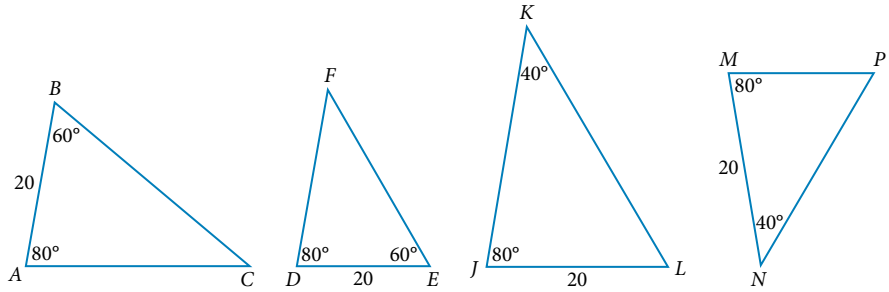
(a)



(b)



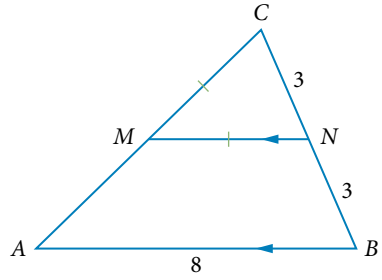
9 Which of the triangles in the diagram below are congruent?



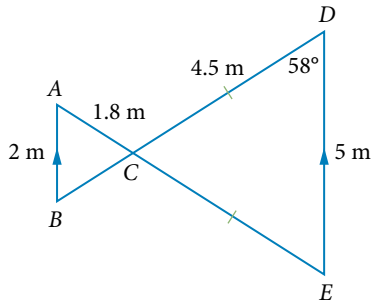
Worked Example

6

10 In ΔABC, N is the midpoint of BC, NM is parallel to BA and equal in length to MC. Calculate the length of AM.



11 Consider the diagram on the right.



(a) Which of the following statements is true?

A $\Delta ABC \cong \Delta DEC$ **B** $|AB| = |BC|$

C $\Delta ABC \sim \Delta EDC$ **D** $\angle ACB = \angle CDE$

(b) Which of the following statements is false?

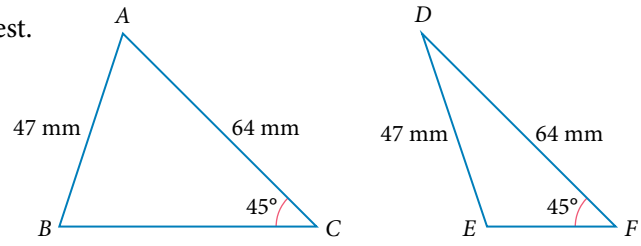
A $\angle ACB = \angle DCE$ **B** $\angle DEC = \angle CAB$

C $\angle CDE = \angle CBA$ **D** $\angle ABC = \angle DCE$

(c) Explain why the other three statements in part **(a)** are false.

(d) What is the common error in the other three statements in part **(b)**?

12 Jireh and Elva are studying for their geometry test. Jireh has drawn two triangles.



Elva looks at the triangles and makes the following argument:

Step 1: $|AB| = |DE|$

Step 2: $|AC| = |DF|$

Step 3: $\angle ACB = \angle DFE$

Step 4: $\Delta ABC \cong \Delta DEF$ (SAS)

Jireh looks at Elva's argument and says it is wrong. Which of the following is a valid reason for Elva's argument being wrong?

A SAS can only be used for similarity and not for congruence.

B SAS can only be used when the angle is between the given sides.

C SAS cannot be used when one of the angles is obtuse.

D There is not enough information—an extra angle is needed.

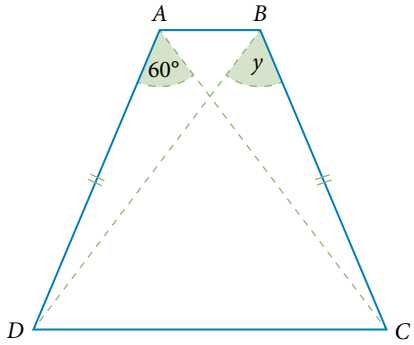
13 On a sunny day, a building casts a shadow of length 15 m. A mathematician of height 1.5 m observes that her shadow at the same time is 25 cm in length. (It is reasonable to assume that both the building and the observer are perpendicular to the ground.) What is the height of the building in metres?

14 Tessa is 1.65 m tall. Standing on a netball court near a light pole 10 m high, she casts a shadow 5.5 m in length.

(a) What is the exact distance from the end of her shadow to the base of the light pole?

(b) What is the exact distance between her and the light pole?

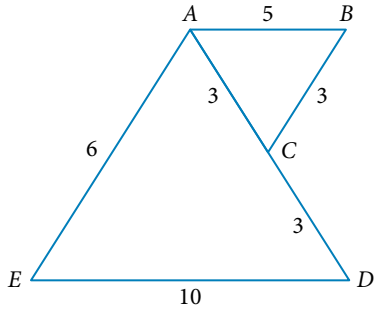
15 In the following diagram, $|AD| = |BC|$, $\angle ADC = \angle BCD$ and $\angle DAC = 60°$. Determine the value of y.



16 Consider the diagram.

(a) Demonstrate that $\Delta ABC \sim \Delta ADE$.

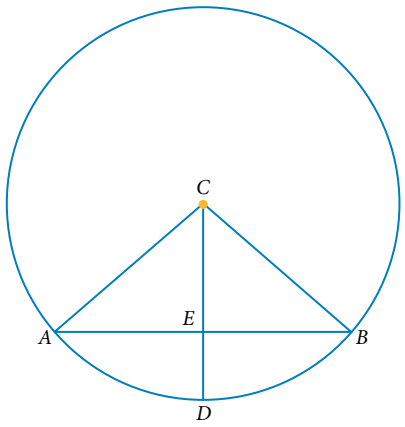
(b) Hence, demonstrate that $AB \parallel DE$



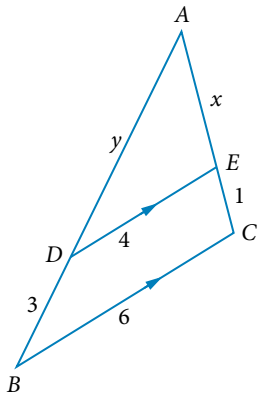
17 In the diagram, C is the centre of the circle and $|AE| = |BE|$.

(a) Demonstrate that $\Delta ACE \cong \Delta BCE$.

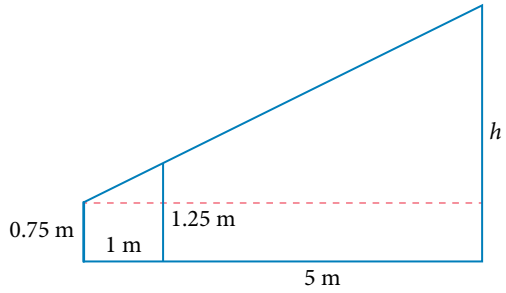
(b) Hence, prove that $AB \perp CD$.



18 Consider the following diagram in which $DE \parallel BC$. Determine the values of x and y.



19 A spotlight is placed on a stand 75 cm above the floor in a large school room with vertical walls. A student of height 1.25 m runs backwards and forwards from the far wall to the spotlight. The spotlight is 5 m away from the far wall. The diagram shows the situation when the student is 1 m from the spotlight. Let h represent the height of the shadow on the wall and assume it reaches the floor.



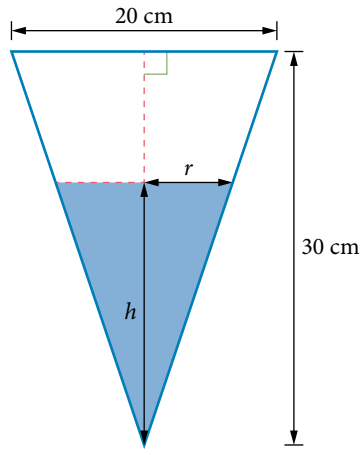
(a) Determine the height of the shadow created when the student is 1 m from the spotlight.

(b) If the height of the shadow is 2 m, how far above the head of the student does it reach?

(c) If the height of the shadow is 2 m, calculate the horizontal distance from the student to the spotlight. (Leave your answer correct to 2 decimal places.)

(d) Let the distance between the student and the spotlight be x m and the height above the student's head of the shadow on the wall be y m. Determine the relationship between x and y. You will need to assume that the wall is very high.

20 The cross-section of an inverted cone is shown. The cone has a diameter of 20 cm and a height of 30 cm. It is filled with water to a height of h cm and the circular surface of the water has a radius of r cm.



(a) Determine an expression for h in terms of r.

(b) Determine an expression for the volume of water in the cone in terms of r.

(c) Determine the radius of the surface of the water if the volume of water in the cone is 400 cm^3.

Geometric proof

4.2

Implication in proof

Implication is when one statement, or set of statements, leads logically to another statement. This is perhaps better understood if the arguments are presented in ordinary language. Consider the following two statements:

- Freda is in Brisbane.
- Brisbane is in Queensland.

These two statements lead to the logical conclusion that Freda is in Queensland. Using the symbol $\Rightarrow$ for implication, the argument above could be written as:

- (Freda is in Brisbane.) and (Brisbane is in Queensland.) $\Rightarrow$ (Freda is in Queensland.)

Now, based on the same two original statements, consider the following conditional statements:

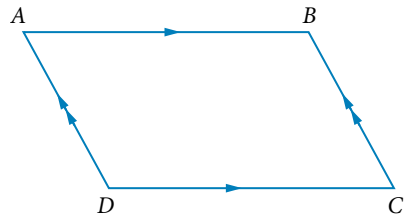
- If Freda is in Brisbane then she is in Queensland.
- If Freda is not in Queensland then she is not in Brisbane.

These two statements are considered to be logically equivalent. Using the symbol $\Leftrightarrow$ for equivalence this could be written as:

- (If Freda is in Brisbane then she is in Queensland.) $\Leftrightarrow$ (If Freda is not in Queensland then she is not in Brisbane.)

Examples in geometry

Consider the diagram on the right.



Using the information in the diagram, the following statements can be made:

- Statement 1: $ABCD$ is a quadrilateral and $AB \parallel DC$ and $AD \parallel BC$.
- Statement 2: $ABCD$ is a parallelogram.

Assuming both statements are correct, which symbol should be used to join them? To answer this question, you should ask yourself the following questions: 'Does Statement 1 imply Statement 2?' and 'Does Statement 2 imply Statement 1?'

As the answer to both is yes, the correct symbol is equivalence.

So, Statement 1 $\Leftrightarrow$ Statement 2.

If $ABCD$ is a parallelogram, then $ABCD$ is a quadrilateral with $AB \parallel CD$ and $AD \parallel BC$.

If $ABCD$ is a quadrilateral with $AB \parallel CD$ and $AD \parallel BC$, then $ABCD$ is a parallelogram.

Consider now a third statement derived from the same diagram:

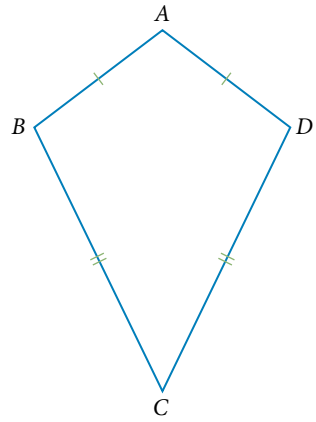
- Statement 3: $ABCD$ is a quadrilateral, and $\angle ABC = \angle ADC$

Consider Statement 2 and Statement 3 together:

- Statement 2: $ABCD$ is a parallelogram.
- Statement 3: $ABCD$ is a quadrilateral with $\angle ABC = \angle ADC$

Statement 2 $\Rightarrow$ Statement 3

If $ABCD$ is a parallelogram, then $ABCD$ is a quadrilateral with $\angle ABC = \angle ADC$.



However, Statement 3 does not imply Statement 2.

If $ABCD$ is a quadrilateral, with $\angle ABC = \angle ADC$ does not imply that $ABCD$ is a parallelogram. $ABCD$ may be the kite shown.

Another way of putting this is to say that it is a necessary condition of being a parallelogram that a pair of opposite angles is equal, but it is not a sufficient condition.

You should note that equivalence is a much stronger statement than implication, and as such will often take longer to prove. To prove equivalence, it is sufficient (and correct) to prove two implications, with the second being the reverse of the first.

Methods of proof in geometry

In this section both direct and indirect methods of proof will be examined. A direct proof starts with a premise and, using valid steps of deduction, reaches the conclusion. There are several forms of indirect proof, one of which is proof by contradiction. For proof by contradiction, you start by making an assumption then work through the proof using that assumption. If this then leads to a contradiction, the assumption must have been wrong, and the alternative must be true.

The majority of proofs looked at in this section will be direct proofs, although it is important to at least be aware of other forms of proof. Another form is proof by contraposition. The example given earlier in the section about the equivalence of statements is an example of proof by contraposition.

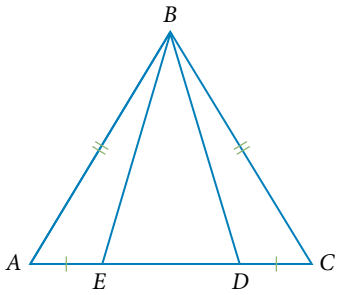
- (If Freda is in Brisbane then she is in Queensland.) $\Leftrightarrow$ (If Freda is not in Queensland then she is not in Brisbane.)

Note that in this case, the negative of both original statements is taken and the direction of the inference has been reversed.

When constructing a direct proof, you should start with a statement of what needs to be proven, then provide the logical sequence of statements that is the basis of the proof and finish with a conclusion that reasserts the initial statement.

8 A simple geometric proof

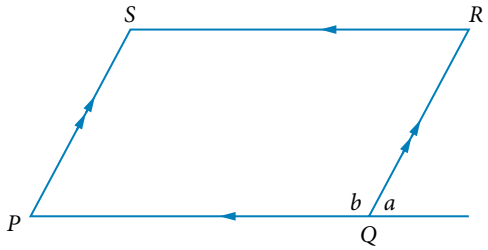
Considering the diagram, prove that $|BE| = |BD|$.

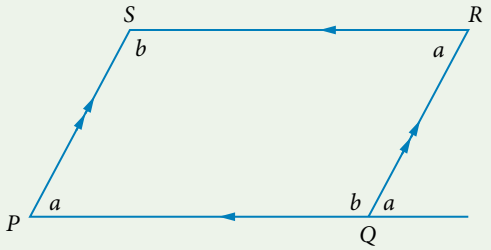


THINKING	WORKING
1 Write the statement that needs to be proven.	To prove: $\|BE\| = \|BD\|$
2 Name the triangles being considered, and write the arguments with reasons that support the statement.	Proof: In ΔBAE and ΔBCD $\angle BAE = \angle BCD$ (Base angles in an isosceles triangle are equal.) $\|BA\| = \|BC\|$ (given) $\|AE\| = \|CD\|$ (given) $\Delta BAE \cong \Delta BCD$ (SAS) $\therefore \|BE\| = \|BD\|$ (Corresponding sides in congruent triangles are equal.)

9 Proof associated with parallelograms

Use the diagram to prove that the sum of the interior angles of a parallelogram is 360°.

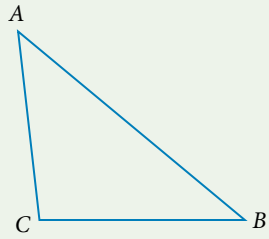


THINKING	WORKING
1 Write the statement that needs to be proven.	To prove: The sum of the interior angles in a parallelogram is 360°. Angles a and b are supplementary, co-interior angles (sum to 180°).
2 Write the relationships between the angles, giving reasons.	

	As co-interior angles between parallel lines are supplementary: $\angle RSP + \angle SPQ = 180°$ $\angle SRQ + \angle PQR = 180°$ $\therefore \angle RSP + \angle SPQ + \angle SRQ + \angle PQR = 360°$
3 Write the conclusion.	The sum of the interior angles of the parallelogram is 360°.

10 Proof by contradiction

Prove that a triangle can have at most one obtuse angle.

THINKING	WORKING
1 Create a statement that is the contradiction of the statement that you wish to prove.	To prove: A triangle can have more than one obtuse angle.
2 Draw a labelled triangle.	
3 Put the statement of the contradiction into algebraic form.	$\angle ACB > 90°$ $\angle BAC > 90°$
4 Write an expression for the sum of the angles in the triangle.	In ΔABC, angles add to 180°. $\angle ABC + \angle BCA + \angle CAB = 180°$
5 Complete the process to see where the line of argument leads.	$\angle ACB + \angle CBA + \angle BAC = 180°$ $(> 90°) + \angle CBA + (> 90°) = 180°$ $\angle CBA + (> 180°) = 180°$ $\angle CBA < 0°$
6 Make a comment about the result and write a concluding statement.	For a triangle to exist, each of the interior angles must be greater than 0°, so the assumption was false. A triangle cannot have more than one obtuse interior angle.

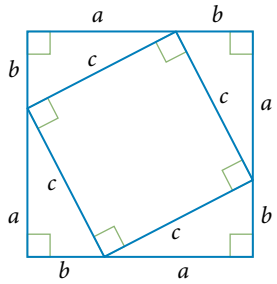
Proof by contradiction is sometimes referred to by the Latin expression *reductio ad absurdum*, which roughly translates as 'reduced to an absurdity'.

Pythagoras' theorem

Pythagoras was a Greek mathematician who lived from 569 to 475 BCE. The following theorem has become known as Pythagoras' theorem as Euclid credited Pythagoras with the proof of this theorem in his famous mathematical text, *The Elements*. In Chinese mathematics, this theorem is known as the Gougu Rule and in Indian mathematics the theorem is known as the Bhaskara Theorem. Pythagoras' theorem states that there is a special relationship between the three side lengths in any right-angled triangle drawn on a flat surface.

> The square on the hypotenuse is equal to the sum of the squares on the other two sides.

Many proofs exist for the theorem of Pythagoras. It is probably true that there are more proofs for this theorem than for any other mathematical theorem. One proof relies on this diagram, as follows:



Area of one triangle: $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times a \times b$

Area of four (congruent) triangles: $\frac{ab}{2} \times 4 = 2ab$

Area of inner square: c^2

Area of outer square in terms of a, b and c: $2ab + c^2$

Area of outer square in terms of a and b only: $(a + b)^2 = a^2 + 2ab + b^2$

Equating two expressions for the area of the outer square and simplifying:

$$2ab + c^2 = a^2 + 2ab + b^2$$
$$c^2 = a^2 + b^2$$

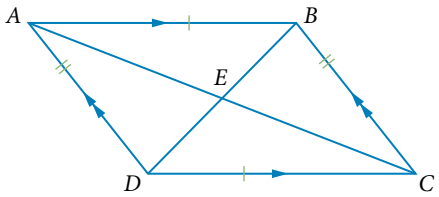
This can clearly be seen as the statement of Pythagoras' theorem.

You will sometimes see QED written at the end of a proof such as this. This is shorthand for the Latin expression *quod erat demonstrandum*, which roughly translates to 'What was required has been shown'.

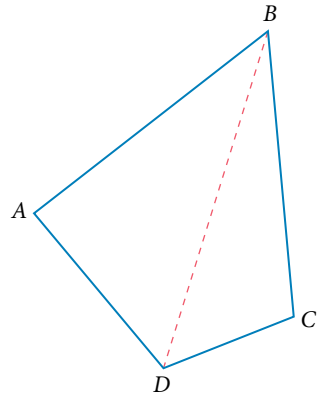
EXERCISE 4.2 Geometric proof

Worked Example 8

1 Use the diagram to prove that the two diagonals of a parallelogram bisect each other.

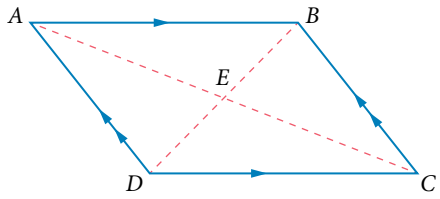


2 Use the diagram to prove that the sum of the interior angles in a quadrilateral is 360°.

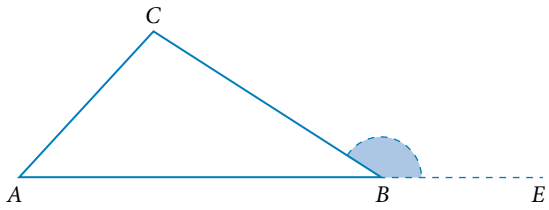


Worked Example 9

3 Use the diagram to prove that opposite angles in a parallelogram are equal.



4 Use the diagram to prove that the external angle of a triangle is equal to the sum of the two opposite interior angles.



5 Decide if each of these statements is true or false.

(a) It is possible for a triangle to have three acute angles.

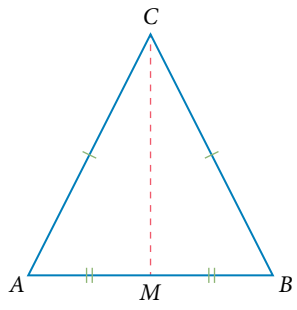
(b) It is possible for a triangle to have a right angle and two acute angles.

(c) It is possible for a triangle to have a reflex angle and two acute angles.

(d) For any triangle, the sum of any two angles must be greater than the size of the third.

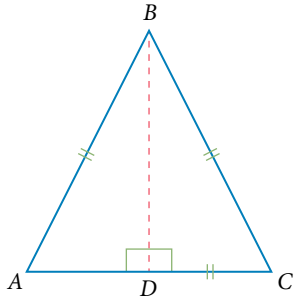
6 Prove that the diagonal of a parallelogram divides it into two congruent triangles.

7 Using the diagram, prove that the line CM, which bisects the base of the isosceles triangle ABC, also bisects $\angle ACB$.

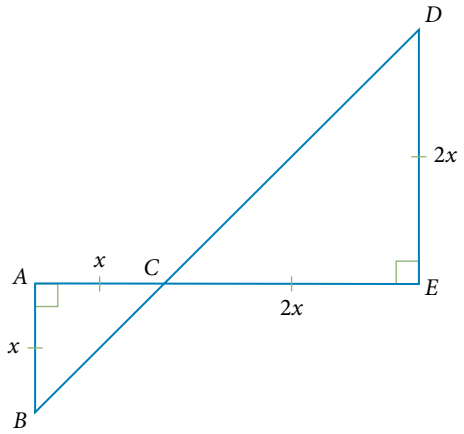


10

8 Use proof by contradiction to demonstrate that the perpendicular drawn from the base of an isosceles triangle to its apex bisects the base of the triangle.



9 Considering the diagram, which of the following statements is false?



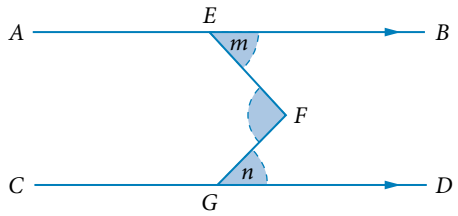
A $\Delta ABC \sim \Delta EDC$ (SAS)

B $\dfrac{|BC|}{|DC|} = 0.5$

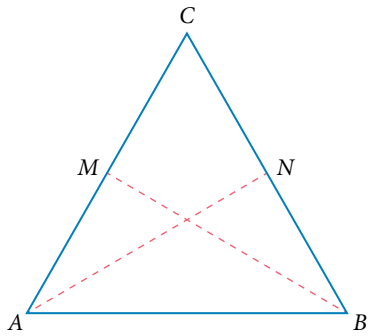
C $|DC| = 2\sqrt{2x^2}$

D $\dfrac{|AB|}{|ED|} = \dfrac{|AC|}{|CD|}$

10 In the diagram, prove that the acute $\angle EFG = m + n$.

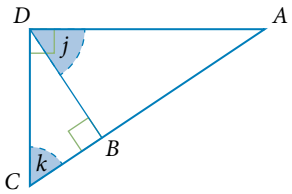


11 M and N are the midpoints of sides AC and BC in the isosceles triangle below. Prove that $|AN| = |BM|$.



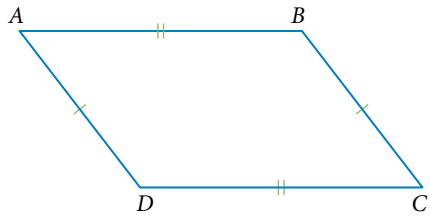
12 Prove that the diagonals of a rhombus meet at right angles.

13 For the diagram below, demonstrate that $j = k$.



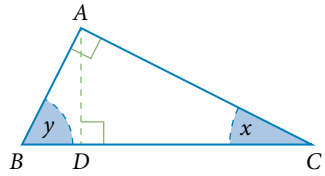
14 Use the method of proof by contradiction to prove that a triangle cannot have more than one right angle.

15 Use the diagram of the quadrilateral $ABCD$ to prove that, if the opposite sides of the quadrilateral are equal, then the opposite sides are parallel. (This is known as a converse theorem for the parallelogram.)



16 Using proof by contradiction, demonstrate that a quadrilateral can have at most one reflex angle.

17 Consider the diagram to the right.



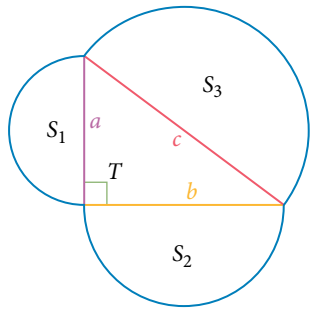
(a) Demonstrate that $\angle CAD = y$ and $\angle BAD = x$.

(b) From part **(a)**, you can see that ΔABC and ΔDBA are similar (AAA). Write an equation indicating the ratio of the sides in ΔDBA correspond to the ratio of sides BC and AB in ΔABC.

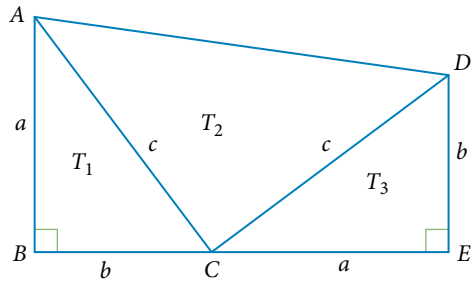
(c) You can also see that ΔABC and ΔDAC are similar (AAA). Write an equation indicating the ratio of the sides in ΔDAC correspond to the ratio of sides BC and AC in ΔABC.

(d) Use the equations you have found in parts **(b)** and **(c)** to prove the Pythagoras' theorem for ΔABC.

18 Consider the following diagram of a right-angled triangle, T, with semi-circles S_1, S_2 and S_3 drawn on each of the three sides. Demonstrate that the area of S_1 + area of S_2 = area of S_3.



19 In 1876 James Garfield produced a proof of Pythagoras' theorem. The following steps will help you reproduce this particular proof of the theorem. Consider the following diagram:



(a) Demonstrate that $\angle ACB = \angle CDE$ and $\angle BAC = \angle ECD$.

(b) Hence, demonstrate that $\angle ACD = 90°$.

(c) Prove that $ABED$ is a trapezium and determine an expression for its area in terms of A and B.

(d) Determine an expression for the area of each of the triangles T_1, T_2 and T_3 in terms of a, b and c.

(e) Hence, determine an expression for the area of T_2 in terms of A and B only.

(f) Hence, prove Pythagoras' theorem.

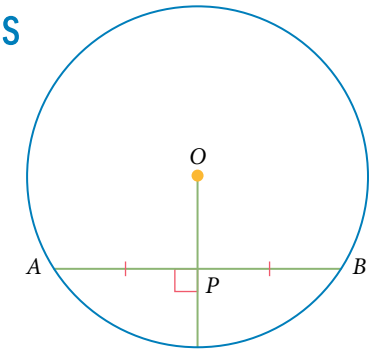
Circle theorems, part 1

4.3

Over the course of the next two sections, several theorems related to circles will be examined.

By convention, the centre of a circle is marked as O.

A radius drawn perpendicular to a chord bisects that chord



The top diagram on the right illustrates this theorem.

The converse of this theorem is that a radius that bisects a chord is perpendicular to that chord.

To prove this theorem, see the lower diagram, where $|AP| = |PB|$

Construct radii OA and OB.

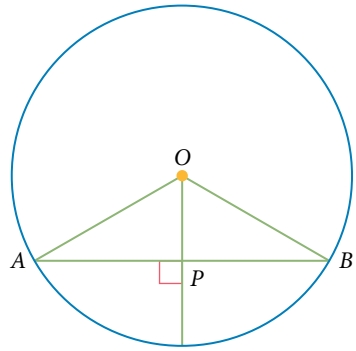
In ΔAOP and ΔBOP, $\angle APO = \angle BPO = 90°$.

$|AO| = |OB|$, as they are equal radii of the circle.

$|OP| = |OP|$, as they are common to ΔAPO and ΔBPO.

$\Delta APO \cong \Delta BPO$ (RHS)

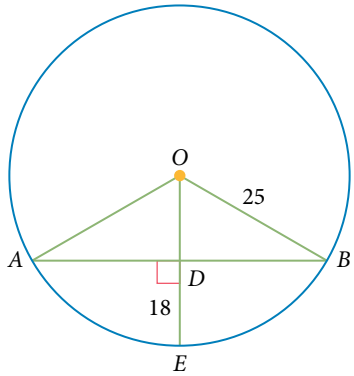
Therefore, $|AP| = |PB|$, as corresponding sides in congruent triangles are equal.



11 Calculating the length of a line segment

Consider the diagram of a circle where the radius is 25 units and $|DE| = 18$ units.

(a) Calculate $|OD|$.

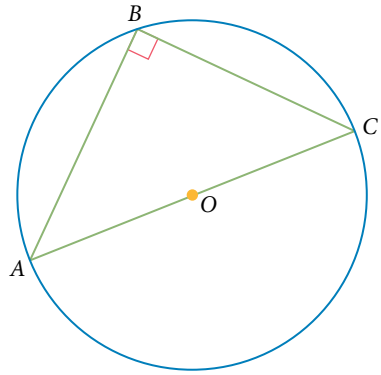


THINKING	WORKING
Identify OD as part of the radius and use this to calculate its length.	$\|OD\| + \|DE\| = 25$ $\|OD\| + 18 = 25$ $\|OD\| = 7$

(b) Determine $|AD|$.

1 Identify a right-angled triangle that contains the unknown value and write the known values.	In ΔAOD, $\lvert AO\rvert = 25$ and $\lvert OD\rvert = 7$.
2 Use Pythagoras' theorem to calculate the required length.	$\lvert AD\rvert^2 + 7^2 = 25^2$ $\lvert AD\rvert^2 = 25^2 - 7^2$ $= 625 - 49$ $= 576$ $\therefore \lvert AD\rvert = \sqrt{576}$ $= 24$

The angle in a semicircle is a right angle



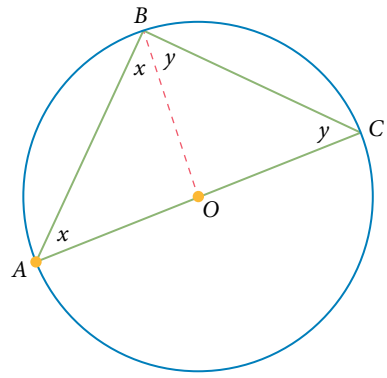
This is also known as the Thales' theorem. Thales was a Greek mathematician who lived at about the same time as Pythagoras. The theorem is sometimes stated as 'The angle at the circumference subtended by a diameter is a right angle'.

The converse of this theorem is that if the angle subtended by a chord is a right angle, then that chord must be a diameter.

Making connections

Angle in a semicircle

Move the slider for vertices ABC and observe the angle at the circumference.



To prove Thales' theorem, see in the diagram, $\angle ABC = 90°$.

Construct radius OB to form two isosceles triangles.

The sum of the angles in $\Delta ABC = 180°$. This gives:

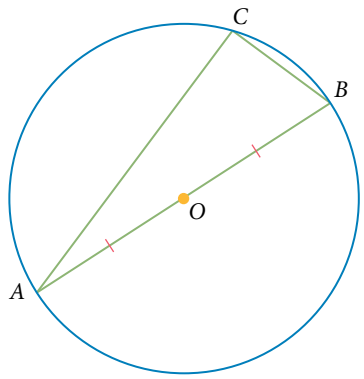
$$\begin{aligned} x + x + y + y &= 180° \\ 2x + 2y &= 180° \\ 2(x + y) &= 180° \\ x + y &= 90° \\ x + y &= \angle ABC \end{aligned}$$

Hence $\angle ABC = 90°$

Thus, the angle in a semicircle is a right angle.

12 Calculating the angle in a semi-circle

In the following diagram, AB is a diameter of a circle with centre O.



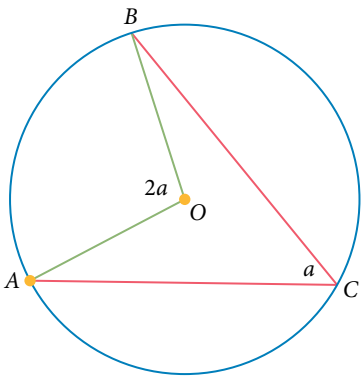
(a) If $\angle CAB = 44°$ calculate the size of $\angle CBA$.

THINKING	WORKING
Determine the size of the known angles, giving reasons, and solve for the unknown angle.	Since the angle in a semicircle is a right angle, $\angle ACB = 90°$. $\angle CAB = 44°$ $\angle ACB + \angle CAB + \angle CBA = 180°$ $90° + 44° + \angle CBA = 180°$ $\angle CBA = 46°$

(b) Determine the size of $\angle ACO$ and $\angle OCB$.

Use equal angles in isosceles triangles to calculate the required angles.	Base angles in an isosceles triangle are equal: $\angle ACO = \angle OAC$ $= 44°$ $\angle OCB = \angle CBO$ $= 46°$

The angle at the centre of a circle subtended by an arc is twice the angle at the circumference subtended by the same arc

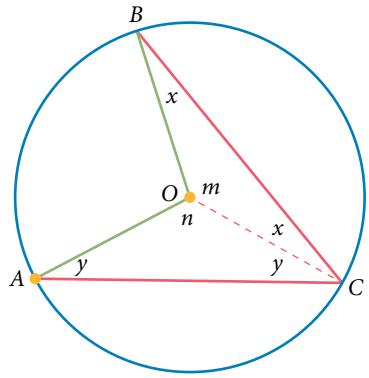


C Making connections

Angles at the centre

Move the sliders and compare the angle at the centre and the angle at the circumference.

To prove that the angle at the centre of a circle is twice the size of an angle at the circumference subtended by the same arc, see in the diagram, $\angle AOB = 2 \times \angle ACB$.



Construct the line segment OC, to form two more isosceles triangles AOC and BOC.

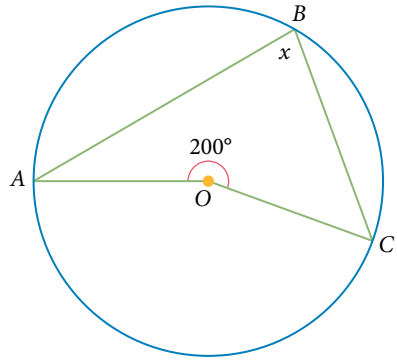
$$\angle BCA = x + y$$
$$2x + m = 180° \qquad 2y + n = 180°$$
$$m = 180° - 2x \qquad n = 180° - 2y$$

$$\begin{aligned} m + n + \angle BOA &= 360° \\ \angle BOA &= 360° - (m + n) \\ &= 360° - (180° - 2x + 180° - 2y) \\ &= 2x + 2y \\ &= 2 \times \angle BCA \end{aligned}$$

Hence the angle at the centre of a circle subtended by an arc is twice the angle at the circumference subtended by the same arc.

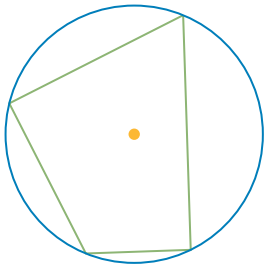
13 Investigating angles at the centre and circumference of a circle subtended by same arc

For the diagram below, calculate the value of x.



THINKING	WORKING
1 Write a relationship between the angle at the centre and the angle at the circumference, giving a reason.	The angle at the centre is twice the angle at the circumference subtended by the same arc. obtuse$\angle AOC = 2 \times \angle ABC$
2 Determine the angle at the centre.	Angles at a point add up to 360°. obtuse$\angle AOC$ + reflex$\angle AOC = 360°$ obtuse$\angle AOC = 360° -$ reflex$\angle AOC$ $= 360° - 200°$ $= 160°$
3 Determine the required angle and state the answer.	$160° = 2x$ $x = 80°$

Opposite angles in a cyclic quadrilateral are supplementary

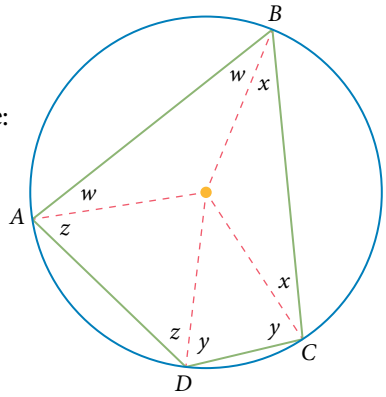


A cyclic quadrilateral is one in which all four vertices lie on the circumference of the same circle.

The opposite angles in a cyclic quadrilateral are supplementary. (They add to 180°.)

The converse of the theorem that opposite angles in a cyclic quadrilateral are supplementary is that if the opposite angles in a quadrilateral are supplementary then it is a cyclic quadrilateral.

Construct a radius to each of the vertices of the cyclic quadrilateral, forming four isosceles triangles.



To prove the theorem, consider the 360° angle at the centre point of the circle:

$$\begin{aligned}(180-2w)+(180-2x)+(180-2y)+(180-2z) &= 360^\circ\\ 720-2w-2x-2y-2z &= 360^\circ\\ 720-2(w+x+y+z) &= 360^\circ\\ -2(w+x+y+z) &= -360^\circ\\ w+x+y+z &= 180^\circ\end{aligned}$$

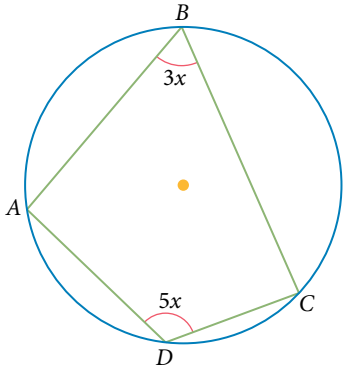
This can also be verified, since the sum of angles in a quadrilateral equal 360°:

$$\begin{aligned}w+w+x+x+y+y+z+z &= 360^\circ\\ 2(w+x+y+z) &= 360^\circ\\ (w+x+y+z) &= 180^\circ\\ \angle DAB+\angle BCD &= w+x+y+z\\ &= 180^\circ\\ \angle ABC+\angle CDA &= w+x+y+z\\ &= 180^\circ\end{aligned}$$

Hence, opposite angles in a cyclic quadrilateral are supplementary.

14 Opposite angles in a cyclic quadrilateral

For the following diagram, calculate the value of x.



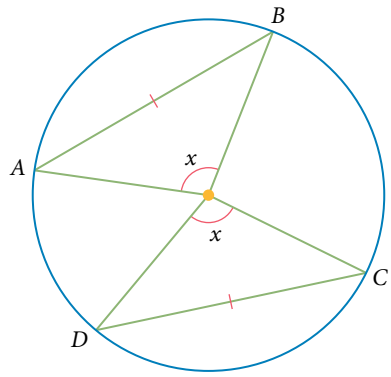
THINKING	WORKING
1 Write an equation involving opposite angles in a cyclic quadrilateral, giving reasons	$3x + 5x = 180°$ Opposite angles in a cyclic quadrilateral are supplementary.
2 Solve the equation to determine the required value.	$8x = 180°$ $x = \frac{180°}{8}$ $= 22.5°$ $x = 22.5°$
3 Evaluate the reasonableness of your answer by calculating the magnitude of $\angle ABC$ and $\angle ADC$.	$\angle ABC = 3x$ $= 3 \times 22.5°$ $= 67.5°$ $\angle ADC = 5x$ $= 5 \times 22.5°$ $= 112.5°$ $\angle ABC + \angle ADC = 67.5° + 112.5°$ $= 180°$ The opposite angles in a cyclic quadrilateral are supplementary, as expected.

C Making connections

Angles in a cyclic quadrilateral

Move the slider to observe that the opposite angles in a cyclic quadrilateral are supplementary.

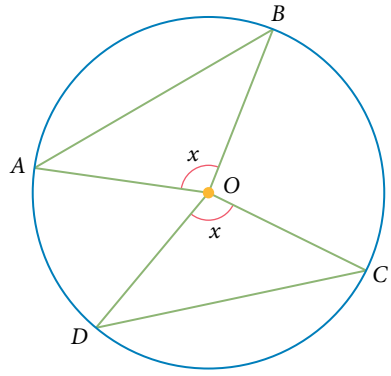
Chords that subtend equal angles at the centre of a circle are equal in length



A chord is a line segment that joins two different points on the circumference of a circle.

The converse of this theorem states that chords that are equal in length subtend equal angles at the centre of a circle.

To prove the theorem, prove that in the diagram, $|AB| = |CD|$.



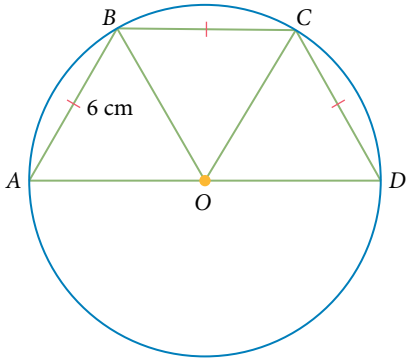
In ΔABO and ΔDCO

$|AO| = |BO|$ (equal radii)
$|CO| = |DO|$ (equal radii)
Hence, $|AO| = |CO|$ and $|BO| = |DO|$
$\angle AOB = \angle COD$ (given)
$\Delta ABO \cong \Delta CDO$
$\therefore |AB| = |CD|$ (corresponding sides in congruent triangles)

Hence chords that subtend equal angles at the centre of a circle are equal in length.

15 Chords of equal length

Consider the following circle:



(a) Determine $\angle BOC$.

THINKING	WORKING
1 Write the relationship between the angles subtended by equal chords.	Since angles subtended by equal chords are equal: $\angle AOB = \angle BOC$ $\angle BOC = \angle COD$
2 Write an equation that can be solved to determine the required angle, and calculate the answer.	Since angles on a straight line are supplementary: $\angle AOB + \angle BOC + \angle COD = 180°$ $\therefore 3\angle BOC = 180°$ $\therefore \angle BOC = 60°$

(b) Determine the diameter of the circle.

1 Determine the other angles in ΔBOC.	ΔBOC is an isosceles triangle. Since base angles of an isosceles triangle are equal: $\angle OBC = \angle OCB$ $= 60°$ Therefore, ΔBOC is an equilateral triangle.
2 Consider the other two triangles.	All three are congruent equilateral triangles (SAS).
3 Calculate the diameter of the circle.	$\lvert AD \rvert = 2 \times \lvert AB \rvert$ $= 2 \times 6$ $= 12$ cm

The theorems covered in this section are as follows:

- A radius drawn perpendicular to a chord bisects that chord.
- The angle in a semicircle is a right angle.
- The angle at the centre subtended by an arc is twice the angle at the circumference subtended by the same arc.
- Opposite angles in a cyclic quadrilateral are supplementary.
- Chords that subtend equal angles at the centre of a circle are equal in length.

C Making connections

Chords of equal length

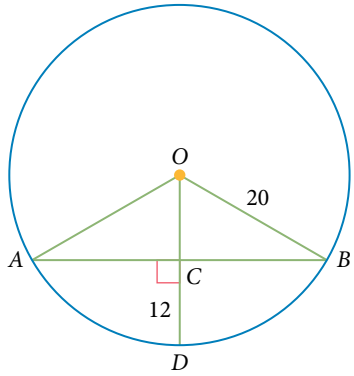
Move the slider to see how chords of equal length subtend equal angles at the centre of a circle and, conversely, how chords subtending equal angles at the centre of a circle have the same length.

EXERCISE 4.3 Circle theorems, part 1

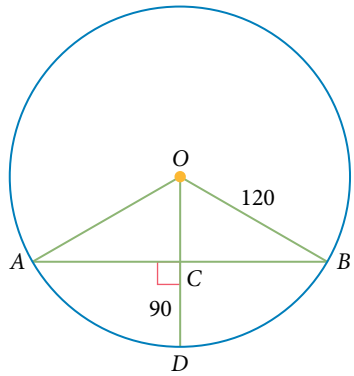
Worked Example

11

1 (a) Consider the diagram of a circle in which the radius is 20 cm and $|CD| = 12$ cm. Determine the exact lengths of $|OC|$ and $|AC|$.

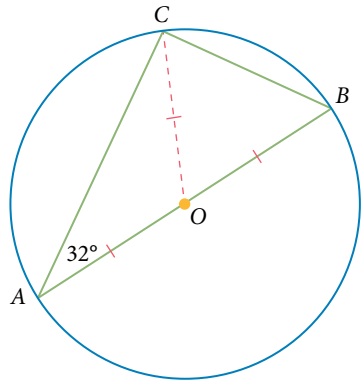


(b) Consider the diagram of a circle in which the radius is 120 cm and $|CD| = 90$ cm. Determine the length of the chord AB.

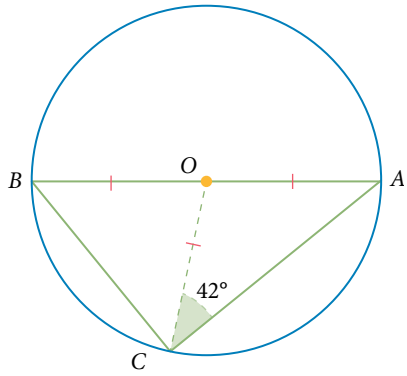


12

2 In the diagram, AB is the diameter of a circle with centre O. Determine the magnitude of $\angle CBA$, $\angle ACO$ and $\angle OCB$.

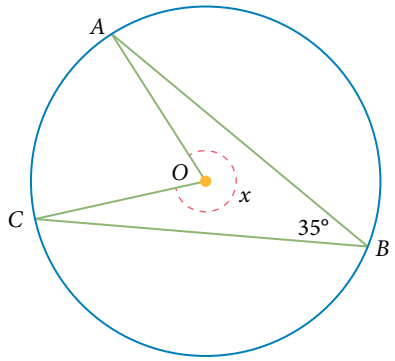


3 In the diagram, AB is the diameter of a circle with centre O. Determine the magnitude of $\angle CAO$ and $\angle OCB$.



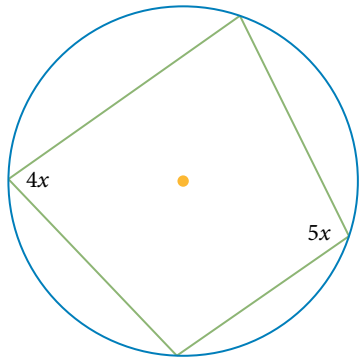
4 In the diagram, calculate the value of x.

13

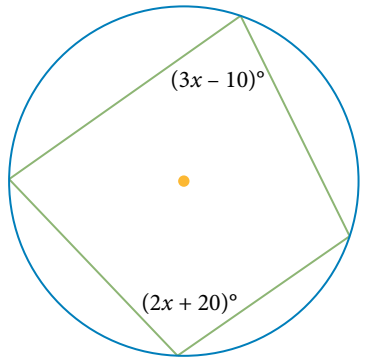


5 For the diagram, calculate the value of x.

14

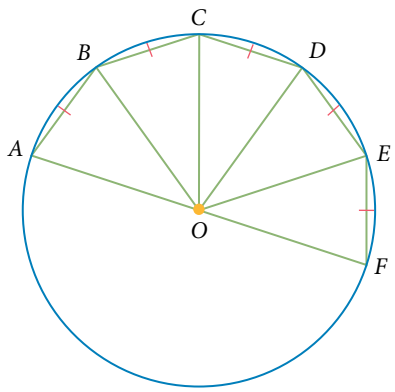


6 For the diagram, calculate the value of x.

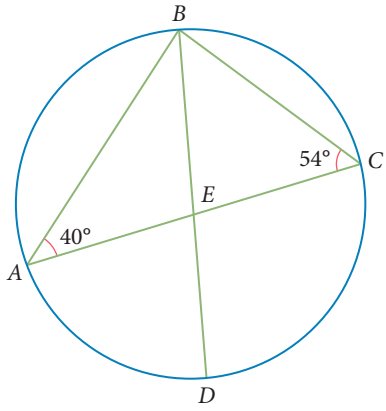


7 Consider the circle, where AF is a diameter. Determine $\angle BOC$.

15

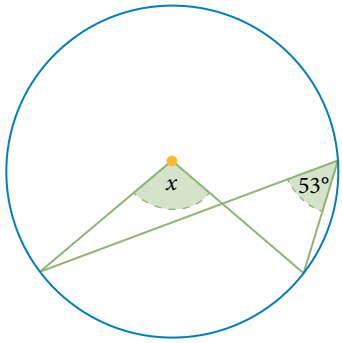


8 Consider the following diagram. Demonstrate that AC is not a diameter of the circle.

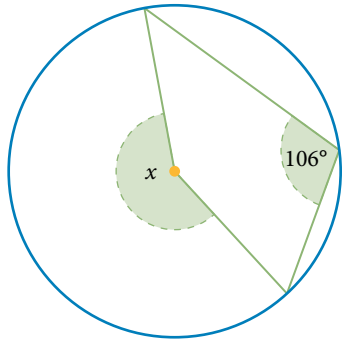


9 Determine the value of the pronumeral in each of the following diagrams.

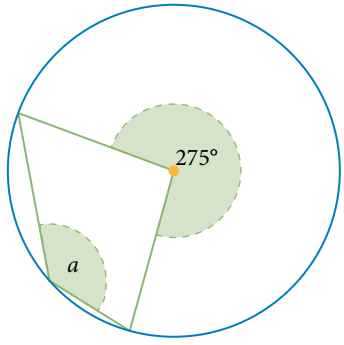
(a)



(b)

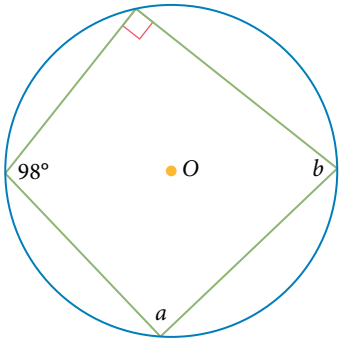


(c)

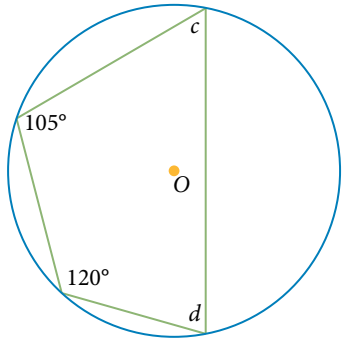


10 Determine the values of the pronumerals in each of the following diagrams.

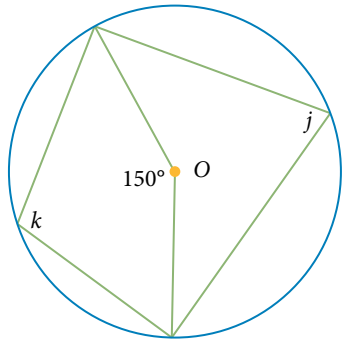
(a)



(b)

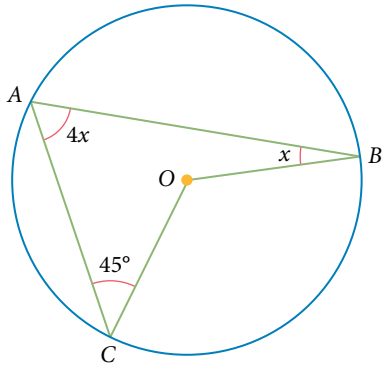


(c)

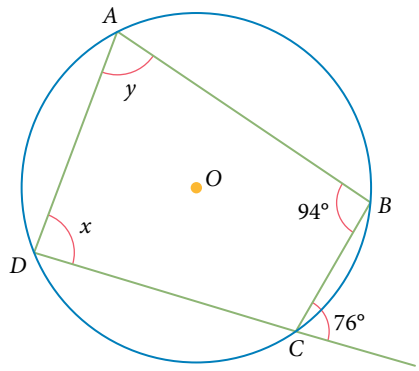


11 Calculate the values of the pronumerals in each of the following diagrams.

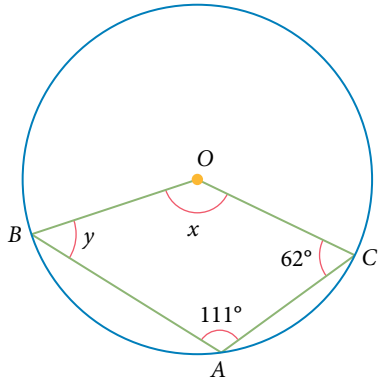
(a)



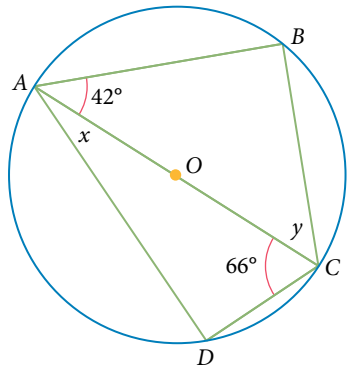
(b)



(c)



(d)

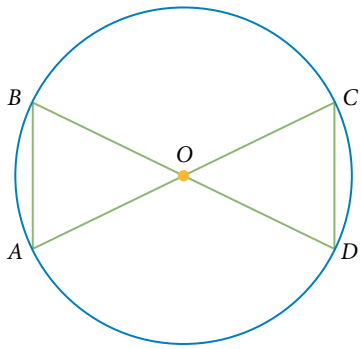


12 Consider the circle with diameter BD. Decide if each of the following statements is true or false.

(a) $|AB| = |DC|$

(b) $\Delta OAB \cong \Delta OCD$

(c) AB is not parallel to CD.



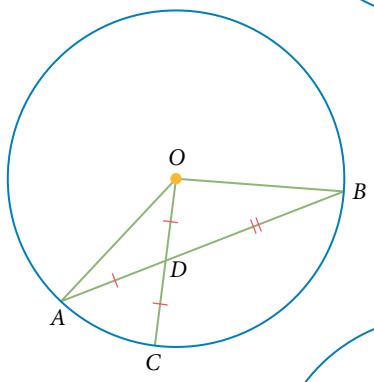
13 Consider the diagram of a circle with centre O. Which of the following statements is false?

A Chord AB bisects radius OC at D.

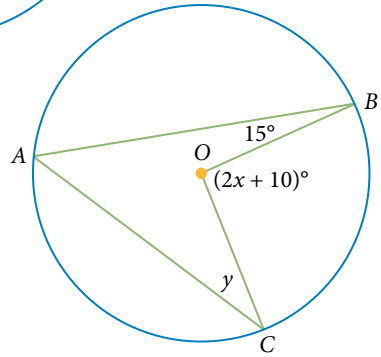
B $\angle CDB$ could be a right angle.

C ΔOBA is isosceles.

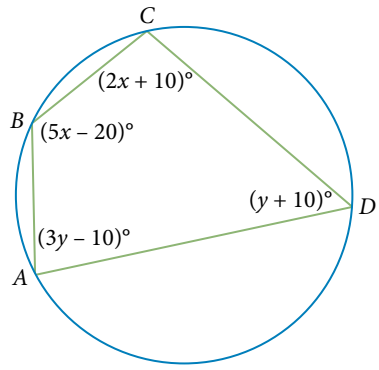
D ΔAOD is isosceles.



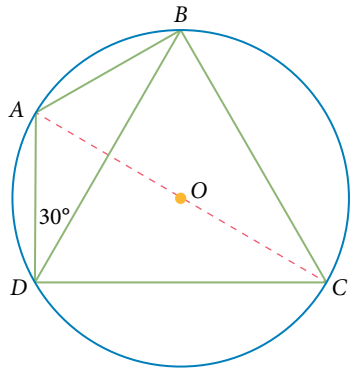
14 Consider the circle with centre O, and points A, B and C on the circumference. Determine an expression for y in terms of x.



15 Determine the values of x and y in the diagram.



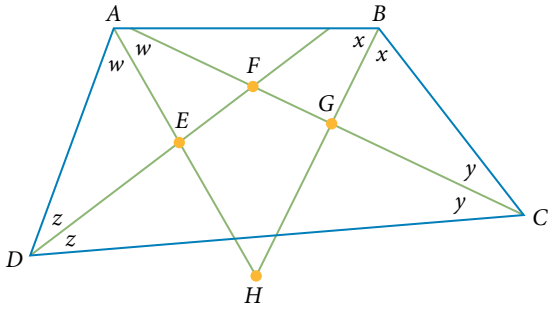
16 In the diagram, $|AD| = |AB|$ and AC is a diameter of the circle. Calculate the value of each of the internal angles of the cyclic quadrilateral $ABCD$.



17 Monique was asked to prove that a parallelogram inscribed in a circle must also be a rectangle. She correctly reasons that, if $ABCD$ is a cyclic quadrilateral with $AB \parallel CD$ and $AD \parallel BC$, she must prove that $AB \perp AD$ (from which it follows that the other three angles must all be right angles, using the fact that co-interior angles are supplementary). Construct a proof using Monique's logic.

18 Use the result you have proved in Question **17** to prove that if a cyclic quadrilateral has two pairs of parallel sides, then its diagonals are diameters of the circle.

19 Sonika is required to prove that the intersection points of the angular bisectors of a quadrilateral form a cyclic quadrilateral. Her first step is to draw a diagram, assuming that the internal angles in the quadrilateral $ABCD$ are $2w$, $2x$, $2y$ and $2z$. Her diagram looks like the one on the right.



(a) Name the quadrilateral that Sonika needs to prove is cyclic.

(b) Express each of the following angles in terms of w, x, y and z.

(i) $\angle HEF$ **(ii)** $\angle EFG$

(iii) $\angle FGH$ **(iv)** $\angle GHE$

(c) Determine the sum of the following pairs of angles in quadrilateral $EFGH$.

(i) $\angle HEF + \angle FGH$ **(ii)** $\angle EFG + \angle GHE$

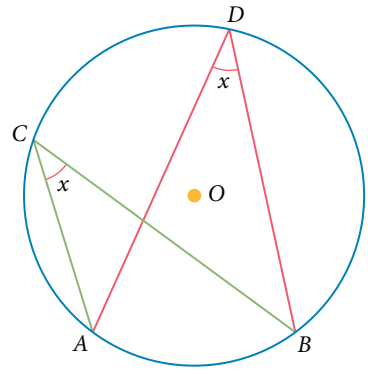
(d) Explain why Sonika has now finished her proof.

Circle theorems, part 2

4.4

In this section, more circle theorems will be examined. You will be provided with proofs and some applications of the theorems.

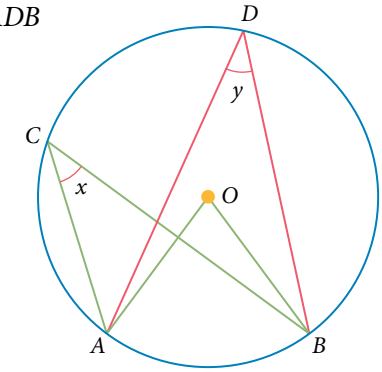
Angles at the circumference of a circle subtended by the same arc are equal



The diagram on the right shows this theorem $\angle ACB$ and $\angle ADB$ are subtended by the same arc and are equal.

This theorem is sometimes stated as, 'Angles in the same segment of a circle are equal'.

In this case, the converse of the theorem is not true. It is quite easy to draw angles at the circumference that are equal but are not subtended by the same arc.



To prove the theorem, see the diagram on the right, showing $\angle ACB = \angle ADB$ where the line segments OA and OB have been added.

Since the angle at the centre is twice the angle at the circumference subtended by the same arc:

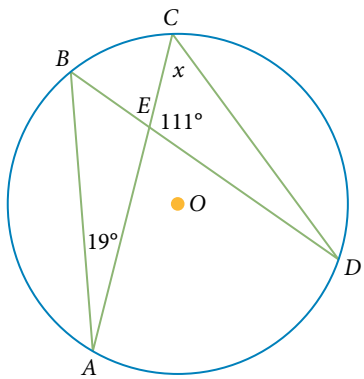
$\angle AOB = 2 \times \angle ACB$ and $\angle AOB = 2 \times \angle ADB$

Therefore, $\angle ACB = \angle ADB$.

Therefore, angles at the circumference of a circle subtended by the same arc are equal.

16 Calculating angles at the circumference subtended by the same arc

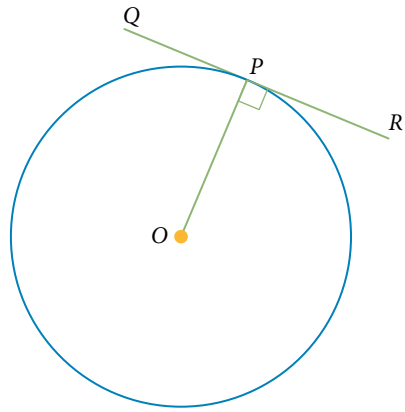
For the circle shown, calculate the value of x.



THINKING	WORKING
1 Determine a relationship between angles giving reasons.	Since angles at the circumference of a circle subtended by the same arc are equal: $\angle BDC = \angle BAC$ $\angle BDC = 19°$
2 Determine the required value.	In ΔECD, since angles in a triangle add up to 180° $\angle ECD + \angle CDE + \angle DEC = 180°$ $x + 19° + 111° = 180°$ $x + 130° = 180°$ $x = 50°$

A tangent drawn to a circle is perpendicular to the radius at the point of contact

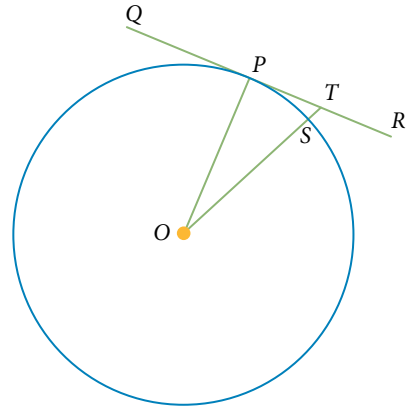
The diagram on the right shows this theorem. The tangent QR is perpendicular to the radius PO.



The converse of the theorem is that if a line intersects with the radius of a circle at right angles at the circumference then that line is a tangent.

The original theorem can be proved by the method of contradiction.

To prove: $\angle OPR = 90°$.



In the diagram to the right, assume OP is not perpendicular to QR. However, there must be some line OT that is perpendicular to QR. Assume $OT \perp QR$.

$\angle PTO = 90°$ and this must be the largest angle in ΔOPT.

Hence, $\angle PTO > \angle OPT$.

So, OP is the longest side of ΔOPT since it represents the hypotenuse, which is always opposite the right angle.

This means $|OP| > |OT|$.

But $|OP| = |OS|$ since they are both radii.

The original assumption is incorrect.

$\Rightarrow OP \perp QR$

A tangent drawn to a circle is perpendicular to the radius at the point of contact.

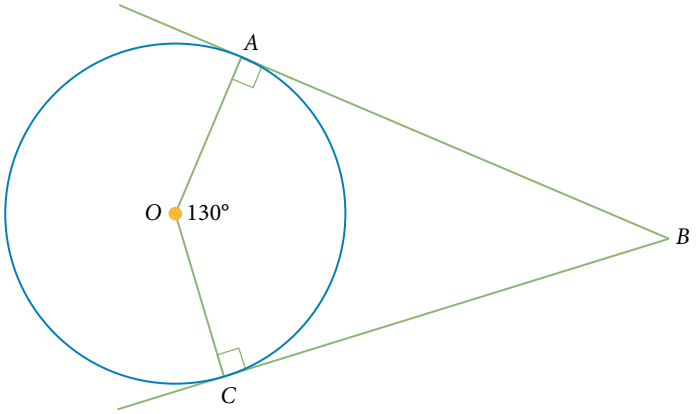
Making connections

Tangent of a circle

Move the slider to see that a tangent drawn to a circle is perpendicular to the radius at the point of contact.

17 Using the tangent perpendicular to the radius

In the following diagram, BA and BC are tangents to a circle with centre O. Calculate the size of $\angle ABC$.



THINKING	WORKING
1 Determine the sizes of known angles and a relationship between the angles.	Since a tangent meets the radius at right angles: $\angle BAO = 90°$ $\angle BCO = 90°$ $\angle AOC = 130°$ (given) Since angles in a quadrilateral add up to 360°: $\angle OAB + \angle ABC + \angle BCO + \angle COA = 360°$
2 Substitute known values and solve.	$90° + \angle ABC + 90° + 130° = 360°$ $\angle ABC + 310° = 360°$ $\angle ABC = 50°$

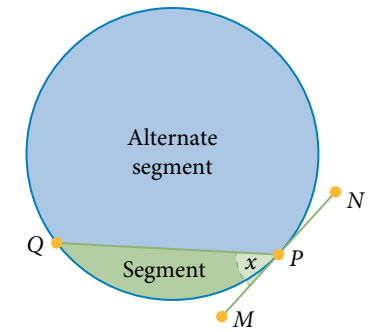
The angle between a tangent and a chord is equal to the angle in the alternate segment

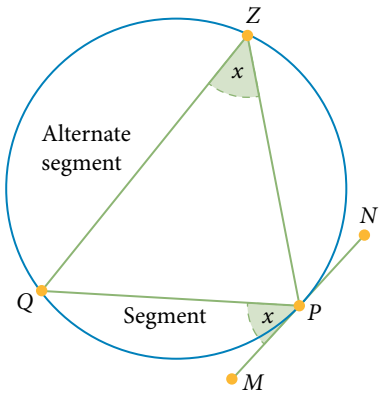
Consider the chord PQ and the tangent drawn at point P, as shown in the diagram.

The chord PQ cuts the circle into two segments: the segment at which the tangent forms $\angle QPM$ and the alternate segment.

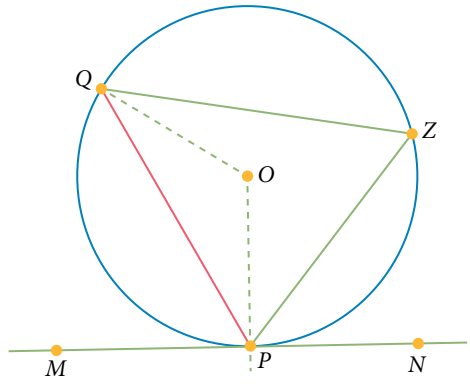
The alternate segment theorem states that the angle between a tangent and a chord, drawn from the point of contact, is equal to any angle formed in the alternate segment.

For any arbitrary point Z on the circumference, the alternate segment theorem states that the angle formed at $\angle MPQ = \angle PZQ$.





To prove the theorem, consider the following diagram to prove $\angle MPQ = \angle QZP$.



Construct two radii, OP and OQ.

$\angle OQP = \angle OPQ$ (Base angles of an isosceles triangle are equal.)

$2 \times \angle OPQ + \angle QOP = 180°$ (Angles in a triangle add up to 180°.)

$2 \times \angle OPQ = 180° - \angle QOP$

Since $\angle QOP = 2 \times \angle QZP$:

$2 \times \angle OPQ = 180° - 2 \times \angle QZP$

$\angle OPQ = 90° - \angle QZP$

Since $\angle OPQ = 90° - \angle MPQ$:

$90° - \angle MPQ = 90° - \angle QZP$

$\angle MPQ = \angle QZP$

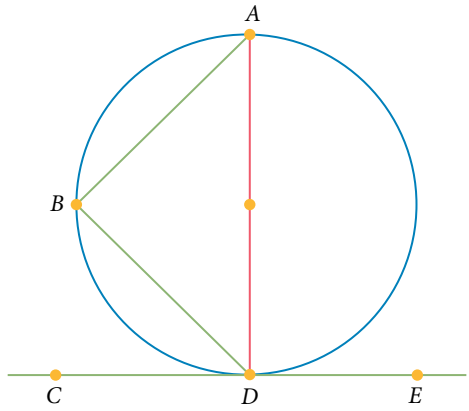
C Making connections

Alternate segment theorem

Move the sliders to see that the angle between a tangent and a chord, drawn from the point of contact, is equal to any angle in the alternate segment.

18 Using the alternate segment theorem

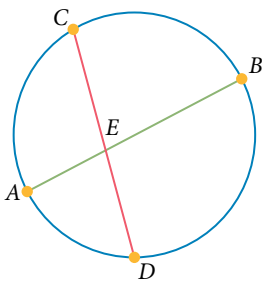
In the following diagram, AD is a diameter of the circle and CE is a tangent to the circle at the point D. Given that $\angle BDC = 31°$, calculate the magnitudes of $\angle BAD$ and $\angle ABD$.



THINKING	WORKING
1 Write a relationship between the given angle and the angle that is formed in the alternate segment.	$\angle BAD = \angle BDC$ $\angle BAD = 31°$
2 Calculate the other angle, giving a reason.	$\angle ABD = 90°$, since the angle in a semicircle is a right angle.

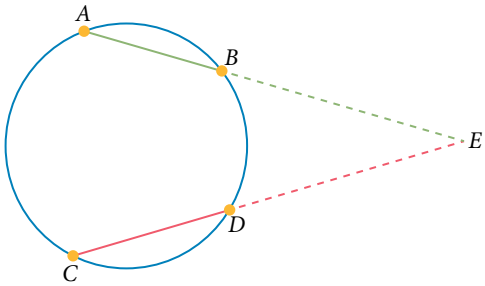
The product of the lengths of the intervals of one chord is equal to the product of the lengths of the intervals of the other chord

Internal intersection



$|EA| \times |EB| = |EC| \times |ED|$

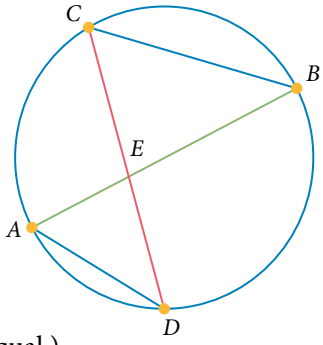
External intersection



$|EA| \times |EB| = |EC| \times |ED|$

This is known as the intersecting chords theorem. It does not matter whether the point of intersection of the two chords is inside or outside the circle, the theorem remains valid. A proof for each case is demonstrated separately.

Internal intersection



When the point of intersection is inside the circle, the diagram is as shown on the right.

To prove $|EA| \times |EB| = |EC| \times |ED|$, construct lines AD and BC.

Consider ΔCEB and ΔAED.

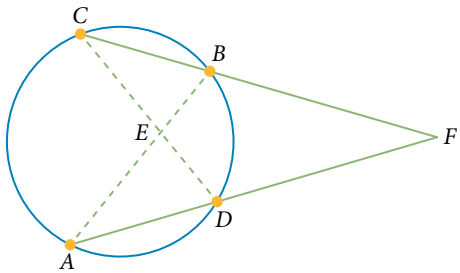
$\angle CEB = \angle AED$ (Vertically opposite angles are equal.)
$\angle BCE = \angle DAE$ (Angles at the circumference subtended by the same arc are equal.)
$\angle EBC = \angle EDA$ (If two corresponding angle pairs are equal, the third angle pair will also be equal.)
$\Delta CEB \sim \Delta AED$ (AAA)

Since corresponding side lengths are in the same ratio in similar triangles:

$$\frac{|EC|}{|EA|} = \frac{|EB|}{|ED|}$$
$$|EC| \times |ED| = |EB| \times |EA|$$

Then it's proved that $|EA| \times |EB| = |EC| \times |ED|$.

When the point of intersection is outside the circle, the diagram is as shown on the right.



To prove $|FC| \times |FB| = |FA| \times |FD|$, construct lines CD and AB.

Consider ΔCDF and ΔABF.

$\angle CFD = \angle AFB$ (They are common angles.)
$\angle FCD = \angle FAB$ (Angles at the circumference subtended by the same arc are equal.)
$\angle CDF = \angle ABF$ (third angle in the triangles)
$\therefore \Delta CDF \sim \Delta ABF$ (AAA)

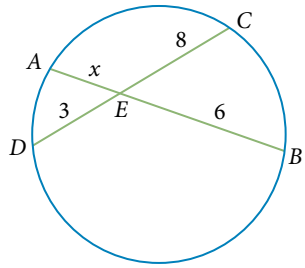
Since corresponding side lengths are in the same ratio in similar triangles:

$$\frac{|FC|}{|FA|} = \frac{|FD|}{|FB|}$$
$$|FC| \times |FB| = |FA| \times |FD|$$

19 Intersecting chords theorem

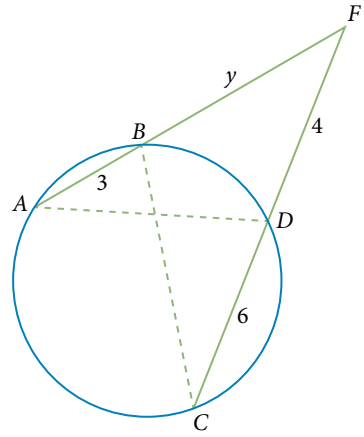
Consider the diagram.

(a) Determine the value of $x = |AE|$.



THINKING	WORKING
1 Determine a relationship using the intersecting chords theorem.	When two chords intersect, the product of the intervals of one chord equals the product of the intervals of the other: $\|EA\| \times \|EB\| = \|EC\| \times \|ED\|$
2 Substitute known values.	$x \times 6 = 8 \times 3$
3 Solve for the required value.	$6x = 24$ $x = \frac{24}{6}$ $\|AE\| = 4$

(b) Determine the value of $y = |BF|$.



1 Determine a relationship using the intersecting chords theorem.	When two chords intersect, the product of the intervals of one equals the product of the intervals of the other: $\|FA\| \times \|FB\| = \|FC\| \times \|FD\|$
2 Substitute known values.	$(y + 3) \times y = 10 \times 4$
3 Solve for the required value.	$y^2 + 3y - 40 = 0$ $(y + 8)(y - 5) = 0$ $\therefore y = 5 \ (y > 0)$

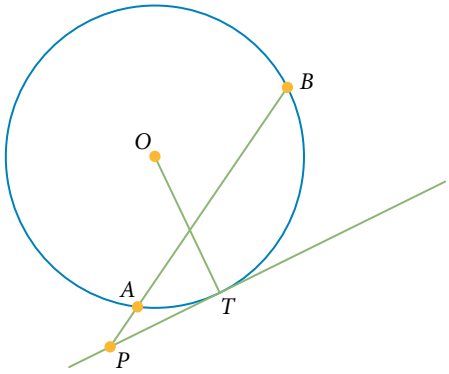
Making connections

Intersection of two chords of a circle

Move the sliders to observe the intersection of two chords of a circle, and how the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord.

The tangent-secant theorem

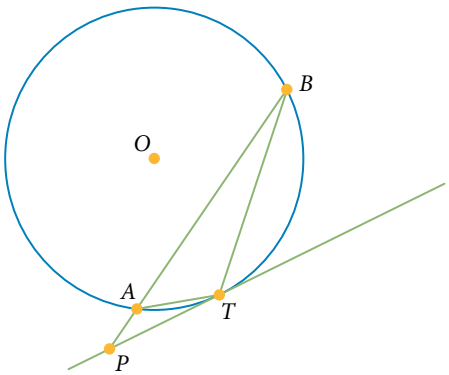
Recall that a secant is a line that passes through a circle, making contact at two distinct points. A chord is the part of a secant that lies on and inside the circle.



When a secant (meeting a circle at A and B) and a tangent (meeting the circle at T) are drawn to a circle from an external point P, the square of the length of the tangent equals the product of the lengths to the circle on the secant. For the top diagram on the right $|PT|^2 = |PA| \times |PB|$.

To prove the theorem, see the lower diagram.

To prove $|PT|^2 = |PA| \times |PB|$, construct TA and TB.



Consider ΔPAT and ΔPTB.

Since an angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment of the circle:

$\angle PTA = \angle PBT$

$\angle APT = \angle BPT$ (common)

$\angle PAT = \angle BTP$ (third angle in triangles)

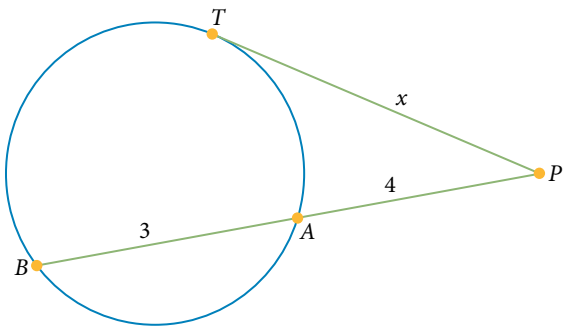
$\Delta PAT \sim \Delta PTB$ (AAA)

$\dfrac{|PT|}{|PB|} = \dfrac{|PA|}{|PT|}$ (The ratio of corresponding sides in similar triangles is equal.)

$|PT|^2 = |PA| \times |PB|$

20 Using the tangent-secant theorem

Consider the diagram, in which PT is a tangent to the circle. Calculate the exact value of x.



THINKING	WORKING
1 Write the relationship giving reasons.	When a secant from P crosses a circle at A and B and touches at T, then $\lvert PT\rvert^2 = \lvert PA\rvert \times \lvert PB\rvert$
2 Substitute values and solve for the unknown.	$x^2 = 4 \times 7$ $= 28$ $x = \sqrt{28}, \ x > 0$ $x = 2\sqrt{7}$

The theorems covered in this section are as follows:

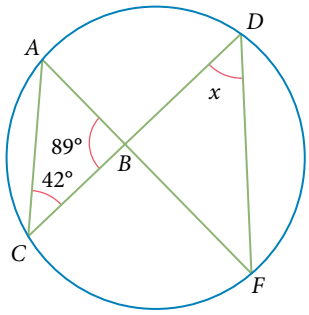
- Angles formed at the circumference, standing on the same arc, are equal.
- The tangent to a circle at any point is perpendicular to the radius at that point.
- The angle between a tangent to a circle and a chord drawn from the point of contact is equal to any angle drawn in the alternate segment of the circle.
- If two chords of a circle intersect, the product of the lengths of the intervals of one chord is equal to the product of the lengths of the intervals of the other; the point of intersection may be inside or outside the circle.
- If, from a point P outside a circle, a secant crosses the circle at A and B, and a tangent touches the circle at T, then $\lvert PT\rvert^2 = \lvert PA\rvert \times \lvert PB\rvert$.

EXERCISE 4.4 Circle theorems, part 2

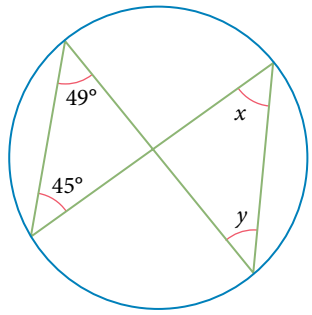
Worked Example 16

1 Determine the values of the pronumerals in the following diagrams.

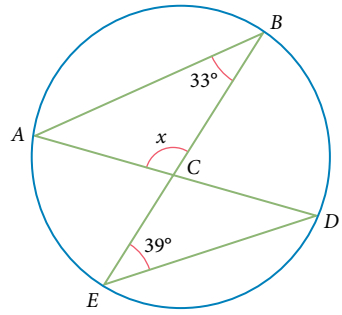
(a)



(b)

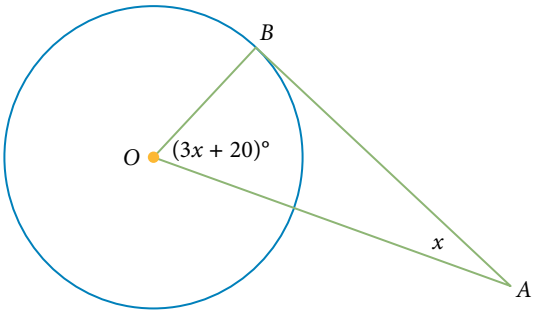


(c)

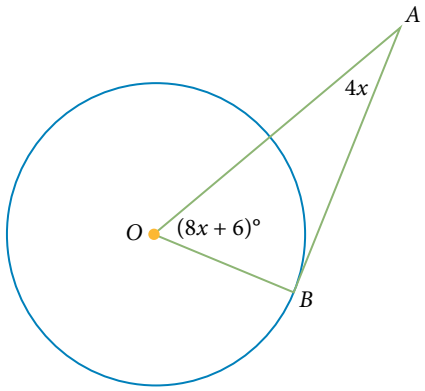


2 Determine the value of the pronumeral in the following diagrams. 17

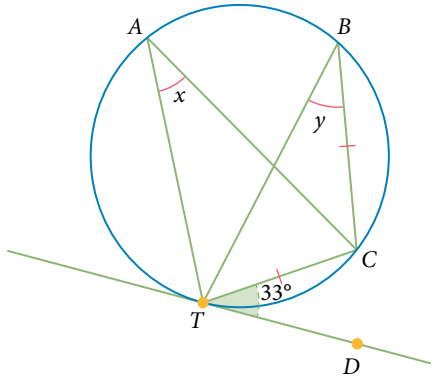
(a) OB is a radius of the circle and AB is a tangent to the circle at B.



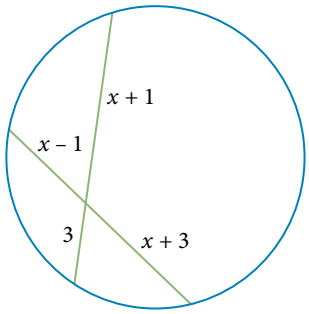
(b) O is the centre of the circle and AB is a tangent to the circle at B.



3 In the diagram, DT is a tangent to the circle at T. Calculate the values of x, y and $\angle BCT$. 18

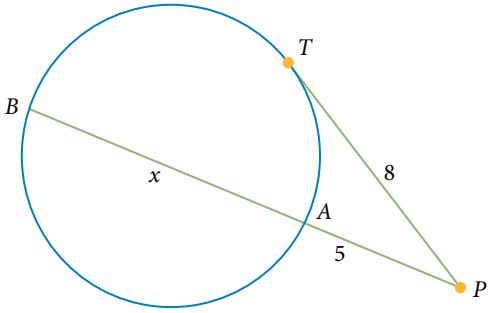


4 Calculate the value of x. 19

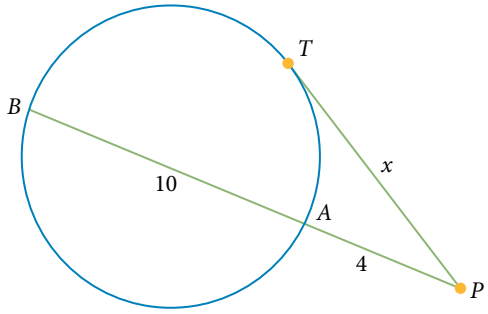


5 In the following diagrams, PT is a tangent to the circle, touching at T. Calculate the exact values of x. 20

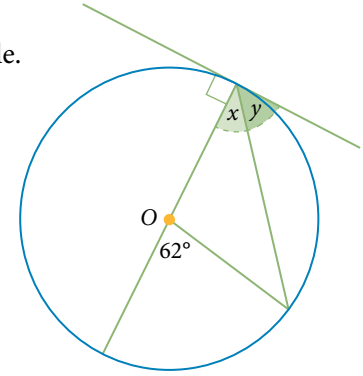
(a)



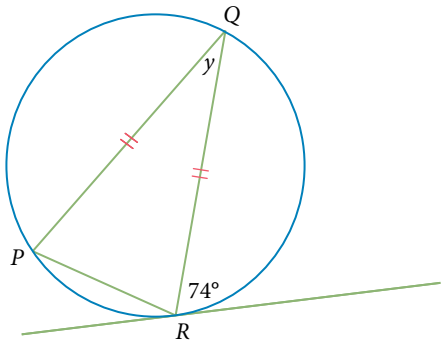
(b)



6 Calculate the values of x and y in the diagram.
The line shown outside the circle is a tangent to the circle.



7 Determine the value of y in the following diagram.



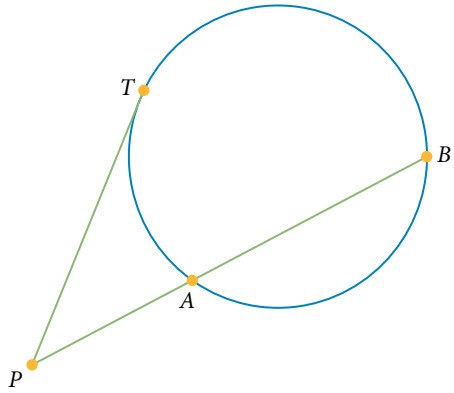
8 Consider the diagram to the right.

(a) If, from a point P outside a circle, a secant crosses the circle at A and B and a tangent touches the circle at T, then which of the following is true?

A $|PT| = |PA| + |PB|$ **B** $|PT| = |PA| \times |PB|$

C $|PB|^2 = |PT|^2 + |TB|^2$ **D** $|PT|^2 = |PA| \times |PB|$

(b) Explain the errors in the false statements.



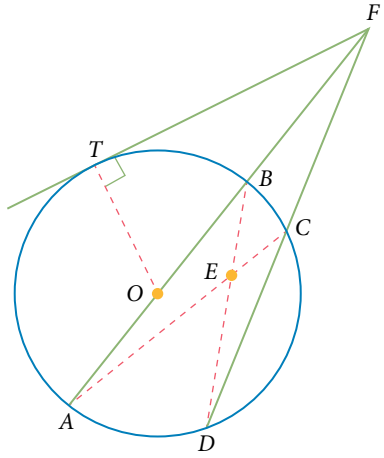
9 Investigate the diagram and identify which of the following statements is false.

A $|FA| \times |FB| = |FD| \times |FC|$

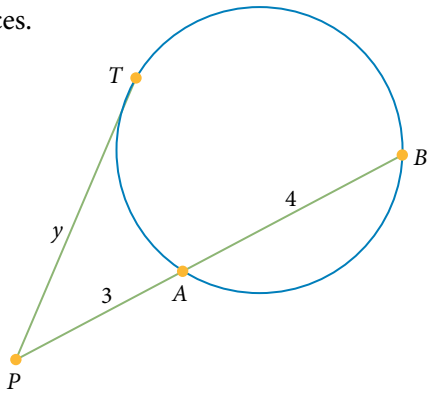
B $|TF|^2 = |FC| \times |CD|$

C $\angle ECD + \angle CDE + \angle DEC = \dfrac{\angle OTF}{2}$

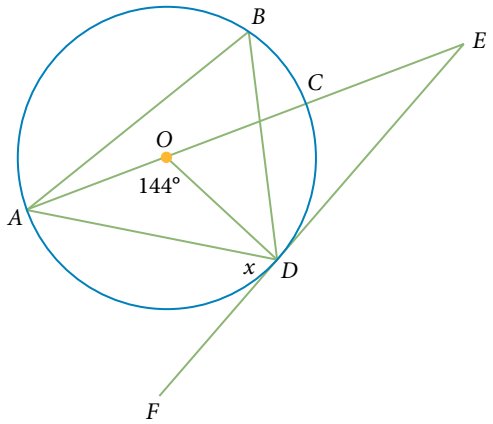
D $\Delta AEB \sim \Delta CED$



10 Determine the value of y, correct to 2 decimal places.

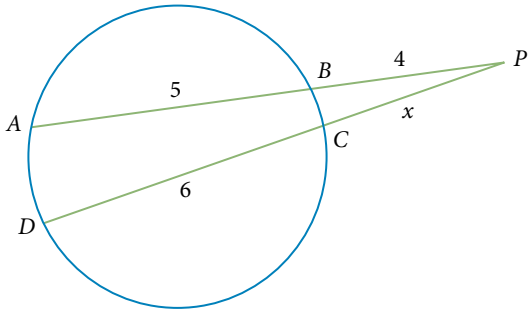


11 In the diagram, D is the point on the circumference where the tangent FE touches the circle. Calculate the value of x.

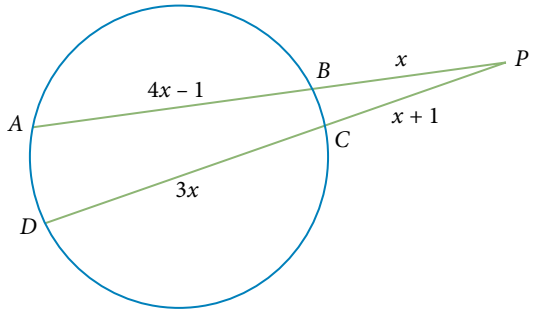


12 Determine the values of x correct to 2 decimal places.

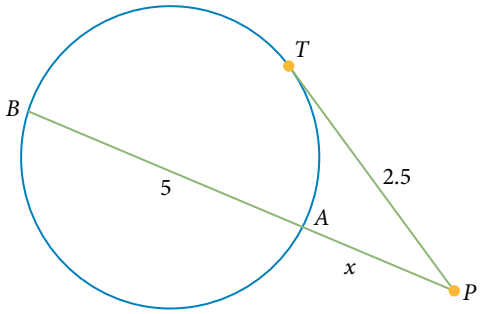
(a)



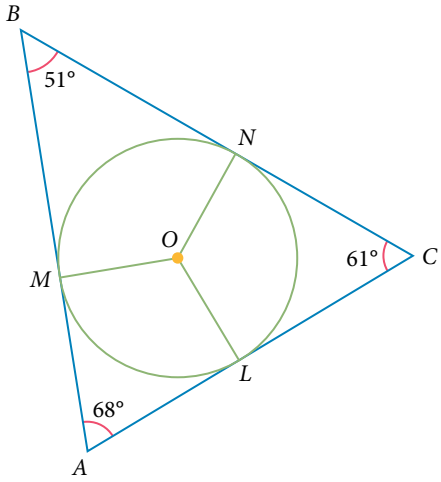
(b)



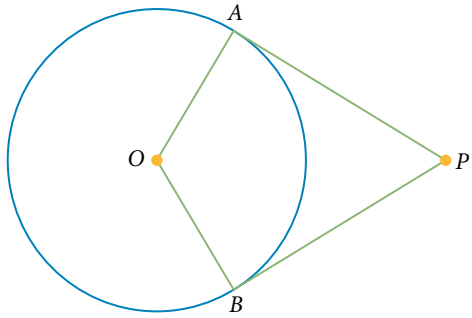
13 Determine the value of x correct to 2 decimal places.



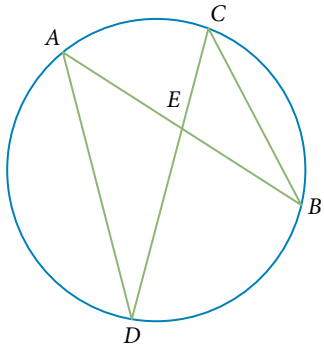
14 Consider the following diagram. Determine the three obtuse angles at the centre of the circle.



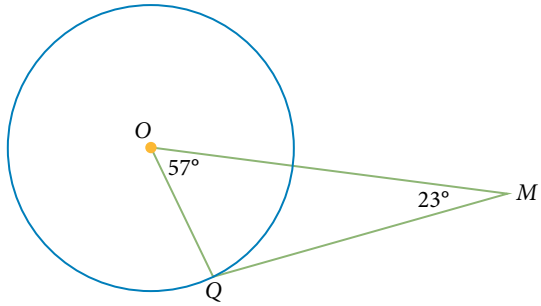
15 In the diagram below, PA and PB are both tangents to the circle, touching at A and B respectively. Prove that $|PA| = |PB|$. This is an illustration of the useful result that two tangents to a circle from a common point are equal in length.



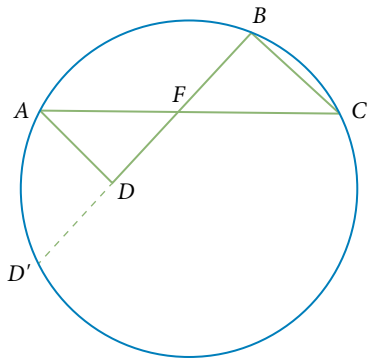
16 In the following diagram, verify that $\Delta AED \sim \Delta CEB$.



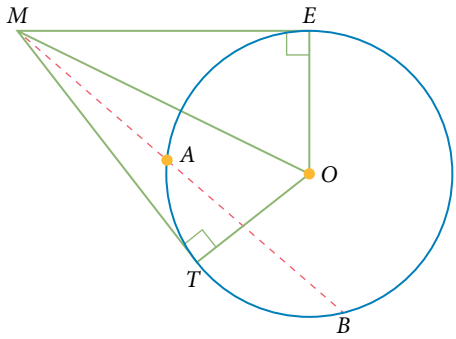
17 Sometimes, circle theorems can be used to prove something is not true. Lachlan drew a circle, with an external point M joined to a point Q on the circumference. Point O is the centre of the circle. Explain how you know MQ is not a tangent to the circle, touching at Q.



18 If line segments AC and BD intersect at F, and $|FC| \times |FA| = |FB| \times |FD|$, then use proof by contradiction to demonstrate that the four points lie on a common circle. (i.e. use the contradiction that points D and D' are the same point and $ABCD$ is a cyclic quadrilateral.) This is the converse of the product of lengths of chord segments theorem.



19 Using the diagram below, prove that if $|AM| \times |BM| = |TM|^2$, then TM is a tangent to the circle at the point T. This is the converse of the tangent-secant theorem.



4

Summary

Congruence and similarity

To prove that two triangles are congruent, it is necessary to demonstrate one of the following conditions:

Test for congruence		Description
SSS	Side, Side, Side	Three pairs of corresponding sides are equal.
SAS	Side, Angle, Side	Two pairs of corresponding sides are equal in magnitude and the included angles are also equal.
ASA	Angle, Side, Angle	Two pairs of angles are equal and one pair of corresponding sides are equal in magnitude.
RHS	Right Angle, Hypotenuse, Side	In a right-angled triangle, corresponding hypotenuses and another pair of corresponding sides are of equal magnitude.

When two triangles are congruent, it is written as $\Delta ABC \cong \Delta DEF$.

To prove that two triangles are similar it is necessary to demonstrate one of the following conditions:

All three pairs of corresponding angles are equal.	AAA
The ratio (scale factor) between all three pairs of corresponding sides is the same.	SSS
Two pairs of corresponding side lengths are in the same ratio and the angles between them are equal.	SAS
The hypotenuse and one other side of a right-angled triangle are in the same ratio to the hypotenuse and corresponding side of another right-angled triangle.	RHS

When two triangles are similar it is written as $\Delta ABC \sim \Delta DEF$.

Geometric proof

A geometric proof consists of a series of steps that lead to a conclusion. Various methods may be used including direct proof and proof by contradiction.

Circle theorems

A radius drawn perpendicular to a chord bisects that chord.

The angle in a semicircle is a right angle.

The angle at the centre subtended by an arc is twice the angle at the circumference subtended by the same arc.

The opposite angles in a cyclic quadrilateral are supplementary.

Chords that subtend equal angles at the centre of a circle are equal in length.

Angles formed at the circumference, subtended by the same arc, are equal.

The tangent to a circle at any point is perpendicular to the radius at that point.

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to any angle drawn in the alternate segment of the circle.

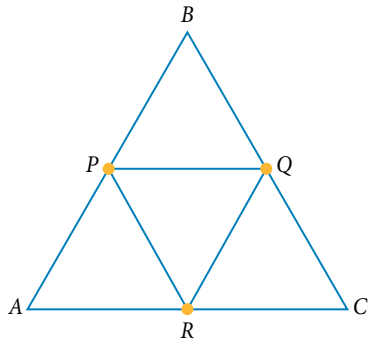
If two chords of a circle intersect, the product of the lengths of the intervals of one chord is equal to the product of the lengths of the intervals of the other; the point of intersection may be inside or outside the circle.

If, from a point P outside a circle, a secant crosses a circle at A and B, and a tangent touches the circle at T, then $|PT|^2 = |PA| \times |PB|$.

Chapter review 4

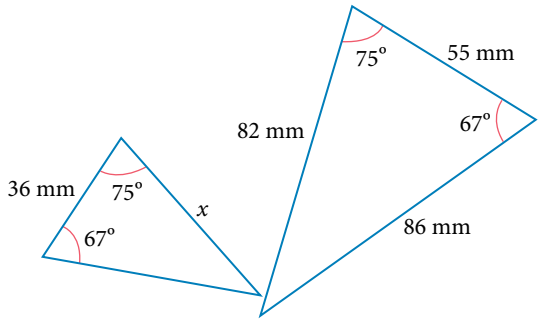
1 Consider the diagram, in which P, Q and R are the midpoints of the sides of equilateral triangle ABC. Name all the triangles that are congruent to ΔBPQ.

Exercise 4.1



2 Consider the diagram of a pair of triangles. Calculate the value of x.

Exercise 4.1

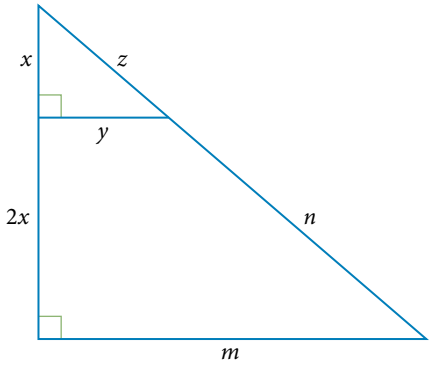


3 Consider the diagram. Two similar triangles are shown.

(a) What is the scale factor?

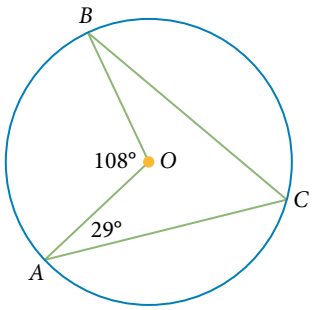
(b) Write two other expressions for the scale factor.

Exercise 4.1



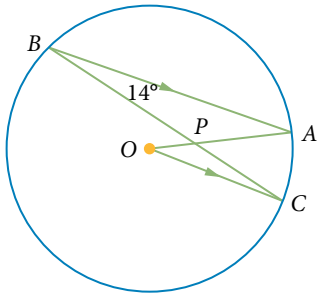
4 In the diagram, A, B and C are points on the circumference of a circle with centre O. Calculate $\angle OBC$.

Exercise 4.3



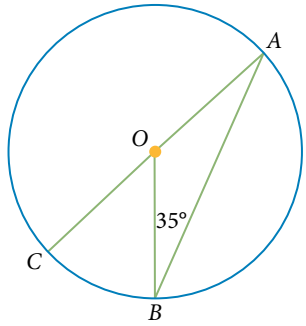
Exercise 4.3

5 Consider the diagram. Determine $\angle APC$.



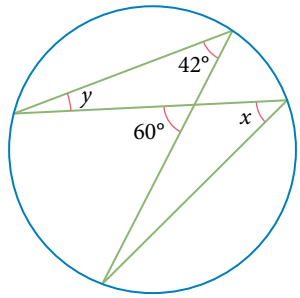
Exercise 4.3

6 Consider the diagram of a circle with centre O. Determine $\angle BOC$.



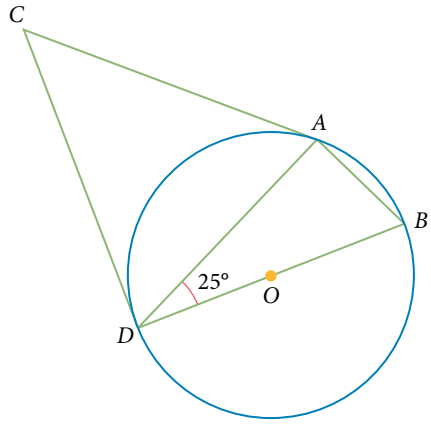
Exercise 4.4

7 In the diagram, calculate the values of x and y.



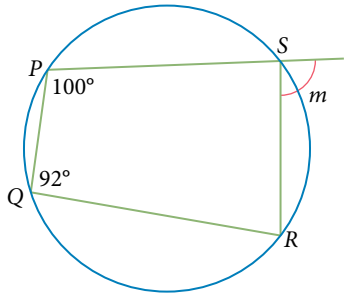
Exercise 4.4

8 Consider the diagram, in which AC and CD are tangents to the circle, touching at A and D respectively. Calculate $\angle ACD$.

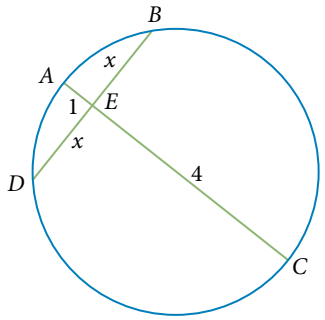


Exercise 4.3

9 Determine the value of m in the diagram below.



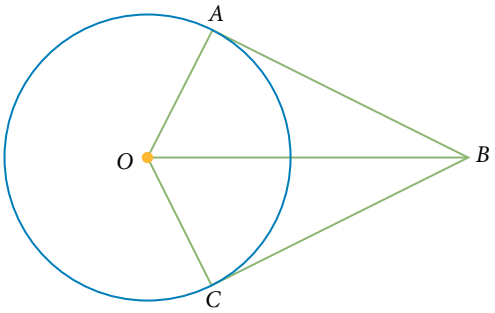
10 In the diagram, determine the value of x. Exercise 4.4



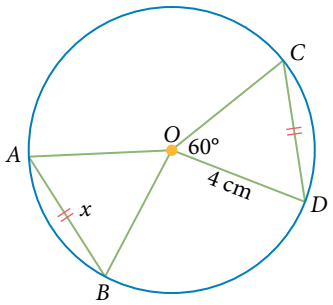
11 In the following diagram, BA and BC are tangents to a circle with centre O, touching at A and C respectively. Agnetha wrote the following statements regarding this diagram: Exercise 4.4

I $|AB| = |BC|$ II $\Delta ABO \cong \Delta CBO$ III $\angle ACB = 90°$

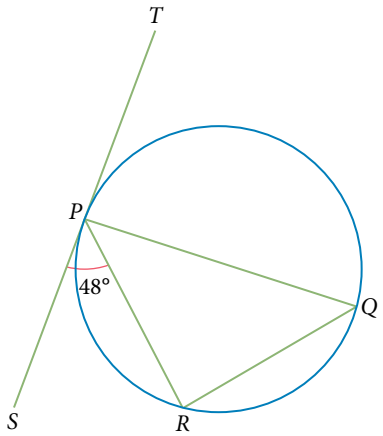
Which of Agnetha's statements must be true?
(There may be more than one.)



12 In the diagram, determine the length of the side marked x. Exercise 4.3



13 In the diagram, SPT is a tangent to the circle, touching at the point P. Determine $\angle PQR$. Exercise 4.4

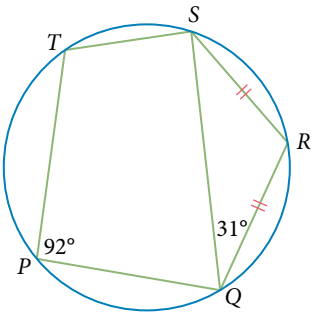


Exercise 4.3

14 Consider the diagram.

(a) Determine $\angle SRQ$.

(b) Determine $\angle TSQ$

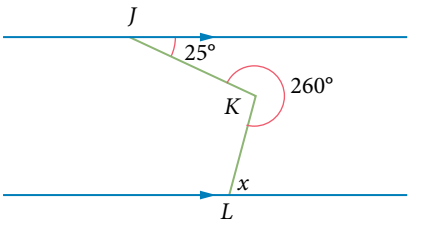


Exercise 4.2

15 Prove that if the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

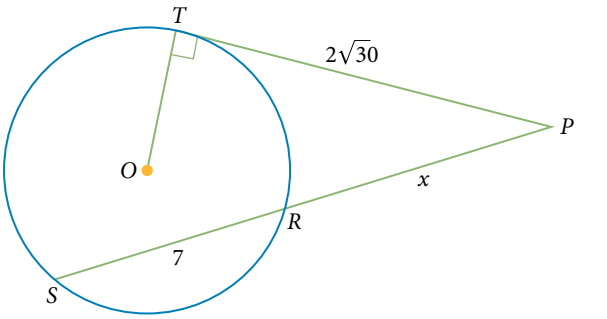
Exercise 4.2

16 In the diagram below, calculate the value of x.



Exercise 4.4

17 In the diagram, calculate the value of x.

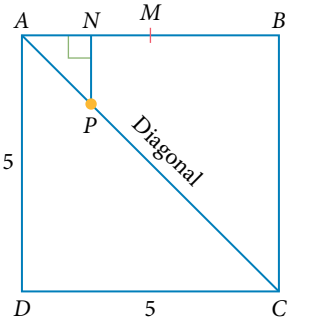


Exercise 4.1

18 A short cricket pitch is 21 m long. The stumps are 1 m high. A bowler, Tanvir, releases the ball from a height of 2 m (his height plus the length of his arm). Assuming that the ball hits the pitch and bounces off it at the same angle, what is the distance from the stumps that the ball should hit the pitch so that it just touches the top of the stumps after it bounces?

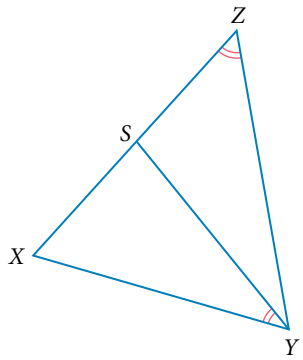
Exercise 4.1

19 In the diagram, $ABCD$ is a square of side length 5 cm. M is the midpoint of AB, and N is the midpoint of AM. From N a line parallel to BC is drawn until it meets the diagonal AC at P. Determine $|AP|$.



20 ΔXYZ is such that $\angle XZY$ is an acute angle. The line segment YS is drawn to meet XZ at the point S so that $\angle XYS = \angle XZY$.
Prove that $\Delta XYS \sim \Delta XZY$.

Exercise 4.2

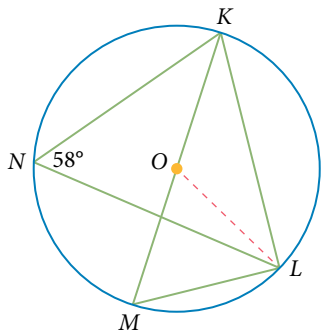


21 Use proof by contradiction to prove that the bisector of the angle at the apex of an isosceles triangle meets the third side at right angles.

Exercise 4.2

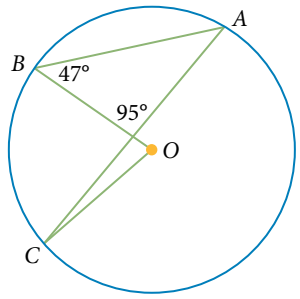
22 For this diagram, determine the angles $\angle KLM$, $\angle KML$ and $\angle KOL$.

Exercise 4.3



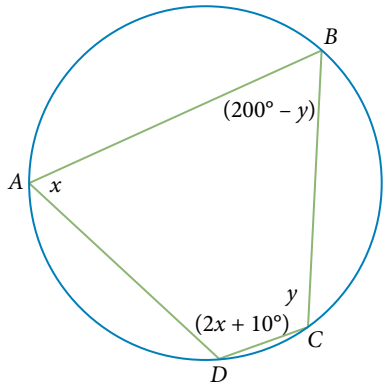
23 For this diagram, determine $\angle ACO$.

Exercise 4.3



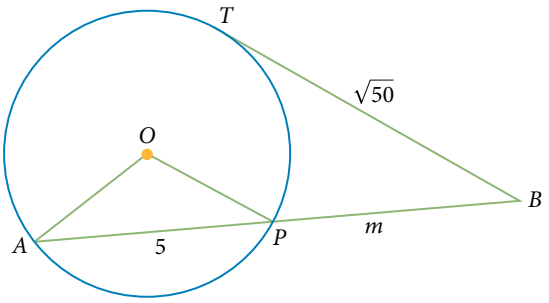
24 For this diagram, calculate the values of x and y.

Exercise 4.3



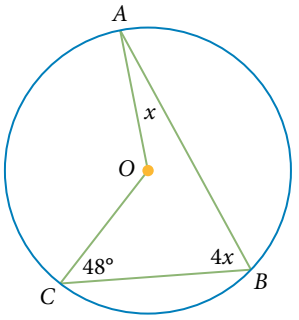
Exercise 4.4

25 For the diagram, determine the value of m.



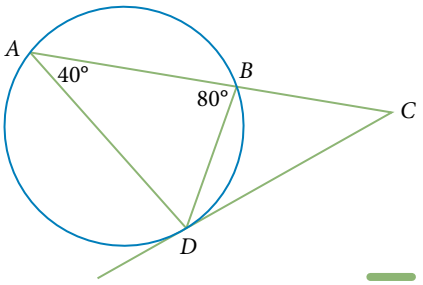
Exercise 4.3

26 In the diagram, O is the centre of the circle. Determine the value of x.



Exercise 4.4

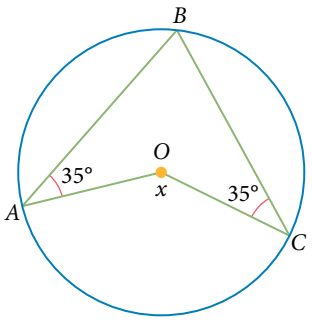
27 In the diagram, CD is a tangent to the circle at point D. Demonstrate that ΔBDC is an isosceles triangle.



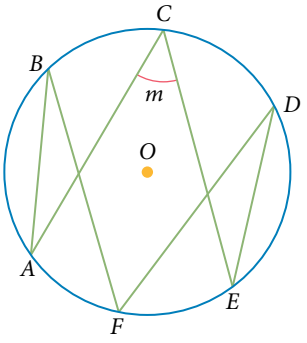
Exercise 4.3

28 Consider each of the following diagrams.

(a) Determine the value of x, leaving your answer correct to the nearest whole degree.

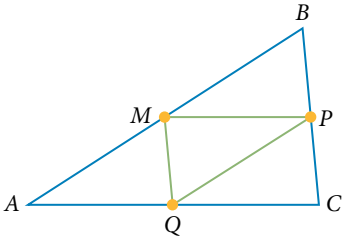


(b) If $\angle ABF = 40°$ and $\angle EDF = 35°$, determine the value of m.



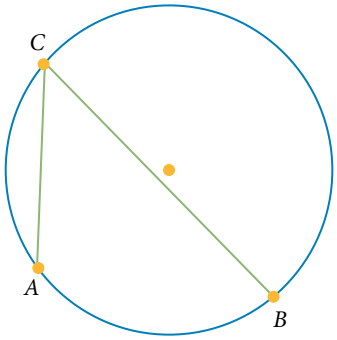
Exercise 4.1

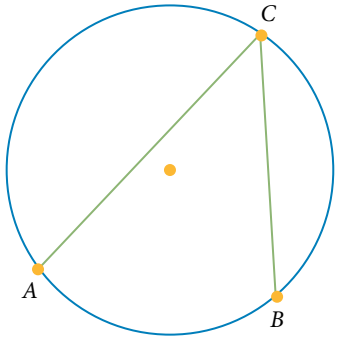
29 Let ABC be any triangle and M, P, Q be the midpoints of the three sides. Prove that $\Delta ABC \sim \Delta PQM$.

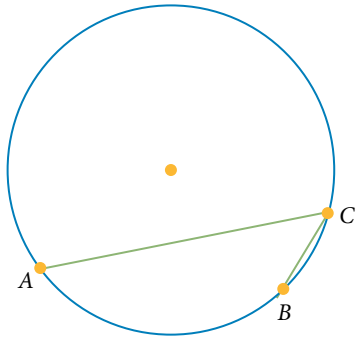


30 Let A, B and C be three points on the circumference of a circle where C is between points A and B. Three different examples of what this could look like are shown in the diagrams. Explain why the magnitude of $\angle ACB$ is the same in each diagram.

Exercise 4.4

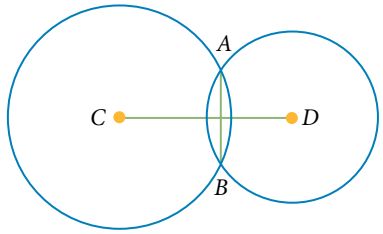






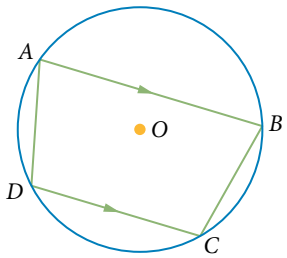
31 Consider the diagram of two circles with centres C and D that intersect at two points A and B. Prove that CD is the perpendicular bisector of AB.

Exercise 4.2



32 Prove that if two sides of a cyclic quadrilateral are parallel then the other two side lengths must be equal.

Exercise 4.3



Exam review

UNIT 1

Paper 1: Technology-free

Exercise 1.1

1 The Year 11 cohort at a particular school contains 100 students; 30 study Art, 50 study Biology, and 5 study both of these subjects. Calculate the number of students who study neither.

Exercise 1.2

2 A phone passcode consists of 4 digits. How many passcodes are possible that end in an odd digit?

Exercise 1.4

3 At a meeting of 10 individuals, every person shakes hands with each other. Calculate the total number of handshakes that take place.

Exercise 1.6

4 Prove that the difference between the entries in row $n + 2$, position 3, and row n, position 3, of Pascal's triangle is equal to n^2.

Exercise 2.2

5 Draw the following vectors on the same Cartesian plane.

(a) $\boldsymbol{a}$, the position vector of (3, 5)

(b) $\overrightarrow{OB}$, where B is (-2, 4)

(c) $\boldsymbol{c}$, the position vector of $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$

(d) $\overrightarrow{OD}$, where D is $\begin{bmatrix} -5 \\ -3 \end{bmatrix}$

Exercise 3.1

6 Write each of the following numbers in the form $\frac{p}{q}$, where p and $q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.

(a) $0.\dot{4}$ **(b)** $0.\overline{23}$ **(c)** $0.1\overline{24}$

Exercise 3.2

7 Write the converse, the contrapositive and the negation of each of the following conditional statements. Determine whether each of the original statements, converse, contrapositive and negation is true or false, justifying your answer where appropriate.

(a) If n is divisible by 3, then n is divisible by 6.

(b) If $x < 0$ or $x > 5$, then $x^2 > 25$.

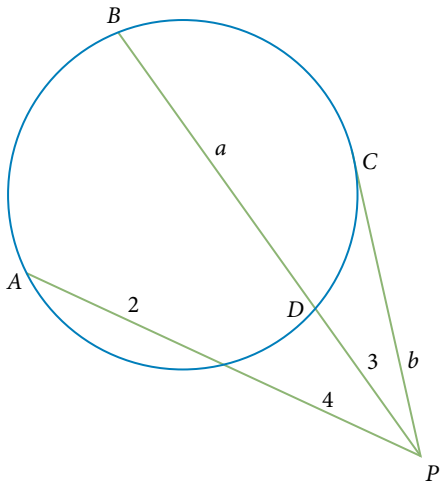
Exercise 3.3

8 Prove that if r is rational and rx is irrational, then x must be irrational.

Exercise 3.3

9 The alternating digit sum of a number is the value obtained by alternately adding and subtracting the digits of a number. For example, the alternating digit sum of 48 716 is $4 - 8 + 7 - 1 + 6 = 8$. Prove that if the alternating digit sum of a three-digit number is divisible by 11, then the number must be divisible by 11.

10 In the following diagram, PC is a tangent to the circle. Calculate the lengths marked a and b. Exercise 4.4



11 Let k be a positive integer. Prove that if $5^{k+1} + 2 \times 3^k + 1$ is divisible by 8, then the number $5^{k+2} + 2 \times 3^{k+1} + 1$ is also divisible by 8. Exercise 3.3

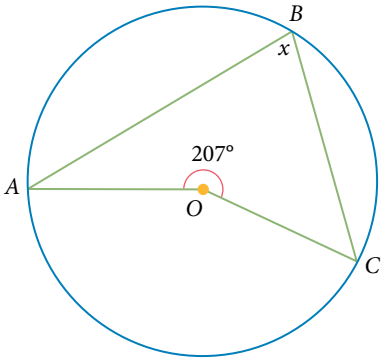
Paper 2: Technology-active

Exercise 2.7

1 Angus can swim in still water at a rate of 8.0 km h^{-1}. Calculate his speed with respect to the river bank if he swims in a river flowing at 6.0 km h^{-1} and keeps his direction (with respect to the water) perpendicular to the current.

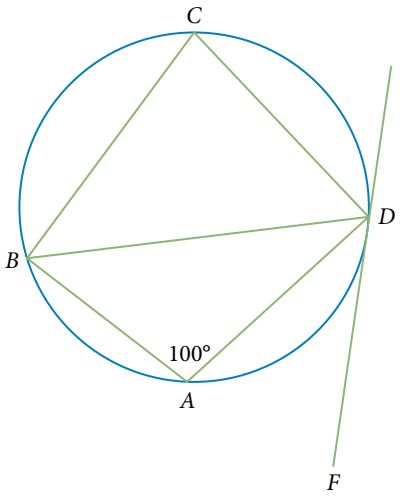
Exercise 4.3

2 Calculate the value of x.



Exercise 4.4

3 Consider the following diagram. Calculate the magnitudes of $\angle BCD$, $\angle DBC$ and $\angle BDF$.



Exercise 2.3

4 Consider the vector $\boldsymbol{a} = 7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$.

(a) Determine the unit vector $\hat{\boldsymbol{a}}$.

(b) Determine vector $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ with a magnitude of 10.

Exercise 2.3

5 Calculate all the values of v and w if $\left(v^2 + 5\right)\hat{\boldsymbol{i}} + \left(w^2 - 7w + 4\right)\hat{\boldsymbol{j}} = 7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

Exercise 2.4

6 Calculate the angle, in degrees correct to 2 decimal places, between vectors $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -5\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}}$.

7 Consider the vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$. Exercise 2.5

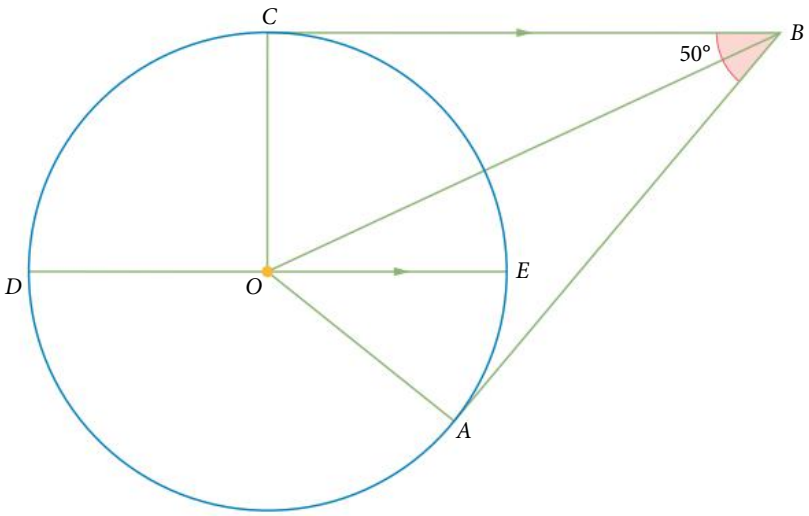
(a) Determine the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(b) Determine the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

8 A particle of mass 15 kg is suspended by two strings attached to two points in the same horizontal plane. The two strings make angles of 22° and 38° respectively to the horizontal. Draw a force diagram and use component form to calculate the magnitude of the tension in each string, in newtons, correct to 2 decimal places. Exercise 2.8

9 A bag contains 100 balls, numbered 1 to 100. A simple game is played whereby a player selects k balls from the bag at once, and wins the game if either the number 1 or the number 100 is one of the balls selected. What is the least value of k such that the probability of winning the game is more than 0.5? Evaluate the reasonableness of your solution. Exercise 1.5

10 In the following diagram, $CB \parallel DE$, O is the centre of the circle and BC and BA are tangents to the circle at the points C and A respectively. Calculate $\angle AOE$. Exercise 4.4







5

Real and complex numbers

Recall 240

5.1 Complex numbers 241

5.2 The complex plane 249

5.3 Radian measure 259

5.4 Complex arithmetic using polar form 263

5.5 Complex solutions of real quadratic functions 271

Summary 277

Chapter review 280

5 Recall

Expand binomial expressions

1 Expand each of the following expressions.

(a) $(x+4)(x-2)$ (b) $(x-8)^2$ (c) $(2x-5y)^2$

(d) $(2a^2b+ab^3)(5ab-2a)$ (e) $(2x+3a)(2x-3a)$ (f) $(x-5)^3$

Simplify surds

2 Simplify the following surds by expressing in the form $a\sqrt{b}$, where a and b are positive integers.

(a) $\sqrt{50}$ (b) $7\sqrt{12}$ (c) $5\sqrt{63}$ (d) $2\sqrt{20}$

Rationalise the denominator

3 Express the following expressions using rational denominators.

(a) $\dfrac{1}{3\sqrt{2}}$ (b) $\dfrac{1}{5-\sqrt{3}}$ (c) $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{7}+\sqrt{5}}$

Calculate distance between two points on the Cartesian plane

4 Calculate the exact distance between the following pairs of coordinate points.

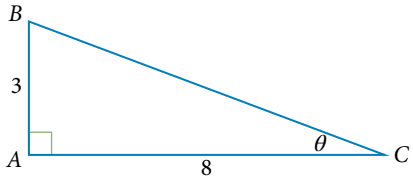
(a) $A(1, 2)$ and $B(8, 5)$ (b) $C(0, -5)$ and $D(6, 1)$

(c) $E(10, -7)$ and $F(9, 3)$ (d) $G(-3, -5)$ and $H(-8, -1)$

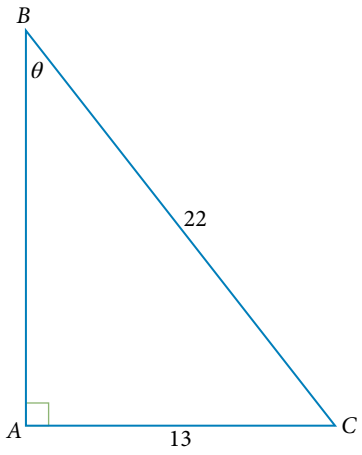
Calculate angles in right-angle triangles

5 Calculate the unknown angle, θ, in each of the following triangles, correct to 2 decimal places where necessary.

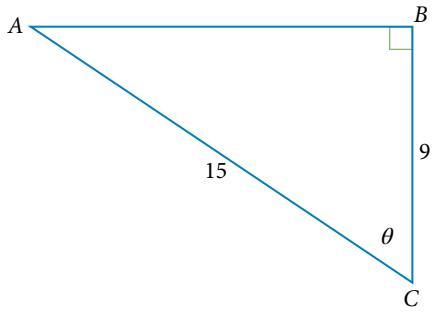
(a)



(b)



(c)



Solve quadratic equations using the quadratic formula

6 Use the quadratic formula to calculate the real solutions, if they exist, to the following equations:

(a) $x^2+x-7=0$ (b) $2x^2-3x-4=0$ (c) $x^2=8x+10$ (d) $2x^2+5x+9=0$

Complex numbers

5.1

When solving an equation for an unknown value, you are trying to determine which numbers can be substituted in place of that value to *satisfy* the equation. For example, consider the equation $x^2 = 4$. One solution to this equation is $x = 2$, since $(2)^2 = 4$. Another solution is $x = -2$, because $(-2)^2 = 4$.

However, consider the equation $x^2 = -4$. This simple-looking equation cannot be solved using the *real number system*, because there is no real number that will multiply by itself to give a negative value. To overcome this problem, mathematicians introduced a new type of number called the *imaginary number*. The imaginary number, denoted i, was defined such that $i^2 = -1$. Imaginary numbers can therefore be used to solve equations that are impossible to solve using real numbers alone.

At first, mathematicians were unsure as to whether there was any practical application for imaginary numbers. However, it turns out that imaginary numbers have proved to be highly useful across a range of very practical applications in engineering and mathematical modelling.

Returning to the example above, $x^2 = -4$ can be rewritten as $x^2 = 4 \times -1$. Using the definition of imaginary numbers, you can substitute $i^2 = -1$ into the equation so that $x^2 = 4i^2$. This equation can now be solved to give $x = \pm 2i$.

Definition of a complex number

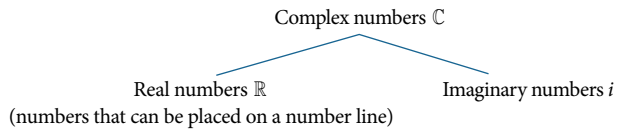
When you are using mathematics that involves the use of both real and imaginary numbers, you are said to be using the *complex number system*, which is denoted by $\mathbb{C}$. A complex number (often denoted by the letter z) consists of a real number added to a multiple of the imaginary number, i. A complex number therefore has the form $x + yi$, where x and y are real numbers.

A complex number consists of two parts, a *real part* and an *imaginary part*, and is defined as:

$\mathbb{C} = \{z : z = x + iy, \text{ where } x, y \in \mathbb{R}, \text{ and } i^2 = -1\}$

The real component of z is $\text{Re}(z) = x$.

The imaginary component of z is $\text{Im}(z) = y$.



1 Definition of a complex number

Determine the real and imaginary components of the following complex numbers.

(a) $z = 5 - 2i$

THINKING	WORKING
1 The real component is the part 'without the i'.	$\text{Re}(z) = 5$
2 The imaginary component is the coefficient of the i term.	$\text{Im}(z) = -2$

(b) $z = 8i$

1 The real component is the part 'without the i'.	$\text{Re}(z) = 0$
2 The imaginary component is the coefficient of the i term.	$\text{Im}(z) = 8$

2 Using imaginary numbers

Use imaginary numbers to write a simplified expression for the following:

(a) $\sqrt{-36}$

THINKING	WORKING
1 Express the square root as a product of a positive and a negative.	$\sqrt{-36} = \sqrt{36} \times \sqrt{-1}$
	$= \sqrt{6^2} \times \sqrt{i^2}$
2 Substitute for $i = \sqrt{-1}$.	$= 6i$

(b) $\sqrt{-7}$

1 Express the square root as a product of a positive and a negative.	$\sqrt{-7} = \sqrt{7} \times \sqrt{-1}$
	$= \sqrt{7} \times \sqrt{i^2}$
2 Substitute for $i = \sqrt{-1}$.	$= \sqrt{7}i$ or $i\sqrt{7}$
3 For clarity, when the imaginary component of a complex number is a surd, write the surd after the i.	$= i\sqrt{7}$

Addition and subtraction of complex numbers

To add two or more complex numbers, add the real components and add the imaginary components. For subtraction, subtract the real components and subtract the imaginary components.

3 Addition and subtraction of complex numbers

For $z_1 = 2 + 3i$ and $z_2 = 4 - 6i$, calculate the following:

(a) $z_1 + z_2$

THINKING	WORKING
Substitute the complex numbers into the expression and simplify.	$z_1 + z_2 = (2 + 3i) + (4 - 6i)$
	$= (2 + 4) + (3 - 6)i$
	$= 6 - 3i$

(b) $z_1 - z_2$

Substitute the complex numbers into the expression and simplify.	$z_1 - z_2 = (2 + 3i) - (4 - 6i)$
	$= (2 - 4) + (3 + 6)i$
	$= -2 + 9i$

Multiplication by a scalar

To multiply a complex number by a constant (or scalar), multiply the real and imaginary components by that scalar.

If $z = a + bi$ and if $k \in \mathbb{R}$, then $kz = k(a + bi) = ka + kbi$.

4 Multiplication by a scalar

For $w = 3 - 4i$ and $z = 5 + 2i$, calculate the following:

(a) $2w + z$

THINKING	WORKING
1 Substitute expressions for w and z.	$2w + z = 2(3 - 4i) + (5 + 2i)$
2 Simplify the expression by expanding the brackets and collecting like terms.	$2w + z = 6 - 8i + 5 + 2i$ $= 11 - 6i$

(b) $4w - 3z$

1 Substitute expressions for w and z.	$4w - 3z = 4(3 - 4i) - 3(5 + 2i)$
2 Simplify the expression by expanding the brackets and collecting like terms.	$4w - 3z = 12 - 16i - 15 - 6i$ $= -3 - 22i$

Multiplication of complex numbers

To multiply two complex numbers, use the distributive law to expand, and wherever an i^2 results, use $i^2 = -1$ to replace the i^2 with -1. For example: $3i^2 = 3 \times -1 = -3$.

5 Multiplication of complex numbers

For $z_1 = 2 + 3i$ and $z_2 = 4 - 2i$, calculate $z_1 \times z_2$.

THINKING	WORKING
1 Use the distributive law to expand the brackets.	$z_1 \times z_2 = (2 + 3i)(4 - 2i)$ $= 2(4 - 2i) + 3i(4 - 2i)$ $= 8 - 4i + 12i - 6i^2$
2 Simplify, by adding like terms.	$= 8 + 8i - 6i^2$
3 Recall the definition $i^2 = -1$, substituting it into the equation.	$= 8 + 8i - 6(-1)$ $= 8 + 8i + 6$
4 Write the solution.	$z_1 \times z_2 = 14 + 8i$

Technology worked example

Complex number arithmetic in Cartesian form

Equality of complex numbers

For two complex numbers to be equal, the real components must be equal and the imaginary components must be equal. Therefore, if $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 = z_2$ if and only if $a = c$ and $b = d$.

6 Equality of complex numbers

Determine the values of a and b that satisfy the equation $(3 - 5i)(a + bi) = 26 + 2i$.

THINKING	WORKING
1 Use the distributive law to expand the brackets.	$(3 - 5i)(a + bi) = (26 + 2i)$ $3a + 3bi - 5ai - 5bi^2 = 26 + 2i$
2 Simplify the expression by substituting $i^2 = -1$.	$3a + 3bi - 5ai - 5b \times (-1) = 26 + 2i$ $3a + 3bi - 5ai + 5b = 26 + 2i$
3 Group and equate the real components and the imaginary components.	$(3a + 5b) + (3bi - 5ai) = 26 + 2i$ $(3a + 5b) + i(3b - 5a) = 26 + 2i$ $3a + 5b = 26 \quad [1]$ $3b - 5a = 2 \quad [2]$
4 Solve simultaneously to calculate a and b.	$5 \times [1]$ $15a + 25b = 130 \quad [3]$ $3 \times [2]$ $-15a + 9b = 6 \quad [4]$ $[3] + [4]$ $34b = 136$ $b = 4$ Substitute $b = 4$ into [1]: $3a + 5 \times (4) = 26$ $3a + 20 = 26$ $3a = 6$ $a = 2$
5 Answer the question.	$a = 2, b = 4$

Complex conjugates

If, for example, $z = 3 + 4i$, then its complex conjugate is $\overline{z} = 3 - 4i$.

z and $\overline{z}$ are called a *conjugate pair*.

For any complex number $z = x + iy$, its complex conjugate, $\overline{z}$ (read: z-bar) is defined as $\overline{z} = x - iy$.

An important property of a conjugate pair is:

$$\begin{aligned} z.\overline{z} &= (x + yi)(x - yi) \\ &= x^2 - y^2 i^2 \\ &= x^2 + y^2 \end{aligned}$$

The product of a conjugate pair is always a real number. Note that real numbers are a subset of the set of complex numbers (meaning they are contained within that set), so the product is still actually a complex number, but one where the imaginary part is zero.

7 Complex conjugates

Consider the complex numbers $a = -2 + 3i$, and $b = 3 + 4i$. Demonstrate that $\overline{a + b} = \overline{a} + \overline{b}$.

THINKING	WORKING
1 Determine $a + b$.	$a + b = (-2 + 3i) + (3 + 4i)$ $= 1 + 7i$
2 Calculate $\overline{a + b}$, the conjugate of $a + b$.	LHS $= \overline{a + b}$ $= 1 - 7i$
3 Determine $\overline{a}$ and $\overline{b}$.	$\overline{a} = -2 - 3i$, $\overline{b} = 3 - 4i$
4 Calculate $\overline{a} + \overline{b}$.	RHS $= \overline{a} + \overline{b}$ $= (-2 - 3i) + (3 - 4i)$ $= 1 - 7i$
5 Write your answer.	Since LHS = RHS, $\overline{a + b} = \overline{a} + \overline{b}$.

Multiplicative inverse of a complex number

Consider the complex number $z = a + bi$. You can determine z^{-1}, the multiplicative inverse of z, such that $z \cdot z^{-1} = 1$. Multiply the numerator and denominator of the fraction by the conjugate, $a - bi$, to rationalise the denominator. Note that this will involve a difference of two squares expansion.

$$\frac{1}{z} = \frac{1}{a + bi} \times \frac{a - bi}{a - bi}$$
$$= \frac{a - bi}{a^2 - b^2 i^2}$$
$$= \frac{a - bi}{a^2 + b^2}$$

If $z = a + bi$, then the multiplicative inverse is $z^{-1} = \frac{a - bi}{a^2 + b^2}$.

8 Multiplicative inverse of a complex number

If $z = 5 - 3i$, determine the multiplicative inverse, z^{-1}. Hence, verify that $z \cdot z^{-1} = 1$.

THINKING	WORKING
1 Determine the multiplicative inverse, z^{-1}, by using the complex conjugate.	$z^{-1} = \frac{1}{5 - 3i} \times \frac{5 + 3i}{5 + 3i}$ $= \frac{5 + 3i}{25 - 9i^2}$ $= \frac{5 + 3i}{25 + 9}$ $= \frac{5}{34} + \frac{3i}{34}$

2 Verify that $z.z^{-1} = 1$.

$$\begin{aligned} z.z^{-1} &= (5-3i)\times\left(\frac{5}{34}+\frac{3i}{34}\right) \\ &= 5\left(\frac{5}{34}+\frac{3i}{34}\right)-3i\left(\frac{5}{34}+\frac{3i}{34}\right) \\ &= \frac{25}{34}+\frac{15i}{34}-\frac{15i}{34}-\frac{9i^2}{34} \\ &= \frac{25}{34}+\frac{9}{34} \\ &= \frac{34}{34} \\ &= 1 \end{aligned}$$

Division of complex numbers

When dividing complex numbers, you can rationalise the denominator in your answer by multiplying the numerator and denominator by the complex conjugate of the divisor.

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\times\frac{c-di}{c-di} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \end{aligned}$$

Technology worked example
Division using complex conjugates

Since $(c+di)(c-di) = c^2+d^2$, you will obtain a denominator that is a real number. This process can also be considered as division of complex numbers.

9 Division of complex numbers

If $w = 3+4i$ and $z = 1+2i$, calculate $\frac{w}{z}$. Express your answer in the form $x+yi$.

THINKING	WORKING
1 Determine the complex conjugate of the denominator.	$\overline{z} = \overline{1+2i}$ $= 1-2i$
2 Multiply numerator and denominator by the conjugate of the denominator.	$\frac{3+4i}{1+2i} = \frac{3+4i}{1+2i}\times\frac{1-2i}{1-2i}$
3 Expand both numerator and denominator, and simplify. Note: Denominator is in the form: $(a+bi)(a-bi) = a^2+b^2$ and $i^2 = -1$	$= \frac{3(1-2i)+4i(1-2i)}{1-4i^2}$ $= \frac{3-6i+4i-8i^2}{1-4i^2}$ $= \frac{3-2i-8(-1)}{1-4(-1)}$ $= \frac{3-2i+8}{1+4}$ $= \frac{11-2i}{5}$
4 Write the answer in the appropriate form.	$\frac{w}{z} = \frac{11}{5}-\frac{2i}{5}$

EXERCISE 5.1 Complex numbers

Worked Example

1 If $z = 5 - 8i$, then $\text{Im}(z) =$ 1

A 5 and 8 **B** only 8 **C** -8 **D** $-8i$

2 Consider the complex number $z = a + bi$.

(a) Which of these is true?

A $\text{Re}(\overline{z}) = -a$ **B** $\text{Im}(\overline{z}) = b$

C $\text{Re}(z.\overline{z}) = a^2 + b^2$ **D** $\text{Im}(z.\overline{z}) = b^2$

(b) Explain why the other alternatives are incorrect.

3 What is $(4i)^2$ equal to?

4 What is $(2 + 3i) - (3 - 2i)$ equal to?

5 Use imaginary numbers to write a simplified expression for each of the following: 2

(a) $\sqrt{-49}$ **(b)** $\sqrt{-11}$ **(c)** $\sqrt{-75}$

6 Determine the conjugates of the following complex numbers. 7

(a) $z = 5 + 6i$ **(b)** $z = 10 - 3i$ **(c)** $z = 8$

7 If $z_1 = 2 - 4i$ and $z_2 = 7 + 2i$, calculate the following: 3

(a) $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $z_2 - z_1$ **(d)** $2z_1 + z_2$

8 If $z_1 = 2 - 4i$ and $z_2 = 7 + 2i$, calculate the following: 4

(a) $2z_1 - 3z_2$ **(b)** $2z_1 + 4z_2$

9 Expand and simplify: 5

(a) $(3 + 2i)(5 + 3i)$ **(b)** $(2 - 4i)(7 + 2i)$ **(c)** $(-2 + 5i)(3 + i)$

(d) $(-3 - 2i)(4 - 3i)$ **(e)** $(2 + 4i)(2 - 4i)$ **(f)** $(6 - \sqrt{2}i)(6 + \sqrt{2}i)$

(g) $(4 + 3i)(2 - i)$ **(h)** $(2 + 2i)(2 - i)$ **(i)** $(3 - i)(3 + i)$

(j) $(2 + \sqrt{6}i)(2 - \sqrt{6}i)$

10 Determine the values of a and b that satisfy the equation $(4 + i)(a + bi) = (14 - 5i)$. 6

11 Calculate the multiplicative inverse, z^{-1}, of the following complex numbers, giving your answer in the form $a + bi$. 8

(a) $z = 1 + 2i$ **(b)** $z = 4 - 5i$

Worked Example 9

12 Express each of the following in the form $x + yi$.

(a) $\dfrac{1+2i}{5-3i}$ (b) $\dfrac{-3+2i}{1+i}$ (c) $\dfrac{-i}{2-i}$ (d) $\dfrac{4+i}{2i}$

13 Simplify each of the following:

(a) $(-3-2i)+(4-3i)$ (b) $(5-i)-(3+2i)$ (c) $(3+2i)(5+3i)$

(d) $(2-4i)(7+2i)$ (e) $(2+3i)^3$ (f) $(5-2i)^3$

(g) $\dfrac{1-2i}{4+3i}$ (h) $\dfrac{3-2i}{4i}$

14 If $z = 2 + 3i$ and $w = 3 - i$, calculate the following:

(a) $z^2 w$ (b) zw^2 (c) $z^3 w^3$

15 If $u = 2 - 7i$, $w = 3 + 2i$ and $z = -2 + 5i$, calculate the following:

(a) $\text{Re}\left(u^2\right)$ (b) $\text{Im}(w.z)$

(c) v^{-1}, where $v = [\text{Im}(z\,.\,w) + \text{Re}\left(u^2\right)i\,]$

16 Calculate z such that $5z + 4 = 14 - 15i$.

17 By using $w = a + bi$ and $z = c + di$, demonstrate that $\overline{wz} = \overline{w} \times \overline{z}$.

18 By using $w = a + bi$ and $z = c + di$, demonstrate that $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

19 By using $w = a + bi$ and $z = c + di$, demonstrate that $(w\,.\,z)^{-1} = w^{-1} z^{-1}$.

20 Calculate z such that $(2 - i) \times z = 16 + 2i$.

21 Calculate z such that $2i.z = -3z + 21 + i$.

22 Calculate z such that $\dfrac{z+2}{z+1} = i$.

23 Determine z such that $z^2 = -45 - 28i$.

The complex plane

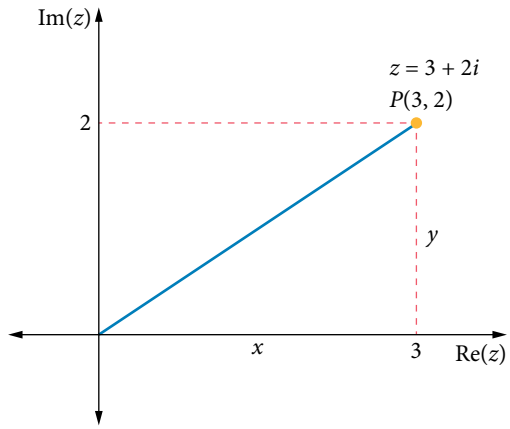
5.2

Graphing complex numbers

You are used to plotting real numbers on number lines and number pairs as a point (x, y) on the Cartesian plane. The complex number $z = x + yi$ can be represented either as the point (x, y) on a complex number plane, or as a vector from the origin to the point (x, y). The complex plane itself is very similar to the Cartesian coordinate axes; the only difference is, instead of labelling the horizontal axis the x-axis, you label it Re(z), and instead of labelling the vertical axis the y-axis, you label it as Im(z). The complex number plane is usually called an *Argand plane* or an *Argand diagram*, after Jean-Robert Argand, the amateur French mathematician who, in 1813, proposed this geometric way of representing a complex number.

For example, consider the complex number $z = 3 + 2i$. This can be represented by the point (3, 2) on the Argand plane. Sometimes, as shown on the right, an interval is drawn from the origin indicating the length, radius or modulus of the complex number, but the complex number is actually the point.

Complex numbers can be plotted on the *Argand plane*, such that the real component of the complex number is plotted on the horizontal axis, labelled Re(z), and the imaginary component is plotted on the vertical axis, labelled Im(z).



10 Graphing complex numbers

Graph the following complex numbers on the Argand plane:

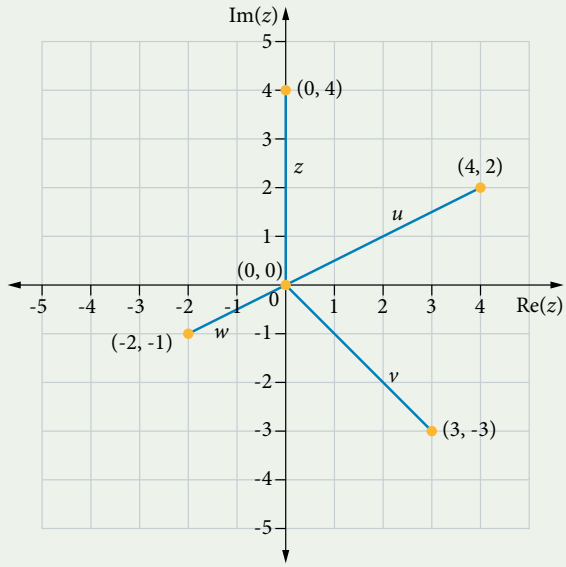
(a) $u = 4 + 2i$ (b) $v = 3 - 3i$ (c) $w = -2 - i$ (d) $z = 4i$

THINKING

1 Construct an Argand diagram with a horizontal Real axis and a vertical Imaginary axis.

2 Draw each complex number as a coordinate pair.

WORKING



Geometric representation of complex number operations

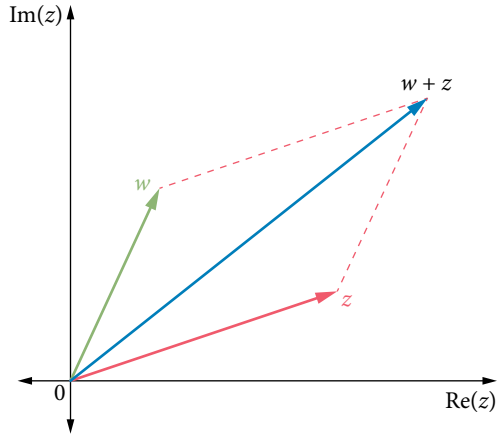
The geometric interpretation of adding or subtracting complex numbers can be seen as the same process as addition of vectors. If you associate the complex number $x + yi$ with the vector $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, addition of complex numbers behaves the same way as addition of vectors in two dimensions.

WARNING

Be careful not to confuse the imaginary number, i, with the horizontal unit vector $\hat{\boldsymbol{i}}$.

Addition of complex numbers

If $w = a + bi$ and $z = c + di$, then $w + z = (a + c) + (b + d)i$. When graphed on the Argand plane, this replicates the parallelogram law of vector addition.



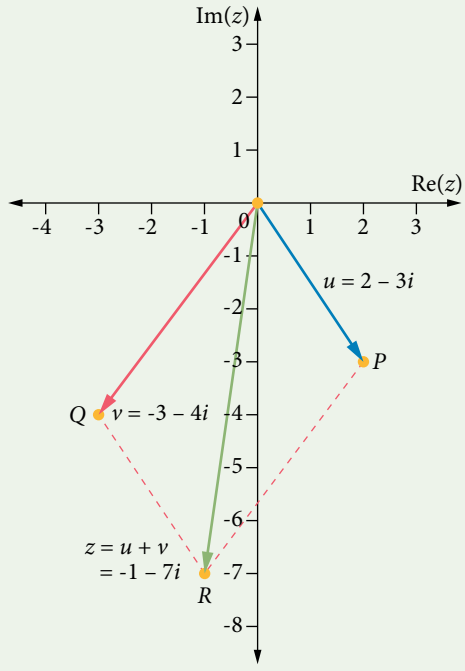
11 Addition of complex numbers

Let $u = 2 - 3i$ and $v = -3 - 4i$. Use geometric addition to calculate $z = u + v$.

THINKING

1 Draw the complex numbers as vectors, complete the parallelogram and label points.

WORKING



2 Read the required point from the graph.

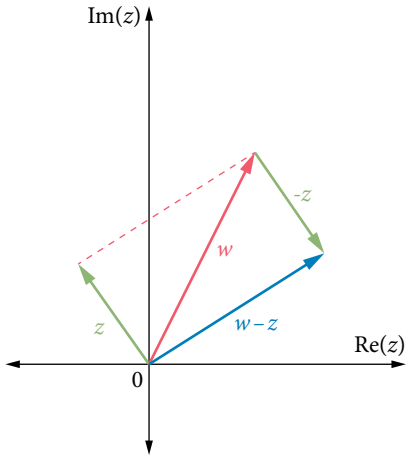
Point R is (-1, -7)

3 Express as a complex number.

$z = -1 - 7i$

Subtraction of complex numbers

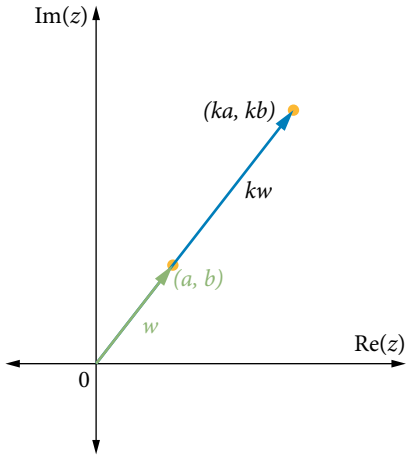
If $w = a + bi$ and $z = c + di$, then $w - z = (a - c) + (b - d)i$. When graphed on the Argand plane, you are essentially adding w and the negative of z.



Multiplication by a scalar

When multiplied by a scalar k, the direction of a complex number is unchanged (unless $k < 0$, in which case the direction is reversed). The magnitude of the complex number increases by a factor of k.

Recall that if $w = a + bi$ and if $k \in \mathbb{R}$, then $kw = k(a + bi) = ka + kbi$, shown in the diagram on the right.



Note that in this diagram, $k > 1$ so the magnitude of the resultant complex number kw is longer than the magnitude of the original complex number w. If $0 < k < 1$, then the magnitude of kw will be shorter than the magnitude of w. It can be observed that multiplying a complex number, z, by a scalar, k, is equivalent to performing a dilation transformation of factor k about the origin.

Multiplication by i

Note that integer powers of i can be simplified. For example:

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 i = -i$

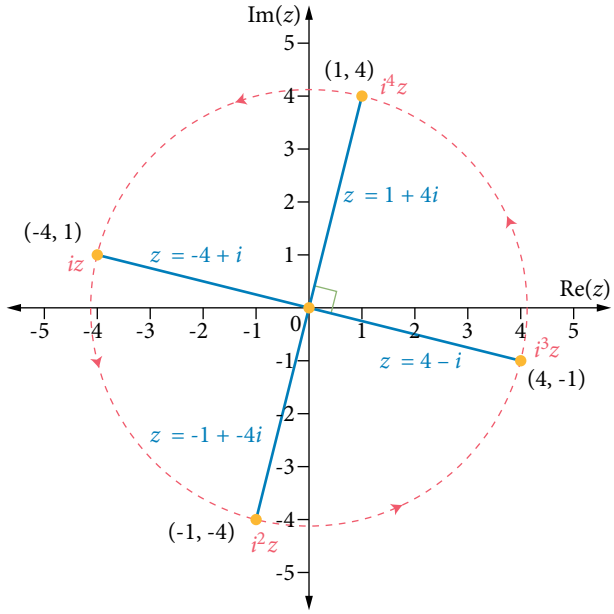
$i^4 = i^2 i^2 = -1 \times -1 = 1$

This pattern repeats such that $i^{4n} = 1$ for integer values of n.

Now consider the complex number $z = a + bi$. You can examine what happens to this complex number if it is multiplied by successive powers of i.

z	$a + bi$
iz	$i(a + bi) = ai + bi^2 = -b + ai$
i^2z	$i^2(a + bi) = ai^2 + bi^3 = -a - bi$
i^3z	$i^3(a + bi) = ai^3 + bi^4 = b - ai$
i^4z	$i^4(a + bi) = ai^4 + bi^5 = a + bi$

For example, consider the complex number $z = 1 + 4i$ represented on the Argand plane as follows:



It can be seen that multiplying a complex number z by i is equivalent to performing a rotation transformation of 90° in an anticlockwise direction about the origin.

12 Multiplication by *i*

If $z = 4 + 2i$, graph the following complex numbers on the same Argand plane:

(a) z **(b)** iz **(c)** i^3z

THINKING

1 Calculate the required complex number by multiplying by the given power of i.

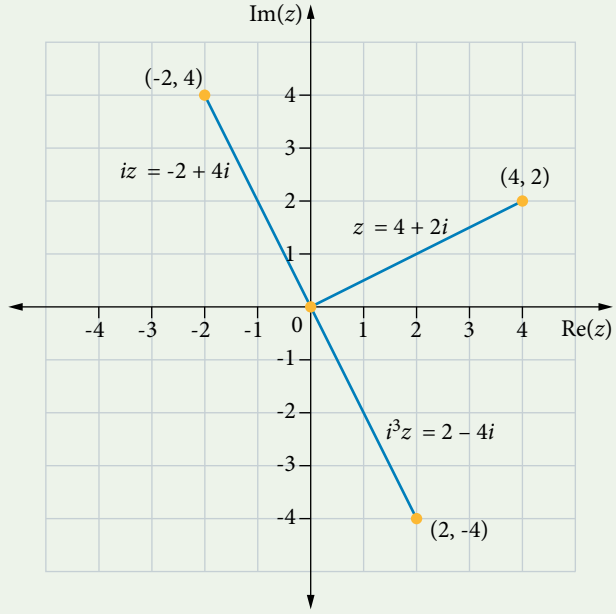
2 Construct an Argand diagram with a horizontal Real axis and a vertical Imaginary axis. Draw each complex number as a coordinate pair.

WORKING

(a) $z = 4 + 2i$

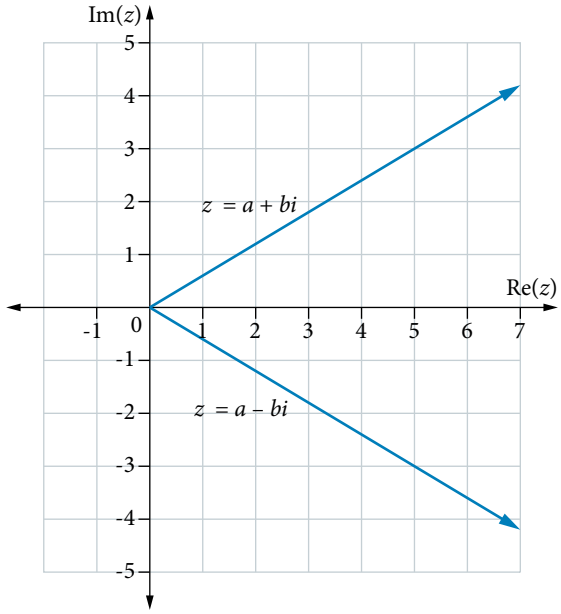
(b) $iz = i(4 + 2i) = 4i + 2i^2 = -2 + 4i$

(c) $i^3z = i^3(4 + 2i) = 4i^3 + 2i^4 = 2 - 4i$



Graphing complex conjugates

You will recall that the conjugate of $z = a + bi$ is $\overline{z} = a - bi$. When graphed on the Argand plane, the conjugate $\overline{z} = a - bi$ appears as a reflection of the original complex number z in the Real axis.



Multiplication as a linear transformation

You have already seen how multiplication by a scalar or multiplication by i can have interesting effects on the graphs of complex numbers in the Argand plane. Generally speaking, multiplying points on the Argand plane by a complex number, $a + bi$, brings about a *linear transformation* of the Argand plane. You will learn more about linear transformations in Chapter 8, but a linear transformation maps a point or set of points according to the rule $T(x, y) = (ax + by, cx + dy)$. Some common linear transformations include rotations around the origin, reflections in lines through the origin, and dilations.

13 Multiplication as a linear transformation

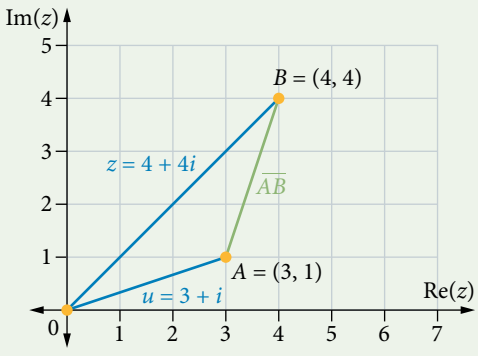
On the Argand plane, point A is defined by the complex number $u = 3 + i$ and point B is defined by the complex number $v = 4 + 4i$.

(a) Calculate the length of line segment $|\overline{AB}|$.

THINKING

1 Draw a labelled graph of the line segment on the Argand plane.

WORKING



2 Use Pythagoras' theorem to calculate the length of line segment $\overline{AB}$.

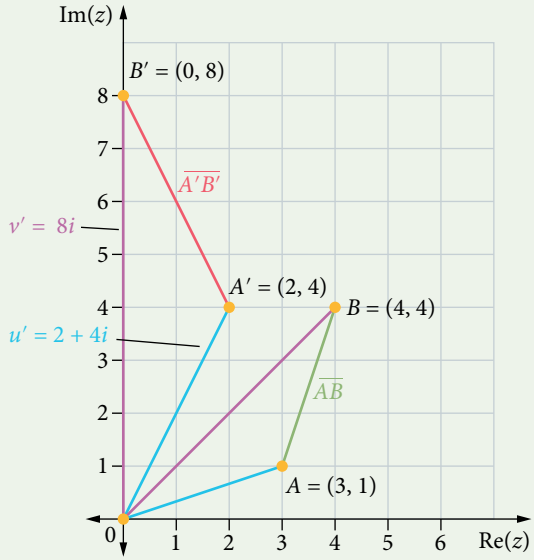
$$\begin{aligned}|\overline{AB}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-3)^2 + (4-1)^2} \\ &= \sqrt{1^2 + 3^2} \\ &= \sqrt{10} \text{ units}\end{aligned}$$

(b) Points A' and B' are created by multiplying complex numbers u and v by $z = 1 + i$. Calculate the length of line segment $\overline{A'B'}$.

1 Multiply complex numbers u and v by the given complex number z. This will determine where to place points A' and B'.

$$\begin{aligned} u' &= (3 + i)(1 + i) \\ &= 3 + 3i + i + i^2 \\ &= 2 + 4i \\ \therefore A' &= (2, 4) \\ v' &= (4 + 4i)(1 + i) \\ &= 4 + 4i + 4i + 4i^2 \\ &= 8i \\ \therefore B' &= (0, 8) \end{aligned}$$

2 Represent points A' and B' on the Argand plane.



3 Use Pythagoras' theorem to calculate the length of line segment $\overline{A'B'}$.

$$\begin{aligned} \left|\overline{A'B'}\right| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0 - 2)^2 + (8 - 4)^2} \\ &= \sqrt{(-2)^2 + 4^2} \\ &= \sqrt{20} \text{ units} \end{aligned}$$

Note that in the previous example, the points A and B were multiplied by the complex number $z = 1 + i$ to produce points A' and B'. The *magnitude* (length) of z is $\sqrt{2}$ and the *angle* that z makes with the positive Real axis is 45°. Note also that the magnitude of the image $\overline{A'B'}$ has increased by a factor of $\sqrt{2}$, and the image $\overline{A'B'}$ has also been rotated anticlockwise by 45°. The properties of magnitude and direction of complex numbers will be discussed later in this chapter, but it is important to understand that multiplying a point on the Argand plane by a complex number, $a + bi$, combines the effects of dilation and rotation as previously discussed.

Locus of a point on the Argand plane

A graph of part of the Argand plane is called a subset of the complex plane. The path traced out by a point obeying a rule is called a locus. For example, the locus of a point moving on a plane so that it is always the same distance from a given fixed point is a circle.

You interpret the modulus $|z|$ as the magnitude of z or its distance from the origin. Hence, from Pythagoras' theorem, $|z| = |x + yi| = \sqrt{x^2 + y^2}$.

For example, $|z| = 3$ represents all the points in the Argand plane that are 3 units from the origin; that is, a circle of radius 3. You can justify this algebraically by letting $z = x + yi$, hence $|z| = |x + yi| = \sqrt{x^2 + y^2}$.

Hence $|z| = 3$, can be expressed as:

$|x + yi| = 3$

$\sqrt{x^2 + y^2} = 3$

$x^2 + y^2 = 9$, which is a circle with centre (0, 0) and radius 3.

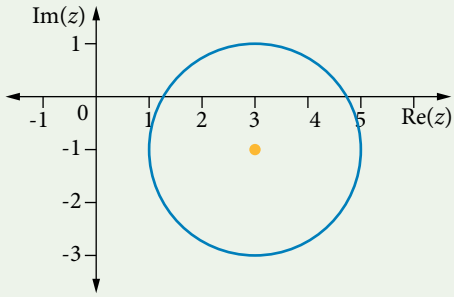
Further, $|z| < 3$ would represent all the points inside the circle (excluding its circumference).

A circle that has been *translated* (moved) will be of the form: $|z - (\text{centre})| = \text{radius}$.

The locus of a point P moving on the Argand plane such that P is a fixed distance from another point is a circle.

14 Complex equation of a circle

Describe and sketch the subset of the complex plane $\{z: |z - 3 + i| = 2\}$.

THINKING	WORKING
1 Let $z = x + yi$.	$\|x + yi - 3 + i\| = 2$
2 Group the real and imaginary components.	$\|(x - 3) + (y + 1)i\| = 2$
3 Apply $\|z\| = \|x + yi\| = \sqrt{x^2 + y^2}$.	$\sqrt{(x - 3)^2 + (y + 1)^2} = 2$
4 Square both sides of the equation.	$(x - 3)^2 + (y + 1)^2 = 4$
5 Describe the subset of the complex plane.	General equation of circle with centre (3, -1), radius 2.
6 Sketch on the Argand plane.	

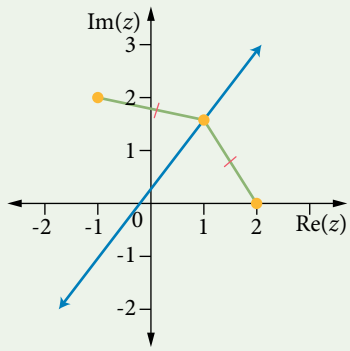
Using the $|z - (\text{centre})| = \text{radius}$ form, the previous example could be expressed as $\{z:|z-(3-i)|=2\}$ giving a circle of centre (3, -1), radius 2.

Note that the application of $|z| = |x + yi| = \sqrt{x^2 + y^2}$ in the previous example will result in an equation involving x and y only—in other words, it produces a graph in the Cartesian plane. This can help you to visualise the complex graph, where the x-axis is equivalent to the Re(z) axis and the y-axis is equivalent to the Im(z) axis, but the graph should still be plotted on the Argand plane.

The locus of a point P moving on the Argand plane such that it is equidistant from two other points is a straight line.

15 Complex equation of a straight line

Describe and sketch the subset of the complex plane $\{z:|z-2| = |z+1-2i|\}$.

THINKING	WORKING
1 Identify the two fixed points.	$\{z:\|z-(2+0i)\| = \|z-(-1+2i)\|\}$ Points are (2, 0) and (-1, 2).
2 Let $z = x + yi$.	$\|x+yi-2\| = \|x+yi+1-2i\|$
3 Group the real and imaginary components.	$\|(x-2)+yi\| = \|(x+1)+(y-2)i\|$
4 Apply $\|z\| = \|x+yi\| = \sqrt{x^2+y^2}$.	$\sqrt{(x-2)^2+y^2} = \sqrt{(x+1)^2+(y-2)^2}$
5 Square both sides of the equation.	$(x-2)^2+y^2 = (x+1)^2+(y-2)^2$
6 Expand both sides of the equation.	$x^2-4x+4+y^2 = x^2+2x+1+y^2-4y+4$
7 Simplify and make y the subject.	$-4x = 2x+1-4y$ $4y = 6x+1$ $y = \frac{1}{4}(6x+1)$
8 Describe and sketch the graph.	Equation represents a straight line. 
9 Interpret the result.	The points (2, 0) and (-1, 2) are equal distances from any point along the line $y = \frac{1}{4}(6x+1)$.

EXERCISE 5.2 The complex plane

Worked Example

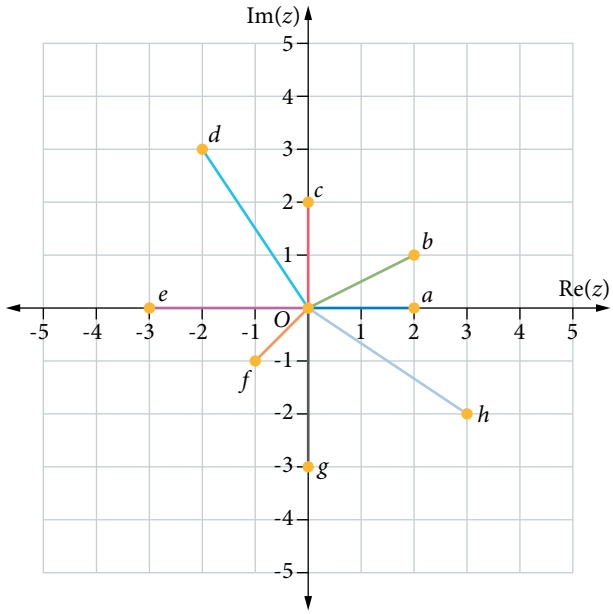
10

1 Represent each of the following complex numbers on an Argand plane.

(a) $a = 2i$ (b) $b = 3$ (c) $c = 3 - i$

(d) $d = -2 - 3i$ (e) $e = c + d$ (f) $f = c - d$

2 Write the complex numbers represented by each of the points indicated on the following Argand diagram.



11

3 Let $u = 3 + i$ and $v = 1 + 2i$. Use geometric addition to graph $z = u + v$.

4 A complex number $u = a + bi$ is multiplied by i^3 to produce a resultant complex number of v. Which of the following is true?

A v is a reflection of u in the x-axis.

B v is a reflection of u in the y-axis.

C v is a rotation of u, 180° anticlockwise about the origin.

D v is a rotation of u, 270° anticlockwise about the origin.

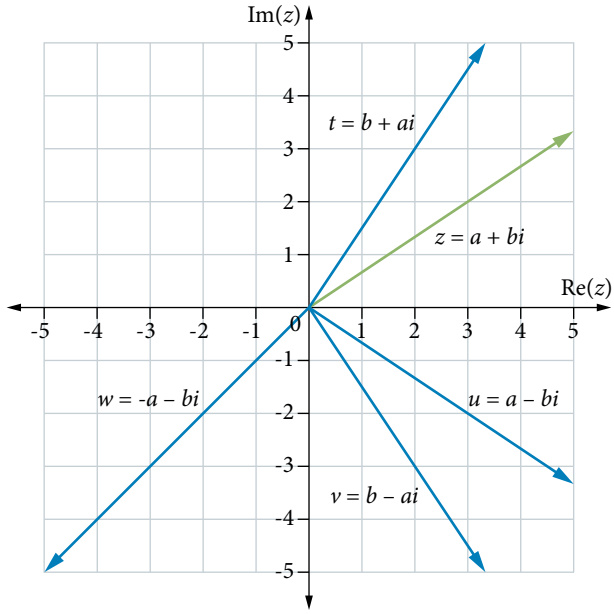
12

5 If $z = 2 + 3i$, graph the following complex numbers on the same Argand plane:

(a) iz (b) $i^2 z$ (c) $i^3 z$

6 In the diagram below, complex number $z = a + bi$. Which of the following complex numbers is equivalent to $i\overline{z}$?

A $t = b + ai$ **B** $u = a - bi$ **C** $v = b - ai$ **D** $w = -a - bi$



7 Describe the locus of the point in the complex plane defined by $|z - 1 + 2i| = 2$.

8 Which of the following does not represent a circle?

A $(z-3)(\overline{z}-3) = 3$ **B** $(z-3)(\overline{z}+3) = 3$ **C** $(z-2)^2 + (y-3)^2 = 4$ **D** $|z-2+i| = 3$

Worked Example

9 If $z = p + qi$, demonstrate that the triangle formed by the origin and the complex numbers z and $\overline{z}$ has an area of pq units.

13

10 Point A is defined by the complex number $u = 2 + 5i$ and point B is defined by the complex number $v = 3 - i$ on the Argand plane.

(a) Calculate the length of line segment $\overline{AB}$.

(b) Points A' and B' are created by multiplying complex numbers u and v by $z = 2 - i$. Calculate the length of line segment $\overline{A'B'}$.

14

11 For each of the following, describe and sketch the defined subset of the complex plane.

(a) $\{z : |z| = 4\}$ **(b)** $\{z : |z - 2 + i| = 3\}$ **(c)** $\{z : |z - 2 - i| = 4\}$

15

12 For each of the following, describe and sketch the defined subset of the complex plane.

(a) $\{z : |z - 2| = |z - 1 + i|\}$ **(b)** $\{z : |z + 1| = |z + 3 - i|\}$

13 A unit square is formed by the complex numbers $a = 0 + 0i$, $b = 1 + 0i$, $c = 1 + 1i$ and $d = 0 + 1i$. Demonstrate that when this unit square is transformed in the Argand plane by multiplying by $z = p + qi$, the resulting shape has an area of $p^2 + q^2$.

Radian measure

5.3

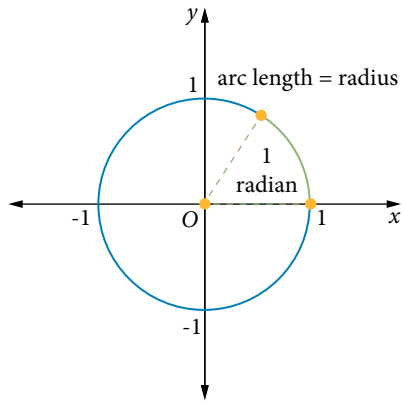
Measuring angles

Before you continue the study of complex numbers, it will be useful to consider an alternative way of measuring angles, which you will use extensively in the remainder of this course. This involves the use of *radians*, a concept that will be studied in greater depth in the Mathematical Methods subject, but which will be introduced here.

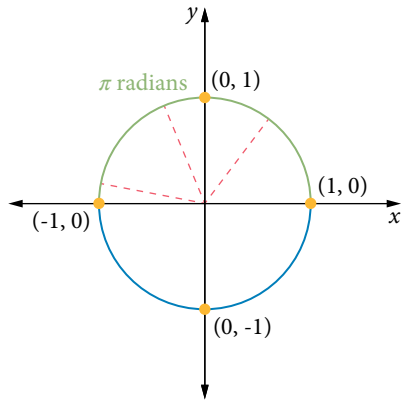
In your studies so far, you have used *degrees* as the unit for measuring angles. By convention, there are 360° in a circle and therefore 90° in a right angle. This convention does not, however, take into account any of the inherent properties of an angle itself. An alternative unit of measure, called the *radian*, is based directly on the properties of an angle within a circle.

Radian measure

The radian is defined as the angle subtended at the centre of a circle by two radii that enclose an arc 1 radius (r) long on the circumference of the circle.



In other words, the product of the angle (measured in radians) and the radius of the circle is equal to the length of the arc on the circumference of the circle. For example, an angle of 1.6 radians encloses an arc that is $1.6r$ long, and an angle of 2π encloses an arc that is $2\pi r$ long. Because $2\pi r$ is also the circumference of a circle, 2π radians is therefore equal to one revolution, which is 360°. From this, a straight angle of 180° will have an equivalent radian measure of π radians, and a right angle (90°) will be equivalent to $\frac{\pi}{2} \approx 1.57$ radians.



Just as (°) is the symbol used for degrees, (^{c}) is the symbol used to represent radians (although, if no symbol is shown, you can assume that the radian measure is being used). It is handy to remember the following basic equation that gives the relationship between degrees and radians:

$2\pi^c = 360°$

$\pi^c = 180°$

WARNING

When using your calculator to perform any kind of angle calculation, especially involving the trigonometric ratios, it is very important to ensure that your calculator is set to the correct angle mode.

Technology worked example

Converting between degrees and radians

Converting between degrees and radians

Starting with $\pi^c = 180°$, note that $1^c = \frac{180°}{\pi^c}$ and $1° = \frac{\pi^c}{180°}$.

To convert an angle measured in degrees to radians, multiply by $\frac{\pi}{180}$. That is, $\theta^c = \frac{\pi}{180} \times \theta°$.

To convert an angle measured in radians to degrees, multiply by $\frac{180}{\pi}$. That is, $\theta° = \frac{180}{\pi} \times \theta^c$.

16 Converting degrees to radians

Convert 50° to radians in terms of π.

THINKING	WORKING
1 Recall the relationship $180° = \pi^c$, write $1°$ in radians by dividing both sides by 180.	$1° = \frac{\pi^c}{180}$
2 Multiply by the given number of degrees.	$50° = \frac{50 \times \pi^c}{180}$
3 Simplify and express in terms of π.	$= \frac{5\pi}{18^c}$

17 Converting radians to degrees

Convert 2.35^c to degrees, correct to 2 decimal places.

THINKING	WORKING
1 Recall the relationship $180° = \pi^c$, write 1^c in radians by dividing both sides by π.	$1^c = \frac{180°}{\pi}$
2 Multiply by the given number of radians.	$2.35^c = \frac{2.35 \times 180°}{\pi}$
3 Use technology to obtain a decimal value, correct to the specified level of accuracy, if cancelling is not possible.	$= 134.65°$ (2 d.p.)

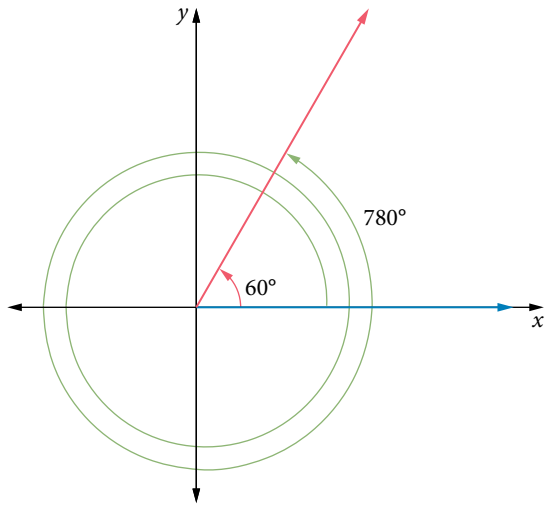
Coterminal angles

Because an angle is simply a measure of rotation, it is possible for two different angles to terminate in the same direction. Consider the angle 0°, which is conventionally expressed in the direction of the positive x-axis. There is an infinite number of angles that will terminate in the same direction. The angles 0°, 360° (or $2\pi^c$), 720° ($4\pi^c$), 1080° ($6\pi^c$), etc. are all *coterminal*, as are the negative angles -360° ($-2\pi^c$), -720° ($-4\pi^c$), etc. It's important to understand that, although these are all different angles, they will end up pointing in the same direction.

Angle $\theta°$ is coterminal with angle $(\theta° + n\,360°)$, where $n \in \mathbb{Z}$.

Angle θ^c is coterminal with angle $(\theta^c + n\,2\pi^c)$, where $n \in \mathbb{Z}$.

In the diagram to the right, you can observe that the angles 60° and 780° are coterminal.



18 Coterminal angles

(a) Calculate the angle that lies in the domain $0° \leq \theta < 360°$ and is coterminal with 960°.

THINKING	WORKING
1 Note that you are working in degrees. Add or subtract multiples of 360° until you end up with an angle in the required domain.	$960° - 360° = 600°$ $600° - 360° = 240°$
2 Give the answer in the required domain.	$\theta = 240°$

(b) Calculate the angle that lies in the domain $-\pi \leq \alpha < \pi$ and is coterminal with $-\frac{22\pi}{3}$.

1 Note that you are working in radians. Add or subtract multiples of 2π until you end up with an angle in the required domain.

$$-\frac{22\pi}{3} + 2\pi = \frac{-22\pi}{3} + \frac{6\pi}{3} = -\frac{16\pi}{3}$$
$$-\frac{16\pi}{3} + 2\pi = \frac{-16\pi}{3} + \frac{6\pi}{3} = -\frac{10\pi}{3}$$
$$-\frac{10\pi}{3} + 2\pi = \frac{-10\pi}{3} + \frac{6\pi}{3} = -\frac{4\pi}{3}$$
$$-\frac{4\pi}{3} + 2\pi = \frac{-4\pi}{3} + \frac{6\pi}{3} = \frac{2\pi}{3}$$

Adding more 2π will result in an angle that exceeds the given domain.

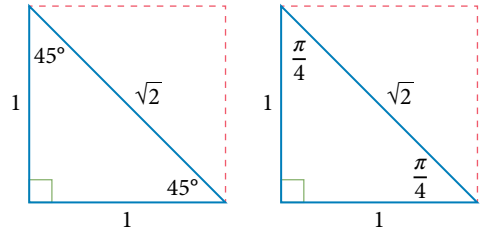
2 Give the answer in the required domain.

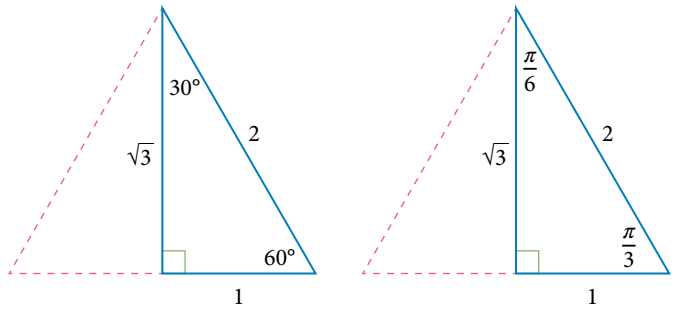
$$\alpha = \frac{2\pi}{3}$$

Some special triangles

Consider these special triangles, which display trigonometric ratios of common angles using exact values. The first pair of triangles is formed by the diagonal bisection of a unit square.

The second pair of triangles is formed by the vertical bisection of an equilateral triangle of side length 2.





From these triangles, and the trigonometric ratios, the following exact values are derived:

$$\sin(30°) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \cos(30°) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \tan(30°) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\sin(45°) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \cos(45°) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \tan(45°) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$\sin(60°) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \cos(60°) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \tan(60°) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

EXERCISE 5.3 Radian measure

Worked Example

16

1 Convert the following degree measures to radians in terms of π.

(a) 30° **(b)** 48° **(c)** 150°

(d) 210° **(e)** 300°

17

2 Convert the following radian measures to degrees, correct to 2 decimal places where necessary.

(a) 0.224 **(b)** 2.436 **(c)** 4.678

(d) $\frac{\pi}{4}$ **(e)** $\frac{2\pi}{9}$ **(f)** $\frac{3\pi}{5}$

18

3 Calculate the angle that lies in the domain $0° \leq \theta < 360^c$ and is coterminal with the following measures:

(a) 370° **(b)** -20° **(c)** 823°

4 Calculate the angle that lies in the domain $-\pi \leq \alpha < \pi$ and is coterminal with the following measures:

(a) $\frac{9\pi}{4}$ **(b)** -5π **(c)** 5.91^c

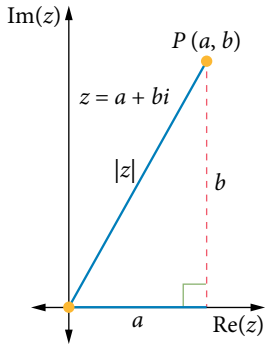
Complex arithmetic using polar form

5.4

The modulus of a complex number

Consider the complex number in Cartesian (rectangular) form, $z = a + bi$. The *modulus* of z is the length of the line segment joining z to the origin.

Pythagoras' theorem can be used to calculate the modulus of a complex number: if $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$



19 The modulus of a complex number

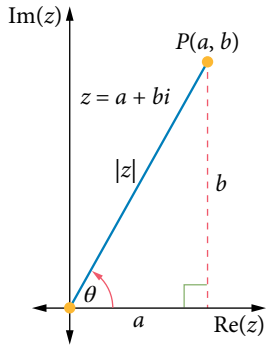
Calculate the modulus of the complex number $z = 5 - 3i$.

THINKING	WORKING
Use Pythagoras' theorem to calculate the modulus of the complex number.	$\|z\| = \sqrt{a^2 + b^2}$ $= \sqrt{5^2 + (-3)^2}$ $= \sqrt{25 + 9}$ $= \sqrt{34}$ units

The argument of a complex number

The argument is the angle of the complex number, measured anticlockwise from the positive Real axis. For any complex number z, the argument is abbreviated $\arg(z)$. In the diagram below, the argument is shown as angle θ, such that $\tan(\theta) = \frac{b}{a}$.

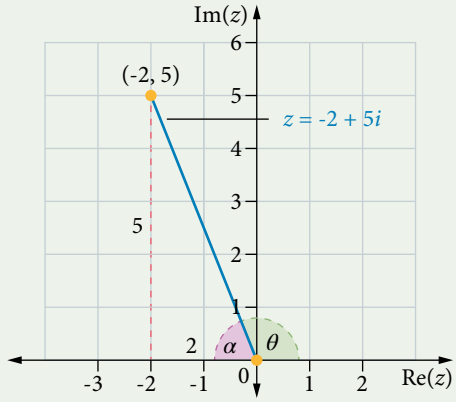
Note that there is an infinite number of angles that are coterminal to θ. This implies that a complex number has an infinite number of arguments that could be used to sketch the complex number on the exact same point on the Argand plane. By convention, you say that every complex number $z = a + bi$ has a *principal argument* in the domain $-\pi < \theta \leq \pi$. The principal argument is referred to using a capital letter: $\text{Arg}(z)$.



If $z = a + bi$ then $\arg(z) = \theta$ such that θ is the angle of the complex number, measured anticlockwise from the positive Real axis. The principal argument of z lies in the domain $-\pi < \theta \leq \pi$ and is denoted $\text{Arg}(z)$.

20 The argument of a complex number

Calculate the Argument of $z = -2 + 5i$ in the domain $-\pi < \theta \leq \pi$.

THINKING	WORKING
1 Sketch the complex number on the Argand plane.	
2 Use the tangent ratio to calculate the acute angle, α.	$\alpha = \tan^{-1}\left(\frac{5}{2}\right)$ $= 1.1903...^{c}$
3 Make the necessary adjustment to calculate θ, the angle from the positive Real axis.	$\theta = \pi - 1.1903...^{c}$ $= 1.95^{c}$ (2 d.p.)
4 Interpret the answer.	$\text{Arg}(z) = 1.95^{c}$

Explore further

Modulus and argument of a complex number

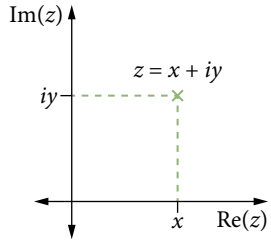
Explore how the modulus and argument of a complex number change when the complex number moves around the Argand plane.

Technology worked example

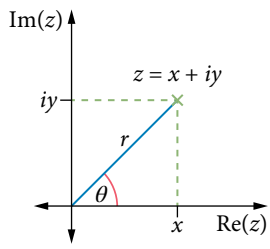
Calculating the modulus and argument of a complex number

Complex numbers in polar form

In your study of complex numbers so far, you have shown that they can be represented as ordered pairs on the Argand plane. The Argand plane uses rectangular axes, as shown on the right.

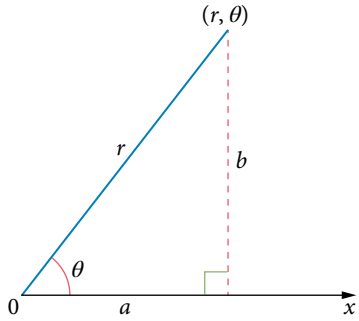


You can also use polar coordinates to represent complex numbers. A polar coordinate system gives the position of a point in terms of the length of a line segment (the radius vector) from a fixed origin (the pole), together with the angle that the line segment makes with a fixed ray (in this case, the positive Re(z) axis).



With knowledge about the modulus and argument of a complex number, you can now represent a complex number, $z = a + bi$, in polar form. Consider the diagram (right), which represents a complex number using a modulus r and an argument θ. Using trigonometry:

$\cos(\theta) = \frac{a}{r}$ hence $a = r\cos(\theta)$ and $\sin(\theta) = \frac{b}{r}$ hence $b = r\sin(\theta)$.



By combining the results above, you get the core conversion that any complex number in the form $z = a + bi$ can be written as $z = r\cos(\theta) + ir\sin(\theta)$.

A complex number written as $r[\cos(\theta) + i\sin(\theta)]$, i.e. using polar coordinates, is said to be in *polar form*. For convenience, polar form is often abbreviated to:

$$\begin{aligned} a + ib &= r\cos(\theta) + ir\sin(\theta) \\ &= r[\cos(\theta) + i\sin(\theta)] \\ &= r\,\text{cis}(\theta) \end{aligned}$$

$$\begin{aligned} a + ib &= r\cos(\theta) + ir\sin(\theta) \\ &= r[\cos(\theta) + i\sin(\theta)] \end{aligned}$$

where the 'cis' (pronounced 'sis') is understood to be short for $[\cos(\theta) + i\sin(\theta)]$.

Converting between rectangular and polar form

To convert from rectangular to polar form, calculate the modulus and argument of the complex number.

WARNING

When calculating the argument of a complex number, it is not sufficient simply to perform the calculation $\tan(\theta) = \frac{b}{a}$, as this will give you two possible values in the domain of $-\pi < \theta \leq \pi$. It is always safest to draw a diagram of the complex number on the Argand plane, calculate the acute angle using the tangent ratio, and then determine the corresponding angle from the positive Real axis in the domain $-\pi < \theta \leq \pi$.

To convert from polar to rectangular form, plot the complex number on an Argand plane and use trigonometry to determine its rectangular coordinates.

21 Converting from rectangular form to polar form

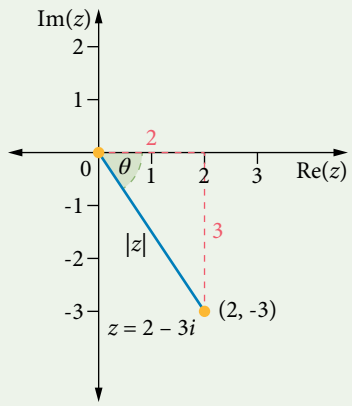
Express the given complex numbers in polar form, giving the Argument in the domain $-\pi < \theta \leq \pi$.

(a) $z = 2 - 3i$

THINKING

1 Plot $z = 2 - 3i$ on an Argand diagram.

WORKING



2 Calculate the modulus, r.

$$r = \sqrt{a^2 + b^2}$$
$$= \sqrt{2^2 + (-3)^2}$$
$$= \sqrt{4 + 9}$$
$$= \sqrt{13}$$

3 Calculate the acute angle θ. (When calculating the acute angle, ignore any negatives).

$$\tan(\theta) = \frac{3}{2}$$
$$\theta = 0.98279...^{c}$$

4 Calculate the argument from the direction of the positive x-axis

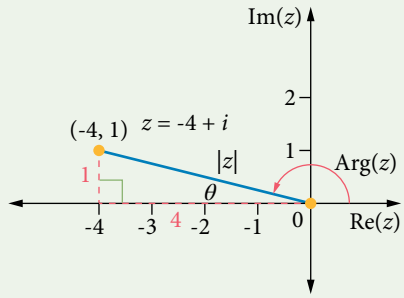
$$\text{Arg}(z) = -0.98^{c} \text{ (2 d.p.)}$$

5 Write in polar form $r\,\text{cis}(\theta)$.

$$2 - 3i = \sqrt{13}\,\text{cis}(-0.98)$$

(b) $z = -4 + i$

1 Plot $z = -4 + i$ on an Argand diagram.



2 Calculate the modulus, r.

$$r = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-4)^2 + 1^2}$$
$$= \sqrt{16 + 1}$$
$$= \sqrt{17}$$

3 Calculate the acute angle, θ, ignoring any negatives.

$$\tan(\theta) = \left(\frac{1}{4}\right)$$
$$\theta = \tan^{-1}\left(\frac{1}{4}\right)$$
$$\theta = 0.24498...^{c}$$

4 Calculate the argument from the direction of the positive x-axis

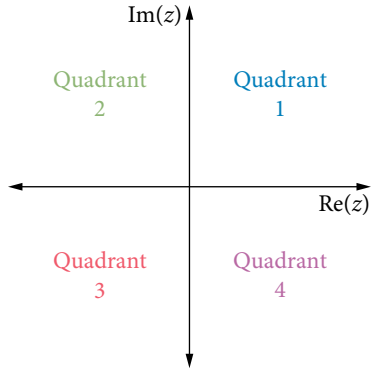
$$\text{Arg}(z) = \pi - 0.98$$
$$= 2.90^{c} \text{ (2 d.p.)}$$

5 Write in polar form $r\,\text{cis}(\theta)$.

$$-4 + i = \sqrt{17}\,\text{cis}(2.90)$$

Note that when calculating angles, it is often more straightforward to use positive values for the x and y components and use the inverse tangent ratio to calculate the *acute* angle (to the nearest Re(z) axis). From here, the argument can be determined as follows:

- For acute angle θ in Quadrant 1, Argument = θ
- For acute angle θ in Quadrant 2, Argument = $\pi - \theta$
- For acute angle θ in Quadrant 3, Argument = $-\pi + \theta$
- For acute angle θ in Quadrant 4, Argument = $-\theta$



22 Converting from polar form to rectangular form

Express $z = 2\text{cis}\left(\frac{\pi}{3}\right)$ in rectangular form, $x + yi$.

THINKING	WORKING
1 Substitute $r = 2$ and $\theta = \frac{\pi}{3}$ into $r\left[\cos(\theta) + i\sin(\theta)\right]$	$2\text{cis}\left(\frac{\pi}{3}\right) = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$
2 Determine the trigonometric expressions and expand.	$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= 1 + i\sqrt{3}$

t Technology worked example

Converting between Cartesian form and polar form

When adding or subtracting complex numbers that are given in polar form, it is best to convert them to rectangular form first.

Multiplication and division of complex numbers in polar form

Consider the two complex numbers, $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$.

Their product $(z_1 z_2)$ can be written as:

$$\begin{aligned} z_1 z_2 &= r_1\text{cis}(\theta_1) \times r_2\text{cis}(\theta_2) \\ &= r_1\left[\cos(\theta_1) + i\sin(\theta_1)\right] \times r_2\left[\cos(\theta_2) + i\sin(\theta_2)\right] \\ &= \left[r_1\cos(\theta_1) + ir_1\sin(\theta_1)\right] \times \left[r_2\cos(\theta_2) + ir_2\sin(\theta_2)\right] \\ &= r_1\cos(\theta_1)\left[r_2\cos(\theta_2) + ir_2\sin(\theta_2)\right] + ir_1\sin(\theta_1)\left[r_2\cos(\theta_2) + ir_2\sin(\theta_2)\right] \\ &= r_1 r_2\cos(\theta_1)\cos(\theta_2) + ir_1 r_2\sin(\theta_2) + ir_1 r_2\sin(\theta_1)\cos(\theta_2) + i^2 r_1 r_2\sin(\theta_1)\sin(\theta_2) \\ &= r_1 r_2\left\{\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] + i\left[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right]\right\} \end{aligned}$$

Later in the course, you will learn about the *addition formulae* for trigonometry. These formulas state that $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.

For complex numbers z_1 and z_2 in polar form, $z_1 z_2 = r_1 r_2\text{cis}(\theta_1 + \theta_2)$

Applying these in reverse to the equation above:

$$\begin{aligned} z_1 z_2 &= r_1 r_2\left[\cos(\theta_1 + \theta_2) - i\sin(\theta_1 + \theta_2)\right] \\ &= r_1 r_2\text{cis}(\theta_1 + \theta_2) \end{aligned}$$

Note that you *multiply* the moduli but *add* the arguments when multiplying complex numbers in polar form.

23 Multiplication of complex numbers in polar form

(a) Calculate the product of $2\text{cis}\left(\frac{\pi}{3}\right) \times 3\text{cis}\left(\frac{\pi}{6}\right)$, giving your answer in polar form.

THINKING	WORKING
Recall the formula for the multiplication of complex numbers in polar form: $z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$	$2\text{cis}\left(\frac{\pi}{3}\right) \times 3\text{cis}\left(\frac{\pi}{6}\right) = (2 \times 3)\text{cis}\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$ $= 6\text{cis}\left(\frac{3\pi}{6}\right)$ $= 6\text{cis}\left(\frac{\pi}{2}\right)$

(b) Calculate the product of $5\text{cis}\left(\frac{\pi}{2}\right) \times 2\text{cis}\left(\frac{5\pi}{6}\right)$, giving your answer in polar form. Ensure that you express the argument in the domain $-\pi < \theta \leq \pi$.

1 Recall the formula for the multiplication of complex numbers in polar form: $z_1 z_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$	$5\text{cis}\left(\frac{\pi}{2}\right) \times 2\text{cis}\left(\frac{5\pi}{6}\right)$ $= (5 \times 2)\text{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right)$ $= 10\text{cis}\left(\frac{3\pi}{6} + \frac{5\pi}{6}\right)$ $= 10\text{cis}\left(\frac{4\pi}{3}\right)$
2 Ensure that the argument is in the domain $-\pi < \theta \leq \pi$, then simplify.	Angle is in the third quadrant, so subtract 2π. $\frac{4\pi}{3} - 2\pi$ $= \frac{4\pi}{3} - \frac{6\pi}{3}$ $= -\frac{2\pi}{3}$ $= 10\text{cis}\left(-\frac{2\pi}{3}\right)$
3 Interpret the answer.	$5\text{cis}\left(\frac{\pi}{2}\right) \times 2\text{cis}\left(\frac{5\pi}{6}\right) = 10\text{cis}\left(\frac{-2\pi}{3}\right)$

In a similar way to multiplication, it can be shown that the quotient (i.e. the *division*) of two complex numbers in polar form can be written as $\frac{z_1}{z_2} = \frac{r_1\text{cis}(\theta_1)}{r_2\text{cis}(\theta_2)} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

Note that you *divide* the moduli but *subtract* the arguments when dividing complex numbers in polar form.

24 Division of complex numbers in polar form

Calculate $4\text{cis}\left(\frac{2\pi}{3}\right) \div 2\text{cis}\left(\frac{\pi}{6}\right)$. Express your answer in polar form with the argument in the domain $-\pi < \theta \leq \pi$.

THINKING	WORKING
Recall the formula for the multiplication of complex numbers in polar form: $z_1 z_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$	$\frac{4\text{ cis}\left(\frac{2\pi}{3}\right)}{2\text{ cis}\left(\frac{\pi}{6}\right)} = \frac{4}{2}\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)$ $= 2\text{cis}\left(\frac{\pi}{2}\right)$

Technology worked example
Multiplying and dividing complex numbers in polar form

Powers of complex numbers in polar form

De Moivre's theorem provides an elegant method for raising complex numbers to integer powers. The theorem applies to complex numbers that are written in polar form. It is an extension of the multiplication rule for complex numbers in polar form.

De Moivre's theorem:
$\left[r\,\text{cis}(\theta)\right]^n = r^n\,\text{cis}(n\theta)$, for $n \in \mathbb{Z}$.

25 Powers of complex numbers in polar form

Simplify $\left[2\,\text{cis}\left(\frac{\pi}{4}\right)\right]^5$ using De Moivre's theorem. Express your answers in polar form, with the argument in the domain $-\pi < \theta \leq \pi$.

THINKING	WORKING
1 Apply De Moivre's theorem and calculate.	$\left[2\,\text{cis}\left(\frac{\pi}{4}\right)\right]^5 = 2^5 \times \text{cis}\left(5 \times \frac{\pi}{4}\right)$ $= 32\,\text{cis}\left(\frac{5\pi}{4}\right)$
2 To determine the argument in principal form, for an angle outside the domain $-\pi < \theta \leq \pi$, calculate a coterminal angle by subtracting (or adding) multiples of 2π.	$\frac{5\pi}{4} - 2\pi = \frac{5\pi}{4} - \frac{8\pi}{4}$ $= -\frac{3\pi}{4}$
3 Interpret the answer.	$\left[2\,\text{cis}\left(\frac{\pi}{4}\right)\right]^5 = 32\,\text{cis}\left(\frac{-3\pi}{4}\right)$

EXERCISE 5.4 Complex arithmetic using polar form

1 When multiplying complex numbers in polar form, what steps are needed?
- A Multiply the moduli and add the arguments.
- B Add the moduli and multiply the arguments.
- C Square the moduli and multiply the arguments.
- D Multiply the moduli and multiply the arguments.

2 Which of the following angles is not in the form of a principal Argument?
A $-\frac{5\pi}{6}$ B $\frac{5\pi}{4}$ C $\frac{2\pi}{3}$ D $-\frac{\pi}{6}$

Worked Example 19

3 Calculate the modulus of the following complex numbers:
(a) $z = 1 + i$ (b) $z = 3 - 4i$ (c) $z = -10 + 3i$
(d) $z = \sqrt{2} + \sqrt{2}i$ (e) $z = -6 - 7i$ (f) $z = 1 + \sqrt{11}i$

Worked Example

20

4 Calculate the Argument of the following complex numbers in the domain $-\pi < \theta \leq \pi$. Give your answer in exact form where possible, otherwise correct to 2 decimal places.

(a) $z = 3 + 3i$ (b) $z = 2 - 4i$ (c) $z = -1 + \sqrt{3}i$ (d) $z = -3 - 4i$

21

5 Express the following complex numbers in polar form. Express the Argument in the domain $-\pi < \theta \leq \pi$. Give your answer in exact form where possible, otherwise correct to 2 decimal places.

(a) $z = 2 + 2i$ (b) $z = -\sqrt{3} - i$ (c) $z = -4 + 3i$ (d) $z = 3 - \sqrt{7}i$

6 Multiply the following complex numbers, expressing your answer in polar form.

(a) $z = (2 - 3i)(1 - i)$ (b) $z = (1 + 5i)(2 + i)$

23

7 Calculate the following complex number products, giving your answer in polar form. Ensure that you express the Argument in the range $-\pi < \theta \leq \pi$.

(a) $3\text{cis}\left(\frac{\pi}{2}\right) \times 4\text{ cis}\left(\frac{\pi}{6}\right)$ (b) $\sqrt{2}\,\text{cis}(\pi) \times \sqrt{3}\ \text{cis}\left(\frac{\pi}{4}\right)$

(c) $6\text{ cis}\left(-\frac{7\pi}{12}\right) \times 2\text{ cis}\left(-\frac{11\pi}{12}\right)$ (d) $10\text{ cis}\left(2.5^{c}\right) \times 4\text{ cis}\left(2.9^{c}\right)$

24

8 Calculate the following complex number quotients, giving your answer in polar form. Ensure that you express the Argument in the range $-\pi < \theta \leq \pi$.

(a) $8\text{cis}\left(\frac{3\pi}{4}\right) \div 4\text{cis}\left(\frac{\pi}{4}\right)$ (b) $12\,\text{cis}(\pi) \div 3\text{ cis}\left(\frac{\pi}{8}\right)$

(c) $7\text{cis}\left(-\frac{3\pi}{4}\right) \div 3\text{cis}\left(\frac{2\pi}{3}\right)$ (d) $\sqrt{20}\,\text{cis}\left(3.1^{c}\right) \div \sqrt{5}\ \text{cis}\left(1.8^{c}\right)$

22

9 Express the following complex numbers in rectangular form $x + yi$. Give your answers in exact form where possible.

(a) $z = 5\,\text{cis}\left(\frac{\pi}{6}\right)$ (b) $z = 2\,\text{cis}\left(\frac{3\pi}{4}\right)$ (c) $z = 4\text{ cis}\,(\pi)$ (d) $z = 10\text{ cis}\left(-0.48^{c}\right)$

25

10 Simplify the following complex number powers using De Moivre's theorem. Express your answers in polar form, with the argument in the domain $-\pi < \theta \leq \pi$.

(a) $\left[3\text{cis}\left(\frac{\pi}{6}\right)\right]^2$ (b) $\left[\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right]^6$ (c) $\left[\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right]^4$

11 Demonstrate that the product of the complex number $z = a + bi$ (where $a \in \mathbb{R}$ and $b \in \mathbb{R}$) and its conjugate $\overline{z}$ will always be a real number.

12 If $z = 3 + 4i$ and $w = 2 - 6i$, express the following in polar form:

(a) $z + w$ (b) $z - 2w$

13 If $z = 2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$ and $w = 5\ \text{cis}\left(\frac{\pi}{2}\right)$, calculate $z + w$, giving your answer in rectangular form $x + yi$.

14 If $z = 10\,\text{cis}\left(\frac{\pi}{3}\right)$ and $w = 6\,\text{cis}\left(-\frac{2\pi}{3}\right)$, calculate $z - w$. Give your answer in polar form.

15 If $z = 1 + \sqrt{3}i$, calculate z^4 using the binomial expansion. Verify your result using De Moivre's theorem.

16 If $z = 2\,\text{cis}\left(\frac{\pi}{4}\right)$, $u = \sqrt{3}\,\text{cis}\left(\frac{5\pi}{6}\right)$ and $v = \sqrt{2}\,\text{cis}\left(\frac{-3\pi}{4}\right)$, simplify the expression $\frac{z^3u^4}{v^2}$. Give your answer in polar form.

Complex solutions of real quadratic functions 5.5

A real quadratic function has the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. For example, $f(x) = 2x^2 - 3x + 4$ is a real quadratic function, because the coefficients a, b, and c are real numbers, whereas $f(x) = 2x^2 - (3 + 2i)x + 4$ is a complex quadratic, because the coefficient of x is a complex number. In this chapter, you are only considering real quadratic functions.

Determining linear factors of real quadratic expressions

The linear factors of a real quadratic expression can be either rational, irrational or complex. If the factors are irrational or complex, then use the technique of completing the square in order to factorise the quadratic. If you are instructed to 'factorise over $\mathbb{R}$', then the imaginary number i cannot be used. If you are instructed to 'factorise over $\mathbb{C}$' then, because Real numbers are a subset of the set of Complex numbers, the imaginary number i can be used if needed to complete the factorisation.

When attempting to factorise a real quadratic expression, $ax^2 + bx + c$, it is useful to calculate the value of the *discriminant*, $b^2 - 4ac$. Depending on the result of this calculation, there are three possible outcomes:

Value of discriminant ($b^2 - 4ac$)	Factorising technique
Discriminant is a perfect square.	Factors will be rational. Use a factorising technique such as 'grouping two and two'.
Discriminant is a positive, non-perfect square.	Factors will be irrational. Use the factorising technique of 'completing the square'.
Discriminant is negative.	Factors will be complex. Use the factorising technique of 'completing the square'.

26 Determining rational linear factors of a quadratic expression

For the expression $2x^2 - 3x + 1$, determine all linear factors over $\mathbb{R}$.

THINKING	WORKING
1 Calculate $b^2 - 4ac$.	$b^2 - 4ac = 9 - 4(2)(1)$ $= 1$
2 Interpret the value of $b^2 - 4ac$ to determine the factorisation technique. $b^2 - 4ac$ is rational, hence factorise by grouping.	$2x^2 - 3x + 1 = 2x^2 - 1x - 2x + 1$ $= x(2x - 1) - 1(2x - 1)$
3 Take out the common factor.	$= (2x - 1)(x - 1)$

27 Determining irrational linear factors of a quadratic expression

For the expression $2x^2 - 6x + 3$, determine all linear factors over $\mathbb{R}$.

THINKING	WORKING
1 Calculate $b^2 - 4ac$.	$b^2 - 4ac = 36 - 4(2)(3)$ $= 12$

2 Interpret the value of $b^2 - 4ac$ to determine the factorisation technique.

$b^2 - 4ac$ is not a perfect square, hence complete the square.

$$2x^2 - 6x + 3 = 2\left[x^2 - 3x + \frac{3}{2}\right]$$
$$= 2\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{3}{2}\right]$$
$$= 2\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{3}{2} \times \frac{2}{2}\right]$$
$$= 2\left[\left(x - \frac{3}{2}\right)^2 - \frac{3}{4}\right]$$

3 Use $a^2 - b^2 = (a + b)(a - b)$ to factorise.

$$= 2\left[\left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right]$$
$$= 2\left(x - \frac{3}{2} + \frac{\sqrt{3}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{3}}{2}\right)$$
$$= 2\left(x - \frac{3 - \sqrt{3}}{2}\right)\left(x - \frac{3 + \sqrt{3}}{2}\right)$$

28 Determining complex linear factors of a quadratic expression

For the expression $4x^2 - 3x + 1$, determine all linear factors over $\mathbb{C}$.

THINKING

1 Calculate $b^2 - 4ac$.

WORKING

$$b^2 - 4ac = 9 - 4(4)(1)$$
$$= -7$$

2 Interpret the value of $b^2 - 4ac$ to determine the factorisation technique.

$b^2 - 4ac$ is negative, hence complete the square.

$$4x^2 - 3x + 1 = 4\left[x^2 - \frac{3}{4}x + \frac{1}{4}\right]$$
$$= 4\left[x^2 - \frac{3}{4}x + \left(\frac{3}{8}\right)^2 - \left(\frac{3}{8}\right)^2 + \frac{1}{4}\right]$$
$$= 4\left[\left(x - \frac{3}{8}\right)^2 - \frac{9}{64} + \frac{1}{4} \times \frac{16}{16}\right]$$
$$= 4\left[\left(x - \frac{3}{8}\right)^2 - \frac{9}{64} + \frac{16}{64}\right]$$
$$= 4\left[\left(x - \frac{3}{8}\right)^2 + \frac{7}{64}\right]$$

3 Use $i^2 = -1$ to produce the difference of perfect squares form $a^2 - b^2$.

$$= 4\left[\left(x - \frac{3}{8}\right)^2 - \frac{7i^2}{64}\right]$$
$$= 4\left[\left(x - \frac{3}{8}\right)^2 - \left(\frac{i\sqrt{7}}{8}\right)^2\right]$$

4 Use $a^2 - b^2 = (a + b)(a - b)$ to factorise.

$$= 4\left(x - \frac{3}{8} + \frac{i\sqrt{7}}{8}\right)\left(x - \frac{3}{8} - \frac{i\sqrt{7}}{8}\right)$$
$$= 4\left(x - \frac{3 - i\sqrt{7}}{8}\right)\left(x - \frac{3 + i\sqrt{7}}{8}\right)$$

Note: Usually if you are asked to 'factorise over $\mathbb{C}$', you may need to introduce the imaginary number, i, in order to generate the $a^2 - b^2$ form. However, since real numbers are a subset of complex numbers, you may not necessarily need to introduce i.

Remember that there is a fundamental difference between an expression and an equation. Both can be factorised, but only an equation can be solved. The solution of an equation is the value or values that can be substituted for the unknown variable in order to satisfy the equation.

General solution of real quadratic equations

Just as the discriminant can be used to determine the correct factorising technique for real quadratic expressions, it can also be used to determine the number of real solutions of a real quadratic equation.

Value of discriminant ($b^2 - 4ac$)	Number of real solutions
Discriminant > 0	Quadratic will have two real solutions.
Discriminant = 0	Quadratic will have one real solution.
Discriminant < 0	Quadratic will have two complex solutions.

29 Solving quadratic equations by completing the square

Solve the equation $4x^2 + 2x + 1 = 0$ over $\mathbb{C}$ by completing the square.

THINKING

1 Calculate $b^2 - 4ac$. This will determine the number of real solutions. In this case, $b^2 - 4ac < 0$, so there will be two complex solutions.

2 Complete the square.

WORKING

$$b^2 - 4ac = 4 - 4(4)(1)$$
$$= -12$$

$$4x^2 + 2x + 1 = 0$$
$$4\left[x^2 + \frac{1}{2}x + \frac{1}{4}\right] = 0$$
$$x^2 + \frac{1}{2}x + \frac{1}{4} = 0$$
$$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{1}{4} = 0$$
$$\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + \frac{1}{4} \times \frac{4}{4} = 0$$
$$\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + \frac{4}{16} = 0$$
$$\left(x + \frac{1}{4}\right)^2 + \frac{3}{16} = 0$$

3 Use $i^2 = -1$ to produce the difference of perfect squares form $a^2 - b^2$.

$$\left(x + \frac{1}{4}\right)^2 - \frac{3i^2}{16} = 0$$
$$\left(x + \frac{1}{4}\right)^2 - \left(\frac{i\sqrt{3}}{4}\right)^2 = 0$$

4 Use $a^2 - b^2 = (a + b)(a - b)$ to factorise.

$$\left(x + \frac{1}{4} - \frac{i\sqrt{3}}{4}\right)\left(x + \frac{1}{4} + \frac{i\sqrt{3}}{4}\right) = 0$$

5 Solve using the null factor law.

$$x = -\frac{1}{4} + \frac{i\sqrt{3}}{4} \text{ or } x = -\frac{1}{4} - \frac{i\sqrt{3}}{4}$$
$$x = \frac{-1 + i\sqrt{3}}{4} \text{ or } x = \frac{-1 - i\sqrt{3}}{4}$$
or
$$x = \frac{-1 \pm i\sqrt{3}}{4}$$

The quadratic formula

Solutions to the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula is also able to be used to calculate complex solutions.

30 Using the quadratic formula to calculate complex solutions to a quadratic equation

Use the quadratic formula to solve the equation $4x^2 + 2x + 1 = 0$ over $\mathbb{C}$.

THINKING	WORKING
1 Calculate $b^2 - 4ac$. This will determine the number of real solutions. In this case, $b^2 - 4ac < 0$, so there will be two complex solutions.	$b^2 - 4ac = 4 - 4(4)(1)$ $= -12$
2 Write the quadratic formula and substitute the given values into the formula. $a = 4, b = 2, c = 1$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-2 \pm \sqrt{-12}}{8}$
3 Use $i^2 = -1$ and simplify.	$x = \frac{-2 \pm \sqrt{12i^2}}{8}$ $= \frac{-2 \pm \sqrt{4 \times 3i^2}}{8}$ $= \frac{-2 \pm 2i\sqrt{3}}{8}$ $= \frac{-1 \pm i\sqrt{3}}{4}$

Complex solutions as conjugate pairs

From the previous worked example, the solutions to $4x^2 + 2x + 1 = 0$ can be expressed as $x = -\frac{1}{4} + \frac{i\sqrt{3}}{4}$ and $x = -\frac{1}{4} - \frac{i\sqrt{3}}{4}$. Note that these two solutions are a complex conjugate pair. This applies to all complex solutions of a quadratic equation with real coefficients. Hence, if you know that $a + bi$ is one of the solutions of a real quadratic equation, then you can conclude that the conjugate $a - bi$ is the other solution.

The conjugate root theorem says that for polynomials, $P(z)$, with real coefficients, the complex roots of $P(z) = 0$ occur in conjugate pairs.

31 Determining solutions using complex conjugates

One of the solutions to $x^2 + 4x + 7 = 0$ is $x = -2 - i\sqrt{3}$. Determine the other solution.

THINKING	WORKING
Since the coefficients of $x^2 + 4x + 7$ are real, the other solution will be the conjugate of $-2 - i\sqrt{3}$.	$x = -2 + i\sqrt{3}$

Solving $(x + 2)(x - 5) = 0$ uses the null factor law to obtain the solutions $x = -2$ and $x = 5$. Reversing this process, if $x = P$ is a solution, then $(x - P)$ will be a factor.

32 Using complex conjugates to determine a quadratic equation

It is known that one solution of the quadratic equation $ax^2 + bx + c = 0$ is $x = 2 + 5i$. Use this information to calculate the values of a, b and c.

THINKING	WORKING
1 Using complex conjugates, write the two solutions to the quadratic equation.	$x = 2 + 5i, 2 - 5i$
2 From the solutions, write the two factors.	$(x - 2 - 5i)(x - 2 + 5i) = 0$
3 Use the difference of two squares to expand the brackets.	$[(x - 2) - 5i][(x - 2) + 5i] = 0$ $(x - 2)^2 - (5i)^2 = 0$ $x^2 - 4x + 4 - 25i^2 = 0$
4 Use $i^2 = -1$ and express in the form $ax^2 + bx + c = 0$.	$x^2 - 4x + 4 + 25 = 0$ $x^2 - 4x + 29 = 0$
5 Identify the values of a, b and c.	$a = 1, b = -4, c = 29$

Technology worked example

Factorising polynomials with complex roots

EXERCISE 5.5 Complex solutions of real quadratic functions

1 Which of the following quadratic equations will have complex conjugate pair solutions?

A $0 = 2x^2 + 3x + 3$ **B** $0 = x^2 - 4x + 2$ **C** $0 = x^2 + 2x + 1$ **D** $0 = 2x^2 + (3 - i)x + 4$

Worked Example

2 It is known that $x = 5 - 2i$ is one of the solutions to a real quadratic equation. What is the other solution?

A $x = 5 - 2i$ **B** $x = -5 - 2i$ **C** $x = -5 + 2i$ **D** $x = 5 + 2i$

26

3 For each of the following, determine all linear factors over $\mathbb{R}$.

(a) $3x^2 + 10x - 8$ **(b)** $5m^2 + 17m - 12$ **(c)** $6p^2 - p - 2$ **(d)** $12n^2 - 16n - 3$

30

4 Use the quadratic formula to solve each of the following over $\mathbb{C}$.

(a) $x^2 - x + 2 = 0$ **(b)** $3x^2 - 2x + 1 = 0$ **(c)** $5z^2 + 3z + 1 = 0$ **(d)** $-3z^2 + 2z - 1 = 0$

31

5 For each of the following quadratic equations, one solution is given. Determine the second solution.

(a) $x^2 - 8x + 19 = 0$; given solution: $x = 4 - i\sqrt{3}$

(b) $x^2 + 4x + 9 = 0$; given solution: $x = -2 + i\sqrt{5}$

6 What is the quadratic equation with roots $2 + 3i$ and $2 - 3i$?

A $z^2 + 4z + 13$ **B** $z^2 - 4z - 5$ **C** $z^2 - 4z - 1$ **D** $z^2 - 4z + 13$

27

7 For each of the following, complete the square and hence determine all linear factors over $\mathbb{R}$.

(a) $x^2 + 4x + 1$ **(b)** $x^2 + 2x - 6$ **(c)** $z^2 + 6z + 3$ **(d)** $4z^2 + 4z - 1$

28

8 For each of the following, determine all linear factors over $\mathbb{C}$.

(a) $x^2 + 4x + 6$ **(b)** $x^2 + 8x + 18$ **(c)** $3z^2 + 4z + 9$ **(d)** $2z^2 - 3z + 2$

29

9 For each of the following, complete the square, and hence solve over $\mathbb{C}$.

(a) $x^2 + 8x + 18 = 0$ **(b)** $2z^2 + 2z + 6 = 0$ **(c)** $3z^2 - 3z + 2 = 0$ **(d)** $2z^2 - 5z + 4 = 0$

10 Demonstrate that $z^2 + 3z + 3$ can be expressed in factorised form as $\frac{1}{4}\left(2z + 3 - i\sqrt{3}\right)\left(2z + 3 + i\sqrt{3}\right)$.

32

11 For each of the following, one root of a quadratic equation with real coefficients is given. Use this information to determine the quadratic equation.

(a) $x = 2 - \sqrt{3}i$ **(b)** $x = 5 + 2i$ **(c)** $z = \frac{5 - 4i}{3}$ **(d)** $z = \frac{7 - i\sqrt{3}}{4}$

12 Determine all solutions for the equation $z^4 - 21z^2 - 100 = 0$.

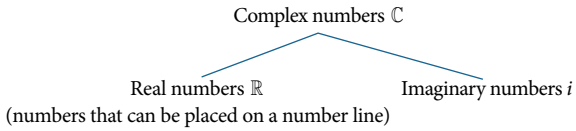
13 Determine all solutions for the equation $z^4 + 10z^2 + 9 = 0$.

Summary

5

Imaginary numbers

The imaginary number i is defined such that $i^2 = -1$.



Definition of a complex number

A complex number consists of two parts, a *real part* and an *imaginary part*, and is defined as $C = \{z : z = x + iy$, where $x, y \in \mathbb{R}$, and $i^2 = -1\}$

The real component of z is $\text{Re}(z) = x$; and the imaginary component of z is $\text{Im}(z) = y$.

Addition and subtraction of complex numbers

To add two or more complex numbers, add the real components and add the imaginary components.

To subtract two or more complex numbers, subtract the real components and subtract the imaginary components.

Multiplication by a scalar

To multiply a complex number by a constant (or scalar), simply multiply the real and imaginary components by that scalar.

If $z = a + bi$ and if $k \in \mathbb{R}$, then $kz = k(a + bi) = ka + kbi$.

Multiplication in rectangular form

To multiply two complex numbers, use the distributive law to expand and, wherever an i^2 results, use $i^2 = -1$ to replace the i^2 with -1.

Equality of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 = z_2$ if and only if $a = c$ and $b = d$.

Complex conjugates

For any complex number $z = x + iy$, its complex conjugate, $\overline{z}$ (read: z-bar) is defined as $\overline{z} = x - iy$.

An important property of a conjugate pair is: $z.\overline{z} = (x + yi)(x - yi) = x^2 - y^2 i^2 = x^2 + y^2$.

When graphed on the Argand plane, the conjugate $\overline{z} = a - bi$ appears as a reflection of the original complex number z in the Real axis.

Multiplicative inverse of a complex number

Consider the complex number $z = a + bi$. You can calculate z^{-1}, the multiplicative inverse of z, such that $z \,.\, z^{-1} = 1$.

If $z = a + bi$, then the multiplicative inverse is: $z^{-1} = \dfrac{a - bi}{a^2 + b^2}$.

Division of complex numbers

To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 + (di)^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\end{aligned}$$

Graphing complex numbers

Complex numbers can be plotted on the Argand plane, such that the real component of the complex number is plotted on the horizontal axis, labelled Re(z), and the imaginary component is plotted on the vertical axis, labelled Im(z).

Linear transformations on the Argand plane

When multiplied by a scalar, k, the direction of a complex number is unchanged (unless $k < 0$, in which case the direction is reversed). The magnitude of the complex number increases by a factor of k.

Multiplication by a scalar results in the *dilation* of the Argand plane.

Multiplication by i results in a *rotation* of 90° in an anticlockwise direction about the origin.

The modulus of a complex number

The *modulus* of z is the length of the line segment joining z to the origin.

If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

The argument of a complex number

The argument is the angle of the complex number, measured anticlockwise from the positive Real axis.

The principal Argument of z lies in the domain $-\pi < \theta \leq \pi$ and is denoted Arg(z).

Complex numbers in polar form

A complex number in rectangular form, $z = a + bi$, can be written as $z = r\cos(\theta) + ir\sin(\theta)$ where r is the modulus of z and θ is the principal Argument of z.

$z = r\cos(\theta) + ir\sin(\theta)$ can be abbreviated to $r\,\mathrm{cis}(\theta)$.

Operations in polar form

For complex numbers $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$:

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

For complex numbers $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$:

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis}(\theta_1)}{r_2 \operatorname{cis}(\theta_2)}$$
$$= \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

De Moivre's theorem states that $\left[r \operatorname{cis}(\theta) \right]^n = r^n \operatorname{cis}(n\theta)$, for $n \in \mathbb{Z}$.

Complex solutions of real quadratic functions

When factorising a real quadratic expression $ax^2 + bx + c$, the discriminant, $b^2 - 4ac$, can be calculated.

If $b^2 - 4ac$ is a perfect square, factors will be rational. If $b^2 - 4ac$ is a positive, non-perfect square, factors will be irrational. If $b^2 - 4ac$ is negative, factors will be complex.

General solution of real quadratic equations

When solving a real quadratic expression $ax^2 + bx + c = 0$, the discriminant, $b^2 - 4ac$, can be calculated.

If $b^2 - 4ac > 0$, the quadratic will have two real solutions. If $b^2 - 4ac = 0$, the quadratic will have one real solution. If $b^2 - 4ac < 0$, the quadratic will have two complex solutions.

The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can be used to solve the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

The conjugate root theorem states that for polynomials, $P(z)$, with real coefficients, the complex roots of $P(z) = 0$ occur in conjugate pairs.

Chapter review 5

Exercise 5.1

1 If $z = 7 + 4i$, then $\text{Im}(z) =$

A 4 **B** 7 **C** -4 **D** $4i$

Exercise 5.1

2 What is $(3i)^2$ equal to?

A 3 **B** -3 **C** 9 **D** -9

Exercise 5.1

3 What is $(8 + 5i) - (7 - 2i)$ equal to?

A $1 + 3i$ **B** $1 - 3i$ **C** $1 + 7i$ **D** $1 - 7i$

Exercise 5.1

4 Determine the conjugates of the following complex numbers.

(a) $z = 2 + 8i$ **(b)** $z = -3 - 6i$ **(c)** $z = 5i$

Exercise 5.1

5 If $z_1 = 5 + i$ and $z_2 = -4 + 2i$, calculate:

(a) $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $z_2 - z_1$ **(d)** $2z_1 + z_2$

Exercise 5.1

6 If $z_1 = 5 - 2i$ and $z_2 = 8 + 4i$, calculate:

(a) $3z_1 - z_2$ **(b)** $5z_1 + 6z_2$

Exercise 5.2

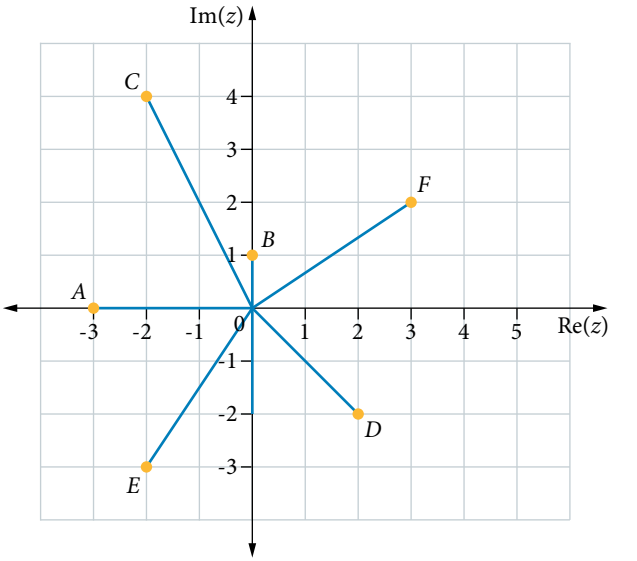
7 Represent each of the following complex numbers on an Argand plane.

(a) $a = 3$ **(b)** $b = -2i$ **(c)** $c = 2 + 2i$

(d) $d = -1 - 3i$ **(e)** $e = a + b$ **(f)** $f = c - d$

Exercise 5.2

8 Determine the complex numbers represented by each of the points indicated on the following Argand diagram.



9 A complex number $u = a + bi$ is multiplied by i^2 to produce a complex number v. Which of the following is true? Exercise 5.2

A v is a reflection of u in the x-axis.

B v is a reflection of u in the y-axis.

C v is a rotation of u, 180° anticlockwise about the origin.

D v is a rotation of u, 270° anticlockwise about the origin.

10 When dividing complex numbers in polar form, what should you do? Exercise 5.4

A Subtract the moduli, divide the Arguments. **B** Divide the moduli, subtract the Arguments.

C Divide the moduli, add the Arguments. **D** Multiply by the complex conjugate.

11 Which of the following angles is not in the form of a principal Argument? Exercise 5.4

A $\frac{5\pi}{6}$ **B** $\frac{2\pi}{3}$ **C** $\frac{8\pi}{7}$ **D** π

12 Calculate the modulus of each of the following complex numbers: Exercise 5.4

(a) $z = 3 + i$ **(b)** $z = 3 + 4i$ **(c)** $z = -3 + 10i$ **(d)** $z = 4 + 7i$

13 Calculate the Argument of the following complex numbers in the domain $-\pi < \theta \leq \pi$. Give your answer in exact form where possible, otherwise correct to 2 decimal places. Exercise 5.4

(a) $z = \sqrt{3} + i$ **(b)** $z = 10 - 10i$ **(c)** $z = -2 + 3i$

14 If $z = 20\,\text{cis}\left(\frac{\pi}{3}\right)$ then which of these is true? Exercise 5.4

A $\overline{z} = 20\,\text{cis}\left(\frac{\pi}{3}\right)$ **B** $\overline{z} = 20\,\text{cis}\left(-\frac{\pi}{3}\right)$ **C** $\overline{z} = 20\,\text{cis}\left(\frac{2\pi}{3}\right)$ **D** $\overline{z} = 20\,\text{cis}\left(-\frac{2\pi}{3}\right)$

15 Express the following complex numbers in rectangular form, $x + yi$. Give your answers in exact form where possible. Exercise 5.4

(a) $z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ **(b)** $z = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

16 It is known that $x = 8 + 6i$ is one of the solutions to a real quadratic equation. What is the other solution? Exercise 5.5

A $x = 8 + 6i$ **B** $x = 8 - 6i$ **C** $x = -8 + 6i$ **D** $x = -8 - 6i$

17 For each of the following, determine all linear factors over $\mathbb{R}$. Exercise 5.5

(a) $15t^2 - 8t - 12$ **(b)** $-2z^2 + z + 15$

18 For each of the following quadratic equations, one solution is given. Determine the second solution. Exercise 5.5

(a) $x^2 - 3x + 3 = 0$; given solution: $x = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

(b) $z^2 - \frac{4}{3}z + 1 = 0$; given solution: $z = \frac{2 - i\sqrt{5}}{3}$

19 Expand and simplify: Exercise 5.1

(a) $(1 + 4i)(2 + 5i)$ **(b)** $(3 - i)(4 - 2i)$ **(c)** $(1 + 2i)^2(-3 - 4i)$ **(d)** $(4 - 3i)^2(4 + 3i)^2$

20 Determine the values of a and b that satisfy the equation $(6 - 3i)(a + bi) = (24 + 18i)$. Exercise 5.1

Exercise 5.1 **21** Calculate the multiplicative inverse, z^{-1}, of the following complex numbers, giving your answer in the form $a + bi$:

(a) $z = 2 + i$ **(b)** $z = -3 + 4i$

Exercise 5.1 **22** Express each of the following in the form $x + yi$:

(a) $\dfrac{2+i}{3-2i}$ **(b)** $\dfrac{2+2i}{-1+i}$

Exercise 5.1 **23** For the complex numbers $a = 2 + 3i$, $b = -1 + 2i$ and $c = 3 - 2i$, demonstrate the following:

(a) $(ab)c = a(bc)$ **(b)** $a(b + c) = ab + ac$ **(c)** $(a + b) + c = a + (b + c)$

Exercise 5.1 **24** Calculate z such that $6z - 3i = 18 - 27i$.

Exercise 5.2 **25** Point A is defined by the complex number $u = 3 + 2i$ and point B is defined by the complex number $v = 2 - 5i$ on the Argand plane.

(a) Calculate the length of line segment $\overline{AB}$.

(b) Points A' and B' are created by multiplying complex numbers u and v by $z = 1 - 2i$. Calculate the length of line segment $\overline{A'B'}$.

Exercise 5.4 **26** Multiply the following complex numbers, expressing your answer in polar form:

(a) $z = (4 + i)(1 - i)$ **(b)** $z = (1 + 2i)(3 + 4i)$

Exercise 5.4 **27** Calculate the following complex number products, giving your answer in polar form. Ensure that you express the Argument in the range $-\pi < \theta \leq \pi$.

(a) $2\,\text{cis}\left(\dfrac{\pi}{3}\right) \times 3\,\text{cis}\left(\dfrac{\pi}{4}\right)$ **(b)** $\sqrt{3}\,\text{cis}\left(\dfrac{5\pi}{6}\right) \times \sqrt{5}\,\text{cis}\left(\dfrac{\pi}{3}\right)$

Exercise 5.4 **28** Calculate the following complex number quotients, giving your answer in polar form. Ensure that you express the Argument in the range $-\pi < \theta \leq \pi$.

(a) $10\,\text{cis}\left(\dfrac{2\pi}{3}\right) \div 2\,\text{cis}\left(\dfrac{\pi}{3}\right)$ **(b)** $6\,\text{cis}\left(-\dfrac{3\pi}{4}\right) \div \text{cis}\left(\dfrac{\pi}{2}\right)$

Exercise 5.4 **29** Simplify the following complex number powers using De Moivre's theorem. Express your answers in polar form, with the argument in the domain $-\pi < \theta \leq \pi$.

(a) $\left[\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{3}\right)\right]^4$ **(b)** $\left[2\,\text{cis}\left(\dfrac{\pi}{8}\right)\right]^5$

Exercise 5.5 **30** What is the quadratic equation with roots $-3 + i$ and $-3 - i$?

A $z^2 + 6z + 8$ **B** $z^2 + 6z + 10$ **C** $z^2 - 6z + 10$ **D** $z^2 - 6z + 8$

Exercise 5.5 **31** For each of the following, complete the square and hence determine all linear factors over $\mathbb{R}$.

(a) $9z^2 - 12z - 2$ **(b)** $x^2 - 6x + 4$

Exercise 5.5 **32** For each of the following, determine all linear factors over $\mathbb{C}$.

(a) $x^2 - 6x + 10$ **(b)** $4z^2 - 3z + 5$

Exercise 5.5 **33** For each of the following, complete the square, and hence solve over $\mathbb{C}$.

(a) $x^2 - 6x + 12 = 0$ **(b)** $x^2 + 2x + 6 = 0$

Exercise 5.5 **34** Use the quadratic formula to solve each of the following over $\mathbb{C}$.

(a) $x^2 + 2x + 3 = 0$ **(b)** $2z^2 - 3z + 4 = 0$

35 Use substitution to verify the following: Exercise 5.5

(a) $2 + 3i$ and $2 - 3i$ are roots of the equation $z^2 - 4z + 13 = 0$.

(b) $-1 + \sqrt{3}\,i$ and $-1 - \sqrt{3}\,i$ are roots of the equation $z^2 + 2z + 4 = 0$.

(c) $\frac{1}{2}(1 + 3i)$ and $\frac{1}{2}(1 - 3i)$ are roots of the equation $2z^2 - 2z + 5 = 0$.

36 Determine z such that $(3 + i) \times z = 10 - 10i$. Exercise 5.1

37 Determine z such that $2z - 12 = iz - i$. Exercise 5.1

38 Determine z such that $z^2 = 7 - 24\,i$. Exercise 5.1

39 If $z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $w = 4\,\text{cis}\left(\frac{\pi}{4}\right)$, calculate $z + w$, giving your answer in rectangular form, $x + yi$. Exercise 5.4

40 If $z = 6\,\text{cis}\left(\frac{\pi}{2}\right)$ and $w = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$, calculate $z - w$. Give your answer in polar form. Exercise 5.4

41 Demonstrate that $z^2 + 5z + \frac{27}{4}$ can be expressed in factorised form as $\frac{1}{4}\left(2z + 5 - i\sqrt{2}\right)\left(2z + 5 + i\sqrt{2}\right)$. Exercise 5.5

42 Determine all solutions for the equation $z^4 + 25z^2 + 144 = 0$. Exercise 5.5

43 For each of the following, one root of a quadratic equation with real coefficients is given. Use this information to determine the quadratic equation. Exercise 5.5

(a) $x = 3 + \sqrt{2}i$

(b) $x = 2 + 8i$





6

Graphing functions

Recall 286

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6.3 Rational functions 315

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6

Recall

Sketch linear functions by first determining axial intercepts

1 Sketch the graphs of each of the following linear functions by first determining the coordinates of the axial intercepts.

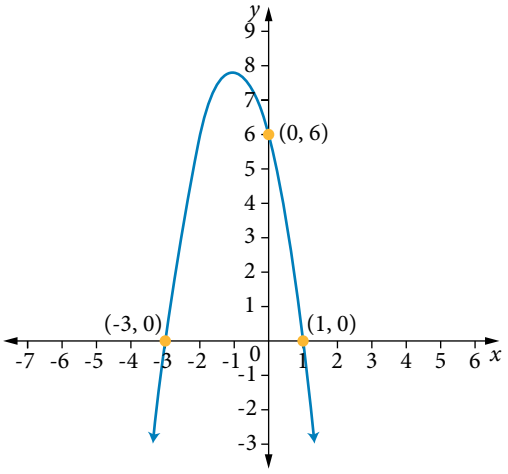
(a) $y = 3x + 6$

(b) $2x - 3y = -12$

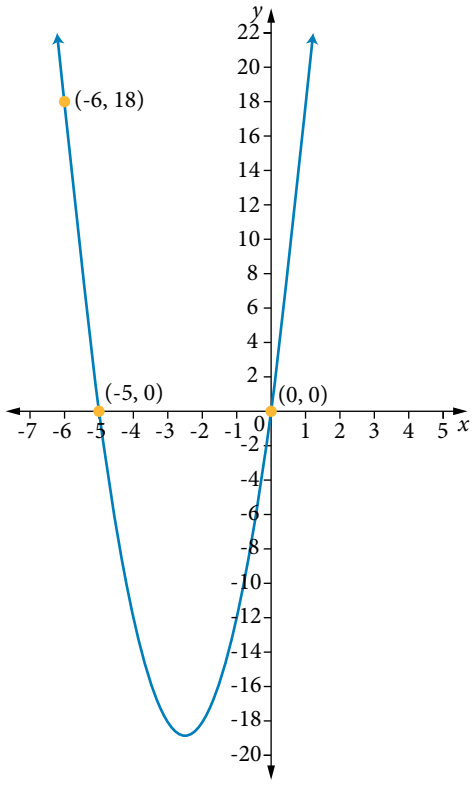
Determine the equation of a parabola given the graph

2 Determine the equation of each of the following parabolas.

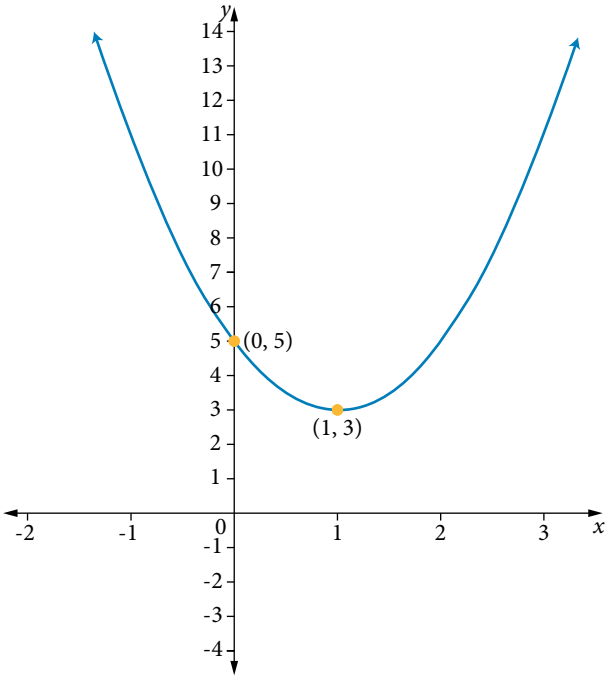
(a)



(b)



(c)



Solve equations using the null factor law

3 Solve the following equations for x.

(a) $x(x-3)(x+2)=0$ **(b)** $0=\dfrac{x(x+5)}{x^2+1}$

Simplify algebraic fractions

4 Simplify the following expressions.

(a) $\dfrac{3x^2+2x}{5x}$ **(b)** $\dfrac{x^3-25x}{3x-15}$ **(c)** $\dfrac{x+3}{x^2-x-12}$

Determine points of intersection of two functions algebraically

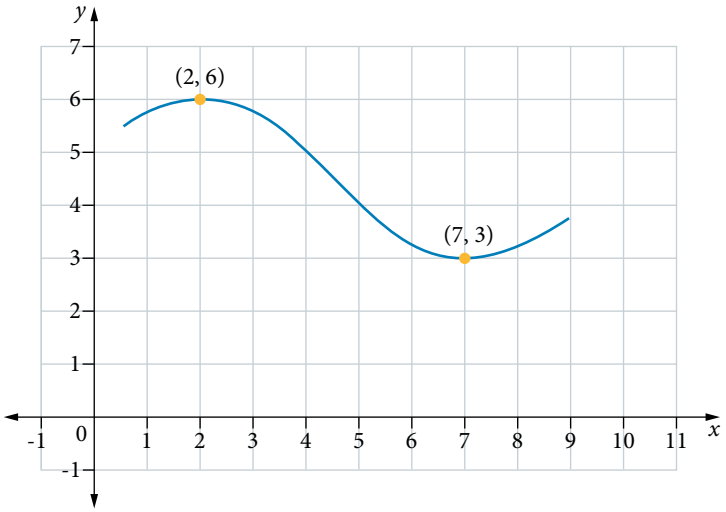
5 Use algebra to determine the coordinates of all points of intersection of the following pairs of functions:

(a) $y=3x$ and $y=x^2$ **(b)** $y=x^2+3$ and $y=4x$

Sketch the graphs of $f(x)+k$, $f(x-h)$, and $f(x-h)+k$ given the graph of $f(x)$

6 Given the graph of the function $y=f(x)$ shown, sketch the following functions:

(a) $y=f(x)-3$ **(b)** $y=f(x-2)$ **(c)** $y=f(x-2)-3$



Use polynomial long division to simplify algebraic fractions

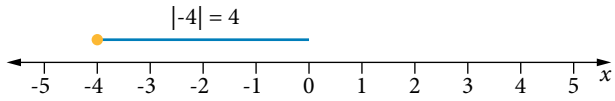
7 Use polynomial long division to simplify each of the following:

(a) $\dfrac{x^2+4x+5}{x-2}$ **(b)** $\dfrac{2x^2-4x+1}{x+1}$ **(c)** $\dfrac{4x^2+1}{x-3}$

6.1 Absolute value functions

The absolute value of a real number

If x is a real number, then the *absolute value* (or *modulus*) *of* x, denoted by $|x|$, is the distance of x from zero on a number line. For example, $|-4| = 4$ as the number -4 is 4 units from 0 on a number line.



Similarly, $|4| = 4$, $|-1.3| = 1.3$, and $|0| = 0$.

The absolute value of a real number x can be defined more formally: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

1 Determining the values of numerical expressions involving absolute values

Determine each of the following.

(a) $|-5|$

THINKING	WORKING
Write the absolute value of the number. If the number is negative, the negative sign is dropped.	$\|-5\| = 5$

(b) $-3|-12 + 2 \times 5|$

Calculate the expression within the vertical bars. Then write the absolute value of the number, and complete any further operations.	$-3\|-12 + 2 \times 5\|$ $= -3\|-2\|$ $= -3 \times 2$ $= -6$

(c) $-5 \times |-6| - |5 \times -6|$

Calculate the expressions within the vertical bars. Then write the absolute value of the numbers, and complete any further operations.	$-5 \times \|-6\| - \|5 \times -6\|$ $= -5 \times \|-6\| - \|-30\|$ $= -5 \times 6 - 30$ $= -60$

When solving simple equations involving absolute values, it is important to remember that there are usually two possible values for the expression within the absolute value bars that give the same result. For example, if $|x - 1| = 3$, then since both $|3| = 3$ and $|-3| = 3$, this equation is satisfied if the expression $x - 1$ is equal to either 3 or -3.

2 Solving simple equations involving absolute values

Solve the following equations.

(a) $|x + 3| = 5$

THINKING	WORKING
1 Write the two possibilities for the expression within the absolute value bars.	$\|x + 3\| = 5$ implies that $x + 3 = 5$ or $x + 3 = -5$.
2 Solve the two equations.	$x + 3 = 5$ $x = 2$ or $-(x + 3) = 5$ $x + 3 = -5$ $x = -8$
3 Answer the question.	Therefore, $x = 2$ or $x = -8$.

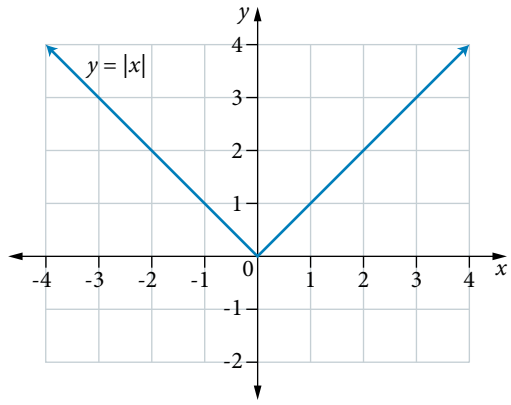
(b) $|10 - 2x| = 4$

1 Write the two possibilities for the expression within the absolute value bars.	$\|10 - 2x\| = 4$ implies that $10 - 2x = 4$ or $10 - 2x = -4$.
2 Solve the two equations.	$10 - 2x = 4$ $-2x = -6$ $x = 3$ or $-(10 - 2x) = 4$ $10 - 2x = -4$ $-2x = -14$ $x = 7$
3 Answer the question.	Therefore, $x = 3$ or $x = 7$.

Note that not all equations of the form of the previous examples will have two solutions. For example, there is only one solution to the equation $|x - 5| = 0$ since if the absolute value of an unknown number is zero, the only possibility is that the unknown number is equal to zero. The equation $|x - 5| = -3$ actually has no solutions, as the absolute value of a number cannot be negative.

Absolute value graphs

The graph of the function $y = |x|$ is shown on the right.



Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, the graph of $y = |x|$ resembles the graph of $y = x$ where $x \geq 0$, and the graph of $y = -x$ where $x < 0$.

The point where the graph changes direction sharply is called the *vertex*. Thus, the vertex of $y = |x|$ has coordinates (0, 0).

In prior courses, you will have learnt how the graph of $y = a f(x - h) + k$ can be obtained by applying transformations to the graph $y = f(x)$, based on the values of a, h, and k.

Applying this to the function $f(x) = |x|$ allows you to readily sketch any graph of the form $y = a|x - h| + k$.

Recall that the values of h and k correspond to the amount by which the original graph is translated horizontally and vertically, respectively. As the vertex of $y = |x|$ has coordinates (0, 0), it follows that the vertex of $y = a|x - h| + k$ has coordinates (h, k).

Also recall that the value of a corresponds to the vertical dilation factor. If a is negative, the graph is reflected in the x-axis. If $a > 1$ or $a < -1$, the new graph will appear vertically narrower than the original. If $-1 < a < 1$, the new graph will appear vertically wider than the original.

Explore further

Absolute value graph

Graph the function $y = a|x - h| + k$ and explore the transformations when you move the sliders for a, h, k.

3 Sketching graphs of simple transformations of $y = |x|$

Sketch the graph of the function $y = -2|x - 3| + 4$, identifying the coordinates of the vertex, and any intercepts with the axes.

THINKING	WORKING
1 Write the values of a, h and k, and, hence, determine the coordinates of the vertex (h, k).	The equation of this function has the form $y = a\|x - h\| + k$ with $a = -2$, $h = 3$, and $k = 4$. The coordinates of the vertex are (3, 4).
2 Describe the shape of the graph based on the value of a.	Since $a = -2$, in comparison to the graph of $y = \|x\|$, the shape of the graph will be narrower, dilated by a factor of 2 parallel to the y-axis and reflected through the horizontal axis.

3 Determine the y-intercept by setting $x = 0$ in the equation. Similarly, determine any x-intercepts by setting $y = 0$ in the equation.

To determine the y-intercept, set x equal to 0 in the equation and calculate the value of y:

$y = -2|0 - 3| + 4$

$= -2$

The coordinates of the y-intercept are (0, -2).

To determine the x-intercepts, set y equal to 0 in the equation and solve for x:

$0 = -2|x - 3| + 4$

$-4 = -2|x - 3|$

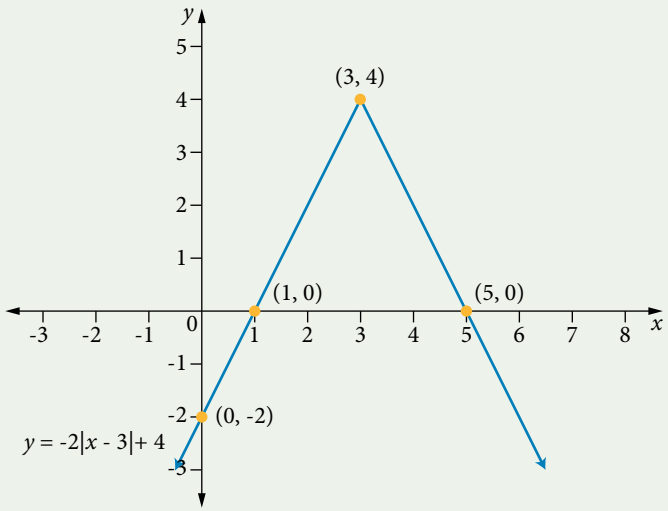
$2 = |x - 3|$

$x - 3 = 2$ or $x - 3 = -2$

Therefore, $x = 5$ or $x = 1$.

The coordinates of the x-intercepts are (1, 0) and (5, 0).

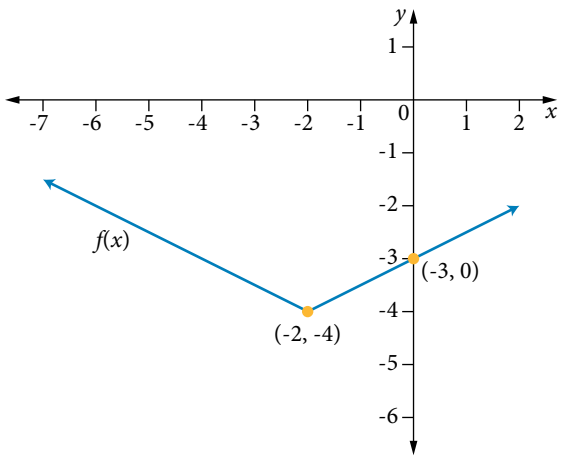
4 Use the information gathered to sketch the graph, indicating the coordinates of the vertex and intercepts.



The following example illustrates how the equation of an absolute value graph can be obtained provided the coordinates of the vertex and one other point are known.

4 Determining the equation of a simple absolute value graph

Determine the equation of the function drawn below.



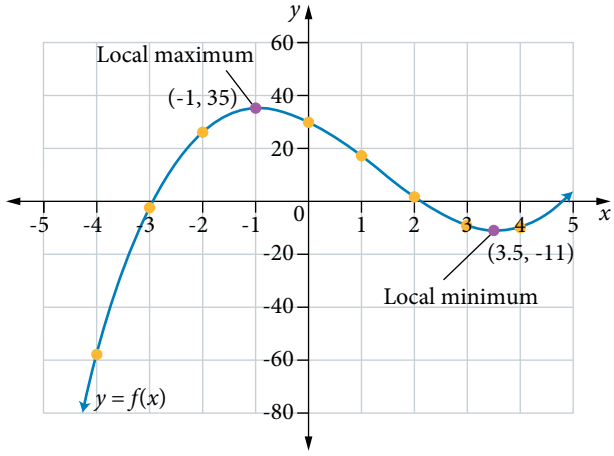
THINKING	WORKING
1 Recall the general form for the equation, and determine the values for h and k if the coordinates of the vertex are given.	The equation of this function has the form $y = a\|x - h\| + k$. The coordinates of the vertex are (-2, -4). Therefore $h = -2$ and $k = -4$. This gives $y = a\|x + 2\| - 4$.
2 Determine the value of a by substituting the x- and y-coordinates of any other point on the graph.	For (0, -3): $-3 = a\|0 + 2\| - 4$ $1 = 2a$ $a = \frac{1}{2}$
3 Write the equation.	The equation of the function is $y = \frac{1}{2}\|x + 2\| - 4$.

Graphs of the form $y = |f(x)|$ and $y = f(|x|)$

Consider the arbitrarily chosen function, $f(x) = x^3 - 4x^2 - 10x + 30$. A table of values has been constructed and used to sketch the graph of $y = f(x)$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-58	-3	26	35	30	17	2	-9	-10

The points on the function called *local maximum* or *local minimum* of $f(x)$ are the points where the function reaches an extreme value (*local extrema*). For example, when $f(x)$ extends upwards from minus infinity and crosses the x-axis, it reaches a local maximum of $f(x) = 35$, turns around and extends downwards to a local minimum of $f(x) = -11.2$.



Suppose you were interested in sketching the graph of $y = |f(x)|$.

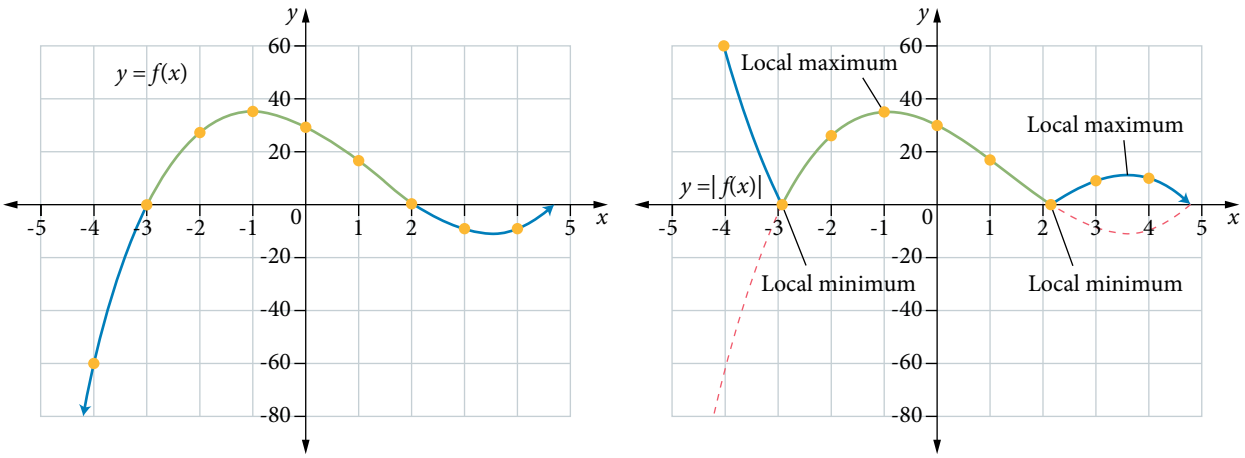
An extra row has been added to the table below in order to do this.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-58	-3	26	35	30	17	2	-9	-10
$\|f(x)\|$	58	3	26	35	30	17	2	9	10

Notice that for a particular value of x, $|f(x)|$ will be equal to $f(x)$ if $f(x) \geq 0$ (the green columns in the table), and equal to $-f(x)$ if $f(x) < 0$ (the blue columns in the table).

This means that if a section of the graph of $f(x)$ is on or above the x-axis, the corresponding section of the graph of $|f(x)|$ will appear identical. Alternatively, if a section of the graph of $f(x)$ is below the x-axis, the corresponding section of the graph of $|f(x)|$ will be a reflection of this section of the graph of $f(x)$ in the x-axis.

The original graph and the new graph are shown side by side below. On the original graph, the part of the graph on or above the x-axis is coloured green, while the parts below the x-axis (that need to be reflected) are shown in blue.



The graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by replicating the part(s) of $y = f(x)$ that are on or above the x-axis, and reflecting the remaining part(s) in the x-axis.

Now suppose you were interested in sketching the graphs of $y = f(|x|)$.

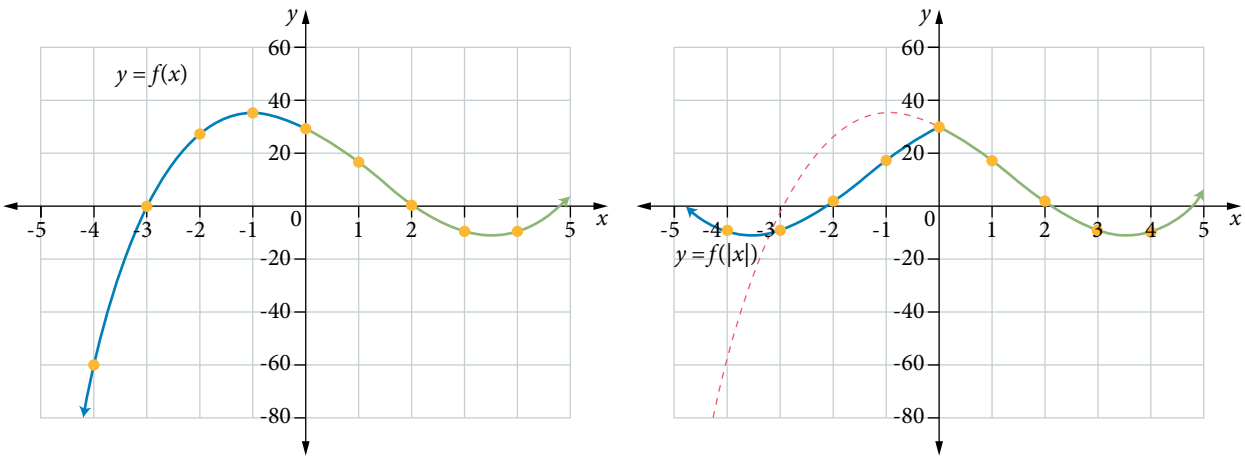
Once again, an extra row has been added to the table in order to do this.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-58	-3	26	35	30	17	2	-9	-10
$f(\|x\|)$	-10	-9	2	17	30	17	2	-9	-10

Notice that for a particular value of x, $f(|x|)$ will be equal to $f(x)$ if $x \geq 0$ (the green columns in the table), and equal to $f(-x)$ if $x < 0$ (the blue columns in the table).

This means that on and to the right of the y-axis, the graph of $f(|x|)$ is identical to the graph of $f(x)$. To the left of the y-axis, the graph of $f(|x|)$ will be a reflection of the part of the graph $f(x)$ that is to the right of the y-axis.

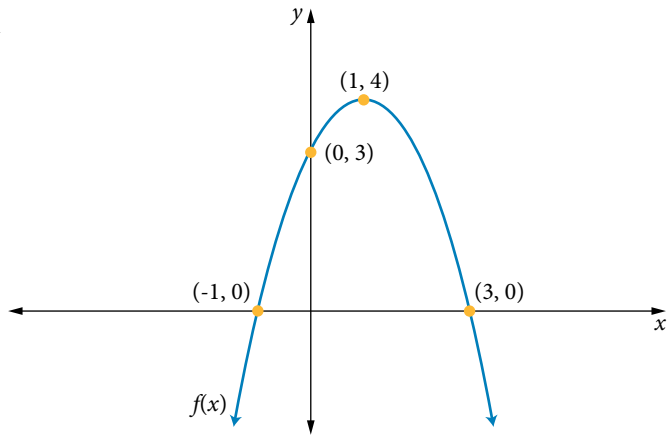
The original graph and the new graph are shown side-by-side below. On the original graph, the part of the graph to the right of the y-axis is coloured green.



The graph of $y = f(|x|)$ can be obtained from the graph of $y = f(x)$ by replicating the part of $y = f(x)$ that is on or to the right of the y-axis, and reflecting the remaining part in the y-axis.

5 Sketching graphs of the form $y = |f(x)|$ and $y = f(|x|)$

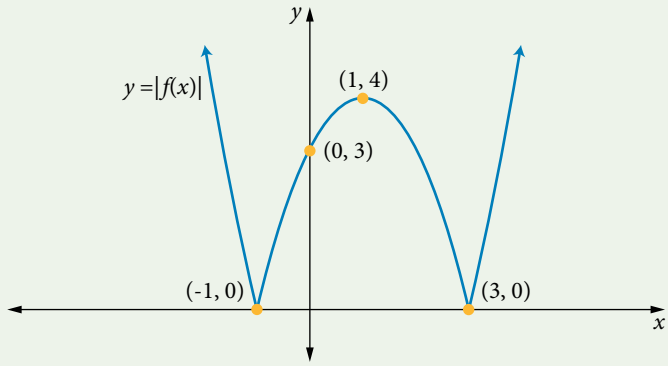
For the function $f(x)$ graphed on the right, sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$. Clearly indicate the location of all intercepts and turning points.



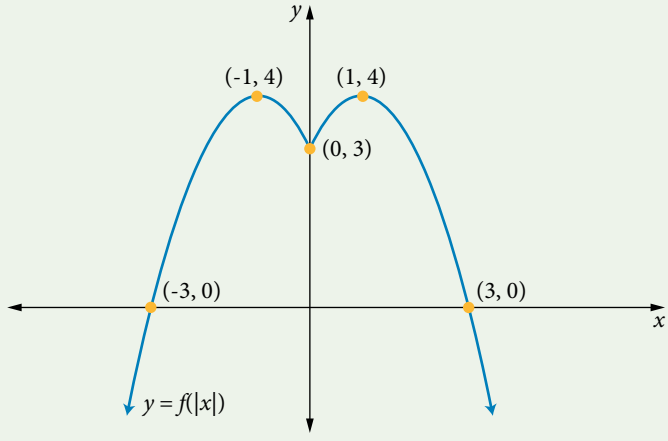
THINKING

1 Replicate the part(s) of $y = f(x)$ that are on or above the x-axis, and reflect the remaining part(s) in the x-axis.

WORKING



2 Replicate the part(s) of $y = f(x)$ that are on or to the right of the y-axis, reflect the remaining part(s) in the y-axis.



Solving more complex equations involving absolute values

Consider the equation $|2x| = x - 3$. It may seem reasonable to conclude that x is a solution to this equation if and only if $2x = x - 3$ or $-2x = x - 3$.

However, you need to be mindful of the domain of each solution given, because $y = |2x|$ represents the function:

$$y = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$2x = x - 3$ or $-2x = x - 3$

$x = -3$ $\quad -3x = -3$

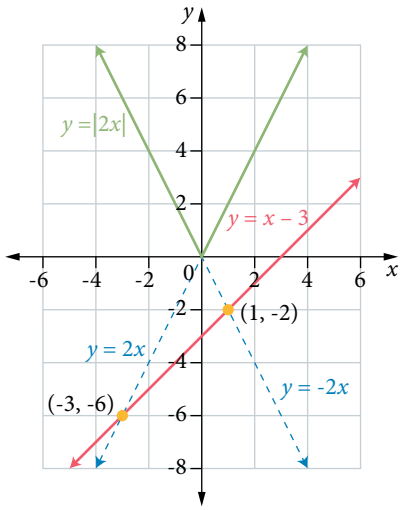
$x = 1$

Considering the functions above, you can see that the solutions to the equations lie outside the given domain. Algebraically, you can demonstrate that $x = 1$ and $x = -3$ are not solutions to $|2x| = x - 3$ by substituting them back into the right-hand side of the equation:

For $x = 1$: $\quad -2 = x - 3$

For $x = -3$: $\quad -6 = x - 3$

These values cannot be solutions to the equation $|2x| = x - 3$ because $|2x| \geq 0$. This can be demonstrated graphically by first sketching the graph of $y = 2x$ and then creating $y = |2x|$ by taking any part of the graph sitting below the x-axis and reflecting it across the x-axis.



6 Solving more complex equations involving absolute values

Solve for x.

$$\left|\frac{1}{2}x - 6\right| = x + 3$$

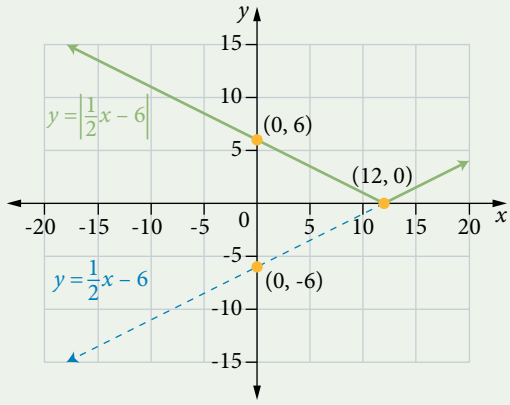
METHOD 1

THINKING	WORKING
1 Write the two possibilities for the expression within the absolute value bars. Since the expression on the right-hand side involves a variable, write the condition that this expression must be greater than or equal to zero.	$\left\|\frac{1}{2}x - 6\right\| = x + 3$ implies that $\frac{1}{2}x - 6 = x + 3$ or $\frac{1}{2}x - 6 = -(x + 3)$. It's also necessary that $x + 3 \geq 0$.

2 Solve the two equations.	$\frac{1}{2}x - 6 = x + 3$ $x - 12 = 2x + 6$ $-18 = x$ or $\frac{1}{2}x - 6 = -(x + 3)$ $x - 12 = -2x - 6$ $3x = 6$ $x = 2$
3 Verify whether each solution satisfies the required inequality, and determine the valid solution(s).	If $x = 2$, $x + 3 \geq 0$. But if $x = -18$, $x + 3 < 0$. Therefore, the only solution is $x = 2$.

METHOD 2

THINKING	WORKING
1 Sketch the graph of the absolute function to determine an appropriate domain.	Start by sketching $y = \frac{1}{2}x - 6$ and then reflecting the portion of the graph below the x-axis. $y = \left\lvert \frac{1}{2}x - 6 \right\rvert$ $= \begin{cases} \frac{1}{2}x - 6, & x \geq 12 \\ -\left(\frac{1}{2}x - 6\right), & x < 12 \end{cases}$



The coordinates of the intercepts are (12, 0) and (0, 6).

2 Determine the appropriate domain of solutions.

For $x < 12$ calculate the solution:

$$-\left(\frac{1}{2}x - 6\right) = x + 3$$
$$\frac{1}{2}x - 6 = -x - 3$$
$$\frac{3}{2}x - 6 = -3$$
$$\frac{3}{2}x = 3$$
$$x = 2$$

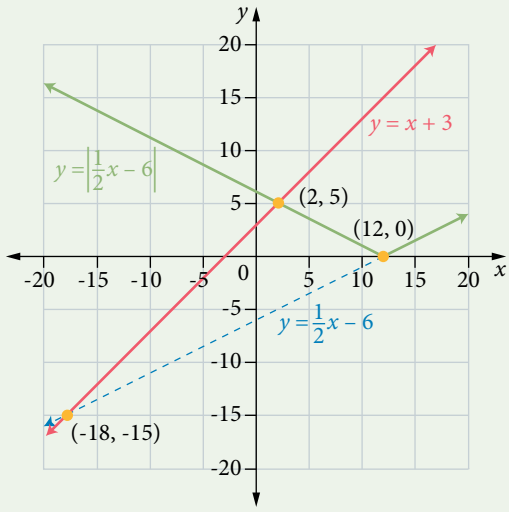
Given that the solution lies within the appropriate domain, $x = 2$ is a solution.

For $x \geq 12$, calculate the solution:

$$\frac{1}{2}x - 6 = x + 3$$
$$-\frac{1}{2}x - 6 = 3$$
$$-\frac{1}{2}x = 9$$
$$x = -18$$

Given that the solution lies outside the domain $x \geq 12$, this is not a solution to the equation.

3 Verify your answer graphically.



The coordinates of the point of intersection of the graphs $y = \left|\frac{1}{2}x - 6\right|$ and $y = x + 3$ are (2, 5).

Therefore the solution to the equation $\left|\frac{1}{2}x - 6\right| = x + 3$ is $x = 2$.

Technology worked example

Solving absolute value equations graphically

EXERCISE 6.1 Absolute value functions

Worked Example

1 Determine the following:

(a) $|-7|$ (b) $|31|$ (c) $|-5-3|$ (d) $-|-4|$

(e) $|-1+5\times 3|$ (f) $-|-3|^3$ (g) $|-5+|-5||$

2 Solve the following equations:

(a) $|x+7|=4$ (b) $|2x-5|=9$ (c) $|4x-10|=2$

(d) $|7-3x|=1$ (e) $2|10+4x|=3$ (f) $5|7-2x|=15$

(g) $|2x+6|=0$ (h) $|7-3x|=0$ (i) $|x-7|=-3$

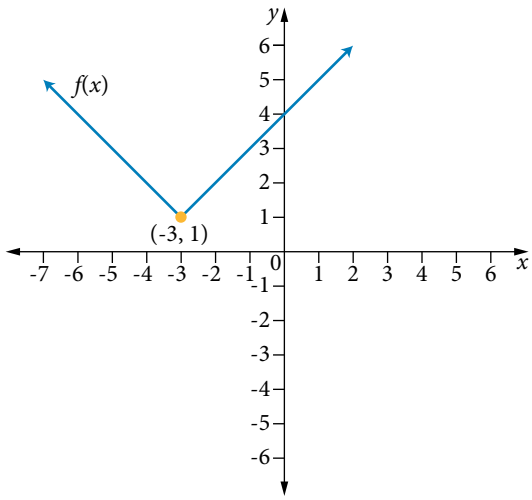
3 Sketch the graph of each of the following functions, identifying the coordinates of the vertex, and any intercepts with the axes.

(a) $y=|x+2|-5$ (b) $y=3|x-1|-6$ (c) $y=2|x+2|+4$ (d) $y=-|x-3|+5$

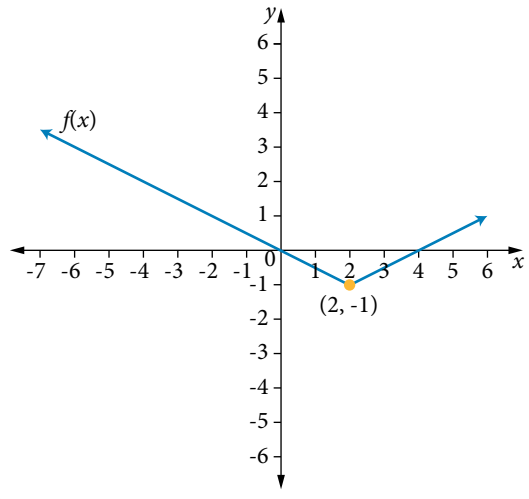
(e) $y=-2|x+1|+3$ (f) $y=-\frac{3}{4}|x-5|$ (g) $y=\frac{2}{3}|x-5|-1$

4 Determine the equation of each of the following functions:

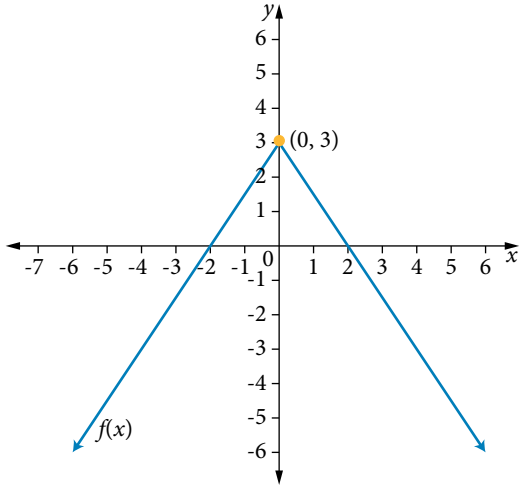
(a)



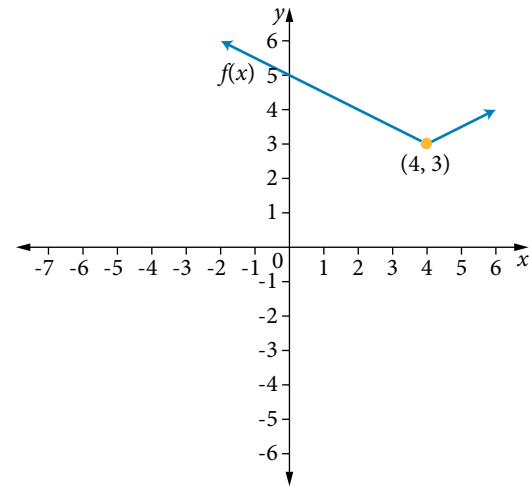
(b)



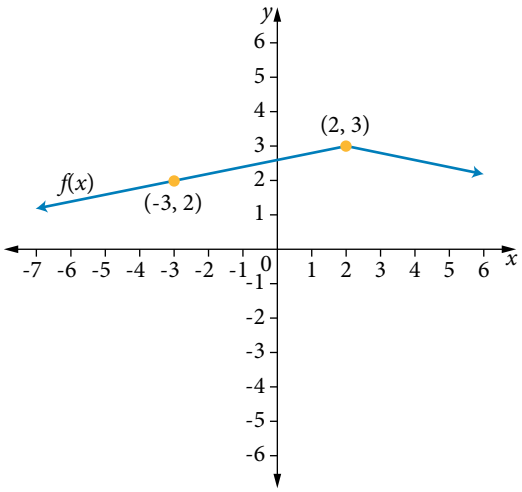
(c)



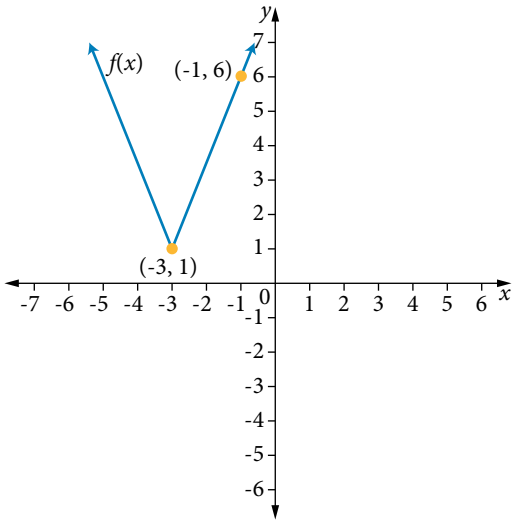
(d)



(e)



(f)

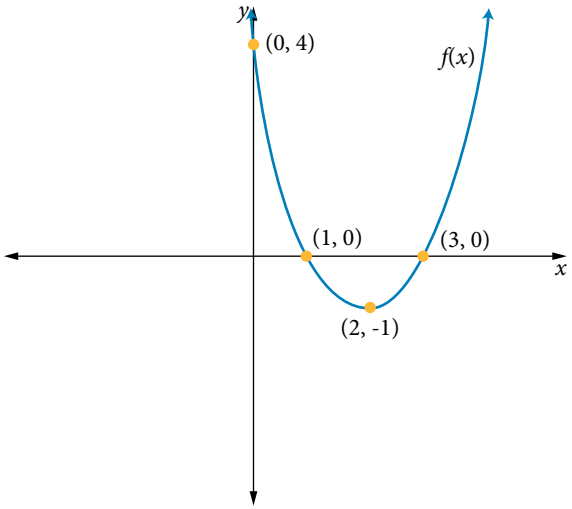


Worked Example

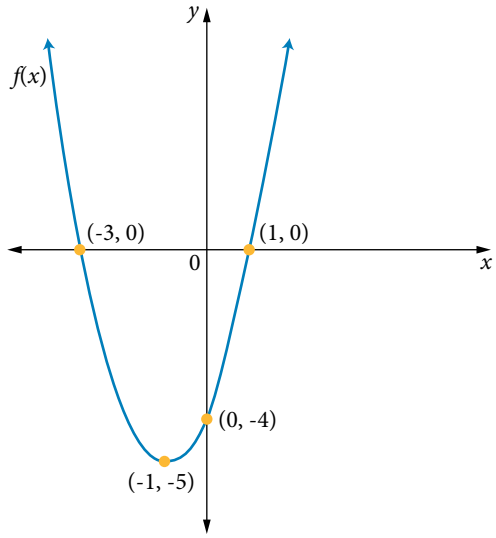
5

5 For each of the functions $f(x)$ drawn below, sketch the graphs of **(i)** $y = |f(x)|$ and **(ii)** $y = f(|x|)$. Clearly indicate the location of all intercepts and turning points.

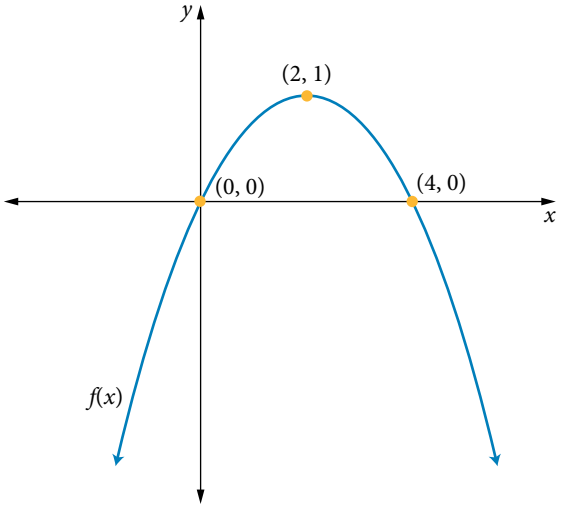
(a)



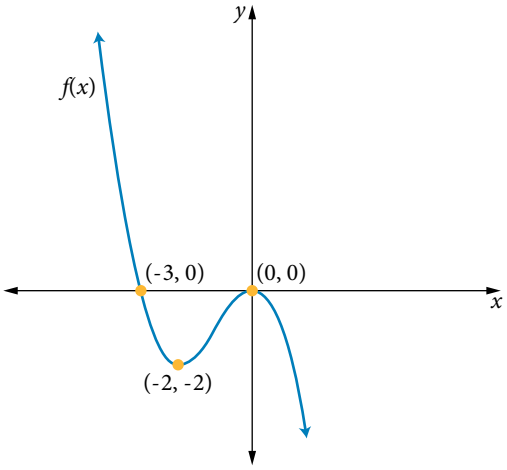
(b)



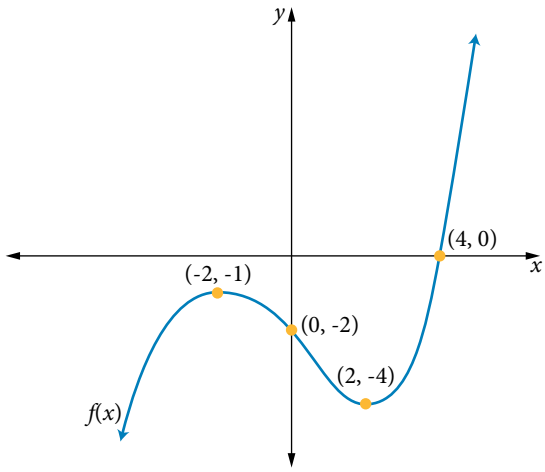
(c)



(d)

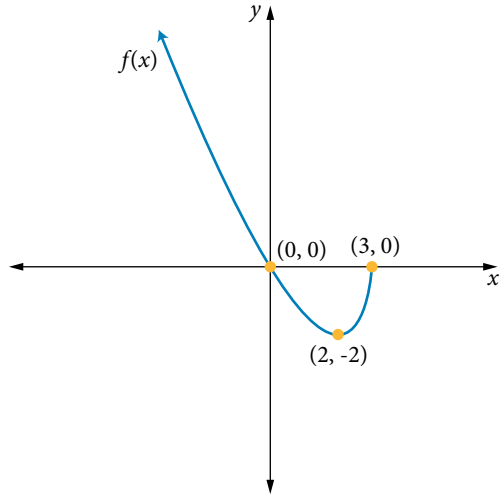


(e)



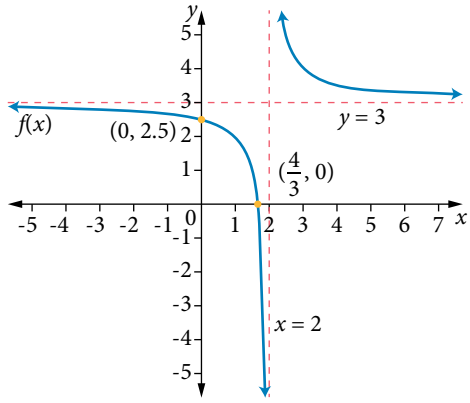
6 The function $y = -x\sqrt{3-x}$ is shown. Sketch the following graphs, clearly indicating all intercepts and turning points.

(a) $y = \left|-x\sqrt{3-x}\right|$ **(b)** $y = -|x|\sqrt{3-|x|}$



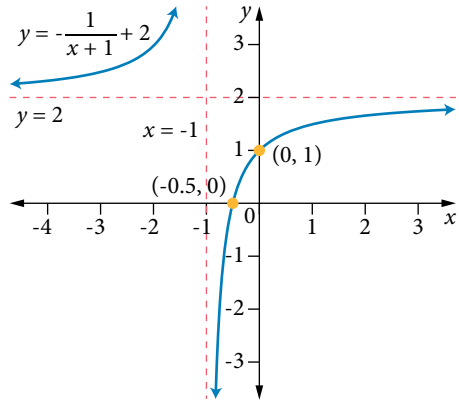
7 The function $y = \dfrac{1}{x-2} + 3$ is shown. Sketch the following graphs, clearly indicating all intercepts and asymptotes.

(a) $y = \left|\dfrac{1}{x-2} + 3\right|$ **(b)** $y = \dfrac{1}{|x|-2} + 3$



8 The function $y = -\dfrac{1}{x+1} + 2$ is shown. Sketch the following graphs, clearly indicating all intercepts and asymptotes.

(a) $y = \left|-\dfrac{1}{x+1} + 2\right|$ **(b)** $y = -\dfrac{1}{|x|+1} + 2$



Worked Example 6

9 Sketch the graphs of both $f(x) = 3|x| - 1$ and $g(x) = -x + 3$ on the same set of axes, clearly indicating any points of intersection. Hence, solve the equation $3|x| - 1 = -x + 3$.

10 Solve the equation $x^2 = |x - 1| + 2$. Express your solution(s) correct to 2 decimal places.

11 Solve the equation $\left|\frac{3}{x} + 2\right| = 2x + 6$. Express your solution(s) correct to 2 decimal places.

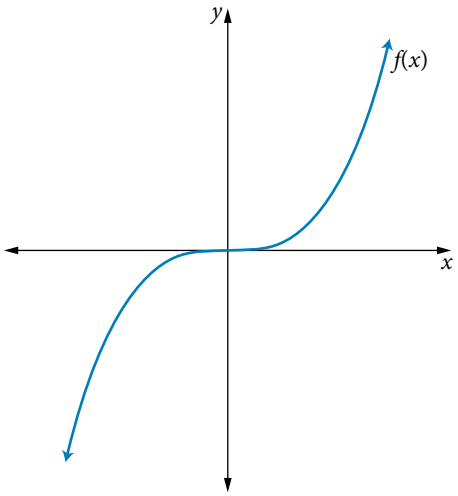
12 **(a)** Which of the following is *not* true for all real numbers a and b?

A $|-ab| = |ab|$ **B** $|a - b| = |b - a|$ **C** $|-ab| = |-a||-b|$ **D** $|a - b| = |a| - |b|$

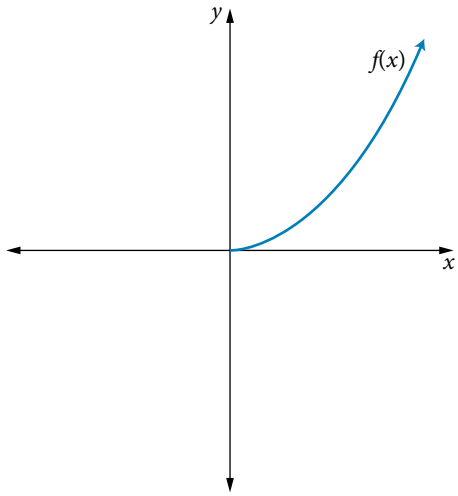
(b) Explain the common errors that might lead to selecting the incorrect alternatives.

13 For which of the following functions would the graph of $y = f(|x|)$ be identical to the graph of $y = f(x)$?

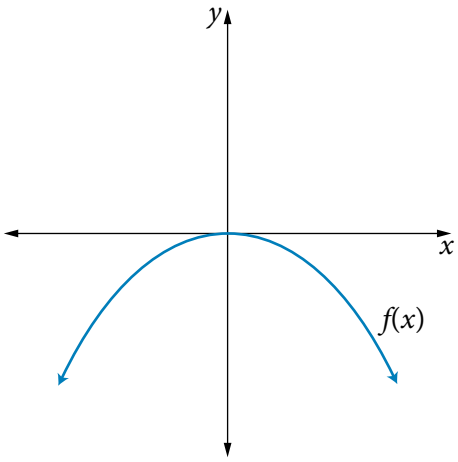
A



B



C

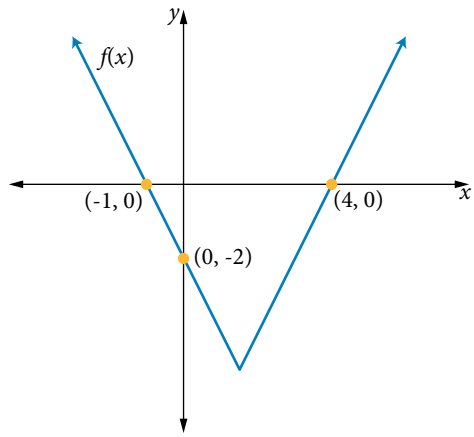


D none of A, B or C

14 Solve the following equations for x:

(a) $|2x - 1| = x + 7$ **(b)** $|2x - 11| = 3x - 4$ **(c)** $|x - 3| = -3x$ **(d)** $|x + 1| = 5 - 3x$

15 Determine the equation of the function shown. Evaluate the reasonableness of your equation by sketching its graph with the aid of your graphics calculator.



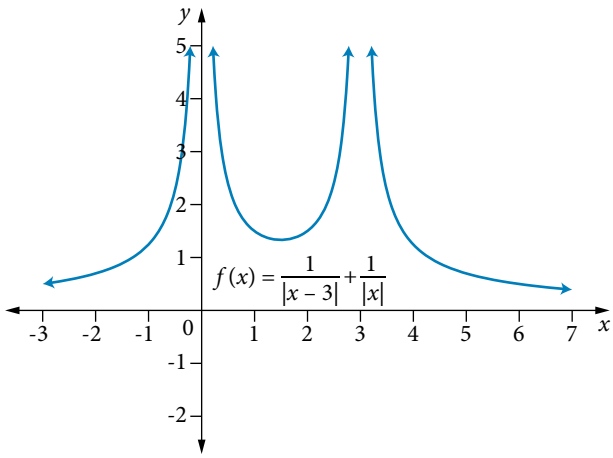
16 A band releases a new single for purchase for a period of 50 days. The number of copies sold, N, during day d, where $d = 1, 2, 3, \ldots, 50$, can be modelled by the function $N = -90|d - 18| + 3000$. For how many days was the number of copies sold greater than or equal to 2000?

17 A remote-controlled helicopter, initially resting on a table 1.4 metres high, ascends upwards at a constant speed of 2.5 m/s for 7.2 seconds, after which it immediately descends at the same speed.

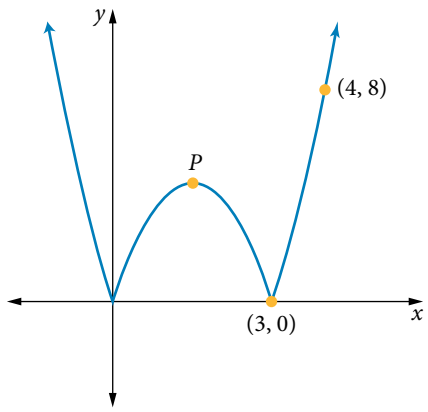
(a) Using absolute value notation, determine a function $h(t)$ where h represents the height of the helicopter in metres above the ground, and t represents the time in seconds since it started ascending.

(b) A remote-controlled plane initially on the ground starts flying at the same instant as the helicopter, with its height above the ground given by the function $h(t) = 0.2t^2$. Determine the time(s) at which the helicopter and plane are the same height above the ground.

18 The diagram shows the graph $y = \dfrac{1}{|x-3|} + \dfrac{1}{|x|}$. Using an algebraic method, determine all solutions to the equation $\dfrac{1}{|x-3|} + \dfrac{1}{|x|} = \dfrac{3}{2}$ such that $0 < x < 3$. Evaluate the reasonableness of your result with reference to the graph.

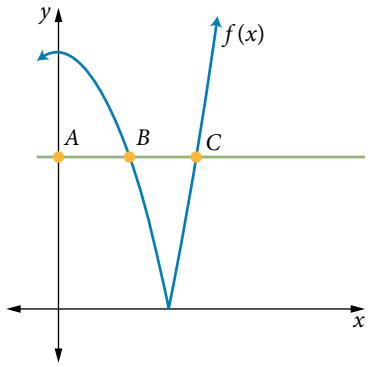


19 The diagram shows the graph of $y = |f(x)|$ where $f(x)$ is a quadratic function. Determine the coordinates of point P.



20 Consider the function $f(x) = x(x - 2)(x + p)$ where p is an unknown real number. Calculate the value of p if the graph of $y = f(|x|)$ intersects the graph of $y = f(x)$ when $x = -3$. Evaluate the reasonableness of your solution.

21 The diagram on the right shows part of the graph of the function $f(x) = |x^2 - 5|$ together with a horizontal line that intersects the y-axis at point A. The part of the horizontal line in the first quadrant intersects the function at points B and C as shown. Determine the coordinates of B given that B is the midpoint of AC.



6.2 Reciprocal functions

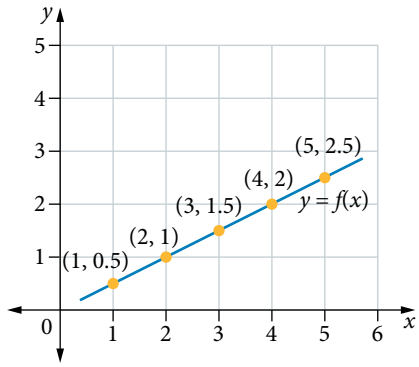
Recall that the *reciprocal* of a non-zero number, y, is the number $\frac{1}{y}$. For example, the reciprocal of 10 is $\frac{1}{10} = 0.1$; the reciprocal of 0.4 is $\frac{1}{0.4} = 2.5$; the reciprocal of 0.001 is $\frac{1}{0.001} = 1000$.

In this section, you will examine the relationship between the graph of a function $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$, known as the reciprocal function of $f(x)$.

If (a, b) is a point on the graph of $y = f(x)$ where $b \neq 0$, then $\left(a, \frac{1}{b}\right)$ will be a point on the graph of $y = \frac{1}{f(x)}$.

7 Sketching reciprocal functions from coordinates

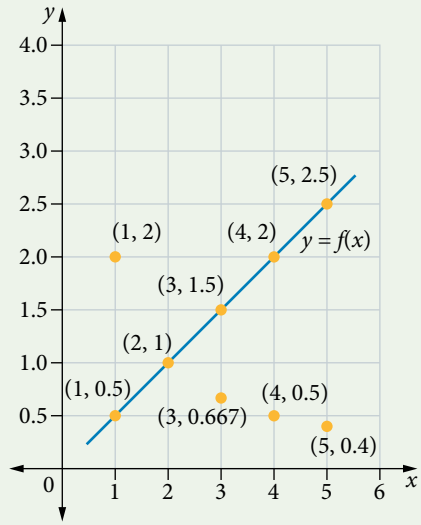
A section of the graph of a function $f(x)$ is shown with the coordinates of several points displayed. Replicate this diagram. Then determine the reciprocal of the y-coordinate of each point given, and hence, sketch the corresponding section of $y = \frac{1}{f(x)}$ on the same set of axes.



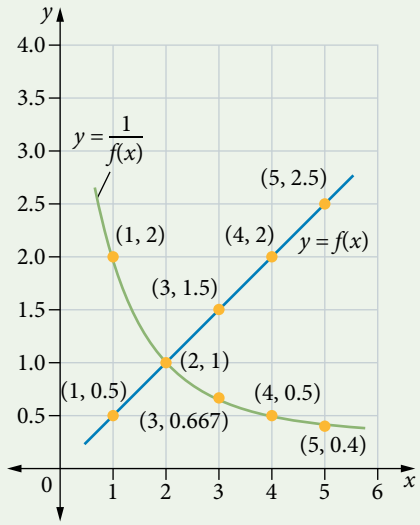
THINKING

1 For each point given, plot a new point with the same x-coordinate, but where the y-coordinate is the reciprocal of the original y-coordinate.

WORKING



2 Join the new points with a smooth curve.



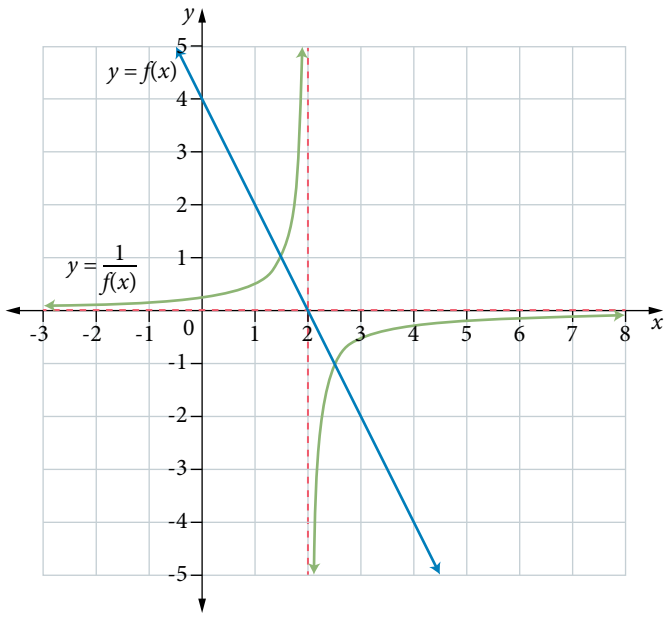
In the previous example, the coordinates of a cluster of points on the original graph was known. Often, however, you will only be given the coordinates of several key points on the graph. The coordinates of the corresponding points on the graph of the reciprocal function can still be obtained in the manner demonstrated, but in order to produce a valid sketch of the reciprocal function that includes all interesting features, it is necessary to understand a few properties of these functions.

Explore further

Reciprocal functions: coordinates

Explore the reciprocal coordinates in a function, moving the dots on the curve.

To the right is a graph of an arbitrary linear function (drawn in blue) and its reciprocal (drawn in green).



Notice first that the x-axis appears to be a horizontal asymptote for the reciprocal function. (Recall from Mathematical Methods that an asymptote for a function is an imaginary straight line that the graph of the function continues to get closer and closer to indefinitely if you follow the line in one or both directions.) This makes sense: in the original function, as you move along the x-axis in either direction, towards ∞ or $-\infty$, the y-coordinate of the original function approaches $-\infty$ or ∞ respectively. But as numbers approach ∞ or $-\infty$, their reciprocals become closer and closer to 0. (For example, $\frac{1}{10} = 0.1$, $\frac{1}{100} = 0.01$, $\frac{1}{1000} = 0.001$ and so on.)

Provided a function approaches $\pm\infty$ as x approaches ∞ or as x approaches $-\infty$, the x-axis will be a horizontal asymptote for the reciprocal function.

WARNING

Note that not all functions approach $\pm\infty$ as x approaches ∞ or as x approaches $-\infty$. In such a case, the x-axis will not be a horizontal asymptote for the reciprocal function.

Next, notice that the reciprocal function has a vertical asymptote at the location where the original function has an x-intercept. This is because as you get closer and closer to an x-intercept (from either the right or the left), the y-coordinate of the original function becomes closer and closer to 0. The closer a number is to 0, the larger the absolute value of its reciprocal. (For example, $\frac{1}{0.1} = 10$, $\frac{1}{0.01} = 100$, $\frac{1}{0.001} = 1000$ and so on.) Hence the graph of the reciprocal function rises sharply (or falls in the negative direction sharply) near an x-intercept of the original function. At the exact location of the x-intercept, the reciprocal function is undefined (since the y-coordinate here would be $\frac{1}{0}$ which is undefined.)

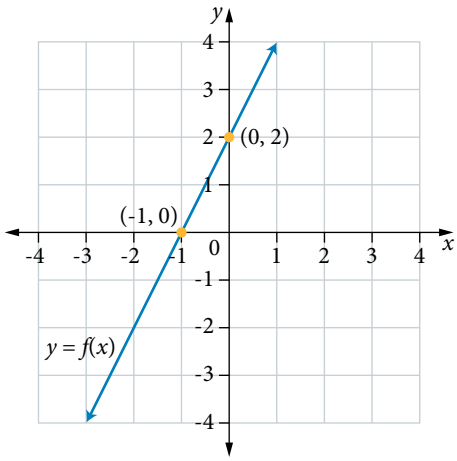
The reciprocal function has vertical asymptotes wherever the original function has x-intercepts.

Statement	Explanation
If a function is increasing in some interval, then the reciprocal function will be decreasing in that interval, and vice versa.	This can be explained by considering any list of numbers that increase in size; say 5, 6, 7 and 8. If the reciprocal of each number is taken, you obtain $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ and $\frac{1}{8}$.
If the original function is above the x-axis on some interval, then the reciprocal function will also be above the x-axis on this interval. The same is true for intervals where the original function is below the x-axis.	Notice that whenever the graph of the original function is above the x-axis, the graph of the reciprocal function is also above the x-axis. The same is true when the original function is below the x-axis. This is because the reciprocal of a positive number is always positive, and the reciprocal of a negative number is always negative.
A function and its reciprocal intersect each other at any points on the original function with y-coordinate equal to 1 or -1.	Notice that the original function and the reciprocal function intersect each other when the y-coordinate of the original function is equal to 1 or -1. This is because the reciprocal of the number 1 is equal to itself, and the same is true for the number -1. There are no other numbers with this property.

Combining these insights should allow you to effectively sketch the reciprocal of any linear function given a sketch of the original. The following example illustrates the specifics of how to do this.

8 Sketching $y = \frac{1}{f(x)}$ from the graph of $y = f(x)$ when $f(x)$ is linear

The graph of the function $f(x)$ is shown on the right. Sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes and intercepts.

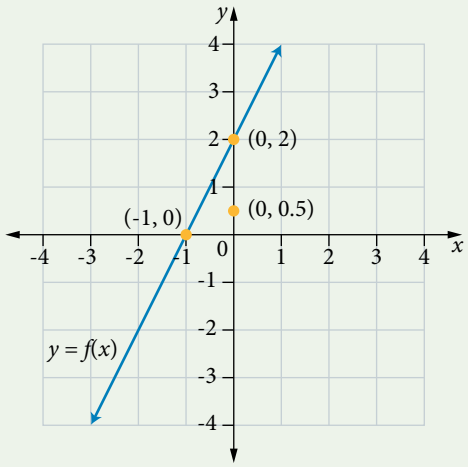


THINKING

1 Determine the y-intercept of the reciprocal function and mark it on the graph. (The y-intercept of the reciprocal function is the reciprocal of the y-intercept of the original function.)

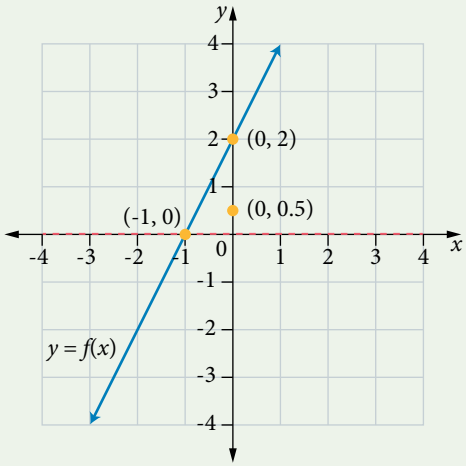
WORKING

Since $f(x)$ has a y-intercept at $(0, 2)$, $\frac{1}{f(x)}$ will have a y-intercept at $(0, 0.5)$.



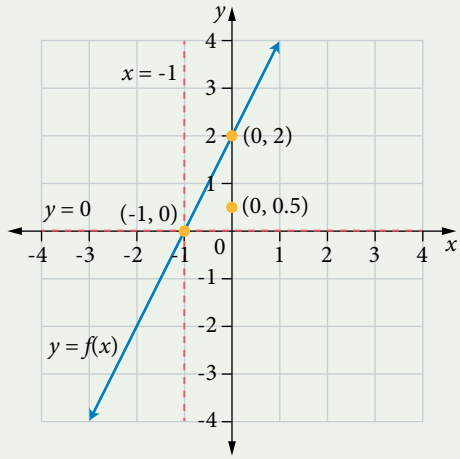
2 Determine whether the x-axis is a horizontal asymptote; if it is, draw a dotted line along the x-axis to indicate this.

Since $f(x)$ approaches ∞ as x approaches ∞ and $f(x)$ approaches $-\infty$ as x approaches $-\infty$, $\frac{1}{f(x)}$ will have a horizontal asymptote along the x-axis.

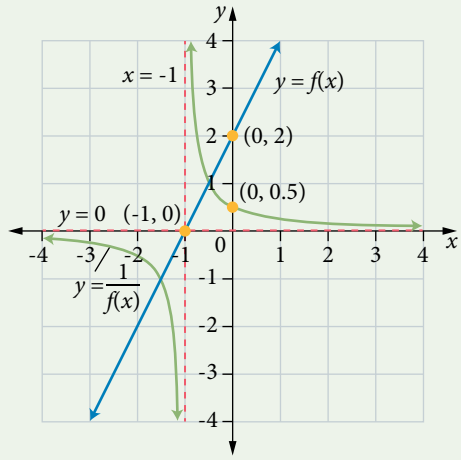


3 Determine the equations of any vertical asymptotes and add the asymptotes to the graph. (The reciprocal function will have vertical asymptotes corresponding to the x-intercepts of the original function.)

Since $f(x)$ has an x-intercept at (-1, 0), $\frac{1}{f(x)}$ will have a vertical asymptote with equation $x = -1$.



4 Sketch the graph consistent with the information already found.



The technique described in the previous example can also be used to sketch reciprocal functions when the original function is non-linear.

If a function contains a local minimum at a particular x-value that is not also an x-intercept, then the reciprocal function will contain a local maximum at this x-value. Similarly, if a function contains a local maximum at a particular x-value that is not also an x-intercept, then the reciprocal function will contain a local minimum at this x-value.

This result follows directly from the earlier observation that if a function is increasing in some interval, then the reciprocal function will be decreasing in that interval, and vice versa. Thus, if a function goes from increasing to decreasing (which corresponds to a local maximum), then the reciprocal function will go from decreasing to increasing (which corresponds to a local minimum).

Note that if a function contains a local minimum or maximum that is also an x-intercept, then the reciprocal function will have an asymptote corresponding to this x-value, rather than a local minimum or maximum.

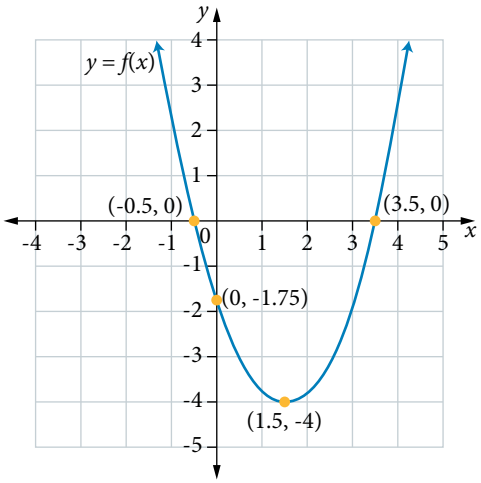
Explore further

Reciprocal functions: non-linear functions

Sketch the reciprocal functions when the original function is non-linear.

9 Sketching $y = \frac{1}{f(x)}$ from the graph of $y = f(x)$ when $f(x)$ contains local extrema

The graph of the function $f(x)$ is shown on the right. Sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes, intercepts, and local extrema.

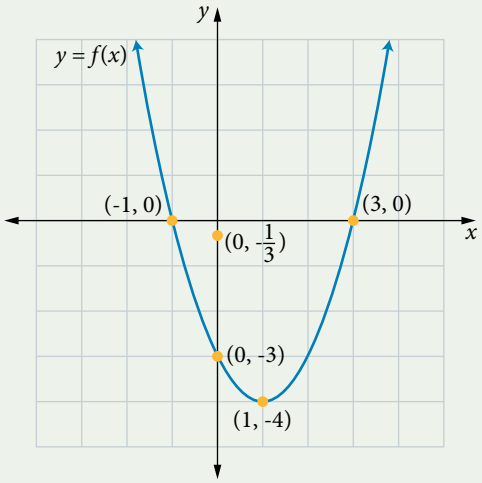


THINKING

1 Determine the y-intercept of the reciprocal function, and mark it on the graph. (The y-intercept of the reciprocal function is the reciprocal of the y-intercept of the original function.)

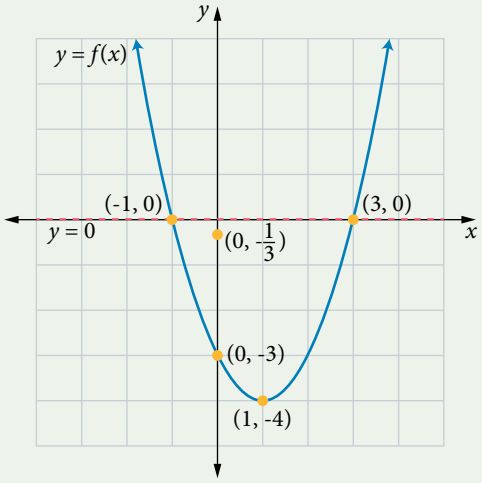
WORKING

Since $f(x)$ has a y-intercept at $(0, -3)$, $\frac{1}{f(x)}$ will have a y-intercept at $\left(0, -\frac{1}{3}\right)$.



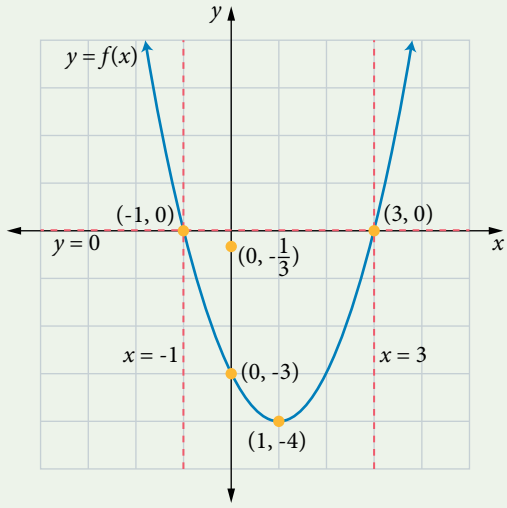
2 Determine whether the x-axis is a horizontal asymptote; if it is, draw a dotted line along the x-axis to indicate this.

Since $f(x)$ approaches ∞ as x approaches $\pm\infty$, $\frac{1}{f(x)}$ will have a horizontal asymptote along the x-axis.



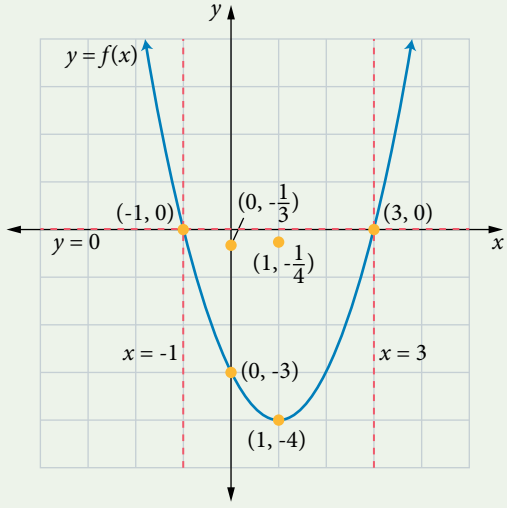
3 Determine the equations of any vertical asymptotes and add the asymptotes to the graph. (The reciprocal function will have vertical asymptotes corresponding to the x-intercepts of the original function.)

Since $f(x)$ has x-intercepts at (-1, 0) and (3, 0), $\frac{1}{f(x)}$ will have vertical asymptotes with equations $x = -1$ and $x = 3$.

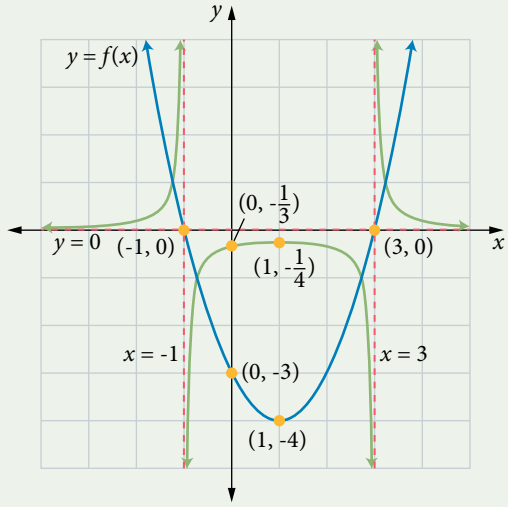


4 Determine the coordinates of any local extrema, and add these points to the graph. Local maxima/minima (that are not also x-intercepts) of the original function will become local minima/maxima respectively of the reciprocal function.

Since $f(x)$ has a local minimum at (1, -4), $\frac{1}{f(x)}$ will have a local maximum at $\left(1, -\frac{1}{4}\right)$.



5 Sketch the graph consistent with the information already found.

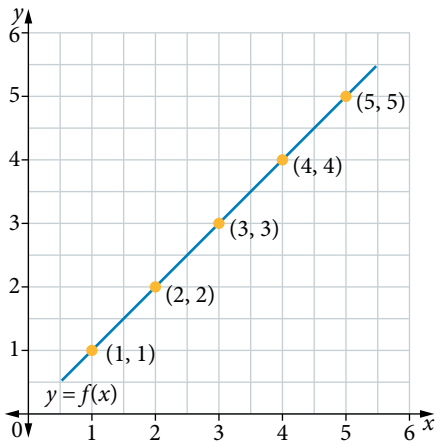


EXERCISE 6.2 Reciprocal functions

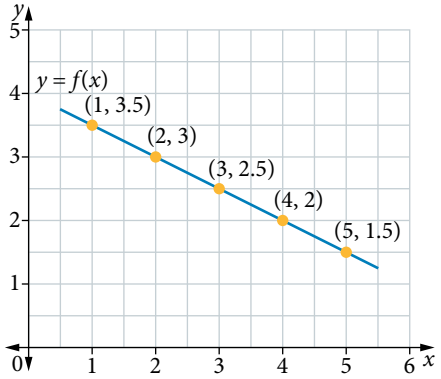
Worked Example 7

1 In each of the following, a section of the graph of a function $f(x)$ is shown with the coordinates of several points displayed. Determine the reciprocal of the y-coordinate of each point given, and hence, sketch the corresponding section of $y = \frac{1}{f(x)}$ on the same set of axes.

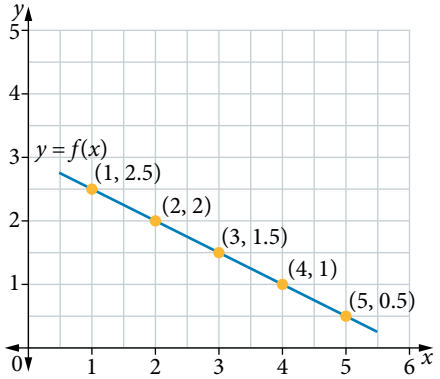
(a)



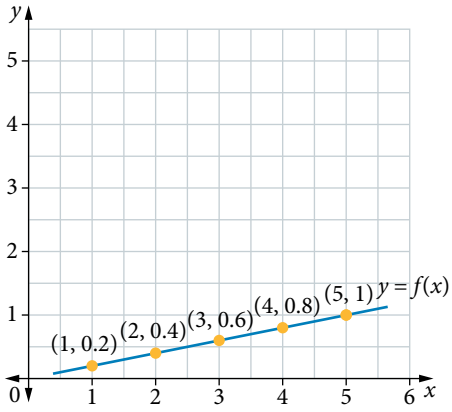
(b)



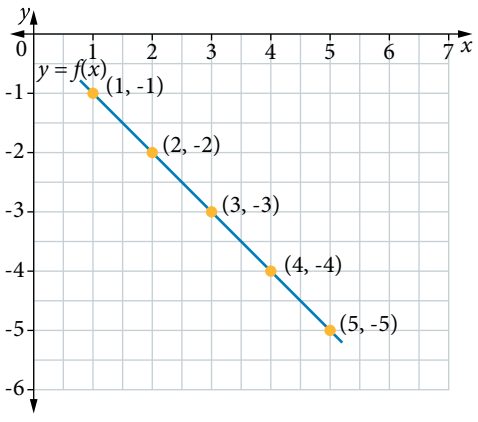
(c)



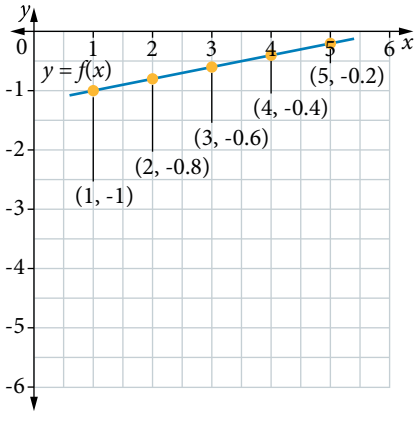
(d)



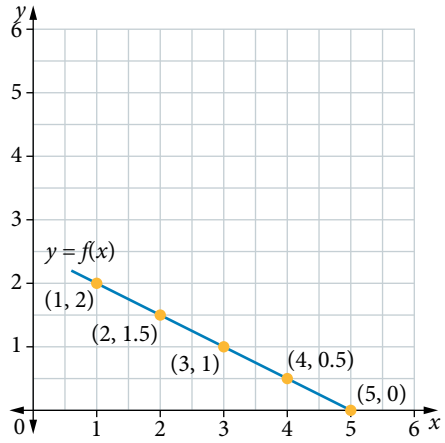
(e)



(f)



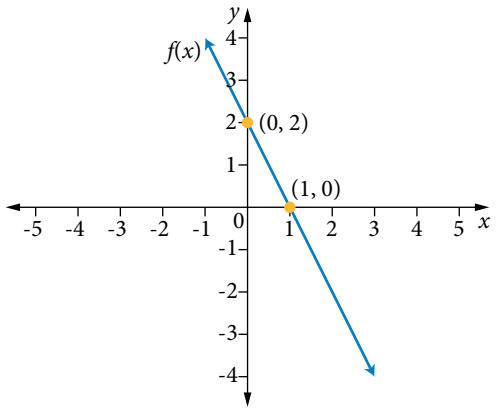
(g)



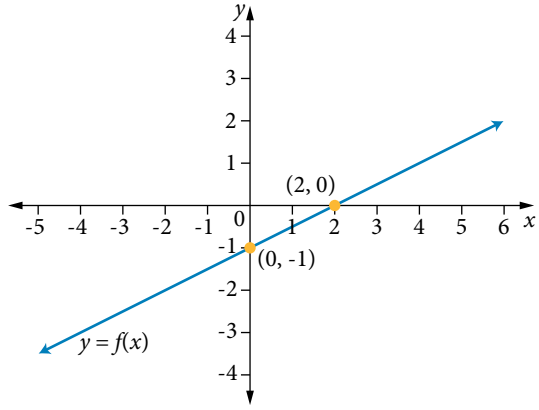
Worked Example

8 2 For each of the following, the graph of the function $f(x)$ is given. Sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes and intercepts.

(a)



(b)

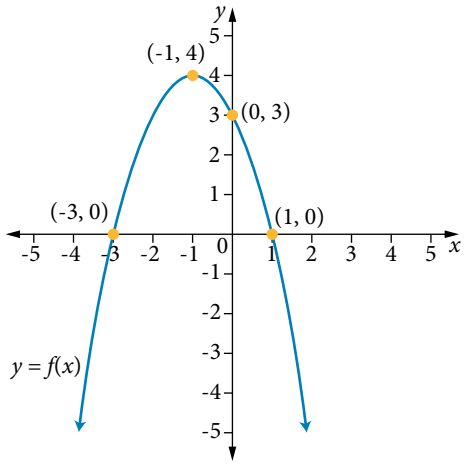


3 Sketch the graph of $y = 2x - 5$, showing all intercepts. Hence, sketch the graph of $y = \frac{1}{2x-5}$ on the same sets of axes, showing all intercepts and asymptotes.

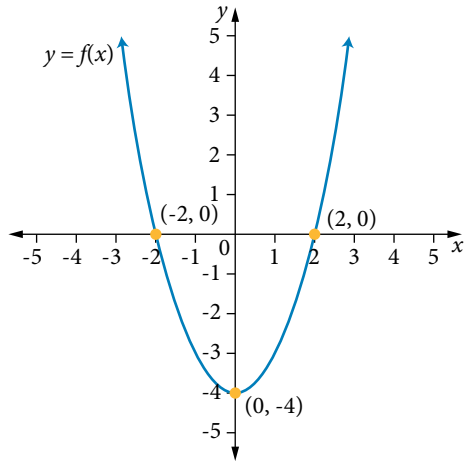
4 Sketch the graph of $y = -\frac{1}{4}x + \frac{1}{2}$, showing all intercepts. Hence, sketch the graph of $y = \frac{1}{-\frac{1}{4}x + \frac{1}{2}}$ on the same sets of axes, showing all intercepts and asymptotes.

9 5 For each of the following functions $f(x)$, redraw the graph, and then sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes, intercepts and turning points.

(a)

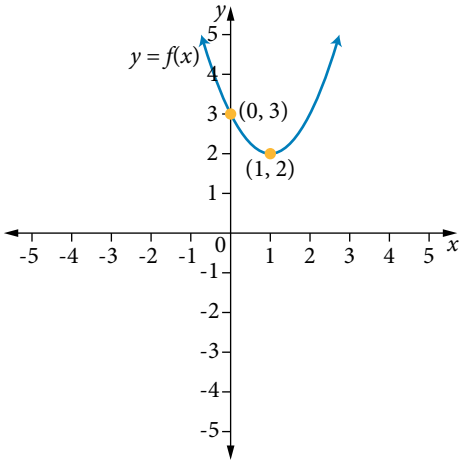


(b)

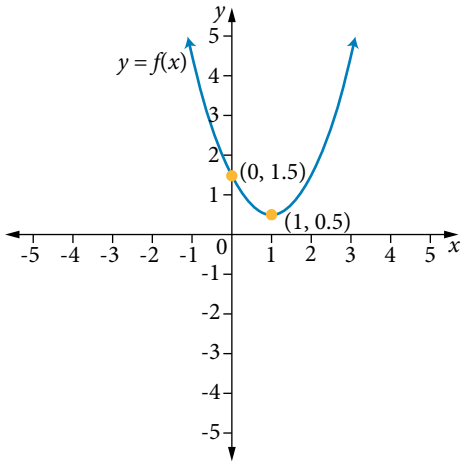


6 For each of the following functions $f(x)$, redraw the graph, and then sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes, intercepts and turning points.

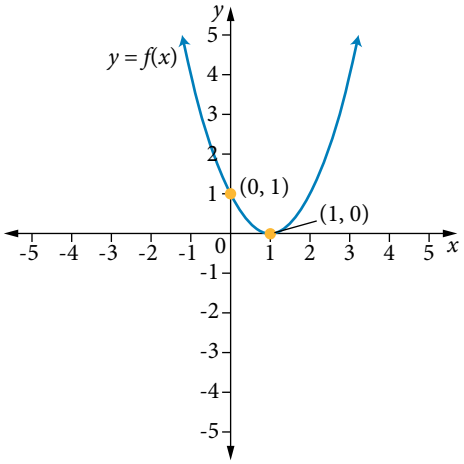
(a)



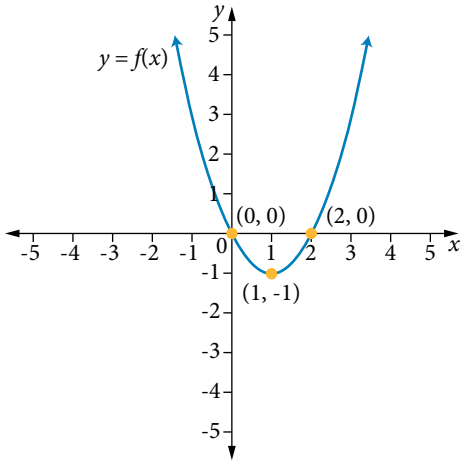
(b)



(c)

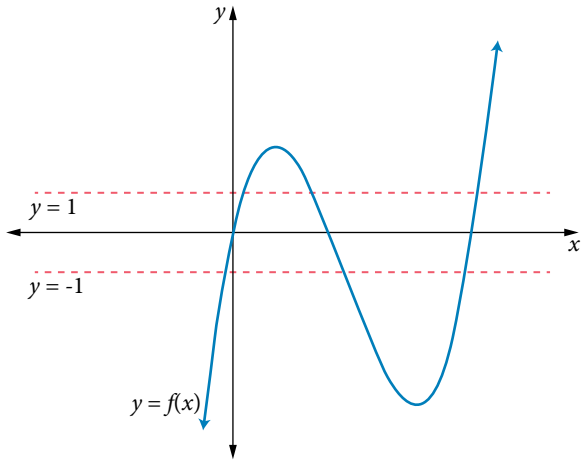


(d)



7 Determine the equations of the vertical asymptotes of the function $y = \frac{1}{2x^3 + 5x^2 - 3x}$.

8 The graph of a function $f(x)$ is shown below, together with the lines $y = \pm 1$. Redraw the graph, and on the same set of axes, sketch the graph of $y = \frac{1}{f(x)}$. Ensure the points where $f(x)$ and $y = \frac{1}{f(x)}$ intersect are positioned accurately.



9 (a) Given a function $f(x)$, which of the following statements is always true?

A If $y = f(x)$ has a local minimum at $x = a$, then $y = \dfrac{1}{f(x)}$ will have a local maximum at $x = a$.

B The x-axis will be a horizontal asymptote of $y = \dfrac{1}{f(x)}$.

C The graph of $y = \dfrac{1}{f(x)}$ will have at least one vertical asymptote.

D If the graph of $y = f(x)$ is entirely above the x-axis, then the graph of $y = \dfrac{1}{f(x)}$ will be entirely above the x-axis.

(b) Explain the errors in the incorrect alternatives.

10 Given a function $f(x)$, which of the following statements is false?

A If $(p, 1)$ is a point on the graph of $y = f(x)$, then $(p, 1)$ is also a point on the graph of $y = \frac{1}{f(x)}$.

B If (p, q) is a point on the graph of $y = f(x)$, then (q, p) is a point on the graph of $y = \frac{1}{f(x)}$.

C If (p, q) is a point on the graph of $y = f(x)$ with $q \neq 0$, then $\left(p, \frac{1}{q}\right)$ is a point on the graph of $y = \frac{1}{f(x)}$.

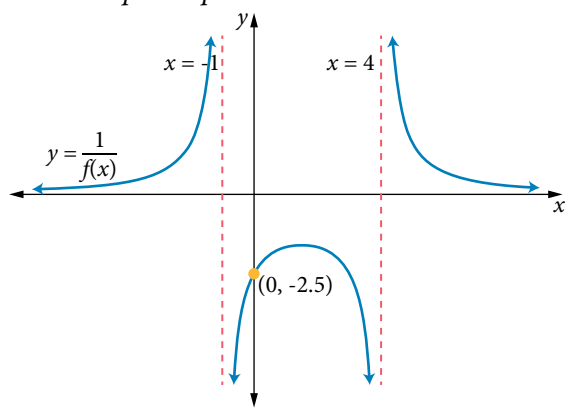
D If (p, q) is a point on the graph of $y = \frac{1}{f(x)}$, with $q \neq 0$, then $\left(p, \frac{1}{q}\right)$ is a point on the graph of $y = f(x)$.

11 Algebraically, determine the coordinates of all points of intersection of $y = 5x + 2$ and $y = \frac{1}{5x+2}$. Evaluate the reasonableness of your solution.

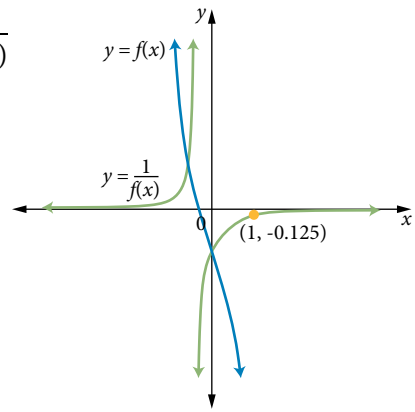
12 Algebraically, determine the coordinates of all points of intersection of $y = x^2 - 3$ and $y = \frac{1}{x^2-3}$. Evaluate the reasonableness of your solution.

13 The graph of $y = \frac{1}{f(x)}$ has a vertical asymptote with equation $x = p$ and a y-intercept of $(0, q)$. If the function $f(x)$ is a straight line, determine its gradient in terms of p and q.

14 The graph of $y = \frac{1}{f(x)}$ where $f(x)$ is a quadratic function is shown on the right. Determine the equation of $f(x)$. Evaluate the reasonableness of your solution.



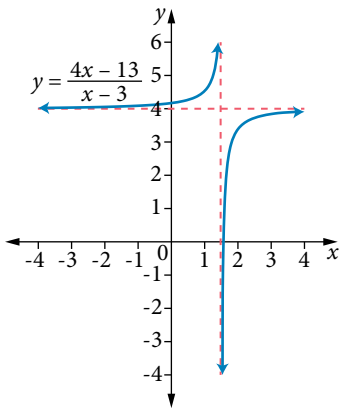
15 The diagram on the right shows the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ where $f(x) = ax^3 - 2x + b$. Determine the values of a and b. Evaluate the reasonableness of your solution.

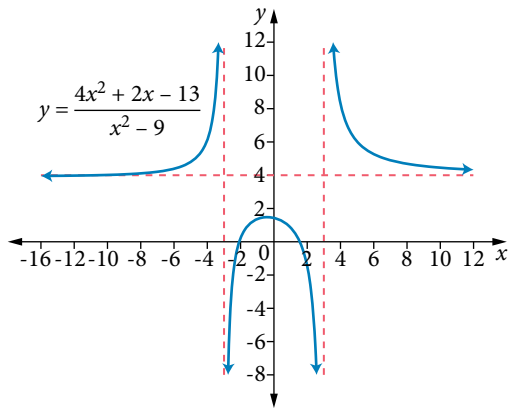


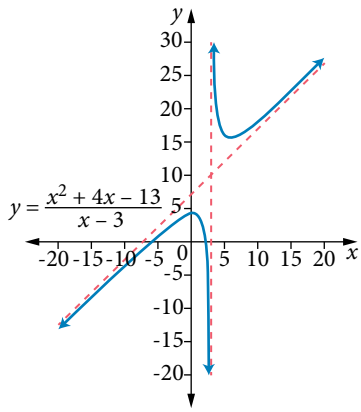
Rational functions

6.3

A rational function is a function of the form $y = \frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are polynomials. Note that a function of the form $y = \frac{c}{g(x)}$ where c is a constant can readily be sketched using the techniques for sketching reciprocal functions described in the previous section. For example, the function $y = \frac{2}{x^2+8}$ could first be rewritten as $y = \frac{1}{\frac{1}{2}x^2+4}$ (dividing the numerator and denominator by 2), which is the reciprocal function of $y = \frac{1}{2}x^2 + 4$. Also, functions of the form $y = \frac{f(x)}{c}$ are polynomials that can be sketched using techniques from Mathematical Methods. Thus, attention will be restricted to rational functions in which $f(x)$ and $g(x)$ are non-constant polynomials of low degree; that is, linear or quadratic. Several such functions are shown below.







Rational functions frequently contain one or more asymptotes. In the previous section, it was explained why a vertical asymptote of a function of the form $y = \frac{1}{g(x)}$ corresponds to a value of x for which $g(x) = 0$. The same reasoning applies to functions of the form $y = \frac{f(x)}{g(x)}$: provided $f(x)$ and $g(x)$ contain no common algebraic factors, the graph of $y = \frac{f(x)}{g(x)}$ will contain a vertical asymptote corresponding to each value of x for which $g(x) = 0$.

Horizontal asymptotes or *oblique* (slanted) asymptotes (such as the one in the third graph pictured above) may also exist. The nature of these depend on the degree of the polynomials $f(x)$ and $g(x)$ and will be discussed later.

In general, a good sketch of a rational function of the form $y = \frac{f(x)}{g(x)}$ may be obtained as follows:

STEP 1 Simplify the function if possible.

STEP 2 Determine the coordinates of all axial intercepts.

STEP 3 Determine the equations of all vertical asymptotes.

STEP 4 Determine the equation of the horizontal or slant asymptote.

STEP 5 Determine the coordinates of additional points if necessary and sketch the graph.

In this section, more specific guidance regarding the process of sketching rational functions of particular types will be discussed.

Explore further

Rational functions

Move the sliders a, b, c, d, e, f to see how the function changes and what happens when both a and d are zero.

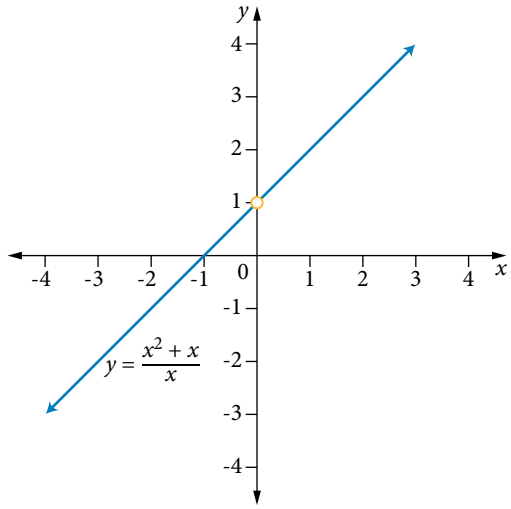
Simplifying rational functions

Occasionally, it is possible to simplify a rational function by factorising the numerator and denominator, and cancelling common factors. This can often result in a much simpler function that can be sketched easily. As an example, consider the function $y = \frac{x^2 + x}{x}$. Factorising the numerator and cancelling x from the numerator and denominator gives:

$$y = \frac{x^2 + x}{x}$$
$$y = \frac{\cancel{x}(x+1)}{\cancel{x}}$$
$$y = x + 1,\ x \neq 0$$

The statement that $x \neq 0$ is necessary since cancelling a factor (that is, dividing the numerator and denominator by that factor) only makes sense if the factor is not equal to 0.

Thus, the function is essentially the same as the straight line $y = x + 1$, except when $x = 0$ where the original function is undefined. Thus, the graph appears identical to that of $y = x + 1$ except where $x = 0$, where there is a gap or a hole (indicated by an open circle on the graph).

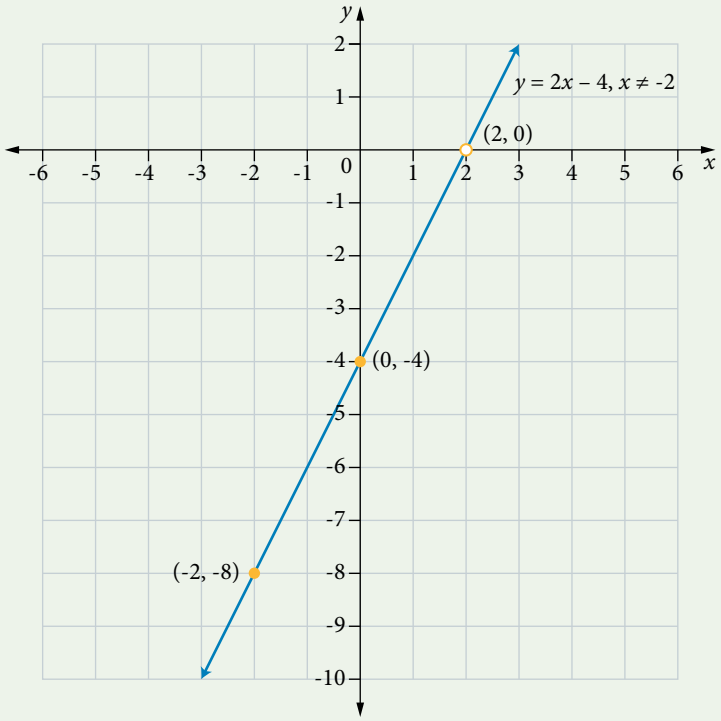


10 Sketching a rational function by first simplifying

Simplify the rational function $y = \frac{2x^2 - 8}{x + 2}$ by identifying common factors in the numerator and denominator. Hence, sketch the graph of this function, clearly identifying any intercepts and discontinuous points.

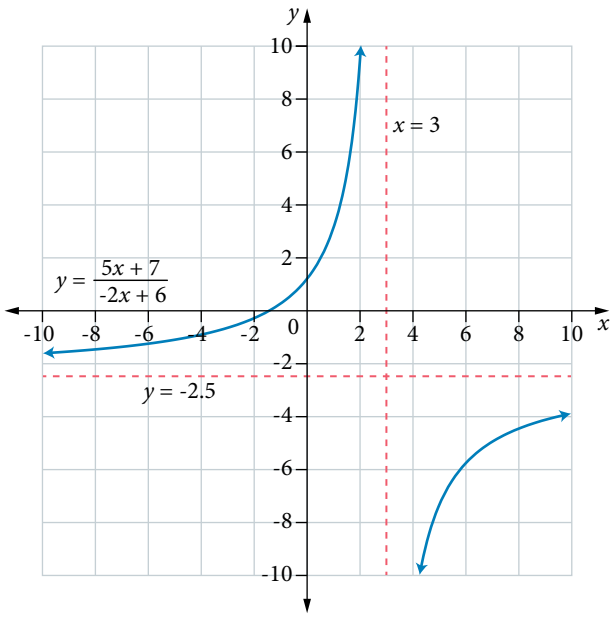
THINKING	WORKING
1 Simplify the expression by factorising the numerator and denominator (if possible). Cancel any common factors, noting the value(s) of x that would result in division by 0.	$y = \frac{2x^2 - 8}{x + 2}$ $= \frac{2(x^2 - 4)}{x + 2}$ $= \frac{2\cancel{(x+2)}(x-2)}{\cancel{(x+2)}}$ $= 2x - 4,\ x \neq -2$
2 Identify the simplified function, and the location of any holes.	Thus, the graph of the function is the same as the graph of $y = 2x - 4$, but with a discontinuity at $x = -2$ (near which y approaches $2 \times -2 - 4 = -8$).

3 Sketch the simplified function. Use an open circle to designate any holes.



Sketching rational functions with linear numerator and denominator

A function of the form $\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are linear expressions is represented by a graph known as a hyperbola. As an example, the diagram on the right shows the graph of $y = \frac{5x+7}{-2x+6}$.



Since the denominator of a hyperbola function is linear, there is always exactly one value of x that makes the denominator equal to 0, and so the graph will always contain a single vertical asymptote.

Using long division or re-expression, you can see the quotient-remainder form:

$$\begin{array}{r} \frac{5}{2} \\ -2x+6\overline{)5x+7} \\ -(5x-15) \\ \hline 0+22 \end{array}$$

Therefore:

$$\begin{aligned} y &= \frac{5x+7}{-2x+6} \\ &= \frac{22}{-2x+6} - \frac{5}{2} \\ &= \frac{11}{-x+3} - \frac{5}{2} \end{aligned}$$

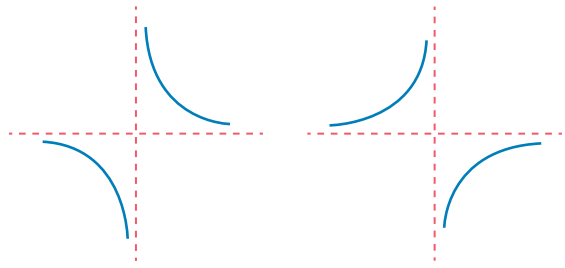
The equation of the vertical asymptote is $x = 3$.

The equation of the horizontal asymptote is $y = -\frac{5}{2}$

Statement	Example
There is always a horizontal asymptote that the graph approaches as x approaches ∞ and as x approaches $-\infty$. To see why this is reasonable, note that the equation can be rewritten as shown, by dividing the numerator and denominator by x.	$y = \frac{5x+7}{-2x+6}$ $= \frac{\frac{5x+7}{x}}{\frac{-2x+6}{x}}$ $= \frac{\frac{5x}{x}+\frac{7}{x}}{\frac{-2x}{x}+\frac{6}{x}}$ $= \frac{5+\frac{7}{x}}{-2+\frac{6}{x}}, x \neq 0$
The same principle extends to rational functions when both the numerator and denominator are higher order polynomials of the same degree.	As x approaches $\pm\infty$, the $\frac{7}{x}$ and $\frac{6}{x}$ terms approach 0, which means that y must approach $\frac{5}{-2} = -2.5$. The equation of the horizontal asymptote is $y = -2.5$.
Consider the function: $y = \frac{8x^3+7x^2+6x+5}{4x^3+3x^2+2x+1}$	Dividing the numerator and denominator by x^3 produces $\frac{8+\frac{7}{x}+\frac{6}{x^2}+\frac{5}{x^3}}{4+\frac{3}{x}+\frac{2}{x^2}+\frac{1}{x^3}}$ which will approach $\frac{8}{4} = 2$ as x approaches $\pm\infty$, leading to a horizontal asymptote with equation $y = 2$.
The position of the horizontal asymptote is given by the ratio of the coefficients of the leading terms of the numerator and denominator. (The leading term of a polynomial is the term containing the highest power of x).	For $y = \frac{5x+7}{-2x+6}$, the coefficients are 5 and -2 respectively, leading to an asymptote with equation $y = \frac{5}{-2}$. For $y = \frac{8x^3+7x^2+6x+5}{4x^3+3x^2+2x+1}$, the coefficients are 8 and 4 respectively, leading to an asymptote with equation $y = \frac{8}{4} = 2$.

If $f(x)$ and $g(x)$ are polynomials of the same degree with no common factors, then $y = \frac{f(x)}{g(x)}$ will have a horizontal asymptote with equation $y = \frac{p}{q}$ where p and q are the coefficients of the leading terms of $f(x)$ and $g(x)$ respectively.

Returning to the situation in which both the numerator and denominator of a rational function are linear, it can also be shown that if the section of the graph on the left of the vertical asymptote is below the horizontal asymptote, then the section on the right is above, and vice versa. Thus, the graph of a hyperbola will resemble one of the two diagrams on the right, where the dotted lines represent the asymptotes.



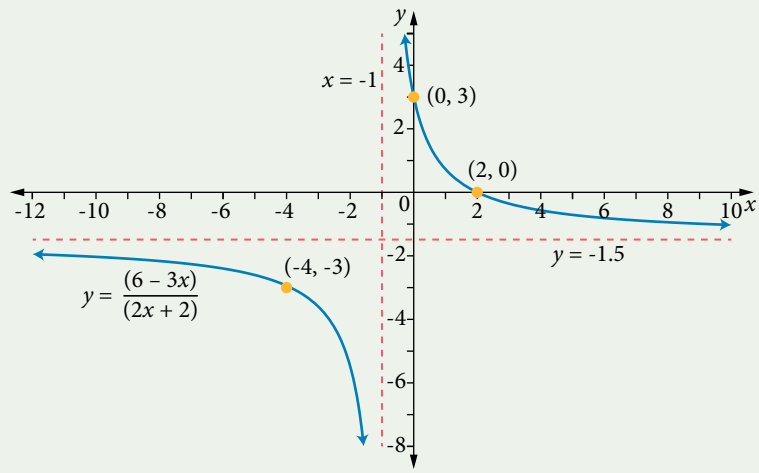
The determination of which of these scenarios is correct for a particular hyperbola is usually indicated by the location of the intercepts, as demonstrated in the following worked example.

11 Sketching functions of the form $y = \frac{ax + b}{cx + d}$

Sketch the function $y = \frac{6 - 3x}{2x + 2}$, clearly indicating all intercepts and asymptotes.

THINKING	WORKING
1 Simplify by factorising the numerator and denominator (if possible) to ensure that there are no common factors that could be cancelled.	$y = \frac{6 - 3x}{2x + 2}$ $= \frac{-3(x - 2)}{2(x + 1)}$
2 Determine the coordinates of all axial intercepts.	For x-intercepts, let $y = 0$ and solve for x: $0 = \frac{-3(x - 2)}{2(x + 1)}$ $0 = -3(x - 2)$ $0 = x - 2$ $x = 2$ The coordinates of the x-intercept are (2, 0). For y-intercepts, let $x = 0$: $y = \frac{-3(0 - 2)}{2(0 + 1)}$ $= 3$ The coordinates of the y-intercept are (0, 3).
3 Determine the equation(s) of any vertical asymptotes. These correspond to values of x for which the denominator is equal to 0.	$0 = 2x + 2$ $= 2(x + 1)$ $x = -1$ Therefore, there is a vertical asymptote with equation $x = -1$.
4 Determine the equation of any horizontal asymptotes.	Since the numerator and denominator are both polynomials of the same degree, and have leading coefficients of -3 and 2 respectively, there is a horizontal asymptote with equation $y = \frac{-3}{2}$. This can be verified using re-expression: $y = \frac{6 - 3x}{2x + 2}$ $= \frac{-3x + 6}{2x + 2}$ $= \frac{-\frac{3}{2}(2x + 2) + 6 + 3}{(2x + 2)}$ $= \frac{9}{2x + 2} - \frac{3}{2}$

5 Mark in the intercepts and asymptotes. Then sketch the hyperbola consistent with the information found. Note that the location of the intercepts means that the section of the hyperbola to the right of the vertical asymptote must be above the horizontal asymptote.



Sketching functions of the form $y = \dfrac{ax + b}{cx + dx + e}$

A rational function of the form $y = \dfrac{ax+b}{cx^2+dx+e}$ (which cannot be simplified) may have either 2, 1 or 0 vertical asymptotes, depending on the number of solutions to $cx^2 + dx + e = 0$.

Further, the x-axis will always be a horizontal asymptote. To see why, note that the equation can be rewritten as follows, by dividing the numerator and denominator by x:

$$y = \frac{ax + b}{cx^2 + dx + e}$$

$$= \frac{\frac{ax}{x} + \frac{b}{x}}{\frac{cx^2}{x} + \frac{dx}{x} + \frac{e}{x}}$$

$$= \frac{a + \frac{b}{x}}{cx + \frac{dx}{x} + \frac{e}{x}}, \; x \neq 0$$

As x approaches $\pm\infty$, the numerator will approach a, while the denominator will continue to grow in magnitude due to the cx term. Thus, the value of y will approach 0.

These insights allow such a function to be sketched.

A rational function of the form $y = \dfrac{ax+b}{cx^2+dx+e}$ (which cannot be simplified) will have the x-axis as a horizontal asymptote.

12 Sketching functions of the form $y = \dfrac{ax + b}{cx^2 + dx + e}$

Sketch the function $y = \dfrac{-2x + 2}{x^2 - 9}$, clearly indicating all intercepts and asymptotes.

THINKING	WORKING
1 Factorise the numerator and denominator (if possible) to ensure that there are no common factors that could be cancelled.	$y = \dfrac{-2x + 2}{x^2 - 9}$ $= \dfrac{-2(x - 1)}{(x - 3)(x + 3)}$

2 Determine the coordinates of all axial intercepts.

For x-intercepts, let $y = 0$ and solve for x:

$$0 = \frac{-2(x-1)}{(x-3)(x+3)}$$
$$0 = -2(x-1)$$
$$x = 1$$

The coordinates of the x-intercept are $(1, 0)$.

For y-intercepts, let $x = 0$ and calculate the value of y:

$$y = \frac{-2(0-1)}{(0-3)(0+3)}$$
$$= -\frac{2}{9}$$

The coordinates of the y-intercept are $\left(0, -\frac{2}{9}\right)$.

3 Determine the equation(s) of any vertical asymptotes. These correspond to values of x for which the denominator is equal to 0.

$$0 = (x-3)(x+3)$$
$$x = \pm 3$$

Therefore, there are vertical asymptotes with equations $x = \pm 3$.

4 Determine the equation of any horizontal asymptotes. (If the numerator is linear and the denominator quadratic, the x-axis will be a horizontal asymptote.)

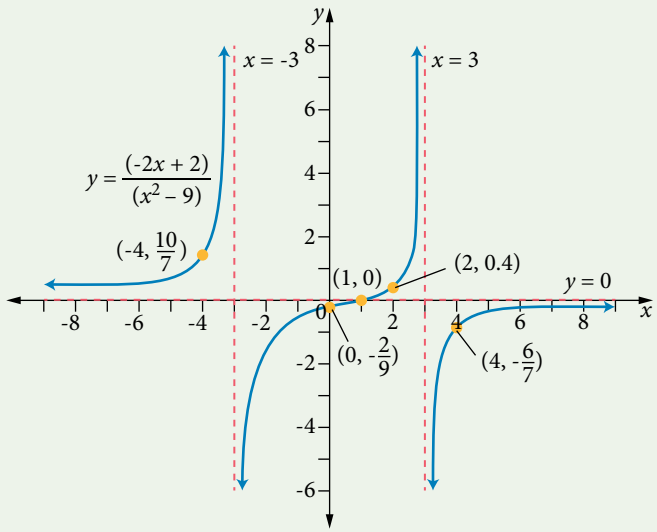
Since the numerator is linear and the denominator quadratic, the x-axis will be a horizontal asymptote.

5 Mark in the intercepts and asymptotes. Then sketch the graph consistent with the information found.

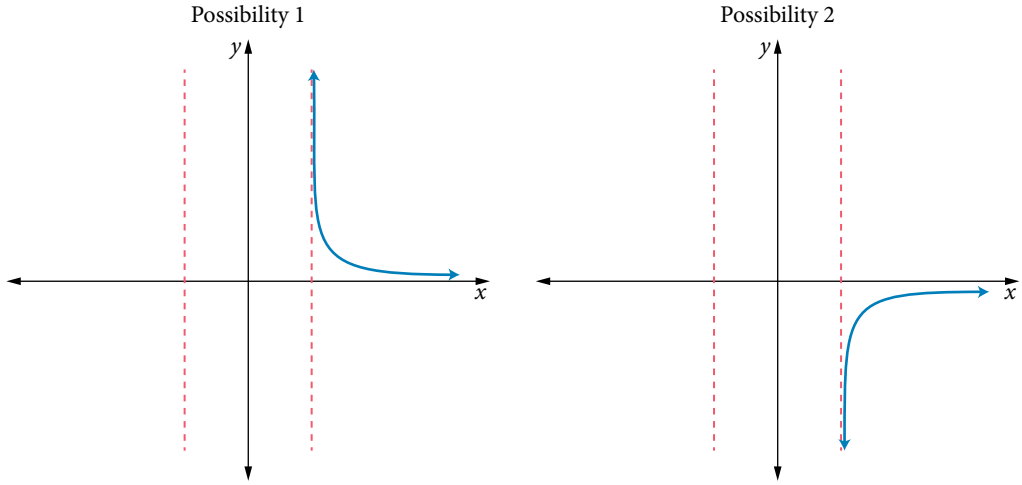
Note that since there are two vertical asymptotes, there will be three sections of the graph.

If there is more than one possibility for the shape of a section, substitute an appropriate value of x into the rule for the function in order to obtain the coordinates of a point that the graph must pass through.

Combining this information, and several other points obtained by substituting suitable values of x into the rule for the function, produces the following sketch:



In Worked Example 12, simply knowing the intercepts and the asymptotes did not provide quite enough information to obtain a good sketch of the graph. For example, just based on this information, it is unclear whether the section of the graph to the right of the $x = 3$ asymptote lies above or below the x-axis. (It cannot pass through the x-axis since the only x-intercept occurred in the middle section of the graph.) Thus, there were two possibilities for the shape of this section:



In order to determine which of these possibilities is correct, it was necessary to determine whether the y-coordinate of a point to the right of the $x = 3$ asymptote was positive or negative. Thus, the next integer value of x greater than 3 was substituted into the equation in order to determine the y-coordinate:

$$y = \frac{-2(4-1)}{(4-3)(4+3)}$$
$$= \frac{-6}{7}$$

The fact that this number is negative means that Possibility 2 is correct.

It should also be noted that the graph actually passed through the horizontal asymptote (in the middle section). It is a common, but incorrect belief that the graph of a function graph cannot cross an asymptote; this is true for vertical asymptotes, but not for horizontal ones, in general.

The graph of a rational function with a horizontal asymptote may cross this asymptote.

Sketching functions of the form $y = \frac{ax^2 + bx + c}{dx + ex + f}$

Rational functions in which both the numerator and denominator are quadratic may be sketched using the ideas previously discussed, recalling that whenever the numerator and denominator are both polynomials of the same degree, there will be a horizontal asymptote, and that the ratio of the leading coefficients gives its location. In this situation, it is also useful to determine if there are any points of intersection between the function and this horizontal asymptote.

13 Sketching functions of the form $y = \frac{ax^2 + bx + c}{dx^2 + ex + f}$

Sketch the function $y = \frac{2x^2 + 2}{x^2 + 2x - 3}$, clearly indicating all intercepts and asymptotes.

THINKING	WORKING
1 Simplify by factorising the numerator and denominator (if possible) to ensure that there are no common factors that could be cancelled.	$y = \frac{2x^2 + 2}{x^2 + 2x - 3}$ $= \frac{2(x+1)}{(x-1)(x+3)}$
2 Determine the coordinates of all axial intercepts.	For x-intercepts, let $y = 0$ and solve for x: $0 = \frac{2(x^2 + 1)}{(x-1)(x+3)}$ $0 = 2(x^2 + 1)$ $x^2 = -1$ This equation has no real solutions; therefore, there are no x-intercepts. For y-intercepts, let $x = 0$ and calculate the value of y: $y = \frac{2(0+1)}{(0-1)(0+3)}$ $= -\frac{2}{3}$ The coordinates of the y-intercept are $\left(0, -\frac{2}{3}\right)$.
3 Determine the equation(s) of any vertical asymptotes. These correspond to values of x for which the denominator is equal to 0.	$0 = (x-1)(x+3)$ $x = 1$ or $x = -3$ Therefore, there are vertical asymptotes with equations $x = 1$ and $x = -3$.
4 Determine the equation of any horizontal asymptotes. When the numerator and denominator are both polynomials of the same degree, the ratio of the leading coefficients gives the position of the horizontal asymptote.	Since the numerator and denominator are both polynomials of the same degree, and have leading coefficients of 2 and 1 respectively, there is a horizontal asymptote with equation $y = \frac{2}{1} = 2$.

5 Determine any points of intersection between the function and the horizontal asymptote. This can be accomplished by equating the expressions for both functions.

Points of intersection between function and horizontal asymptote:

$$\frac{2x^2+2}{x^2+2x-3} = 2$$
$$2x^2+2 = 2\left(x^2+2x-3\right)$$
$$2x^2+2 = 2x^2+4x-6$$
$$8 = 4x$$
$$2 = x$$

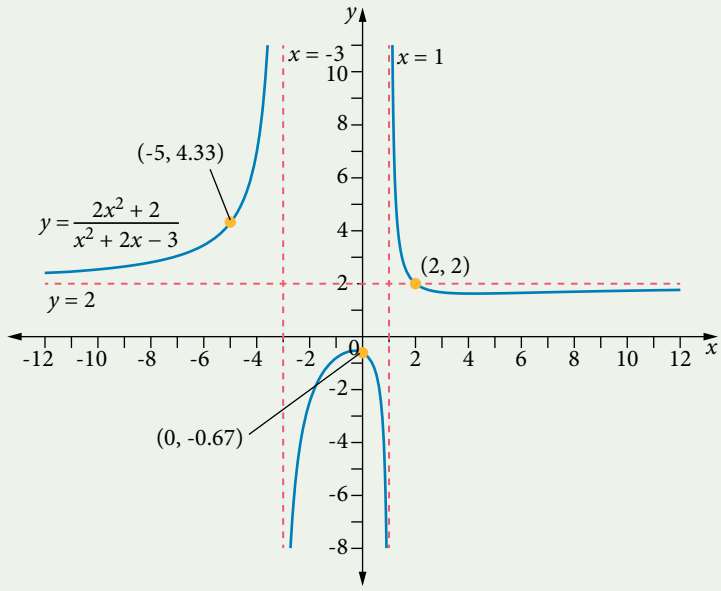
Therefore, the function and horizontal asymptote intersect at (2, 2).

6 Mark in the intercepts and asymptotes. Then sketch the graph consistent with the information found.

Note that since there are two vertical asymptotes, there will be three sections of the graph.

If there is more than one possibility for the shape of a section, substitute an appropriate value of x into the rule for the function in order to obtain the coordinates of a point that the graph must pass through.

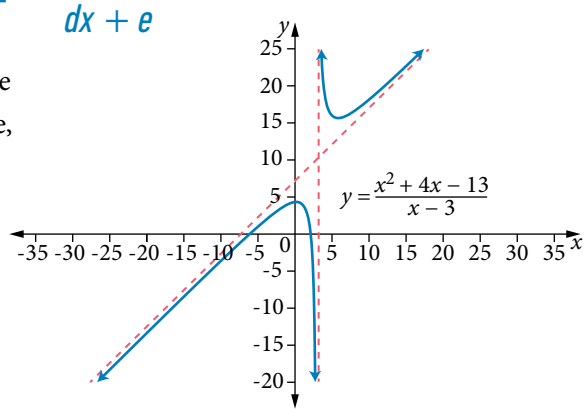
Combining this information, and several other points obtained by substituting suitable values of x into the rule for the function, produces the following sketch:



In Worked Example 13, the extra point in the left-most section of the graph was not actually needed. To see this, notice that in this region there are no x-intercepts, and the graph cannot pass through the horizontal asymptote (since the only point of intersection was found to be in the right-most region). This means the graph must be above the horizontal asymptote in this region. Of course, it certainly doesn't hurt to determine the coordinates of a point, just to be sure.

Sketching functions of the form $y = \frac{ax^2+bx+c}{dx+e}$

If the numerator of a rational function is quadratic, and the denominator linear, then instead of a horizontal asymptote, there will be an oblique or slant asymptote that the graph approaches as x approaches ∞, and as x approaches $-\infty$. As an example, consider the function $y = \frac{x^2+4x-13}{x-3}$, the graph of which is shown on the right.



To see why this function has an oblique asymptote (and to determine the equation of this asymptote), it is necessary to use polynomial long division to rewrite the function in *quotient-remainder form*.

$$
\begin{array}{r}
x+7 \\
x-3\overline{)\,x^2+4x-13} \\
\underline{x^2-3x} \\
7x-13 \\
\underline{7x-21} \\
8
\end{array}
$$

As x approaches $\pm\infty$, the $\frac{8}{x-3}$ term will approach 0, and y will approach the value $x+7$.

Graphically, this means that the graph will approach the line with equation $y = x + 7$.

Thus,

$$
\begin{aligned}
y &= \frac{x^2+4x-13}{x-3} \\
&= x+7+\frac{8}{x-3}
\end{aligned}
$$

A rational function of the form $y = \frac{ax^2+bx+c}{dx+e}$ (which cannot be simplified) will have an oblique asymptote instead of a horizontal one. The equation of this asymptote is given by the quotient when the function is rewritten in quotient-remainder form using polynomial long division.

14 Sketching functions of the form $y = \frac{ax^2+bx+c}{dx+e}$

Sketch the function $y = \frac{x^2+3x}{x-1}$, clearly indicating all intercepts and asymptotes.

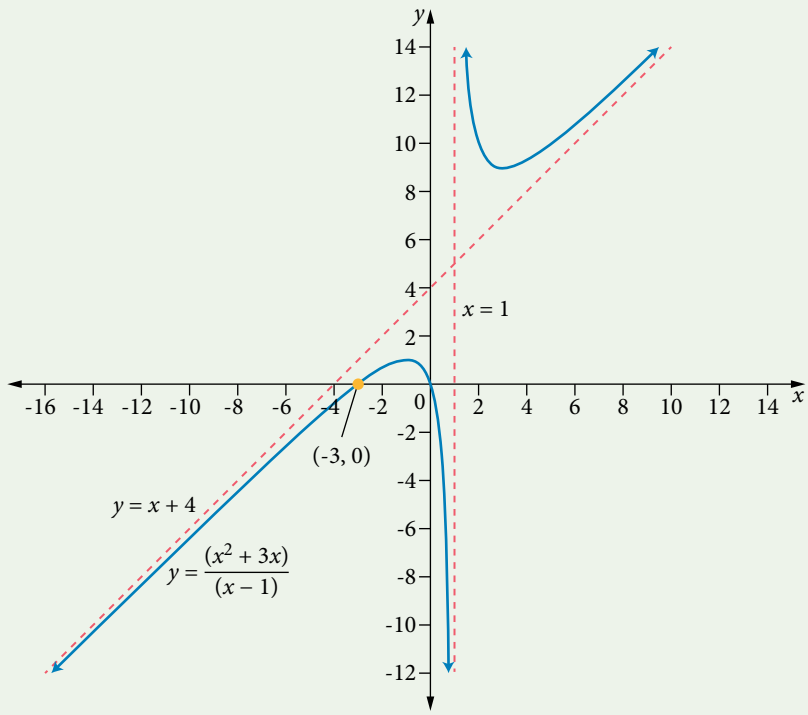
THINKING	WORKING
1 Simplify by factorising the numerator and denominator (if possible) to ensure that there are no common factors that could be cancelled.	$y = \frac{x^2+3x}{x-1}$ $= \frac{x(x+3)}{x-1}$
2 Determine the coordinates of all axial intercepts.	For x-intercepts, let $y = 0$ and solve for x: $0 = \frac{x(x+3)}{x-1}$ $0 = x(x+3)$ $x = 0$ or $x = -3$ The coordinates of the x-intercepts are (0, 0) (which is also the y-intercept) and (-3, 0).
3 Determine the equation(s) of any vertical asymptotes. These correspond to values of x for which the denominator is equal to 0.	$0 = x - 1$ $x = 1$ Therefore, there is a vertical asymptote with equation $x = 1$.

4 Determine the equation of the oblique asymptote: its equation is the quotient when the numerator is divided by the denominator using long division.

$$
\begin{array}{r}
x+4 \\
x-1\overline{)\,x^2+3x+0} \\
\underline{x^2-x} \\
4x+0 \\
\underline{4x-1} \\
1
\end{array}
$$

Therefore, the equation of the oblique asymptote is $y = x + 4$.

5 Mark in the intercepts and asymptotes. Then sketch the graph consistent with the information found.



EXERCISE 6.3 Rational functions

Worked Example

10

1 Simplify each of the following rational functions by identifying common factors in the numerator and denominator. Hence, sketch the graph of each, clearly identifying any intercepts and holes.

(a) $y = \dfrac{3x^2 - 3}{x - 1}$ (b) $y = \dfrac{-2x^2 + 18}{4x - 12}$ (c) $y = \dfrac{3x^3 + 2x}{x}$

(d) $y = \dfrac{x^3 - 5x^2}{10 - 2x}$ (e) $y = \dfrac{x^3 - 4x^2 + 3x}{3x - x^2}$

11

2 Sketch each of the following functions, clearly indicating all intercepts and asymptotes.

(a) $y = \dfrac{6x + 3}{2x + 8}$ (b) $y = \dfrac{2x + 5}{8 - 4x}$ (c) $y = \dfrac{3 - 3x}{2x - 6}$

3 Sketch each of the following functions, clearly indicating all intercepts and asymptotes. 12

(a) $y = \dfrac{5x+5}{x^2}$ (b) $y = \dfrac{5x+5}{x^2+1}$ (c) $y = \dfrac{2x+3}{x^2-1}$

(d) $y = \dfrac{2x+3}{x^2-4}$ (e) $y = \dfrac{-2x+4}{x^2+2x-3}$

4 Sketch each of the following functions, clearly indicating all intercepts and asymptotes. 13

(a) $y = \dfrac{3x^2+8}{x^2-3x-4}$ (b) $y = \dfrac{x^2-10x}{x^2-9}$ (c) $y = \dfrac{2x^2-2}{x^2}$

(d) $y = \dfrac{2x^2-2}{x^2+1}$ (e) $y = \dfrac{3x^2-6x-9}{x^2}$

5 Sketch each of the following functions, clearly indicating all intercepts and asymptotes. 14

(a) $y = \dfrac{x^2-2x}{x-3}$ (b) $y = \dfrac{x^2-9}{3x+6}$ (c) $y = \dfrac{2x^2-5x}{4x+4}$

(d) $y = \dfrac{x^2+4}{1-x}$ (e) $y = \dfrac{x^2-4x+3}{4-2x}$

6 Sketch the graph of the function $y = \dfrac{x^2-x-6}{x^3-3x^2-10x}$.

7 Sketch the graph of the function $y = \dfrac{x^5+x^6}{x^7+x^8}$.

8 Which of the following statements about the graph of the function $y = \dfrac{10-2x}{x^2-3x-10}$ is true?

A There is an x-intercept at $(5, 0)$.

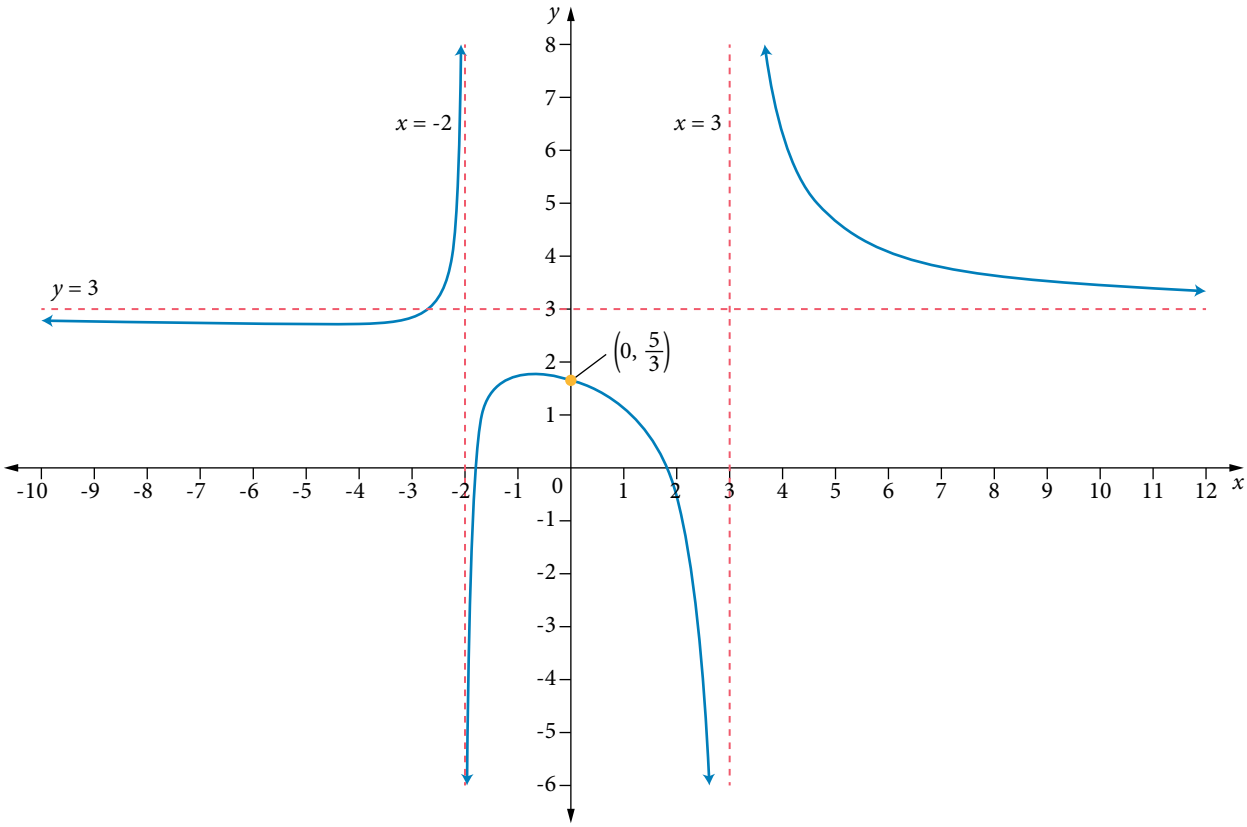
B There is a vertical asymptote with equation $x = 5$.

C The function is equivalent to the function $y = -\dfrac{2}{x+2}$.

D There is a hole in the graph at $x = 5$.

9 Use your graphics calculator to sketch the graph of $y = \dfrac{x}{x^5-4x+5}$. Determine the equation of the vertical asymptote correct to 4 decimal places.

10 The graph of the function $y = \dfrac{px^2 + q}{x^2 + bx + c}$ is shown below. Determine the values of p, q, b and c. Justify the reasonableness of your solution.



11 Consider the function $y = \dfrac{5x^2 + 4x + 3}{x + 2}$. Prove that this function never intersects with its oblique asymptote.

12 Consider a general function of the form $y = \dfrac{ax^2 + bx + c}{dx + e}$ where $a \neq 0$, $d \neq 0$, and where the numerator and denominator have no common factors.

(a) Prove that $\dfrac{ax^2 + bx + c}{dx + e} = \dfrac{a}{d}x + \dfrac{bd - ae}{d^2} + \dfrac{cd^2 - bde + ae^2}{d^2(dx + e)}$ provided $d \neq 0$.

(b) Hence, determine the equation of the oblique asymptote of such a function (in terms of a, b, c, d and e), clearly justifying your reasoning.

Summary

Absolute value functions

The absolute value of a real number, x, is defined as follows: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

The graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by replicating the part(s) of $y = f(x)$ that are on or above the x-axis, and reflecting the remaining part(s) in the x-axis.

The graph of $y = f(|x|)$ can be obtained from the graph of $y = f(x)$ by replicating the part(s) of $y = f(x)$ that is on or to the right of the y-axis, and reflecting those parts in the y-axis.

Reciprocal functions

If (a, b) is a point on the graph of $y = f(x)$ where $b \neq 0$, then $\left(a, \frac{1}{b}\right)$ will be a point on the graph of $y = \frac{1}{f(x)}$.

Provided a function approaches $\pm\infty$ as x approaches ∞ or as x approaches $-\infty$, the x-axis will be a horizontal asymptote for the reciprocal function.

The reciprocal function has vertical asymptotes wherever the original function has x-intercepts.

If a function is increasing in some interval, then the reciprocal function will be decreasing in that interval, and vice versa.

If the original function is above the x-axis on some interval, then the reciprocal function will be also be above the x-axis on that interval. The same is true for intervals where the original function is below the x-axis.

A function and its reciprocal intersect each other at any points on the original function with y-coordinate equal to 1 or -1.

If a function contains a local minimum at a particular x-value that is not also an x-intercept, then the reciprocal function will contain a local maximum at this x-value. Similarly, if a function contains a local maximum at a particular x-value that is not also an x-intercept, then the reciprocal function will contain a local minimum at this x-value.

Rational functions

In order to sketch a function of the form $y = \frac{f(x)}{g(x)}$:

STEP 1 Simplify the function if possible.
STEP 2 Determine the coordinates of all axial intercepts.
STEP 3 Determine the equations of all vertical asymptotes.
STEP 4 Determine the equation of the horizontal or slant asymptote.
STEP 5 Determine the coordinates of additional points if necessary and sketch the graph.

If $f(x)$ and $g(x)$ are polynomials of the same degree with no common factors, then $y = \frac{f(x)}{g(x)}$ will have a horizontal asymptote with equation $y = \frac{p}{q}$ where p and q are the coefficients of the leading terms of $f(x)$ and $g(x)$ respectively.

A rational function of the form $y = \frac{ax + b}{cx^2 + dx + e}$ (which cannot be simplified) will have the x-axis as a horizontal asymptote.

A rational function of the form $y = \frac{ax^2 + bx + c}{dx + e}$ (which cannot be simplified) will have an oblique asymptote instead of a horizontal one. The equation of this asymptote is given by the quotient when the function is rewritten in quotient-remainder form using polynomial long division.

Chapter review 6

Exercise 6.1

1 Determine the following:

(a) $|2 \times -5| \times -5$ **(b)** $-|-4|^2$ **(c)** $|-2 - |-2||$

Exercise 6.1

2 Solve the following equations:

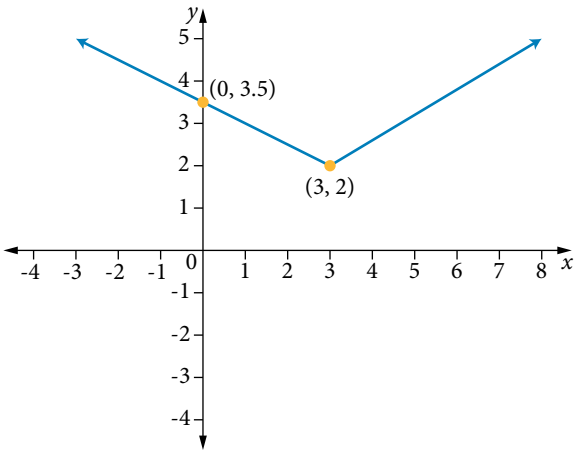
(a) $|2x + 3| = 7$ **(b)** $3|5 - 2x| = 12$

Exercise 6.1

3 Sketch the graph of $y = -2|x + 5| + 2$, identifying the coordinates of the vertex, and any intercepts with the axes.

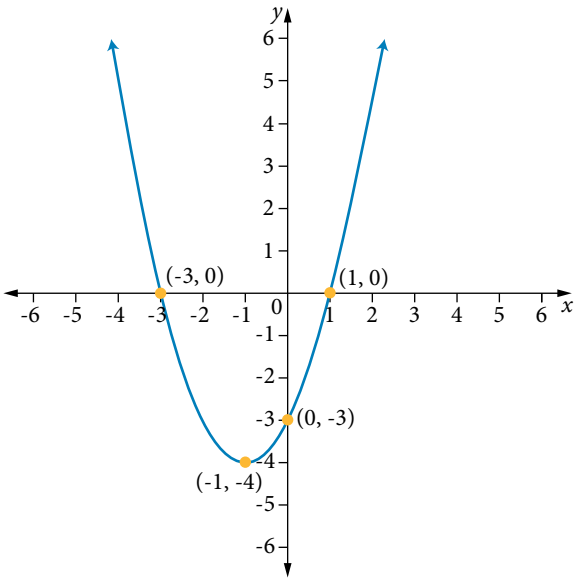
Exercise 6.1

4 Determine the equation of the function shown.

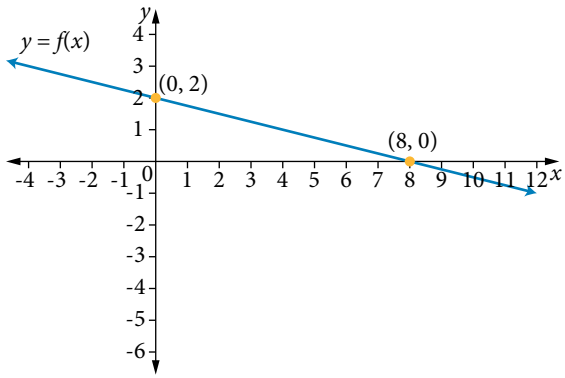


Exercise 6.1

5 For the function $f(x)$ shown on the graph below, sketch the graphs of **(i)** $y = |f(x)|$ and **(ii)** $y = f(|x|)$. Clearly indicate the location of all intercepts and local extrema.



6 The graph of the function $f(x)$ is shown below. Redraw the graph, and sketch the graph of $y = \dfrac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes and intercepts. Exercise 6.2



7 Determine the equations of the vertical asymptotes of the function $y = \frac{1}{f(x)}$ if $f(x) = x^3 - 2x$. Exercise 6.2

8 Sketch the graph of $y = -\dfrac{3}{4}x + 6$, showing all intercepts. Hence, sketch the graph of $y = \dfrac{1}{-\frac{3}{4}x + 6}$ on the same sets of axes, showing all intercepts and asymptotes. Exercise 6.2

9 Simplify each of the following rational functions by identifying common factors of the numerator and denominator. Hence, sketch the graph of each, clearly identifying any intercepts and holes. Exercise 6.3

(a) $y = \dfrac{2x^2 + x - 6}{x + 2}$ **(b)** $y = \dfrac{3x^2 - x^3}{x - 3}$

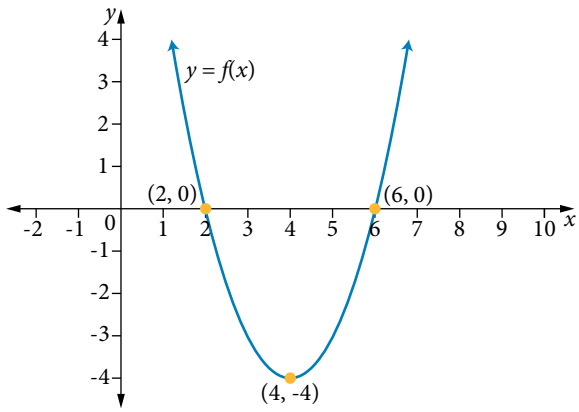
10 Sketch each of the following functions, clearly indicating all intercepts and asymptotes. Exercise 6.3

(a) $y = \dfrac{6x + 3}{2x + 8}$ **(b)** $y = \dfrac{3x + 6}{x^2 - 1}$ **(c)** $y = \dfrac{x^2 - 3x}{4 - x^2}$

11 Solve the following equations. Exercise 6.1

(a) $|2x - 1| = x + 5$ **(b)** $|3x - 4| = -5x$

12 The graph of the function $f(x)$ is shown. Redraw the graph, and sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes, intercepts and turning points. Exercise 6.2



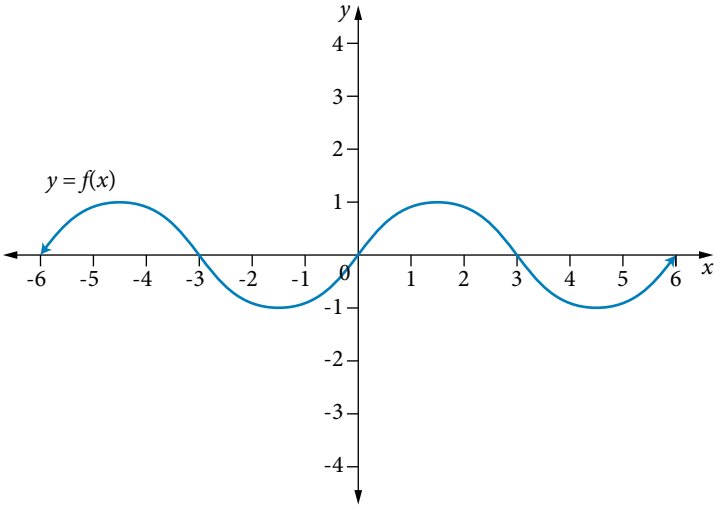
Exercise 6.1

13 If $a < b < c$, then the expression $|b - a| + |a - c| + |b - c|$ is equal to:

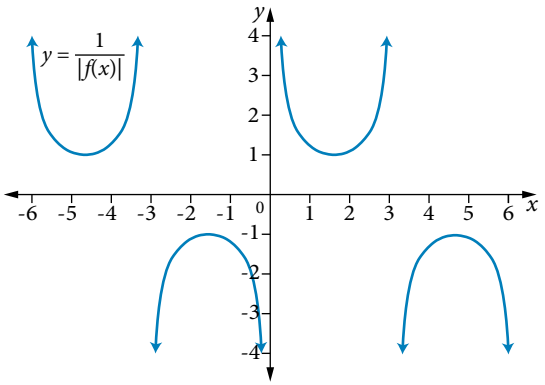
A 0 **B** $2c - 2a$ **C** $2a - 2c$ **D** $2b - 2c$

Exercise 6.2

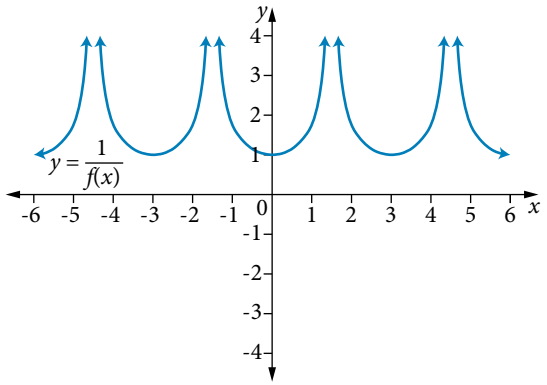
14 Consider the graph of the function $f(x)$ shown. Which of the following is the graph of $y = \dfrac{1}{|f(x)|}$?



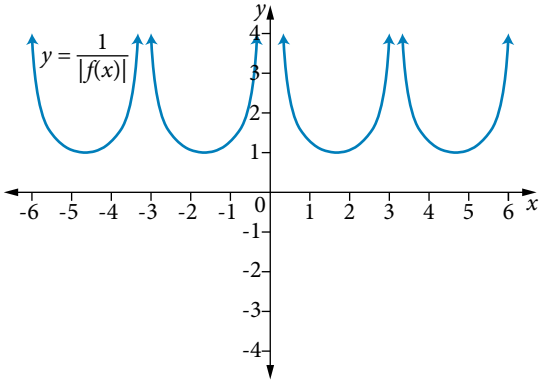
A



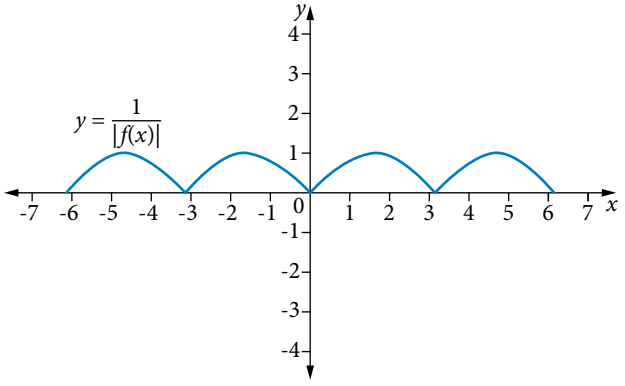
B



C



D



15 Algebraically, determine the coordinates of all points of intersection of $y = 2x^3 + 1$ and $y = \frac{1}{2x^3+1}$. Justify the reasonableness of your solution. Exercise 6.2

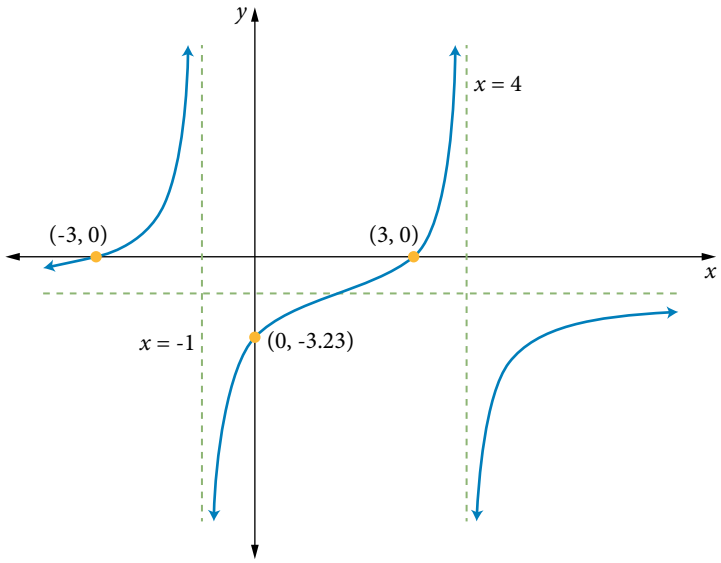
16 Determine the equation of the oblique asymptote of the function $y = \frac{-2x^2+3x+4}{x+5}$. Justify the reasonableness of your solution. Exercise 6.3

17 Determine all solutions to the equation $||x-4|-3| = 2$. Exercise 6.1

18 Consider a function of the form $f(x) = \frac{ax^2+bx+c}{dx^2+ex+f}$. Prove that such a function intersects its horizontal asymptote only once, at most. Also, describe the conditions under which the function does not intersect its horizontal asymptote. Exercise 6.3

19 Determine the value of p given that the graph of $y = \frac{px^2+1}{x^2+10x+p}$ has a single vertical asymptote. Evaluate the reasonableness of your solution. Exercise 6.3

20 The graph shown is of a rational function with quadratic polynomials for both the numerator and denominator. Determine the equation of the horizontal asymptote. Exercise 6.3





7

Trigonometry

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7 Recall

Calculate the sum of fractions of π

1 Express the following in terms of π in simplest form.

(a) $\frac{\pi}{3}+\frac{\pi}{3}$ (b) $\frac{\pi}{3}+\frac{\pi}{4}$ (c) $\frac{11\pi}{12}+\frac{\pi}{2}$ (d) $\frac{2\pi}{3}+\pi$

Calculate the difference between two fractions of π

2 Calculate the following in terms of π, expressing your answers in simplest form.

(a) $\frac{\pi}{3}-\frac{\pi}{4}$ (b) $\pi-\frac{\pi}{6}$ (c) $\frac{5\pi}{6}-\frac{\pi}{12}$ (d) $\frac{4\pi}{3}-\frac{\pi}{4}$

Solve linear equations

3 Solve the following equations for x.

(a) $2x-1=5$ (b) $x+\frac{\pi}{4}=\frac{\pi}{3}$ (c) $x-\frac{\pi}{4}=\frac{\pi}{2}$ (d) $\pi-3x=\frac{2\pi}{3}$

Determine exact values of special trigonometric expressions with angles in degrees

4 Determine the exact values for the following trigonometric expressions.

(a) $\cos(60°)$ (b) $\tan(30°)$ (c) $\sin(45°)$ (d) $\cos(45°)$

(e) $\sin(90°)$ (f) $\tan(225°)$ (g) $\cos(180°)$ (h) $\cos(120°)$

Convert angles from degrees to radians

5 Calculate the values of these angles in radians.

(a) $90°$ (b) $45°$ (c) $135°$ (d) $330°$

Convert angles from radians to degrees

6 Determine the values of these angles in degrees.

(a) $\frac{\pi}{3}$ (b) $\frac{5\pi}{4}$ (c) $\frac{7\pi}{6}$ (d) $\frac{8\pi}{3}$

Determine exact values of trigonometric expressions with angles in radians

7 Determine the exact values for the following trigonometric expressions.

(a) $\cos\left(\frac{\pi}{3}\right)$ (b) $\tan\left(\frac{\pi}{4}\right)$ (c) $\sin\left(\frac{3\pi}{4}\right)$ (d) $\cos\left(\frac{7\pi}{6}\right)$

(e) $\sin(\pi)$ (f) $\tan\left(\frac{5\pi}{3}\right)$ (g) $\cos\left(\frac{11\pi}{6}\right)$ (h) $\sin\left(\frac{5\pi}{3}\right)$

Substitute values in expressions

8 For each of the following, determine the values of x for $n=0, 1, 2, 3, 4$.

(a) $x=2n+1$ (b) $x=\pi\left(n+\frac{1}{2}\right)$

Use the symmetry properties of the circle

9 If $\sin(x)=0.4$, $\cos(y)=0.8$ and $\tan(w)=2$, calculate the value of the following.

(a) $\sin(-x)$ (b) $\sin(\pi-x)$ (c) $\cos(2\pi-y)$ (d) $\cos(\pi+y)$

(e) $\cos(y-360°)$ (f) $\sin(x+180°)$ (g) $\tan(\pi-w)$ (h) $\tan(180°+w)$

7.1 Solving trigonometric equations

Trigonometry is a mathematical topic that has many practical applications, as trigonometric functions model many situations that involve periodic motion. The solution of the associated equations provides important information that can be applied to areas as diverse as the navigation of sea-going vessels and the calculation of the orbits of satellites above the Earth's surface.

In this section, you will solve equations involving trigonometric functions. In some cases, the domain will be restricted so that there are a limited number of solutions. You will also solve equations for which there is an infinite number of solutions when the domain is not restricted. Graphical methods for solution of equations will be used in a later section.

Both radians and degrees will be used in this chapter, but unless otherwise specified, angles are assumed to be in radians.

It is important to be able to solve trigonometric equations using the inverse trigonometric functions to produce a reference angle, which can be used to determine further solutions for the specified domain.

WARNING

The inverse function of $\sin(\theta)$ is $\sin^{-1}(\theta)$, but note that $\sin^{-1}(\theta) \neq \frac{1}{\sin(\theta)}$, i.e. it is not the reciprocal function. This applies to all the inverse trigonometric functions.

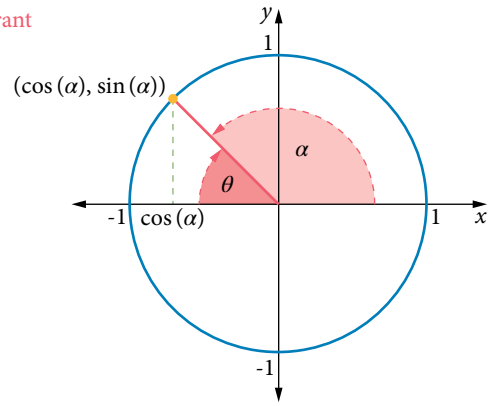
For example, $\cos^{-1}(-1) = \pi$, because $\cos(\pi) = -1$.

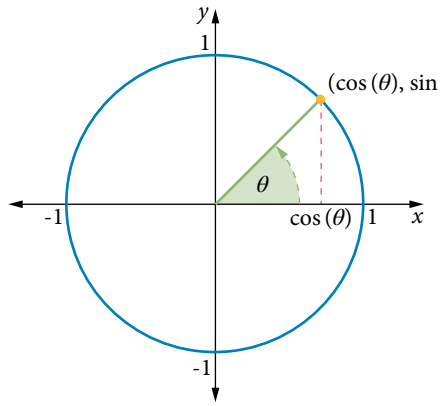
It is important, when solving equations, to remember the quadrants in which each of the trigonometric functions are positive and negative. The diagram below will be helpful.

Angles in quadrants 2, 3 and 4 can be expressed in terms of the reference angle in the first quadrant:

2nd Quadrant
$\sin(\alpha) > 0$
$\cos(\alpha) < 0$
$\tan(\alpha) < 0$

$\alpha = \pi - \theta$

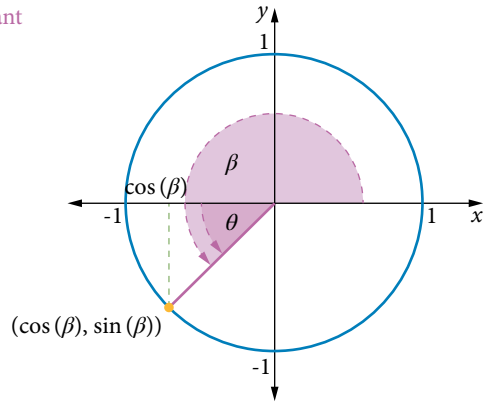


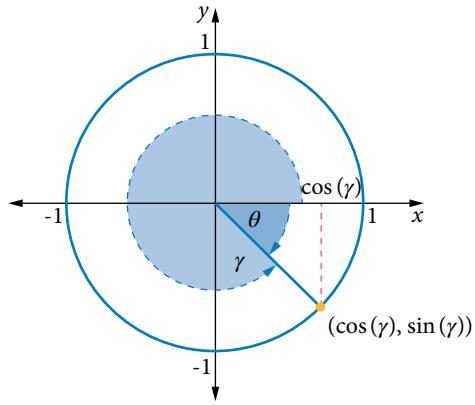


1st Quadrant
$\sin(\theta) > 0$
$\cos(\theta) > 0$
$\tan(\theta) > 0$

3rd Quadrant
$\sin(\beta) < 0$
$\cos(\beta) < 0$
$\tan(\beta) > 0$

$\beta = \pi + \theta$





4th Quadrant
$\sin(\gamma) < 0$
$\cos(\gamma) > 0$
$\tan(\gamma) < 0$

$\gamma = 2\pi - \theta$

1 Solving an equation of the type $\sin(x) = c$

Solve the following equations for the given domain.

(a) Solve the equation $\sin(x) = \frac{1}{2}$, $0 \leq x \leq 2\pi$.

THINKING	WORKING
1 Recall the quadrants in which $\sin(x)$ is positive.	The sine function is positive in the first and second quadrants.
2 Determine the value of the reference angle in the first quadrant.	$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$
3 Determine the equivalent angles in the quadrants you have named and state the solutions.	$x = \frac{\pi}{6}$ and $x = \pi - \frac{\pi}{6}$ $= \frac{5\pi}{6}$

(b) $\sin(x) = -\frac{\sqrt{2}}{2}$, $0 \leq x \leq 2\pi$

1 Recall the quadrants in which $\sin(x)$ is negative.	The sine function is negative in the third and fourth quadrants.
2 Determine the value of the reference angle in the first quadrant.	$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
3 Determine the equivalent angles in the quadrants you have named and state the solutions.	$x = \pi + \frac{\pi}{4}$ and $x = 2\pi - \frac{\pi}{4}$ $= \frac{5\pi}{4}$ $= \frac{7\pi}{4}$

Making connections

Vertical translation of $g(x) = \sin(x) + k$

Explore how the value of c changes the position of the graph and the solution to $g(x) = 0$.

Explore further

Solving trigonometric equations

Use a graphical approach to determine the solutions to trigonometric equations.

2 Solving an equation of the type $\cos(x) = c$

Solve the equation $2\cos(x) + \sqrt{3} = 0$, $0 \leq x \leq 2\pi$.

THINKING	WORKING
1 Rearrange the equation so that is in the form $\cos(x) = c$.	$\cos(x) = -\frac{\sqrt{3}}{2}$
2 Recall the quadrants in which $\cos(x)$ is negative.	The cosine function is negative in the second and third quadrants.
3 Calculate the value of the reference angle in the first quadrant.	$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

4 Determine the equivalent angles in the quadrants you have named and state the solutions.	$x = \pi - \frac{\pi}{6}$ and $x = \pi + \frac{\pi}{6}$ $= \frac{5\pi}{6}$ $= \frac{7\pi}{6}$

3 Solving an equation of the type tan(x) = c

Solve the equation $\tan(x) - 1 = 0$, $0° \leq x \leq 360°$.

THINKING	WORKING
1 Rearrange the equation so that is in the form $\tan(x) = c$.	$\tan(x) = 1$
2 Recall the quadrants in which $\tan(x)$ is positive.	This is true for the first and third quadrants.
3 Calculate the value of the reference angle in the first quadrant.	$\tan^{-1}(1) = 45°$
4 Determine the equivalent angles in the quadrants you have named and state the solutions. (Note that since the domain was given in degrees, the solutions should also be in degrees.)	$x = 45°$ and $x = 180° + 45°$ $= 225°$

4 Solving equations with different domains

Solve the following equations for the given domain.

(a) $\tan(x) = -\sqrt{3}$, $-\pi \leq x \leq \pi$

THINKING	WORKING
1 Recall the quadrants in which $\tan(x)$ is negative.	This is true for the second and fourth quadrants.
2 Calculate the value of the reference angle in the first quadrant.	$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
3 Determine the equivalent angles in the quadrants you have named.	$x = \pi - \frac{\pi}{3}$ and $x = 0 - \frac{\pi}{3}$ $= \frac{2\pi}{3}$ $= -\frac{\pi}{3}$
4 Determine the solutions that are within the required domain.	$x = \frac{2\pi}{3}$ and $x = -\frac{\pi}{3}$

(b) $\sin(x) = \frac{1}{2}, \frac{\pi}{2} \leq x \leq \pi$

1 Recall the quadrants in which $\sin(x)$ is positive.	This is true for the first and second quadrants, but the domain only includes the second quadrant.
2 Calculate the value of the reference angle in the first quadrant.	$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$
3 Determine the equivalent angles in the quadrants needed and state the solution.	$x = \pi - \frac{\pi}{6}$ $= \frac{5\pi}{6}$

Technology worked example

Solving trigonometric equations with a restricted domain

5 Solving equations of the form $\sin(ax) = c$, $\cos(ax) = c$ or $\tan(ax) = c$

Solve the following equations.

(a) $\cos(2x) = \frac{1}{2}, 0 \leq x \leq 2\pi$

THINKING	WORKING
1 Recall the quadrants in which $\cos(ax)$ is positive.	This is true for the first and fourth quadrants.
2 Adjust the domain to match the argument (the angle in the bracket).	$0 \leq x \leq 2\pi$ $0 \leq 2x \leq 4\pi$
3 Calculate the value of the reference angle in the first quadrant.	$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$
4 Determine the possible values of ax inside the adjusted domain, (add 2π to each of the first two values) and state the solutions.	$2x = \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi + \frac{\pi}{3}, 2\pi + \frac{5\pi}{3}$ $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$
5 Solve for x and list the answers. (Check that your final answers are within the original domain given.)	$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) $\sin\left(\frac{x}{3}\right) = 0.6,\ 0 \le x \le 6\pi$ (Give your answers correct to 2 decimal places.)

1 Recall the quadrants in which $\sin(ax)$ is positive.	This is true for the first and second quadrants.
2 Adjust the domain to match the argument (the angle in the bracket).	$0 \le x \le 6\pi$ $0 \le \frac{x}{3} \le 2\pi$
3 Calculate the value of the reference angle in the first quadrant.	$\frac{x}{3} = \sin^{-1}(0.6)$ $= 0.6435$
4 Determine the possible values of ax inside the adjusted domain. Check that you have found all of them.	$\frac{x}{3} = 0.6435$ and $\frac{x}{3} = \pi - 0.6435$ $= 2.4981$
5 Calculate the values of x and list the answers. (Check that your final answers are within the original domain given.)	$x = 1.93$ and 7.49

6 Solving equations of the form $\sin(x - b) = c$, $\cos(x - b) = c$ or $\tan(x - b) = c$

Solve the following equations.

(a) $\cos(x - 30°) = \frac{\sqrt{2}}{2},\ 0° \le x \le 360°$

THINKING	WORKING
1 Recall the quadrants in which $\cos(x - b)$ is positive.	This is true for the first and fourth quadrants.
2 Adjust the domain to match the argument (the angle in the bracket).	$0° \le x \le 360°$ $-30° \le x - 30° \le 330°$
3 Calculate the value of the reference angle in the first quadrant.	$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$
4 Determine the possible values of $x - b$ inside the adjusted domain.	$x - 30° = 45°$ and $x - 30° = 360° - 45°$ $= 315°$
5 Solve for x and list the answers. (Check that your final answers are within the original domain given.)	$x = 75°$ and $345°$

(b) $\sin\left(x+\frac{5\pi}{3}\right)=\frac{1}{2}, 0\le x\le 2\pi$

1 Recall the quadrants in which $\sin(x-b)$ is positive.	This is true for the first and second quadrants.
2 Adjust the domain to match the argument (the angle in the bracket).	$0\le x\le 2\pi$ $0+\frac{5\pi}{3}\le x+\frac{5\pi}{3}\le 2\pi+\frac{5\pi}{3}$
3 Calculate the value of the reference angle in the first quadrant.	$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$
4 Determine the possible values of $x-b$ inside the adjusted domain.	$x+\frac{5\pi}{3}=\frac{\pi}{6},\pi-\frac{\pi}{6},2\pi+\frac{\pi}{6},3\pi-\frac{\pi}{6}\ldots$ $=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}\ldots$ In the domain $\frac{5\pi}{3}\le x+\frac{5\pi}{3}\le\frac{11\pi}{3}$: $x+\frac{5\pi}{3}=\frac{13\pi}{6},\frac{17\pi}{6}$
5 Solve for x and list the answers. (Check that your final answers are within the original domain given, remembering that a negative angle represents a clockwise rotation.)	$x=\frac{13\pi}{6}-\frac{10\pi}{6},\frac{17\pi}{6}-\frac{10\pi}{6}$ $=\frac{3\pi}{6},\frac{7\pi}{6}$ $=\frac{\pi}{2},\frac{7\pi}{6}$ $\sin\left(x+\frac{5\pi}{3}\right)=\frac{1}{2}$, over $0\le x\le 2\pi$, the solutions are $x=\frac{1}{2}$ or $x=\frac{7\pi}{6}$.

7 Solving more complex equations

Solve the equation $\tan\left(3\left(x-\frac{\pi}{4}\right)\right)=-1,\ 0\le x\le 2\pi$.

THINKING	WORKING
1 Recall the quadrants in which $\tan(a(x-b))$ is negative.	This is true for the second and fourth quadrants.
2 Calculate the value of the reference angle in the first quadrant.	$\tan^{-1}(1)=\frac{\pi}{4}$
3 Adjust the domain of values to match the argument.	$0\le x\le 2\pi$ $-\frac{\pi}{4}\le x-\frac{\pi}{4}\le\frac{7\pi}{4}$ $-\frac{3\pi}{4}\le 3\left(x-\frac{\pi}{4}\right)\le\frac{21\pi}{4}$

4 Determine the possible values of $a(x-b)$.

$$3\left(x-\frac{\pi}{4}\right)=-\frac{\pi}{4},\frac{3\pi}{4},\frac{7\pi}{4},\frac{15\pi}{4},\frac{19\pi}{4}$$

5 Determine all solutions, checking that all values lie in the given domain.

$$x-\frac{3\pi}{4}=-\frac{\pi}{12},\frac{3\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12},\frac{15\pi}{12},\frac{19\pi}{12}$$
$$x=\frac{2\pi}{12},\frac{6\pi}{12},\frac{10\pi}{12},\frac{14\pi}{12},\frac{18\pi}{12},\frac{22\pi}{12}$$
$$x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2},\frac{11\pi}{6}$$

General solutions

When solving a trigonometric equation over an unrestricted domain, infinite solutions exist. You can write the solutions in general form by considering the positions of coterminal angles.

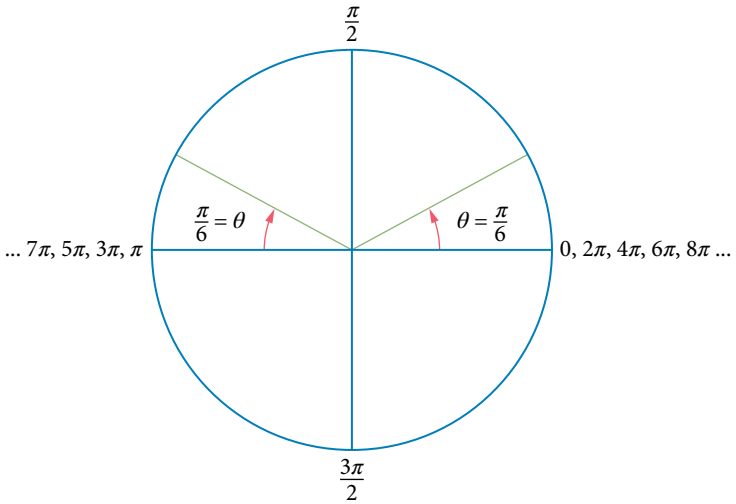
Consider the solutions to the equation $\sin(x)=\frac{1}{2}$.

In the first quadrant, the reference angle is $x=\theta=\frac{\pi}{6}$ and in the second quadrant $\sin(x)$ is also positive.

Therefore, the equivalent angle is:

$$x=\pi-\theta$$
$$=\pi-\frac{\pi}{6}$$
$$=\frac{5\pi}{6}$$

To calculate additional solutions, determine the coterminal angles in the circle.



The solutions to the equation in the first quadrant	The solutions to the equation in the third quadrant
$0+\frac{\pi}{6}=\frac{\pi}{6}$	$\pi-\frac{\pi}{6}=\frac{5\pi}{6}$
$2\pi+\frac{\pi}{6}=\frac{13\pi}{6}$	$3\pi-\frac{\pi}{6}=\frac{17\pi}{6}$
$4\pi+\frac{\pi}{6}=\frac{25\pi}{6}$	$5\pi-\frac{\pi}{6}=\frac{29\pi}{6}$
$6\pi+\frac{\pi}{6}=\frac{37\pi}{6}$	$7\pi-\frac{\pi}{6}=\frac{41\pi}{6}$

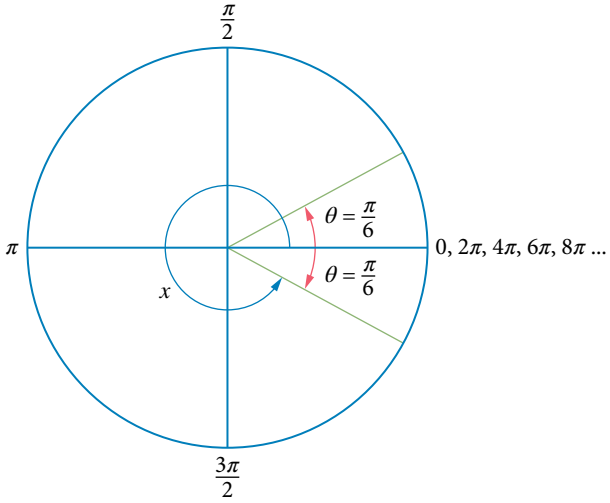
The general solution to the equation $\sin(x)=a$, where $\sin^{-1}(a)=\theta$ represents the reference angle in the first quadrant, can be written as $x=2n\pi+\theta$ or $x=(2n+1)\pi-\theta$ for $n\in\mathbb{Z}$.

These equations can be combined to give $x=n\pi+(-1)^n\theta$ for $n\in\mathbb{Z}$.

Consider the solutions to the equation $\cos(x) = \frac{\sqrt{3}}{2}$. In the first quadrant, the first solution and reference angle is $\theta = \frac{\pi}{6}$, and $\cos(\theta)$ is also positive in the fourth quadrant, where the equivalent angle is:

$$\begin{aligned} x &= 2\pi - \theta \\ &= 2\pi - \frac{\pi}{6} \\ &= \frac{11\pi}{6} \end{aligned}$$

To calculate additional solutions, determine the coterminal angles around the circle.



The solutions to the equation in the first quadrant	The solutions to the equation in the fourth quadrant
$0 + \frac{\pi}{6} = \frac{\pi}{6}$	$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$
$2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$	$4\pi - \frac{\pi}{6} = \frac{23\pi}{6}$
$4\pi + \frac{\pi}{6} = \frac{25\pi}{6}$	$6\pi - \frac{\pi}{6} = \frac{35\pi}{6}$
$6\pi + \frac{\pi}{6} = \frac{37\pi}{6}$	$8\pi - \frac{\pi}{6} = \frac{47\pi}{6}$

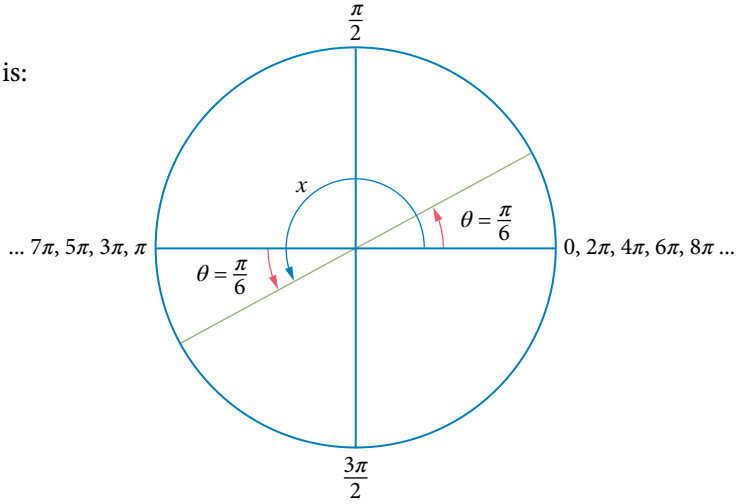
The general solution to the equation $\cos(x) = a$, where $\cos^{-1}(a) = \theta$ represents the reference angle in the first quadrant, can be written as $x = 2n\pi \pm \theta$ for $n \in \mathbb{Z}$.

Consider the solutions to the equation $\tan(x) = \frac{1}{\sqrt{3}}$

In the first quadrant the reference angle is $x = \theta = \frac{\pi}{6}$ and in the third quadrant $\tan(x)$ is also positive, the equivalent angle is:

$$\begin{aligned} x &= \pi + \theta \\ &= \pi + \frac{\pi}{6} \\ &= \frac{7\pi}{6} \end{aligned}$$

To calculate additional solutions, determine the coterminal angles around the circle.



The solutions to the equation in the first quadrant:	The solutions to the equation in the third quadrant:
$0+\frac{\pi}{6}=\frac{\pi}{6}$	$\pi+\frac{\pi}{6}=\frac{7\pi}{6}$
$2\pi+\frac{\pi}{6}=\frac{13\pi}{6}$	$3\pi+\frac{\pi}{6}=\frac{19\pi}{6}$
$4\pi+\frac{\pi}{6}=\frac{25\pi}{6}$	$5\pi+\frac{\pi}{6}=\frac{31\pi}{6}$
$6\pi+\frac{\pi}{6}=\frac{37\pi}{6}$	$7\pi+\frac{\pi}{6}=\frac{43\pi}{6}$

The general solution to the equation $\tan(x)=a$, where $\tan^{-1}(a)=\theta$ represents the reference angle in the first quadrant, can be written as: $x=n\pi+\theta$ for $n\in\mathbb{Z}$.

8 Solving equations with an unrestricted domain

Solve these equations, leaving your answer in general form.

(a) $\sin(x)=\frac{\sqrt{3}}{2}$

THINKING	WORKING
1 Recall the quadrants in which $\sin(x)$ is positive.	This is true for the first and second quadrants.
2 Calculate the reference angle in the first quadrant.	$\theta=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$
3 Write a general solution for all possible values of x.	$x=\frac{\pi}{3}+2n\pi, n\in\mathbb{Z}$ or $x=(2n+1)\pi-\frac{\pi}{3}, n\in\mathbb{Z}$ These solutions may be combined: $x=n\pi+(-1)^n\frac{\pi}{3}, n\in\mathbb{Z}$

(b) $\tan(x)=\frac{1}{\sqrt{2}}$

1 Recall the quadrants in which $\tan(x)$ is positive.	Q1 and Q3
2 Calculate the reference angle in the first quadrant.	$\theta=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$
3 Write a general solution for all possible values of x.	$x=n\pi+\frac{\pi}{4}, n\in\mathbb{Z}$

9 Solving equations with an unrestricted domain along the vertical axis

Solve $\cos(2x) = 0$ for x.

1 Recall the quadrants in which $\cos(ax)$ is positive.	This will be along the vertical axis.
2 Calculate the reference angle for ax that is appropriate for the equation.	$\cos^{-1}(0) = \frac{\pi}{2}$
3 Write a general solution for ax.	$2x = 2n\pi \pm \frac{\pi}{2},\ n \in \mathbb{Z}$
4 Write a general solution for x.	$x = n\pi \pm \frac{\pi}{4},\ n \in \mathbb{Z}$

EXERCISE 7.1 Solving trigonometric equations

Worked Example

1 Solve the following equations in the domain $0 \le x \le 2\pi$.

(a) $\sin(x) = -\frac{1}{2}$ (b) $\sin(x) = \frac{\sqrt{2}}{2}$

2 Solve the following equations in the domain $0 \le x \le 2\pi$.

(a) $\cos(x) = \frac{\sqrt{3}}{2}$ (b) $\cos(x) = -\frac{1}{2}$

3 Solve the following equations in the domain $0 \le x \le 2\pi$.

(a) $\tan(x) = -\sqrt{3}$ (b) $\tan(x) = 0$

4 Solve each of these equations in the given domain.

(a) $\cos(x) = -\frac{\sqrt{3}}{2},\ -\pi \le x \le \pi$ (b) $\sin(\theta) = 1,\ 0 \le \theta \le 6\pi$ (c) $\tan(2x) = 1,\ 0 \le x \le \pi$

5 Solve the following equations for $0 \le x \le 2\pi$. Give your answers correct to 2 decimal places.

(a) $\tan(x) = 1.2$ (b) $\sin(x) = 0.2$ (c) $\cos(x) = -0.4$ (d) $\sin(x) = -\frac{1}{4}$

6 Solve each of these equations in the given domain.

(a) $\cos(3x) = -\frac{\sqrt{2}}{2},\ 0 \le x \le 2\pi$ (b) $\sin(4x) = \frac{1}{2},\ 0 \le x \le 4\pi$

(c) $\tan(2x) = -1,\ 0 \le x \le 2\pi$ (d) $\cos\left(\frac{\theta}{2}\right) = \frac{1}{2},\ 0° \le \theta \le 360°$

7 Consider the function $f(x) = \sin(3x)$.

(a) Estimate the number of solutions for $f(x) = \frac{\sqrt{3}}{2}, 0 \le x \le 2\pi$.

(b) Solve $f(x) = \frac{\sqrt{3}}{2}, 0 \le x \le 2\pi$.

(c) Estimate the number of solutions for $f(x) = -1, 0 \le x \le 2\pi$.

(d) Solve the equation $f(x) = -1, 0 \le x \le 2\pi$.

8 Solve each of the following equations for the given domain. 6

(a) $\cos\left(x+\frac{5\pi}{6}\right)=\frac{\sqrt{3}}{2}, 0\le x\le 2\pi$ **(b)** $\sin\left(x-\frac{\pi}{3}\right)=\frac{1}{2}, 0\le x\le 2\pi$

(c) $\tan(x+135°)=-1, 0°\le x\le 360°$

9 Solve these equations for $[0, 2\pi]$.

(a) $\sin(2A)=\cos\left(\frac{5\pi}{6}\right)$ **(b)** $\cos\left(\frac{\theta}{2}\right)=-\sin\left(\frac{\pi}{6}\right)$

10 Use technology to solve the equation $\cos\left(2\left(x+\frac{\pi}{4}\right)\right)=-0.45$, writing your answers in general form. Give answers correct to 2 decimal places.

11 Solve these equations in the given domain.

(a) $\tan^2(x)-3=0, 0\le x\le 2\pi$ **(b)** $4\cos^2(\theta)=1, 0\le\theta\le 2\pi$ **(c)** $\sin^2(x)=1, -\pi\le x\le\pi$

12 Solve the following equations, giving your answers in exact form or correct to 2 decimal places.

(a) $\sin(x)(3\sin(x)-1)=0, 0\le x\le\pi$ **(b)** $2\cos^2(x)-\cos(x)-1=0, 0°\le x\le 360°$

13 What are the solutions to the equation $\cos(2x)+\frac{1}{2}=0, 0\le x\le\pi$?

A $\frac{\pi}{6}, \frac{5\pi}{6}$ **B** $\frac{\pi}{12}, \frac{5\pi}{12}$ **C** $\frac{\pi}{3}, \frac{2\pi}{3}$ **D** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

14 If $\sin(x)=\frac{1}{a}$ where $a>1$, what is x equal to?

A $\sin^{-1}\left(\frac{1}{a}\right)$ only **B** $\sin^{-1}\left(\frac{1}{a}\right)$ or $\pi-\sin^{-1}\left(\frac{1}{a}\right)$

C $\pi-\sin^{-1}\left(\frac{1}{a}\right)$ or $\pi+\sin^{-1}\left(\frac{1}{a}\right)$ **D** $\frac{1}{a}$ or $\pi-\frac{1}{a}$

15 Solve the following for $0\le x\le 2\pi$. 7

(a) $\tan\left(3\left(x+\frac{\pi}{4}\right)\right)=1$ **(b)** $\cos\left(2\left(x-\frac{\pi}{3}\right)\right)=\frac{1}{2}$

(c) $\sin\left(2\left(x+\frac{\pi}{6}\right)\right)=\frac{\sqrt{3}}{2}$ **(d)** $\sin\left(3\left(x-\frac{\pi}{2}\right)\right)=-\frac{1}{2}$

16 Use technology to solve the equation $\cos(2(x-0.4))=0.71, 0\le x\le\pi$, giving your answers correct to 2 decimal places.

17 Solve the following equations, expressing your solutions in general form. You will notice that the general solutions found in this question are in a different form from those found using technology. You can substitute values of n to check that they represent the same solutions. 8

(a) $\tan(2x)=\sqrt{3}$ **(b)** $\sin(x)=\frac{\sqrt{2}}{2}$ **(c)** $\cos(x)=-\frac{1}{2}$ **(d)** $\sin(x)=-\frac{\sqrt{3}}{2}$

18 Solve the equation $\sin^2(2x)-3\sin(2x)=4, 0\le x\le 2\pi$.

19 Solve $5\cos(x)=2\sin(x), 0\le x\le 2\pi$. Give your answer correct to 2 decimal places. Hint: $\tan(x)=\frac{\sin(x)}{\cos(x)}$

Worked Example

9

20 A particle moves in a straight line so that its position, x cm, at time t seconds after it begins to move from a point O is $x = 2 - 4\sin(2t)$. When the particle travels to the right of O, x is considered to be positive.

(a) Determine the position of the particle when it starts to move.

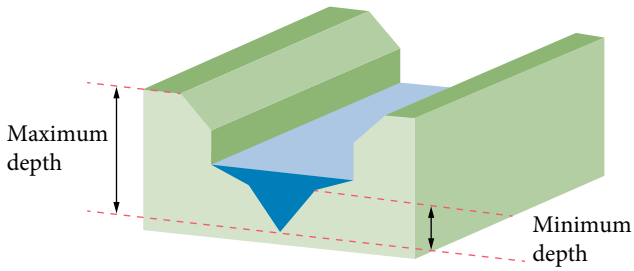
(b) Solve the equation $2 - 4\sin(2t) = 0$ to determine the first time that the particle passes through O.

(c) Calculate the next three times that the particle passes the point O. Leave your answer in exact form.

(d) Solve an equation to determine the first time that the particle is 3 cm to the right of O. Give your answer correct to 2 decimal places.

(e) Does the particle ever get to a position 3 cm to the left of O?

21 The depth of water in a tidal river is modelled by the function $D = 2\cos\left(\frac{\pi t}{6} + 3\right) + 10$, where D is the depth of the water in metres and t is the time in hours after midnight.



(a) Calculate the maximum and minimum depth of the water.

(b) After how many hours does the first high tide occur? Give your answer correct to 2 decimal places.

(c) Determine the time between the first two low tides.

22 Solve $\cos^4(x) - \sin^4(x) = 0, 0 \leq x \leq 2\pi$. (Hint: $\cos^2(A) + \sin^2(A) = 1$ and $\frac{\sin(A)}{\cos(A)} = \tan(A)$, for any angle A.)

23 Consider the following equation $5\sin\left(\pi x - \frac{1}{2}\right) + 3 = p$.

(a) Calculate values of p for which the equation has no solutions.

(b) If $p = 3$, determine the first four positive solutions, in terms of π.

(c) Suggest a general solution for this equation when $p = 3$.

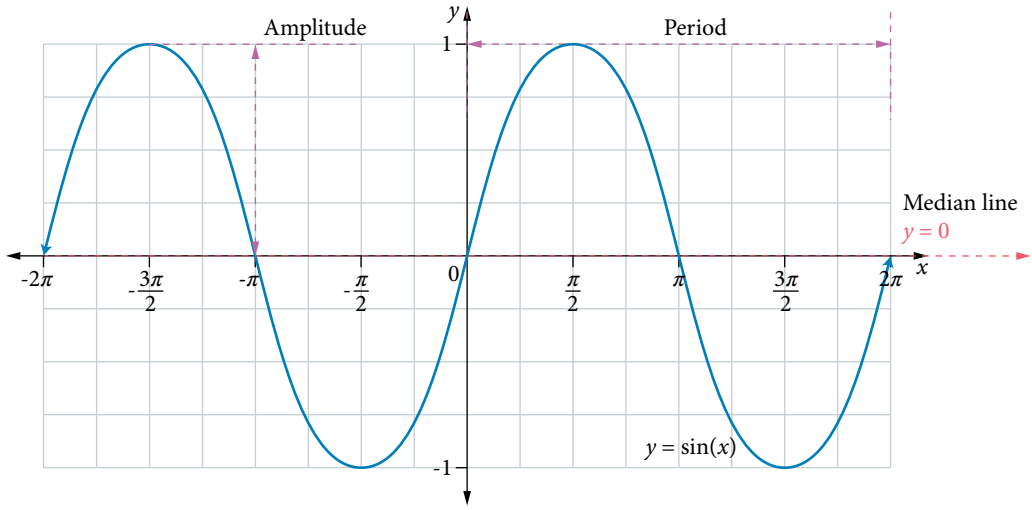
Graphing and modelling with trigonometric functions

7.2

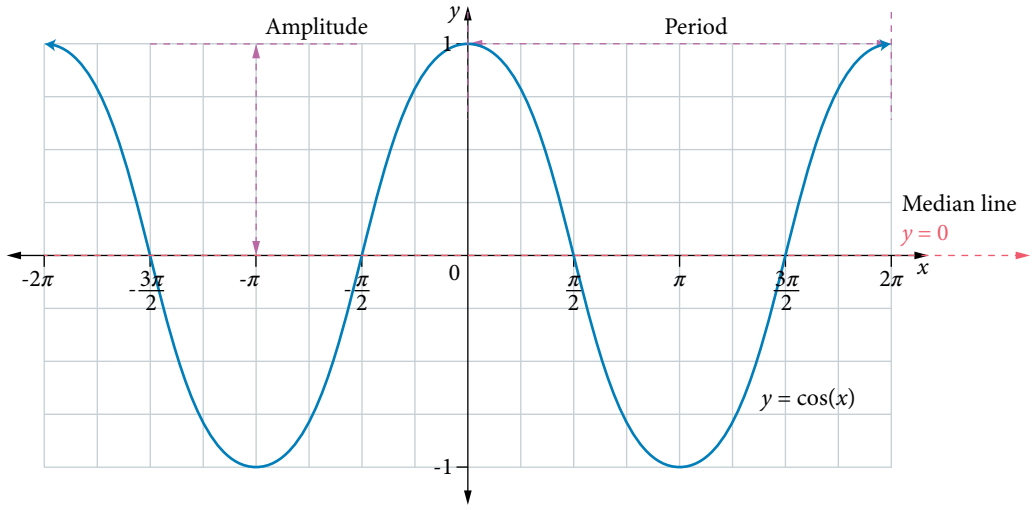
Trigonometric graphs are periodic and therefore repeat at regular intervals. The intervals at which these graphs repeat is called the period. The maximum vertical distance from the middle of the graph or median line is called the amplitude, which is always positive.

Graphs of the sine and cosine functions

The graph of $y = \sin(x)$ is shown below. It is drawn by transferring directly the values of the y-coordinate of a point P as it moves anticlockwise around a unit circle. It has been plotted here for $-2\pi \leq x \leq 2\pi$, where two cycles or periods are shown. The amplitude is 1 unit and the median line is the x-axis with the equation $y = 0$.



The graph of $y = \cos(x)$ is shown below. It is drawn by transferring directly the values of the x-coordinate of a point P as it moves anticlockwise around a unit circle. It has been plotted here for $-2\pi \leq x \leq 2\pi$, where two cycles or periods are shown. The amplitude is 1 unit and the median line is the x-axis with the equation $y = 0$.



For the graphs of both $f(x) = \sin(x)$ and $f(x) = \cos(x)$:

- The period is 2π.
- The amplitude is 1 unit.
- The range is $[-1,1]$.

Making connections

The unit circle and the graph of the sine function

Move the slider to see the relationship between the unit circle and the graph of the sine and cosine functions.

Explore further

Modelling trigonometric functions

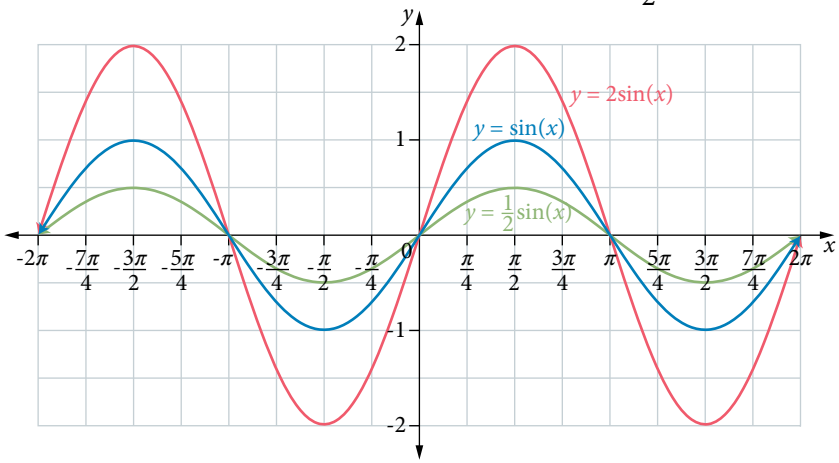
Explore how the graphs of $y = \sin(x)$ and $y = \cos(x)$ change under dilation, reflection and translation.

Transformations of sine and cosine graphs

Vertical dilations: functions of the form $y = Af(x)$

When a graph is dilated from the x-axis it is stretched vertically, which affects the amplitude of the graph. It does not affect the period.

For $f(x) = \sin(x)$ and $f(x) = \cos(x)$, graphs of the form $y = Af(x)$ are dilated by factor A from the x-axis. The following graphs show how the shapes of the graphs of $f(x) = 2\sin(x)$ and $f(x) = \frac{1}{2}\sin(x)$ are related to the shape of $f(x) = \sin(x)$. Except for the x-axis intercepts, for any given x-value, the distance of the curve from the median line to $f(x) = 2\sin(x)$ is twice that of the curve $f(x) = \sin(x)$. At any given x-value, the height of the curve $f(x) = \frac{1}{2}\sin(x)$ is half that of the curve $f(x) = \sin(x)$.



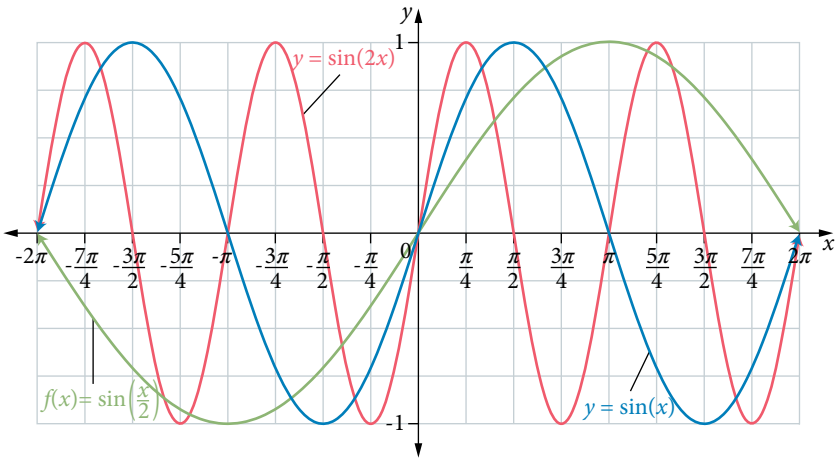
If A is positive, for both $f(x) = A\sin(x)$ and $f(x) = A\cos(x)$:

- The period is 2π.
- The amplitude is $|A|$ units.
- The range is $[-|A|,|A|]$.

Horizontal dilations: functions of the form $y = f(ax)$

When a graph is dilated from the y-axis it is stretched or contracted horizontally, which affects the period of the graph. It does not affect the amplitude.

The graph on the next page shows how the shapes of the graphs of $f(x) = \sin(2x)$ and $f(x) = \sin\left(\frac{x}{2}\right)$ are related to the shape of the graph with equation $f(x) = \sin(x)$.



The graph of $f(x) = \sin(x)$ has period 2π, the period of the graph of $f(x) = \sin(2x)$ is π and the period of the graph of $f(x) = \sin\left(\frac{x}{2}\right)$ is 4π.

In general, for $f(x) = \sin(x)$ and $f(x) = \cos(x)$, graphs of the form $y = f(a(x))$ are dilated by factor $\frac{1}{a}$ from the y-axis. The period, T, for graphs of the form $y = f(a(x))$ is $\frac{2\pi}{a}$.

For example, if $f(x) = \cos(3x)$, the period is $\frac{2\pi}{3}$.

If $f(x) = \cos\left(\frac{x}{5}\right)$, the period is $\frac{2\pi}{\frac{1}{5}} = 10\pi$.

For functions of the form $f(x) = \sin(ax)$ and $f(x) = \cos(ax)$:

- The period is $\frac{2\pi}{a}$.
- The amplitude is 1 unit.
- The range is [-1, 1].

10 Graphing functions of the form $f(x) = A\cos(ax)$

Consider the function $f(x) = 3\cos(2x)$, $0 \leq x \leq 2\pi$.

(a) Write the amplitude and period of $f(x)$.

THINKING	WORKING
The amplitude is the coefficient of $f(x)$ and the period is 2π divided by the coefficient of x.	The amplitude is 3 units. The period is π.

(b) Calculate the x-intercepts, by solving the equation $f(x) = 0$.

Solve $f(x) = 0$ for $0 \leq x \leq 2\pi$.

$0 \leq x \leq 2\pi$

$0 \leq 2x \leq 4\pi$

$\cos(2x) = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

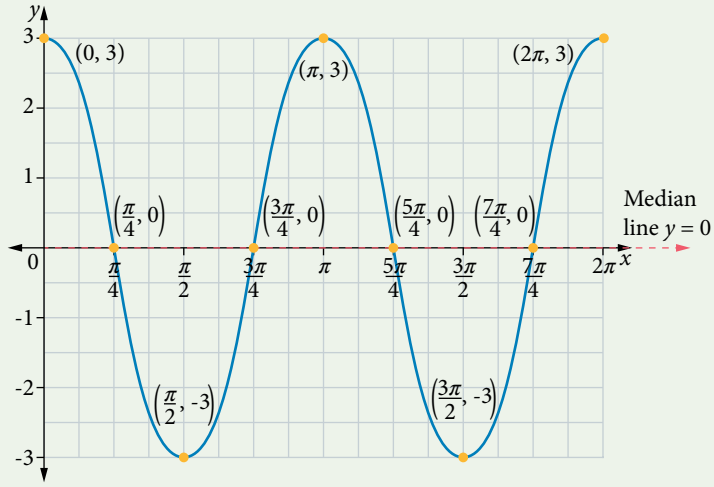
(c) Determine the equation of the median line.

The median line is the horizontal line midway between the maximum and minimum points on the graph.

$$y = \frac{-3+3}{2}$$
$$= 0$$

(d) Sketch the graph of the function, labelling all key features, including the coordinates of the maximum and minimum points and the median line.

1 Mark and label the maximum and minimum points and axis intercepts in coordinate form. Draw the median line labelled with its equation and draw a smooth curve through the points.



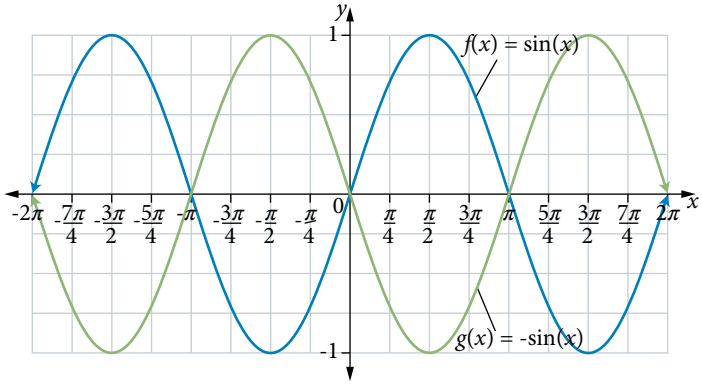
2 Maximum and minimum points are found midway between the x-intercepts.

Maximum points: $(0, 3), (\pi, 3), (2\pi, 3)$

Minimum points: $\left(\frac{\pi}{2}, -3\right), \left(\frac{3\pi}{2}, -3\right)$

Reflection: Functions of the form $f(x) = A\sin(ax)$ and $f(x) = A\cos(ax)$ for $A < 0$

A negative value for the coefficient A reflects the graph in the x-axis. The graphical relationship between $f(x) = \sin(x)$ and $g(x) = -\sin(x)$ is shown on the graph below where each curve is the reflection of the other in the x-axis.



For functions of the form $f(x) = A\sin(ax)$ and $f(x) = A\cos(ax)$, $A < 0$:

- The period is $\frac{2\pi}{a}$.
- The amplitude is $|A|$ units.
- The range is $[-|A|, |A|]$.

11 Graphing functions of the form $f(x) = A\sin(ax)$ for $A < 0$

Consider the function $f(x) = -3\sin(2x)$, $0 \le x \le 2\pi$

(a) Determine the period, amplitude and range of the function.

THINKING	WORKING
1 Calculate the period of the function in the form $y = A\sin(ax)$ using the formula: period $= \frac{2\pi}{a}$	period $= \frac{2\pi}{2}$ $= \pi$ The length of each cycle in the graph will span a horizontal length of π units.
2 Determine the amplitude $\lvert A\rvert$ of the function in the form $y = A\sin(ax)$.	$A = -3$ For $A < 0$, the graph of $f(x) = \sin(x)$ is reflected in the x-axis. $\lvert A\rvert = 3$ The graph spans 3 units in each direction from the median line.
3 Determine the range of the function $[-\lvert A\rvert, \lvert A\rvert]$.	The range of the function is $y \in [-3, 3]$.

(b) Determine the coordinates of the x-intercepts for the graph of $y = f(x)$.

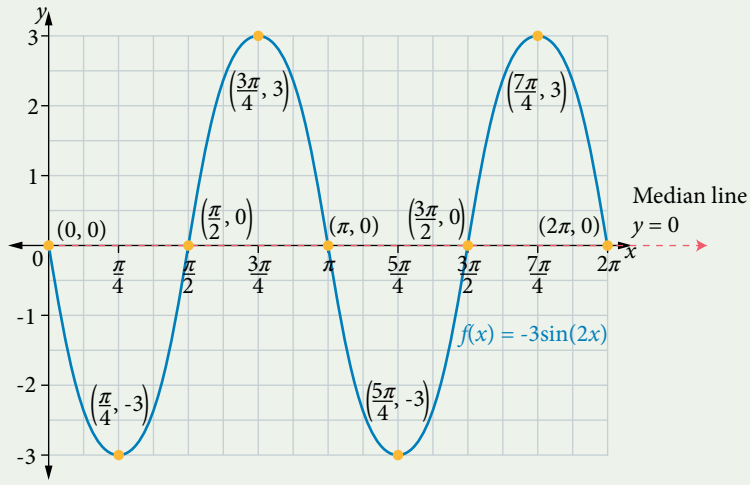
1 Solve $f(x) = 0$ for $0 \le x \le 2\pi$.	$-3\sin(2x) = 0$ $2x = 0, \pi, 2\pi, 3\pi, 4\pi$ $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$
2 Interpret the solution.	The coordinates of the x-intercepts are $(0,0)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, 0\right)$ and $(2\pi, 0)$.

(c) Determine the equation of the median line.

The median line is the horizontal line midway between the maximum and minimum points on the graph.	$y = \frac{-3 + 3}{2}$ $y = 0$

(d) Sketch the graph of the function, labelling all major points in coordinate form, including the maximum and minimum points and showing the median line.

Draw a smooth curve through the major points. Maximum and minimum points are found midway between the x-intercepts.



Maximum points: $\left(\frac{3\pi}{4}, 3\right), \left(\frac{7\pi}{4}, 3\right)$

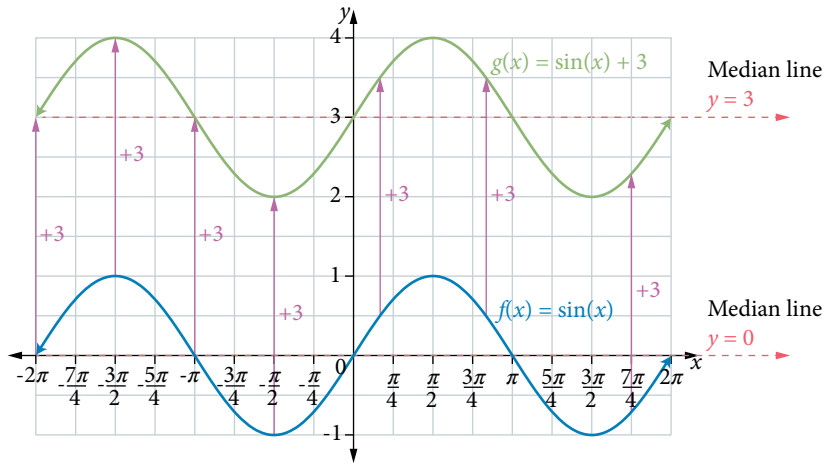
Minimum points: $\left(\frac{\pi}{4}, -3\right), \left(\frac{5\pi}{4}, -3\right)$

Vertical translations: Functions of the form $f(x) = A\,f(ax) + c$

When a curve is translated, each point is moved the same amount in the same direction so that its shape is unchanged.

A vertical translation is one that moves a curve parallel to the y-axis.

The diagram below shows the curve $f(x) = \sin(x)$ translated up 3 units. It has the equation $g(x) = \sin(x) + 3$.



For $f(x) = A\,f(ax) + c$ the vertical translation is c. When a trigonometric curve is translated in this way, the amplitude and period are left unchanged, but the range of the function is different.

For functions of the form $f(x) = A\sin(ax) + c$ and $f(x) = A\cos(ax) + c$:

- The period is $\frac{2\pi}{a}$.
- The amplitude is $|A|$ units.
- The range is $[-|A| + c, |A| + c]$.

12 Graphing a trigonometric function with a vertical translation

Consider the function $f(x) = 3\sin(2x) + 4$, $0 \leq x \leq 2\pi$.

(a) Determine the amplitude, period and range of the function.

THINKING	WORKING
1 Calculate the period of the function in the form $y = A\sin(ax)$ using the formula: $\text{period} = \frac{2\pi}{a}$	$\text{period} = \frac{2\pi}{2}$ $= \pi$ The length of each cycle in the graph will span a horizontal length of π units.
2 Determine the amplitude $\|A\|$ of the function in the form $y = A\sin(ax)$.	$A = 3$ $\|A\| = 3$ The graph spans 3 units in each direction from the median line.
3 Determine the range of the function $[-\|A\| + c, \|A\| + c]$.	The range of the function with a vertical shift of 4 units: $y \in [-3 + 4, 3 + 4]$ $y \in [1, 7]$

(b) Calculate the intercepts of $f(x)$ with the median line.

Solve $f(x) = 4$ for $0 \leq x \leq 2\pi$.

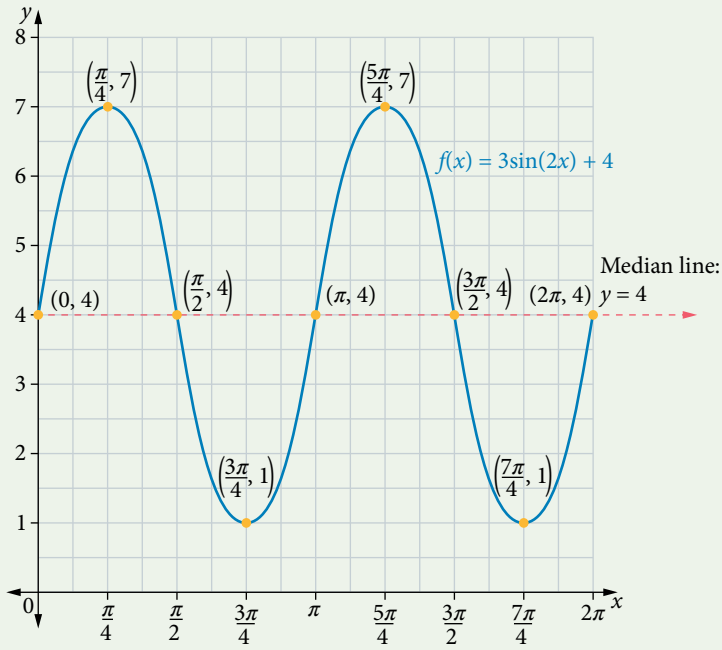
$$3\sin(2x) + 4 = 4$$
$$\sin(2x) = 0$$
$$2x = 0, \pi, 2\pi, 3\pi, 4\pi$$
$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

The graph meets the median line at $(0, 4), \left(\frac{\pi}{2}, 4\right), (\pi, 4), \left(\frac{3\pi}{2}, 4\right)$ and $(2\pi, 4)$.

(c) Sketch the graph of the function, labelling all key points in coordinate form and showing the median line.

In this example, the median line is not the x-axis. The median line is midway between the maximum and minimum value over one period. The maximum and minimum points are halfway between the x-intercepts.

$$\text{median} = \frac{\min + \max}{2}$$
$$= \frac{7+1}{2}$$
$$= 4$$



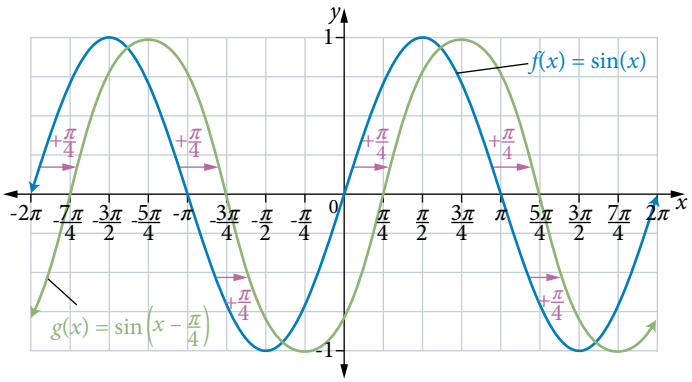
Maximum points: $\left(\frac{\pi}{4}, 7\right), \left(\frac{5\pi}{4}, 7\right)$

Minimum points: $\left(\frac{3\pi}{4}, 1\right), \left(\frac{\pi}{4}, 1\right)$

Horizontal translations: Functions of the form $f(x) = \sin(x - b)$ and $f(x) = \cos(x - b)$

A horizontal translation is one that moves a curve either right or left, parallel to the x-axis. The translation is b units to the right if $b > 0$, and b units to the left if $b < 0$. When a trigonometric curve is translated in this way, the amplitude, range and period remain unchanged. However, intersections with the median line will probably change.

The graph below shows the function $f(x) = \sin(x)$ translated $\frac{\pi}{4}$ units to the right. It has the equation $g(x) = \sin\left(x - \frac{\pi}{4}\right)$.



For functions of the form $f(x) = A\sin(a(x - b)) + c$ and $f(x) = A\cos(a(x - b)) + c$:

- The period is $\frac{2\pi}{a}$.
- The amplitude is $|A|$ units.
- The range is $[-|A| + c, |A| + c]$.

The graphs of $y = \sin(x)$ and $y = \cos(x)$ have been translated b units horizontally and c units vertically.

When determining horizontal translations, the function needs to be expressed in the form $y = A\sin(a(x - b))$ with the angle written in factorised form to determine the size and direction of horizontal translation.

For example, the equation $f(x) = \sin\left(2x + \frac{\pi}{4}\right)$ needs to be written as $f(x) = \sin\left(2\left(x + \frac{\pi}{8}\right)\right)$ where the graph of $f(x)$ can be produced from the graph of $g(x) = \sin(2x)$ by translating it $\frac{\pi}{8}$ units to the left.

13 Graphing a trigonometric function with a horizontal translation

Consider the function $f(x) = 3\sin\left(2x - \frac{\pi}{2}\right)$, $0 \leq x \leq 2\pi$.

(a) Determine the amplitude, period and range of the function.

THINKING	WORKING
1 Express $f(x)$ in the form $y = A\sin(a(x - b))$.	$f(x) = 3\sin\left(2x - \frac{\pi}{2}\right)$ $= 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$
2 The amplitude is the magnitude of the coefficient of $f(x)$. The period is 2π divided by the coefficient of x. The range is the set of y-values.	The amplitude is 3 units. The period is π. The range is $y \in [-3, 3]$.

(b) Calculate the coordinates of the x-intercepts for the graph of $y = f(x)$.

Solve $3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) = 0$ for $0 \leq x \leq 2\pi$.

$$3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) = 0$$

$$2\left(x - \frac{\pi}{4}\right) = 0, \pi, 2\pi, 3\pi, 4\pi$$

$$x - \frac{\pi}{4} = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

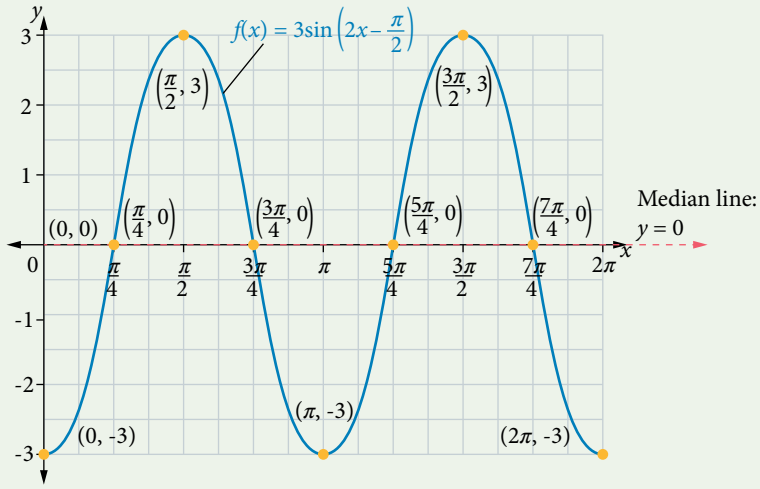
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

The intercepts with the x-axis:

$$\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$$

(c) Sketch the graph of the function, labelling all key features.

Mark the major points and draw the median line. Draw a smooth curve through the points. The maximum and minimum points are halfway between the x-intercepts.



Maximum points: $\left(\frac{\pi}{2}, 3\right), \left(\frac{3\pi}{2}, 3\right)$

Minimum points: $(0, -3), (\pi, -3), (2\pi, -3)$

Sketching graphs using transformations

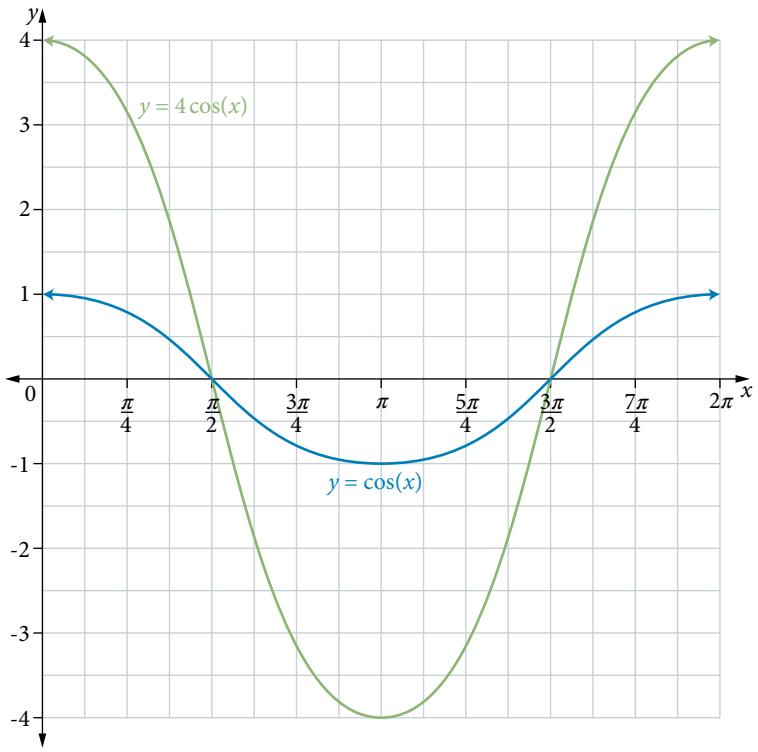
Trigonometric graphs can be sketched using transformations. The order in which the transformations are applied is dilation, reflection and then translation.

In general, the graphs of functions of the form $y = A\,f(a(x - b)) + c$ compared to $y = f(x)$ are:

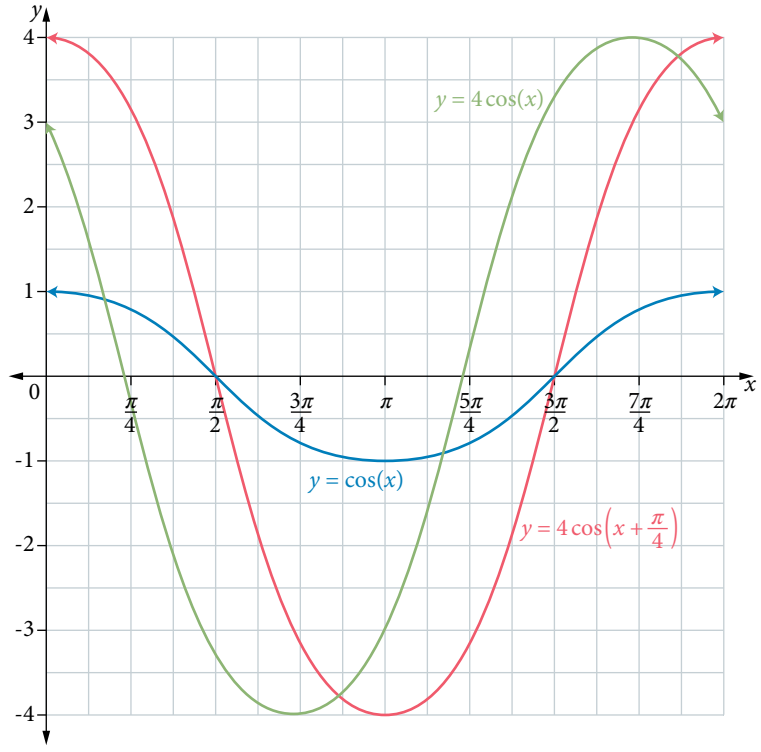
- dilated by a factor $|A|$ from the x-axis, where $(x, y) \to (x, |A|y)$
- dilated by factor of $\frac{1}{a}$ from the y-axis, where $(x, y) \to \left(\frac{1}{a}x, y\right)$
- reflected in the x-axis for $A < 0$, where $(x, y) \to (x, -y)$
- translated b units horizontally, where $(x, y) \to (x + b, y)$
- translated c units vertically, where $(x, y) \to (x, y + c)$

For example, the graph of $y = 4\cos\left(x + \frac{\pi}{4}\right)$ can be produced by:

- dilating $y = \cos(x)$ by a factor of 4 from the x-axis to produce $y = 4\cos(x)$



- translating $y = 4\cos(x)$ by $\frac{\pi}{4}$ units to the left to produce $y = 4\cos\left(x + \frac{\pi}{4}\right)$.



The x-intercepts, minima and maxima can be deduced using the sequence of transformations. The y-intercept can be found using algebra.

14 Determining transformations of trigonometric functions

Determine the transformations that have taken place to transform either $f(x) = \sin(x)$ or $f(x) = \cos(x)$ into the given function. Also list the period and amplitude of each function.

(a) $f(x) = \cos\left(3x + \frac{\pi}{6}\right)$

THINKING	WORKING
1 Write in factorised form.	$f(x) = \cos\left(3\left(x + \frac{\pi}{18}\right)\right)$
2 List the transformations in the correct order.	Dilate by a factor of $\frac{1}{3}$ from the y-axis. Translate $\frac{\pi}{18}$ units to the left.
3 Determine the period and amplitude of the graph.	Period is $\frac{2\pi}{3}$; amplitude is 1.

(b) $f(x) = -\sin(\pi x) + 2$

1 List the transformations in the correct order.	Dilate by a factor of $\frac{1}{\pi}$ from the y-axis. Reflect in the x-axis and translate 2 units up.
2 Determine the period and amplitude of the graph.	Period is $\frac{2\pi}{\pi} = 2$; amplitude is 1.

(c) $f(x) = 6\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

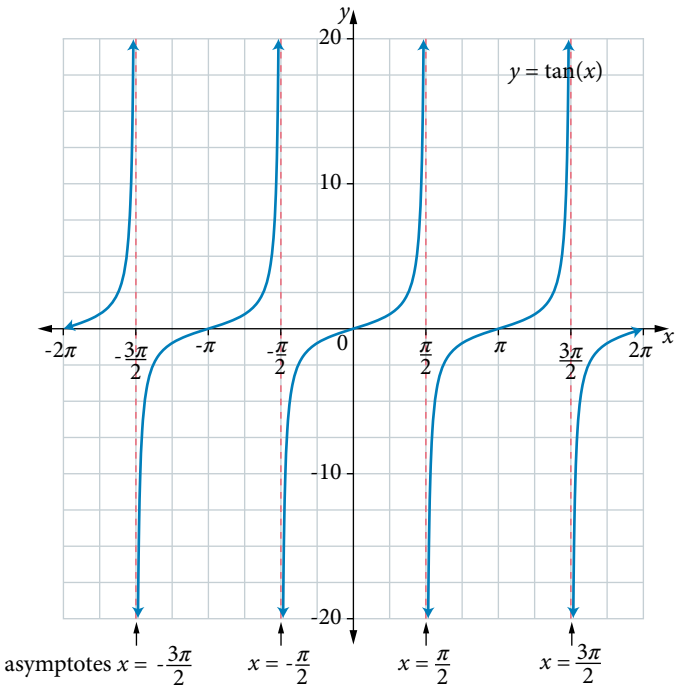
1 Write in factorised form.	$f(x) = 6\cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$
2 List the transformations in the correct order.	Dilate by a factor of 6 from the x-axis. Dilate by a factor of 2 from the y-axis. Translate $\frac{\pi}{2}$ units to the right.
3 Determine the period and amplitude of the graph.	Period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$; amplitude is 6.

Graphs of the tangent function

The graph of $y = \tan(x)$ behaves rather differently from the sine and cosine function graphs. It has an undefined amplitude, vertical asymptotes and a period of π.

The relationship mentioned in the previous section, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, explains some of these characteristics. The vertical asymptotes occur where $\cos(x) = 0$, for $x = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.
The x-intercepts occur where $\tan(x) = \sin(x) = 0$, for $x = n\pi$, $n \in \mathbb{Z}$.

The graph below shows the function $f(x) = \tan(x)$, $-2\pi \leq x \leq 2\pi$ with asymptotes at $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The x-intercepts are halfway between the asymptotes at $(-2\pi, 0)$, $(-\pi, 0)$, $(0,0)$, $(\pi, 0)$ and $(2\pi, 0)$.



For $f(x) = A\tan(ax)$:

- The period is $\frac{\pi}{a}$.
- The asymptotes are at $x = \frac{(2k+1)\pi}{2a}$ for $k \in \mathbb{Z}$.
- The range is $\mathbb{R}$.

For $f(x) = A\tan(a(x - b))$

- The period is $\frac{\pi}{a}$.
- The dilation is by a factor of $|A|$ from the x-axis.
- The asymptotes are at $x = \frac{(2k+1)}{a}\frac{\pi}{2} + b$ for $k \in Z$.
- The range is $\mathbb{R}$.

For the graph of $y = \tan(x)$, $-2\pi \leq x \leq 2\pi$:

- The x-intercepts can be found by solving the equation $\tan(x) = 0$, giving $x = -2\pi, -\pi, 0, \pi, 2\pi$.
- The asymptotes can be found by solving the equation $x = (2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$, giving $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

The dilations, reflections and translations for graphs of the sine and cosine functions also apply to graphs of the tangent function.

15 Graphing functions of the form $f(x) = \tan(ax)$

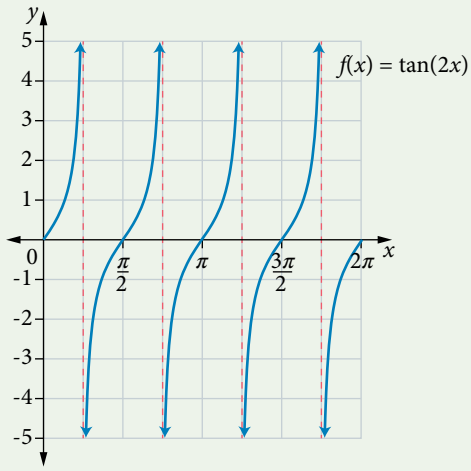
Consider the function $f(x) = \tan(2x)$, $0 \leq x \leq 2\pi$.

(a) Determine the period, equations of the asymptotes and x-intercepts of the function.

THINKING	WORKING
1 Determine the period.	Period is $\frac{\pi}{2}$.
2 Determine the equations of the asymptotes.	$x = (2k+1)\frac{\pi}{4}$ For $k = 0$, $x = \frac{\pi}{4}$; for $k = 1$, $x = \frac{3\pi}{4}$; $k = 2$, $x = \frac{5\pi}{4}$; for $k = 3$, $x = \frac{7\pi}{4}$.

3 Solve $f(x) = 0$ for the given domain to determine the x-intercepts.	$\tan(2x) = 0$ $2x = 0, \pi, 2\pi, 3\pi, 4\pi$ $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(b) Sketch the graph of $y = f(x)$.

Use the asymptotes and axis intercepts to sketch the graph of $y = f(x)$.	

(c) List the transformations that have taken place to transform $y = \tan(x)$ into $y = \tan(2x)$.

Write the transformations in the correct order.	Dilation by a factor of $\frac{1}{2}$ from the y-axis.

16 Graphing functions of the form $f(x) = \tan(a(x - b))$

Consider $h(x) = \tan\left(2\left(x - \frac{\pi}{3}\right)\right), 0 \leq x \leq \pi$.

(a) Sketch the graph of $y = h(x)$, showing the asymptotes and the coordinates of key points.

THINKING	WORKING
1 Write in factorised form.	$h(x) = \tan\left(2\left(x - \frac{\pi}{3}\right)\right)$
2 Determine the period.	Period is $\frac{\pi}{2}$.
3 Identify the horizontal translation.	The translation is $\frac{\pi}{3}$ units to the right.
4 Determine the equations of the asymptotes.	Use $x = (2k+1)\frac{\pi}{4} + \frac{\pi}{3}$. For $k = -1$, $x = \frac{\pi}{12}$; for $k = 0$, $x = \frac{7\pi}{12}$.

5 Calculate the x-intercepts solving $h(x) = 0$ for the given domain.

$$\tan\left(2\left(x - \frac{\pi}{3}\right)\right) = 0$$
$$2\left(x - \frac{\pi}{3}\right) = 0, \pi$$
$$x - \frac{\pi}{3} = 0, \frac{\pi}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{6}$$

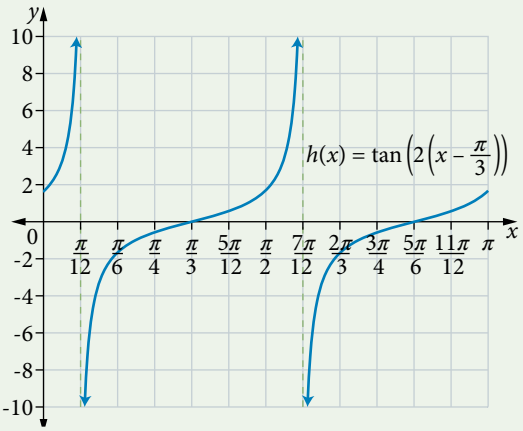
The coordinates of the x-intercepts are $\left(\frac{\pi}{3}, 0\right)$ and $\left(\frac{5\pi}{6}, 0\right)$.

6 Calculate the y-intercept by evaluating $h(0)$.

$$h(0) = \tan\left(2\left(0 - \frac{\pi}{3}\right)\right)$$
$$= \sqrt{3}$$

The coordinates of the y-intercept are $\left(0, \sqrt{3}\right)$.

7 Sketch the graph of $y = h(x)$.



(b) List the transformations in sequential order to produce the graph of $y = h(x)$ from the graph of $f(x) = \tan(x)$.

Determine the transformations in the correct order.

Dilate $f(x)$ by a factor of $\frac{1}{2}$ from the y-axis, then translate it $\frac{\pi}{3}$ units to the right to obtain $h(x)$.

Modelling with trigonometric functions

It is the periodic and bounded nature of the sine and cosine functions that are useful in modelling practical situations. It is important to use mathematical models so that future events can be predicted.

17 Modelling with a trigonometric function

The height of a carriage on a Ferris wheel, h metres above a platform from which people will enter and exit the ride, t minutes after the ride starts, is modelled by the function $h(t) = -10\cos\left(\frac{\pi t}{9}\right) + 10$.

(a) Demonstrate that one complete cycle of the ride takes 18 minutes.

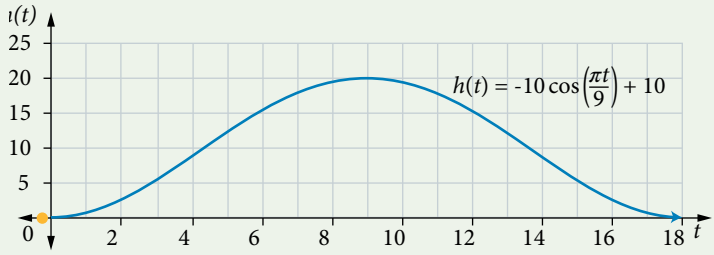
THINKING	WORKING
1 Determine the period of the function.	$T = \dfrac{2\pi}{\frac{\pi}{9}}$ $= 2\pi \times \dfrac{9}{\pi}$ $= 18$
2 Interpret the answer.	As the period is 18 minutes then one complete cycle of the ride takes 18 minutes.

(b) Calculate the maximum height of the ride above the platform.

1 Use the amplitude and centre of motion to calculate the maximum and minimum height of the carriage above the platform.	For $h(t) = -10\cos\left(\dfrac{\pi t}{9}\right) + 10$, the amplitude is 10, and the centre of motion is also 10.
2 Interpret the answer.	The maximum height is $10 + 10 = 20$ metres.

(c) Sketch the graph of $h(t)$ for $0 \leq t \leq 18$.

Use the information from parts **(a)** and **(b)** to sketch the graph.



(d) In a single ride, determine when the carriage is 5 metres above the platform.

1 Solve the equation $h(t) = 5$ for t.	$-10\cos\left(\dfrac{\pi t}{9}\right) + 10 = 5$ $-10\cos\left(\dfrac{\pi t}{9}\right) = -5$ $\cos\left(\dfrac{\pi t}{9}\right) = \dfrac{1}{2}$ $\dfrac{\pi t}{9} = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ $t = 3,\ 15$
2 Interpret the answer.	The carriage is 5 metres above the platform 3 minutes and 15 minutes after the ride starts.

EXERCISE 7.2 Graphing and modelling with trigonometric functions

Worked Example

1 Consider the function $f(x) = 2\cos(3x)$, $0 \leq x \leq 2\pi$. 10

(a) Determine the amplitude and period.

(b) Solve the equation $f(x) = 0$ to determine the x-intercepts.

(c) Determine the equation of the median line.

(d) Sketch the graph of the function, labelling intercepts with the axes, and maximum and minimum points.

2 Consider the function $f(x) = -3\sin(2x)$, $0 \leq x \leq 2\pi$. 11

(a) Determine the period, amplitude and range.

(b) Solve the equation $f(x) = 0$ to determine the x-intercepts.

(c) Determine the equation of the median line.

(d) Sketch the graph of the function, labelling the coordinates of all key points.

3 Consider the function $f(x) = -3\cos(2x) + 3$, $0 \leq x \leq 2\pi$. 12

(a) Determine the period, amplitude and range. (b) Determine the x-intercepts.

(c) Sketch the graph of the function, labelling key points in coordinate form and the median line.

4 Consider the function $f(x) = 2\cos\left(2x + \frac{\pi}{2}\right)$, $0 \leq x \leq 2\pi$. 13

(a) Determine the period, amplitude and range.

(b) Solve the equation $f(x) = 0$ to determine the x-intercepts.

(c) Sketch the graph of the function, labelling all key points in coordinate form and showing the median line.

5 What are the period and amplitude of $f(x) = 2\sin\left(4\left(x - \frac{\pi}{3}\right)\right)$ respectively?

A $\frac{\pi}{3}, 2$ B $\frac{\pi}{2}, 4$ C $\frac{\pi}{3}, 4$ D $\frac{\pi}{2}, 2$

6 Consider the transformation of $f(x) = \sin(x)$ into $g(x) = \sin(2x - \pi)$. 14

(a) What is the correct set of transformations that have transformed the function $f(x)$?

A dilation by a factor of 2 from the y-axis, translation of π units parallel to the x-axis

B dilation by a factor of 2 from the y-axis, translation of $\frac{\pi}{2}$ units parallel to the x-axis

C dilation by a factor of $\frac{1}{2}$ from the y-axis, translation of $\frac{\pi}{2}$ units parallel to the x-axis

D dilation by a factor of $\frac{1}{2}$ from the y-axis, translation of π units parallel to the x-axis

(b) What common errors might lead to selecting the first of the incorrect alternatives?

7 Consider the graph with equation $f(x) = -\sin\left(2\left(x + \frac{\pi}{4}\right)\right), 0 \leq x \leq 2\pi$.

Worked Example

(a) Determine the coordinates of the intercepts with the axes.

(b) Sketch the graph of the function, showing all the key features.

15

8 Consider the function $f(x) = \tan(x), 0 \leq x \leq \pi$.

(a) Determine the period, the equation of the asymptote and the x-intercepts.

(b) Sketch the graph of the function.

9 Consider the function $g(x) = \tan(3x), 0 \leq x \leq \pi$.

(a) Determine the period, the equations of the asymptotes and the x-intercepts.

(b) Sketch the graph of the function.

10 Determine the period, amplitude and range for the following functions.

(a) $f(x) = -\cos(x) + 1$ (b) $f(x) = 2\sin(\pi x - 0.5)$ (c) $f(x) = 2\tan(3x)$

(d) $f(x) = 11\sin\left(\frac{x}{6}\right) + 4$ (e) $f(t) = \frac{1}{2}\cos\left(t + \frac{\pi}{6}\right) - 7$

16

11 Consider $g(x) = \tan\left(2\left(x + \frac{\pi}{4}\right)\right)$ for $0 \leq x \leq 2\pi$.

(a) List the sequence of transformations needed to produce $g(x)$ from $f(x) = \tan(x)$.

(b) Determine the period, the equations of the asymptotes and the x-intercepts.

(c) Draw the graph.

12 Determine the equations of the vertical asymptotes for the following functions in the domain $0 \leq x \leq 2\pi$.

(a) $f(x) = -3\tan(x)$ (b) $f(x) = 5\tan\left(\frac{x}{2}\right)$

(c) $f(x) = \tan(2x) + 6$ (d) $f(x) = \tan\left(\frac{x}{2} + \frac{\pi}{2}\right)$

13 Sketch the graph of the function $f(x) = \tan\left(3\left(x + \frac{\pi}{6}\right)\right)$ for three complete cycles.

17

14 A string on a guitar is plucked so that the distance, d mm, of the midpoint of the string from its resting position at time t seconds after it is plucked is given by $d(t) = \sin(200\pi t)$.

(a) Calculate the maximum total distance that the string moves in one cycle.

(b) Calculate the number of cycles that the string makes in 1 second.

15 Use graphing technology to solve $3\sin(2x) = 1, 0 \leq x \leq 2\pi$. Give your answers correct to 2 decimal places.

16 If the graph of the function $f(x) = \sin(x)$ has been transformed in the following ways, determine the new equation of the function $g(x)$ in each case, and its period and amplitude.

(a) Reflection in the x-axis, followed by a translation of $\frac{\pi}{3}$ units to the right.

(b) Dilation by a factor of 4 from the x-axis, and dilation by a factor of 5 from the y-axis.

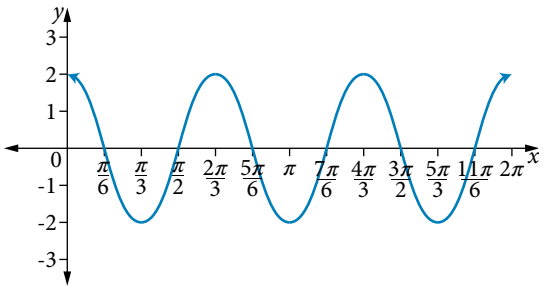
(c) Dilation by a factor of $\frac{1}{3}$ from the y-axis, followed by a translation of $\frac{\pi}{2}$ units to the left.

17 For the following functions, sketch the graph, where $0 \leq x \leq \pi$. Label the coordinates of all key points.

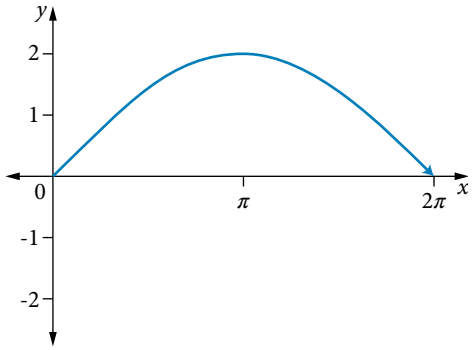
(a) $y = \sin\left(3x - \frac{\pi}{2}\right)$ **(b)** $y = 4\cos\left(2\left(x + \frac{\pi}{3}\right)\right)$

18 For each of the graphs shown below, determine a possible equation.

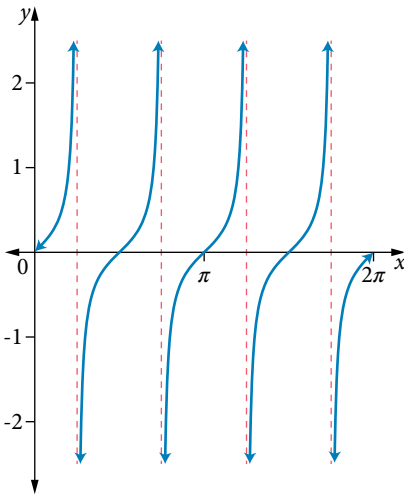
(a)



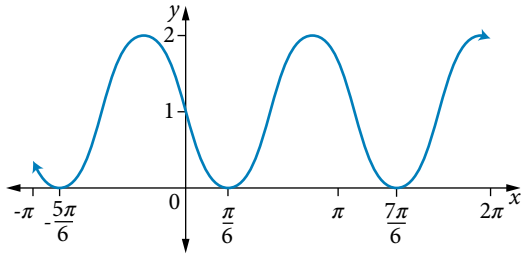
(b)



(c)



(d)



19 Use graphs of trigonometric functions to solve these inequations, giving your answers correct to 2 decimal places where necessary.

(a) $\cos(x) \leq -0.8,\ 0 \leq x \leq 2\pi$ **(b)** $\tan(2x) > \sqrt{3},\ -\pi \leq x \leq \pi$

20 The depth of water d metres in a harbour, t hours after midnight, is modelled by the equation $d(t) = 2\sin\left(\frac{\pi t}{6}\right) + 5$ for $0 \leq t \leq 24$.

(a) Demonstrate that the period of the function is 12 hours.

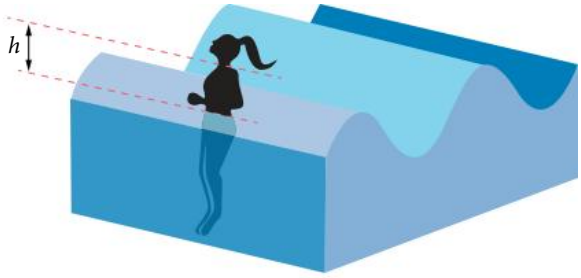
(b) Demonstrate that the maximum depth of the water in the harbour is 7 metres and determine the time that this occurs.

(c) Demonstrate that the minimum depth of the water in the harbour is 3 metres and determine when this occurs.

(d) Sketch the graph of $y = d(t)$ for the specified domain, showing the coordinates of the endpoints and the maximum and minimum depths.

(e) Calculate when the depth of water in the harbour is 4 metres.

21 The progress of an endurance wave pool in a water park that goes for 12 minutes is modelled by $h(t) = \sin\left(\frac{\pi}{2}t\right) + 0.5$, where h metres is the height of each person's shoulders above the water, t minutes after the ride commences.



(a) Calculate the period of $h(t)$ and demonstrate that each rider experiences 3 cycles in 12 minutes.

(b) Calculate the maximum and minimum height of the rider's shoulders during the ride.

(c) Sketch the graph of $y = h(t)$ for $0 \leq t \leq 12$.

(d) Calculate the times when the rider's shoulders are level with the surface of the water for the first two times.

22 The temperature, T_1 °C inside a slow cooker, t minutes after 1 pm is given by $T_1(t) = 40 - 10\sin\left(\frac{\pi}{12}t - 12\right)$ for $0 \leq t \leq 60$. The temperature, T_2 °C, in the kitchen, t minutes after midday, is given by $T_2(t) = 30 + 5\sin\left(\frac{\pi}{12}t - 12\right)$ for $0 \leq t \leq 60$.

(a) Determine the temperature to the nearest degree inside the slow cooker at 1:30 pm.

(b) Calculate the temperature to the nearest degree in the kitchen at 1:45 pm.

(c) Sketch the graphs of T_1 and T_2 on the same set of axes for $0 \leq t \leq 60$.

(d) Calculate when the temperatures inside the slow cooker and the kitchen are the same, expressed to the nearest minute.

(e) Write an expression for D, the difference between the temperature inside the slow cooker and the kitchen where $D = T_1 - T_2$.

(f) Sketch the graph of D for $0 \leq t \leq 60$.

(g) Calculate when D reaches its maximum value, expressed to the nearest minute.

7.3 The reciprocal trigonometric functions

The sine, cosine and tangent functions can be used to define three additional functions with rules as follows:

- $\text{secant}(x) = \sec(x) = \dfrac{1}{\cos(x)}, \cos(x) \neq 0$
- $\text{cosecant}(x) = \text{cosec}(x) = \dfrac{1}{\sin(x)}, \sin(x) \neq 0$
- $\text{cotangent}(x) = \cot(x) = \dfrac{1}{\tan(x)}, \sin(x) \neq 0$

Since these new functions involve the reciprocals of the three standard circular functions, they are known as the reciprocal circular functions.

Since $\cos(x) = 0$, if x is an odd multiple of $\dfrac{\pi}{2}$, that is, for $x = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$, it follows that the domain of secant(x) is $\mathbb{R} \setminus \left\{(2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}\right\}$.

Since $\sin(x) = 0$, if x is an integer multiple of π, the domain of both cosec(x) and cotan(x) is $\mathbb{R} \setminus \{n\pi, n \in \mathbb{Z}\}$.

Note that the range of cosec(x) and secant(x) is $(-\infty, -1] \cup [1, \infty)$. This will be evident from the graphs you will see in this section.

$\text{cosec}(x) = \dfrac{1}{\sin(x)}$, for $\sin(x) \neq 0$ where $x \neq n\pi$ for $n \in \mathbb{Z}$

$\sec(x) = \dfrac{1}{\cos(x)}$, for $\cos(x) \neq 0$ where $x \neq \dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$

$\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$, for $\sin(x) \neq 0$ where $x \neq n\pi$ for $n \in \mathbb{Z}$

These functions should not be confused with the inverse functions such as $\sin^{-1}(x)$. A reciprocal function with index notation, for example, would be $(\sin(x))^{-1}$. However, to avoid confusion, this notation is not often used.

Calculating values of reciprocal functions

The reciprocal trigonometric functions can be calculated using their related functions. For instance, as $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$, then

$$\text{cosec}\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$$
$$= \frac{1}{\frac{1}{\sqrt{2}}}$$
$$= \sqrt{2}$$

18 Calculating reciprocal trigonometric functions for exact values

Calculate the exact value of each of the following.

(a) $\sec(\pi)$

THINKING	WORKING
1 Express as a reciprocal function.	$\sec(\pi) = \dfrac{1}{\cos(\pi)}$
2 Substitute and calculate.	As $\cos(\pi) = -1$, then $\sec(\pi) = \dfrac{1}{\cos(\pi)}$ $= \dfrac{1}{-1}$ $= -1$

(b) $\operatorname{cosec}\left(\dfrac{\pi}{2}\right)$

1 Express as a reciprocal function.	$\operatorname{cosec}\left(\dfrac{\pi}{2}\right) = \dfrac{1}{\sin\left(\frac{\pi}{2}\right)}$
2 Substitute and calculate.	As $\sin\left(\dfrac{\pi}{2}\right) = 1$, then $\operatorname{cosec}\left(\dfrac{\pi}{2}\right) = \dfrac{1}{\sin\left(\frac{\pi}{2}\right)}$ $= \dfrac{1}{1}$ $= 1$

(c) $\operatorname{cosec}(\pi)$

1 Express as a reciprocal function.	$\operatorname{cosec}(\pi) = \dfrac{1}{\sin(\pi)}$
2 Substitute and calculate.	As $\sin(\pi) = 0$, then $\operatorname{cosec}(\pi) = \dfrac{1}{\sin(\pi)}$ $= \dfrac{1}{0}$ $\therefore \operatorname{cosec}(\pi)$ is undefined

(d) $\cot\left(\dfrac{\pi}{3}\right)$

1 Express as a reciprocal function.	$\cot\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\tan\left(\frac{\pi}{3}\right)}$
2 Substitute and calculate.	As $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$, then $\cot\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\tan\left(\frac{\pi}{3}\right)}$ $= \dfrac{1}{\sqrt{3}}$

Technology can be used to calculate the value of reciprocal trigonometric expressions.

Technology worked example
Calculating reciprocal trigonometric values

19 Solving equations involving reciprocal trigonometric functions

Solve the equation secant$(x) = 2$, for $0 \leq x \leq 2\pi$.

THINKING	WORKING
1 Recall the reciprocal form of the function.	$\sec(x) = \frac{1}{\cos(x)}$
2 Rewrite the equation.	$\frac{1}{\cos(x)} = 2$
3 Solve the equation.	$\cos(x) = \frac{1}{2}$ $x = \cos^{-1}\left(\frac{1}{2}\right)$ $x = \frac{\pi}{3}$
4 Determine the other solutions in the required domain.	$x = 2\pi - \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$
5 Simplify and give all the solutions.	$x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

Technology worked example
Solving reciprocal trigonometric equations

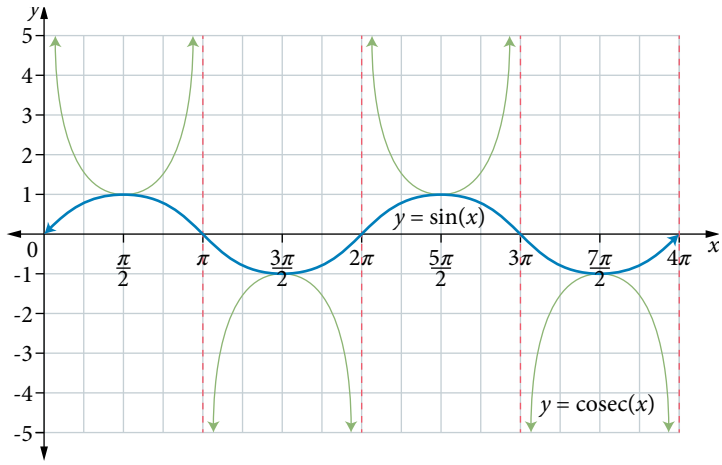
Graphs of reciprocal trigonometric functions

In this section, you will graph functions with rules of the type $y = \frac{1}{f(x)}$. In particular, you will consider reciprocals of the trigonometric functions, where $f(x)$ is $\sin(x)$, $\cos(x)$ or $\tan(x)$. Since you graphed these functions in the previous section, it is helpful when sketching the reciprocal graph to consider its relationship to the original graph of the trigonometric function.

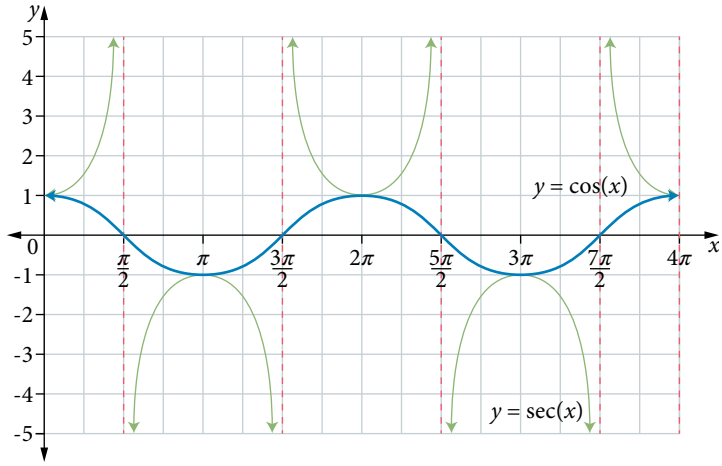
Some features can be seen quite easily. Because $\frac{a}{0}$ is undefined, the reciprocal function will have vertical asymptotes where the original function has x-intercepts.

If $g(x) = \frac{1}{f(x)}$, then there are vertical asymptotes for $g(x)$ where $f(x) = 0$, at the x-intercepts of $f(x)$.

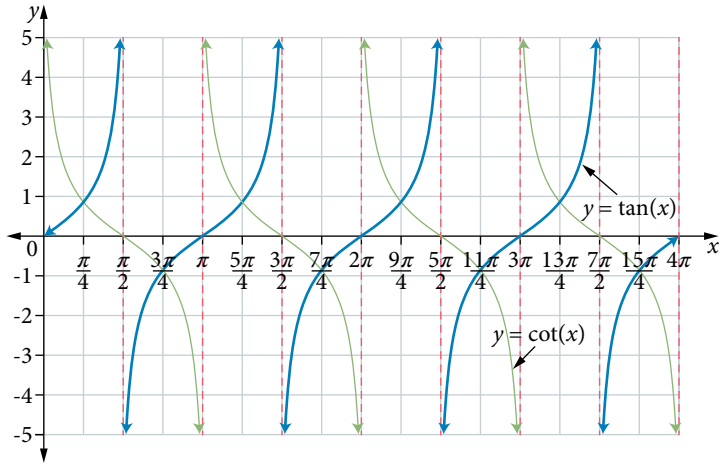
The graph of $y = \text{cosec}(x)$ is shown here in green, with the graph of $y = \sin(x)$ in blue. Note that the vertical asymptotes are at $x = n\pi$, $n \in \mathbb{Z}$, where the x-intercepts are on the graph of $y = \sin(x)$.The domain is $\mathbb{R} \setminus \{n\pi\}$, $n \in \mathbb{Z}$. The local minimum and maximum points occur where $\sin(x) = \pm 1$, and the range is $\mathbb{R} \setminus (-1, 1)$.



The graph of $y = \sec(x)$ is shown here in green, with the graph of $y = \cos(x)$ in blue. Note that the vertical asymptotes are at $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$, where the x-intercepts are on the graph of $y = \cos(x)$.The domain is $\mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in \mathbb{Z}$. The local minimum and maximum points occur where $\cos(x) = \pm 1$, and the range is $\mathbb{R} \setminus (-1, 1)$.



The graph of $y = \cot(x)$ is shown here in green, with the graph of $y = \tan(x)$ in blue. Note that the vertical asymptotes are at $x = n\pi$, $n \in \mathbb{Z}$, where the x-intercepts are on the graph of $y = \tan(x)$. The domain is $\mathbb{R} \setminus \{n\pi\}$, $n \in \mathbb{Z}$, and the range is $\mathbb{R}$.



Transformations of the reciprocal trigonometric functions

Dilation, reflection and translation, studied in Section 7.2, can be used in the same way to help sketch the reciprocal functions. Algebra can also be used to calculate the coordinates of end-points.

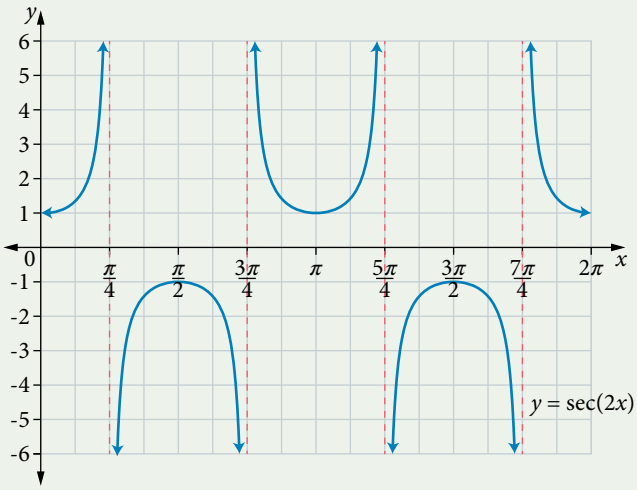
Explore further

Reciprocal functions

Determine the effect of dilations and translations on the graph of $g(x)$ where $g(x) = \frac{1}{f(x)}$.

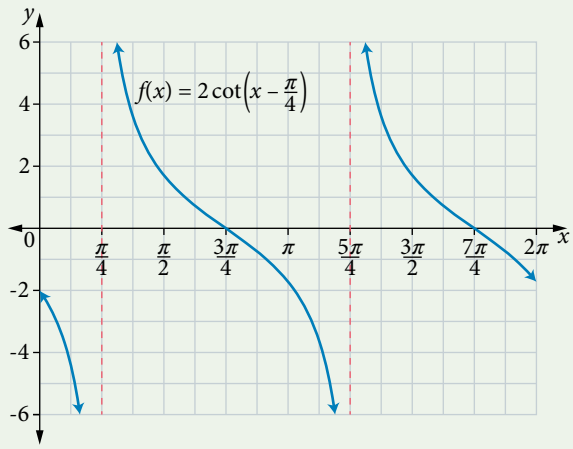
20 Sketching graphs of the reciprocal trigonometric functions of the form $f(ax)$

Sketch the graph of $f(x) = \sec(2x)$ for $0 \leq x \leq 2\pi$, giving the domain, range and period. Also show the asymptotes and any intercepts with the axes.

THINKING	WORKING
1 Determine the vertical asymptotes.	Solve $\cos(2x) = 0$ for $0 \leq x \leq 2\pi$ $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
2 List the transformations that the graph of $y = \text{secant}(x)$ has undergone.	Dilation by a factor of $\frac{1}{2}$ from the y-axis.
3 Determine the y-intercept.	$f(0) = \frac{1}{\cos(2 \times 0)}$ $= \frac{1}{1}$ $= 1$ The coordinates of the y-intercept are (0, 1).
4 Determine the x-intercepts.	$\sec(2x) = 0$ $\frac{1}{\cos(2x)} = 0$ $1 = 0 \times \cos(2x)$ As there are no solutions for this equation then there are no x-intercepts.
5 Sketch the graph.	
6 Write the domain, range and period. The period is the same as the original trigonometric function.	domain $[0, 2\pi] / \left\{(2n+1)\frac{\pi}{4}\right\}, n = 0, 1, 2, 3;$ range $\mathbb{R} / (-1, 1)$; period $\frac{2\pi}{2} = \pi$

21 Sketching reciprocal trigonometric functions of the form $Af(x-b)$.

Sketch the graph of $f(x) = 2\cot\left(x - \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$, giving the domain, range and period. Also show the asymptotes and intercepts with the axes.

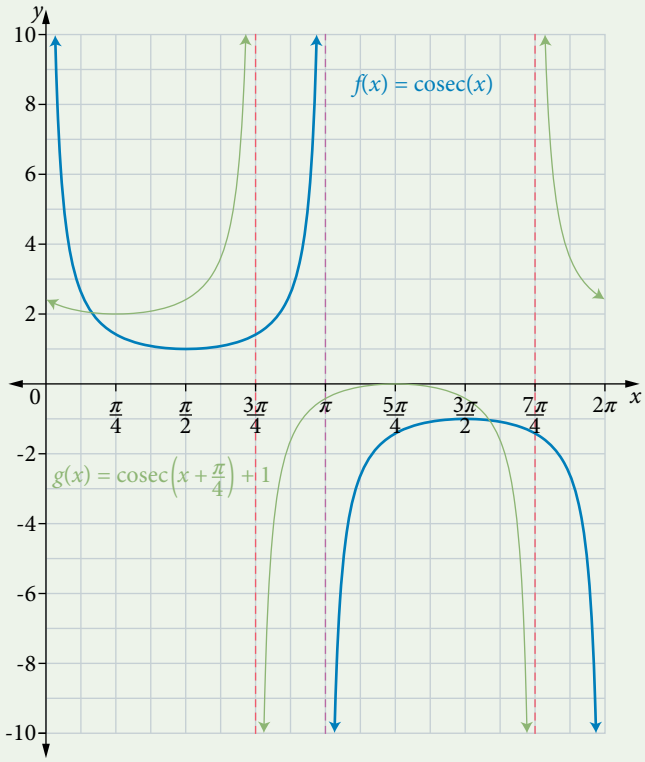
THINKING	WORKING
1 Determine the equations of the asymptotes.	The equations of the asymptotes were originally $x = 0$, $x = \pi$ and $x = 2\pi$. The asymptotes are now at $x = 0 + \frac{\pi}{4}$ and $x = \pi + \frac{\pi}{4}$ $= \frac{\pi}{4}$ $= \frac{5\pi}{4}$ The equations of the asymptotes are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.
2 Determine the y-intercept.	$f(0) = \frac{2}{\tan\left(0 - \frac{\pi}{4}\right)}$ $= \frac{2}{-1}$ $= -2$ The coordinates of the y-intercept are (0,-2).
3 Determine the x-intercepts.	The intercepts were originally at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, and have been translated $\frac{\pi}{4}$ units to the right. So, they are at: $x = \frac{\pi}{2} + \frac{\pi}{4}, \frac{3\pi}{2} + \frac{\pi}{4}$ $= \frac{3\pi}{4}, \frac{7\pi}{4}$ The coordinates of the x-intercepts are $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.
4 Sketch the graph.	
5 Write the domain, range and period. The period is the same as the original trigonometric function.	domain $[0, 2\pi]$ / $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$; range $\mathbb{R}$; period π

22 Sketching reciprocal functions of the form $f(a(x - b)) + c$

Sketch the graph of $g(x) = \text{cosec}\left(x + \frac{\pi}{4}\right) + 1$ for $0 \leq x \leq 2\pi$, giving the domain, range and period. Also show the asymptotes and the intercepts with the axes.

THINKING	WORKING
1 Calculate the x-intercept by solving the equation $g(x) = 0$.	$g(x) = 0$ $\text{cosec}\left(x + \frac{\pi}{4}\right) + 1 = 0$ $\text{cosec}\left(x + \frac{\pi}{4}\right) = -1$ $\frac{1}{\sin\left(x + \frac{\pi}{4}\right)} = -1$ $\sin\left(x + \frac{\pi}{4}\right) = -1$ $x + \frac{\pi}{4} = \sin^{-1}(-1)$ $x + \frac{\pi}{4} = \frac{3\pi}{2}$ $x = \frac{5\pi}{4}$ The coordinates of the x-intercept are $\left(\frac{5\pi}{4}, 0\right)$.
2 Determine the y-intercept by evaluating $g(0)$.	$g(x) = \text{cosec}\left(x + \frac{\pi}{4}\right) + 1$ $g(0) = \text{cosec}\left(\frac{\pi}{4}\right) + 1$ $= \frac{1}{\sin\left(\frac{\pi}{4}\right)} + 1$ $= \frac{1}{\frac{1}{\sqrt{2}}} + 1$ $= \sqrt{2} + 1$ The coordinates of the y-intercept are $(0, \sqrt{2} + 1)$.
3 Determine the transformations the graph of $f(x) = \text{cosec}(x)$ has undergone.	Translation $\frac{\pi}{4}$ units to the left. Translation 1 unit up.
4 Determine the equations of the asymptotes.	The equations of the asymptotes of $y = \text{cosec}(x)$ are $x = \pi$ and $x = 2\pi$. The equations of the asymptotes of $g(x)$ are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. The vertical translation does not affect the position of the asymptotes.

5 Sketch the graph.



The graph of $f(x) = \text{cosec}(x)$ is shown in blue for comparison, and $g(x) = \text{cosec}\left(x + \frac{\pi}{4}\right) + 1$ is shown in green.

6 Identify the domain, range and period.

domain $[0, 2\pi] / \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$; range $\mathbb{R} / (0, 2)$; period 2π

Graphs of the reciprocal trigonometric functions may be sketched using technology.

EXERCISE 7.3 The reciprocal trigonometric functions

Worked Example 18

1 Calculate exact values for the following:

(a) $\sec(2\pi)$ (b) $\text{cosec}(0)$ (c) $\sec\left(\frac{\pi}{4}\right)$

(d) $\text{cosec}\left(\frac{3\pi}{4}\right)$ (e) $\cot(2\pi)$ (f) $\sec\left(\frac{5\pi}{6}\right)$

2 Calculate each of the following, correct to 2 decimal places:

(a) $\sec(1.2)$ (b) $\sec(5)$ (c) $\text{cosec}(3.05)$

(d) $\text{cosec}(2.8)$ (e) $\cot(6)$ (f) $\cot(0.09)$

3 Calculate the following, simplifying where possible:

(a) $\operatorname{cosec}\left(\frac{\pi}{2}\right)+3\cot\left(\frac{5\pi}{4}\right)$ (b) $\sec\left(\frac{\pi}{3}\right)-\cot\left(\frac{7\pi}{4}\right)$

(c) $\operatorname{cosec}\left(\frac{\pi}{6}\right)+\sec\left(\frac{7\pi}{4}\right)+\sec\left(\frac{5\pi}{3}\right)$

Worked Example

4 Solve each of the following equations in the given domain: 19

(a) $\cot(x)+\sqrt{3}=0, 0\le x\le 2\pi$ (b) $\operatorname{cosec}\left(\frac{x}{2}\right)=\frac{2}{\sqrt{3}}, 0\le x\le 2\pi$

(c) $\sec(\theta-30°)=2, 0°\le\theta\le 360°$ (d) $\cot^2(x)=\frac{1}{3}, 0\le x\le 2\pi$

5 Solve the following equations:

(a) $\cot\left(2x-\frac{\pi}{2}\right)=0.68, 0\le x\le\pi$ (b) $1-\operatorname{cosec}(x)=4, 0\le x\le 2\pi$

6 Consider the graph of $g(x)=4\sec(x)$ in the domain $0\le x\le 2\pi$. 20

(a) By first considering the graph of $f(x)=4\cos(x)$, determine the equations of the asymptotes of $g(x)=4\sec(x)$.

(b) Determine the transformations that the graph of $f(x)=4\cos(x)$ has undergone.

(c) Determine any intercepts with the axes.

(d) Sketch the graphs of $y=4\cos(x)$ and $y=4\sec(x)$ on the same axes.

(e) Determine the domain, range and period of $g(x)$.

7 Consider the function $g(x)=2\operatorname{cosec}\left(x+\frac{\pi}{4}\right), 0\le x\le 2\pi$. 21

(a) Determine the equations of the asymptotes of $g(x)$.

(b) Determine the coordinates of the intercepts with the axes.

(c) Sketch the graph of $g(x)=2\operatorname{cosec}\left(x+\frac{\pi}{4}\right), 0\le x\le 2\pi$.

(d) Determine the domain, range and period.

8 Determine the transformations, in the correct order, that have transformed the basic reciprocal trigonometric functions, $f(x)=\operatorname{cosec}(x)$, $f(x)=\sec(x)$ and $f(x)=\cot(x)$, into each of the following:

(a) $g(x)=5\cot\left(\frac{x}{\pi}\right)$ (b) $g(x)=-\operatorname{cosec}(2x-3)$ (c) $g(x)=\sec\left(x-\frac{\pi}{6}\right)-4$

9 Determine the period and range of each of the functions.

(a) $g(x)=5\cot\left(\frac{x}{\pi}\right)$ (b) $g(x)=-\operatorname{cosec}(2x-3)$ (c) $g(x)=\sec\left(x-\frac{\pi}{6}\right)-4$

10 Calculate the general solution for these equations, giving your answers correct to 2 decimal places where necessary.

(a) $\operatorname{cosec}(x)=2$ (b) $\cot(x)=\sqrt{3}$ (c) $\sec(x)=-\sqrt{2}$

11 Sketch the graph of $f(x)=\cot(x)-1$ for $0\le x\le 2\pi$, showing the asymptotes and intercepts with the axes. Also give the domain, range and period of the graph. 22

12 Which of the following is correct?

A $\operatorname{cosec}(x) = \sin^{-1}(x)$ **B** $\sec(x) = \cos^{-1}(x)$

C $\dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$ **D** $\tan^{-1}(x) = \dfrac{\cos(x)}{\sin(x)}$

13 **(a)** Which of the following is not true?

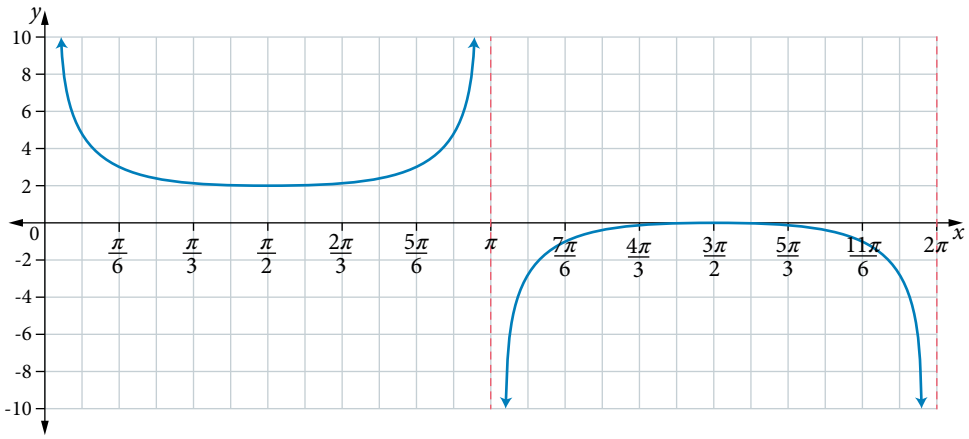
A $\operatorname{cosec}^2(\theta) = \left(\dfrac{1}{\sin(\theta)}\right)^2$ **B** $\cot^2(\theta) = \dfrac{1}{\tan^2(\theta)}, \sin(\theta) \neq 0$

C $\operatorname{cosec}^2(\theta)$, for $\theta \in (-\infty, 0] \cup [0, \infty)$ **D** $\operatorname{cosec}^2(\theta) = \dfrac{1}{\cos^2(\theta)}$

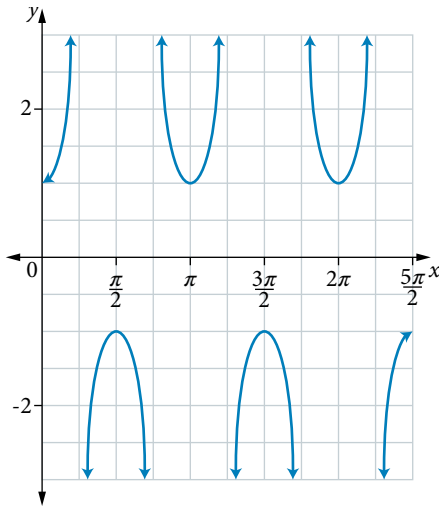
(b) What common error could lead to choosing an incorrect alternative as the answer in part **(a)**?

14 Determine a possible equation for each of these graphs. In each case, assume that there is no dilation from the x-axis.

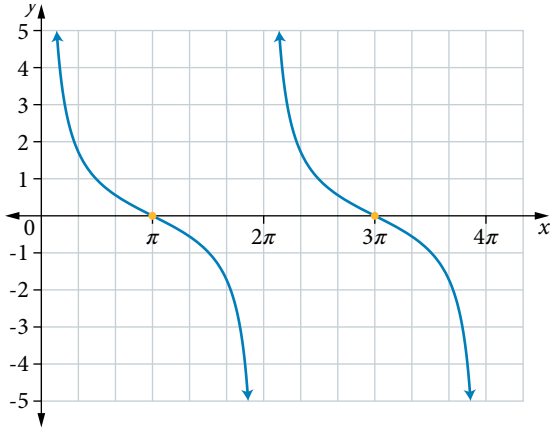
(a)



(b)



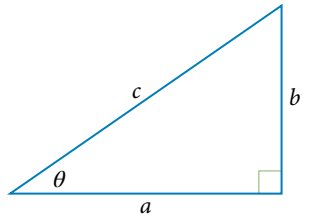
(c)



15 Calculate the following:

(a) $\sec^2\left(\dfrac{\pi}{3}\right) - \tan^2\left(\dfrac{\pi}{3}\right)$ **(b)** $\sec^2\left(\dfrac{\pi}{4}\right) - \tan^2\left(\dfrac{\pi}{4}\right)$

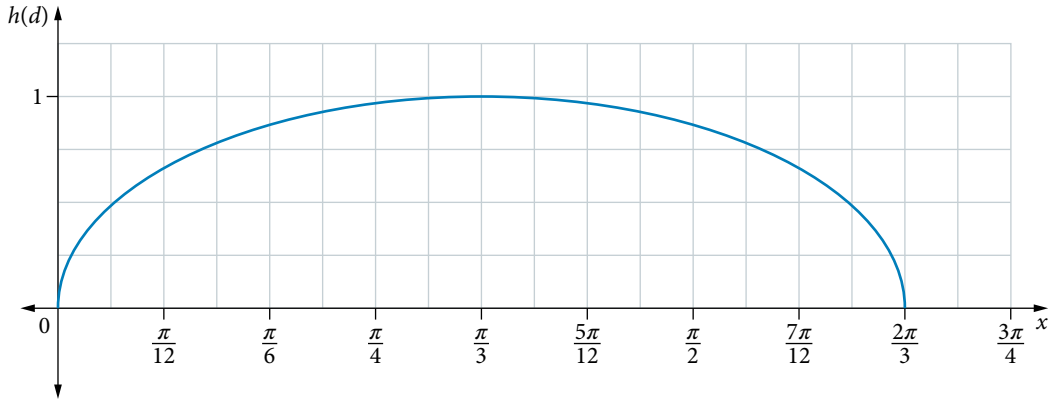
(c) Use the right-angled triangle shown and Pythagoras' theorem to demonstrate that $\sec^2(\theta) - \tan^2(\theta) = 1$ for $0 \leq \theta \leq \dfrac{\pi}{2}$.



16 Calculate the general solutions for the following equations in terms of n, where $n \in \mathbb{Z}$.

(a) $\cot\left(2\left(x+\frac{\pi}{4}\right)\right)-\sqrt{3}=0$ **(b)** $\sqrt{3}\sec\left(3\left(x-\frac{\pi}{6}\right)\right)+2=0$

17 A thin-wire sculpture in the shape of an arch has been produced. It is modelled by the relationship $h(d)=\sec(d+a)+b$, where the height of the sculpture, h metres, is expressed in terms of the distance, d metres, from an origin. The values a and b are positive constants. A cross-section of the sculpture is shown below.



(a) Determine the values of a and b, and hence write the rule for $h(d)$.

(b) Calculate the exact height of the sculpture for the following distances to the right of the origin.

(i) $\frac{\pi}{12}$ metres **(ii)** $\frac{\pi}{6}$ metres

(c) Calculate the distances to the right of the origin to the nearest centimetre, where the height of the sculpture is half a metre.

(d) Another thin-wire sculpture is placed $\frac{\pi}{12}$ metres to the right of the original sculpture. It is modelled by the function $H(d)$. Write the rule for the sculpture $H(d)$.

(e) Use technology to calculate the distance of the point where the sculptures intersect from the origin.

18 Use graphs to solve these inequations for $0 \le x \le 2\pi$.

(a) $\cot(x) < 1$ **(b)** $\operatorname{cosec}(x) \le 2$

19 Consider the function $f(x)=\sec(\theta)$.

(a) Demonstrate that $\sec(-\theta)=\sec(\theta)$.

(b) Demonstrate that $\sec(\theta)$ and $\operatorname{cosec}(\theta)$ are complementary functions. This requires you to prove that $\operatorname{cosec}(\theta)=\sec\left(\frac{\pi}{2}-\theta\right)$.

(c) Determine the transformations that have taken place to transform $\sec(\theta)$ into $\sec\left(\frac{\pi}{2}-\theta\right)$.

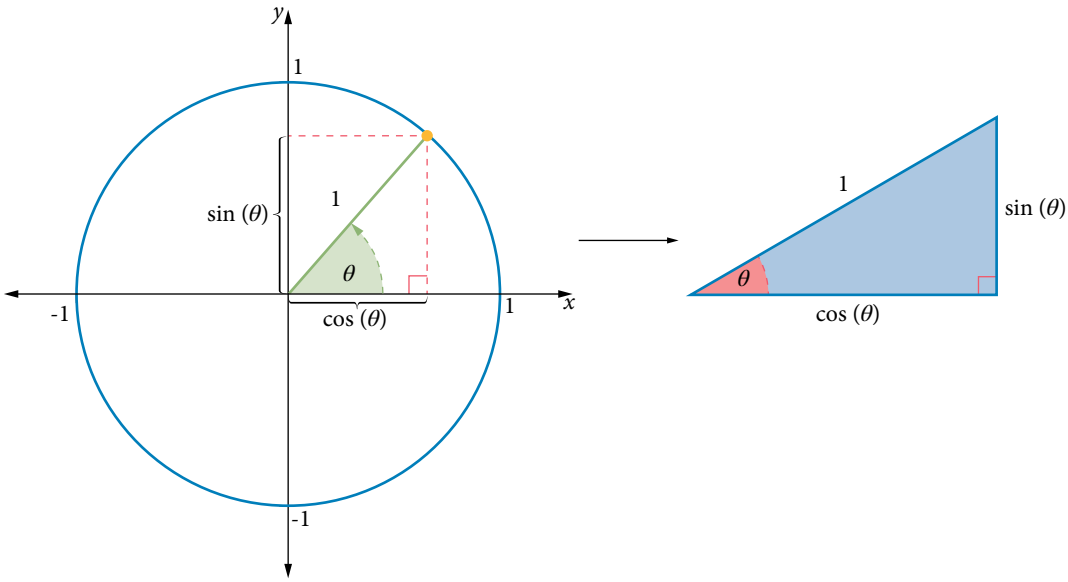
(d) Explain the relationship between the graphs of $y=\sec(\theta)$ and $y=\operatorname{cosec}(\theta)$.

(e) Sketch the graph of $y=\sec(\theta)$, $0 \le \theta \le 2\pi$ and use the information you have found to sketch the graph of $y=\operatorname{cosec}(\theta)$, $0 \le \theta \le 2\pi$.

7.4 The Pythagorean trigonometric identities

An equation that is true for every value of the variable is called an identity. An example of an identity that should be familiar is $(a+b)^2 = a^2 + 2ab + b^2$ where the equality holds for all values of a and b. You will sometimes see an identity written, for example, as $(a+b)^2 \equiv a^2 + 2ab + b^2$.

You will recall the definition of $\sin(\theta)$ as the y-value on the unit circle for an anticlockwise rotation of θ from the positive x-axis. Similarly, $\cos(\theta)$ is the corresponding x-value. In the diagram below, considering the right-angled triangle formed, Pythagoras' theorem can be used to give the identity $\sin^2(\theta) + \cos^2(\theta) = 1$. This remains true in all four quadrants, as negative values of $\sin(\theta)$ and $\cos(\theta)$ will become positive when squared.



Two more identities can be derived from $\sin^2(\theta) + \cos^2(\theta) = 1$.

Dividing both sides by $\cos^2(\theta)$ gives $\dfrac{\sin^2(\theta)}{\cos^2(\theta)} + \dfrac{\cos^2(\theta)}{\cos^2(\theta)} = \dfrac{1}{\cos^2(\theta)}$

Using the ratio $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, $\cos(\theta) \neq 0$ gives $\tan^2(\theta) + 1 = \sec^2(\theta)$.

Similarly, dividing both sides by $\sin^2(\theta)$:

$\dfrac{\sin^2(\theta)}{\sin^2(\theta)} + \dfrac{\cos^2(\theta)}{\sin^2(\theta)} = \dfrac{1}{\sin^2(\theta)}$, $\sin(\theta) \neq 0$ which gives $1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

The Pythagorian trigonometric identities:

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

Explore further

The unit circle and trigonometric properties

Explore Pythagorean properties using the unit circle.

Each of the equations in the box above is an identity because it is true for every value of θ in the implied domain, not just for some values of θ.

23 Calculating the values of trigonometric expressions using identities

(a) Calculate the value of $\sin(\theta)$ for $\frac{\pi}{2} \leq \theta \leq \pi$ if $\cos(\theta) = -\frac{7}{25}$.

THINKING	WORKING
1 Recall the appropriate identity.	$\sin^2(\theta) + \cos^2(\theta) = 1$
2 Substitute the given value.	$\sin^2(\theta) + \left(-\frac{7}{25}\right)^2 = 1$
3 Rearrange and simplify.	$\sin^2(\theta) = 1 - \frac{49}{625}$ $= \frac{576}{625}$
4 Calculate.	$\sin(\theta) = \pm\sqrt{\frac{576}{625}}$ $= \pm\frac{24}{25}$
5 Interpret the answer.	Given $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin(\theta) = \frac{24}{25}$.

(b) Calculate $\cos(\theta)$ for $\pi \leq \theta \leq 2\pi$ if $\tan(\theta) = 3$.

1 Recall the appropriate identity.	$1 + \tan^2(\theta) = \sec^2(\theta)$
2 Substitute the given value.	$1 + 9 = \sec^2(\theta)$
3 Rearrange and simplify.	$\sec^2(\theta) = 10$
4 Calculate.	$\sec(\theta) = \pm\sqrt{10}$
5 Select the answer(s) that are consistent with the given quadrant(s).	The angle is in quadrant 3 (where $\tan(\theta) > 0$): $\sec(\theta) = -\sqrt{10}$.
6 Answer the question.	$\cos(\theta) = -\frac{1}{\sqrt{10}}$

The next three worked examples illustrate the ways the Pythagorean identities can be used to simplify expressions, solve equations and prove other identities.

24 Simplifying trigonometric expressions

Simplify the following.

(a) $\cos^2(A) + \sin^2(A) + \frac{1}{\cot^2(A)}$

THINKING	WORKING
1 Recall the appropriate identities.	$\cos^2(A) + \sin^2(A) = 1$ $\frac{1}{\cot^2(A)} = \tan^2(A)$

2 Substitute in the expression.	$\cos^2(A)+\sin^2(A)+\dfrac{1}{\cot^2(A)}=1+\tan^2(A)$
3 Simplify and write the final answer.	$1+\tan^2(A)=\sec^2(A)$ $\cos^2(A)+\sin^2(A)+\dfrac{1}{\cot^2(A)}=\sec^2(A)$

(b) $\dfrac{\sin(\theta)}{1-\cos(\theta)}+\dfrac{\sin(\theta)}{1+\cos(\theta)}$, $\cos(\theta)\neq\pm1$

1 Write the expression with a common denominator.	$\dfrac{\sin(\theta)}{1-\cos(\theta)}+\dfrac{\sin(\theta)}{1+\cos(\theta)}=\dfrac{\sin(\theta)(1+\cos(\theta))+\sin(\theta)(1-\cos(\theta))}{(1+\cos(\theta))(1-\cos(\theta))}$
2 Expand the numerator and denominator of the fraction.	$=\dfrac{\sin(\theta)+\cos(\theta)\sin(\theta)+\sin(\theta)-\sin(\theta)\cos(\theta)}{1-\cos^2(\theta)}$
3 Collect the terms in the numerator.	$=\dfrac{2\sin(\theta)}{1-\cos^2(\theta)}$
4 Use an identity to simplify the denominator.	$=\dfrac{2\sin(\theta)}{\sin^2(\theta)}$
5 Cancel using a common factor.	$=\dfrac{2}{\sin(\theta)}$
	Note that $\sin(\theta)\neq0$ because $\cos(\theta)\neq\pm1$.
6 Express as a single trigonometric expression.	$=2\text{cosec}(\theta)$, $\theta\neq n\pi$ where $n\in\mathbb{Z}$

25 Using trigonometric identities to solve equations

Solve the following equations in the given domain.

(a) $\cos^2(x)-\sin^2(x)=\dfrac{1}{2}$, $0\leq x\leq 2\pi$

THINKING	WORKING
1 Recall an appropriate identity and rearrange it if necessary.	$\cos^2(x)+\sin^2(x)=1$ $\cos^2(x)=1-\sin^2(x)$
2 Substitute so that there is only one variable in the equation and simplify.	$1-\sin^2(x)-\sin^2(x)=\dfrac{1}{2}$ $1-2\sin^2(x)=\dfrac{1}{2}$ $2\sin^2(x)=\dfrac{1}{2}$

3 Solve the equation and write the answers.	$\sin^2(x) = \frac{1}{4}$ $\sin(x) = \pm\frac{1}{2}$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) $\operatorname{cosec}^2(x) - \sqrt{3}\cot(x) = 1, 0 \leq x \leq \pi$

1 Recall an appropriate identity and rearrange if necessary.	$1 + \cot^2(x) = \operatorname{cosec}^2(x)$
2 Substitute so that there is only one variable in the equation and simplify.	$1 + \cot^2(x) - \sqrt{3}\cot(x) = 1$ $\cot^2(x) - \sqrt{3}\cot(x) = 0$
3 Solve the equation and write the answers.	$\cot(x)\left(\cot(x) - \sqrt{3}\right) = 0$ $\cot(x) = 0$ or $\cot(x) = \sqrt{3}$ $\tan(x)$ is undefined when $\cot(x) = 0$ or $\tan(x) = \frac{1}{\sqrt{3}}$ $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

26 Proving trigonometric identities

Prove that $\dfrac{\operatorname{cosec}(\theta)}{\tan(\theta) + \cot(\theta)} = \cos(\theta)$.

THINKING	WORKING
1 Express the left side of the equation in terms of $\sin(\theta)$ and $\cos(\theta)$.	$\dfrac{\operatorname{cosec}(\theta)}{\tan(\theta) + \cot(\theta)} = \dfrac{\frac{1}{\sin(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}}$
2 Simplify the left-hand side.	$\text{LHS} = \dfrac{\frac{1}{\sin(\theta)}}{\frac{\sin^2(\theta) + \cos^2(\theta)}{\sin(\theta)\cos(\theta)}}$ $= \dfrac{\frac{1}{\sin(\theta)}}{\frac{1}{\sin(\theta)\cos(\theta)}}$ $= \dfrac{1}{\sin(\theta)} \times \dfrac{\sin(\theta)\cos(\theta)}{1}$ $= \cos(\theta)$
3 Make a concluding statement.	LHS = RHS, as required

The following approaches are helpful when proving trigonometric identities.

- Change terms to expressions in $\sin(x)$ and $\cos(x)$ rather than $\tan(x)$, $\sec(x)$, etc.
- Work on one side of the given equation until it equals the other side. It might be necessary to work on both sides separately until they are shown to be equal.

27 Proving a more complex trigonometric identity

Prove that $\frac{\cos(x)}{1-\sin(x)} = \sec(x) + \tan(x)$.

THINKING	WORKING
1 Work with the left-hand side. Multiply both the numerator and the denominator by an appropriate expression and simplify.	$\frac{\cos(x)}{1-\sin(x)} \times \frac{(1+\sin(x))}{(1+\sin(x))} = \frac{\cos(x)(1+\sin(x))}{1-\sin^2(x)}$ $= \frac{\cos(x)(1+\sin(x))}{\cos^2(x)}$ $= \frac{1+\sin(x)}{\cos(x)}$
2 Rearrange until the required expression is reached.	$\frac{1+\sin(x)}{\cos(x)} = \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}$ $= \sec(x) + \tan(x)$
3 Make a concluding statement.	LHS = RHS, as required.

EXERCISE 7.4 The Pythagorean trigonometric identities

Worked Example

23

1 If $\cos(\theta) = \frac{3}{5}$, and $0 \le \theta \le \frac{\pi}{2}$, determine the value of $\sin(\theta)$.

2 If $\sin(\theta) = -\frac{12}{13}$, and $\pi \le \theta \le 2\pi$, determine $\cos(\theta)$ and $\tan(\theta)$.

3 If $\cos(\theta) = -\frac{3}{\sqrt{13}}$, and $\frac{\pi}{2} \le \theta \le \pi$, determine the values of $\text{cosec}(\theta)$ and $\cot(\theta)$.

24

4 Simplify the following:

(a) $\cos(\theta)\tan(\theta)$

(b) $\text{cosec}^2(\theta) - \cot^2(\theta)$

(c) $\sin^3(\theta) + \sin(\theta)\cos^2(\theta)$

(d) $\frac{\sin(\theta)}{\tan(\theta) \times \cos(\theta)}$

(e) $\frac{\cos(\theta)}{\cot(\theta)\sin(\theta)} - \cos^2(\theta)$

(f) $\frac{\cos(\theta)}{1+\cos(\theta)} \times \frac{\cos(\theta)}{1-\cos(\theta)}$

(g) $\frac{\cos(\theta)}{1+\sin(\theta)} + \tan(\theta)$

5 Simplify the following expressions for the given conditions:

(a) If $x = a\cos(\theta)$ and $y = b\sin(\theta)$, simplify $\frac{x^2}{a^2} + \frac{y^2}{b^2}$.

(b) If $x = a\sec(\theta)$ and $y = b\tan(\theta)$, simplify $\frac{x^2}{a^2} - \frac{y^2}{b^2}$.

(c) If $x = a\cot(\theta)$, $a \neq 0$, simplify $\frac{a^2}{x^2 + a^2}$.

6 Solve the following equations in the given domain: 25

(a) $1+\tan^2(x)-\sec(x)=0,\ 0\le x\le 2\pi$

(b) $3\sin(\theta)-2\cos^2(\theta)=0,\ 0\le x\le 2\pi$

(c) $\cos^4(x)-\sin^4(x)=0,\ \frac{\pi}{2}\le x\le\frac{3\pi}{2}$

(d) $\cos^3(x)+\cos(x)\sin^2(x)=-\frac{\sqrt{3}}{2},\ \frac{\pi}{2}\le x\le\frac{3\pi}{2}$

7 Prove the following identities: 26

(a) $1+\frac{\sin^2(\theta)}{\cos^2(\theta)}=\sec^2(\theta)$

(b) $\left(1-\sin^2(\theta)\right)\left(\tan^2(\theta)+1\right)=1$

(c) $\sin^2(\theta)+\cot^2(\theta)\sin^2(\theta)=1$

(d) $\tan^2(\theta)-\sin^2(\theta)=\sin^4(\theta)\sec^2(\theta)$

(e) $\text{cotan}(\theta)(\cos(\theta)-\sec(\theta))=-\sin(\theta)$

8 If $x=3a\tan(\theta)$ and $a\neq 0$, simplify $\frac{9a^2}{x^2+9a^2}$.

9 If $x=a\sin(\theta)$, $a>0$, $\frac{\pi}{2}<x<\frac{3\pi}{2}$, simplify $\frac{x}{\sqrt{a^2-x^2}}$.

10 If $\cos(\theta)=\frac{x}{3}$ and $0<\theta<\pi$, determine $\sin(\theta)$.

11 If $\cos(\theta)=\frac{1}{2x}$, $\frac{3\pi}{2}<\theta<2\pi$, determine $\tan(\theta)$.

12 If $\sin(x)=-\frac{3}{5}$, $\frac{3\pi}{2}\le x\le 2\pi$, then what is $\tan(x)$ equal to?

A $\frac{3}{4}$ B $-\frac{3}{4}$ C $\frac{4}{3}$ D $-\frac{4}{3}$

13 (a) Given $x=a\sin(\theta)$, what is another way that a^2-x^2 can be expressed?

A $a^2\sin^2(\theta)$ B $a^2\tan^2(\theta)$ C $a^2\sec^2(\theta)$ D $\frac{a^2}{\sec^2(\theta)}$

(b) What common error might lead to selecting the first of the incorrect alternatives?

14 Prove the following identities: 27

(a) $\frac{\cos^2(\theta)}{1-\sin(\theta)}-\frac{\cos^2(\theta)}{1+\sin(\theta)}=2\sin(\theta)$

(b) $\frac{1+\cos(\theta)}{\sin(\theta)}=\frac{\sin(\theta)}{1-\cos(\theta)}$

(c) $\frac{2(1+\sin(\theta))}{\cos(\theta)}+\frac{2\cos(\theta)}{1+\sin(\theta)}=4\sec(\theta)$

15 Express the following in terms of $\cos(\theta)$.

(a) $\sin^2(\theta)-\cos^2(\theta)$

(b) $2\sin^2(\theta)+\cos^2(\theta)$

(c) $\tan^2(\theta)$

(d) $\sec^2(\theta)+\tan^2(\theta)$

16 Calculate the value of each of the expressions found in Question 15 if $\cos(\theta)=\frac{1}{2}$.

(a) $1-2\cos^2(\theta)$

(b) $2-3\cos^2(\theta)$

(c) $\frac{1-\cos^2(\theta)}{\cos^2(\theta)}$

(d) $\frac{2-\cos^2(\theta)}{\cos^2(\theta)}$

17 Solve the equation $2\sec^2(2x) = 5\tan(2x)$ for $0 \leq x \leq \frac{\pi}{2}$, leaving your answer in exact form.

18 Express the following for $\{n, m, k\} \in \mathbb{R}^+$:

(a) If $n\sin^2(\theta) - m\cos^2(\theta) = k$, express $\sin^2(\theta)$ in terms of n, m and k.

(b) If $n\sin^2(\theta) - m\cos^2(\theta) = k$, express $\cos^2(\theta)$ in terms of n, m and k.

19 **(a)** Simplify the expression $\dfrac{1+\cot(x)}{\operatorname{cosec}(x)} - \dfrac{\sec(x)}{\tan(x)+\cot(x)}$.

(b) Hence solve $\dfrac{1+\cot(x)}{\operatorname{cosec}(x)} - \dfrac{\sec(x)}{\tan(x)+\cot(x)} = \pm\dfrac{1}{2}$, giving your solution in general form.

20 $x = b\sin(\theta)$ and $y = a\cos(\theta)$, where a and b are constants.

(a) Express x in terms of y, a and b.

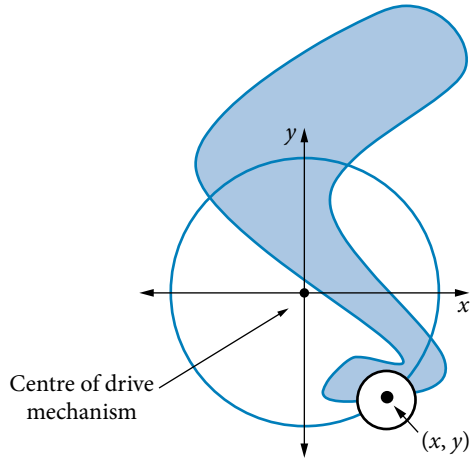
(b) Express y in terms of x, a and b.

(c) For $x = y = 2$ and $\theta = \frac{\pi}{3}$, calculate the values of a and b.

(d) For $x = y = 1$, express $\tan(\theta)$ in terms of b.

(e) For $x = y = 1$, express $\tan(\theta)$ in terms of a.

21 William and Lucinda are riding exercise bikes. The position of the pedals is described in terms of x, the horizontal distance, and y, the vertical distance, from the centre of the drive mechanism, and t is the number of seconds after the start of the exercise session. The values of x and y are in centimetres. The motion of the pedals of William's exercise bike is modelled by the equations $x = 12\cos(6\pi t)$ and $y = 12\sin(6\pi t)$.



(a) Calculate $x^2 + y^2$, and hence describe the path of the pedals. Hint: Use technology to draw a graph if you do not recognise the shape.

(b) How quickly does William pedal, in revolutions per second?

(c) The equations that model the motion of the pedals of Lucinda's exercise bike are $x = 14\cos(4\pi t)$ and $y = 10\sin(4\pi t)$. Calculate $x^2 + y^2$, hence describe the path of the pedals.

(d) How quickly does Lucinda pedal, in revolutions per second?

Compound and double-angle identities

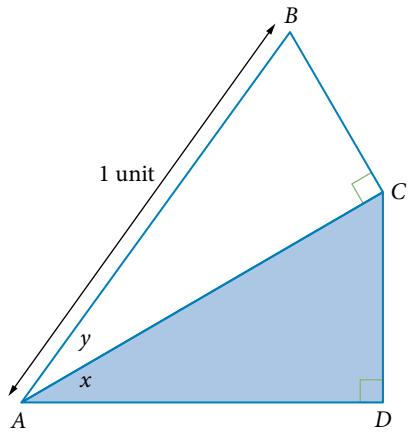
7.5

In this section, you will establish identities involving trigonometric expressions that include the sum or difference of angles.

Sums and differences

Consider two right-angled triangles ABC and ACD. The side length AC is common to ΔABC and ΔACD.

$\angle CAD = x$ and $\angle BAC = y$ and the length of AB is 1 unit.



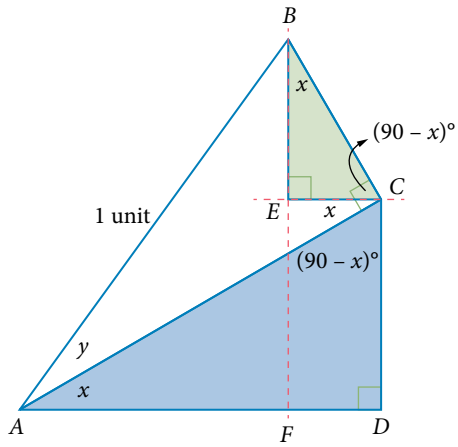
Lines are drawn vertically through B and horizontally through C, as shown on the right, to form the right-angled triangle BCE.

$\angle ACD = (90 - x)°$

$\angle ACE = x$ (alternate angle to CAD)

$\angle ECB = (90 - x)°$ (as $\angle ACB$ is a right angle)

$\angle EBC = x$



The right-angled triangle AFB contains the angle $(x + y)$ and is used to determine an expression for the trigonometric expression $\sin(x + y)$.

Using trigonometry, the following lengths are found:

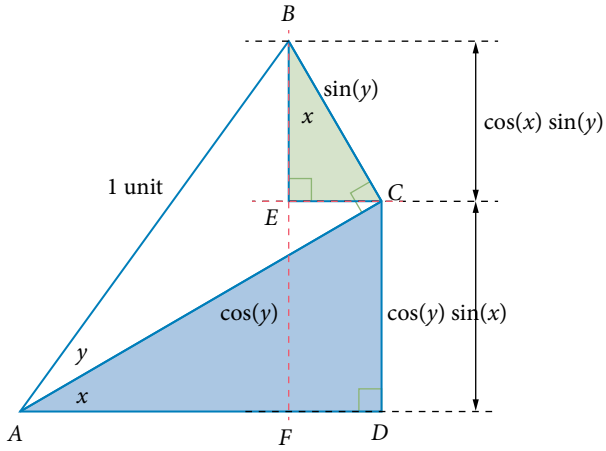
In ΔABC: $BC = \sin(y)$, $AC = \cos(y)$

In ΔACD:

$$\sin(x) = \frac{CD}{AC}$$

$$= \frac{CD}{\cos(y)}$$

$\therefore\ CD = \sin(x)\cos(y)$



In ΔBCE:

$$\cos(x) = \frac{BE}{BC} = \frac{BE}{\sin(y)}$$
$$\therefore BE = \cos(x)\sin(y)$$

$$BF = BE + EF$$

$$BF = \cos(x)\sin(y) + \cos(y)\sin(x)$$

In ΔABF:

$$\sin(x + y) = \frac{BF}{AB}$$
$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{1}$$
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

If y is replaced by $-y$ in the identity above, then it can be written:

$$\sin\left(x + \left(-y\right)\right) = \sin(x)\cos\left(-y\right) + \cos(x)\sin\left(-y\right)$$
$$\sin\left(x - y\right) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

because $\cos(-y) = \cos(y)$, $\sin(-y) = -\sin(-y)$.

Using the relationships $\cos(A) = \sin\left(\frac{\pi}{2} - A\right)$, and $\sin(A) = \cos\left(\frac{\pi}{2} - A\right)$, and the identity for $\sin(x - y)$, you can deduce the following:

$$\cos\left(x + y\right) = \sin\left(\frac{\pi}{2} - \left(x + y\right)\right)$$
$$= \sin\left(\left(\frac{\pi}{2} - x\right) - y\right)$$
$$= \sin\left(\frac{\pi}{2} - x\right)\cos(y) - \cos\left(\frac{\pi}{2} - x\right)\sin(y)$$
$$= \cos(x)\cos(y) - \sin(x)\sin(y)$$

If y is replaced by $-y$ in the identity above, then it can be written:

$$\cos\left(x + (-y)\right) = \cos(x)\cos(-y) - \sin(x)\sin(-y)$$
$$= \cos(x)\cos(y) + \sin(x)\sin(y)$$

The sum and difference identities for the tangent function can be generated as follows:

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)}$$
$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

Now, dividing numerator and denominator by $\cos(x)\cos(y)$:

$$\tan(x+y) = \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

$$= \frac{\frac{\sin(x)\cos(y)}{\cos(x)\cos(y)} + \frac{\cos(x)\sin(y)}{\cos(x)\cos(y)}}{\frac{\cos(x)\cos(y)}{\cos(x)\cos(y)} - \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}}$$

$$= \frac{\frac{\sin(x)}{\cos(x)} + \frac{\sin(y)}{\cos(y)}}{1 - \frac{\sin(x)}{\cos(x)} \times \frac{\sin(y)}{\cos(y)}}$$

$$= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

Similarly, with the substitution of $-y$ for y: $\tan(x-y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \qquad \sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \qquad \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} \qquad \tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

28 Using identities to simplify trigonometric expressions

Demonstrate that $\sin\left(\frac{3\pi}{2} + x\right) = -\cos(x)$.

THINKING	WORKING
1 Use an appropriate identity and substitute the given angles.	$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ $\sin\left(\frac{3\pi}{2} + x\right) = \sin\left(\frac{3\pi}{2}\right)\cos(x) + \cos\left(\frac{3\pi}{2}\right)\sin(x)$
2 Substitute the known values and simplify.	$= -1 \times \cos(x) + 0 \times \sin(x)$ $= -\cos(x)$

29 Using a compound angle identity to determine a trigonometric expression

If x and y are acute angles such that $\sin(x) = \frac{1}{4}$ and $\cos(y) = \frac{3}{4}$, determine $\sin(x+y)$ without calculating x or y.

THINKING	WORKING
1 Recall the identities required.	$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ $\sin^2(x) + \cos^2(x) = 1$

2 Determine the expressions needed.	$\sin^2(x)+\cos^2(x)=1$ $\left(\frac{1}{4}\right)^2+\cos^2(x)=1$ $\cos^2(x)=\frac{15}{16}$ $\cos(x)=\frac{\sqrt{15}}{4}$ $\sin^2(y)=1-\left(\frac{3}{4}\right)^2$ $\sin^2(y)=\frac{7}{16}$ $\sin(y)=\frac{\sqrt{7}}{4}$
3 Substitute the values and simplify.	$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ $=\frac{1}{4}\times\frac{3}{4}+\frac{\sqrt{15}}{4}\times\frac{\sqrt{7}}{4}$ $=\frac{3}{16}+\frac{\sqrt{105}}{16}$ $=\frac{1}{16}(3+\sqrt{105})$

30 Determining the exact value of a trigonometric expression using a compound angle identity

Determine the exact value of $\cos(15°)$.

THINKING	WORKING
1 Express the given angle as the sum or difference of two angles for which exact values are known.	$15°=60°-45°$
2 Write the appropriate identity and substitute the angles.	$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$ $\cos(15°)=\cos(60°-45°)$ $=\cos(60°)\cos(45°)+\sin(60°)\sin(45°)$
3 Substitute the known exact values and simplify.	$\cos(15°)=\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$ $=\frac{1+\sqrt{3}}{2\sqrt{2}}$ $=\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$

31 Simplifying trigonometric expressions with compound angles

Simplify $\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)+\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.

THINKING	WORKING
1 Choose the identity that is appropriate for the given expression.	Expression is in the form: $\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ where: $x=\frac{a+b}{2}$ and $y=\frac{a-b}{2}$.
2 Substitute into the identity and simplify.	$\sin\left(\frac{a+b}{2}+\frac{a-b}{2}\right)=\sin(a)$

Double-angle identities

When $y=x$ in any compound angle formula, the following double-angle identities can be derived:

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ $\quad$ $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\sin(x+x)=\sin(x)\cos(x)+\cos(x)\sin(x)$ $\quad$ $\cos(x+x)=\cos(x)\cos(x)-\sin(x)\sin(x)$

$\sin(2x)=2\sin(x)\cos(x)$ $\quad$ $\cos(2x)=\cos^2(x)-\sin^2(x)$

Two further forms of this one can be found using the identity $\cos^2(x)+\sin^2(x)=1$, and substituting for $\sin^2(x)$ and $\cos^2(x)$.

$$\begin{aligned}\cos(2x)&=\cos^2(x)-\left(1-\cos^2(x)\right)\\&=2\cos^2(x)-1\end{aligned}$$

$$\begin{aligned}\cos(2x)&=\left(1-\sin^2(x)\right)-\sin^2(x)\\\cos(2x)&=1-2\sin^2(x)\end{aligned}$$

$\sin(2x)=2\sin(x)\cos(x)$

$\cos(2x)=\cos^2(x)-\sin^2(x)$

$\cos(2x)=2\cos^2(x)-1$

$\cos(2x)=1-2\sin^2(x)$

$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

A double-angle identity for $\tan(2x)$ can be found by letting $y=x$ and substituting:

$$\begin{aligned}\tan(x+y)&=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\\&=\frac{\tan(x)+\tan(x)}{1-\tan(x)\tan(x)}\\&=\frac{2\tan(x)}{1-\tan^2(x)}\end{aligned}$$

The double-angle formula for the sine function is $\sin(2x)=2\sin(x)\cos(x)$, but note that $\sin(2x)\neq 2\sin(x)$. Similarly, $\cos(2x)\neq 2\cos(x)$ and $\tan(2x)\neq 2\tan(x)$.

32 Simplifying trigonometric expressions using double-angle identities

Simplify $\left(\sin(3x)-\cos(3x)\right)^2$.

THINKING	WORKING
1 Expand the brackets and rearrange.	$\left(\sin(3x)-\cos(3x)\right)^2=\sin^2(3x)-2\sin(3x)\cos(3x)+\cos^2(3x)$ $=\left(\sin^2(3x)+\cos^2(3x)\right)+2\sin(3x)\cos(3x)$
2 Use appropriate identities to simplify the expression.	$=1+2\sin(3x)\cos(3x)$ using $\sin^2(\theta)+\cos^2(\theta)=1$ $=1+\sin(6x)$ using $\sin(2\theta)=2\sin(\theta)\cos(\theta)$

33 Using a double-angle formula to calculate the value of trigonometric expressions

Given that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, use a double-angle identity to demonstrate that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

THINKING	WORKING
1 Recall the required identity.	$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$
2 Substitute the given angle into the identity.	$\tan\left(2\times\frac{\pi}{6}\right) = \frac{2\tan\left(\frac{\pi}{6}\right)}{1-\tan^2\left(\frac{\pi}{6}\right)}$
3 Substitute the known value and simplify.	$\tan\left(\frac{\pi}{3}\right) = \frac{2\times\frac{1}{\sqrt{3}}}{1-\left(\frac{1}{\sqrt{3}}\right)^2}$ $= \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}}$ $= \frac{2}{\sqrt{3}}\times\frac{3}{2}$ $= \frac{3}{\sqrt{3}}$ $= \sqrt{3}$

34 Using half-angle identities

(a) Use the identity $\cos(2x) = 1-2\sin^2(x)$, demonstrate that $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$.

THINKING	WORKING
1 Recall the identity.	$\cos(2x) = 1-2\sin^2(x)$
2 Rearrange to make the required function the subject of the equation.	$2\sin^2(x) = 1-\cos(2x)$ $\sin(x) = \pm\sqrt{\frac{1-\cos(2x)}{2}}$
3 Assign a pronumeral to the double angle.	Let $2x = A$ $x = \frac{A}{2}$
4 Substitute into the rearranged identity.	$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$

(b) If $\cos(A) = \frac{3}{5}$ for $\frac{3\pi}{2} \leq A \leq 2\pi$, determine the exact value of $\sin\left(\frac{A}{2}\right)$ without calculating A.

1 Recall the half-angle identity found in part **(a)**.	$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$
2 Substitute the given value and simplify.	$= \pm\sqrt{\frac{1-\frac{3}{5}}{2}}$ $= \pm\frac{\sqrt{5}}{5}$
3 Consider the sign of the answer, for the given domain, and write the answer.	In the fourth quadrant, the sine function is negative. $\sin\left(\frac{A}{2}\right) = -\frac{\sqrt{5}}{5}$

35 Using double-angle identities in a proof

(a) Prove that $\left(\cos(\theta) - \sin(\theta)\right)^2 = 1 - \sin\left(2\theta\right)$.

THINKING	WORKING
1 Expand the left-hand side.	$\left(\cos(\theta) - \sin(\theta)\right)^2 = \cos^2(\theta) - 2\sin(\theta)\cos(\theta) + \sin^2(\theta)$
2 Use identities to simplify.	$\cos^2(\theta) + \sin^2(\theta) = 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ Therefore, $\cos^2(\theta) - 2\sin(\theta)\cos(\theta) + \sin^2(\theta) = 1 - \sin\left(2\theta\right)$

(b) Prove that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$

1 Rewrite the left-hand side so that a compound angle identity can be used.	$\cos(3x) = \cos(2x + x)$
2 Use an identity to expand the right-hand side.	$\cos(2x + x) = \cos(2x)\cos(x) - \sin(2x)\sin(x)$
3 Use double-angle identities to express the right-hand side in terms of single angles.	$\cos(2x)\cos(x) - \sin\left(2x\right)\sin(x)$ $= \left(2\cos^2(x) - 1\right)\cos(x) - 2\sin\left(x\right)\cos(x)\sin(x)$ $= 2\cos^3(x) - \cos(x) - 2\sin^2(x)\cos(x)$ $= 2\cos^3(x) - \cos(x) - 2\left(1 - \cos^2(x)\right)\cos(x)$ $= 2\cos^3(x) - \cos(x) - 2\cos(x) + 2\cos^3(x)$ $= 4\cos^3(x) - 3\cos(x)$

36 Solving equations using double-angle identities

(a) Use the expansion found in Worked Example 35 for $\cos(3x)$ to solve the equation $4\cos^3(x) - 3\cos(x) = \frac{1}{2}$ for $0 \leq x \leq 2\pi$.

THINKING	WORKING
1 Use the simplified expression to rewrite the equation.	$4\cos^3(x) - 3\cos(x) = \frac{1}{2}$ $\cos(3x) = \frac{1}{2}$
2 Solve the equation.	$3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$
3 Write the solutions for x in the specified domain.	$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

(b) Determine the value of θ such that $\sqrt{2}\sin(2\theta) = 2\cos(\theta)$, $0 \leq \theta \leq 2\pi$.

1 Use a double-angle identity to simplify the equation.	$\sqrt{2}\sin(2\theta) = 2\cos(\theta)$ $\sqrt{2} \times 2\sin(\theta)\cos(\theta) = 2\cos(\theta)$
2 Rearrange to leave 0 on the right-hand side.	$\sqrt{2}\sin(\theta)\cos(\theta) - \cos(\theta) = 0$
3 Factorise and solve.	$\cos(\theta)\left(\sqrt{2}\sin(\theta) - 1\right) = 0$ $\cos(\theta) = 0$ or $\sin(\theta) = \frac{1}{\sqrt{2}}$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Changing products of trigonometric expressions to sums and differences

When the sum and difference identities are added or subtracted, another set of identities can be found. These enable you to convert products of trigonometric expressions to sums or differences.

Adding the two identities:

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

gives $2\sin(x)\cos(y) = \sin(x + y) + \sin(x - y)$.

Subtracting the same two identities gives $2\cos(x)\sin(y) = \sin(x + y) - \sin(x - y)$.

Now, adding the two identities:

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

gives $2\cos(x)\cos(y) = \cos(x + y) + \cos(x - y)$.

Subtracting the same two identities gives:

$-2\sin(x)\sin(y) = \cos(x+y) - \cos(x-y)$ or
$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$

$2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$

$2\cos(x)\sin(y) = \sin(x+y) - \sin(x-y)$

$2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$

$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$

37 Converting products to sums or differences of trigonometric expressions

Express $\sin(\theta)\cos(3\theta)$ as a sum or difference.

THINKING	WORKING
1 Recall an appropriate identity.	$2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$
2 Substitute the given angles.	$2\sin(\theta)\cos(3\theta) = \sin(\theta+3\theta) + \sin(\theta-3\theta)$
3 Simplify.	$= \sin(4\theta) + \sin(-2\theta)$ $= \sin(4\theta) - \sin(2\theta)$
4 Write your answer.	$\sin(\theta)\cos(\theta) = \frac{1}{2}(\sin(4\theta) - \sin(2\theta))$

EXERCISE 7.5 Compound and double-angle identities

Worked Example

1 Use compound angle identities to prove the following: 28

(a) $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos(\theta)$

(b) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$

(c) $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1+\tan(\theta)}{1-\tan(\theta)}$

(d) $\cos\left(\frac{5\pi}{3} - \theta\right) = -\frac{1}{2}\left(\cos(\theta) + \sqrt{3}\sin(\theta)\right)$

2 If $\sin(x) = \frac{3}{5}$ and $\cos(y) = \frac{3}{4}$ and both x and y are acute angles, determine $\sin(x+y)$. 29

3 If $\sin(x) = \frac{1}{2}$ and $\cos(y) = \frac{5}{13}$ and both x and y are acute angles, determine the value of $\cos(x+y)$.

4 If x and y are angles such that $\sin(x) = -\frac{12}{13}$ and $\cos(y) = \frac{4}{5}$, determine the value of $\tan(x+y)$ if $\frac{3\pi}{2} < x, y < 2\pi$.

5 Use identities to express the following in simplest exact form:

(a) $\sin(60° - x)$ (b) $\cos(x + 45°)$ (c) $\tan(x - 30°)$ (d) $\sin\left(x - \frac{\pi}{6}\right)$

6 Determine the exact value of $\sin(15°)$ by calculating the following: 30

(a) $\sin(60° - 45°)$ (b) $\sin(45° - 30°)$

7 Use an identity to determine the exact values of the following:

(a) $\sin(75°)$ (b) $\tan(15°)$ (c) $\cos\left(\frac{\pi}{12}\right)$ (d) $\cos\left(\frac{7\pi}{12}\right)$

31

8 Simplify each of the following expressions:

(a) $\sin(3\alpha)\cos(\alpha) - \cos(3\alpha)\sin(\alpha)$

(b) $\cos(\beta)\cos(4\beta) + \sin(4\beta)\sin(\beta)$

(c) $\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)$

(d) $\sin\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) - \cos\left(\frac{x}{2}\right)\sin\left(\frac{3x}{2}\right)$

(e) $\sin(\alpha)\cos(\alpha-\beta) + \cos(\alpha)\sin(\alpha-\beta)$

(f) $\cos(\theta+\alpha)\cos(\theta-\alpha) + \sin(\theta+\alpha)\sin(\theta-\alpha)$

(g) $\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

(h) $\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

33

9 Use a double-angle identity to verify the following:

(a) $\sin(90°) = 1$

(b) $\cos(90°) = 0$

(c) $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

(d) $\tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$

32

10 Simplify the following expressions:

(a) $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

(b) $-1 + 2\cos^2(2A)$

(c) $\left(\sin(x) + \cos(x)\right)^2$

(d) $\left(\sin(3x) - \cos(3x)\right)^2$

35

11 Prove the following identities:

(a) $\cos^4(x) - \sin^4(x) = \cos(2x)$

(b) $\frac{1}{2}\left(\cos(2x) + 1\right) = \cos^2(x)$

(c) $\frac{1+\cos(2A)}{\sin(2A)} = \cot(A)$

36

12 Solve these equations for $0 \le x \le 2\pi$.

(a) $\sin(2x) = \sin(x)$

(b) $\sin(2x) = \cos(x)$

13 (a) Prove the identity $\sin(3x) = 3\sin(x) - 4\sin^3(x)$.

(b) Hence, solve the equation $3\sin(x) - 4\sin^3(x) = \frac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$.

14 If $\tan(\theta) = \frac{1}{2}$, determine the value of $\tan(2\theta)$ and $\tan(4\theta)$.

34

15 Consider the identity $\cos(2x) = 2\cos^2(x) - 1$.

(a) Use the identity to demonstrate that $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$.

(b) If $\cos(A) = \frac{4}{5}$ for $0 \le A \le \frac{\pi}{2}$, determine the exact value of $\cos\left(\frac{A}{2}\right)$.

16 Prove the following identities.

(a) $\cot(x) - \tan(x) = 2\cot(2x)$

(b) $\frac{\cos(2\theta) + 2\sin^2(\theta)}{\cos(\theta)} = \sec(\theta)$

(c) $\left(1 - \tan(\theta)\right)^2 + \left(1 + \tan(\theta)\right)^2 = 2\sec^2(\theta)$

(d) $\frac{1}{1-\sin(x)} + \frac{1}{1+\sin(x)} = 2\sec^2(x)$

17 Solve these equations for $0 \le x \le 2\pi$.

(a) $1 - \sin(x) = \cos(2x)$

(b) $\cos^2(x) + 2\cos(x)\sin(x) - \sin^2(x) = 0$

18 Use technology to solve $\dfrac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = 3.150$ for $0 \le x \le 2\pi$, giving your answer correct to 3 decimal places.

Worked Example 37

19 Express as sums or differences:

(a) $2\cos(A+2B)\cos(A-2B)$ **(b)** $\sin(3A+B)\cos(A-B)$

20 Verify the identity $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$ if $x = \dfrac{2\pi}{3}$, $y = \dfrac{7\pi}{6}$

21 Consider the identity $2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$.

(a) Let $a+b=X$ and $a-b=Y$. Determine expressions for a and b in terms of X and Y.

(b) Express the identity in terms of X and Y.

(c) Hence express $\sin(5\theta) + \sin(\theta)$ as a product.

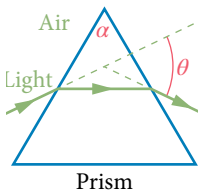
22 Determine expressions for the following, in terms of powers of $\cos(x)$ and $\sin(x)$.

(a) $\cos(4x)$ **(b)** $\sin(4x)$

23 Prove the identity $\sin^2(A)\cos^2(B) - \cos^2(A)\sin^2(B) = \sin^2(A) - \sin^2(B)$.

24 The index of refraction of a transparent material is the ratio of the speed of light in a vacuum to the speed of light while travelling through the material. Triangular prisms can be used to measure the index of refraction. For the triangular prism with uppermost internal angle α, where a beam of light has deviated angle θ from its path, the index of refraction, n, is given by the formula:

$$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$



(a) For a prism with $\alpha = \dfrac{\pi}{3}$, express n in terms of $\cot\left(\dfrac{\theta}{2}\right)$.

(b) If the index of refraction of glass is 1.5, calculate θ, correct to 2 decimal places.

(c) Where $\theta = \dfrac{\pi}{4}$, calculate the index of refraction of the material, correct to 2 decimal places.

25 The path travelled by an object projected at ground level with an initial velocity, v m/s, at an initial angle θ is given by $y = -\dfrac{5x^2}{v^2\cos^2(\theta)} + x\tan(\theta)$, where x is the horizontal distance and y the vertical distance travelled, in metres. A rugby ball is kicked from ground level with an initial velocity, v m/s, at an initial angle θ.

(a) Express the maximum horizontal distance, x, travelled by the ball, in terms of v and θ.

(b) Calculate the maximum horizontal distance travelled by the rugby ball when it is kicked with an initial velocity of 10 m/s at an angle of 45°.

(c) Using the formula found in part **(a)**, calculate the initial angle that the ball should be projected to travel the longest horizontal distance.

(d) The longest authenticated rugby penalty kick, of 78 metres, was recorded in 1954. Calculate the initial velocity to the nearest metre per second with which the ball was kicked if the ball was projected at an angle of 45°.

7.6 Transformations of trigonometric expressions

Transforming the expression $a\sin(x) + b\cos(x)$ into the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$ is very useful for sketching graphs and solving equations. It enables you to write an expression as a single term, rather than having to deal with two different functions. With careful selection of the form of the transformation, R and α are positive constants, with $0 \leq \alpha \leq \frac{\pi}{2}$.

$$\begin{aligned} a\sin(x) + b\cos(x) &= R\sin(x + \alpha) \\ &= R\big(\sin(x)\cos(\alpha) + \cos(x)\sin(\alpha)\big) \\ &= R\sin(x)\cos(\alpha) + R\cos(x)\sin(\alpha) \end{aligned}$$

Equating coefficients:

$$R\cos(\alpha) = a \quad [1]$$
$$R\sin(\alpha) = b \quad [2]$$

Squaring and adding [1] and [2]:

$$\begin{aligned} R^2\sin^2(\alpha) + R^2\cos^2(\alpha) &= a^2 + b^2 \\ R^2\big(\sin^2(\alpha) + \cos^2(\alpha)\big) &= a^2 + b^2 \\ R^2 &= a^2 + b^2 \\ R &= \sqrt{a^2 + b^2} \end{aligned}$$

Dividing [2] by [1]: $\tan(\alpha) = \frac{b}{a}$

The same method can be used to express the other transformations.

$$a\sin(x) + b\cos(x) = R\sin(x + \alpha)$$
$$a\sin(x) - b\cos(x) = R\sin(x - \alpha)$$
$$a\cos(x) + b\sin(x) = R\cos(x - \alpha)$$
$$a\cos(x) - b\sin(x) = R\cos(x + \alpha)$$

$$R = \sqrt{a^2 + b^2}\ ,\ R > 0$$
$$\tan(\alpha) = \frac{b}{a}, 0 \leq \alpha \leq \frac{\pi}{2}$$

It is a good idea to work out these transformations using all the steps, as it is very easy to confuse the different forms.

38 Expressing a sum or difference in the form $R\sin(x \pm \alpha)$

Express $\sin(x) - \cos(x)$ in the form $R\sin(x - \alpha)$, where R is a positive constant and $0 \leq \alpha \leq \frac{\pi}{2}$.

THINKING	WORKING
1 Express $R\sin(x - \alpha)$ in expanded form	$\begin{aligned} R\sin(x - \alpha) &= R\big(\sin(x)\cos(\alpha) - \cos(x)\sin(\alpha)\big) \\ &= R\sin(x)\cos(\alpha) - R\cos(x)\sin(\alpha) \end{aligned}$
2 Equate the expressions.	$\sin(x) - \cos(x) = R\sin(x)\cos(\alpha) - R\cos(x)\sin(\alpha)$
3 Equate the coefficients.	For $\cos(x)$: $R\sin(\alpha) = 1$ For $\sin(x)$: $R\cos(\alpha) = 1$
4 Solve the equations for α.	$\frac{R\sin(\alpha)}{R\cos(\alpha)} = \frac{1}{1}$ $\tan(\alpha) = 1$ $\alpha = \frac{\pi}{4}$

5 Square each equation and add them to solve for R.	$R^2\sin^2(\alpha)+R^2\cos^2(\alpha)=2$ $R^2\left(\sin^2(\alpha)+\cos^2(\alpha)\right)=2$ $R^2=2$ $R=\sqrt{2}$, where $R>0$
6 Express in the form $R\sin(x-\alpha)$.	$\sin(x)-\cos(x)=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

39 Expressing a sum or difference in the form $R\cos(x\pm\alpha)$ to solve an equation

Consider the expression $\sqrt{3}\cos(x)-\sin(x)$.

(a) Express $\sqrt{3}\cos(x)-\sin(x)$ in the form $R\cos(x+\alpha)$ where $R>0$ and $0\le\alpha\le\frac{\pi}{2}$.

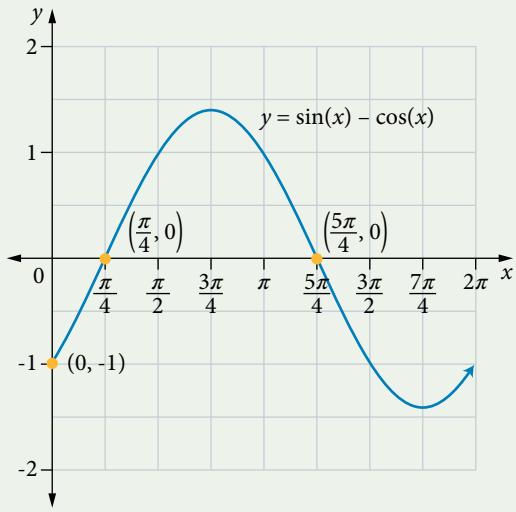
THINKING	WORKING
1 Express $R\cos(x+\alpha)$ in expanded form.	$R\cos(x+\alpha)=R\cos(x)\cos(\alpha)-R\sin(x)\sin(\alpha)$
2 Equate the expressions.	$\sqrt{3}\cos(x)-\sin(x)=R\cos(x)\cos(\alpha)-R\sin(x)\sin(\alpha)$
3 Equate coefficients.	$\sqrt{3}=R\cos(\alpha)$ $1=R\sin(\alpha)$
4 Calculate the value of R.	$R=\sqrt{\left(\sqrt{3}\right)^2+1^2}$ $R=2$, where $R>0$
5 Calculate the value of α.	$\tan(\alpha)=\frac{1}{\sqrt{3}}=\frac{\pi}{6}$
6 Express in the form $R\cos(x+\alpha)$.	$\sqrt{3}\cos(x)-\sin(x)=2\cos\left(x+\frac{\pi}{6}\right)$

(b) Solve the equation $\sqrt{3}\cos(x)-\sin(x)=1$, $0\le x\le 2\pi$.

1 Write the equation using the transformed expression.	$\sqrt{3}\cos(x)-\sin(x)=1$ $2\cos\left(x+\frac{\pi}{6}\right)=1$
2 Solve the equation in the given domain.	$\cos\left(x+\frac{\pi}{6}\right)=\frac{1}{2}$ $x+\frac{\pi}{6}=\frac{\pi}{3},\frac{5\pi}{3}$ $x=\frac{\pi}{6},\frac{9\pi}{6}$ $=\frac{\pi}{6},\frac{3\pi}{2}$
3 Write the solutions.	$x=\frac{\pi}{6},\frac{3\pi}{2}$

40 Expressing a sum or difference in the form $R\sin(x \pm \alpha)$ to sketch a graph

In Worked Example 38, the expression $\sin(x) - \cos(x)$ was found to be equal to $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$.
Use this relationship to sketch the graph of $y = \sin(x) - \cos(x)$.

THINKING	WORKING
1 Write the equation using the transformed expression.	$y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$
2 Determine the period and amplitude.	period 2π; amplitude $\sqrt{2}$
3 Calculate the intercepts with the axes.	Let $x = 0$ $y = \sqrt{2}\sin\left(0 - \frac{\pi}{4}\right) = -1$ The coordinates of the y-intercept are $(0, -1)$. Let $y = 0$ $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 0$ $x - \frac{\pi}{4} = 0, \pi$ $x = \frac{\pi}{4}, \frac{5\pi}{4}$ The coordinates of the x-intercepts are $\left(\frac{\pi}{4}, 0\right)$ and $\left(\frac{5\pi}{4}, 0\right)$.
4 Sketch the graph in the given domain.	

41 Solving a problem using the transformation of an expression to the form $R\sin(x+\alpha)$

A rectangle $OABC$ is formed with one vertex on the origin O, point A at $(0, 3\cos(\theta))$ and point C at $(4\sin(\theta), 0)$.

(a) Express $4\sin(\theta) + 3\cos(\theta)$ in the form $f(\theta) = R\sin(\theta+\alpha)$ where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$.

THINKING	WORKING
1 Equate the two forms of the expression.	$4\sin(\theta) + 3\cos(\theta) = R\sin(\theta+\alpha)$
2 Expand the new form and equate coefficients.	$4\sin(\theta) + 3\cos(\theta) = R\sin(\theta)\cos(\alpha) + R\cos(\theta)\sin(\alpha)$ $R\cos(\alpha) = 4$ $R\sin(\alpha) = 3$
3 Calculate R.	$R = \sqrt{4^2 + 3^2}$ $= \sqrt{25}$ $= 5$
4 Calculate α.	$\tan(\alpha) = \frac{b}{a} = \frac{3}{4}$ $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$ $= 0.6435$ (4 d.p.)
5 Write the final answer.	$f(\theta) = 4\sin(\theta) + 3\cos(\theta)$ $= 5\sin(\theta + 0.6435)$

(b) State the coordinates of B in terms of θ.

Write the position of B in coordinate form.	$(4\sin(\theta), 3\cos(\theta))$

(c) Express the perimeter of the rectangle in terms of $f(\theta)$.

Write the perimeter in terms of $f(\theta)$.	perimeter $= 2\left(4\sin(\theta) + 3\cos(\theta)\right)$ $= 2f(\theta)$

(d) Calculate the maximum value of the perimeter of the rectangle.

1 Use the expression found for the perimeter.	The amplitude of $f(\theta)$ is 5, so the maximum value of $2f(\theta)$ is 10.
2 Determine the maximum value of the perimeter.	maximum perimeter = 10

EXERCISE 7.6 Transformations of trigonometric expressions

Worked Example

38

1 Express the following in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0, 0 \le \alpha \le \frac{\pi}{2}$.

(a) $\sin(x) + \sqrt{3}\cos(x)$ (b) $\sqrt{3}\sin(x) - \cos(x)$ (c) $\cos(x) + \sin(x)$

(d) $5\cos(2x) - 5\sin(2x)$ (e) $\sin(x) - 2\cos(x)$

2 Express the following in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, giving the value of α correct to 2 decimal places.

(a) $3\cos(x) + 2\sin(x)$ (b) $\sin(x) - 7\cos(x)$

39

3 Solve the following equations by first transforming the left-hand-side expressions to the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$.

(a) $3\sin(x) + \sqrt{3}\cos(x) = 3, 0 \le x \le \pi$ (b) $\cos(x) - \sin(x) = -1, 0 \le x \le 2\pi$

(c) $\sqrt{3}\sin(x) - \cos(x) = -\sqrt{2}, 0 \le x \le 2\pi$ (d) $\sqrt{2}\cos(x) - \sqrt{2}\sin(x) = 1, 0 \le x \le 2\pi$

4 Use a transformation into the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, to determine the period and amplitude of the following expressions.

(a) $3\sin(\theta) + 4\cos(\theta)$ (b) $\cos\left(\frac{\pi x}{4}\right) - 6\sin\left(\frac{\pi x}{4}\right)$ (c) $\sqrt{2}\sin(3x) - \sqrt{5}\cos(3x)$

5 For each of the expressions in question 4, write the maximum and minimum value of the expression. Determine the smallest positive value of the variable for which they occur. Where necessary, give your answers correct to 3 decimal places.

(a) $3\sin(\theta) + 4\cos(\theta)$ (b) $\cos\left(\frac{\pi x}{4}\right) - 6\sin\left(\frac{\pi x}{4}\right)$ (c) $\sqrt{2}\sin(3x) - \sqrt{5}\cos(3x)$

40

6 Transform the function $f(x) = \sin(x) - \sqrt{3}\cos(x)$ into the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, and hence sketch the graph of the function for $0 \le x \le 2\pi$, showing key features.

7 Transform the function $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x)$ into the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, and hence sketch the graph of the function for $0 \le x \le 2\pi$, showing key features.

8 What is the amplitude of the graph of the function $f(x) = m\sin(x) - n\cos(x)$?

A $m - n$ B $m + n$ C $m^2 + n^2$ D $\sqrt{m^2 + n^2}$

9 (a) Express the function $f(x) = 6\sin(x) - 2\sqrt{3}\cos(x)$ in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$.

(b) Draw a graph of both the original and the transformed function on the same axes to check that your answer is correct.

10 Write each of these expressions as a single trigonometric function.

(a) $\dfrac{1}{6\cos(x) + 6\sin(x)}$ (b) $\dfrac{\sin(5x) + \sqrt{3}\cos(5x)}{2\cos(5x) - 2\sqrt{3}\sin(5x)}$

11 Consider the expression $\sin(x) - \cos(x)$.

(a) Transform the expression into the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$.

(b) Starting with the transformed expression, use a compound angle identity to demonstrate that it is equal to the original expression.

12 Rearrange the expression $\sin(x) - \cos(x)$ so that it can be expressed in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$.

Worked Example 41

13 A rectangle $OABC$ is formed with one vertex the origin O, point A at $(0, 7\cos(\theta))$ and point C at $(24\sin(\theta), 0)$.

(a) Express $f(\theta) = 24\sin(\theta) + 7\cos(\theta)$ in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$. Give your answers exact or correct to 3 decimal places.

(b) Express the coordinates of B in terms of θ.

(c) Express the perimeter in terms of $f(\theta)$.

(d) Using technology, calculate the greatest perimeter of the rectangle as θ varies in the interval $0 \leq \theta \leq \frac{\pi}{2}$.

(e) Write the coordinates of B for which the perimeter is greatest, expressed correct to 2 decimal places.

14 Determine the solutions to the equation $\cos(x) + \sqrt{3}\sin(x) = 1$ and write them in general form.

15 The position of a particle y metres from an origin O at time t seconds after it is released is $y = 5\cos(4t) - 12\sin(4t)$.

(a) Determine the starting position of the particle.

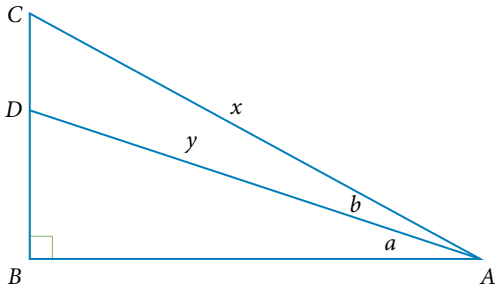
(b) Express $5\cos(4t) - 12\sin(4t)$ in the form $R\cos(4t + \alpha)$, giving the value of α correct to 3 decimal places.

(c) Sketch the graph of $y = 5\cos(4t) - 12\sin(4t)$, $0 \leq t \leq \pi$.

(d) Determine the maximum distance from O the particle is able to travel.

(e) Determine the total distance travelled by the particle in the first π seconds

16 The right-angled triangle ABC is shown with $AC = x$, $AD = y$, $\angle BAD = a$ and $\angle CAD = b$.



(a) Demonstrate that the length of BC can be expressed as $x\sin(a + b)$.

(b) Hence, express BC in fully expanded form.

(c) Demonstrate that AB can be expressed as $x\cos(a + b)$ and express AB in fully expanded form.

(d) Hence, express $\frac{y}{x}$ in terms of a and b in fully expanded form.

(e) Determine y in exact form for $a = 45°$, $b = 30°$ and $x = 10$.

(f) Express CD in terms of x, y, a and b in fully expanded form.

(g) Hence, determine CD when $a = 45°$, $b = 30°$, $x = 10$, using the y-value that you found in part **(e)**.

7

Summary

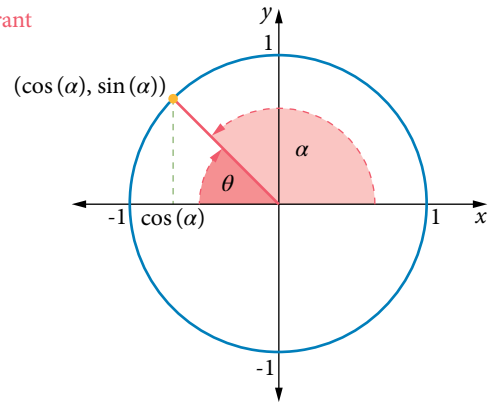
Solving trigonometric equations

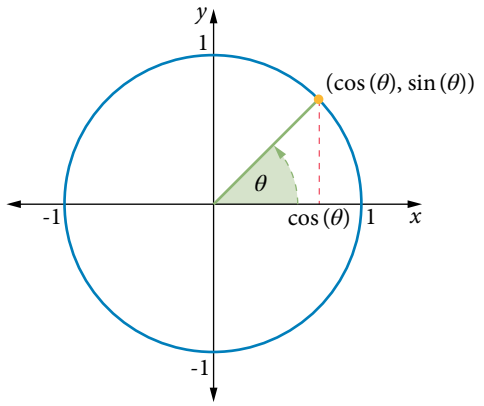
Based on the definitions of the trigonometric functions, they have different signs in different quadrants.

Angles in quadrants 2, 3 and 4 can be expressed in terms of the reference angle θ in the first quadrant:

2nd Quadrant
$\sin(\alpha) > 0$
$\cos(\alpha) < 0$
$\tan(\alpha) < 0$

$\alpha = \pi - \theta$

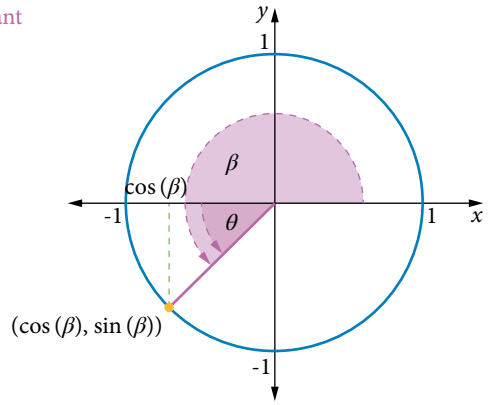


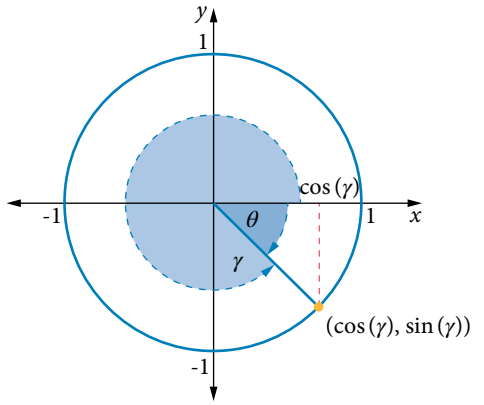


1st Quadrant
$\sin(\theta) > 0$
$\cos(\theta) > 0$
$\tan(\theta) > 0$

3rd Quadrant
$\sin(\beta) < 0$
$\cos(\beta) < 0$
$\tan(\beta) > 0$

$\beta = \pi + \theta$





4th Quadrant
$\sin(\gamma) < 0$
$\cos(\gamma) > 0$
$\tan(\gamma) < 0$

$\gamma = 2\pi - \theta$

In quadrant 1: all functions are positive.

In quadrant 2: only the sine function is positive.

In quadrant 3: only the tangent function is positive.

In quadrant 4: only the cosine function is positive.

Solutions to equations can be found using the symmetry properties of the trigonometric functions in the four quadrants.
They can be written in general form or given within a stated domain.

General solutions

Where θ represents the reference angle in the first quadrant:

For $\sin(x) = \alpha$: $x = 2n\pi + \theta$ or $x = (2n+1)\pi - \theta$ for $n \in \mathbb{Z}$

These can be combined to give: $x = n\pi + (-1)^n \theta$ for $n \in \mathbb{Z}$

For $\cos(x) = \alpha$: $x = 2n\pi \pm \theta$ for $n \in \mathbb{Z}$

For $\tan(x) = \alpha$: $x = n\pi + \theta$

Graphing and modelling with trigonometric functions

Graphs of the function $f(x) = A\sin(a(x - b)) + c$ or $f(x) = A\cos(a(x - b)) + c$ have the following features:

- amplitude: $|A|$
- period: $\frac{2\pi}{a}$
- They have been transformed by a dilation by factor $|A|$ from the x-axis, and by a factor of $\frac{1}{a}$ from the y-axis.
- There has been a translation of b units parallel to the x-axis and a translation of c units parallel to the y-axis.
- The median line is at $y = c$.
- The range is $[c - |A|, c + |A|]$.
- The graph of the tangent function behaves in a similar way under transformations, but it has a period of $\frac{\pi}{a}$ and its range is $\mathbb{R}$.

Reciprocal trigonometric functions

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\cot(x) = \frac{1}{\tan(x)}$$

The graphs of these reciprocal functions have the following features:

- The period is the same as the original function.
- Where there was an x-intercept, there will be a vertical asymptote.
- Where there was a vertical asymptote, there will be an x-intercept.
- The range of the cosec and sec functions is not continuous. The range of the cot graph is $\mathbb{R}$.

Pythagorean trigonometric identities

$$\sin^2(x) + \cos^2(x) = 1$$

$$1 + \tan^2(x) = \sec^2(x)$$

$$1 + \operatorname{cosec}^2(x) = \cot^2(x)$$

These identities can be used to simplify expressions, solve equations and prove other identities.

Compound angle and double-angle formulas

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$
$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

When $x = y$ in the above identities, the double-angle identities are found.

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

Products of sines and cosines can be expressed as sums and differences with the following identities.

$$2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$$
$$2\cos(x)\sin(y) = \sin(x+y) - \sin(x-y)$$
$$2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$$
$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$$

Transformations of trigonometric expressions

Sums and differences of $\sin(x)$ and $\cos(x)$ can be expressed as a single trigonometric function.

$$a\sin(x) + b\cos(x) = R\sin(x+\alpha)$$
$$a\sin(x) - b\cos(x) = R\sin(x-\alpha)$$
$$a\cos(x) + b\sin(x) = R\cos(x-\alpha)$$
$$a\cos(x) - b\sin(x) = R\cos(x+\alpha)$$

$$R = \sqrt{a^2+b^2},\ R > 0$$
$$\tan(\alpha) = \frac{b}{a},\ 0 \le (\alpha) \le \frac{\pi}{2}$$

Chapter review 7

1 Solve the following equations for $0 \le x \le 2\pi$. Exercise 7.1

(a) $2\cos\left(x + \frac{\pi}{6}\right) = -1$ **(b)** $\sqrt{2}\cos\left(2\left(x - \frac{\pi}{4}\right)\right) = 1$ **(c)** $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$

(d) $2\sin\left(3\left(x - \frac{\pi}{6}\right)\right) = \sqrt{3}$ **(e)** $\tan\left(x + \frac{\pi}{3}\right) = -1$ **(f)** $\sqrt{3}\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1$

2 Solve the following for $0 \le x \le 2\pi$, expressing your answers correct to 2 decimal places. Exercise 7.1

(a) $2\cos(2(x - 0.5)) = 1$ **(b)** $\sin(3(x + 1.2)) = 0.4$ **(c)** $\cos(2(x + 0.5)) = 0.8$

3 For the following, determine the period, amplitude and the axis intercepts for $0 \le x \le 2\pi$. Sketch each graph for the given domain, showing the key features. Exercise 7.2

(a) $y = 2\cos\left(3\left(x - \frac{\pi}{3}\right)\right)$ **(b)** $y = -3\sin\left(2\left(x + \frac{\pi}{6}\right)\right)$

(c) $y = 2\cos\left(2\left(x + \frac{\pi}{4}\right)\right) + 1$ **(d)** $y = 4\sin\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$

4 For the function $g(x) = \tan\left(2\left(x - \frac{3\pi}{4}\right)\right)$, $0 \le x \le 2\pi$: Exercise 7.2

(a) Determine the period and range.

(b) Determine the transformations needed to produce $g(x)$ from $f(x) = \tan(x)$.

(c) Determine the equations of the asymptotes and x-intercepts.

(d) Sketch the graph of the function $g(x)$, labelling all major points in coordinate form.

5 Solve the following equations for $0 \le x \le 2\pi$.

(a) $\cot(x) = \sqrt{3}$ **(b)** $\sqrt{3}\sec(2x) = 2$

(c) $\text{cosec}(2x) = 2$ **(d)** $\cot\left(2\left(x - \frac{\pi}{4}\right)\right) = 1$

6 **(a)** **(i)** For the graph of the function $f(x) = \sec(x) + 1$, determine the range and identify any axis intercepts and vertical asymptotes for $0 \le x \le 2\pi$. Exercise 7.2

(ii) Sketch the graph of $y = f(x)$, $0 \le x \le \pi$.

(b) **(i)** For the function $f(x) = \text{cosec}(x) - 1$, determine the range and identify any axis intercepts and vertical asymptotes for $0 \le x \le \pi$.

(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le \pi$.

(c) **(i)** For the function $f(x) = \cot\left(x + \frac{\pi}{2}\right)$, determine the range and identify any axis intercepts and vertical asymptotes for $0 \le x \le \pi$.

(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 2\pi$.

7 Simplify the following: Exercise 7.4

(a) $\tan(\theta) + \sec(\theta)$ **(b)** $\cot(\theta) + \text{cosec}(\theta)$ **(c)** $\frac{\sin^2(\theta) - \cos^2(\theta)}{\cos(\theta) + \sin(\theta)}$

(d) $\cos^4(\theta) - \sin^4(\theta)$ **(e)** $\frac{\sin(\theta)}{1 + \cos(\theta)} \times \frac{\sin(\theta)}{1 - \cos(\theta)}$ **(f)** $\frac{\sin(\theta)}{1 + \sin(\theta)} \times \frac{\sin(\theta)}{1 - \sin(\theta)}$

Exercise 7.4

8 Prove the following identities.

(a) $\sec^2(\theta) - \tan^2(\theta) = 1$

(b) $\cos^3(\theta) + \cos(\theta)\sin^2(\theta) = \cos(\theta)$

(c) $\cot^2(\theta)\sin^2(\theta) + \sin^2(\theta) = 1$

(d) $\dfrac{2}{1+\sin(\theta)} + \dfrac{2}{1-\sin(\theta)} = 4\sec^2(\theta)$

(e) $\sin^4(\theta) - \cos^4(\theta) = 2\sin^2(\theta) - 1$

(f) $\tan^2(\theta) - \sin^2(\theta) = \sin^4(\theta)\sec^2(\theta)$

Exercise 7.5

9 Express the following in expanded form:

(a) $\sin(x - 3y)$ **(b)** $\cos(2x + y)$ **(c)** $\tan(x + 3y)$

Exercise 7.5

10 Write the following expressions in simplest exact form.

(a) $\sin(30° + x)$ **(b)** $\cos\left(x - \dfrac{\pi}{3}\right)$ **(c)** $\tan\left(x + \dfrac{\pi}{4}\right)$

Exercise 7.5

11 Simplify the following expressions:

(a) $\cos(5x)\cos(3x) + \sin(5x)\sin(3x)$

(b) $\sin(5x)\cos(x) + \cos(5x)\sin(x)$

(c) $\sin\left(\dfrac{5x}{2}\right)\cos\left(\dfrac{3x}{2}\right) + \cos\left(\dfrac{5x}{2}\right)\sin\left(\dfrac{3x}{2}\right)$

(d) $\cos\left(\dfrac{x}{2}\right)\cos\left(\dfrac{5x}{2}\right) + \sin\left(\dfrac{x}{2}\right)\sin\left(\dfrac{5x}{2}\right)$

(e) $\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right) - \cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

(f) $\dfrac{\tan\left(\frac{x+y}{2}\right) + \tan\left(\frac{x-y}{2}\right)}{1 - \tan\left(\frac{x+y}{2}\right)\tan\left(\frac{x-y}{2}\right)}$

Exercise 7.5

12 Simplify the following using a double-angle identity:

(a) $\sin(7\theta)\cos(7\theta)$ **(b)** $\cos^2\left(\dfrac{A}{4}\right) - \sin^2\left(\dfrac{A}{4}\right)$ **(c)** $\dfrac{2\tan(5x)}{1 - \tan^2(5x)}$

Exercise 7.5

13 Express these products as a sum or difference of trigonometric functions.

(a) $2\sin(5A)\cos(A)$ **(b)** $2\cos(6A)\sin(2A)$ **(c)** $4\cos(3x)\cos(5x)$ **(d)** $8\sin(8x)\sin(2x)$

Exercise 7.5

14 Solve these equations for $0 \le x \le 2\pi$.

(a) $\sin(2x) - \cos(x) = 0$

(b) $\cos(2x) + \cos(x) = 0$

(c) $\cos^2(x) + \sin(x) = 1$

(d) $2\cot^2(x) - 2 + \operatorname{cosec}^2(x) = 0$

Exercise 7.6

15 Express the following in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$, $0 \le \alpha \le \dfrac{\pi}{2}$.

(a) $\sin(x) - \cos(x)$ **(b)** $\cos(x) + \sqrt{3}\sin(x)$ **(c)** $\sqrt{3}\cos(x) - \sin(x)$ **(d)** $\sin(x) + \cos(x)$

Exercise 7.6

16 Express $7\sin(x°) + \cos(x°)$ in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, where $R > 0$, $0° \le \alpha \le 90°$. Give your answer correct to 2 decimal places where necessary.

Exercise 7.1

17 Solve these equations for $0 \le x \le 2\pi$.

(a) $\sin(x) - \cos(x) = \dfrac{1}{\sqrt{2}}$

(b) $\cos(x) - \sqrt{3}\sin(x) = 2$

Exercise 7.2

18 Determine the period and amplitude for the graphs of these functions. Give your answers correct to 2 decimal places where necessary.

(a) $2\sin(x) + 5\cos(x)$

(b) $6.2\cos(3x) - 1.8\sin(3x)$

Exercise 7.6

19 Consider the function $f(x) = \sqrt{3}\sin(x) + \cos(x)$.

(a) Express it as a single trigonometric term.

(b) Hence, solve $f(x) = \sqrt{3}\sin(x) + \cos(x) = 0$, $0 \le x \le 2\pi$, expressing the solutions in exact form.

(c) Solve $f(x) = \pm 2$, $0 \le x \le 2\pi$.

(d) Sketch the graph of $f(x) = \sqrt{3}\sin(x) + \cos(x)$, $0 \le x \le 2\pi$, including the axis intercepts and end points in coordinate form. Write the period and amplitude of the function.

20 If $\sin(x) = \frac{3}{5}$ and $0 \le x \le \frac{\pi}{2}$, determine the value of the following, without calculating the value of x. Exercise 7.5

(a) $\cos(x)$ (b) $\text{cosec}(x)$ (c) $\cot(x)$ (d) $\cos(2x)$

21 (a) Use technology to determine, correct to 3 decimal places, the value of $2\sin(61°)\cos(43°)$ Exercise 7.4

(b) Using an identity, write another form of the expression $2\sin(61°)\cos(43°)$.

(c) Verify that the two forms from parts (a) and (b) are equal.

22 If $\cos(x) = -\frac{2}{3}$, $\frac{\pi}{2} \le x \le \pi$, determine the value of the following. Exercise 7.5

(a) $\sin(x)$ (b) $\tan(x)$ (c) $\sin\left(\frac{x}{2}\right)$

23 For $f(x) = 4\tan\left(2\left(x - \frac{\pi}{4}\right)\right)$, $0 \le x \le 2\pi$, the graph of $y = f(x)$ has asymptotes at: Exercise 7.2

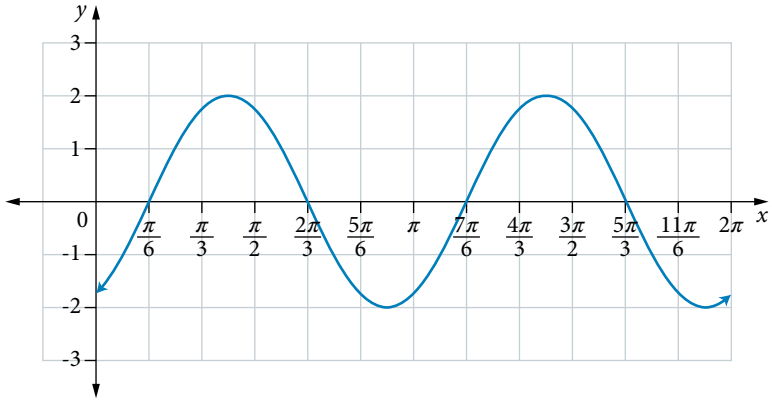
A $x = 0, x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi$

B $x = 0, x = \frac{\pi}{4}, x = \pi, x = \frac{3\pi}{4}, x = 2\pi$

C $x = 0, x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \pi, x = \frac{5\pi}{4}, x = \frac{7\pi}{4}, x = 2\pi$

D $x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$

24 The graph shown could be described by which equation? Exercise 7.2



A $y = 2\sin\left(x - \frac{\pi}{6}\right)$

B $y = 2\sin\left(2\left(x + \frac{\pi}{6}\right)\right)$

C $y = 2\sin\left(2\left(x - \frac{\pi}{6}\right)\right)$

D $y = 2\sin\left(x + \frac{\pi}{6}\right)$

25 Prove the following identities. Exercise 7.4

(a) $\sec(\theta) + \tan(\theta) = \frac{\cos(\theta)}{1 - \sin(\theta)}$ (b) $\frac{\sin(\theta) + \tan(\theta)}{1 + \cos(\theta)} = \tan(\theta)$

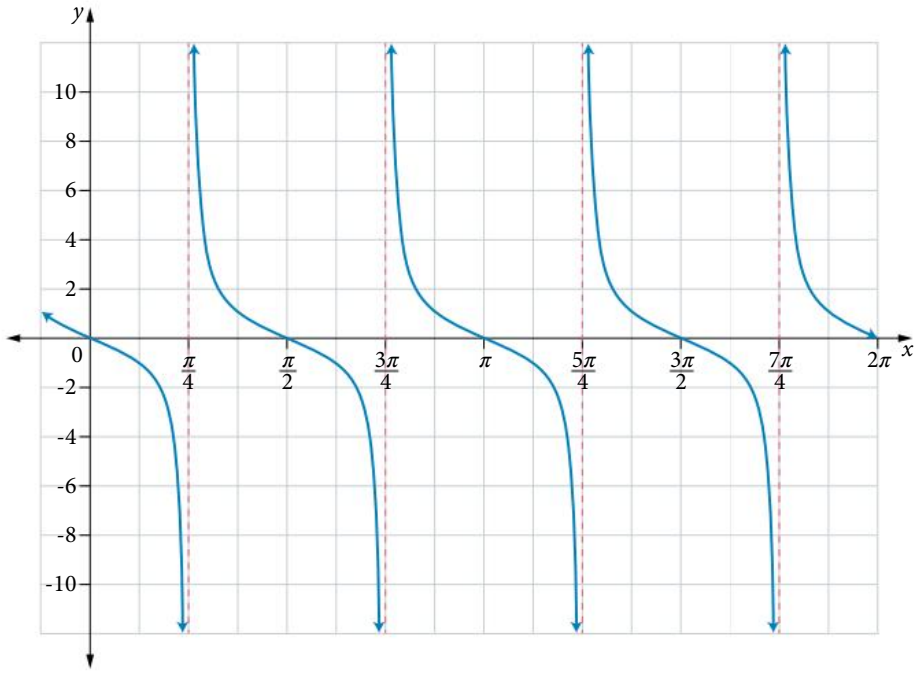
26 Determine the exact value of $\tan\left(\frac{\pi}{12}\right)$ by calculating $\tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$. Exercise 7.5

27 (a) Write two forms of the expression $\sin\left(\frac{5\pi}{12}\right)$ that can be used to determine its exact value using compound angle identities. Exercise 7.5

(b) Use either one of the possible compound angle identities to determine the exact value of $\sin\left(\frac{5\pi}{12}\right)$.

Exercise 7.2

28 Determine a possible equation for the graph shown below.



Exercise 7.3

29 **(a)** For the functions $f(x) = \text{cosec}(x)$ and $g(x) = \text{cosec}\left(x + \frac{\pi}{4}\right) + 1$, $0 \leq x \leq 2\pi$, determine the axis intercepts expressed in coordinate form and the equations of the asymptotes.

(b) Sketch the graph of $y = f(x)$ for $0 \leq x \leq 2\pi$ showing all the key features.

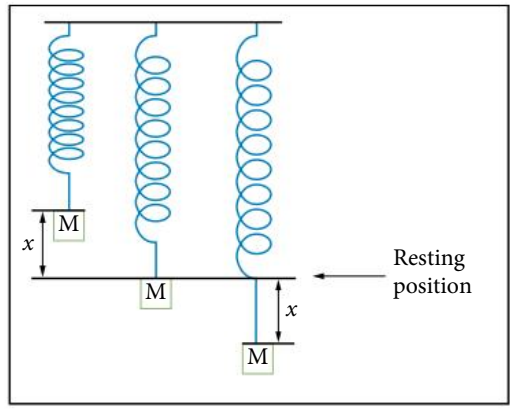
(c) For $g(x)$, determine the axis intercepts expressed in coordinate form and the equations of the asymptotes.

(d) Write the domain, range and period of $g(x)$.

(e) Sketch the graph of $y = g(x)$ for $0 \leq x \leq 2\pi$ showing all the key features.

Exercise 7.6

30 A weight attached to a spring is pulled down 20 cm from its resting position and released. The weight takes 4 seconds to return to its resting position. The equation that can be used to model the motion of the weight when it is x cm from its resting position, t seconds after it is released, is $x = -20\cos\left(\frac{\pi t}{2}\right)$.



(a) Determine the first six times that the weight passes its resting position ($x = 0$) after it has been released from its lowest position.

(b) Determine the first four times when the weight is positioned 10 cm from its resting position after it is released.

(c) Determine the first four times that the weight is positioned 20 cm from its resting position after it is released.

(d) Determine the first time interval after the weight is released that it is within 5 cm from its resting position, expressed to 2 decimal places.

(e) Determine the total vertical distance that the weight has travelled 15 seconds after it is released.

31 The temperature in a desert, T°C at time t hours after midnight is modelled by the function $T = -30\cos\left(\frac{\pi t}{12} - \frac{2}{3}\right) + 20$. Exercise 7.2

(a) What were the highest and lowest temperatures for the day and at what times were they recorded?

(b) Sketch a graph to demonstrate the way that the temperature changes for the 24-hour period from midnight.

(c) A group of hikers is able to walk when the temperature is in the interval $T \in [10, 30]$. Between what times can the hikers walk during the day?

(d) Determine the times to the nearest minute that the temperature was 0.

32 Prove the following identities. Exercise 7.4

(a) $\left(\cot(\theta) + \operatorname{cosec}(\theta)\right)^2 = \frac{1 + \cos(\theta)}{1 - \cos(\theta)}$

(b) $\frac{2}{\sec(\theta) + \operatorname{cosec}(\theta)} \times \frac{1 + \sin(2\theta)}{\sin(\theta) + \cos(\theta)} = \sin(2\theta)$

33 **(a)** If $x = b\sin(\theta)$ and $y = a\cos(\theta)$, determine the following by eliminating θ:

(i) x in terms of y, a and b **(ii)** y in terms of x, a and b

(b) If $\tan^2(\theta) + \sec^2(\theta) = 3$, calculate the value of the following:

(i) $\sin^2(\theta)$ **(ii)** $\cot^2(\theta)$

34 For the following equations, determine a general solution for an unrestricted domain. Exercise 7.1

(a) $\sqrt{3}\tan(x) - 1 = 0$ **(b)** $\cos(x) = -\frac{1}{\sqrt{2}}$ **(c)** $\sin(2x) - \cos(x) = 0$

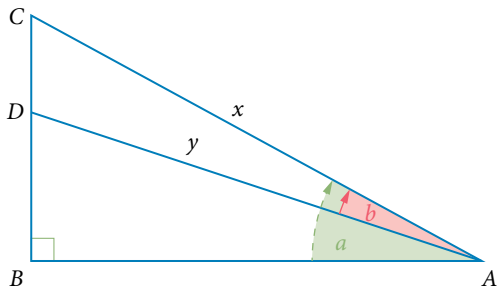
35 Consider the identity $2\cos(a)\sin(b) = \sin(a + b) - \sin(a - b)$. Exercise 7.5

(a) Let $a + b = X$ and $a - b = Y$. Determine expressions for a and b in terms of X and Y.

(b) Rewrite the identity in terms of X and Y.

(c) Hence, express $\sin(\theta) - \sin(3\theta)$ as a product of trigonometric functions.

36 The right-angled triangle ABC is shown with $AC = x$, $AD = y$, $\angle BAC = a$ and $\angle CAD = b$. Exercise 7.5



(a) Write an expression for AB in terms of a and x.

(b) Write an expression for AB in terms of a, b and y.

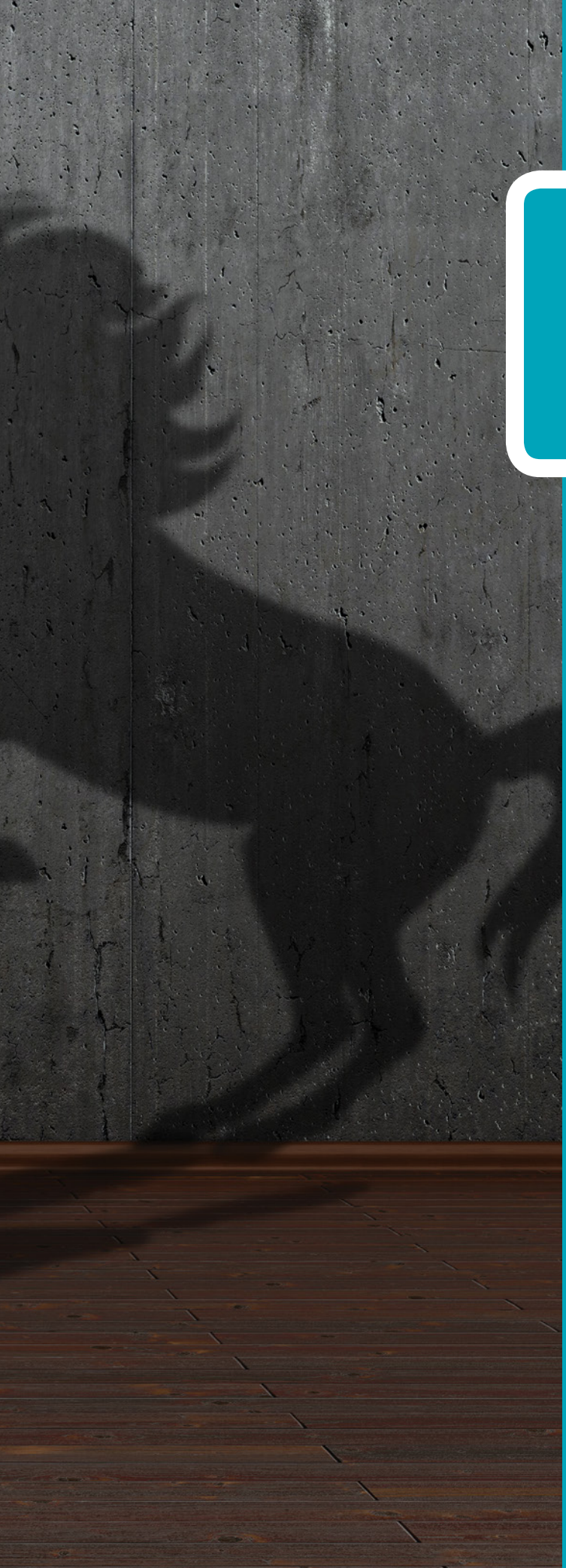
(c) Equate the expressions for AB in parts **(a)** and **(b)** to demonstrate $\tan(a) = \operatorname{cosec}(b)\left(\frac{x}{y} - \cos(b)\right)$.

(d) Determine the value of $\tan(a)$ for $x = 2$, $y = \sqrt{2}\left(\sqrt{3} - 1\right)$ and $b = \frac{\pi}{4}$.

(e) For $y = \frac{1}{2}x$, $b = \frac{\pi}{6}$, demonstrate that $\tan(a) = 4 - \sqrt{3}$.

(f) For $y = \frac{1}{2}x$, $b = \frac{\pi}{3}$, determine the value of a and comment on the result.





8

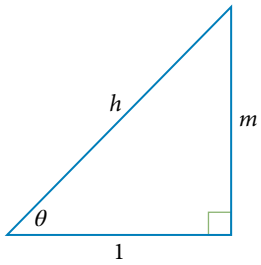
Introduction to matrices

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8 Recall

Identify and use the basic trigonometric ratios and the Pythagorean identity

1 Consider the triangle below.



(a) Express the value of $\sin(\theta)$, $\cos(\theta)$ in terms of m.

(b) Hence, demonstrate that $\cos^2(\theta) + \sin^2(\theta) = 1$.

Identify and transpose linear rules in specific forms

2 A line in the plane is defined uniquely by two points $A(x_1, y_1)$ and $B(x_2, y_2)$. The equation of this line has four equivalent expressions:

- General form: $ax + by + c = 0$
- Point-gradient form: $y - y_1 = m(x - x_1)$
- Gradient-intercept form: $y = mx + c$
- Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$

(a) Write $y + 1 = \frac{1}{2}(x - 4)$ in the gradient-intercept form.

(b) Write $\frac{x}{2} + \frac{y}{3} = 1$ in the general form.

(c) If a line passes through the point (2, 1) with a slope of 3, express its equation in the point-gradient form.

(d) Write $2x - 3y - 6 = 0$ in the intercept form.

Determining relevant geometric features of a linear graph

3 Let $A(3, 4)$ and $B(5, 2)$ be two points in the plane.

(a) Calculate the distance between A and B.

(b) Calculate coordinates of the midpoint M of the segment AB.

(c) Calculate the gradient of the line through the points A and B.

(d) Determine the equation of the line through the points A and B. (Express your answer in the point-gradient form).

(e) Hence, determine the equation of the line in gradient-intercept form that passes through the points A and B.

Determine collinear points

4 Determine whether the three points in each of the questions below are collinear.

(a) $A(1, 2)$, $B(4, 8)$ and $C(5, 10)$

(b) $A(2, 3)$, $B(5, 9)$ and $C(6, 11)$

(c) $A(-2, -3)$, $B(-5, -9)$ and $C(-6, -11)$

(d) $A(2, 3)$, $B(6, 8)$ and $C(7, 11)$

Solve linear equations

5 Solve the following linear equations for the unknown variable.

(a) $4a - 3 = 1$

(b) $2a - 3 = 1 - a$

(c) $\frac{6}{x} + 3 = 1$

(d) $2x + 4 = 3 - x$

Solve quadratic equations

6 Solve the following quadratic equations for the unknown variable.

(a) $x^2 = 5x$

(b) $x^2 - 8x + 12 = 0$

(c) $a^2 + a - 1 = 0$

8.1 Matrix properties and arithmetic

Introduction to matrices

This section introduces the concept of a *matrix* and matrix operations, and illustrates how *matrices* (plural of matrix) can be applied in a variety of different contexts.

Matrices can be used to represent and process tables of numerical data. They can also be used to:

- solve equations
- determine solutions to problems in social or road networks
- encode and decode information.

A *matrix* is a rectangular array of numbers, usually enclosed by square brackets and denoted by a bold capital letter. A matrix is displayed as an array with rows (horizontal) and columns (vertical) of elements.

Sometimes a matrix is used to represent a finite, simple, labelled diagram or network, where the rows and columns of the matrix are labelled to match the diagram vertices or vice versa. This is called an *adjacency matrix*.

Representing information in matrix form

A table of information can be represented in a matrix. Suppose a furniture company sells sofas, lounge chairs and coffee tables at two different retail outlets. In May, the Gold Coast store sold 5 sofas, 8 lounge chairs and 10 coffee tables. In the same month, the Brisbane store sold 7 sofas, 12 lounge chairs and 20 coffee tables.

This sales information can be summarised in a table:

	Sofas (S)	Lounge chairs (L)	Coffee tables (C)
Gold Coast store	5	8	10
Brisbane store	7	12	20

The above table can then be expressed as the matrix **A**.

$$\mathbf{A} = \begin{matrix} \\ \text{Store 1} \\ \text{Store 2} \end{matrix} \begin{matrix} \begin{matrix} \text{S} & \text{L} & \text{C} \end{matrix} \\ \begin{bmatrix} 5 & 8 & 10 \\ 7 & 12 & 20 \end{bmatrix} \end{matrix}$$

Without the labels, matrix **A** can be represented as:

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 10 \\ 7 & 12 & 20 \end{bmatrix}$$

As you will see later in the chapter, the information is now in a form where it can be used as the basis of various mathematical calculations.

Order of a matrix

The *order* (or *size* or *dimensions*) of a matrix is determined by the number of rows and columns in the matrix, in that order. For example, the matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$ has 2 rows and 3 columns.

Such a matrix is described as a 'two-by-three' matrix or a 2×3 matrix.

A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Two matrices are said to have the same order (or size) if they have the same number of rows and columns.

For example, $\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{bmatrix}$ are two matrices with the same order.

They are both 2×3 matrices.

The order or size of a matrix that has m rows and n columns is $m \times n$.

Elements of a matrix

The numbers within a matrix are known as its *elements*.

For example, $\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$ is a matrix with six elements.

Each matrix element can be described according to its position in the matrix, in terms of its row and column.

The symbol a_{ij} represents the element in the i th row and the j th column of matrix $\mathbf{A}$.

For example, a general 2×3 matrix is $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

So for the matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$, a_{23} is the element in the *second* row and *third* column, which is 12.

Note: The '23' is read as 'two, three', not 'twenty-three'.

a_{ij} represents the element in the i th row and the j th column of matrix $\mathbf{A}$.

1 Order and elements of a matrix

Answer the following questions for the matrix $\mathbf{A} = \begin{bmatrix} 16 & 6 & 12 \\ 6 & 7 & 5 \end{bmatrix}$.

(a) Determine the order of $\mathbf{A}$.

THINKING	WORKING
Count the number of rows and columns and express the order in the form rows × columns.	$\mathbf{A}$ has 2 rows and 3 columns, so $\mathbf{A}$ is a 2×3 matrix.

(b) Determine the value of a_{13}.

For element a_{ij}, the value of i corresponds to the row position and the value of j corresponds to the column position.	a_{13} is the element in the first row and the third column. $a_{13} = 12$

(c) Which element of matrix **A** is the number 7?

Identify the row that the element is in and then identify the column that the element is in.	The number 7 is in the second row and the second column. 7 is a_{22}.

2 Representing data in matrix form

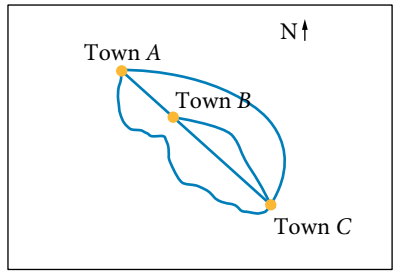
Hannah keeps a record of her LOTE test results. Her first four test results in French are 85%, 76%, 88% and 71%. Her first four test results in German are 73%, 68%, 76% and 73%. Express Hannah's test results in the form of a 2×4 matrix **A**.

THINKING	WORKING
The first number in a matrix order corresponds to the number of rows, the second number corresponds to the number of columns.	A 2×4 matrix means 2 rows and 4 columns. $\mathbf{A} = \begin{bmatrix} 85 & 76 & 88 & 71 \\ 73 & 68 & 76 & 73 \end{bmatrix}$

3 Creating an adjacency matrix from a diagram

Consider the map showing the number of roads connecting three towns, *A*, *B* and *C*.

Construct a matrix **R** that shows the number of direct roads connecting each town.



THINKING	WORKING
1 A matrix depicting the connecting roads between towns is a square matrix with the names of each town on both the horizontal and vertical headers.	$\mathbf{R} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$

2 Count the roads directly connecting one town to another (without travelling through another town) and fill in the matrix with the correct numerical values.

$$\mathbf{R} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \end{array}$$

Some special types of matrices

Some special types of matrices are defined below.

Special matrix	Definition	Example
Row matrix	A matrix that has only one row is known as a *row matrix*. In general, a row matrix is a $1 \times n$ matrix.	$\mathbf{A} = \begin{bmatrix} 4 & -1 & 8 \end{bmatrix}$
Column matrix	A matrix that has only one column is known as a *column matrix*. In general, a column matrix is an $m \times 1$ matrix.	$\mathbf{B} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$
Square matrix	A *square matrix* is a matrix with the same number of rows and columns. In general, a square matrix is an $n \times n$ matrix.	$\mathbf{C} = \begin{bmatrix} 4 & 7 & -2 \\ 3 & 4 & 8 \\ 5 & 1 & -6 \end{bmatrix}$
Equal matrices	Two matrices are equal if they have the same order and their corresponding elements are equal. If $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and $\mathbf{A} = \mathbf{B}$, then $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$ and $a_{22} = b_{22}$.	The two matrices below are equal. $\begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$ The two matrices below have the same order, however, the corresponding elements are different. They are not equal. $\begin{bmatrix} 16 & 9 \\ 4 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$
Zero matrix	A *zero matrix* $\mathbf{O}$ is a matrix where each element is 0. Adding or subtracting a zero matrix does not change the original matrix. Subtracting a matrix from itself gives a zero matrix.	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
Identity matrix	An *identity matrix* $\mathbf{I}$ is a square matrix made up of zeros and ones. Ones make up the elements of the main diagonal (running from top left to bottom right), while all remaining elements are zeroes. The identity matrix has some special properties, which you will study later in this chapter.	$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4 Equal matrices

Consider the following two matrices, $\mathbf{A} = \begin{bmatrix} 4u & v \\ 7 & w+4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -4 & 0 \\ -x & 1 \end{bmatrix}$. If $\mathbf{A} = \mathbf{B}$, calculate the values of u, v, w and x.

THINKING	WORKING
1 Identify and equate the corresponding elements from both matrices to form and then solve equations.	$4u = -4$ $u = -1$ $v = 0$ $7 = -x$ $x = -7$ $w + 4 = 1$ $w = -3$
2 Interpret the answer.	If $\mathbf{A} = \mathbf{B}$, then $u = -1$, $v = 0$, $x = -7$ and $w = -3$.

Matrix addition and subtraction

Matrices can be added and subtracted. To add and subtract matrices, the order of each matrix must be the same. For addition, add the corresponding elements. For subtraction, subtract the corresponding elements.

In general, if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$.

In general, if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\mathbf{A} - \mathbf{B} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$.

Like whole number arithmetic, matrix addition is commutative (in other words, $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$), but matrix subtraction is not. Consider the following examples:

Let $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 8 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$.

Matrix addition

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 4 & 1 \\ 2 & 6 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+8 & 1+(-2) \\ 2+1 & 6+0 \\ 3+1 & -4+2 \end{bmatrix} = \begin{bmatrix} 12 & -1 \\ 3 & 6 \\ 4 & -2 \end{bmatrix}$$

$$\mathbf{B} + \mathbf{A} = \begin{bmatrix} 8 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 2 & 6 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 8+4 & -2+1 \\ 1+2 & 0+6 \\ 1+3 & 2+(-4) \end{bmatrix} = \begin{bmatrix} 12 & -1 \\ 3 & 6 \\ 4 & -2 \end{bmatrix}$$

This shows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Matrix subtraction

$$\mathbf{A}-\mathbf{B}=\begin{bmatrix}4 & 1\\ 2 & 6\\ 3 & -4\end{bmatrix}-\begin{bmatrix}8 & -2\\ 1 & 0\\ 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4-8 & 1-(-2)\\ 2-1 & 6-0\\ 3-1 & -4-2\end{bmatrix}$$

$$=\begin{bmatrix}-4 & 3\\ 1 & 6\\ 2 & -6\end{bmatrix}$$

$$\mathbf{B}-\mathbf{A}=\begin{bmatrix}8 & -2\\ 1 & 0\\ 1 & 2\end{bmatrix}-\begin{bmatrix}4 & 1\\ 2 & 6\\ 3 & -4\end{bmatrix}$$

$$=\begin{bmatrix}8-4 & -2-1\\ 1-2 & 0-6\\ 1-3 & 2-(-4)\end{bmatrix}$$

$$=\begin{bmatrix}4 & -3\\ -1 & -6\\ -2 & 6\end{bmatrix}$$

This shows that $\mathbf{B}-\mathbf{A}=-\mathbf{A}+\mathbf{B}$
$=-(\mathbf{A}-\mathbf{B})$

and $\mathbf{A}-\mathbf{B}\neq\mathbf{B}-\mathbf{A}$.

When adding or subtracting two matrices, the order of each matrix must be the same. Furthermore, it is not possible to add or subtract a matrix and a scalar.

5 Adding matrices

The sales of sofas (S), lounge chairs (L) and coffee tables (C) at a company's three furniture stores for the month of May is represented by the matrix $\mathbf{M}$.

$$\mathbf{M}=\begin{array}{c}\begin{matrix}S & L & C\end{matrix}\\ \begin{bmatrix}5 & 8 & 10\\ 7 & 12 & 20\\ 4 & 11 & 15\end{bmatrix}\end{array}\begin{matrix}\\ \text{Store 1}\\ \text{Store 2}\\ \text{Store 3}\end{matrix}$$

The corresponding sales at the three stores for June is represented by the matrix $\mathbf{J}$.

$$\mathbf{J}=\begin{array}{c}\begin{matrix}S & L & C\end{matrix}\\ \begin{bmatrix}7 & 6 & 9\\ 8 & 14 & 17\\ 3 & 11 & 13\end{bmatrix}\end{array}\begin{matrix}\\ \text{Store 1}\\ \text{Store 2}\\ \text{Store 3}\end{matrix}$$

(a) Calculate the sales of sofas, lounge chairs and coffee tables for the three furniture stores for the two months combined. Give your answer in matrix form.

THINKING

1 Add the corresponding elements of each matrix.

WORKING

$$\mathbf{M}+\mathbf{J}=\begin{bmatrix}5 & 8 & 10\\ 7 & 12 & 20\\ 4 & 11 & 15\end{bmatrix}+\begin{bmatrix}7 & 6 & 9\\ 8 & 14 & 17\\ 3 & 11 & 13\end{bmatrix}$$

$$=\begin{bmatrix}5+7 & 8+6 & 10+9\\ 7+8 & 12+14 & 20+17\\ 4+3 & 11+11 & 15+13\end{bmatrix}$$

$$=\begin{bmatrix}12 & 14 & 19\\ 15 & 26 & 37\\ 7 & 22 & 28\end{bmatrix}$$

2 Complete the matrix adding the labels.

$$\mathbf{M}+\mathbf{J}=\begin{matrix} \\ \text{Store 1} \\ \text{Store 2} \\ \text{Store 3} \end{matrix}\begin{matrix} \begin{matrix} S & L & C \end{matrix} \\ \begin{bmatrix} 12 & 14 & 19 \\ 15 & 26 & 37 \\ 7 & 22 & 28 \end{bmatrix} \end{matrix}$$

(b) Interpret the matrix element $(\mathbf{M}+\mathbf{J})_{12}$.

Identify the row value and the column value.

$(\mathbf{M}+\mathbf{J})_{12}=14$

This indicates that over the two months of May and June, store 1 sold 14 lounge chairs.

6 Subtracting matrices

Consider the following two matrices, $\mathbf{A}=\begin{bmatrix} -3 & 2 & 5 \\ 6 & 1 & 3 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 1 & 4 & 7 \\ -2 & -6 & 0 \end{bmatrix}$.

(a) Calculate $\mathbf{A}-\mathbf{B}$.

THINKING

Subtract the corresponding elements.

WORKING

$$\begin{aligned}\mathbf{A}-\mathbf{B}&=\begin{bmatrix} -3 & 2 & 5 \\ 6 & 1 & 3 \end{bmatrix}-\begin{bmatrix} 1 & 4 & 7 \\ -2 & -6 & 0 \end{bmatrix}\\&=\begin{bmatrix} -3-1 & 2-4 & 5-7 \\ 6-(-2) & 1-(-6) & 3-0 \end{bmatrix}\\&=\begin{bmatrix} -4 & -2 & -2 \\ 8 & 7 & 3 \end{bmatrix}\end{aligned}$$

(b) Calculate $\mathbf{B}-\mathbf{A}$.

Subtract the corresponding elements.

$$\begin{aligned}\mathbf{B}-\mathbf{A}&=\begin{bmatrix} 1 & 4 & 7 \\ -2 & -6 & 0 \end{bmatrix}-\begin{bmatrix} -3 & 2 & 5 \\ 6 & 1 & 3 \end{bmatrix}\\&=\begin{bmatrix} 1-(-3) & 4-2 & 7-5 \\ -2-6 & -6-1 & 0-3 \end{bmatrix}\\&=\begin{bmatrix} 4 & 2 & 2 \\ -8 & -7 & -3 \end{bmatrix}\end{aligned}$$

Technology worked example
Adding and subtracting matrices

Scalar multiplication of matrices

When a matrix is multiplied by a real number, known as a *scalar*, each element in that matrix is multiplied by that scalar. This is known as *scalar multiplication*. Scalar multiplication does not change the order of the matrix.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $k\mathbf{A} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ where k is a scalar.

A scalar works like a scale factor to increase or decrease every element in that matrix by the same factor.

7 Scalar multiplication

Consider the two matrices, $\mathbf{A} = \begin{bmatrix} 4 & -8 \\ 2 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & -8 \end{bmatrix}$.

Calculate $3\mathbf{A} - 2\mathbf{B}$.

THINKING	WORKING
1 Write in expanded matrix form.	$3\mathbf{A} - 2\mathbf{B} = 3\begin{bmatrix} 4 & -8 \\ 2 & 6 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 3 & -8 \end{bmatrix}$
2 Multiply the elements in **A** by 3 and the elements in **B** by 2.	$3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 3\times 4 & 3\times -8 \\ 3\times 2 & 3\times 6 \end{bmatrix} - \begin{bmatrix} 2\times 1 & 2\times 0 \\ 2\times 3 & 2\times -8 \end{bmatrix}$ $= \begin{bmatrix} 12 & -24 \\ 6 & 18 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 6 & -16 \end{bmatrix}$
3 Subtract the elements in corresponding positions.	$3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 10 & -24 \\ 0 & 34 \end{bmatrix}$

Technology worked example
Scalar multiplication and matrix arithmetic

EXERCISE 8.1 Matrix properties and arithmetic

Worked Example 1

1 Consider matrix $\mathbf{A} = \begin{bmatrix} 9 & 3 \\ 4 & 1 \\ 8 & 2 \end{bmatrix}$.

(a) What is the order of matrix **A**?
(b) Determine the value of a_{22}.
(c) Which element of matrix **A** is the number that has a value of 2?

2

2 Anne keeps a record of her LOTE test results. Her first three test results in Indonesian were 68%, 71% and 79%. Her first three test results in Japanese were 59%, 63% and 70%. Express Anne's test results in the form of a 2×3 matrix **A**.

3 Supermarket 1 sells cartons of apple juice for \$4.90 each and cartons of orange juice for \$4.65 each. Supermarket 2 sells the same cartons of apple juice for \$5.15 each and the same cartons of orange juice for \$4.95 each. Construct a matrix to represent the cost of fruit juice at these two supermarkets, where the rows represent the supermarket.

4 Four students were surveyed to determine the number of males (M) and females (F) living in the family home. The results are shown in matrix **G**. Based on this information, which statement is *not* correct?

$$\mathbf{G} = \begin{matrix} \\ \text{Eva} \\ \text{Ray} \\ \text{Marc} \\ \text{Maria} \end{matrix} \begin{matrix} M & F \\ \end{matrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}$$

A There is one male in Maria's family.

B There is an equal number of children in Eva's family and Ray's family.

C There are more males than females in all four families combined.

D There are three times as many males in Marc's family compared to Eva's family.

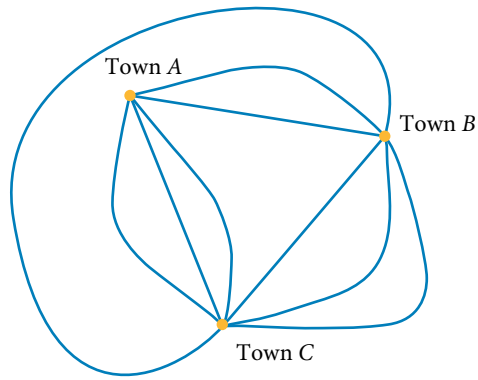
4

5 Consider the following matrices. If $\mathbf{A} = \mathbf{B}$, calculate the values of t, u, v and w.

$$\mathbf{A} = \begin{bmatrix} 2t & -2u \\ v+3 & 7 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 8 & -6 \\ 2 & w \end{bmatrix}$$

3

6 The diagram below shows the number of roads that connect three towns, A, B and C. Construct a 3×3 adjacency matrix to show the number of roads connecting each town to the others.



7 The following table shows average maximum daily temperatures in degrees Celsius for the cities of Stanthorpe, Toowoomba, Cairns and Warwick for the months of June, July and August.

	June	July	August
Stanthorpe	17.0	16.8	17.3
Toowoomba	21.5	20.6	21.1
Cairns	26.2	25.6	29.9
Warwick	19.6	19.4	20.1

(a) Express the above information in a matrix **T** of order 4×3. Give headings for the rows and columns.

(b) Which city has the highest average maximum daily temperature in July?

(c) Which city has the lowest average maximum daily temperature in August?

(d) What does t_{32} represent?

(e) According to matrix **T**, on average, how much warmer is Warwick in August compared with June?

8 Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ -2 & -1 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 5 & -9 & 6 \\ 7 & 2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$.

Which of the following operations is *not* possible?

A $\mathbf{D} - \mathbf{B}$ B $\mathbf{C} + 3\mathbf{E}$ C $3\mathbf{A} - 0.5\mathbf{A}$ D $\mathbf{A} + 2\mathbf{C}$

Worked Example

9 The sales of party hats (P), streamers (S) and cake toppers (C) at three convenience stores for the month of July is represented by the matrix **J**.

The corresponding sales at the three convenience stores for November are represented by the matrix **N**.

$$\mathbf{J} = \begin{matrix} \\ \text{Store 1} \\ \text{Store 2} \\ \text{Store 3} \end{matrix} \begin{matrix} P & S & C \\ \end{matrix} \begin{bmatrix} 100 & 160 & 80 \\ 140 & 240 & 65 \\ 175 & 220 & 90 \end{bmatrix}$$

$$\mathbf{N} = \begin{matrix} \\ \text{Store 1} \\ \text{Store 2} \\ \text{Store 3} \end{matrix} \begin{matrix} P & S & C \\ \end{matrix} \begin{bmatrix} 75 & 125 & 80 \\ 125 & 160 & 55 \\ 135 & 150 & 95 \end{bmatrix}$$

5

(a) Determine the combined sales of party hats, streamers and cake toppers for the three convenience stores over the two months. Give your answer in matrix form.

(b) Determine the combined sales of streamers for store 2 over the two months.

(c) Determine the combined sales of cake toppers for store 1 and store 2 over the two months.

10 Let $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 2 & 5 \\ -1 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -5 & 0 \\ 3 & 6 \\ -4 & 1 \end{bmatrix}$.

6

(a) Calculate $\mathbf{A} - \mathbf{B}$. (b) Calculate $\mathbf{B} - \mathbf{A}$.

11 Let $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 5 & -4 \\ 2 & 10 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4 & 1 \\ -2 & -6 \end{bmatrix}$.

(a) Calculate $\mathbf{A} + \mathbf{B}$. (b) Calculate $\mathbf{B} + \mathbf{A}$. (c) Calculate $\mathbf{B} + \mathbf{C}$. (d) Calculate $\mathbf{C} + \mathbf{B} + \mathbf{A}$.

12 Let $\mathbf{A} = \begin{bmatrix} -1 & 4 \\ 3 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ 0 & -5 \end{bmatrix}$. Calculate the following.

7

(a) $-2\mathbf{A}$ (b) $4\mathbf{B}$ (c) $3\mathbf{A} - 2\mathbf{B}$

(d) $\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}$ (e) $\frac{1}{2}(\mathbf{A} + \mathbf{B})$ (f) $3\mathbf{B} - 6\mathbf{A}$

13 Let $\mathbf{A} = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 1 \\ 0 & -1 & -3 \end{bmatrix}$. Calculate the following.

(a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{B} - \mathbf{A}$ (c) $-(3\mathbf{A} - 2\mathbf{B})$

14 The monthly sales figures for juice and salads at two cafes for March and August are shown in the table below.

	March		August	
	Juice	Salads	Juice	Salads
Cafe 1	80	180	240	120
Cafe 2	160	200	300	80

(a) Construct matrix **M** to represent the March sales and matrix **A** to represent the August sales.

(b) Determine **M** + **A** and what it represents.

(c) Which cafe had the better sales figures for each of the items for the two months combined?

(d) If sales in December for both cafes increased by 20% from August's sales figures, determine a matrix **D** that shows December's sales figures for both cafes.

(e) Construct a matrix that shows the total sales for March, August and December.

15 Determine the values of a and b in the matrix equation below:

$$\begin{bmatrix} 10 & 5 \\ 6 & 1 \end{bmatrix} + 2\begin{bmatrix} (a+b) & 2 \\ 1 & (b-a) \end{bmatrix} = \begin{bmatrix} 20 & 9 \\ 8 & 3 \end{bmatrix}$$

16 Determine the values of all unknowns in the following matrix equation:

$$\begin{bmatrix} 10 & d^2 \\ 4b & 1 \end{bmatrix} - 3\begin{bmatrix} 4 & d \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} a & 10 \\ b & 2c \end{bmatrix}$$

Matrix multiplication

8.2

The previous section showed that scalar multiplication involves the multiplication of a matrix by a scalar (a real number). The effect of scalar multiplication is that each element in the matrix is multiplied by that scalar and this does not change the order of the matrix.

Matrix multiplication is different from scalar multiplication. It involves the multiplication of a matrix by another matrix, and the rules for performing this calculation are very different.

Consider the following scenario. An organic berry company produces two different berry packages consisting of strawberries, blackberries and raspberries. Each strawberry is sold for \$0.85, each blackberry is sold for \$0.50 and each raspberry is sold for \$0.65.

Package 1 contains 4 strawberries, 5 blackberries and 4 raspberries.

Package 2 contains 6 strawberries, 3 blackberries and 6 raspberries.

Given this information, you can calculate the total selling price of each package.

Price for package 1: $(4 \times 0.85) + (5 \times 0.50) + (4 \times 0.65) = \$8.50$

Price for package 2: $(6 \times 0.85) + (3 \times 0.50) + (6 \times 0.65) = \$10.50$

This can easily be calculated with matrices, where your aim is to calculate the cost of each package.

The resultant, or product, matrix will be in the form:

$$\begin{matrix} \$ \\ \begin{bmatrix} & \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix}$$

The two berry packages can be represented by a 2×3 matrix **N**.

$$\mathbf{N} = \begin{matrix} \begin{matrix} \text{S} & \text{B} & \text{R} \end{matrix} \\ \begin{bmatrix} 4 & 5 & 4 \\ 6 & 3 & 6 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix}$$

The cost for each type of berry can be represented by the column matrix **S**.

$$\mathbf{S} = \begin{matrix} \$ \\ \begin{bmatrix} 0.85 \\ 0.50 \\ 0.65 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{S} \\ \text{B} \\ \text{R} \end{matrix}$$

Note that the number of columns in the initial matrix **N** is equal to the number of rows in the second matrix **S**.

Observe the steps to arrange the data to multiply the matrices.

STEP 1: Multiplying the elements in row 1 of **N** by the elements in **S** gives the cost for package 1.

As you move across row 1 of matrix **N**, move down the column of matrix **S**, forming each product as you go. Then add each product to obtain the cost of package 1 ($8.50). This forms the element in the first row and first column of the matrix (**N** × **S**).

$$\mathbf{N} \times \mathbf{S} = \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \text{S} & \text{B} & \text{R} \end{matrix} \\ \begin{bmatrix} 4 & 5 & 4 \\ \square & \square & \square \end{bmatrix} \end{matrix} \times \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} 0.85 \\ 0.50 \\ 0.65 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{S} \\ \text{B} \\ \text{R} \end{matrix}$$

$$= \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} (4 \times 0.85) + (5 \times 0.50) + (4 \times 0.65) \\ \square \end{bmatrix} \end{matrix}$$

$$= \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} 8.50 \\ \square \end{bmatrix} \end{matrix}$$

STEP 2: Multiplying the elements in row 2 of **N** by the elements in **S** gives the cost for package 2.

Move across row 2 of matrix **N** and down the column of matrix **S**, forming each product as you go. Then add each product to obtain the cost of package 2 ($10.50). This forms the element in the second row and first column of the matrix (**N** × **S**).

$$\mathbf{N} \times \mathbf{S} = \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \\ \begin{bmatrix} \square & \square & \square \\ 6 & 3 & 6 \end{bmatrix} \end{matrix} \times \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} 0.85 \\ 0.50 \\ 0.65 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{S} \\ \text{B} \\ \text{R} \end{matrix}$$

$$= \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} \square \\ (6 \times 0.85) + (3 \times 0.50) + (6 \times 0.65) \end{bmatrix} \end{matrix}$$

$$= \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} \square \\ 10.50 \end{bmatrix} \end{matrix}$$

In matrix form, the calculations can be summarised in the following way:

$$\mathbf{N} \times \mathbf{S} = \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \\ \begin{bmatrix} 4 & 5 & 4 \\ 6 & 3 & 6 \end{bmatrix} \end{matrix} \times \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} 0.85 \\ 0.50 \\ 0.65 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{S} \\ \text{B} \\ \text{R} \end{matrix}$$

$$= \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} (4 \times 0.85) + (5 \times 0.50) + (4 \times 0.65) \\ (6 \times 0.85) + (3 \times 0.50) + (6 \times 0.65) \end{bmatrix} \end{matrix}$$

$$= \begin{matrix} \\ \text{Package 1} \\ \text{Package 2} \end{matrix} \begin{matrix} \begin{matrix} \$ \end{matrix} \\ \begin{bmatrix} 8.50 \\ 10.50 \end{bmatrix} \end{matrix}$$

The cost of package 1 is $8.50.

The cost of package 2 is $10.50.

Rules for matrix multiplication

Matrix multiplication is only possible under certain conditions: the number of columns in the first matrix must equal the number of rows in the second matrix. Such matrices are said to be *conformable* for multiplication.

In general, if an $m \times n$ matrix is multiplied by an $n \times p$ matrix, the product matrix will be of order $m \times p$ (see right).

order of matrix product

$(m \times n) \times (n \times p) = m \times p$

must be the same

If the number of columns in the first matrix does not equal the number of rows in the second matrix, the matrix product cannot be found and the matrix product is said to be *undefined*.

You can use the order of matrices to determine the order of the resultant matrix. Using the matrices from the previous example, **N** and **S**, the number of columns (3) in matrix **N** is equal to the number of rows (3) in matrix **S**, meaning that the matrices can be multiplied in the order $\mathbf{N} \times \mathbf{S}$.

$$\mathbf{N} \times \mathbf{S} = \begin{bmatrix} 4 & 5 & 4 \\ 6 & 3 & 6 \end{bmatrix} \begin{bmatrix} 0.85 \\ 0.50 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 8.50 \\ 10.50 \end{bmatrix}$$

$2 \times 3 \quad 3 \times 1 \quad 2 \times 1$

The order of the resultant matrix is given by the number of rows in matrix **N** and by the number of columns in matrix **S**. That is, the resultant matrix has order 2×1.

For 2×2 matrices, if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\mathbf{A} \times \mathbf{B} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$.

To calculate the element $\mathbf{AB}_{ij}$ of the product **AB**, multiply the corresponding elements from row i of matrix **A** and column j of matrix **B**, and then add together.

8 Determining matrix products

Let $\mathbf{A} = \begin{bmatrix} 12 & -3 \\ 7 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}$.

Determine whether or not the following matrix products can be found. Where the matrix product can be found, determine the order of the matrix product.

(a) BA

THINKING	WORKING
1 Determine the order of each matrix.	**B** is a 2×2 matrix and **A** is a 2×2 matrix.
2 Determine the number of columns in the first matrix and the number of rows in the second matrix.	**B** has 2 columns and **A** has 2 rows. The matrix product **BA** can be found. **BA** is a 2×2 matrix.

(b) AC

1 Determine the order of each matrix.	**A** is a 2×2 matrix and **C** is a 2×1 matrix.
2 Determine the number of columns in the first matrix and the number of rows in the second matrix.	**A** has 2 columns and **C** has 2 rows. The matrix product **AC** can be found. **AC** is a 2×1 matrix.

(c) BD

1 Determine the order of each matrix.	**B** is a 2×2 matrix and **D** is a 3×1 matrix.
2 Determine the number of columns in the first matrix and the number of rows in the second matrix.	**B** has 2 columns and **D** has 3 rows. The matrix product **BD** *cannot* be found. **BD** is undefined.

(d) CD

1 Determine the order of each matrix.	**C** is a 2×1 matrix and **D** is a 3×1 matrix.
2 Determine the number of columns in the first matrix and the number of rows in the second matrix.	**C** has 1 column and **D** has 3 rows. The matrix product **CD** *cannot* be found. **CD** is undefined.

9 Matrix multiplication

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 6 & 2 \\ 1 & -2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Determine the following matrix products (if they exist).

(a) AB

THINKING	WORKING
1 Determine the order of each matrix.	**A** is a 2×2 matrix and **B** is a 2×2 matrix.
2 Determine the number of columns in the first matrix and the number of rows in the second matrix.	**A** has 2 columns and **B** has 2 rows, therefore the matrix exists. The matrix product **AB** is a 2×2 matrix.

3 Move across each row of the first matrix and down each column of the second matrix. Form each product and add.

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} (1\times 6)+(2\times 1) & (1\times 2)+(2\times -2) \\ (4\times 6)+(-3\times 1) & (4\times 2)+(-3\times -2) \end{bmatrix}$$
$$= \begin{bmatrix} 6+2 & 2-4 \\ 24-3 & 8+6 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & -2 \\ 21 & 14 \end{bmatrix}$$

(b) BA

1 Determine the order of each matrix.

B is a 2×2 matrix and **A** is a 2×2 matrix.

2 Determine the number of columns in the first matrix and the number of rows in the second matrix.

B has 2 columns and **A** has 2 rows, therefore the matrix exists. The matrix product **BA** is a 2×2 matrix.

3 Move across each row of the first matrix and down each column of the second matrix. Form each product and add the results.

$$\mathbf{BA} = \begin{bmatrix} 6 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} (6\times 1)+(2\times 4) & (6\times 2)+(2\times -3) \\ (1\times 1)+(-2\times 4) & (1\times 2)+(-2\times -3) \end{bmatrix}$$
$$= \begin{bmatrix} 6+8 & 12-6 \\ 1-8 & 2+6 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ -7 & 8 \end{bmatrix}$$

(c) DC

1 Determine the order of each matrix.

D is a 3×2 matrix and **C** is a 2×1 matrix.

2 Determine the number of columns in the first matrix and the number of rows in the second matrix.

D has 2 columns and **C** has 2 rows, therefore the matrix exists. The matrix product **DC** is a 3×1 matrix.

3 Move across each row of the first matrix and down each column of the second matrix. Form each product and add.

$$\mathbf{DC} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} (2\times 1)+(3\times 5) \\ (1\times 1)+(0\times 5) \\ (0\times 1)+(4\times 5) \end{bmatrix} = \begin{bmatrix} 17 \\ 1 \\ 20 \end{bmatrix}$$

Technology worked example
Multiplying matrices

In normal arithmetic, multiplication is commutative (order is not important); i.e. $a \times b = b \times a$.

Matrix multiplication must be performed in the required order. In general, matrix multiplication is not commutative.

AB ≠ BA

In Worked Example 9, you saw that $\mathbf{AB} = \begin{bmatrix} 8 & -2 \\ 21 & 14 \end{bmatrix}$ and $\mathbf{BA} = \begin{bmatrix} 14 & 6 \\ -7 & 8 \end{bmatrix}$.

That is, **AB** ≠ **BA**.

Applications of matrix multiplication

Matrix multiplication can be used to calculate overall results in cases where the total has been split into several weighted parts, as you can see in the following example.

10 Matrix multiplication to calculate a semester result

Larry scores the following marks (out of 100) for his assessment components in Chemistry: topic tests 80; practical work 86; semester examination 70.

The final semester result for Chemistry is weighted as follows: topic tests 30%; practical work 30%; semester examination 40%.

Use matrix multiplication to calculate Larry's final semester result (out of 100) for Chemistry.

THINKING	WORKING
1 Represent the marks for each assessment component as a 1×3 matrix.	Let $\mathbf{A} = \begin{bmatrix} 80 & 86 & 70 \end{bmatrix}$.
2 Represent the weighting of each assessment component as a 3×1 matrix.	Let $\mathbf{W} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$.
3 Calculate **AW**.	**A** is a 1×3 matrix and **W** is a 3×1 matrix. So the matrix product **AW** is a 1×1 matrix. $\mathbf{AW} = \begin{bmatrix} 80 & 86 & 70 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$ $= [77.8]$
4 Give the final answer. The result needs to be interpreted.	Larry's final semester result for Chemistry was 77.8 out of 100. *Note*: [77.8] is not the same as 77.8. [77.8] is a *matrix* with one element in one row and one column, 77.8 is the *element* of the matrix in this row and column.

Matrix multiplication can be used in scenarios where different product types have different costs or attributes.

11 Matrix multiplication to calculate construction costs

A manufacturer makes three products A, B and C, each requiring different amounts of construction time at four different stages P, Q, R and S. Matrix $\mathbf{X}$ shows the construction time, in hours, required to make one of each product.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 1.5 \\ 0.5 & 1 & 2 & 1 \\ 0 & 1.5 & 3 & 0 \end{bmatrix} \end{array}$$

Each stage of construction, P, Q, R and S, costs the manufacturer \$100, \$50, \$80 and \$250 per hour respectively.

(a) Express each stage of hourly construction costs as a column matrix $\mathbf{Y}$.

THINKING	WORKING
Represent each stage of hourly construction costs as a 4×1 matrix. Representing this as a 4×1 matrix rather than a 1×4 matrix will allow for multiplication by $\mathbf{X}$.	$\mathbf{Y} = \begin{bmatrix} 100 \\ 50 \\ 80 \\ 250 \end{bmatrix}$

(b) Calculate $\mathbf{XY}$ and determine what information this product represents.

1 Calculate $\mathbf{XY}$.

$\mathbf{X}$ is a 3×4 matrix and $\mathbf{Y}$ is a 4×1 matrix.

So the matrix product $\mathbf{XY}$ is a 3×1 matrix.

$$\mathbf{XY} = \begin{bmatrix} 1 & 2 & 0 & 1.5 \\ 0.5 & 1 & 2 & 1 \\ 0 & 1.5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \\ 80 \\ 250 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \times 100) + (2 \times 50) + (0 \times 80) + (1.5 \times 250) \\ (0.5 \times 100) + (1 \times 50) + (2 \times 80) + (1 \times 250) \\ (0 \times 100) + (1.5 \times 50) + (3 \times 80) + (0 \times 250) \end{bmatrix}$$

$$= \begin{bmatrix} 575 \\ 510 \\ 315 \end{bmatrix}$$

2 Interpret the information in $\mathbf{XY}$.

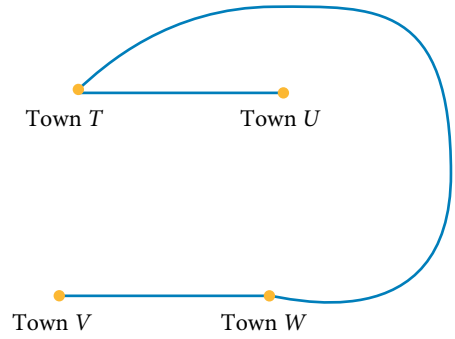
Product A costs \$575, product B costs \$510 and product C costs \$315 to manufacture.

Powers of matrices to analyse a transport network

In the previous section, matrices were used to indicate the number of direct roads or paths from one town to another. Consider a hypothetical road network involving four towns, T, U, V and W. Let matrix $\mathbf{C}$ represent the number of direct connections between the towns.

$$\mathbf{C} = \begin{array}{c} \begin{matrix} T & U & V & W \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} T \\ U \\ V \\ W \end{matrix}$$

From this matrix, a diagram of the roads between each town can be constructed.



From the diagram, you can see that there is a total of three roads connecting the towns. You can also determine the number of connections by adding all the elements in matrix $\mathbf{C}$ and dividing by 2. The division by 2 is required because each road has been counted twice. So from matrix $\mathbf{C}$, you can see that the number of roads equals $\frac{2+1+1+2}{2} = 3$.

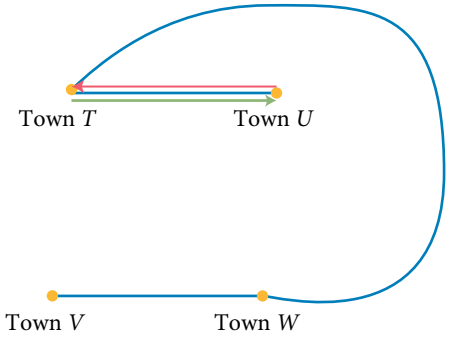
Now consider what happens when you square matrix $\mathbf{C}$. (For higher-order matrices such as this, it is often more convenient to use technology to carry out these operations):

$$\mathbf{C}^2 = \begin{array}{c} \begin{matrix} T & U & V & W \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \times \begin{array}{c} \begin{matrix} T & U & V & W \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} T \\ U \\ V \\ W \end{matrix}$$

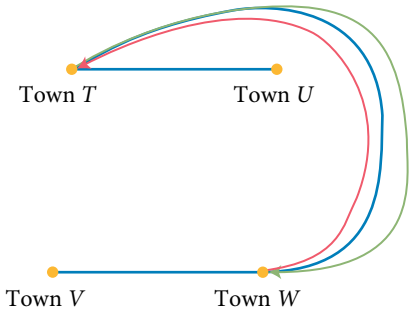
$$\mathbf{C}^2 = \begin{array}{c} \begin{matrix} T & U & V & W \end{matrix} \\ \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} \begin{matrix} T \\ U \\ V \\ W \end{matrix}$$

The elements in the $\mathbf{C}^2$ matrix represent? how many ways you can travel from one town to another town via one other town, as illustrated below.

The element $c_{11} = 2$ of matrix $\mathbf{C}^2$ informs you that there are two ways to start and finish at town T, passing through one other town. The first way is to start at town T, travel to town U and then travel back to town T, i.e. follow the path TUT.

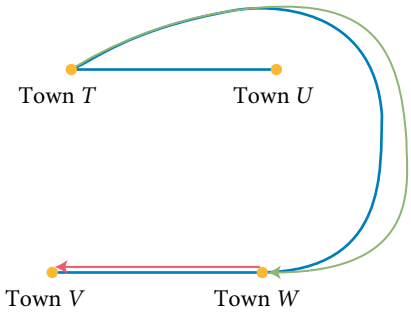


The second way is to start at town T, travel to town W and then travel back to town T, i.e. follow the path TWT.



The element $c_{12} = 0$ of matrix $\mathbf{C}^2$ informs you that it is not possible to travel from town T to town U via one other town.

The element $c_{13} = 1$ of matrix $\mathbf{C}^2$ informs you that there is one way to travel from town T to town V via another town, namely via town W, i.e. follow the path TWV.



The element $c_{14} = 0$ of matrix $\mathbf{C}^2$ informs you that it is not possible to travel from town T to town W via another town.

Extending the analysis to the remaining rows of matrix $\mathbf{C}^2$, you can conclude that calculating the second power of matrix $\mathbf{C}$, i.e. $\mathbf{C}^2$, represents the number of direct road links between two towns via another (or third) town.

The matrix $\mathbf{C}^3$ will tell you the number of ways you can travel from one town to another via two other towns.

What do the elements in the first row of matrix $\mathbf{C}^3$ show?

The element $c_{11} = 0$ of matrix $\mathbf{C}^3$ informs you that it is not possible to start and finish at town T via two other towns.

The element $c_{12} = 2$ of matrix $\mathbf{C}^3$ informs you that there are two ways to travel from town T to town U via two other towns. The first way is to start at town T, travel to town U, then travel back to town T and finally travel back to town U, i.e. follow the path $TUTU$. The second way is to start at town T, travel to town W, then travel back to town T and then travel to town U, i.e. follow the path $TWTU$.

$$\mathbf{C}^3 = \mathbf{C}^2 \times \mathbf{C}^1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{matrix} T \\ U \\ V \\ W \end{matrix} \begin{matrix} \begin{bmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} \end{matrix} \begin{matrix} T \\ U \\ V \\ W \end{matrix}$$

The element $c_{13} = 0$ of matrix $\mathbf{C}^3$ informs you that it is not possible to travel from town T to town V via two other towns.

The element $c_{14} = 3$ of matrix $\mathbf{C}^3$ informs you that there are three ways to travel from town T to town W via two other towns, i.e. follow the path $TUTW$ or $TWVW$ or $TWTW$.

Extending the analysis to the remaining rows of matrix $\mathbf{C}^3$, you can conclude that matrix $\mathbf{C}^3$ represents the number of direct road links between two towns via two other towns.

Matrix **C** is an example of a matrix known as an *incidence matrix*. The element in row i and column j is 1 if i and j are related (incident) and 0 if they are not.

Matrix **C** represents the number of direct links between two objects, e.g. towns.

In general, when considering networks, the matrix **C** raised to the power of n represents the number of links between two objects via $n - 1$ other objects.

WARNING

If $\mathbf{C} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, $\mathbf{C}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} aa + bc & ab + bd \\ ca + dc & cb + dd \end{bmatrix}$

Note: This is not equal to simply squaring each element.

12 Using matrices to analyse a transport network

The matrix **C** displays direct air routes between Warwick (W), Toowoomba (T) and Cairns (C).

$$\mathbf{C} = \begin{array}{c} \begin{matrix} W & T & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} W \\ T \\ C \end{matrix}$$

(a) Calculate matrix $\mathbf{C}^2$.

THINKING

Multiply matrix **C** by itself.

WORKING

$$\mathbf{C}^2 = \begin{array}{c} \begin{matrix} W & T & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \begin{matrix} W & T & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} W \\ T \\ C \end{matrix}$$

$$= \begin{array}{c} \begin{matrix} W & T & C \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} W \\ T \\ C \end{matrix}$$

(b) What is represented by element c_{13} of matrix $\mathbf{C}^2$?

1. Look at element c_{13} of matrix $\mathbf{C}^2$.
2. Interpret the results and explain your reasoning.

$c_{13} = 1$

There is one air route with a single stopover between Warwick and Cairns. It flies from Warwick to Toowoomba and then from Toowoomba to Cairns.

EXERCISE 8.2 Matrix multiplication

Worked Example 8

1 For the matrices given, determine whether or not the following matrix products can be found.

$$\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 2 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

(a) **AD** (b) **BD** (c) **AC**

9

2 For the matrices given, determine the following matrix products (if they exist).

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 4 & 7 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} -2 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

(a) **QP** (b) **PQ** (c) **SR**

3 For the matrices given, use technology to calculate the following matrix products (if they exist).

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 1 & 1 \\ 3 & -1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

(a) $\mathbf{A}^2$ (b) **AB** (c) **BA**

4 Observe the order in matrices **A**, **B** and **C** and select the matrix product that will result in a matrix of order 2×2.

$$\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

A **AB** B **BA** C **AC** D **BC**

5 Calculate the following matrix products:

(a) $\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 5 & 1 \\ -3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 8 & 9 \\ 0 & 3 & 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 4 & 3 \\ -2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ -1 & -2 \\ 4 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} -1 & 0 & 4 \\ 3 & -1 & 0 \\ 6 & 2 & 0 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (h) $\begin{bmatrix} 2 \\ -5 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix}$

11

6 A manufacturer makes three products X, Y and Z, each requiring different amounts of construction time at stages A, B, C, D and E. Matrix $\mathbf{M}$ shows the different amounts of construction time, in hours, required to make one of each product. The stages of construction, A, B, C, D and E, cost the manufacturer \$300, \$400, \$50, \$100 and \$200 per hour respectively.

$$\mathbf{M} = \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 3 & 0 & 3 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} X \\ Y \\ Z \end{matrix}$$

(a) Express each stage of hourly construction costs as a column matrix $\mathbf{R}$.

(b) Calculate matrix $\mathbf{MR}$ and determine what information this product represents.

7 Apples cost \$5.50 per kilogram, bananas cost \$5.30 per kilogram and pears cost \$6.10 per kilogram. Charlie buys 5 kg of apples, 3.5 kg of bananas and 2 kg of pears. Construct appropriate matrices and a matrix equation to determine the total cost of the fruit Charlie bought.

10

8 Liz scored the following marks for her assessment components (out of 100) in Biology: topic tests 76; practical work 90; semester examination 72. The final semester result for Biology is weighted as follows: topic tests 20%; practical work 30%; semester examination 50%. Use a matrix multiplication method to calculate Liz's final semester result (out of 100) for Biology. Give your answer correct to the nearest whole number.

12

9 The matrix $\mathbf{C}$ on the right displays direct air routes between Longreach, Mt Isa and Townsville.

$$\mathbf{C} = \begin{array}{c} \begin{matrix} L & M & T \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} L \\ M \\ T \end{matrix}$$

(a) Calculate matrix $\mathbf{C}^2$.

(b) What does matrix $\mathbf{C}^2$ represent?

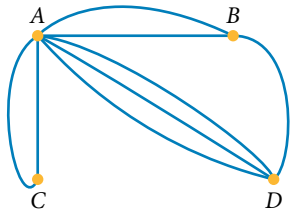
(c) Describe the one-stopover route from Longreach to Townsville.

10 When squared, the incidence matrix representing the number of direct flights between town A, town B, town C and town D is represented by the matrix $\mathbf{C}^2$.

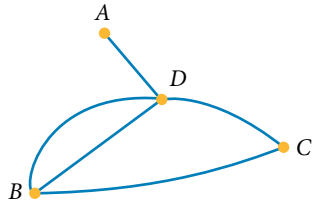
$$\mathbf{C}^2 = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 6 \end{bmatrix}$$

(a) Based on this information, which of the following diagrams best represents the direct flight connections between the four towns?

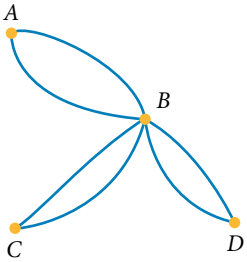
A



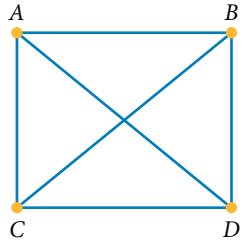
B



C



D



(b) Explain why the other answers are incorrect.

11 Gary, Patricia and Winston repair electrical equipment. The matrix $\mathbf{T}$ shows the time it takes (in minutes) for each of Gary (G), Patricia (P) and Winston (W) to repair a microwave (M), a DVD player (D) and a CD player (C).

$$\mathbf{T} = \begin{array}{c} \begin{matrix} G & P & W \end{matrix} \\ \begin{bmatrix} 20 & 25 & 15 \\ 40 & 45 & 30 \\ 60 & 70 & 70 \end{bmatrix} \end{array} \begin{matrix} M \\ D \\ C \end{matrix}$$

The matrix **E** shows the number of microwaves, DVD players and CD players repaired in three suburbs, Birkdale (*B*), Ferny Grove (*F*), and Northgate (*N*) over a period of a week.
Construct a matrix that displays the total time (in minutes) it would take for each of Gary, Patricia and Winston to repair the electrical equipment in Birkdale (*B*), Ferny Grove (*F*) and Northgate (*N*).

$$\mathbf{E} = \begin{array}{c} \\ B \\ F \\ N \end{array} \begin{array}{c} \begin{array}{ccc} M & D & C \end{array} \\ \begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 5 \\ 1 & 4 & 3 \end{bmatrix} \end{array}$$

12 A gift shop sells bathroom gift packs for Father's Day. Gift pack 1 (*GP1*) contains two bottles of talcum powder, one roll-on deodorant and two bottles of shower gel. Gift pack 2 (*GP2*) contains one bottle of talcum powder, two roll-on deodorants and one bottle of shower gel. Gift pack 3 (*GP3*) contains two roll-on deodorants and two bottles of shower gel.

(a) Express this information in a matrix **G** where each row represents a different gift pack.

(b) Individually, each bottle of talcum powder (*T*) sells for $12, each roll-on deodorant (*D*) sells for $7 and each bottle of shower gel (*S*) sells for $15. Represent these individual sales prices in a 3×1 matrix **S**.

(c) Ignoring packaging costs and potential discounts, determine the following.

(i) **GS** **(ii)** the selling price for each bathroom gift pack

(d) As a goodwill gesture, a local business orders a total of nine gift packs, one for each of its male employees. The breakdown of the order is: three of gift pack 1, two of gift pack 2 and four of gift pack 3. Use a matrix method to determine the total cost of the order.

13 An ice-cream shop sells ice-cream in three different sizes. A single-scoop (*SS*) sells for $3, a double-scoop (*DS*) sells for $5 and a mega-scoop (*MS*) sells for $6.

(a) Express these prices in a 3×1 matrix **S**.

(b) In a single day, the shop sells 80 single-scoops, 50 double-scoops and 30 mega-scoops. Express these sales in a 1×3 matrix **N**.

(c) The product of the two earlier matrices is **NS**. Calculate **NS** and interpret the results.

(d) The table below displays the number of single-scoop (*SS*), double-scoop (*DS*) and mega-scoop (*MS*) ice-creams sold at four ice-cream shops in a single day. Express this information in a 4×3 matrix **T**.

	Shop A	Shop B	Shop C	Shop D
Single-scoop (*SS*)	50	30	100	80
Double-scoop (*DS*)	40	90	60	20
Mega-scoop (*MS*)	0	10	20	15

(e) The product of the two earlier matrices is **TS**. Calculate **TS** and interpret the result.

(f) Which ice-cream shop made the most money from sales?

(g) In one week's time, the selling price of each type of ice-cream is due to rise by 50c per ice-cream. Determine the new selling price matrix **S**.

(h) Calculate how much each ice-cream shop will make on sales of ice-cream in a single day if they sell the same amount in one week's time.

(i) Calculate the increase in the amount of money each ice-cream shop will make on sales in a single day if they sell the same amount of ice-cream in one week's time.

14 If $\mathbf{A} = \begin{bmatrix} (x-1) & 1 \\ -2 & (x+1) \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} x \\ y \end{bmatrix}$, determine the values of x and y that would satisfy the equation $\mathbf{A}^2\mathbf{B} = -2\mathbf{C}$. Use technology to verify that your solution is correct.

8.3 Inverse matrices

The process for dividing by a matrix is not as straightforward as the process of multiplying two matrices. It is worth remembering how the multiplication and division operations work for scalars.

Consider the simple linear equation $5x = 10$. To solve this equation, you need to divide both sides of the equation by 5. Another way to think about this is to multiply both sides of the equation by $\frac{1}{5}$, which is the *multiplicative inverse* of 5. The equation then becomes:

$$\begin{aligned} 5x &= 10 \\ \frac{1}{5} \times 5x &= \frac{1}{5} \times 10 \\ 1x &= 2 \end{aligned}$$

Multiplying a number by its *multiplicative inverse* results in the *identity element* for multiplication—for scalars, this is 1. The identity element for multiplication is 1, because multiplying any scalar by 1 leaves the scalar unchanged.

Understanding this concept is important for understanding matrix division. Rather than dividing matrices, you can instead multiply by the *inverse* of a matrix.

The identity matrix

Before you attempt to calculate inverse matrices, you must first establish the identity element for matrix multiplication. This will depend on the order of the matrix.

The 2×2 identity matrix

Let $\mathbf{A} = \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore: $\mathbf{AI} = \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (-3 \times 1) + (5 \times 0) & (-3 \times 0) + (5 \times 1) \\ (-1 \times 1) + (2 \times 0) & (-1 \times 0) + (2 \times 1) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix} = \mathbf{A}$$

Notice $\mathbf{AI} = \mathbf{A}$.

Try in the reverse order: $\mathbf{IA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (1 \times -3) + (0 \times -1) & (1 \times 5) + (0 \times 2) \\ (0 \times -3) + (1 \times -1) & (0 \times 5) + (1 \times 2) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix} = \mathbf{A}$$

So $\mathbf{IA} = \mathbf{A}$.

Therefore, you can observe that the matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ functions as an identity matrix for 2×2 matrix multiplication.

The 3×3 identity matrix

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore: $\mathbf{AI} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \mathbf{A}$$

So $\mathbf{AI} = \mathbf{A}$

And similarly, following the same procedure as earlier $\mathbf{IA} = \mathbf{A}$.

In general, $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$.

This is an exception to 'the rule' that, in general for matrices, $\mathbf{AB} \neq \mathbf{BA}$.

This is the unique mathematical quality of the identity matrix.

The identity matrix **I**, is a square matrix with ones on the main diagonal and zero elements elsewhere:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where n is the order of the square matrix.

The inverse of a matrix

The inverse of matrix **A** is the matrix that, when multiplied by **A**, gives the identity matrix, **I**. In other words, $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$.

Consider a general 2×2 matrix, $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let the inverse matrix be $\mathbf{A}^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

From the definition of the inverse matrix: $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equate corresponding elements in the matrices to form these six equations.

$$ap + br = 1 \quad [1]$$
$$aq + bs = 0 \quad [2]$$
$$cp + dr = 0 \quad [3]$$
$$cq + ds = 1 \quad [4]$$

Multiply [1] by c and [3] by a to get a system of equations with a common term.

$$acp + bcr = c \quad [5]$$
$$acp + adr = 0 \quad [6]$$

Subtract [5] from [6].

$$adr - bcr = -c$$
$$r(ad - bc) = -c$$
$$r = \frac{-c}{(ad - bc)}$$

Multiply [2] by c and [4] by a to get a system of equations with a common term.

$$acq + bcs = 0 \quad [7]$$
$$acq + ads = a \quad [8]$$

Subtract [7] from [8].

$$ads - bcs = a$$
$$s(ad - bc) = a$$
$$s = \frac{a}{(ad - bc)}$$

Continue this process to get p and q.

$$p = \frac{d}{(ad - bc)} \text{ and } q = \frac{-b}{(ad - bc)}$$

Replace the values in the inverse matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{(ad - bc)} & \frac{-b}{(ad - bc)} \\ \frac{-c}{(ad - bc)} & \frac{a}{(ad - bc)} \end{bmatrix}$$
$$= \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $(ad - bc) \neq 0$

The value of $(ad - bc)$ is important as it indicates whether or not a matrix can have an inverse. If the value of $(ad - bc)$ is 0, then $\frac{1}{(ad - bc)}$ is undefined, and, therefore, the matrix is *singular* and has no inverse.

The inverse of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
The value $(ad - bc)$ is known as the *determinant* of matrix $\mathbf{A}$ and is written det $\mathbf{A}$ or $|\mathbf{A}|$.

13 Calculating the determinant and inverse of a matrix

(a) Let $\mathbf{A} = \begin{bmatrix} -4 & -4 \\ 3 & 1 \end{bmatrix}$. Calculate $\mathbf{A}^{-1}$, if it exists.

THINKING	WORKING
1 Define the matrix.	$\mathbf{A} = \begin{bmatrix} -4 & -4 \\ 3 & 1 \end{bmatrix}$
2 Identify the values of a, b, c and d and calculate the determinant $(ad - bc)$.	$a = -4$, $b = -4$, $c = 3$ and $d = 1$ det $\mathbf{A} = ad - bc$ $= -4 \times 1 - (-4) \times 3$ $= -4 + 12$ $= 8$
3 If det $\mathbf{A} \neq 0$, then the inverse exists.	$\mathbf{A}^{-1}$ exists.
4 Determine the inverse matrix $\mathbf{A}^{-1}$ by swapping the elements on the main diagonal, change the sign of the elements on the secondary diagonal and divide by the determinant. $\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$	$\mathbf{A}^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 4 \\ -3 & -4 \end{bmatrix}$ $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{2} \end{bmatrix}$

(b) Let $\mathbf{B} = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix}$. Calculate $\mathbf{B}^{-1}$, if it exists.

1 Define the matrix.	$\mathbf{B} = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix}$
2 Identify the values of a, b, c and d and calculate the determinant	$a = 5$, $b = -10$, $c = 2$ and $d = -4$ $\det \mathbf{B} = ad - bc$ $= (5 \times -4) - (-10 \times 2)$ $= -20 - (-20)$ $= -20 + 20$ $= 0$
3 If $\det \mathbf{B} \neq 0$, then the inverse exists.	$\det \mathbf{B} = 0$, so the inverse does not exist.

Singular matrices

If $\det \mathbf{A} = 0$, then matrix $\mathbf{A}$ is singular and its inverse $\mathbf{A}^{-1}$ does not exist.

Technology worked example
Calculating the determinant and inverse of a 2×2 matrix

14 Calculating an unknown element of a singular matrix

Determine the value of p for which $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 6 & p \end{bmatrix}$ is a singular matrix.

THINKING	WORKING
1 Define the matrix.	$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 6 & p \end{bmatrix}$
2 Calculate $\det \mathbf{A}$.	$a = -2$, $b = 3$, $c = 6$ and $d = p$ $\det \mathbf{A} = ad - bc$ $= -2p - 18$
3 Determine the value of p such that $\det \mathbf{A} = 0$.	$\det \mathbf{A} = 0$ as $\mathbf{A}$ is a singular matrix. $(ad - bc) = 0$ $-2p - 18 = 0$ $-2p = 18$ $p = -9$

Inverses of higher order matrices

In this course, you are required to calculate the inverse of a 2×2 matrix manually, that is, without using technology. Inverses of higher order matrices can be calculated using technology.

Technology worked example
Calculating the determinant and inverse of larger matrices

Using inverse matrices to solve matrix equations

Consider again the simple equation at the beginning of this section: $5x = 10$. You calculate the value of x as shown on the right.

$$5x = 10$$
$$\frac{1}{5} \times 5x = \frac{1}{5} \times 10$$
$$x = 2$$

In this example, you multiply both sides of the equation by the multiplicative inverse of 5, i.e. $\frac{1}{5}$.

You can use the same technique to solve matrix equations such as the one shown on the right.

$$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}$$

However, you need to be careful about the order in which you are multiplying. Because matrix multiplication is not commutative, both sides of the equation must either be *pre-multiplied* or *post-multiplied* by the inverse.

The matrix equation $\mathbf{AX} = \mathbf{B}$ can be solved by pre-multiplying by the inverse of $\mathbf{A}$:

$$\mathbf{AX} = \mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

The matrix equation $\mathbf{XA} = \mathbf{B}$ can be solved by post-multiplying by the inverse of $\mathbf{A}$:

$$\mathbf{XA} = \mathbf{B}$$
$$\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$$
$$\mathbf{XI} = \mathbf{BA}^{-1}$$
$$\mathbf{X} = \mathbf{BA}^{-1}$$

WARNING

Remember that matrix multiplication is not commutative, so ensure that you multiply in the correct order.

15 Solving matrix equations with pre-multiplying

Determine the matrix $\mathbf{X}$ that satisfies the equation $\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}$.

THINKING

1 Determine the form of the equation.

2 Calculate the inverse matrix, $\mathbf{A}^{-1}$.

WORKING

The equation is in the form $\mathbf{AX} = \mathbf{B}$,

where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{ab - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{(1 \times 0) - (1 \times -2)} \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.5 \\ 1 & 0.5 \end{bmatrix}$$

3 Solve the equation.

The equation is in the form $\mathbf{AX} = \mathbf{B}$, hence you need to pre-multiply both sides of the equation by the inverse.

$$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -0.5 \\ 1 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 0 & -0.5 \\ 1 & 0.5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

16 Solving matrix equations with post-multiplying

Determine the matrix $\mathbf{X}$ that satisfies the equation $\mathbf{X}\begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -10 \end{bmatrix}$.

THINKING

1 Determine the form of the equation.

2 Calculate the inverse matrix, $\mathbf{A}^{-1}$.

3 Solve the equation.

WORKING

The equation is in the form $\mathbf{XA} = \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{(0 \times 4) - (1 \times -1)}\begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$$

The equation is in the form $\mathbf{XA} = \mathbf{B}$, so you need to post-multiply both sides of the equation by the inverse.

$$\mathbf{X}\begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -10 \end{bmatrix}$$

$$\mathbf{X}\begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -10 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathbf{X}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 5 & 1 \\ 2 & -2 \end{bmatrix}$$

Technology worked example
Solving matrix equations

Explore further
Encryption and decryption with matrices
Investigate how to encode and decode a message using inverse matrices.

EXERCISE 8.3 Inverse matrices

1 Let $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 6 & -1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) Calculate **AI**.

(b) Calculate **IA**.

2 Let $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}$.

(a) Calculate **AB**.

(b) Calculate **BA**.

(c) What do the results in parts (a) and (b) tell you about matrices **A** and **B**?

3 $\mathbf{A} = \begin{bmatrix} u & v \\ x & y \end{bmatrix}$

(a) Which of the following represents the inverse of **A**?

A $\mathbf{A}^{-1} = \dfrac{1}{uy - vx}\begin{bmatrix} -y & v \\ x & -u \end{bmatrix}$

B $\mathbf{A}^{-1} = \dfrac{1}{uy - vx}\begin{bmatrix} y & v \\ x & u \end{bmatrix}$

C $\mathbf{A}^{-1} = \dfrac{1}{uy - vx}\begin{bmatrix} y & -v \\ -x & u \end{bmatrix}$

D $\mathbf{A}^{-1} = \dfrac{1}{uv - xy}\begin{bmatrix} y & -v \\ -x & u \end{bmatrix}$

(b) Explain the common errors in the incorrect alternatives.

Worked Example 13

4 Let $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 12 & -4 \\ 3 & -1 \end{bmatrix}$.

(a) Calculate det **A**.

(b) Explain why $\mathbf{A}^{-1}$ exists.

(c) Calculate $\mathbf{A}^{-1}$.

(d) Calculate det **B**.

(e) Explain why $\mathbf{B}^{-1}$ does not exist.

5 Calculate the determinant of each of the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} -3 & 7 \\ 1 & 2 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}$

(e) $\mathbf{E} = \begin{bmatrix} -3 & -3 \\ 1 & -1 \end{bmatrix}$

(f) $\mathbf{F} = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$

6 For each of the matrices in question 5, calculate the inverse (if it exists).

7 Prove each of the following matrices does not have an inverse.

(a) $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} 6 & 1 \\ 18 & 3 \end{bmatrix}$ (c) $\mathbf{C} = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$

8 Calculate the inverse of each of the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} 12 & 7 \\ -4 & 6 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} -15 & -9 \\ 11 & 7 \end{bmatrix}$ (c) $\mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

9 Calculate the inverse of each of the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

Worked Example 14

10 Let $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ p & -4 \end{bmatrix}$ where p is a real number and $\det \mathbf{A} = 6$.

(a) Calculate the value of p. (b) Calculate $\mathbf{A}^{-1}$. (c) Use technology to verify that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

11 Let $\mathbf{A} = \begin{bmatrix} x & 2 \\ 2 & x \end{bmatrix}$. If $\mathbf{A}$ is a singular matrix, determine the value of x.

12 Let $\mathbf{B} = \begin{bmatrix} -2 & 3 \\ -2 & 4 \end{bmatrix}$. Calculate $\mathbf{B}^{-1} + \mathbf{B}$.

13 Solve the following matrix equations to determine $\mathbf{X}$. 15

(a) $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -4 & -2 \\ 7 & 2 \end{bmatrix}$

14 Solve the following matrix equations to determine $\mathbf{X}$. 16

(a) $\mathbf{X}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 11 \end{bmatrix}$ (b) $\mathbf{X}\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -10 & 16 \\ 3 & -4 \end{bmatrix}$

15 Solve the following matrix equation.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 & -3 & 7 \\ 1 & -3 & 3 \\ 1 & 0 & 3 \end{bmatrix}$$

16 If $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ p^2 & p \end{bmatrix}$, determine p such that $\det\left(\mathbf{A}^{-1}\right) = -1$.

8.4 Solving simultaneous linear equations with matrices

Representing systems of linear equations in matrix form

The system of linear equations $x + 2y = 3$ and $4x + 5y = 6$ can be represented by the matrix equation

$$\begin{bmatrix} x + 2y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

This is equivalent to the matrix multiplication $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

You can represent the two linear equations in the form $\mathbf{AX} = \mathbf{B}$.

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

$\mathbf{A}$ is called the *coefficient matrix* because it represents the coefficients of x and y in the linear equations.

$$\begin{bmatrix} 1 \times x + 2 \times y \\ 4 \times x + 5 \times y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

A system of linear equations can be represented as the matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is the coefficient matrix.

17 Representing equations in matrix form

(a) Express the simultaneous linear equations $-x + 3y = 8$ and $4x - y = 2$ in matrix form.

THINKING	WORKING
1 Write the equations and the constants as 2×1 matrices.	$\begin{bmatrix} -1x + 3y \\ 4x - y \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$
2 Express in the form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.	$\begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

(b) Write the system of linear equations represented by this matrix equation.

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

1 Write the equations and the constants as 2×1 matrices.	$\begin{bmatrix} 3 \times x - 1 \times y \\ 2 \times x + 5 \times y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ $\begin{bmatrix} 3x - y \\ 2x + 5y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$
2 Express as a system of linear equations.	$3x - y = 3$ $2x + 5y = -4$

Using matrices to solve a system of linear equations

Consider again the pair of simultaneous linear equations: $x + 2y = 3$ and $4x + 5y = 6$.

As seen previously, they can be expressed in the matrix form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X}$ is a column vector of the unknowns.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Simultaneous linear equations can be solved using inverse matrices. Note that although you solve systems of two equations with two unknowns by hand, technology can be used to solve larger systems.

$\mathbf{AX} = \mathbf{B}$

$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ (pre-multiply both sides by $\mathbf{A}^{-1}$)

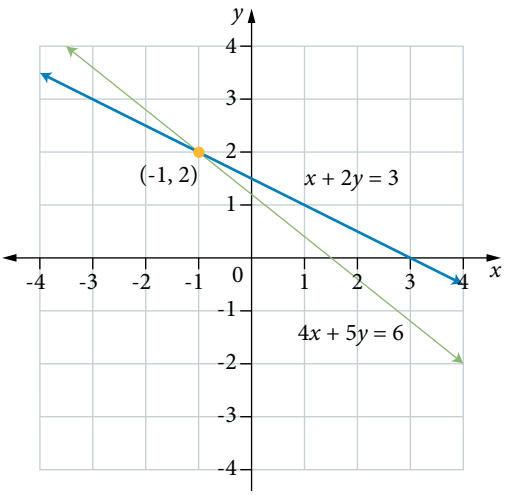
$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ (since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$)

t Technology worked example

Solving a system of linear equations with two unknowns

Graphical representation of simultaneous solutions of linear equations

For the previous example, the lines given by the two equations intersect at the point (-1, 2) as shown below.



The solution to a system of simultaneous linear equations is represented graphically as the point of intersection of the lines.

18 Solving simultaneous linear equations using inverse matrices

Solve the following pairs of simultaneous equations using matrices, and sketch a graphical interpretation of the solution.

(a) $2x + 3y = -5$ and $5x - 2y = 16$

THINKING	WORKING
1 Write the system of linear equations in the form $\mathbf{AX} = \mathbf{B}$.	$\mathbf{AX} = \mathbf{B}$ $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 16 \end{bmatrix}$

2 Determine whether $\mathbf{A}^{-1}$ exists.

$$\det \mathbf{A} = (2 \times -2) - (3 \times 5)$$
$$= -4 - 15$$
$$= -19$$

$\therefore \mathbf{A}^{-1}$ exists.

3 Pre-multiply both sides of $\mathbf{AX} = \mathbf{B}$ by $\mathbf{A}^{-1}$.

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ 16 \end{bmatrix}$$

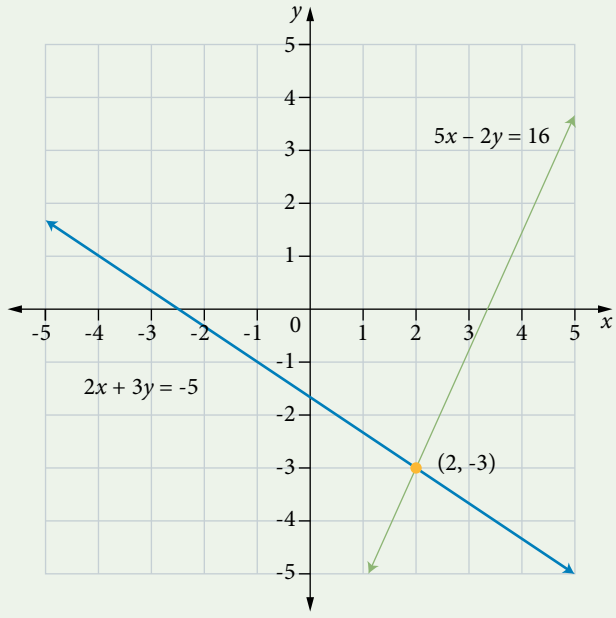
4 Calculate $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\mathbf{X} = -\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} -5 \\ 16 \end{bmatrix}$$
$$= -\frac{1}{19}\begin{bmatrix} (-2 \times -5) + (-3 \times 16) \\ (-5 \times -5) + (2 \times 16) \end{bmatrix}$$
$$= -\frac{1}{19}\begin{bmatrix} 10 - 48 \\ 25 + 32 \end{bmatrix}$$
$$= -\frac{1}{19}\begin{bmatrix} -38 \\ 57 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

5 Interpret the result.

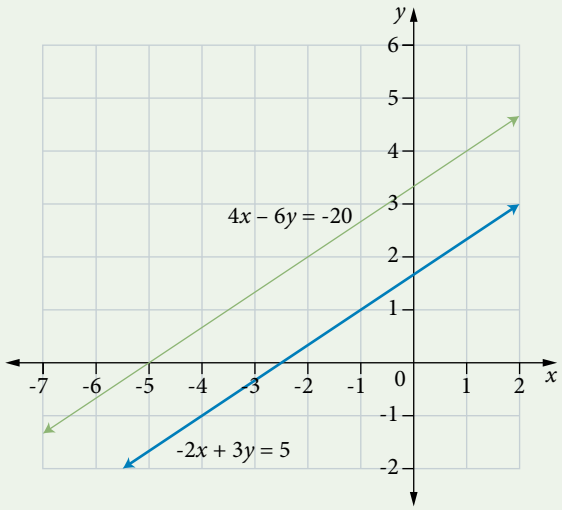
The simultaneous solution is $x = 2$ and $y = -3$.

Graphically, the lines given by the two equations intersect at the point (2, -3).



(b) $-2x + 3y = 5$ and $4x - 6y = -20$

1 Write the system of linear equations in the form $\mathbf{AX} = \mathbf{B}$.	$\mathbf{AX} = \mathbf{B}$ $\begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -20 \end{bmatrix}$
2 Determine whether $\mathbf{A}^{-1}$ exists.	$\det \mathbf{A} = (-2 \times -6) - (3 \times 4)$ $= 12 - 12$ $= 0$ $\therefore \mathbf{A}^{-1}$ does not exist.
3 Interpret the result.	As $\mathbf{A}^{-1}$ does not exist, you cannot solve the equation. Graphically, these two lines are parallel.



Technology worked example

Solving a system of linear equations with three unknowns

19 Using matrices to solve problems involving simultaneous equations

(a) At a bakery, two sausage rolls and three vegetarian pasties cost $20.30, while three sausage rolls and five vegetarian pasties cost $32.90. Use a matrix method to determine the cost of a sausage roll and the cost of a vegetarian pastie.

THINKING	WORKING
1 Define variables.	Let x be the cost of a sausage roll and let y be the cost of a vegetarian pastie.
2 Write the two equations.	$2x + 3y = 20.30$ and $3x + 5y = 32.90$
3 Categorise the coefficient matrix **A** and the column matrix **B**.	$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 20.30 \\ 32.90 \end{bmatrix}$

4 Write the system of linear equations in the form $\mathbf{AX} = \mathbf{B}$.

$$\mathbf{AX} = \mathbf{B}$$
$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20.30 \\ 32.90 \end{bmatrix}$$

5 Pre-multiply both sides of $\mathbf{AX} = \mathbf{B}$ by $\mathbf{A}^{-1}$.

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 20.30 \\ 32.90 \end{bmatrix}$$

6 Calculate $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. Technology can assist in this calculation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.80 \\ 4.90 \end{bmatrix}$$

7 Interpret the result.

The cost of a sausage roll is \$2.80 and the cost of a vegetarian pastie is \$4.90.

(b) Three carpenters visit a hardware store to buy materials. Adie buys five boxes of nails, two screwdrivers and one measuring tape. She pays \$80.20 in total. Brodie buys two boxes of nails and one screwdriver. He pays \$29.70 in total. Cressida buys one box of nails and two measuring tapes. She pays \$29.30 in total. Calculate the individual prices of a box of nails, a screwdriver and a measuring tape.

THINKING	WORKING
1 Define variables.	Let x be the price of a box of nails. Let y be the price of a screwdriver. Let z be the price of a measuring tape.
2 Write three equations for the purchases of three people. Be especially careful of zero values or if items are listed in a different order.	$5x + 2y + 1z = 80.20$ $2x + 1y + 0z = 29.70$ $1x + 0y + 2z = 29.30$
3 Convert the information into a matrix of the form $\mathbf{AX} = \mathbf{B}$.	$\mathbf{AX} = \mathbf{B}$ $\begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 80.20 \\ 29.70 \\ 29.30 \end{bmatrix}$
4 Solve using technology.	$\mathbf{X} = \begin{bmatrix} 12.30 \\ 5.10 \\ 8.50 \end{bmatrix}$
5 Interpret the result.	Therefore, $x = 12.30$, $y = 5.10$ and $z = 8.50$. A box of nails costs \$12.30. A screwdriver costs \$5.10. A measuring tape costs \$8.50.

Technology worked example

Matrix applications with three unknowns

EXERCISE 8.4 Solving simultaneous linear equations with matrices

Worked Example 17

1 Express the following simultaneous linear equations in matrix form:

(a) $x + y = 4$ and $x - y = 8$

(b) $2x + y = 6$ and $x + 3y = -4$

(c) $x - 4y = 1$ and $-2x + y = 0$

(d) $5x - 3y = 1$ and $-x + 6y = 10$

2 Express the following matrix equations as a system of linear equations.

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

3 Consider the following system of linear equations: $2x - 4z = -2$, $y + 2x + z = 3$ and $5y - 8x - 3 = z$.

(a) What is the correct way of representing these equations using matrices?

A $\begin{bmatrix} 2 & 0 & -4 \\ 1 & 2 & 1 \\ 5 & -8 & -3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

B $\begin{bmatrix} 2 & -4 & 0 \\ 1 & 2 & 1 \\ 5 & -8 & -3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

C $\begin{bmatrix} 2 & 0 & -4 \\ 2 & 1 & 1 \\ -8 & 5 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$

D $\begin{bmatrix} 2 & 0 & -4 \\ 2 & 1 & 1 \\ -8 & 5 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$

(b) Identify the matrix $\mathbf{X}$ for the system of linear equations.

A $\mathbf{X} = \begin{bmatrix} x & y & z \end{bmatrix}$ **B** $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **C** $\mathbf{X} = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$ **D** $\mathbf{X} = \begin{bmatrix} -x \\ y \\ -z \end{bmatrix}$

(c) What are the common errors that might lead to selecting the incorrect alternatives in parts **(a)** and **(b)**?

4 Solve the following pairs of simultaneous equations and in each case give a graphical interpretation of the solution. 18

(a) $x - y = 6$ and $x + y = 10$

(b) $x + y = 4$ and $2x - y = 11$

(c) $x + 2y = 1$ and $2x + y = 2$

(d) $2x + 3y = 1$ and $4x + 5y = 1$

(e) $2x + 3y = -11$ and $-5x - y = 21$

(f) $x + y = -13$ and $6x - 4y = 2$

5 Use matrices to solve the following systems of linear equations, if possible.

(a) $2x + y = 6$ and $4x + 2y = 12$

(b) $x - 3y = 6$ and $-2x + 6y = 8$

6 Solve the following simultaneous linear equations. Express each solution in the form of a fraction.

(a) $3x - 5y = -2$ and $4x + y = 1$

(b) $-7x + 2y = -10$ and $x - 5y = 6$

(c) $-3x + \frac{1}{2}y = 4$ and $\frac{1}{2}x - 5y = -9$

(d) $5x - 8y = 15$ and $-2x + 4y = 15$

7 Use technology to solve the following simultaneous linear equations.

Worked Example

(a) $x + y = 2$, $y + z = 1$ and $x + y + z = 4$

(b) $2x + y - z = 2$, $x - 2y + 3z = 7$ and $3x + 5y - z = 0$

19

8 The sum of two positive whole numbers is 18 and their difference is 4. Use a matrix method to determine these two whole numbers.

9 At a café, the total cost of two milkshakes and four cookies is $20, while the total cost of three milkshakes and five cookies is $28. Use a matrix method to calculate the individual cost of a cookie and a milkshake.

10 Jimmy and Sally each have a number of stamps. Sally has three times as many stamps as Jimmy. The total number of stamps is 68. Use a matrix method to determine the number of stamps that Jimmy and Sally have.

11 At a bakery, three meat pies and four muffins cost $25.80, while one meat pie and seven muffins cost $25.60. Use a matrix method to determine the cost of a meat pie and the cost of a muffin.

12 A street vendor sells hot dogs, buckets of chips and cans of soft drinks. Trang buys 2 hot dogs, 5 buckets of chips and 2 cans of soft drink for $29.50. Paul buys 3 hot dogs, 2 buckets of chips and 1 can of soft drink for $22.00. Surabhi buys 4 hot dogs, 2 buckets of chips and 4 cans of soft drink for $31.00. Use a matrix method to determine the cost of a hot dog, the cost of a bucket of chips and the cost of a can of soft drink.

13 Lola, Thomas and Arjun bought tickets for three separate events. The table below shows the number of tickets bought by each person.

	Lola	Thomas	Arjun
NRL Grand Final	3	2	1
AFL Grand Final	2	3	5
Concert	5	1	4

The total cost for Lola, Thomas and Arjun was $1850, $1300 and $2050 respectively.

(a) Represent this information as a system of linear equations.

(b) Calculate how much it would cost Isa to purchase 4 NRL Grand Final tickets, 1 AFL Grand Final ticket and 2 concert tickets.

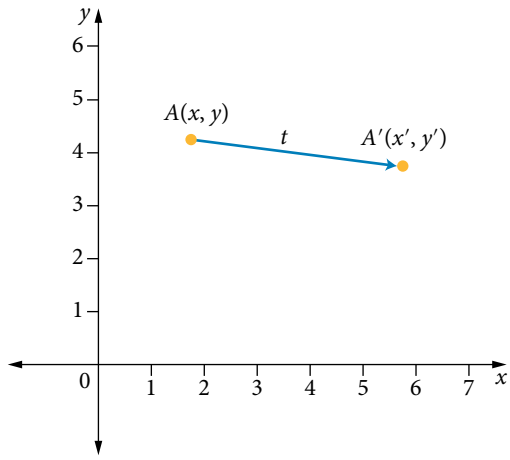
14 Let $\mathbf{A} = \begin{bmatrix} 2 & x \\ 3 & -2 \end{bmatrix}$ where x is a real number.

(a) Calculate the value of x for which $\mathbf{A} = \mathbf{A}^{-1}$.

(b) Determine the value of x for which $\mathbf{A}$ is singular.

8.5 Translations in the plane

A geometric transformation is an operation that maps each point on the Cartesian plane onto another point on the plane. There are many different types of geometric transformation, including translation, dilation, rotation and reflection, and each of these transformations can be performed using matrix methods.

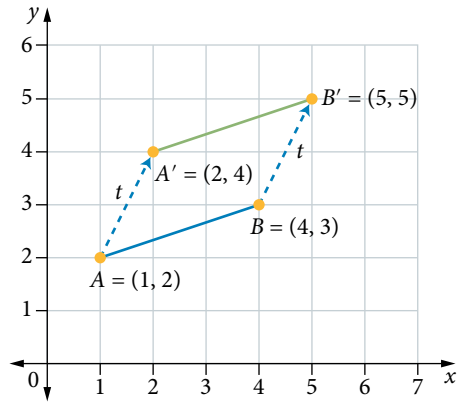


In the diagram above, a transformation t has resulted in point $A(x, y)$ moving or being 'mapped onto' point $A'(x', y')$ on the Cartesian plane. You say that point $A(x, y)$ is the *pre-image* point, and point $A'(x', y')$ is the *image* point.

Different geometric transformations affect the Cartesian plane in different ways. In this and the next section, you will look at two basic types of transformation: the translations and the linear transformations.

Translations

A translation is a basic geometric transformation in which every point in the Cartesian plane is translated (or moved) by a certain fixed distance. For example, consider the line segment $\overline{AB}$ connecting the points $A(1, 2)$ and $B(4, 3)$. When these points undergo a translation of 1 unit in the positive x-direction and 2 units in the positive y-direction, then you can say that the image of these points will be
$A'(2, 4)$ and $B'(5, 5)$.



Mathematically, this translation can be expressed using column matrix addition:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

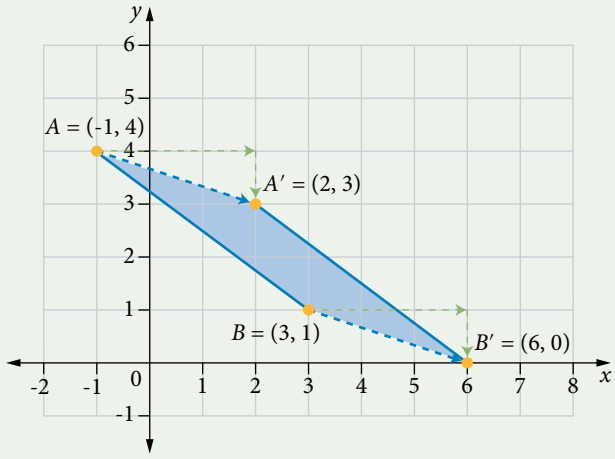
Generally, a translation that maps an original point $P(x, y)$ onto an image point $P'(x', y')$ can be expressed using column matrices:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

where $\mathbf{T} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a translation vector corresponding to a movement of a units in the positive x-direction and b units in the positive y-direction.

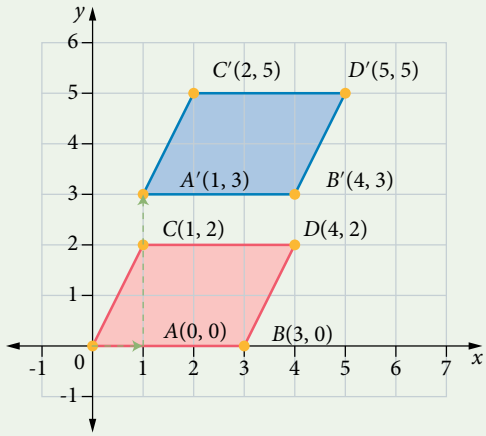
20 Translations

Determine the image of a line segment connecting points $A(-1, 4)$ and $B(3, 1)$ after undergoing a translation defined by $\mathbf{T} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Verify the translation graphically.

THINKING	WORKING
1 Recall the general translation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$
2 Determine the image of each point using the matrix equation.	For point $A(-1, 4)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ For point $B(3, 1)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$
3 Interpret your result.	The two image points are $A'(2, 3)$ and $B'(6, 0)$.
4 Sketch your result.	

21 Translations and area

The parallelogram defined by the points $A(0, 0)$, $B(3, 0)$, $C(1, 2)$ and $D(4, 2)$ undergoes a translation defined by $\mathbf{T} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Sketch the pre-image and image of this translation, and verify that parallelograms $ABCD$ and $A'B'C'D'$ have the same area.

THINKING	WORKING
1 Determine the general translation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$
2 Determine the image of each point using the matrix equation.	For point $A(0, 0)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ For point $B(3, 0)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ For point $C(1, 2)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ For point $D(4, 2)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$
3 Interpret your result.	Therefore, the image points are $A'(1, 3)$, $B'(4, 3)$, $C'(2, 5)$ and $D'(5, 5)$.
4 Sketch your result.	
5 Calculate the area of the parallelograms.	Parallelogram $ABCD$: base $= (3 - 0)$ $= 3$ units; height $= (2 - 0)$ $= 2$ units area $=$ base $\times$ height $= 3 \times 2$ $= 6$ units^2 Parallelogram $A'B'C'D'$: base $= (4 - 1)$ $= 3$ units; height $= (5 - 3)$ $= 2$ units area $=$ base $\times$ height $= 3 \times 2$ $= 6$ units^2
6 Interpret your result.	Therefore, the areas of the parallelograms are equal.

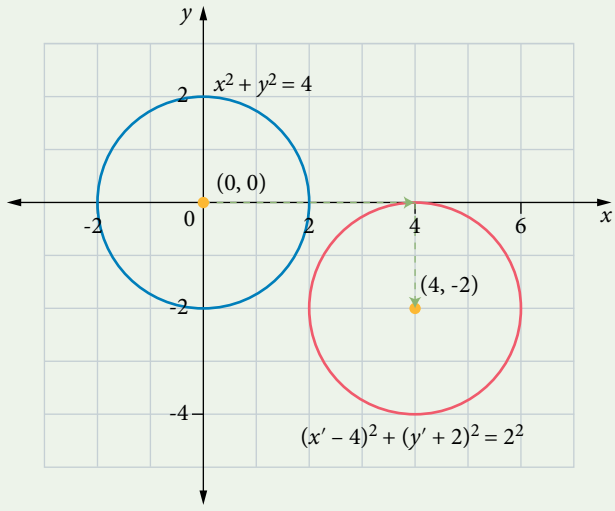
You will notice in Worked Example 21 that the area of the pre-image object and image object are equal. This is because a translation does not alter the shape or size of the pre-image in any way; it simply changes the position of the original image. You can say that under a translation, the pre-image and image are *congruent*.

Translations of graphs of relations on the plane

A relation is a set of ordered pairs that can be expressed graphically on the Cartesian plane as a relationship between x and y values. Graphs of relations can undergo translations using the matrix techniques described.

22 Translations of graphs of relations on the plane

A circle is defined by the relation $x^2 + y^2 = 4$. Determine and sketch the image of this circle after undergoing a translation of $\mathbf{T} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$.

THINKING	WORKING
1 Determine the general translation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$
2 Obtain an expression for $\begin{bmatrix} x \\ y \end{bmatrix}$. This will allow you to substitute x and y in terms of your general image coordinates x' and y'.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $x = x' - 4$ $y = y' + 2$
3 Substitute into the given relation.	$x^2 + y^2 = 4$ $(x' - 4)^2 + (y' + 2)^2 = 2^2$
4 Interpret your result.	The image is a circle of radius 2 and centre at (4, -2).
5 Sketch your result.	

Combining translations

If a point or curve undergoes two successive translations, this can be represented by a single translation.

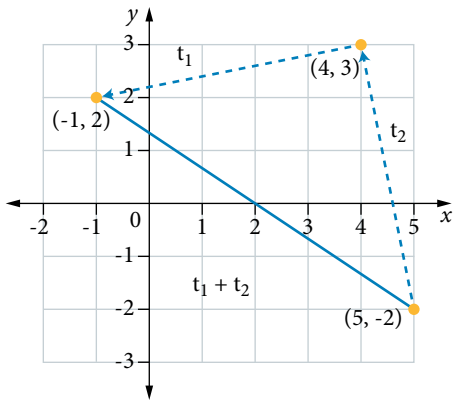
23 Combining translations

Determine image of the point $A(5, -2)$ under the successive translations $\mathbf{T}_1 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\mathbf{T}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$.

Demonstrate that this image is the same as the point obtained when point A undergoes a single translation of $(\mathbf{T}_1 + \mathbf{T}_2)$.

THINKING	WORKING
1 Determine the general translation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$
2 Determine the image of the point after undergoing the two translations. *Note*: A'' is the image of A'.	For point $A(5, -2)$ under translation $\mathbf{T}_1$: $A' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ For point $A'(4, 3)$ under translation $\mathbf{T}_2$: $A'' = \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ Therefore, the image after $\mathbf{T}_1$ and $\mathbf{T}_2$ is $(-1, 2)$.
3 Determine the combined translation matrix $(\mathbf{T}_1 + \mathbf{T}_2)$.	$(\mathbf{T}_1 + \mathbf{T}_2) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$
4 Determine the image of the point after undergoing the combined translation.	For point $A(5, -2)$ under translation $(\mathbf{T}_1 + \mathbf{T}_2)$: $A' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ Therefore, the image of A after $(\mathbf{T}_1 + \mathbf{T}_2)$ is $(-1, 2)$.
5 Interpret the result.	The image of A after undergoing successive translations of $\mathbf{T}_1$ and $\mathbf{T}_2$ is the same as the image after undergoing a translation of $(\mathbf{T}_1 + \mathbf{T}_2)$.

Considering this result geometrically, it makes intuitive sense that any number of successive translations can be represented by a single translation matrix. It makes no difference how an image is moved, as long as it ends up at the same point.



C Making connections

Transformations on the plane

Investigate the effect of different translation matrices on different shapes on the Cartesian plane.

EXERCISE 8.5 Translations in the plane

Worked Example

20

1 Determine the image of the following points after undergoing the stated translations.

(a) $A(7, -4)$, $\mathbf{T} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ (b) $B(-1, 0)$, $\mathbf{T} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ (c) $C(-7, -8)$, $\mathbf{T} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ (d) $D\left(-\frac{1}{2}, \frac{3}{4}\right)$, $\mathbf{T} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

2 Determine the translation matrix $\mathbf{T}$ that would transform the given points onto their images.

(a) $A(10, -3) \rightarrow A'(2, 5)$ (b) $B(5, 6) \rightarrow B'(7, 8)$ (c) $C(-5, 3) \rightarrow C'(0, 0)$ (d) $D(1, -1) \rightarrow D'(8, 10)$

3 A circle $(x-5)^2 + (y-8)^2 = 16$ undergoes a translation of $\mathbf{T} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. What is the resultant image?

A $(x'-3)^2 + (y'-5)^2 = 16$ B $(x'-7)^2 + (y'-11)^2 = 16$

C $(x'-8)^2 + (y'-10)^2 = 16$ D $(x'-2)^2 + (y'-6)^2 = 16$

4 A point $A(-3, 7)$ undergoes a translation $\mathbf{T}_1 = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$ followed by another translation $\mathbf{T}_2$.

(a) If the final image of point A is $A' = (3, -7)$, then what must $\mathbf{T}_2$ be equal to?

A $\mathbf{T}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ B $\mathbf{T}_2 = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$ C $\mathbf{T}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ D $\mathbf{T}_2 = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

(b) Explain the common errors that might lead to selecting the wrong alternatives.

23

5 Determine the image of the point $A(-1, 8)$ under the successive translations $\mathbf{T}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{T}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Demonstrate that this image is the same as the point obtained when point A undergoes a single translation of $(\mathbf{T}_1 + \mathbf{T}_2)$.

21

6 Determine the image of the parallelogram $ABCD$ defined by the points $A(-1, 3)$, $B(4, 3)$, $C(6, 7)$, $D(1, 7)$ after undergoing the translation $\mathbf{T} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. Verify that the area of the original parallelogram $ABCD$ is the same as its image.

22

7 Determine the equation of the image of each of the following curves after undergoing the stated translation.

(a) $y = x^2$, $\mathbf{T} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (b) $y = 2x - 5$, $\mathbf{T} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$

(c) $y = \sqrt{x}$, $\mathbf{T} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (d) $y = x^2 - 5x + 1$, $\mathbf{T} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

8 A point $A(-8, -4)$ undergoes three successive equal translations. Determine the matrix of these translations if the image of point A is $A' = (-17, 2)$.

9 A linear function $y = mx + c$ undergoes a general translation defined by $\mathbf{T} = \begin{bmatrix} p \\ q \end{bmatrix}$, where p and q are scalars. Demonstrate that the gradient of the image of the function is unchanged, and that the y-intercept is increased by $(q - mp)$.

10 A translation of $\mathbf{T} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is applied to point $A(a^2, b - 8)$ to create an image point $A'(5a - 4, 2b + 1)$. Determine the possible values of a and b.

11 A translation is applied to the curve $y = x^2$ such that the image of the curve has x-intercepts of $(1, 0)$ and $(5, 0)$. Determine the translation matrix $\mathbf{T}$ that would make this possible.

Linear transformations in the plane

8.6

In the previous section, you observed how matrix addition could be used to translate points, shapes and curves in the Cartesian plane. These translations had the effect of *moving* the pre-image to a different part of the plane, but the associated image retained its original size, shape and orientation. Mathematically, a translation involves the simple addition of scalars to the x and y values of a pre-image, such that:

- $x' = x + a$
- $y' = y + b$

A *linear transformation* is a different kind of geometric transformation, which has the potential to twist, stretch and warp the Cartesian plane in many interesting ways. Mathematically, a linear transformation is defined as a mapping of the form:

- $x' = ax + by$
- $y' = cx + dy$

This operation can be performed using matrix multiplication.

A linear transformation maps a point $P(x, y)$ onto an image point $P'(x', y')$ according to the matrix equation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, where the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is known as the transformation matrix, **L**.

24 Types of transformations

Determine which of the following are linear transformations. For those transformations that are linear, determine the transformation matrix, **L**.

(a) $x' = 2x + y$ and $y' = 5x - y$

THINKING	WORKING
1 Determine whether the given transformation meets the general form of a linear equation.	General form: $x' = ax + by$ and $y' = cx + dy$ Given transformation: $x' = 2x + y$ and $y' = 5x - y$ This is a linear transformation.
2 Establish the transformation matrix, **L**.	$x' = 2x + y$ $y' = 5x - y$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x + 1y \\ 5x - 1y \end{bmatrix}$ $= \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\mathbf{L} = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}$

(b) $x' = x^2 + 5y$ and $y' = x - 3y$

1 Determine whether the given transformation meets the general form of a linear equation.	General form: $x' = ax + by$ and $y' = cx + dy$ Given transformation: $x' = x^2 + 5y$ and $y' = x - 3y$
2 Interpret the answer.	This is *not* a linear transformation, because the x-image includes an x^2 value.

(c) $x' = y$ and $y' = 2x$

1 Determine whether the given transformation meets the general form of a linear equation.	General form: $x' = ax + by$ and $y' = cx + dy$ Given transformation: $x' = 0x + y$ and $y' = 2x + 0y$ This is a linear transformation. Note that the zero coefficients will result in zero elements in the matrix.
2 Establish the transformation matrix, **L**.	$x' = y$ $y' = 2x$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0x + 1y \\ 2x + 0y \end{bmatrix}$ $= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\mathbf{L} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

(d) $x' = x + 4$ and $y' = y - 1$

1 Determine whether the given transformation meets the general form of a linear equation.	General form: $x' = ax + by$ and $y' = cx + dy$ Given transformation: $x' = x + 4$ and $y' = y - 1$
2 Interpret the answer.	This is *not* a linear transformation—it is a translation.

25 Linear transformations

Use a matrix method to determine the image A' of the point $A(-4, 3)$ after undergoing a linear transformation according to the rule $x' = y - 2x$, $y' = x + 3y$.

THINKING	WORKING
1 Determine the transformation equation from the rules given, ensuring that all elements are in the correct order.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2x + 1y \\ 1x + 3y \end{bmatrix}$ $= \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2 Multiply the transformation matrix by the given point.	For $A(-4,3)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$
3 Interpret the answer.	The coordinates of the image point are $A'(11, 5)$.
4 Verify the solution algebraically.	Given that $x = -4$ and $y = 3$: $x' = y - 2x = 3 - 2 \times -4 = 3 + 8 = 11$ and $y' = x + 3y = -4 + 3 \times 3 = -4 + 9 = 5$ Therefore $(x', y') \rightarrow (11, 5)$

26 Linear transformations of multiple points

(a) Determine the image of the square $ABCD$ defined by the points $A(0, 0)$, $B(2, 0)$, $C(2, 2)$, $D(0, 2)$ after undergoing a linear transformation defined by the matrix $\mathbf{L} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Sketch the original square $ABCD$ as well as the image $A'B'C'D'$.

THINKING	WORKING
1 Determine the transformation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

2 Multiply the transformation matrix by the given points and interpret the result.

For point $A(0, 0)$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The coordinates of the image point A' are $(0, 0)$.

For point $B(2, 0)$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

The coordinates of the image point B' are $(2, 0)$.

For point $C(2, 2)$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

The coordinates of the image point C' are $(0, 4)$.

For point $D(0, 2)$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

The coordinates of the image point D' are $(-2, 4)$.

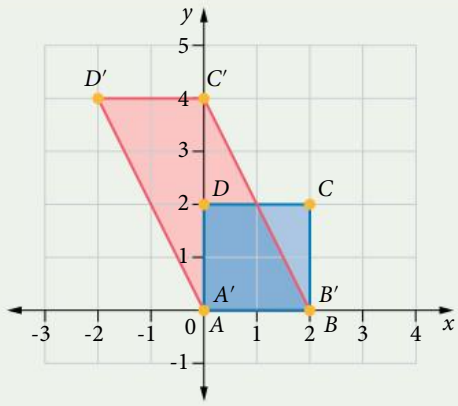
3 Alternatively, create a single image matrix in the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{L} \times \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$ and interpret the answer.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_A & x_B & x_C & x_D \\ y_A & y_B & y_C & y_D \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

From left to right, the coordinates of the image points can be read from the resultant matrix:

$A'(0, 0)$, $B'(2, 0)$, $C'(0, 4)$ and $D'(-2, 4)$.

4 Sketch the original square and its image.



(b) Calculate the determinant of the transformation matrix, **L**.

1 Identify key values and calculate the determinant using the formula

$\det \mathbf{A} = ad - bc$ for matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

For matrix $\mathbf{L} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$:

$a = 1$, $b = -1$, $c = 0$ and $d = 2$

$$\begin{aligned} \det \mathbf{L} &= ad - bc \\ &= 1 \times 2 - (-1) \times 0 \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

2 Interpret your answer.

The determinant of the transformation matrix, **L**, is 2.

(c) Determine the equations of the linear transformations using $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{L}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$.

1 Calculate the inverse matrix:

$$\mathbf{L}^{-1} = \frac{1}{\det \mathbf{L}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\mathbf{L} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ and $\det \mathbf{L} = 2$

Then:

$$\begin{aligned} \mathbf{L}^{-1} &= \frac{1}{2} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

2 Recall the transformation equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{L} \times \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \begin{bmatrix} x' + \frac{1}{2}y' \\ 0x' + \frac{1}{2}y' \end{bmatrix} \end{aligned}$$

3 Write the equations of the transformation with the pre-image as the subject.	$x = x' + \frac{1}{2}y'$ $y = \frac{1}{2}y'$
4 Express the linear transformation equations in terms of x' and y'.	$y' = 2y$ $x' = x - \frac{1}{2}y'$ $= x - \frac{1}{2} \times (2y)$ $= x - y$
5 Verify the equations using the points found in part (a).	$A(0,0) \rightarrow A'(0-0, 2\times 0) \Rightarrow (0,0)$ $B(2,0) \rightarrow B'(2-0, 2\times 0) \Rightarrow (2,0)$ $C(2,2) \rightarrow C'(2-2, 2\times 2) \Rightarrow (0,4)$ $D(0,2) \rightarrow D'(0-2, 2\times 2) \Rightarrow (-2,4)$

(d) Calculate the area of the original square $ABCD$ as well as the area of its image $A'B'C'D'$.

1 Calculate the area of the original shape.	$ABCD$ forms a square, with side lengths of 2 units. area $= s^2$ $= 2^2$ $= 4$ units^2
2 Calculate the area of the image shape.	$A'B'C'D'$ is a parallelogram with a base length of 2 units and a perpendicular height of 4 units. area $= b \times h$ $= 2 \times 4$ $= 8$ units^2

In Worked Example 26, the area of the image shape was double the area of the original shape. The determinant of the transformation matrix **L** is equal to 2. This is not a coincidence; there is a direct link between the determinant of a transformation matrix and the area ratio of the image shape to the original, and you will explore this relationship in more detail later in this chapter.

You will also notice that it can be time-consuming to perform a separate matrix multiplication for every point that is undergoing a transformation. In Worked Example 26, four image points needed to be calculated independently. This procedure can be simplified by multiplying the transformation matrix **L** by a two-row matrix containing every pre-image point. Consider the operation:

$$\begin{matrix} & & & A & B & C & D \end{matrix}$$
$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{matrix} A' & B' & C' & D' \end{matrix}$$
$$\begin{bmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

A linear transformation can be applied to multiple points using a single matrix equation. For the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the pre-image points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$..., image points $A'(x_1', y_1')$, $B(x_2', y_2')$, $C(x_3', y_3')$... can be obtained as follows:

$$\begin{bmatrix} x_1' & x_2' & x_3' & \dots \\ y_1' & y_2' & y_3' & \dots \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & \dots \\ y_1 & y_2 & y_3 & \dots \end{bmatrix}$$

Technology can also be used to determine image points of linear transformations.

Making connections

Transformation matrices

Move the sliders a, b, c, d to see the linear transformation of multiple points.

Technology worked example

General linear transformations with matrices

Determining the transformation matrix

Consider again the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and consider what happens when this transformation is applied to the points $A(1, 0)$ and $B(0, 1)$ to determine:

$$A': \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad B': \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

This means that under a linear transformation defined by $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $(1, 0) \rightarrow (a, c)$ and $(0, 1) \rightarrow (b, d)$. In other words, the first column of the transformation matrix is the image of the point (1, 0) and the second column is the image of the point (0, 1).

Given the linear transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the first column $\begin{bmatrix} a \\ c \end{bmatrix}$ represents the image of the point (1, 0) and the second column $\begin{bmatrix} b \\ d \end{bmatrix}$ represents the image of the point (0, 1).

27 Determining the transformation matrix

A certain linear transformation results in the point $A(1, 0)$ being mapped to $A'(5, -1)$ and the point $B(0, 1)$ being mapped to $B'(-7, -3)$. Determine the image of the point $C(2, 5)$ under this same transformation.

THINKING

1 Determine the transformation matrix, $\mathbf{L}$ from the mapping of the points (1, 0) and (0, 1).

2 Determine the transformation equation.

WORKING

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \begin{bmatrix} 0 \\ 1 \end{bmatrix}'$$

$$\downarrow \quad \downarrow$$

$$\mathbf{L} = \begin{bmatrix} 5 & -7 \\ -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

3 Multiply the transformation matrix by the given point.	For point $C(2, 5)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -25 \\ -17 \end{bmatrix}$
4 Interpret your result.	The image point is $C'(-25, -17)$.

EXERCISE 8.6 Linear transformations in the plane

Worked Example

24

1 Determine which of the following are linear transformations. For those transformations that are linear, determine the transformation matrix, **L**.

(a) $x' = x - 9$ and $y' = y + 3$

(b) $x' = 2x - 7y$ and $y' = -3x - y$

(c) $x' = y - x$ and $y' = x - y$

(d) $x' = x^3 - 2x^2 + 1$ and $y' = y^2 - 1$

(e) $x' = 5x$ and $y' = 10y$

25

2 Use a matrix method to determine the image of each of the following points after undergoing the stated linear transformation.

(a) Point $A(-3, 5)$; transformation: $x' = -x + y$ and $y' = x - 2y$

(b) Point $B(1, 1)$; transformation: $x' = 2x - 3y$ and $y' = 5x + y$

(c) Point $C(2, -1)$; transformation: $x' = 2x$ and $y' = y - x$

(d) Point $D(-4, 9)$; transformation: $x' = -5y$ and $y' = 4x$

3 A certain linear transformation maps the point (0, 1) onto (-3, 4) and the point (1, 0) onto (2, 7). Which of the following matrices can be used to describe this linear transformation?

A $\mathbf{L} = \begin{bmatrix} -3 & 4 \\ 2 & 7 \end{bmatrix}$ **B** $\mathbf{L} = \begin{bmatrix} -3 & 2 \\ 4 & 7 \end{bmatrix}$ **C** $\mathbf{L} = \begin{bmatrix} 2 & 7 \\ -3 & 4 \end{bmatrix}$ **D** $\mathbf{L} = \begin{bmatrix} 2 & -3 \\ 7 & 4 \end{bmatrix}$

4 Under certain linear transformations, points on the plane may be *invariant*—that is, the positions of the points are not affected by the transformation. (The image point is the same as the pre-image point.)

(a) Without attempting any calculations, determine which of the following linear transformation matrices will result in the point $P(0, 1)$ being invariant.

A $\mathbf{L} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **B** $\mathbf{L} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\mathbf{L} = \begin{bmatrix} 15 & 0 \\ -18 & 1 \end{bmatrix}$ **D** $\mathbf{L} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) Explain why the alternatives are incorrect.

Worked Example 26

5 Consider the square $ABCD$, defined by the points $A(0, 0)$, $B(4, 0)$, $C(4, 4)$ and $D(0, 4)$. The square undergoes a linear transformation defined by $\mathbf{L} = \begin{bmatrix} 1 & -1 \\ 0.5 & 0 \end{bmatrix}$.

(a) Determine the vertices of the image $A'B'C'D'$, then sketch the original square $ABCD$ and the image $A'B'C'D'$ on the same axes.

(b) Calculate the determinant of the transformation matrix, $\mathbf{L}$.

(c) Determine the equations of the linear transformations using $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{L}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$.

(d) Calculate the area of the original square $ABCD$ as well as the area of its image $A'B'C'D'$.

6 Calculate the image of ΔABC, defined by the points $A(0, 0)$, $B(5, 2)$ and $C(3, 4)$ after undergoing a linear transformation defined by $\mathbf{L} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

27

7 **(a)** A certain linear transformation results in the point $A(1, 0)$ being mapped to $A'(2, -3)$ and the point $B(0, 1)$ being mapped to $B'(-1, 0)$. Determine the image of the point $C(4, -2)$ under this same transformation.

(b) A certain linear transformation results in the point $A(1, 0)$ being mapped to $A'(1, -1)$ and the point $B(0, 1)$ being mapped to $B'(-3, 3)$. Determine the image of the point $C(-5, 3)$ under this same transformation.

8 The line segment $\overline{AB}$ defined by $A(1, 1)$ and $B(5, 4)$ undergoes a translation defined by $\mathbf{T} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ followed by a linear transformation defined by $\mathbf{L} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$. Determine the image of this line segment, $\overline{A''B''}$.

9 The unit square with vertices $A(0, 0)$, $B(1, 0)$, $C(1, 1)$ and $D(0, 1)$ is transformed according to the linear transformation matrix $\mathbf{L} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Calculate the area of the image of square $ABCD$ in terms of a and d.

10 The line $y = \frac{1}{3}x$ is transformed according to the linear transformation matrix $\mathbf{L} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Demonstrate that the original line is perpendicular to its image.

11 The line $y = 2x + 1$ undergoes a linear transformation defined by $\mathbf{L} = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$. Determine the equation of its image.

8.7 Reflections, rotations and dilations

In the previous section, you investigated general linear transformations in the Cartesian plane, which could be defined by the matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

It is now possible to study several specific types of linear transformations: reflections, rotations and dilations. For all of these different transformations, it will be useful to remember the general technique for constructing the linear transformation matrix **L**, by considering the images of the points (1, 0) and (0, 1), shown on the right.

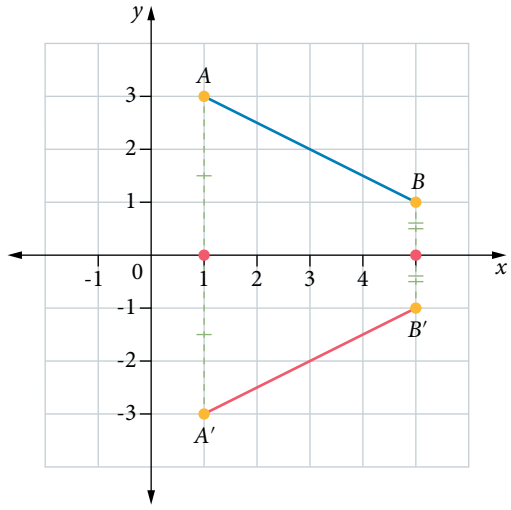
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}'$$

$$\downarrow \qquad \downarrow$$

$$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Reflections

A reflection is a linear transformation in which the pre-image is 'reflected' to create an image on the opposite side of the *line of reflection* (or 'mirror line'). Corresponding points of the pre-image and image are equidistant from and perpendicular to the line of reflection, and any points that were originally on the line of reflection remain unchanged. (The line of reflection is an *invariant line* under a reflection transformation.)



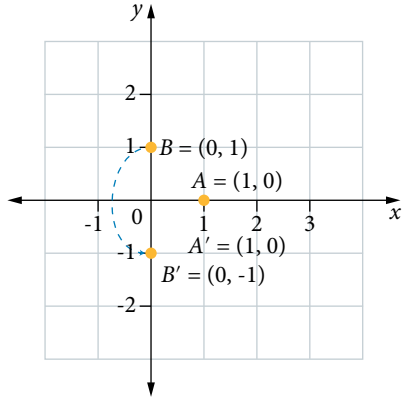
The transformation matrix associated with reflections is often labelled **M**.

For example, consider the image created when a line segment $\overline{AB}$ is reflected in the x-axis, shown on the right.

The image points, A' and B', are equidistant from and perpendicular to the line of reflection.

Reflection in the x-axis

To determine the transformation matrix for a reflection in the x-axis, consider the images of the points (1, 0) and (0, 1) under the transformation shown in the diagram on the right.



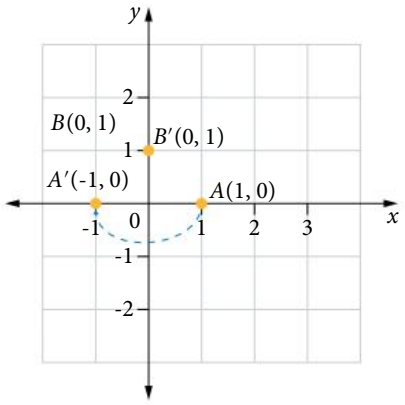
The point (1, 0) is unchanged (invariant), while the point (0, 1) maps to (0, -1).

Therefore, the matrix for a reflection in the x-axis ($y = 0$) is

$$\mathbf{M}_{y=0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Reflection in the *y*-axis

Now consider the images of the points (1, 0) and (0, 1) if they are reflected in the *y*-axis, as shown right on the.



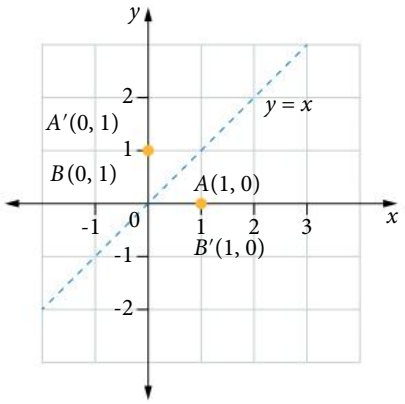
This time the point (1, 0) maps to the point (-1, 0) and the point (0, 1) is invariant.

Therefore, the matrix for a reflection in the *y*-axis ($x = 0$) is

$$\mathbf{M}_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Reflection in the line $y = x$

Reflecting in the line $y = x$ has the effect of swapping the x and y values of every point in the Cartesian plane.



The point (1, 0) maps to (0, 1) and the point (0, 1) maps to (1, 0).

Therefore, the matrix for a reflection in the line $y = x$ is

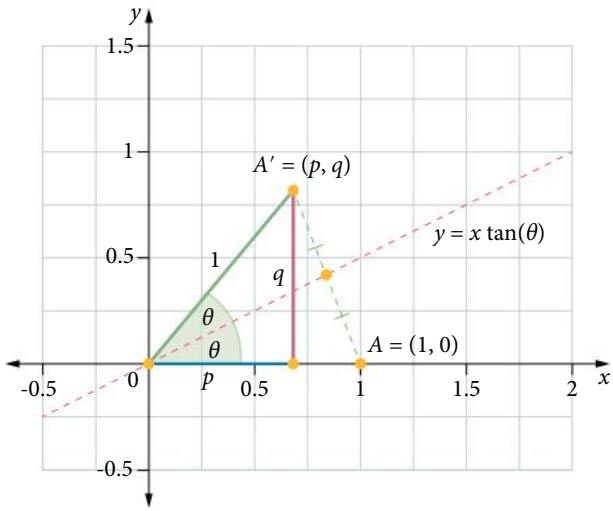
$$\mathbf{M}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Reflection in the line $y = mx$

When reflecting in a straight line of gradient m that passes through the origin, your transformation is less straightforward. Recall the gradient of a straight line, $m = \tan(\theta)$. With this in mind, you can consider what happens to the points (1, 0) and (0, 1) when they are reflected in a line passing through the origin and forming angle θ with the positive x-axis.

First, consider the point (1, 0).

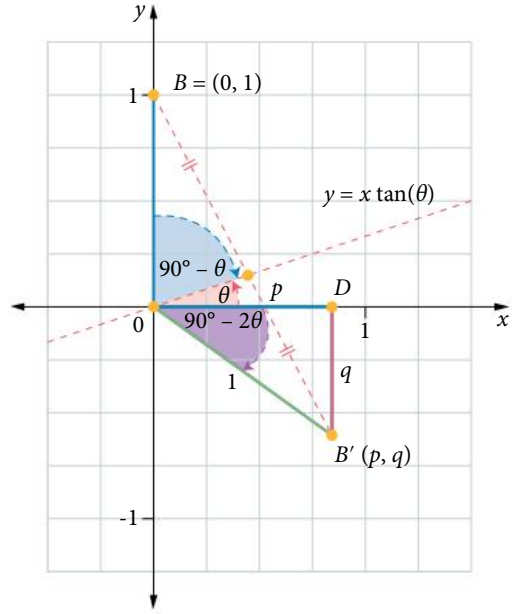
The image point A' makes an angle of 2θ with the positive x-axis. Therefore, using basic trigonometry, the coordinates of the image point A' are $A'(\cos(2\theta), \sin(2\theta))$.



The image of point (0, 1) is even more complex.

Using trigonometry, the coordinates of the image point B' are $B'(\cos(90° - 2\theta), -\sin(90° - 2\theta))$. Since $\sin(90° - x) = \cos(x)$ and $\cos(90° - x) = \sin(x)$, this simplifies to $B'(\sin(2\theta), -\cos(2\theta))$.

Therefore, the matrix for a reflection in the line $y = x\tan(\theta)$ is $\mathbf{M}_{y=x\tan(\theta)} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.



Transformation	Matrix	Invariant lines
The matrix for a reflection in the x-axis	$\mathbf{M}_{y=0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$y = 0$ and $x = k$
The matrix for a reflection in the y-axis	$\mathbf{M}_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$x = 0$ and $y = k$
The matrix for a reflection in the line $y = x$	$\mathbf{M}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$y = x$
The matrix for a reflection in the line $y = \tan(\theta)$	$\mathbf{M}_{y=x\tan(\theta)} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	$y = x\tan(\theta)$

28 Reflections

Use a matrix method to determine the image of ΔABC defined by the points $A(2, 0)$, $B(5, 1)$ and $C(6, 6)$ when reflected in the line $y = x$.

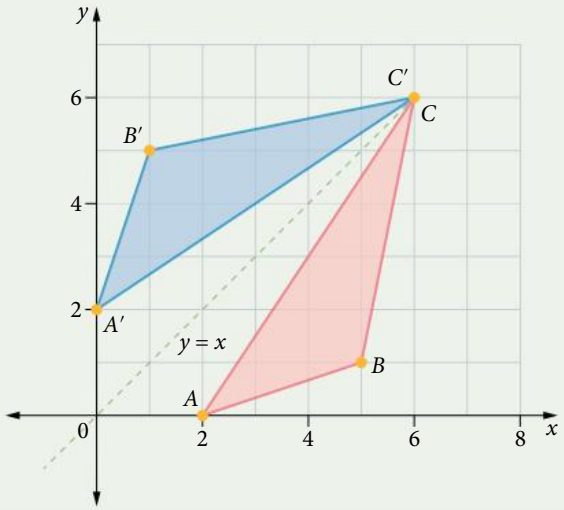
THINKING	WORKING
1 Determine the transformation matrix.	$\mathbf{M}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2 Recall the transformation equation.	$\begin{bmatrix} x_1' & x_2' & x_3' & \dots \\ y_1' & y_2' & y_3' & \dots \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & \dots \\ y_1 & y_2 & y_3 & \dots \end{bmatrix}$

3 Multiply the transformation matrix by the given point. Since there are multiple points, these can be combined into a single equation.

$$\begin{bmatrix} x_1' & x_2' & x_3' & \cdots \\ y_1' & y_2' & y_3' & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 6 \\ 2 & 5 & 6 \end{bmatrix}$$

4 Interpret your result and sketch the graph.

The image points are $A'(0, 2)$, $B'(1, 5)$ and $C'(6, 6)$.



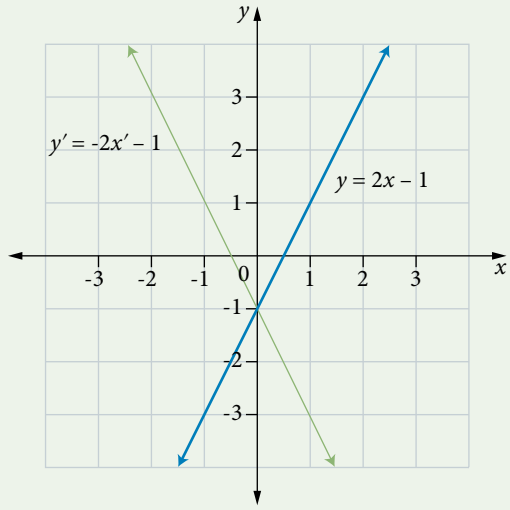
29 Reflection of a curve

Use a matrix method to determine the image of the line $y = 2x - 1$ when reflected in the y-axis.

THINKING	WORKING
1 Determine the transformation matrix.	$\mathbf{M}_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
2 Recall the transformation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3 Pre-multiply by the inverse to determine an equation for $\begin{bmatrix} x \\ y \end{bmatrix}$.	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\frac{1}{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $x = -x'$ $y = y'$
4 Substitute into the equation of the pre-image.	$y = 2x - 1$ $y' = 2(-x') - 1$ $y' = -2x' - 1$

5 Interpret your result and sketch the graph.

The equation of the image of the line $y = 2x - 1$ when reflected in the y-axis is $y' = -2x' - 1$.



Rotations

A rotation is a transformation in which the entire plane spins around a certain fixed point, known as the *centre of rotation*. When a rotation is performed using a linear transformation matrix method, the centre of rotation is always the origin, and by convention the angle of rotation is measured anticlockwise from the positive x-axis. Transformation matrices for rotations are usually named **R**.

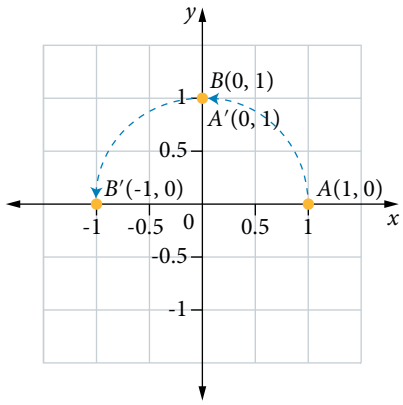
Again, to determine the transformation matrix that defines a rotation, consider the image of the points (1, 0) and (0, 1) under the rotation.

Rotations of multiples of 90°

Consider what happens to the points (1, 0) and (0, 1) following a number of common rotations. First, consider a rotation of 90° (see top diagram on the right).

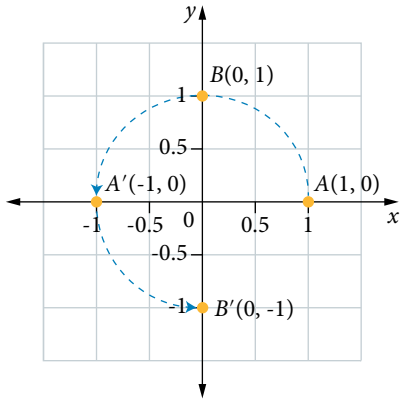
In this rotation, the points $A(1, 0)$ and $B(0, 1)$ are mapped onto $A'(0, 1)$ and $B'(-1, 0)$.

Therefore, the matrix for a rotation of 90° is $\mathbf{R}_{90°} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.



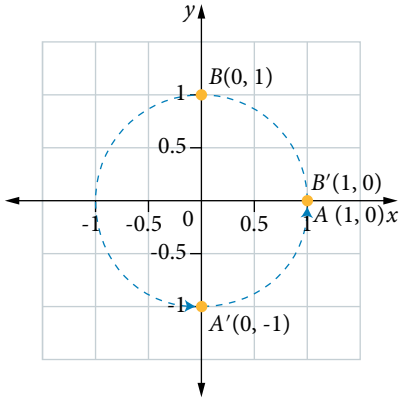
For a rotation of 180°, the points $A(1, 0)$ and $B(0, 1)$ are mapped onto $A'(-1, 0)$ and $B'(0, -1)$ as shown in the lower diagram on the right.

Therefore, the matrix for a rotation of 180° is $\mathbf{R}_{180°} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.



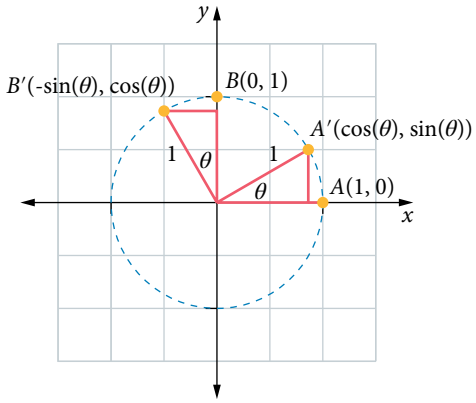
Finally, for a rotation of 270°, the points $A(1, 0)$ and $B(0, 1)$ are mapped onto $A'(0, -1)$ and $B'(1, 0)$.

Therefore, the matrix for a rotation of 270° is $\mathbf{R}_{270°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.



Rotation of θ

Consider the images of points (1, 0) and (0, 1) under a general rotation of θ°:



Therefore, the matrix for a general rotation of θ° is $\mathbf{R}_{\theta°} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

The matrix for a rotation of 90°: $\mathbf{R}_{90°} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The matrix for a rotation of 180°: $\mathbf{R}_{180°} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

The matrix for a rotation of 270°: $\mathbf{R}_{270°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

The matrix for a general rotation of θ°:

$\mathbf{R}_{\theta°} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

The centre of rotation for all of these linear transformations is the origin, and angles are measured anticlockwise from the positive x-axis.

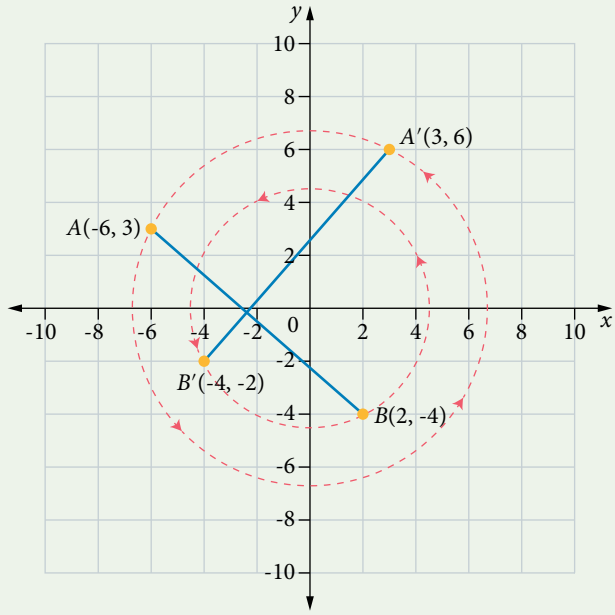
30 Rotations

(a) Use a matrix method to determine the image of the line segment $\overline{AB}$ defined by the points $A(-6, 3)$ and $B(2, -4)$ when rotated 270° anticlockwise around the origin.

THINKING	WORKING
1 Determine the transformation matrix.	$\mathbf{R}_{270°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
2 Remember the transformation equation.	$\begin{bmatrix} x_1' & x_2' & x_3' & \dots \\ y_1' & y_2' & y_3' & \dots \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & \dots \\ y_1 & y_2 & y_3 & \dots \end{bmatrix}$
3 Multiply the transformation matrix by the given point. Since there are multiple points, these can be combined into a single equation.	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -6 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 6 & -2 \end{bmatrix}$

4 Interpret your result. It is helpful to draw a diagram.

The image points are $A'(3, 6)$ and $B'(-4, -2)$.



(b) Verify that line segment $\overline{AB}$ and its image $\overline{A'B'}$ are the same length.

1 Use Pythagoras' theorem to calculate the length of both line segments.

For line segment $\overline{AB}$:

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-(-6))^2 + ((-4)-3)^2} \\ &= \sqrt{8^2 + (-7)^2} \\ &= 10.63 \text{ (2 d.p.)} \end{aligned}$$

For line segment $\overline{A'B'}$:

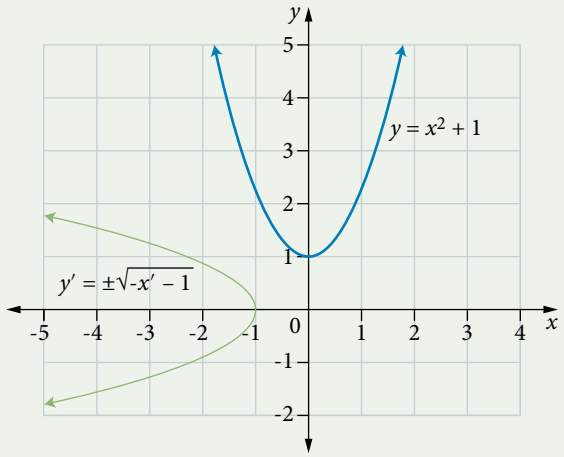
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{((-4)-3)^2 + ((-2)-6)^2} \\ &= \sqrt{(-7)^2 + (-8)^2} \\ &= 10.63 \text{ (2 d.p.)} \end{aligned}$$

2 Interpret your result.

The lengths of line segment $\overline{AB}$ and its image $\overline{A'B'}$ are the same.

31 Rotation of a curve

Determine the image of the curve $y = x^2 + 1$ when rotated 90° anticlockwise around the origin.

THINKING	WORKING
1 Determine the transformation matrix.	$\mathbf{R}_{90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
2 Recall the transformation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3 Pre-multiply by the inverse to determine an equation for $\begin{bmatrix} x \\ y \end{bmatrix}$.	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $x = y'$ $y = -x'$
4 Substitute into the original equation.	$y = x^2 + 1$ $-x' = (y')^2 + 1$ $(y')^2 = -x' - 1$ $y' = \pm\sqrt{-x' - 1}$
5 Interpret your result and sketch the graph.	The equation of the image is $y' = \pm\sqrt{-x' - 1}$. 

Dilations

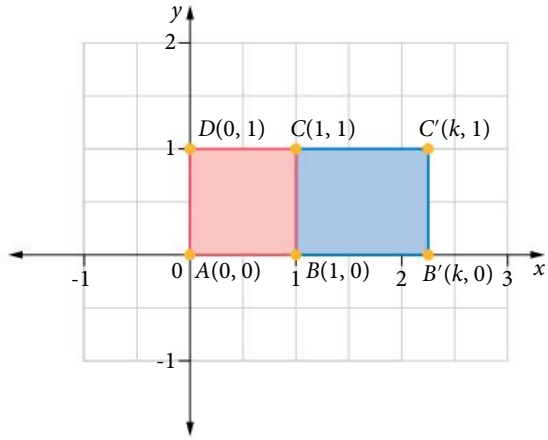
A dilation is a linear transformation that either enlarges or shrinks the pre-image on the plane. Dilations can be performed in either a horizontal (parallel to the x-axis) or a vertical (parallel to the y-axis) direction, or can be applied to the entire plane. The best way to imagine the effect of a dilation is to consider a shape drawn onto the rubber surface of a balloon, which is then stretched in different directions.

Usually you would refer to a dilation factor, k. When $k > 1$, an enlargement occurs, and when $0 < k < 1$, the effect is a reduction in size. If $k < 0$, you begin to combine the effects of a reflection and a dilation. Note that the transformation matrix defining a dilation is often labelled $\mathbf{D}$.

The easiest way to understand a dilation is to consider what happens to a unit square when it is dilated in various directions.

Horizontal dilations

Consider the unit square defined by the points $A(0, 0)$, $B(1, 0)$, $C(1, 1)$, $D(0, 1)$. When the plane undergoes a horizontal dilation of factor k, this unit square expands (or shrinks) horizontally as shown in the diagram on the right.



In this transformation, the square has been stretched horizontally to become a rectangle, and the points $B(1, 0)$ and $C(1, 1)$ have been mapped to $B'(k, 0)$ and $C(k, 1)$ to accommodate the horizontal dilation factor of k. You will notice that the points originally on the y-axis are unchanged.

In this case, the images of points (1, 0) and (0, 1) are mapped onto $(k, 0)$ and (0, 1) respectively.

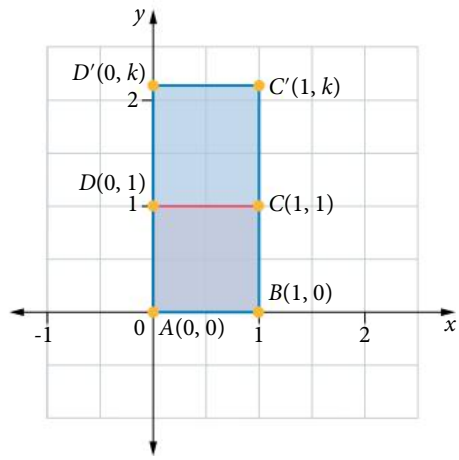
Therefore, the matrix for a horizontal dilation of factor k is $\mathbf{D}_{k,x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.

Vertical dilations

A vertical dilation is also straightforward.

For this transformation, the images of points (1, 0) and (0, 1) are mapped onto (1, 0) and $(0, k)$ respectively. This time, all points on the x-axis are invariant.

Therefore, the matrix for a vertical dilation of factor k is $\mathbf{D}_{k,y} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

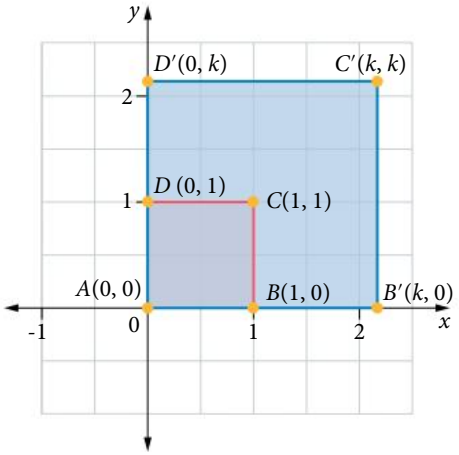


Dilations about the origin

A dilation can be applied to the entire plane at once. When this occurs, the distance of every point on the plane from the origin is increased by a factor of k.

The only invariant point for a general dilation is the origin, (0, 0). For the general dilation transformation, the images of points (1, 0) and (0, 1) are mapped onto (k, 0) and (0, k) respectively.

Therefore, the matrix for a dilation of factor k about the origin is $\mathbf{D}_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.



The matrix for a horizontal dilation of factor k: $\mathbf{D}_{k,x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

The matrix for a vertical dilation of factor k: $\mathbf{D}_{k,y} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

The matrix for a dilation of factor k about the origin: $\mathbf{D}_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

32 Dilation of a curve

Determine the image of the relation $x^2 + y^2 = 4$ when dilated vertically by a factor of 2.

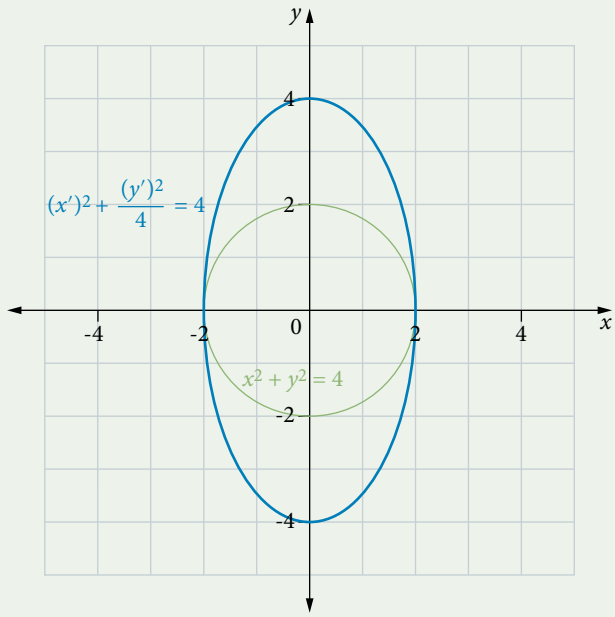
THINKING	WORKING
1 Determine the transformation matrix.	$\mathbf{D}_{2,y} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
2 Remember the transformation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3 Pre-multiply by the inverse to determine an equation for $\begin{bmatrix} x \\ y \end{bmatrix}$.	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $x = x'$ $y = \frac{y'}{2}$

4 Substitute into the original equation.

$$x^2 + y^2 = 4$$
$$(x')^2 + \left(\frac{y'}{2}\right)^2 = 4$$
$$(x')^2 + \frac{(y')^2}{4} = 4$$

5 Interpret your result and sketch the graph.

The equation of the image is $(x')^2 + \frac{(y')^2}{4} = 4$.



t Technology worked example
Transformations using matrices

C Making connections
Combining transformations
Investigate how changing reflection, rotation and dilation transformation matrices alters an image on the Cartesian plane.

EXERCISE 8.7 Reflections, rotations and dilations

1 Determine the transformation matrix that could be used to perform the following linear transformations. Give your answer using exact values where possible.

(a) a rotation of 135° anticlockwise about the origin

(b) a rotation of 210° anticlockwise about the origin

Worked Example

(c) a reflection in the line $y = \sqrt{3}x$

28

2 Use a matrix method to determine the image of ΔABC defined by the points $A(0, 0)$, $B(2, 4)$ and $C(-1, 3)$ when reflected in the following:

(a) the x-axis (b) the y-axis (c) the line $y = x$

3 Without performing any calculations, determine which of the following matrices could be used to produce a reflection in the line $y = -x$.

A $\mathbf{M}_{y=-x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

B $\mathbf{M}_{y=-x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

C $\mathbf{M}_{y=-x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

D $\mathbf{M}_{y=-x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

4 **(a)** Without performing any calculations, determine which of the following transformations would leave the line $y = x$ unaffected.

A a dilation of factor 10 about the origin

B a horizontal dilation of factor 2

C a rotation of 90° clockwise

D a reflection in the x-axis

(b) Explain why the alternatives are incorrect.

5 Determine the image of line segment $\overline{AB}$, defined by the points $A(4, 0)$ and $B(8, 2)$, when reflected in the line $y = \frac{1}{\sqrt{3}}x$. Verify your solution graphically.

Worked Example 29

6 Determine the image of the line $y = (x-1)^2$ when reflected in the line $y = x$. Evaluate the reasonableness of your answer by drawing a graph.

7 Use a matrix method to determine the image of the line segment $\overline{AB}$ defined by the points $A(1, 5)$ and $B(-1, 3)$ when undergoing the following rotations:

30

(a) 90° anticlockwise around the origin

(b) 180° anticlockwise around the origin

8 Determine the image of ΔABC defined by the points $A(0, 0)$, $B(0, 5)$ and $C(2, 3)$ when rotated 60° anticlockwise around the origin. Verify your solution graphically.

9 Determine the image of the line $y = 7 - 2x$ when rotated 270° anticlockwise around the origin. Evaluate the reasonableness of your answer by drawing a graph.

31

10 Determine the image of parallelogram $ABCD$ defined by the points $A(0, 0)$, $B(0, 4)$, $C(2, 6)$ and $D(2, 2)$ after undergoing a horizontal dilation of factor $k = 2$. Demonstrate that the area of the image is double the area of the original shape. Verify your solution graphically.

11 Determine the image of the curve $y = 2x^2$ after undergoing a vertical dilation of factor $k = \frac{1}{4}$. Verify your answer graphically.

32

12 The line segment $\overline{AB}$ is defined by the points $A(1, 1)$ and $B(3, 5)$. What vertical dilation factor k (where $k > 1$) must be applied so that the image line segment $\overline{A'B'}$ has a length of 10 units?

13 The line $y = 10 - 2x$ is rotated anticlockwise about the origin such that its image has an x-intercept of $(p, 0)$ and a y-intercept of $(0, q)$. Determine the angle of rotation θ (where $0° \leq \theta \leq 90°$) such that $p = q$. Give your answer correct to 2 decimal places.

8.8 Applying transformations in the plane

Inverse linear transformations

Recall from the previous section that a linear transformation in the Cartesian plane can be defined by the matrix equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

In many cases, it is possible to *reverse* the effects of a linear transformation—that is, to calculate the inverse of the transformation. Some of the common transformations you have encountered, and the process required to reverse the effects of that transformation, are shown in the box below.

Example of linear transformation	Process required for inverse transformation
rotation of angle θ anticlockwise	rotation of angle θ clockwise
reflection in x-axis	reflection in x-axis
vertical dilation of factor k	vertical dilation of factor $\frac{1}{k}$

It is possible to determine the inverse of all of these common transformations, because the linear transformations themselves are *one-to-one* mappings. Every point on the plane (x, y) is mapped onto a unique image point (x', y'), and this unique mapping allows the transformation to be reversed. Some transformations, however, map two different pre-image points onto the same image point. When this occurs, you lose information about the original points and you are no longer able to 'undo' the transformation. Such transformations are said to be *non-invertible*.

As you will see, there is an important relationship between the inverse of a transformation and the inverse of the transformation matrix. But first, consider a linear transformation that is non-invertible.

33 Non-invertible transformations

Consider the linear transformation defined by the matrix $\mathbf{L} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

(a) Determine the image of the points $A(1, 3)$ and $B(7, 0)$ under this transformation. Use this information to explain why this linear transformation is non-invertible.

THINKING	WORKING
1 Determine the transformation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
2 Apply the transformation equation to the given points.	For point $A(1, 3)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $= \begin{bmatrix} 7 \\ 21 \end{bmatrix}$ The coordinates of image point A' are $(7, 21)$.

	For point $B(7, 0)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$ The coordinates of image point B' are $(7, 21)$.
3 Interpret your result.	The image of $A(1, 3)$ is $A'(7, 21)$ and the image of $B(7, 0)$ is $B'(7, 21)$. Because original points A and B map to the same point under the transformation, it is not possible to reverse the transformation. The transformation is non-invertible.

(b) Determine the equation of the image of the plane under this transformation.

1 Determine the transformation equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
2 Multiply the matrices to determine an expression for $\begin{bmatrix} x' \\ y' \end{bmatrix}$.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $x' = 1x + 2y$ and $y' = 3x + 6y$
3 Determine a relationship between x' and y'.	$y' = 3x'$
4 Interpret your result.	Under the transformation defined by matrix $\mathbf{L} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, the entire plane is mapped onto the line $y' = 3x'$.

Note from Worked Example 33 that the transformation matrix $\mathbf{L} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is singular—that is, $\det \mathbf{L} = (1 \times 6) - (2 \times 3) = 0$. In general, a linear transformation is invertible as long as the transformation matrix $\mathbf{L}$ is not singular.

A linear transformation defined by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{L} \begin{bmatrix} x \\ y \end{bmatrix}$ is invertible if and only if there exists a matrix $\mathbf{L}^{-1}$ such that $\mathbf{L}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{L}^{-1}\mathbf{L} \begin{bmatrix} x \\ y \end{bmatrix}$, where $\mathbf{L}^{-1}\mathbf{L} = \mathbf{I}$, the identity matrix.

34 Inverse transformations

(a) Demonstrate that the transformation defined by matrix $\mathbf{R}_{\theta=-45°}$ (a rotation of -45° about the origin) represents the inverse of the transformation defined by matrix $\mathbf{R}_{\theta=45°}$ (a rotation of 45° about the origin).

THINKING	WORKING
1 Define the two transformation matrices.	$\mathbf{R}_{\theta=-45°} = \begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix}$ $= \begin{bmatrix} \cos(45°) & \sin(45°) \\ -\sin(45°) & \cos(45°) \end{bmatrix}$ $= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $\mathbf{R}_{\theta=45°} = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix}$ $= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
2 Multiply the two matrices together.	$\mathbf{R}_{\theta=-45°} \times \mathbf{R}_{\theta=45°} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $= \begin{bmatrix} \left(\frac{1}{2}+\frac{1}{2}\right) & \left(-\frac{1}{2}+\frac{1}{2}\right) \\ \left(-\frac{1}{2}+\frac{1}{2}\right) & \left(\frac{1}{2}+\frac{1}{2}\right) \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3 Interpret your result.	Since $\mathbf{R}_{\theta=0-45°} \times \mathbf{R}_{\theta=45°} = \mathbf{I}$, the rotation of -45° about the origin is the inverse of a rotation of 45° about the origin.

(b) A rotation of 45° anticlockwise about the origin has resulted in the image point $A'(5, -2)$. Determine the original point A.

1 Define the transformation equation.

Using matrix $\mathbf{R}_{\theta=45°}$ from the previous question:

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2 Pre-multiply by the inverse matrix.

Using the inverse matrix $\mathbf{R}_{\theta=-45°}$ from the previous question:

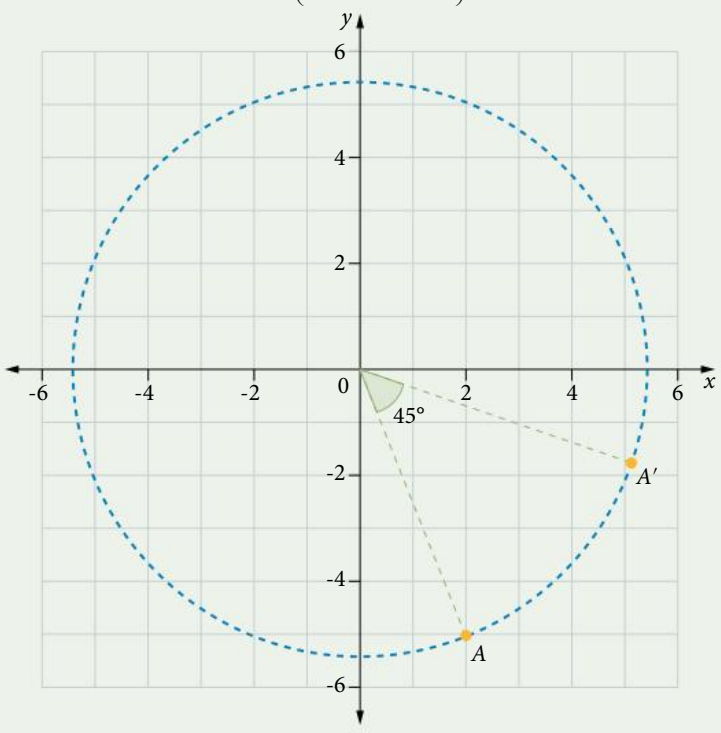
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{5}{\sqrt{2}} - \frac{2}{\sqrt{2}} = x \quad \text{and} \quad -\frac{5}{\sqrt{2}} - \frac{2}{\sqrt{2}} = y$$

$$\frac{3}{\sqrt{2}} = x \qquad -\frac{7}{\sqrt{2}} = y$$

3 Interpret your result and sketch the graph.

Therefore, the original point is $A\left(\frac{3}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$. Rationalising the denominator gives $A\left(\frac{3\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)$.



35 Inverse transformations

A linear transformation defined by the transformation matrix $\mathbf{L} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ has transformed the plane to produce the image of a triangle. The image $\Delta A'B'C'$ is defined by the points $A'(0, 0)$, $B'(12, 4)$ and $C'(19, 7)$. Determine the original ΔABC.

THINKING

1 Determine the transformation equation including the transformation matrix and the image points.

2 Pre-multiply by the inverse matrix $\mathbf{L}^{-1}$.

3 Multiply to calculate the original points.

4 Interpret your result and sketch the graph.

WORKING

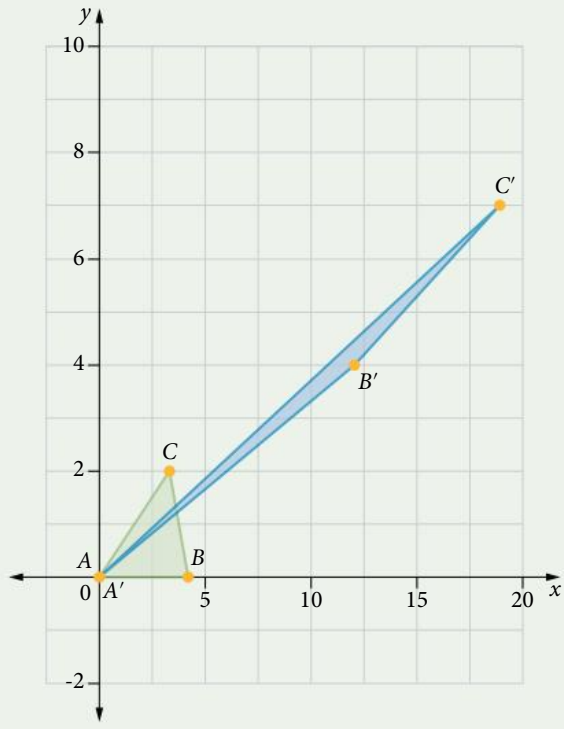
$$\begin{bmatrix} 0 & 12 & 19 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 12 & 19 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 12 & 19 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 12 & 19 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore, the original ΔABC is defined by the points $A(0, 0)$, $B(4, 0)$ and $C(3, 2)$.



Invariant lines

In a linear transformation, the origin (0, 0) always remains unchanged by the transformation. This point is *invariant* under a linear transformation.

Some linear transformations on the plane will also leave a certain line or lines unchanged. Such lines are referred to as *invariant lines*, and they will always pass through the origin. When you consider some of the special linear transformations you have studied in this chapter, you will recognise which lines will be invariant under these transformations. For example, under a reflection in the x-axis, both the lines $y = 0$ and $x = 0$ will be invariant.

To determine whether a linear transformation will give rise to an invariant line or lines, consider the equation for the straight line with gradient m passing through the origin: $y = mx$. If this line maps to an image of $y' = mx'$ under the transformation, then the line is invariant.

Consider the image of the line $y = mx$ under the linear transformation $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. To determine the image of this line, set up the transformation equation and calculate the inverse matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{ad - bc}(dx' - by') = x$$

$$\frac{1}{ad - bc}(-cx' + ay') = y$$

Now, substitute into the original equation to determine the image:

$$y = mx$$

$$\frac{1}{ad - bc}(-cx' + ay') = \frac{m}{ad - bc}(dx' - by')$$

Simplify this equation so that it is in the form $y' = mx'$:

$$\frac{1}{ad - bc}(-cx' + ay') = \frac{m}{ad - bc}(dx' - by')$$

$$(-cx' + ay') = m(dx' - by')$$

$$-cx' + ay' = dmx' - bmy'$$

$$ay' + bmy' = dmx' + cx'$$

$$y'(a + bm) = x'(c + dm)$$

$$y' = \frac{(c + dm)}{(a + bm)}x'$$

The equation $m = \frac{(c + dm)}{(a + bm)}$ is used to calculate the gradient of a line that is invariant under the linear transformation defined by the matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

For this image line to be the same as the original line, the gradients must be equal. Hence:

$$m = \frac{(c + dm)}{(a + bm)}$$

36 Invariant lines

(a) Determine the equation of any lines that are invariant under the transformation defined by $\mathbf{L} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

THINKING	WORKING
1 Determine the equation used to calculate the gradient of invariant lines, and determine values for a, b, c and d.	$m = \dfrac{(c+dm)}{(a+bm)}$ $\mathbf{L} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ so $a=2, b=1, c=3, d=2$
2 Substitute values from the transformation matrix and solve the equation for m.	$m = \dfrac{(3+2m)}{(2+1m)}$ $m(2+1m) = 3+2m$ $2m+m^2 = 3+2m$ $m^2 = 3$ $m = \pm\sqrt{3}$
3 Interpret your solution.	The lines that are invariant under the transformation defined by $\mathbf{L} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ are the lines $y = \sqrt{3}x$ and $y = -\sqrt{3}x$.

(b) Verify your answer to part **(a)** by calculating the images of the invariant lines under the linear transformation.

1 Create the transformation equation and solve for x and y.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $2x' - y' = x$ $-3x' + 2y' = y$

2 Substitute into the equations of the invariant lines found in part (a).

For $y = \sqrt{3}x$:

$$-3x' + 2y' = \sqrt{3}(2x' - y')$$
$$-3x' + 2y' = 2\sqrt{3}x' - \sqrt{3}y'$$
$$2y' + \sqrt{3}y' = 2\sqrt{3}x' + 3x'$$
$$(2 + \sqrt{3})y' = (2\sqrt{3} + 3)x'$$
$$y' = \frac{(2\sqrt{3} + 3)}{(2 + \sqrt{3})}x'$$

Factorising out $\sqrt{3}$ from the numerator:

$$y' = \frac{\sqrt{3}(2 + \sqrt{3})}{(2 + \sqrt{3})}x'$$
$$y' = \sqrt{3}x'$$

For $y = -\sqrt{3}x$:

$$-3x' + 2y' = -\sqrt{3}(2x' - y')$$
$$-3x' + 2y' = -2\sqrt{3}x' + \sqrt{3}y'$$
$$2y' - \sqrt{3}y' = -2\sqrt{3}x' + 3x'$$
$$(2 - \sqrt{3})y' = (-2\sqrt{3} + 3)x'$$
$$y' = \frac{(-2\sqrt{3} + 3)}{(2 - \sqrt{3})}x'$$

Factorising out $-\sqrt{3}$ from the numerator:

$$y' = \frac{-\sqrt{3}(2 - \sqrt{3})}{(2 - \sqrt{3})}x'$$
$$y' = -\sqrt{3}x'$$

3 Interpret your solution.

The line $y = \sqrt{3}x$ maps to $y' = \sqrt{3}x'$ and the line $y = -\sqrt{3}x$ maps to $y' = -\sqrt{3}x'$, therefore these lines are invariant.

Technology worked example

Investigating invariant lines under general linear transformations

EXERCISE 8.8 Applying transformations in the plane

Worked Example

33

1 Determine which of the following transformation matrices will lead to non-invertible linear transformations. For those transformations that are non-invertible, determine the line onto which the plane is transformed.

(a) $\mathbf{L} = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$ (b) $\mathbf{L} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\mathbf{L} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ (d) $\mathbf{L} = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$

2 Which of the following transformations will be non-invertible?

A a reflection in the line $y = 0.2x$
B a horizontal dilation of factor 0
C a rotation of 400° clockwise about the origin
D a translation of 3 units up and 5 units to the right

3 (a) Which of the following transformations will not result in at least one invariant line?

A a vertical dilation of factor 5
B a reflection in the line $y = 2x$
C a rotation of $\frac{\pi}{8}$ anticlockwise about the origin
D the transformation defined by the matrix $\mathbf{L} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

(b) Explain why the alternatives are incorrect.

34

4 (a) Demonstrate that the transformation defined by matrix $\mathbf{M}_{y=x}$ (a reflection in the line $y = x$) is its own inverse.

(b) A reflection in the line $y = x$ has resulted in the image point $A'(-9, 1)$. Determine the original point A.

35

5 A linear transformation defined by the transformation matrix $\mathbf{L} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$ has transformed the plane to produce the image $\Delta A'B'C'$ defined by the points $A'(2, 5)$, $B'(1, 1)$ and $C'(5, 11)$. Determine the original ΔABC and demonstrate your answer graphically.

6 A rotation of 180° about the origin has resulted in the image curve $y' = -(x'+1)^2$. Determine the equation of the original curve, and verify your answer graphically.

36

7 Determine the equation of any lines that are invariant under linear transformations defined by the following matrices:

(a) $\mathbf{L} = \begin{bmatrix} -2 & 0 \\ 1 & 6 \end{bmatrix}$ (b) $\mathbf{L} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

8 Determine value(s) for a such that the matrix $\mathbf{L} = \begin{bmatrix} a & 2a \\ 5 & a \end{bmatrix}$ represents a non-invertible linear transformation.

9 Demonstrate that a reflection in the line $y = x\tan(\theta)$ is always its own inverse.

10 Demonstrate that a rotation of θ about the origin, where $0° < \theta < 360°$, will never result in an invariant line.

Determining the transformation matrix 8.9

Determining the transformation matrix based on images of the standard points

Previously in this chapter, you saw that the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, used to define a linear transformation, could be constructed by considering the images of points (1, 0) and (0, 1), such that $(1, 0) \rightarrow (a, c)$ and $(0, 1) \rightarrow (b, d)$.

Therefore, if you are given information about the images of points (1, 0) and (0, 1), it is easy to establish the transformation matrix and solve related problems.

37 Determining the transformation matrix based on images of the standard points

Determine the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of a certain linear transformation given that $(1, 0) \rightarrow (-3, 5)$ and $(0, 1) \rightarrow (1, 7)$.

THINKING	WORKING
1 Recall the standard transformation matrix.	$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2 Use the information about the images of the standard points to construct the transformation matrix.	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \begin{bmatrix} 0 \\ 1 \end{bmatrix}'$ $\downarrow \quad \downarrow$ $\mathbf{L} = \begin{bmatrix} -3 & 1 \\ 5 & 7 \end{bmatrix}$
3 Interpret your result.	The transformation matrix for this linear transformation is $\mathbf{L} = \begin{bmatrix} -3 & 1 \\ 5 & 7 \end{bmatrix}$.

38 Determining the image of a function from the images of given points

A certain linear transformation maps the points $(1, 0) \rightarrow (-1, 2)$ and $(0, 1) \rightarrow (0, -1)$. Determine the image of the parabola $y = x^2 + 3x - 4$ under this transformation. Verify your answer graphically.

THINKING	WORKING
1 Recall the standard transformation matrix.	$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2 Use the information about the images of the standard points to construct the transformation matrix.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \begin{bmatrix} 0 \\ 1 \end{bmatrix}'$$
$$\downarrow \quad \downarrow$$
$$\mathbf{L} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

3 Create the transformation equation and rearrange to determine expressions for x and y.

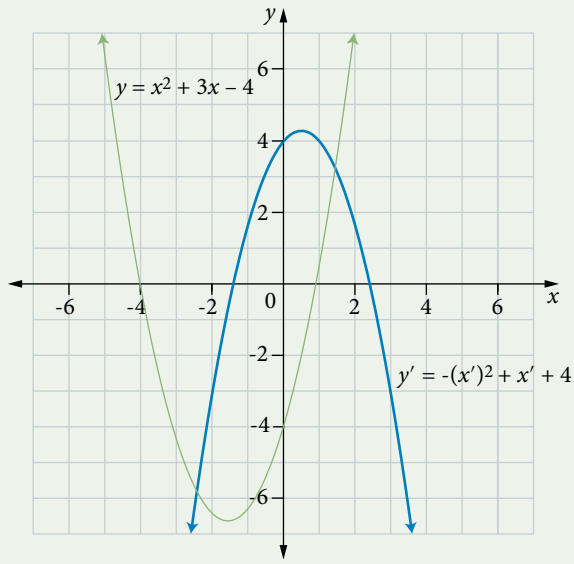
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$-x' = x$$
$$-2x' - y' = y$$

4 Substitute into the original equation.

$$y = x^2 + 3x - 4$$
$$-2x' - y' = (-x')^2 + 3(-x') - 4$$
$$-2x' - y' = (x')^2 - 3x' - 4$$
$$y' = -(x')^2 + x' + 4$$

5 Interpret and sketch the graph of your result.

The equation of the image of $y = x^2 + 3x - 4$ is $y' = -(x')^2 + x' + 4$.



Determining the transformation matrix based on images of non-standard points

When you are given non-standard points and their images, the procedure for determining the transformation matrix of a linear transformation is not as straightforward. In general, the images of any two points P and Q that are away from and non-collinear with the origin (that is, not on the same line through the origin) will be sufficient to determine the matrix representation of the transformation.

WARNING

When using two non-standard points to determine the transformation matrix, do not use the origin (0, 0) or pairs of points collinear with the origin.

39 Determining the transformation matrix using simultaneous solutions

A certain linear transformation maps the points $(2, 3) \rightarrow (2, -3)$ and $(-1, 5) \rightarrow (-14, 8)$. Determine the transformation matrix associated with this linear equation by setting up and solving simultaneous equations.

THINKING	WORKING
1 Recall the standard transformation matrix.	$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2 Create a matrix equation for each point and multiply through to determine equations involving a, b, c and d.	For point $(2, 3) \rightarrow (2, -3)$: $\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $2 = 2a + 3b$ $-3 = 2c + 3d$ For point $(-1, 5) \rightarrow (-14, 8)$: $\begin{bmatrix} -14 \\ 8 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ $-14 = -a + 5b$ $8 = -c + 5d$
3 Combine these equations to create a matrix equation involving a coefficient matrix.	$2a + 3b + 0c + 0d = 2$ $0a + 0b + 2c + 3d = -3$ $-1a + 5b + 0c + 0d = -14$ $0a + 0b - c + 5d = 8$ $\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ -1 & 5 & 0 & 0 \\ 0 & 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -14 \\ 8 \end{bmatrix}$

4 Solve this equation for a, b, c and d. Use technology to calculate the inverse of the coefficient matrix.

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ -1 & 5 & 0 & 0 \\ 0 & 0 & -1 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -14 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ -1 & 5 & 0 & 0 \\ 0 & 0 & -1 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -3 \\ -14 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

5 Interpret your result.

Therefore, the transformation matrix associated with this linear transformation is $\mathbf{L} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

40 Determining the transformation matrix using the inverse matrix

A certain linear transformation maps the points $(2, 3) \rightarrow (2, -3)$ and $(-1, 5) \rightarrow (-14, 8)$. Determine the transformation matrix associated with this linear equation by calculating an inverse matrix.

THINKING	WORKING
1 Recall the standard transformation matrix.	$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2 Create a matrix equation involving both transformed points in the same equation.	$\begin{bmatrix} x_1' & x_2' \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ $\begin{bmatrix} 2 & -14 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$
3 Post-multiply by the inverse to determine an expression for the transformation matrix. You can use technology to calculate the inverse.	$\begin{bmatrix} 2 & -14 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
4 Interpret your result.	Therefore, the transformation matrix associated with this linear transformation is $\mathbf{L} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Technology worked example
Determining a 2×2 transformation matrix from points

EXERCISE 8.9 Determining the transformation matrix

Worked Example 37

1 Determine the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for the following linear transformations, given the indicated standard point mappings:

(a) $(1, 0) \rightarrow (1, 1)$ and $(0, 1) \rightarrow (-5, 2)$

(b) $(1, 0) \rightarrow (0, 2)$ and $(0, 1) \rightarrow (-1, 3)$

(c) $(1, 0) \rightarrow \left(\sqrt{2}, \sqrt{5}\right)$ and $(0, 1) \rightarrow \left(-\sqrt{2}, \sqrt{3}\right)$

(d) $(1, 0) \rightarrow \left(\frac{1}{2}, -\frac{3}{5}\right)$ and $(0, 1) \rightarrow \left(-\frac{1}{5}, -\frac{1}{2}\right)$

2 Which of the following sets of non-standard points could be used to determine the transformation matrix for a certain linear transformation?

A $A(1, 5)$ and $B(3, 15)$ B $A(0, 0)$ and $B(2, -4)$ C $A(-2, 6)$ and $B(1, -3)$ D $A(2, 0)$ and $B(1, 1)$

38

3 A certain linear transformation maps the points $(1, 0) \rightarrow \left(1, \sqrt{2}\right)$ and $(0, 1) \rightarrow \left(\sqrt{2}, -1\right)$. Determine the image of the line $y = 2x + 1$ under this transformation. Demonstrate your answer graphically.

4 A certain linear transformation maps the points $(1, 0) \rightarrow (-1, 3)$ and $(0, 1) \rightarrow (1, 2)$. Determine the image of the square $ABCD$ defined by the points $A(0, 0)$, $B(2, 0)$, $C(2, 2)$, $D(0, 2)$ under this transformation. Demonstrate your answer graphically.

39

5 A certain linear transformation maps the points $(1, 1) \rightarrow (4, -3)$ and $(4, -1) \rightarrow (11, -12)$. Determine the transformation matrix associated with this linear equation by setting up and solving simultaneous equations.

40

6 A certain linear transformation maps the points $\left(\sqrt{2}, 1\right) \rightarrow \left(2, \sqrt{2} + 1\right)$ and $\left(1, \sqrt{3}\right) \rightarrow \left(2\sqrt{3}, 1 + \sqrt{3}\right)$. Determine the transformation matrix associated with this linear equation by calculating an inverse matrix.

7 The transformation matrix for a certain linear equation is $\mathbf{L} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix}$. It is known that this transformation maps the point $A(2, 1)$ to the image point $A'(10b - 2, 8b + 12)$. Use this information to solve for a and b.

8 The matrix equation for rotation of angle $\theta°$ anticlockwise around the origin is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Demonstrate that $\cos(\theta) = \dfrac{xx' + yy'}{x^2 + y^2}$.

(b) Demonstrate that $\sin(\theta) = \dfrac{xy' - yx'}{x^2 + y^2}$.

(c) The image of the point $A(2, 4)$ after a rotation about the origin is $A'\left(1 - 2\sqrt{3}, 2 + \sqrt{3}\right)$. Use your answers to (a) and (b) to determine the transformation matrix $\mathbf{R}_{\theta°}$ for this transformation.

8.10 Combining transformations

Composition of linear transformations

Consider the application of two successive linear transformations on a set of points or a curve. It is possible to express this combined transformation using a single linear transformation matrix.

Suppose that two separate linear transformations are applied to the plane. Let the first transformation be defined by the transformation matrix $\mathbf{L}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let the second transformation be defined by the transformation matrix $\mathbf{L}_2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

The image points for the first transformation can be calculated:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$x' = ax + by$ and $y' = cx + dy$

The image points for the second transformation can then be calculated:

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} p(ax + by) + q(cx + dy) \\ r(ax + by) + s(cx + dy) \end{bmatrix}$$

$x'' = apx + bpy + cqx + dqy$ and $y'' = arx + bry + csx + dsy$

The two equations can be rewritten:

$x'' = (ap + cq)x + (bp + dq)y$ and $y'' = (ar + cs)x + (ds + br)y$

Reforming these back into a matrix equation:

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \mathbf{L}_2\mathbf{L}_1 \begin{bmatrix} x \\ y \end{bmatrix}$$

Two successive linear transformations defined by the transformation matrices $\mathbf{L}_1$ (representing the first transformation) and $\mathbf{L}_2$ (representing the second transformation) can be combined into a single transformation defined by the matrix $\mathbf{L}_2\mathbf{L}_1$.

The second transformation matrix must be multiplied first.

This result is important, because it demonstrates how two successive linear transformations can be defined by a single linear transformation matrix. However, as matrix multiplication is not commutative, the order in which the two transformation matrices are multiplied is important.

41 Determining the composition transformation matrix

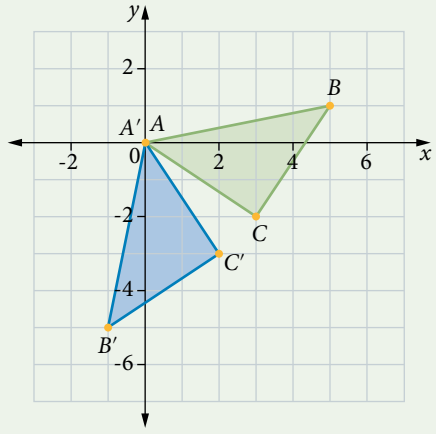
Determine the transformation matrix $\mathbf{L}_c$ that would represent a transformation $\mathbf{L}_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ followed by a transformation $\mathbf{L}_2 = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$.

THINKING	WORKING
1 Recall the equation for determining the composite transformation matrix.	$\mathbf{L}_c = \mathbf{L}_2\mathbf{L}_1$
2 Multiply the two transformation matrices in the correct order to determine the composite matrix.	$\mathbf{L}_c = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}$
3 Interpret your result.	The composite transformation matrix for this linear transformation is $\mathbf{L}_c = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}$.

42 Performing a combined linear transformation

Determine the image of ΔABC, defined by points $A(0, 0)$, $B(5, 1)$ and $C(3, -2)$, after undergoing a reflection in the y-axis followed by a rotation of 90° anticlockwise around the origin.

THINKING	WORKING
1 Determine the two transformation matrices.	For a reflection in the y-axis: $\mathbf{L}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ For a rotation of 90°: $\mathbf{L}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
2 Multiply the two transformation matrices in the correct order to determine the composite matrix.	$\mathbf{L}_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
3 Multiply the transformation matrix by the original set of points to determine the image points.	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -5 & -3 \end{bmatrix}$
4 Interpret and graph your result. (*Note*: There is no need to use the double-prime notation A'').	The image is a triangle, defined by points $A'(0, 0)$, $B'(-1, -5)$ and $C'(2, -3)$.



Technology worked example
Combining two linear transformations into a single 2×2 matrix

Irregular transformations

Linear transformations by their very nature are oriented about the origin (0, 0). For instance, the rotation transformations you have encountered have all used the origin as the point of rotation, and the reflection transformations you have encountered have been based around a line of reflection that passes through the origin. It is also possible to perform a rotation about a point *other* than the origin, or a reflection in a different line, but it is not straightforward. The following example will demonstrate how it can be achieved.

43 Irregular transformations

Determine the image of the curve $y = x^2$ when it is rotated 180° anticlockwise about the point (1, 2).

THINKING	WORKING
1 Determine the individual transformations required to perform the required operation.	To rotate 180° anticlockwise about the point (1, 2), the entire plane should be translated 2 units down and 1 unit to the left so that the point of rotation is now the origin. The rotation of 180° anticlockwise about the origin is performed, then the plane should be translated 2 units up and 1 unit to the right to return to its former position.
2 Perform the first transformation. (*Note*: For ease of reference, the prime markings x' and y' can be removed after each intermediate transformation to create a new 'original' equation.)	First, translate 2 units down and 1 unit to the left: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $x = x' + 1$ and $y = y' + 2$ Substitute into the original equation: $y = x^2$ $(y' + 2) = (x' + 1)^2$ $y' = (x')^2 + 2x' - 1$ $\therefore \ y = x^2 + 2x - 1$

3 Perform the second transformation.

Next, rotate 180° anticlockwise about the origin.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = -x'$$

$$y = -y'$$

Substitute into the equation from step 2:

$$y = x^2 + 2x - 1$$

$$-y' = (-x')^2 - 2x' - 1$$

$$y' = -(x')^2 + 2x' + 1$$

$$\therefore\ y = -x^2 + 2x + 1$$

4 Perform the third transformation.

Translate 2 units up and 1 unit to the right:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = x' - 1$ and $y = y' - 2$

Substitute into the equation from step 3:

$$y = -x^2 + 2x + 1$$

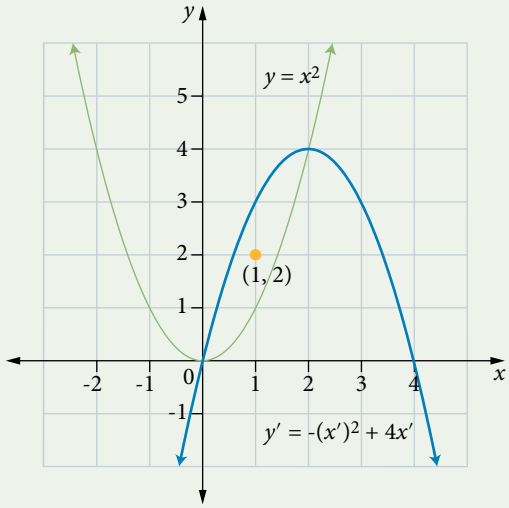
$$y' - 2 = -(x' - 1)^2 + 2(x' - 1) + 1$$

$$y' = -\left((x')^2 - 2x' + 1\right) + 2x' - 2 + 1 + 2$$

$$y' = -(x')^2 + 4x'$$

5 Interpret and sketch the graph of your result.

The image is $y' = -(x')^2 + 4x'$



EXERCISE 8.10 Combining transformations

Worked Example

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1 Determine the transformation matrix $\mathbf{L}_c$ that would combine, in the correct order, the following transformation matrices.

(a) $\mathbf{L}_1 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ followed by $\mathbf{L}_2 = \begin{bmatrix} 2 & -2 \\ 5 & -1 \end{bmatrix}$. (b) $\mathbf{L}_1 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ followed by $\mathbf{L}_2 = \begin{bmatrix} -2 & -1 \\ -4 & 0 \end{bmatrix}$.

(c) $\mathbf{L}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ followed by $\mathbf{L}_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

2 Determine the transformation matrix $\mathbf{L}_c$ that would combine the following transformations in the correct order.

(a) a reflection in the x-axis followed by a reflection in the y-axis

(b) a rotation of 180° about the origin followed by a reflection in the line $y = x$

(c) a horizontal dilation of factor $k = 3$ followed by a rotation of 270° about the origin

3 (a) For which of the following combinations of transformations would the *order* of the transformation matrix multiplication be unimportant?

A a rotation of 90° anticlockwise and a reflection in the x-axis

B a reflection in the line $y = x$ and a reflection in the x-axis

C a rotation of 90° anticlockwise and a rotation of 180° anticlockwise

D a reflection in the y-axis and a rotation of 270° about the origin

(b) Use the incorrect alternatives in part (a) to demonstrate that the order of the transformation matrix multiplication is important.

4 A transformation defined by $\mathbf{L}_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ followed by $\mathbf{L}_2 = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ can be represented by which single composite transformation matrix?

A $\mathbf{L}_c = \begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}$ B $\mathbf{L}_c = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$ C $\mathbf{L}_c = \begin{bmatrix} 4 & 2 \\ -2 & 5 \end{bmatrix}$ D $\mathbf{L}_c = \begin{bmatrix} -2 & -2 \\ 0 & -3 \end{bmatrix}$

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5 Determine the image of ΔABC, defined by points $A(1, 1)$, $B(1, 6)$ and $C(-2, 4)$ after undergoing a reflection in the x-axis followed by a vertical dilation of factor $k = 2$. Demonstrate your answer graphically.

6 Determine the image of line segment $\overline{AB}$ defined by points $A(7, 3)$ and $B(8, 2)$ after undergoing successive transformations defined by $\mathbf{L}_1 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{L}_2 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. Demonstrate your answer graphically.

7 Determine the image of the line $y = 5 - 3x$ after being reflected in the line $y = x$ and then reflected in the x-axis. Demonstrate your answer graphically.

8 Determine the image of the circle $x^2 + y^2 = 9$ after undergoing a horizontal dilation of factor $k = 2$ followed by a rotation of 270° about the origin. Demonstrate your answer graphically.

9 Determine the image of ΔABC, defined by points $A(0, 0)$, $B(2, -1)$ and $C(1, 5)$ after being rotated 90° anticlockwise around point B. Verify your answer graphically.

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10 Determine the image of the curve $y = x^2 - 1$ after being reflected in the line $x = 1$. Verify your answer graphically.

Establishing geometric results using matrix multiplication

8.11

Calculating points of intersection using matrix multiplication

You have seen earlier in the chapter how matrices can be used to solve simultaneous equations. The geometric equivalent of this process involves calculating points of intersection between two linear functions on the Cartesian plane.

44 Calculating points of intersection using matrix multiplication

Use a matrix method to calculate the point of intersection of the linear functions $5x + 2y = 1$ and $y = 2 - x$. Verify your answer graphically.

THINKING

1 Express the two functions in the form $ax + by = c$.

2 Set up a matrix equation involving a coefficient matrix.

3 Solve the matrix equation by calculating the inverse of the coefficient matrix.

4 Interpret and graph your result.

WORKING

$5x + 2y = 1$ and $x + y = 2$

$$\begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

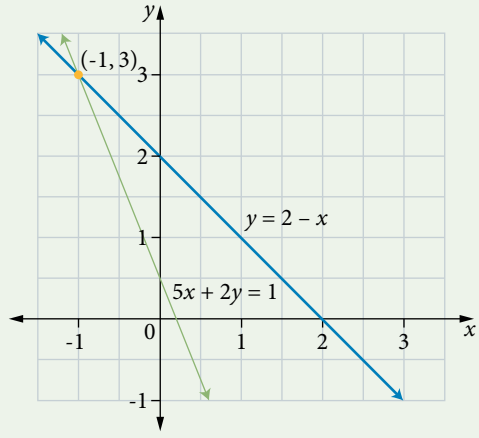
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = \frac{1}{3} - \frac{4}{3} = -1$$

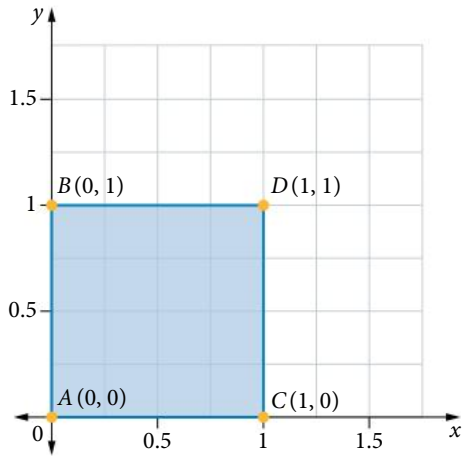
$$y = -\frac{1}{3} + \frac{10}{3} = 3$$

The point of intersection is (-1, 3).



Relationship between the determinant and area of image

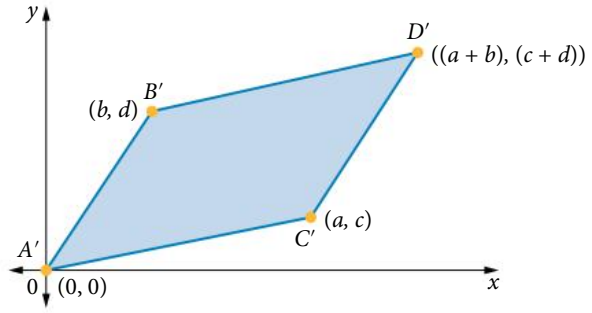
Consider a unit square $ABCD$ defined by the points $A(0, 0)$, $B(0, 1)$, $C(1, 0)$ and $D(1, 1)$. Note that by definition, the unit square has an area of 1 unit^2.



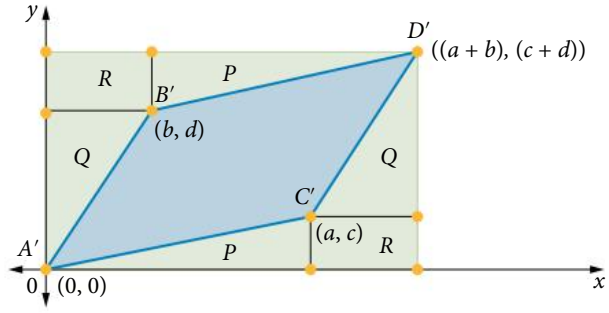
When this unit square undergoes a linear transformation in the plane based on the general transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the image points can be found as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b & a & a+b \\ 0 & d & c & c+d \end{bmatrix}$$

Therefore, the image points are $A'(0, 0)$, $B'(b, d)$, $C'(a, c)$ and $D'(a + b, c + d)$. The image of the unit square is a parallelogram $A'B'C'D'$ (see right).



The area of this parallelogram can be calculated by using composite regular polygons as follows:



$$\begin{aligned}
\text{area}_{(\text{whole rectangle})} &= (a+b)(c+d) \\
&= ac + ad + bc + bd \\
2 \times \text{area}_{(\text{P})} &= 2 \times \frac{1}{2}(ac) = ac \\
2 \times \text{area}_{(\text{Q})} &= 2 \times \frac{1}{2}(bd) = bd \\
2 \times \text{area}_{(\text{R})} &= 2 \times (bc) = 2bc \\
\text{area}_{(A'B'D'C')} &= ac + ad + bc + bd - ac - bd - 2bc \\
&= ad - bc \\
&= \det \mathbf{L}
\end{aligned}$$

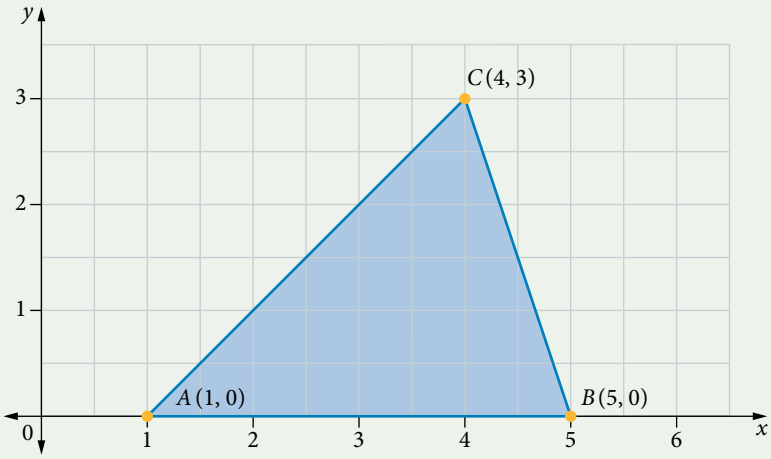
Therefore, the unit square has been transformed under the linear transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that the area of the unit square is $\det \mathbf{L}$. This suggests a relationship between the areas of the original and image shapes, and the determinant of the transformation matrix.

If an original (pre-image) shape undergoes a linear transformation defined by $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{area}_{image} = |\det \mathbf{L}| \times \text{area}_{pre\text{-}image}$.

The absolute value of the determinant gives the area ratio. If the determinant is negative, the image may be a reflection of the original shape.

45 Relationship between determinant and area of image

(a) ΔABC is formed by the vertex points $A(1, 0)$, $B(5, 0)$ and $C(4, 3)$. Determine the area of the image of this shape after undergoing a transformation defined by $\mathbf{L} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$.

THINKING	WORKING
1 Calculate the area of the original shape.	 $\text{area} = \dfrac{b \times h}{2}$ $= \dfrac{4 \times 3}{2}$ $= 6 \text{ units}^2$
2 Calculate the determinant of the transformation matrix and use this to calculate the area of the image.	$\mathbf{L} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ $\det \mathbf{L} = (-2 \times -1) - (0 \times 0)$ $= 2$ $\text{area}_{image} = \lvert \det \mathbf{L} \rvert \times \text{area}_{pre\text{-}image}$ $= \lvert 2 \rvert \times 6$ $= 12 \text{ units}^2$
3 Interpret your result.	The area of the image $\Delta A'B'C' = 12 \text{ units}^2$.

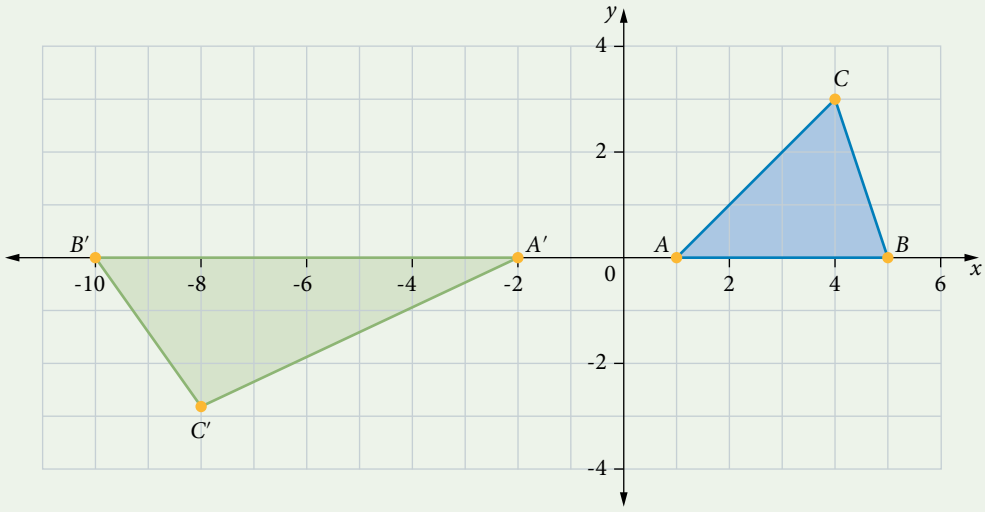
(b) Evaluate the reasonableness of your answer to part **(a)** by determining the image $\Delta A'B'C'$ and calculating its area directly.

1 Determine the image of $\Delta A'B'C'$ by matrix multiplication.

$$\begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -10 & -8 \\ 0 & 0 & -3 \end{bmatrix}$$

2 Interpret and graph your result.

The image is $\Delta A'B'C'$ defined by the points $A'(-2, 0)$, $B'(-10, 0)$ and $C'(-8, -3)$.



3 Calculate the area of the image.

$$\text{area} = \frac{b \times h}{2}$$
$$= \frac{8 \times 3}{2}$$
$$= 12 \text{ units}^2$$

This verifies the answer from **(a)**.

Establishing trigonometric identities using matrix transformations

Transformation matrices can be used to establish many of the trigonometric identities you have studied in previous chapters. Most of these proofs utilise the composition of different rotations to establish the trigonometric identity.

46 Establishing trigonometric identities using matrix transformations

(a) Use the composition of rotation matrices $\mathbf{R}_\theta$ and $\mathbf{R}_{-\theta}$ to prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$.

THINKING	WORKING
1 Remember the matrices $\mathbf{R}_\theta$ and $\mathbf{R}_{-\theta}$.	$\mathbf{R}_{\theta^\circ}=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$ and $\mathbf{R}_{-\theta^\circ}=\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}$ $=\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}$ (*Note*: $\cos(-\theta)=\cos(\theta)$ and $\sin(-\theta)=-\sin(\theta)$.)
2 Establish a rule for the composition of rotations.	$\mathbf{R}_\theta$ and $\mathbf{R}_{-\theta}$ represent inverse transformations. When inverse transformations are combined, they leave the original image unchanged. In other words, $\mathbf{R}_{-\theta}\times\mathbf{R}_\theta=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
3 Multiply the transformation matrices.	$\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\times\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\begin{bmatrix}\cos^2(\theta)+\sin^2(\theta) & -\sin(\theta)\cos(\theta)+\sin(\theta)\cos(\theta)\\ -\sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta) & \sin^2(\theta)+\cos^2(\theta)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\begin{bmatrix}\cos^2(\theta)+\sin^2(\theta) & 0\\ 0 & \sin^2(\theta)+\cos^2(\theta)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
4 Equate the relevant matrix element to complete the proof.	$\therefore \sin^2(\theta)+\cos^2(\theta)=1$

(b) Use composition of rotations to prove the identity $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$.

THINKING	WORKING
1 Recall the individual rotation matrices.	$\mathbf{R}_\alpha=\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$ $\mathbf{R}_\beta=\begin{bmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{bmatrix}$ $\mathbf{R}_{\alpha+\beta}=\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$
2 Establish a rule for the composition of rotations.	A rotation of $(\alpha+\beta)$ is the same as a rotation of α followed by a rotation of β. A rotation of α followed by a rotation of β can be combined into a single transformation matrix using the rule $\mathbf{L}_c=\mathbf{L}_2\mathbf{L}_1$: $\mathbf{L}_c=\begin{bmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{bmatrix}\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$ $=\begin{bmatrix}\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta)\end{bmatrix}$ A rotation of $(\alpha+\beta)$ can also be represented by the matrix $\mathbf{R}_{\alpha+\beta}=\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$ Therefore: $\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$ $=\begin{bmatrix}\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta)\end{bmatrix}$
3 Equate the relevant matrix element to complete the proof.	From element$_{11}$ of both matrices: $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$.

EXERCISE 8.11 Establishing geometric results using matrix multiplication

1 The area of an original shape on the Cartesian plane is 2 units^2. After undergoing a linear transformation defined by the matrix **L**, the area of its image is 3 units^2. Which of the following transformation matrices **L** could have brought about this transformation?

A $\mathbf{L} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\mathbf{L} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ **C** $\mathbf{L} = \begin{bmatrix} -2 & \frac{1}{2} \\ -1 & 1 \end{bmatrix}$ **D** $\mathbf{L} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -1 & 2 \end{bmatrix}$

2 Which of the following matrix equations could be used to describe the simultaneous equations $y = 2x + 1$ and $y = 5 - x$?

A $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

3 ΔABC has an area of 15 units^2. Calculate the area of the image of this triangle, $\Delta A'B'C'$, after undergoing linear transformations defined by the following transformation matrices.

(a) $\mathbf{L} = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$ **(b)** $\mathbf{L} = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix}$ **(c)** $\mathbf{L} = \begin{bmatrix} \sqrt{2} & \sqrt{3} \\ -\sqrt{3} & -\sqrt{2} \end{bmatrix}$ **(d)** $\mathbf{L} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

Worked Example 44

4 Use matrix methods to calculate the point of intersection of the following pairs of linear functions, and verify your answer graphically.

(a) $x - y = -5$ and $2x - y = 1$ **(b)** $y = 11 - 2x$ and $y = x - 3$ **(c)** $y = x + 5$ and $y = 2x - 7$

5 **(a)** ΔABC is formed by the vertex points $A(0, 8)$, $B(4, 8)$ and $C(4, 3)$. Determine the area of the image of this shape after undergoing a transformation defined by $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

45

(b) Verify your answer to part **(a)** by determining the image $\Delta A'B'C'$ and calculating its area directly.

6 Using matrix methods, demonstrate that the rotation of a line $y = mx + c$ by 90° or 270° about the origin results in an image line with gradient $-\frac{1}{m}$.

7 Use composition of rotations to prove the identity $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

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8 Use composition of rotations to prove the identity $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

9 Use composition of rotations to prove the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

10 Use matrix methods to prove that $\cos(3\theta) = \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta)$, hence demonstrate that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

8

Summary

Introduction to matrices

A *matrix* is a rectangular array of numbers enclosed by square brackets and denoted by a bold, capital letter. A matrix is displayed in the form of rows (horizontal) and columns (vertical).

The *order* (or *size* or *dimensions*) of a matrix is determined by the number of rows and columns in the matrix, in that order. A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

The symbol a_{ij} represents the element occurring in the i th row and the j th column of matrix **A**.

Matrix arithmetic

To add and subtract matrices, the order of each matrix must be the same. For addition, add the corresponding elements. For subtraction, subtract the corresponding elements.

In general, if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$.

In general, if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\mathbf{A} - \mathbf{B} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$.

When a matrix is multiplied by a real number, known as a *scalar*, each element in that matrix is multiplied by that scalar. This is known as *scalar multiplication*.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $k\mathbf{A} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ where k is a scalar.

Matrix multiplication

For matrix multiplication to be defined, the number of columns in the first matrix must equal the number of rows in the second matrix. In general, if an $m \times n$ matrix is multiplied by an $n \times p$ matrix, the product matrix will be of order $m \times p$.

For 2×2 matrices, if $\mathbf{A} = \begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \boxed{e} & f \\ \boxed{g} & h \end{bmatrix}$, then $\mathbf{A} \times \mathbf{B} = \begin{bmatrix} \boxed{ae+bg} & af+bh \\ ce+dg & cf+dh \end{bmatrix}$.

To calculate the element in the first row and first column of a product **AB**, multiply the first row of **A** by the first column of **B** and add up, $\mathbf{AB} \neq \mathbf{BA}$.

Matrix multiplication is not commutative.

For an incidence matrix **C**, the element in row i and column j is 1 if i and j are related (incident) and is 0 if they are not. Such matrices can be used to represent the number of direct links between two objects, e.g. towns.

In general, when considering networks, the matrix **C** raised to the power of n represents the number of links between two objects via $(n-1)$ other objects.

Inverse matrices

In general, the identity matrix, **I**, is a square matrix with ones on the main diagonal and zero elements elsewhere:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ where n is the order of the square matrix.

The inverse of matrix **A** is the matrix that will give the identity matrix, **I**, when multiplied by **A**. In other words, $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$.

The inverse of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

In a 2×2 matrix, the value $(ad - bc)$ is known as the *determinant* of matrix **A** and is written det **A** or $|\mathbf{A}|$.

Inverses of order 3×3 and higher can be calculated using technology.

If det **A** = 0 then matrix **A** is singular and its inverse does not exist.

The matrix equation **AX** = **B** can be solved by pre-multiplying by the inverse of **A**:

$$\mathbf{AX} = \mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

The matrix equation **XA** = **B** can be solved by post-multiplying by the inverse of **A**:

$$\mathbf{XA} = \mathbf{B}$$
$$\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$$
$$\mathbf{X} = \mathbf{BA}^{-1}$$

Solving systems of linear equations using matrices

A system of linear equations can be represented as the matrix equation **AX** = **B**, where **A** is the coefficient matrix.

Translations in the plane

A translation involves the simple addition of scalars to the x and y values of a pre-image, such that $x' = x + a$ and $y' = y + b$

A translation that maps an original point $P(x, y)$ onto an image point $P'(x', y')$ can be expressed using column matrices:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$ where $\mathbf{T} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a translation vector corresponding to a movement of a units in the positive x-direction and b units in the positive y-direction.

Linear transformations on the plane

A linear transformation is defined as a mapping of the form: $x' = ax + by$ and $y' = cx + dy$.

A linear transformation maps a point $P(x, y)$ onto an image point $P'(x', y')$ according to the matrix equation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ where the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is known as the transformation matrix, **L**.

Reflections

A reflection is a linear transformation in which the pre-image is 'reflected' to create an image on the opposite side of the *line of reflection* (or 'mirror line').

The matrix for a reflection in the x-axis: $\mathbf{M}_{y=0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The matrix for a reflection in the y-axis: $\mathbf{M}_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

The matrix for a reflection in the line $y = x$: $\mathbf{M}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The matrix for a reflection in the line $y = x\tan(\theta)$: $\mathbf{M}_{y=x\tan(\theta)} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Rotations

A rotation is a transformation in which the entire plane spins around a certain fixed point, known as the *centre of rotation*.

The matrix for a rotation of 90°: $\mathbf{R}_{90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The matrix for a rotation of 180°: $\mathbf{R}_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

The matrix for a rotation of 270°: $\mathbf{R}_{270^\circ} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

The matrix for a general rotation of θ°: $\mathbf{R}_{\theta^\circ} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Dilations

A dilation is a linear transformation that either enlarges or shrinks the pre-image on the plane. Dilations can be performed in either a horizontal (parallel to the x-axis) or a vertical (parallel to the y-axis) direction, or can be applied to the entire plane.

The matrix for a horizontal dilation of factor k:

$$\mathbf{D}_{k,x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix for a vertical dilation of factor k:

$$\mathbf{D}_{k,y} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

The matrix for a dilation of factor k about the origin:

$$\mathbf{D}_{k} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Applying linear transformations

A linear transformation defined by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix}$ is invertible if and only if there is a matrix $\mathbf{L}^{-1}$ such that $\mathbf{L}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{L}^{-1}\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix}$, where $\mathbf{L}^{-1}\mathbf{L} = \mathbf{I}$, the identity matrix.

Some linear transformations on the plane will also leave a certain line or lines unchanged. Such lines are referred to as *invariant lines*. Invarient lines in reflections through the y-axis are in the form $y = k$ or $x = 0$. Invariant lines in reflections through the x-axis are in the form $x = k$ or $y = 0$. Invariant lines under a linear transformation $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ pass through the origin and the gradient can be calculated using $m = \frac{(c + dm)}{(a + bm)}$.

Determining the transformation matrix

The transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, used to define a linear transformation, can be constructed by considering the images of points (1, 0) and (0, 1), such that $(1, 0) \rightarrow (a, c)$ and $(0, 1) \rightarrow (b, d)$.

When given non-standard points, there are two methods for determining the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of a linear transformation. The first method is to set up and solve simultaneous equations; the second method is to set up a transformation equation and use inverse matrices to calculate $\mathbf{L}$.

Combining linear transformations

Two successive linear transformations defined by the transformation matrices $\mathbf{L}_1$ (representing the first transformation) and $\mathbf{L}_2$ (representing the second transformation) can be combined into a single transformation defined by the matrix $\mathbf{L}_2\mathbf{L}_1$.

Irregular transformations, such as rotation about a point that is *not* the origin or reflection in a line *not* passing through the origin, can be performed by first translating the plane, then performing the linear transformation, then translating the plane back to its original position.

Establishing geometric results using matrix multiplication

If an original (pre-image) shape undergoes a linear transformation defined by $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{area}_{image} = |\det \mathbf{L}| \times \text{area}_{pre\text{-}image}$.

Trigonometric identities can be established by calculating the composition of successive rotations or reflections in the plane.

Chapter review 8

Exercise 8.1

1 Consider $\mathbf{A} = \begin{bmatrix} 3 & -6 & 5 \\ 0 & -4 & 0 \\ -2 & 0 & 2 \end{bmatrix}$.

(a) Determine the order of $\mathbf{A}$.

(b) Determine the value of a_{31}.

(c) Which element of matrix $\mathbf{A}$ is the number -6?

Exercise 8.1

2 Consider the following two matrices. If $\mathbf{A} = \mathbf{B}$, determine the values of a and b.

$$\mathbf{A} = \begin{bmatrix} 2a-1 & a+3 \\ -2 & -6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & \frac{7}{2} \\ b+2 & b-2 \end{bmatrix}$$

Exercise 8.1

3 The sales of sofas (S), lounge chairs (L) and coffee tables (C) at a company's three furniture stores for the month of August are represented by the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{array}{c} \begin{matrix} \text{S} & \text{L} & \text{C} \end{matrix} \\ \begin{bmatrix} 7 & 4 & 9 \\ 3 & 8 & 15 \\ 6 & 11 & 12 \end{bmatrix} \end{array} \begin{matrix} \text{Store 1} \\ \text{Store 2} \\ \text{Store 3} \end{matrix}$$

The corresponding sales at the three stores for September are represented by the matrix $\mathbf{S}$.

$$\mathbf{S} = \begin{array}{c} \begin{matrix} \text{S} & \text{L} & \text{C} \end{matrix} \\ \begin{bmatrix} 5 & 6 & 8 \\ 9 & 12 & 14 \\ 2 & 7 & 13 \end{bmatrix} \end{array} \begin{matrix} \text{Store 1} \\ \text{Store 2} \\ \text{Store 3} \end{matrix}$$

(a) Determine the sales of sofas, armchairs and coffee tables for the three furniture stores for the two months combined. Give your answer in matrix form $(\mathbf{A} + \mathbf{S})$.

(b) What does the matrix element $(\mathbf{A} + \mathbf{S})_{31}$ represent?

Exercise 8.1

4 Consider the following two matrices, $\mathbf{M} = \begin{bmatrix} 0 & -3 \\ 7 & 5 \\ 5 & -2 \\ 9 & -4 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & -2 \\ 6 & 7 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$.

(a) Calculate $\mathbf{N} - \mathbf{M}$.

(b) Calculate $\mathbf{M} - \mathbf{N}$.

Exercise 8.1

5 If $\mathbf{A} = \begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$, calculate $5\mathbf{A} - 3\mathbf{B}$.

Exercise 8.1

6 If $\mathbf{A} = \begin{bmatrix} 6 & -9 \\ -3 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$, calculate $\frac{1}{3}\mathbf{A} + 4\mathbf{B}$.

Exercise 8.2

7 Let $\mathbf{P} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 0 & 2 \\ -1 & 2 \\ -2 & 3 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 4 & 7 \end{bmatrix}$ and $\mathbf{T} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Determine whether the following matrix products can be calculated.

(a) $\mathbf{PQ}$ **(b)** $\mathbf{QP}$ **(c)** $\mathbf{PS}$ **(d)** $\mathbf{RT}$

8 For $\mathbf{A} = \begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 7 \\ 2 & -1 \end{bmatrix}$: Exercise 8.2

(a) Calculate **AB**.

(b) Calculate **BA**.

9 (a) If $\mathbf{A} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$, calculate $\mathbf{A}^{-1}$ without using technology. Exercise 8.3

(b) If $\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, calculate $\mathbf{B}^{-1}$ using technology.

10 The matrix equation $\mathbf{XA} = \mathbf{B}$ can be solved using which of the following equations? Exercise 8.3

A $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ B $\mathbf{X} = \mathbf{BA}^{-1}$ C $\mathbf{X} = \mathbf{AB}^{-1}$ D $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}$

11 Consider the following system of linear equations: $x + 3y - 2z = -2$ and $2x + 3 - 5z = 6y$ and $y - x - z = 3$. What is the correct way of representing these equations using matrices? Exercise 8.4

A $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 3 & -5 \\ 1 & -1 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$

B $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -6 & -5 \\ -1 & 1 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}$

C $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -6 & -5 \\ 1 & -1 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}$

D $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -5 \\ -1 & 1 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$

12 Calculate the inverse, $\mathbf{A}^{-1}$, of each of the following matrices. Exercise 8.3

(a) $\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 1 & -1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 2 & -2 \end{bmatrix}$

13 Express the following simultaneous linear equations in matrix form. Exercise 8.4

(a) $y = 2x - 3$ and $2x - 3y = 4$

(b) $5x - 2y = z$, $8y + z = 10$ and $x + 2y + 3z = 4$

14 Determine the image of line segment $\overline{AB}$ connecting points $A(2, -5)$ and $B(-1, 2)$ after undergoing a translation defined by $\mathbf{T} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Demonstrate your answer graphically. Exercise 8.5

15 Determine the image of the point $A(3, -2)$ after undergoing a linear transformation defined by each of the following matrices: Exercise 8.6

(a) $\mathbf{L} = \begin{bmatrix} 5 & 1 \\ -1 & 2 \end{bmatrix}$

(b) $\mathbf{L} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$

(c) $\mathbf{L} = \begin{bmatrix} -3 & 6 \\ -2 & 1 \end{bmatrix}$

16 Determine the image of ΔABC defined by the points $A(1, 1)$, $B(6, 3)$ and $C(4, 5)$ when reflected in the y-axis. Verify your answer graphically. Exercise 8.7

Exercise 8.8

17 For which of the following transformations does the inverse exist? If the inverse does not exist, determine the line onto which the plane is transformed.

(a) $\mathbf{L} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ **(b)** $\mathbf{L} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

Exercise 8.8

18 Determine the value of p such that the transformation matrix $\mathbf{L} = \begin{bmatrix} 3 & -1 \\ 4 & 2p \end{bmatrix}$ is non-invertible.

Exercise 8.10

19 Determine the single transformation matrix $\mathbf{L}_c$ that could be used to perform a composite of the following linear transformations.

(a) $\mathbf{L}_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ followed by $\mathbf{L}_2 = \begin{bmatrix} -5 & -4 \\ 1 & -1 \end{bmatrix}$

(b) $\mathbf{L}_1 = \begin{bmatrix} 0 & -3 \\ 1 & -1 \end{bmatrix}$ followed by an anticlockwise rotation of 90° about the origin

(c) a reflection in the line $y = x$ followed by $\mathbf{L}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Exercise 8.2

20 Martin scores the following marks (out of 100) for his assessment components in Chemistry: topic tests 74; practical work 81; semester examination 85. The final semester result for Chemistry is weighted as follows: topic tests 25%; practical work 25%; semester examination 50%. Use a matrix multiplication method to calculate Martin's final semester result (out of 100) for Chemistry.

Exercise 8.2

21 A manufacturer makes three products A, B and C, each requiring different amounts of construction time P, Q, R and S. Matrix $\mathbf{X}$ shows the construction time, in hours, required to make one of each product. The stages of construction P, Q, R and S cost the manufacturer \$120, \$65, \$40 and \$100 per hour respectively.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 2 & 3 & 1 & 2.5 \\ 0 & 1.5 & 1 & 3 \\ 1 & 2.5 & 4 & 2 \end{bmatrix} \end{array}$$

(a) Express the stages of hourly construction costs as a column matrix $\mathbf{Y}$.

(b) Calculate $\mathbf{XY}$ and determine what information it displays.

Exercise 8.3

22 Let $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$. Calculate $\mathbf{B}^{-1} + \mathbf{B}$.

Exercise 8.3

23 Solve the following matrix equations to determine $\mathbf{X}$.

(a) $\begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -2 & 3 \\ -3 & 2 \end{bmatrix}$ **(b)** $\mathbf{X}\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$

Exercise 8.4

24 Solve the following pairs of simultaneous equations and verify your answer graphically.

(a) $5x + 3y = 1$ and $x - 2y = 8$ **(b)** $y = 3 - x$ and $-5x - y = 1$

25 Use technology to solve the following simultaneous linear equations. Exercise 8.4

(a) $3x + 2y + z = -1$ and $x - 3y - z = 6$ and $2x - y + z = 1$

(b) $5x - 3y + z = 7$ and $-2x + y - z = -1$ and $3x + 5y + 2z = 0$

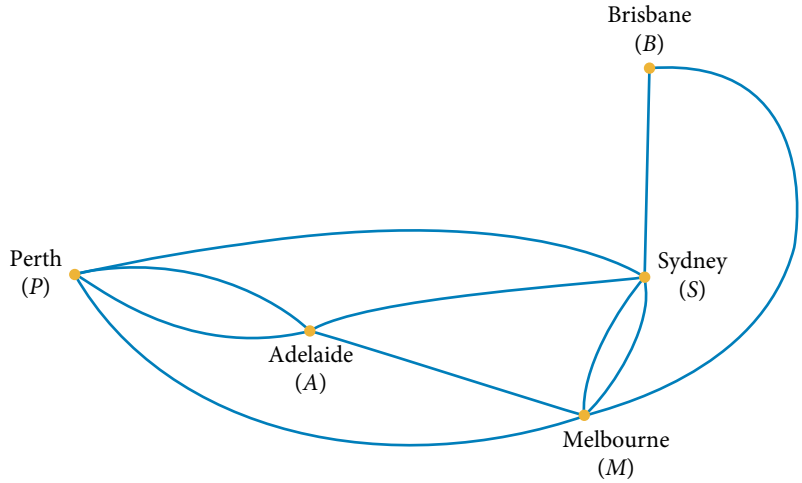
26 Determine and sketch the image of the curve $y = (x-1)^2$ after it is translated according to the matrix $\mathbf{T} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Exercise 8.5

27 Calculate and graph the image of the curve $y = x^2 - 3$ when rotated 90° anticlockwise about the origin. Exercise 8.7

28 A reflection in the x-axis has resulted in the image curve $y' = (x')^2 - 5x' + 6$. Determine the equation of the original curve, and verify your answer graphically. Exercise 8.8

29 A certain linear transformation maps the points (2, 6)→(10, -12) and (-5, 1)→(-9, 14). Determine the transformation matrix associated with this linear equation by setting up and solving simultaneous equations. Exercise 8.9

30 The following diagram shows the flight paths offered by an airline. Exercise 8.1



(a) Construct a matrix $\mathbf{R}$ showing all available direct routes. Order the cities alphabetically in your rows and columns.

(b) Use technology to calculate the matrix $\mathbf{R}^2$.

(c) How many routes are there from Melbourne to Sydney with one stopover?

(d) The element $r_{12} = 2$ in the matrix $\mathbf{R}^2$ describes which two routes?

31 A bakery sells loaves of bread, pies and sausage rolls. Jessica buys 1 loaf of bread, 2 pies and 1 sausage roll, and pays $14.70. Anh buys 2 loaves of bread and 5 pies, and pays $26.60. Greta buys 2 loaves of bread, 2 pies and 3 sausage rolls, and pays $24.50. Use a matrix method to calculate the individual cost of each item. Exercise 8.4

32 The line $y = 5 - 2x$ undergoes a translation defined by $\mathbf{T} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ followed by a linear transformation defined by $\mathbf{L} = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$. Determine and graph the image of the line. Exercise 8.6

Exercise 8.8 **33** Demonstrate that a rotation about the origin is always invertible. Hence, demonstrate that the rotation matrix $\mathbf{R}_{-\theta}$ is the inverse of the rotation matrix $\mathbf{R}_{\theta}$.

Exercise 8.8 **34** The linear transformation matrix $\mathbf{L} = \begin{bmatrix} p^2 & p \\ 0 & -1 \end{bmatrix}$ produces an invariant line of $y = 2x$. Determine the possible value(s) of p.

Exercise 8.10 **35** Determine the image of the curve $x^2 + y^2 = 16$ after undergoing a rotation of 270° around the point (-1, -1). Verify your answer graphically.

Exercise 8.10 **36** Determine the image of the curve $y = x^2$ after it is reflected in the line $y = x + 2$. Verify your answer graphically.

Exercise 8.11 **37** Use composition of rotations to prove the identity $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

Exercise 8.11 **38** Use composition of rotations to prove the identity $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$.

Exercise 8.11 **39** A regular polygon of area 12 units^2 lies on the Cartesian plane. When it undergoes a linear transformation defined by $\mathbf{L} = \begin{bmatrix} p & 5 \\ p & p \end{bmatrix}$, its new area is 48 units^2. Calculate the possible value(s) of p that would bring about such a transformation.

Exam review UNIT 2

Paper 1: Technology-free

1 If $u = 3 - 6i$ and $v = 1 + 3i$, then calculate the following: Exercise 5.1, 5.4

(a) $\text{Re}(v) - \text{Im}(u)$ (b) $\overline{u}$

(c) $|v|$ (d) Arg (u) (correct to 2 decimal places)

2 Simplify the following expressions: Exercise 5.1

(a) $7(2 + 3i)$ (b) $2(3 - 4i) - 8(1 + 5i)$ (c) $(3 + i)(5 - i)$

3 Given $z_1 = 5 - i$ and $z_2 = 3 + 2i$, calculate the following: Exercise 5.1

(a) $z_1 - z_2$ (b) $2z_1 + 3z_2$ (c) $z_1 z_2$

(d) $(z_2)^2$ (e) $(\overline{z}_1)^2$ (f) $\overline{z}_1 z_1$

(g) $\overline{z}_2 z_2$ (h) $(z_1 + z_2)(\overline{z}_1 - \overline{z}_2)$

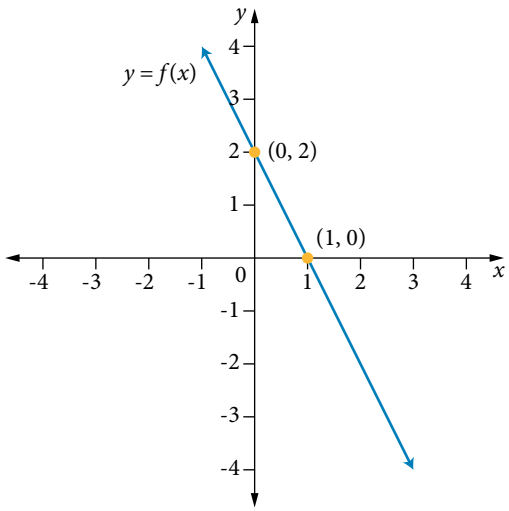
4 Simplify the following, expressing your answer in polar form: Exercise 5.4

(a) $2\text{ cis}\left(\frac{\pi}{8}\right) \times 5\text{ cis}\left(\frac{\pi}{4}\right)$ (b) $6\text{ cis}(\pi) \div 2\text{ cis}\left(\frac{\pi}{5}\right)$ (c) $\left[\sqrt{3}\text{ cis}\left(\frac{\pi}{12}\right)\right]^4$

5 Determine the following: Exercise 6.1

(a) $-|3 - 4| \times 2$ (b) $\frac{|-3|^2}{(-3)^2}$ (c) $||-5|-|5||$

6 The graph of the function $f(x)$ is shown below. Redraw the graph, and sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes and intercepts. Exercise 6.2



Exercise 6.3

7 Sketch each of the following functions, clearly indicating all intercepts and asymptotes.

(a) $y = \dfrac{3-3x}{2x-4}$ **(b)** $y = \dfrac{7-2x}{x^2-5x+4}$ **(c)** $y = \dfrac{x^2-4x+3}{x^2+1}$

Exercise 7.3

8 Determine the value of the following:

(a) $\sec(x)$ if $\tan(x) = \dfrac{1}{2\sqrt{2}}$, $\pi \leq x \leq \dfrac{3\pi}{2}$

(b) $\cos(2x)$ if $\sin(x) = \dfrac{1}{3}$, $\dfrac{\pi}{2} \leq x \leq \pi$

(c) $\sin(6x)$ if $\cos(3x) = -\dfrac{1}{4}$, $\pi \leq 3x \leq \dfrac{3\pi}{2}$

Exercise 7.1

9 Solve the following equations for $0 \leq x \leq 2\pi$.

(a) $2\tan(2x) + 2 = 0$ **(b)** $2\sin^2(x) + \sin(x) = 0$ **(c)** $\sin(x) - \cos(x) = 1$

Exercise 7.6

10 **(a)** Express $f(x) = \sqrt{3}\sin(x) + \cos(x)$ in the form $R\sin(x \pm \alpha)$.

(b) Hence, solve $f(x) = \sqrt{3}\sin(x) + \cos(x) = 0$, $0 \leq x \leq 2\pi$, expressing the solutions in exact form.

(c) Solve $f(x) = \pm 2$, $0 \leq x \leq 2\pi$.

(d) Calculate the coordinates of the end points of the domain.

(e) Sketch the graph of $f(x) = \sqrt{3}\sin(x) + \cos(x)$, $0 \leq x \leq 2\pi$, showing the axial intercepts and end points in coordinate form. Determine the period and amplitude of the function.

Exercise 8.3

11 If $\mathbf{A} = \begin{bmatrix} 4 & -8 \\ 2 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & -8 \end{bmatrix}$, calculate the following:

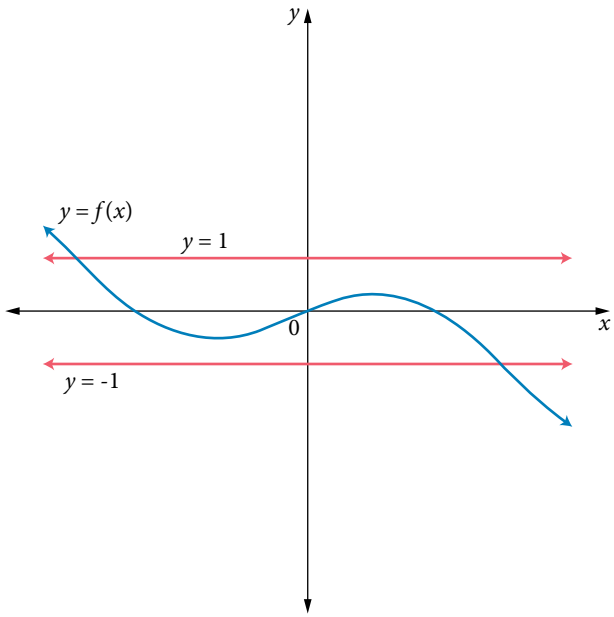
(a) $\dfrac{1}{2}\mathbf{A} + 3\mathbf{B}$ **(b)** $\mathbf{A}^2$ **(c)** $\mathbf{B}^{-1}$

Exercise 5.1

12 Express each of the following in the form $x + yi$.

(a) $\dfrac{-2+i}{-2+3i}$ **(b)** $\dfrac{3-i}{2-3i}$ **(c)** $\dfrac{2i}{3+i}$ **(d)** $\dfrac{4-3i}{3i}$

13 The graph of a function $f(x)$ is shown below, together with the lines $y = \pm 1$. Redraw the graph, and on the same set of axes, sketch the graph of $y = \dfrac{1}{f(x)}$. Ensure the points where $y = f(x)$ and $y = \dfrac{1}{f(x)}$ intersect are positioned accurately. Exercise 6.2



14 Use a matrix technique to solve the following system of simultaneous equations: $2x + 5y = -1$ and $y = x + 4$. Exercise 8.4

15 Determine all complex solutions to the equation $z^2 = i$. Exercise 5.5

16 Demonstrate that $z^2 + 3z + \dfrac{17}{2}$ can be expressed in factorised form as $\dfrac{1}{4}(2z + 3 - 5i)(2z + 3 + 5i)$. Exercise 5.5

Paper 2: Technology-active

Exercise 7.5

1 Write these expressions as the sum or difference of two trigonometric functions.

(a) $2\sin(5A)\cos(A)$ **(b)** $2\cos(3A)\cos(7A)$

Exercise 8.1

2 Consider the matrix $\mathbf{B} = \begin{bmatrix} 9 & 7 \\ 11 & -4 \\ 6 & 8 \end{bmatrix}$.

(a) What is the order of $\mathbf{B}$? **(b)** What is the value of b_{32}?

(c) What element is represented by the number 11?

Exercise 5.5

3 For the quadratic expression $5x^2 - 10x + 6$, determine all linear factors over $\mathbb{C}$.

Exercise 7.3

4 If $\cos(x) = 0.7$ and $\sin(y) = 0.2$, determine the value of each of the following, giving your answers correct to 2 decimal places.

(a) $\cos(x+y)$ **(b)** $\text{cosec}(2x)$ **(c)** $\tan(y)$

Exercise 7.4

5 Consider the identity $\left(\tan(\theta) + \sec(\theta)\right)^2 = \dfrac{1+\sin(\theta)}{1-\sin(\theta)}$.

(a) Substitute the value $\theta = \dfrac{\pi}{12}$ to verify the identity. **(b)** Prove that the identity is true for any angle.

Exercise 8.3

6 Solve the following matrix equations to determine $\mathbf{X}$.

(a) $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ **(b)** $\mathbf{X}\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -6 & -2 \end{bmatrix}$

Exercise 8.4

7 Solve the following system of simultaneous equations using a matrix technique: $2x + 2y - z = 1$, $x + 4y + z = -7$ and $x + 8z = -6$.

Exercise 8.6

8 ΔABC is formed by the vertex points $A(1,-4)$, $B(2,3)$ and $C(-2,1)$. Determine the image of this triangle after undergoing a linear transformation defined by the matrix $\mathbf{L} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

Exercise 8.7

9 Determine the image of the curve $y = x^2 - 4$ after undergoing a reflection in the line $y = x$.

Exercise 7.4

10 **(a)** If $x = \cos(\theta)$, determine an expression in terms of x for each of the following:

(i) $\sin^2(\theta) - \cos^2(\theta)$ **(ii)** $\tan^2(\theta)$ **(iii)** $\tan(2\theta)$

(b) If $\theta = 84°$, determine the value of $\tan(2\theta)$, giving your answer correct to 2 decimal places.

Exercise 6.1

11 Use algebra to determine all real solutions to the equation $|x| + |x+3| + |x+5| = 4x + 20$. Use technology to justify the reasonableness of your solution.

Exercise 8.7

12 **(a)** Determine the image of $y = 2 - x$ when reflected in $y = 0.1x$. Verify your answer graphically.

(b) Determine the gradient of any invariant line(s) under the transformation described in part **(a)**.

Exercise 8.10

13 Determine mathematically whether reflection in the y-axis followed by reflection in the x-axis is the *inverse* of a single rotation of 180° about the origin.

Exam review

UNITS 1 & 2

Paper 1: Technology-free

1 Consider the sets $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 7, 8\}$, $C = \{4, 6, 7\}$, and $\xi = \{1, 2, 3, ..., 10\}$. Verify the inclusion–exclusion principle for these sets. Exercise 1.1

2 Simplify the following expressions: Exercise 1.3

(a) $\dfrac{n!}{2(n-2)!}$ (b) $\dfrac{(n+1)!}{(n-2)!}$

3 Draw rows 0 to 6 of Pascal's triangle and use it to determine the values of the following: Exercise 1.6

(a) ${}^{6}C_{3}$ (b) ${}^{4}C_{4}$

4 If $\boldsymbol{a} = 5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 8\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, calculate $3\boldsymbol{a} - 2\boldsymbol{b}$. Exercise 2.3

5 Calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ given the following pairs of vectors. Exercise 2.4

(a) $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = 8\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

6 If $z_1 = 3 + 7i$ and $z_2 = -2 + i$, calculate the following: Exercise 5.1

(a) $5z_1 - 2z_2$ (b) $\dfrac{z_1}{z_2}$

7 If $z = \sqrt{2} - \sqrt{2}i$, calculate the following: Exercise 5.4

(a) $|z|$ (b) $\text{Arg}(z)$

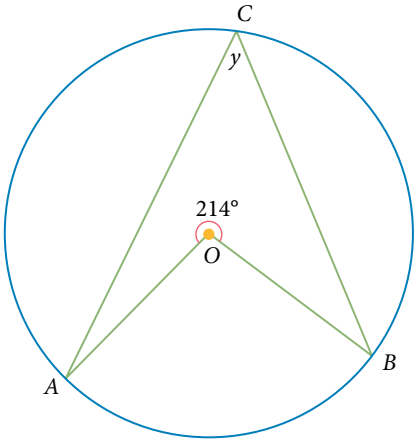
8 If $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 2 \end{bmatrix}$, calculate $2\mathbf{A} - 4\mathbf{B}$. Exercise 8.1

9 Write each of the following numbers in the form $\dfrac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1: Exercise 3.1

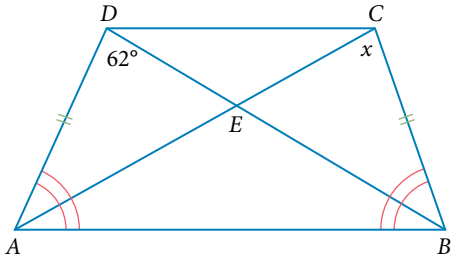
(a) $0.\dot{5}$ (b) $0.12\dot{5}$

10 Determine the values of the pronumerals in the following: Exercise 4.3

(a)

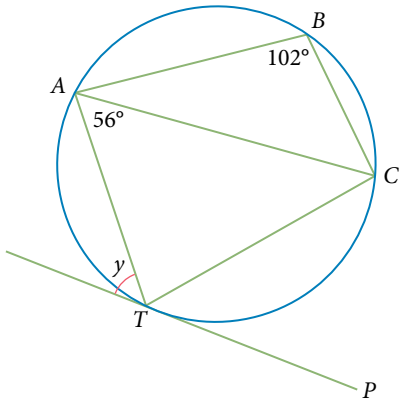


(b)



Exercise 4.3

11 Determine the value of y in the diagram shown below.



Exercise 7.1

12 Determine the values of the following.

(a) $\sin\left(\frac{11\pi}{6}\right)$ **(b)** $\tan\left(\frac{14\pi}{3}\right)$ **(c)** $\sec\left(\frac{3\pi}{4}\right)$

Exercise 7.4

13 If $\cos(\theta) = -\frac{7}{25}$, $\frac{\pi}{2} \leq \theta \leq \pi$, determine the following values:

(a) $\tan(\theta)$ **(b)** $\operatorname{cosec}(\theta)$

Exercise 6.1

14 Solve the following equations:

(a) $|x - 3| = 11$ **(b)** $|12 - 3x| = 9$

Exercise 7.1

15 Solve the equation $\cos\left(2\left(x - \frac{\pi}{4}\right)\right) = \frac{1}{2}$, $0 \leq x \leq 2\pi$.

Exercise 1.7

16 Prove that, having 50 whole numbers, you can choose 9 of them so that the difference of any two is divisible by 6.

Exercise 3.2

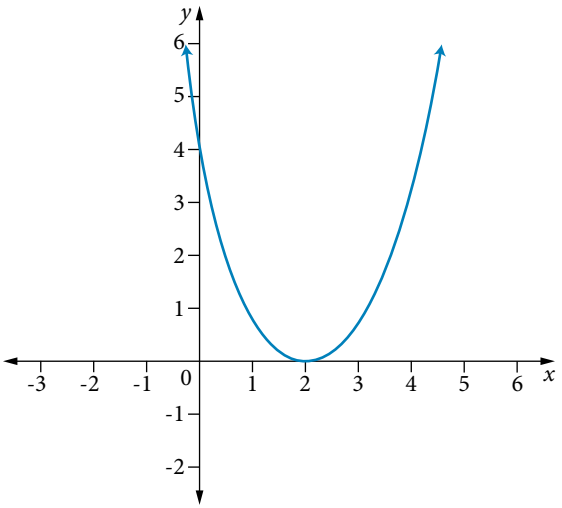
17 Write the negation of the following statement, where x represents a real number:
$x > 6$ or $x < -3 \Rightarrow (x - 6)(x + 3) > 0$. Also, determine whether the original or the negation is true.

Exercise 3.3

18 Prove that a four-digit number is divisible by 19 if and only if the number obtained by removing the final digit, and adding twice this digit to the remainder of the number, is divisible by 19.

Exercise 6.2

19 The graph of the function $f(x)$ is shown. Redraw the graph and sketch the graph of $y = \frac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes, intercepts and turning points.



20 Consider the vector $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. Exercise 2.2

(a) Calculate $\hat{\boldsymbol{a}}$.

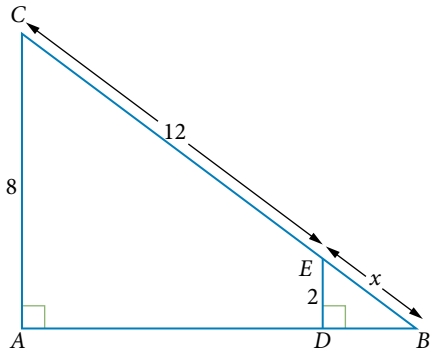
(b) Determine the vector $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ with a magnitude of 12.

21 Calculate the following, giving your answer in polar form. Ensure that you express the Argument in the domain $-\pi < \theta \leq \pi$. Exercise 5.4

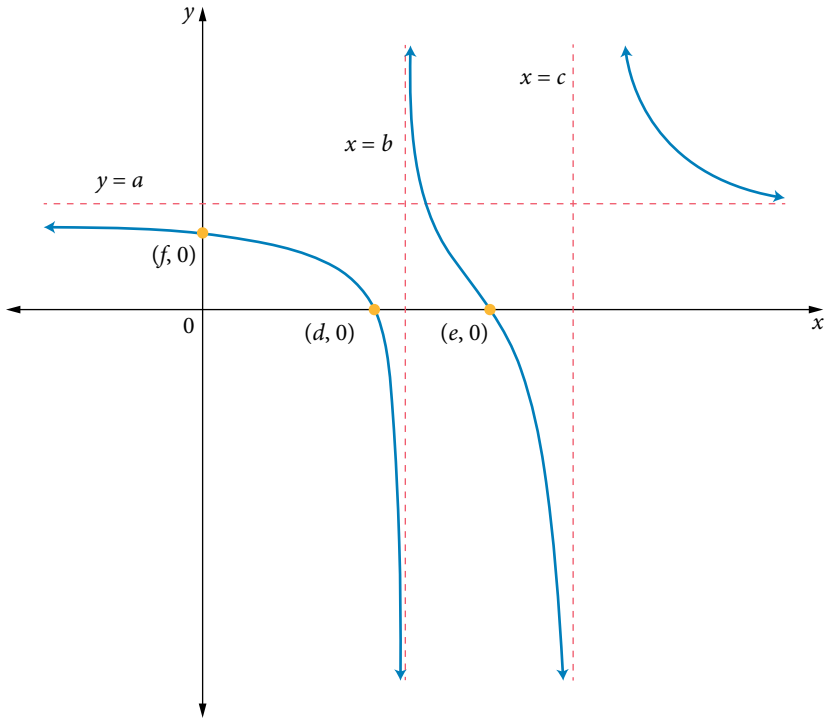
(a) $5\,\text{cis}\left(\frac{2\pi}{3}\right) \times 3\,\text{cis}\left(\frac{\pi}{2}\right)$

(b) $12\,\text{cis}\left(\frac{\pi}{2}\right) \div 4\,\text{cis}\left(\frac{\pi}{3}\right)$

22 Determine the value of x. Exercise 4.1



23 The rational function graphed below has a quadratic numerator and denominator. Determine the equation for this function in terms of the constants a, b, c, d, e and f. Exercise 6.3



Paper 2: Technology-active

Exercise 1.2

1 Using the numbers 1, 2, 3,…, 9, how many three-digit numbers can be formed under the following conditions?

(a) There are no restrictions.

(b) The digits are distinct.

(c) The digits are distinct and the number is greater than 300.

Exercise 1.4

2 A team of 4 tennis players is to be chosen from a squad of 8 players. How many different teams are possible under the following conditions?

(a) There are no restrictions.

(b) Two particular players who do not get on with each other cannot both be on the team together.

Exercise 8.4

3 Use a matrix method to solve the following simultaneous linear equations.

$$\begin{aligned} x - y + z &= 6 \\ -3x - 2y + 4z &= 4 \\ 2x + y + z &= 2 \end{aligned}$$

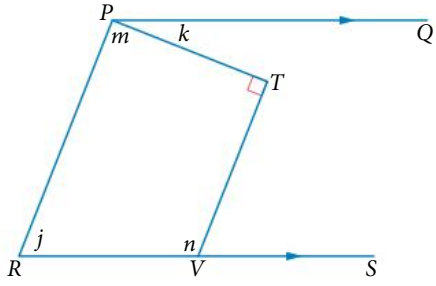
Exercise 1.5

4 Determine the minimum number of times a coin needs to be tossed in order for the probability of obtaining at least three heads to be greater than 0.9.

5 Calculate the exact values of the unknown pronumerals in the following vector equation: $\left(a^2 - 2a\right)\hat{\boldsymbol{i}} + \left(b^2 + 10\right)\hat{\boldsymbol{j}} = (4a - 5)\hat{\boldsymbol{i}} - (10b + 6)\hat{\boldsymbol{j}}$.

Exercise 4.2

6 Consider the following diagram, in which $PQ \parallel RS$ and $\angle PTV = 90°$.



(a) Demonstrate that $m + j = 180° - k$.

(b) Demonstrate that $n - k = 90°$.

7 Determine z such that $z^2 = 8 + 6i$. Exercise 5.5

8 Consider the function $f(x) = 3\sin\left(\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right)$. Exercise 7.6

(a) Determine the period and amplitude of the function.

(b) Determine the transformations, in the correct order, required to transform $f(x)$ into the simple function $g(x) = \sin(x)$.

(c) Determine the range of the function $h(x) = \dfrac{1}{3\sin\left(\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right)}$.

9 Determine the image of the curve $y = -x^2$ after being reflected in the line $y = x - 1$. Verify your answer graphically. Exercise 8.7

Answers

Chapter 1 Combinatorics

RECALL

1 (a) 6, 8, 10 (b) 2, 4, 6, 7, 8, 9, 10 (c) 1, 3, 5

2 (a) 61 (b) 9 (c) 88

(d) 52 (e) 22 (f) 79

3 (a) 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43

(b) 11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44

4 (a) $\frac{5}{26}$ (b) $\frac{3}{10}$ (c) $\frac{1}{3}$

5 (a) 190 (b) 9900 (c) 15

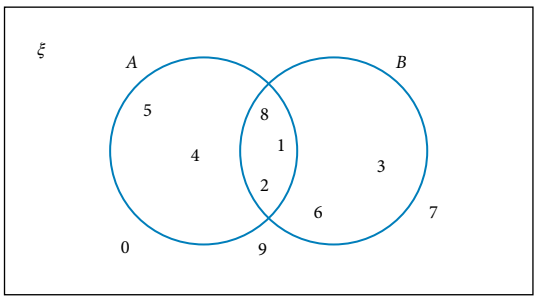
6 (a) 120 (b) 10 (c) 20

7 (a) a (b) $\frac{n(n-1)}{2}$ (c) $\frac{n^2+n}{2}$

8 (a) $x = 6$ or -2 (b) $x = -2.35$ or 0.85

EXERCISE 1.1

1 (a)



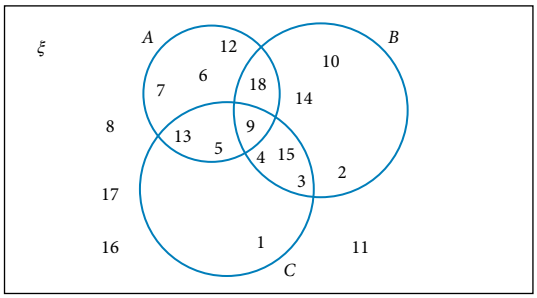
(b) (i) {1, 2, 3, 4, 5, 6, 8} (ii) {1, 2, 8} (iii) {0, 3, 6, 7, 9}

(iv) {0, 4, 5, 7, 9} (v) {0, 1, 2, 3, 6, 7, 8, 9}

(vi) {0, 1, 2, 4, 5, 7, 8, 0} (vii) {0, 7, 9}

(viii) {0, 3, 4, 5, 6, 7, 9}

2 (a)



(b) (i) {3, 4, 9, 15} (ii) {2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 18}

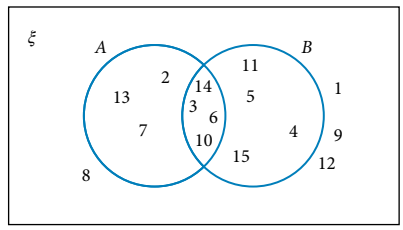
(iii) {9} (iv) {1, 5, 13}

(v) {1, 2, 3, 4, 8, 9, 10, 11, 14, 15, 16, 17, 18}

(vi) {1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 18}

(vii) {18} (viii) {8, 11, 16, 17}

3 (a)



(b) (i) 11 (ii) 7 (iii) 8 (iv) 4

(c) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

LHS $= n(A \cup B) = 11$

RHS $= n(A) + n(B) - n(A \cap B) = 7 + 8 - 4 = 11$

LHS = RHS

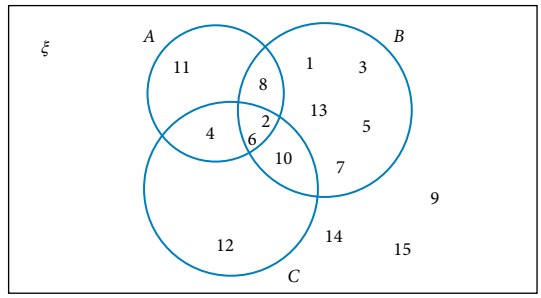
4 $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

LHS $= n(A \cup B) = 10$

RHS $= n(A) + n(B) - n(A \cap B) = 7 + 7 - 4 = 10$

LHS = RHS

5 (a)



(b) (i) 12 (ii) 5 (iii) 9 (iv) 5

(v) 3 (vi) 3 (vii) 3 (viii) 2

(c) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

LHS $= n(A \cup B \cup C) = 12$

RHS $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 5 + 9 + 5 - 3 - 3 - 3 + 2 = 12$

LHS = RHS

6 $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

LHS $= n(A \cup B \cup C) = 9$

RHS $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 4 + 3 + 5 - 1 - 2 - 1 + 1 = 9$

LHS = RHS

7 5

8 (a) 9 (b) 1

9 690

10 (a) D

(b) A common error is to assume that $(A \cap B)'$ is equal to $A' \cap B'$. However, drawing a Venn diagram makes it clear that the set $(A \cap B)'$ is actually equivalent to $A' \cup B'$.

11 C **12** 7 **13** 7 **14** 7

15 10 **16** 6 **17** 3999 **18** 12

EXERCISE 1.2

1 30

2 18

3 26 000

4 (a) 125 (b) 60 (c) 36

5 (a) 4896 (b) 544 (c) 4352

6 (a) 1296 (b) 20

7 720 **8** 133

9 (a) C

(b) The number of paths that go through Y is equal to 4×3. Alternatives A and D are wrong, since there are 4 ways to get from X to Z, and then 3 ways to get from Y to Z. There are also 2 paths that do not go through Y, hence alternative B is wrong.

10 D **11** 576 **12** 972

13 8 **14** 222 **15** 224

16 41 minutes **17** 18 students

EXERCISE 1.3

1 (a) 120 (b) 24 (c) 5

(d) 720 (e) 9900 (f) 90

2 (a) $n(n-1)(n-2)$ (b) n (c) $2(n+1)$

(d) $3(n-1)$ (e) $\frac{(n+2)(n+1)}{5}$ (f) 1

(g) $(n+2)(n+1)n$ (h) $\frac{1}{n(n+1)}$ (i) $\frac{1}{(n-5)(n-6)}$

3 (a) (i) 20

(ii) AB, BA, AC, CA, AD, DA, AE, EA, BC, CB, BD, DB, BE, EB, CD, DC, CE, EC, DE, ED

(b) (i) 6 (ii) ABC, ACB, BAC, BCA, CAB, CBA

4 (a) ${}^7P_3 = 210$ (b) ${}^5P_5 = 120$ (c) ${}^4P_4 = 24$

(d) ${}^4P_3 = 24$ (e) ${}^6P_2 = 30$ (f) ${}^5P_3 = 60$

5 (a) n (b) $n(n-1)$ (c) $n(n-1)(n-2)$

6 (a) 30 (b) 210 (c) 120

7 (a) $\frac{n!}{10!}$ (b) $\frac{(n+r)!}{n!}$ (c) $\frac{(2n)!}{n!}$ (d) $\frac{(5n+1)!}{(3n-6)!}$

8 (a) 5040 (b) 95 040

9 (a) D

(b) The expression $(3n)!$ is equal to $3n \times (3n-1) \times (3n-2) \times (3n-3) \times \ldots \times 2 \times 1$.

Alternative A is equal to $3 \times n! = 3 \times n \times (n-1) \times (n-2) \times \ldots 2 \times 1$. Note that the factorial operator binds to the number or variable immediately preceding it, so, the factorial operator only applies to the n in the expression $3n!$.

Alternative B is equal to $3! \times n! = 3 \times 2 \times 1 \times n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$. A common misconception is that the factorial operator distributes over the variables inside a set of brackets, leading students to erroneously to conclude that $(3n)!$ is equal to $3!n!$. It is clear that the expanded forms of these expressions are not equivalent.

Alternative C is equal to $3n(n-1)! = 3 \times n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1$, which happens to be equivalent to alternative A.

10 B

11 (a) 120 (b) 48 (c) 72 (d) 108

12 3600

13 (a) 720 (b) 240 (c) 48 (d) 96

14 4320

15 960

16 (a) 3360 (b) 840 (c) 180

17 277 200

18 56

19 (a) 2 522 520 (b) 27 720 (c) 12 600

20 (a) 27 720 (b) 11 550

21 5

22 $\left(\frac{n}{2}\right)! \times 2^{\frac{n}{2}}$

23 96

24 11 520

25 (a) 5040 (b) 2520

EXERCISE 1.4

1 (a) 10; AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

(b) 4; ABC, ABD, ACD, BCD

(c) 6; A, B, C, D, E, F

(d) 1; ABCDEFG

2 (a) ${}^6C_3$, 20 (b) ${}^7C_5$, 21

3 (a) $\binom{10}{3}$, 120 (b) $\binom{9}{7}$, 36

4 (a) 1 (b) n (c) $\frac{n(n-1)}{2}$

(d) $\frac{n(n-1)(n-2)}{6}$ (e) $\frac{n(n-1)}{2}$ (f) $\frac{(n+1)n}{2}$

5 (a) $\frac{(n+1)!}{(r+1)!(n-r)!}$ (b) $\frac{(2n)!}{n!n!}$ (c) $\frac{(3n+1)!}{(n+4)!(2n-3)!}$

6 (a) 1287 (b) 184 756 (c) 497 420 (d) 273 438 880

7 (a) 462 (b) 84 (c) 378

8 (a) 210 (b) 126 (c) 42

9 3640

10 (a) B

(b) It seems reasonable that the number of ways to choose 0 objects from a set of n objects might be 0, leading to alternative A. However, using the formula for ${}^{n}C_{r}$ gives ${}^{n}C_{0} = 1$.

11 (a) A

(b) In alternative D, it is only the identities of the three individuals that matter. There is no order to them, or special roles that they are assigned. Hence, this option is not correct.

12 (a) 220 (b) 105 (c) 36

13 (a) 3003 (b) 980 (c) 1176

14 (a) 6188 (b) 2520

15 17 **16** 22 506 **17** 7150 **18** 12

EXERCISE 1.5

1 $\frac{1}{11}$ **2** $\frac{1}{10}$ **3** $\frac{1}{30}$ **4** $\frac{1}{3}$

5 (a) $\frac{4}{35}$ (b) $\frac{1}{35}$

6 $\frac{3}{11}$ **7** $\frac{1}{3360}$ **8** D **9** B

10 (a) $\frac{572}{20\,825}$ (b) $\frac{2759}{4165}$

11 0.000 2401 **12** $\frac{1}{10}$ **13** $\frac{15}{128}$

14 (a) $\frac{1}{8\,145\,060}$ (b) $\frac{1}{678\,755}$

15 0.411 **16** 0.021 13 **17** 40

EXERCISE 1.6

1 (a) 10 (b) 1 (c) 35 (d) 3

2 (a) 1 (b) 120 (c) 36

3 2

4 32 768

5 28 920

6 To prove: ${}^{n}C_{3} = {}^{n-1}C_{2} + {}^{n-1}C_{3}$

$$\begin{aligned}\text{LHS} &= {}^{n}C_{3}\\ &= \frac{n!}{3!(n-3)!}\\ &= \frac{n(n-1)(n-2)\,\cancel{(n-3)!}}{3!\,\cancel{(n-3)!}}\\ &= \frac{n(n-1)(n-2)}{6}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= {}^{n-1}C_{2} + {}^{n-1}C_{3}\\ &= \frac{(n-1)!}{2!(n-3)!} + \frac{(n-1)!}{3!(n-4)!}\\ &= \frac{(n-1)(n-2)\,\cancel{(n-3)!}}{2\,\cancel{(n-3)!}} + \frac{(n-1)(n-2)(n-3)\,\cancel{(n-4)!}}{6\,\cancel{(n-4)!}}\\ &= \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)(n-3)}{6}\\ &= \frac{3(n-1)(n-2)}{6} + \frac{(n-1)(n-2)(n-3)}{6}\\ &= \frac{(n-1)(n-2)\big(3+(n-3)\big)}{6}\\ &= \frac{(n-1)(n-2)(n)}{6}\\ &= \frac{n(n-1)(n-2)}{6}\\ &= \text{LHS}\end{aligned}$$

7 B

8 A

9 In general, the number in row n that is in the first diagonal is in position $n - 1$, and is therefore equal to ${}^{n}C_{n-1}$. The number immediately to its left is in position $n - 2$ and is therefore equal to ${}^{n}C_{n-2}$. Finally, the number in the row below slightly to the left is in row $n + 1$, position $n - 1$, and is therefore equal to ${}^{n+1}C_{n-1}$.

Thus, it is necessary to prove that $\left({}^{n}C_{n-1}\right)^{2} = {}^{n}C_{n-2} + {}^{n+1}C_{n-1}$

$$\begin{aligned}\text{LHS} &= \left({}^{n}C_{n-1}\right)^{2}\\ &= \left(\frac{n!}{1!(n-1)!}\right)^{2}\\ &= \left(\frac{n\,\cancel{(n-1)!}}{1!\,\cancel{(n-1)!}}\right)^{2}\\ &= n^{2}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= {}^{n}C_{n-2} + {}^{n+1}C_{n-1}\\ &= \frac{n!}{(n-2)!2!} + \frac{(n+1)!}{(n-1)!\big(n+1-(n-1)\big)!}\\ &= \frac{n!}{(n-2)!21} + \frac{(n+1)!}{(n-1)!2!}\\ &= \frac{n(n-1)\,\cancel{(n-2)!}}{\cancel{(n-2)!}\,2} + \frac{(n+1)n\,\cancel{(n-1)!}}{\cancel{(n-1)!}\,2}\\ &= \frac{n(n-1)}{2} + \frac{(n+1)n}{2}\\ &= \frac{n^{2} - n + n^{2} + n}{2}\\ &= \frac{2n^{2}}{2}\\ &= n^{2}\\ &= \text{LHS}\end{aligned}$$

10 (a) (i) row 2, position 0 (ii) row 3, position 1
(iii) row 4, position 2 (iv) row 5, position 3
(v) row 6, position 4 (vi) row 7, position 5

(b) row 8, position 5

(c) $\frac{(n+2)(n+1)n}{6}$

11 $\frac{(n+3)(n+2)(n+1)n}{24}$

12 If the numbers in row n of Pascal's triangle are $1, a_1, a_2, a_3, \ldots, a_{n-1}, 1$, then the numbers in row $n+1$ must be: $1, 1+a_1, a_1+a_2, a_2+a_3, \ldots, a_{n-1}+1, 1$.

Let the sum of the numbers in row n be S. Thus,

$S = 1 + a_1 + a_2 + a_3 + \ldots + a_{n-1} + 1$
$= a_1 + a_2 + a_3 + \ldots + a_{n-1} + 2$

The sum of the numbers in row $n+1$ is:

$1 + (1 + a_1) + (a_1 + a_2) + (a_2 + a_3) + \ldots + (a_{n-2} + a_{n-1}) + (a_{n-1} + 1) + 1$
$= 2a_1 + 2a_2 + 2a_3 + \ldots + 2a_{n-1} + 4$
$= 2(a_1 + a_2 + a_3 + \ldots + a_{n-1} + 2)$
$= 2S$

as required.

13 To prove: ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$

LHS $= {}^{n}C_{r}$
$= \frac{n!}{r!(n-r)!}$

RHS $= {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$
$= \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} + \frac{(n-1)!}{r!(n-1-r)!}$
$= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}$
$= \frac{(n-1)!}{(r-1)!(n-r)!} \times \frac{r}{r} + \frac{(n-1)!}{r!(n-r-1)!} \times \frac{(n-r)}{(n-r)}$
$= \frac{r(n-1)!}{r!(n-r)!} + \frac{(n-r)(n-1)!}{r!(n-r)!}$
$= \frac{r(n-1)! + (n-r)(n-1)!}{r!(n-r)!}$
$= \frac{(n-1)!(r+(n-r))}{r!(n-r)!}$
$= \frac{(n-1)!(n)}{r!(n-r)!}$
$= \frac{n!}{r!(n-r)!}$
$=$ LHS

EXERCISE 1.7

1 (a) 6 (b) 7 (c) 2
(d) 4 (e) 9

2 There must be at least one box containing at least $\left\lceil \frac{8}{7} \right\rceil = 2$ objects. This means that at least two students will have scored the same number of goals.

3 There must be at least one box containing at least $\left\lceil \frac{11}{10} \right\rceil = 2$ objects. This means that at least two of the phone numbers will have the same final digit.

4 There must be at least one box containing at least $\left\lceil \frac{25}{4} \right\rceil = 7$ objects. This means that at least that at least seven students would have received the same grade.

5 (a) There must be at least one box containing at least $\left\lceil \frac{27}{26} \right\rceil = 2$ objects. This means that at least that at least two students have first names starting with the same letter.

(b) There must be at least one box containing at least $\left\lceil \frac{27}{12} \right\rceil = 3$ objects. This means that at least that at least three students have the same birth month.

(c) There must be at least one box containing at least $\left\lceil \frac{22}{7} \right\rceil = 4$ objects. This means that at least four students were born on the same day of the week.

6 There are 10 possible numbers of mistakes made. Placing 22 students in boxes representing the 10 possible numbers of mistakes means that at least three students made the same number of mistakes.

7 A

8 D

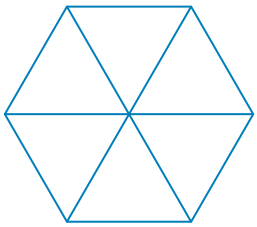
9 The positive even integers less than 21 can be broken up into the following five groups that add to 22: [2,20], [4,18], [6,16], [8,14], and [10,12]. Think of these five groups as boxes, and the six chosen integers as objects assigned to boxes based on their value (for example, the number 16 would be placed in the third box listed since it is the one labelled with 16). Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{6}{5} \right\rceil = 2$ objects. Since each box contains a pair of integers that add to 22, if a box contains 2 objects, it means that there must be 2 numbers chosen that add to 22.

10 The possible remainders after division by 3 are: 0, 1 and 2. This means that at least two of the four integers that must be assigned to these must have the same remainder when divided by 3, and hence their difference must be divisible by 3.

11 If two numbers have the same remainder when divided by 7, then their difference will be divisible by 7. Think of the possible remainders upon division by 7 as boxes (there are 7 of them: 0, 1,…, 6), and the 100 integers as objects that must be assigned to these boxes based on their remainders. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{100}{7} \right\rceil = 15$ objects. This means that at least 7 of the integers must have the same remainder when divided by 7, and hence their difference must be divisible by 7.

12 The rectangle can be divided into 3 squares of side length 1 cm. Think of these squares as boxes, and the 4 points as objects to be placed in these boxes. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{4}{3} \right\rceil = 2$ objects. This means that at least 2 of the points must lie inside (or on the edge of) the same square. Since the maximum distance between any two points inside (or on the edge of) a square is the length of the diagonal ($\sqrt{2}$ cm in this case), the result follows.

13 The regular hexagon can be divided into 6 equilateral triangles of side length 1 cm, as shown below.



Think of these triangles as boxes, and the 13 points as objects to be placed in these boxes. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{13}{6} \right\rceil = 3$ objects. This means that at least 3 of the points must lie inside (or on the edge of) the same triangle. Since the maximum distance between any two points inside (or on the edge of) an equilateral triangle is less than the length of the triangle, the result follows.

14 The possible digit sums of three digit numbers are 1, 2, …, 27. Think of these as boxes, and the 168 integers as objects to be placed in the boxes according to their digit sum. Note, however, that some digit sums cannot possibly occur 8 times. Specifically, there is only one three-digit number with a digit sum of 1 (100), two that have a digit sum of 2 (110 and 101), three that have a digit sum of 3 (111, 120, 102), one that has a digit sum of 27 (999), and three that have a digit sum of 26 (998, 989, 899). Thus, these five boxes could potentially hold a total of 10 of the 168 numbers. This leaves 11 boxes, and 158 numbers to be placed in them. Using the Pigeon-hole principle, at least one must contain $\left\lceil \frac{158}{22} \right\rceil = 8$ of the numbers, all of which necessarily contain the same digit sum.

15 Pick any two of the points and draw a diameter through these points. This partitions the sphere into two hemispheres. By the Pigeon-hole principle, 2 of the remaining 3 points must lie in one of the hemispheres. These two points, along with the original two points, lie in a hemisphere.

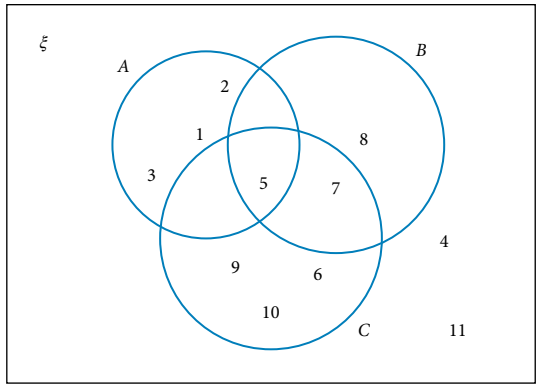
16 (a) Let $a, a+1, a+2, \ldots, a+19$ be a set of 20 consecutive integers. These integers can be broken up into the following groups: $[a, a+10], [a+1, a+11], \ldots, [a+9, a+19]$. (Notice that the two integers in each category differ by 10.) Think of these 10 groups as boxes, and the 11 selected integers as objects to be assigned to these boxes based on their value. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{11}{10} \right\rceil = 2$ objects. This means that at least two of the points must differ by 10.

(b) The integers from 1 to 100 can be grouped into five groups containing 20 consecutive integers as follows: $[1, 2, 3, \ldots, 20], [21, 22, 23, \ldots, 40], [41, 42, 43, \ldots, 60], [61, 62, 63, \ldots, 80], [81, 82, 83, \ldots, 100]$. Think of these as boxes, and assign the 51 integers to these. Using the Pigeon-hole principle, at least one will contain $\left\lceil \frac{51}{5} \right\rceil = 11$ integers. From part (a), choosing any 11 integers from a set of 20 consecutive integers guarantees that at least two will differ by 10.

CHAPTER REVIEW 1

1 27

2 First, draw a Venn diagram.



From the Venn diagram:

$n(A \cup B \cup C) = 9$

$n(A) = 4$

$n(B) = 3$

$n(C) = 5$

$n(A \cap B) = 1$

$n(B \cap C) = 2$

$n(A \cap C) = 1$

$n(A \cap B \cap C) = 1$

The inclusion–exclusion principle states:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Using the values determined for the specific sets:

LHS $= n(A \cup B \cup C)$

$= 9$

RHS $= 4+3+5-1-2-1+1$

$= 9$

3 A

4 C

5 (a) 720 (b) 4 (c) 90

6 (a) $(n+1)n$ (b) $3n(n-1)$

7 (a) $\dfrac{n!}{5!}$ (b) $\dfrac{(n+r-1)!}{n!}$

8 (a) ${}^6C_4$, 15 (b) ${}^5C_4$, 5

9 (a) 1 (b) $\dfrac{(n+1)n(n-1)}{6}$

10 $\dfrac{7}{10}$

11 $\dfrac{(n+2)(n+1)}{2}$

12 There must be at least one box containing at least $\left\lceil \dfrac{46}{21} \right\rceil = 3$ objects. This means that at least three students would have received the same mark.

13 7 436 000

14 (a) 720 (b) 336 (c) 656

15 (a) 6720 (b) 720 (c) 192

16 $\dfrac{1}{30}$

17 $\dfrac{11}{1105}$

18 6188

19 524 287

20 Think of the schools as boxes, and the children as objects to be placed in these boxes according to the school that they attend. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \dfrac{813}{4} \right\rceil = 204$ objects. This means that at least one of the schools contains at least 204 students.

21 7

22 92

23 To prove: ${}^5C_r = {}^4C_{r-1} + {}^4C_r$

LHS $= {}^5C_r$

$= \dfrac{5!}{r!(5-r)!}$

RHS $= {}^4C_{r-1} + {}^4C_r$

$= \dfrac{4!}{(r-1)!(4-(r-1))!} + \dfrac{4!}{r!(4-r)!}$

$= \dfrac{4!}{(r-1)!(5-r)!} + \dfrac{4!}{r!(4-r)!}$

$= \dfrac{4!}{(r-1)!(5-r)!} \times \dfrac{r}{r} + \dfrac{4!}{r!(4-r)!} \times \dfrac{(5-r)}{5-r}$

$= \dfrac{4!r}{r!(5-r)!} + \dfrac{4!(5-r)}{r!(5-r)!}$

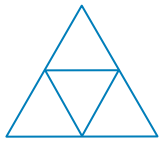
$= \dfrac{4!(r+(5-r))}{r!(5-r)!}$

$= \dfrac{4!(5)}{r!(5-r)!}$

$= \dfrac{5!}{r!(5-r)!}$

= LHS

24 An equilateral triangle can be broken up into 4 smaller equilateral triangles as shown.



Since the original triangle has sides of length 2 cm, each small triangle has sides of length 1 cm. Think of the smaller triangles as boxes, and the 9 points as objects to be placed in the boxes according to where the points are placed. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \dfrac{9}{4} \right\rceil = 3$ objects. This means that at least 3 of the points must lie inside (or on the edge of) the same triangle. Since the maximum distance between any two points inside (or on the edge of) an equilateral triangle is less than the length of the triangle, the result follows.

25 140

26 (a) 6188 (b) 630 (c) 686

27 $\dfrac{5}{9}$

28 10

29 20 160

30 26

31 280

32 $\dfrac{9216}{2598960}$

33 Let the initial number chosen be in row n, position r. This number is therefore ${}^{n}C_{r}$.

The circled numbers in the surrounding hexagon are ${}^{n}C_{r+1}$, ${}^{n+1}C_{r}$, and ${}^{n-1}C_{r-1}$.

The numbers in the squares are ${}^{n}C_{r-1}$, ${}^{n+1}C_{r+1}$, and ${}^{n-1}C_{r}$.

To prove ${}^{n}C_{r+1} \times {}^{n+1}C_{r} \times {}^{n-1}C_{r-2} = {}^{n}C_{r-1} \times {}^{n+1}C_{r+1} \times {}^{n-1}C_{r}$:

$$\text{LHS} = \frac{n!}{(r+1)!(n-r-1)!} \times \frac{(n+1)!}{r!(n-r+1)!} \times \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$\text{RHS} = \frac{n!}{(r-1)!(n-r+1)!} \times \frac{(n+1)!}{(r+1)!(n-r)!} \times \frac{(n-1)!}{r!(n-r-1)!}$$

The product of the terms in the numerators and denominators of both sides is equal (since the same terms appear, but in a different order). Hence the left-hand side is equal to the right-hand side, as required.

34 Think of the possible digit sets as boxes and the five-digit numbers as objects to be assigned to boxes based on their digit set. The number of boxes is the number of ways to choose 5 digits from 9 possibilities, that is, ${}^{9}C_{5} = 126$. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \frac{379}{126} \right\rceil = 4$ objects. This means that at least 4 numbers will have the same digits.

Chapter 2 Vectors in the plane

RECALL

1 (a) $x = 2$ (b) $t = -\frac{6}{5}$ (c) $k = 2\sqrt{2}$

(d) $x = -7$ (e) $p = -\frac{1}{18}$

2 (a) $x = 5, y = 1$ (b) $x = 2, y = 3$

(c) $x = -2$ and $y = -6$ (d) $x = \frac{1}{2}$ and $y = -\frac{7}{2}$

3 (a) $2\sqrt{7}$ (b) $10\sqrt{3}$

4 (a) $\overline{CD} = \sqrt{29}$ units (b) $\overline{EF} = 10$ units

(c) $\overline{GH} = 4\sqrt{5}$ units (d) $\overline{IJ} = \sqrt{130}$ units

5 (a) 0.9063 (4 d.p.) (b) -0.9135 (4 d.p.)

6 (a) $\theta = 64.32°$, $\theta = 115.68°$ (2 d.p.)

(b) $\theta = 110.04°$, $\theta = 249.96°$ (2 d.p.)

7 (a) $c = 39$, $A = 22.62°$, $B = 67.38°$

(b) $a = 23.01$, $b = 9.77$, $B = 23°$

8 (a) $B = 45.83°$, $C = 61.17°$, $a = 29.31$

(b) $b = 8.37$, $c = 10.52$, $C = 59°$

(c) $c = 43.01$, $A = 68.78°$, $B = 48.22°$

(d) $A = 53.13°$, $B = 67.38°$, $C = 59.49°$

9 (a) Point A: S60°E; Point B: N15°W

(b) Point A: 120°T; Point B: 345°T

(c) Point A: 300°T; Point B: 165°T

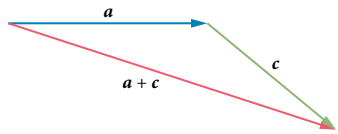
EXERCISE 2.1

1 (a) 10 km/h south-west is a vector quantity.

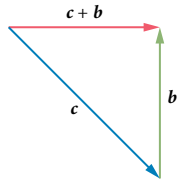
(b) 50 N towards the ground is a vector quantity.

(c) 15 km is a scalar quantity.

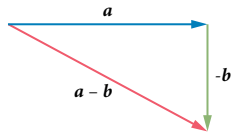
2 (a)



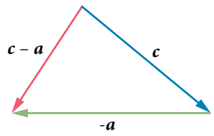
(b)



(c)



(d)



3 (a) $\overrightarrow{CD} = -\boldsymbol{a}$ (b) $\overrightarrow{AD} = \boldsymbol{b}$

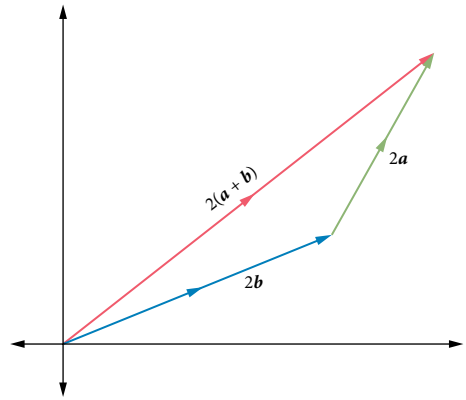
(c) $\overrightarrow{CA} = -\boldsymbol{a} - \boldsymbol{b}$ (d) $\overrightarrow{DB} = \boldsymbol{a} - \boldsymbol{b}$

4 (a) $\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$ (b) $\overrightarrow{AD} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$

(c) $\overrightarrow{DA} = -\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}$ (d) $\overrightarrow{DB} = -\boldsymbol{b} - \boldsymbol{c}$

5 (a) $\overrightarrow{ON} = 2\boldsymbol{a} + 3\boldsymbol{b}$ (b) $\overrightarrow{CP} = \boldsymbol{a} + 3\boldsymbol{b}$ (c) $\overrightarrow{PB} = -3\boldsymbol{a} - 2\boldsymbol{b}$

6



7 (a) D

(b) For alternatives A and B there is no vector defined as $\boldsymbol{b}$, therefore $\boldsymbol{b}$ cannot be part of the answer. For alternative C using the parallelogram rule of addition, you need to subtract $\boldsymbol{b}$ from $\boldsymbol{a}$ rather than add it.

8 (a) D

(b) The relationship $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ is correct, as it conforms to the parallel law of addition. For option B, as C and D are the midpoints of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are in the

ratio of 1 : 2. For option **C**, any three non-collinear points will form a triangle. So option D is incorrect because the correct relationship would be $2\overrightarrow{CD} = \overrightarrow{AB}$.

9 (a) $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{a}, \overrightarrow{DE} = \frac{1}{2}\boldsymbol{c} - \frac{1}{2}\boldsymbol{a}$ (b) $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

(c) The line joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.

(d) $\overrightarrow{BF} = \frac{1}{3}(\boldsymbol{c} - \boldsymbol{a}), \overrightarrow{FC} = \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})$

(e) $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$

$= \boldsymbol{a} + \frac{1}{3}(\boldsymbol{c} - \boldsymbol{a})$

$= \boldsymbol{a} + \frac{1}{3}\boldsymbol{c} - \frac{1}{3}\boldsymbol{a}$

$= \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{c}$

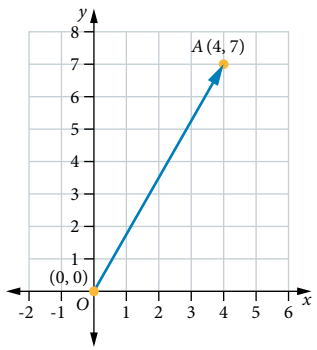
$= \frac{1}{3}(2\boldsymbol{a} + \boldsymbol{c})$

10 (a) $\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{d}$ (b) $\overrightarrow{AE} = \boldsymbol{b} + \frac{1}{2}\boldsymbol{d}$ (c) $\overrightarrow{DE} = \boldsymbol{b} - \frac{1}{2}\boldsymbol{d}$

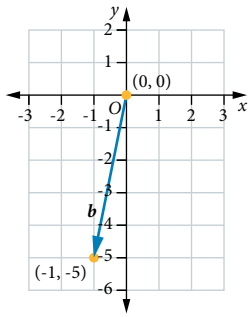
(d) $\overrightarrow{DF} = \frac{2}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{d}$ (e) $\overrightarrow{AF} = \frac{2}{3}\overrightarrow{AC}$ (f) $\overline{AF} : \overline{FC} = 2 : 1$

EXERCISE 2.2

1 (a)



(b)



2 $\boldsymbol{a} = (5, 5), \boldsymbol{b} = (4, -5), \boldsymbol{c} = (0, 9), \boldsymbol{d}$ (-13, -5), $\boldsymbol{e} = (10, 2), \boldsymbol{f} = (-3, 5)$

3 (a) $\boldsymbol{a} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}; \boldsymbol{b} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} -10 \\ -6 \end{bmatrix}, \boldsymbol{d} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \boldsymbol{e} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}, \boldsymbol{f} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

(b) $|\boldsymbol{a}| = 4\sqrt{2}, |\boldsymbol{b}| = 4\sqrt{2}, |\boldsymbol{c}| = 2\sqrt{34}, |\boldsymbol{d}| = 8, |\boldsymbol{e}| = 5\sqrt{5}, |\boldsymbol{f}| = \sqrt{65}$

4 (a) (6, -18) (b) (1, -3) (c) (-2, 6) (d) (0.8, -2.4)

5 (a) D

(b) Starting at point B, you need to trace back to O and then forward to A to determine $\overrightarrow{BA}$. This corresponds to a vector sum $\overrightarrow{BO} + \overrightarrow{OA}$. Alternative A is incorrect since $\overrightarrow{OB}$ is not the same as $\overrightarrow{BO}$. Alternative B is incorrect since $\overrightarrow{AO}$ is not the same as $\overrightarrow{OA}$ and alternative C is incorrect because $\overrightarrow{OB}$ is not the same as $\overrightarrow{BO}$.

6 C

7 (a) The coordinates of the terminal point are (13, -2).

(b) The coordinates of the initial point are (19, -15).

8 (a) $\overrightarrow{AB} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}; \overrightarrow{BC} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}; \overrightarrow{AC} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

(b) $|\overrightarrow{AB}| = 2\sqrt{13}; |\overrightarrow{BC}| = 2\sqrt{13}; |\overrightarrow{AC}| = 2\sqrt{26}$

(c) ΔABC is a right-angled triangle if and only if Pythagoras' theorem holds true.

$a^2 + b^2 = c^2$

LHS: $a^2 + b^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$

$= \left(\sqrt{52}\right)^2 + \left(\sqrt{52}\right)^2$

$= 104$

RHS: $c^2 = |\overrightarrow{AC}|^2$

$= \left(2\sqrt{26}\right)^2$

$= 104$

LHS = RHS

$\therefore |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{AC}|^2$

Therefore, ΔABC is a right-angled triangle.

(d) The coordinates of D are (4, -7).

(e) The coordinates of the point of intersection of the diagonals of the square $ABCD$ are (3, -2).

9 $q = 6$ or $q = 2$

10 17.46 km (2 d.p.)

11 (a) $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{c}; \overrightarrow{OX} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ (b) $\overrightarrow{CX} = \boldsymbol{a} - \frac{1}{2}\boldsymbol{c}$

(c) $\overrightarrow{CY} = \frac{2}{3}\boldsymbol{a} - \frac{1}{3}\boldsymbol{c}$ (d) $\overrightarrow{OY} = \frac{2}{3}\overrightarrow{OB}$

(e) $\overline{OY} : \overline{YB} = 2 : 1$

12 (a) The position vector of the midpoint of $\overline{AB}$ is $\left(-\frac{1}{2}, \frac{15}{2}\right)$

(b) The position vector of the midpoint of $\overline{AB}$ is $\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$.

(c) The position vectors of the points of trisection of $\overline{DE}$ are $\left(\frac{7}{3}, \frac{17}{3}\right)$ and $\left(\frac{11}{3}, \frac{13}{3}\right)$.

(d) The position vectors of the points of trisection of $\overline{DE}$ are $\frac{2}{3}\boldsymbol{d} + \frac{1}{3}\boldsymbol{e}$ and $\frac{1}{3}\boldsymbol{d} + \frac{2}{3}\boldsymbol{e}$.

(e) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= \overrightarrow{OB} - \overrightarrow{OA}$

$= \boldsymbol{b} - \boldsymbol{a}$

$\overrightarrow{AP} = m$ and $\overrightarrow{AB} = m + n$

$\overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \boldsymbol{a} + \frac{m}{m+n}(\boldsymbol{b} - \boldsymbol{a})$

$= \left(1 - \frac{m}{m+n}\right)\boldsymbol{a} + \frac{m}{m+n}\boldsymbol{b}$

$= \left(\frac{m+n}{m+n} - \frac{m}{m+n}\right)\boldsymbol{a} + \frac{m}{m+n}\boldsymbol{b}$

$= \frac{n}{m+n}\boldsymbol{a} + \frac{m}{m+n}\boldsymbol{b}$

EXERCISE 2.3

1 $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -3\hat{\boldsymbol{j}}$, $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, $\boldsymbol{d} = -3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, $\boldsymbol{e} = 2\hat{\boldsymbol{i}}$, $\boldsymbol{f} = 7\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$

2 $|\boldsymbol{a}| = 3\sqrt{2}$ units and the direction of $\boldsymbol{a} = 45°$; $\boldsymbol{a} = (3\sqrt{2}, 45°)$

$|\boldsymbol{b}| = 2\sqrt{5}$ units and the direction of $\boldsymbol{b} = 116.57°$ (2 d.p.);

$\boldsymbol{b} = (2\sqrt{5}, 116.57°)$

$|\boldsymbol{c}| = \sqrt{41}$ units and the direction of $\boldsymbol{c} = 218.66°$ (2 d.p.);

$\boldsymbol{c} = (\sqrt{41}, 218.66°)$

$|\boldsymbol{d}| = \sqrt{13}$ units and the direction of $\boldsymbol{d} = 303.69°$ (2 d.p.);

$\boldsymbol{d} = (\sqrt{13}, 303.69°)$

3 (a) $\boldsymbol{a} = (2\sqrt{26}, 348.69°)$ (b) $\boldsymbol{b} = (\sqrt{73}, 20.56°)$

(c) $\boldsymbol{c} = (\sqrt{5}, 243.43°)$ (d) $\boldsymbol{d} = (5, 126.87°)$

4 (a) $\boldsymbol{a} = 12.29\hat{\boldsymbol{i}} + 8.60\hat{\boldsymbol{j}}$ (b) $\boldsymbol{b} = -11.85\hat{\boldsymbol{i}} + 19.71\hat{\boldsymbol{j}}$

(c) $\boldsymbol{c} = 7.07\boldsymbol{i} - 7.07\hat{\boldsymbol{j}}$ (d) $\boldsymbol{d} = -35.86\hat{\boldsymbol{i}} - 3.14\hat{\boldsymbol{j}}$

5 (a) $7\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ (b) $-\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$

(c) $29\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ (d) $-3\hat{\boldsymbol{i}} - 25\hat{\boldsymbol{j}}$

6 $m = -1, n = -1$

7 (a) C

(b) Alternative A can't be a unit vector because $|\boldsymbol{a}| = \sqrt{x^2 + y^2} = \sqrt{0.5} \approx 0.71$. Alternative B is not a unit vector because $|\boldsymbol{b}| = \sqrt{x^2 + y^2} = \sqrt{2} \approx 1.41$. Because $|\hat{\boldsymbol{i}}| = 1$ and $|\hat{\boldsymbol{j}}| = 1$ then $\boldsymbol{d} = 1$, which is a scalar quantity, so alternative B is not a vector.

8 B

9 $\boldsymbol{a} = -28\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$

10 (a) $a = \frac{5}{3}, b = -2$ (b) $x = -17, y = 7$

11 (a) $\hat{\boldsymbol{a}} = \frac{1}{10}(8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}})$ (b) $\hat{\boldsymbol{a}} = \frac{\sqrt{5}}{15}(3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}})$

(c) $\hat{\boldsymbol{a}} = \frac{\sqrt{85}}{85}(-2\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}})$ (d) $\hat{\boldsymbol{a}} = \frac{\sqrt{145}}{145}(-\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}})$

12 (a) $\hat{\boldsymbol{b}} = \frac{\sqrt{10}}{30}(3\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}})$ (b) $\boldsymbol{c} = \frac{\sqrt{10}}{2}(3\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}})$

13 (a) $g = 0, h = \frac{1}{3}$ (b) $p = 2, q = 8$

14 $\hat{\boldsymbol{c}} = \frac{\sqrt{277}}{277}(-14\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}})$

15 $\hat{\boldsymbol{c}} = \frac{\sqrt{130}}{13}(-3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$

16 (a) $\overrightarrow{DC} = -21\hat{\boldsymbol{i}} + 35\hat{\boldsymbol{j}}$ (b) $|\overrightarrow{DC}| = 7\sqrt{34}$

17 (a) $-3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ (b) $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ (c) $5\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$

(d) 5 (e) $\sqrt{2}$ (f) $\sqrt{106}$

18 (a) $5\sqrt{5}$ (b) 169.70° (2 d.p.)

19 (a) $p = \frac{2}{5}$ (b) $\frac{3}{5}\hat{\boldsymbol{i}} + \frac{4}{5}\hat{\boldsymbol{j}}$

(c) $6\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ (d) $q = \frac{4}{25}$ (e) $\frac{7}{25}\hat{\boldsymbol{i}} + \frac{24}{25}\hat{\boldsymbol{j}}$

(f) $14\hat{\boldsymbol{i}} + 48\hat{\boldsymbol{j}}$ (g) $\frac{10\sqrt{221}}{221}(5\hat{\boldsymbol{i}} + 14\hat{\boldsymbol{j}})$

20 (a) $x = -3, -2; y = 2$ (b) $x = 2 - \sqrt{6}, x = 2 + \sqrt{6}; y = -1, y = 1$

21 (a) $\hat{\boldsymbol{a}} = \frac{1}{5}(3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}})$ and $\hat{\boldsymbol{b}} = \frac{1}{5}(-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$

(b) $x = -\frac{5}{2}, y = -\frac{5}{3}$

22 (a) $\hat{\boldsymbol{c}} = \frac{\sqrt{5}}{5}(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}})$ and $\hat{\boldsymbol{d}} = \frac{\sqrt{5}}{5}(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$

(b) $v = -\frac{\sqrt{5}}{10}, w = \frac{\sqrt{5}}{5}$

EXERCISE 2.4

1 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 23.67$ (2 d.p.) (b) $\boldsymbol{a} \cdot \boldsymbol{b} = -23.67$ (2 d.p.)

(c) $\boldsymbol{a} \cdot \boldsymbol{b} = 23.67$ (2 d.p.)

(d) $\boldsymbol{a} \cdot \boldsymbol{b} = 56$ (e) $\boldsymbol{a} \cdot \boldsymbol{b} = -56$ (f) $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

2 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 21$ (b) $\boldsymbol{a} \cdot \boldsymbol{b} = 14$ (c) $\boldsymbol{a} \cdot \boldsymbol{b} = -52$

(d) $\boldsymbol{a} \cdot \boldsymbol{b} = -10$ (e) $\boldsymbol{a} \cdot \boldsymbol{b} = 50$ (f) $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

3 A

4 (a) B

(b) If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then their dot product would not equal zero, because the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is zero, the vectors are perpendicular. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can hold $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ regardless of their magnitudes. If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal, then they are also parallel. Their dot products would not equal zero.

5 Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

$\boldsymbol{a} \cdot \boldsymbol{b} = 3 \times 7 + 7 \times (-3)$

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$

6 $x = -\frac{15}{7}$

7 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel and in the same direction, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel and in the opposite direction, then $\boldsymbol{a} \cdot \boldsymbol{b} = -|\boldsymbol{a}||\boldsymbol{b}|$.

8 (a) The angle, correct to the nearest degree, between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 25°.

(b) The angle, correct to the nearest degree, between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 96°.

(c) The angle, correct to the nearest degree, between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 67°.

(d) The angle, correct to the nearest degree, between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 153°.

9 $\boldsymbol{d} = \pm\left(6\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}\right)$

10 $p = \dfrac{5q}{2}$

11 $B = \left(\dfrac{1-\sqrt{33}}{2}, \dfrac{\left(1-\sqrt{33}\right)^2}{4}\right)$ or $B = \left(\dfrac{1+\sqrt{33}}{2}, \dfrac{\left(1+\sqrt{33}\right)^2}{4}\right)$

In approximate form,

$B \approx (-2.3723, 5.6278)$ or $B \approx (3.3723, 11.3724)$

12 Begin by calculating vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ in component form.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \left(-2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)\\ &= 4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= \overrightarrow{OC} - \overrightarrow{OB}\\ &= 8\boldsymbol{i} - \hat{\boldsymbol{j}} - \left(2\boldsymbol{i} + 3\hat{\boldsymbol{j}}\right)\\ &= 6\boldsymbol{i} - 4\hat{\boldsymbol{j}}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= \overrightarrow{OC} - \overrightarrow{OA}\\ &= 8\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \left(-2\underset{\sim}{i} - 3\hat{\boldsymbol{j}}\right)\\ &= 10\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\end{aligned}$$

ΔABC is isosceles if two sides are of equal magnitude:

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{4^2 + 6^2}\\ &= \sqrt{52}\\ &= 2\sqrt{13}\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{BC}\right| &= \sqrt{6^2 + (-4)^2}\\ &= \sqrt{52}\\ &= 2\sqrt{13}\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{AC}\right| &= \sqrt{10^2 + 2^2}\\ &= \sqrt{104}\\ &= 2\sqrt{26}\end{aligned}$$

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$, therefore ΔABC is isosceles.

ΔABC is a right-angled triangle if $\overrightarrow{AB} \perp \overrightarrow{BC}$ and $\angle ABC = 90°$

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{BC} &= 4 \times 6 + 6 \times (-4)\\ &= 0\end{aligned}$$

$\therefore \angle ABC = 90°$

13 (a) Use vector addition to calculate the vectors of the quadrilateral's sides, then compare the magnitudes of those side vectors. Adjacent sides $\overline{AB}$ and $\overline{DA}$ are equal in length and adjacent sides $\overline{BC}$ and $\overline{CD}$ are equal in length. Thus, $ABCD$ is a kite.

(b) Use vector addition to calculate the vectors of the kite's diagonals, then calculate the dot product.

EXERCISE 2.5

1 (a) $\dfrac{8}{5}$

(b) $\dfrac{18\sqrt{13}}{13}$

(c) 0

(d) $-\dfrac{3\sqrt{61}}{61}$

(e) $-\dfrac{23\sqrt{41}}{41}$

(f) $\dfrac{47\sqrt{53}}{53}$

2 (a) C

(b) Although $\left|\hat{\boldsymbol{a}}\right| = 1$, there is no reason why $\left|\boldsymbol{c}\right| = 1$. This would only occur if the angle $\theta = 0°$, and this information is not given, so alternative A is incorrect. Alternative B could be considered almost the correct answer, but note the inequality signs, so $\left|\boldsymbol{c}\right|$ could still be -1 or 1. Because as $\left|\boldsymbol{c}\right| = \cos(\theta)$, there is no possible way that alternative D is correct.

3 (i) (a) $\dfrac{18}{13}\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)$

(b) $\dfrac{8}{25}\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$

(c) 0

(d) $\dfrac{-3}{61}\left(5\boldsymbol{i} + 6\hat{\boldsymbol{j}}\right)$

(e) $\dfrac{-29}{65}\left(4\hat{i} + 7\hat{\boldsymbol{j}}\right)$

(f) $\dfrac{47}{53}\left(-7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

(ii) (a) $-\dfrac{1}{13}\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$

(b) $\dfrac{1}{25}\left(76\hat{i} - 57\hat{j}\right)$

(c) $8\hat{i} + 4\hat{j}$

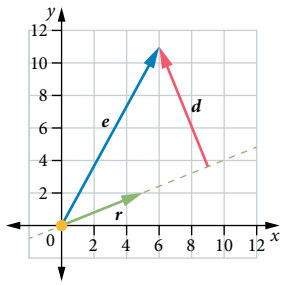
(d) $\dfrac{-4}{61}\left(42\hat{\boldsymbol{i}} - 35\hat{\boldsymbol{j}}\right)$

(e) $\dfrac{-2}{65}\left(7\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right)$

(f) $\dfrac{32}{53}\left(2\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}\right)$

4 $k = \dfrac{5\sqrt{17} - 4}{2} \approx 8.31$

5 (a)



(b) $\boldsymbol{e} - (\boldsymbol{e} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} = -\frac{86}{29}\hat{\boldsymbol{i}} + \frac{215}{29}\hat{\boldsymbol{j}}$

(c) The distance = 7.98 km (2 d.p.) and the bearing = 158° T.

(d) This answer assumes that the road continues along a straight line, and that the ground is perfectly flat. Any changes in elevation of the terrain would result in a longer distance from the explorer to the road.

6 (a) $\boldsymbol{a} = |\boldsymbol{a}|\cos(\alpha)\hat{\boldsymbol{i}} + |\boldsymbol{a}|\sin(\alpha)\hat{\boldsymbol{j}}$

Since $\boldsymbol{a}$ is a unit vector $|\boldsymbol{a}| = 1$.

$\boldsymbol{a} = 1 \times \cos(\alpha)\hat{\boldsymbol{i}} + 1 \times \sin(\alpha)\hat{\boldsymbol{j}}$

$= \cos(\alpha)\hat{\boldsymbol{i}} + \sin(\alpha)\hat{\boldsymbol{j}}$

(b) $\boldsymbol{b} = \cos(\beta)\hat{\boldsymbol{i}} + \sin(\beta)\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = \cos(\beta)\hat{\boldsymbol{i}} - \sin(\beta)\hat{\boldsymbol{j}}$

(c) $\boldsymbol{a} \cdot \boldsymbol{b} = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$;

$\boldsymbol{a} \cdot \boldsymbol{c} = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

(d) Now $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Here $\theta = \alpha - \beta$ and $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$.

Therefore $\boldsymbol{a} \cdot \boldsymbol{b} = \cos(\alpha - \beta)$.

Thus, $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Similarly, $\boldsymbol{a} \cdot \boldsymbol{c} = |\boldsymbol{a}||\boldsymbol{c}|\cos(\theta)$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$.

Here $\theta = \alpha + \beta$ and $|\boldsymbol{a}| = |\boldsymbol{c}| = 1$.

Therefore $\boldsymbol{a} \cdot \boldsymbol{c} = \cos(\alpha + \beta)$.

Thus, $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

EXERCISE 2.6

1 If $\boldsymbol{p} \cdot \boldsymbol{q} = 0$ $\boldsymbol{p} \perp \boldsymbol{q}$

$\boldsymbol{p} \cdot \boldsymbol{q} = (a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}).(-b\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}})$

$= (a.(-b)) + (b.a)$

$= -ab + ab$

$= 0$

$\boldsymbol{p} \cdot \boldsymbol{q} = 0$, therefore $\boldsymbol{p} \perp \boldsymbol{q}$

2 $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= (0\hat{\boldsymbol{i}} + 1\hat{\boldsymbol{j}}) - (-2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}})$

$= 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$

$= (3\boldsymbol{i} + 7\hat{\boldsymbol{j}}) - (-2\boldsymbol{i} - 3\hat{\boldsymbol{j}})$

$= 5\boldsymbol{i} + 10\hat{\boldsymbol{j}}$

If $\overrightarrow{AC} = k.\overrightarrow{AB}$, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$.

Now

$\overrightarrow{AC} = 5\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}$

$= 2.5(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$

$= 2.5\overrightarrow{AB}$

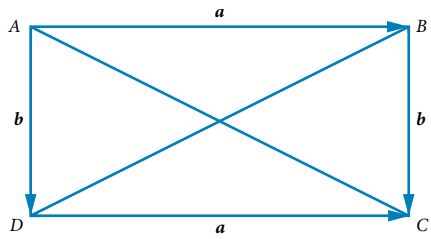
Since $\overrightarrow{AC} = 2.5\overrightarrow{AB}$ and both vectors include point A, therefore points $A(-2,-3)$, $B(0,1)$ and $C(3,7)$ are collinear.

3 D

4 (a) D

(b) Since $\boldsymbol{a} = \frac{1}{2}\boldsymbol{b}$, vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are in different directions, this option doesn't apply to the diagram.

5



$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{BD} = \boldsymbol{b} - \boldsymbol{a}$

$|\overrightarrow{AC}|^2 = (\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} + \boldsymbol{b})$

$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b}$

$|\overrightarrow{BD}|^2 = (\boldsymbol{b} - \boldsymbol{a})(\boldsymbol{b} - \boldsymbol{a})$

$= |\boldsymbol{b}|^2 + |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b}$

Now, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ since $\boldsymbol{a} \perp \boldsymbol{b}$.

So, $|\overrightarrow{AC}|^2 = |\overrightarrow{BD}|^2$.

Hence, the diagonals of the rectangle are equal in length.

6 Let $\overrightarrow{OA} = \overrightarrow{CB} = \boldsymbol{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \boldsymbol{b}$.

Then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{CO}$.

$\overrightarrow{OB}.\overrightarrow{CA} = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$

$= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b}$

$= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$

Now if $\overrightarrow{OB} \cdot \overrightarrow{CA} = 0$, then the diagonals $\overrightarrow{OB}$ and $\overrightarrow{CA}$ are perpendicular.

$\overrightarrow{OB} \perp \overrightarrow{CA}$

$\Rightarrow \overrightarrow{OB}.\overrightarrow{CA} = 0$

$\Rightarrow |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 = 0$

$\Rightarrow |\boldsymbol{a}|^2 = |\boldsymbol{b}|^2$

Therefore, the diagonals of a parallelogram meet at right angles if and only if it is a rhombus.

7 To prove that *EFGH* is a parallelogram you need to show that $\boldsymbol{p} = \boldsymbol{q}$ and $\boldsymbol{r} = \boldsymbol{s}$.

By the definition of vector addition, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ and $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$.

$$\begin{aligned}\boldsymbol{p} &= \overrightarrow{EB} + \overrightarrow{BF} \\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \\ &= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) \\ &= \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

$$\begin{aligned}\boldsymbol{q} &= \overrightarrow{HD} + \overrightarrow{DG} \\ &= \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} \\ &= \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{DC}\right) \\ &= \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

Hence $\boldsymbol{q} = \boldsymbol{p}$, as they are both equal to $\frac{1}{2}\overrightarrow{AC}$.

Similarly, by the definition of vector addition, $-\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{BD}$ and $\overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{BD}$.

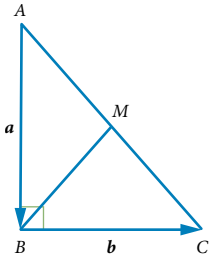
$$\begin{aligned}\boldsymbol{r} &= -\overrightarrow{AE} + \overrightarrow{AH} \\ &= -\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD} \\ &= \frac{1}{2}\left(-\overrightarrow{AB} + \overrightarrow{AD}\right) \\ &= \frac{1}{2}\overrightarrow{BD}\end{aligned}$$

$$\begin{aligned}\boldsymbol{s} &= \overrightarrow{FC} - \overrightarrow{GC} \\ &= \frac{1}{2}\overrightarrow{BC} - \frac{1}{2}\overrightarrow{DC} \\ &= \frac{1}{2}\left(\overrightarrow{BC} - \overrightarrow{DC}\right) \\ &= \frac{1}{2}\overrightarrow{BD}\end{aligned}$$

Hence $\boldsymbol{s} = \boldsymbol{r}$, as they are both equal to $\frac{1}{2}\overrightarrow{BD}$.

Therefore *EFGH* is a parallelogram.

8



$$\begin{aligned}\overrightarrow{BM} &= \boldsymbol{b} - \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \\ &= -\frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{AM}\right|^2 &= \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} + \boldsymbol{b}) \\ &= \frac{1}{4}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b}\right)\end{aligned}$$

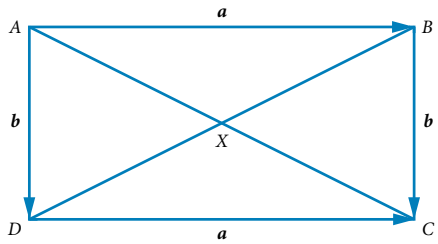
$$\begin{aligned}\left|\overrightarrow{BM}\right|^2 &= \frac{1}{4}(a - b)(a - b) \\ &= \frac{1}{4}\left(|a|^2 + |b|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b}\right)\end{aligned}$$

But, $\boldsymbol{a} \perp \boldsymbol{b}$, so $\boldsymbol{a}\cdot\boldsymbol{b} = 0$

Hence, $\left|\overrightarrow{AM}\right|^2 = \left|\overrightarrow{BM}\right|^2$

Hence, the midpoint of the hypotenuse is equidistant from all three vertices.

9



$$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$$
$$\overrightarrow{DB} = \boldsymbol{a} - \boldsymbol{b}$$
$$\overrightarrow{AX} = x\overrightarrow{AC} = x(\boldsymbol{a} + \boldsymbol{b})$$
$$\overrightarrow{DX} = y\overrightarrow{DB} = y(\boldsymbol{a} - \boldsymbol{b})$$

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AX} - \overrightarrow{DX} \\ &= x(\boldsymbol{a} + \boldsymbol{b}) - y(\boldsymbol{a} - \boldsymbol{b}) \\ &= x\boldsymbol{a} + x\boldsymbol{b} - y\boldsymbol{a} + y\boldsymbol{b} \\ &= (x - y)\boldsymbol{a} + (x + y)\boldsymbol{b}\end{aligned}$$

But $\overrightarrow{AD} = \boldsymbol{b}$ so,

$0\boldsymbol{a} + \boldsymbol{b} = (x - y)\boldsymbol{a} + (x + y)\boldsymbol{b}$

$$x - y = 0$$
$$x + y = 1$$
$$2x = 1$$
$$x = \frac{1}{2}$$
$$y = \frac{1}{2}$$

10 First, calculate $\overrightarrow{BX}$ and $\overrightarrow{AX}$.

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} \\ &= -\boldsymbol{a} + \boldsymbol{b}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BX} &= \frac{1}{2}\overrightarrow{BC} \\ &= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{q})\end{aligned}$$

$$\begin{aligned}\overrightarrow{AX} &= \overrightarrow{AB} + \overrightarrow{BX} \\ &= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) \\ &= \boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a} \\ &= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} \\ &= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$

Now calculate $2(\overrightarrow{BX}\cdot\overrightarrow{BX} + \overrightarrow{AX}\cdot\overrightarrow{AX})$:

$$\begin{aligned}\overrightarrow{BX}\cdot\overrightarrow{BX} &= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\cdot\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) \\ &= \frac{1}{4}(\boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a}) \\ &= \frac{1}{4}(\boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{b}\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{a}) \\ &= \frac{1}{4}\left(|\boldsymbol{b}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{a}|^2\right)\end{aligned}$$

$$\overrightarrow{AX}\cdot\overrightarrow{AX}=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\cdot\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$
$$=\frac{1}{4}(\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b})$$
$$=\frac{1}{4}(\boldsymbol{a}\cdot\boldsymbol{a}+2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{b})$$
$$=\frac{1}{4}\left(|\boldsymbol{a}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{b}|^2\right)$$

$$2\left(\overrightarrow{BX}\cdot\overrightarrow{BX}+\overrightarrow{AX}\cdot\overrightarrow{AX}\right)=\frac{1}{2}\left(\left(|\boldsymbol{b}|^2-2\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{a}|^2\right)+\left(|\boldsymbol{a}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{b}|^2\right)\right)$$
$$=\frac{1}{2}\left(2|\boldsymbol{a}|^2+2|\boldsymbol{b}|^2\right)$$
$$=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2$$

$$\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{AC}\right|^2=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2$$
$$=2\left(\overrightarrow{BX}\cdot\overrightarrow{BX}+\overrightarrow{AX}\cdot\overrightarrow{AX}\right)$$
$$=2\left(\left|\overrightarrow{AX}\right|^2+\left|\overrightarrow{BX}\right|^2\right)$$

11 Start by determining the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$\overrightarrow{OA}=\overrightarrow{CB}=\boldsymbol{a}$
$OC=\overrightarrow{AB}=\boldsymbol{b}$
$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\boldsymbol{a}+\boldsymbol{b}$
$\overrightarrow{AC}=\overrightarrow{AB}-\overrightarrow{BC}=\boldsymbol{b}-\boldsymbol{a}$

Then calculate the sum of the squares of the lengths of the sides of the parallelogram:

$\left|\overrightarrow{OA}\right|^2+\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{OC}\right|^2+\left|\overrightarrow{CB}\right|^2$
$=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+|\boldsymbol{b}|^2+|\boldsymbol{a}|^2$
$=2\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2\right)$

Determine the sum of the squares of the lengths of the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2$
$=|\boldsymbol{a}+\boldsymbol{b}|^2+|\boldsymbol{b}-\boldsymbol{a}|^2$
$=(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})+(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a})$
$=\boldsymbol{a}\cdot\boldsymbol{a}+2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{b}-2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{a}$
$=2\boldsymbol{a}\cdot\boldsymbol{a}+2\boldsymbol{b}\cdot\boldsymbol{b}$
$=2|\boldsymbol{a}|^2+2|\boldsymbol{b}|^2$
$=2\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2\right)$

Now $\left|\overrightarrow{OA}\right|^2+\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{OC}\right|^2+\left|\overrightarrow{CB}\right|^2=2\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2\right)$

and $\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2=2\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2\right)$

Therefore $\left|\overrightarrow{OA}\right|^2+\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{OC}\right|^2+\left|\overrightarrow{CB}\right|^2=\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2$

EXERCISE 2.7

1 **(a)** $3\,\text{km h}^{-1}$ **(b)** $0\,\text{km h}^{-1}$

2 **(a)** The distance Louise travelled from O to B is 850 m.

(b) Louise's displacement from O to B, correct to 1 decimal place, is 680.1 m from O in a direction of N72.9°E.

(c) Louise's average speed, in metres per second, correct to 1 decimal place is $1.2\,\text{m s}^{-1}$.

(d) Louise's average velocity, in metres per second, correct to 1 decimal place is $0.9\,\text{m s}^{-1}$ in a direction of N72.9°E.

3 Kudzie's displacement, correct to 1 decimal place, for his entire walk is 510.9 m in the direction 140°T.

4 The displacement of Camp B from the base camp is 14 543.5 m at an angle of elevation of 10.5°.

5 $2\,\text{m s}^{-1}$ south

6 C

7 D

8 **(a)** The velocity of the aircraft is $110\,\text{m s}^{-1}$ north.

(b) The velocity of the aircraft is $115\,\text{m s}^{-1}$ north.

(c) The velocity of the aircraft, correct to 1 decimal place, is $103.1\,\text{m s}^{-1}$ in the direction 346°T.

9 The magnitude and direction of Mark's resultant displacement, correct to the nearest whole number, is 152 m in the direction 119°T.

10 **(a)** Eylece's final displacement from the cave entrance is 401 m in the direction 66°T.

(b) Eylece's average velocity is $0.53\,\text{km h}^{-1}$ (2 d.p.) in the direction 66°T.

11 **(a)** The time it takes for the boat to reach the opposite shore is 29.06 seconds (2 d.p.).

(b) The distance between the points of embarkation and disembarkation is 70.44 m.

12 The magnitude and direction of Ben's velocity is $0.7\,\text{m s}^{-1}$ at a direction of 45°T.

13 The rain is falling at an angle 81.48° to the horizontal (to 2 d.p.). This answer assumes that the rain falls at a constant velocity for both legs of the journey, and that the car travels along a road which is completely horizontal. If the road is not horizontal, the angle of the rain will be affected.

EXERCISE 2.8

1 **(a)** $\boldsymbol{F}_1+\boldsymbol{F}_2=30$ N north **(b)** $\boldsymbol{F}_1+\boldsymbol{F}_2=16$ N west

(c) $\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3=250$ N east **(d)** $\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3=56$ N south

2 **(a)** 200.00 N at 323.13°T **(b)** 120.42 N at 138.37°T

(c) 151.16 N at 235.78°T **(d)** 357.35 N at 17.93°T

3 The resultant force being applied to the container ship is 3941 N in the direction S54°E.

4 **(a)** $\boldsymbol{R}=150.86$ N acting in the direction N4.81°E.

(b) $\boldsymbol{R}=231.06$ N acting in the direction N81.82°W.

(c) $\boldsymbol{R}=28.11$ N acting in the direction 97.2°T.

(d) $\boldsymbol{R}=564.66$ N acting in the direction 011.39°T.

5 $\boldsymbol{F}_3 = 480\,\text{N}$ north

6 N40°W

7 Required force is 470.4 N.

8 D

9 (a) D

(b) Because the object's motion will remain unchanged, the forces are in equilibrium. $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$ is true because an object in equilibrium has a resultant force of zero. The object experiences no acceleration because it has a non-zero resultant force.

10 $\mu = 0.11$ (2 d.p.)

11 (a) $\boldsymbol{F}_1 = 167.73\hat{\boldsymbol{i}} + 108.93\hat{\boldsymbol{j}}$; $\boldsymbol{F}_2 = 38.20\hat{\boldsymbol{i}} - 216.66\hat{\boldsymbol{j}}$; $\boldsymbol{F}_3 = -135.95\hat{\boldsymbol{i}} - 63.39\hat{\boldsymbol{j}}$

(b) $\boldsymbol{R} = 69.99\hat{\boldsymbol{i}} - 171.12\hat{\boldsymbol{j}}$. Therefore, $\boldsymbol{R} = 184.88$ N at 157.76°T.

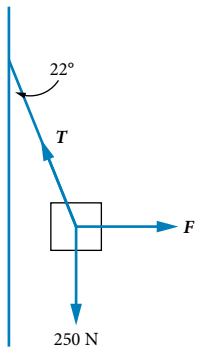
12 (a) $\boldsymbol{F}_1 = 176.78\hat{\boldsymbol{i}} + 176.78\hat{\boldsymbol{j}}$; $\boldsymbol{F}_2 = -154.87\hat{\boldsymbol{i}} - 221.17\hat{\boldsymbol{j}}$; $\boldsymbol{F}_3 = 147.92\hat{\boldsymbol{i}} - 317.21\hat{\boldsymbol{j}}$

(b) $\boldsymbol{R} = 169.83\hat{\boldsymbol{i}} - 361.60\hat{\boldsymbol{j}}$. Therefore, $\boldsymbol{R} = 399.50$ N at S25.16°E

13 Force $\boldsymbol{F}_1$: $|\boldsymbol{F}_1| = 152.92\,\text{N}$; Force $\boldsymbol{F}_2$: $|\boldsymbol{F}_2| = 203.22$ N

14 $|\boldsymbol{F}| = 49.01\,\text{N}; |\boldsymbol{T}| = 85.45\,\text{N}$

15 (a)



(b) $|\boldsymbol{F}| = 101.01\,\text{N}; |\boldsymbol{T}| = 269.63\,\text{N}$

16 $\boldsymbol{F} = 10\,209$ N

17 $|\boldsymbol{F}_2| = 3279; |\boldsymbol{F}_2| = 2977$

18 (a) $\boldsymbol{F} = 1162$ N at 345°T on a bearing is 345.26°T

(b) $|\boldsymbol{R}| = 1151.22$ N on a bearing of 343°T

19 $|\boldsymbol{G}| = 57.9$ N (2 d.p.)

20 $\theta = 31.0°$ (1 d.p.)

CHAPTER REVIEW 2

1 B

2 $\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

3 (6, -9)

4 $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$

5 C

6 The two vectors are $-6\hat{\boldsymbol{i}} + 14\hat{\boldsymbol{j}}$ and $6\hat{\boldsymbol{i}} - 14\hat{\boldsymbol{j}}$.

7 $-9\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$

8 $\boldsymbol{a} = (2\sqrt{5}, 333.43°)$

9 $\frac{6}{5}$

10 $190\,\text{km h}^{-1}$ in the direction of the car

11 500 m

12 $\boldsymbol{F}_3 = 440$ N east

13 (a) $\overrightarrow{CD} = -\frac{1}{3}\boldsymbol{a}$ (b) $\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$

(c) $\overrightarrow{AD} = \frac{2}{3}\boldsymbol{a} + \boldsymbol{b}$ (d) $\overrightarrow{DB} = \frac{1}{3}\boldsymbol{a} - \boldsymbol{b}$

14 (a) $\overrightarrow{OP} = \boldsymbol{a} + \boldsymbol{b}$ (b) $\overrightarrow{OG} = 2\boldsymbol{a} + 2\boldsymbol{b}$

(c) $\overrightarrow{OQ} = 2\boldsymbol{a} + \boldsymbol{b}$ (d) $\overrightarrow{CE} = \boldsymbol{a} + \boldsymbol{b}$

(e) $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$ (f) $\overrightarrow{DI} = -3\boldsymbol{a} + 2\boldsymbol{b}$

(g) $\overrightarrow{FQ} = -\boldsymbol{a} - \boldsymbol{b}$ (h) $\overrightarrow{DE} + \overrightarrow{EO} = -3\boldsymbol{a}$

15 (a) $4\boldsymbol{c} - 3\boldsymbol{d} = \begin{bmatrix} 29 \\ -16 \end{bmatrix}$ (b) $7\boldsymbol{d} - 8\boldsymbol{c} = \begin{bmatrix} -61 \\ 40 \end{bmatrix}$

16 (a) $\boldsymbol{a} + \boldsymbol{b} = \hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$ (b) $3\boldsymbol{b} - 2\boldsymbol{a} = 13\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$

(c) $-2\boldsymbol{a} - 7\boldsymbol{b} = -17\hat{\boldsymbol{i}} + 17\hat{\boldsymbol{j}}$

17 (a) $|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{37}$ (b) $|-2\boldsymbol{a} - 7\boldsymbol{b}| = 17\sqrt{2}$

18 (a) $\overrightarrow{AB} = -21\boldsymbol{i} - 12\hat{\boldsymbol{j}}$ (b) $|\overrightarrow{AB}| = 3\sqrt{65}$

19 (a) $|\boldsymbol{a}| = 26$ (b) $|\boldsymbol{b}| = 3\sqrt{5}$

20 (a) $10.71\hat{\boldsymbol{i}} + 11.89\hat{\boldsymbol{j}}$ (b) $\boldsymbol{b} = -20.35\hat{\boldsymbol{i}} + 12.72\hat{\boldsymbol{j}}$

21 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = -1$ (b) $\boldsymbol{a} \cdot \boldsymbol{b} = 25$

22 $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ hence $\boldsymbol{a} \perp \boldsymbol{b}$

23 (a) The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \frac{2\sqrt{5}}{5}$.

(b) The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \frac{-19\sqrt{2}}{10}$.

24 (a) The resultant force is 75 N south.

(b) The resultant force is $100\sqrt{2}$ N S45°E.

25 A

26 $\frac{4}{13}\left(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)$

27 (a) $a = \frac{23}{2}$ and $b = 6$ (b) $f = \frac{59}{19}$ and $g = \frac{18}{19}$

28 (a) $\hat{\boldsymbol{a}} = \frac{\sqrt{10}}{30}\left(-9\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$ (b) $\boldsymbol{b} = \frac{\sqrt{10}}{6}\left(-9\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$

29 75°

30 177°

31 **(a)** The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\frac{3}{17}\left(4\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}\right)$.

(b) The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is $\frac{22}{17}\left(\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}\right)$.

32 At this time, Lawton and Ted are 71.1 km from the school and in the direction 302.7°T.

33 $|\boldsymbol{F}_1| = |\boldsymbol{F}_2| = 490$ N

34 Force $\boldsymbol{F}_1$: $|\boldsymbol{F}_1| = 191.2$ N; Force $\boldsymbol{F}_2$: $|\boldsymbol{F}_2| = 254.0$ N

35 Coefficient of friction is 891.8 N. This assumes that the floor is horizontal. If the floor is inclined at an angle, then the force Nick must exert will be greater; if the floor is declined at an angle, then the force Nick must exert will be less.

36 $a = -1$, 10 and $b = \sqrt[3]{-3}$

37 **(a)** The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\frac{7\sqrt{2}}{2}$.

(b) The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\frac{7}{2}\hat{\boldsymbol{i}}-\frac{7}{2}\hat{\boldsymbol{j}}$.

(c) The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is $-\frac{1}{2}\hat{\boldsymbol{i}}-\frac{1}{2}\hat{\boldsymbol{j}}$.

(d) $\boldsymbol{a} = \left(\frac{7}{2}\hat{\boldsymbol{i}}-\frac{7}{2}\hat{\boldsymbol{j}}\right)+\left(-\frac{1}{2}\hat{\boldsymbol{i}}-\frac{1}{2}\hat{\boldsymbol{j}}\right)$

38 By Pythagoras' theorem:

$$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{CB}\right|^2$$
$$|\boldsymbol{c}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{a}|^2$$

But,

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$
$$|\boldsymbol{c}|^2 = \boldsymbol{c}\cdot\boldsymbol{c} = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})$$
$$|\boldsymbol{c}|^2 = \boldsymbol{a}\cdot\boldsymbol{a} + 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{b}$$
$$= |\boldsymbol{a}|^2 + |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{c}$$

Therefore,

$$2\boldsymbol{a}\cdot\boldsymbol{b} = 0$$
$$\Rightarrow \boldsymbol{a}\cdot\boldsymbol{b} = 0$$
$$\Rightarrow \boldsymbol{a} \perp \boldsymbol{b}$$

39 **(a)** $\boldsymbol{a} + \boldsymbol{b}$

(b)
$$\overrightarrow{AC} = x\overrightarrow{AE} = x(\boldsymbol{a}+\boldsymbol{b})$$
$$\overrightarrow{BC} = y\overrightarrow{BD} = y\boldsymbol{b}$$
$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = 2\boldsymbol{a}$$
$$\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = 2\boldsymbol{a}$$
$$2\boldsymbol{a} = x(\boldsymbol{a}+\boldsymbol{b}) - y\boldsymbol{b}$$
$$2\boldsymbol{a} = x\boldsymbol{a} + x\boldsymbol{b} - y\boldsymbol{b}$$
$$2\boldsymbol{a} + 0\boldsymbol{b} = x\boldsymbol{a} + (x-y)\boldsymbol{b}$$
$$x = 2, y = 2$$
$$\overrightarrow{BC} = 2\overrightarrow{BD}$$
$$\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$$

40 **(a)** The velocity of the boat relative to the bank is 2.24 m s^{-1} in a direction of 63.43°. This assumes that the current is constant across the whole span of the river. If it is not constant, the relative velocity will change.

(b) Freddie will get to the opposite bank 100 m downstream from point A. This assumes that the riverbanks are parallel, straight lines and the river is fixed width. If this is not the case, then Freddie will arrive at a different point.

(c) Freddie's final displacement from starting point D is 223.61 m at 63.43° relative to the bank.

(d) Rohanne will need to head upstream at an angle of 60° to the bank, and it will take her 115.47 seconds to cross the river.

41 Lachie must apply a force of 500.74 N up the plane (2 d.p.).

Chapter 3 Introduction to proof

RECALL

1 **(a)** 0, 2, 4, 6, 8, 10 **(b)** 1, 3, 5, 7, 9 **(c)** 2, 3, 5, 7
(d) 0, 4, 8 **(e)** 1, 2, 4 **(f)** 0, 1, 4, 9

2 **(a)** 2^3 **(b)** $2^2 \times 3$ **(c)** $2 \times 3^2 \times 5$
(d) $3^2 \times 7$ **(e)** 2×5^2

3 **(a)** $5\sqrt{2}$ **(b)** $3\sqrt{7}$ **(c)** 17

4 **(a)** $\frac{5\sqrt{6}}{6}$ **(b)** $-\frac{6+4\sqrt{5}}{11}$ **(c)** $\frac{19+8\sqrt{3}}{13}$

5 **(a)** x^3 **(b)** $y^3\sqrt{y}$ **(c)** $x + 2\sqrt{xy} + y$

6 **(a)** 3^{2n} **(b)** 3^{n-1} **(c)** 3^{n+1}

7 **(a)** $x \leq 4$ **(b)** $x < -3$

EXERCISE 3.1

1 **(a)** $\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{Z}^+,\mathbb{N}$ **(b)** $\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{Z}^-$ **(c)** $\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{N}$
(d) $\mathbb{R},\mathbb{Q}$ **(e)** $\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{N},\mathbb{Z}^+$ **(f)** $\mathbb{R},\mathbb{Q}$
(g) $\mathbb{R},\mathbb{Q}$ **(h)** $\mathbb{R},\mathbb{Q}$ **(i)** $\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{N},\mathbb{Z}^+$
(j) $\mathbb{R},\mathbb{Q}'$ **(k)** $\mathbb{R},\mathbb{Q}$ **(l)** $\mathbb{R},\mathbb{Q}'$
(m) $\mathbb{R},\mathbb{Q}$ **(n)** $\mathbb{R},\mathbb{Q}'$
(o) $\sqrt{-9}$ does not belong to any of the sets
(p) $\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{N}$

2 **(a)** 0.4 **(b)** 2.13 **(c)** 0.075 **(d)** 0.583
(e) $0.\overline{18}$ **(f)** $0.\overline{63}$ **(g)** $0.3\dot{6}$ **(h)** $3.\dot{7}$

3 **(a)** terminating, as the denominator contains no prime factors other than 2 or 5

(b) recurring, as the denominator contains a prime factor other than 2 or 5 (namely, 3)

(c) terminating, as the denominator contains no prime factors other than 2 or 5

(d) recurring, as the denominator contains a prime factor other than 2 or 5 (namely, 7)

4 (a) $\frac{9}{5}$ (b) $\frac{7}{50}$ (c) $\frac{4111}{1000}$ (d) $\frac{1}{3}$

(e) $\frac{4}{9}$ (f) $\frac{14}{99}$ (g) $\frac{125}{999}$ (h) $\frac{7}{30}$

(i) $\frac{6}{11}$ (j) $\frac{14}{45}$ (k) $\frac{313}{900}$ (l) $\frac{127}{990}$

5 A

6 (a) C

(b) Alternatives B and D are clearly different numbers from 0.49. It's a common error to think these are the same. Alternative A is also different. Although the final digit repeats several times, it does not do so indefinitely. It is a common misconception that the number $0.4\dot{9}$ (alternative C) is less than 0.5. To demonstrate that it is, in fact, equal to 0.5, consider the following:

Let $x = 0.499\,999...$

$10x = 4.999\,999...$

$9x = 4.5$

$x = \frac{4.5}{9}$

$= \frac{1}{2}$

$= 0.5$

7 Follow the same process as if you had 0.2222...

Let $x = 0.aaaa...$

$10x = a.aaaa...$

$9x = a$

$x = \frac{a}{9}$

8 $0.\overline{ab} = \frac{10a + b}{99}$

9 $0.a\overline{b} = \frac{9a + b}{90}$

10 112 and 40

EXERCISE 3.2

1 (a) Either P is odd or Q is odd.

(b) $x \leq 5$ and $x \geq -5$ (equivalently, $-5 \leq x \leq 5$)

(c) x is divisible by neither 7 nor 8.

(d) Neither x nor y is zero. Equivalently, both x and y are non-zero.

2 (a) Doubling any integer and adding 3 results in an odd number. This is true, as doubling any integer results in an even number, and the sum of any even number and any odd number is odd.

(b) There is a real number that is equal to its reciprocal. This is true, as the number 1 has this property.

(c) The square of a real number is always greater than zero. This is false, since the square of zero is zero.

(d) There is a real number whose square is -1. This is false.

(e) The product of any integer and the next is divisible by 3. This is false (since, for example, 4×5 is not divisible by 3).

(f) The difference between any two real numbers is positive. This is false, since $3 - 5$ is negative, for example.

(g) For all real numbers, there is a real number that can be added to it to obtain an answer of zero. This is true since adding the negative of any number to itself results in an answer of zero.

(h) There is a real number with the property that multiplying it by any other real number (y) gives an answer of y. This is true, as the number 1 has this property.

3 (a) $\forall$ integers n, $n^2 > n$. (Equivalently, $\forall n \in \mathbb{Z}$, $n^2 > n$.) This is false as $0^2 = 0$.

(b) $\exists$ a real number x, such that $5x = 0$. (Equivalently, $\exists x \in \mathbb{R}$ such that $5x = 0$.) This is true as the number 0 has this property.

(c) $\forall$ integers n, $n + n + 1$ is odd. (Equivalently, $\forall n \in \mathbb{Z}$, $n + n + 1$ is odd.) This is true.

(d) $\exists$ a real number x, such that $x = x^2$. (Equivalently, $\exists x \in \mathbb{R}$ such that $x = x^2$.) This is true as both 0 and 1 have this property.

(e) $\forall$ real numbers x and y, $x^2 + y^2 < xy$. (Equivalently, $\forall x, y \in \mathbb{R}$, $x^2 + y^2 < xy$.) This is false (for example, when $x = -1$ and $y = 1$).

(f) $\exists$ a real number x, such that for all real numbers, y, $\frac{y}{x} = y$. (Equivalently, $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, \frac{y}{x} = y$.) This is true. The number 1 has this property.

(g) $\forall$ non-zero integers n, $\exists$ an integer m such that n is divisible by m. (Equivalently, $\forall n \in \mathbb{Z}^- \cup \mathbb{Z}^+, \exists m \in \mathbb{Z}$ such that n is divisible by m.) This is true as every integer is divisible by itself.

4 (a) $\exists$ a real number x such that $x^2 \leq 0$. Negation is true (since $0^2 \leq 0$).

(b) $\forall$ real numbers x, $x^2 \neq x$. Original is true (since $1^2 = 1$).

(c) $\exists$ a positive integer n such that $10n \leq n$. Original is true since 10 times any positive number is greater than the number.

(d) $\exists$ a real number x such that x is neither positive nor negative. Negation is true (as the number 0 has this property).

(e) $\forall$ integers n, either $n = 0$ or $n^2 \geq 1$. Negation is true as squaring any integer gives a positive integer, except for the integer 0.

(f) $\exists$ an integer n such that $(-1)^n = 1$ and $(-1)^n \neq -1$. Original is true because raising -1 to any integer power always results in 1 or -1.

5 (a) $x > 3 \Rightarrow x^2 > 9$

(b) n is divisible by $9 \Rightarrow n$ is divisible by 3

(c) $n > 5 \Rightarrow n > 4$

(d) $P > 3 \Rightarrow 7P$ is positive

(e) $2Q$ is a perfect square $\Rightarrow Q$ is even

(f) m is a multiple of $6 \Rightarrow m$ is divisible by 3

(g) $x < -2 \Rightarrow x^2 > 2$

(h) n even and $n > 2 \Rightarrow n$ is not prime

6 (a) original: This is true, since a number divisible by 20 must be divisible by any factor of 20.

converse: If n is divisible by 5, then n is divisible by 20. This is false. For example, 10 is divisible by 5 but not by 20.

contrapositive: If n is not divisible by 5, then n is not divisible by 20. This must be true as the original statement is true.

negation: There exists an integer n with the property that n is divisible by 20 and n is not divisible by 5. This must be false as the original statement was true.

(b) original: This is true, since the square of any integer is divisible by all factors of the integer.

converse: If n^2 is divisible by 3, then n is divisible by 3. This is actually true as the square of any integer must have the same prime factors as the integer.

contrapositive: If n^2 is not divisible by 3, then n is not divisible by 3. This must be true as the original statement is true.

negation: There exists an integer n with the property that n is divisible by 3 but n^2 is not divisible by 3. This must be false as the original statement was true.

(c) original: This is true, as the second inequality can be obtained by multiplying both sides of the first by 10.

converse: If $10x > 70$, then $x > 7$. This is true as the first inequality can be obtained by dividing both sides of the second inequality by 10.

contrapositive: If $10x \leq 70$, then $x \leq 7$. This must be true as the original statement is true.

negation: There exists a real number x with the property that $x > 7$ and $10x \leq 70$. This must be false as the original statement was true.

(d) original: This is true. (It is the null factor law.)

converse: If $x = 0$ or $y = 0$, then $xy = 0$. This is true.

contrapositive: If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.This must be true as the original statement is true.

negation: There exist real numbers x and y with the property that $xy = 0$ and $x \neq 0$ and $y \neq 0$. This must be false as the original statement was true.

(e) original: This is false. For example, 20 is divisible and its final digit is not 5.

converse: If the final digit of n is 5, then n is divisible by 5. This is true.

contrapositive: If the final digit of n is not 5, then n is not divisible by 5. This must be false as the original statement is false.

negation: There exists an integer n that is divisible by 5, and with the final digit of n not 5. This must be true as the original statement was false.

(f) original: This is clearly true.

converse: If $xy = 16$, then $x = 4$ and $y = 4$. This is false. For example, $8 \times 2 = 16$.

contrapositive: If $xy \neq 16$, then $x \neq 4$ or $y \neq 4$. This must be true as the original statement is true.

negation: There exists real numbers x and y with the property that $x = 4$ and $y = 4$, and $xy \neq 16$. This must be false as the original statement was true.

(g) original: This is true, as any number divisible by 24 must contain factors of 2 and 3.

converse: If n is even and n is divisible by 3, then n is divisible by 24. This is false. For example, 12 is even and divisible by 3 but not by 24.

contrapositive: If n is odd or n is not divisible by 3, then n is not divisible by 24. This must be true as the original statement is true.

negation: There exists an integer n with the property that n is divisible by 24, and n is either odd or not divisible by 3. This must be false as the original statement was true.

7 (a) n is even $\Leftrightarrow n^2$ is even

(b) $x + y = 0 \Leftrightarrow x = -y$

(c) n is even and divisible by 3 $\Leftrightarrow n$ is divisible by 6

8 (a) Examples: 7, 14, 28; Counter example: 21

(b) Examples: 2, 3, 4; Counter example: 0.5

(c) Examples: 2, 3, 5; Counter example: 11

(d) Examples: 60, 120, 180; Counter example: 30

(e) Examples: $x = 10$, $y = 0$; $x = 10$, $y = 1$; $x = 10$, $y = 2$; Counter example: $x = 4$, $y = 100$

(f) Examples: $x = 3$, $y = 3$; $x = 3$, $y = 4$; $x = 3$, $y = 5$; Counter example: $x = 0.1$, $y = 10$

9 (a) D

(b) Alternative A, in everyday language, says that for all real numbers x, there exists a real number y such that $xy = 6$. But this is not true for $x = 0$. Alternative B, in everyday language, says that there exists a real number x, such that for all real numbers y, $xy = 6$. There is no single number (for x) that multiplies with every real number to give a result of 6. Alternative C, in everyday language, says that there exists a real number x, such that for all real numbers y, $x + y = 6$. There is no single number (for x) that adds to every real number to give a result of 6.

10 B

11 $\exists$ a real number x such that $x > 0$ and $x < 10$, but $x < 0$ or $x > 10$. It's clear that the original statement is true.

12 (a) $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

(b) $13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

(c) $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

EXERCISE 3.3

1 (a) Let p and q be odd integers.

Then $p = 2k + 1$ and $q = 2l + 1$ for integers k and l.

$$\begin{aligned} p + q &= 2k + 1 + 2l + 1 \\ &= 2k + 2l + 2 \\ &= 2(k + l + 1) \end{aligned}$$

As $k + l + 1$ must be an integer, it follows that $p + q$ is even.

(b) Let p be an odd integer and q be an even integer.

Then $p = 2k + 1$ and $q = 2l$ for integers k and l.

$$\begin{aligned} p + q &= 2k + 1 + 2l \\ &= 2(k + l) + 1 \end{aligned}$$

As $k + l$ must be an integer, it follows that $p + q$ is odd.

(c) Let p and q be odd integers.

Then $p = 2k + 1$ and $q = 2l + 1$ for integers k and l.

$$\begin{aligned} pq &= (2k + 1)(2l + 1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

As $2kl + k + l$ must be an integer, it follows that pq is odd.

(d) Let p and q be two consecutive odd integers.

Then $p = 2k + 1$ and $q = 2k + 3$ for some integer k.

$$\begin{aligned} p + q &= 2k + 1 + 2k + 3 \\ &= 4k + 4 \\ &= 4(k + 1) \end{aligned}$$

As $k + l$ must be an integer, it follows that $p + q$ is divisible by 4.

(e) Let $n, n + 1, n + 2, n + 3, n + 4$ be five consecutive integers.

$$\begin{aligned} &n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2 + (n + 4)^2 \\ &= n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + \\ &\quad 9 + n^2 + 8n + 16 \\ &= 5n^2 + 20n + 30 \\ &= 5(n^2 + 4n + 6) \end{aligned}$$

As $n^2 + 4n + 6$ must be an integer, the result follows.

(f) Let a and b be rational numbers. Then $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where p, q, r, s are integers with q, s not equal to 0.

$$\begin{aligned} ab &= \frac{p}{q} \times \frac{r}{s} \\ &= \frac{pr}{qs} \end{aligned}$$

Since pr and qs must both be integers, and qs cannot be zero (since neither q nor s is 0), it follows that ab is rational.

(g) Let a and b be rational numbers. Then $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where p, q, r, s are integers with q, s not equal to 0.

$$\begin{aligned} a + b &= \frac{p}{q} + \frac{r}{s} \\ &= \frac{ps + qr}{qs} \end{aligned}$$

Since $ps + qr$ and qs must both be integers, and qs cannot be zero (since neither q nor s is 0), it follows that $a + b$ is rational.

(h) If n is odd, then $n = 2k + 1$ for some integer k.

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since $2k^2 + 2k$ must be an integer, it follows that n^2 is odd.

(i) If n is divisible by 7, then $n = 7k$ for some integer k.

$$\begin{aligned} n^2 &= (7k)^2 \\ &= 49k^2 \\ &= 7(7k^2) \end{aligned}$$

Since $7k^2$ must be an integer, it follows that n^2 is divisible by 7.

(j) If $m + n$ and $n + p$ are even, then $m + n = 2k$ and $n + p = 2l$ for integers k and l

For the first equation, $n = 2k - m$. Substituting this into the other equation gives

$$\begin{aligned} n + p &= 2l \\ 2k - m + p &= 2l \\ -m + p &= 2l - 2k \\ m + p &= 2l - 2k + 2m \\ &= 2(l - k + m) \end{aligned}$$

Since $l - k + m$ must be an integer, it follows that $m + p$ is even.

2 (a) The contrapositive statement is: if n is odd, then $3n + 2$ is odd.

Suppose that n is odd. Then $n = 2k + l$ for some integer k.

$$\begin{aligned} 3n + 2 &= 3(2k + 1) + 2 \\ &= 6k + 3 + 2 \\ &= 6k + 5 \\ &= 2(3k + 2) + 1 \end{aligned}$$

As the expression in the brackets must be an integer, it follows that $3n + 2$ is odd.

(b) The contrapositive statement is: if neither a nor b is even, then ab is odd.

Suppose that neither a nor b is even. This means that both are odd.

Then $a = 2k + 1$ and $b = 2l + 1$ for integers k and l.

$$\begin{aligned} ab &= (2k + 1)(2l + 1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

As $2kl + k + l$ must be an integer, it follows that ab is odd.

(c) The contrapositive statement is: if n is odd, then $n^3 + 5$ is even.

Suppose that n is odd. Then $n = 2k + 1$ for some integer k.

$$\begin{aligned} n^3 + 5 &= (2k + 1)^3 + 5 \\ &= (2k + 1)(4k^2 + 4k + 1) + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \end{aligned}$$

As the expression in the brackets must be an integer, it follows that $n^3 + 5$ is even.

(d) The contrapositive statement is: if $\sqrt{x}$ is rational, then x is rational.

Suppose $\sqrt{x}$ is rational. Then $\sqrt{x} = \frac{p}{q}$ for integers p, q with $q \neq 0$. Squaring both sides gives $x = \frac{p^2}{q^2}$. Since p^2 and q^2 are integers with $q^2 \neq 0$, it follows that x is rational.

(e) The contrapositive statement is: if $\frac{1}{x}$ is rational, then x is rational.

Suppose $\frac{1}{x}$ is rational. Then $\frac{1}{x} = \frac{p}{q}$ for integers p, q with $q \neq 0$.

Note also that $p \neq 0$ since this would mean that $\frac{1}{x} = 0$, which is false for all numbers x.

Therefore, $x = \frac{q}{p}$. Since p and q are integers with $p \neq 0$, it follows that x is rational.

3 (a) Suppose, for a contradiction, that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{p}{q}$ for integers p, q with $q \neq 0$, and with p, q having no common factor other than 1.

$$3 = \frac{p^2}{q^2}$$

$p^2 = 3q^2$, hence p^2 is divisible by 3.

If p^2 is divisible by 3, then p is divisible by 3.

Therefore $p = 3m$ for some integer m.

$$9m^2 = 3q^2$$
$$3m^2 = q^2$$

Hence q^2 is also divisible by 3, so q is divisible by 3.

Since p and q are both divisible by 3 then we have a contradiction since p and q have no common factors other than 1.

It follows that $\sqrt{3}$ is an irrational number.

(b) Suppose, for a contradiction, that $\sqrt{5}$ is rational. Then $\sqrt{5} = \frac{p}{q}$ for integers p, q with $q \neq 0$, and with p, q having no common factor other than 1.

$$5 = \frac{p^2}{q^2}$$

$p^2 = 5q^2$, hence p^2 is divisible by 5.

If p^2 is divisible by 5, then p is divisible by 5.

Therefore $p = 5m$ for some integer m.

$$25m^2 = 5q^2$$
$$5m^2 = q^2$$

Hence q^2 is also divisible by 5, so q is divisible by 5.

Since p and q are both multiples of 5 then we have a contradiction since p and q have no common factors other than 1.

It follows that $\sqrt{5}$ is an irrational number.

(c) Let a be a rational number and b an irrational number. Suppose, for a contradiction, that $a + b$ is rational.

Since a and $a + b$ are both rational, $a = \frac{p}{q}$ and $a + b = \frac{r}{s}$ where p, q, r, s are integers with $q \neq 0$ and $s \neq 0$.

$$a + b = \frac{r}{s}$$
$$\frac{p}{q} + b = \frac{r}{s}$$
$$b = \frac{r}{s} - \frac{p}{q}$$
$$= \frac{rq - ps}{sq}$$

This is a contradiction as b is irrational. Hence, $a + b$ must be irrational.

(d) Let a be a non-zero rational number and b an irrational number. Suppose, for a contradiction, that ab is rational.

Since a and ab are both rational, $a = \frac{p}{q}$ and $ab = \frac{r}{s}$ where p, q, r, s are integers with $q \neq 0$ and $s \neq 0$. Note also that $p \neq 0$ since a is non-zero.

$$ab = \frac{r}{s}$$
$$\frac{p}{q}b = \frac{r}{s}$$
$$b = \frac{rq}{sp}$$

This is a contradiction as b is irrational. Hence, ab must be irrational.

(e) Suppose, for a contradiction, that there exists integers a and b such that $18a + 6b = 1$.

$$6(3a + b) = 1$$
$$3a + b = \frac{1}{6}$$

This is a contradiction as $3a + b$ is an integer. Therefore, there cannot be integers a and b such that $18a + 6b = 1$.

4 (a) First suppose that $n + 9$ is even. Then $n + 9 = 2k$ for some integer k.

$$n + 6 = n + 9 - 3$$
$$= 2k - 3$$
$$= 2(k - 2) + 1$$

Thus, $n + 6$ is odd.

Conversely, suppose that $n + 6$ is odd. Then $n + 6 = 2k + 1$ for some integer k.

$$n + 9 = n + 6 + 3$$
$$= 2k + 1 + 3$$
$$= 2k + 4$$
$$= 2(k + 2)$$

Thus, $n + 9$ is even.

(b) First suppose that $n - 3$ is odd. Then $n - 3 = 2k + 1$ for some integer k.

$$n + 2 = n - 3 + 5$$
$$= 2k + 1 + 5$$
$$= 2k + 6$$
$$= 2(k + 3)$$

Thus, $n + 2$ is even.

Conversely, suppose that $n+2$ is even. Then $n+2=2k$ for some integer k.

$$\begin{aligned} n-3 &= n+2-5 \\ &= 2k-5 \\ &= 2(k-3)+1 \end{aligned}$$

Thus, $n-3$ is odd.

(c) First suppose that n is even. Then $n=2k$ for some integer k.

$$\begin{aligned} 13n+4 &= 13\times 2k+4 \\ &= 26k+4 \\ &= 2(13k+2) \end{aligned}$$

Thus, $13n+4$ is even.

Conversely, suppose that $13n+4$ is even. Suppose, for a contradiction, that n is odd. Then $n=2k+1$ for some integer k.

$$\begin{aligned} 13n+4 &= 13(2k+1)+4 \\ &= 26k+17 \\ &= 2(13k+8)+1 \end{aligned}$$

Thus, $13n+4$ is odd—a contradiction. Hence, if $13n+4$ is even, n must be even.

(d) First suppose that n is odd. Then $n=2k+1$ for some integer k.

$$\begin{aligned} 7n+6 &= 7(2k+1)+6 \\ &= 14k+13 \\ &= 2(7k+6)+1 \end{aligned}$$

Thus, $7n+6$ is odd.

Conversely, suppose that $7n+6$ is odd. Suppose, for a contradiction, that n is even. Then $n=2k$ for some integer k.

$$\begin{aligned} 7n+6 &= 7\times 2k+6 \\ &= 14k+6 \\ &= 2(7k+3) \end{aligned}$$

Thus, $7n+6$ is even—a contradiction. Hence, if $7n+6$ is odd, n must be odd.

(e) First suppose that n is even. Then $n=2k$ for some integer k.

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

Thus, n^2 is even.

To prove the converse, suppose that n^2 is even. Suppose, for a contradiction, that n is not even. Then this would mean that n is odd, meaning $n=2k+1$ for some integer k.

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1 \end{aligned}$$

But this would mean that n^2 is odd—a contradiction. Hence, if n^2 is even, n must be even.

5 (a) C

(b) Alternative A is the converse of the given statement, which is not logically equivalent to it (despite being a true statement in this case). Alternative B is how a proof by contradiction would proceed. Alternative D is how a proof by contradiction would proceed, hence won't work to prove this by contraposition.

6 Suppose, for a contradiction, that this number is rational.

Then $1-5\sqrt{2}=\dfrac{p}{q}$ for integers p, q with $q\neq 0$.

$$\begin{aligned} -5\sqrt{2} &= \frac{p}{q}-1 \\ \sqrt{2} &= -\frac{p}{5q}+\frac{1}{5} \\ \sqrt{2} &= \frac{-p+q}{5q} \end{aligned}$$

This is a contradiction as $\sqrt{2}$ is irrational, and thus cannot be expressed as the ratio of two integers. Therefore, the number $1-5\sqrt{2}$ must be irrational.

7 Suppose, for a contradiction, that both a and b are greater than $\sqrt{c}$.

Then

$$\begin{aligned} a\times b &> \sqrt{c}\times\sqrt{c} \\ ab &> c \end{aligned}$$

This is a contradiction as $ab=c$. Thus, it must be the case that at least one of a and b is less than or equal to $\sqrt{c}$.

8 This is most readily proved by proving the contrapositive statement: if a is not odd, then a^2-2a+7 is not even.

If a is not odd, then it is even, and can be expressed as $a=2k$ for some integer k.

$$\begin{aligned} a^2-2a+7 &= 4k^2-4k+7 \\ &= 2(2k^2-2k+3)+1 \end{aligned}$$

Thus, a^2-2a+7 is odd (not even), as required.

9 Let a, b, c, d be the digits, in order, of a four-digit number, N. The actual number is therefore $N=1000a+100b+10c+d$. First suppose that N is divisible by 9. Thus,

$1000a+100b+10c+d=9k$ for some integer k. Then

$$\begin{aligned} 999a+99b+9c+a+b+c+d &= 9k \\ a+b+c+d &= 9k-999a-99b-9c \\ a+b+c+d &= 9(k-111a-11b-c) \end{aligned}$$

Thus, the sum of the digits is divisible by 9.

Conversely, suppose the sum of the digits is divisible by 9.

Then $a+b+c+d=9k$ for some integer k. Then

$$\begin{aligned} N &= 1000a+100b+10c+d \\ &= 999a+99b+9c+d+a+b+c \\ &= 999a+99b+9c+9k \\ &= 9(111a+11b+c+k) \end{aligned}$$

Thus, N is divisible by 9.

10 (a) Squaring a real number produces a non-negative real number. Hence,

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2-2ab+b^2 &\geq 0 \\ a^2+b^2 &\geq 2ab \end{aligned}$$

(b) $$\begin{aligned} (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) &= 1+\frac{a}{b}+\frac{b}{a}+1 \\ &= 2+\frac{a^2+b^2}{ab} \end{aligned}$$

From part (a), $a^2 + b^2 \geq 2ab$, and so, provided that ab is positive (which it is in this case, since a and b are positive), it follows:

$$\frac{a^2+b^2}{ab} \geq \frac{2ab}{ab}$$
$$\frac{a^2+b^2}{ab} \geq \frac{2\cancel{ab}}{\cancel{ab}}$$
$$\frac{a^2+b^2}{ab} \geq 2$$

Combining this inequality with the earlier one yields:

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4.$$

(c) From part (a):

$$a^2+b^2 \geq 2ab$$
$$a^2+2ab+b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a+b \geq \sqrt{4ab} \quad \text{(provided } a, b \text{ positive)}$$
$$a+b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

11 Let n be an odd integer. Then $n = 2k+1$ for some integer k.

Consider the rule: $A^2 - B^2 = (A-B)(A+B)$. If it is possible to show that n can be expressed as a product of the form $(A-B)(A+B)$ where A, B are integers, then it will follow that it can be expressed as the difference between two perfect squares.

$$n = 2k+1$$
$$= \big((1+k)+k\big)\big((1+k)-k\big)$$
$$= (1+k)^2 - k^2$$

12 For a contradiction, suppose there exist integers a and b such that $a^2 - 4b - 3 = 0$, and so $a^2 = 4b+3$. From this equation, a^2 must be odd, and so a must be odd. Hence, $a = 2k+1$ for some integer k. Then

$$(2k+1)^2 = 4b+3$$
$$4k^2+4k+1 = 4b+3$$
$$4k^2+4k-4b = 2$$
$$4\left(k^2+k-b\right) = 2$$
$$k^2+k-b = \frac{1}{2}$$

The left-hand side of this equation is an integer, while the right-hand side is not. This is a contradiction. Therefore, there cannot possibly exist integers a and b such that $a^2 - 4b - 3 = 0$. The original statement is true.

13 If $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+2} + 3^{3k} = 5A$ where A is an integer. Therefore,

$$2^{k+2} = 5A - 3^{3k}$$
$$2^{k+3} = 2\left(5A - 3^{3k}\right)$$
$$2^{k+3} = 10A - 2\times 3^{3k}$$

Thus,

$$2^{k+3} + 3^{3k+3} = 10A - 2\times 3^{3k} + 3^{3k+3}$$
$$= 10A - 2\times 3^{3k} + 3^3 \times 3^{3k}$$
$$= 10A - 2\times 3^{3k} + 27\times 3^{3k}$$
$$= 10A + 25\times 3^{3k}$$
$$= 5\left(2A + 5\times 3^{3k}\right)$$

As the expression in the brackets is an integer, it follows that $2^{k+3} + 3^{3k+3}$ is divisible by 5.

14 Suppose, for a contradiction, that $\sqrt{6} + \sqrt{10}$ is rational.

Then $\sqrt{6} + \sqrt{10} = \frac{p}{q}$ for integers p, q with $q \neq 0$.

$$\left(\sqrt{6}+\sqrt{10}\right)^2 = \left(\frac{p}{q}\right)^2$$
$$6 + 2\sqrt{6}\sqrt{10} + 10 = \frac{p^2}{q^2}$$
$$16 + \sqrt{60} = \frac{p^2}{q^2}$$
$$\sqrt{60} = \frac{p^2}{q^2} - 16$$
$$= \frac{p^2 - 16q^2}{q^2}$$

This is a contradiction, as $\sqrt{60}$ is irrational (since 60 is not a perfect square). The result follows.

15 Suppose, for a contradiction, that $r = \frac{p}{q}$ is a rational solution to the equation, where p, q are integers with no common factor other than 1 and with $q \neq 0$. Then

$$\left(\frac{p}{q}\right)^3 + \frac{p}{q} + 1 = 0$$
$$\frac{p^3}{q^3} + \frac{p}{q} + 1 = 0$$
$$p^3 + pq^2 + q^3 = 0$$

If p is even and q is odd, then p^3 must be even, pq^2 must be even, and q^3 must be odd; this would mean the left-hand side of the equation must be an odd number—a contradiction (since the right-hand side is 0).

If p is odd and q is even, then p^3 must be odd, pq^2 must be even, and q^3 must be even; this would mean again that the left-hand side of the equation must be an odd number—a contradiction (since the right-hand side is 0).

Finally, if both p and q are odd, then p^3 must be odd, pq^2 must be odd, and q^3 must be odd; this would mean again that the left-hand side of the equation must be an odd number—a contradiction (since the right-hand side is 0).

(Note that both p and q cannot be even as it was assumed that p and q have no common factor other than 1.)

Thus, in all cases, a contradiction is obtained. The result follows.

CHAPTER REVIEW 3

1 **(a)** $\mathbb{R}, \mathbb{Q}$ **(b)** $\mathbb{R}, \mathbb{Q}$ **(c)** $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{Z}^-$

(d) $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ **(e)** $\mathbb{R}, \mathbb{Q}'$

2 **(a)** 0.17 **(b)** 0.275 **(c)** $0.41\dot{6}$

3 **(a)** $\frac{19}{50}$ **(b)** $\frac{5}{9}$ **(c)** $\frac{229}{990}$

4 C

5 **(a)** n is not divisible by 5 and n is not divisible by 6. Equivalently, n is divisible by neither 5 nor 6.

(b) $x < 0$ or $x > 10$ **(c)** p is not prime, or $p = 2$.

6 **(a)** Multiplying any integer by 10 results in a number that is divisible by 4. This is false, as $3 \times 10 = 30$ and 30 is not divisible by 4.

(b) There is a real number which, when squared, gives an answer of 2. This is true as both $\sqrt{2}$ and $-\sqrt{2}$ have this property.

(c) For all real numbers, there is a real number that can be multiplied by it that gives an answer of 1. This is false, as the number zero multiplied by any other real number cannot possibly produce 1. Note that this statement would be true if it began with '$\forall$ non-zero real numbers x…'.

7 **(a)** $\forall$ real numbers x, $x^2 \geq 0$. Negation is true as the square of any real number is non-negative.

(b) $\exists$ an integer n such that $5n + 3$ is even. Negation is true since if $n = 3$, for example, $5n + 3 = 18$ which is even.

(c) $\exists$ an integer n such that $\forall$ integers m, $2m \neq n$. Negation is true, as the integer 3, for example, cannot be expressed as 2 multiplied by any integer.

8 **(a)** $x < -10 \Rightarrow 3x$ is negative

(b) n ends in a zero $\Rightarrow$ n is a multiple of 5

(c) n is a multiple of 4 $\Rightarrow$ n is even **(d)** $x > 3 \Rightarrow x^2 > 9$

9 **(a)** original: False, since, for example, 8 is divisible by 4, but not by 12.

converse: If n is divisible by 12, then n is divisible by 4. This is true, since any multiple of 12 has 4 as a factor, and thus, must be divisible by 4.

contrapositive: If n is not divisible by 12, then n is not divisible by 4. This must be false as the original statement is false.

negation: There exists an integer n with the property that n is divisible by 4 but n is not divisible by 12. This must be true as the original statement was false.

(b) original: False, since, for example, $0.5 \times 2 = 1$, but neither 0.5 nor 2 are equal to 1.

converse: If $x = 1$ or $y = 1$, then $xy = 1$. This is false (for example, when $x = 1$ and $y = 2$).

contrapositive: If $x \neq 1$ and $y \neq 1$, then $xy \neq 1$. This must be false as the original statement is false.

negation: There exist real numbers x and y with the property that $xy = 1$ but $x \neq 1$ and $y \neq 1$. This must be true as the original statement was false.

(c) original: This is clearly true.

converse: if $x^2 < 9$, then $x > 0$ and $x < 3$. This is false. For example, if $x = -2$, then $x^2 < 9$, but $x < 0$.

contrapositive: if $x^2 \geq 9$, then either $x \leq 0$ or $x \geq 3$. This is true as the original statement is true.

negation: There exists a real number x with the property that $x > 0$ and $x < 3$, but $x^2 \geq 9$. This must be false as the original statement was true.

10 C

11 Let p and q be odd integers.

Then $p = 2k + 1$ and $q = 2l + 1$ for integers k and l.

$$\begin{aligned} p^2 + q^2 &= (2k+1)^2 + (2l+1)^2 \\ &= 4k^2 + 4k + 1 + 4l^2 + 4l + 1 \\ &= 4k^2 + 4k + 4l^2 + 4l + 2 \\ &= 2(2k^2 + 2k + 2l^2 + 2l + 1) \end{aligned}$$

As $2k^2 + 2k + 2l^2 + 2l + 1$ must be an integer, it follows that $p^2 + q^2$ is even.

12 The contrapositive statement is: if n is odd, then $n^3 - n$ is divisible by 4.

Suppose that n is odd. This means that $n = 2k + 1$ for some integer k.

$$\begin{aligned} n^3 - n &= (2k+1)^3 - (2k+1) \\ &= (2k+1)(4k^2 + 4k + 1) - 2k - 1 \\ &= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 - 2k - 1 \\ &= 8k^3 + 12k^2 + 4k \\ &= 4(2k^3 + 3k^2 + k) \end{aligned}$$

As $2k^3 + 3k^2 + k$ must be an integer, it follows that $n^3 - n$ is divisible by 4.

13 Suppose, for a contradiction, that $\sqrt{7}$ is rational. Then $\sqrt{7} = \frac{p}{q}$ for integers p, q with $q \neq 0$, and with p,q having no common factor other than 1.

$$7 = \frac{p^2}{q^2}$$

$p^2 = 7q^2$, hence p^2 is divisible by 7.

If p^2 is divisible by 7, then p is divisible by 7.

Therefore $p = 7m$ for some integer m.

$$49m^2 = 7q^2$$
$$7m^2 = q^2$$

Hence q^2 is also divisible by 7, so q is divisible by 7.

Since p and q are both divisible by 7 then we have a contradiction since p and q have no common factors other than 1.

It follows that $\sqrt{7}$ is an irrational number.

14 Suppose, for a contradiction, that there exist integers a and b such that $12a - 9b = 20$.

Now $20 = 3(4a - 3b)$, and since a and b are integers, it follows that $4a - 3b$ is an integer. It follows that 20 is divisible by 3, which is a contradiction. Therefore, there cannot possibly be a pair of integers a and b for which $12a - 9b = 20$.

15 First suppose that n is odd. Then $n = 2k + 1$ for some integer k.

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= (2k+1)(4k^2 + 4k + 1) \\ &= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

Thus, n^3 is odd.

To prove n^3 odd $\Rightarrow n$ is odd, the contrapositive statement, n even $\Rightarrow n^3$ is even, will be proved. Suppose that n is even. Then $n = 2k$ for some integer k.

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \end{aligned}$$

Thus, n^3 is even, as required.

16 $0.a\overline{3b} = \dfrac{99a + 30 + b}{990}$

17 Since the numbers are consecutive, it follows that $b = a + 1$, $c = a + 2$, and $d = a + 3$. The left-hand side of the equation to be proved is:

$$\begin{aligned} -a^2 + b^2 - c^2 + d^2 &= -a^2 + (a+1)^2 - (a+2)^2 + (a+3)^2 \\ &= -a^2 + a^2 + 2a + 1 - a^2 - 4a - 4 + \\ &\quad a^2 + 6a + 9 \\ &= 4a + 6 \end{aligned}$$

The right-hand side is:

$$\begin{aligned} a + b + c + d &= a + a + 1 + a + 2 + a + 3 \\ &= 4a + 6 \end{aligned}$$

Since the left-hand side is equal to the right-hand side, the result is demonstrated.

18 Suppose, for a contradiction, that $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2} = \dfrac{p}{q}$ for integers p, q with $q \neq 0$, and with p, q having no common factor other than 1.

$$2 = \frac{p^3}{q^3}$$

$p^3 = 2q^3$, hence p^3 is divisible by 2.

If p^3 is divisible by 2, then p is divisible by 2.

Therefore $p = 2m$ for some integer m.

$$\begin{aligned} (2m)^3 &= 2q^3 \\ 4m^3 &= q^3 \end{aligned}$$

Hence q^3 is also divisible by 2, so q is divisible by 2.

Since p and q are both divisible by 2 then we have a contradiction since p and q have no common factors other than 1.

It follows that $\sqrt[3]{2}$ is an irrational number.

19 For a contradiction, suppose there exist integers a and b such that $a^2 - 4b - 2 = 0$, and so $a^2 = 4b + 2$. From this equation, a^2 must be even, and so a must be even. Hence, $a = 2k$ for some integer k. Then

$$\begin{aligned} (2k)^2 &= 4b + 2 \\ 4k^2 &= 4b + 2 \\ 2k^2 &= 2b + 1 \end{aligned}$$

The left-hand side of this equation is an even number, while the right-hand side is an odd number. This is a contradiction. Therefore, there cannot possibly exist integers a and b such that $a^2 - 4b - 2 = 0$. The stated result follows.

20 Assume that $6^k + 4$ is divisible by 5. Then $6^k + 4 = 5A$ for some integer A.

$$\begin{aligned} 6^{k+1} + 4 &= 6 \times 6^k + 4 \\ &= 6 \times (5A - 4) + 4 \quad \text{since } 6^k + 4 = 5A \\ &= 30A - 24 + 4 \\ &= 30A - 20 \\ &= 10(A - 2) \end{aligned}$$

21 Suppose, for a contradiction, that there exists a pair of positive real numbers x and y such that $\dfrac{1}{7} + \dfrac{x}{y} = \dfrac{1+x}{7+y}$.

Rearranging this equation:

$$\frac{1}{7} + \frac{x}{y} = \frac{1+x}{7+y}$$

$$y + 7x = \frac{7y(1+x)}{7+y}$$

$$(y + 7x)(7 + y) = 7y(1 + x)$$

$$7y + y^2 + 49x + 7xy = 7y + 7xy$$

$$y^2 + 49x = 0$$

But since x and y are both positive, y^2 and $49x$ are positive, and so $y^2 + 49x > 0$. This is a contradiction. Therefore, there cannot possibly exist a pair of positive real numbers x and y such that $\dfrac{1}{7} + \dfrac{x}{y} = \dfrac{1+x}{7+y}$. The stated result follows.

22 Let a, b, c, d be the digits, in order, of a four-digit number, N. The actual number is therefore $N = 1000a + 100b + 10c + d$. First suppose that N is divisible by 7. Thus, $1000a + 100b + 10c + d = 7k$ for some integer k.

The number obtained by removing the final digit and subtracting it twice from the original number is

$$\begin{aligned} &100a + 10b + c - 2d \\ &= 100a + 10b + c - 2(7k - 1000a - 100b - 10c) \\ &= 100a + 10b + c - 14k + 2000a + 200b + 20c \\ &= 2100a + 210b + 21c - 14k \\ &= 7(300a + 30b + 3c - 2k) \end{aligned}$$

Thus, this new number is divisible by 7.

Conversely, suppose the new number described is divisible by 7. Then $100a + 10b + c - 2d = 7k$ for some integer k. Then

$$\begin{aligned} N &= 1000a + 100b + 10c + d \\ &= 1000a + 100b + d + 10(7k - 100a - 10b + 2d) \\ &= 1000a + 100b + d + 70k - 1000a - 100b + 20d \\ &= 21d + 70k \\ &= 7(3d + 10k) \end{aligned}$$

Thus, N is divisible by 7.

23 Suppose, for a contradiction, that there does exist a rational solution to the equation given. Let such a rational solution be $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.

Since $\frac{p}{q}$ is a solution to the equation given,

$$\begin{aligned} a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c &= 0 \\ \frac{ap^2}{q^2} + \frac{bp}{q} + c &= 0 \\ ap^2 + bpq + cq^2 &= 0 \end{aligned}$$

If one of p and q is odd and the other even, then since a, b, c are all odd, there will be exactly one term on the left-hand side of the previous equation that will be odd, resulting in an odd sum; this is impossible as the sum is 0, an even number. If both p and q are odd, then there will be exactly three terms on the left-hand side that will be odd, also resulting in an odd sum. Therefore, the only possibility is that both p and q are even. But this contradicts the fact that p and q have no common factor other than 1. Therefore, there cannot possibly be a rational solution to the given equation.

24 $\frac{1+\sqrt{5}}{2}$

25

$$\begin{aligned} \frac{1}{4-3P} &= \frac{1}{4-3\frac{\left(\frac{1}{3}\right)^n+1}{\left(\frac{1}{3}\right)^n+3}} \\ &= \frac{\left(\frac{1}{3}\right)^n+3}{4\left(\left(\frac{1}{3}\right)^n+3\right)-3\left(\left(\frac{1}{3}\right)^n+1\right)} \end{aligned}$$

(multiplying numerator and denominator by $\left(\frac{1}{3}\right)^n + 3$)

$$\begin{aligned} &= \frac{\left(\frac{1}{3}\right)^n+3}{4\left(\frac{1}{3}\right)^n+12-3\left(\frac{1}{3}\right)^n-3} \\ &= \frac{\left(\frac{1}{3}\right)^n+3}{\left(\frac{1}{3}\right)^n+9} \times \frac{\frac{1}{3}}{\frac{1}{3}} \\ &= \frac{\left(\frac{1}{3}\right)^{n+1}+1}{\left(\frac{1}{3}\right)^{n+1}+3} \end{aligned}$$

Chapter 4 Geometric properties and proofs

RECALL

1 (a) $a = b = c = d = 130°$ (b) $e = 110°, f = 70°$

(c) $x = 120°, y = 50°, w = 50°$

2 (a) $k = 105°, l = 75°, m = 75°$ (b) $x = 33°$

3 (a) $x = 20°$ (b) $m = 40°$

4 (a) 360° (b) 120°

5 (a) 70° (b) 80°

6 $a + c = 180°$ (Angles on a straight line are supplementary.)

$c = 180° - a$

$b = c + 90°$ (Exterior angle of a triangle equals the sum of the two interior opposite angles.)

$c = b - 90°$

Therefore, $180° - a = b - 90°$ (Angles in a triangle add up to 180°.)

$a + b = 270°$

EXERCISE 4.1

1 $x = 4.5, y = 3$

2 In ΔABD and ΔCBD

$|AB| = |CB|$
$|AD| = |CD|$ (given)

BD is common

$\Delta ABD \cong \Delta CBD$ (SSS)

3 In ΔABO and ΔCDO

$|OA| = |OD|$
$|OB| = |OC|$ (equal radii)

Since vertically opposite angles on straight lines are equal:

$\angle AOB = \angle DOC$

$\Delta ABO \cong \Delta CDO$ (SAS)

4 In ΔACB and ΔECD, since alternate angles between parallel lines are equal:

$\angle BAC = \angle DEC$

$|AC| = |EC|$ (given)
$|AB| = |ED|$ (given)

$\therefore \Delta ABC \cong \Delta ECD$ (SAS)

5 The triangles are not congruent.

6 In ΔABC and ΔDBE, $\angle ABC = \angle DBE$

Since corresponding angles on parallel lines are equal:

$\angle BAC = \angle BDE$
$\angle BCA = \angle BED$

$\Delta ABC \sim \Delta DBE$ (AAA)

7 (a) In ΔABC and ΔDEF

$$\frac{|ED|}{|AB|} = \frac{70}{35} = 2$$
$$\frac{|FE|}{|CB|} = \frac{90}{45} = 2$$
$$\frac{|FD|}{|AC|} = \frac{108}{54} = 2$$

So $\Delta ABC \sim \Delta DEF$ (SSS)

(b) $x = 84°$

8 (a) $x = 27°$ **(b)** $x = 3$

9 $\Delta ABC \cong \Delta DEF \cong \Delta JKL$ (ASA)

10 $|AM| = 4$

11 (a) C **(b)** D

(c) In part **(a)** $\Delta ABC \cong \Delta DEC$ is false because the triangles are not congruent, as the corresponding sides are of different lengths. For the statement $|AB| = |BC|$ the two side lengths are not necessarily equal.

(d) In part **(b)** the statement $\angle ABC = \angle DCE$ is false since these two angles are different.

12 B **13** 90 m

14 (a) $33\frac{1}{3}$ m **(b)** $27\frac{5}{6}$ m

15 $y = 60°$

16 (a)
$$\frac{|AB|}{|DE|} = \frac{5}{10} = \frac{1}{2}$$
$$\frac{|AC|}{|AE|} = \frac{3}{6} = \frac{1}{2}$$
$$\frac{|CB|}{|AD|} = \frac{3}{6} = \frac{1}{2}$$

$\Delta ABC \sim \Delta ADE$ (SSS)

(b) $\angle BAC = \angle ADE$, since corresponding angles in similar triangles are equal. These are alternate angles, therefore $AB \parallel DE$.

17 (a) In ΔACE and ΔBCE

$|CA| = |CB|$ (equal radii)
$|AE| = |BE|$ (given)
$|CE| = |CE|$ (common)

$\Delta ACE \cong \Delta BCE$ (SSS)

(b) $\Delta ACE \cong \Delta BCE$

$\angle AEC = \angle BEC$ (corresponding angles in congruent triangles)

Since angles on a straight line are supplementary:

$\angle AEC + \angle BEC = 180°$

$\therefore CE \perp AE \quad \therefore \angle AEC = 90°$
$\therefore AB \perp CD \quad \therefore CE \perp AE$
$\therefore AB \perp CD$

18 $x = 2, y = 6$

19 (a) 3.25 m **(b)** 0.75 m **(c)** 3.33 m **(d)** $xy = 2.5$

20 (a) $h = 3r$ **(b)** $V = \pi r^3$ **(c)** 5.03 cm

EXERCISE 4.2

1 In ΔAED and ΔBEC

$\angle ABE = \angle CDE$ (Alternate angles on parallel lines are equal.)
$\angle BAE = \angle DCE$ (Alternate angles on parallel lines are equal.)
$|AB| = |CD|$ (given)
$\Delta ABE \cong \Delta CDE$ (ASA)
$|BE| = |DE|$ (corresponding sides in congruent triangles)
$|AE| = |CE|$ (corresponding sides in congruent triangles)

2 As interior angle sum of any triangle is 180°:

In ΔABD, $\angle BAD + \angle ADB + \angle DBA = 180°$

In ΔCBD, $\angle BCD + \angle CDB + \angle CBD = 180°$

The interior angle sum of quadrilateral $ABCD$ is equal to $\angle ABC + \angle BCD + \angle CDA + \angle DAB$.

This is the same as the sum of the interior angles in the two triangles ΔABD and ΔBCD.

The angle sum of a quadrilateral is $180° + 180° = 360°$.

3 As alternate angles on parallel lines are equal:

$\angle ADB = \angle DBC$
$\angle BDC = \angle DBA$

Adding these two equations gives

$\angle ADC = \angle ABC$

Also, as co-interior angles on parallel lines are supplementary:

$\angle DAB + \angle ABC = 180°$
$\angle DCB + \angle ABC = 180°$
Hence $\angle DCB = \angle DAB$

Opposite angles in a parallelogram are equal.

4 $\angle CBA + \angle CBE = 180°$ (Angles on a straight line are supplementary.)
$\angle CBA + \angle CAB + \angle ACB = 180°$ (Angles in a triangle add up to 180°.)
Hence $\angle CAB + \angle ACB = \angle CBE$

The external angle of a triangle is equal to the sum of the two opposite interior angles.

5 (a) True **(b)** True **(c)** False **(d)** False

6 Since alternate angles on parallel lines are equal, in ΔABD and ΔBCD

$\angle ABD = \angle BDC$
$\angle ADB = \angle DBC$
$|BD| = |BD|$ (common)
$\Delta ABD \cong \Delta CDB$ (ASA)

The diagonal of a parallelogram divides it into two congruent triangles.

7 $|AM| = |BM|$ (given)
$|CM| = |CM|$ (common)
$|AC| = |BC|$ (equal sides in an isosceles triangle)
$\Delta ACM \cong \Delta BCM$ (SSS)
Hence $\angle ACM = \angle BCM$

as congruent triangles are equal.

Therefore, the bisector of the base of the triangle bisects $\angle ACB$.

The bisector of the base of an isosceles triangle bisects the angle at the apex.

8 Contradictory assumption: BD does not bisect AC.
$\angle BDA = \angle BDC$ (both right angles)
$|AB| = |BC|$ (given)
$|BD| = |BD|$ (common)
$\Delta ADB \cong \Delta BDC$ (RHS)
$|AD| = |DC|$ (corresponding sides in congruent triangles)

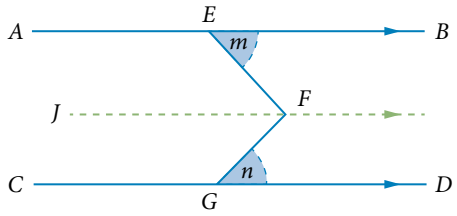
This contradicts the initial assumption; therefore, the initial assumption is false.

So, BD does bisect AC.

The perpendicular drawn from the base of an isosceles triangle to its apex bisects the base of the triangle.

9 D

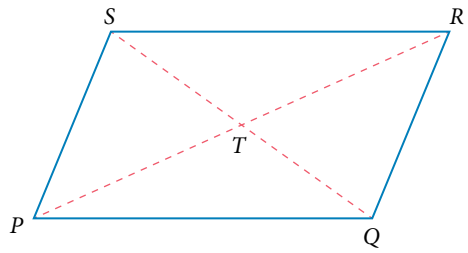
10 Construct a line FJ parallel to AB and CD through the point F.



As alternate angles on parallel lines are equal:
$\angle EFJ = m$
$\angle GFJ = n$
$\angle EFJ + \angle GFJ = m + n$
Hence $\angle EFG = m + n$

11 In ΔACN and ΔBCM, $\angle ACB$ is common.
$|CA| = |CB|$ (given equal sides of an isosceles triangle)
$|CM| = |CN|$ (M and N are midpoints)
$\Delta ACN \cong \Delta BCM$ (SAS)
Hence $|AN| = |BM|$ (corresponding sides of congruent triangles)

12 Construct a rhombus and draw the two diagonals.



In ΔSTR and ΔQTR, RT is common.
$|SR| = |QR|$ (equal sides of a rhombus)
$|ST| = |QT|$ (diagonals of a rhombus bisect each other)
$\Delta STR \cong \Delta QTR$ (SSS)
$\angle STR = \angle QTR$ (corresponding angles in congruent triangles)

Now, since these two angles add to 180°, each one must be equal to 90°.

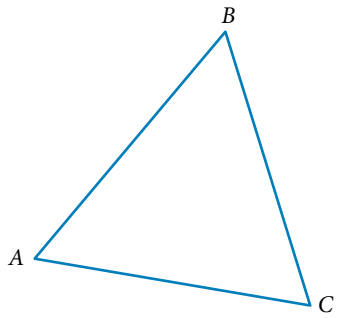
The diagonals of a rhombus meet at right angles.

13 As angles in a triangle add to 180°: $k = 90° - \angle BDC$

As the angles are complementary: $j = 90° - \angle BDC$

Therefore $j = k$

14 Draw a labelled diagram.



Assume there are two right angles, for example, $\angle ABC$ and $\angle BCA$.

$$\angle ABC + \angle BCA + \angle CAB = 180°$$
$$90° + 90° + \angle CAB = 180°$$
$$180° + \angle CAB = 180°$$
$$\angle CAB = 0°$$

This leads to an impossible situation, namely that one of the angles is 0°. Thus, the assumption is false.

Conclusion: No more than one angle in a triangle can be a right angle.

15 Construct a diagonal, for example BD.
In ΔABD and ΔCDB
$|AD| = |BC|$ (given)
$|AB| = |DC|$ (given)
$|DB| = |DB|$ (common to ΔABD and ΔCDB)
$\Delta ABD \cong \Delta CDB$ (SSS)

So,

$\angle ABD = \angle CDB$ (corresponding angles in congruent triangles)

They are alternate angles which means $AB \parallel DC$.

Similarly $AD \parallel BC$.

Thus, the two pairs of opposite sides are parallel.

16 Contradictory assumption: At least two of the angles are reflex (greater than 180°)

In a quadrilateral $ABCD$, assume $\angle ADC > 180°$ and $\angle ABC > 180°$.
$\angle ABC + \angle BCD + \angle ADC + \angle DAB = 360°$
$> 180° + \angle BCD + > 180° + \angle DAB = 360°$
$\therefore \angle BCD + \angle DAB < 0°$

Thus, the assumption leads to a contradiction.

Conclusion: At most one internal angle in a quadrilateral is reflex.

17 (a) In ΔBAD

$$\angle BAD + 90° + y° = 180° \text{ (supplementary angles)}$$
$$\angle BAD = 90° - y°$$

Given $\angle CAD + \angle BAD = 90°$

$$\angle CAD + (90° - y°) = 90°$$

$\therefore \angle CAD = y°$

Also ΔACD

$$\angle CAD + 90° + x° = 180°$$
$$\angle CAD = 90° - x°$$

Given $(90° - x°) + \angle BAD = 90°$

$\therefore \angle BAD = x°$

(b) $\dfrac{|BC|}{|AB|} = \dfrac{|AB|}{|DB|}$

(c) $\dfrac{|BC|}{|AC|} = \dfrac{|AC|}{|CD|}$

(d) The answer to part **(b)** can be rearranged to give $|BC| \times |BD| = |AB|^2$.

The answer to part **(c)** can be rearranged to give $|BC| \times |CD| = |AC|^2$.

Adding these two results:

$$|BC| \times |BD| + |BC| \times |CD| = |AB|^2 + |AC|^2$$
$$|BC|(|BD| + |CD|) = |AB|^2 + |AC|^2$$

But $|BD| + |CD| = |BC|$

Therefore $|BC|^2 = |AB|^2 + |AC|^2$, thus demonstrating Pythagoras' theorem.

18 For a circle: Area $= \pi r^2$

$$\text{Area of } S_1 = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 = \frac{\pi}{8}a^2$$

$$\text{Area of } S_2 = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 = \frac{\pi}{8}b^2$$

$$\text{Area}(S_1 + S_2) = \frac{\pi}{8}(a^2 + b^2) = \frac{\pi}{8}c^2 \quad \text{(Using Pythagoras' theorem)}$$

$$= \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = S_3$$

19 (a) $\Delta ABC \cong \Delta CED$ (SSS)

$\angle ACB = \angle CDE$ (corresponding angles in congruent triangles)

$\angle BAC = \angle ECD$ (corresponding angles in congruent triangles)

(b) $\angle BAC$ and $\angle ACB$ are the two acute angles in ΔABC.

$$\angle BAC + \angle ACB + \angle ABC = 180°$$
$$\angle BAC + \angle ACB = 90°$$

But $\angle BAC = \angle ECD$

$$\angle ECD + \angle ACB = 90°$$
$$\angle ECD + \angle ACB + \angle ACD = 180°$$
$$\therefore \angle ACD = 90°$$

(c) $\angle ABE = 90°$ (given)

$\angle DEB = 90°$ (given)

$\angle ABE + \angle DEB = 180°$

These are co-interior angles:

$\therefore AB \parallel DE$

$\therefore ABED$ is a trapezium

$$\frac{1}{2}(a^2 + 2ab + b^2)$$

(d) Area of $T_1 = \dfrac{1}{2}ab$

Area of $T_2 = \dfrac{1}{2}c^2$

Area of $T_3 = \dfrac{1}{2}ab$

(e) $\dfrac{a^2}{2} + \dfrac{b^2}{2}$

(f) Equating the two expressions for the area of T_2.

$$\frac{c^2}{2} = \frac{a^2}{2} + \frac{b^2}{2}$$
$$c^2 = a^2 + b^2$$

This is the same as $a^2 + b^2 = c^2$.

EXERCISE 4.3

1 (a) $|OC| = 8$ cm, $|AC| = 4\sqrt{21}$ **(b)** $|AB| = 60\sqrt{15}$

2 $\angle CBA = 58°$, $\angle ACO = 32°$, $\angle OCB = 58°$

3 $\angle CAO = 42°$, $\angle OCB = 48°$

4 $x = 290°$

5 $x = 20°$

6 $x = 34°$

7 $\angle BOC = 36°$

8 Since angles in a triangle add up to 180°

$$\angle ABC + \angle BAC + \angle BCA = 180°$$
$$\angle ABC = 180° - \angle BAC + \angle BCA$$
$$= 180° - 40° + 54°$$
$$= 86°$$

This is the angle at the circumference but does not equal 90°. Therefore AC cannot be a diameter of the circle.

9 (a) $x = 106°$ **(b)** $x = 212°$ **(c)** $a = 137.5°$

10 (a) $a = 90°, b = 82°$ **(b)** $c = 60°, d = 75°$

(c) $j = 75°, k = 105°$

11 (a) $x = 15°$ **(b)** $x = 86°, y = 76°$

(c) $x = 138°, y = 49°$ **(d)** $x = 24°, y = 48°$

12 (a) True **(b)** True **(c)** False

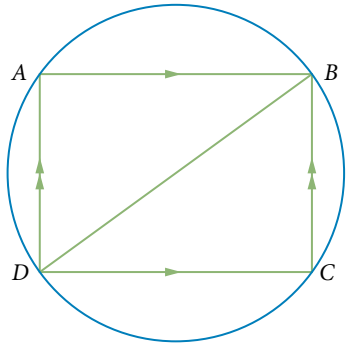
13 B

14 $y = (x - 10)°$

15 $x = 30, y = 40$

16 $\angle BAD = 120°$, $\angle ABC = 90°$, $\angle BCD = 60°$, $\angle ADC = 90°$

17 Draw a diagram and construct the diagonal DB.



Since alternate angles between parallel lines are equal:
$\angle ADB = \angle DBC$
$\angle ABD = \angle BDC$
$|BD| = |BD|$ (common side)
$\Delta ABD \cong \Delta CDB$ (ASA)

Since opposite angles in a cyclic quadrilateral are supplementary:
$\angle DAB + \angle BCD = 180°$
But, $\angle DAB = \angle BCD$ (corresponding angles in congruent triangles)
$\angle DAB = 90°$

$AB \perp AD$ or $\angle DAB = 90°$

18 You have demonstrated that this cyclic quadrilateral is a rectangle, and it therefore contains four right angles. You know that the angle in a semicircle is a right angle, so the two diagonals must be diameters. (This is an example of the converse of the theorem.)

19 **(a)** $EFGH$

(b) **(i)** $(180 - w - z)°$ **(ii)** $(180 - y - z)°$
(iii) $(180 - x - y)°$ **(iv)** $(180 - w - x)°$

(c) **(i)** $(360 - w - x - y - z)°$ **(ii)** $(360 - w - x - y - z)°$

(d) $\angle EFG + \angle GHE = 180°$
$\angle HEF + \angle FGH = 180°$

Therefore, the quadrilateral $EFGH$ is cyclic.

EXERCISE 4.4

1 **(a)** $x = 49°$ **(b)** $x = 49°, y = 45°$ **(c)** $x = 108°$

2 **(a)** $x = 17.5$ **(b)** $x = 7$

3 $x = 33°, y = 33°, \angle BCT = 114°$

4 $x = 3$

5 **(a)** $x = \frac{39}{5}$ **(b)** $x = 2\sqrt{14}$

6 $x = 31°, y = 59°$

7 $y = 32°$

8 **(a)** D

(b) For the statement $|PT| = |PA| + |PB|$, $\overline{PB}$ is longer than $\overline{PT}$ $|PT| = |PB|$ only if PB is a tangent to the circle, hence this is false. $|PT| = |PA| \times |PB|$ is incorrect since the lines are not intersecting chords. If you consider $|PB|^2 = |PT|^2 + |TB|^2$, a common error is to consider that $\angle PTB$ is a right angle, which is incorrect.

9 C

10 $y = 4.58$

11 $x = 72°$

12 **(a)** $x = 3.71$ **(b)** $x = 6.16$

13 $x = 1.04$

14 $\angle MOL = 112°, \angle MON = 129°, \angle NOL = 119°$

15 $|PA| = |PB|$

16 Since angles at the circumference subtended by the same arc are equal:
$\angle DAE = \angle BCE$
$\angle ADE = \angle CBE$

Since vertically opposite angles on straight lines are equal:
$\angle AED = \angle CEB$
$\Delta AED \sim \Delta CEB$ (AAA)

17 Assume MQ is a tangent to the circle.

Since the tangent meets the radius at right angles:
$\angle MQO = 90°$
$90° + 57° + 23° = 180°$ (Angles in a triangle add up to 180°.)
$170° = 180°$

This is not possible so the assumption is false.

Therefore, MQ is not a tangent to the circle at Q.

18 Assume that D and D' are distinct points.
In ΔFCB and ΔFDA

$|FC| \times |FA| = |FB| \times |FD|$ (given)
$\frac{|FA|}{|FB|} = \frac{|FD|}{|FC|}$
$\angle AFD = \angle BFC$ (Vertically opposite angles on straight lines are equal.)

$\Delta ADF \sim \Delta BCF$ (SAS)
$\angle ADF = \angle BCF$ (Corresponding angles in similar triangles are equal.)
but $\angle AD'F = \angle BCF$ (Angles at the circumference subtended by the same arc are equal.)
thus $\angle AD'F = \angle ADF$

This means that D and D' must be the same point.

This is a contradiction, therefore the four points lie on the circle.

19 To prove TM is a tangent to the circle at the point T, construct a tangent, ME, to the circle, the two radii OT and OE, and the line segment MO.

$|AM| \times |BM| = |TM|^2$ (given)

$|AM| \times |BM| = |EM|^2$ (tangent-secant theorem for the known tangent)

In ΔMTO and ΔMEO

$|TM| = |EM|$ (from above)

$|OT| = |OE|$ (equal radii)

MO is common

$\Delta MTO \cong \Delta MEO$ (SSS)

$\angle MTO = \angle MEO$ (corresponding angles in congruent triangles)

$\angle MTO = 90^\circ$

Thus TM is a tangent to the circle at the point T.

CHAPTER REVIEW 4

1 ΔAPR, ΔQPR, ΔQCR

2 $x = 53.7$ cm

3 **(a)** 3 or $\frac{1}{3}$ **(b)** $\frac{m}{y}$ and $\frac{n+z}{n}$

4 $\angle OBC = 25^\circ$

5 $\angle APC = 42^\circ$

6 $\angle BOC = 70^\circ$

7 $x = 42^\circ$, $y = 18^\circ$

8 $\angle ACD = 50^\circ$

9 $m = 92^\circ$

10 $x = 2$

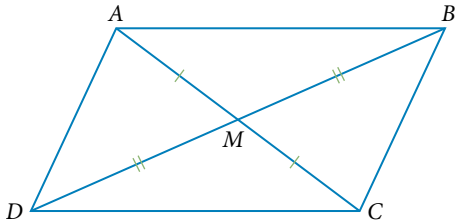
11 I and II only are true.

12 $x = 4$ cm

13 $\angle PQR = 48^\circ$

14 **(a)** $\angle SRQ = 118^\circ$ **(b)** $\angle TSQ = 88^\circ$

15



In ΔAMB and ΔCBD, we are given $|AM| = |MC|$ and $|DM| = |MB|$.

Since vertically opposite angles on a straight line are equal:

$\angle AMB = \angle CMD$

$\Delta AMB \cong \Delta CMD$ (SAS)

$\angle CAB = \angle ACD$ as corresponding angles in congruent triangles are equal.

But these are alternate angles, so AB must be parallel to CD.

A similar argument can be used in ΔAMD and ΔBMC to show that AD is parallel to BC.

Hence $ABCD$ is a parallelogram.

16 $x = 75^\circ$

17 $x = 8$

18 7 m

19 $|AP| = \frac{5\sqrt{2}}{4}$

20 In ΔXYS and ΔXZY:

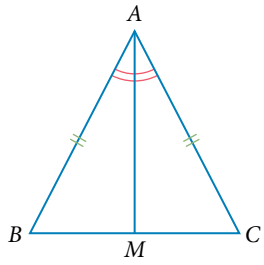
$\angle XYS = \angle XZY$ (given)

$\angle ZXY = \angle YXS$ (common angle)

$\angle XSY = \angle XYZ$ (third angle of the triangles)

Hence $\Delta XYS \sim \Delta XZY$ (AAA)

21



To prove: $\angle BMA = 90^\circ$

Assume: $\angle BMA \neq 90^\circ$

In ΔABM and ΔAMC:

$|AB| = |AC|$

$\angle BAM = \angle CAM$

AM is common.

$\Delta ABM \cong \Delta ACM$ (SAS)

Since corresponding angles in congruent triangles are equal:

$\angle BMA = \angle CMA$

Since angles on a straight line are supplementary:

$\angle BMA = 90^\circ$

The assumption must be false, so $\angle BMA = 90^\circ$.

The angle bisector meets the base at right angles.

22 $\angle KLM = 90^\circ$, $\angle KML = 58^\circ$, $\angle KOL = 116^\circ$

23 $\angle ACO = 9^\circ$

24 $x = 50^\circ$, $y = 130^\circ$

25 $m = 5$

26 $x = 16^\circ$

27 Since the angle between a tangent to a circle and a chord is equal to the angle in the alternate segment: $\angle BDC = 40^\circ$

Since angles on a straight line are supplementary:

$\angle DBC = 100^\circ$

Since angles in a triangle add up to 180°:

$$\begin{aligned} \angle BDC + \angle BCD + \angle DBC &= 180^\circ \\ 40^\circ + 100^\circ + \angle BCD &= 180^\circ \\ \angle BCD &= 40^\circ \\ &= \angle BDC \end{aligned}$$

These are the base angles of ΔBDC, which is therefore isosceles.

28 **(a)** $x = 167^\circ$ **(b)** $m = 75^\circ$

29 In ΔABC and ΔAMQ

$\frac{|AB|}{|AM|} = \frac{2}{1}$ (M is the midpoint of AB)

$\frac{|AC|}{|AQ|} = \frac{2}{1}$ (Q is the midpoint of AC)

$\angle BAC = \angle MAQ$ (common)

$\Delta ABC \sim \Delta AMQ$ (SAS)

Therefore, since corresponding sides in similar triangles are in the same ratio:

$\frac{|BC|}{|MQ|} = \frac{2}{1}$

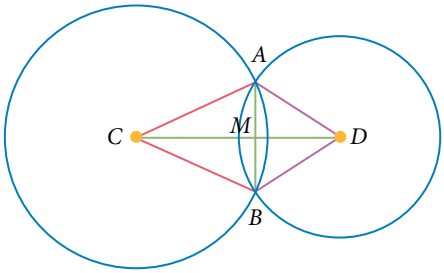
Similarly,

$\frac{|AB|}{|PQ|} = \frac{2}{1}$ and $\frac{|AC|}{|MP|} = \frac{2}{1}$

$\Delta ABC \sim \Delta PQM$ (SSS)

30 Angles formed at the circumference, subtended by the same arc, are equal. In each case $\angle ACB$ is subtended on the minor arc AB. Therefore $\angle ACB$ is the same in each diagram.

31 Construct radii CA, CB, DA and DB. M is the point of intersection of AB and CD.



Consider ΔACD and ΔBCD.

$|CD|$ is common to both triangles.

$|AC| = |BC|$ (equal radii)

$|AD| = |BD|$ (equal radii)

$\Delta ACD \cong \Delta BCD$ (SSS)

Since corresponding angles in congruent triangles are equal:

$\angle ACD = \angle BCD$

Consider ΔACM and ΔBCM.

$\angle ACM = \angle BCM$

CM is common to both triangles.

$|AC| = |BC|$ as they are equal radii.

$\Delta ACM \cong \Delta BCM$ (SAS)

$|AM| = |BM|$ since corresponding sides in congruent triangles are equal.

CD bisects AB.

Since corresponding sides in congruent triangles are equal:

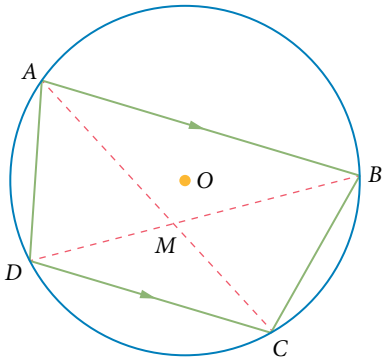
$\angle AMC = \angle BMC$

$\angle AMC + \angle BMC = 180°$

$\angle AMC = 90°$

Therefore CD is the perpendicular bisector of AB.

32 On the diagram, draw the diagonals and mark the point of intersection as M.



In ΔAMB and ΔCMD:

Since vertically opposite angles on a straight line are equal: $\angle AMB = \angle CMD$.

Since alternate angles between parallel lines are equal: $\angle ABM = \angle MDC$.

As the third angles in the triangles: $\angle BAM = \angle DCM$.

Therefore, $\Delta ABM \sim \Delta CDM$ (AAA)

Also

In ΔAMD and ΔBMC:

Since vertically opposite angles on a straight line are equal: $\angle AMD = \angle BMC$.

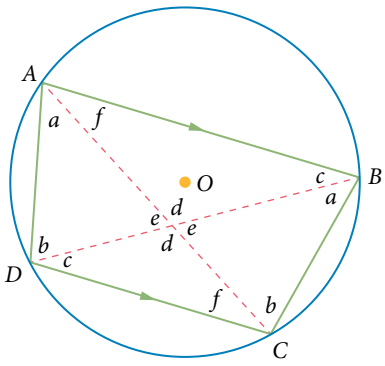
Since angles at the circumference subtended by the same arc are equal: $\angle DAM = \angle MBC$

As the third angles in the triangles: $\angle ADM = \angle BCM$.

Therefore, $\Delta AMD \sim \Delta BMC$ (AAA).

Now, to show $\angle MAB = \angle MBA$ and $\angle MDC = \angle MCD$.

To make this easier, use this version of the diagram where angles have been given letter names.



Since opposite angles in a cyclic quadrilateral are supplementary:

$a + f + f + b = 180°$

$b + c + c + a = 180°$

$2f = 2c$

$f = c$

Therefore $\angle MAB = \angle MBA$ and $\angle MDC = \angle MCD$.

These are base angles of a triangle, so ΔAMD and ΔBMC must be isosceles triangles.

$|MA| = |MB|$ as they are equal sides of an isosceles triangle.

$|MC| = |MD|$ as they are equal sides of an isosceles triangle.

Since vertically opposite angles on a straight line are equal:
$\angle AMD = \angle BMC$
$\Delta AMD \cong \Delta BMC$ (SAS)

Since corresponding sides in congruent triangles are equal:
$|AD| = |BC|$

Exam review Unit 1

PAPER 1: TECHNOLOGY-FREE

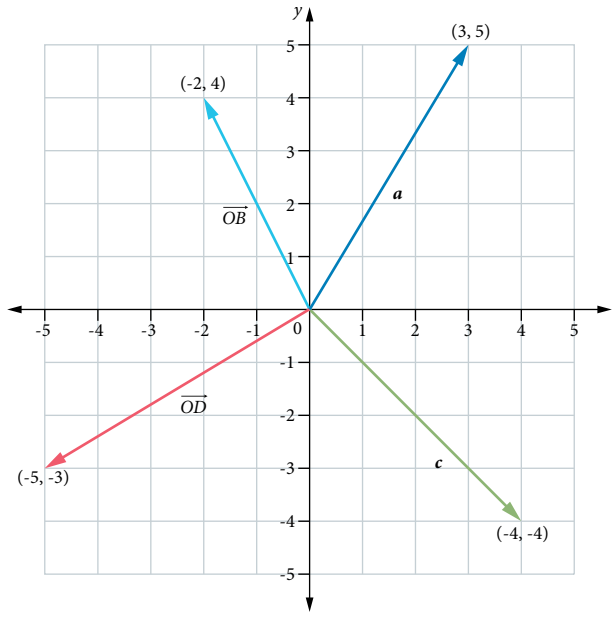
1 25

2 5000

3 45

4 $${}^{n+2}C_3 - {}^{n}C_3 = \frac{(n+2)!}{3!(n-1)!} - \frac{n!}{3!(n-3)!}$$
$$= \frac{(n+2)!}{3!(n-1)!} - \frac{n!}{3!(n-3)!} \times \frac{(n-1)(n-2)}{(n-1)(n-2)}$$
$$= \frac{(n+2)!}{3!(n-1)!} - \frac{n!(n-1)(n-2)}{3!(n-1)!}$$
$$= \frac{(n+2)(n+1)n! - n!(n-1)(n-2)}{3!(n-1)!}$$
$$= \frac{n!\big((n+2)(n+1) - (n-1)(n-2)\big)}{3!(n-1)!}$$
$$= \frac{n! \times 6n}{6(n-1)!}$$
$$= \frac{n\,\cancel{(n-1)!} \times \cancel{6}n}{\cancel{6}\,\cancel{(n-1)!}}$$
$$= n^2$$

5



6 (a) $\frac{4}{9}$ (b) $\frac{23}{99}$ (c) $\frac{41}{330}$

7 (a) original: This is false, since, for example, 9 is divisible by 3, but not by 6.

converse: If n is divisible by 6, then n is divisible by 6. This is true, since any multiple of 6 has 3 as a factor, and thus, must be divisible by 3.

contrapositive: If n is not divisible by 6, then n is not divisible by 3. This must be false as the original statement is false.

negation: There exists an integer n with the property that n is divisible by 3 but n is not divisible by 6. This must be true as the original statement was false.

(b) original: This is false. For example, if $x = -1$, then $x^2 \leq 25$.

converse: If $x^2 > 25$, then $x < 0$ or $x > 5$. This is true. If the square of a number is greater than 25 then the number must be either less than -5 (in which case it is less than 0) or greater than 5.

contrapositive: If $x^2 \leq 25$, then $x \geq 0$ and $x \leq 5$. This is false, as the original statement is false.

negation: There exists a real number x with the property that $x < 0$ or $x > 5$, but $x^2 \leq 25$. This must be true as the original statement was false.

8 Suppose, for a contradiction, that r is rational and rx is irrational, but x is rational. Since r and x are both rational, $r = \frac{a}{b}$ and $x = \frac{c}{d}$ for integers a, b, c, d with $b, d \neq 0$.
Then
$$rx = \frac{a}{b} \times \frac{c}{d}$$
$$= \frac{ac}{bd}$$

Thus, rx is rational. But this contradicts the fact that rx is irrational. Hence, it must be the case that x is irrational.

9 Let A, B, C be the digits, in order, of a three-digit number, N. The actual number is therefore $N = 100A + 10B + C$, and the alternating digit sum is $A - B + C$. Suppose that the alternating digit sum is divisible by 11. Then $A - B + C = 11k$ for some integer k. Then
$$N = 100a + 10b + c$$
$$= 99a + 11b + a - b + c$$
$$= 99a + 11b + 11k$$
$$= 11(9a + b + k)$$

Thus, N must be divisible by 11.

10 $a = 5, b = 2\sqrt{6}$

11 If $5^{k+1} + 2 \times 3^k + 1$ is divisible by 8, then $5^{k+1} + 2 \times 3^k + 1 = 8A$ where A is an integer. Therefore:
$$5^{k+1} = 8A - 2 \times 3^k - 1$$
$$5^{k+2} = 5\left(8A - 2 \times 3^k - 1\right)$$
$$5^{k+2} = 40A - 10 \times 3^k - 5$$

Thus:

$$\begin{aligned}5^{k+2}+2\times3^{k+1}+1&=40A-10\times3^k-5+2\times3^{k+1}+1\\&=40A-10\times3^k+2\times3\times3^k-4\\&=40A-10\times3^k+6\times3^k-4\\&=40A-4\times3^k-4\\&=4\left(10A-3^k-1\right)\end{aligned}$$

The number 3^k is odd (as k is an integer), and hence, the expression in the brackets must be even (since it is the sum of three terms, two of which are odd and the other even). As an even number multiplied by 4 is necessarily a multiply of 8, the result follows.

PAPER 2: TECHNOLOGY-ACTIVE

1 10 km h^{-1}

2 76.5°

3 $\angle BCD = 80°$, $\angle DBC = 38°$, $\angle BDF = 80°$

4 (a) $\hat{\boldsymbol{a}} = \frac{\sqrt{74}}{74}\left(7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}\right)$ (b) $\boldsymbol{b} = \frac{5\sqrt{74}}{37}\left(7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}\right)$

5 $v = \pm\sqrt{2}$ and $w = 1$ or 6

6 124.80° (2 d.p.)

7 (a) $-\frac{5}{13}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ (b) $\frac{14}{13}\left(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$

8 force $\boldsymbol{F}_1$: $\left|\boldsymbol{F}_1\right| = 133.76$ N; force $\boldsymbol{F}_2$: $\left|\boldsymbol{F}_2\right| = 157.38$ N

9 30

10 $\angle AOE = 40°$

Chapter 5 Real and complex numbers

RECALL

1 (a) $x^2 + 2x - 8$ (b) $x^2 - 16x + 64$ (c) $4x^2 - 20xy + 25y^2$

(d) $10a^3b^2 - 4a^3b + 5a^2b^4 - 2a^2b^3$

(e) $4x^2 - 9a^2$ (f) $x^3 - 15x^2 + 75x - 125$

2 (a) $5\sqrt{2}$ (b) $14\sqrt{3}$

(c) $15\sqrt{7}$ (d) $4\sqrt{5}$

3 (a) $\frac{\sqrt{2}}{6}$ (b) $\frac{\left(5+\sqrt{3}\right)}{22}$

(c) $\frac{\sqrt{14}-\sqrt{10}-\sqrt{21}+\sqrt{15}}{2}$

4 (a) $AB = \sqrt{58}$ units (b) $CD = 6\sqrt{2}$ units

(c) $EF = \sqrt{101}$ units (d) $GH = \sqrt{41}$ units

5 (a) $\theta = 20.56°$ (b) $\theta = 36.22°$ (c) $\theta = 53.13°$

6 (a) $x = \frac{-1+\sqrt{29}}{2}$, $x = \frac{-1-\sqrt{29}}{2}$

(b) $x = \frac{3+\sqrt{41}}{4}$, $x = \frac{3-\sqrt{41}}{4}$

(c) $x = 4 + \sqrt{26}$, $x = 4 - \sqrt{26}$

(d) There are no real solutions.

EXERCISE 5.1

1 C

2 (a) C

(b) For alternative A, $\overline{z} = a - bi$, so $\mathrm{Re}\left(\overline{z}\right) = a$. For alternative B, $\overline{z} = a - bi$, so $\mathrm{Im}\left(\overline{z}\right) = -b$. For alternative D, $z.\overline{z} = (a+bi)(a-bi) = a^2 + b^2$. There is no imaginary component.

3 -16

4 $-1 + 5i$

5 (a) $7i$ (b) $i\sqrt{11}$ (c) $5i\sqrt{3}$

6 (a) $\overline{z} = 5 - 6i$ (b) $\overline{z} = 10 + 3i$ (c) $\overline{z} = 8$

7 (a) $9 - 2i$ (b) $-5 - 6i$

(c) $5 + 6i$ (d) $11 - 6i$

8 (a) $-17 - 14i$ (b) 32

9 (a) $9 + 19i$ (b) $22 - 24i$ (c) $-11 + 13i$ (d) $-18 + i$

(e) 20 (f) 38 (g) $11 + 2i$ (h) $6 + 2i$

(i) 10 (j) 10

10 $a = 3$, $b = -2$

11 (a) $\frac{1}{5} - \frac{2}{5}i$ (b) $\frac{4}{41} + \frac{5}{41}i$

12 (a) $-\frac{1}{34} + \frac{13}{34}i$ (b) $-\frac{1}{2} + \frac{5}{2}i$

(c) $\frac{1}{5} - \frac{2}{5}i$ (d) $\frac{1}{2} - 2i$

13 (a) $1 - 5i$ (b) $2 - 3i$ (c) $9 + 19i$

(d) $22 - 24i$ (e) $-46 + 9i$ (f) $65 - 142i$

(g) $-\frac{2}{25} + \frac{11}{25}i$ (h) $-\frac{1}{2} - \frac{3}{4}i$

14 (a) $-3 + 41i$ (b) $34 + 12i$ (c) $-594 + 1358i$

15 (a) $\mathrm{Re}\left(u^2\right) = -45$ (b) $\mathrm{Im}(w, z) = 11$ (c) $\frac{11}{2146} + \frac{45}{2146}i$

16 $z = 2 - 3i$

17 $$\begin{aligned}wz&=(a+bi)(c+di)\\&=ac+adi+bci+bdi^2\\&=ac+adi+bci-bd\\&=(ac-bd)+(ad+bc)i\end{aligned}$$

Therefore, $\overline{wz} = (ac - bd) - (ad + bc)i$

$\overline{w} = a - bi$ and $\overline{z} = c - di$, therefore:

$$\begin{aligned}\overline{w}.\overline{z}&=(a-bi)(c-di)\\&=ac-bci-adi+bdi^2\\&=ac-bci-adi-bd\\&=(ac-bd)-(ad+bc)i\end{aligned}$$

Therefore, $\overline{wz} = \overline{w} \times \overline{z}$.

18 $\frac{z}{w} = \frac{c+di}{a+bi} \times \frac{a-bi}{a-bi}$

$= \frac{ac-bci+adi-bdi^2}{a^2-b^2i^2}$

$= \frac{ac-bci+adi+bd}{a^2+b^2}$

$= \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}$

$= \frac{(ac+bd)}{(a^2+b^2)} + \frac{(ad-bc)i}{(a^2+b^2)}$

Therefore, $\overline{\left(\frac{z}{w}\right)} = \frac{(ac+bd)}{(a^2+b^2)} - \frac{(ad-bc)i}{(a^2+b^2)}$

$\frac{\overline{z}}{\overline{w}} = \frac{c-di}{a-bi} \times \frac{a+bi}{a+bi}$

$= \frac{ac+bci-adi-bdi^2}{a^2-b^2i^2}$

$= \frac{ac+bd+bci-adi}{a^2+b^2}$

$= \frac{(ac+bd)}{(a^2+b^2)} + \frac{(bc-ad)i}{(a^2+b^2)}$

$= \frac{(ac+bd)}{(a^2+b^2)} - \frac{(ad-bc)i}{(a^2+b^2)}$

Therefore, $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

19 $w.z = (a+bi)(c+di)$

$= ac+adi+bci+bdi^2$

$= (ac-bd)+(ad+bc)i$

Therefore,

$(w.z)^{-1} = \frac{1}{(ac-bd)+(ad+bc)i} \times \frac{(ac-bd)-(ad+bc)i}{(ac-bd)-(ad+bc)i}$

$= \frac{(ac-bd)-(ad+bc)i}{(ac-bd)^2+(ad+bc)^2}$

$w^{-1}z^{-1} = \frac{1}{(a+bi)} \times \frac{1}{(c+di)}$

$= \frac{1}{(a+bi)(c+di)}$

$= \frac{1}{ac+adi+bci+bdi^2}$

$= \frac{1}{(ac-bd)+(ad+bc)i} \times \frac{(ac-bd)-(ad+bc)i}{(ac-bd)-(ad+bc)i}$

$= \frac{(ac-bd)-(ad+bc)i}{(ac-bd)^2+(ad+bc)^2}$

Therefore, $(w.z)^{-1} = w^{-1}z^{-1}$

20 $z = 6+4i$

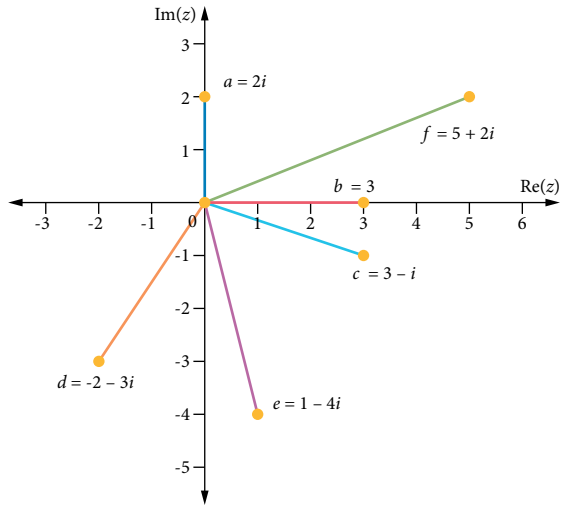
21 $z = 5-3i$

22 $z = -\frac{3}{2} - \frac{i}{2}$

23 $z = 2-7i$ or $z = -2+7i$

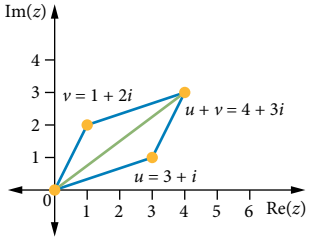
EXERCISE 5.2

1



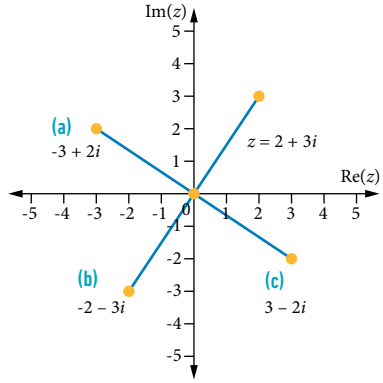
2 $A(2, 0)$, $B(2, 1)$, $C(0, 2)$, $D(-2, 3)$, $E(-3, 0)$, $F(-1, -1)$, $G(0, -3)$ and $H(3, -2)$

3



4 D

5

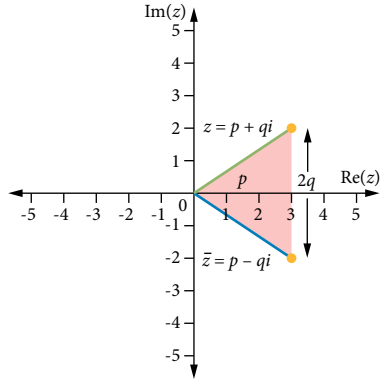


6 A

7 circle with centre (1, -2), radius 2

8 B

9



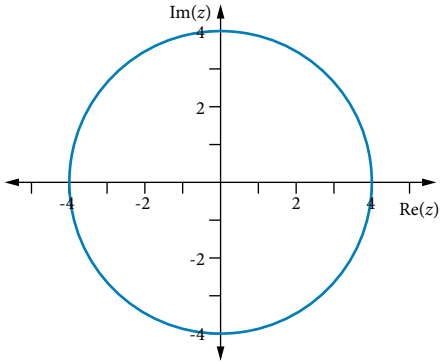
The triangle has a base of $2q$ and a perpendicular height of p.

$$\text{Area} = \frac{b \times h}{2}$$
$$= \frac{2pq}{2}$$
$$= pq$$

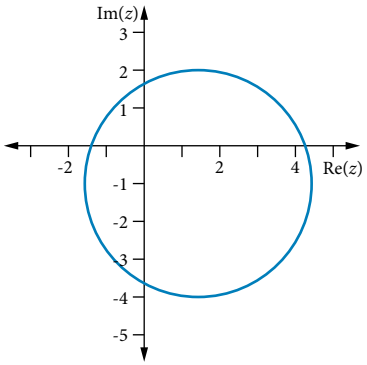
10 (a) $d = \sqrt{37}$ units

(b) $\sqrt{185}$ units

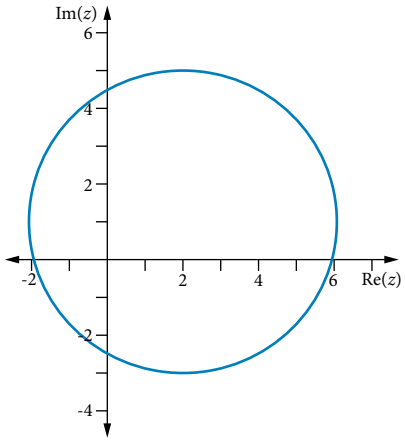
11 (a) circle with centre (0, 0), radius 4



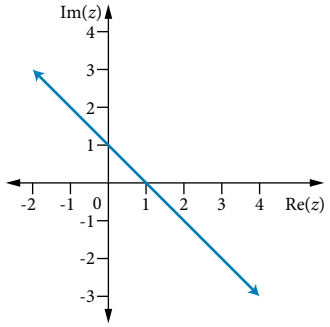
(b) circle with centre (2, -1), radius 3



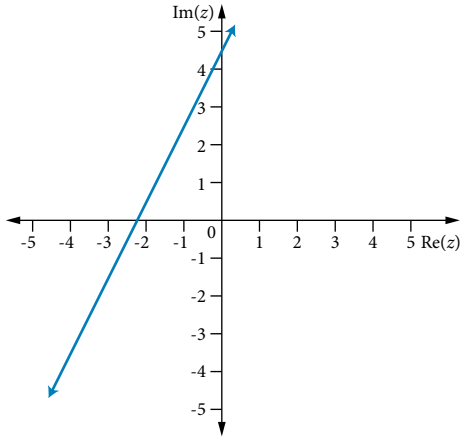
(c) circle with centre (2, 1), radius 4



12 (a) $y = -x + 1$



(b) $y = \frac{1}{2}(4x + 9)$



13 Determine the image of the complex numbers after the transformation is applied.

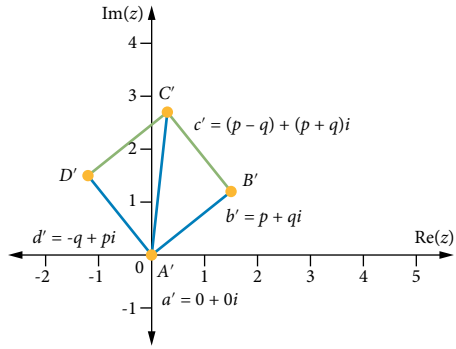
$a' = az = (0 + 0i)(p + qi) = 0 + 0i$

$b' = bz = (1 + 0i)(p + qi) = p + qi$

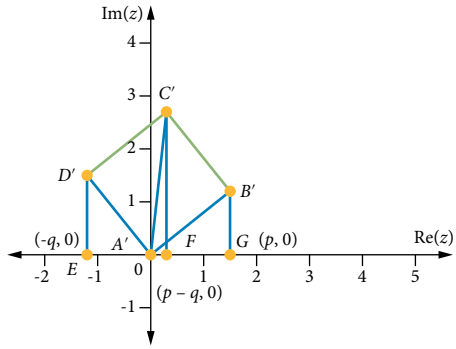
$c' = cz = (1 + 1i)(p + qi) = p + qi + pi + i^2q$
$= (p - q) + (p + q)i$

$d' = dz = (0 + 1i)(p + qi) = -q + pi$

The new shape defined by these transformed points is graphed below:



One way to determine the area of this shape is to construct vertical lines from B', C' and D' to the x-axis to create points $E(-q, 0)$, $F((p - q), 0)$ and $G(p, 0)$. Then, calculate the area of trapezia $ED'C'F$ and $FC'B'G$ before subtracting the area of triangles $\Delta ED'A'$ and $\Delta A'B'G$.



$$\text{Area of } ED'C'F = \frac{p+(p+q)}{2}\times\left[(p-q)-(-q)\right]$$
$$= \frac{(2p+q)\times p}{2}$$
$$= \frac{2p^2+pq}{2}$$

$$\text{Area of } FC'B'G = \frac{q+(p+q)}{2}\times\left[p-(p-q)\right]$$
$$= \frac{(p+2q)\times q}{2}$$
$$= \frac{pq+2q^2}{2}$$

$$\text{Sum of areas of trapeziums} = \frac{2p^2+pq}{2}+\frac{pq+2q^2}{2}$$
$$= \frac{2p^2+2pq+2q^2}{2}$$
$$= p^2+pq+q^2$$

Finally, subtract the area of triangles $\Delta ED'A'$ and $\Delta A'B'G$:

$$\text{Area of } A'B'C'D' = p^2+pq+q^2-\frac{qp}{2}-\frac{pq}{2}$$
$$= p^2+pq+q^2-pq$$
$$= p^2+q^2$$

EXERCISE 5.3

1 (a) $\frac{\pi}{6}$ (b) $\frac{4\pi}{15}$ (c) $\frac{5\pi}{6}$
(d) $\frac{7\pi}{6}$ (e) $\frac{5\pi}{3}$

2 (a) 12.83° (b) 139.57° (c) 268.03°
(d) 45° (e) 40° (f) 108°

3 (a) 10° (b) 340° (c) 103°

4 (a) $\frac{\pi}{4}$ (b) $-\pi$ (c) -0.37 (2 d.p.)

EXERCISE 5.4

1 A

2 B

3 (a) $\sqrt{2}$ units (b) 5 units (c) $\sqrt{109}$ units
(d) 2 units (e) $\sqrt{85}$ units (f) $2\sqrt{3}$ units

4 (a) $\frac{\pi}{4}$ (b) -1.11 (2 d.p.) (c) $\frac{2\pi}{3}$
(d) -2.21 (2 d.p.)

5 (a) $z = 2\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$ (b) $z = 2\ \text{cis}\left(-\frac{5\pi}{6}\right)$
(c) $z = 5\ \text{cis}\ (2.50)$ (2 d.p.) (d) $z = 4\ \text{cis}(-0.72)$ (2 d.p.)

6 (a) $z = \sqrt{26}\ \text{cis}\ (-1.77)$ (2 d.p.) (b) $z = \sqrt{130}\ \text{cis}\ (1.84)$ (2 d.p.)

7 (a) $12\ \text{cis}\left(\frac{2\pi}{3}\right)$ (b) $\sqrt{6}\,\text{cis}\left(-\frac{3\pi}{4}\right)$
(c) $12\ \text{cis}\left(\frac{\pi}{2}\right)$ (d) $40\ \text{cis}\left(-0.88^c\right)$

8 (a) $2\ \text{cis}\left(\frac{\pi}{2}\right)$ (b) $4\ \text{cis}\left(\frac{7\pi}{8}\right)$
(c) $\frac{7}{3}\text{cis}\left(\frac{7\pi}{12}\right)$ (d) $2\ \text{cis}\left(1.3^c\right)$

9 (a) $z = \frac{5\sqrt{3}}{2}+\frac{5i}{2}$ (b) $z = -\sqrt{2}+\sqrt{2}i$
(c) $z = -4+0i$ (d) $z = 8.87-4.62i$ (2 d.p.)

10 (a) $9\ \text{cis}\left(\frac{\pi}{3}\right)$ (b) $8\ \text{cis}\left(\frac{\pi}{2}\right)$ (c) $9\ \text{cis}\left(-\frac{2\pi}{3}\right)$

11 If $z = a+bi$ then $\overline{z} = a-bi$.
$(a+bi)(a-bi) = a^2+abi-abi-b^2i^2$
$= a^2+b^2$
This will always be a real number, as $a \in R$ and $b \in R$.

12 (a) $z-w = \sqrt{29}\ \text{cis}\ (-0.38)$ (2 d.p.)
(b) $z-2w = \sqrt{257}\ \text{cis}\ (1.63)$ (2 d.p.)

13 $z+w = 2+3i$

14 $z-w = 16\ \text{cis}\left(\frac{\pi}{3}\right)$

15 Use Pascal's triangle to expand the binomial.
$z^4 = \left(1+\sqrt{3}i\right)^4$
$= 1^4\left(\sqrt{3}i\right)^0+4\times1^3\left(\sqrt{3}i\right)^1+6\times1^2\left(\sqrt{3}i\right)^2+4\times1^1\left(\sqrt{3}i\right)^3$
$+1^0\left(\sqrt{3}i\right)^4$
$= 1+4\sqrt{3}i-18-12\sqrt{3}i+9$
$= -8-8\sqrt{3}i$

Convert to polar form:
$\left|z^4\right| = \sqrt{(-8)^2+\left(-8\sqrt{3}\right)^2}$
$= \sqrt{64+192}$
$= \sqrt{256}$
$= 16$

$\theta = \tan^{-1}\left(\frac{8\sqrt{3}}{8}\right)$
$= \tan^{-1}\left(\sqrt{3}\right)$
$= \frac{\pi}{3}$

Angle is in fourth quadrant, so:

$$\text{Arg}\left(z^4\right) = -\pi + \frac{\pi}{3}$$
$$= -\frac{2\pi}{3}$$

Therefore, $z^4 = 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$

We also know that $z = 1 + \sqrt{3}i$. Converting this to polar form, and applying De Moivre's theorem:

$$|z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$$

$$z^4 = 2^4\,\text{cis}\left(\frac{4\pi}{3}\right)$$
$$= 16\,\text{cis}\left(\frac{4\pi}{3} - 2\pi\right)$$
$$= 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

The answer is thus verified.

16 $36\,\text{cis}\left(\frac{-5\pi}{12}\right)$

EXERCISE 5.5

1 A

2 D

3 (a) $(3x-2)(x+4)$ (b) $(5m-3)(m+4)$

(c) $(3p-2)(2p+1)$ (d) $(2n-3)(6n+1)$

4 (a) $x = \frac{1 \pm i\sqrt{7}}{2}$ (b) $x = \frac{1 \pm i\sqrt{2}}{3}$

(c) $z = \frac{-3 \pm i\sqrt{11}}{10}$ (d) $z = \frac{1 \pm i\sqrt{2}}{3}$

5 (a) $x = 4 + i\sqrt{3}$ (b) $x = -2 - i\sqrt{5}$

6 D

7 (a) $\left(x+2-\sqrt{3}\right)\left(x+2+\sqrt{3}\right)$ (b) $\left(x+1-\sqrt{7}\right)\left(x+1+\sqrt{7}\right)$

(c) $\left(z+3-\sqrt{6}\right)\left(z+3+\sqrt{6}\right)$

(d) $\left(2z+1-\sqrt{2}\right)\left(2z+1+\sqrt{2}\right)$

8 (a) $\left(x+2-i\sqrt{2}\right)\left(x+2+i\sqrt{2}\right)$

(b) $\left(x+4-i\sqrt{2}\right)\left(x+4+i\sqrt{2}\right)$

(c) $\left(3z+2-i\sqrt{23}\right)\left(z+\frac{2}{3}+\frac{i\sqrt{23}}{3}\right)$

(d) $2\left(z-\frac{3}{4}-\frac{i\sqrt{7}}{4}\right)\left(z-\frac{3}{4}+\frac{i\sqrt{7}}{4}\right)$

9 (a) $x = -4 \pm i\sqrt{2}$ (b) $z = -\frac{1 \pm i\sqrt{11}}{2}$

(c) $z = \frac{3 \pm i\sqrt{15}}{6}$ (d) $z = \frac{5 \pm i\sqrt{7}}{4}$

10 $z^2 + 3z + 3 = z^2 + 3z + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 3$

$$= \left(z+\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{12}{4} = \left(z+\frac{3}{2}\right)^2 + \frac{3}{4}$$
$$= \left(z+\frac{3}{2}\right)^2 - \frac{3i^2}{4} = \left(z+\frac{3}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^2$$
$$= \left(z+\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)\left(z+\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)$$
$$= \frac{1}{2}\left(2z+3-i\sqrt{3}\right)\frac{1}{2}\left(2z+3+i\sqrt{3}\right)$$
$$= \frac{1}{4}\left(2z+3-i\sqrt{3}\right)\left(2z+3+i\sqrt{3}\right)$$

11 (a) $x^2 - 4x + 7 = 0$ (b) $x^2 - 10x + 29 = 0$

(c) $z^2 - \frac{10}{3}z + \frac{41}{9} = 0$ (d) $z^2 - \frac{7}{2}z + \frac{13}{4} = 0$

12 $z = -5, z = 5, z = -2i, z = 2i$

13 $z = 3i, z = -3i, z = -i, z = i$

CHAPTER REVIEW 5

1 A

2 D

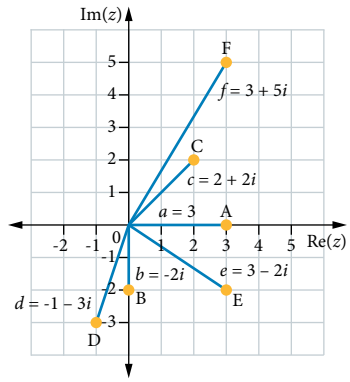
3 C

4 (a) $\overline{z} = 2 - 8i$ (b) $\overline{z} = -3 + 6i$ (c) $\overline{z} = -5i$

5 (a) $1 + 3i$ (b) $9 - i$ (c) $-9 + i$ (d) $6 + 4i$

6 (a) $7 - 10i$ (b) $73 + 14i$

7



8 A: -3, B: i, C: $-2 + 4i$, D: $2 - 2i$, E: $-2 - 3i$, F: $3 + 2i$

9 C

10 B

11 C

12 (a) $\sqrt{10}$ units (b) 5 units (c) $\sqrt{109}$ units (d) $\sqrt{65}$ units

13 (a) Argument $= \frac{\pi}{6}$ (b) Argument $= -\frac{\pi}{4}$

(c) Argument $= 2.16^c$ (2 d.p.)

14 B

15 (a) $1+\sqrt{3}i$ (b) $-1-i$

16 B

17 (a) $(5t-6)(3t+2)$ (b) $(2z+5)(3-z)$

18 (a) $x=\frac{3}{2}-\frac{i\sqrt{3}}{2}$ (b) $z=\frac{2+i\sqrt{5}}{3}$

19 (a) $-18+13i$ (b) $10-10i$ (c) 25 (d) 625

20 $a=2, b=4$

21 (a) $\frac{2}{5}-\frac{i}{5}$ (b) $\frac{-3}{25}-\frac{4i}{25}$

22 (a) $\frac{4}{13}+\frac{7}{13}i$ (b) $-2i$

23 (a) RHS $= a(bc)$
$= (2+3i)\times(-1+2i)(3-2i)$
$= (2+3i)\times(-3+2i+6i-4i^2)$
$= (2+3i)(1+8i)$
$= 2+16i+3i+24i^2$
$= -22+19i$

LHS $= (ab)c$
$= (2+3i)(-1+2i)\times(3-2i)$
$= (-2+4i-3i+6i^2)\times(3-2i)$
$= (-8+i)\times(3-2i)$
$= -24+16i+3i-2i^2$
$= -22+19i$
$=$ RHS as required

(b) RHS $= ab+ac$
$= (2+3i)(-1+2i)+(2+3i)(3-2i)$
$= (-2+4i-3i+6i^2)+(6-4i+9i-6i^2)$
$= 4+6i$

LHS $= (2+3i)(-1+2i+3-2i)$
$= (2+3i)(2)$
$= 4+6i$
$=$ RHS as required

(c) RHS $= a+(b+c)$
$= (2+3i)+(-1+2i+3-2i)$
$= (2+3i)+(2)$
$= 4+3i$

LHS $= (a+b)+c$
$= (2+3i-1+2i)+(3-2i)$
$= (1+5i)+(3-2i)$
$= 4+3i$
$=$ RHS as required

24 $z=3-4i$

25 (a) $\overline{AB}=\sqrt{50}$ units (b) $\overline{A'B'}=5\sqrt{10}$ units

26 (a) $z=\sqrt{34}\,\text{cis}\,(-0.54)$ (2 d.p.) (b) $z=5\sqrt{5}\,\text{cis}\,(2.03)$ (2 d.p.)

27 (a) $6\,\text{cis}\left(\frac{7\pi}{12}\right)$ (b) $\sqrt{15}\,\text{cis}\left(-\frac{5\pi}{6}\right)$

28 (a) $5\,\text{cis}\left(\frac{\pi}{3}\right)$ (b) $6\,\text{cis}\left(\frac{3\pi}{4}\right)$

29 (a) $4\,\text{cis}\left(-\frac{2\pi}{3}\right)$ (b) $32\,\text{cis}\left(\frac{5\pi}{8}\right)$

30 B

31 (a) $(3z-2-\sqrt{6})(3z-2+\sqrt{6})$
(b) $(x-3-\sqrt{5})(x-3+\sqrt{5})$

32 (a) $(x-3-i)(x-3+i)$
(b) $4\left(z-\frac{3}{8}-\frac{i\sqrt{71}}{8}\right)\left(z-\frac{3}{8}+\frac{i\sqrt{71}}{8}\right)$

33 (a) $x=3\pm i\sqrt{3}$ (b) $x=-1\pm i\sqrt{5}$

34 (a) $x=-1\pm i\sqrt{2}$ (b) $z=\frac{3\pm i\sqrt{23}}{4}$

35 (a) If $P(z)=z^2-4z+13$, and $2+3i$ and $2-3i$ are roots of the equation $z^2-4z+13=0$, then $P(2+3i)=0$ and $P(2-3i)=0$

$(2+3i)^2-4(2+3i)+13 = 4+12i+9i^2-8-12i+13$
$= 4+12i-9-8-12i+13$
$= 0$

Hence $P(2+3i)=0$

$(2-3i)^2-4(2-3i)+13 = 4-12i+9i^2-8+12i+13$
$= 4-12i-9-8+12i+13$
$= 0$

Hence $P(2-3i)=0$

Since $P(2+3i)=0$ and $P(2-3i)=0$, then $2+3i$ and $2-3i$ are roots of the equation $z^2-4z+13=0$.

(b) If $P(z)=z^2+2z+4$, and $-1+\sqrt{3}i$ and $-1-\sqrt{3}i$ are roots of the equation $z^2+2z+4=0$ then $P(-1+\sqrt{3}i)=0$ and $P(-1-\sqrt{3}i)=0$

$(-1+\sqrt{3}i)^2+2(-1+\sqrt{3}i)+4$
$= 1-2\sqrt{3}i+3i^2-2+2\sqrt{3}i+4$
$= 1-2\sqrt{3}i-3-2+2\sqrt{3}i+4$
$= 0$

Hence $P(-1+\sqrt{3}i)=0$

$(-1-\sqrt{3}i)^2+2(-1-\sqrt{3}i)+4$
$= 1+2\sqrt{3}i+3i^2-2-2\sqrt{3}i+4$
$= 1+2\sqrt{3}i-3-2-2\sqrt{3}i+4$
$= 0$

Hence $P(-1-\sqrt{3}i)=0$

Since $P(-1+\sqrt{3}i)=0$ and $P(-1-\sqrt{3}i)=0$, then $-1+\sqrt{3}i$ and $-1-\sqrt{3}i$ are roots of the equation $z^2+2z+4=0$.

(c) If $P(z)=2z^2+2z+5$, and $\frac{1}{2}(1+3i)$ and $\frac{1}{2}(1-3i)$ are roots of the equation $2z^2-2z+5=0$ then

$P\left(\frac{1}{2}(1+3i)\right)=0$ and $P\left(\frac{1}{2}(1-3i)\right)=0$.

If $z=\frac{1}{2}(1+3i)$, then

$z^2 = \frac{1}{4}(1+6i+9i^2)$
$= \frac{1}{4}(-8+6i)$

$$2z^2 - 2z + 5 = 2 \times \frac{1}{4}(-8 + 6i) - 2 \times \frac{1}{2}(1 + 3i) + 5$$
$$= \frac{1}{2}(-8 + 6i) - (1 + 3i) + 5$$
$$= -4 + 3i - 1 - 3i + 5$$
$$= 0$$

If $z = \frac{1}{2}(1 - 3i)$, then

$$z^2 = \frac{1}{4}(1 - 6i + 9i^2)$$
$$= \frac{1}{4}(-8 - 6i)$$

$$2z^2 - 2z + 5 = 2 \times \frac{1}{4}(-8 - 6i) - 2 \times \frac{1}{2}(1 - 3i) + 5$$
$$= \frac{1}{2}(-8 - 6i) - (1 - 3i) + 5$$
$$= -4 - 3i - 1 + 3i + 5$$
$$= 0$$

Since $P\left(\frac{1}{2}(1 + 3i)\right) = 0$ and $P\left(\frac{1}{2}(1 - 3i)\right) = 0$, then

$\frac{1}{2}(1 + 3i)$ and $\frac{1}{2}(1 - 3i)$ are roots of $2z^2 - 2z + 5 = 0$.

36 $2 - 4i$

37 $z = 5 + 2i$

38 $z = 4 - 3i$ or $z = -4 + 3i$

39 $z + w = \left(2\sqrt{2} + 1\right) + \left(\sqrt{3} + 2\sqrt{2}\right)i$

40 $z - w = \sqrt{20}\,\text{cis}\left(2.03^c\right)$ (2 d.p.)

41 $z^2 + 5z + \frac{27}{4} = z^2 + 5z + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + \frac{27}{4}$
$$= \left(z + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{27}{4}$$
$$= \left(z + \frac{5}{2}\right)^2 + \frac{2}{4}$$
$$= \left(z + \frac{5}{2}\right)^2 - \frac{2i^2}{4}$$
$$= \left(z + \frac{5}{2}\right)^2 - \left(\frac{i\sqrt{2}}{2}\right)^2$$
$$= \left(z + \frac{5}{2} - \frac{i\sqrt{2}}{2}\right)\left(z + \frac{5}{2} + \frac{i\sqrt{2}}{2}\right)$$
$$= \frac{1}{2}\left(2z + 5 - i\sqrt{2}\right)\frac{1}{2}\left(2z + 5 + i\sqrt{2}\right)$$
$$= \frac{1}{4}\left(2z + 5 - i\sqrt{2}\right)\left(2z + 5 + i\sqrt{2}\right)$$

42 $z = -4i$, $z = 4i$, $z = -3i$, $z = 3i$

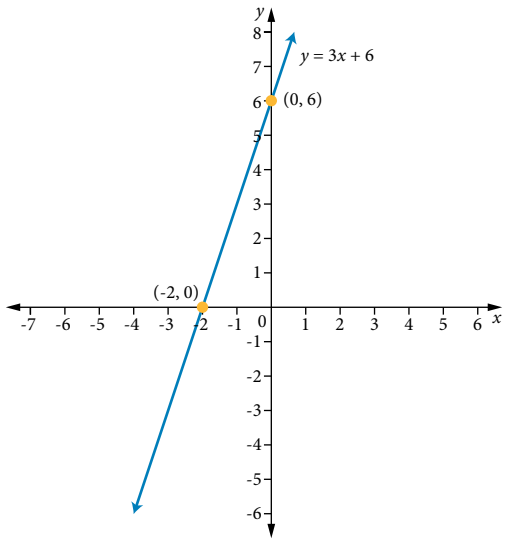
43 (a) $x^2 - 6x + 11 = 0$

(b) $x^2 - 4x + 68 = 0$

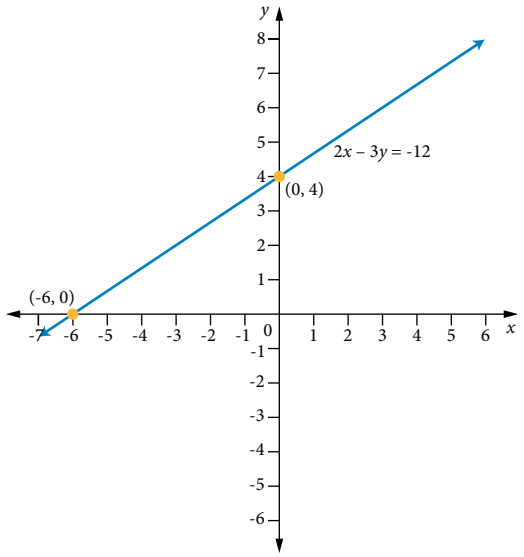
Chapter 6 Graphing functions

RECALL

1 (a)



(b)



2 (a) $y = -2(x + 3)(x - 1)$ (b) $y = 3x(x + 5)$

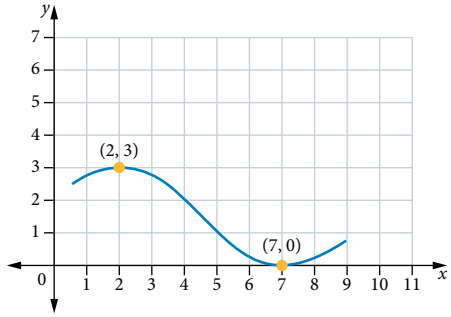
(c) $y = 2(x - 1)^2 + 3$

3 (a) $x = 0, 3$, or -2 (b) $x = 0$ or -5

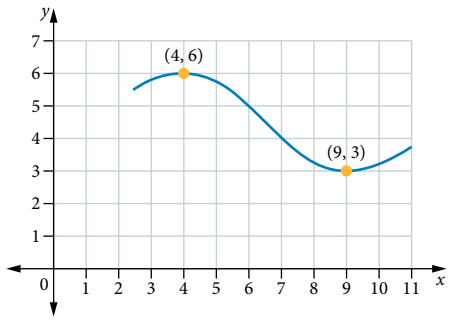
4 (a) $\frac{3x + 2}{5}$ (b) $\frac{x(x + 5)}{3}$ (c) $\frac{1}{x - 4}$

5 (a) (0, 0) and (3, 9) (b) (3, 12) and (1, 4)

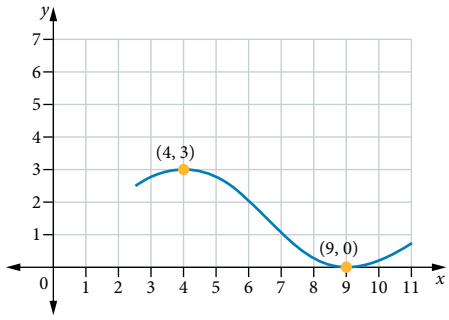
6 (a)



(b)



(c)



7 (a) $\dfrac{x^2+4x+5}{x-2} = x+6+\dfrac{17}{x-2}$

(b) $\dfrac{2x^2-4x+1}{x+1} = 2x-6+\dfrac{7}{x+1}$

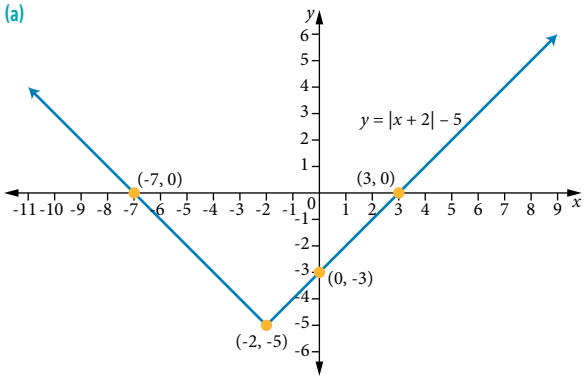
(c) $\dfrac{4x^2+1}{x-3} = 4x+12+\dfrac{37}{x-3}$

EXERCISE 6.1

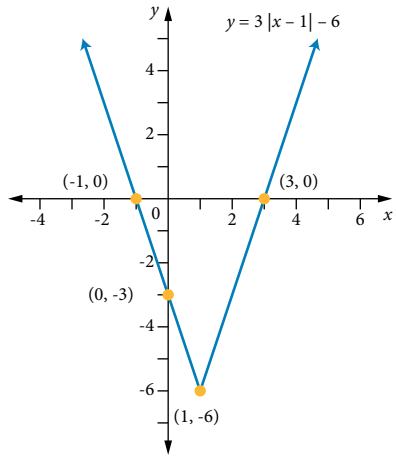
1 (a) 7 (b) 31 (c) 8 (d) -4
(e) 14 (f) -27 (g) 0

2 (a) $x = -3$ or $x = -11$ (b) $x = 7$ or $x = -2$
(c) $x = 3$ or $x = 2$ (d) $x = 2$ or $x = \dfrac{8}{3}$
(e) $x = -\dfrac{17}{8}$ or $x = -\dfrac{23}{3}$ (f) $x = 2$ or $x = 5$
(g) $x = -3$ (h) $x = \dfrac{7}{3}$
(i) No solution, as the absolute value of any number is always non-negative.

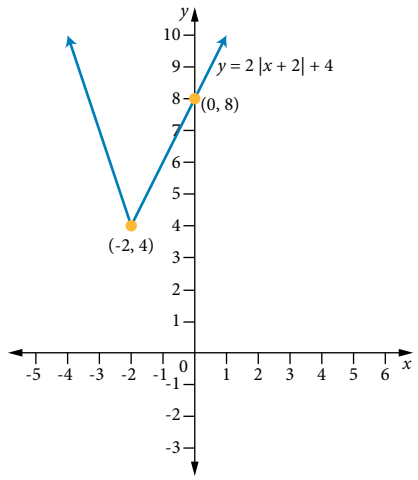
3 (a)



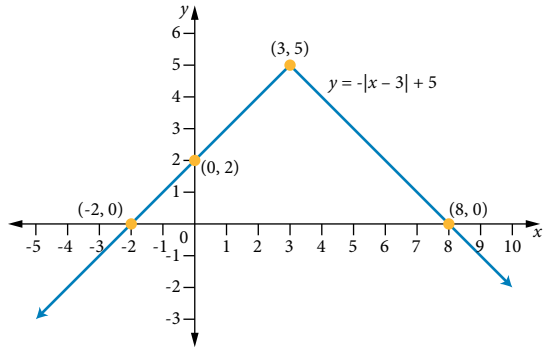
(b)



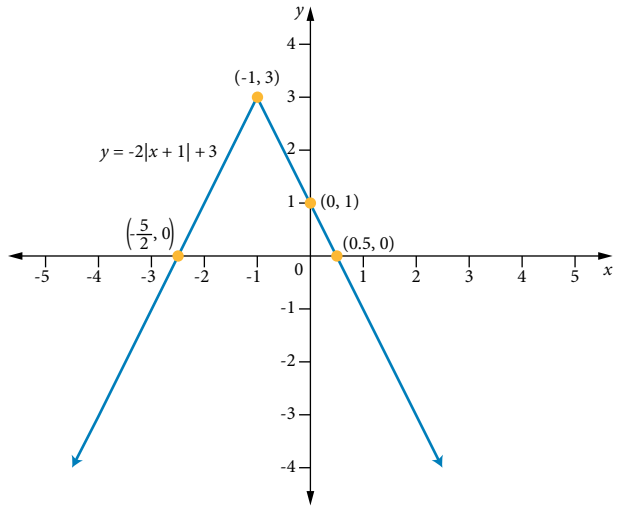
(c)



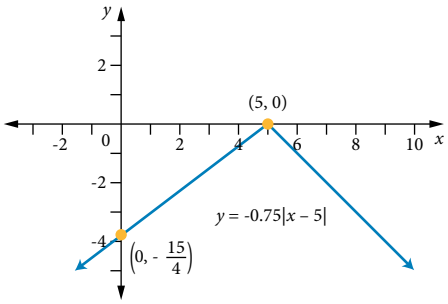
(d)



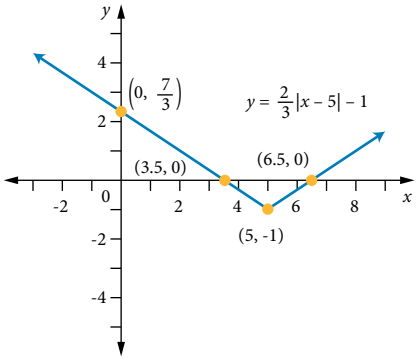
(e)



(f)



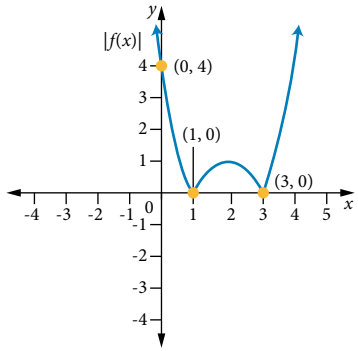
(g)



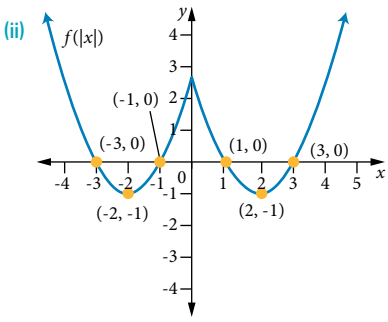
4 **(a)** $y = |x + 3| + 1$ **(b)** $y = \frac{1}{2}|x - 2| - 1$ **(c)** $y = -\frac{3}{2}|x| + 3$

(d) $y = \frac{1}{2}|x - 4| + 3$ **(e)** $y = -\frac{1}{5}|x - 2| + 3$ **(f)** $y = \frac{5}{2}|x + 3| + 1$

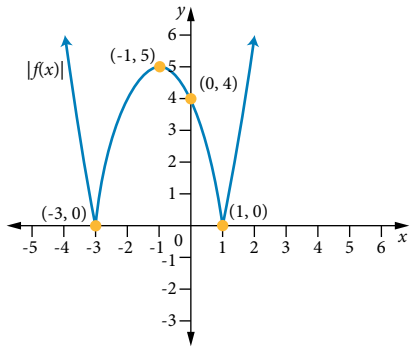
5 **(a)** **(i)**



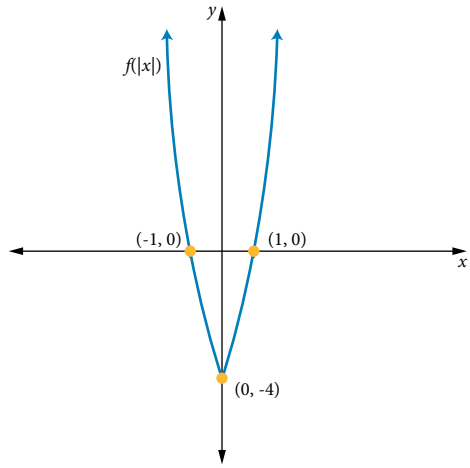
(ii)



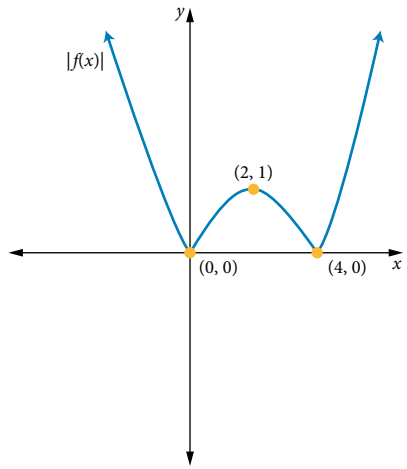
(b) **(i)**



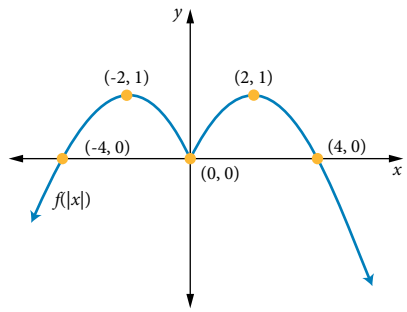
(ii)



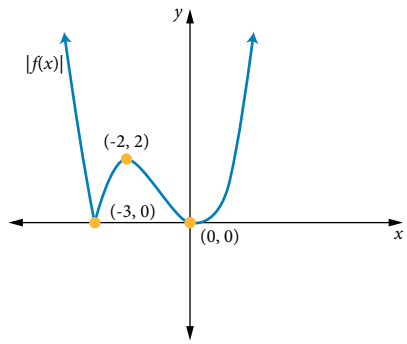
(c) **(i)**



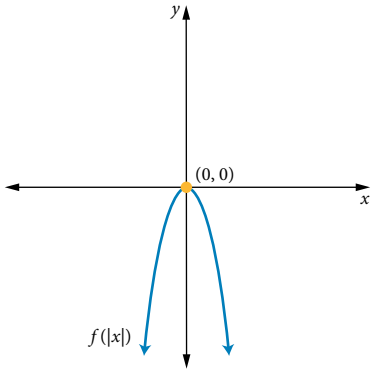
(ii)



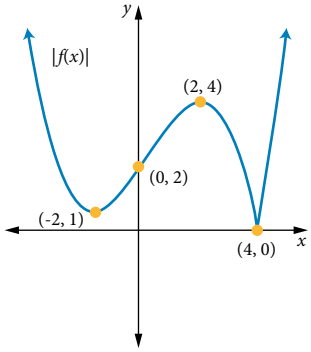
(d) **(i)**



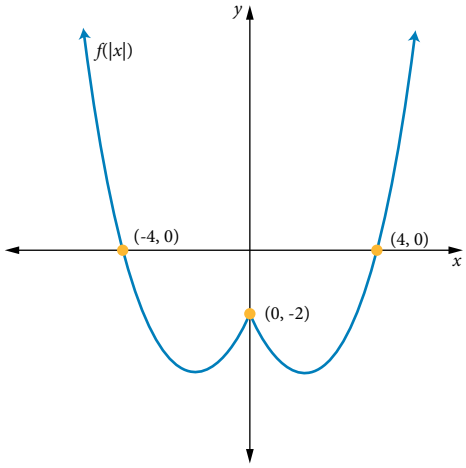
(ii)



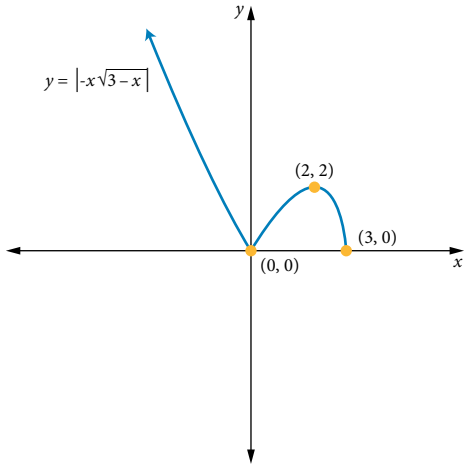
(e) (i)



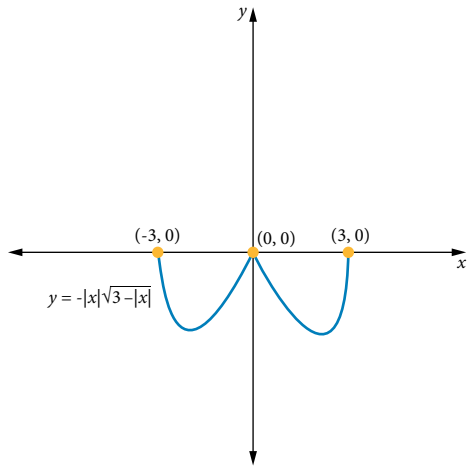
(ii)



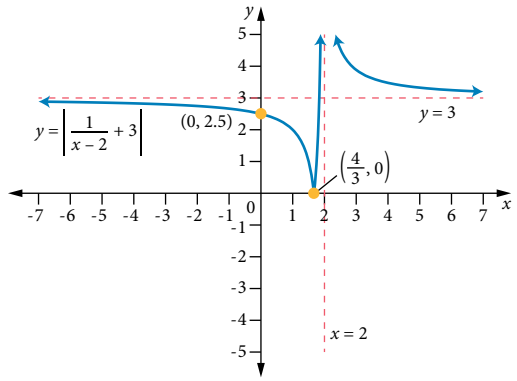
6 (a)



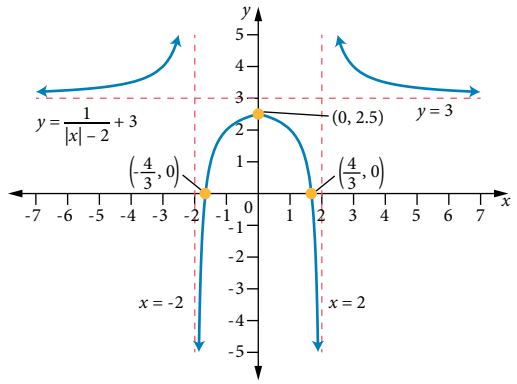
(b)



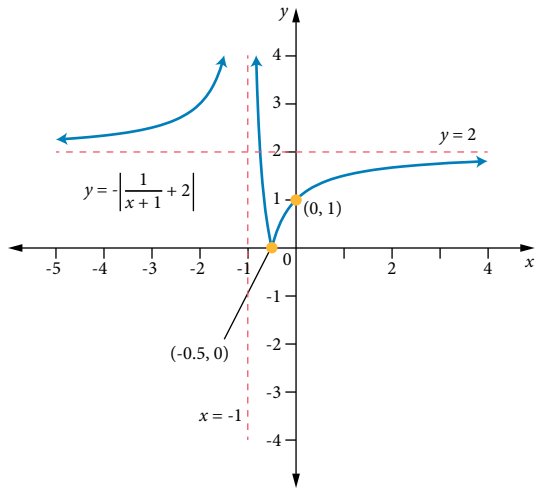
7 (a)



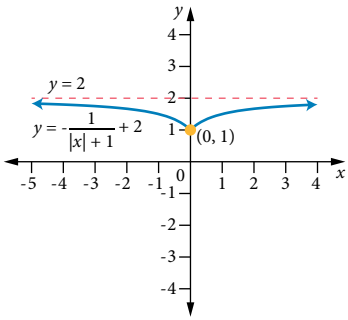
(b)



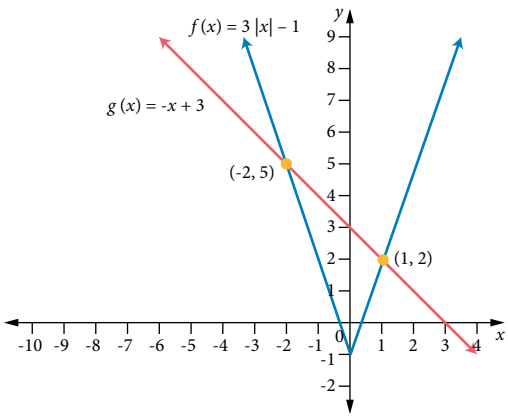
8 (a)



(b)



9



$x = -2$ or $x = 1$

10 $x = -2.30$ or $x = 1.62$ (2 d.p.)

11 $x = -2.58$, -0.42, or 0.58 (2 d.p.)

12 (a) D

(b) $|x| = |-x|$ is true in general. $|a - b| = |b - a|$ may appear to be false since $a - b$ and $b - a$ are not equivalent, but since $b - a = -1(a - b)$, this statement is actually true. For $|-ab| = |-a||-b|$ even though there is only one negative sign on the left-hand side, but two on the right-hand side, both sides are equal to $|ab|$, and so this statement is true.

13 C

14 (a) $x = -2$ or $x = 8$ (b) $x = 3$ (c) $x = -\frac{3}{2}$ (d) $x = 1$

15 $y = 2\left|x - \frac{3}{2}\right| - 5$

16 23 days

17 (a) $h(t) = -2.5|t - 7.2| + 19.4$ (b) $t = 8.79$ seconds

18 $x = 1$ or $x = 2$

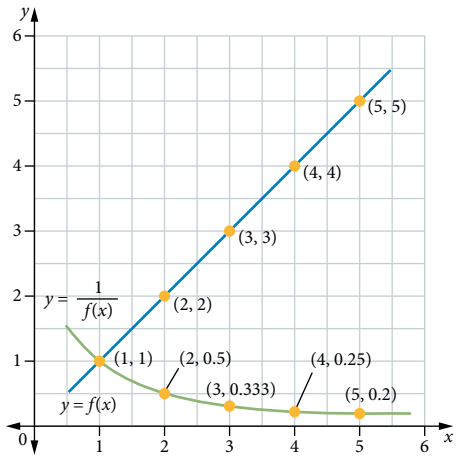
19 (1.5, 4.5)

20 $p = 4.5$

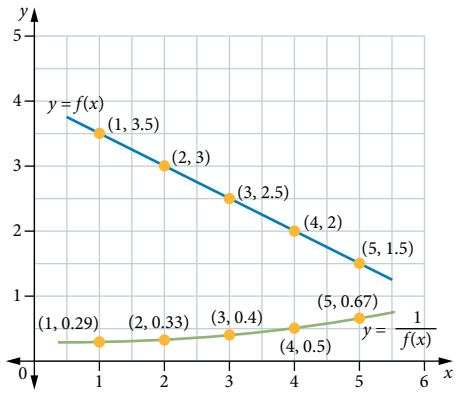
21 $\left(\sqrt{2}, 3\right)$

EXERCISE 6.2

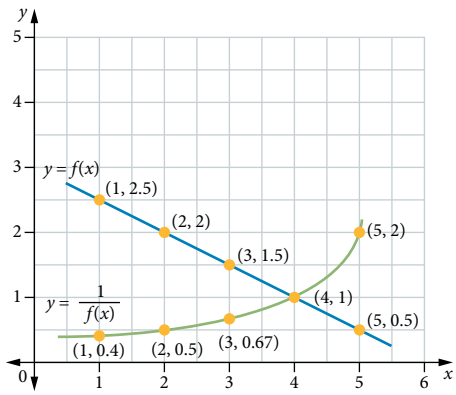
1 (a)



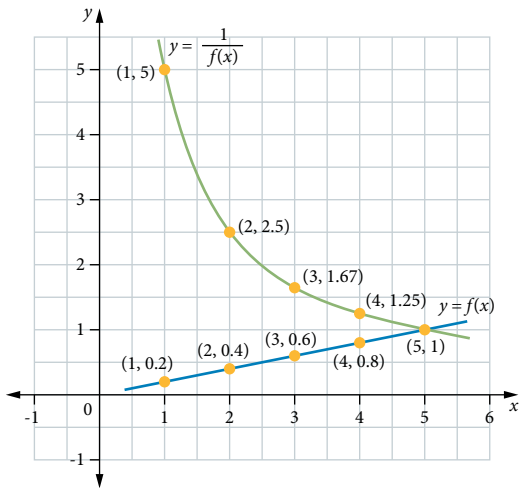
(b)



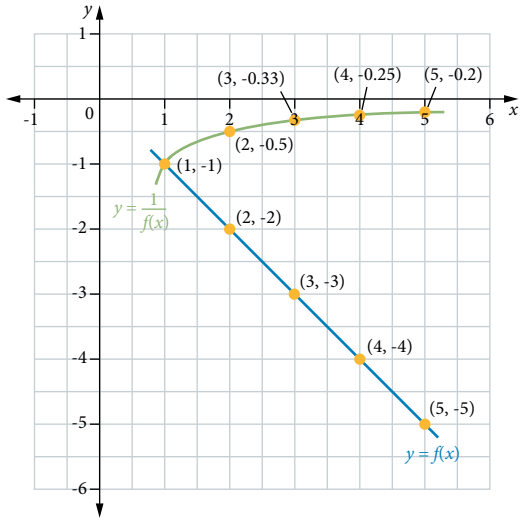
(c)



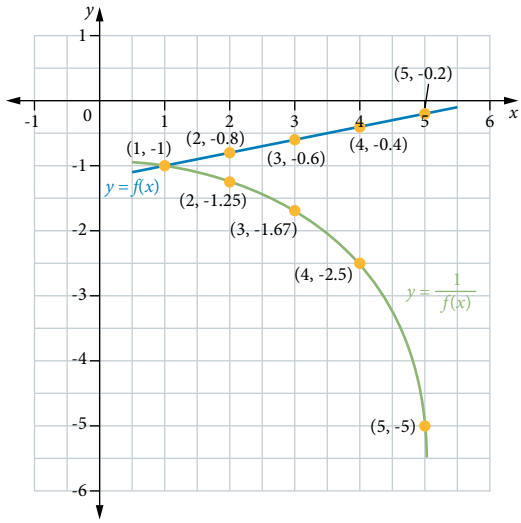
(d)



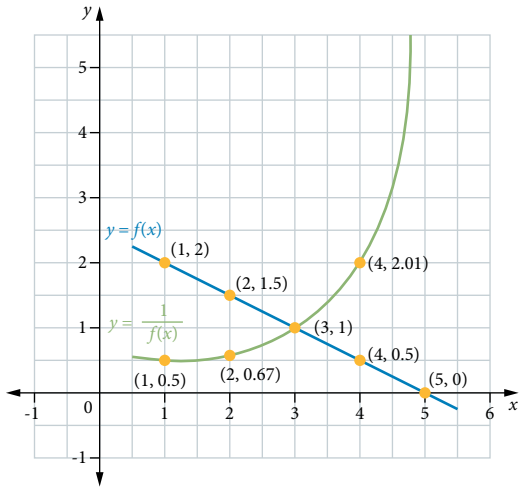
(e)


(3, -0.33)
(4, -0.25)
(5, -0.2)
(2, -0.5)
(1, -1)
$y = \frac{1}{f(x)}$
(2, -2)
(3, -3)
(4, -4)
(5, -5)
$y = f(x)$

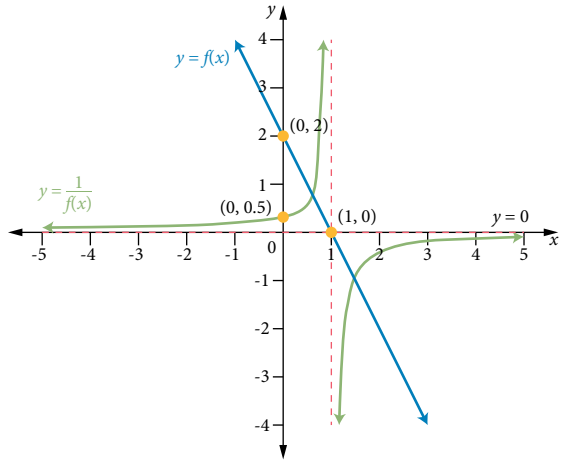
(f)


(5, -0.2)
(1, -1)
(2, -0.8)
(3, -0.6)
(4, -0.4)
$y = f(x)$
(2, -1.25)
(3, -1.67)
(4, -2.5)
$y = \frac{1}{f(x)}$
(5, -5)

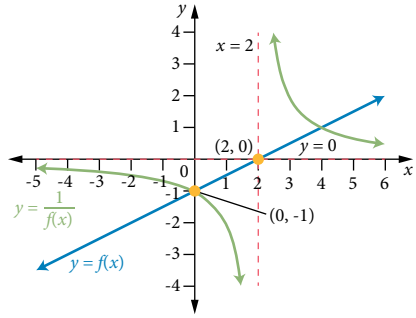
(g)


$y = f(x)$
(1, 2)
(2, 1.5)
(4, 2.01)
$y = \frac{1}{f(x)}$
(3, 1)
(4, 0.5)
(1, 0.5)
(2, 0.67)
(5, 0)

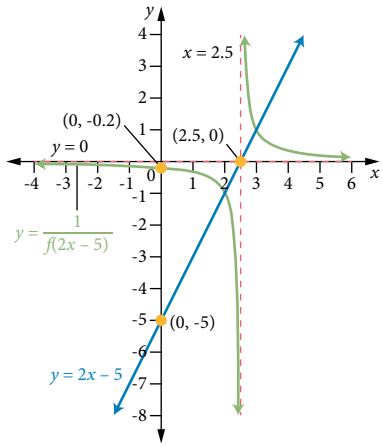
2 (a)


$y = f(x)$
(0, 2)
$y = \frac{1}{f(x)}$
(0, 0.5)
(1, 0)
$y = 0$

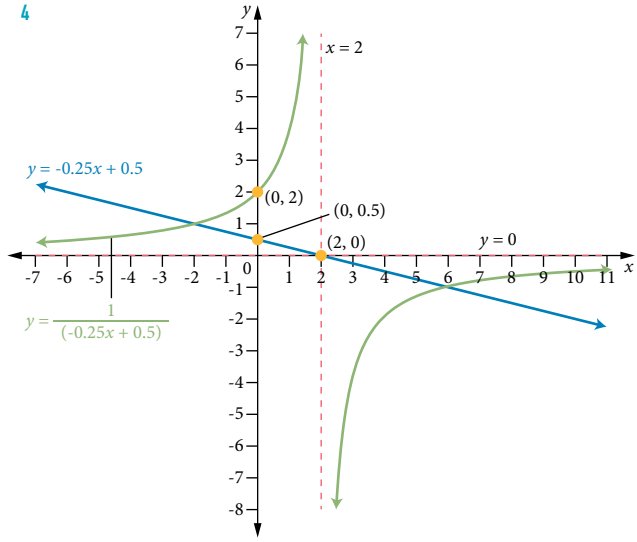
(b)


$x = 2$
(2, 0)
$y = 0$
$y = \frac{1}{f(x)}$
(0, -1)
$y = f(x)$

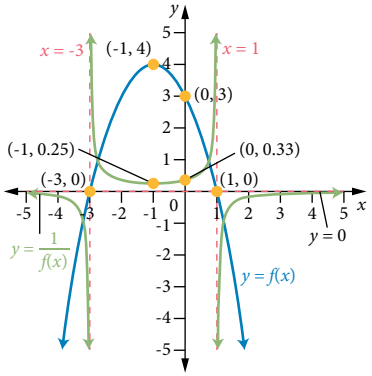
3


$x = 2.5$
(0, -0.2)
(2.5, 0)
$y = 0$
$y = \frac{1}{f(2x - 5)}$
(0, -5)
$y = 2x - 5$

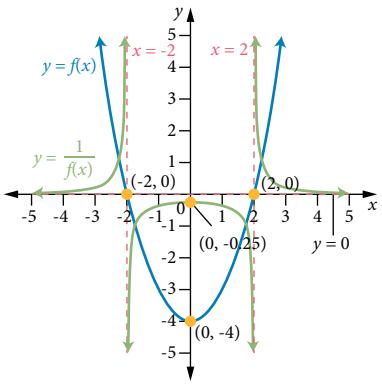
4


$x = 2$
$y = -0.25x + 0.5$
(0, 2)
(0, 0.5)
(2, 0)
$y = 0$
$y = \frac{1}{(-0.25x + 0.5)}$

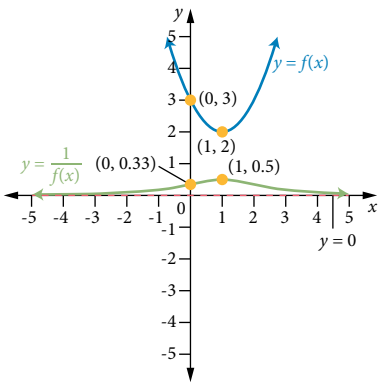
5 (a)



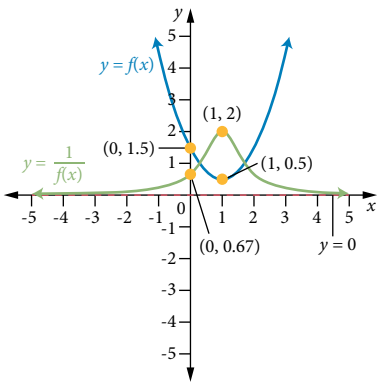
(b)



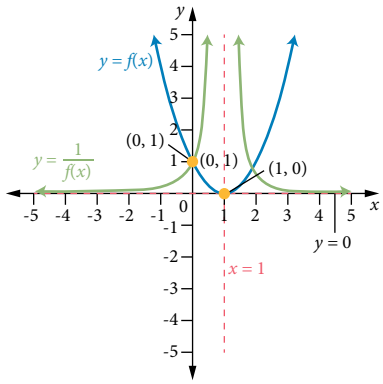
6 (a)



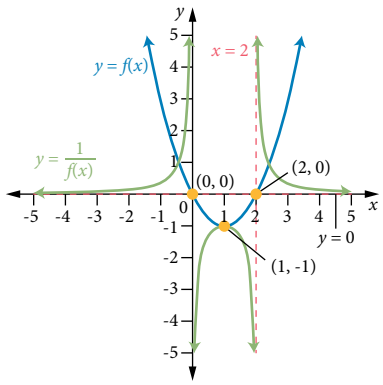
(b)



(c)

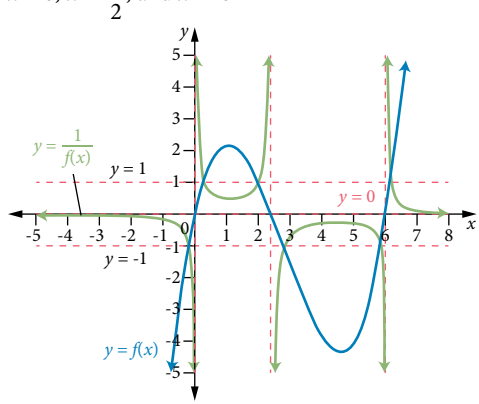


(d)



7 $x = 0$, $x = \frac{1}{2}$, and $x = -3$

8



9 (a) D

(b) Alternative A is true in most cases, but it is not true when the local minimum is also an x-intercept; in this case, the graph of $y = \frac{1}{f(x)}$ will have an asymptote at the corresponding x-value. For alternative B, the x-axis will be a horizontal asymptote of $y = \frac{1}{f(x)}$ whenever $f(x)$ approaches $\pm\infty$ as x approaches $\pm\infty$; however, there exist functions for which this is not the case (for example, $f(x) = \sqrt{1 - x^2}$). In alternative C, the graph of $y = \frac{1}{f(x)}$ will not have any vertical asymptotes if $f(x)$ does not have any x-intercepts, which often happens.

10 B

11 $\left(-\frac{1}{5}, 1\right)$ and $\left(-\frac{3}{5}, -1\right)$

12 $(2, 1), (-2, 1), \left(\sqrt{2}, -1\right)$ and $\left(-\sqrt{2}, -1\right)$

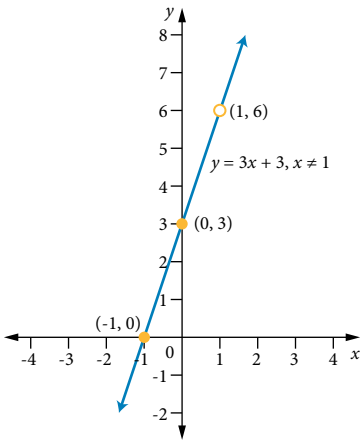
13 gradient $= -\frac{1}{pq}$

14 $f(x) = 0.1(x + 1)(x - 4)$

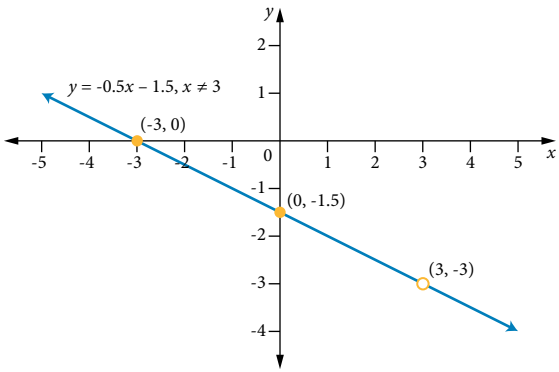
15 $a = -5$ and $b = -1$

EXERCISE 6.3

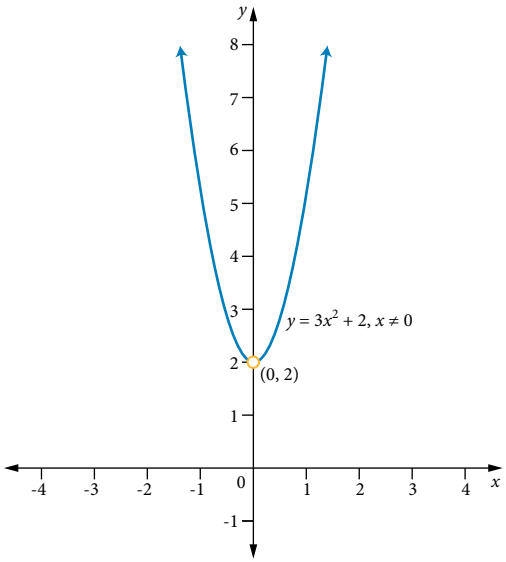
1 (a)



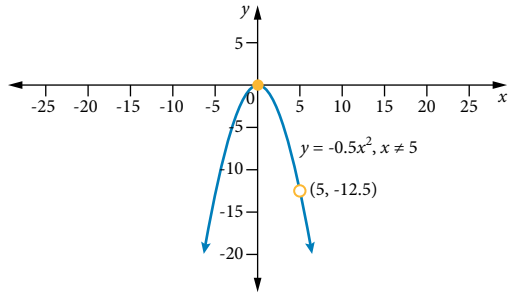
(b)



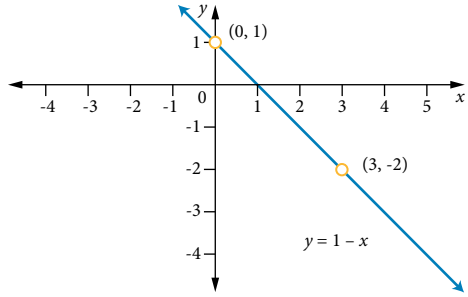
(c)



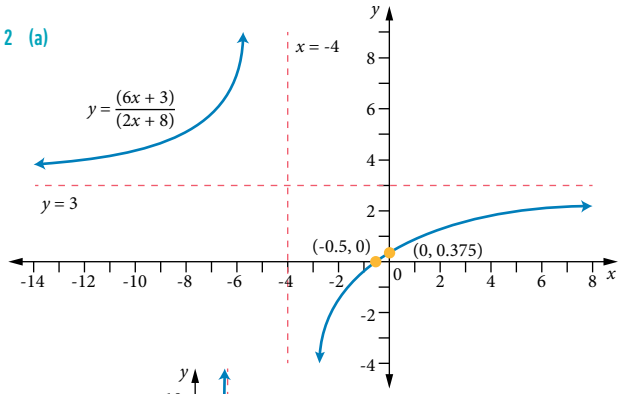
(d)



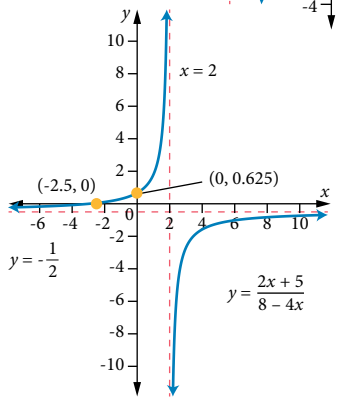
(e)



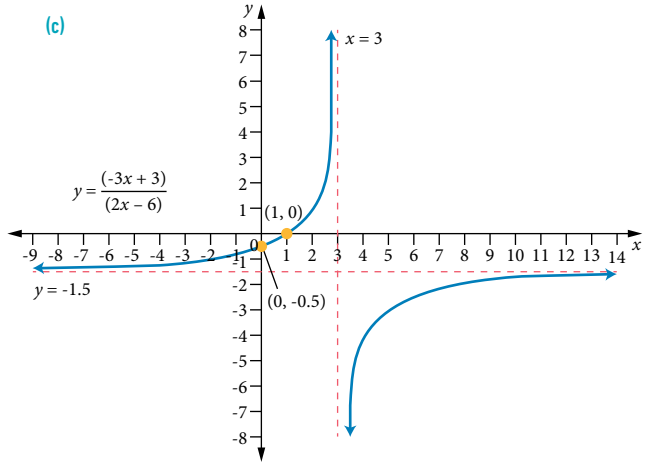
2 (a)



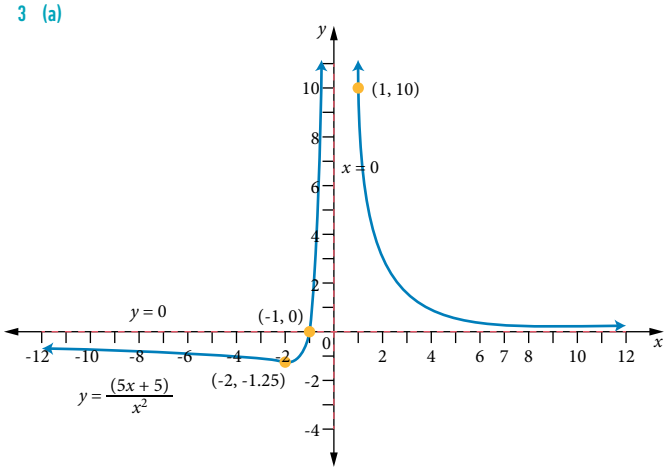
(b)



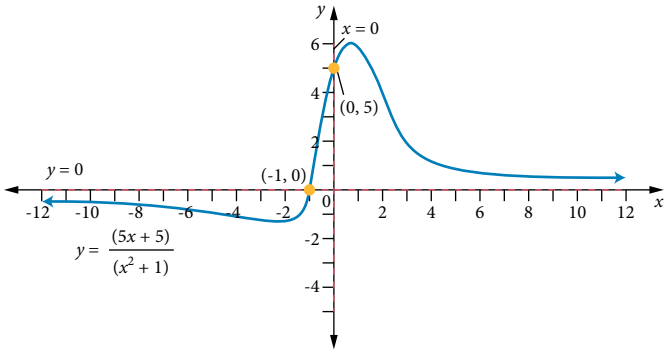
(c)



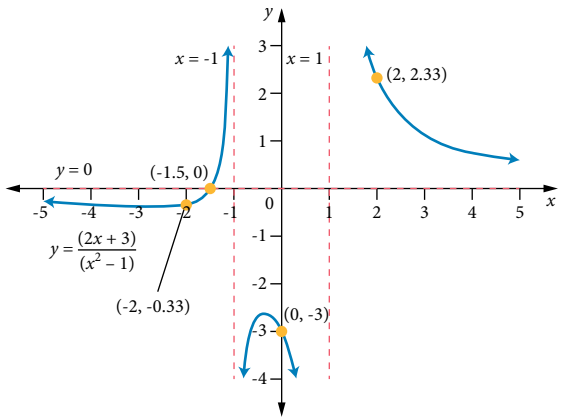
3 (a)


y
x
(1, 10)
x = 0
y = 0
(-1, 0)
(-2, -1.25)
y = (5x + 5) / x²

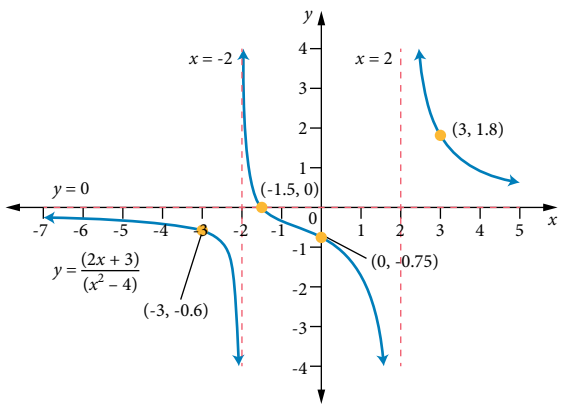
(b)


y
x
x = 0
(0, 5)
y = 0
(-1, 0)
y = (5x + 5) / (x² + 1)

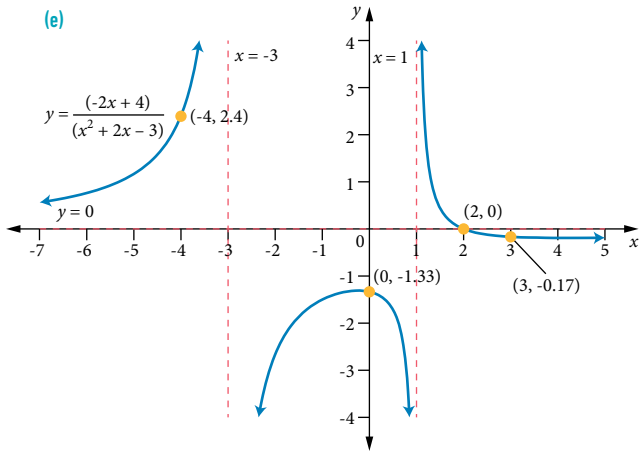
(c)


y
x
x = -1
x = 1
(2, 2.33)
y = 0
(-1.5, 0)
y = (2x + 3) / (x² - 1)
(-2, -0.33)
(0, -3)

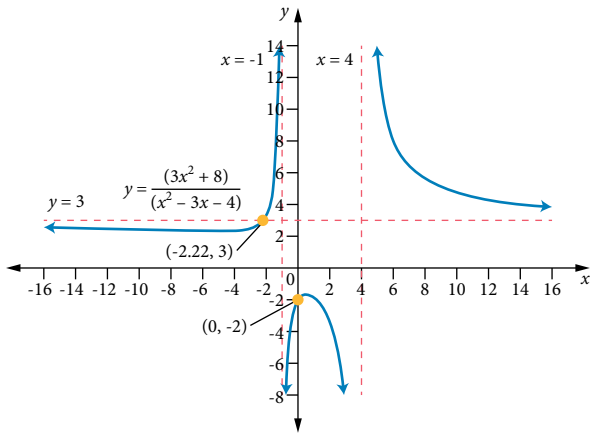
(d)


y
x
x = -2
x = 2
(3, 1.8)
y = 0
(-1.5, 0)
y = (2x + 3) / (x² - 4)
(-3, -0.6)
(0, -0.75)

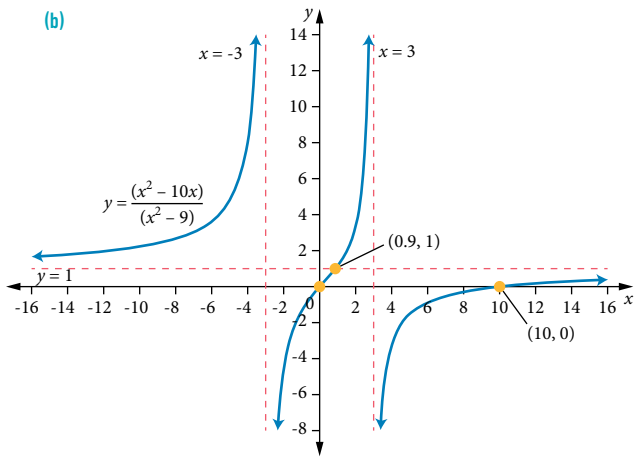
(e)


y
x
x = -3
x = 1
y = (-2x + 4) / (x² + 2x - 3)
(-4, 2.4)
y = 0
(2, 0)
(0, -1.33)
(3, -0.17)

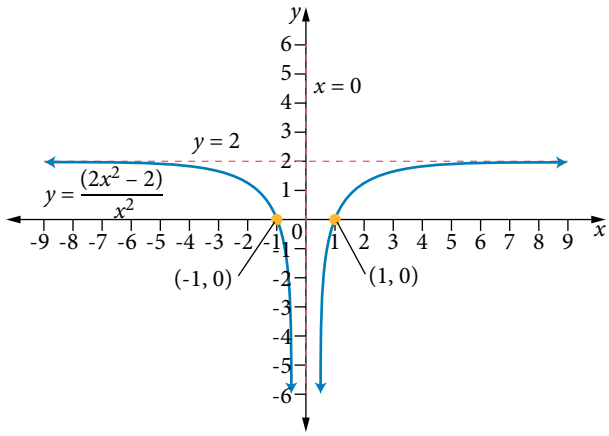
4 (a)


y
x
x = -1
x = 4
y = 3
y = (3x² + 8) / (x² - 3x - 4)
(-2.22, 3)
(0, -2)

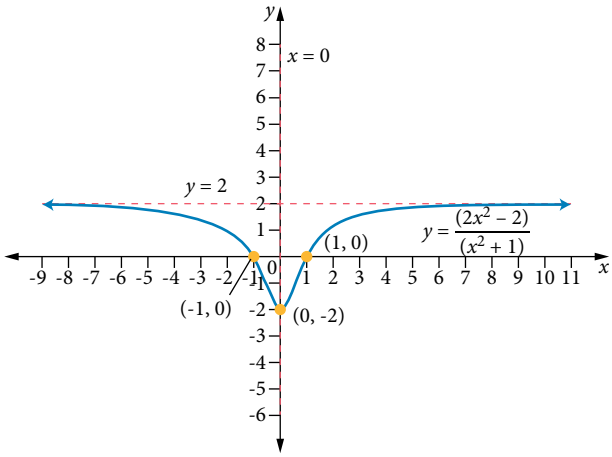
(b)


y
x
x = -3
x = 3
y = (x² - 10x) / (x² - 9)
(0.9, 1)
y = 1
(10, 0)

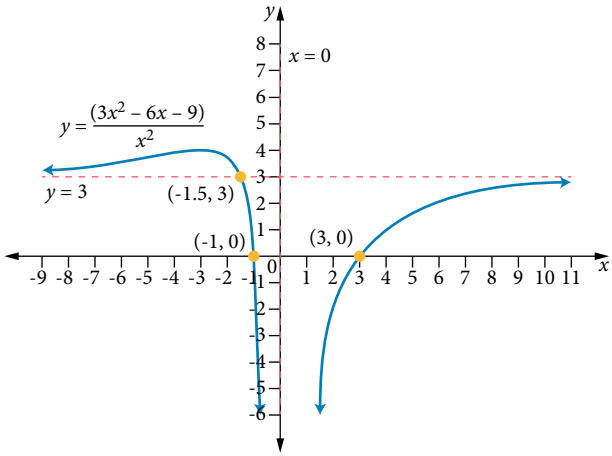
(c)


y
x
x = 0
y = 2
$y = \frac{(2x^2 - 2)}{x^2}$
(-1, 0)
(1, 0)

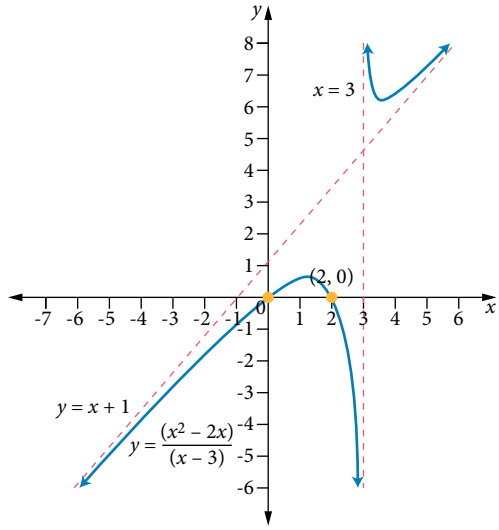
(d)


y
x
x = 0
y = 2
$y = \frac{(2x^2 - 2)}{(x^2 + 1)}$
(1, 0)
(-1, 0)
(0, -2)

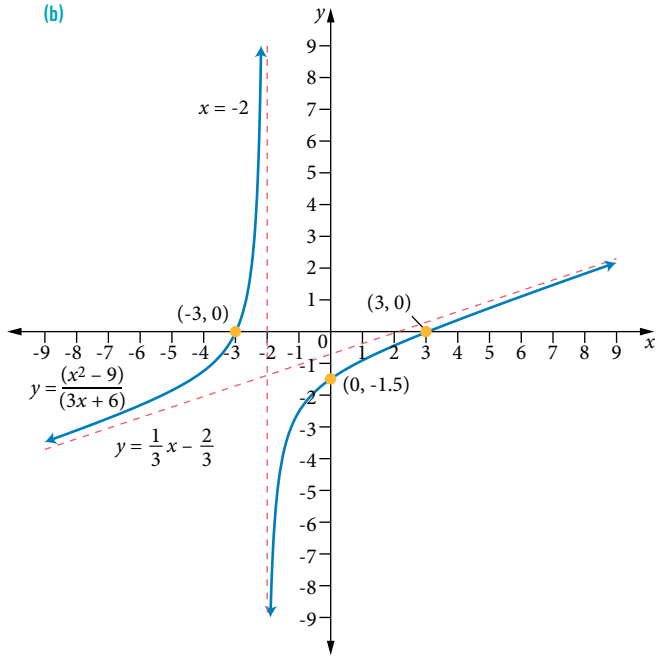
(e)


y
x
x = 0
$y = \frac{(3x^2 - 6x - 9)}{x^2}$
y = 3
(-1.5, 3)
(-1, 0)
(3, 0)

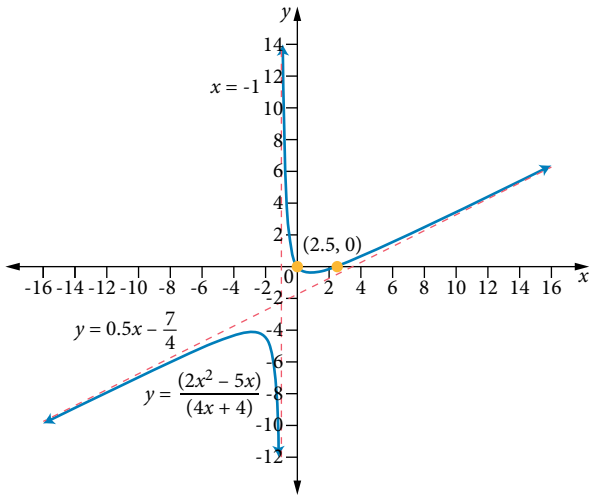
5 (a)


y
x
x = 3
(2, 0)
y = x + 1
$y = \frac{(x^2 - 2x)}{(x - 3)}$

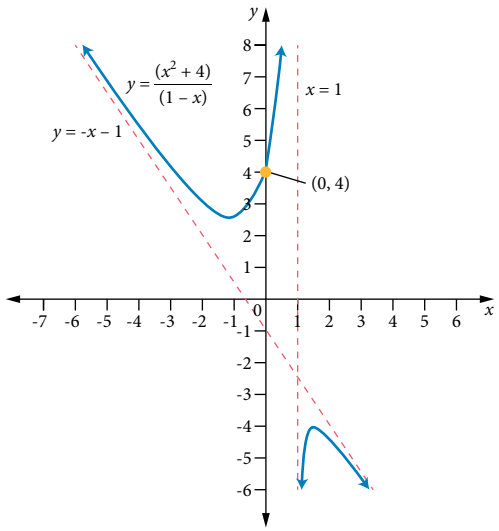
(b)


y
x
x = -2
(-3, 0)
(3, 0)
(0, -1.5)
$y = \frac{(x^2 - 9)}{(3x + 6)}$
$y = \frac{1}{3}x - \frac{2}{3}$

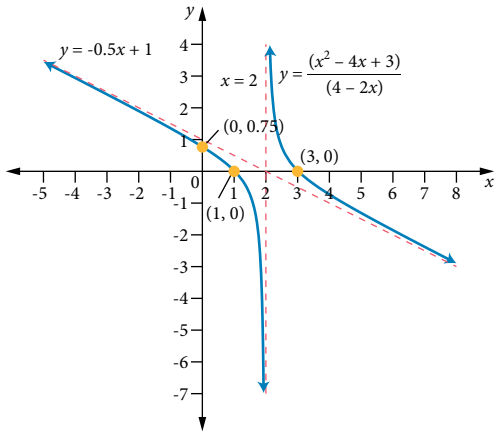
(c)


y
x
x = -1
(2.5, 0)
$y = 0.5x - \frac{7}{4}$
$y = \frac{(2x^2 - 5x)}{(4x + 4)}$

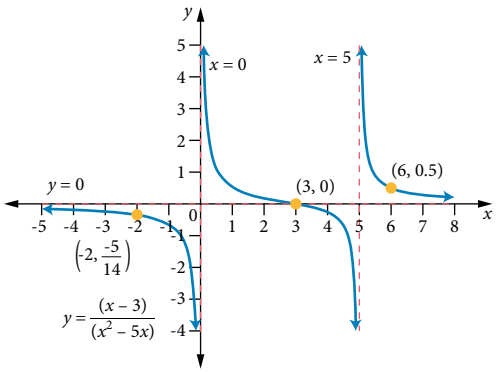
(d)



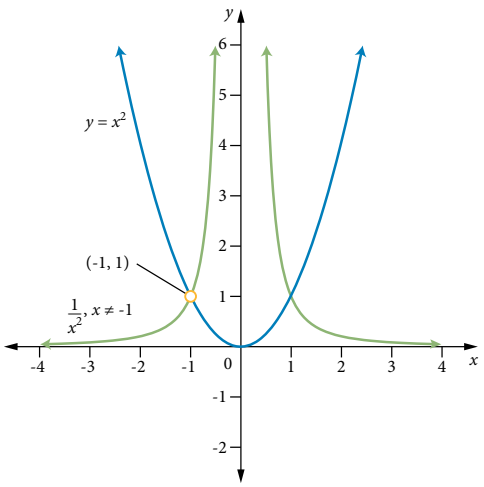
(e)



6

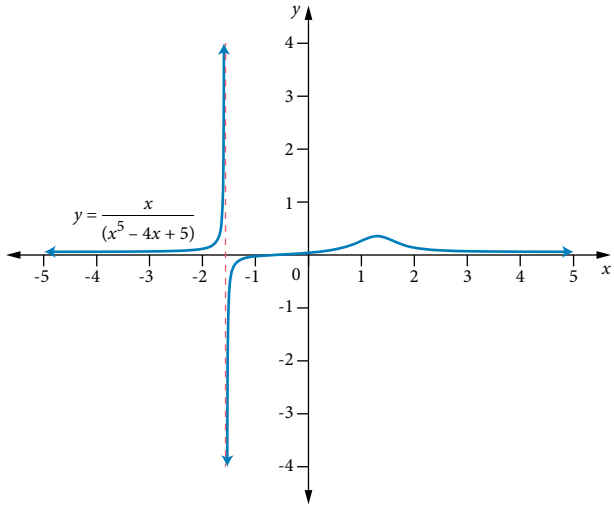


7



8 D

9 equation of vertical asymptote: $x = -1.6304$ (4 d.p.)



10 $p = 3, q = -10, b = -1, c = -6$

11 The equation of the oblique asymptote is the quotient when the numerator is divided by the denominator using long division:

$$\begin{array}{r} 5x - 6 \\ x + 2 \overline{)\,5x^2 + 4x + 3} \\ \underline{5x^2 + 10x} \\ -6x + 3 \\ \underline{-6x - 12} \\ 15 \end{array}$$

Therefore, the equation of the oblique asymptote is $y = 5x - 6$. Any points of intersection with the original function can be found by equating both expressions for y:

$$\frac{5x^2 + 4x + 3}{x + 2} = 5x - 6$$
$$5x^2 + 4x + 3 = (5x - 6)(x + 2)$$
$$5x^2 + 4x + 3 = 5x^2 + 10x - 6x - 12$$
$$5x^2 + 4x + 3 = 5x^2 + 4x - 12$$
$$3 = -12$$

Thus, there is no solution to this equation. This means that there are no points of intersection between the original function and its oblique asymptote.

12 (a)

$$\text{RHS} = \frac{a}{d}x + \frac{bd - ae}{d^2} + \frac{cd^2 - bde + ae^2}{d^2(dx + e)}$$
$$= \frac{ad(dx + e)x}{d^2(dx + e)} + \frac{(bd - ae)(dx + e)}{d^2(dx + e)} + \frac{cd^2 - bde + ae^2}{d^2(dx + e)}$$
$$= \frac{ad^2x^2 + adex + bd^2x + bde - adex - ae^2 + cd^2 - bde + ae^2}{d^2(dx + e)}$$
$$= \frac{d^2(ax^2 + bx + c)}{d^2(dx + e)}$$
$$= \text{LHS}$$

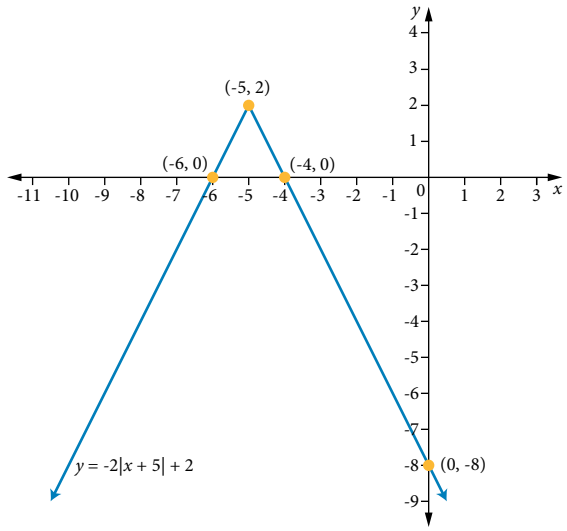
(b) $y = \frac{a}{d}x + \frac{bd - ae}{d^2}$

CHAPTER REVIEW 6

1 **(a)** -50 **(b)** -16 **(c)** 4

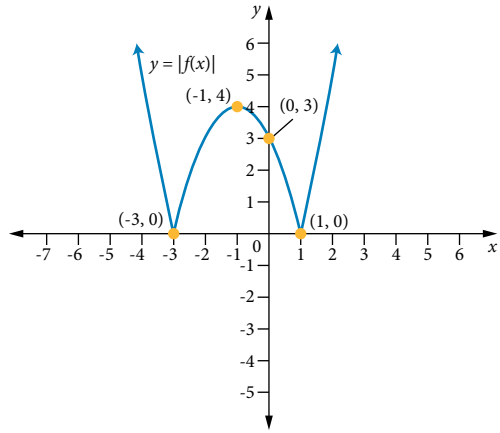
2 **(a)** $x = 2$ or $x = -5$ **(b)** $x = \frac{1}{2}$ or $x = \frac{9}{2}$

3

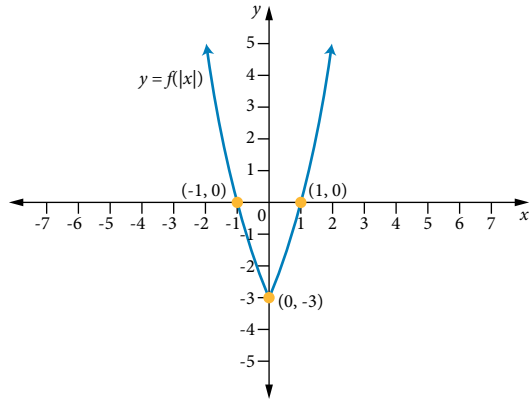


4 $y = 0.5|x - 3| + 2$

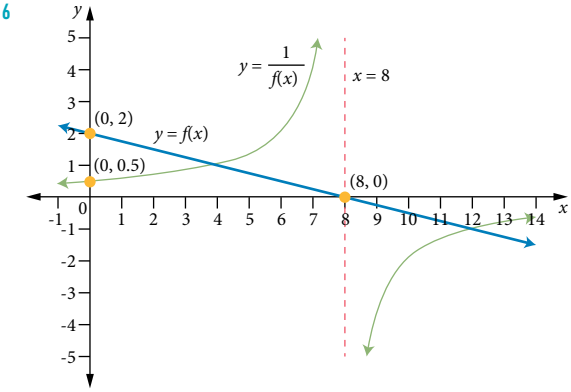
5 **(i)** The graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by replicating the part(s) of $y = f(x)$ that are on or above the x-axis, and reflecting the remaining part(s) in the x-axis.



(ii) The graph of $y = f(|x|)$ can be obtained from the graph of $y = f(x)$ by replicating the part(s) of $y = f(x)$ that are on or to the right of the y-axis, reflecting the remaining part(s) in the y-axis.

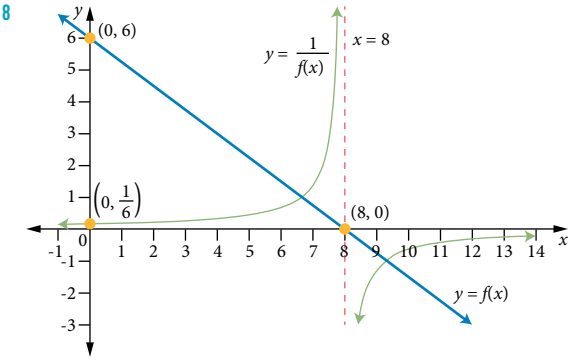


6

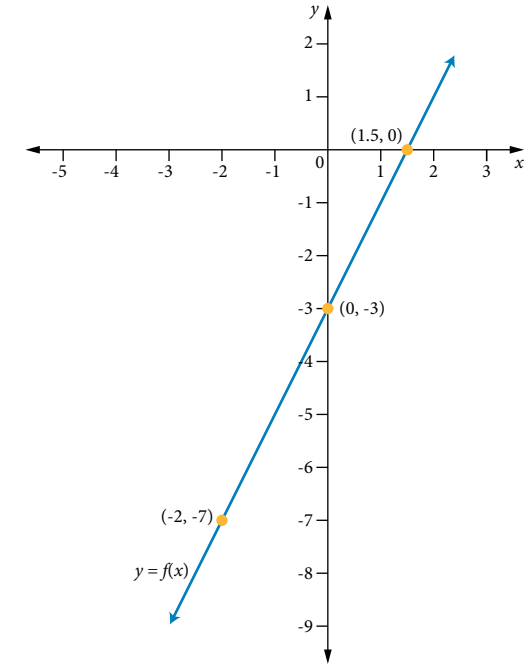


7 $x = 0$, $x = \sqrt{2}$, $x = -\sqrt{2}$

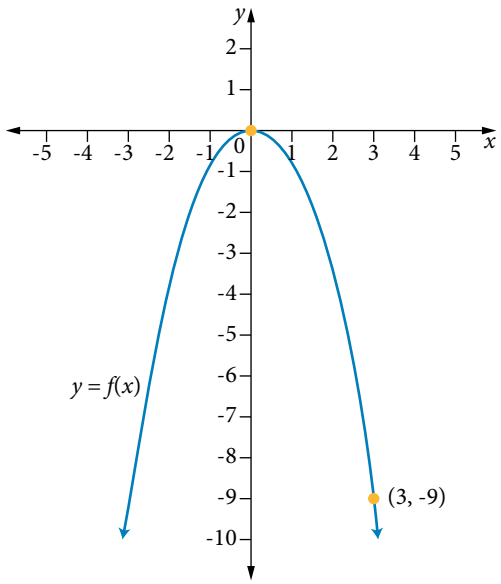
8



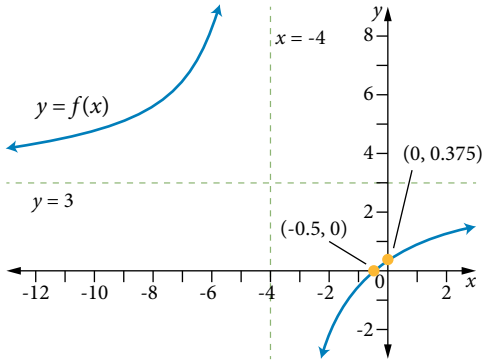
9 **(a)**



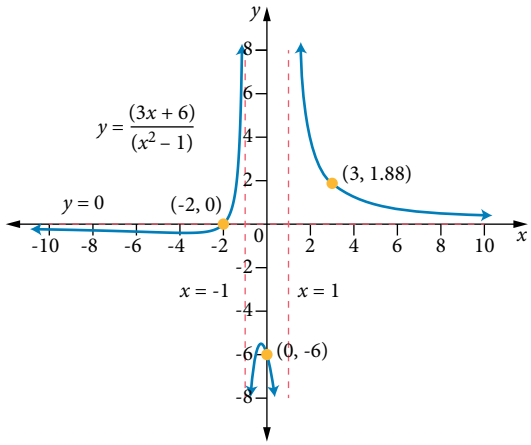
(b)



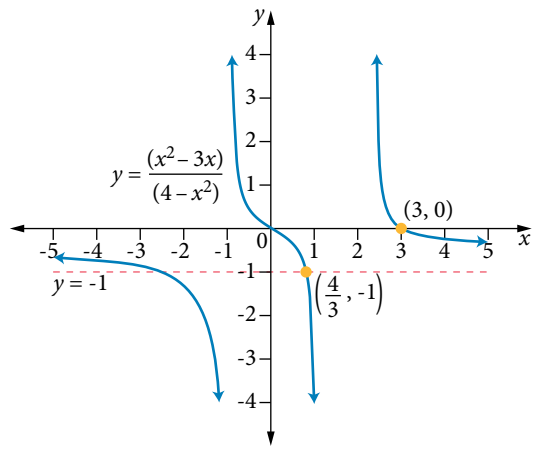
10 (a)



(b)

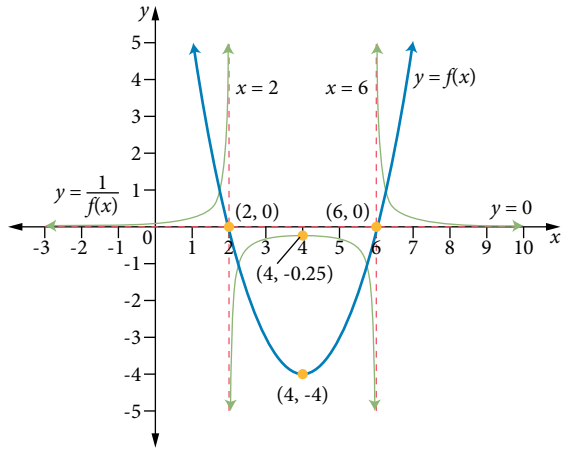


(c)



11 (a) $x = 6$ or $x = -\frac{4}{3}$ (b) $x = -2$

12



13 B 14 C

15 (0, 1) and (-1, -1)

16 $y = -2x + 13$

17 $x = -1$, 3, 5 or 9

18 The equation of the horizontal asymptote for such a function is $y = \frac{a}{d}$. Points of intersection of the function and this line occur when $f(x) = \frac{a}{d}$:

$$\frac{ax^2 + bx + c}{dx^2 + ex + f} = \frac{a}{d}$$
$$d(ax^2 + bx + c) = a(dx^2 + ex + f)$$
$$adx^2 + bdx + cd = adx^2 + aex + af$$
$$bdx + cd = aex + af$$
$$bdx - aex = af - cd$$
$$x(bd - ae) = af - cd$$
$$x = \frac{af - cd}{bd - ae}$$

Thus, there is, at most, one solution to this equation, and, at most, one point of intersection. There will be no solution (and hence, no point of intersection), when the denominator for the expression for x found previously is equal to 0; that is, if $bd = ae$.

19 $p = 25$ 20 $y = -1.44$ (2 d.p.)

Chapter 7 Trigonometry

RECALL

1 (a) $\frac{2\pi}{3}$ (b) $\frac{7\pi}{12}$ (c) $\frac{17\pi}{12}$ (d) $\frac{5\pi}{3}$

2 (a) $\frac{\pi}{12}$ (b) $\frac{5\pi}{6}$ (c) $\frac{3\pi}{4}$ (d) $\frac{13\pi}{12}$

3 (a) 3 (b) $\frac{\pi}{12}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{9}$

4 (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

(e) 1 (f) 1 (g) -1 (h) $-\frac{1}{2}$

5 (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{11\pi}{6}$

6 (a) 60° (b) 225° (c) 210° (d) 480°

7 (a) $\frac{1}{2}$ (b) 1 (c) $\frac{\sqrt{2}}{2}$ (d) $-\frac{\sqrt{3}}{2}$

(e) 0 (f) $-\sqrt{3}$ (g) $\frac{\sqrt{3}}{2}$ (h) $\frac{-\sqrt{3}}{2}$

8 (a) $x = 1, 3, 5, 7, 9$ (b) $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

9 (a) -0.4 (b) 0.4 (c) 0.8 (d) -0.8

(e) 0.8 (f) -0.4 (g) -2 (h) 2

EXERCISE 7.1

1 (a) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{\pi}{4}, \frac{3\pi}{4}$

2 (a) $\frac{\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{2\pi}{3}, \frac{4\pi}{3}$

3 (a) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (b) $x = 0, \pi$

4 (a) $\pm\frac{5\pi}{6}$ (b) $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ (c) $\frac{\pi}{8}, \frac{5\pi}{8}$

5 (a) 0.88, 4.02 (b) 0.20, 2.94 (c) 1.98, 4.30 (d) 3.39, 6.03

6 (a) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}$

(b) $\frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}, \frac{41\pi}{24}, \frac{49\pi}{24}, \frac{53\pi}{24}, \frac{61\pi}{24}, \frac{65\pi}{24}, \frac{73\pi}{24}, \frac{77\pi}{24}, \frac{85\pi}{24}, \frac{89\pi}{24}$

(c) $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

(d) 120°

7 (a) 6

(b) $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$

(c) 3 (d) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

8 (a) $\pi, \frac{4\pi}{3}$ (b) $\frac{\pi}{2}, \frac{7\pi}{6}$ (c) 0°, 180°, 360°

9 (a) $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$ (b) $\frac{4\pi}{3}$

10 $x = n\pi + 0.23$ or $n\pi + 1.34$

11 (a) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (b) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(c) $-\frac{\pi}{2}, \frac{\pi}{2}$

12 (a) 0, 0.34, 2.80, π (b) 0°, 120°, 240°, 360°

13 C

14 B

15 (a) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ (b) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$

(c) $0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$ (d) $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

16 0.01, 0.79

17 (a) $\frac{n\pi}{2} + \frac{\pi}{6}, n \in \mathbb{Z}$ (b) $n\pi + (-1)^n \frac{\pi}{4}, n \in \mathbb{Z}$

(c) $(2n+1)\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$ (d) $n\pi + (-1)^{n+1} \frac{\pi}{3}, n \in \mathbb{Z}$

18 $\frac{3\pi}{4}, \frac{7\pi}{4}$

19 1.19, 4.33

20 (a) 2 cm to the right of O (b) $\frac{\pi}{12}$ seconds

(c) $\frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ seconds (d) 1.70 seconds

(e) No

21 (a) maximum = 12 m, minimum = 8 m

(b) 6.27 hours (c) 12 hours

22 $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

23 (a) $p < -2$ or $p > 8$

(b) $\frac{1}{2\pi}, 1+\frac{1}{2\pi}, 2+\frac{1}{2\pi}, 3+\frac{1}{2\pi}$

(c) $x = n + \frac{1}{2\pi}, n \in \mathbb{Z}$

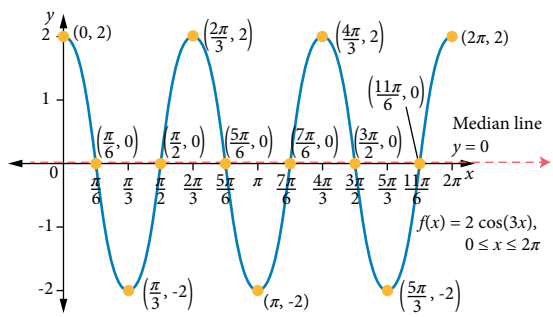
EXERCISE 7.2

1 (a) amplitude 2, period $\frac{2\pi}{3}$

(b) $\left(\frac{\pi}{6}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

(c) The median line has the equation $y = 0$.

(d)

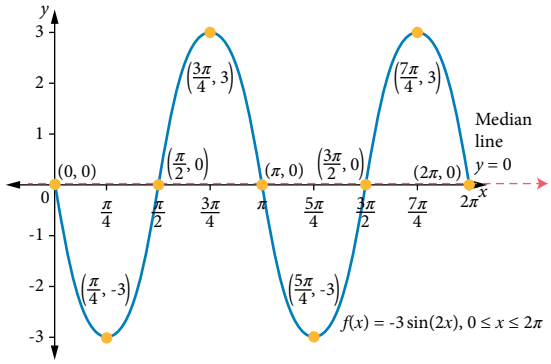


2 **(a)** period π, amplitude 2, range [-3,3]

(b) $(0,0), \left(\frac{\pi}{2}, 0\right), (\pi, 0), \left(\frac{3\pi}{2}, 0\right), (2\pi, 0)$

(c) The median line has the equation $y = 0$.

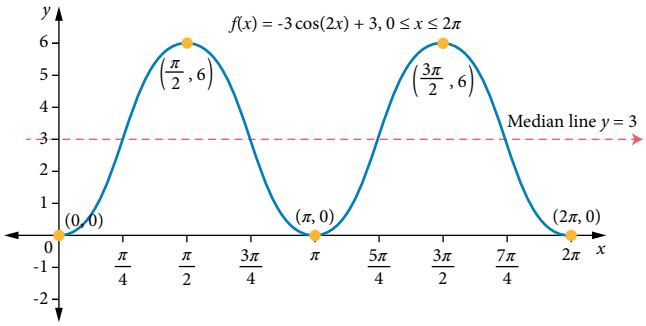
(d)



3 **(a)** period π, amplitude 3, range [0,6]

(b) $(0,0), (\pi, 0), (2\pi, 0)$

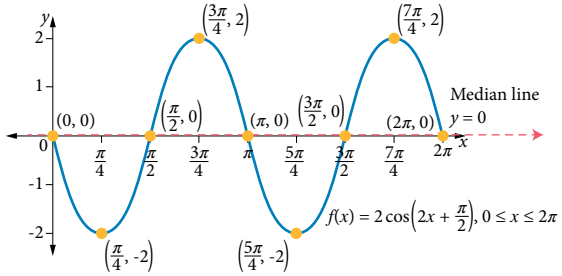
(c)



4 **(a)** period π, amplitude 2, range [-2,2]

(b) $(0,0), \left(\frac{\pi}{2}, 0\right), (\pi, 0), \left(\frac{3\pi}{2}, 0\right), (2\pi, 0)$

(c)



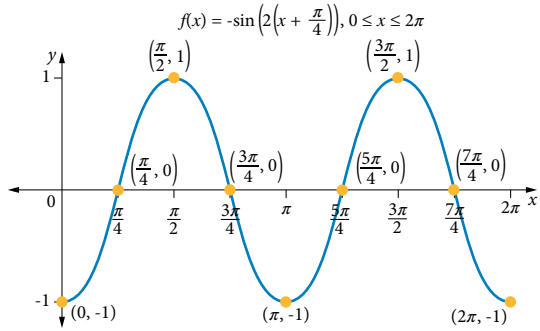
5 D

6 **(a)** C

(b) A common error will be to think that the dilation from the y-axis is equal by a factor of 2 because $g(x) = \sin(2x - \pi)$.

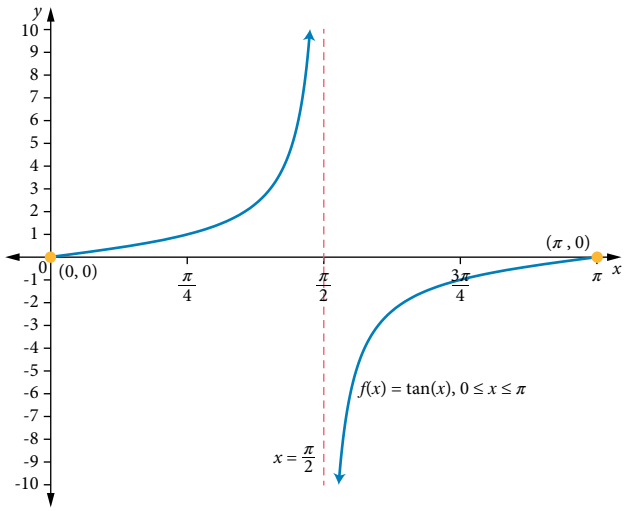
7 **(a)** x-intercepts: $\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$;
y-intercept: (0,-1)

(b)



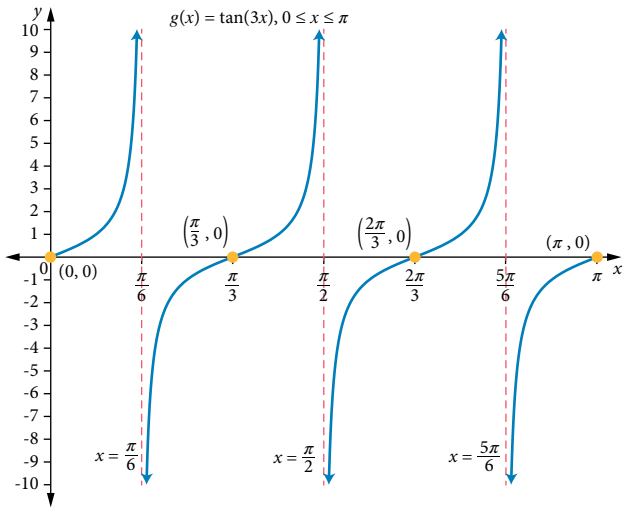
8 **(a)** period π; asymptote $x = \frac{\pi}{2}$; x-intercepts (0,0) and $(\pi, 0)$

(b)



9 **(a)** period $\frac{\pi}{3}$; asymptotes $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$;
x-intercepts $(0,0), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right), (\pi, 0)$

(b)



10 **(a)** period 2π; amplitude 1; range [0,2]

(b) period 2; amplitude 2; range [-2,2]

(c) period $\frac{\pi}{3}$; amplitude undefined; range $\mathbb{R}$

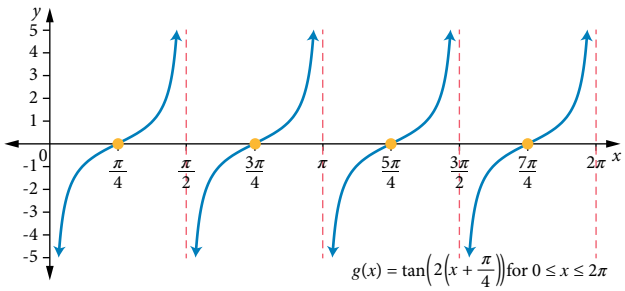
(d) period 12π; amplitude 11; range [-7,15]

(e) period 2π; amplitude $\frac{1}{2}$; range $\left[-\frac{15}{2}, -\frac{13}{2}\right]$

11 (a) Dilation by a factor of $\frac{1}{2}$ from the y-axis. Translation $\frac{\pi}{4}$ units to the left.

(b) period $\frac{\pi}{2}$; asymptotes $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$, $x = 2\pi$; x-intercepts $\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

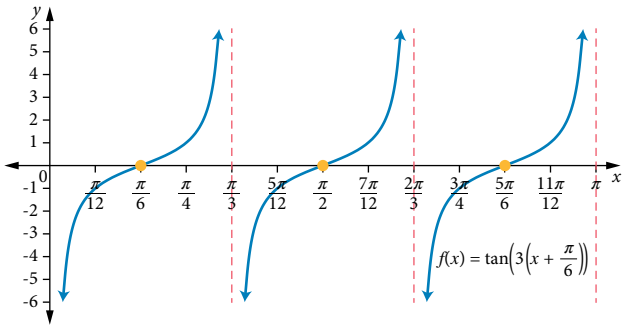
(c)



12 (a) $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ (b) $x = \pi$

(c) $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{7\pi}{4}$ (d) $x = 0$, $x = 2\pi$

13



14 (a) 4 mm (b) 100

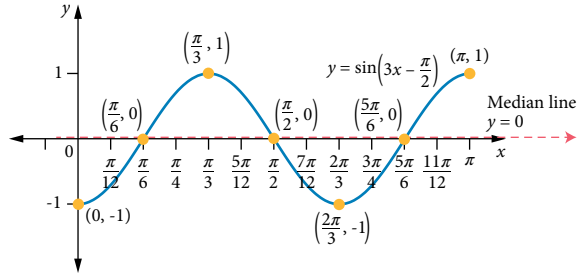
15 $x = 0.17, 1.40, 3.31, 4.54$

16 (a) $g(x) = -\sin\left(x - \frac{\pi}{3}\right)$; period 2π; amplitude 1

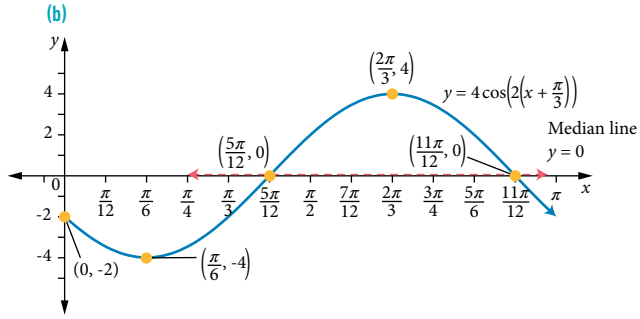
(b) $g(x) = 4\sin\left(\frac{x}{5}\right)$; period 10π; amplitude 4

(c) $g(x) = \sin\left(3x + \frac{\pi}{2}\right)$; period $\frac{2\pi}{3}$; amplitude: 1

17 (a)



(b)



18 (a) $f(x) = 2\cos(3x)$; note that the equation could also be $f(x) = 2\sin\left(3\left(x + \frac{\pi}{6}\right)\right)$

(b) $f(x) = 2\sin\left(\frac{x}{2}\right)$

(c) $f(x) = A\tan(2x)$

(d) $f(x) = -\cos\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$

19 (a) [2.50, 3.79]

(b) $\left(-\frac{5\pi}{6}, -\frac{3\pi}{4}\right), \left(-\frac{\pi}{3}, -\frac{\pi}{4}\right), \left(\frac{\pi}{6}, \frac{\pi}{4}\right), \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$

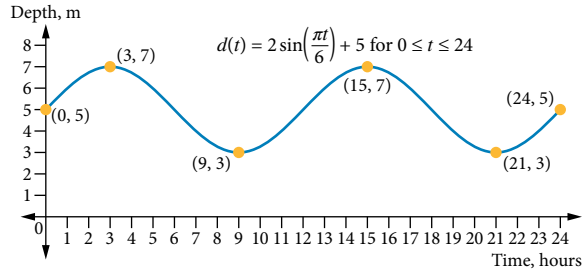
20 (a) The period is $\frac{2\pi}{\frac{\pi}{6}} = 12$. The period of the function is 12 hours.

(b) The function has an amplitude of 2 with a median line at $d = 5$.

Hence, maximum value is 5 + 2 = 7. Maximum values occur one-quarter and three-quarters of the way through one period, at 3 am and 3 pm.

(c) The minimum value is 5 − 2 = 3. This occurs half a period after each maximum, at 9 am and 9 pm.

(d)

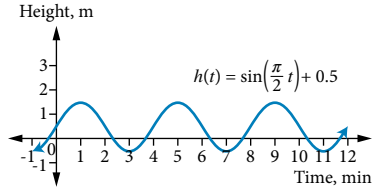


(e) 7 am, 11 am, 7 pm, 11 pm

21 (a) The period is 4 minutes. Therefore, three cycles take 12 minutes.

(b) maximum: 1.5 m; minimum: -0.5 m

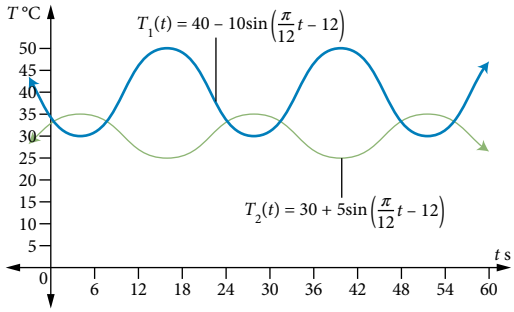
(c)



(d) 2 minutes, 20 seconds and 3 minutes, 40 seconds after the start

22 **(a)** 32°C **(b)** 29°C

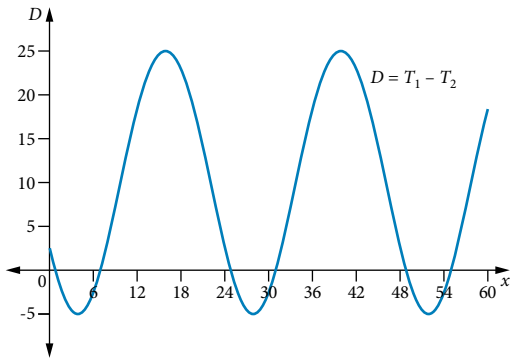
(c)



(d) 1:01 pm, 1:07 pm, 1:25 pm, 1:31 pm, 1:49 pm, 1:55 pm

(e) $D = 10 - 15\sin\left(\frac{\pi}{12}t - 12\right)$

(f)



(g) 1:16 pm, 1:40 pm

EXERCISE 7.3

1 **(a)** 1 **(b)** undefined **(c)** $\sqrt{2}$
(d) $\sqrt{2}$ **(e)** undefined **(f)** $-\frac{2}{\sqrt{3}}$

2 **(a)** 2.76 **(b)** 3.53 **(c)** 10.93
(d) 2.99 **(e)** -3.44 **(f)** 11.08

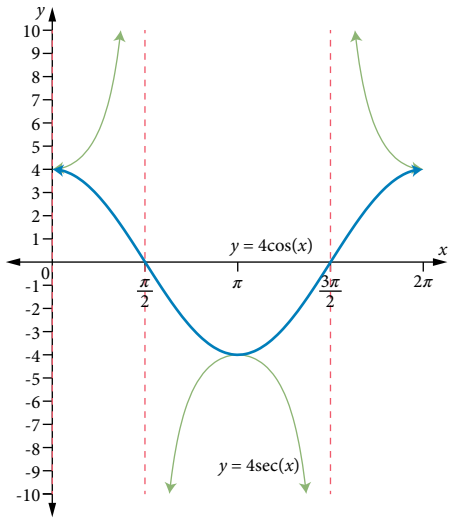
3 **(a)** 4 **(b)** 3 **(c)** $4 + \sqrt{2}$

4 **(a)** $\frac{5\pi}{6}, \frac{11\pi}{6}$ **(b)** $\frac{2\pi}{3}, \frac{4\pi}{3}$
(c) 90°, 330° **(d)** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

5 **(a)** 1.27, 2.84 **(b)** 3.48, 5.94

6 **(a)** $x = \frac{\pi}{2}, \frac{3\pi}{2}$ **(b)** Dilation by a factor of 4 from the y-axis.
(c) $(0, 4)$; no x-intercepts

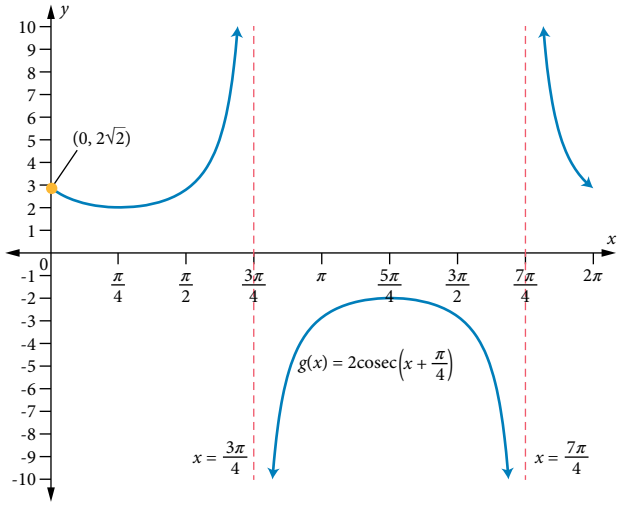
(d)



(e) domain $[0, 2\pi] / \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$; range $\mathbb{R} / (-4, 4)$; period 2π

7 **(a)** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **(b)** $(0, 2\sqrt{2})$; no x-intercepts

(c)



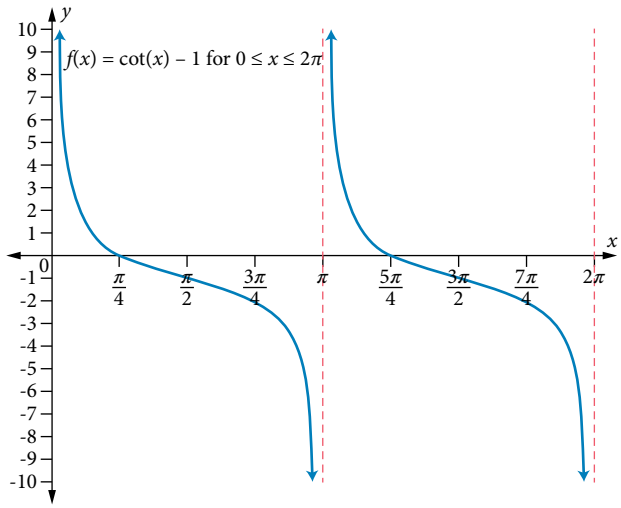
(d) domain $[0, 2\pi] / \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$; range $\mathbb{R} / (-2, 2)$; period: 2π

8 **(a)** Dilation by a factor of 5 from the x-axis. Dilation by a factor of π from the y-axis.
(b) Reflection in the x-axis. Dilation by a factor of $\frac{1}{2}$ from the y-axis. Translation of $\frac{3}{2}$ units to the right.
(c) Translation of $\frac{\pi}{6}$ units to the right. Translation of 4 units down.

9 **(a)** period π^2; range $\mathbb{R}$ **(b)** period π; range $\mathbb{R} / (-1, 1)$
(c) period 2π; range $\mathbb{R} / (-5, -3)$

10 **(a)** $2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}$ or $2n\pi + \frac{\pi}{6}, (2n+1)\pi - \frac{\pi}{6}, n \in \mathbb{Z}$
(b) $n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$ **(c)** $2n\pi \pm \frac{3\pi}{4}, n \in \mathbb{Z}$

11 domain $\mathbb{R} / \{0, \pi, 2\pi\}$; range $\mathbb{R}$; period π



12 C

13 **(a)** D

(b) A common error could be to confuse cosec (θ) with $\cos(\theta)$, and hence to think that $\text{cosec}^2(\theta) = \dfrac{1}{\cos^2(\theta)}$ is correct.

14 **(a)** $y = \text{cosec}(x) + 1$ or $y = \sec\left(x - \dfrac{\pi}{2}\right) + 1$

(b) $y = \sec(2x)$ or $y = \text{cosec}\left(2\left(x + \dfrac{\pi}{4}\right)\right)$

(c) $y = \cot\left(\dfrac{x}{2}\right)$ or $y = -\tan\left(\dfrac{1}{2}(x - \pi)\right)$

15 **(a)** 1 **(b)** 1

(c) $\sec(\theta) = \dfrac{c}{a}$

$\tan(\theta) = \dfrac{b}{a}$

$\sec^2(\theta) - \tan^2(\theta) = \left(\dfrac{c}{a}\right)^2 - \left(\dfrac{b}{a}\right)^2$

$= \dfrac{c^2 - b^2}{a^2}$

Pythagoras' theorem tells you that $c^2 - b^2 = a^2$.

Therefore,

$\sec^2(\theta) - \tan^2(\theta) = \dfrac{a^2}{a^2}$

$= 1$

This can also be proved for any angle θ, not just in the first quadrant.

16 **(a)** $x = \dfrac{(3n-1)\pi}{6}$ **(b)** $x = \dfrac{(6n-2)\pi}{9}, x = \dfrac{(6n-1)\pi}{9}$

17 **(a)** $a = \dfrac{2\pi}{3}, b = 2; h(d) = \sec\left(d + \dfrac{2\pi}{3}\right) + 2$

(b) **(i)** $-\sqrt{2} + 2$ metres **(ii)** $\dfrac{6 - 2\sqrt{3}}{3}$ metres

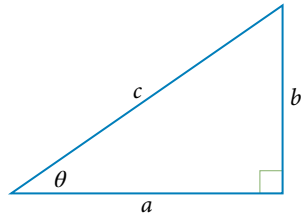
(c) 21 cm and 189 cm, to the nearest centimetre

(d) $H(d) = \sec\left(d + \dfrac{7\pi}{12}\right) + 2$ **(e)** $\dfrac{3\pi}{8}$ metres

18 **(a)** $\left(\dfrac{\pi}{4}, \pi\right) \cup \left(\dfrac{5\pi}{4}, 2\pi\right)$ **(b)** $\left[\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right] \cup (\pi, 2\pi)$

19 **(a)** $\sec(-\theta) = \dfrac{1}{\cos(-\theta)} = \dfrac{1}{\cos(\theta)} = \sec(\theta)$

(b)



In the right-angled triangle shown,

$\sin(\theta) = \dfrac{b}{c}$

$\text{cosec}(\theta) = \dfrac{c}{a}$

and

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{a}{c}$

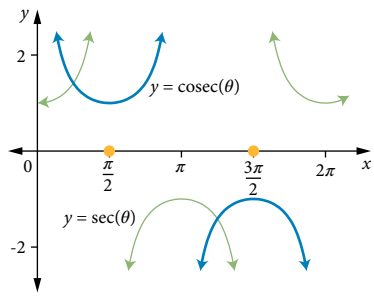
$\sec\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{c}{a}$

$= \text{cosec}(\theta)$

(c) translation of $\dfrac{\pi}{2}$ units to the right

(d) The second graph is found from the first by performing the translation in part **(c)**.

(e)



EXERCISE 7.4

1 $\sin(\theta) = \dfrac{4}{5}$

2 $\cos(\theta) = \pm\dfrac{5}{13}; \tan(\theta) = \pm\dfrac{12}{5}$

3 $\text{cosec}(\theta) = \dfrac{\sqrt{13}}{2}; \cot(\theta) = \dfrac{-3}{2}$

4 **(a)** $\sin(\theta)$ **(b)** 1 **(c)** $\sin(\theta)$

(d) 1 **(e)** $\sin^2(\theta)$ **(f)** $\cot^2(\theta)$

(g) $\sec(\theta)$

5 **(a)** 1 **(b)** 1 **(c)** $\sin^2(\theta)$

6 **(a)** $x = 0, 2\pi$ **(b)** $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

(c) $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$ **(d)** $x = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

7 **(a)** Start with the left-hand side:

$$1+\frac{\sin^2(\theta)}{\cos^2(\theta)}=1+\tan^2(\theta)$$
$$=\sec^2(\theta)$$

which is equal to the right-hand side.

(b) Start with the left-hand side:

$$\left(1-\sin^2(\theta)\right)\left(\tan^2(\theta)+1\right)=\cos^2(\theta)\times\sec^2(\theta)$$
$$=\frac{\cos^2(\theta)}{\cos^2(\theta)}$$
$$=1$$

which is equal to the right-hand side.

(c) Start with the left-hand side:

$$\sin^2(\theta)+\cot^2(\theta)\sin^2(\theta)=\sin^2(\theta)\left(1+\cot^2(\theta)\right)$$
$$=\sin^2(\theta)\operatorname{cosec}^2(\theta)$$
$$=\frac{\sin^2(\theta)}{\sin^2(\theta)}$$
$$=1$$

which is equal to the right-hand side.

(d) Start with the left-hand side:

$$\tan^2(\theta)-\sin^2(\theta)=\frac{\sin^2(\theta)}{\cos^2(\theta)}-\sin^2(\theta)$$
$$=\frac{\sin^2(\theta)-\cos^2(\theta)\sin^2(\theta)}{\cos^2(\theta)}$$
$$=\frac{\sin^2(\theta)\left(1-\cos^2(\theta)\right)}{\cos^2(\theta)}$$
$$=\frac{\sin^2(\theta)\sin^2(\theta)}{\cos^2(\theta)}$$
$$=\sin^4(\theta)\sec^2(\theta)$$

which is equal to the right-hand side.

(e) Start with the left-hand side:

$$\cot(\theta)\left(\cos(\theta)-\sec(\theta)\right)=\frac{\cos(\theta)}{\sin(\theta)}\left(\cos(\theta)-\frac{1}{\cos(\theta)}\right)$$
$$=\frac{\cos(\theta)}{\sin(\theta)}\left(\frac{\cos^2(\theta)-1}{\cos(\theta)}\right)$$
$$=\frac{\cos(\theta)}{\sin(\theta)}\left(\frac{-\sin^2(\theta)}{\cos(\theta)}\right)$$
$$=-\sin(\theta)$$

which is equal to the right-hand side.

8 $\cos^2(\theta)$

9 $\tan(\theta)$

10 $\frac{\sqrt{9-x^2}}{3}$

11 $\tan(\theta)=-\sqrt{4x^2-1}$

12 B

13 (a) D

(b) A common error could be to confuse the expressions $a^2\sin^2(\theta)$ and $a^2\left(1-\sin^2(\theta)\right)$.

14 (a) Start with the left-hand side:

$$\frac{\cos^2(\theta)}{1-\sin(\theta)}-\frac{\cos^2(\theta)}{1+\sin(\theta)}$$
$$=\frac{\cos^2(\theta)\left(1+\sin(\theta)\right)-\cos^2(\theta)\left(1-\sin(\theta)\right)}{\left(1-\sin(\theta)\right)\left(1+\sin(\theta)\right)}$$
$$=\frac{\cos^2(\theta)+\cos^2(\theta)\sin(\theta)-\cos^2(\theta)+\cos^2(\theta)\sin(\theta)}{1-\sin^2(\theta)}$$
$$=\frac{2\cos^2(\theta)\sin(\theta)}{\cos^2(\theta)}$$
$$=2\sin(\theta)$$

which is equal to the right-hand side.

(b) Start with the left-hand side:

$$\frac{1+\cos(\theta)}{\sin(\theta)}=\frac{1+\cos(\theta)}{\sin(\theta)}\times\frac{1-\cos(\theta)}{1-\cos(\theta)}$$
$$=\frac{1-\cos^2(\theta)}{\sin(\theta)\left(1-\cos(\theta)\right)}$$
$$=\frac{\sin^2(\theta)}{\sin(\theta)\left(1-\cos(\theta)\right)}$$

which is equal to the right-hand side.

(c) Start with the left-hand side:

$$\frac{2\left(1+\sin(\theta)\right)}{\cos(\theta)}+\frac{2\cos(\theta)}{1+\sin(\theta)}$$
$$=\frac{2\left(1+\sin(\theta)\right)\left(1+\sin(\theta)\right)+2\cos(\theta)\cos(\theta)}{\cos(\theta)\left(1+\sin(\theta)\right)}$$
$$=\frac{2\left(1+2\sin(\theta)+\sin^2(\theta)\right)+2\cos^2(\theta)}{\cos(\theta)\left(1+\sin(\theta)\right)}$$
$$=\frac{2+4\sin(\theta)+2\sin^2(\theta)+2\cos^2(\theta)}{\cos(\theta)\left(1+\sin(\theta)\right)}$$
$$=\frac{2+4\sin(\theta)+2}{\cos(\theta)\left(1+\sin(\theta)\right)}$$
$$=\frac{4+4\sin(\theta)}{\cos(\theta)\left(1+\sin(\theta)\right)}$$
$$=\frac{4\left(\sin(\theta)+1\right)}{\cos(\theta)\left(1+\sin(\theta)\right)}$$
$$=\frac{4}{\cos(\theta)}$$
$$=4\sec(\theta)$$

which is equal to the right-hand side.

15 (a) $1-2\cos^2(\theta)$ **(b)** $2-\cos^2(\theta)$

(c) $\frac{1-\cos^2(\theta)}{\cos^2(\theta)}$ **(d)** $\frac{2-\cos^2(\theta)}{\cos^2(\theta)}$

16 (a) $\frac{1}{2}$ **(b)** $\frac{5}{4}$ **(c)** 3 **(d)** 7

17 $x=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right),\frac{1}{2}\tan^{-1}(2)$

18 (a) $\frac{k+m}{n+m}$ **(b)** $\frac{n-k}{n+m}$

19 (a) $\cos(x)$ (b) $x = n\pi \pm \frac{\pi}{3}$

20 (a) $x = \pm\frac{b\sqrt{(a-y)(a+y)}}{a}$ (b) $y = \pm\frac{a\sqrt{(b-x)(b+x)}}{b}$

(c) $a = 4,\ b = \frac{4\sqrt{3}}{3}$ (d) $\tan(\theta) = \pm\frac{\sqrt{b^2-1}}{b^2-1}$

(e) $\tan(\theta) = \pm\sqrt{a^2-1}$

21 (a) $x^2 + y^2 = 144$. The path of the pedals is a circle, centred at the drive mechanism, radius 12 cm.

(b) 3 revolutions per second

(c) $\frac{x^2}{196} + \frac{y^2}{100} = 1$. The path of the pedals is an ellipse with horizontal axis 28 cm and vertical axis 20 cm.

(d) 2 revolutions per second

EXERCISE 7.5

1 (a) $\sin\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)\cos\left(\frac{\pi}{2}\right) - \cos(\theta)\sin\left(\frac{\pi}{2}\right)$
$= \sin(\theta)\times 0 - \cos(\theta)\times 1$
$= -\cos(\theta)$

(b) $\cos\left(\frac{\pi}{2} + \theta\right) = \cos\left(\frac{\pi}{2}\right)\cos(\theta) - \sin\left(\frac{\pi}{2}\right)\sin(\theta)$
$= 0\times\cos(\theta) - 1\times\sin(\theta)$
$= -\sin(\theta)$

(c) $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan(\theta) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan(\theta)\tan\left(\frac{\pi}{4}\right)}$
$= \frac{\tan(\theta)+1}{1-\tan(\theta)\times 1}$
$= \frac{1+\tan(\theta)}{1-\tan(\theta)}$

(d) $\cos\left(\frac{5\pi}{3} - \theta\right) = \cos\left(\frac{5\pi}{3}\right)\cos(\theta) + \sin\left(\frac{5\pi}{3}\right)\sin(\theta)$
$= \frac{1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta)$
$= \frac{1}{2}\left(\cos(\theta) + \sqrt{3}\sin(\theta)\right)$

2 $\frac{1}{20}\left(9 + 4\sqrt{7}\right)$

3 $\frac{1}{26}\left(5\sqrt{3} - 12\right)$

4 $\frac{63}{16}$

5 (a) $\frac{1}{2}\left(\sqrt{3}\cos(x) - \sin(x)\right)$ (b) $\frac{\sqrt{2}}{2}\left(\cos(x) - \sin(x)\right)$

(c) $\frac{\sqrt{3}\tan(x) - 1}{\sqrt{3} + \tan(x)}$ (d) $\frac{1}{2}\left(\sqrt{3}\sin(x) - \cos(x)\right)$

6 (a) $\frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$ (b) $\frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

7 (a) $\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$ (b) $2 - \sqrt{3}$

(c) $\frac{\sqrt{2}\,(1+\sqrt{3}\,)}{4}$ (d) $\frac{\sqrt{2}\,(1-\sqrt{3}\,)}{4}$

8 (a) $\sin(2\alpha)$ (b) $\cos(3\beta)$ (c) $\cos(2x)$

(d) $-\sin(x)$ (e) $\sin(2\alpha - \beta)$ (f) $\cos(2\alpha)$

(g) $\sin(y)$ (h) $\cos(y)$

9 (a) 1 (b) 0 (c) $-\frac{\sqrt{3}}{2}$ (d) $-\sqrt{3}$

10 (a) $\frac{1}{2}\sin(\theta)$ (b) $\cos(4A)$ (c) $1 + \sin(2x)$ (d) $1 - \sin(6x)$

11 (a) $\cos^4(x) - \sin^4(x)$
$= \left(\cos^2(x) + \sin^2(x)\right)\left(\cos^2(x) - \sin^2(x)\right)$
$= 1\times\left(\cos^2(x) - \sin^2(x)\right)$
$= \cos(2x)$

(b) $\frac{1}{2}\left(\cos(2x) + 1\right) = \frac{1}{2}\left(2\cos^2(x) - 1 + 1\right)$
$= \frac{1}{2}\times\left(2\cos^2(x)\right)$
$= \cos^2(x)$

(c) $\frac{1+\cos(2A)}{\sin(2A)} = \frac{1 + 2\cos^2(A) - 1}{2\sin(A)\cos(A)}$
$= \frac{2\cos^2(A)}{2\sin(A)\cos(A)}$
$= \frac{\cos(A)}{\sin(A)}$
$= \cot(A)$

12 (a) $x = 0, \pi, 2\pi$ or $\frac{\pi}{3}, \frac{5\pi}{3}$ (b) $x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\frac{\pi}{6}, \frac{5\pi}{6}$

13 (a) Consider the left-hand side:
$\sin(3x) = \sin(2x + x)$
$= \sin(2x)\cos(x) + \cos(2x)\sin(x)$
$= 2\sin(x)\cos(x)\cos(x) + (1 - 2\sin^2(x))\sin(x)$
$= 2\sin(x)\cos^2(x) + \sin(x) - 2\sin^3(x)$
$= 2\sin(x)(1 - \sin^2(x)) + \sin(x) - 2\sin^3(x)$
$= 2\sin(x) - 2\sin^3(x) + \sin(x) - 2\sin^3(x)$
$= 3\sin(x) - 4\sin^3(x)$

(b) $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$

14 $\tan(2\theta) = \frac{4}{3};\ \tan(4\theta) = -\frac{24}{7}$

15 (a) $\cos(2x) = 2\cos^2(x) - 1$
$2\cos^2(x) - 1 = \cos(2x)$
$2\cos^2(x) = 1 + \cos(2x)$
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
$\cos(x) = \pm\sqrt{\frac{1+\cos(2x)}{2}}$

(b) $\frac{3\sqrt{10}}{10}$

16 (a) Consider the left-hand side:

$$\begin{aligned}\cot(x)-\tan(x) &= \frac{\cos(x)}{\sin(x)}-\frac{\sin(x)}{\cos(x)}\\ &= \frac{\cos^2(x)-\sin^2(x)}{\sin(x)\cos(x)}\\ &= \frac{\cos(2x)}{\frac{1}{2}\sin(2x)}\\ &= 2\times\frac{\cos(2x)}{\sin(2x)}\\ &= 2\cot(2x)\end{aligned}$$

(b) Consider the left-hand side:

$$\begin{aligned}\frac{\cos(2\theta)+2\sin^2(\theta)}{\cos(\theta)} &= \frac{2\cos^2(\theta)-1+2\sin^2(\theta)}{\cos(\theta)}\\ &= \frac{2\left(\cos^2(\theta)+\sin^2(\theta)\right)-1}{\cos(\theta)}\\ &= \frac{2-1}{\cos(\theta)}\\ &= \sec(\theta)\end{aligned}$$

(c) Consider the left-hand side:

$$\begin{aligned}&\left(1-\tan(\theta)\right)^2+\left(1+\tan(\theta)\right)^2\\ &= 1-2\tan(\theta)+\tan^2(\theta)+1+2\tan(\theta)+\tan^2(\theta)\\ &= 2+2\tan^2(\theta)\\ &= 2\left(1+\tan^2(\theta)\right)\\ &= 2\sec^2(\theta)\end{aligned}$$

(d) Consider the left-hand side:

$$\begin{aligned}&\frac{1}{1-\sin(x)}+\frac{1}{1+\sin(x)}\\ &= \frac{1+\sin(x)+1-\sin(x)}{1-\sin^2(x)}\\ &= \frac{2}{\cos^2(x)}\\ &= 2\sec^2(x)\end{aligned}$$

17 (a) $x=0,\frac{\pi}{6},\frac{5\pi}{6},\pi,2\pi$ **(b)** $x=\frac{3\pi}{8},\frac{7\pi}{8},\frac{11\pi}{8},\frac{15\pi}{8}$

18 $x=1.263, 4.405$

19 (a) $\cos(2A)+\cos(4B)$ **(b)** $\frac{1}{2}\left(\sin(4A)+\sin(2A+2B)\right)$

20 $$\begin{aligned}\text{LHS} &= 2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{7\pi}{6}\right)\\ &= 2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \cos\left(\frac{-3\pi}{6}\right)-\cos\left(\frac{11\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{11\pi}{6}\right)\\ &= 0-\frac{\sqrt{3}}{2}\\ &= -\frac{\sqrt{3}}{2}\\ &= \text{LHS}\end{aligned}$$

21 (a) $a=\frac{X+Y}{2}, b=\frac{X-Y}{2}$

(b) $2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)=\sin(X)+\sin(Y)$

(c) $2\sin(3\theta)\cos(2\theta)$

22 (a) $8\cos^4(x)-8\cos^2(x)+1$

(b) $4\sin(x)\cos(x)-8\sin^3(x)\cos(x)$

23 $$\begin{aligned}&\sin^2(A)\cos^2(B)-\cos^2(A)\sin^2(B)\\ &= (\sin(A)\cos(B)-\cos(A)\sin(B))(\sin(A)\cos(B)+\cos(A)\sin(B))\\ &= \sin(A-B)\sin(A+B)\\ &= \frac{1}{2}(\cos(A-B-A-B)-\cos(A-B+A+B))\\ &= \frac{1}{2}(\cos(2B)-\cos(2A)) \text{ since } \cos(-2B)=\cos(2B)\\ &= \frac{1}{2}\left(1-2\sin^2(B)-1+2\sin^2(A)\right)\\ &= \sin^2(A)-\sin^2(B)\end{aligned}$$

24 (a) $n=\frac{1}{2}\left(\sqrt{3}+\cot\left(\frac{\theta}{2}\right)\right)$

(b) $\theta=1.34$ **(c)** 2.07

25 (a) $x=\frac{1}{10}v^2\sin(2\theta)$ **(b)** 10 metres

(c) $\theta=45°$ **(d)** $28\,\text{m s}^{-1}$

EXERCISE 7.6

1 (a) $2\sin\left(x+\frac{\pi}{3}\right)$ **(b)** $2\sin\left(x-\frac{\pi}{6}\right)$

(c) $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ **(d)** $5\sqrt{2}\cos\left(2x+\frac{\pi}{4}\right)$

(e) $\sin(x)-2\cos(x)=\sqrt{5}\sin\left(x-\tan^{-1}(2)\right)$

2 (a) $\sqrt{13}\cos(x-0.59)$ **(b)** $5\sqrt{2}\sin(x-1.43)$

3 (a) $x=\frac{\pi}{6},\frac{\pi}{2}$ **(b)** $x=\frac{\pi}{2},\pi$

(c) $x=\frac{17\pi}{12},\frac{23\pi}{12}$ **(d)** $x=\frac{\pi}{12},\frac{17\pi}{12}$

4 (a) period 2π; amplitude 5 **(b)** period 8; amplitude $\sqrt{37}$

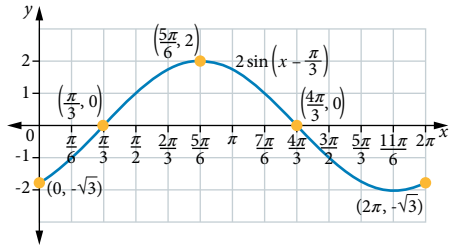
(c) period $\frac{2\pi}{3}$; amplitude $\sqrt{7}$

5 (a) maximum (0.644, 5); minimum (3.785, -5)

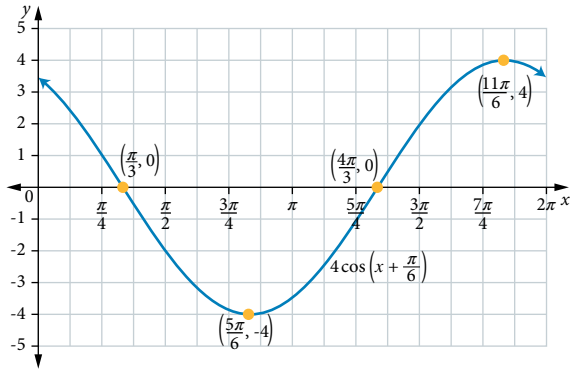
(b) maximum (6.21, 6.083), minimum (2.21, -6.083)

(c) maximum (0.859, 2.646), minimum (1.906, -2.646)

6 $2\sin\left(x-\frac{\pi}{3}\right)$; period 2π; amplitude 2



7 $4\cos\left(x+\frac{\pi}{6}\right)$; period 2π; amplitude 4



8 D

9 (a) $4\sqrt{3}\sin\left(x-\frac{\pi}{6}\right)$ (b) The two graphs are identical.

10 (a) $\frac{\sqrt{2}}{12}\sec\left(x-\frac{\pi}{4}\right)$ (b) $\frac{1}{2}\tan\left(5x+\frac{\pi}{3}\right)$

11 (a) $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

(b) $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin(x)\cos\left(\frac{\pi}{4}\right)-\sqrt{2}\cos(x)\sin\left(\frac{\pi}{4}\right)$

$=\frac{\sqrt{2}}{\sqrt{2}}\sin(x)-\frac{\sqrt{2}}{\sqrt{2}}\cos(x)$

$=\sin(x)-\cos(x)$

12 $-\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

13 (a) $f(\theta)=25\sin(\theta+0.284)$ (b) $(24\sin(\theta), 7\cos(\theta))$

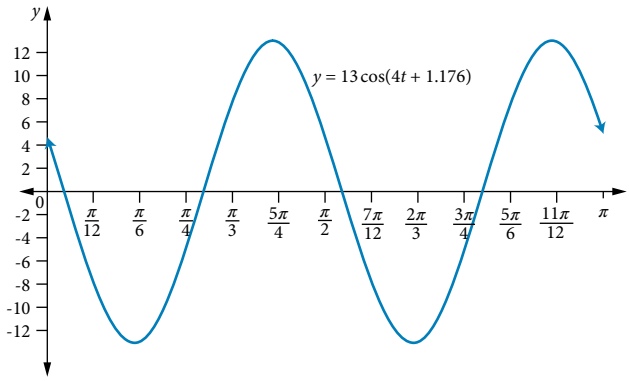
(c) perimeter $=2f(\theta)$ (d) 50 (e) $B=(23.04, 1.96)$

14 $x=\frac{2\pi}{3}, 2\pi, \frac{8\pi}{3}, 4\pi, \ldots$

$x=2n\pi+\frac{2\pi}{3}$ or $2n\pi, n\in\mathbb{Z}$

15 (a) 5 metres in the positive direction (b) $y=13\cos(4t+1.176)$

(c)



(d) 13 metres (e) 104 metres

16 (a) $\sin(a+b)=\frac{BC}{AC}$

$=\frac{BC}{x}$

$BC=x\sin(a+b)$

(b) $BC=x\sin(a)\cos(b)+x\cos(a)\sin(b)$

(c) $AB=x(\cos(a)\cos(b)-\sin(a)\sin(b))$

(d) $\frac{y}{x}=\frac{\cos(a)\cos(b)-\sin(a)\sin(b)}{\cos(a)}$ (e) $5\left(\sqrt{3}-1\right)$

(f) $CD=x\sin(a)\cos(b)+x\cos(a)\sin(b)-y\sin(a)$ (g) $5\sqrt{2}$

CHAPTER REVIEW 7

1 (a) $x=\frac{\pi}{2}, \frac{7\pi}{6}$ (b) $x=\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

(c) $x=0, \frac{\pi}{2}, 2\pi, \ldots$ (d) $x=\frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$

(e) $x=\frac{5\pi}{12}, \frac{17\pi}{12}$ (f) $x=\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

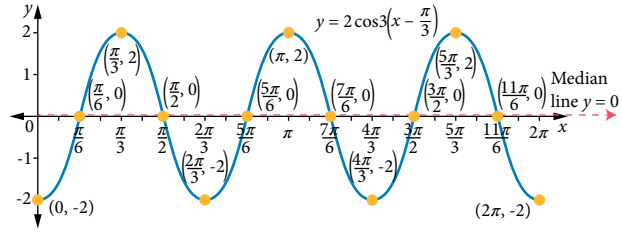
2 (a) $x=1.02, 3.12, 4.17, 6.26$

(b) $x=1.03, 1.80, 3.13, 3.90, 5.22, 5.99$

(c) $x=2.32, 2.96, 5.46, 6.10$

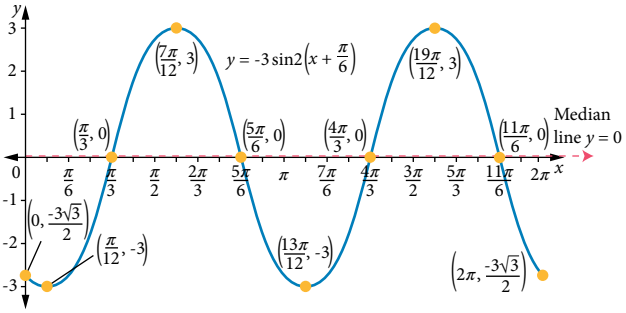
3 (a) period $\frac{2\pi}{3}$; amplitude 2; axis intercepts $(0, -2)\left(\frac{\pi}{6}, 0\right)$,

$\left(\frac{\pi}{2}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

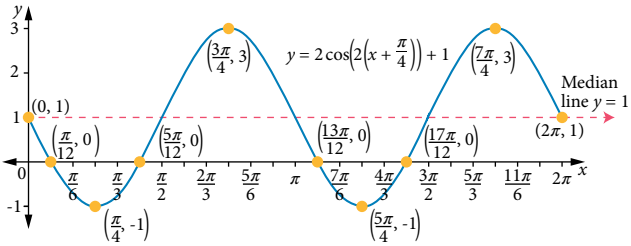


(b) period π; amplitude 3; axis intercepts $\left(0, -\frac{3\sqrt{3}}{2}\right)$,

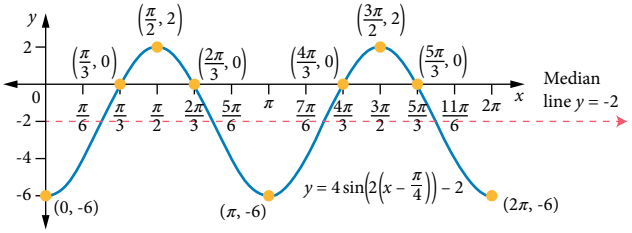
$\left(\frac{\pi}{3}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{4\pi}{3}, 0\right), \left(\frac{11\pi}{6}, 0\right)$



(c) period π; amplitude 2; axis intercepts (0,1) $\left(\frac{\pi}{12}, 0\right)$, $\left(\frac{5\pi}{12}, 0\right)$, $\left(\frac{13\pi}{12}, 0\right)$, $\left(\frac{17\pi}{12}, 0\right)$



(d) period π; amplitude 4; axis intercepts (0, -6) $\left(\frac{\pi}{3}, 0\right)$, $\left(\frac{2\pi}{3}, 0\right)$, $\left(\frac{4\pi}{3}, 0\right)$, $\left(\frac{5\pi}{3}, 0\right)$

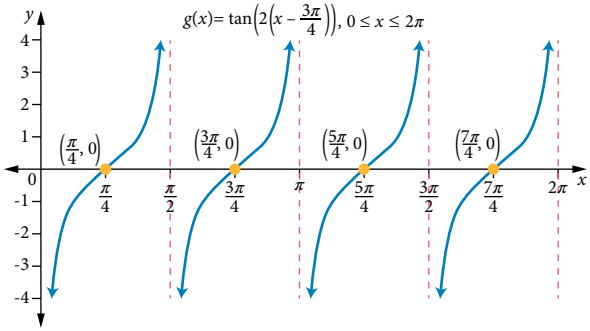


4 (a) period $\frac{\pi}{2}$; range: $\mathbb{R}$

(b) Dilation by a factor of $\frac{1}{2}$ from the y-axis. Translation $\frac{3\pi}{4}$ units to the right.

(c) asymptotes $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$, $x = 2\pi$; x-intercepts $\left(\frac{\pi}{4}, 0\right)$, $\left(\frac{3\pi}{4}, 0\right)$, $\left(\frac{5\pi}{4}, 0\right)$, $\left(\frac{7\pi}{4}, 0\right)$

(d)

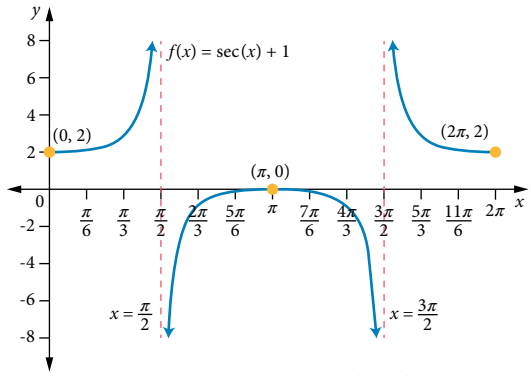


5 (a) $x = \frac{\pi}{6}, \frac{7\pi}{6}$ (b) $x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

(c) $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ (d) $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

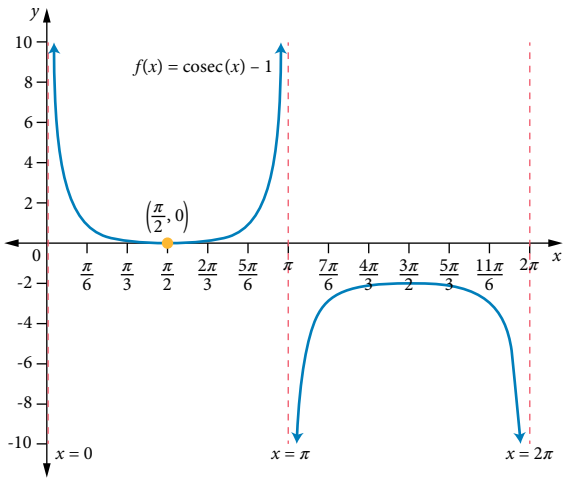
6 (a) (i) range $(-\infty, 0] \cup [2, \infty)$; x-intercept $(\pi, 0)$; y-intercept (0, 2); asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(ii)



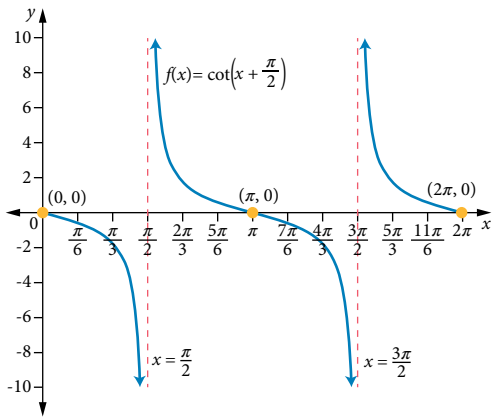
(b) (i) range $(-\infty, -2] \cup [0, \infty)$; x-intercept $\left(\frac{\pi}{2}, 0\right)$; asymptotes $x = 0$, $x = \pi$, $x = 2\pi$

(ii)



(c) (i) range $\mathbb{R}$; x-intercept (0, 0), $(\pi, 0)$, $(2\pi, 0)$; y-intercept (0, 0); asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(ii)



7 (a) $\dfrac{1+\sin(\theta)}{\cos(\theta)}$ (b) $\dfrac{1+\cos(\theta)}{\sin(\theta)}$ (c) $\sin(\theta)-\cos(\theta)$

(d) $2\cos^2(\theta)-1$ or $1-2\sin^2(\theta)$ or $\cos(2\theta)$

(e) 1 (f) $\tan^2(\theta)$

8 (a) $\sec^2(\theta)-\tan^2(\theta)=\dfrac{1}{\cos^2(\theta)}-\dfrac{\sin^2(\theta)}{\cos^2(\theta)}$

$=\dfrac{1-\sin^2(\theta)}{\cos^2(\theta)}$

$=\dfrac{\cos^2(\theta)}{\cos^2(\theta)}$

$=1$

(b) $\cos^3(\theta)+\cos(\theta)\sin^2(\theta)=\cos(\theta)\left(\cos^2(\theta)+\sin^2(\theta)\right)$

$=\cos(\theta)\times 1$

$=\cos(\theta)$

(c) $\cot^2(\theta)\sin^2(\theta)+\sin^2(\theta)=\dfrac{\cos^2(\theta)}{\sin^2(\theta)}\times\sin^2(\theta)+\sin^2(\theta)$

$=\cos^2(\theta)+\sin^2(\theta)$

$=1$

(d) $\dfrac{2}{1+\sin(\theta)}+\dfrac{2}{1-\sin(\theta)}=\dfrac{2\left(1-\sin(\theta)\right)+2\left(1+\sin(\theta)\right)}{1-\sin^2(\theta)}$

$=\dfrac{2-2\sin(\theta)+2+2\sin(\theta)}{\cos^2(\theta)}$

$=\dfrac{4}{\cos^2(\theta)}$

$=4\sec^2(\theta)$

(e) $\sin^4(\theta)-\cos^4(\theta)$

$=\left(\sin^2(\theta)+\cos^2(\theta)\right)\left(\sin^2(\theta)-\cos^2(\theta)\right)$

$=1\times\left(\sin^2(\theta)-\cos^2(\theta)\right)$

$=-1\left(\cos^2(\theta)-\sin^2(\theta)\right)$

$=-1\left(1-2\sin^2(\theta)\right)$

$=2\sin^2(\theta)-1$

(f) $\tan^2(\theta)-\sin^2(\theta)=\dfrac{\sin^2(\theta)}{\cos^2(\theta)}-\sin^2(\theta)$

$=\dfrac{\sin^2(\theta)-\sin^2(\theta)\cos^2(\theta)}{\cos^2(\theta)}$

$=\dfrac{\sin^2(\theta)\left(1-\cos^2(\theta)\right)}{\cos^2(\theta)}$

$=\dfrac{\sin^2(\theta)\times\sin^2(\theta)}{\cos^2(\theta)}$

$=\dfrac{\sin^4(\theta)}{\cos^2(\theta)}$

$=\sin^4(\theta)\sec^2(\theta)$

9 (a) $\sin(x)\cos(3y)-\cos(x)\sin(3y)$

(b) $\cos(2x)\cos(y)-\sin(2x)\sin(y)$

(c) $\dfrac{\tan(x)+\tan(3y)}{1-\tan(x)\tan(3y)}$

10 (a) $\dfrac{1}{2}\left(\cos(x)+\sqrt{3}\sin(x)\right)$ (b) $\dfrac{1}{2}\left(\cos(x)+\sqrt{3}\sin(x)\right)$

(c) $\dfrac{\tan(x)+1}{1-\tan(x)}$

11 (a) $\cos(2x)$ (b) $\sin(6x)$ (c) $\sin(4x)$

(d) $\cos(2x)$ (e) $-\cos(x)$ (f) $\tan(x)$

12 (a) $\dfrac{1}{2}\sin(14\theta)$ (b) $\cos\left(\dfrac{A}{2}\right)$ (c) $\tan(10x)$

13 (a) $\sin(6A)+\sin(4A)$ (b) $\sin(8A)-\sin(4A)$

(c) $2(\cos(8x)+\cos(2x))$ (d) $4(\cos(6x)-\cos(10x))$

14 (a) $x=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ or $x=\dfrac{\pi}{6},\dfrac{5\pi}{6}$ (b) $x=\pi$ or $x=\dfrac{\pi}{3},\dfrac{5\pi}{3}$

(c) $x=0,\pi,2\pi$ or $x=\dfrac{\pi}{2}$ (d) $x=\dfrac{\pi}{3},\dfrac{2\pi}{3},\dfrac{4\pi}{3},\dfrac{5\pi}{3}$

15 (a) $\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)$ (b) $2\cos\left(x-\dfrac{\pi}{3}\right)$

(c) $2\cos\left(x+\dfrac{\pi}{6}\right)$ (d) $\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)$

16 $5\sqrt{2}\sin\left(x^\circ+8.13^\circ\right)$

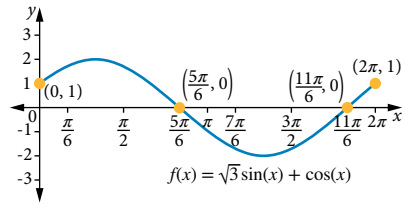
17 (a) $x=\dfrac{5\pi}{12},\dfrac{13\pi}{12}$ (b) $x=\dfrac{5\pi}{3}$

18 (a) period 2π; amplitude $\sqrt{29}$ (b) period $\dfrac{2\pi}{3}$; amplitude 6.46

19 (a) $f(x)=2\sin\left(x+\dfrac{\pi}{6}\right)$ (b) $x=\dfrac{5\pi}{6}$, $x=\dfrac{11\pi}{6}$

(c) $x=\dfrac{\pi}{3}$, $x=\dfrac{4\pi}{3}$

(d) period 2π; amplitude 2



20 (a) $\dfrac{4}{5}$ (b) $\dfrac{5}{3}$ (c) $\dfrac{4}{3}$ (d) $\dfrac{7}{25}$

21 (a) 1.279 (b) $\sin\left(104^\circ\right)+\sin\left(18^\circ\right)$

(c) $2\sin\left(61^\circ\right)\cos\left(43^\circ\right)=1.279\,31$

$\sin\left(104^\circ\right)+\sin\left(18^\circ\right)=1.279\,31$

22 (a) $\dfrac{\sqrt{5}}{3}$ (b) $-\dfrac{\sqrt{5}}{2}$ (c) $\dfrac{\sqrt{5}}{6}$

23 A

24 C

25 (a) $\sec(\theta)+\tan(\theta)=\dfrac{1}{\cos(\theta)}+\dfrac{\sin(\theta)}{\cos(\theta)}$

$=\dfrac{1+\sin(\theta)}{\cos(\theta)}$

$=\dfrac{\cos(\theta)\left(1+\sin(\theta)\right)}{\cos^2(\theta)}$

$=\dfrac{\cos(\theta)\left(1+\sin(\theta)\right)}{1-\sin^2(\theta)}$

$=\dfrac{\cos(\theta)\left(1+\sin(\theta)\right)}{\left(1+\sin(\theta)\right)\left(1-\sin(\theta)\right)}$

$=\dfrac{\cos(\theta)}{1-\sin(\theta)}$

(b) $\dfrac{\sin(\theta)+\tan(\theta)}{1+\cos(\theta)}=\dfrac{\sin(\theta)+\frac{\sin(\theta)}{\cos(\theta)}}{1+\cos(\theta)}$

$=\dfrac{\frac{\sin(\theta)\cos(\theta)+\sin(\theta)}{\cos(\theta)}}{1+\cos(\theta)}$

$=\dfrac{\sin(\theta)\left(\cos(\theta)+1\right)}{\cos(\theta)\left(1+\cos(\theta)\right)}$

$=\dfrac{\sin(\theta)}{\cos(\theta)}$

$=\tan(\theta)$

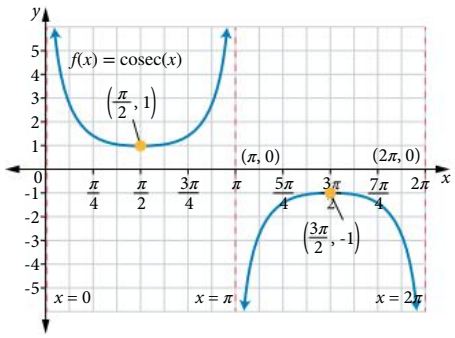
26 $2-\sqrt{3}$

27 (a) $\sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right)$, $\sin\left(\dfrac{2\pi}{3}-\dfrac{\pi}{4}\right)$ **(b)** $\dfrac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$

28 $y=-\tan(2x)$ or $y=\cot\left(2\left(x+\dfrac{\pi}{4}\right)\right)$

29 (a) There are no axis intercepts. Asymptotes at $x=0,\pi,2\pi$.

(b)



(c) axis intercepts $\left(\dfrac{5\pi}{4},0\right)$, $\left(0,\ 1+\sqrt{2}\right)$;

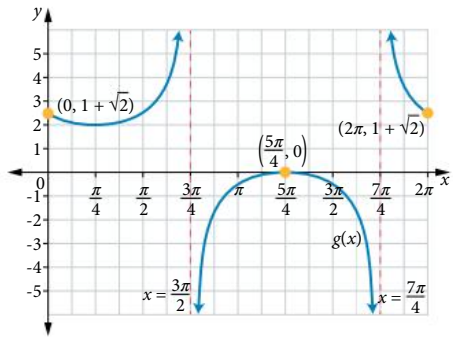
asymptotes $x=\dfrac{3\pi}{4}$, $x=\dfrac{7\pi}{4}$

(d) Domain: $[0,2\pi]\setminus\left\{\dfrac{3\pi}{4},\dfrac{7\pi}{4}\right\}$

Range $(-\infty,0]\cup[2,\infty)$

Period: 2π

(e)



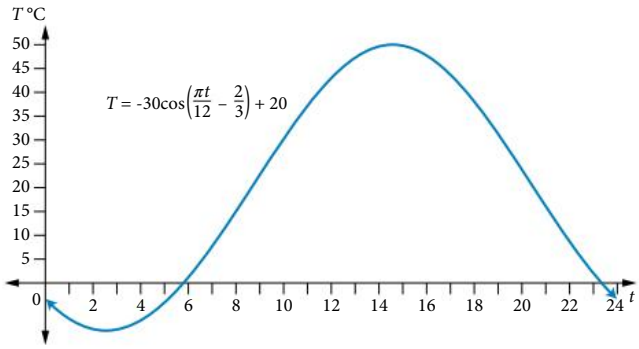
30 (a) 1, 3, 5, 7, 9, 11 seconds **(b)** $\dfrac{2}{3},\dfrac{4}{3},\dfrac{8}{3},\dfrac{10}{3}$ seconds

(c) 0, 2, 4, 6 seconds **(d)** $t\in[0.84,1.16]$

(e) 300 cm

31 (a) Highest temperature is 50°C at 2:33 pm. Lowest temperature is -10°C at 2:33 am.

(b)



(c) from 7:15 am to 9:51 am and from 7:15 pm to 9:51 pm

(d) 5:46 am, 11:20 pm

32 (a) $\left(\cot(\theta)+\operatorname{cosec}(\theta)\right)^2$

$=\cot^2(\theta)+2\cot(\theta)\operatorname{cosec}(\theta)+\operatorname{cosec}^2(\theta)$

$=\dfrac{\cos^2(\theta)}{\sin^2(\theta)}+\dfrac{2\cos(\theta)}{\sin^2(\theta)}+\dfrac{1}{\sin^2(\theta)}$

$=\dfrac{\left(1+\cos(\theta)\right)^2}{\sin^2(\theta)}$

$=\dfrac{\left(1+\cos(\theta)\right)^2}{\left(1-\cos(\theta)\right)\left(1+\cos(\theta)\right)}$

$=\dfrac{1+\cos(\theta)}{1-\cos(\theta)}$

(b) $\dfrac{2}{\sec(\theta)+\operatorname{cosec}(\theta)} \times \dfrac{1+2\sin(2\theta)}{\sin(\theta)+\cos(\theta)}$

$= \dfrac{2}{\frac{1}{\cos(\theta)}+\frac{1}{\sin(\theta)}} \times \dfrac{1+2\sin(2\theta)}{\sin(\theta)+\cos(\theta)}$

$= \dfrac{2}{\frac{\sin(\theta)+\cos(\theta)}{\sin(\theta)\cos(\theta)}} \times \dfrac{1+2\sin(2\theta)}{\sin(\theta)+\cos(\theta)}$

$= \dfrac{2\sin(\theta)\cos(\theta)}{\sin(\theta)+\cos(\theta)} \times \dfrac{1+2\sin(2\theta)}{\sin(\theta)+\cos(\theta)}$

$= \dfrac{2\sin(\theta)\cos(\theta)\left(1+2\sin(2\theta)\right)}{\left(\sin(\theta)+\cos(\theta)\right)^2}$

$= \dfrac{2\sin(\theta)\cos(\theta)\left(1+2\sin(2\theta)\right)}{\sin^2(\theta)+2\sin(\theta)\cos(\theta)+\cos^2(\theta)}$

$= \dfrac{2\sin(\theta)\cos(\theta)\left(1+2\sin(2\theta)\right)}{1+2\sin(\theta)\cos(\theta)}$

$= 2\sin(\theta)\cos(\theta)$

$= \sin(2\theta)$

33 (a) (i) $x = \pm\dfrac{b}{a}\sqrt{a^2-y^2}$ (ii) $y = \pm\dfrac{a}{b}\sqrt{b^2-x^2}$

(b) (i) $\dfrac{1}{2}$ (ii) 1

34 (a) $x = n\pi + \dfrac{\pi}{6},\ n \in \mathbb{Z}$ (b) $x = (2n+1)\pi \pm \dfrac{\pi}{4},\ n \in \mathbb{Z}$

(c) $x = (2n+1)\dfrac{\pi}{2}$ or $x = 2n\pi + \dfrac{\pi}{6}, (2n+1)\pi - \dfrac{\pi}{6},\ n \in \mathbb{Z}$

35 (a) $a = \dfrac{X+Y}{2}, b = \dfrac{X-Y}{2}$

(b) $2\cos\left(\dfrac{X+Y}{2}\right)\sin\left(\dfrac{X-Y}{2}\right) = \sin(X) - \sin(Y)$

(c) $-2\cos(2\theta)\sin(\theta)$

36 (a) $x\cos(a)$ (b) $y\cos(a-b)$

(c) $x\cos(a) = y\cos(a-b)$

$= y\left(\cos(a)\cos(b)+\sin(a)\sin(b)\right)$

$\dfrac{x\cos(a)}{y} = \cos(a)\cos(b)+\sin(a)\sin(b)$

$\dfrac{x}{y} = \cos(b) + \dfrac{\sin(a)\sin(b)}{\cos(a)}$

$= \cos(b) + \tan(a)\sin(b)$

$\dfrac{x}{y} - \cos(b) = \tan(a)\sin(b)$

$\dfrac{\frac{x}{y}-\cos(b)}{\sin(b)} = \tan(a)$

$\tan(a) = \operatorname{cosec}(b)\left(\dfrac{x}{y} - \cos(b)\right)$

(d) $\tan(a) = \sqrt{3}$

(e) $\tan(a) = \operatorname{cosec}(b)\left(\dfrac{x}{y} - \cos(b)\right)$

$= \operatorname{cosec}\left(\dfrac{\pi}{6}\right)\left(\dfrac{x}{\frac{1}{2}x} - \cos\left(\dfrac{\pi}{6}\right)\right)$

$= 2\left(2 - \dfrac{\sqrt{3}}{2}\right)$

$= 4 - \sqrt{3}$

(f) $\sqrt{3}$; a and b are equal, and each has the value $\dfrac{\pi}{3}$.

Chapter 8 Introduction to matrices

RECALL

1 (a) $\cos(\theta) = \dfrac{1}{\sqrt{1+m^2}}$; $\sin(\theta) = \dfrac{m}{\sqrt{1+m^2}}$

(b) LHS $= \cos^2(\theta) + \sin^2(\theta)$

$= \left(\dfrac{1}{\sqrt{1+m^2}}\right)^2 + \left(\dfrac{m}{\sqrt{1+m^2}}\right)^2$

$= \dfrac{1}{1+m^2} + \dfrac{m^2}{1+m^2}$

$= \dfrac{1+m^2}{1+m^2}$

$= 1$

$=$ RHS

2 (a) $y = \dfrac{1}{2}x - 3$ (b) $3x + 2y - 6 = 0$

(c) $y - 1 = 3(x-2)$ (d) $\dfrac{x}{3} - \dfrac{y}{2} = 1$

3 (a) $2\sqrt{2}$ (b) $M = (4, 3)$ (c) $m = -1$

(d) $y - 4 = -(x-3)$ (e) $y = -x + 7$

4 (a) A, B, C are collinear. (b) A, B, C are collinear.

(c) A, B, C are collinear. (d) A, B, C are not collinear.

5 (a) $a = 1$ (b) $a = \dfrac{4}{3}$

(c) $x = -3$ (d) $x = -\dfrac{1}{3}$

6 (a) $x = 0, x = 5$ (b) $x = 6, x = 2$

(c) $a = \dfrac{1}{2}\left(-1+\sqrt{5}\right), a = \dfrac{1}{2}\left(-1-\sqrt{5}\right)$

EXERCISE 8.1

1 (a) 3×2 (b) 1 (c) a_{32}

2 $\mathbf{A} = \begin{bmatrix} 68 & 71 & 79 \\ 59 & 63 & 70 \end{bmatrix}$

3 apple orange

$\begin{bmatrix} 4.90 & 4.65 \\ 5.15 & 4.95 \end{bmatrix}$ Supermarket 1
Supermarket 2

4 C

5 $t = 4, u = 3, v = -1, w = 7$

6 $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{bmatrix}$

7 (a) $\mathbf{T} = \begin{matrix} \\ S \\ T \\ C \\ W \end{matrix} \begin{matrix} \begin{matrix} \text{Jun} & \text{Jul} & \text{Aug} \end{matrix} \\ \begin{bmatrix} 17.0 & 16.8 & 17.3 \\ 21.5 & 20.6 & 21.1 \\ 26.2 & 25.6 & 29.9 \\ 19.6 & 19.4 & 20.1 \end{bmatrix} \end{matrix}$

(b) Cairns (c) Stanthorpe

(d) $t_{32} = 25.6$ represents the average maximum daily temperature (in °C) in Cairns in July.

(e) 0.5°C

8 D

9 (a) $\begin{matrix} \\ \text{Mon} \\ \text{Tue} \\ \text{Wed} \end{matrix} \begin{matrix} \begin{matrix} P & S & C \end{matrix} \\ \begin{bmatrix} 175 & 285 & 160 \\ 265 & 400 & 120 \\ 310 & 370 & 185 \end{bmatrix} \end{matrix}$ (b) 400 (c) 280

10 (a) $\begin{bmatrix} 3 & 4 \\ -1 & -1 \\ 3 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & -4 \\ 1 & 1 \\ -3 & -7 \end{bmatrix}$

11 (a) $\begin{bmatrix} 6 & 1 \\ 4 & 14 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 1 \\ 4 & 14 \end{bmatrix}$

(c) $\begin{bmatrix} 9 & -3 \\ 0 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 2 \\ 2 & 8 \end{bmatrix}$

12 (a) $\begin{bmatrix} 2 & -8 \\ -6 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & -16 \\ 0 & -20 \end{bmatrix}$ (c) $\begin{bmatrix} -7 & 20 \\ 9 & 19 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & -1 \end{bmatrix}$ (e) $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & -1 \end{bmatrix}$ (f) $\begin{bmatrix} 12 & -36 \\ -18 & -33 \end{bmatrix}$

13 (a) $\begin{bmatrix} -1 & -4 & 4 \\ 5 & 3 & 5 \\ 6 & 6 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 & 4 \\ 1 & -3 & -3 \\ -6 & -8 & -11 \end{bmatrix}$

(c) $\begin{bmatrix} 8 & -3 & 8 \\ 0 & -9 & -10 \\ -18 & -23 & -30 \end{bmatrix}$

14 (a) $\mathbf{M} = \begin{matrix} \\ \text{Cafe 1} \\ \text{Cafe 2} \end{matrix} \begin{matrix} \begin{matrix} J & S \end{matrix} \\ \begin{bmatrix} 80 & 180 \\ 160 & 200 \end{bmatrix} \end{matrix} \begin{matrix} 1 \\ 2 \end{matrix}$; $\mathbf{A} = \begin{matrix} \\ \text{Cafe 1} \\ \text{Cafe 2} \end{matrix} \begin{matrix} \begin{matrix} J & S \end{matrix} \\ \begin{bmatrix} 240 & 120 \\ 300 & 80 \end{bmatrix} \end{matrix} \begin{matrix} 1 \\ 2 \end{matrix}$

(b) $\mathbf{M} + \mathbf{A} = \begin{bmatrix} 320 & 300 \\ 460 & 280 \end{bmatrix}$

This matrix represents the total sales figures for juice and salads for each cafe for the two months.

(c) Cafe 2 had better sales for juice; cafe 1 had better sales for salads.

(d) $\mathbf{D} = \begin{bmatrix} 288 & 144 \\ 360 & 96 \end{bmatrix}$ (e) $\begin{bmatrix} 608 & 444 \\ 820 & 376 \end{bmatrix}$

15 $b = 3$ and $a = 2$

16 $a = -2$, $b = 3$, $c = -1$, $d = 5$ or $d = -2$

EXERCISE 8.2

1 (a) **AD** can be found. (b) **BD** cannot be found.

(c) **AC** can be found.

2 (a) $\mathbf{QP} = \begin{bmatrix} -1 & -7 \\ -9 & -7 \end{bmatrix}$ (b) $\mathbf{PQ} = \begin{bmatrix} 3 & -1 \\ -23 & -11 \end{bmatrix}$ (c) $\mathbf{SR} = \begin{bmatrix} -11 \\ -1 \\ 1 \end{bmatrix}$

3 (a) $\mathbf{A}^2 = \begin{bmatrix} -1 & -3 & -1 \\ 18 & -5 & -1 \\ -5 & -2 & 3 \end{bmatrix}$ (b) $\mathbf{AB} = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 6 & 3 & 1 & 6 \\ 2 & -4 & -2 & -1 \end{bmatrix}$

(c) It is not possible to calculate the product **BA**.

4 A

5 (a) $\begin{bmatrix} 22 \\ 4 \end{bmatrix}$ (b) $[-14]$ (c) $\begin{bmatrix} -17 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 9 & -16 & -9 \end{bmatrix}$ (e) $\begin{bmatrix} 14 \\ -2 \\ 8 \end{bmatrix}$

(f) $\begin{bmatrix} 23 & 9 \end{bmatrix}$ (g) $\begin{bmatrix} -5 \\ 1 \\ 10 \\ 9 \end{bmatrix}$ (h) $\begin{bmatrix} 6 & 2 \\ -15 & -5 \end{bmatrix}$

6 (a) $\mathbf{R} = \begin{bmatrix} 300 \\ 400 \\ 50 \\ 100 \\ 200 \end{bmatrix}$ (b) $\mathbf{MR} = \begin{matrix} \\ X \\ Y \\ Z \end{matrix} \begin{bmatrix} 2000 \\ 1400 \\ 1150 \end{bmatrix}$

Product X costs \$2000 to manufacture. Product Y costs \$1400 to manufacture. Product Z costs \$1150 to manufacture.

7 $\begin{bmatrix} 5.50 & 5.30 & 6.10 \end{bmatrix} \begin{bmatrix} 5 \\ 3.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 58.25 \end{bmatrix}$

The fruit cost a total of \$58.25

8 78 out of 100

9 (a) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(b) The matrix $\mathbf{C}^2$ represents the number of air routes there are with one stopover.

(c) The one-stopover route from Longreach (L) to Townsville (T) is through Mt Isa (M). The path followed is LMT.

10 (a) B

(b) In alternative A, the number of pathways appears to correspond to the number of pathways between some towns, but because we are dealing with *indirect* flights, this diagram contains far too many connections.

For alternative C, the number of direct flights in this network corresponds to some numbers in the matrix, but the numbers in the matrix are too small for *indirect* flights.

For alternative D, if this diagram were correct, you would expect to have more consistent numbers in the matrix because of the symmetry of the patterns—the numbers in the matrix are also not high enough to support this option.

11 $\begin{bmatrix} 560 & 650 & 595 \\ 580 & 670 & 560 \\ 360 & 415 & 345 \end{bmatrix}$

12 (a) $\mathbf{G} = \begin{matrix} \\ GP1 \\ GP2 \\ GP3 \end{matrix}\begin{matrix} \begin{matrix} T & D & S \end{matrix} \\ \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \end{matrix}$ (b) $\mathbf{S} = \begin{matrix} \\ T \\ D \\ S \end{matrix}\begin{matrix} \$ \\ \begin{bmatrix} 12 \\ 7 \\ 15 \end{bmatrix} \end{matrix}$

(c) (i) $\mathbf{GS} = \begin{matrix} \\ GP1 \\ GP2 \\ GP3 \end{matrix}\begin{matrix} \$ \\ \begin{bmatrix} 61 \\ 41 \\ 44 \end{bmatrix} \end{matrix}$

(ii) The selling price for gift pack 1 (*GP1*) is \$61.The selling price for gift pack 2 (*GP2*) is \$41. The selling price for gift pack 3 (*GP3*) is \$44.

(d) \$441

13 (a) $\mathbf{S} = \begin{matrix} \\ SS \\ DS \\ MS \end{matrix}\begin{matrix} \$ \\ \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \end{matrix}$ (b) $\mathbf{N} = \begin{matrix} \begin{matrix} SS & DS & MS \end{matrix} \\ \begin{bmatrix} 80 & 50 & 30 \end{bmatrix} \end{matrix}$

(c) **NS** is the total sales made from selling 80 single-scoops, 50 double-scoops and 30 mega-scoops. **NS** = [670]
The total sales are \$670.

(d) $\mathbf{T} = \begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}\begin{matrix} \begin{matrix} SS & DS & MS \end{matrix} \\ \begin{bmatrix} 50 & 40 & 0 \\ 30 & 90 & 10 \\ 100 & 60 & 20 \\ 80 & 20 & 15 \end{bmatrix} \end{matrix}$

(e) $\mathbf{TS} = \begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}\begin{matrix} \$ \\ \begin{bmatrix} 350 \\ 600 \\ 720 \\ 430 \end{bmatrix} \end{matrix}$

The matrix product **T** gives the total sales at each store. Shop A total sales is \$350. Shop B total sales is \$600. Shop C total sales is \$720. Shop D total sales is \$430.

(f) Shop C (g) $\mathbf{S} = \begin{matrix} \\ SS \\ DS \\ MS \end{matrix}\begin{matrix} \$ \\ \begin{bmatrix} 3.50 \\ 5.50 \\ 6.50 \end{bmatrix} \end{matrix}$

(h) Shop A makes \$395. Shop B makes \$665. Shop C makes \$810. Shop D makes \$487.50.

(i) Increase in sales of shop A is \$45. Increase in sales of shop B is \$65. Increase in sales of shop C is \$90. Increase in sales of shop D is \$57.50.

14 $[x = -1, y = -2]$ and $[x = 1, y = 2]$

EXERCISE 8.3

1 (a) $\mathbf{AI} = \begin{bmatrix} -2 & 3 \\ 6 & -1 \end{bmatrix}$ (b) $\mathbf{IA} = \begin{bmatrix} -2 & 3 \\ 6 & -1 \end{bmatrix}$

2 (a) $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) **A** and **B** are inverses of each other.

3 (a) C

(b) For alternative A, the signs within the matrices are all reversed. For alternative B, the x and v should have negative signs. For alternative D, the determinant should be $uy - vx$.

4 (a) $\det \mathbf{A} = 2$

(b) The inverse of **A**, $\mathbf{A}^{-1}$, exists because $\det \mathbf{A}$ doesn't equal 0.

(c) $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & 3 \end{bmatrix}$ (d) $\det \mathbf{B} = 0$

(e) $\det \mathbf{B} = 0$, therefore $\mathbf{B}^{-1}$ does not exist.

5 (a) $\det \mathbf{A} = 9$ (b) $\det \mathbf{B} = -13$ (c) $\det \mathbf{C} = 16$
(d) $\det \mathbf{D} = -6$ (e) $\det \mathbf{E} = 6$ (f) $\det \mathbf{F} = 0$

6 (a) $\mathbf{A}^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -5 \\ -1 & 2 \end{bmatrix}$ (b) $\mathbf{B}^{-1} = -\frac{1}{13}\begin{bmatrix} 2 & -7 \\ -1 & -3 \end{bmatrix}$

(c) $\mathbf{C}^{-1} = \frac{1}{16}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (d) $\mathbf{D}^{-1} = -\frac{1}{6}\begin{bmatrix} 0 & -2 \\ -3 & 3 \end{bmatrix}$

(e) $\mathbf{E}^{-1} = -\frac{1}{6}\begin{bmatrix} -1 & 3 \\ -1 & -3 \end{bmatrix}$ (f) **F** does not have an inverse.

7 (a) $\det \mathbf{A} = (2 \times 3) - (6 \times 1)$
$= 6 - 6$
$= 0$

(b) $\det \mathbf{B} = (6 \times 3) - (1 \times 18)$
$= 18 - 18$
$= 0$

(c) $\det \mathbf{C} = (2 \times -6) - (4 \times -3$
$= 0$

8 (a) $\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{50} & -\frac{7}{100} \\ \frac{1}{25} & \frac{3}{25} \end{bmatrix}$ (b) $\mathbf{B}^{-1} = \begin{bmatrix} -\frac{7}{6} & -\frac{3}{2} \\ \frac{11}{6} & \frac{5}{2} \end{bmatrix}$

(c) $\mathbf{C}^{-1} = \begin{bmatrix} 12 & -20 \\ -15 & 30 \end{bmatrix}$

9 (a) $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 11 & 14 \\ 0 & -1 & -1 \\ 0 & -3 & -4 \end{bmatrix}$ (b) $\mathbf{B}^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$

10 (a) $p = -15$ (b) $\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} -4 & -2 \\ 15 & 6 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ -15 & -4 \end{bmatrix}$

Input $\mathbf{A} \times \mathbf{A}^{-1}$

$\mathbf{A} \times \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

11 $x = -2, x = 2$

12 $\mathbf{B}^{-1} + \mathbf{B} = \begin{bmatrix} -4 & \frac{9}{2} \\ -3 & 5 \end{bmatrix}$

13 (a) $\mathbf{X} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ (b) $\mathbf{X} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$

14 (a) $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\mathbf{X} = \begin{bmatrix} 4 & -6 \\ -2 & 1 \end{bmatrix}$

15 $\mathbf{X} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

16 $p = \frac{1}{2}\left(1 - \sqrt{5}\right), p = \frac{1}{2}\left(1 + \sqrt{5}\right)$

EXERCISE 8.4

1 (a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & -3 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

2 (a) $y = -1$ and $x + y = -2$

(b) $4x + y = 6$ and $-5x + 2y = -2$

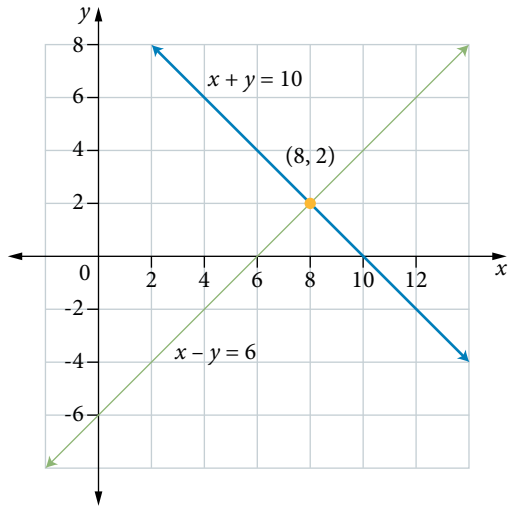
(c) $7x - y = 7$ and $2x - 3y = -4$

(d) $-2x - 2y = -3$ and $x + 4y = 8$

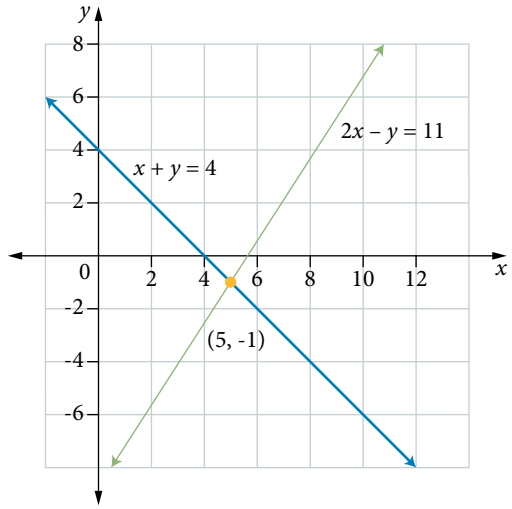
3 (a) C (b) B

(c) In part (a): for alternative A the elements in row 2 and row 3 are not in the correct order. For alternative B the elements in every row are in incorrect order. For alternative D the last row is incorrect as the z and 3 need to change sign when they are swapped to different sides of the equation. In part (b): for alternative A the matrix is not a column matrix. For alternative C the elements should be x, y and z, not numbers. For alternative D the elements x, y and z should be positive.

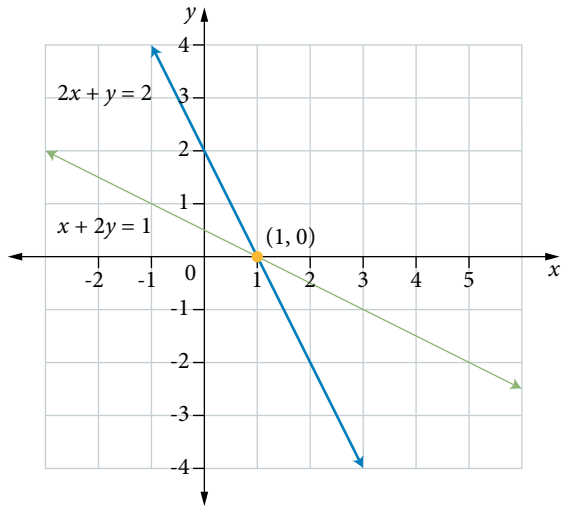
4 (a) $x = 8$ and $y = 2$



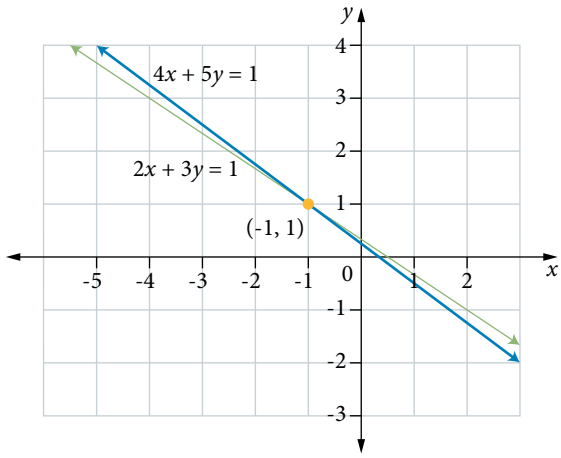
(b) $x = 5$ and $y = -1$



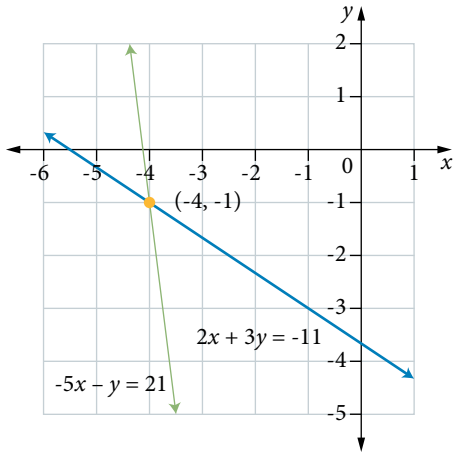
(c) $x = 1$ and $y = 0$



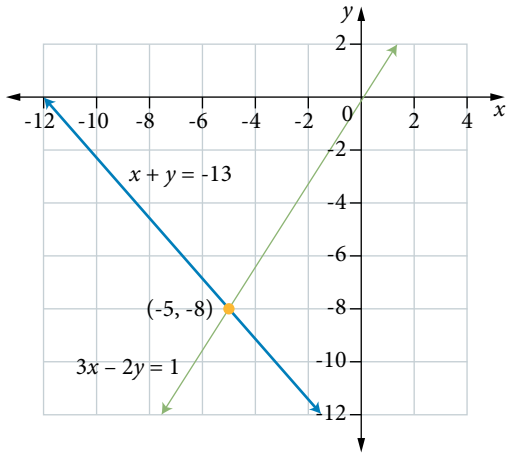
(d) $x = -1$ and $y = 1$



(e) $x = -4$ and $y = -1$



(f) $x = -5$ and $y = -8$



5 (a) There is no unique solution, as the equations represent the same line.

(b) There is no unique solution as the equations represent parallel lines.

6 (a) $x = \frac{3}{23}$ and $y = \frac{11}{23}$

(b) $x = \frac{38}{33}$ and $y = -\frac{32}{33}$

(c) $x = -\frac{62}{59}$ and $y = \frac{100}{59}$

(d) $x = 45$ and $y = \frac{105}{4}$

7 (a) $x = 3$, $y = -1$ and $z = 2$ (b) $x = 2$, $y = -1$ and $z = 1$

8 The two whole numbers are 7 and 11.

9 A milkshake costs \$6 and a cookie costs \$2.

10 Jimmy has 17 stamps and Sally has 51 stamps.

11 A meat pie costs \$4.60. A muffin costs \$3.00.

12 A hot dog costs \$4.50. A bucket of chips costs \$3.50. A can of soft drink costs \$1.50

13 (a) $3x + 2y + 5z = 1850$, $2x + 3y + z = 1300$ and $x + 5y + 4z = 2050$

(b) Isa's tickets will cost \$1350.

14 (a) $x = -1$ (b) $x = -\frac{4}{3}$

EXERCISE 8.5

1 (a) The image is $A'(4, 1)$. (b) The image is $B'(0, -4)$.

(c) The image is $C'(-1, -9)$. (d) The image is $D'\left(\frac{5}{2}, -\frac{5}{4}\right)$.

2 (a) $\mathbf{T} = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$ (b) $\mathbf{T} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(c) $\mathbf{T} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ (d) $\mathbf{T} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

3 B

4 (a) D

(b) In alternative A, the translation represents matrix addition, not matrix multiplication. In alternative B, $\mathbf{T}_2$ would result in an image $A' = (-3, -7)$. In alternative C, $\mathbf{T}_2$ would result in an image $A' = (3, 7)$.

5 For point $A(-1, 8)$ under translation $\mathbf{T}_1$:

$$A' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

For point $A'(2, 9)$ under translation $\mathbf{T}_2$:

$$A'' = \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

Therefore, the image after $\mathbf{T}_1$ and $\mathbf{T}_2$ is (7, 11).

$$(\mathbf{T}_1 + \mathbf{T}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

For point $A(-1, 8)$ under translation $(\mathbf{T}_1 + \mathbf{T}_2)$:

$$A' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix} + \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

Therefore, the image of A after $(\mathbf{T}_1 + \mathbf{T}_2)$ is (7, 11).

The image of A after undergoing successive translations of $\mathbf{T}_1$ and $\mathbf{T}_2$ is the same as the image after undergoing a translation of $(\mathbf{T}_1 + \mathbf{T}_2)$.

6 The image is defined by points $A'(-5, 5)$, $B'(0, 5)$, $C'(2, 9)$ and $D'(-3, 9)$. For both parallelograms, area = 20 units^2.

7 (a) $y' = (x')^2 - 6x' + 10$ (b) $y' = 2x' - 4$

(c) $y' = \sqrt{x' - 1} + 4$ (d) $y' = (x')^2 - 19x' + 82$

8 $\mathbf{T} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

9 The original function has a gradient of m and a y-intercept of c.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} p \\ q \end{bmatrix}$$

$$x = x' - p$$
$$y = y' - q$$
$$y = mx + c$$
$$y' - q = m(x' - p) + c$$
$$y' = mx' - mp + c + q$$
$$y' = mx' + c + (q - mp)$$

Therefore, the image has a gradient of m and a y-intercept of $c + (q - mp)$.

10 $a = 2$ or $a = 3$, $b = -4$

11 $\mathbf{T} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

EXERCISE 8.6

1 (a) This is not a linear transformation.

(b) This is a linear transformation defined by $\mathbf{L} = \begin{bmatrix} 2 & -7 \\ -3 & -1 \end{bmatrix}$.

(c) This is a linear transformation defined by $\mathbf{L} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

(d) This is not a linear transformation.

(e) This is a linear transformation defined by $\mathbf{L} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

2 (a) $A' = (8, -13)$ (b) $B' = (-1, 6)$

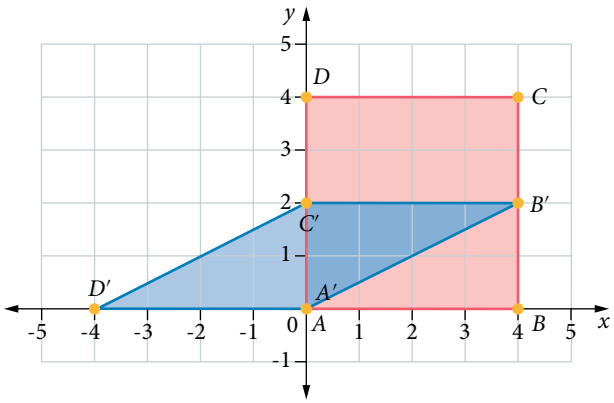
(c) $C' = (4, -3)$ (d) $D' = (-45, -16)$

3 D

4 (a) C

(b) Alternative A: $P(0, 1)$ will be mapped onto $P'(1, 0)$ as seen in the second column. Alternative B: $P(0, 1)$ will be mapped onto $P'(0, -1)$ as seen in the second column. Alternative D: $P(0, 1)$ will be mapped onto $P'(1, -1)$ as seen in the second column.

5 (a) The image points are $A'(0, 0)$, $B'(4, 2)$, $C'(0, 2)$ and $D'(-4, 0)$.



(b) $\det \mathbf{L} = 0.5$

(c) $y' = \frac{1}{2}x$

$x' = x - y$

(d) area of original square = 16 units^2; area of image = 8 units^2

6 The image $\Delta A'B'C'$ is defined by the points $A'(0,0)$, $B'(7,16)$ and $C'(7,18)$.

7 (a) The image is point $C'(10, -12)$.

(b) The image is point $C'(-14, 14)$.

8 The line segment $\overline{A''B''}$ is defined by $A''(-9, 4)$ and $B''(-4, 15)$.

9 The area of image $A'B'C'D'$ is ad units^2.

10 The gradient of the original line $= \frac{1}{3}$.

To determine the image of the line, remember that the transformation matrix $\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is used to describe the linear transformation $x' = ax + by$ and $y' = cx + dy$. In this case, the transformation is $x' = 0x + -1y$ and $y' = 1x + 0y$ or $x' = -y$ and $y' = x$. Use these equations to substitute into the original line:

$$-x' = \frac{1}{3}y'$$
$$-3x' = y'$$
$$y' = -3x'$$

The equation of the image line is therefore $y' = -3x'$.

The image line has a gradient of -3.

Since $m_{\text{image}} = -\frac{1}{m_{\text{original}}}$, the original line and its image are perpendicular.

11 $y' = -\frac{3}{8}x' + \frac{1}{8}$

EXERCISE 8.7

1 (a) $\mathbf{R}_{135^\circ} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(b) $\mathbf{R}_{210^\circ} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$ (c) $\mathbf{M}_{y=\sqrt{3}x} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

2 (a) The image is $\Delta A'B'C'$ defined by the points $A'(0, 0)$, $B'(2, -4)$ and $C'(-1, -3)$.

(b) The image is $\Delta A'B'C'$ defined by the points $A'(0, 0)$, $B'(-2, 4)$ and $C'(1, 3)$.

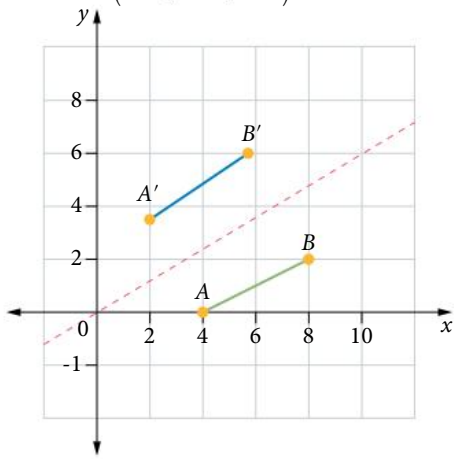
(c) The image is $\Delta A'B'C'$ defined by the points $A'(0, 0)$, $B'(4, 2)$ and $C'(3, -1)$.

3 B

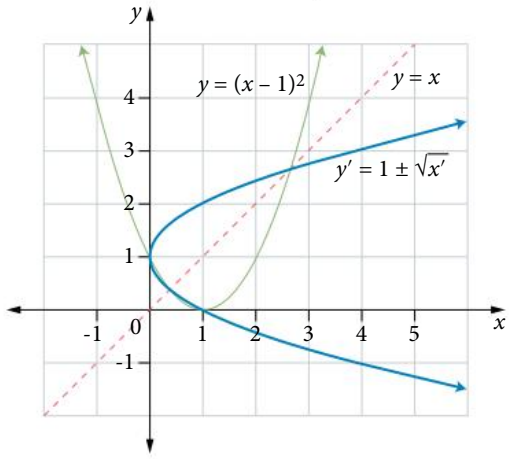
4 (a) A

(b) For alternative B, the gradient of the line would decrease. For alternative C, a rotation of 90° clockwise would map to $y = -x$. For alternative D, a reflection in the x-axis would map to $y = -x$.

5 The image is line segment $\overline{A'B'}$ defined by $A'\left(2, 2\sqrt{3}\right)$ and $B'\left(4+\sqrt{3}, 4\sqrt{3}-1\right)$.



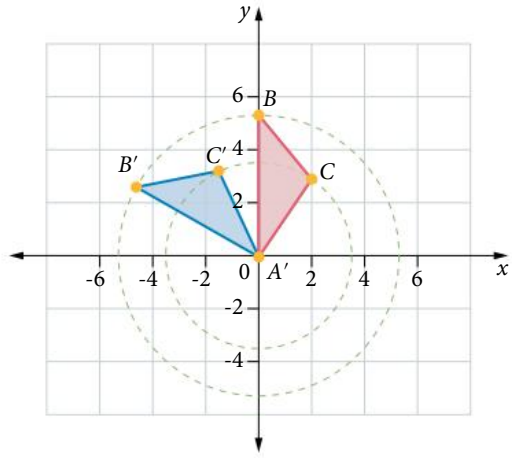
6 The image equation is $y' = 1 \pm \sqrt{x'}$.



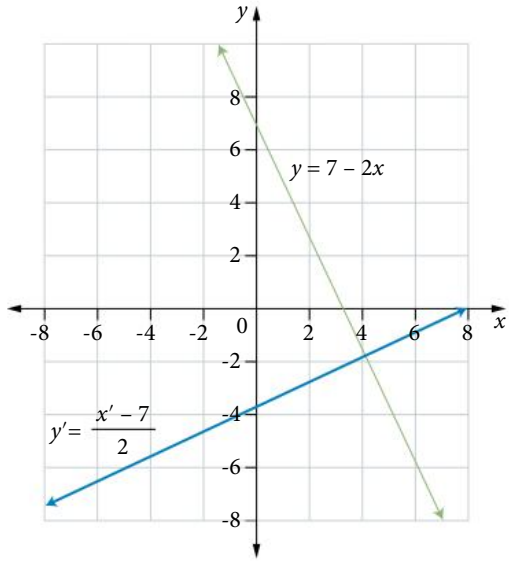
7 **(a)** The image is $\overline{A'B'}$, defined by the points $A'(-5, 1)$ and $B'(-3, -1)$.

(b) The image is $\overline{A'B'}$, defined by the points $A'(-1, -5)$ and $B'(1, -3)$.

8 The image is $\Delta A'B'C'$ defined by the points $A(0, 0)$, $B\left(-\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$ and $C\left(1-\frac{3\sqrt{3}}{2}, \sqrt{3}+\frac{3}{2}\right)$.



9 The image of the line is $y' = \frac{x'-7}{2}$.

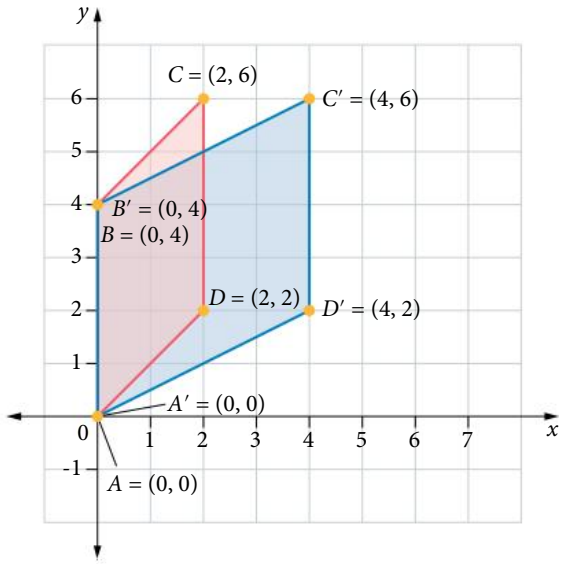


10 The image is a parallelogram $A'B'C'D'$ defined by the points $A'(0, 0)$, $B'(0, 4)$, $C'(4, 6)$ and $D'(4, 2)$.

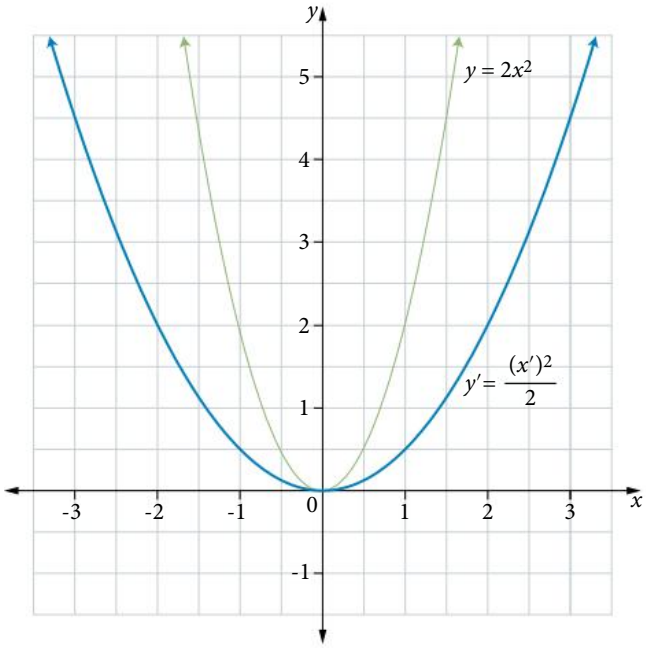
area (original) = base × height
$= 4 \times 2$
$= 8$ units^2

area (image) = base × height
$= 4 \times 4$
$= 16$ units^2

Therefore, the image is double the area of the original parallelogram.



11 The image is $y' = \frac{(x')^2}{2}$.



12 $k = \sqrt{6}$

13 $\theta = 18.43°$ (2 d.p.)

EXERCISE 8.8

1 (a) $\det \mathbf{L} = 0$, so the transformation defined by $\mathbf{L} = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$ is non-invertible. The plane is mapped on to the image line $y' = \frac{-x'}{2}$.

(b) $\det \mathbf{L} = -1$, so the transformation defined by $\mathbf{L} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ is invertible.

(c) $\det \mathbf{L} = 0$, so the transformation defined by $\mathbf{L} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ is non-invertible. The plane is mapped onto the image line $y' = 2x'$.

(d) $\det \mathbf{L} = 0$, so the transformation defined by $\mathbf{L} = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$ is non-invertible. The plane is mapped onto the image line $y' = \frac{x'}{4}$.

2 B

3 (a) C

(b) Alternative A will result in the invariant lines $y = 0$ and $x = 0$. For alternative B, the reflection in the line $y = 2x$ will be invariant. For alternative D, the formula $m = \frac{(3+m)}{(2+0)}$ has a solution $m = 3$, which gives the invariant line $y = 3x$ for the transformation defined by the matrix $\mathbf{L}$.

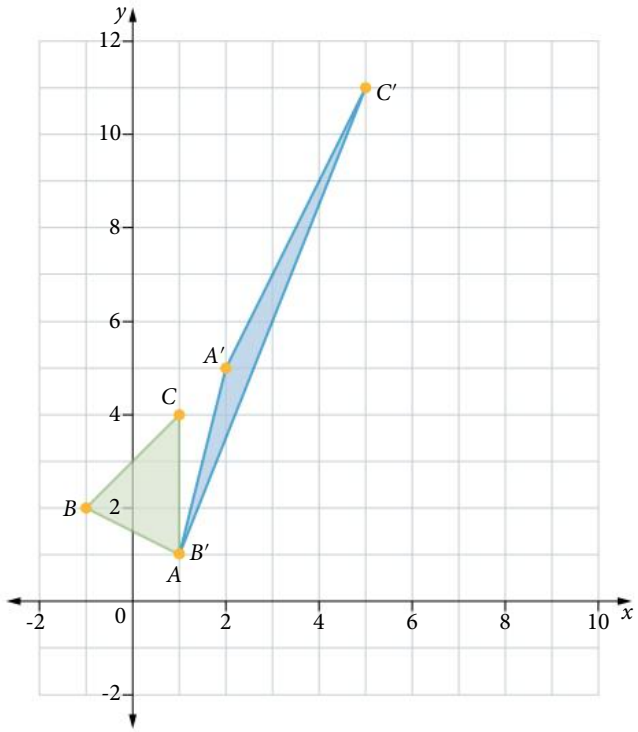
4 (a) $\mathbf{M}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\mathbf{M}_{y=x} \times \mathbf{M}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

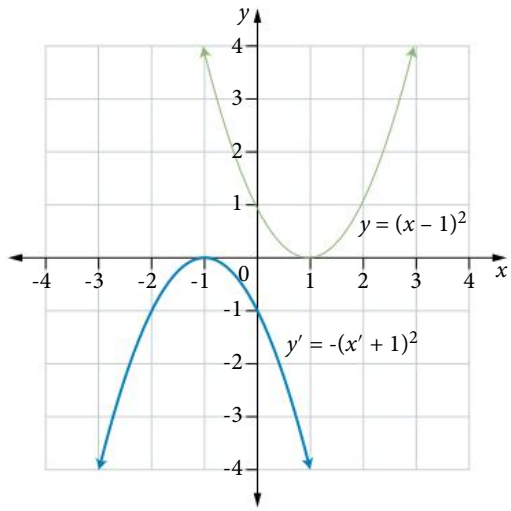
$\mathbf{M}_{y=x} \times \mathbf{M}_{y=x} = \mathbf{I}$, therefore $\mathbf{M}_{y=x}$ is the inverse of $\mathbf{M}_{y=x}$.

(b) The original point is $A(1, -9)$.

5 The original ΔABC is defined by the points $A(1, 1)$, $B(-1, 2)$ and $C(1, 4)$.



6 The original curve is $y = (x-1)^2$.



7 (a) The invariant line is $y = -\frac{1}{8}x$.

(b) The invariant lines are $y = -1.21x$ and $y = 0.21x$ (2 d.p.).

8 $a = 0$ and $a = 10$

9 The matrix for reflection in the line $y = x\tan(\theta)$ is

$$\mathbf{M}_{y=x\tan(\theta)} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

If $\mathbf{M}_{y=x\tan(\theta)} \times \mathbf{M}_{y=x\tan(\theta)} = \mathbf{I}$, then $\mathbf{M}_{y=x\tan(\theta)}$ is its own inverse.

Multiplying the transformation matrices:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \times \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(2\theta) + \sin^2(2\theta) & \cos(2\theta)\sin(2\theta) - \sin(2\theta)\cos(2\theta) \\ \cos(2\theta)\sin(2\theta) - \sin(2\theta)\cos(2\theta) & \sin^2(2\theta) + \cos^2(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(2\theta) + \sin^2(2\theta) & 0 \\ 0 & \sin^2(2\theta) + \cos^2(2\theta) \end{bmatrix}$$

Using the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, this matrix simplifies to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$.

Therefore, a reflection in the line $y = x\tan(\theta)$ is always its own inverse.

10 The matrix for a general rotation of $\theta°$ is

$$\mathbf{R}_{\theta°} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

To demonstrate that there is no invariant line under this transformation, there must be no possible gradient m such that $y = mx$ is equivalent to $y' = mx'$. In other words, there will be no solutions to the equation $m = \dfrac{(c + dm)}{(a + bm)}$ for m.

For this particular transformation matrix

$\mathbf{R}_{\theta°} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, $a = \cos(\theta)$, $b = -\sin(\theta)$, $c = \sin(\theta)$, $d = \cos(\theta)$. Therefore:

$$m = \frac{(c + dm)}{(a + bm)}$$

$$m = \frac{(\sin(\theta) + m\cos(\theta))}{(\cos(\theta) - m\sin(\theta))}$$

$$m(\cos(\theta) - m\sin(\theta)) = (\sin(\theta) + m\cos(\theta))$$

$$m\cos(\theta) - m^2\sin(\theta) = \sin(\theta) + m\cos(\theta)$$

$$-m^2\sin(\theta) = \sin(\theta)$$

$$-m^2 = \frac{\sin(\theta)}{\sin(\theta)}$$

$$-m^2 = 1$$

$$m^2 = -1$$

There are no real solutions for this equation, therefore no invariant lines exist.

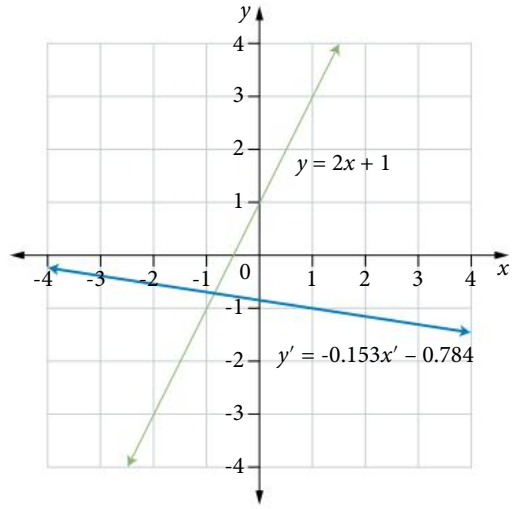
EXERCISE 8.9

1 (a) $\mathbf{L} = \begin{bmatrix} 1 & -5 \\ 1 & 2 \end{bmatrix}$ (b) $\mathbf{L} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

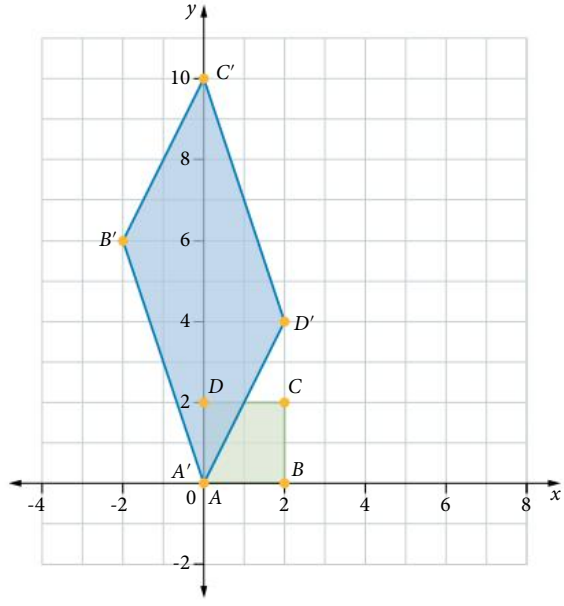
(c) $\mathbf{L} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{5} & \sqrt{3} \end{bmatrix}$ (d) $\mathbf{L} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{5} \\ -\frac{3}{5} & -\frac{1}{2} \end{bmatrix}$

2 D

3 $y' = -0.153x' - 0.784$ (3 d.p.)



4 The image is a parallelogram $A'B'C'D'$ defined by image points $A'(0, 0)$, $B'(-2, 6)$, $C'(0, 10)$, $D'(2, 4)$.



5 The transformation matrix is $\mathbf{L} = \begin{bmatrix} 3 & 1 \\ -3 & 0 \end{bmatrix}$.

6 The transformation matrix is $\mathbf{L} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

7 $a = 2$ and $b = 1$

8 (a) From the transformation equation:

$$x' = x\cos(\theta) - y\sin(\theta) \quad [1]$$
$$y' = x\sin(\theta) + y\cos(\theta) \quad [2]$$

From [1]:

$$y\sin(\theta) = x\cos(\theta) - x'$$
$$\sin(\theta) = \frac{x\cos(\theta) - x'}{y} \quad [3]$$

From [2]:

$$x\sin(\theta) = y' - y\cos(\theta)$$
$$\sin(\theta) = \frac{y' - y\cos(\theta)}{x} \quad [4]$$

Solving equations [3] and [4] simultaneously:

$$\frac{x\cos(\theta) - x'}{y} = \frac{y' - y\cos(\theta)}{x}$$
$$x\left(x\cos(\theta) - x'\right) = y\left(y' - y\cos(\theta)\right)$$
$$x^2\cos(\theta) - xx' = yy' - y^2\cos(\theta)$$
$$x^2\cos(\theta) + y^2\cos(\theta) = xx' + yy'$$
$$\cos(\theta)\left(x^2 + y^2\right) = xx' + yy'$$
$$\cos(\theta) = \frac{xx' + yy'}{\left(x^2 + y^2\right)}$$

(b) Proceeding in a similar manner to part (a):

From [1]:

$$y\sin(\theta) + x' = x\cos(\theta)$$
$$\cos(\theta) = \frac{y\sin(\theta) + x'}{x} \quad [5]$$

From [2]:

$$-x\sin(\theta) + y' = y\cos(\theta)$$
$$\cos(\theta) = \frac{y' - x\sin(\theta)}{y} \quad [6]$$

Solving equations [5] and [6] simultaneously:

$$\frac{y\sin(\theta) + x'}{x} = \frac{y' - x\sin(\theta)}{y}$$
$$x\left(y' - x\sin(\theta)\right) = y\left(y\sin(\theta) + x'\right)$$
$$xy' - x^2\sin(\theta) = y^2\sin(\theta) + yx'$$
$$xy' - yx' = x^2\sin(\theta) + y^2\sin(\theta)$$
$$xy' - yx' = \sin(\theta)\left(x^2 + y^2\right)$$
$$\sin(\theta) = \frac{xy' - yx'}{x^2 + y^2}$$

(c) $\mathbf{R}_{\theta^\circ} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

EXERCISE 8.10

1 (a) $\mathbf{L}_c = \begin{bmatrix} 4 & 2 \\ 6 & 1 \end{bmatrix}$ **(b)** $\mathbf{L}_c = \begin{bmatrix} -6 & -3 \\ -12 & -4 \end{bmatrix}$ **(c)** $\mathbf{L}_c = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

2 (a) $\mathbf{L}_c = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **(b)** $\mathbf{L}_c = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **(c)** $\mathbf{L}_c = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$

3 (a) C

(b) For alternative A: A rotation of 90° anticlockwise and a reflection in the x-axis, the order is important because:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

For alternative B: A reflection in the line $y = x$ and a reflection in the x-axis, the order is important because:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

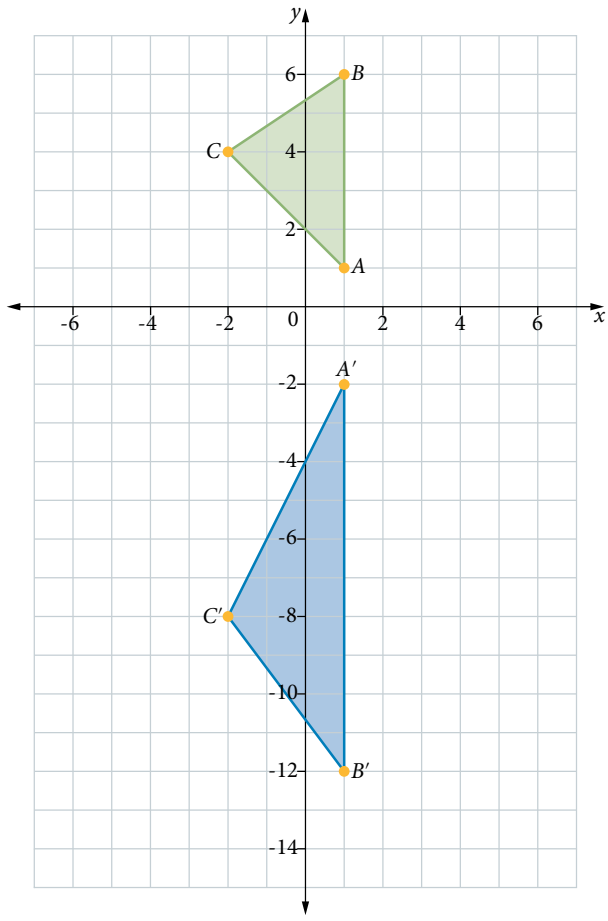
For alternative D: A reflection in the y-axis and a rotation of 270° about the origin, the order is important because:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \text{ and}$$

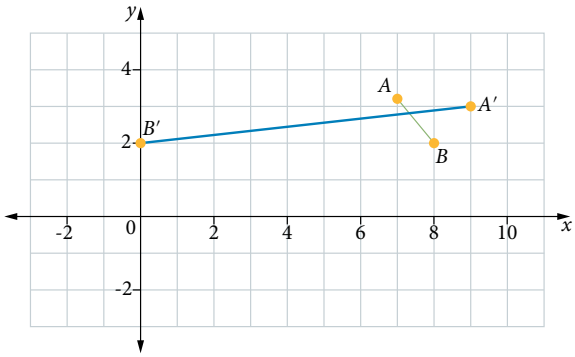
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4 A

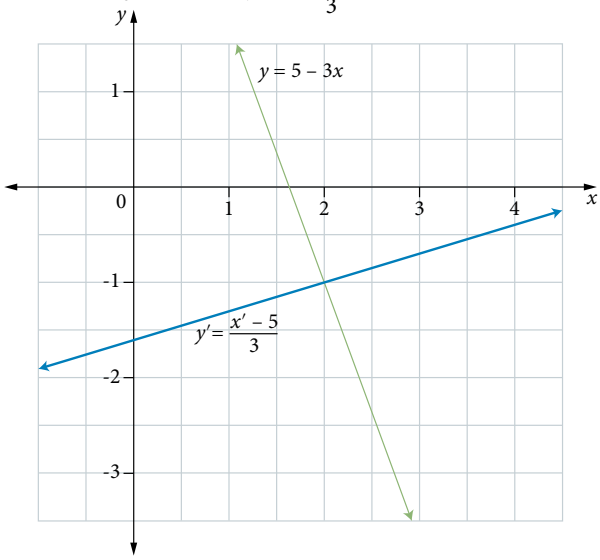
5 The image is $\Delta A'B'C'$, defined by points $A'(1, -2)$, $B'(1, -12)$ and $C'(-2, -8)$.



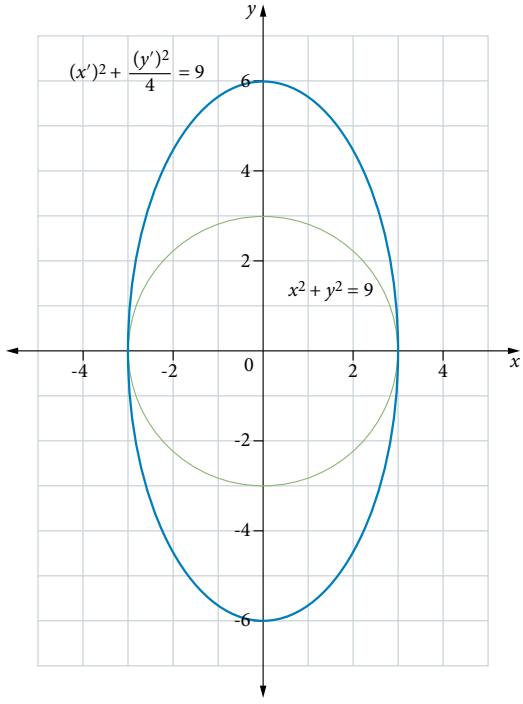
6 The image is a line segment $\overline{A'B'}$ defined by points $A'(10, 3)$ and $B'(0, 2)$.



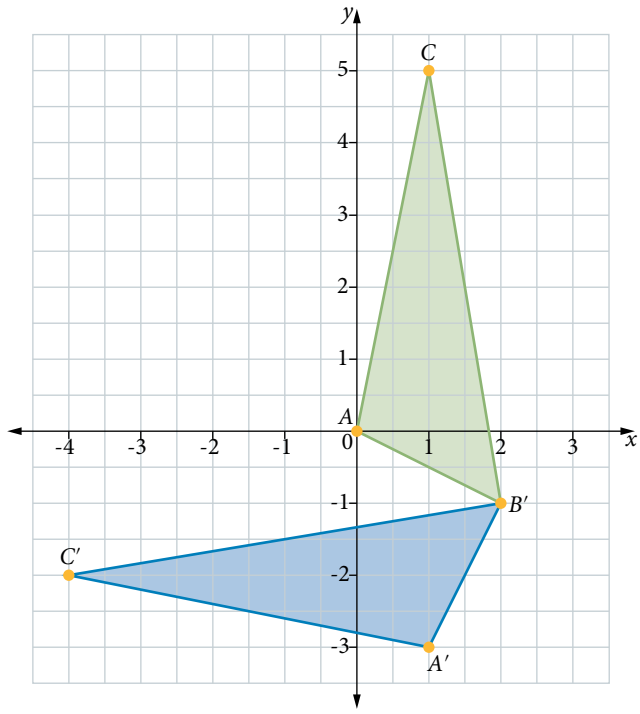
7 The image is the line $y' = \dfrac{x' - 5}{3}$.



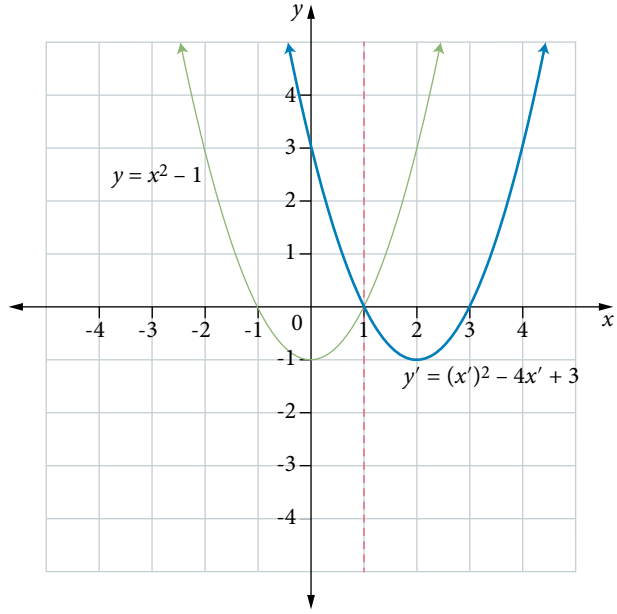
8 The image of the circle is $(x')^2 + \dfrac{(y')^2}{4} = 9$.



9 The image is $\Delta A'B'C'$, defined by points $A'(1, -3)$, $B'(2, -1)$ and $C'(-4, -2)$.



10 The image is the curve $y' = (x')^2 - 4x' + 3$.



EXERCISE 8.11

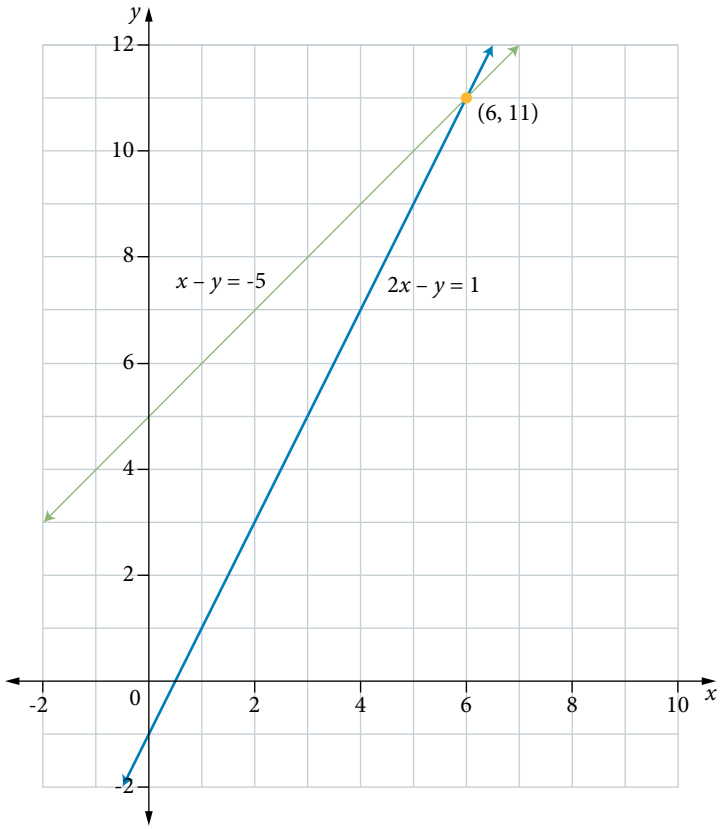
1 C

2 A

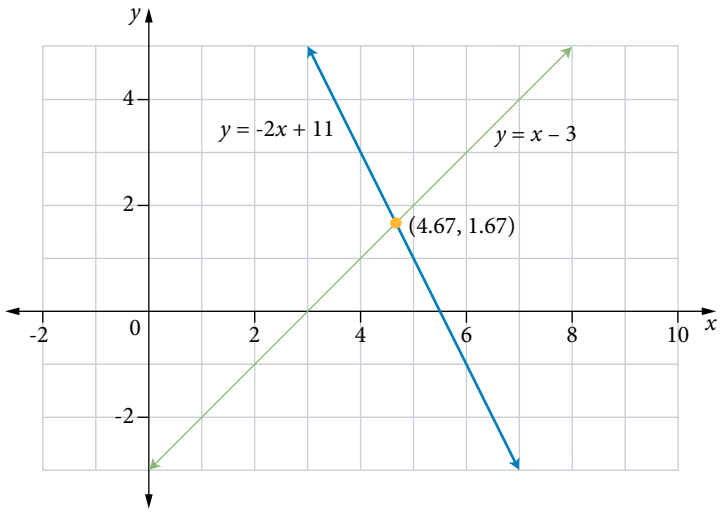
3 **(a)** $\text{area}_{(\Delta A'B'C')} = 30 \text{ units}^2$ **(b)** $\text{area}_{(\Delta A'B'C')} = 75 \text{ units}^2$

(c) $\text{area}_{(\Delta A'B'C')} = 15 \text{ units}^2$ **(d)** $\text{area}_{(\Delta A'B'C')} = 0 \text{ units}^2$

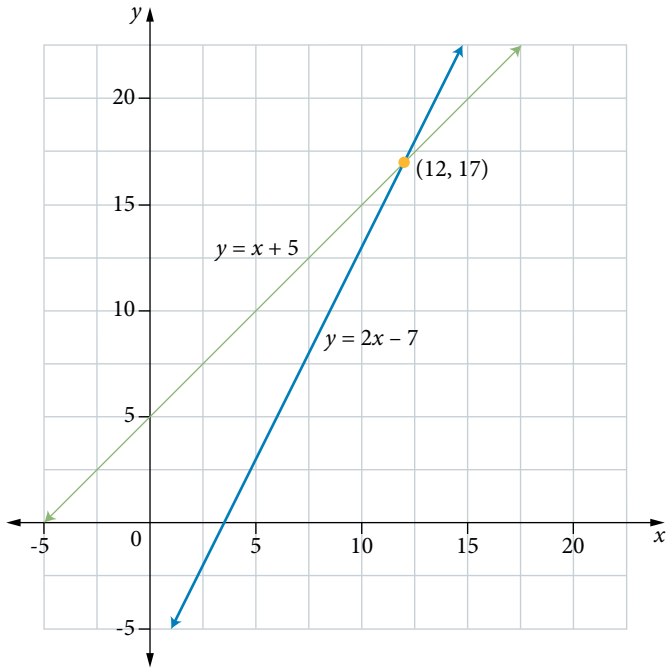
4 (a) Point of intersection is (6, 11).



(b) Point of intersection is $\left(\frac{14}{3}, \frac{5}{3}\right)$.



(c) Point of intersection is (12, 17).



5 (a) The area of the image $\Delta A'B'C'$ is 5 units^2.

(b) The image is $\Delta A'B'C'$ defined by the points $A'(0, -4)$, $B'(4, -4)$ and $C'\left(4, -\frac{3}{2}\right)$. The area of image $\Delta A'B'C'$ is 5 units^2. This verifies the answer of part **(a)**.

6 Consider the general equation of the straight line $y = mx + c$.

Under a rotation of 90° anticlockwise about the origin, $\mathbf{R}_{90°} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = y'$ and $y = -x'$

Substitute into original equation:

$y = mx + c$

$-x' = my' + c$

$my' = -x' - c$

$$y' = -\frac{1}{m}x' - \frac{c}{m}$$

Therefore, the image line $y' = -\frac{1}{m}x' - \frac{c}{m}$ has a gradient of $-\frac{1}{m}$.

Under a rotation of 270° anticlockwise about the origin,

$$\mathbf{R}_{270°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = -y'$ and $y = x'$

Substitute into original equation:

$y = mx + c$

$x' = -my' + c$

$my' = -x' + c$

$y' = -\frac{1}{m}x' + \frac{c}{m}$

Therefore, the image line $y' = -\frac{1}{m}x' + \frac{c}{m}$ has a gradient of $-\frac{1}{m}$.

7

$$\mathbf{R}_{\alpha} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$\mathbf{R}_{\beta} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}$$

$$\mathbf{R}_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

A rotation of $(\alpha+\beta)$ is the same as a rotation of α followed by a rotation of β.

A rotation of α followed by a rotation of β can be combined into a single transformation matrix using the rule $\mathbf{L}_c = \mathbf{L}_2\mathbf{L}_1$:

$$\mathbf{L}_c = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix}$$

A rotation of $(\alpha+\beta)$ can also be represented by the matrix $\mathbf{R}_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

Therefore:

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix}$$

From element$_{21}$ of both matrices:

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

8 Now, $\cos(\alpha-\beta) = \cos(\alpha+(-\beta))$:

$$\mathbf{R}_{\alpha} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$\mathbf{R}_{-\beta} = \begin{bmatrix} \cos(-\beta) & -\sin(-\beta) \\ \sin(-\beta) & \cos(-\beta) \end{bmatrix} = \begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}$$

$$\mathbf{R}_{\alpha-\beta} = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}$$

A rotation of α followed by a rotation of $-\beta$ can be combined into a single transformation matrix using the rule $\mathbf{L}_c = \mathbf{L}_2\mathbf{L}$:

$$\mathbf{L}_c = \begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ -\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) & \sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta) \end{bmatrix}$$

A rotation of $(\alpha + (-\beta))$ can also be represented by the matrix

$$\mathbf{R}_{\alpha-\beta} = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ -\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) & \sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta) \end{bmatrix}$$

From element$_{11}$ of both matrices, $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

9 Now, $\sin(2\theta) = \sin(\theta+\theta)$:

$$\mathbf{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{2\theta} = \begin{bmatrix} \cos(\theta+\theta) & -\sin(\theta+\theta) \\ \sin(\theta+\theta) & \cos(\theta+\theta) \end{bmatrix}$$

A rotation of θ followed by another rotation of θ can be combined into a single transformation matrix using the rule $\mathbf{L}_c = \mathbf{L}_2\mathbf{L}$:

$$\mathbf{L}_c = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta)-\sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta)-\sin^2(\theta) \end{bmatrix}$$

A rotation of $(\theta+\theta)$ can also be expressed by the matrix

$$\mathbf{R}_{2\theta} = \begin{bmatrix} \cos(\theta+\theta) & -\sin(\theta+\theta) \\ \sin(\theta+\theta) & \cos(\theta+\theta) \end{bmatrix}.$$

Therefore:

$$\begin{bmatrix} \cos(\theta+\theta) & -\sin(\theta+\theta) \\ \sin(\theta+\theta) & \cos(\theta+\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta)-\sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta)-\sin^2(\theta) \end{bmatrix}$$

From element$_{21}$ of both matrices:

$\sin(\theta+\theta) = 2\sin(\theta)\cos(\theta)$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

10 Now, $\cos(3\theta) = \cos(\theta+\theta+\theta)$:

$$\mathbf{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{3\theta} = \begin{bmatrix} \cos(3\theta) & -\sin(3\theta) \\ \sin(3\theta) & \cos(3\theta) \end{bmatrix}$$

Three successive rotations of θ can be combined into a single transformation matrix using the rule $\mathbf{L}_c = \mathbf{L}_3\left(\mathbf{L}_2\mathbf{L}_1\right)$.

First, calculate $\mathbf{L}_1\mathbf{L}_2$:

$$\mathbf{L}_1\mathbf{L}_2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

Then multiply the answer by $\mathbf{L}_3$:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right] - \sin(\theta)\left[2\sin(\theta)\cos(\theta)\right] & \cos(\theta)\left[-2\sin(\theta)\cos(\theta)\right] - \sin(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right] \\ \sin(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right] + \cos(\theta)\left[2\sin(\theta)\cos(\theta)\right] & \sin(\theta)\left[-2\sin(\theta)\cos(\theta)\right] + \cos(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right] \end{bmatrix}$$

$$= \begin{bmatrix} \cos^3(\theta) - \cos(\theta)\sin^2(\theta) - 2\sin^2(\theta)\cos(\theta) & -2\sin(\theta)\cos^2(\theta) - \sin(\theta)\cos^2(\theta) + \sin^3(\theta) \\ \sin(\theta)\cos^2(\theta) - \sin^3(\theta) + 2\sin(\theta)\cos^2(\theta) & -2\sin^2(\theta)\cos(\theta) + \cos^3(\theta) - \sin^2(\theta)\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) & \sin^3(\theta) - 3\sin(\theta)\cos^2(\theta) \\ -\sin^3(\theta) + 3\sin(\theta)\cos^2(\theta) & \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} \cos(3\theta) & -\sin(3\theta) \\ \sin(3\theta) & \cos(3\theta) \end{bmatrix} = \begin{bmatrix} \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) & \sin^3(\theta) - 3\sin(\theta)\cos^2(\theta) \\ -\sin^3(\theta) + 3\sin(\theta)\cos^2(\theta) & \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) \end{bmatrix}$$

Equating element$_{11}$ of both matrices: $\cos(3\theta) = \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta)$

Therefore, using the Pythagorean identity:

$\cos(3\theta) = \cos^3(\theta) - 3\left(1 - \cos^2(\theta)\right)\cos(\theta)$

$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\left(1 - \cos^2(\theta)\right)$

$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta) + 3\cos^3(\theta)$

$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$

CHAPTER REVIEW 8

1 **(a)** $\mathbf{A}$ is a 3×3 matrix. **(b)** $a_{31} = -2$

(c) a_{12}

2 $a = \frac{1}{2}$, $b = -4$

3 **(a)** $\mathbf{A} + \mathbf{S} = \begin{bmatrix} 12 & 10 & 17 \\ 12 & 20 & 29 \\ 8 & 18 & 25 \end{bmatrix}$

(b) $(\mathbf{A} + \mathbf{S})_{31}$ indicates that store 3 sold 8 sofas over the two months of August and September.

4 **(a)** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -6 & 2 \\ -7 & 8 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & -1 \\ 1 & -2 \\ 6 & -2 \\ 7 & -8 \end{bmatrix}$

5 $\begin{bmatrix} 33 & -22 \\ 14 & 31 \end{bmatrix}$

6 $\begin{bmatrix} -2 & 13 \\ 7 & -27 \end{bmatrix}$

7 **(a)** The matrix product **PQ** cannot be calculated.

(b) The matrix product **QP** can be calculated. It will be a 2×3 matrix.

(c) The matrix product **PS** cannot be calculated.

(d) The matrix product **RT** can be calculated. It will be a 3×1 matrix.

8 **(a)** $\mathbf{AB} = \begin{bmatrix} 2 & -29 \\ 2 & 20 \end{bmatrix}$ **(b)** $\mathbf{BA} = \begin{bmatrix} 21 & 7 \\ -11 & 1 \end{bmatrix}$

9 **(a)** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ **(b)** $\mathbf{B}^{-1} = \begin{bmatrix} 5 & -2 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$

10 B

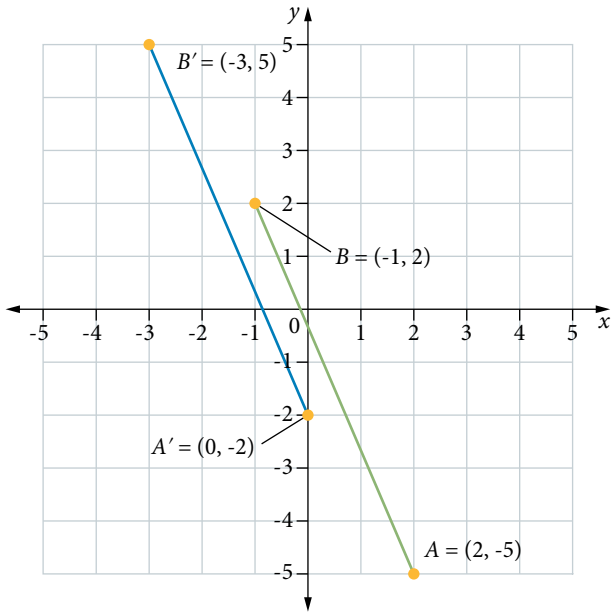
11 B

12 (a) $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$ (b) $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$

13 (a) $\begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

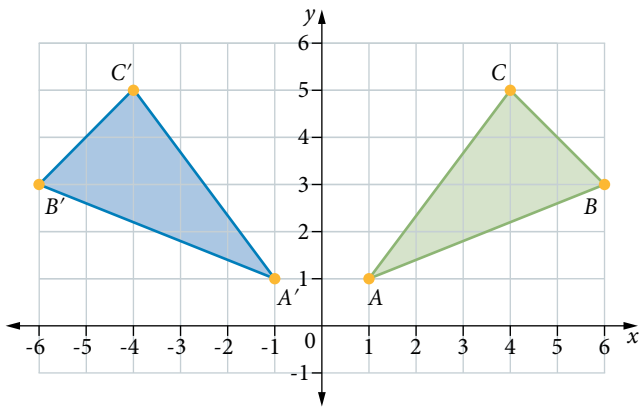
(b) $\begin{bmatrix} 5 & -2 & -1 \\ 0 & 8 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 4 \end{bmatrix}$

14 Image is line segment $\overline{A'B'}$ between points $A'(0, -2)$ and $B'(-3, 5)$.



15 (a) $A'(13, -7)$ (b) $A'(-6, 1)$ (c) $A'(-21, -8)$

16 The image is $\Delta A'B'C'$ defined by the points $A'(-1, 1)$, $B'(-6, 3)$ and $C'(-4, 5)$.



17 (a) $\det \mathbf{L} = 0$ so the transformation defined by $\mathbf{L} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ is non-invertible. The plane is mapped onto the image line $y' = \frac{x'}{2}$.

(b) $\det \mathbf{L} = 1$ so the transformation defined by $\mathbf{L} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ is invertible.

18 $p = -\frac{2}{3}$

19 (a) $\mathbf{L}_c = \begin{bmatrix} -9 & -14 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{L}_c = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$

(c) $\mathbf{L}_c = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

20 Martin's final semester result (out of 100) for Chemistry is 81.25.

21 (a) $\mathbf{Y} = \begin{bmatrix} 120 \\ 65 \\ 40 \\ 100 \end{bmatrix}$

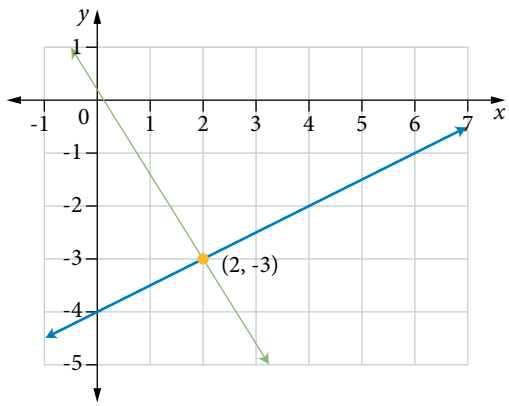
(b) $\mathbf{XY} = \begin{bmatrix} 725.00 \\ 437.50 \\ 642.50 \end{bmatrix}$

Product A costs \$725, product B costs \$437.50 and product C costs \$642.50 to manufacture.

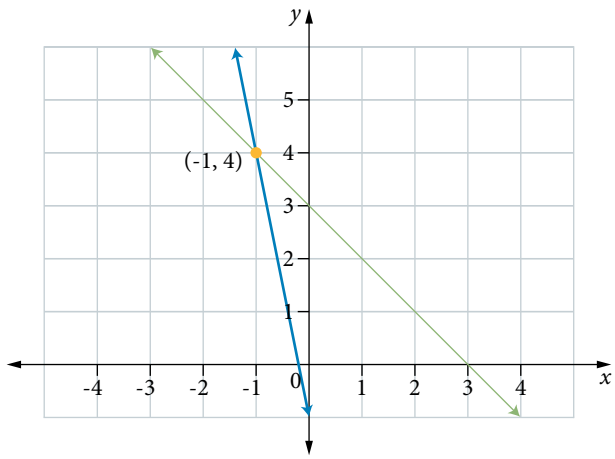
22 $\mathbf{B}^{-1} + \mathbf{B} = \begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$

23 (a) $\mathbf{X} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\mathbf{X} = \begin{bmatrix} 8 & -4 \\ 2 & 0 \end{bmatrix}$

24 (a) $x = 2$ and $y = -3$

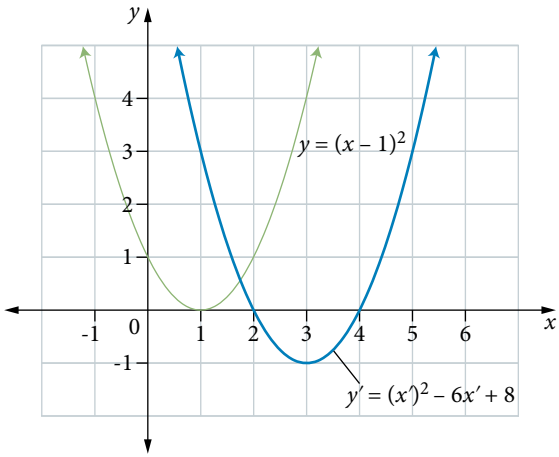


(b) $x = -1$ and $y = 4$

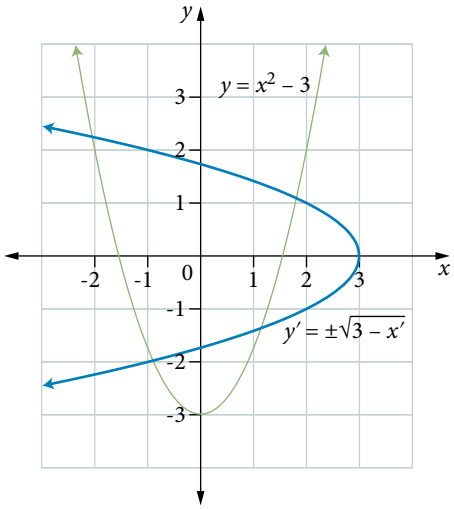


25 (a) $x = 1$, $y = -1$ and $z = -2$ (b) $x = 2$, $y = 0$ and $z = -3$

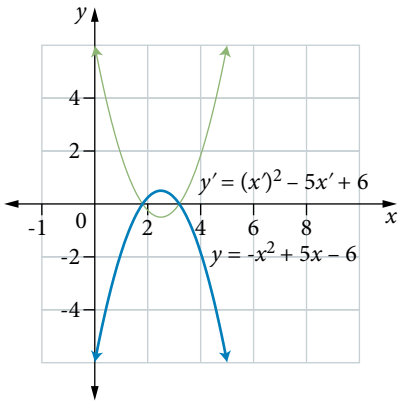
26 The image is $y' = (x')^2 - 6x' + 8$.



27 $y' = \pm\sqrt{3 - x'}$



28 Original curve is $y = -x^2 + 5x - 6$.



29 $\mathbf{L} = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$

30 (a)

$$\mathbf{R} = \begin{array}{c} \\ A \\ B \\ M \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & M & P & S \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

(b)

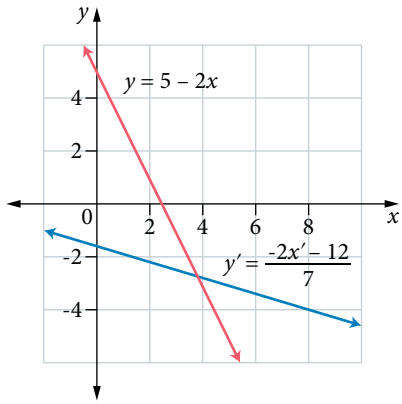
$$\mathbf{R}^2 = \begin{array}{c} \\ A \\ B \\ M \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & M & P & S \end{array} \\ \begin{bmatrix} 6 & 2 & 4 & 2 & 4 \\ 2 & 2 & 2 & 2 & 2 \\ 4 & 2 & 7 & 4 & 3 \\ 2 & 2 & 4 & 6 & 4 \\ 4 & 2 & 3 & 4 & 7 \end{bmatrix} \end{array}$$

(c) 3

(d) AMB and ASB

31 A loaf of bread costs \$2.80, a pie costs \$4.20 and a sausage roll costs \$3.50.

32 $y' = \dfrac{-2x' - 12}{7}$



33 Consider the rotation matrix $\mathbf{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

The inverse of this matrix is

$$\mathbf{R}_\theta^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Using the Pythagorean identity:

$$\mathbf{R}_\theta^{-1} = \frac{1}{1} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The inverse matrix $\mathbf{R}_\theta^{-1}$ will be defined for all angles θ, hence the rotation matrix is always invertible.

The inverse matrix $\mathbf{R}_\theta^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$.

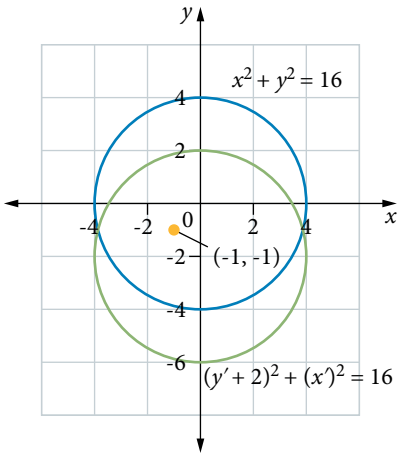
Also:

$$\mathbf{R}_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \mathbf{R}_\theta^{-1}$$

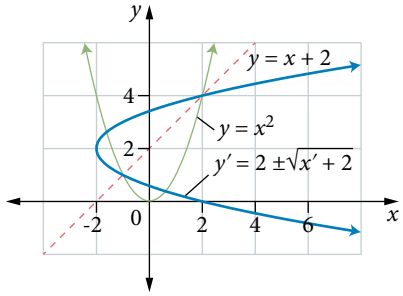
Hence, $\mathbf{R}_{-\theta}$ is the inverse of the rotation matrix $\mathbf{R}_\theta$.

34 $p = -1$

35 The image is the curve $(y'+2)^2+(x')^2=16$.



36 The image is the curve $y'=2\pm\sqrt{x'+2}$.



37 $\cos(2\theta)=\cos(\theta+\theta)$

$$\mathbf{R}_\theta=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$\mathbf{R}_{2\theta}=\begin{bmatrix}\cos(2\theta) & -\sin(\theta+\theta)\\ \sin(\theta+\theta) & \cos(\theta+\theta)\end{bmatrix}$$

A rotation of θ followed by another rotation of θ can be combined into a single transformation matrix using the rule $\mathbf{L}_c=\mathbf{L}_2\mathbf{L}$:

$$\mathbf{L}_c=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2(\theta)-\sin^2(\theta) & -2\sin(\theta)\cos(\theta)\\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta)-\sin^2(\theta)\end{bmatrix}$$

A rotation of $(\theta+\theta)$ can also be expressed by the matrix

$$\mathbf{R}_{2\theta}=\begin{bmatrix}\cos(2\theta) & -\sin(\theta+\theta)\\ \sin(\theta+\theta) & \cos(\theta+\theta)\end{bmatrix}.$$

Therefore:

$$\begin{bmatrix}\cos(2\theta) & -\sin(\theta+\theta)\\ \sin(\theta+\theta) & \cos(\theta+\theta)\end{bmatrix}=\begin{bmatrix}\cos^2(\theta)-\sin^2(\theta) & -2\sin(\theta)\cos(\theta)\\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta)-\sin^2(\theta)\end{bmatrix}$$

From element$_{11}$ of both matrices:

$\cos(\theta+\theta)=\cos^2(\theta)-\sin^2(\theta)$

$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$

38 $\sin(\alpha-\beta)=\sin(\alpha+(-\beta))$

$$\mathbf{R}_\alpha=\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$$

$$\mathbf{R}_{-\beta}=\begin{bmatrix}\cos(-\beta) & -\sin(-\beta)\\ \sin(-\beta) & \cos(-\beta)\end{bmatrix}=\begin{bmatrix}\cos(\beta) & \sin(\beta)\\ -\sin(\beta) & \cos(\beta)\end{bmatrix}$$

$$\mathbf{R}_{\alpha-\beta}=\begin{bmatrix}\cos(\alpha-\beta) & -\sin(\alpha-\beta)\\ \sin(\alpha-\beta) & \cos(\alpha-\beta)\end{bmatrix}$$

A rotation of α followed by a rotation of $-\beta$ can be combined into a single transformation matrix using the rule $\mathbf{L}_c=\mathbf{L}_2\mathbf{L}$:

$$\mathbf{L}_c=\begin{bmatrix}\cos(\beta) & \sin(\beta)\\ -\sin(\beta) & \cos(\beta)\end{bmatrix}\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ -\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) & \sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta)\end{bmatrix}$$

A rotation of $(\alpha+(-\beta))$ can also be represented by the matrix

$$\mathbf{R}_{\alpha-\beta}=\begin{bmatrix}\cos(\alpha-\beta) & -\sin(\alpha-\beta)\\ \sin(\alpha-\beta) & \cos(\alpha-\beta)\end{bmatrix}$$

Therefore:

$$\begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ -\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) & \sin(\alpha)\sin(\beta)+\cos(\alpha)\cos(\beta) \end{bmatrix}$$

From element$_{21}$ of both matrices:

$\sin(\alpha-\beta) = -\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta)$

$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

39 $p = \frac{1}{2}\left(5+\sqrt{41}\right)$, $p = \frac{1}{2}\left(5-\sqrt{41}\right)$, $p = 4$, $p = 1$

Exam review: Unit 2

PAPER 1: TECHNOLOGY-FREE

1 **(a)** 7 **(b)** $\overline{u} = 3+6i$

(c) $|v| = \sqrt{10}$ **(d)** $\text{Arg}(u) = -1.11$

2 **(a)** $14+21i$ **(b)** $-2-48i$ **(c)** $16+2i$

3 **(a)** $2-3i$ **(b)** $19+4i$ **(c)** $17+7i$

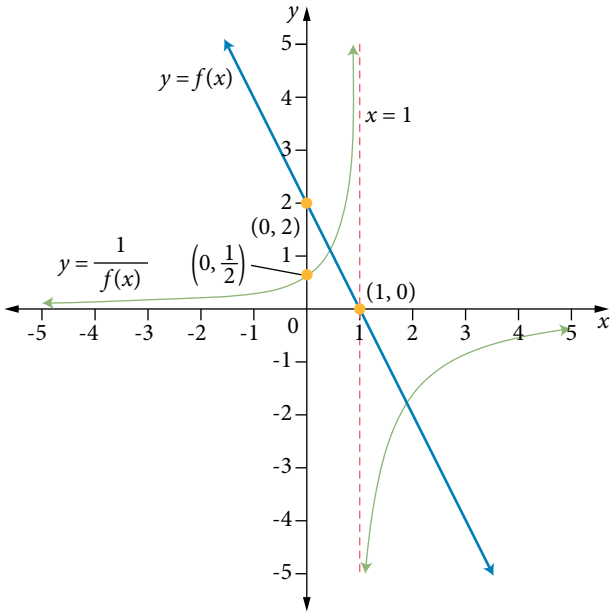
(d) $5+12i$ **(e)** $24+10i$ **(f)** 26

(g) 13 **(h)** $13+26i$

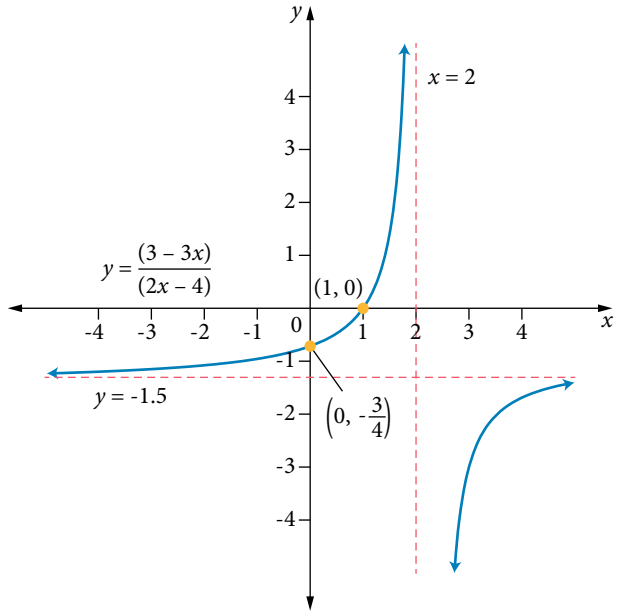
4 **(a)** $10\,\text{cis}\left(\frac{3\pi}{8}\right)$ **(b)** $3\,\text{cis}\left(\frac{4\pi}{5}\right)$ **(c)** $9\,\text{cis}\left(\frac{\pi}{3}\right)$

5 **(a)** -2 **(b)** 1 **(c)** 0

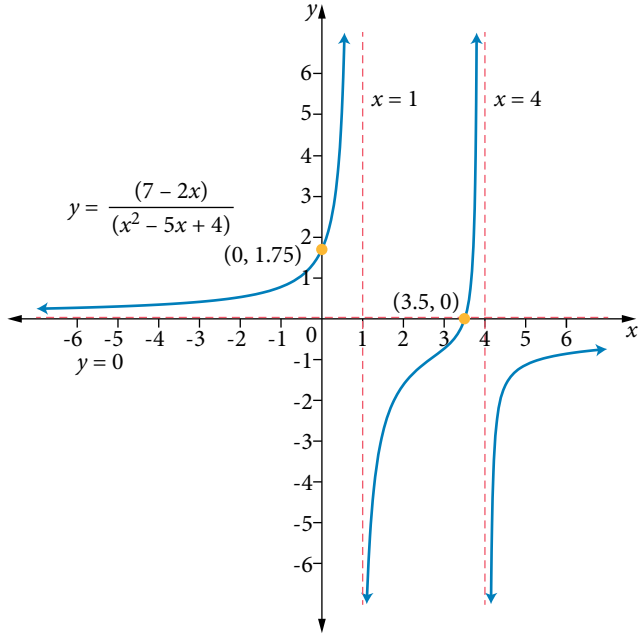
6



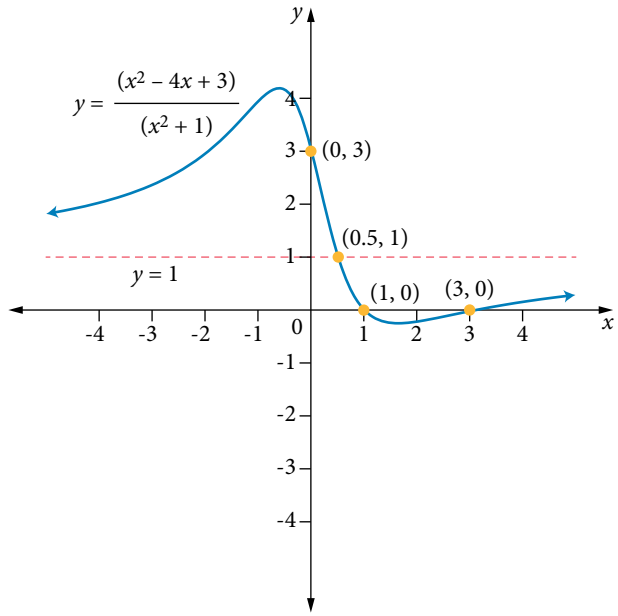
7 **(a)**



(b)



(c)



8 **(a)** $\frac{3\sqrt{2}}{4}$ **(b)** $\frac{7}{9}$ **(c)** $\frac{-\sqrt{15}}{8}$

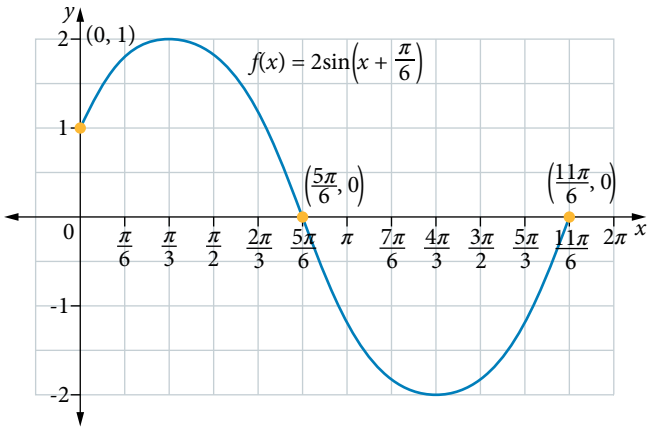
9 **(a)** $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

(b) $x = 0, \pi, 2\pi$ or $\frac{7\pi}{6}, \frac{11\pi}{6}$ **(c)** $x = \frac{\pi}{2}, \pi$

10 **(a)** $f(x) = 2\sin\left(x + \frac{\pi}{6}\right)$ **(b)** $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

(c) $x = \frac{\pi}{3}, \frac{4\pi}{3}$ **(d)** $(0, 1)\ (2\pi, 1)$

(e) period 2π; amplitude 2

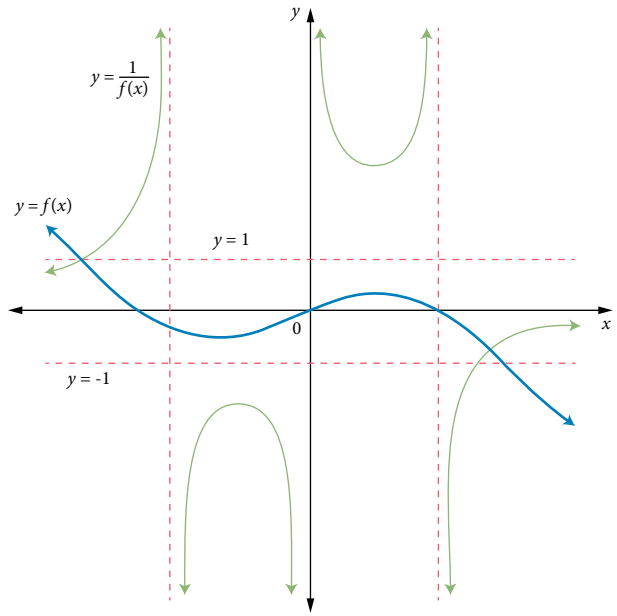


11 **(a)** $\frac{1}{2}\mathbf{A} + 3\mathbf{B} = \begin{bmatrix} 5 & -4 \\ 10 & -21 \end{bmatrix}$ **(b)** $\mathbf{A}^2 = \begin{bmatrix} 0 & -80 \\ 20 & 20 \end{bmatrix}$

(c) $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix}$

12 **(a)** $\frac{7}{13} + \frac{4}{13}i$ **(b)** $\frac{9}{13} + \frac{7}{13}i$ **(c)** $\frac{1}{5} + \frac{3}{5}i$ **(d)** $-1 - \frac{4}{3}i$

13



14 $x = -3$ and $y = 1$

15 $z = \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$ and $z = -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$

16
$$z^2 + 3z + \frac{17}{2} = z^2 + 3z + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{17}{2}$$
$$= \left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{17}{2}$$
$$= \left(z + \frac{3}{2}\right)^2 + \frac{25}{4}$$
$$= \left(z + \frac{3}{2}\right)^2 - \frac{25i^2}{4}$$
$$= \left(z + \frac{3}{2}\right)^2 - \left(\frac{5i}{2}\right)^2$$
$$= \left(z + \frac{3}{2} - \frac{5i}{2}\right)\left(z + \frac{3}{2} + \frac{5i}{2}\right)$$
$$= \frac{1}{2}(2z + 3 - 5i)\frac{1}{2}(2z + 3 + 5i)$$
$$= \frac{1}{4}(2z + 3 - 5i)(2z + 3 + 5i)$$

PAPER 2: TECHNOLOGY-ACTIVE

1 **(a)** $\sin(6A) + \sin(4A)$ **(b)** $\cos(10A) + \cos(4A)$

2 **(a)** 3×2 matrix **(b)** $b_{32} = 8$ **(c)** 11 is b_{21}.

3 $5\left[\left(x - \frac{\sqrt{5} - i}{\sqrt{5}}\right)\left(x - \frac{\sqrt{5} + i}{\sqrt{5}}\right)\right]$

4 **(a)** 0.54 **(b)** 1.00 **(c)** 0.20

5 **(a)** Left-hand side:

$$\left(\tan\left(\frac{\pi}{12}\right)+\sec\left(\frac{\pi}{12}\right)\right)^2$$

$= 1.6984\ldots$

Right-hand side:

$$\frac{1+\sin\left(\frac{\pi}{12}\right)}{1-\sin\left(\frac{\pi}{12}\right)} = 1.6984\ldots$$

The identity is correct for $\theta = \frac{\pi}{12}$.

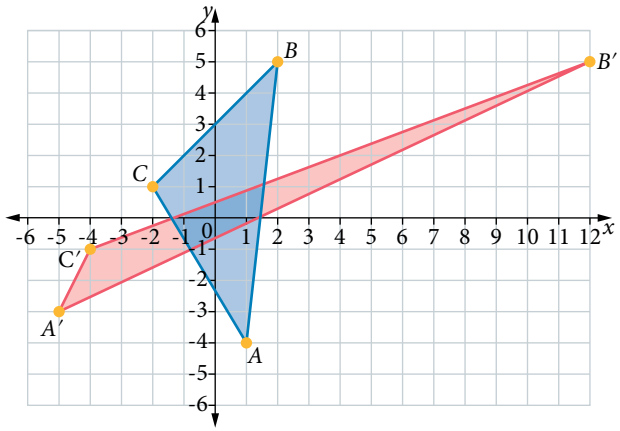
(b) Left-hand side:

$$\begin{aligned}\left(\tan(\theta)+\sec(\theta)\right)^2 &= \tan^2(\theta)+2\tan(\theta)\sec(\theta)+\sec^2(\theta)\\ &= \frac{\sin^2(\theta)+2\sin(\theta)+1}{\cos^2(\theta)}\\ &= \frac{\left(1+\sin(\theta)\right)^2}{1-\sin^2(\theta)}\\ &= \frac{\left(1+\sin(\theta)\right)^2}{\left(1-\sin(\theta)\right)\left(1+\sin(\theta)\right)}\\ &= \frac{1+\sin(\theta)}{1-\sin(\theta)}\end{aligned}$$

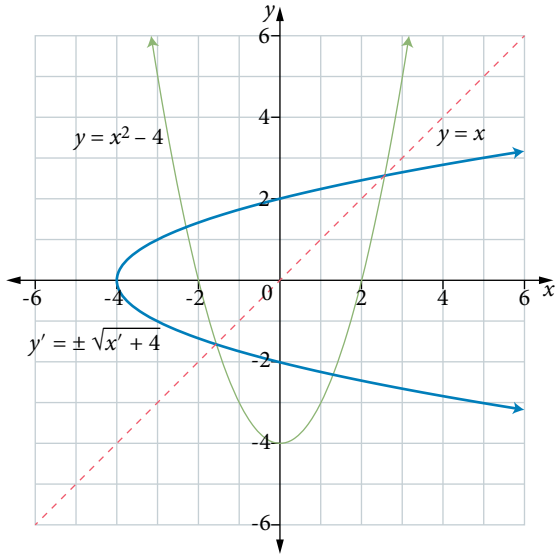
6 **(a)** $\mathbf{X} = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$ **(b)** $\mathbf{X} = \begin{bmatrix} 4 & -1 \\ -7 & 1 \end{bmatrix}$

7 $x = 2, y = -2, z = -1$

8 The image is $\Delta A'B'C'$ formed by points $A'(-5, -3)$, $B'(12, 5)$ and $C'(-4, -1)$.



9 $y' = \pm\sqrt{x'+4}$

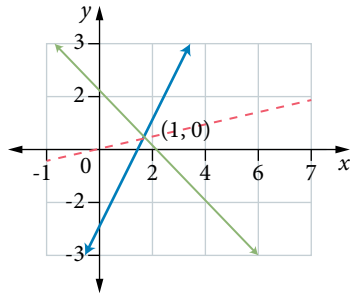


10 **(a)** **(i)** $1-2x^2$ **(ii)** $\frac{1-x^2}{x^2}$ **(iii)** $\frac{2x\sqrt{1-x^2}}{\left(2x^2-1\right)}$

(b) $\tan(2\theta) = -0.21$

11 $x = -3.6$

12 **(a)** $1.178x' - 0.782y' = 2$ (3 d.p.)



(b) $m = 0.100$, $m = -9.999$ (3 d.p.)

13 If two transformations **A** and **B** are the inverse of one another, then $\mathbf{AB} = \mathbf{I}$.

First, consider a reflection in the y-axis followed by reflection in the x-axis. When combined, this transformation is:

$$\begin{aligned}\mathbf{L}_c &= \mathbf{L}_2\mathbf{L}_1\\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned}$$

Next, consider a rotation of 180° about the origin. The transformation representing this linear transformation is:

$$\mathbf{R}_{180°} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now:

$$\mathbf{L}_c \times \mathbf{R}_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Therefore, a reflection in the y-axis followed by reflection in the x-axis *does* represent the inverse of a single rotation of 180° about the origin.

Exam review: Units 1 & 2

PAPER 1: TECHNOLOGY-FREE

1 $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$\text{LHS} = n(A \cup B \cup C)$
$= 8$
$\text{RHS} = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
$= 5 + 4 + 3 - 2 - 2 - 1 + 1$
$= 8$
LHS = RHS

2 (a) $\dfrac{n(n-1)}{2}$ (b) $\dfrac{(n+1)n(n-1)}{6}$

3 (a) 20 (b) 1

4 $-\hat{\boldsymbol{i}} - 11\hat{\boldsymbol{j}}$

5 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 18$ (b) $\boldsymbol{a} \cdot \boldsymbol{b} = 2$

6 (a) $19 + 33i$ (b) $\dfrac{1}{5} - \dfrac{17}{5}i$

7 (a) $|z| = 2$ (b) $\text{Arg}(z) = -\dfrac{\pi}{4}$

8 $\begin{bmatrix} -18 & 8 \\ 4 & -10 \end{bmatrix}$

9 (a) $\dfrac{5}{9}$ (b) $\dfrac{113}{900}$

10 (a) $y = 73^\circ$ (b) $x = 62^\circ$

11 $y = 46^\circ$

12 (a) $-\dfrac{1}{2}$ (b) $-\sqrt{3}$ (c) $-\sqrt{2}$

13 (a) $\tan(\theta) = -\dfrac{24}{7}$ (b) $\text{cosec}(\theta) = \dfrac{25}{24}$

14 (a) $x = 14$ or $x = -8$ (b) $x = 7$ or $x = -1$

15 $x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}$

16 If two numbers have the same remainder when divided by 6, then their difference will be divisible by 6.

Think of the possible remainders after division by 6 as boxes (there are 6 of them: 0, 1,…, 5), and the 50 integers as objects that must be assigned to these boxes based on their remainders. Using the Pigeon-hole principle, there must be at least one box containing at least $\left\lceil \dfrac{50}{6} \right\rceil = 9$ objects. This means that at least 9 of the integers must have the same remainder when divided by 6, and hence, their difference must be divisible by 6.

17 $\exists$ a real number x such that $x > 6$ or $x < -3$, but $(x-6)(x+3) \leq 0$.
The original statement is true because if $x > 6$, both $x - 6$ and $x + 3$ must be positive, while if $x < -3$, both $x - 6$ and $x + 3$ must be negative; in either case, the product is positive.

18 Let $n = 1000a + 100b + 10c + d$ be a four-digit number with digits a, b, c and d. The number obtained by removing the final digit, and adding twice this digit to the remainder of the number is $p = 100a + 10b + c + 2d$.

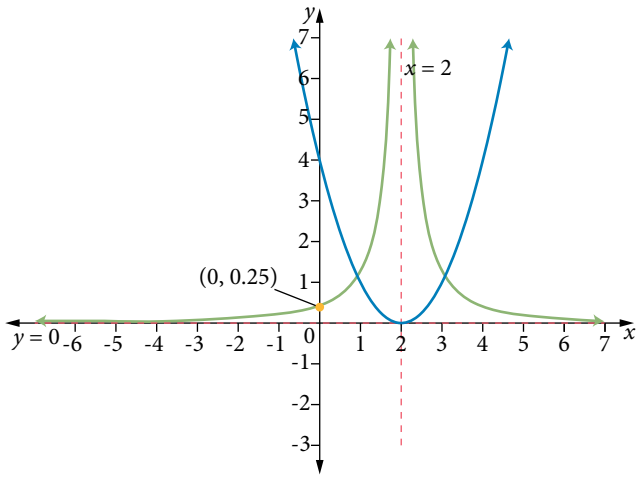
First assume that n is divisible by 19.
Thus, $1000a + 100b + 10c + d = 19k$, where k is an integer.
$p = 100a + 10b + c + 2d$
$= 100a + 10b + c + 2(19k - 1000a - 100b - 10c)$
$= 100a + 10b + c + 38k - 2000a - 200b - 20c$
$= -1900a - 190b - 19c + 38k$
$= 19(-100a + 10b + c - 2k)$

Therefore, p is divisible by 19.

Conversely, suppose that p is divisible by 19.
Then $100a + 10b + c + 2d = 19k$, where k is an integer.
$n = 1000a + 100b + 10c + d$
$= 1000a + 100b + 10(19k - 100a - 10b - 2d) + d$
$= 1000a + 100b + 190k - 1000a - 100b - 20d + d$
$= 190k - 19d$
$= 19(10 - d)$

Therefore, n is divisible by 19.

19



20 (a) $\hat{\boldsymbol{a}} = \dfrac{1}{5}(3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}})$ (b) $\boldsymbol{b} = -\dfrac{36}{5}\hat{\boldsymbol{i}} - \dfrac{48}{5}\hat{\boldsymbol{j}}$

21 (a) $6\,\text{cis}\left(-\dfrac{5\pi}{6}\right)$ (b) $3\,\text{cis}\left(\dfrac{\pi}{6}\right)$

22 $x = 4$

23 $y = \dfrac{fbc(x-d)(x-e)}{de(x-b)(x-c)}$

PAPER 2: TECHNOLOGY-ACTIVE

1 (a) 729 (b) 504 (c) 392

2 (a) 70 (b) 55

3 $x = 2$, $y = -3$ and $z = 1$

4 9

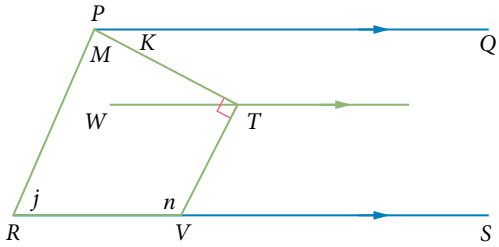
5 $a = 5$ and 1; $b = -8$ and -2

6 **(a)** As cointerior angles between parallel lines are equal:

$m + j + k = 180°$

$m + j = 180° - k$

(b) Construct TW parallel to PQ and RS.



$\angle PTW = k$ (Alternate angles on parallel lines are equal.)

$\angle TVS + n = 180°$ (Angles on a straight line are supplementary.)

$\angle TVS = 180° - n$

$\angle WTV = 180° - n$ (Alternate angles on parallel lines are equal.)

$180° - n + k = 90°$ (given $\angle PTV = 90°$)

$n - k = 90°$

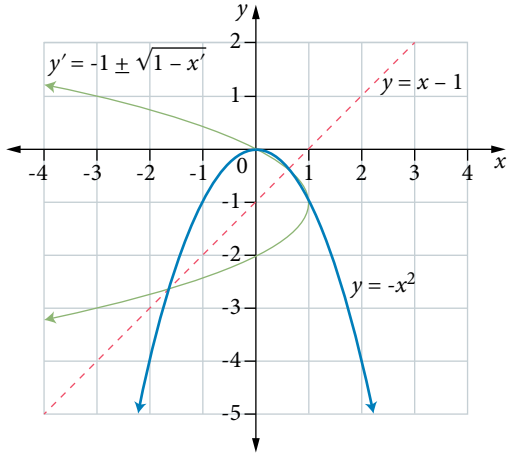
7 $z = 3 + i$ or $z = -3 - i$

8 **(a)** period 4π; amplitude 3

(b) Dilation by a factor of $\frac{1}{3}$ from the x-axis. Dilation by a factor of $\frac{1}{2}$ from the y-axis. Translation of $\frac{\pi}{4}$ units to the right (parallel to the x-axis)

(c) $\mathbb{R} / \left(-\frac{1}{3}, \frac{1}{3}\right)$

9 The image is the curve $y' = -1 \pm \sqrt{1 - x'}$.



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