ACADEMIC ASSOCIATES

STUDY GUIDE

MATHEMATICS METHODS

YEAR 12 ATAR COURSE – UNITS 3 & 4

REVISED EDITION

Greg Hill

ACADEMIC ASSOCIATES

WACE Study Guide

MATHEMATICS METHODS

YR 12 ATAR COURSE

Greg Hill

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PREFACE

The purpose of this book is to assist students with their preparation for tests and examinations in the new Mathematics Methods course for Units 3 and 4.

The ***Syllabus Checklist*** indicates to students which skills they must have acquired and the objectives they need to meet under each of the major headings of the course.

The ***Worked Examples*** are presented in a detailed manner, with brief notes and explanations being used to amplify the understanding required for the particular question. Some of these worked examples could be used in the written notes that students are permitted to take into an examination.

The ***Problems to Solve*** section in each chapter provide students with a broad range of questions without the repetitive nature of problems usually associated with a course textbook.

The ***Trial Tests*** are an additional component to this book, and allow students to familiarise themselves with test questions. Suggested times are given for these tests, and students should be encouraged to adhere to these times to prepare properly for final examinations. Fully worked solutions are provided for students to receive immediate, accurate and useful feedback on their performance. Tests are provided for both the resource free (no calculator) and the resource rich (calculator allowed) components of the assessment.

The **Examination Stye Questions** are provided enabling students to practice questions similar to those found in examinations. A wide variety of resource free and resource rich questions on each of the topics in this course are available for students. A fully worked set of solutions are given for each of these questions.

I hope this study guide will help students to better understand the concepts they will encounter and to achieve greater success in this course.

Greg Hill
April 2021

DIFFERENTIATION

1

Syllabus Checklist

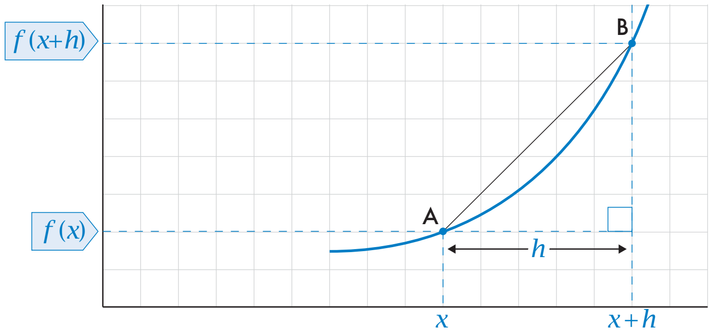
By the end of this chapter, you should be able to:

- ☐ examine and use the product and quotient rules
- ☐ examine the notion of composition of functions and use the chain rule for determining the derivatives of composite functions.

FORMULAE AND DEFINITIONS

Mathematicians have developed a method to find the gradient at a particular point on a curve.

Consider the point A(x, $f(x)$) on a given curve as shown on the diagram below:



Suppose point B is located further along the curve – an increase of 'h' units. Point B will have coordinates B (($x + h$), $f(x + h)$)

The straight line AB is called a chord of the curve.

The slope of this curve can be calculated by:

$$\frac{rise}{run} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

As the distance 'h' becomes smaller point B moves towards point A and chord AB approaches a limiting value at point A.

ie. As $h \to 0$ the gradient of chord AB approaches the gradient of the tangent line at point A.

This limiting function is known as the **derivative**.

ie $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Example

Given $f(x) = x^2$, find $f'(x)$ using the limiting chord process

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If $f(x) = x^2$ then $f(x+h) = (x+h)^2$

$$\therefore f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h$$

As $h \to 0$ the **gradient function** $f'(x) = 2x$

Derivative Notation

The derivative of a function is commonly represented by:

$f'(x) \qquad f' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad y' \qquad \frac{d}{dx}[f(x)]$

Derivative Rules

The following table lists the rules needed to find the derivative of a function.

	Original Function		Derivative Function
1.	‘c’ is a constant	$f(x) = c$	$f'(x) = 0$
2.		$f(x) = mx$	$f'(x) = m$
3.		$f(x) = x^n$	$f'(x) = nx^{n-1}$
4.		$f(x) = ax^n$	$f'(x) = nax^{n-1}$
5.	Sum rule	$y = f(x) \pm g(x)$	$\frac{dy}{dx} = f'(x) \pm g'(x)$
6.	Product rule	$y = f(x) \cdot g(x)$	$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$
7.	Quotient rule	$y = \frac{f(x)}{g(x)}$	$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
8.	Chain rule	$y = [f(x)]^n$	$\frac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$

Second Derivative

A second derivative of a function can be found by differentiating twice. The notation for second derivative includes

$f''(x)$ and $\frac{d^2y}{dx^2}$

Worked Examples

1.1 Find the derivatives of each of the following:

(a) $y = 3x^7$

(b) $y = \dfrac{10}{7x^3}$

(c) $y = \dfrac{-5}{\sqrt[3]{x}}$

(a) $y = 3x^7$ *use rule:* $y = ax^n$

$\dfrac{dy}{dx} = 21x^6$ $\dfrac{dy}{dx} = nax^{n-1}$

(b) $y = \dfrac{10}{7x^3}$ *firstly rearrange and move x^3 to the numerator*

$y = \dfrac{10x^{-3}}{7}$

$\dfrac{dy}{dx} = \dfrac{-30x^{-4}}{7}$ *use above rule to differentiate*

$\dfrac{dy}{dx} = \dfrac{-30}{7x^4}$ *ensure indices are positive*

(c) $y = \dfrac{-5}{\sqrt[3]{x}}$

$y = \dfrac{-5}{x^{\frac{1}{3}}}$ *change to power form*

$y = -5x^{-\frac{1}{3}}$ *move $x^{\frac{1}{3}}$ to the numerator*

$\dfrac{dy}{dx} = \dfrac{5}{3}x^{-\frac{4}{3}}$ *use above rule to differentiate*

$\dfrac{dy}{dx} = \dfrac{5}{3x^{\frac{4}{3}}}$ *ensure indices are positive*

1.2 Find $f'(x)$ for each of the following:

(a) $f(x) = \sqrt{x}\,(x^2 + 2)$

(b) $f(x) = \dfrac{x^4 + 7}{x}$

(a) $f(x) = \sqrt{x}\,(x^2 + 2)$

$f(x) = x^{\frac{5}{2}} + 2x^{\frac{1}{2}}$ *multiply out brackets*

$f'(x) = \dfrac{5}{2}x^{\frac{3}{2}} + x^{-\frac{1}{2}}$ *differentiate using the sum rule*

$f'(x) = \dfrac{5x^{\frac{3}{2}}}{2} + \dfrac{1}{\sqrt{x}}$ *ensure indices are positive*

(b) $f(x) = \dfrac{x^4 + 7}{x}$

$f(x) = \dfrac{x^4}{x} + \dfrac{7}{x}$ *separate into individual fractions*

$f(x) = x^3 + 7x^{-1}$ *rearrange*

$f'(x) = 3x^2 - 7x^{-2}$ *differentiate using the sum rule*

$f'(x) = 3x^2 - \dfrac{7}{x^2}$ *ensure indices are positive*

1.3 Find $\frac{dy}{dx}$ for each of the functions below:

(a) $y = (x^2 - 6)(x^3 + 4x^2)$

(b) $y = \frac{5x}{x^3 + 7}$

(a) $y = (x^2 - 6)(x^3 + 4x^2)$

$\frac{dy}{dx} = (2x)(x^3 + 4x^2) + (x^2 - 6)(3x^2 + 8x)$ *use the product rule to differentiate*

$\frac{dy}{dx} = 2x^4 + 8x^3 + 3x^4 + 8x^3 - 18x^2 - 48x$ *simplify by expanding brackets*

$\frac{dy}{dx} = 5x^4 + 16x^3 - 18x^2 - 48x$ *simplify*

The final answer has been simplified fully but this may not always be necessary.

(b) $y = \frac{5x}{x^3 + 7}$

$\frac{dy}{dx} = \frac{5(x^3 + 7) - (5x)(3x^2)}{(x^3 + 7)^2}$ *use quotient rule to differentiate*

$\frac{dy}{dx} = \frac{5x^3 + 35 - 15x^3}{(x^3 + 7)^2}$ *simplify by expanding brackets*

$\frac{dy}{dx} = \frac{-10x^3 + 35}{(x^3 + 7)^2}$ *simplify*

1.4 Differentiate the following functions:

(a) $y = (x + 7)^5$

(b) $y = \frac{4}{(2x - 3)^3}$

(a) $y = (x + 7)^5$

$\frac{dy}{dx} = 5(x + 7)^4$ *use the chain rule to differentiate*

(b) $y = \frac{4}{(2x - 3)^3}$

$y = 4(2x - 3)^{-3}$ *move $(2x - 3)^3$ to the numerator*

$\frac{dy}{dx} = -12(2x - 3)^{-4}(2)$ *differentiate using the chain rule*

$\frac{dy}{dx} = \frac{-24}{(2x - 3)^4}$ *simplify and ensure positive indices*

1.5 Find the equation of the tangent line to the curve $y = \sqrt{3x + 1}$ at the point where $x = 5$.

To find the gradient we require the gradient function

$y = (3x + 1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3x + 1)^{-\frac{1}{2}} 3$

$\frac{dy}{dx} = \frac{3}{2\sqrt{3x + 1}}$

Substitute in x = 5 to find the gradient

$\frac{dy}{dx} = \frac{3}{8}$

To find the equation of the tangent line a coordinate point is required.

when $x = 5$: $y = \sqrt{3(5)+1}$

$y = 4$

(5, 4)

Equation of tangent line

$y = mx + c$

Substitute in gradient

$y = \frac{3}{8}x + c$

Substitute in coordinate to find 'c'

$y = \frac{3}{8}x + c$

$4 = \frac{3}{8}(5) + c$

$c = \frac{17}{8}$

$\therefore$ equation of tangent line is $y = \frac{3}{8}x + \frac{17}{8}$

1.6 The tangent to the curve $y = -x^2 - 5x + 6$ at point T is perpendicular to the line $x = -5y + 10$. Find the coordinates of T.

Gradient of line

$-5y = x - 10$

$y = -\frac{1}{5}x + 2$

$\text{gradient} = -\frac{1}{5}$

Perpendicular gradient = 5

Gradient function of curve $y = -x^2 - 5x + 6$ is:

$\frac{dy}{dx} = -2x - 5$

As gradient = 5

$5 = -2x - 5$

$2x = -10$

$x = -5$

Substitute x value into original equation to find y value

When $x = -5$

$y = -(-5)^2 - 5(-5) + 6$

$y = -25 + 25 + 6$

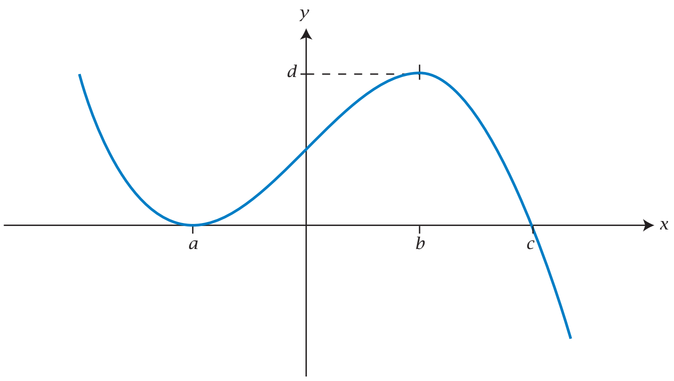
$y = 6$

Coordinate T = (–5, 6)

PROBLEMS TO SOLVE

Chapter 1: Differentiation

1. Use the graph below to answer the following questions.



(a) Determine the x intercept(s).

(b) Which points have a gradient of zero?

(c) Where on the curve is the gradient

i. negative?

ii. positive?

2. Use $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find the gradient function of $y = 2x^3$

3. Find the gradient function of

(a) $y = 3x^2$

(b) $y = \dfrac{7}{x}$

(c) $y = 4x$

(d) $y = 2$

(e) $= -5x^3$

(f) $y = \dfrac{1}{\sqrt{x}}$

(g) $y = \dfrac{2}{5x^2}$

(h) $y = -\dfrac{6}{\sqrt[3]{x}}$

(i) $y = x^{-\frac{3}{4}}$

(j) $y = 7\sqrt{x}$

4. Determine the gradient on the curve at the indicated point:

(a) $y = 4x^3$ $(-1, -4)$

(b) $y = \dfrac{2}{x}$ $(2, 1)$

(c) $y = \sqrt{x}$ $(4, 2)$

(d) $y = -\dfrac{3}{\sqrt[3]{x}}$ $(27, -1)$

5. Differentiate with respect to x.

(a) $y = 4x + 5$

(b) $y = 3x^2 - 6x$

(c) $y = 5x^3 + 9x^2 - 6$

(d) $y = \frac{x^5}{2} + 7x^4 - x$

(e) $y = \sqrt{x} + x$

(f) $y = x - \frac{3}{x}$

(g) $y = 3\sqrt{x} + \frac{1}{\sqrt{x}}$

(h) $y = 2x^{\frac{5}{2}} - 4x^{\frac{7}{2}} + 2x^{\frac{1}{2}}$

6. Find the gradient of the tangent to the curve $y = x^2 - 3x + 2$ where the curve cuts the x-axis.

7. Find the coordinates of the points on the curve $y = 9x - 3x^3$ where the tangent is parallel to $y = -1$.

8. The curve $y = kx^2 - 7x + 6$ has a gradient of 11 when $x = 3$. Find k.

9. Find

(a) $\frac{dv}{dt}$ given $v = 5t^3 + 4t$

(b) $\frac{dc}{dm}$ given $c = 5m + \frac{4}{m^2}$

(c) $\frac{df}{dg}$ given $f = \frac{1}{\sqrt[4]{g}}$

(d) $\frac{da}{db}$ given $a = \frac{5}{12b^4}$

10. Given $m = 5v$, $v = 3h^2 - 2$ and $h = 2x^3$ find $\frac{dm}{dx}$ using the chain rule.

11. Find the derivatives of the following:

(a) $y = (3x - 2)^2$

(b) $y = \sqrt{4x - 5}$

(c) $y = \frac{1}{4x^2 + 2}$

(d) $y = \frac{3}{(2x^2 - 5)}$

(e) $y = \frac{1}{2 + \sqrt{x}}$

(f) $y = \frac{1}{(\sqrt{x^2 + 1})^{-1}}$

12. Determine the gradient of the following curves at the indicated coordinate point:

(a) $y = x^3 + 2x^2$ (2, 16)

(b) $y = \sqrt{1 - 3x}$ (0, 1)

(c) $y = -\frac{4}{x}$ (2, –2)

(d) $y = (x^2 + 2)^2$ (1, 9)

(e) $y = \frac{1}{(3x+1)^2}$ $(1, \frac{1}{16})$

13. Use the product rule to differentiate the following:

(a) $y = (4x + 2)(2x - 6)$

(b) $y = (3x^2 - 5)(5x + 1)$

(c) $y = (2x^2 + x)(7x - 1)$

(d) $y = x(4x + 1)^2$

(e) $y = (x - 1)^3 (3x + 2)^2$ *(Do not simplify)*

(f) $y = \sqrt{x}\,(2x^2 - 1)^3$ *(Do not simplify)*

14. Use the quotient rule to differentiate the following: *(Do not simplify)*

(a) $y = \frac{7x}{2x+1}$

(b) $y = \frac{x+4}{x-6}$

(c) $y = \frac{3x^2}{7-x}$

(d) $y = \frac{5x^3}{x^2+4}$

(e) $y = \frac{(x+3)^2}{(2x-5)^3}$

(f) $y = \frac{\sqrt{x}}{(3x-5)}$

15. Find $\frac{d^2y}{dx^2}$ for the following:

(a) $y = 4x^3 - 5x^2$

(b) $y = (2x - 9)^4$

(c) $y = 3x^2 - \sqrt{x}$

(d) $y = \frac{x^4}{8} - \frac{x^3}{3} + 2x^2 - 6x$

(e) $y = \frac{1}{\sqrt{x^2 - 9}}$

(f) $y = \frac{x-3}{x}$

16. Find the gradient of the curve $y = 3(2x-5)^{\frac{3}{2}}$ when $x = 3$

17. Find the gradient of the curve

$$y = \left(\sqrt{(3x+1)^2} - \frac{12}{\sqrt[3]{(3x+1)^2}}\right)^2$$

where the curve cuts the y axis.

18. Given $t = 6v$, $v = \frac{1}{p}$ and $p = 4x^3$, use the chain rule to determine $\frac{dt}{dx}$.

19. Determine the value of the constant k if $f(x) = (2x + k)^2$ and $f'(1) = 16$.

20. The curve $y = x^3 - ax^2 + bx + c$ has a y intercept of 5 and at that point the gradient of the curve is 6. If the curve passes through the point (2, 13) find the values of a, b, and c.

21. The curve $y = (2x + 1)^2 (cx + d)$ passes through the point (1, 9) and has a gradient of 3 at that point. Find the values of c and d.

22. Determine the equation of the tangent to $y = \sqrt{x+5}$ at $x = -1$.

23. Find the coordinates of all the points on the curve defined by $y = \frac{6}{(3x+1)^2}$ where the gradient has the value of -2.

24. Given $y = 3f$, $f = 2g^2 + 4$, $g = 6x^2$ determine $\frac{dy}{dx}$ using the chain rule.

25. The function $f(x) = x^3 + ax^2 + 6x + b$ where a and b are constants has tangents at $A(0, b)$ and $B(3, 4)$ that are parallel. Find the values of a and b.

26. Given $y = \frac{2x+3}{x-3}$. Find:

(a) the gradient of the curve when $x = 4$

(b) $\frac{d^2y}{dx^2}$ when $x = 4$.

27. The point $(2, b)$ lies on the curve $y = \frac{a+4x}{3x-5}$ and the gradient at that point is 9. Find the values of a and b.

28. If $y = x + \sqrt{x^2 - 4}$ show that $\left(x^2 - 4\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

29. The function g is defined such that:

- $g(0) = 2$
- $g'(0) = 16$

(a) If $f(x) = g(x) + x$ calculate $f'(x)$ and hence $f'(0)$

(b) If $h(x) = xg(x) - \frac{1}{g(x)}$ calculate $h'(x)$ and hence $h'(0)$.

30. A tangent to the curve $y = 8\sqrt{x} + \frac{1}{2}x - 4$ is drawn at point T. If the tangent is parallel to the line $-3x + 2y = 7$ find the equation of the tangent to the curve at point T.

2 APPLICATIONS OF DIFFERENTIATION

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ use the increments formula: $\delta y = \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x
- ☐ apply the concept of the second derivative as the rate of change of the first derivative function
- ☐ identify acceleration as the second derivative of position with respect to time
- ☐ examine the concepts of concavity and points of inflection and their relationship with the second derivative
- ☐ apply the second derivative test for determining local maxima and minima
- ☐ sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection
- ☐ solve optimisation problems from a wide variety of fields using first and second derivatives.

FORMULAE AND DEFINITIONS

Average Rate of Change

The average rate of change of y with respect to x is calculated over an interval of time i.e. $a \le x \le b$.

If $y = f(x)$ then the average rate of change is calculated by: $\frac{f(b) - f(a)}{b - a}$

Instantaneous Rate of Change

The derivative $\frac{dy}{dx}$ is said to be the instantaneous rate of change of y with respect to x.

i.e. $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

Small Change

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

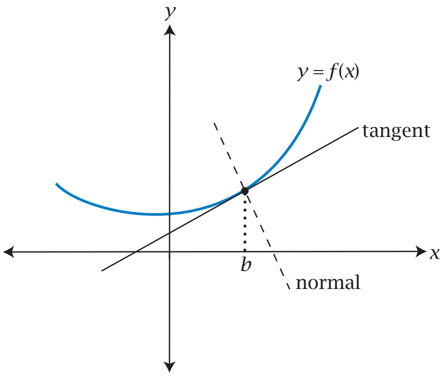
Where δy is the small change in y and δx is the corresponding small change in x.

If δx is small then

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$\therefore \delta y = \frac{dy}{dx} \cdot \delta x$$

Tangents



Equation of tangent line: $y = mx + c$

- $f'(b) = m_{\text{tangent}}$
- $-\dfrac{1}{f'(b)} = m_{\text{normal}}$

Optimisation

Calculus is used to determine the optimum solution – the maximum profit, minimum cost or other similar quantities.

Method

1. Draw a diagram if possible.
2. Define the variables.
3. Define the function to be optimised.
4. Use relationships between variables to express the function in terms of one variable. i.e. $y = f(x)$
5. Determine stationary points using
   - Calculator **or**
   - Calculus techniques
     - Solve the equation $f'(x) = 0$
6. Determine the nature of the stationary point using either
   - sign test
   - second derivative test
     - if $f''(x) > 0$ stationary point is a local minimum.
     - if $f''(x) < 0$ stationary point is a local maximum.
7. If the domain is restricted to $a \leq x \leq b$ then check end points for global maximum or minimum.
8. State your conclusion in words.

Economic Applications

Three important functions are:

the cost function $C(x)$

the revenue function $R(x)$

the profit function $P(x) = R(x) - C(x)$

If $R(x) > C(x)$ i.e. $P(x) > 0 \to$ profit

If $R(x) < C(x)$ i.e. $P(x) < 0 \to$ loss

If $R(x) = C(x)$ i.e. $P(x) = 0 \to$ breakeven

Rates of Change

The rate of change of profit with respect to the number of units sold is called **Marginal Profit**: $P'(x)$.

Marginal Revenue: $R'(x)$

Marginal Cost: $C'(x)$ is the rate of change of total cost with respect to the total number of units produced and sold.

$C'(x)$ gives the approximate cost of producing one more unit after the x^{th} unit has been produced and sold

Average Cost

The average cost of producing x items is given by: $\dfrac{C(x)}{x}$.

Rectilinear Motion

Rectilinear motion involves the motion of a particle in a straight line.

Displacement is the distance the particle is from the origin as a function of time. Denoted as $x(t)$

Velocity is the rate of change of displacement. i.e. $v(t) = \dfrac{dx}{dt}$

Acceleration is the rate of change of velocity. i.e. $a(t) = \dfrac{dv}{dt}$

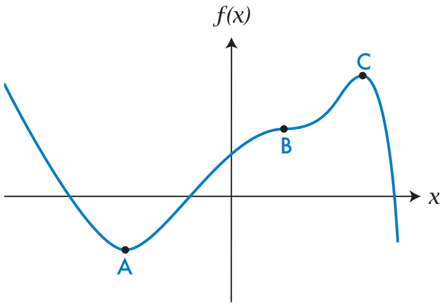
If $v(t) > 0$	particle is moving in a positive direction to the right. Speed is $\lvert v \rvert$.
If $v(t) = 0$	particle is stationary and may change direction.
If $v(t) < 0$	particle is moving in a negative direction to the left. Speed is $\lvert v \rvert$.

If $v(t) > 0$,	$a(t) > 0$	particle is accelerating to the right.
If $v(t) > 0$,	$a(t) < 0$	particle is decelerating to the right.
If $v(t) < 0$,	$a(t) > 0$	particle is decelerating to the left.
If $v(t) < 0$,	$a(t) < 0$	particle is accelerating to the left.
If $a(t) = 0$		particle is moving with constant velocity.

Graphing Features

Graphing a function requires all the key features of a function to be determined. Calculus and a graphics calculator will assist in finding these features.

Stationary points



Stationary points are located when the first derivative is zero. i.e. $f'(x) = 0$.

Types of stationary points:

- Local maximum – point C
- Local minimum – point A
- Horizontal point of inflection – point B

Classifying Stationary Points

There are two methods for determining the nature of the stationary points.

- **Sign test**
  Determine the slope of the function (positive or negative) either side of where $f'(x) = 0$.

negative slope	$f'(x) = 0$	positive slope	Indicates a minimum turning point
positive slope	$f'(x) = 0$	negative slope	Indicates a maximum turning point
	$f'(x) = 0$	positive slope	Indicates a horizontal point of inflection
	$f'(x) = 0$	negative slope	Indicates a horizontal point of inflection

- **Second Derivative Test**
  ** If $f'(x) = 0$ and $f''(x) < 0$ the stationary point is a MAXIMUM (Concave down)*

  ** If $f'(x) = 0$ and $f''(x) > 0$ the stationary point is a MINIMUM (Concave up)*

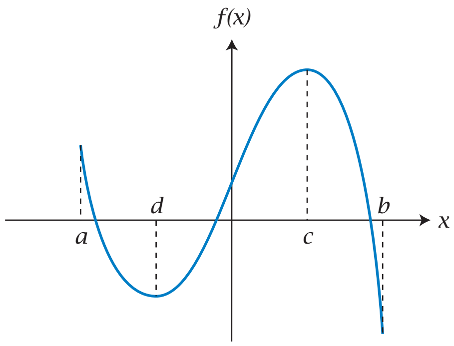
  ** If $f'(x) = 0$ and $f''(x) = 0$ the stationary point is a possible HORIZONTAL POINT OF INFLECTION – further investigation needed using Sign Test.*

Points of inflection

A point of inflection occurs when there is a change in concavity. These are found when $f''(x) = 0$.

Global Maximum and Minimum

The global maximum and minimum can be found when a function is defined over a given closed interval. This global maximum or minimum can occur at the end points or turning points.



The function $f(x)$ is defined over the interval $a \leq x \leq b$.

- Global maximum occurs when $x = c$.
- Global minimum occurs when $x = b$.

2.1 Stationary Points

Determine the stationary points and points of inflection for:

$y = 2x^5 - 5x^4 + 7$

Stationary points: when $\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = 10x^4 - 20x^3$$

$$\therefore 10x^4 - 20x^3 = 0$$

$$10x^3(x - 2) = 0$$

$$\therefore x = 0, \ x = 2$$

Substitute into the **original** equation to find y

$\therefore$ Coordinates of stationary points are (0, 7) and (2, –9)

Nature of stationary points

Use sign test or second derivative test.

Coordinate (0, 7) – Sign test

x	-1	0	1
$\frac{dy}{dx}$	+ ve	0	– ve

$\therefore$ local maximum at (0, 7)

Coordinate (2, –9) – Second derivative test

$$\frac{dy}{dx} = 10x^4 - 20x^3$$

$$\frac{d^2y}{dx^2} = 40x^3 - 60x^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 80 > 0$$

$\therefore$ (2, –9) is a local minimum.

Points of inflection

For point(s) of inflection determine when $\frac{d^2y}{dx^2} = 0$

i.e. $40x^3 - 60x^2 = 0$

$$20x^2(2x - 3) = 0$$

$$\therefore x = 0, \ x = \frac{3}{2}$$

Substitute x values into the original equation to determine the y values.

Points are $(0, 7)$ and $\left(\frac{3}{2}, \frac{-25}{8}\right)$. As $(0, 7)$ is a local maximum the point of inflection is $\left(\frac{3}{2}, \frac{-25}{8}\right)$.

2.2 The function $f(x) = x^3 + ax^2 + bx + 270$ with constants a and b has an inflection point when $x = -1$ and a stationary point when $x = -6$. Find a and b and the remaining stationary point.

$f(x) = x^3 + ax^2 + bx + 270$

Stationary points when $f'(x) = 0$

$f'(x) = 3x^2 + 2ax + b$

Inflection points when $f''(x) = 0$

$f''(x) = 6x + 2a$

Inflection point when $x = -1$

$\therefore 6x + 2a = 0$

$6(-1) + 2a = 0$

$2a = 6$

$a = 3$

Stationary point when $x = -6$

$3x^2 + 2ax + b = 0$

$3(-6)^2 + 2(3)(-6) + b = 0$

$108 - 36 + b = 0$

$72 + b = 0$

$b = -72$

To find the remaining stationary point $f'(x) = 0$

$\therefore 3x^2 + 6x - 72 = 0$

$3(x^2 + 2x - 24) = 0$

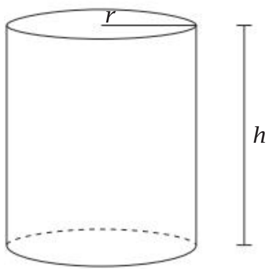
$(x + 6)(x - 4) = 0$

$\therefore x = -6, \; x = 4$

Remaining stationary point is (4, 94)

2.3 A can of soft drink has a volume of 400 cm^3. Find the height and radius which will minimise the surface area using calculus methods. Determine the minimum surface area.

Steps: Draw a diagram



Determine the given equation

$v = \pi r^2 h$

$\therefore 400 = \pi r^2 h \; \ldots$ ①

Determine the equation to be minimised

$SA = 2\pi r^2 + 2\pi rh \; \ldots$ ②

Rearrange equation ① and substitute into equation ② to obtain SA in terms of r

i.e. $\therefore\ 400 = \pi r^2 h \ \ldots$ ①

$$\frac{400}{\pi r^2} = h$$

$$SA = 2\pi r^2 + 2\pi r h$$

$$SA = 2\pi r^2 + 2\pi r\left(\frac{400}{\pi r^2}\right) \quad \textit{Simplify}$$

$$SA = 2\pi r^2 + \frac{800}{r}$$

To find the minimum requires the location of stationary points i.e. $SA' = 0$

$$SA = 2\pi r^2 + 800r^{-1}$$

$$SA' = 4\pi r - 800r^{-2}$$

i.e. $4\pi r - \dfrac{800}{r^2} = 0$

$$r^3 = \frac{800}{4\pi}$$

$$r = \sqrt[3]{\frac{800}{4\pi}}$$

$$r \approx 3.993$$

To determine whether this is a local minimum use the second derivative test

$$SA' = 4\pi r - 800r^{-2}$$

$$SA'' = 4\pi + 1600r^{-3}$$

Substitute $r = 3.993$

$$SA'' = 4\pi + 1600(3.993)^{-3}$$

$$\approx 37.7 \text{ which is} > 0$$

$$\therefore \text{ a minimum value}$$

Dimensions $r \approx 3.993\,\text{cm}$

$h \approx 7.986\,\text{cm}$

Minimum surface area $\approx 300.53\,\text{cm}^2$

2.4 Optimisation

The sum of one number and two times another number is 40. Find the numbers if their product is to be a maximum.

Let the numbers be x and y

Condition: $x + 2y = 40$

Product: $P = xy$ $\qquad x = 40 - 2y$

i.e. $P = (40 - 2y)y$

i.e. $P = 40y - 2y^2$

To calculate maximum find the stationary points i.e. $\dfrac{dP}{dy} = 0$

$$P = 40y - 2y^2$$

$$\frac{dP}{dy} = 40 - 4y$$

$\therefore\ 40 - 4y = 0$

$$y = 10$$

Check whether $y = 10$ is a maximum.

Second Derivative test

$$\left.\frac{d^2P}{dy^2}\right|_{y=10} = -4$$

As $-4 < 0$, $y = 10$ is a local maximum

Substituting $y = 10$ results in $x = 20$

The numbers are 20 and 10.

2.5 Equations of Tangent Lines

Find the equations of the tangent line and the normal to the curve $y = x^2 + 2$ at the point (2, 6).

Equation of tangent line

Determine the gradient when $x = 2$

$$y = x^2 + 2$$

$$\frac{dy}{dx} = 2x$$

when $x = 2$ $\quad \frac{dy}{dx} = 4 \leftarrow$ *gradient*

$$\therefore y = mx + c$$

$$y = 4x + c$$

Substitute (2, 6) to find 'c'

$$6 = 4(2) + c$$

$$c = -2$$

$\therefore$ equation of tangent is $y = 4x - 2$

Equation of normal

Gradient of normal equation is $-\frac{1}{4}$ (perpendicular to the equation of the tangent line)

$$\therefore y = -\frac{1}{4}x + c$$

Substitute (2, 6) to find 'c'

$$6 = -\frac{1}{4}(2) + c$$

$$\frac{13}{2} = c$$

$$\therefore y = -\frac{1}{4}x + \frac{13}{2}$$

2.6 Rates of Change

The number of new bacteria (B), t hours after observations commenced is given by $B = t^3 + 30t + 200$.

Determine

(a) the initial number of bacteria

(b) the average number of bacteria observed in the first 10 hours

(c) the instantaneous rate of change of B when $t = 10$ hours.

(a) The initial number of bacteria is when $t = 0$

i.e. $B = t^3 + 30t + 200$

$B = 200$

Initial number of bacteria is 200.

(b) Determine $\frac{B(10)}{10} - \frac{B(0)}{10}$

$$= \frac{1000 + 300 + 200}{10} - \left(\frac{200}{10}\right)$$

$= 130$ bacteria/hour

(c) $B = t^3 + 30t + 200$

$$\frac{dB}{dt} = 3t^2 + 30$$

Substitute in $t = 10$

$$\frac{dB}{dt} = 330 \text{ bacteria/hour}$$

2.7 Find the approximate change in the area of a square when the sides are increased from 15 cm to 15.5 cm.

$$A = x^2$$

$$\frac{dA}{dx} = 2x$$ *differentiate the function*

$$\delta A \approx \frac{dA}{dx} \cdot \delta x$$

δx = corresponding small change in length

δA = small change in area

$$\delta A \approx 2x \cdot \delta x$$

Substitute when $x = 15$ $\quad \delta x = 0.5$

$$\delta A \approx 2(15) \cdot 0.5$$

$$\approx 15$$

Approximate change in the area is 15 cm^2

2.8 Determine the approximate percentage change in v when x changes by 1% given $v = 3x^4$.

If x changes by 1% then

$$\frac{\delta x}{x} = 0.01$$

Require $\frac{\delta v}{v}$

$$v = 3x^4$$

$$\therefore \frac{dv}{dx} = 12x^3$$

$$\delta v \approx \frac{dv}{dx} \cdot \delta x$$

$$\delta v \approx 12x^3 \cdot \delta x$$

$$\therefore \frac{\delta v}{v} \approx \frac{12x^3 \cdot \delta x}{3x^4}$$

$$\frac{\delta v}{v} \approx \frac{4\delta x}{x}$$

$$\frac{\delta v}{v} \approx 4\frac{\delta x}{x}$$

$$\frac{\delta v}{v} \approx 4(0.01)$$

$$\frac{\delta v}{v} \approx 0.04$$

When x changes by 1%
v changes by 4%.

2.9 A school drama group intends to sell tickets to the annual production at a price per ticket of $\$(16 - 0.05t)$. If t tickets are sold, the cost for running the production is given by $C(t) = 1000 + 5t$.

(a) Find an expression for the revenue.

(b) Determine the expression for the total profit if t tickets are sold.

(c) Find the average profit/loss per ticket if 100 tickets are sold.

(d) By how much will the total revenue increase due to the sale of the 101st ticket?

(e) Calculate the marginal profit if 100 tickets are sold.

(a) The price per ticket is $(16 - 0.05t)$

∴ The revenue

$$R(t) = \underbrace{t}_{\text{number of tickets}}\underbrace{(16 - 0.05t)}_{\text{price per ticket}}$$

$$R(t) = 16t - 0.05t^2$$

(b) Profit = Revenue - Cost

$$P(t) = R(t) - C(t)$$
$$= (16t - 0.05t^2) - (1000 + 5t)$$
$$= -0.05t^2 + 11t - 1000$$

(c) Substitute $t = 100$ into $P(t)$

$P(t) = -400$

Average profit $= \dfrac{-400}{100} \quad \left(\dfrac{\text{profit}}{100}\right)$

$= -4$

Loss of \$4 per ticket.

(d) $R(t) = 16t - 0.05t^2$

$$\frac{dR}{dt} = 16 - 0.1t$$

when $t = 100$

$$\frac{dR}{dt} = 16 - 0.1(100)$$
$$= 6$$

Revenue will increase by \$6.00 on the sale of the 101st ticket.

(e) Marginal profit is $\dfrac{dP}{dt}$

$$P(t) = -0.05t^2 + 11t - 1000$$

$$\frac{dP}{dt} = -0.1t + 11$$

when $t = 100$

$$\frac{dP}{dt} = -0.1(100) + 11$$
$$= \$1$$

2.10 Rectilinear Motion

A particle moves in such a way that its displacement x in metres from the origin is given by $x = t^3 - 6t^2 + 9t - 1$ where t is time in seconds.

Determine:

(a) where the particle is initially.

(b) an expression for the velocity of the particle in terms of t.

(c) when the particle is at rest.

(d) an expression for the acceleration of the particle and the acceleration when $t = 2$ secs.

(e) the distance travelled in the first 3 seconds.

(a) To determine where the particle is initially let $t = 0$

i.e. $x = t^3 - 6t^2 + 9t - 1$

$x = -1$

$\therefore$ the particle is 1 unit to the left of origin.

(b) $v = x'(t)$

$\therefore v = 3t^2 - 12t + 9$

(c) When a particle is at rest $v = 0$

$\therefore 3t^2 - 12t + 9 = 0$

$t = 1, \quad t = 3$ secs

(d) $a = v'(t)$

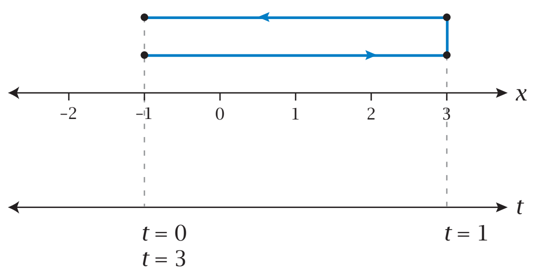
$\therefore a = 6t - 12$

when $t = 2$

$a = 6(2) - 12$

$= 0$ m/s^2

(e) To calculate the distance draw a diagram of the motion.



Total distance travelled is 8 m.

PROBLEMS TO SOLVE

Chapter 2: Applications of Differentiation

1. Sketch a function $y = f(x)$ with all of the following features.

    - $f(1) = f(-1) = 0$
    - $f'(0) = f'(2) = 0$
    - $f''(2) = 0$
    - $f'(x) < 0, \quad x < 0$
    - $f'(x) > 0, \quad x > 2, \quad 0 < x < 2$

2. Find the maximum and minimum values of $f(x) = x^3 - 9x^2 + 15x + 20$ for $0 \leq x \leq 8$.

3. Given $y = \dfrac{x^2}{(3-x)}$ find the coordinates on the curve where the gradient is 0.

4. Find the equations of the tangent and normal to the curve with equation $y = x(3-x)^2$ at $(2, 2)$.

5. The cost of producing an item is given by $C(x) = \$2000 + 8x + \dfrac{x^2}{300}$. Each item sells for \$300. Determine:

    (a) the revenue function $R(x)$.

    (b) the marginal cost function.

    (c) the average cost if 300 items are produced.

    (d) the profit function $P(x)$.

    (e) the profit if 300 items are produced and sold.

    (f) the number of items that need to be produced and sold in order to give a maximum profit and hence calculate the maximum profit.

6. Given $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are constants
and $f(0) = 2$

    $f'(1) = 2$

    $f''\left(\dfrac{1}{2}\right) = 4$

    find a family of cubic functions to satisfy all of the above conditions.

7. A function is defined as $f(x) = (x-1)(x^2 - 3x + 1)$.

    Determine, using calculus where necessary,

    (a) the x intercepts.

    (b) the y intercept.

    (c) the stationary point(s) and their nature.

    (d) the point(s) of inflection.

8. When estimating the volume of a planet – a perfect sphere, Dr Spock estimated the radius to be 4172 ± 0.1 km. Determine the error in the calculation of the volume given the error in the radius.

9. An old clock has an oscillating pendulum of length L. The period of oscillation (in minutes) is determined according to the rule: $T = k\sqrt{L}$ where k is a constant.

(a) Determine how much the period would change if the length of the pendulum was reduced by 2%.

(b) If $k = 2$ and the length of the pendulum decreased from 150 cm to 148 cm determine the percentage change in T.

10. The displacement of a particle x is given by $x = t^3 - 6t^2 + 9t$ where t is in seconds and x is in metres. Determine:

(a) the displacement after 2 seconds.

(b) the velocity at time t.

(c) the velocity after 3 seconds.

(d) when the particle is at rest.

(e) when the particle is moving in a positive direction.

(f) the total distance travelled in the first 5 seconds.

11. A particle initially at the origin moves in such a way that its displacement at any given time t is given by:

$$x(t) = \begin{cases} 2t^3 - 3t^2 & 0 \le t \le 2 \\ 20t - 3t^2 - 24 & 2 < t \le 4 \end{cases}$$

Find over the given interval:

(a) the velocity of the particle at any time t.

(b) when and where the particle is at rest after $t = 0$.

(c) the distance travelled in the first three seconds.

(d) the minimum velocity.

12. The ratio of the radius (r) to height (h) is 5:3 for a specific cone.

(a) Show that the volume of the cone is given by $V = \dfrac{25\pi h^3}{27}$.

(b) Find the approximate increase in the volume of liquid in the cone if the depth of water changes from 5 to 5.02 cm, using the incremental formula: $\delta V \approx \dfrac{dV}{dh} \cdot \delta h$.

13. The cost in dollars of producing x items is given by: $C(x) = (3000 + 5x)$

The revenue per item sold is given by $(40 – 0.02x)$.

(a) State the revenue function $R(x)$ for the number of items sold.

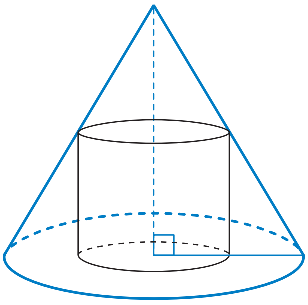
(b) Give an expression for the profit function $P(x)$.

(c) Determine how many items are needed to make a maximum profit and state the maximum profit.

(d) Explain clearly if a loss occurred and when it occurred.

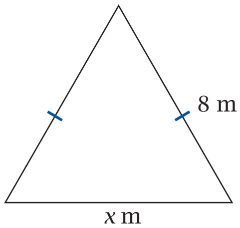
(e) Determine the marginal profit of the 250th item sold.

14. A right circular cone has a radius of 18 cm and a height of 12 cm. Determine the volume of the largest cylinder which will fit inside the cone.



15. Determine all the stationary points and points of inflection for the function $f(x) = x^3 - 6x^2 + 12x - 3$.

16. A garden bed has dimension according to the following diagram.

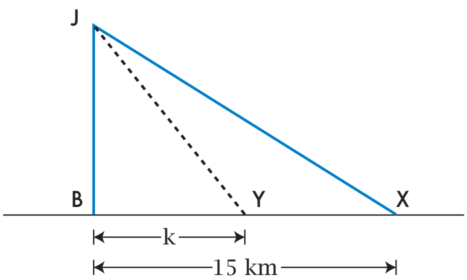


(a) Show that the area of the garden bed, A, as a function of x is given by $A(x) = \frac{x}{4}\sqrt{256 - x^2}$.

(b) Find the approximate change in the area when x is increased from 5 to 5.01 m.

17. The volume V of a sphere is measured as 500 cm^3 ± 10 cm^3. Use differentiation to find the radius of the sphere in the form $r \pm t$ cm.

18. Jeremy (J) on a surfski is 10 km from the beach (B) and needs to be at point (X), 15 km from B in the least possible time. The surfski has a maximum speed of 5 km/h and Jeremy can walk at a speed of 8 km/h.



If he paddles to Y and $BY = k$, show that $JY = \sqrt{100 + k^2}$

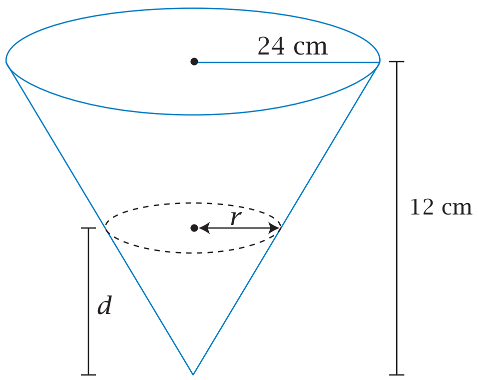
(a) Determine YX in terms of k.

(b) Find in terms of k the shortest time for the journey from J to X via Y.

(c) Find the least amount of time Jeremy would take for the journey from J to X via Y to the nearest minute.

19. Water is being poured into an inverted cone at a rate of 15 cm^3/minute. The cone has a height of 12 cm and a radius of 24 cm.

Let the depth of the water be 'd' and the radius of the surface of the water 'r'.



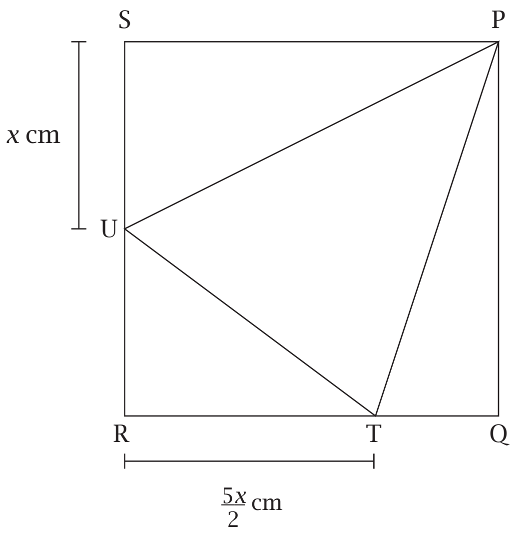
(a) Determine the volume of water formed for any value of 'r'.

(b) Determine the percentage change in the volume of water if the radius of the cone is reduced by 10%.

20. Calculate all roots, stationary points and points of inflection for the function $y = x^3(2 + x)$. Justify using the second derivative test the nature of the stationary points.

21. Estimate the change in the volume of a sphere when the radius increases by 0.06 mm and $V = 34$ cm^3.

22. The square below is of side length 20 cm.



$SU = x$ cm and $RT = \dfrac{5x}{2}$ cm

(a) Show that the area of ΔPUT is: $200 - 10x + \frac{5x^2}{4}$

(b) Use calculus to find the value(s) of x that will minimise the area of ΔPUT.

(c) State the area of ΔPUT.

23. (a) A balloon with a radius of 5 cm is expanding such that its volume increases by 3 cm³. Use differentiation to determine the approximate change in the radius which results in such an increase.

(b) The balloon continues to expand. Use differentiation to determine the approximate percentage change to the radius which would cause a 4% increase in the surface area.

24. An open wooden toy box is to be constructed from second hand wood panelling costing 2.5 cents/cm². The rectangular box has dimensions height (h); width (x) and length twice the width i.e. ($2x$). Overall cost to build the box is \$15.

(a) Show that the cost in dollars of the wood panelling is: $C = \frac{x^2 + 3xh}{20}$.

(b) Show that the volume of the box is given by: $V = 200x - \frac{2x^3}{3}$.

(c) Use calculus to determine the dimensions and maximum volume.

25. If $y = \sqrt[3]{x}$, use the incremental formula $\delta y \approx \frac{dy}{dx}\delta x$ to determine the approximate value of $\sqrt[3]{1003}$.

26. The amount A mg of medicine remaining in the body of patients after t hours is given by: $A = \frac{50}{1+t^2}$.

(a) Find the remaining amount after 3 hours.

(b) Calculate the rate of change of A when $t = 3$.

27. The mass M in grams of an item after t hours is given by: $M = \frac{20t}{1+2t}$.

(a) Find the average rate of change of M from $t = 2$ to $t = 3$.

(b) Calculate the instantaneous rate of change of M when $t = 2$.

3 INTEGRATION

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ identify anti-differentiation as the reverse of differentiation
- ☐ use the notion $\int f(x)dx$ for anti-derivatives or indefinite integrals
- ☐ establish and use the formula $\int x^n dx = \frac{1}{n+1}x^{n+1} + c \quad n \neq -1$
- ☐ identify and use linearity of anti-differentiation
- ☐ determine indefinite integrals of the form $\int f(ax - b)dx$
- ☐ identify families of curves with the same derivative function
- ☐ determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$
- ☐ examine the area problem and use sums of the form $\Sigma_i f(x_i)\delta x_i$ to estimate the area under the curve $y = f(x)$
- ☐ identify the definite integral $\int_a^b f(x)dx$ as the limit of the sums of the form $\Sigma_i f(x_i)\delta x_i$
- ☐ interpret the definite integral $\int_a^b f(x)dx$ as the area under the curve $y = f(x)$
- ☐ interpret $\int_a^b f(x)dx$ as the sum of signed areas
- ☐ apply the additive and linearity of definite integrals
- ☐ examine the concept of the signed area function $F(x) = \int_a^x f(t)dt$
- ☐ apply the theorem $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$
- ☐ develop the formula $\int_a^b f'(x)dx = f(b) - f(a)$ and calculate definite integrals.

FORMULAE AND DEFINITIONS

Antidifferentiation

Antidifferentiation is the opposite process to differentiation (commonly called integration). Integration is the process of finding a primitive or antiderivative of a function.

We have found:

$$\left.\begin{aligned} \frac{d}{dx}(x^3) &= 3x^2 \\ \frac{d}{dx}(x^3 - 6) &= 3x^2 \\ \frac{d}{dx}(x^3 + 2) &= 3x^2 \end{aligned}\right\} \qquad \text{In general: } \frac{d}{dx}(x^3 + c) = 3x^2$$

If $\quad f'(x) = 3x^2$

then $f(x) = x^3 + c$

Any two primitives of a function differ by a constant.

Notation

In Calculus we use the symbol $\int$ to indicate the process of integration.

i.e. $\int 3x^2\,dx = x^3 + c$

The integral of $3x^2$ with respect to x is $x^3 + c$.

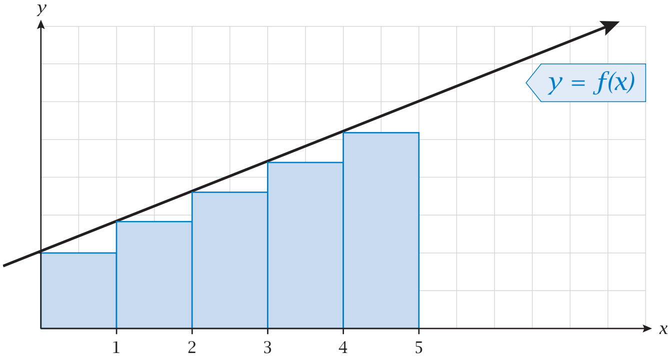
Commonly Used Antiderivatives

$\frac{dy}{dx}$	y
x^n	$\frac{x^{n+1}}{n+1} + c$
ax^n	$\frac{ax^{n+1}}{n+1} + c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$

Area Under a Curve

Technique of Upper and Lower Sums

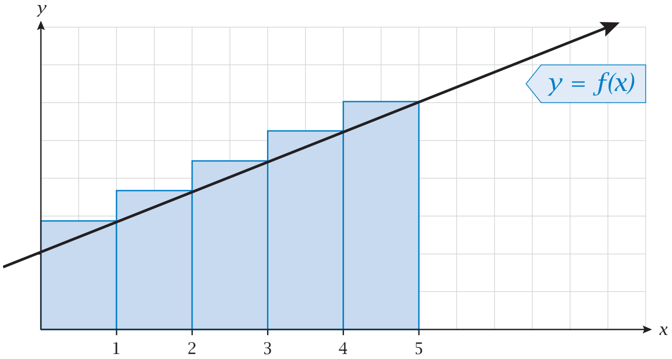
The approximate area under a curve – between the curve and the x axis can be found by summing the areas of rectangles.



The Sum of the Lower Rectangles (L) provides an underestimate of the area under the curve.

Width of Rectangle – depends on the question.

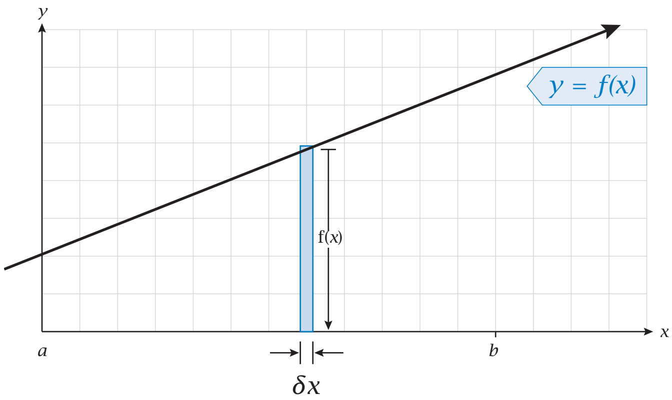
Height of Rectangle – fixed by the curve.



The Sum of the Upper Rectangles (U) provides an overestimate of the area under the curve.

If L = sum of the Lower Rectangle
A = true area
U = sum of the Upper Rectangle
then $L < A < U$

To improve on this estimate divide the area into a large number of strips (n) each of width δx and height $f(x)$.



Area of rectangle strip $= f(x)\delta x$

The more strips and the smaller δx the greater the accuracy.

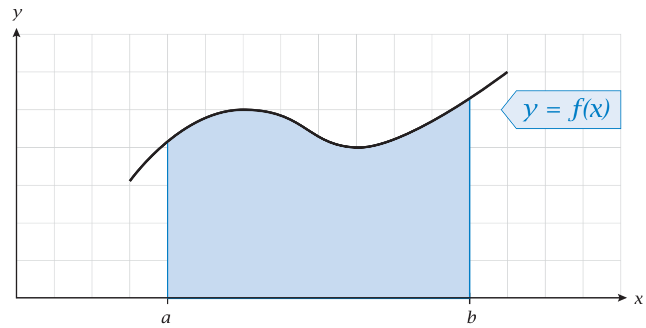
$$\text{Exact Area} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x$$

$$= \int_a^b f(x)\mathrm{d}x$$

Where $\int$ is used to represent the limit of a sum.

Integrals

$\int_a^b f(x)dx$ denotes the area under the curve $y = f(x)$ from $x = a$ to $x = b$.

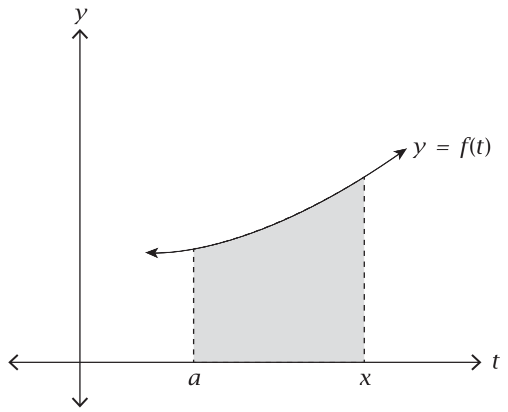


$f(x)$ is a non-negative function where $a \leq x \leq b$

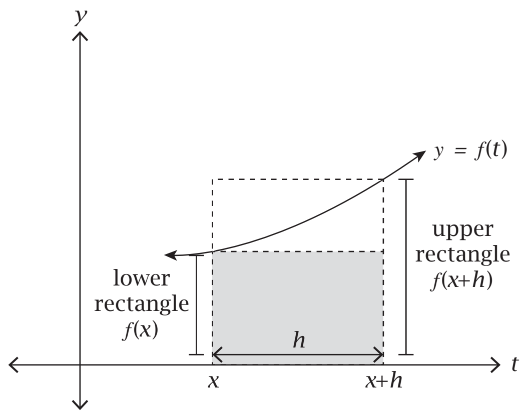
$\int_a^b f(x)dx$	$f(x)$	is the integrand
	$a,\ b$	are the limits of integration
	dx	"with respect to x"
	$\int$	ancient letter S = Sum

Fundamental Theorem of Calculus I

Let $A(x) = \int_a^x f(t)\,dt$ be the area between the lower limit a, the upper limit x, the x axis and $y = f(t)$.



For a small change in x of h the change in the area between the lower and upper rectangles is $A(x + h) - A(x)$.



Area lower rectangle is $hf(x)$

Area upper rectangle is $hf(x + h)$

i.e. $\quad hf(x) < A(x + h) - A(x) < hf(x + h)$

$$f(x) < \frac{A(x+h)-A(x)}{h} < f(x+h)$$

$$\lim_{h\to 0} f(x) < \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} < \lim_{h\to 0} f(x+h)$$

Both outside limits are equal to $f(x)$ thus by the sandwich concept:

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = f(x)$$

$$\therefore A'(x) = f(x)$$

If $A(x) = \int_a^x f(t)\,dt$ then

$$A'(x) = f(x)$$

i.e. $\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$

As $\quad A'(x) = f(x)$

$\quad A(x) = F(x) + c \quad F(x)$ is the anti-derivative of $f(x)$

Area from $x = a$ to $x = a$ must $= 0$.

$$F(a) = A(a) + c$$

$$F(a) = 0 + c$$

$$c = F(a)$$

$$\therefore F(x) = A(x) + F(a)$$

When $\quad a = b$

$$A(b) = \int_a^b f(t)\,dt$$

$$F(b) = A(b) + F(a)$$

$$A(b) = F(b) - F(a)$$

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

This fundamental theorem shows that differentiation and integration are inverse processes.

Fundamental Theorem of Calculus II

$\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is the anti-derivative of $f(x)$.

This theorem allows the area under a curve to be calculated.

The Linearity Property of Anti-differentiation

- ◇ $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
- ◇ $\int kf(x)\,dx = k\int f(x)\,dx$
- ◇ $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

Indefinite Integral

$\int f'(x)\,dx = f(x) + c$ 'c' is a constant

$$\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Definite Integral

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Worked Examples

3.1 Find the antiderivative of the following:

(a) $x^3 + 7x - 2$

(b) $-\dfrac{5}{x^4}$

(c) $5(3x+7)^4$

(d) $\dfrac{x - x^4}{2x^3}$

(a) $\int x^3 + 7x - 2 \; dx$ — *Use:* $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$

$= \dfrac{x^4}{4} + \dfrac{7x^2}{2} - 2x + c$

(b) $\int -\dfrac{5}{x^4}\,dx$ — *Rearrange using index laws.*

$= \int -5x^{-4}\,dx$

$= \dfrac{-5x^{-3}}{-3} + c$

$= \dfrac{5}{3x^3} + c$

(c) $\int 5(3x+7)^4\,dx$ — $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$

$= \dfrac{5(3x+7)^5}{5(3)} + c$

$= \dfrac{(3x+7)^5}{3} + c$

(d) $\int \dfrac{(x - x^4)}{2x^3}\,dx$ — *No quotient rule. Break into separate fractions.*

$= \int \dfrac{x}{2x^3} - \dfrac{x^4}{2x^3}\,dx$

$= \int \dfrac{1}{2x^2} - \dfrac{x}{2}\,dx$ — *Simplify fractions.*

$= \int \dfrac{x^{-2}}{2} - \dfrac{x}{2}\,dx$ — *Rearrange using index laws.*

$= \dfrac{x^{-1}}{-2} - \dfrac{x^2}{4} + c$

$= \dfrac{1}{-2x} - \dfrac{x^2}{4} + c$

3.2 If $\dfrac{dy}{dx} = 5(x+4)^4$ and $y = 35$ when $x = -2$ determine y in terms of x.

If $\dfrac{dy}{dx} = 5(x+4)^4$

$y = \dfrac{\cancel{5}(x+4)^5}{\cancel{5}} + c$

$y = (x+4)^5 + c$

Substitute in $y = 35$ when $x = -2$

$35 = (-2+4)^5 + c$

$35 = 32 + c$

$c = 3$

$\therefore y = (x+4)^5 + 3$

3.3 Evaluate the following definite integrals.

(a) $\int_{3}^{5} (4x+3)\,dx$

(b) $\int_{-1}^{4} (x-2)^2\,dx$

(a) $\int_{3}^{5} (4x + 3)\,dx$

$= \left[\dfrac{{}^{2}\cancel{4}\,x^2}{\cancel{2}} + 3x\right]_3^5$ *antidifferentiate*

$= [2x^2 + 3x]_3^5$

$= \left[2(5)^5 + 3(5)\right] - \left[2(3)^2 + 3(3)\right]$ *Substitute in upper and lower limits*

$= [65] - [27]$

$= 38$

(b) $\int_{-1}^{4} (x-2)^2\,dx$

$= \left[\dfrac{(x-2)^3}{3}\right]_{-1}^{4}$

$= \left[\dfrac{(4-2)^3}{3}\right] - \left[\dfrac{(-1-2)^3}{3}\right]$

$= \left[\dfrac{8}{3}\right] - \left[\dfrac{-27}{3}\right]$

$= \dfrac{35}{3}$

PROBLEMS TO SOLVE

Chapter 3: Integration

1. Find the antiderivatives of the following:

(a) $3x^4 - 6$

(b) $\frac{4}{x^5} + 2x$

(c) $\sqrt{x} + 3$

(d) 4

(e) $x(x-6)$

(f) $\frac{1}{\sqrt{x}} + \sqrt[3]{x}$

2. If $\frac{dy}{dx} = 4 - \frac{2}{x^2}$ and $x = -2$ when $y = 2$ find:

(a) y in terms of x

(b) y when $x = \frac{1}{2}$

(c) x when $y = \frac{-5}{3}$

3. Find the antiderivatives of the following:

(a) $(2x-3)^3$

(b) $(4-3x)^4$

(c) $2(4x+1)^3$

(d) $\frac{3}{(2x+5)^2}$

(e) $\sqrt{3x-1}$

(f) $\frac{2}{\sqrt[3]{5x-6}}$

4. Find the following:

(a) $\int 10x\left(x^2-6\right)^4 dx$

(b) $\int (2x-1)\left(x^2-x+4\right)^3 dx$

(c) $\int 15x\left(2+x^2\right)^5 dx$

(d) $\int x^2\left(x^3+4\right)^9 dx$

(e) $\int \frac{x}{\left(5x^2+1\right)^3} dx$

(f) $\int 2x(3+x)^2 dx$

5. Find the antiderivative of each of the following indefinite integrals:

(a) $\int (3x^2 + x)dx$

(b) $\int \left(\frac{10}{(2x+3)^2}\right)dx$

(c) $\int \left(\frac{2-x^5}{x^2}\right)dx$

PROBLEMS TO SOLVE

(d) $\int 2x(x^2 - 5x)\,dx$

(e) $\int(\sqrt{5x^3} + e^{-x})\,dx$

6. Evaluate the following definite integrals:

(a) $\int_0^4 (2x-3)^3\,dx$

(b) $\int_3^4 (3x - x^3)\,dx$

(c) $\int_0^7 \sqrt{3x+4}\,dx$

(d) $\int_1^3 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$

(e) $\int_0^2 (2x+1)\left(x^2+x-4\right)^2 dx$

7. Determine the following:

(a) $\frac{d}{dx}\int_2^{x^2}\left(\frac{t^2}{4}\right)dt$

(b) $\frac{d}{dx}\int_0^{x}\frac{3t}{\left(1-t^2\right)^2}\,dt$

(c) $\frac{d}{dx}\int_{3x^2}^{0}\left(\frac{2+t}{t-3}\right)dt$

(d) $\frac{d}{dx}\int_0^{2x^3}\sqrt{1+t^2}\,dt$

(e) $\int_1^2 \frac{d}{dx}\left[\frac{x^3}{x^2+1}\right]dx$

8. Find the exact value of $\int_1^4 \sqrt{x}\,(x+3)\,dx$

9. If $\frac{dy}{dx} = 12(3x-2)^2$ and $y = 3$ when $x = 0$ find:

(a) y in terms of x

(b) y when $x = 2$

10. The gradient of a curve at any given point is $5x - 8$. If the curve passes through the point (2, 1) find the equation of the curve.

11. Find the exact value of $\int_{-\pi}^{\pi}(4x+1)\,dx$

12. If $\frac{dy}{dx} = (-3-2x)(x^2 + 3x - 7)^5$ and $y = -100$ when $x = 1$ find y in terms of x.

13. If $f'(x) = 3x + 1 - \frac{1}{x^2}$ and $f(1) = 4$ find:

(a) $f(x)$

(b) $f(3)$

14. Find the value of t if $\int_t^1 (4x+5)\,dx = 4$ where t is a positive constant.

15. Given $\int_2^5 f(x)\,dx = 15$ evaluate:

(a) $\int_5^2 3f(x)\,dx$

(b) $\int_2^4 [f(x)+3]\,dx + \int_4^5 f(x)\,dx$

16. Any coordinate point on a curve has a gradient given by $\frac{dy}{dx} = \frac{1}{\sqrt{1-4x}}$. Find the equation of the curve which passes through the point (–2, 1).

17. The gradient of the curve is given by $\frac{dy}{dx} = k(x-a)(x-b)$ where k, a and b are constants and $a < b$. The curve has stationary points at (–2, 11) and (3, 6).

(a) Find the values of k, a and b.

(b) Determine the equation of the curve.

4 APPLICATIONS OF INTEGRATION

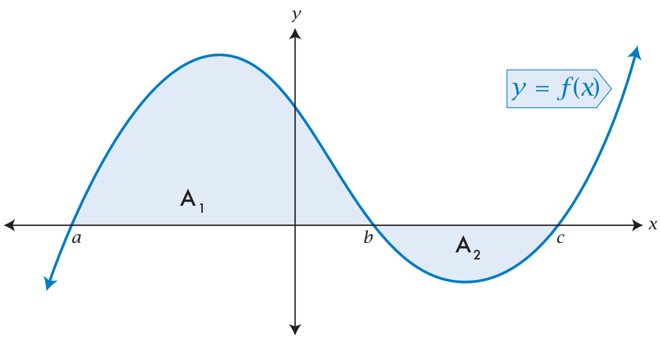
Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ calculate total change by integrating instantaneous or marginal rate of change
- ☐ calculate the area under the curve
- ☐ calculate the area between curves
- ☐ determine the displacement given velocity in linear motion problems
- ☐ determine positions given linear acceleration and initial values of position and velocity.

FORMULAE AND DEFINITIONS

Area Rule



Areas below the x-axis are **negative**.

Area between the curve and the **x axis** is:

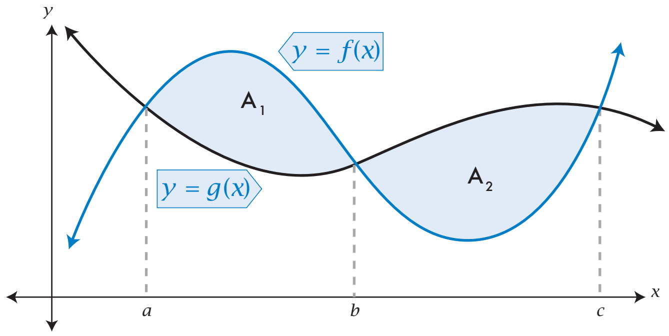
<u>Area</u> = $A_1 + A_2$

$$= \int_a^b f(x)\,dx - \int_b^c f(x)\,dx$$

or

$$\int_a^b f(x)\,dx + \left|\int_b^c f(x)\,dx\right|$$

Area between two curves



To determine the shaded area find the points of intersection of the two curves i.e. at $x = a$, b and c.

top function *bottom function*

Area $A_1 = \int_a^b [f(x) - g(x)]\, dx$

top function *bottom function*

Area $A_2 = \int_b^c [g(x) - f(x)]\, dx$

<u>Total Area</u> $= A_1 + A_2$

Rectilinear Motion

Rectilinear motion is an application of differentiation and also an application of integration.

Displacement: $x = f(t)$

Velocity: $v = f'(t)$

Acceleration: $a = f''(t)$

Reversing this process yields:

$v = \int a\, dt$

$x = \int v\, dt$

The constant c can be found by substituting conditions.

Over an interval of time i.e. $a \le t \le b$

Displacement: $x = \int_a^b v\, dt$

Distance can be found by calculating the area under the velocity – time graph.

Total Change

Given the rate of change of a function we must integrate in order to find the original **function**.

Remember:

- Marginal cost: $\frac{dC}{dx}$
- Marginal revenue: $\frac{dR}{dx}$
- Marginal profit: $\frac{dP}{dx}$

If no units are sold then $R(0) = 0$

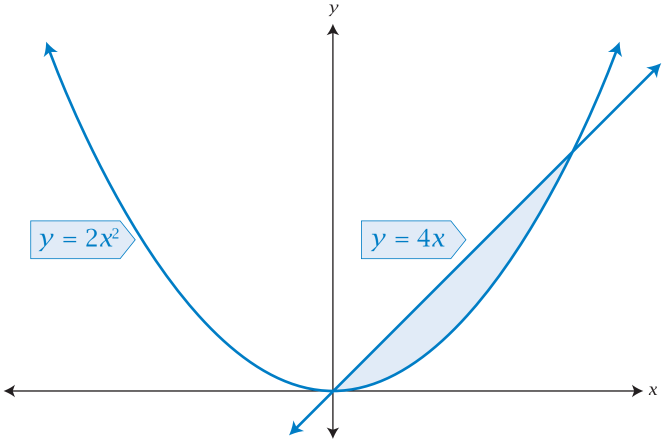
The **total change** over a period of time can be found from the rate of change of a function.

To calculate the extra cost of producing b units rather than a units then the total change is calculated by: $\int_a^b \frac{dC}{dx}\, dx$

Worked Examples

4.1 Areas

Find the area enclosed by $y = 2x^2$ and $y = 4x$.



Determine the intersection points.

Solve $\quad 2x^2 = 4x$

$$2x^2 - 4x = 0$$

$$2x(x-2) = 0$$

$$\therefore x = 0,\ 2$$

Calculate Area

curve above *curve below*

$$\int_0^2 (4x - 2x^2)\,dx$$

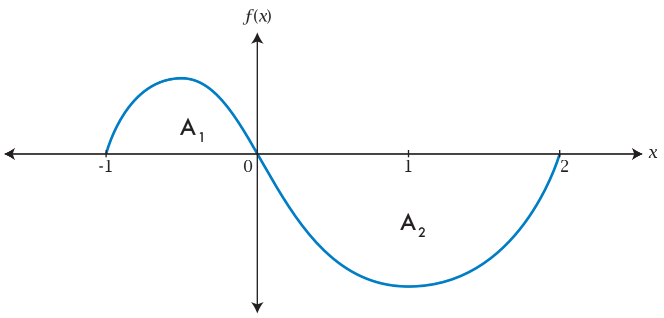
$$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$$

$$= \left(8 - 5\frac{1}{3}\right) - 0$$

$$= 2\frac{2}{3} \text{ units}^2$$

Remember: Curve above subtract curve below.

4.2 Find the area between the curve $f(x) = x(x-2)(x+1)$ and the x axis.



Expand and simplify

$y = x(x-2)(x+1)$

$y = x(x^2 - x - 2)$

$y = x^3 - x^2 - 2x$

subtract as curve is below the x axis

$$\text{Area} = \underbrace{\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx}_{A1} - \underbrace{\int_{0}^{2} (x^3 - x^2 - 2x)\, dx}_{A2}$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= -\left(\frac{-5}{12}\right) - \left(-2\frac{2}{3}\right)$$

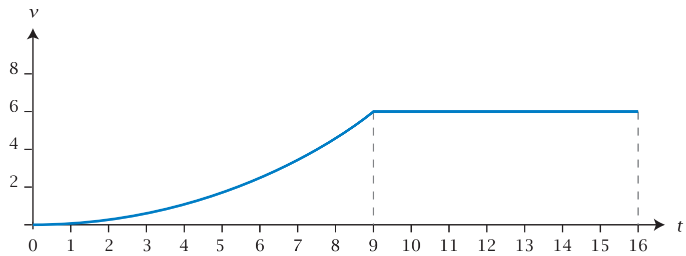
$$= 3\frac{1}{12} \text{ units}^2$$

Note: Using classpad a simplified approach can be used

$$\text{Total Area} = \int_{-1}^{2} \left| x(x-2)(x+1) \right| dx$$

4.3 Rectilinear Motion (Velocity – Time Graph)

Given the following velocity – time graph determine the answers to the questions below.



(a) Between $t = 0$ and $t = 9$ the velocity equation is $v(t) = \dfrac{2t^2}{27}$
Find the acceleration at $t = 5$ seconds.

(b) Determine the distance travelled in the first 16 seconds.

(a)
$$v(t) = \frac{2t^2}{27}$$

$$a(t) = \frac{4t}{27} \qquad a(t) = v'(t)$$

when $t = 5$

$$a(5) = \frac{4(5)}{27}$$

$$\approx 0.741 \text{ m/s}^2$$

(b) To calculate distance requires the antidifferentiation of $v(t)$ i.e. the <u>area</u> under the curve.

$$\text{Distance} = \int_{0}^{9} v(t)\, dt + \text{Area rectangle}$$

length × width

$$= \left[\frac{2t^3}{81}\right]_0^9 + (6 \times 7)$$

$$= [18] + [42]$$

$$= 60 \text{ m}$$

PROBLEMS TO SOLVE

Chapter 4: Applications of Integration

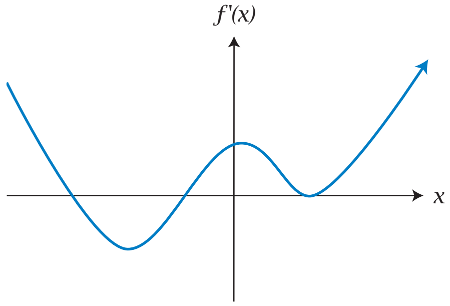
1. Find the area between the two functions

   $f(x) = x^3 + 1$

   $g(x) = x + 1$

2. Find the total area enclosed between $y = x^2 + 2x + 1$ and $y = 2 + \frac{x}{2}$

3. Given the graph of $f'(x)$, sketch a possible graph of $f(x)$ and $f''(x)$

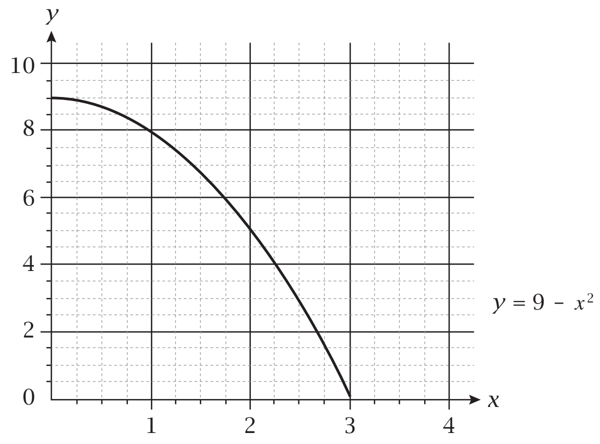


4. If a water tank is initially full and the volume of water (v) in the tank (t) minutes after draining commences is given by $\frac{dv}{dt} = 0.2t^2 - 30$, where v is in kL, calculate:
   (a) how much water is in the tank initially if it takes 15 minutes to empty?
   (b) how much water was drained in the 8th minute?

5. The velocity of a particle is given by $v(t) = t(t-2)(t-5)$ where v is measured in metres per second. Given $v(0) = 0$ determine:
   (a) the displacement of the particle from the origin after 2 seconds
   (b) the distance travelled in the first 3 seconds.

6. A company has determined that the marginal cost in \$ per unit for producing x units is given by $\frac{dC}{dx} = 60\sqrt{x}$ where $C(0) = \$1500$. Determine:
   (a) $C(x)$
   (b) $C(150)$
   (c) the average cost of producing 150 units
   (d) The total profit if 150 units are produced and sold, if the revenue received is \$600 per unit.

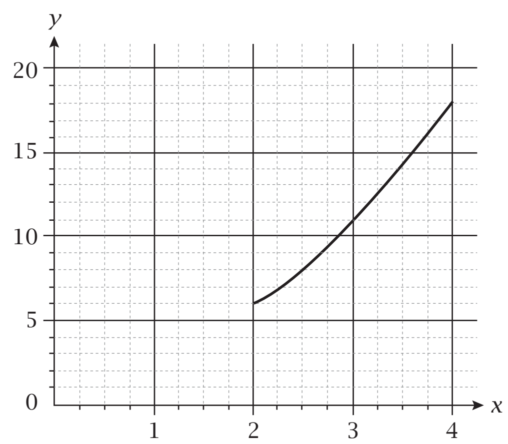
7. The diagram below is the function $y = 9 - x^2$ for $0 \leq x \leq 3$.



Calculate the approximate area trapped between the curve $y = f(x)$, the x and y axes by using rectangles of width 0.5 and averaging the underestimates and overestimates.

Rectangle	$0 \leq x < 0.5$	$0.5 \leq x < 1$	$1 \leq x < 1.5$	$1.5 \leq x < 2$	$2 \leq x < 2.5$	$2.5 \leq x \leq 3$
Underestimate						
Overestimate						

8. (a) Consider the function $y = x^2 + 2$ over the domain $2 \leq x \leq 4$.



By using 4 rectangles, determine the approximate area between the curve and the x axis.

(b) How can the accuracy of the area be improved?

9. The velocity $v(t)$ in metres per second at time t seconds of an object moving in a straight line is given by:

$v(t) = 3t^2 - 6t$ where $0 \leq t \leq 5$

(a) Find $x(t)$, the displacement at time t given $x(0) = 0$

(b) At what time t in the interval $0 \leq t \leq 5$, does the object return to its starting point?

(c) At what time t in the interval $0 \leq t \leq 5$ is the object furthest from its starting point?

10. The marginal cost of producing x units is $\frac{dC}{dx} = 125 - 0.003x^2$ dollars per day.

If the manufacturers fixed costs are \$4000 per day, calculate:

(a) the total cost $\$C$ of producing 40 units per day

(b) the average cost of producing 40 units per day per unit

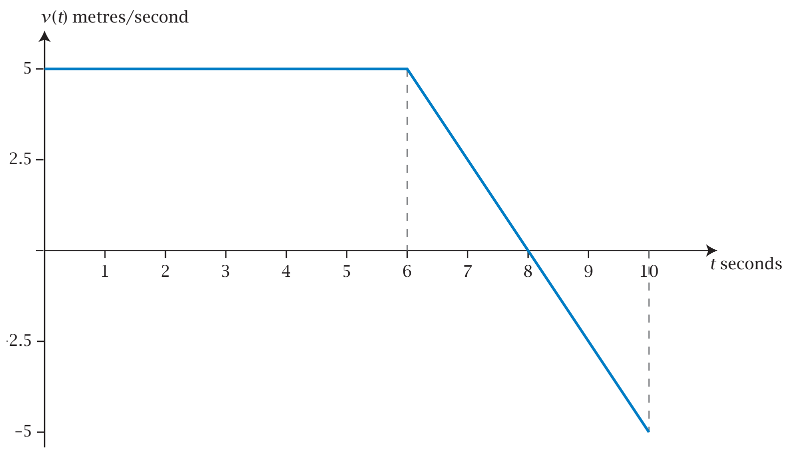
(c) the change in the cost if the production per day is changed from 40 to 50 units.

11. Find the area bounded by the following functions:

$$y = 0$$
$$x = 3$$
$$2y = 4 - x$$
$$e^{\frac{x}{2}} = y + 1$$

12. Calculate the area bounded by the function $y = 3x - x^2$, the lines $x = -1$ and $x = 3$ and the x axis.

13. A particle moves along a straight line. The velocity – time graph is shown below.



(a) Describe using the graph above the movement of the particle over the 10 seconds.

(b) Find the velocity of the particle when $t = 3$.

(c) Write an expression for v in terms of t for $6 \leq t \leq 10$.

(d) Find the velocity when $t = 9$.

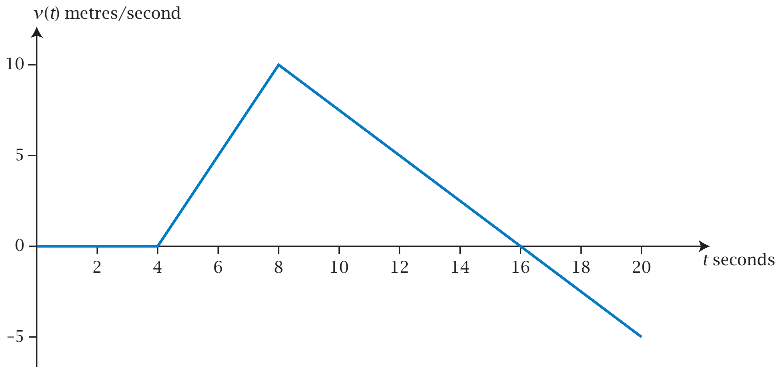
(e) Find the acceleration of the particle when

i. $t = 1$

ii. $t = 7$

(f) Find the change in displacement for $0 \leq t \leq 10$.

14. The following diagram represents a velocity – time graph. The particle travels in a straight line and is at the origin when $t = 0$.



(a) Calculate the velocity of the particle when:

i. $t = 8$

ii. $t = 12$

(b) Find the instantaneous rate of change of the particles velocity in the time periods below and comment on your answer.

i. $4 < t < 8$

ii. $8 < t < 16$

(c) Find the position of the particle when

i. $t = 4$

ii. $t = 8$

iii. $t = 12$

(d) Describe the motion of the particle in the interval $12 \le t \le 20$.

15. The rate of change of the cost per printer, in dollars, of a company producing x printers is given by: $C'(x) = x(-3 + 0.03x)$.

(a) Determine the rate of change of the cost of producing 110 printers.

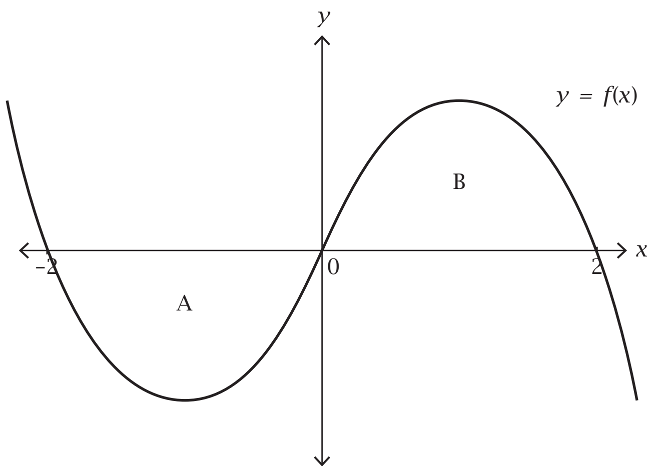
(b) If the initial costs are \$250 determine $C(x)$. The revenue for each printer is \$750.

(c) Determine the profit function $P(x)$.

(d) Use calculus to find the number of printers so that a maximum profit is obtained.

16. The area enclosed by the line $y = mx$ and the parabola $y = x^2$ is 24.813.
Find the value of m (m is positive).

17. The areas of the bounded regions A and B for the graph of $y = f(x)$ are 10 and 7 units respectively.



Evaluate:

(a) $\int_{-2}^{2} f(x)\,dx$

(b) $\int_{0}^{2} f(-x)\,dx$

(c) $\int_{0}^{-2} f(x)\,dx$

(d) $\int_{-2}^{2} \left[f(x)+3\right]dx$

18. A particle moving along a straight line has a velocity of 25 m/s as it passes 0, the origin. The acceleration t seconds later is $10t - 30$ cm/sec^2.

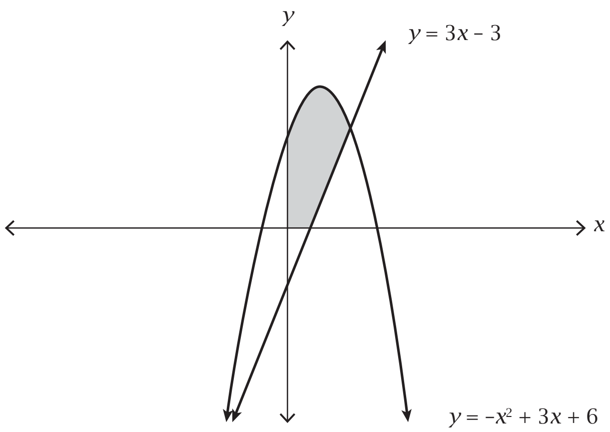
    Calculate the total distance travelled from 0 when the particle rests for the second time.

19. The marginal costs, in dollars, involved in producing x copies of a new magazine are given by $C'(x) = \dfrac{14.5}{\sqrt[3]{x}} + 7$

    (a) State an expression used to determine the extra cost in producing 500 copies rather than 400.

    (b) Determine the average cost per magazine in producing the extra 100 magazines.

20. Find the area of the shaded region below.



21. The marginal cost for producing x items of a product is given by $\dfrac{dC}{dx} = 0.02x + 150$ where \$C is the cost of producing x items of the product.

    (a) Given the fixed cost is \$2500, find the cost of producing 50 items.

    (b) Find the total change in cost of producing 200 items rather than 100 items.

22. The velocity $v(t)$ in metres per second of a particle moving in a straight line is given by $v(t) = t^2 - 4t + 3$.

    (a) Determine the distance travelled in the 3rd second.

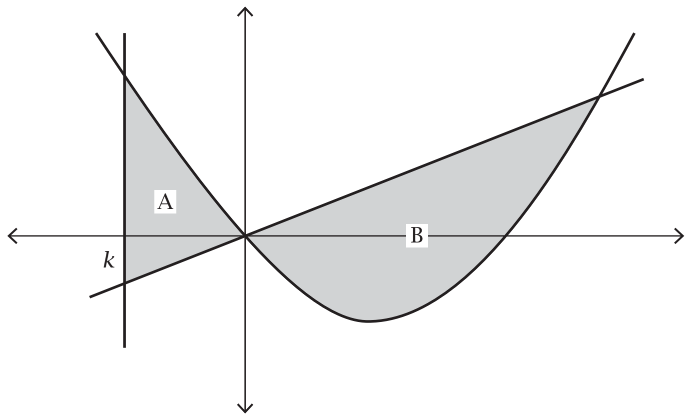
    (b) Determine the distance travelled in the interval $1 \le t \le 3$.

    (c) If initially the particle had a displacement of $x = -15$ what is the displacement when $t = 3$.

    (d) Calculate the acceleration when $t = 2$.

23. The graph below consists of the following functions:

- $y = x^2 - 3x$
- $y = x$
- $x = k$ where k is a constant.



(a) State an integral which represents the area of Region B and calculate the area.

(b) State an integral which represents the area of Region A.

(c) Find the value of k for which the area of Region A is equal to the area of Region B.

5

TRIGONOMETRIC FUNCTIONS

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ establish the formula $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ by graphical treatment, numerical estimations of the limits and informal proofs based on geometric constructions
- ☐ use trigonometric functions and their derivatives to solve practical problems
- ☐ apply the product, quotient and chain rule to differentiate functions such as $\tan x$, $x \sin x$, $e^{-x} \sin x$ and $f(ax - b)$
- ☐ establish and use the formulas $\int \sin x \, dx = -\cos x + c$ and $\int \cos x \, dx = \sin x + c$.

FORMULAE AND DEFINITIONS

To determine $\frac{d}{dx}(\sin x)$ and $\frac{d}{dx}(\cos x)$ the following information is needed:

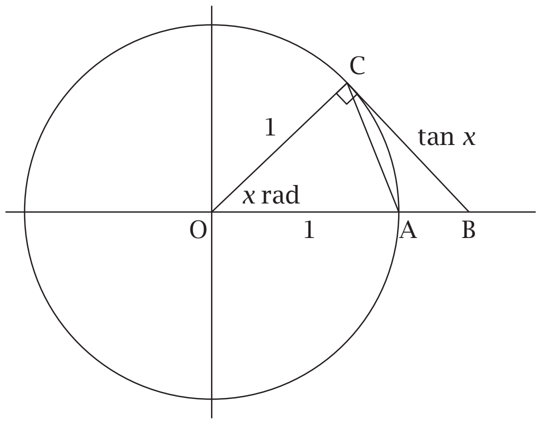
$$\lim_{x \to 0} \frac{\sin x}{x}$$

Table of Values	
x (radians)	$\frac{\sin x}{x}$
0.1	0.9983342
0.01	0.9999833
0.001	0.9999998
0.0001	1.0000000
0.00001	1.0000000
-0.1	0.9983342
-0.01	0.9999833
-0.001	0.9999998
-0.0001	1.0000000
-0.00001	1.0000000

$$\therefore \boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Geometrical Proof

Consider a unit circle, centre O and BC tangent to the circle at C.



$$\tan x = \frac{CB}{OC}$$

$\therefore\ CB = OC \tan x$

$\therefore\ CB = \tan x$

Then:

$$\begin{array}{llll}
\text{Area } \triangle\, OAC & < \text{Area sector } OAC & < \text{Area } \triangle\, OBC & \\
\frac{1}{2}r^2 \sin x & < \frac{1}{2}r^2 x & < \frac{1}{2}r(r \tan x) & \\
\frac{1}{2}(1)^2 \sin x & < \frac{1}{2}(1)^2 x & < \frac{1}{2}(1 \tan x) & \\
\frac{1}{2}\sin x & < \frac{1}{2}x & < \frac{1}{2}\tan x & \\
 & & & \times 2 \\
\sin x & < x & < \tan x & \\
\frac{1}{\sin x} & > \frac{1}{x} & > \frac{\cos x}{\sin x} & \\
 & & & \times \sin x \\
1 & > \frac{\sin x}{x} & > \cos x & \\
\cos x & < \frac{\sin x}{x} & < 1 &
\end{array}$$

$$\boxed{\lim_{x\to 0} \cos x \quad < \lim_{x\to 0} \frac{\sin x}{x} \quad < \lim_{x\to 0} 1}$$

As $x \to 0$, $\frac{\sin x}{x}$ is sandwiched or squeezed between $\cos x = 1$ and 1. Therefore $\frac{\sin x}{x}$ must also equal 1.

i.e. $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

$$
\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\left(\frac{1-\cos x}{x}\bullet\frac{1+\cos x}{1+\cos x}\right)\\
&= \lim_{x\to 0}\left(\frac{1-\cos^2 x}{x\left(1+\cos x\right)}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin^2 x}{x\left(1+\cos x\right)}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin x}{x}\bullet\frac{\sin x}{1+\cos x}\right)\\
&= \lim_{x\to 0}\frac{\sin x}{x}\bullet\lim_{x\to 0}\frac{\sin x}{1+\cos x}\\
&= 1\times\frac{\sin\left(0\right)}{1+\cos\left(0\right)}\\
&= 1\times 0\\
&= 0
\end{aligned}
$$

$$\therefore \quad \boxed{\lim_{x\to 0}\frac{1-\cos x}{x}=0}$$

Trigonometric Identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Derivative of $y = \sin x$

From First Principles

$$
\begin{aligned}
f\left(x\right) &= \sin x\\
f'\left(x\right) &= \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}\\
&= \lim_{h\to 0}\frac{\sin\left(x+h\right)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\
&= \lim_{h\to 0}\frac{-\sin x\left(1-\cos h\right)+\sin h\cos x}{h}\\
&= -\lim_{h\to 0}\sin x\left(\frac{1-\cos h}{h}\right)+\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\bullet\cos x\\
&= -\sin x\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)+\cos x\bullet\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\
&= -\sin x\left(0\right)+\cos x\left(1\right)\\
&= \cos x
\end{aligned}
$$

$$\therefore \quad \boxed{\begin{array}{ll}\text{If} & y=\sin x\\ \text{then} & \dfrac{dy}{dx}=\cos x\end{array}}$$

Derivative of $y = \cos x$

From First Principles

$$
\begin{aligned}
f(x) &= \cos x \\
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\cos x}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h} \\
&= -\cos x\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\
&= -\cos x(0)-\sin x(1) \\
&= -\sin x
\end{aligned}
$$

$\therefore$ If $y = \cos x$

then $\dfrac{dy}{dx} = -\sin x$

Integration of Trigonometric Functions

$$\int \cos ax\,dx = \frac{\sin ax}{a}+c$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a}+c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x)+c$$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x)+c$$

Worked Examples

5.1 Differentiate

(a) $y = x^3 + \cos x$

(b) $y = 2\sin x$

(c) $y = (\cos x + 1)(\sin x - 2)$

(d) $y = \dfrac{\sin x}{\cos x}$

(e) $y = \sin^3 x$

(f) $y = \cos(2x - 3)$

(g) $y = \sin(5x + 1)$

(a) $y = x^3 + \cos x$ — *sum and difference rule*

$$\frac{dy}{dx} = 3x^2 - \sin x$$

(b) $y = 2\sin x$ — *product rule*

$$\begin{aligned}\frac{dy}{dx} &= (0\times\sin x) + (2\times\cos x)\\ &= 2\cos x\end{aligned}$$

(c) $y = (\cos x + 1)(\sin x - 2)$ *product rule*

$$\frac{dy}{dx} = (-\sin x)(\sin x - 2) + (\cos x + 1)(\cos x)$$
$$= -\sin^2 x + 2 \sin x + \cos^2 x + \cos x$$

(d) $y = \dfrac{\sin x}{\cos x}$ *quotient rule*

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$\cos^2 x + \sin^2 x = 1$
$$= \frac{1}{\cos^2 x}$$

$\therefore$ If $y = \tan x$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

(e) $y = \sin^3 x$ *chain rule*

$$\frac{dy}{dx} = 3(\sin x)^2(\cos x)$$
$$= 3 \sin^2 x \cos x$$

(f) $y = \cos (2x - 3)$ *chain rule*

$$\frac{dy}{dx} = [-\sin (2x - 3)] \times 2$$
$$= -2 \sin (2x - 3)$$

$\dfrac{d}{dx}(\cos (2x - 3))$
$= - \sin (2x - 3)$
$\dfrac{d}{dx}(2x - 3) = 2$

(g) $y = \sin (5x + 1)$

$$\frac{dy}{dx} = [\cos (5x + 1)] \times 5$$ *chain rule*
$$= 5 \cos (5x + 1)$$

5.2 Find the equation of the tangent to the curve $y = 2x + \cos 2x$ at the point where $x = \dfrac{\pi}{3}$.

$$y = 2x + \cos 2x$$
$$\frac{dy}{dx} = 2 - 2 \sin 2x$$
$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = 2 - \sqrt{3}$$

Equation of tangent at $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3} - \dfrac{1}{2}\right)$

$$y = mx + c$$
$$\frac{2\pi}{3} - \frac{1}{2} = \left(2 - \sqrt{3}\right)\left(\frac{\pi}{3}\right) + c$$
$$c = \frac{\sqrt{3}\pi}{3} - \frac{1}{2}$$
$$\therefore\ y = (2 - \sqrt{3})x + \frac{\sqrt{3}\pi}{3} - \frac{1}{2}$$

5.3 Find:

(a) $\int [2 \sin(x)]\, dx$
(b) $\int [\sin (2x + 1)]\, dx$
(c) $\int [\cos (3x)]\, dx$
(d) $\int \left[6 \cos \left(\frac{2x}{3}\right)\right] dx$
(e) $\int [(\sin^3 x)(\cos x)]\, dx$
(f) $\int [(\sin 2x \cos 4x + \cos 2x \sin 4x)]\, dx$

(a) $\int [2\ \sin(x)]\ dx$

$= 2 \int \sin(x)\ dx$

$= 2(-\cos x) + c$

$= -2\cos(x) + c$

(b) $\int [\sin(2x+1)]\ dx$

$= \frac{1}{2} \int [2\sin(2x+1)]\ dx$

$= \frac{1}{2}(-\cos(2x+1)) + c$

$= -\frac{1}{2}\cos(2x+1) + c$

(c) $\int [\cos(3x)]\ dx$

$= \frac{1}{3}\int [3\cos(3x)]\ dx$

$= \frac{1}{3}(\sin 3x) + c$

$= \frac{1}{3}\sin(3x) + c$

(d) $\int \left[6\cos\left(\frac{2x}{3}\right)\right] dx$

$= 9 \int \left[\frac{2}{3}\cos\left(\frac{2x}{3}\right)\right] dx$

$= 9\sin\left(\frac{2x}{3}\right) + c$

(e) $\int [(\sin^3 x)(\cos x)]\ dx$

$= \int [(\cos x)(\sin^3 x)]\ dx$

$= \frac{\sin^4(x)}{4} + c$

(f) $\int [(\sin 2x \cos 4x + \cos 2x \sin 4x)]\ dx$

$= \int [\sin(2x+4x)]\ dx$

$= \int [\sin(6x)]\ dx$

$= \frac{1}{6}\int [6\sin(6x)]\ dx$

$= \frac{1}{6}(-\cos 6x) + c$

$= -\frac{1}{6}\cos(6x) + c$

5.4 Evaluate the following:

(a) $\int_0^{\pi} [\sin(x)]dx$

(b) $\int_0^{\frac{\pi}{2}} [\cos(2x)]dx$

(a) $\int_0^{\pi} [\sin(x)]dx = [-\cos(x)]_0^{\pi}$

$= -\cos(\pi) - (-\cos(0))$

$= -(-1) - (-1)$

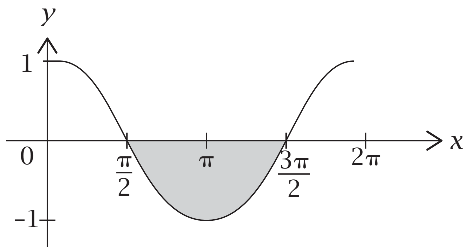
$= 2$

(b) $\int_0^{\frac{\pi}{2}} \left[\cos(2x)\right] dx = \left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{1}{2}\sin\left(2x\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}\sin 2(0)\right)$

$= 0$

5.5 Find the area enclosed by $y = \cos x$ and the x axis from $x = \frac{\pi}{2}$ to $\frac{3\pi}{2}$.



$$\text{Area} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left|\cos(x)\right| dx$$

$$= \left[\,\left|\sin(x)\right|\,\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left|\left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right)\right|$$

$= 2$ square units

PROBLEMS TO SOLVE

Chapter 5: Trigonometric Functions

1. Use a table of values to find $\lim_{x \to 0} \frac{1 - \cos x}{x}$.

2. Find the derivative of each of the following:

   (a) $y = \cos(x) + \sin(x)$

   (b) $y = 3x - \sin(x)$

   (c) $y = \tan(x)$

3. Find $\frac{dy}{dx}$ for each of the following:

   (a) $y = 5\sin(x)$

   (b) $y = x\cos(x)$

   (c) $y = x^3 \sin(x)$

4. Differentiate each of the following with respect to x.

   (a) $y = \frac{\sin(x)}{x}$

   (b) $y = \frac{x^2}{\cos(x)}$

   (c) $y = \frac{3\cos(x)}{\sin(2x)}$

5. Determine $f'(x)$ for each of the following:

   (a) $f(x) = \sin(6x)$

   (b) $f(x) = 2\cos(3x)$

   (c) $f(x) = 2x + 4\cos(5x)$

   (d) $f(x) = \sqrt{\cos(x)}$

   (e) $f(x) = 2x\sin(3x)$

6. Determine $\frac{dy}{dx}$ for each of the following:

   (a) $y = 2\cos x + 3\sin(2x)$

   (b) $y = \frac{3 + \sin(x)}{x^3}$

   (c) $y = 6\sin^2(x)$

   (d) $y = 3\cos^3(x)$

   (e) $y = \frac{4\sin(2x)}{\cos(3x)}$

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following:

   (a) $y = \sin(3x)$

   (b) $y = x\cos(x)$

   (c) $y = \sin^2(x)$

8. Find the gradient of the curve $y = \sin(x)$ when $x = \frac{\pi}{4}$.

9. Find the gradient of the curve $y = \frac{4\sin(2x)}{1 + \cos(x)}$ when $x = \frac{\pi}{4}$.

10. Find the equation of the tangent to the curve $y = \sin(2x)$ when $x = \frac{\pi}{2}$.

11. Find the points on the curve $y = \sin^2 x$ for $0 \le x \le 2\pi$ when the gradient of the curve is 1.

12. Find the equation of the tangent to the curve $y = x\cos(3x)$ when $x = \frac{\pi}{3}$.

13. Find the anti-derivative of the following:

(a) $6\sin(x)$

(b) $-2\cos(3x)$

(c) $15\sin(3x)$

(d) $\sin\left(\frac{2x}{3}\right)$

(e) $\cos(-x)$

14. Determine:

(a) $\int [5\sin(2x + 3)]\, dx$

(b) $\int [4\cos(4x - 3)]\, dx$

(c) $\int [\sin(2x + \frac{\pi}{3})]\, dx$

(d) $\int [5\cos(2x) + 6\sin(3x)]\, dx$

(e) $\int [\sin(x)\cos^3(x)]\, dx$

(f) $\int [15\cos(x)\sin^2(x)]\, dx$

(g) $\int [\cos(2x)\cos(4x) - \sin(2x)\sin(4x)]\, dx$

15. Evaluate:

(a) $\int_0^{\frac{\pi}{2}} \left[\cos(x)\right] dx$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\sin\left(\frac{x}{2}\right)\right] dx$

16. A particle moves along a straight line such that its displacement, y metres at time t seconds is given by $y = 3\sin(2t) + 4$. Determine:

(a) an expression for the velocity of the particle at time t.

(b) the maximum velocity of the particle.

(c) an expression for the acceleration of the particle at time t.

(d) the velocity of the particle when $t = \frac{\pi}{2}$.

17. The length x cm of a spring at time t seconds is given by $x = 15 + 2\sin(3t)$.

(a) Find the shortest length of the spring.

(b) Determine the rate of change of x when $t = \frac{\pi}{6}$.

18. Find the stationary points and point(s) of inflection for:

(a) $y = \sin^2(x)$ for $0 \le x \le 2\pi$

(b) $y = \sin(x) + \cos(x) \quad 0 \le x \le 2\pi$

19. The ferris wheel 'London Eye' contains 32 capsules. A person enters the capsule at the lowest point above the ground. The height h metres above the ground after t minutes is given by:

$$h = 75 - 60\cos\left(\frac{2\pi t}{25}\right).$$

Calculate:

(a) the maximum height of the capsule above the ground.

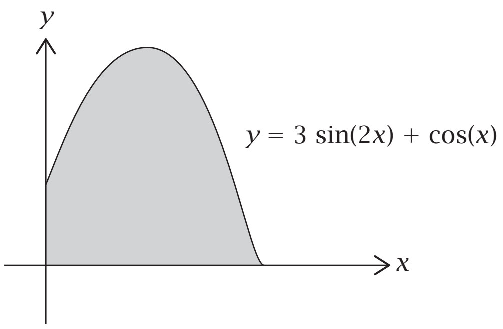
(b) the minimum height of the capsule.

(c) the time taken to complete one revolution.

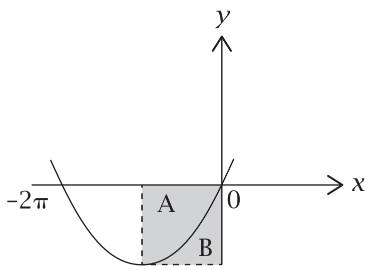
(d) the time(s) when the capsule is 100 metres above the ground in one revolution.

(e) the rate of change when $t = 10.5$.

20. Find the area of the shaded region bounded by the x and y axes and the curve $y = 3\sin(2x) + \cos(x)$.



21. The graph below shows the curve $y = 4\sin\left(\frac{x}{2}\right)$ for $-2\pi \le x \le 0$.



Find:

(a) the coordinates of the minimum point.

(b) the area of region A.

(c) the area of region B.

22. A particle moves along a straight line so that its acceleration 'a' in m/s^2 at time t seconds is given by:

$$a = -\frac{3\pi^2}{4} \cos\left(\frac{\pi t}{2}\right)$$

Initial velocity is 0 m/s
Initial displacement is 3 metres from the origin.

Determine:

(a) an expression for the velocity of the particle at time (t).

(b) an expression for the displacement of the particle at time (t).

(c) the velocity and displacement when $t = 3$.

23. Find the derivative of each of the following:

(a) $y = e^x \sin(3x)$

(b) $y = \cos(x) \cdot e^{-2x}$

(c) $y = xe^{\sin(x)}$

(d) $y = \cos\left(\ln \frac{1}{x}\right)$

(e) $y = (\sin(x) + \ln x)^2$

LOGARITHMS

6

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a b$ i.e. $a^{\log_a b} = b$
- ☐ establish and use the algebraic properties of logarithms
- ☐ examine the inverse relationship between logarithms and exponentials $y = a^x$ is equivalent to $x = \log_a y$
- ☐ interpret and use logarithmic scales
- ☐ solve equations involving indices using logarithms
- ☐ identify the qualitative features of the graph of $y = \log_a x$ including asymptotes, and of its translations
  $y = \log_a x + b$ and
  $y = \log_a (x - c)$
- ☐ solve simple equations involving logarithmic functions algebraically and graphically
- ☐ identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems
- ☐ define the natural logarithm $\ln x = \log_e x$
- ☐ examine and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$
- ☐ establish and use the formula $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- ☐ establish and use the formula $\int \frac{1}{x} dx = \ln x + c,\ x > 0$
- ☐ determine the derivatives of the form $\frac{d}{dx}(\ln f(x))$ and integrals of the form $\int \frac{f'(x)}{f(x)}\, dx$, $f(x) > 0$
- ☐ use logarithmic functions and their derivatives to solve practical problems.

FORMULAE AND DEFINITIONS

Definition

If $y = b^x$ *(Exponential form)*

$\Updownarrow$

then $x = \log_b y$ *(Logarithmic form)*

$\therefore \quad y = b^{\log_b y}$

$\therefore \quad x = \log_b b^x$

Laws of Logarithms

① $\log_b (AB) = \log_b A + \log_b B$

② $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$

③ $\log_b A^B = B \log_b A$

④ $\log_b \left(\frac{1}{A}\right) = -\log_b A$

⑤ $\log_b 1 = 0$

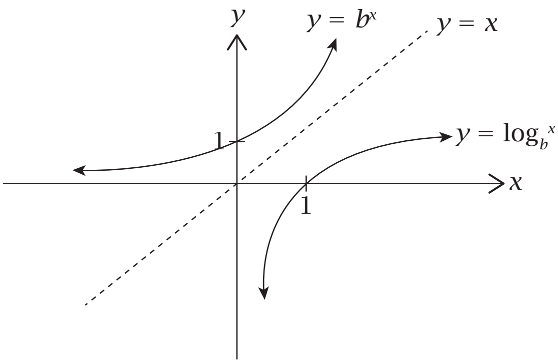
⑥ $\log_b b = 1$

Change of base rule

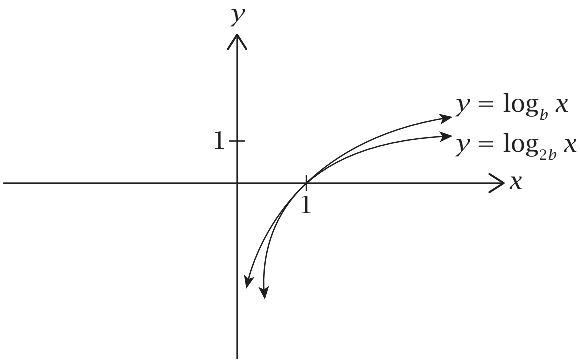
$$\log_b a = \frac{\log_{10} a}{\log_{10} b}$$

Logarithmic graph

If $y = b^x$ then $x = \log_b y$. Interchanging the x and y results in the equation becoming $y = \log_b x$.



The graph of $y = b^x$ is reflected about the line $y = x$ to obtain $y = \log_b x$.



As 'b' increases the graph of $y = \log_b x$ becomes 'flatter'.

Differentiation of logarithmic functions

If $\quad f(x) = \log_e x$

then $\quad f(x) = \ln x \quad \leftarrow$ natural logarithm base 'e'

$$y = \ln x$$

$$y = \log_e x$$

$$\therefore x = e^y$$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}$$

If $\quad y = \ln x$

then $\dfrac{dy}{dx} = \dfrac{1}{x}$

In general:

If $\quad g(x) = \ln [f(x)]$

then $\quad g'(x) = \dfrac{f'(x)}{f(x)}$

If $\quad y = \ln [f(x)]$

then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $\quad f(x) = \log_b x$

$= \dfrac{\ln x}{\ln b}$ *change of base rule*

$= \dfrac{1}{\ln b} \cdot \ln x$

then $\quad f'(x) = \dfrac{1}{\ln b} \cdot \dfrac{1}{x}$

$= \dfrac{1}{x \ln b}$

If $\quad f(x) = \log_b x$

then $\quad f'(x) = \dfrac{1}{x \ln b}$

Integration of logarithmic functions

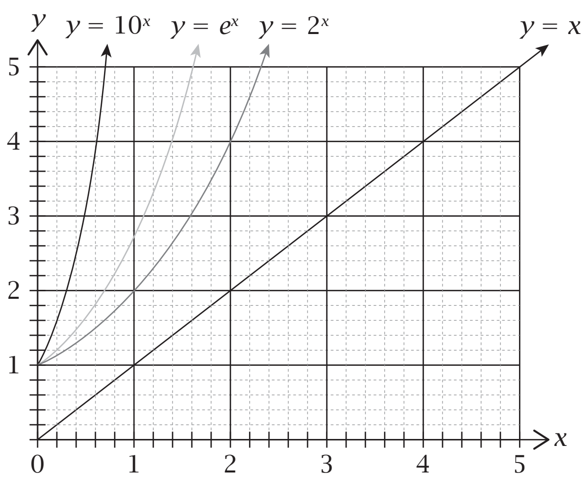
$\int \frac{1}{x}\, dx = \ln x + c \quad$ for $x > 0$

In general:

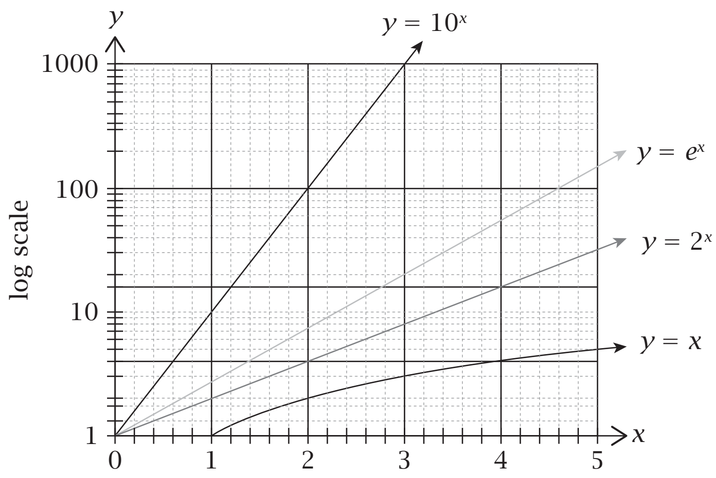
$\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + c \quad$ for $f(x) > 0$

Logarithmic scales

A logarithmic scale is a way of representing information on a graph especially when the magnitude of the numbers change exponentially. The units on the axis are powers or logarithms of a base number. If one axis uses a logarithmic scale then it is *semi-log*, both axes will be a *log-log* graph.

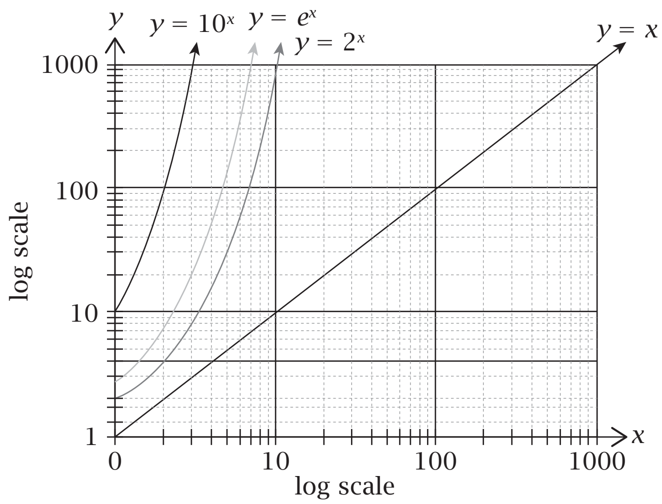


Semi-log graph



The graph of an exponential function of the form $y = k^x$ will appear as a straight line.

Log-log graph



Worked Examples

6.1 Write each as an exponential equation

(a) $\log_4 16 = 2$

(b) $\log_x 5 = t$

(a) $\log_4 16 = 2$ *(logarithmic form)*

$4^2 = 16$ *(exponential form)*

(b) $\log_x 5 = t$

$x^t = 5$

6.2 Rewrite the following in logarithmic form

(a) $2^7 = 128$

(b) $3^{-2} = \frac{1}{9}$

(a) $2^7 = 128$ *(exponential form)*

$\log_2 128 = 7$ *(logarithmic form)*

(b) $3^{-2} = \frac{1}{9}$

$\log_3 \left(\frac{1}{9}\right) = -2$

6.3 Without using a calculator, evaluate the following

(a) $\log_5 125$

(b) $\log_{0.1} 10$

(a) Let $y = \log_5 125$

then $5^y = 125$

$5^y = 5^3$

$\therefore$ $y = 3$

i.e. $\log_5 125 = 3$

(b) Let $P = \log_{0.1} 10$

then $0.1^P = 10$

$10^{-P} = 10$

$\therefore$ $P = -1$

i.e. $\log_{0.1} 10 = -1$

6.4 Evaluate the following without the use of a calculator

(a) $2 \log_6 3 + \log_6 4$

(b) $\log_{10} 500 - \frac{1}{2}\log_{10} 25$

(a) $2 \log_6 3 + \log_6 4$

$= \log_6 3^2 + \log_6 4$ *(Law 3)*

$= \log_6 (3^2 \times 4)$ *(Law 1)*

$= \log_6 36$

$= \log_6 6^2$

$= 2 \log_6 6$ *(Law 6)*

$= 2$ $\log_6 6 = 1$

(b) $\log_{10} 500 - \frac{1}{2}\log_{10} 25$

$= \log_{10} 500 - \log_{10} 25^{\frac{1}{2}}$ $25^{\frac{1}{2}} = \sqrt{25}$

$= \log_{10} 500 - \log_{10} 5$

$= \log_{10} \left(\frac{500}{5}\right)$ *(Law 2)*

$= \log_{10} (100)$

$= \log_{10} 10^2$

$= 2 \log_{10} 10$ $\log_{10} 10 = 1$

$= 2$

6.5 If $4\log_{10}\left(x\sqrt[4]{y^2}\right) = 2 - \log_{10} x + \log_{10} y$ find y in terms of x.

$4\log_{10}\left(x\sqrt[4]{y^2}\right) = 2 - \log_{10} x + \log_{10} y$ *Convert 2 into $\log_{10}$ 100*

$\log_{10}\left(x\sqrt[4]{y^2}\right)^4 = \log_{10} 100 - \log_{10} x + \log_{10} y$

$\log_{10}\left(x^4 y^2\right) = \log\left(\frac{100}{x}y\right)$

$\therefore x^4 y^2 = \frac{100y}{x}$

$y = \frac{100}{x^5}$

6.6 If $a = \log_t 3$ and $b = \log_t 5$ express each of the following in terms of a and/or b.

(a) $\log_t 15$

(b) $\log_t 0.6$

(c) $\log_t 45$

(a) $\log_t 15 = \log_t (3 \times 5)$
$= \log_t 3 + \log_t 5$
$= a + b$

(b) $\log_t 0.6 = \log_t \left(\frac{3}{5}\right)$
$= \log_t 3 - \log_t 5$
$= a - b$

(c) $\log_t 45 = \log_t (3^2 \times 5)$
$= \log_t 3^2 + \log_t 5$
$= 2 \log_t 3 + \log_t 5$
$= 2a + b$

6.7 Solve the following *exactly* without the use of a calculator

(a) $3^x = 11$

(b) $5^{3x-1} = 7^{x+3}$

(a) $3^x = 11$
$\log 3^x = \log 11$ *take logs of both sides*
$x \log 3 = \log 11$ *law 3*
$\therefore\ x = \dfrac{\log 11}{\log 3}$

(b) $5^{3x-1} = 7^{x+3}$ *take logs of both sides*
$\log 5^{3x-1} = \log 7^{x+3}$
$(3x - 1) \log 5 = (x + 3) \log 7$ *law 3*
$3x \log 5 - \log 5 = x \log 7 + 3 \log 7$
$3x \log 5 - x \log 7 = 3 \log 7 + \log 5$
$x(3 \log 5 - \log 7) = 3 \log 7 + \log 5$
$x = \dfrac{3 \log 7 + \log 5}{3 \log 5 - \log 7}$

6.8 Solve the following *exactly* without the use of a calculator

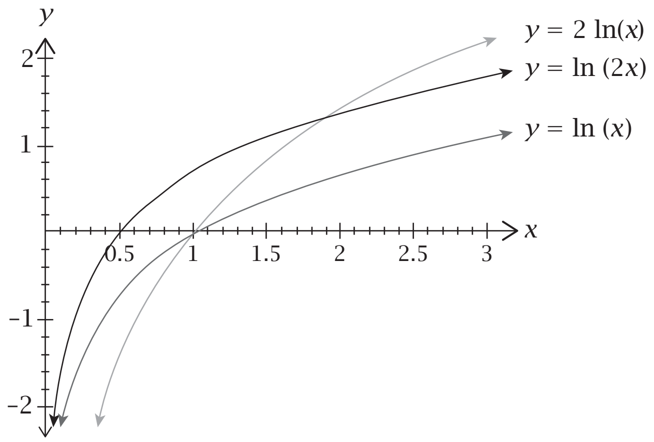
(a) $e^{x+2} = 7$

(b) $2e^{5x+1} = 1000$

(a) $e^{x+2} = 7$ *Take the* natural *log of both sides as*
$\ln e^{x+2} = \ln 7$ *equation involves 'e'*
$(x + 2) \ln e = \ln 7$ *ln e = 1*
$x + 2 = \ln 7$
$x = \ln 7 - 2$

(b) $2e^{5x+1} = 1000$
$e^{5x+1} = \dfrac{1000}{2}$
$e^{5x+1} = 500$ *take ln of both sides*
$\ln e^{5x+1} = \ln 500$
$(5x + 1) \ln e = \ln 500$ *ln e = 1*
$5x + 1 = \ln 500$
$5x = \ln 500 - 1$
$x = \dfrac{\ln 500 - 1}{5}$

6.9 Graph on the same set of axes $y = \ln x$, $y = 2 \ln x$, and $y = \ln 2x$. Find the coordinates of the x intercept.



	x intercept
$y = \ln x$	(1,0)
$y = 2 \ln x$	(1,0)
$y = \ln 2x$	$(\frac{1}{2},0)$

6.10 Differentiate each of the following:

(a) $y = \ln (6x - 2)$

(b) $y = x^3 \ln 3x$

(c) $y = \ln \left(\frac{2x - 3}{x + 4}\right)$

(a)

$$y = \ln (6x - 2)$$
$$\frac{dy}{dx} = \frac{1}{6x - 2} \times 6$$
$$\frac{dy}{dx} = \frac{6}{6x - 2}$$

(b)

$$y = x^3 \ln (3x)$$
$$\frac{dy}{dx} = (3x^2)(\ln (3x)) + (x^3)\left(\frac{3}{3x}\right) \qquad \textit{Product Rule}$$
$$\frac{dy}{dx} = (3x^2)(\ln (3x)) + (x^3)\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = 3x^2 \ln (3x) + x^2$$

(c)

$$y = \ln \left(\frac{2x - 3}{x + 4}\right)$$
$$y = \ln (2x - 3) - \ln (x + 4) \qquad \textit{Log Law 2}$$
$$\frac{dy}{dx} = \left(\frac{1}{2x - 3} \times 2\right) - \left(\frac{1}{x + 4}\right)$$
$$\frac{dy}{dx} = \frac{2}{2x - 3} - \frac{1}{x + 4}$$
$$\frac{dy}{dx} = \frac{11}{(2x - 3)(x + 4)}$$

6.11 Find the gradient of $y = \ln x$ at (1,0)

$$y = \ln x$$
$$\frac{dy}{dx} = \frac{1}{x}$$

when $x = 1$

$$\textit{gradient } \frac{dy}{dx} = \frac{1}{1}$$
$$= 1$$

6.12 Determine the following integrals

(a) $\int \left[\frac{15}{3x-2}\right] dx$, $3x - 2 > 0$

(b) $\int \left[\frac{16x^2}{2x^3-4}\right] dx$, $2x^3 - 4 > 0$

(c) $\int_1^3 \left[\frac{2}{x}\right] dx$ (as an exact value), $x > 0$

(a) $\int \left[\frac{15}{3x-2}\right] dx$

$= 5\ln(3x-2) + c$

(b) $\int \left[\frac{16x^2}{2x^3-4}\right] dx$

$= \frac{8}{3}\ln(2x^3-4) + c$

(c) $\int_1^3 \left[\frac{2}{x}\right] dx$

$= \left[2\ln x\right]_1^3$

$= 2\ln 3 - 2\ln 1$ (ln 1 = 0)

$= 2\ln 3$

PROBLEMS TO SOLVE

Chapter 6: Logarithms

1. Write the following in exponential form:

   (a) $\log_6 36 = 2$

   (b) $\log_4 \left(\frac{1}{16}\right) = -2$

   (c) $\log_a b = c$

2. Write the following in logarithmic form:

   (a) $2^5 = 32$

   (b) $7^{-3} = \frac{1}{343}$

   (c) $p^r = q$

3. Solve the following without using a calculator:

   (a) $\log_3 81$

   (b) $\log_{10} 10^6$

   (c) $\log_2 \left(\frac{1}{32}\right)$

   (d) $\log_6 1$

   (e) $\log_5 0.04$

   (f) $\ln (e^4)$

   (g) $\ln \frac{1}{\sqrt{e}}$

   (h) $\ln \sqrt[5]{e}$

4. Calculate the following without the use of a calculator:

   (a) $\log_3 8 + \log_3 \left(\frac{1}{8}\right)$

   (b) $2 \log 5 + \log 8 - \frac{1}{2}\log 4$

   (c) $\frac{\log 16}{\log 2}$

   (d) $\frac{2}{3}\log_2 8 + 6\log_2 \sqrt[3]{2} - \frac{1}{2}\log_2 \frac{1}{4}$

   (e) $\log_a (a^4) + \log_b (b^3) - \log_c (c^2)$

5. Express the following as a single logarithm:

   (a) $3 \log a - \log b - \log c$

   (b) $\log a + \frac{1}{2} \log c - 2 \log b$

   (c) $2 + \log a^3 - 2 \log a$

6. Given p and q are positive solve the following simultaneous equations.

$\log(pq) = 5$

$\log\left(\frac{p}{q}\right) = 1$

7. If $p = \log_c 3$ and $q = \log_c 2$ express each of the following in terms of p and/or q.

(a) $\log_c 6$

(b) $\log_c 1.5$

(c) $\log_c 1\frac{1}{8}$

(d) $\log_c \sqrt[4]{36}$

8. State the following as y in terms of x.

(a) $3\log_2 y = 2\log_2 x$

(b) $\log_2 y + 2 = \log_2 x^3$

(c) $2\log_2(xy) = 5\log_2 x$

9. Solve these equations exactly:

(a) $3^x = 8$

(b) $5^{2x+1} = 9$

(c) $6^{1-x} = 2^{3x+5}$

(d) $2 \times 2^{2x} - 11 \times 2^x + 5 = 0$

(e) $e^{5x+1} = 3$

(f) $6e^{1-2x} = 360$

10. Differentiate the following:

(a) $y = \ln(2x + 3)$

(b) $y = \ln(x^4)$

(c) $y = x\ln(3x)$

(d) $y = \sqrt{x}\ln\left(\frac{x}{3}\right)$

(e) $y = \frac{\ln x}{x}$

(f) $y = \frac{1}{10 - \ln x}$

(g) $y = (\ln x)^2$

11. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = e^{2x}\ln(2x)$

(b) $y = \ln(\sin x)$

(c) $y = \ln(e^{-2x} + 4)$

(d) $y = \frac{\cos^2 x}{\ln x}$

(e) $y = \ln\left[\frac{(x+4)^2}{(3x-1)}\right]$

12. Find the following integrals (Note: Denominators are greater than or equal to 0)

(a) $\int \left[\frac{6}{x}\right] dx$

(b) $\int \left[\frac{1}{6x+5}\right] dx$

(c) $\int \left[\frac{4x}{2-x^2}\right] dx$

(d) $\int \left[\frac{2x-1}{x^2-x}\right] dx$

(e) $\int \left[\frac{x-3}{x^2-6x+1}\right] dx$

(f) $\int \left[\frac{\cos x}{\sin x}\right] dx$

(g) $\int \left[\frac{\sin 2x}{1+\cos 2x}\right] dx$

(h) $\int \left[\frac{\sin x - \cos x}{\sin x + \cos x}\right] dx$

13. Evaluate each of the following exactly.

(a) $\int_2^3 \left(\frac{5}{x}\right) dx$

(b) $\int_1^4 \left(\frac{3}{2x-1}\right) dx$

(c) $\int_2^4 \left(\frac{1}{x} + e^x\right) dx$

14. If $\int_1^p \left[\frac{3}{2x-1}\right] dx = 2$ find the value of p if $p > 1$.

15. Find the equation of the tangent to the curve $y = x^2 \ln(x)$ at the point where $x = 1$.

16. Sketch the following graphs identifying any intercepts and asymptotes.

(a) $y = \log_2 x$

(b) $y = \log_2 x + 1$

(c) $y = \log_2 (x + 1)$

17. The tangent to the curve $y = \ln (kx - 1)$ has a gradient of 1 when $x = 2$. Determine the value of k.

18. Find the equations of the functions according to the following:

(a) translation of $y = \log_2 (x)$: 3 units left and 2 units up

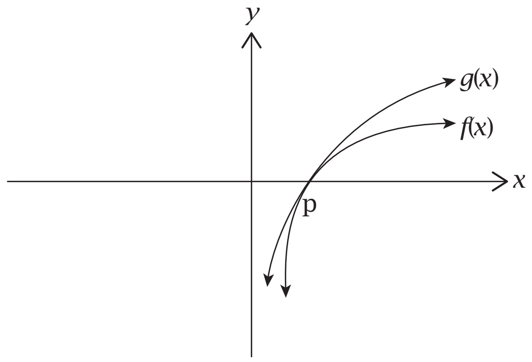
(b) translation of $y = \log_3 (x)$: 2 units right and 1 unit down

19. Find the coordinates on the curve $y = x \ln x$ where the gradient is 1.

20. Determine the minimum value of $y = \ln x + \frac{1}{x}$.

21. The total cost, \$C, for producing x units of clothing is given by the equation $C = 500 + 300 \ln(x + 1)$. Calculate the average cost per unit when the marginal cost is \$3.

22. Given
$f(x) = \ln x$
$g(x) = \log_a x$



(a) determine the possible values of a

(b) state the coordinate of p

23. Find the equation of the tangent to the curve $y = x \ln x$ at the point (e, e).

24. A particle moves in a straight line such that its displacement from point 0, at time t seconds is given by $x = 15 \ln(2t - 5) - 5t$ where x is in metres. Find the time(s):

(a) when the numerical value of the velocity is equal to its acceleration

(b) when the particle is at rest.

25. Use calculus to determine the exact coordinates of the stationary point(s) for $y = -\ln x + 2x^2$ for $x > 0$. Determine the nature of these stationary point(s).

26. The formula $\text{pH} = -\log[H^+]$ calculates the pH level where H^+ is the hydrogen ion concentration in moles/L.

Calculate:

(a) the hydrogen ion concentration if the pH is 6.89

(b) the pH if the hydrogen ion concentration is 1.25×10^{-8}

27. The brightness of a star from Earth can be calculated using the formula:

$m - M = 5(\log d - 1)$

where m = apparent magnitude
M = absolute magnitude
d = distance to the star in parsecs

Calculate:

(a) the absolute magnitude, M, of a star whose apparent magnitude, m, is 0.25 if the star is 450 parsecs from Earth.

(b) the distance to Earth from Betelgeuse if it has an absolute magnitude $M = 0.9$ and an apparent magnitude, $m = 7.4$.

28. A radioactive material decays such that the amount A in grams, present after time t years is given by:

$A = 400e^{-0.2t}$

Calculate:

(a) the amount present after 3 years

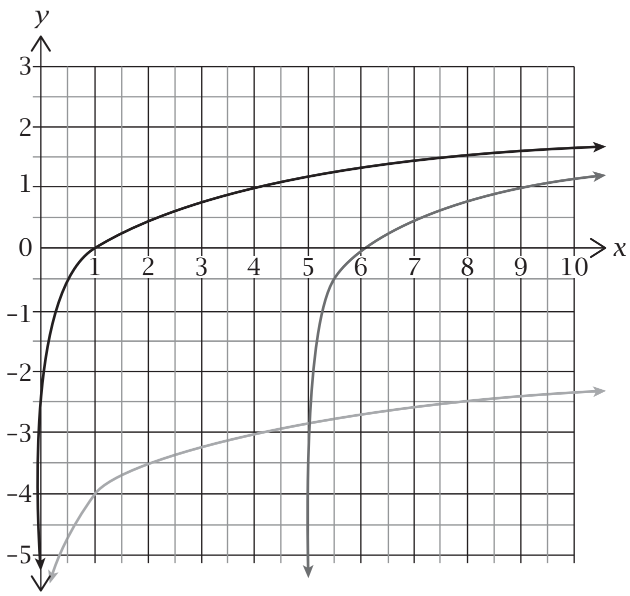
(b) the half life of the material

(c) the time taken for 100 grams of material to remain

29. Given:

$y = \log_a (x)$
$y = \log_a (x) + p$
$y = \log_a (x - q)$

find the values of a, p and q from the graph below.



30. \$150 000 is invested in an account earning 6% p.a. interest compounded annually. The amount \$A after t years is:

$A = 150\,000(1.06)^t$

Determine:

(a) the value of A after 4 years

(b) the length of time required for the account to reach \$275 000

(c) the interest rate if the \$150 000 triples in value after 10 years.

31. A radioactive compound decays over a 50 year period according to the following table:

t	0	10	20	30	40	50
A	700	573	469	384	315	258

(a) Graph t against $\ln A$ to show the relationship between t and $\ln A$ is linear.

(b) The amount of radioactivity is given by $A = A_0e^{kt}$. Find A_0 and k.

(c) When will the radioactivity first fall below 150?

7 EXPONENTIAL FUNCTIONS

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$ for various values of $a > 0$
- ☐ identify that e is the unique number for a for which the above limit is 1
- ☐ establish and use the formula $\frac{d}{dx}(e^x) = e^x$
- ☐ use exponential functions of the form Ae^{kx} and their derivatives to solve practical problems
- ☐ apply the product, quotient and chain rule to differentiate functions such as xe^x, $e^{-x} \sin x$ and $f(ax - b)$
- ☐ establish and use the formula $\int e^x \, dx = e^x + c$.

FORMULAE AND DEFINITIONS

The Exponential Function

A table of values will enable the numerical value of the expression $\left(1 + \frac{1}{n}\right)^n$ to be found as $n \to \infty$.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815
100000	2.71827
1000000	2.71828
10000000	2.71828

As n becomes larger the expression $\left(1 + \frac{1}{n}\right)^n$ approaches 2.71828

The letter 'e' is used to represent the $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

i.e. $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

where $e \approx 2.71828$

'e' is commonly used in questions dealing with exponential growth and decay where the general equation is: $A = A_0 e^{kt}$
Where: A_0 is the initial amount
t is time
k is constant

Use a graphics calculator to investigate the limiting value of $h \to 0$

If $h = 2$ $\quad \dfrac{e^h - 1}{h} \approx 3.194528$

If $h = 1$ $\quad \dfrac{e^h - 1}{h} \approx 1.718282$

If $h = 0.1$ $\quad \dfrac{e^h - 1}{h} \approx 1.051709$

If $h = 0.0001$ $\quad \dfrac{e^h - 1}{h} \approx 1.00005$

$$\therefore \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Therefore using * (next page)

$$\begin{aligned} f'(x) &= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\ &= e^x \cdot 1 \\ &= e^x \end{aligned}$$

Given $f(x) = e^x$

$$\therefore f'(x) = e^x$$

Derivative Rules

$y = e^{f(x)}$ $\quad \dfrac{dy}{dx} = f'(x)e^{f(x)}$

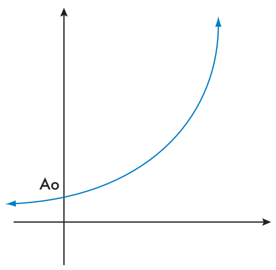
Second Derivative

A second derivative of a function can be found by differentiating twice. The notation for second derivative includes

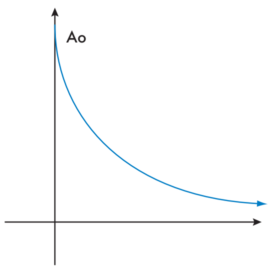
$f''(x)$ and $\dfrac{d^2y}{dx^2}$

Graphs

Graphs of exponential functions are:



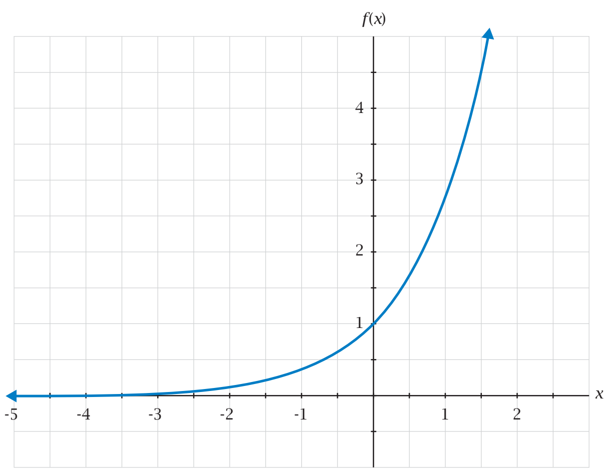
exponential growth



exponential decay

Derivative of the Exponential function $f(x) = e^x$

The graph of $f(x) = e^x$ is drawn below



The limiting chord process can also be applied to this function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{e^{x+h} - e^x}{h}$$

$$= \frac{e^x \cdot e^h - e^x}{h}$$

$$= \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0} \frac{(e^h - 1)}{h} \quad *$$

Integral of the Exponential function $f(x) = e^x$

$$\int e^x dx = e^x + c$$

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + c$$

Geometric Growth and Decay

Given $\frac{dA}{dt} = kA$ where t represents time

then $\boxed{A = A_0 e^{kt}}$

A_0 is the initial value when $t = 0$

- $k > 0$ represents exponential growth
- $k < 0$ represents exponential decay.

Worked Examples

7.1 The population, P, of mice grows according to the formula $P = 15e^{0.2t}$ where t is the time in months.

Determine

(a) the initial size of the population.

(b) how many mice there will be after 8 months.

(c) when the population of mice doubles.

(a) the initial size of the population is when $t = 0$

ie $P = 15e^{0.2t}$

$P = 15e^0$

$P = 15$

$\therefore$ The initial population is 15 mice.

(b) When $t = 8$

$P = 15e^{0.2(8)}$

≈ 74.3

$\therefore$ There are 74 mice after 8 months.

(c) Double the initial number of mice ie. $P = 30$

$$P = 15e^{0.2t}$$

$\therefore 30 = 15e^{0.2t}$

Using calculator

$t \approx 3.466$

$\therefore$ the number of mice doubles after 3.466 months.

7.2 Describe how each of the graphs below can be obtained by transforming $y = e^x$

(a) $y = e^x + 2$

(b) $y = e^{x-1}$

(c) $y = 2e^x$

(d) $y = -3e^{2x+3} - 4$

If the graph of $y = f(x)$ is transformed by $y = af(bx - c) + d$ the following occur in **order**:

'c'	· translation c units right or left
'b'	· dilation parallel to the x axis scale factor $\frac{1}{\lvert b \rvert}$ · if b is negative graph is reflected in the y axis
'a'	· dilation parallel to the y axis scale factor $\lvert a \rvert$ · if a is negative graph is reflected in the x axis.
'd'	· translation vertically up or down d units.

(a) translation vertically up 2 units

(b) translation right 1 unit

(c) dilation parallel to the y axis scale factor of 2

(d) – translation 3 units left

– dilation parallel to the x axis scale factor of $\frac{1}{2}$

– dilation parallel to the y axis scale factor of 3 and reflection in the x axis

– translation 4 units down.

7.3 Find the derivatives of each of the following:

(a) $y = 4e^{5x}$

(b) $f(x) = \dfrac{4}{e^{3x}} + 2x^3$

(c) $y = xe^{2x}$

(d) $y = \dfrac{e^{4x}}{2 + e^{3x}}$

(e) $y = \sqrt{1 - 3e^{4x}}$

(a) $y = 4e^{5x}$

$\dfrac{dy}{dx} = 20e^{5x}$

use rule: $y = e^{f(x)}$

$\dfrac{dy}{dx} = f'(x)e^{f(x)}$

(b) $f(x) = \dfrac{4}{e^{3x}} + 2x^3$

$f(x) = 4e^{-3x} + 2x^3$ *rearrange*

$f'(x) = -12e^{-3x} + 6x^2$ *differentiate using sum rule and* $y = e^{f(x)}$ $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

$f'(x) = \dfrac{-12}{e^{3x}} + 6x^2$ *ensure indices are positive*

(c) $y = xe^{2x}$

$\dfrac{dy}{dx} = (1)e^{2x} + x2e^{2x}$ *use product rule to differentiate*

$\dfrac{dy}{dx} = e^{2x} + 2xe^{2x}$

(d) $y = \dfrac{e^{4x}}{2 + e^{3x}}$

$\dfrac{dy}{dx} = \dfrac{(4e^{4x})(2 + e^{3x}) - (e^{4x})(3e^{3x})}{(2 + e^{3x})^2}$ *use quotient rule to differentiate*

This answer has not been simplified.

(e) $y = \sqrt{1 - 3e^{4x}}$

$y = (1 - 3e^{4x})^{\frac{1}{2}}$ *rearrange*

$\dfrac{dy}{dx} = \dfrac{1}{2}(1 - 3e^{4x})^{-\frac{1}{2}}(-12e^{4x})$ *use chain rule to differentiate*

$\dfrac{dy}{dx} = \dfrac{-6e^{4x}}{\sqrt{1 - 3e^{4x}}}$ *simplify and ensure positive indices*

7.4 Find the equation of the tangent line to the curve $y = x - e^{2x}$ at the point where $x = 0$.

To find the gradient we require the gradient function

$y = x - e^{2x}$

$y' = 1 - 2e^{2x}$

Substitute in $x = 0$ *to find the gradient*

$y' = 1 - 2e^{2(0)}$

$y' = -1$

To find the equation of the tangent line a coordinate point is required.

when $x = 0$: $y = x - e^{2x}$

$y = 0 - e^{2(0)}$

$y = -1$

(0, –1)

Equation of tangent line

$y = mx + c$

Substitute in gradient

$y = -1x + c$

Substitute in coordinate to find ‘c’

$y = -1x + c$

$-1 = -1(0) + c$

$-1 = c$

$\therefore$ equation of tangent line is $y = -x - 1$

7.5 Find the exact coordinates of the two points on the curve $y = \frac{e^x}{x^2-8}$ where the gradient is 0.

Find the gradient function

$$y = \frac{e^x}{x^2-8}$$

$$y' = \frac{e^x(x^2-8) - e^x(2x)}{(x^2-8)^2}$$

As gradient is 0

$$0 = \frac{e^x(x^2-8) - e^x(2x)}{(x^2-8)^2}$$

$$0 = e^x(x^2-8-2x)$$

$$\therefore \cancel{e^x = 0} \qquad x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4, \; x = -2$$

Substitute x values into original equation to find the y values

When $x = 4$ $\quad y = \frac{e^4}{4^2-8}$

$$y = \frac{e^4}{8}$$

When $x = -2$ $\quad y = \frac{e^{-2}}{4-8}$

$$y = \frac{e^{-2}}{-4}$$

Coordinates are $(4, \frac{e^4}{8})$ and $\left(-2, -\frac{1}{4e^2}\right)$

7.6 Exponential Application

A colour dye with initial concentration of 0.6 units is placed into a tub of water and the rate of change of the dye is given by:

$\frac{dC}{dt} = -0.72C$ units per minute

Where $C = C(t)$ is the concentration of the dye at any time t minutes after being placed into the tub.

(a) Find C as a function of t.

(b) Determine the initial rate of change of C.

(c) Calculate the concentration of dye 2 minutes after the dye was placed in the water.

(d) How long does it take for the concentration of the dye to be 0.2 units?

(a) $$\frac{dC}{dt} = -0.72\underbrace{C}_{\text{increasing or decreasing constant}}$$

$$\therefore C = \underbrace{C_0}_{\text{initial value}} e^{kt}$$

$$C = 0.6e^{-0.72t}$$

(b) $$C = 0.6e^{-0.72t}$$

$$\therefore C' = -0.432e^{-0.72t}$$

when $t = 0$

$$C' = -0.432$$

Decreasing at a rate of 0.432 units/min

(c) $C = 0.6\,e^{-0.72(2)}$

$C \approx 0.142$ units

(d) $C = 0.6\,e^{-0.72t}$

Substitute in $C = 0.2$ to find t

$0.2 = 0.6\,e^{-0.72t}$

$\frac{0.2}{0.6} = e^{-0.72t}$

$\frac{1}{3} = e^{-0.72t}$ * *use calculator*

$t \approx 1.526$ minutes

7.7 Find the antiderivative of:

e^{2x-7}

$\int e^{2x-7}\,dx$ $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$

$= \frac{e^{2x-7}}{2} + c$

7.8 Evaluate the definite integrals:

$\int_1^3 (e^{3x} + e^{-3x})\,dx$

$\int_1^3 (e^{3x} + e^{-3x})\,dx$

$= \left[\frac{e^{3x}}{3} - \frac{e^{-3x}}{3}\right]_1^3$

$= \left[\frac{e^{3(3)}}{3} - \frac{e^{-3(3)}}{3}\right] - \left[\frac{e^{3(1)}}{3} - \frac{e^{-3(1)}}{3}\right]$

$= \left[\frac{e^{9}}{3} - \frac{e^{-9}}{3}\right] - \left[\frac{e^{3}}{3} - \frac{e^{-3}}{3}\right]$

$\approx [2701.0279] - [6.6786]$

≈ 2694.35

PROBLEMS TO SOLVE

Chapter 7: Exponential Functions

1. Use your graphics calculator to investigate the effects of changing the values of a, b, c and d on the graph of $y = ae^{bx-c} + d$.

2. (a) Describe the effect on the graph of $y = e^x$ if it is transformed to obtain each of the following:

   i. $y = e^{-x}$

   ii. $y = -e^x$

   iii. $y = e^{2x}$

   iv. $y = 2e^x$

   v. $y = \frac{1}{3}e^{3x+4} - 1$

   (b) Find the transformed equation if $y = e^x$ is:

   - translated 3 units right
   - dilated parallel to the x axis scale factor 2
   - reflected in the x axis
   - translated 4 units up.

3. The amount of money, \$A, accrued by investing \$8000 at 6% per annum compounded continuously for t years is given by $A = 8000e^{0.06t}$. Find the amount of money after:

   (a) 2 years.

   (b) 10 years.

4. A radioactive element disintegrates according to the formula $R = 2000e^{-0.02t}$ where t is the time in years.

   (a) Find the value of R when $t = 10$ years.

   (b) What is the half-life of the element?

5. The population P of a bacterial culture is modelled by the rule: $P = \frac{700}{1 + e^{-0.2t}}$ where t is the time in days.

   (a) What is the initial population?

   (b) What is the population after 5 days?

   (c) How long will it take for the population to reach 650?

6. Differentiate the following with respect to x:

   (a) $y = 2e^{2x}$

   (b) $y = \frac{5}{e^x + 1}$

   (c) $y = \frac{1}{3e^{4x}}$

   (d) $y = \sqrt{e^{4x}}$

   (e) $y = (e^{3x} + 1)^2$

   (f) $y = e^{\sqrt{x}}$

7. Differentiate the following with respect to x: *(Simplify where necessary)*

(a) $y = xe^x$

(b) $y = e^x(10 - x)^4$

(c) $y = (3x + 1)^2 e^{-x}$

(d) $y = \dfrac{e^{3x-2}}{e^{x-2}}$

(e) $y = \dfrac{x^3 - 4x^2 + 3x}{x^2}$

(f) $y = \sqrt{x}\, e^x$

8. Find the equation of the tangent to the curve $y = e^{2x+1}$ at the point where $x = -1$. Answer in exact form.

9. Determine the equation of the tangent line to the curve $y = \dfrac{e^x}{x}$ at the point $(1, e)$.

10. Find and classify all stationary points for the function: $y = x^2 e^x$.

11. A curve $y = ae^{-k^2x^2}$ where a and k are positive constants, has a single turning point at $(0, a)$.

(a) Prove that this turning point is correct and establish using calculus the nature of the turning point.

(b) Determine the point(s) of inflection.

12. Antidifferentiate the following:

(a) e^{3x+1}

(b) $4e^{2x}$

(c) $\dfrac{6}{e^{5x}}$

(d) $3\sqrt{e^{4x}}$

(e) $e^{5x} + e^{-5x}$

(f) $2xe^{x^2+1}$

13. Find each of the following indefinite integrals:

(a) $\int 4e^{3x}\,dx$

(b) $\int 3xe^{x^2-6}\,dx$

(c) $\int (\sqrt{5x^3} + e^{-x})\,dx$

(d) $\int \dfrac{8e^{2x} + e^{-x}}{e^{-x}}\,dx$

14. Evaluate the following definite integrals:

(a) $\displaystyle\int_{-1}^{4} e^{2x}\,dx$ (in exact form)

(b) $\displaystyle\int_{0}^{1} \left(e^{3x} - \frac{4}{(x+1)^2}\right) dx$

15. Given $\dfrac{dP}{dt} = e^{4-2t}$

(a) Express P in terms of t if when $t = 1$, $P = \dfrac{e^2}{2}$

(b) Find P when $t = 2$

16. An experiment found the rate at which a certain drug was absorbed by the body was proportional to the concentration 'C' of the drug at any time t (in hours).

i.e. $\dfrac{dC}{dt} = -kC$ for some constant k.

If the original concentration was 175 units and 60 minutes later was 125 units find:

(a) when the concentration of the drug reaches 95 units.

(b) the concentration of the drug after 4½ hours.

17. The rate at which bacteria increases is proportional to the size of the bacteria population P.

i.e. $\dfrac{dP}{dt} = kP$

If t is in weeks and the population increases by 150 in the third week and 350 in the fourth week determine:

(a) the initial size of the population

(b) the population of bacteria after 6 weeks.

18. Find the derivative of each of the following:

(a) $y = e^{x^2}$

(b) $y = x^5 e^{-x}$

(c) $y = e^{-x} \sin x$

(d) $y = \dfrac{e^{2x}}{\ln x}$

(e) $y = e^{-2x} \cos x$

(f) $y = (e^x - e^{-x})^2$

(g) $y = \dfrac{e^{2x}}{x^3 - x}$

19. Determine:

(a) $\displaystyle\int \left(\frac{e^{5x}}{3} + \pi\right) dx$

(b) $\displaystyle\int_0^2 \left(2x\, e^{2x^2+3}\right) dx$

(c) $\displaystyle\int \left(e^{-4x}\pi - e\right) dx$

(d) $\displaystyle\int_0^3 6\left(\sqrt{e^x} + x\right) dx$

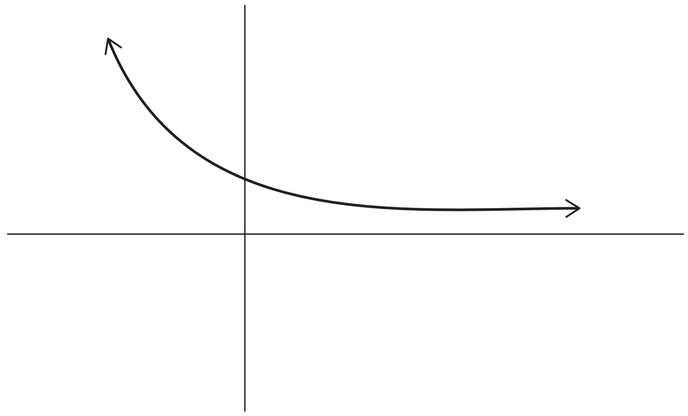
as an exact value

(e) $\displaystyle\int e^{x^2-5x+2}\left(2x-5\right) dx$

20. If $f'(x) = (x-2)\left(e^{x^2-4x+3}\right)$ and $f(1) = 2$ determine $f(x)$.

21. State the sequence of transformations that transforms $y = e^{3x+2}$ into $y = -e^{x+1}$.

22. The graph of $y = ae^{-bx}$ is shown below, where a and b are constants.



Sketch on the same axes $y = -ae^{bx}$.

23. The population, P (in thousands), at time t years from January 1st 2014 is modelled by the equation $P = 17.45e^{kt}$ for $t \geq 0$.

(a) Express the rate of change of this population over time, in terms of P.

(b) What is the size of the population on January 1st 2014?

(c) The population for the first year increases by 125. Determine the value of k.

(d) During which year is the population expected to double.

24. The median house price (H) changes according to $\dfrac{dH}{dt} = 0.065H$, where t is the number of years since January 1st 2000.

(a) How long will it take for house prices to triple in value?

The median house price in 2012 was \$550 000.

(b) Determine the instantaneous rate of change in 2012.

(c) Calculate the median house price in 2002.

(d) Calculate the average rate of change in the median house price from 2002 to 2012.

DISCRETE RANDOM VARIABLES AND BINOMIAL DISTRIBUTIONS

8

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ develop the concepts of a discrete random variable and its associated probability function, and their use in modelling data
- ☐ use relative frequencies obtained from data to obtain point estimates of probabilities associated with a discrete random variable
- ☐ identify uniform discrete random variables and use them to model random phenomena with equally likely outcomes
- ☐ examine simple examples of non-uniform discrete random variables
- ☐ identify the mean or expected value of a discrete random variable as a measurement of centre and evaluate it in simple cases
- ☐ identify the variance and standard deviation of a discrete random variable as measures of spread, and evaluate them using technology
- ☐ examine the effects of linear changes of scale and origin on the mean and standard deviation
- ☐ use discrete random variables and associated probabilities to solve practical problems
- ☐ use a Bernoulli random variable as a model for two-outcome situations
- ☐ identify contexts suitable for modelling by Bernoulli random variables
- ☐ determine the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p
- ☐ use Bernoulli random variables and associated probabilities to model data and solve practical problems
- ☐ examine the concept of Bernoulli trials and the concept of a binomial random variable as the number of successes 'n' with the probability of success 'p' in each trial
- ☐ identify contexts suitable for modelling by binomial random variables
- ☐ determine and use $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ and also the mean $\mu = np$ and variance $\sigma^2 = np(1-p)$
- ☐ use binomial distributions and associated probabilities to solve practical problems.

FORMULAE AND DEFINITIONS

Random Variable

Suppose a coin is tossed three times. The resulting sample space is:

H H H	T H H
H T H	T H T
H H T	T T H
H T T	T T T

Considering only the number of heads, a value of 0, 1, 2 or 3 will be assigned to each outcome in the sample space.

These numbers 0, 1, 2, or 3 are random quantities determined from an outcome of an experiment.

A **random variable** denoted by X can take a value or range of values. It is a measurable quantity from the outcomes of a random experiment.

Discrete Random Variable

A **discrete random variable** can take only certain specific values (discrete) within its given domain.

A **discrete probability distribution** is the values of a discrete random variable and its associated probabilities.

The function f defined by $f(x) = P(X = x)$ is called the **probability function** of the discrete random variable x.

The following **conditions must also hold**:

(a) The sum of all the probabilities must be equal to 1

(b) Negative probabilities are not allowed

Expected Value

The expected value (mean) of a discrete probability distribution is $E(X) = \mu = \Sigma(X \cdot P(X = x))$

Standard Deviation/Variance

The variance of a discrete probability distribution is $\sigma^2 = \Sigma(x - \mu)^2 \cdot P(X = x)$

The standard deviation of a discrete probability distribution is $\sigma = \sqrt{\Sigma(x - \mu)^2 \bullet P(X = x)}$.

Bernoulli Distribution

A Bernoulli distribution is the probability distribution of a discrete random variable which has a value of 1 with 'success' probability of **p** and the value of 0 with 'failure' probability of **1 – p**.

- Expected value $E(x) = p$
- Standard deviation $\sigma = \sqrt{p(1 - p)}$

Binomial Distribution

A binomial distribution is a discrete distribution which has the following properties

- Each trial has two possible outcomes – 'success' or 'failure'. ie. each trial is a **Bernoulli trial**.
- The probability of success, p, remains constant from trial to trial
- There are n repeated independent trials

If $X \sim B(n, p)$ represents a random variable X binomial distribution with n trials and probability of success P then:

$P(X = r) = {}^{n}C_{r}\, p^{r}\,(1-p)^{n-r}$ or $P(X = r) = \binom{n}{r} p^{r}\,(1-p)^{n-r}$

If X is a random variable and $X \sim B(n, p)$ then to approximate the mean and standard deviation use the following:

- Expected value (mean) $\mu = np$
- Standard Deviation $\sigma = \sqrt{np(1-p)}$

Change of Scale/Origin

If a random variable X with:

- Expected value (mean) = μ
- Standard Deviation = σ
- Variance = σ^2

then the random variable $aX + b$ will have:

- Expected value (mean) = $a\mu + b$
- Standard Deviation = $|a|\sigma$
- Variance = $a^2\sigma^2$

Worked Examples

8.1 Which of the following tables represent a discrete probability distribution? Give reasons for your answer.

(a)

x	0	1	2	3
$P(X = x)$	0.1	0.3	0.5	0.1

(b)

x	2	4	5	6	-7
$P(X = x)$	0.1	-0.2	0.4	0.5	0.2

(a) Yes, this is a discrete probability distribution.

- All probabilities add to 1
- There are no negative probabilities

(b) No this is not a discrete probability distribution since $P(X = 4) = -0.2$ – a negative probability.

8.2 The probability function of a discrete random variable X is given by:

$$P(X = x) = \begin{cases} kx^2 & x = 1, 2, 3, 4, 5, 6 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the value of k and hence draw up a table to represent the probability distribution of X.

(b) Determine:

(i) $P(X = \text{odd})$

(ii) $P(X < 4)$

(iii) $P(X > 5 \mid X \geq 2)$

(c) Calculate the value of $E(X)$

(a) $P(X = 1) = 1k$ $P(X = 3) = 9k$ $P(X = 5) = 25k$

$P(X = 2) = 4k$ $P(X = 4) = 16k$ $P(X = 6) = 36k$

All probabilities must add to 1

$\therefore\ k + 4k + 9k + 16k + 25k + 36k = 1$

$$91k = 1$$

$$k = \frac{1}{91}$$

Probability Distribution

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{91}$	$\frac{4}{91}$	$\frac{9}{91}$	$\frac{16}{91}$	$\frac{25}{91}$	$\frac{36}{91}$

(b) (i) $P(X = \text{odd})$

$= P(X = 1) + P(X = 3) + P(X = 5)$

$= \frac{1}{91} + \frac{9}{91} + \frac{25}{91}$

$= \frac{35}{91}$

$= \frac{5}{13}$

(ii) $P(X < 4)$

$= P(X = 1) + P(X = 2) + P(X = 3)$

$= \frac{1}{91} + \frac{4}{91} + \frac{9}{91}$

$= \frac{14}{91}$

$= \frac{2}{13}$

(iii) $P(X > 5 \mid X \geq 2)$

$= \dfrac{P(X > 5)}{P(X \geq 2)}$

$= \dfrac{\frac{36}{91}}{\frac{90}{91}}$

$= \frac{36}{90}$

$= \frac{2}{5}$

(c) $E(X) = \left(1 \times \frac{1}{91}\right) + \left(2 \times \frac{4}{91}\right) + \left(3 \times \frac{9}{91}\right) + \left(4 \times \frac{16}{91}\right) + \left(5 \times \frac{25}{91}\right) + \left(6 \times \frac{36}{91}\right)$

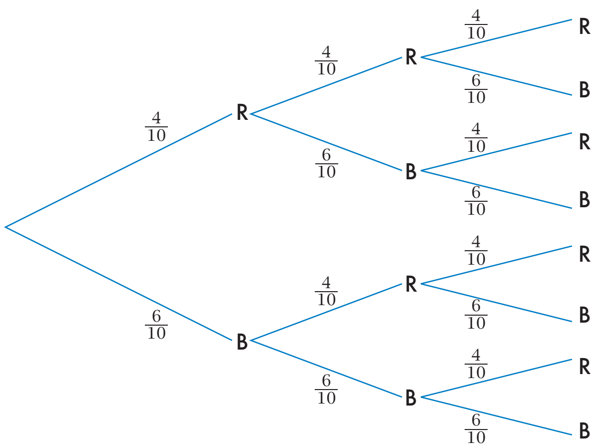
$= \frac{63}{13}$

8.3 Three balls are drawn from a bag containing four red balls and six black balls. If the ball is replaced after each selection find:

(a) the probability distribution for the random variable X, the number of red balls drawn.

(b) the probability that 2 red balls are selected given that at least one ball was red.

(a) Draw a tree diagram to find all possible outcomes and their probabilities.



Outcomes	Probabilities
RRR	$\frac{4}{10}\times\frac{4}{10}\times\frac{4}{10}=\frac{64}{1000}$
RRB	$\frac{4}{10}\times\frac{4}{10}\times\frac{6}{10}=\frac{96}{1000}$
RBR	$\frac{4}{10}\times\frac{6}{10}\times\frac{4}{10}=\frac{96}{1000}$
RBB	$\frac{4}{10}\times\frac{6}{10}\times\frac{6}{10}=\frac{144}{1000}$
BRR	$\frac{6}{10}\times\frac{4}{10}\times\frac{4}{10}=\frac{96}{1000}$
BRB	$\frac{6}{10}\times\frac{4}{10}\times\frac{6}{10}=\frac{144}{1000}$
BBR	$\frac{6}{10}\times\frac{6}{10}\times\frac{4}{10}=\frac{144}{1000}$
BBB	$\frac{6}{10}\times\frac{6}{10}\times\frac{6}{10}=\frac{216}{1000}$

The probabilities are shown in the table below

x	0	1	2	3
$P(X = x)$	$\frac{216}{1000}$	$\frac{432}{1000}$	$\frac{288}{1000}$	$\frac{64}{1000}$

(b) $P(X = 2 \mid X \geq 1)$

$$= \frac{P(X = 2)}{P(X \geq 1)}$$

$$= \frac{288}{784}$$

$$= \frac{18}{49}$$

8.4 A spinner numbered 1 to 5 has the following probability distribution

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.1	0.3	0.3

The spinner is spun twice. Determine the probability of obtaining

(a) the same number twice

(b) a 1 and a 4 in any order

(c) a total of 6 when the two numbers are added together

(d) a 3 on the first spin given the total obtained from both spins is 6

(a) $P(\text{same number}) = P(1\text{ and }1) + P(2\text{ and }2) + P(3\text{ and }3) + P(4\text{ and }4) + P(5\text{ and }5)$

$$= (0.1\times 0.1) + (0.2 \times 0.2) + (0.1\times 0.1) + (0.3\times 0.3) + (0.3\times 0.3)$$

$$= 0.24$$

(b) $P(1\text{ and }4\text{ in any order}) = P(1\text{ and }4) + P(4\text{ and }1)$

$$= (0.1\times 0.3) + (0.3\times 0.1)$$

$$= 0.06$$

(c) $P(\text{a total of }6) = P(1\text{ and }5) + P(2\text{ and }4) + P(3\text{ and }3) + P(4\text{ and }2) + P(5\text{ and }1)$

$$= (0.1\times 0.3) + (0.2\times 0.3) + (0.1\times 0.1) + (0.3\times 0.2) + (0.3\times 0.1)$$

$$= 0.19$$

(d) $P(3 \text{ on first} \mid \text{total is } 6)$

$$= \frac{P(3 \text{ and } 3)}{0.19}$$

$$= \frac{0.01}{0.19}$$

$$= \frac{1}{19}$$

8.5 A coin is tossed 10 times. Determine the probability of obtaining

(a) exactly 4 heads

(b) at least 8 heads

This is a binomial probability distribution with $P = \frac{1}{2}$, $n = 10$

$$\therefore X \sim B\left(10, \frac{1}{2}\right)$$

(a) $P(X = 4) = \binom{10}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6$

≈ 0.2051

(b) $P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10)$

$$= \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + \binom{10}{10}\left(\frac{1}{2}\right)^{10}$$

≈ 0.0547

8.6 The probability a certain drug will cure a disease is 0.7.
7 patients who have the disease are treated with this drug. Determine the probability that:

(a) exactly two will be cured

(b) at least 4 will be cured

(c) all 7 will be cured given that at least 4 were cured

This is a binomial probability distribution with $X \sim B(7, 0.7)$

(a) $P(X = 2) = \binom{7}{2}(0.7)^2(0.3)^5$

≈ 0.025

(b) $P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$

$$= \binom{7}{4}(0.7)^4(0.3)^3 + \binom{7}{5}(0.7)^5(0.3)^2 + \binom{7}{6}(0.7)^6(0.3) + \binom{7}{7}(0.7)^7$$

≈ 0.874

(c) $P(X = 7 \mid X \geq 4)$

$$= \frac{P(X = 7)}{P(X \geq 4)}$$ *answer from part (b)*

≈ 0.0942

8.7 A multiple choice test consists of 10 questions with 5 choices of answers per question. A student randomly chooses an answer by guessing.

(a) Write down the probability distribution function of X, the number of correct answers.

(b) Find the expected value (mean) and standard deviation of X.

(a) $X \sim B\left(10, \frac{1}{5}\right)$

$P(X = x) = \binom{10}{x}\left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{10-x}$, $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

(b) Expected Value (Mean)

$\mu = np$

$= 10 \times \frac{1}{5}$

$= 2$

Standard Deviation

$\sigma = \sqrt{np(1-p)}$

$= \sqrt{10\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)}$

≈ 1.265

8.8 How many throws of a die are required so that the probability of obtaining at least one even number exceeds 0.95?

The random variable X = the number of times an even number occurs in the throwing of a die.

Known: $P(X \geq 1) = 0.95$

$P = \frac{1}{2}$ ← *the probability of obtaining an even number on throwing a die once.*

Required: $n = ?$

$$P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=n) > 0.95$$

$$1 - P(X=0) > 0.95$$

$$P(X=0) < 0.05$$

$$\therefore \binom{n}{0}\left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^n < 0.05$$

$$n > 4.32$$

$$n \geq 5$$

$\therefore$ At least 5 throws are required

8.9 A discrete random variable X has the following probability distribution.

x	1	2	3
$P(X = x)$	0.4	0.1	0.5

Calculate:

(a) $E(X)$

(b) Variance (X)

If Y = 2x + 3, determine:

(c) $E(Y)$

(d) Variance (Y)

(a) $E(X) = 1 \times 0.4 + 2 \times 0.1 + 3 \times 0.5$
$= 2.1$

(b) Variance $(X) = (1 - 2.1)^2(0.4) + (2 - 2.1)^2(0.1) + (3 - 2.1)^2(0.5)$
$= 0.89$

(c) $E(Y) = 2(2.1) + 3$
$= 7.2$

$Y = 2x + 3$
change of scale and origin

(d) Variance $(Y) = (0.89) \times 2^2$
$= 3.56$

change of scale only

PROBLEMS TO SOLVE

Chapter 8: Discrete Random Variables and Binomial Distribution

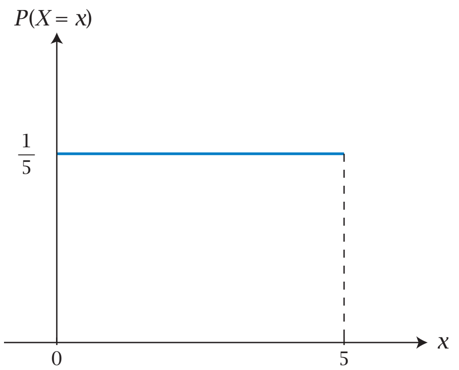
1. Determine if the following are examples of Bernoulli trials.
   (a) Flipping a coin and noting if it lands on heads or tails.
   (b) Selecting a ball from a bag containing red, green and yellow balls.
   (c) Rolling a die and noting whether the uppermost face is a 4.
   (d) Picking a card from a standard pack and noting whether the card is red or black.

2. Calculate the probability of success for each Bernoulli distribution below
   (a) Rolling a 4 on a 10 sided die.
   (b) Choosing the correct answer on a multiple choice question containing five possible answers, one of which is correct.
   (c) Picking a red marble from a bag containing three red, two green and 1 yellow marble.

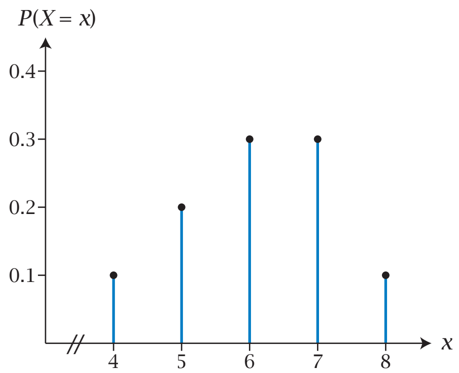
3. Calculate the mean and standard deviation of the Bernoulli distributions in question 2 above.

4. Determine whether the graphs below represent discrete probability distribution. Give reasons for your answer.

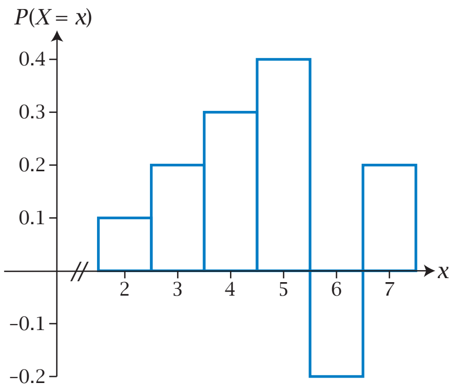
(a)



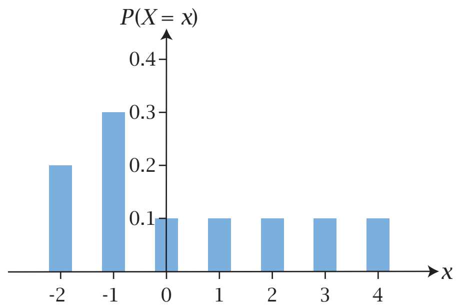
(b)



(c)



(d)



5. Justify whether each of the following represents a discrete probability distribution.

(a) $f(x) = \begin{cases} \dfrac{x-1}{10}, & x = 0,\ 1,\ 2,\ 3,\ 4,\ 5 \\ 0 & , \text{ elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{x^2}{55}, & x = 0,\ 1,\ 2,\ 3,\ 4,\ 5 \\ 0 & , \text{ elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x+3}{x-3}, & x = 0,\ 1,\ 2,\ 3,\ 4, \\ 0 & , \text{ elsewhere} \end{cases}$

6. Each of the following tables represent a probability distribution of a discrete random variable X. Find the value(s) of k.

(a)

x	-3	-2	1	3
$P(X=x)$	k	$2k$	$3k$	$\frac{1}{6}$

(b)

x	1	2	3	4	5	6
$P(X=x)$	0.1	0.2	0.1	k	0.1	0.2

(c)

x	1	2	3	4	5	6
$P(X=x)$	k	k	$2k$	$1-8k$	k	$3k$

7. Complete the following tables showing the cumulative probabilities for the random variables X in Questions 6(a) and 6(b).

(a)

x	-3	-2	1	3
$P(X \leq x)$				

(b)

x	1	2	3	4	5	6
$P(X \leq x)$						

8. Calculate the mean and standard deviation for each of the discrete random variables X in Question 7.

9. The probability distribution of a discrete random variable X is shown in the table below.

x	0	1	2	3	4	5	6
$P(X=x)$	0.42	0.22	0.14	0.12	0.07	0.02	0.01

Determine which is greater – the mean of X or the median of X.

10. Given $X \sim \text{Bin}(20,\ 0.6)$ determine:

(a) $P(X = 12)$

(b) $P(X \geq 15)$

(c) $P(X = 12 \mid X < 16)$

(d) $P(9 \leq X \leq 11)$

11. A discrete random variable X has a binomial distribution with $n = 6$ and $p = 0.2$.

Determine:

(a) the mean 'μ' of X

(b) the standard deviation 'σ' of X

12. A binomial distribution has a mean of 3.6 and a standard deviation of 1.2. Find the number of trials 'n' and the probability of success 'p'.

13. The probability function for a discrete random variable X is given by:

$$P(X = x) = \begin{cases} k(4-x), & x = 0,\ 1,\ 2,\ 3 \\ 0 & , \text{elsewhere} \end{cases}$$

Determine:

(a) k

(b) $P(X = \text{even})$

(c) $P(X = \text{odd})$

14. The probability of each outcome 1, 2, 3, 4 and 5 on a spinner is shown in the probability distribution below:

n	1	2	3	4	5
$P(N = n)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{3}{10}$

(a) Determine:

i. $P(N < 3)$

ii. $P(N = \text{odd})$

(b) Suppose a spinner is spun three successive times and the score noted. Determine the probability the total score is equal to 5.

15. Experience has shown that 30% of trains are late departing the city station. Determine the probability that of the next ten train departures:

(a) at most three will be late departing.

(b) at least six will be late departing.

16. A farmer claims that his bananas are ready and ripe to eat 90% of the time. Find the probability that of the next 18 bananas:

(a) at least 14 are ready to eat.

(b) exactly 18 are ripe and ready to eat.

(c) between 12 and 15 inclusive are ready to eat.

17. A mine site is beginning a safety in the workplace programme and investigates the number of workplace injuries and time lost over a 40 week period. The results are shown in the table below:

Number of workplace injuries	Number of weeks
0	4
1	16
2	12
3	6
4	2

(a) Construct a probability distribution for the number of weekly workplace injuries.

(b) Calculate the expected value and standard deviation of weekly workplace injuries.

18. It has been found that 25% of all people who take medication 'v' get sleepy within 5 minutes. Find the probability that from 15 people:

(a) at most three are sleepy within 5 minutes.

(b) at least ten will be sleepy within 5 minutes.

(c) at most six will be sleepy if at least three are sleepy within 5 minutes.

(d) seven or eight will be sleepy within 5 minutes.

19. Suppose you play a game with a biased coin. The game is played by tossing the coin once. The probability of throwing a head is $\frac{2}{3}$. If you throw a head, you lose \$10. If you throw a tail, you win \$15. Will you make a profit if you play this game in the long run?

20. The expected value of a distribution is 2.3. The table below shows the discrete probability distribution with missing values of 'a' and 'b'.

x	1	2	3	4
$P(X = x)$	0.4	0.2	a	b

Determine the values of 'a' and 'b'.

21. A couple make a decision to continue to have children until they have both a boy and a girl or until they have four children in total. Boys and girls are equally likely.

(a) If the number of children in the family is represented by the random variable X, construct a probability distribution.

(b) Calculate the expected size of the family.

22. A card game consists of placing five cards face down on a table. A player picks up two cards without replacing them. The player is then paid the sum of the amounts on the cards. If two of the cards are \$10 and three of the cards are marked \$5, how much should the player pay to play the game if it is to be fair?

23. A bag contains 2 black balls and 5 red balls. Three balls are randomly selected without replacement. Let x represent the number of black balls selected.

(a) Construct a probability distribution for x.

(b) What is the most likely number of black balls drawn from the bag?

24. The probability function for n students lining up at the school canteen is given by $a_n = k(0.2)^n$ where k is a constant and $n \geq 0$. There is a probability of 15% that there is only one person lined up at the canteen at any time.

(a) Find the constant k.

(b) Complete the following table.

a_0	a_1	a_2	a_3

(c) Determine the probability of more than 3 students waiting in line at the canteen.

25. An electrical company produces switches which are packed in boxes each containing 20 switches. From past experience it is known that 4% of switches produced are defective.

(a) Find the probability that a randomly selected box has at least one defective switch.

(b) Find the probability that 5 randomly selected boxes each contain:

i. at least one defective switch.

ii. exactly two defective switches.

(c) If Paul the electrician purchases 50 boxes of switches, find the expected number of boxes that contain exactly two defective switches.

26. 'D' Realty determined that the number of vacant houses during the winter months in Dunsborough was 40%. The realty company decided to advertise and organised a pamphlet drop. If the pamphlet was delivered to 12 random houses in Dunsborough in winter determine the probability that:

(a) exactly two houses were vacant.

(b) at most 5 houses were vacant.

(c) at least 7 houses were vacant.

The 'D' Realty company decides to increase the number of houses included in the pamphlet drop.

(d) What is the smallest number of houses that must be selected so that there is an 80% chance that a least 8 houses are not vacant?

27. If a discrete random variable has an expected value $E(X) = 12$ and a standard deviation of 2 determine:

(a) $E(X + 3)$

(b) Standard deviation of $(X + 3)$

(c) $E(2X - 4)$

(d) Standard deviation of $(2X - 4)$

28. Given the following discrete random variable X with probability distribution

x	1	2	3	4
$P(X = x)$	0.3	0.2	0.4	0.1

Calculate:

(a) $E(X)$

(b) Variance (X)

If $Y = 4X$ determine:

(c) $E(Y)$

(d) Variance (Y)

If $Z = 3X + 5$ determine:

(e) $E(Z)$

(f) Variance (Z)

9 CONTINUOUS RANDOM VARIABLES AND NORMAL DISTRIBUTION

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable
- ☐ examine the concepts of a probability density function, cumulative distribution function and those given by integrals; examine simple types
- ☐ identify the expected value, variance and standard deviation of a continuous random variable and evaluate them using technology
- ☐ examine the effects of linear changes of scale and origin on the mean and the standard deviation
- ☐ identify contexts for modelling by normal random variables
- ☐ identify features of the graph of the pdf of the normal distribution with mean μ and standard deviation σ and the use of the standard normal distribution
- ☐ calculate probabilities and quantiles associated with a given normal distribution using technology and use these to solve practical problems.

FORMULAE AND DEFINITIONS

A **continuous random variable** is when the variable is not restricted to a specific value but can take on any value over a specified domain.

Continuous random variables represent quantities that can be **measured** and include such variables as distance, height and weight.

If X is a continuous random variable then the probability can be found by finding the area under a curve known as a probability density function (pdf).

i.e. $P(a \leq X \leq b) = \int_a^b f(x)dx$

where $a \leq b$

The following **conditions must also hold**:

(a) The probability density function (pdf) must be greater than or equal to zero for all values of x.
i.e. $f(x) \geq 0$, for all x

(b) The total area under the curve must be 1

i.e. $\int_{-\infty}^{\infty} f(x)dx = 1$

Note: For a continuous random variable

- $P(X = x) = 0$ for all x values
- $P(a \leq X \leq b) = P(a < X < b)$

The cumulative distribution function (cdf) defined as $f(x)$ for a continuous random variable X over the interval $a \le X \le b$ is:

$$F(x) = \int_a^x f(x)\,dx$$

This allows probabilities to be determined *without* integration but by *substitution*.

i.e. $P(m \le x \le n) = F(n) - F(m)$

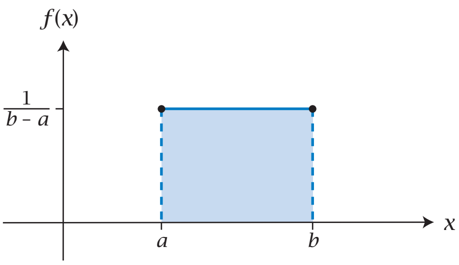
Uniform (Rectangular) Distribution

A continuous random variable is **uniformly distributed** when the random variable X is **equally likely** to take any value in the interval $[a,\ b]$.

The probability density function $f(x)$ is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le X \le b \\ 0 & , \text{elsewhere} \end{cases}$$

The graph of a uniformly distributed continuous random variable is shown below.



Expected Value $E(X) = \dfrac{a+b}{2}$

The Normal Distribution

The normal distribution represents a continuous random variable with the following properties:

- half of the population is above the mean and half is below the mean.
- the curve is symmetrical.
- just over $\frac{2}{3}$ (68%) of the population is within one standard deviation of the mean, 95% within two standard deviations and 99.7% within three standard deviations of the mean.
- the area under the curve is 1.
- when standardised, the distribution has a mean of 0 and standard deviation of 1.

The probability density function for a normal distribution with a mean of μ and a standard deviation of σ is:

$X \sim N(\mu,\ \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

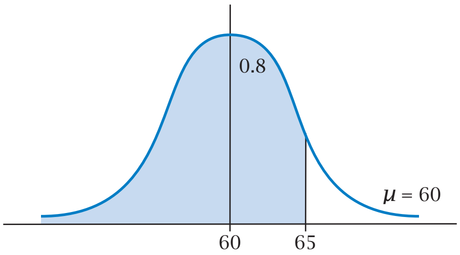
Standard Scores (z scores)

Scores are standardised when they have a mean of 0 and a standard deviation of 1. The value of z obtained determines the number of standard deviations above or below the mean a particular score is.

If X has a mean of μ and a standard deviation σ then the z score $= \dfrac{x-\mu}{\sigma}$.

Note: all values can be obtained from tables or a calculator.

Quantiles and Percentiles



Area under the curve below 65 is 0.8. This is known as the 80th percentile and 65 is the 0.8 quantile.

Well known quantiles are:

- 0.25 quantile known as the lower quartile.
- 0.75 quantile known as the upper quartile.

Expected Value

The expected value (mean) of a continuous random variable X within a probability density function $f(x)$ over the interval $[a, b]$ is:

$$E(X) = \int_a^b \left(x \cdot f(x)\right) dx$$

Standard Deviation/Variance

The variance of a continuous random variable X with a probability density function $f(x)$ over the interval $[a, b]$ is:

$$Var(x) = \int_a^b \left[f(x) \cdot (x - \mu)^2\right] dx$$

$$\text{Standard Deviation} = \sqrt{Var(x)}$$

Change of Scale/Origin

If a random variable X with:

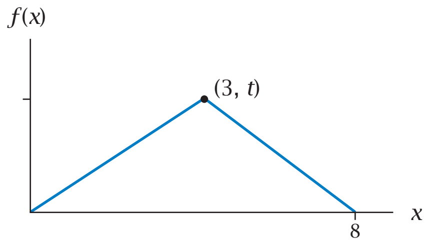
- Expected value (mean) = μ
- Standard deviation = σ
- Variance = σ^2

then the random variable $aX + b$ will have:

- Expected value = $a\mu + b$
- Standard deviation = $|a|\sigma$
- Variance = $a^2\sigma^2$

Worked Examples

9.1 For the following graph which represents a continuous probability distribution, find t.



For this to be a continuous probability distribution, the area under the curve must equal 1

i.e. Area $\triangle = 1$

$$\frac{1}{2}\text{base} \times p.h = 1$$

$$\frac{1}{2}(8) \times t = 1$$

$$t = \frac{1}{4}$$

9.2 A continuous random variable has a probability density function defined by:

$$f(x) = \begin{cases} \frac{x^2}{9}, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Determine:

(a) $P(1 \leq X \leq 3)$

(b) the value of k such that $P(X \geq k) = 0.875$

(a) Use integration to find the area under the curve from $1 \leq x \leq 3$

$$P(1 \leq X \leq 3) = \int_1^3 \frac{x^2}{9}\,dx$$

$$= \left[\frac{x^3}{27}\right]_1^3$$

$$= \frac{3^3}{27} - \frac{1^3}{27}$$

$$= \frac{26}{27}$$

(b) $P(X \geq k) = 0.875$

$\therefore P(k \leq X \leq 3) = 0.875$

Use integration

$$\int_k^3 \frac{x^2}{9}\,dx = 0.875$$

$$\left[\frac{x^3}{27}\right]_k^3 = 0.875$$

$$\therefore \frac{3^3}{27} - \frac{k^3}{27} = 0.875$$

$$\frac{k^3}{27} = 0.125$$

$$k^3 = 3.375$$

$$k = 1.5$$

9.3 A continuous random variable X has the probability density function given as:

$$f(x) = \begin{cases} \frac{1}{4}, & 3 \leq x \leq 7 \\ 0, & \text{otherwise} \end{cases}$$

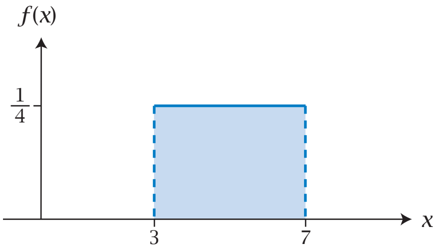
Calculate:

(a) $P(X = 4)$

(b) $P(X \geq 6)$

(c) $P(3.2 \leq X \leq 5.1)$

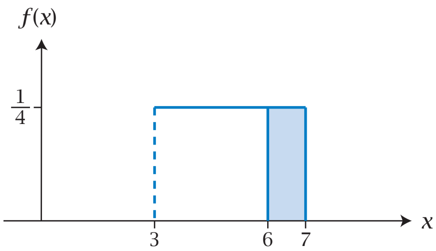
This is a uniform distribution. The graph of the pdf is shown below:



The area under the curve is 1.

(a) $P(X = 4) = 0$

(b) $P(X \geq 6)$

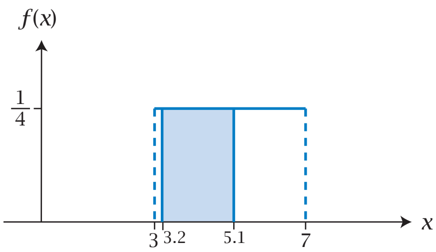


$$\text{Area rectangle} = 1 \times \frac{1}{4}$$

$$= \frac{1}{4}$$

$$\therefore P(X \geq 6) = \frac{1}{4}$$

(c) $P(3.2 \leq X \leq 5.1)$



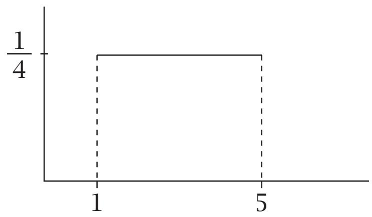
$$\text{Area} = 1.9 \times \frac{1}{4}$$

$$= \frac{19}{40}$$

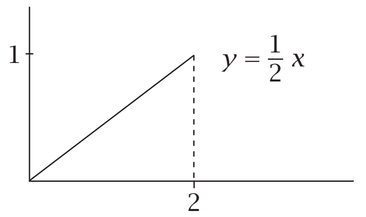
$$\therefore P(3.2 \leq X \leq 5.1) = \frac{19}{40}$$

9.4 Express the following continuous random variables as cumulative distribution functions (cdf).

(a)



(b)



Find $P(0.5 \le x \le 1)$

(a) pdf is:

$$f(x) = \begin{cases} 0.25 & , \quad 1 \le x \le 5 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

for $1 \le k \le 5$

$$\begin{aligned} P(X \le k) &= \int_1^k 0.25\,dx \\ &= \left[0.25x\right]_1^k \\ &= 0.25k - 0.25 \\ &= 0.25(k-1) \end{aligned}$$

Hence cdf is:

$$P(X \le x) = \begin{cases} 0 & , \quad x < 1 \\ 0.25(x-1) & , \quad 1 \le x \le 5 \\ 1 & , \quad x > 5 \end{cases}$$

(b) cdf is

$$\begin{aligned} P(X \le k) &= \int_0^k \left(\frac{1}{2}x\right) dx \\ &= \left[\frac{x^2}{4}\right]_0^k \\ &= \frac{k^2}{4} \end{aligned}$$

$$\therefore P(X \le x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{x^2}{4} & , \quad 0 \le x \le 2 \\ 1 & , \quad x > 2 \end{cases}$$

Hence $P(0.5 \le x \le 1)$

$$\begin{aligned} &= P(x \le 1) - P(x < 0.5) \\ &= \frac{(1)^2}{4} - \frac{(0.5)^2}{4} \\ &= \frac{1}{4} - \frac{1}{16} \\ &= \frac{3}{16} \end{aligned}$$

9.5 Trains arrive at the local station every 20 minutes. If my arrival at the station every day is a uniform random variable and T represents the waiting time before the train arrives determine:

(a) the pdf of T.

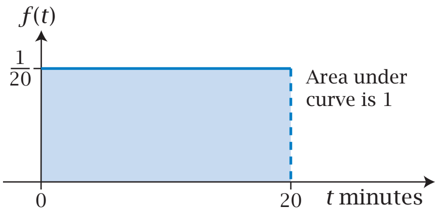
(b) the probability of waiting less than 5 minutes.

(c) the probability of waiting less than 15 minutes given I have waited more than 10 minutes.

(a) The pdf of T is presented as:

$$f(t) = \begin{cases} \frac{1}{20}, & 0 \le t \le 20 \\ 0, & \text{otherwise} \end{cases}$$

The graph of the pdf is drawn below:



(b) $P(T < 5) = 5 \times \frac{1}{20}$

$= 0.25$

(c) $P(T < 15 \mid T > 10) = \dfrac{P(10 < T < 15)}{P(T > 10)}$

$= \dfrac{0.25}{0.5}$

$= 0.5$

9.6 Let X be a random variable which is normally distributed with a mean of 10 and a standard deviation of 2. Calculate the following:

(a) $P(X \le 11)$

(b) $P(8 \le X \le 12)$

(c) $P(X \ge 8.5)$

(d) k, given $P(X \ge k) = 0.9192$

(e) the 0.25 quantile

(f) the 75th percentile

Use calculator to obtain solutions

$x \sim N(10,\ 2^2)$

(a) $P(X \le 11) \approx 0.6915$

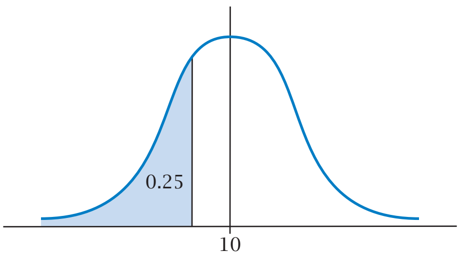
(b) $P(8 \le X \le 12) \approx 0.6827$

(c) $P(X \ge 8.5) \approx 0.7734$

(d) k given $P(X \ge k) \approx 0.9192$

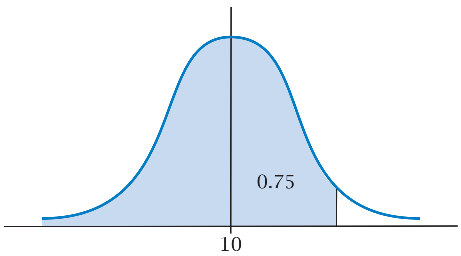
$k \approx 7.2006$

(e)



Using calculator: 0.25 quantile is 8.6510

(f)



Using calculator: 75th percentile is 11.3490

9.7 John has joined the local 'Little Athletics Club' as a specialist discus thrower. His throws are normally distributed with a mean of 26.0 metres and a standard deviation of 2 metres.

(a) What is the probability that John throws between 23 and 30 metres?

(b) Determine the probability he throws no less than 25 metres.

(c) If the standard deviation remains unchanged and the mean needs to be adjusted so that 25% of the throws are above 25 metres calculate the new mean.

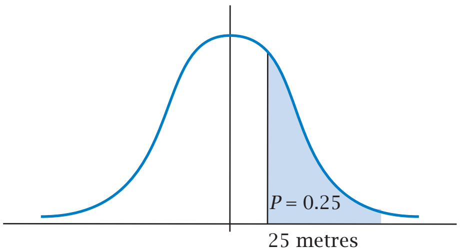
Use calculator to determine the solutions

$X \sim N(26,\ 2^2)$

(a) $P(23 \leq X \leq 30) \approx 0.9104$

(b) $P(X \geq 25) \approx 0.6915$

(c)



$\therefore$ area under curve is 0.25

$\sigma = 2$

$\mu = ?$

(i) Calculate z value when $\mu = 0,\ \sigma = 1,\ p = 0.25$

$$z \approx 0.6744$$

(ii) Use standard scores formula.

$$z = \frac{x - \mu}{\sigma}$$

$$0.6744 = \frac{25 - \mu}{2}$$

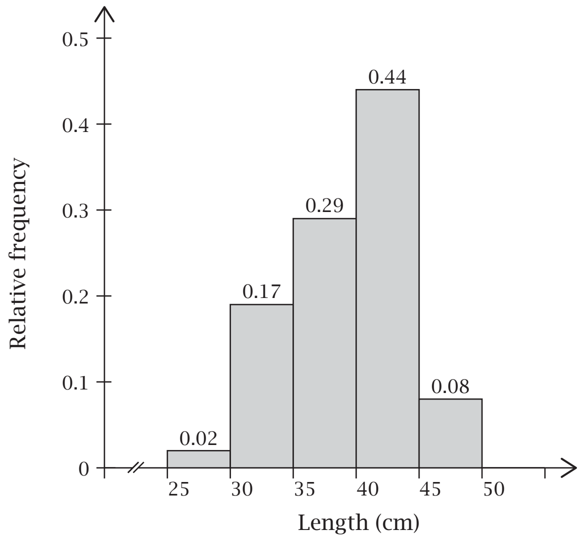
$$\mu = 23.6512$$

The mean is altered to 23.6512 m.

PROBLEMS TO SOLVE

Chapter 9: Continuous Random Variables and Normal Distribution

1. From an experiment, the length of leaves were measured and the results shown in the relative frequency histogram below:



A leaf is randomly selected and measured. Calculate the probability the length (L cm) of the leaf is:

(a) $P(L < 35)$

(b) $P(L > 40)$

(c) $P(35 \leq L < 45)$

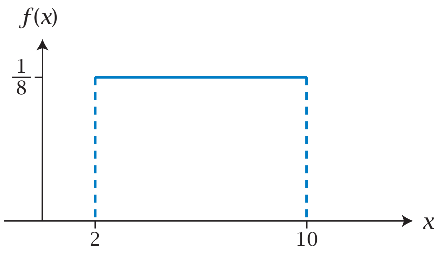
(d) $P(L > 40 \mid L < 45)$

2. Determine whether the following represents continuous probability distributions. Justify your response.

(a) $f(x) = \begin{cases} 0.25, & 1 \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{2x}{25} - \dfrac{2}{5}, & 5 \leq x \leq 10 \\ 0, & \text{elsewhere} \end{cases}$

3. The graph of a uniform random variable is shown below:



Find

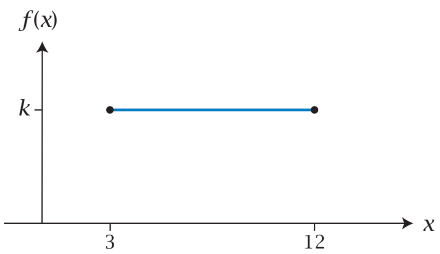
(a) $P(X < 5)$

(b) $P(3 < X < 8)$

(c) $P(X < 7 \mid X > 5)$

(d) $P(X = 5)$

4. A continuous random variable X has a distribution as shown below.



Determine

(a) k

(b) $P(X \geq 4)$

(c) $P(X < 7)$

(d) $P(X \leq 8 \mid x \leq 11)$

(e) the expected value of X.

(f) the standard deviation of X.

(g) the probability density function.

5. Which of the following represent continuous random variables?

Justify your response.

(a) $f(x) = \begin{cases} 0.5(4 - x^2), & -0.25 \leq x \leq 0.25 \\ 0 & , \text{ elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} 2, & 0.5 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

(c)

x	0	1	2	3
$f(x)$	0.1	0.2	0.5	0.2

(d) $f(x) = \begin{cases} -x, & -1 \leq x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

(e) $f(x) = \begin{cases} \frac{1}{2}(x+3)(x-1), & 3 \le x \le 5 \\ 0 & , \text{ elsewhere} \end{cases}$

6. Find the value of k for the following probability density functions:

(a) $f(x) = \begin{cases} k(1-x) & , -1 \le x \le 1 \\ 0 & , \text{ elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} ke^{-0.5x} & , x \ge 0 \\ 0 & , \text{ elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} k(x+2) & , 1 \le x \le 2 \\ 0 & , \text{ elsewhere} \end{cases}$

7. Calculate the mean and standard deviation of the following continuous random variables:

(a) $f(x) = \begin{cases} 3x^2 & , 0 \le x \le 1 \\ 0 & , \text{ elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} 0.4 & , 0 \le x \le 2.5 \\ 0 & , \text{ elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} 0.25e^{-0.25x} & , x \ge 0 \\ 0 & , \text{ elsewhere} \end{cases}$

8. Find the cumulative distribution functions (cdf) for each of the following pdf's:

(a) $f(x) = \begin{cases} 3x^2 & , 0 \le x \le 1 \\ 0 & , \text{ elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} 0.4 & , 0 \le x \le 2.5 \\ 0 & , \text{ elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} 0.25e^{-0.25x} & , x \ge 0 \\ 0 & , \text{ elsewhere} \end{cases}$

9. A continuous random variable X has a mean of 10 and a standard deviation of 5. It is transformed to the random variable Y by each of the following:

(a) $Y = 4X$

(b) $Y = 2X - 5$

(c) $Y = 3X + 1$

Find the expected value and standard deviation of Y in each situation above.

10. A uniform continuous random variable X is defined over the interval $0 \le x \le 140$. It is transformed to the random variable Y by $Y = 3X + 4$.

Calculate:

(a) $E(X)$

(b) Var (X)

(c) SD(X)

(d) $E(Y)$

(e) Var (Y)

(f) SD (Y)

11. If $X \sim N(100,\ 10^2)$ determine:

(a) $P(X \geq 90)$

(b) $P(85 \leq X \leq 101)$

(c) $P(X \leq 80 \mid X \leq 96)$

12. A random variable X, is normally distributed with mean of 200 and a standard deviation of 35.

Calculate

(a) $P(190 \leq X \leq 210)$

(b) $P(X \geq 220)$

(c) $P(X \leq 205)$

(d) $P(X \geq 210 \mid X \leq 220)$

13. For each of the following find k, given the mean and standard variation deviation of the random variable X which is normally distributed.

(a) $\mu = 15 \quad \sigma = 2$

$P(X \geq k) = 0.927$

(b) $\mu = 105 \quad \sigma = 30$

$P(X \leq k) = 0.246$

(c) $\mu = 20 \quad \sigma = 5$

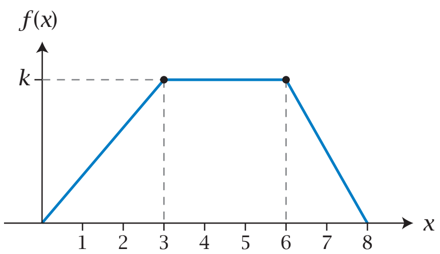
$P(16 \leq X \leq k) = 0.7$

14. If $X \sim N(20,\ 5^2)$ determine:

(a) the 24th percentile

(b) the 0.52 quantile

15. A continuous random variable X is shown in the diagram below:



(a) Find k

(b) Give the probability density function for the above continuous distribution.

(c) Find $P(1 < X < 5)$

16. The travelling time between Perth and Bunbury is uniformly distributed between 90 and 120 minutes.

(a) What is the probability density function for the travelling time?

(b) Calculate the probability the travelling time will be:

i. less than 100 minutes

ii. exactly 110 minutes

iii. between 95 and 104 minutes

iv. more than 92 minutes given that it will take less than 100 minutes.

(c) Calculate the expected travel time and standard deviation.

17. Train arrival times are uniformly distributed between 9.05 am and 9.20 am with the scheduled arrival time being 9.05 am.

(a) What is the expected value of the arrival time?

(b) The train is running late if it arrives more than 3 minutes after the scheduled arrival time. Determine the probability that the train is late.

(c) If Melanie arrives at the train station at 9.13 am what is the probability that she will catch the train?

18. A 750 ml labelled bottle of water is filled uniformly by an electronic filling machine according to the probability density function:

$$P(x = x) = \begin{cases} \frac{1}{10}, & 742 \leq x \leq 752 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine the probability that a bottle will be filled with between 750 and 752 ml of water.

(b) Determine the probability that a bottle will be filled with more than 748 ml of water.

(c) Government regulations state that bottles of water must be filled within 2 ml of the stated quantity. What is the probability that a bottle selected will fail to meet Government regulations?

19. The probability density function of a random variable x is given by:

$$f(x) = \begin{cases} kx(4-x), & 0 \le x \le 4 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) Find the value of k.

The distribution above is the lifetime in years, x, of a switch in a sprinkler system. If all switches are independent of each other and the sprinkler system was fitted with two new switches find:

(b) the probability that neither switch fails in the first year.

(c) the probability that exactly one switch fails within the first two years.

(d) t such that $P(0 < X < t) = 0.4$

20. A certain species of spider has a length which is normally distributed with a mean length of 17 mm and a standard deviation of 2.25 mm.

(a) What is the probability the spider is more than 20 mm?

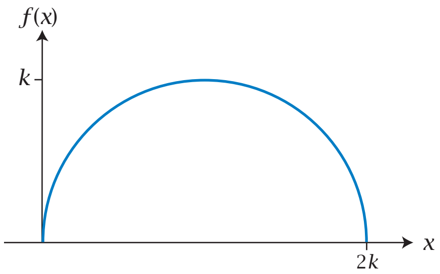
(b) What is the probability that the spider's length is within 3 mm of the mean length?

(c) What is the maximum length of 95% of the spiders?

(d) If 8 spiders are caught, what is the probability that at least one is less than 12 mm?

21. The probability density function shown below is defined as:

$$f(x) = \begin{cases} \sqrt{-x^2 + 2xk}, & 0 \le x \le 2k \\ 0 & , \text{otherwise} \end{cases}$$



(a) Determine k

(b) Calculate $P(0 \le X \le 0.4)$

22. A company manufactures batteries with a lifetime which is normally distributed with a mean of 1500 hours and a standard deviation of 70 hours.

(a) What proportion of batteries have lifetimes exceeding 1480 hours?

(b) The company decides to improve the mean lifetime of the batteries, but retain the same standard deviation. If 98% of all batteries last more than 1480 hours, determine the new mean lifetime.

(c) The company can improve the manufacturing process by reducing the standard deviation. If the mean lifetime remains at 1500 hours, calculate the new standard deviation if 98% of all batteries last more than 1480 hours.

23. Salmon are caught off the coast of Tasmania. The weights are normally distributed with a mean of 18 kg. The lightest 3% are returned to the sea. The heaviest 7% are used in pet food. The remainder weighing between 11.981 kg and 22.723 kg are used for human consumption as fresh fish.

Determine the probability that a randomly selected salmon:

(a) i. will be used for human consumption purposes.

ii. will be used for human consumption given that it is not returned to the sea.

(b) Calculate the standard deviation of the salmon.

24. The waiting time W (in minutes) for a bus at a particular bus stop is a uniformly distributed random variable. A bus departs every 40 minutes from the bus stop.

(a) State the probability density function of W.

(b) Calculate the probability that a passenger who arrives at the bus stop waits no more than 15 minutes for the bus to depart.

(c) What is the probability that less than three passengers from a random selection of 8 will have to wait at least 15 minutes for the bus to depart?

10

RANDOM SAMPLE, PROPORTIONS AND CONFIDENCE INTERVALS

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ examine the concept of a random sample
- ☐ discuss sources of bias in samples
- ☐ use graphical displays of simulated data to discuss variability of random samples from various types of distributions
- ☐ examine the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples and the formulas for the mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$ of the sample proportion $\hat{p}$.
- ☐ examine the approximate normality of the distribution of $\hat{p}$ for large samples
- ☐ simulate repeated random sampling for a variety of p values and range of sample sizes to illustrate the distribution of $\hat{p}$ and the approximate standard normality of $\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}$
- ☐ examine the concept of an interval estimate for a parameter associated with a random variable
- ☐ use the approximate confidence interval $\left(\hat{p} - z\sqrt{\left(\frac{\hat{p}(1-\hat{p})}{n}\right)}, \hat{p} + z\sqrt{\left(\frac{\hat{p}(1-\hat{p})}{n}\right)}\right)$ as an interval estimate for p, z is the appropriate quantile for the standard normal distribution
- ☐ define the approximate margin of error $E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ and understand the trade-off between margin of error and level of confidence
- ☐ use simulation to illustrate variations in confidence intervals between samples and show that most, but not all, confidence intervals contain p.

FORMULAE AND DEFINITIONS

Population

A population in statistics represents **all** measurements or objects being studied

Sample

A sample is a subset or portion of the population.

All items produced by a production process is the **population** while a quality control selection is the **sample**.

Types of Sampling

- Random sampling – each element in the population has an equal chance of occurring. Examples include: random number generator, putting everyone's name in a hat and drawing out several names.
- Systematic sampling – the list of elements are ordered and every k^{th} element is selected.
- Convenience sampling – readily available data is used.
- Cluster sampling – divide the population into groups called clusters. The clusters are randomly selected and each element in the selected cluster is used.
- Stratified sampling – the population is divided into groups called strata and then a sample is randomly selected from each strata.

Bias

A sample is collected such that some elements of the population are less or more likely to be included than others.

Proportions

The *proportion* in the population who have a certain characteristic is the *population proportion* – **p**.

The *proportion* in the *random sample* who have that particular characteristic is the *sample proportion* – $\hat{\mathbf{p}}$.

The number of successes (x) in n trials follows a binomial distribution i.e. $X \sim B(n, p)$.

The proportion of 'successes' in the sample is a random variable given by:

$$\hat{p} = \frac{x}{n} = \frac{x \text{ successes}}{n \text{ trials}}$$

$$\text{Mean} = \text{Expected Value} = E(\hat{p}) = p$$

$$\text{Standard Deviation} = \sqrt{\frac{p(1-p)}{n}}$$

Central Limit Theorem

The Central Limit Theorem states that as n becomes large the distribution of the sample proportions will approach a *normal distribution* with mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.

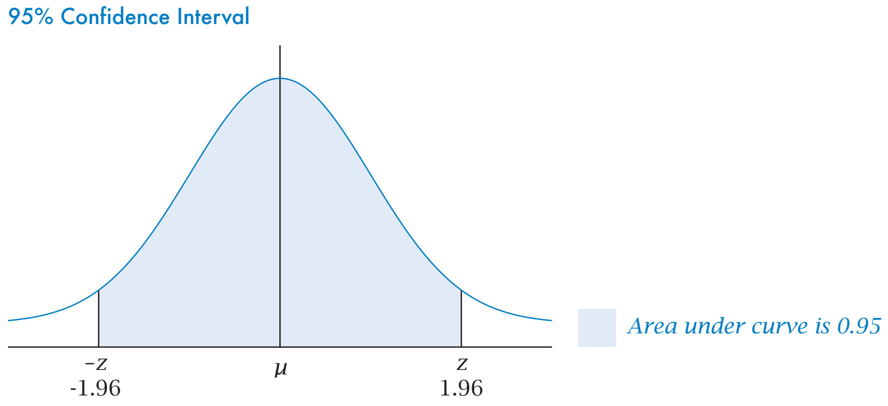
The values of np and $n(1 - p)$ should be greater than or equal to 10 for the sample distribution to approximate a normal distribution.

Confidence Intervals

A **confidence interval** determined from data obtained from a sample estimates an unknown parameter. The confidence interval also indicates how accurate the estimate is. A confidence interval consists of two parts – an interval calculated from the sample data and a confidence level.

A **confidence level** determines the probability that the interval will contain the unknown parameter. The three most common confidence intervals are 90%, 95% and 99%. If a 95% confidence interval is used then there is a 95% confidence that the population proportion is contained within that interval when the values are normally distributed in the population.

The value of z for the confidence interval of 95% is 1.96. This is obtained by using the standard normal distribution as shown in the diagram below:



The central limit theorem states that there is approximately a 95% confidence the sample proportion will be within ±1.96 standard errors of the population proportion when n is large.

Common z scores and associated confidence intervals are

Confidence Interval	90%	95%	99%
z score	1.645	1.960	2.576

Confidence Interval for a Population Proportion 'p'

The interval estimate for the population proportion p, from a sample proportion $\hat{p}$ is:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where z is the standard score for a 90%, 95% or 99% confidence interval.

Margin of Error

Margin of error is the maximum amount p differs from $\hat{p}$.

$$\text{Margin of Error (ME)} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Sample Size

To calculate the sample size use the formula above:

$$\text{i.e. ME} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Note: Always round n up to the next whole number.

Worked Examples

10.1 A school has a population of 1200 students divided into the following year groups.

Year 8: 240
Year 9: 300
Year 10: 360
Year 11: 120
Year 12: 180

A student council of 60 students is to be selected using a stratified sample. How many of each year group will be selected?

Year 8: $\dfrac{240}{1200} \times 60 = 12$

Year 9: $\dfrac{300}{1200} \times 60 = 15$

Year 10: $\dfrac{360}{1200} \times 60 = 18$

Year 11: $\dfrac{120}{1200} \times 60 = 6$

Year 12: $\dfrac{180}{1200} \times 60 = \underline{9}$

$\phantom{\text{Year 12: } \dfrac{180}{1200} \times 60 =}\ 60$

10.2 A die was tossed 240 times. It landed showing a '3' on the uppermost face 15 times.

(a) What is the value of p – the population proportion: obtaining a '3' on the uppermost face?

(b) Calculate $\hat{p}$ the sample proportion: obtaining a '3' on the uppermost face.

(c) Calculate the mean and standard deviation for $\hat{p}$ in a sample of 240.

(a) p = population proportion

In the long term a '3' occurs $\dfrac{1}{6}$ occasions hence $p = \dfrac{1}{6}$.

(b) $\hat{p}$ = sample proportion

In the sample of 240 a '3' occurs 15 times.

Hence: $\hat{p} = \dfrac{15}{240}$

$= 0.0625$

(c) *Mean*

$\hat{p}$ has a mean equal to p

$\therefore$ Mean $= \dfrac{1}{6}$

Standard Deviation

$$\text{SD} = \sqrt{\frac{p(1-p)}{n}}$$

$$= \sqrt{\frac{\frac{1}{6}\left(1-\frac{1}{6}\right)}{240}}$$

$$= 0.0241$$

10.3 A survey of 750 teachers found that 270 were satisfied with the amount of curriculum support. Determine the 90% confidence interval for the population proportion. Interpret your result.

$$\hat{p} = \frac{270}{750} = 0.36$$

Using formula

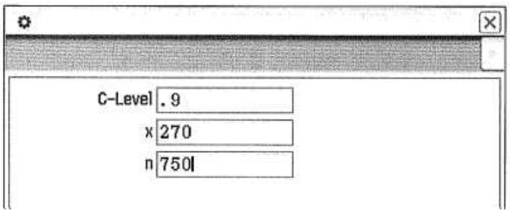
Confidence Interval with $z = 1.645$

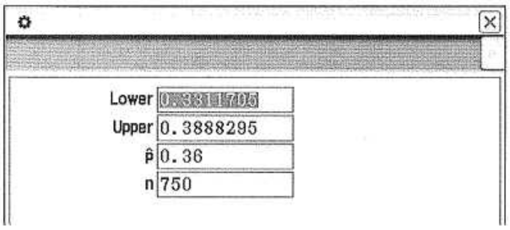
$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.36 - 1.645\sqrt{\frac{0.36(1-0.36)}{750}} \le p \le 0.36 + 1.645\sqrt{\frac{0.36(1-0.36)}{750}}$$

$$0.3312 \le p \le 0.3888$$

Using CAS calculator





We are 90% confident that the population proportion occurs between 0.3312 and 0.3888 i.e. with 90% confidence – between 33.12% and 38.88% of all teachers are satisfied with the amount of curriculum support.

10.4 For a sample of 1000, a 90% confidence level and a sample proportion of 0.3 find the margin of error.

Use the formula:

$$\text{ME} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\text{ME} = 1.645\sqrt{\frac{0.3(1-0.3)}{1000}}$$

$$\text{ME} = 0.0238$$

10.5 A survey is conducted resulting in a sample proportion of 0.4. Given a 95% confidence level and a margin of error of 0.02 calculate the sample size.

Use the formula:

$$\text{ME} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.2 = 1.96\sqrt{\frac{0.4(1-0.4)}{n}}$$

$$n = 2304.96$$

$$n \approx 2305$$

PROBLEMS TO SOLVE

Chapter 10: Random Sample, Proportions and Confidence Intervals

1. A large pilates class for women is held at the local fitness centre each week. A survey is to be completed by a sample of 15 of these women. State the sampling method used if:

   (a) names are placed in a hat and 15 women selected.

   (b) the first 15 women to arrive at the pilates class are selected.

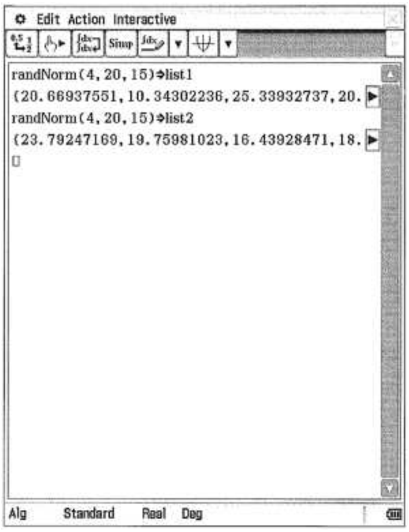
   (c) names of the women are selected at regular intervals from the class list.

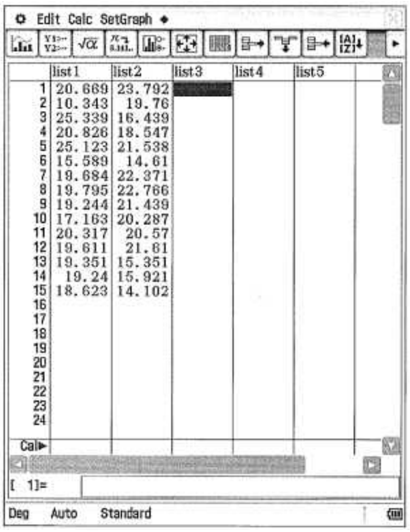
2. Identify if the following sample is biased. Explain.

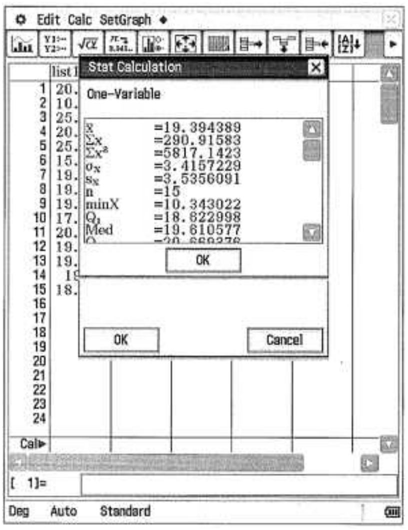
   (a) John is outside a Holden car dealership and conducts a survey on the most popular car. His question to all those who enter: 'What is your favourite make of car?'

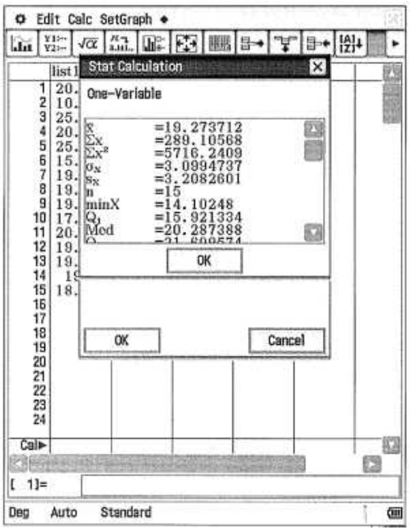
   (b) The manager of an apartment building surveys residents on the lowest floor 'if they think noise is a problem?'

3. Two random samples of 15 items from a normal distribution i.e. $N(20, 4^2)$ were generated with the following results.

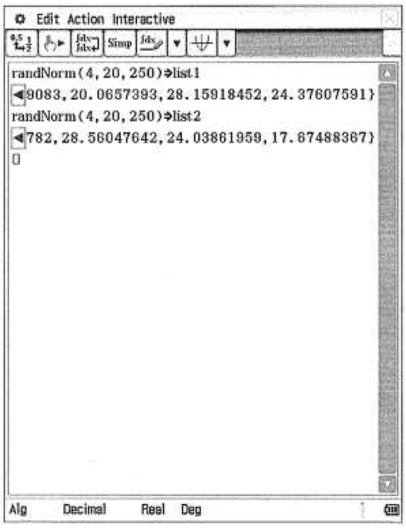


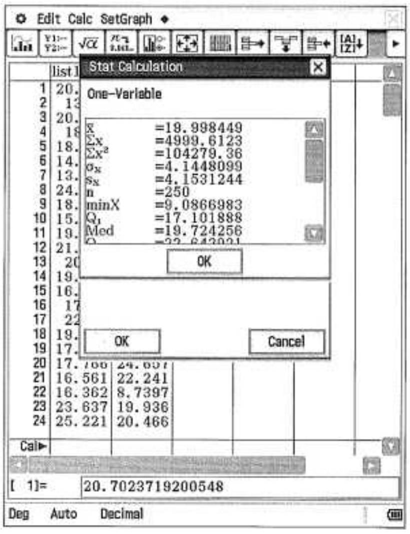


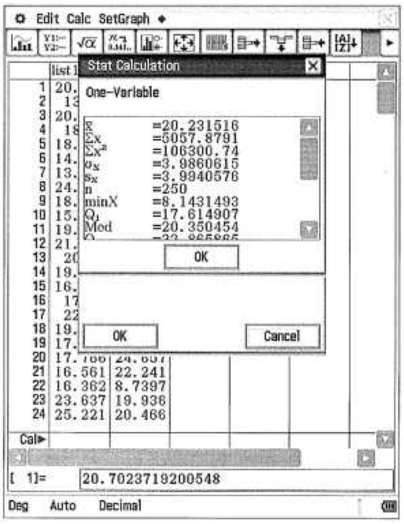




Another two samples of 250 items were generated with the following results:







Comment on the variation in the samples with respect to:

- sample mean and population mean
- sample standard deviation and population standard deviation.

4. In a survey of 1234 people, 543 stated they have home security. Calculate the sample proportion, $\hat{p}$, the proportion of people who have home security.

5. A survey was conducted to estimate the proportion of people who preferred Weetbix over Vitabrix. It was found that 720 out of 900 surveyed preferred Weetbix.
   (a) Calculate the sample proportion $\hat{p}$ of those who preferred Weetbix.
   (b) Estimate the standard deviation of $\hat{p}$.

6. A survey was conducted on the number of tomato seeds that germinate. The packaging describes a 96% success rate for all seeds. From a sample of 950 tomato seeds 874 germinated.
   (a) Determine the population proportion p – the number of tomato seeds germinated.
   (b) Calculate the sample proportion $\hat{p}$, the number of tomato seeds that germinate.
   (c) Calculate the mean and standard deviation of $\hat{p}$.

7. In a survey of 250 households, 75 shop regularly at the local supermarket. Construct a 95% confidence interval for the population proportion of households who use the supermarket.

8. A random sample of 400 people are given the flu injection. Of these, 172 experience pain in their arm. Construct a 95% confidence interval, p, for the population proportion who experience pain.

9. A coin is tossed 1000 times and a tail occurs 570 times. At the 90% confidence level does the result indicate the coin is biased?

10. A survey of 500 residents in a Perth suburb indicated that 320 had a smart phone.

    (a) Calculate the sample proportion, $\hat{p}$, the proportion who owned a smart phone.

    (b) Construct a 95% confidence interval for, p, the population proportion.

    (c) Find the margin of error.

11. A sample proportion, $\hat{p}$ is 0.7, the margin of error, 0.04, and the confidence interval is 99%. Find the sample size.

12. A pharmaceutical company promises that a new tablet will assist migraine sufferers 92% of the time. The tablet is given to 250 migraine sufferers and of these 200 indicated that the tablet relieved symptoms. Can the pharmaceutical company's claim be upheld at the 95% confidence level?

13. If a random sample of 200 items is taken from a population having $p = 0.3$, what is the probability that the proportion of successes, $\hat{p}$, is less than or equal to 0.27.

14. (a) If the population proportion, p, is 0.5, the sample size, 200, determine the proportion of successes in the sample that lie between 0.47 and 0.51.

    (b) What happens if the sample size is reduced to 100?

15. A survey conducted at a major department store found that of 350 people, 280 were women and the rest men.

    (a) What percentage of people surveyed were women?

    (b) Calculate a 90% confidence interval for the population percentage of women.

    (c) Calculate the margin of error in (a) above.

16. A random sample of 520 people found that 130 were left-handed.

    (a) Determine the sample proportion, $\hat{p}$, of those who were left-handed.

    (b) Calculate the standard deviation of $\hat{p}$.

    (c) Another survey is to be taken. With a 95% confidence interval, a sample proportion from (a) above and a margin of error of 0.03, find the sample size.

    A 95% confidence interval for the sample proportion $\hat{p}$ in (a) is: $0.2128 \le \hat{p} \le 0.2872$. Comment on the following samples with respect to the above interval.

    (d) A random sample of 1200 students from which 256 are left-handed.

    (e) A random sample of 360 architects found that 205 are left-handed.

17. 25% of students at university are graduates. A random sample of 175 students are selected. Determine the probability that at least 28% of the sample are graduate students.

18. A quality control manager for a card printing company notes that some cards fail to print i.e. they are blank. A sample of 320 cards finds 20 blank. Calculate:

(a) the sample proportion $\hat{p}$, the number of blank cards.

(b) the standard deviation $\hat{p}$

Another sample of 360 cards was taken.

(c) If the true proportion of blank cards is 0.03, what is the probability that the sample proportion is at most 0.04?

19. 7% of all gift vouchers are never redeemed. One day, 15 gift vouchers were sold. Let X be the number of gift vouchers not redeemed.

(a) Define the probability distribution X

(b) Calculate $P(X = 3)$

(c) Calculate $P(X > 3)$

A random sample of 150 people found that 12 had not redeemed their gift voucher.

(d) Determine a 90% confidence interval for the population proportion p.

(e) A sample of 300 people were surveyed. Determine a range of values for the number of people who have not redeemed their gift voucher.

DIFFERENTIATION AND APPLICATIONS: TRIAL TEST 1 – CALCULATOR ASSUMED

Calculators allowed
Time Allowed: 40 minutes
Total Marks: 40

1. Use the increments formula to find the approximate change in the curved surface area of a cone with perpendicular height 5 cm if the base radius changes from 3 cm to 3.03 cm.

[4]

2. A company selling computer desks determines that the cost function, $C(x)$ and revenue function $R(x)$, in dollars when producing x desks is given by:

$C(x) = 50(x^2 - x) + 2000$

$R(x) = 250x^2 - 4x^3$

(a) Determine the profit function $P(x)$.

(b) Calculate the cost of producing 10 computer desks.

(c) Find $C'(6)$ and explain its meaning.

(d) Find the average profit on each desk when 12 computer desks are produced and sold.

(e) Calculate the number of desks that need to be sold in order for there to be a profit.

(f) Calculate the maximum profit and the number of desks that need to be sold to gain this profit.

[10]

3. Given $m = 5n^2 + 2$, $m = \frac{g}{3}$ *and* $f = \sqrt{g}$

Determine $\frac{df}{dn}$

[3]

4. The cost of manufacturing a computer chip since the year 2000 has changed due to improved technology and is modelled according to $C = 40\sqrt{2 - 0.1t}$, $0 \le t \le 15$ where C is the cost in dollars and t is the number of years since 2000.

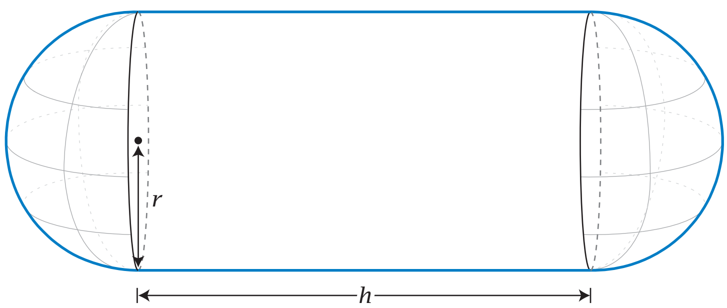
(a) Calculate the average change in the cost of manufacturing the computer chip between the year 2000 and 2010.

(b) When does a decrease in the instantaneous rate of change of the cost of manufacturing the computer chip become greater than \$2.00 per year?

[5]

5. A tank is to be built with a capacity of $200\pi \text{ m}^3$. It is to be built of metal at a cost of \$$p$ per m^2 of cylindrical surface and \$$2p$ per m^2 of hemispherical surface.

The tank is to be built according to the following:



(a) Show that the cost function $C = \frac{400\pi p}{r} + \frac{16\pi r^2 p}{3}$.

(b) Find the radius for the minimum cost of the tank.

(c) Find the cost when $p = 10$.

[10]

6. A body moves from the origin according to its displacement equation, x metres, at time t seconds given as:

$$x(t) = \frac{2t^4 - t^3 - 15t^2}{t + 3} + 4 \quad , \qquad t \geq 0$$

Determine:

(a) The initial position of the body.

(b) The velocity equation $v(t)$.

(c) When the body is at rest.

(d) The acceleration of the body at time $t = 4$ seconds.

(e) The distance the body has travelled from $t = 1$ *to* $t = 4$ seconds.

[8]

DIFFERENTIATION AND APPLICATIONS: TRIAL TEST 1 – CALCULATOR FREE

Calculators NOT allowed
Time Allowed: 15 minutes
Total Marks: 18

1. (a) If $f(x) = (6x^3 - 5)^4$ determine $f'(x)$.

(b) Calculate $g'(-1)$ given $g(x) = (5 - 2x)(2 - 3x)^2$

(c) Determine $\frac{d}{dx}\left(\frac{2x^3 - \pi}{\sqrt{4x + 1}}\right)$ Do not simplify

[5]

2. Given the function $f(x) = \frac{x^2 - 12}{x - 4}$ determine:

(a) $f'(x)$

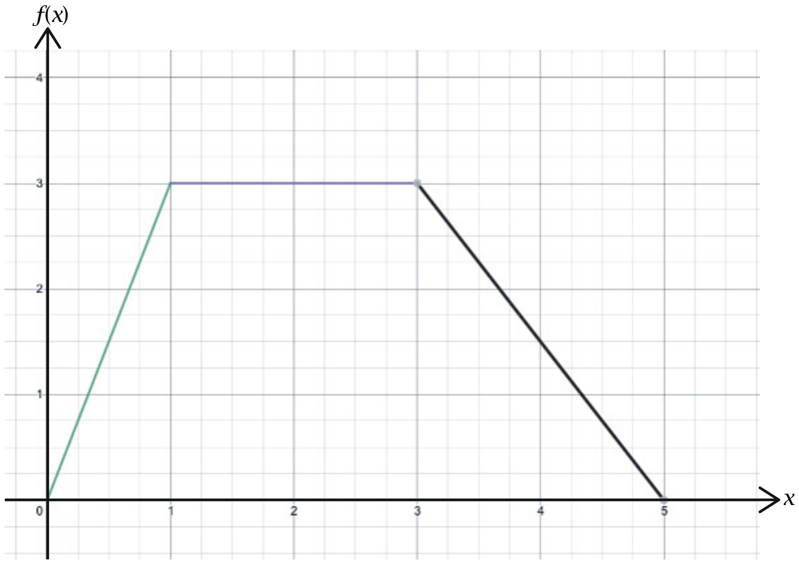
(b) the coordinates of the stationary point(s).

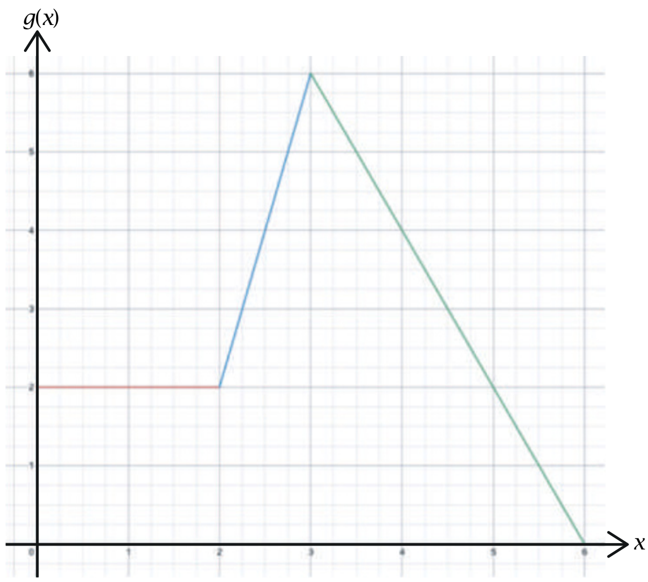
(c) the nature of the stationary point(s) using the second derivative.

(d) where the function is concave up.

[8]

3. The graphs of $f(x)$ and $g(x)$ are shown below.





Evaluate the following:

(a) $g'(5)$

(b) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$ when $x = 2.5$

(c) $\dfrac{d}{dx}[g(f(x))]$ at $x = 4$

[5]

INTEGRATION AND APPLICATIONS: TRIAL TEST 2 – CALCULATOR ASSUMED

Calculators allowed

Time Allowed: 30 minutes

Total Marks: 31

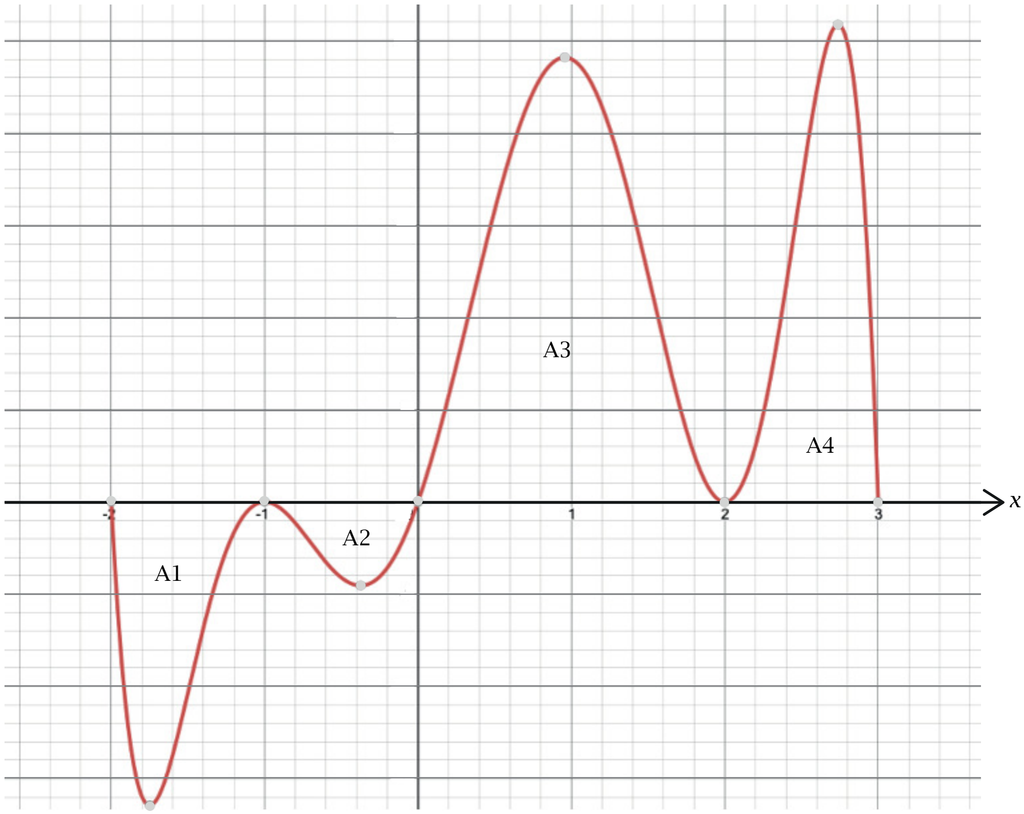
1. Water flows into a rainwater tank which initially contains 30 litres at a rate of $r'(t) = \frac{100}{2t+1}\,dt$ litres/min where t is the time in minutes and $t \geq 0$.Calculate the total amount of water in the tank after 45 minutes.

(a) Calculate the amount of time for the rainwater tank to be $\frac{2}{3}$ full if the tank is completely full after 3 hours.

[5]

2. The areas in square units between the $y = f(x)$ axis and the curve are as follows:

- A1 = 18
- A2 = 10
- A3 = 21
- A4 = 17



Calculate the value of:

(a) $\int_{-2}^{-1} f(x)\,dx$

(b) $\int_{-2}^{3} f(x)\,dx$

(c) $\int_{0}^{-1} 2f(x)\,dx$

(d) $\int_{-2}^{2} (2 + f(x))\,dx$

(e) $\int_{-1}^{0} f'(x)\,dx + \int_{0}^{3} f(x)\,dx$

[7]

3. The graph below illustrates the velocity of a body $v(t)$ m/s over the interval $0 \le t \le 16$ where t is the time in seconds.

(a) Determine the displacement of the body at

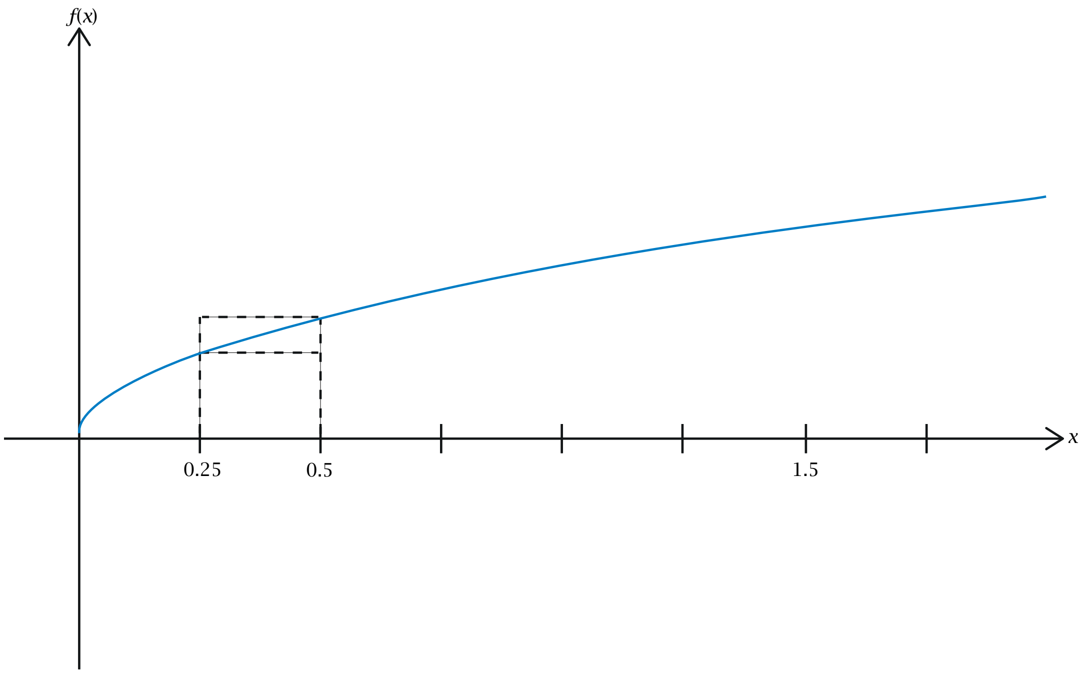
(i) $t = 6$

(ii) $t = 16$

(b) Determine the time(s) the body has a maximum displacement and state this displacement.

[7]

4. The graph of $y = f(x)$ is shown below.



A table of values for the above graph is as follows:

x	0.25	0.5	0.75	1	1.25	1.5
$f(x)$	0.71	1	1.22	1.41	1.58	1.73

Using the table of values and the sum of the rectangles based on $\Sigma_i\, f(x_i)\delta x_i$ where $\delta x = 0.25$ determine over the interval $0.25 \le x \le 1.5$.

(a) An overestimate for the area of the function.

(b) An underestimate for the area of the function.

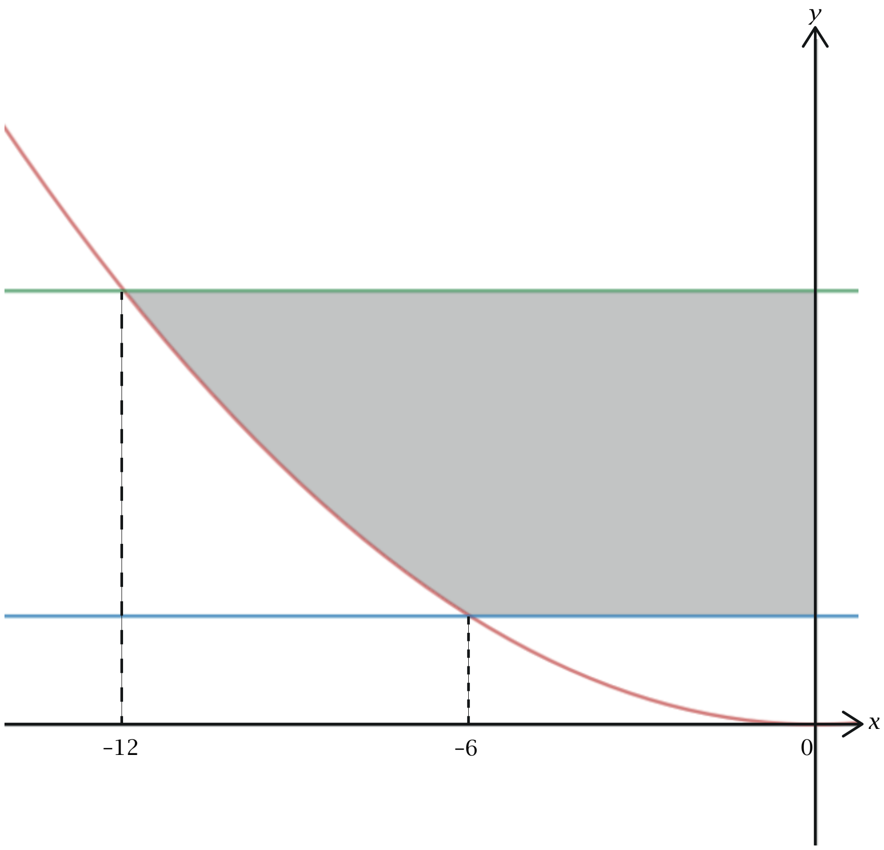
(c) A numerical approximation for the area.

(d) Verify whether the approximation for the area is smaller or larger than the actual area. Give a reason for your answer.

(e) How can the approximation be improved? Explain.

[7]

5. A shaded region shown below is enclosed by the graph of $y = \frac{x^2}{36}$ along with two lines parallel to the x axis.



Determine the area of the shaded region.

[5]

INTEGRATION AND APPLICATIONS: TRIAL TEST 2 – CALCULATOR FREE

Calculators NOT allowed
Time Allowed: 20 minutes
Total Marks: 20

1. Determine the following:

(a) $\int \left(1 - \frac{1}{\sqrt{x}}\right) dx$

(b) $\int_0^2 \left(\frac{2}{(2x+1)^3}\right) dx$

(c) $\frac{d}{dx} \int_x^0 \left(\frac{2t}{(1-t^2)^3}\right) dt$

[6]

2. State the velocity and displacement equations of an object given the following information:

- The acceleration of an object in *m/sec*^{2} travelling in a straight line is $a(t) = -\dfrac{3}{\sqrt{(2t-1)}}$
- The object is at a displacement of 16 cm at $t = 1$ *second.*
- The displacement of the object 4 *seconds* <u>later</u> is 38 cm.

[6]

3. (a) Evaluate $\frac{d}{dx}(2x\sqrt{x+1})$

(b) Use your answer in part (a) to determine $\frac{x}{\sqrt{x+1}}\,dx$

[5]

4. Calculate the area enclosed between and $y = 2x$ and $y = \sqrt{4x}$.

[3]

EXPONENTIALS, TRIGONOMETRY, BINOMIAL & DISCRETE DISTRIBUTIONS: TRIAL TEST 3 – CALCULATOR ASSUMED

Calculators allowed

Time Allowed: 45 minutes

Total Marks: 44

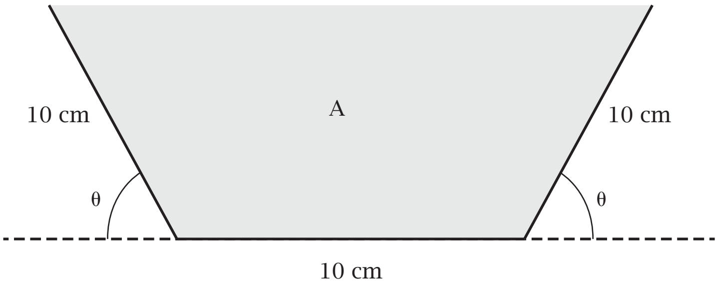
1. Scientists study a population of bats over a 6 week period and conclude that the population is increasing by a rate given by $B'(t) = 4.1e^{0.4t}$ where t is the number of weeks since the study began.

(a) What is the change in the population during the third week of the study?

(b) The study concludes at the end of the sixth week. A further study has been commissioned when the population of bats reaches 1750. When will the second study begin?

[4]

2. Galvanised steel is bent to form guttering for homes. The following represents a cross-section of a typical gutter.



Determine the value of θ which gives the maximum area A.

[8]

3. A particle moves along a straight line such that its velocity in metres per second is given by: $v = -2\sin(2t) + 2,\ 0 \le t \le 5$.

(a) Calculate the distance equation given that the particle is initially at the origin.

(b) Calculate the distance travelled in the first three seconds.

[4]

4. A bag contains two blue and eight yellow balls. Two balls are selected at random from the bag without replacement. Let X be the number of blue balls in the sample.

(a) Determine the probability distribution of X.

(b) Find

(i) $P(X = 0)$

(ii) $P(X \geq 1)$

(iii) $P(X = 0 \mid X \leq 1)$

[6]

5. Under normal use, 8% of switches from a large batch manufactured by a company are found to be defective. A random sample of 20 switches are selected. Determine the probability:

(a) there are no defective switches in the sample.

(b) there are 15 non-defective switches in the sample.

(c) A major electrical supplier will either accept or reject the batch of switches. If the sample of 20 switches contain no more than one defective switch the supplier will accept the batch, otherwise it will reject the batch of switches. Determine the probability the batch will be accepted.

(d) The electrical supplier also makes additional conditions on the supply of the switches.

- If the first sample of 20 switches contain no more than 1 defective switch, the batch is accepted.
- If the first sample contains 2 defective switches another sample of 20 is selected. If this contains no defective switches the batch is accepted.

What is the probability the batch is accepted?

(e) If a batch of switches is rejected with a least one defective switch the probability is at least 0.688. Determine the size of the sample for this batch of switches.

[7]

6. Abbie has \$35 to play with at the Casino and she tries her luck at a heads or tails coin game, where if she wins, she receives double her bet. She bets \$5 on heads and quits if she wins. If she loses, she bets \$10 on heads and quits if she wins. If she loses again, she bets her final \$20 on the next toss, again on heads.

(a) Find the probability that she wins \$5

(b) Determine the expected loss or gain

[5]

7. A student sits for a test which contains 5 multiple choice questions. He has a probability of 0.8 of knowing the correct answer to each question but if he does not know the answer he will guess. There are 5 alternatives for each question and he has attempted each question.

Let X be the number of correct answer(s) obtained.

Determine

(a) $P(X = 3)$

(b) $E(X)$

(c) $\text{Var}(X)$

(d) If the student knows the correct answer to 3 questions, what is the probability that the student will obtain full marks?

(e) 2 marks will be awarded for a correct answer and 1 mark deducted for an incorrect answer. If Y is the total score obtained find:

(i) $E(Y)$

(ii) $\text{Var}(Y)$

[10]

EXPONENTIALS, TRIGONOMETRY, BINOMIAL & DISCRETE DISTRIBUTIONS: TRIAL TEST 3 – RESOURCE FREE

Calculators NOT allowed
Time Allowed: 25 minutes
Total Marks: 25

1. Determine each of the following:

(a) $f'(x)$ given $f(x) = \sin^4 x$

(b) $\int_0^1 \left(\frac{e^{4x} - e^x}{e^{2x}}\right) dx$

[4]

2. If a discrete random variable has a mean, $E(X) = 15$ and a standard deviation of 3 determine:

(a) $E(2X - 1)$

(b) Standard Deviation of $(2X - 1)$

(c) Variance $(2X - 1)$

[3]

3. A particle moves in a straight line according to the function

$$f(t) = \frac{e^{\sin(t)}}{t+1}$$

where t is in seconds and $f(t)$ in metres.

(a) Determine the velocity function.

(b) Find the velocity when $t = 0$.

[3]

4. A binomial distribution X has a mean of 10 and a standard deviation of 3.

(a) Determine the values of n and p.

(b) Determine the mean and variance of the distribution Y where $Y = 3 - 2X$

[4]

5. (a) Show that the function $y = 2k - 3ke^{x^3}$, where k is a positive constant, has only one stationary point at $(0, -k)$

(b) Determine the nature of the stationary point using the second derivative.

[6]

6. Given the function $y = -x\cos(x)$

(a) Determine $\dfrac{dy}{dx}$

(b) Using part (a), show how you would determine $\int x\sin(x)\,dx$

[5]

EXPONENTIALS & LOGARITHMS: TRIAL TEST 4 – CALCULATOR ASSUMED

Calculators allowed
Time Allowed: 45 minutes
Total Marks: 44

1. A particle A moves along a straight line. Its velocity in metres per second is given by $v(t) = \ln(t + 1) \quad t \geq 0$.

   (a) Calculate the initial velocity and acceleration.

   (b) The acceleration of A when the velocity is 2 m/s.

   (c) Describe the acceleration of A for large values of t.

   [7]

2. Consider the function $y = -e^{4x^2 - \ln(3 - 2x)} + 1$

Determine the equation of the tangent using calculus at the point where $x = 2$, giving the answer in exact form. Show working.

[4]

3. Given the function $f(x) = 2x\ln(x) - 2x + 3$ *where* $x > 0$ and showing calculus techniques:

(a) Calculate $f'(x)$

(b) Hence determine an expression for $\int \ln(x)\, dx$. Show working.

(c) From part (b), determine an expression for $\int \ln(x^3)\, dx$. Show working.

[4]

4. The loudness, S, (in decibels) of a sound of intensity, I, is defined to be:

$$S = 10 \log \left(\frac{I}{I_0}\right)$$

where I_0 is the minimum intensity of sound.

(a) The loudness of the sound of a lawn mower is 90 decibels. Calculate the intensity of this sound in terms of I_0.

(b) The intensity of sound of a loud car horn is $10^{7.6}\,I_0$. How many times is the intensity of sound of the lawn mower greater than the intensity of the sound of a loud car horn?

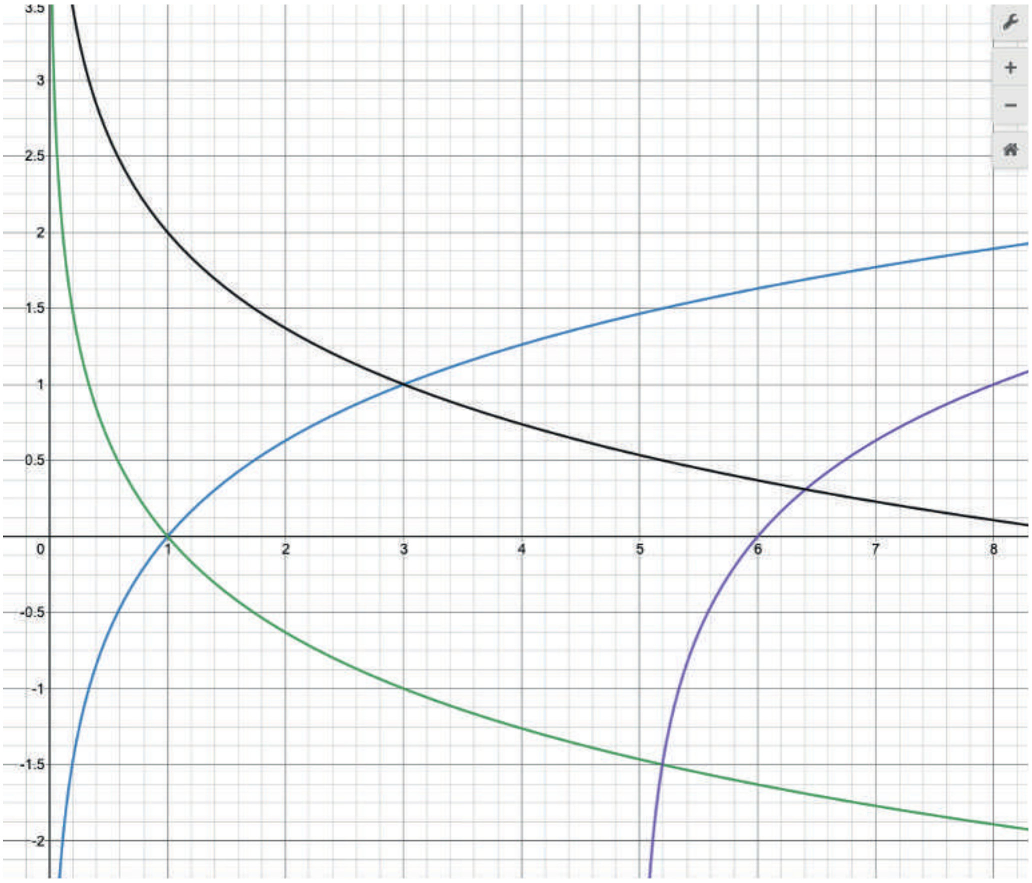
(c) Determine the loudness (in decibels) of the sound of the loud car horn in part (b)

[6]

5. The four graphs below have functions:

- $f(x)$
- $af(x)$
- $f(x + d)$
- $bf(x) + c$

where $f(x) = \log_p x$ where $x > 0$ and $p > 1$.



(a) State the values of a, b, c, d and p.

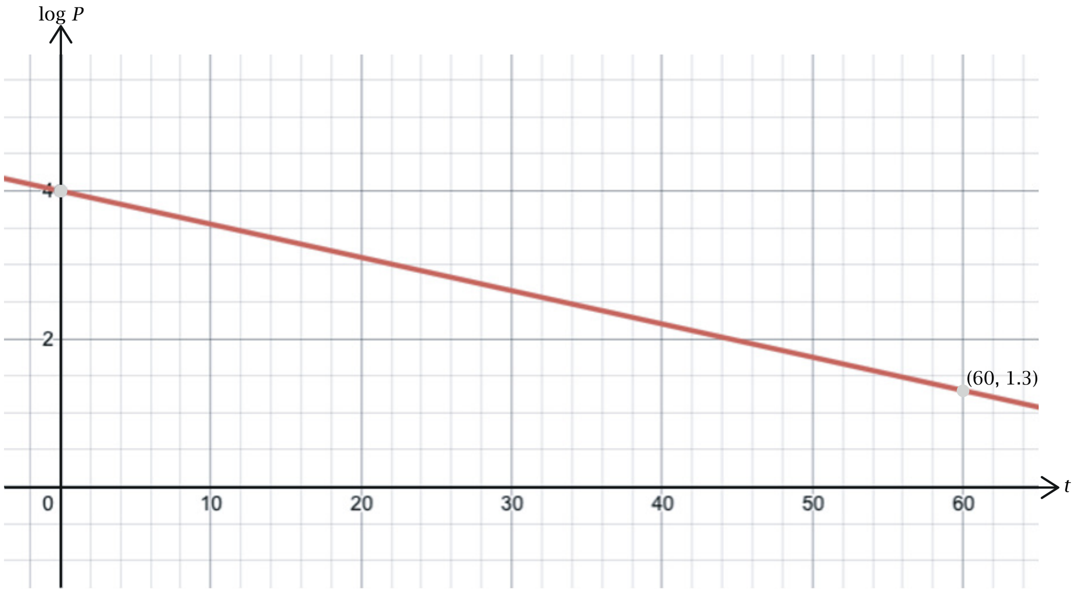
(b) Add the graph of the function $g(x) = -\log_p(x - 2) + 3$ on the axes above.

(i) State the asymptote of the graph of $g(x) = -\log_p(x - 2) + 3$.

(ii) Determine the root of the graph of $g(x) = -\log_p(x - 2) + 3$.

[8]

6. The size of a decreasing population, P after t years is according to the equation $P = ab^t$. The graph below shows the linear relationship between t and $\log P$ through the points (0, 4) and (60, 1.3)



(a) Using $P = ab^t$ **show** that the relationship between t and $\log P$ is linear.

(b) Determine the equation between t and $\log P$.

(c) Calculate the values of a and b in the equation $P = ab^t$ and interpret these values.

(d) When will the population reach 2000?

(e) Calculate the population after 30 years.

[8]

7. At the start of the year in 2012 a colony of native marsupials had a population of 2200. The population was growing continuously such that $\frac{dP}{dt} = 0.05P$ where P is the number of marsupials in the colony t years after the start of 2012.

(a) Calculate the number of marsupials in the colony at the start of 2018.

(b) In which year will the colony reach 4000 marsupials?

(c) Calculate the rate of change when the population of the colony is 3000.

(d) A new disease at the start of 2018 threatens the population of the colony of marsupials. This disease causes the population to decrease at a rate of 9% per year. Determine in which year and month the population of the colony will be less than 1300.

[7]

LOGARITHMS: TRIAL TEST 4 – RESOURCE FREE

Calculators NOT allowed
Time Allowed: 25 minutes
Total Marks: 26

1. (a) Differentiate the following: $y = \ln(x^2)\, e^{3x+1}$

   (b) $\int \frac{8x}{1 + 3x^2}\, dx$

[4]

2. Using calculus, determine the exact coordinates of:

   (a) the maximum point on the graph of $y = \dfrac{\ln x}{\frac{1}{2}x}$

(b) the equation of the tangent to the curve when $x = 1$

[6]

3. Find the *exact* solution to the following equation: $50\,e^{2x+1} = 3000$

[3]

4. (a) Express $2\log_2 3 + \log_2 4 - 1$ as a single logarithm.

(b) Express y in terms of x: $\log_4 y = \log_4 (4x) - 2$.

[5]

5. If $\log_p q = r$
$\log_p t = f$
$\log_p g = h$

then find in terms of r, f and h.

(a) $\log_p (tg)$

(b) $\log_p \left(\frac{q}{t}\right)$

(c) $\log_p \left(\sqrt{qg}\right)$

(d) $2\log_p \left(\frac{qt}{\sqrt{g}}\right)$

[8]

CONTINUOUS DISTRIBUTIONS & SAMPLE PROPORTIONS: TRIAL TEST 5 – CALCULATOR ASSUMED

Calculators allowed
Time Allowed: 40 minutes
Total Marks: 40

1. A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \dfrac{k - x}{2} & , \quad 3 \le x \le 5 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

(a) Find k

(b) Determine $P(4 \le x \le 5)$

[3]

2. Simone takes anywhere from 14 to 26 minutes to travel from home to work each day dependent upon the road conditions.

Let Y be the time taken to travel to work each day

(a) Determine the probability density function Y.

(b) Find the probability that it takes Simone less than 18 minutes to travel to work.

(c) Find the probability that it takes between 16 and 25 minutes for Simone to travel to work.

(d) Determine the median travelling time.

(e) Simone starts work at 9.00 am. If she leaves home at 8.45 am everyday, determine the probability that she will be late?

(f) Calculate the probability that Simone will be late on Thursday and Friday.

(g) Determine the probability that out of 5 working days, Simone will be late on at least 3 days.

[7]

3. The weights of trout in a certain lake are normally distributed with a mean of 1.8 kg and a standard deviation of 0.28 kg.

(a) If a trout is selected at random determine the probability that it weighs more than 1.6 kg given it weighs less than 1.9 kg.

(b) If 20% of the trout in the lake are underweight, determine the actual weight of an underweight trout?

[4]

4. A survey conducted at a local supermarket found that from 500 people, 350 preferred salmon to barramundi.

(a) Find a 95% confidence interval for the population proportion of those who prefer salmon.

(b) If another survey is to be conducted with the sample proportion, $\hat{p} = 0.8$, the margin of error = 0.04 and the confidence interval 99%, determine the sample size.

[4]

5. A new drug, Type 1 in liquid form is designed to help overweight people lose weight. It was found the weight loss was normally distributed with a mean of 30 kg and a standard deviation of 7.2 kg.

Find the probability that

(a) the weight loss was less than 25 kg.

(b) the weight loss was between 27 kg and 32 kg.

(c) If the weight loss was less than 15 kg, Drug Type 2 was administered. If 1000 individuals use the initial drug, Type 1, how many will need to switch to Drug Type 2?

The doctors require 90% of all individuals to have a weight loss greater than 25 kg.

(d) If the current standard deviation is maintained, determine the new mean.

(e) If the original mean is maintained, determine the new standard deviation.

[7]

6. A large sample of university students were asked to estimate the time it took to walk from the Library to the Cafeteria. The times varied from 8 minutes to 16 minutes. Calculate the probability that a randomly chosen university student estimated a time less than 11 minutes using a symmetrical triangular distribution.

[3]

7. From a sample of 40 Year 7 students, 25 said they used a social media app on their phone.

(a) Calculate the approximate probability of Year 7 students using a social media app on their phone.

(b) Determine an approximate standard deviation of the sampling distribution.

(c) Calculate an approximate 95% confidence interval for the probability of using a social media app on a phone.

(d) Explain in context what this confidence interval means.

[7]

8. 58% of Australians have travelled to Europe. Calculate the probability that in a sample of 100 Australians, 20 to 60 of them have travelled to Europe.

[5]

EXAMINATION STYLE QUESTIONS

1. *Calculator Free*

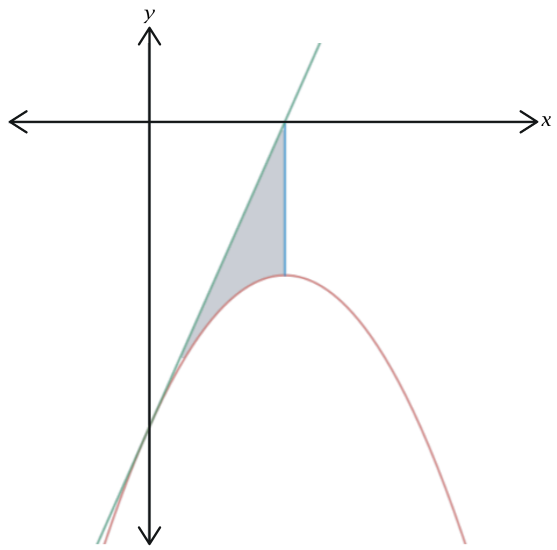
A curve has a gradient function: $\frac{dy}{dx} = 4x^2 - 5$.

(a) Determine the gradient of the curve when $x = -2$.

(b) For what value(s) of x is the rate of change of y with respect to x equal to 1.

2. *Calculator Assumed*

The shaded region below is enclosed by a curve with equation $y = -x^2 + 6x - 18$ and two lines, one parallel to the y axis and the other a tangent to the curve at the y axis.



(a) Determine the coordinates of the maximum turning point of the curve.

(b) Find the equation of the tangent to the curve at the y axis.

(c) Calculate the area of the shaded region.

3. *Calculator Free*

(a) A curve has an equation $y = x^3 - 6x^2 + 25x - 12$. Determine $\frac{dy}{dx}$.

(b) Given $y = \frac{2x(3x^6 - 7x^2)}{x^3}$. Calculate $\frac{dy}{dx}$.

(c) If $y = x^{\frac{1}{3}}\left(x^{\frac{5}{3}} - x^{\frac{1}{3}}\right)$ determine $\frac{dy}{dx}$.

4. *Calculator Assumed*

A probability distribution is shown in the table below for a discrete random variable X.

x	-3	-2	-1	0	1	2
$P(X = x)$	0.1	0.2	0.1	0.3	0.1	0.2

(a) Determine

(i) $P(X > -2)$

(ii) $P(X \geq -2 \mid X < 2)$

(b) Evaluate

(i) $E(X)$

(ii) $E(2X - 3)$

(c) Calculate

(i) $Var(X)$

(ii) $Var(2X - 3)$

5. *Calculator Free*

Determine each of the following:

(a) $\frac{d}{dx}(x^3\ e^{2x})$

(b) $\frac{d}{dx}(\cos(3x - 5))$

(c) $f'(x)$ when $f(x) = \sin(2x)\ \ln(2x)$

6. *Calculator Assumed*

The random variable X has a probability distribution as follows:

$$f(x) = \begin{cases} a(x-2)(x-4), & 0 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$$

(a) Determine the exact value of a.

(b) Evaluate $E(X)$ and $Var(X)$.

(c) If $Y = 3 - 2X$ determine $E(Y)$ and $Var(Y)$.

(d) Calculate the median value correct to 3 decimal places of the distribution.

(e) Determine k if $P(0.05 \leq X \leq k) = \frac{3}{4}$

(f) State the cumulative distribution function $F(x)$.

(g) Hence state the value of $P(X \leq 1.25)$

7. *Calculator Free*

(a) A curve has equation $y = x^2(x - 4)$. Determine the gradient of the curve at $(3, -9)$.

(b) A curve has an equation $y = f(x)$ with $\frac{dy}{dx} = -px^2 + \frac{3}{4}x + p$ where p is a constant.

When $x = -4$ the gradient of the curve is 21. Find the value of p.

8. *Calculator Assumed*

A random sample n of light bulbs are selected from a production line. Light bulbs are known to be defective with probability p, which are independent of each other.

The random variable T is the number of defective light bulbs in the sample.

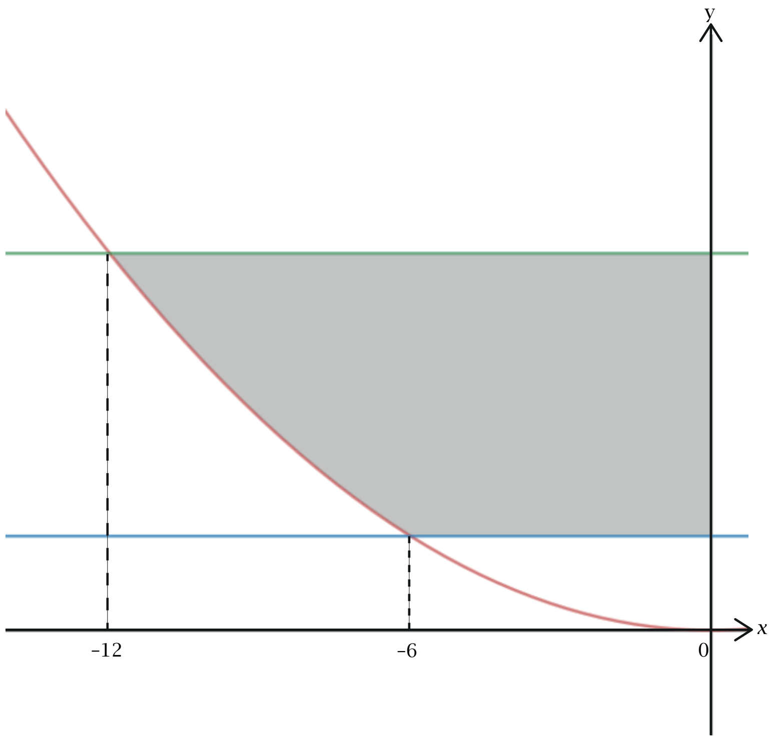
The mean and standard deviation of T are 90.75 and 8.25 respectively.

(a) Calculate the values of n and p.

(b) Determine $P(T \geq 99)$

9. *Calculator Assumed*

A shaded region shown below is enclosed by the graph of $y = \frac{x^2}{36}$ along with two lines parallel to the x axis.

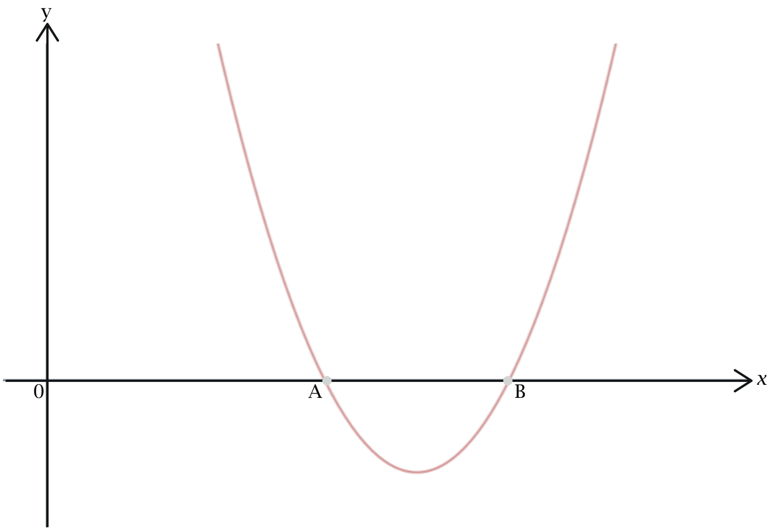


Determine the area of the shaded region.

10. *Calculator Free*

The curve below intersects the x axis at A and B. The graph has the equation $y = (x - 3)(x - 5)$.

Find the equations of the tangent lines to the curve at A and B.



11. *Calculator Free*

Determine the following:

(a) $\int \frac{\sin(3t)}{2 + \cos(3t)}\, dt$

(b) $\int_1^4 \left(2 + \frac{3}{x}\right) dx$ (Leave in exact form)

(c) $\int \frac{-3}{e^{-5x+2}}\, dx$

12. *Calculator Free*

(a) Determine the exact solution to $2^{3x-1} = 3^{2x}$

(b) Simplify $\log_3 27 \times \log_2 16^3$

(c) Write the following in the form $\log_2 k$ where k is a constant:

$2 - \log_2 25 + 2\log_2 10$ and hence evaluate.

13. *Calculator Free*

For a continuous random variable X, the probability density function is:

$$f(x) = \begin{cases} \frac{1}{2}\sin(x), & 0 \le x \le \pi \\ 0 & \textit{elsewhere} \end{cases}$$

Determine:

(a) $P\left(X \le \frac{\pi}{6}\right)$

(b) $P\left(\frac{\pi}{2} \le X \le \frac{2\pi}{3}\right)$

14. *Calculator Free*

Given a curve with an equation $y = f(x)$ where $\frac{dy}{dx} = 2x - \frac{6}{\sqrt{x}} + 3$ and a point $(1, 2)$ which lies on the curve find:

(a) $f(x)$

(b) the equation of the tangent to the curve at (1, 2) giving your answer in the form $ax + by + c = 0$ where a, b and c are integers.

15. *Calculator Assumed*

The continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} ke^{3x}, & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

(a) Show that $k = \dfrac{3}{e^3 - 1}$.

(b) Determine $P(X \le 0.3)$

(c) Calculate $P(X > 0.1 \mid X \le 0.8)$.

(d) Determine the mean and variance of X.

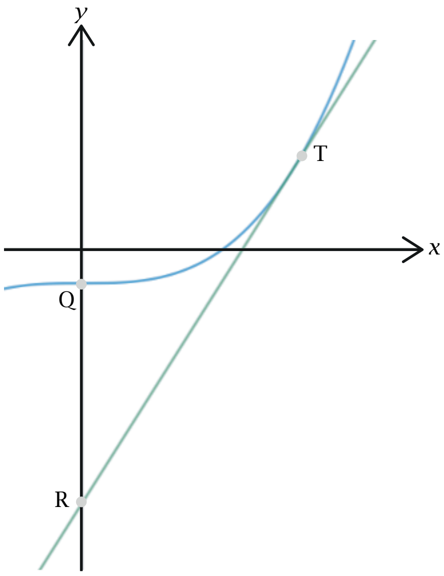
16. *Calculator Free*

If $\frac{dy}{dx} = 3x^{-\frac{1}{2}} + \frac{x}{\sqrt{x}}$ and given that $y = 27$ when $x = 4$ find y in terms of x.

17. *Calculator Free*

A tangent to the curve $y = 2x^3 - 5$ at the point $T(2, 11)$ intersects the y axis at R. The curve has a y intercept at the point Q.

Calculate the length of RQ.



18. *Calculator Assumed*

A leaf is picked from a eucalyptus tree and its mass, M, in grams t days later is given by: $M = M_0 e^{-kt}$

Initially the mass of the leaf is 6.25 grams and 3 days later is 2.75 grams.

(a) Determine the value of M_0.

(b) Calculate k.

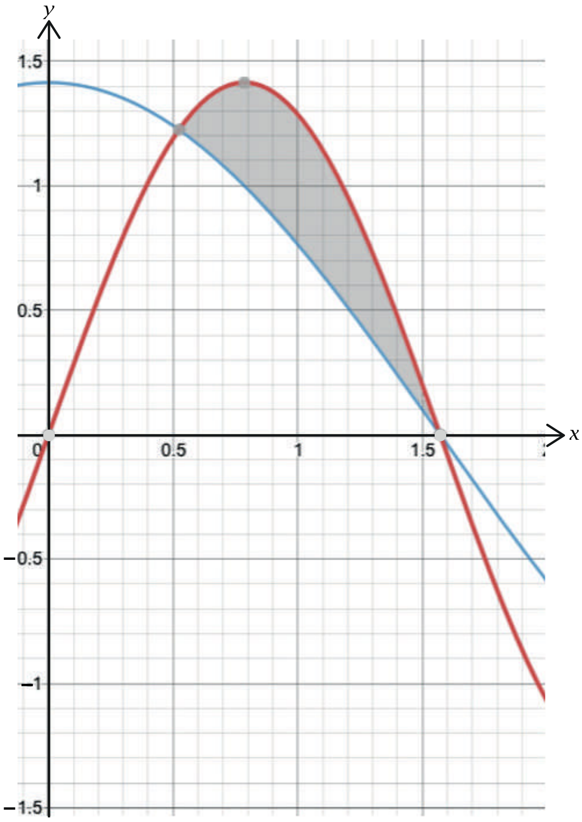
(c) How long does it take for the mass of a leaf to drop to 1.4 grams?

(d) At what time, t, was the instantaneous rate of the mass of a leaf $-0.6 \ln(2)$?

(e) Calculate the rate of change of M when the mass is 5 grams.

19. *Calculator Free*

The graphs of $y = \sqrt{2}\sin(2x)$ and $y = \sqrt{2}\cos(x)$ are shown below.



Consider the shaded region only.

(a) The x coordinates of the points of intersection are $\frac{\pi}{6}$ and $\frac{\pi}{2}$. Explain with working why this is the case.

(b) Calculate the area of the shaded region.

20. *Calculator Free*

A curve with equation $y = f(x)$ passes through the point (1, 8).

Given that $f'(x) = 6x^2 - 24x + 18$:

(a) Determine $f(x)$.

(b) Sketch the graph of $f(x)$ showing all important features.

21. *Calculator Free*

Determine the coordinates of the stationary points and their nature using the second derivative on the curve $y = x^3 - 12x$.

22. *Calculator Free*

Given $y = 2xe^{-0.5x}$ determine:

(a) the exact coordinates of the stationary point(s).

(b) the exact coordinates of the point of inflection.

23. *Calculator Assumed*

(a) A random sample of 42 students is taken from the population of all Mathematical Methods students, for which the overall proportion who enjoy the subject is 0.6. Calculate the standard deviation of $\hat{p}$.

(b) Determine the size of the sample required if $p = 0.4$ and the standard deviation of $\hat{p} \leq 0.04$.

24. *Calculator Assumed*

A continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} \dfrac{k}{x+2}, & 0.3 \leq x \leq 0.9 \\ 0 & \textit{elsewhere} \end{cases}$$

(a) Determine the exact value of k.

(b) Calculate $P(X \leq 0.6)$ correct to 2 decimal places.

(c) Determine $E(X)$.

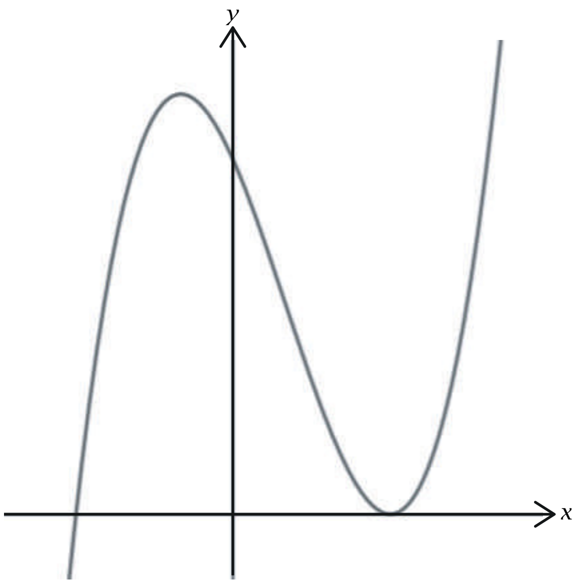
(d) Calculate $E(Y)$ where $Y = (1 - 3X)$.

25. *Calculator Free*

The graph of $y = \frac{1}{3}x^3 - x^2 - 3x + k$ is shown below where k is a constant.

A minimum point exists on the x axis and is a tangent to the curve.

State the value of k.



26. *Calculator Free*

A dietician decides to randomly survey the amount of 'fast food' eaten in preparation for her weight loss study. The survey will be conducted by interviewing every 10^{th} person who leave a fast-food restaurant.

(a) State the type of sampling method used.

(b) Discuss the main source of bias that exists with this survey and how this could be improved.

27. *Calculator Free*

A body is moving in a straight line and its velocity (v) in metres at time t seconds is given by $v = \frac{32}{(t+3)^2}$ where t is positive.

(a) Determine the distance travelled in the first 3 seconds.

(b) If acceleration is -1 m/s^2, determine the velocity of the body.

28. *Calculator Free*

(a) Expand $(2\sqrt{x} + 3)^2$ and hence find k where k is a constant in the expression $4x + k\sqrt{x} + 9$.

(b) Determine $\int(2\sqrt{x} + 3)^2 dx$.

29. *Calculator Assumed*

Lolly manufacturer 'Froggys' has created two types of frogs – red and green. Due to demand it produces 4 times as many red frogs as green. Packets are filled with a random selection of 30 frogs.

The random variable X is the number of red frogs in each packet of 30.

(a) Describe the distribution of X.

(b) Determine the probability that in a packet of frogs there are:

(i) at least 20 red frogs.

(ii) exactly 5 green frogs.

(c) 10 packets of frogs are randomly chosen. Determine the probability that exactly 9 will contain at least 20 red frogs.

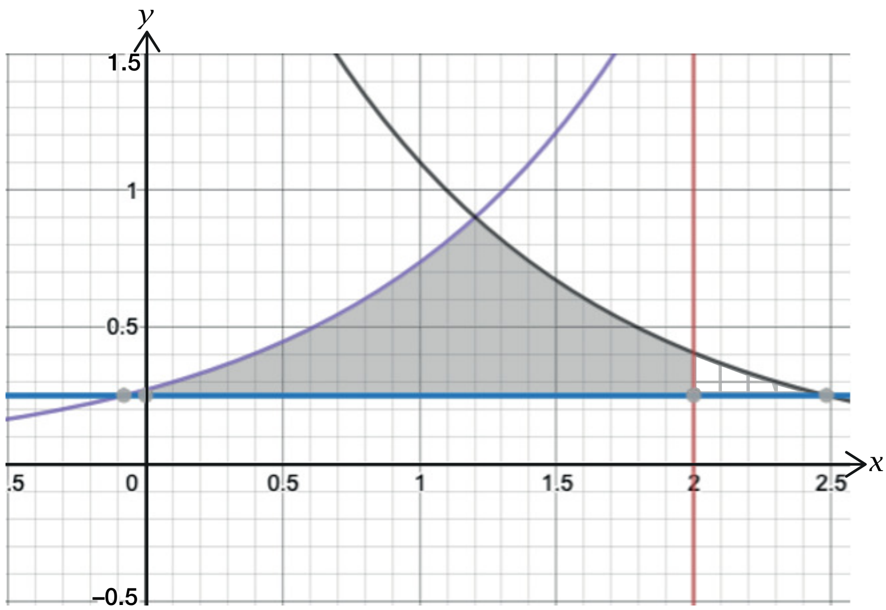
(d) Demand for green frogs decrease to 8% and hence the production process is altered. Determine the smallest sample size required in order that the probability that the sample contains at least one green frog is at least 95%.

(e) Calculate the mean and standard deviation of red frogs in a packet of 30.

30. *Calculator Assumed*

The shaded region below is enclosed by four functions, two of which are:

$f(x) = 2e^{x-2}$ and $g(x) = 3e^{-x}$



(a) State the equations of the remaining two functions.

(b) Calculate the area of the shaded region enclosed by the four functions.

31. *Calculator Free*

(a) Given that $y = 3x^3 - 7x - 2\pi$ find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(b) Determine $\int\left(3 + 2\sqrt{x} - \dfrac{1}{x^2}\right)dx$.

32. *Calculator Free*

A discrete random variable X, is given by:

$$P(X = x) = \begin{cases} ke^{2x+1}, & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

(a) Determine the exact value of k.

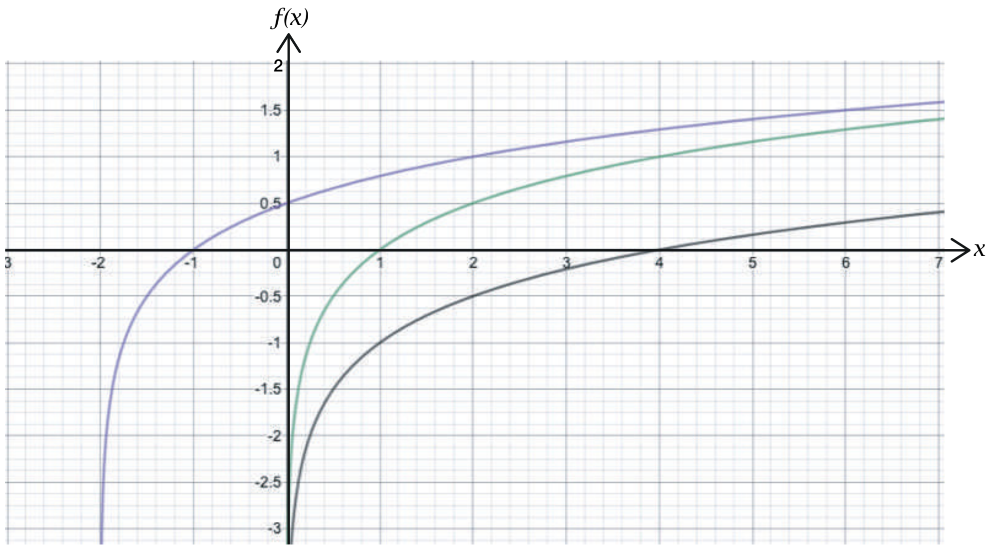
(b) Calculate the exact value of the mean of X.

33. *Calculator Free*

The three graphs below have functions:

- $f(x)$
- $f(x) + p$
- $f(x + q)$

where $f(x) = \log_r x$ where $x > 0$ and $r > 1$.



(a) State the values of p, q and r.

(b) Add the graph of the function $g(x) = -\log_r(x - 1) + 1$ on the axes above.

(i) State the asymptote of the graph of $g(x) = -\log_r(x - 1) + 1$.

(ii) Determine the root of the graph of $g(x) = -\log_r(x - 1) + 1$.

34. *Calculator Free*

A random variable X for a probability distribution is given by the formula:

$P(X = t) = k(t^2 - 1)$ for $t = 2,3,5,11$

(a) Calculate the value of k.

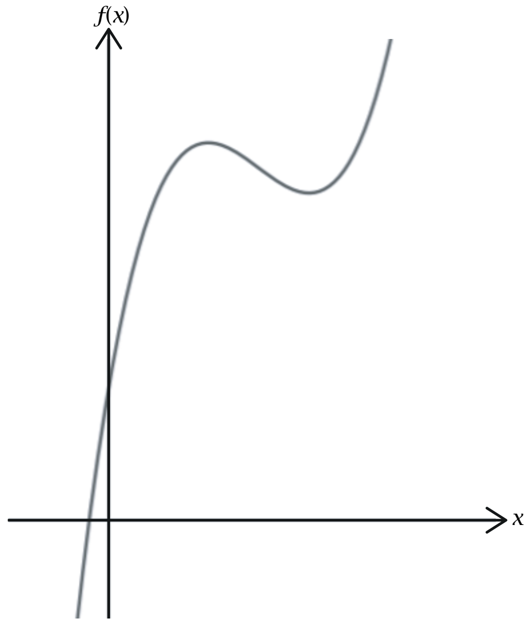
(b) Complete a probability distribution table.

35. *Calculator Free*

Use calculus to find the exact value of $\int_1^2 \left(4 + 6x^2 - \frac{5}{x^2}\right)dx$.

36. *Calculator Free*

The graph of $f(x)$, a cubic function is shown below.



The equation of the curve is $y = ax^3 - 3x^2 + 8x + b$ where a and b are constants.

The graph has two stationary points with a minimum point located at (2, 10).

Calculate the values of a and b.

37. *Calculator Assumed*

A game consists of rolling two tetrahedral die. The two dice, one white and the other red, have four faces numbered 0, 1, 2 and 3 respectively. The numbers face down on the two dice are recorded. The random variable T is the number on the white die multiplied by the number on the red die.

(a) Draw a probability distribution table for the random variable T.

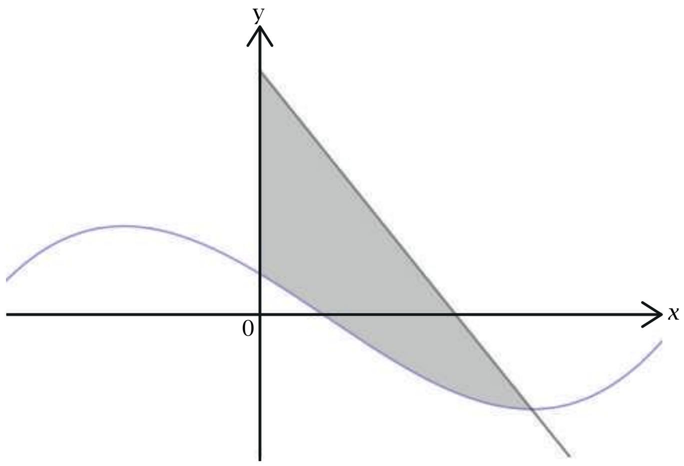
The cost to play a game is \$2.50. A prize of \$15 is awarded if the result is at least a four, \$5 if the result is a 1, 2 or a 3 and nothing for any other result.

(b) Determine the expected profit or loss for a game.

38. *Calculator Free*

The graph below has a curve with an equation $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$ and a straight line which intersects the curve at its minimum value and also passes through the y axis at (0, 6).

Determine the exact shaded area.



39. *Calculator Free*

A function $y = f(x)$ is such that $\frac{dy}{dx} = 3x^2 - 7x + k$ where k is a constant, has a maximum turning point at (1, –3).

(a) Calculate k.

(b) Determine y.

40. *Calculator Free*

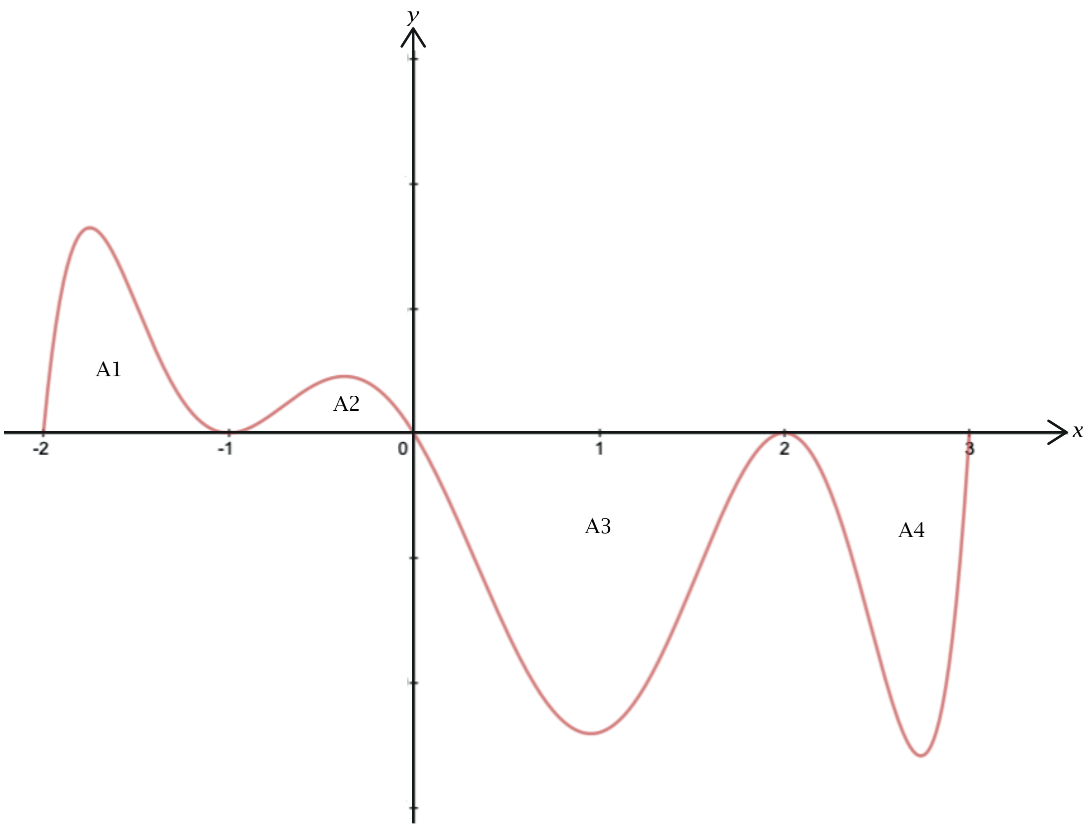
(a) Differentiate with respect to x: $y = \int_x^3 (t^3 - t^4)\,dt$

(b) Determine $\int 28(2x - 5)^3 dx$

41. *Calculator Free*

The areas in square units between the x axis and the curve $y = f(x)$ are as follows:

- A1 = 15
- A2 = 8
- A3 = 24
- A4 = 18



Calculate the value of:

(a) $\int_{2}^{3} f(x)dx$

(b) $\int_{-2}^{3} f(x)dx$

(c) $2\int_{0}^{-2} f(x)dx$

(d) $\int_{-1}^{2} (3 + f(x))dx$

(e) $\int_{-2}^{0} f'(x)dx + \int_{0}^{3} f(x)dx$

42. *Calculator Free*

A continuous curve $y = f(x)$ has exactly two stationary points with the following:

- A horizontal point of inflection at M when $x = m$.
- A minimum point occurs at N when $x = n$.
- $m < n$.

Which of these are correct?

1.

- *If* $m < x < n$, *then* $\frac{dy}{dx}$ *is positive and*
- *when* $x < m$ *is positive.*

2.

- *If* $m < x < n$, *then* $\frac{dy}{dx}$ *is positive and*
- *when* $x < m$ *is negative.*

3.

- *If* $m < x < n$, *then* $\frac{dy}{dx}$ *is negative and*
- *when* $x < m$ *is positive.*

4.

- *If* $m < x < n$, *then* $\frac{dy}{dx}$ *is negative and*
- *when* $x < m$ *is negative.*

43. *Calculator Free*

Determine the value(s) of x for which $y = 18x - 2x^3$ an increasing function?

44. *Calculator Assumed*

A body is moving in a straight line according to its displacement equation:
$x = 2\sin(2t - 3) + 2$, $0 \le t \le 4$ where x is in metres and t is in seconds.

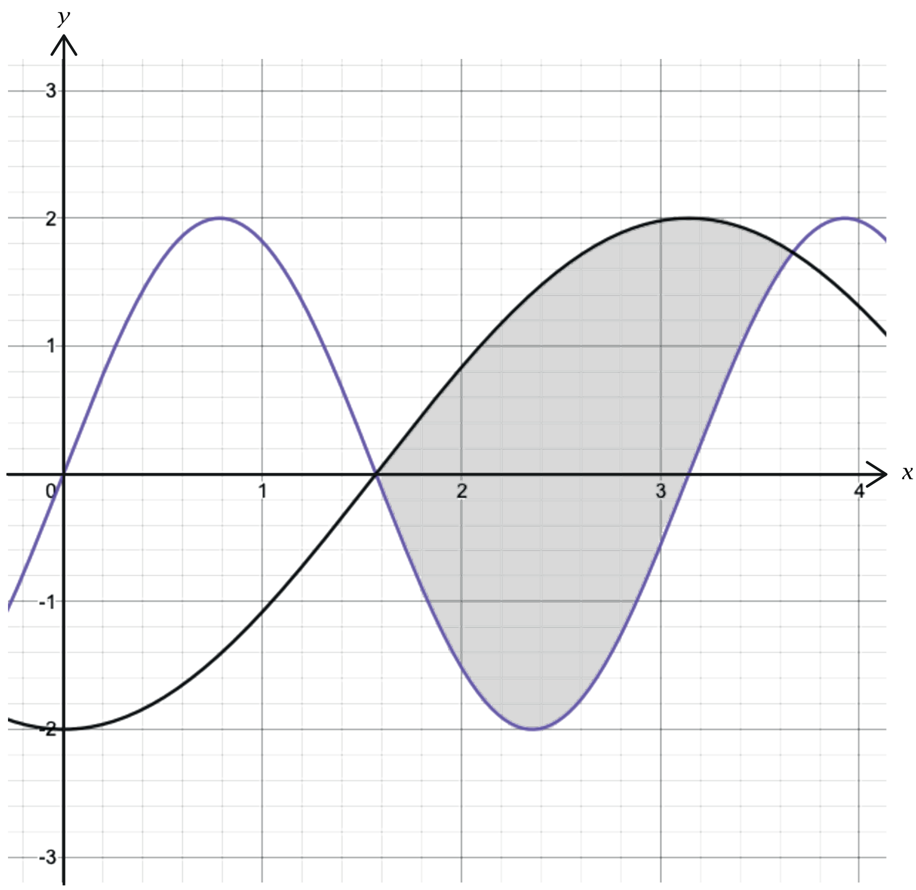
(a) Calculate the maximum displacement and when this occurs. Show use of the second derivative.

(b) Determine when the body is stationary.

(c) Calculate the acceleration of the body when $t = \pi$.

45. *Calculator Assumed*

The graphs of $f(x) = 2\sin(2x)$ and $g(x) = -2\cos(x)$ are shown below.



Determine the exact area of the shaded region.

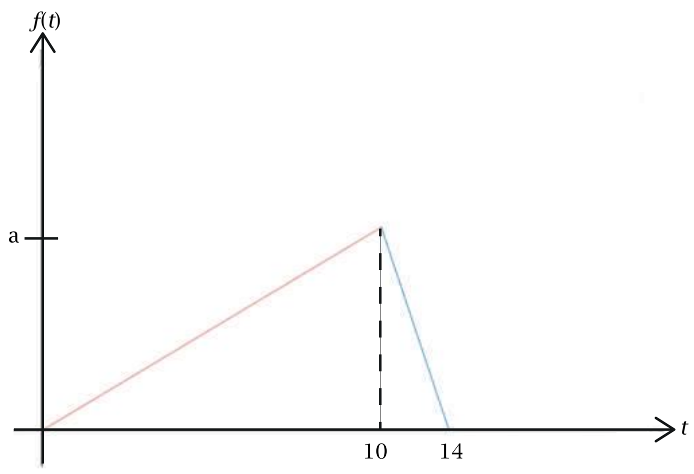
46. *Calculator Free*

State the velocity and displacement equations of an object given the following information:

- The acceleration of an object in *cm/sec*^{2} travelling in a straight line is $a(t) = -\dfrac{3}{\sqrt{(2t+7)}}$
- The object is at a displacement of 6 cm at $t = 1$ *second*
- The displacement of the object 8 *seconds* later is 28 cm.

47. *Calculator Assumed*

The length of time taken, T, in hours for 'Oz Post' to deliver a parcel is a random variable defined by the following triangular probability density function.



(a) State the value of a.

(b) Calculate the probability that the parcel will be delivered in less than 8 hours and 15 minutes.

(c) Determine $E(T)$ and $Var(T)$ giving your answer in hours and minutes.

'Oz Post' revamps its service improving the time for parcels to be delivered. New sorting technology will decrease time by 20% and new vans will decrease time by a further 90 minutes.

(d) Calculate the new values of $E(T)$ and $Var(T)$ giving your answer in hours and minutes.

48. *Calculator Assumed*

A random sample of 120 students at a school found that 97 use the app 'toktik'.

(a) Determine the sample proportion of students who use the 'toktik' app.

(b) State an approximate margin of error for a 95% confidence interval for the proportion of students who use the 'toktik' app.

(c) Calculate a 95% confidence interval for the proportion of students who use the 'toktik' app.

49. *Calculator Free*

Given $P(X = x) = \begin{cases} k \log_2 x & x = 2, 4, 8, 16 \\ 0 & \textit{elsewhere} \end{cases}$

(a) Calculate the value of k and hence construct a probability distribution table.

(b) Determine $P(X < 8)$

(c) Calculate $P(X > 2 \mid X < 8)$

50. *Calculator Free*

Given $f(x) = x^2 \ln(x)$ where $x > 0$

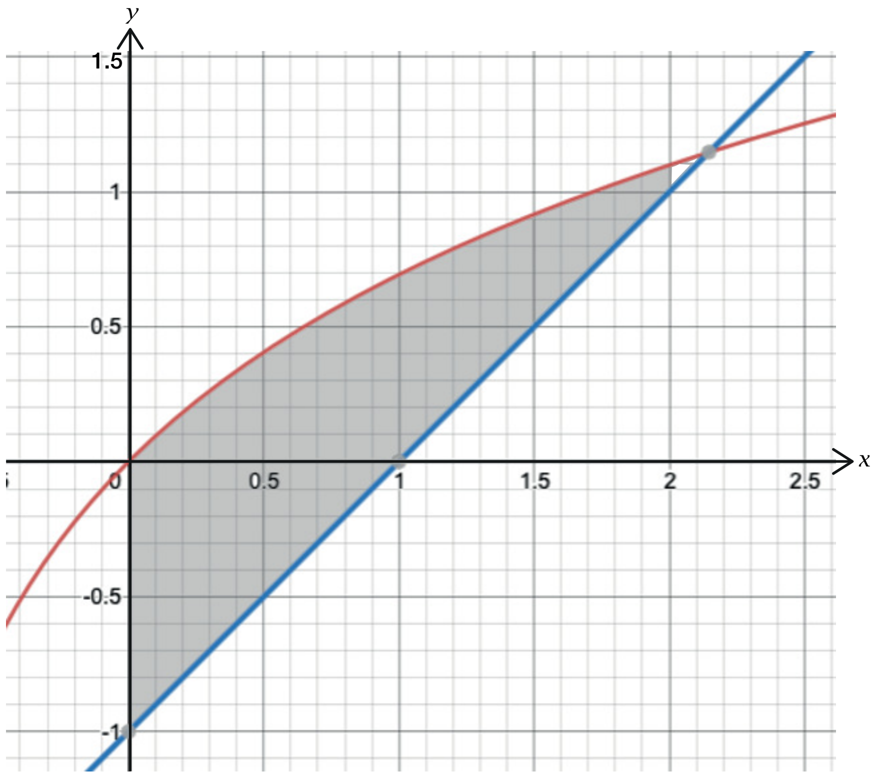
(a) Show that $f'(x) = 0$ when $x = e^{-\frac{1}{2}}$.

(b) Determine the nature of the stationary point(s) for $f(x) = x^2 \ln(x)$.

51. *Calculator Free*

(a) Show $\frac{d}{dx}((\ln(x+1))(x+1) - x - 1) = \ln(x+1)$

The graphs of $y = x - 1$ and $y = \ln(x + 1)$ are shown below.



(b) Use your result from (a) above to determine the exact area of the shaded region bounded by the two graphs above and the lines $x = 0$ and $x = 2$.

52. *Calculator Assumed*

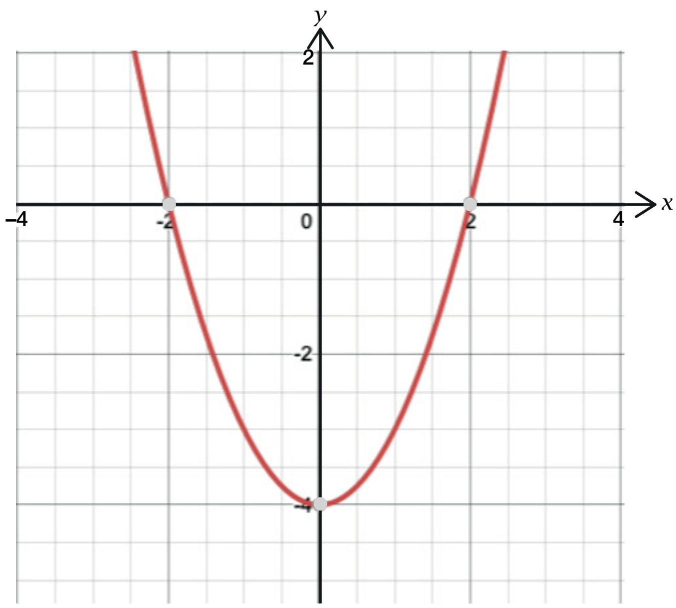
A survey finds that 144 people support the local country football team from a random sample of n people. The true population proportion of supporters has a symmetric confidence interval of $0.5576 \le p \le 0.7224$.

(a) Calculate the value of n.

(b) Calculate the value of the confidence level for this interval.

53. *Calculator Assumed*

The graph of $y = x^2 - 4$ is shown below.



Using six rectangles between $x = -2$ and $x = 2$ determine:

(a) an ***underestimate*** approximation of the area.

(b) the difference between the actual area and the approximation in part (a).

54. *Calculator Free*

(a) A curve has an equation $y = x^2 - 3x - 18\sqrt{x}$, with $x > 0$.

Determine $\frac{dy}{dx}$.

(b) If $f(x) = (2 - x^2)(\sqrt{x} - 5x)$. Find $f'(x)$. Do not simplify

(c) If $\frac{\sqrt{x^3 - 4x}}{y = (5x^2 + 2)}$ determine $\frac{dy}{dx}$. Do not simplify

55. *Calculator Assumed*

The bus journey between Sydney and Brisbane costs the company $\$C$ when driven at a steady speed of v *km/h*. This can be modelled by the following equation.

$$C = \frac{1350}{v} + \frac{3v}{14} + 50$$

Determine v using calculus which minimises the cost of the bus journey and hence calculate the cost.

56. *Calculator Assumed*

The process of 'capture – recapture' is used to estimate the size of an endangered frog population in a lake. A random sample of 40 frogs are captured, a tag added and then each frog is placed back into the lake. At a later date, another random sample of 15 frogs are captured, of which 6 are tagged.

(a) Determine a point estimate for the frog population.

(b) Using only the proportion of frogs from the population that are tagged determine a 95% confidence interval.

(c) Construct an approximate 95% confidence interval for the frog population in the lake.

57. *Calculator Assumed*

Water flows into a rainwater tank which initially contains 25 litres at a rate of $r'(t) = \frac{50}{2t+3}dt$ litres/min where t is the time in minutes and $t \geq 0$.

(a) Calculate the total amount of water in the tank after 30 minutes.

(b) Calculate the amount of time for the rainwater tank to be $\frac{1}{2}$ full if the tank is completely full after 2 hours.

58. *Calculator Assumed*

The length of time, T hours, of the lifetime of a lightbulb is a random variable with a probability density function represented by:

$$f(t) = \begin{cases} \frac{1}{20}e^{-\frac{t}{20}}, & t \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

(a) Calculate the probability that the lifetime of a lightbulb is:

(i) less than 20 hours.

(ii) between 15 and 22 hours.

(b) Find the expected median time of the lifetime of the lightbulb.

(c) Calculate the expected mean of the lifetime of the lightbulb.

(d) Find the probability that the lifetime of the lightbulb will last no more than 25 hours given that has been working correctly for 20 hours.

(e) A random sample of 15 lightbulbs are selected. Determine the probability that at least 7 will fail after less than 20 hours.

59. *Calculator Free*

(a) If $p = \log_k 2$ and $q = \log_k 3$

(i) give an expression in terms of p and q for $\log_k \frac{27}{8}$

(ii) evaluate k^{2q}.

(b) Given: $\log_t 5 + \log_t x = 0$ determine x.

(c) Calculate the exact solution(s) for x: $12^{1-x} + 1 = 12^x$

60. *Calculator Assumed*

A burger store wants to expand its range of burgers by introducing crocodile to its menu. A survey will help to determine the introduction of this new burger by selecting an appropriate sample from the population.

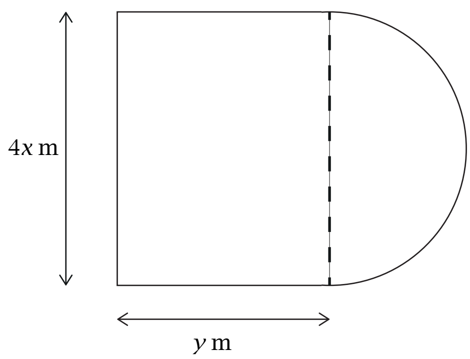
Discuss whether bias exists in any of these survey methods.

(a) Ask an employee of the burger company to obtain a response from the first 10 customers that enters the store.

(b) Ask customers on the burger store online ordering website to answer a short questionnaire on the introduction of crocodile to their menu.

61. *Calculator Assumed*

A swimming pool is to be constructed according to the following diagram consisting of a rectangle and a semicircle. The area of the pool is 360 m^2.



(a) Show that the perimeter of the pool can be written as: $P = 4x + \frac{180}{x} + \pi x$

(b) Using calculus determine the value of x which minimises the perimeter of the pool, giving your answer to 2 decimal places and hence state the perimeter.

62. *Calculator Assumed*

There are 3 students in a class of 25 students who wear spectacles. A single student is selected at random. The random variable X is the number of students who do not wear spectacles.

(a) Describe the distribution of X.

(b) Calculate $E(X)$.

From the class, three students are selected at random. The random variable R is the number of students who wear spectacles.

(c) Draw a probability distribution table for the random variable R.

(d) Calculate $E(R)$.

63. *Calculator Assumed*

A body is moving in a straight line at time t seconds where $t \geq 0$ according to the following:

$v(t) = \sqrt{4t}\ m/s$

(a) Calculate the distance the body travels in the first 9 seconds.

(b) Determine the acceleration of the body after 2 seconds.

(c) Calculate the velocity of the body when acceleration is 2 m/s^2.

64. *Calculator Free*

A curve has an equation $y = 10 - 4\sqrt{x} + 3x$ where $x \geq 0$ and passes through point P with an x coordinate of $\frac{1}{4}$.

Find the equation of the tangent to the curve at the point P

65. *Calculator Free*

Find $f'(x)$ for each of the following:

(a) $f(x) = \sqrt{4 - x^2}$

(b) $f(x) = \dfrac{5x + 1}{x^2 + 3}$

(c) $f(x) = (3x^2 - 1)^{\frac{1}{3}}$

(d) $f(x) = (x - 5)\sqrt{x + 3}$

66. *Calculator Assumed*

Given the function $R(x) = \int_{-12}^{x} f(r)\, dr$ where $f(r) = -\frac{r}{3} - 2$ determine:

(a) the value of $R(x)$ when

(i) $x = -4$

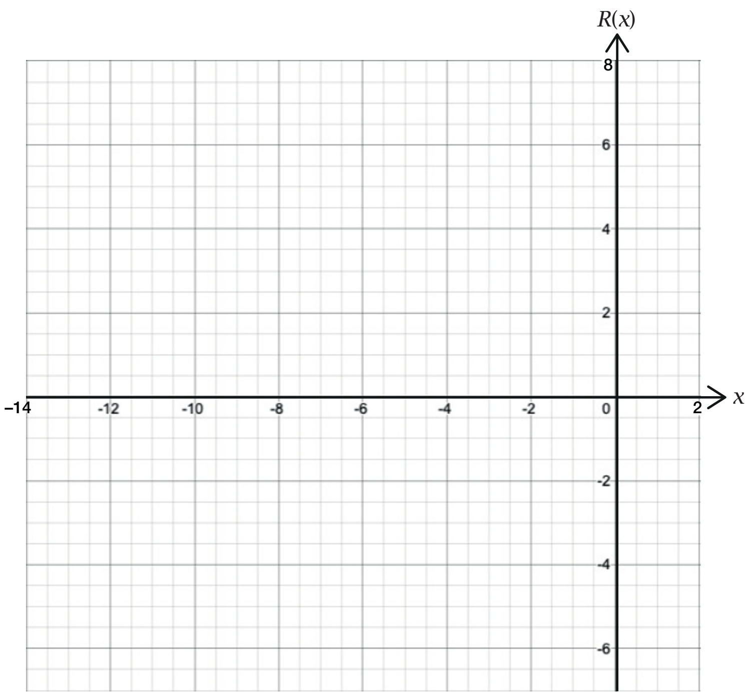
(ii) $x = 0$

(b) the value of x when $R(x)$ is decreasing.

(c) the value of x when $R'(x) = 0$.

(d) $R(x)$ in terms of x.

(e) the graph of $y = R(x)$ for $-12 \le x \le 0$.



67. *Calculator Assumed*

A function is defined as $g(x) = x^3 + ax^2 + b$, where a and b are constants.

(a) Show that $g(x)$ has a stationary point at $x = 0$ for all values of a and b.

(b) Calculate the values of a and b if $g(x)$ has another stationary point at $(-4, 5)$.

(c) If $g(x)$ has a stationary point at (m, n) where $m \neq 0$.

(i) State an equation for a in terms of m.

(ii) State an equation for b only in terms of m and n.

68. *Calculator Free*

Determine the following:

(a) $\frac{dy}{dx}[3 \ln(6 - 4x^3)]$

(b) $\frac{dy}{dx}\left[\ln\left(\frac{6}{2 - x}\right)\right]$

(c) $\int \frac{x - 2}{x^2 - 4x - 5}\, dx$

(d) (i) Show that $\frac{2}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1}$

(ii) Hence determine $\int \frac{2}{x^2 - 1}\, dx$

(e) If $\ln p = 0.45$ evaluate $\ln \frac{p^2}{\sqrt[3]{p}}$

69. *Calculator Assumed*

Given the function $R(x) = \frac{1}{\cos(x) + 1}$ where $0 \le x \le 2$

Four rectangles of width 0.5 are shown on the diagram above.

(a) Complete the table below:

x interval	0 – 0.5	0.5 – 1	1 – 1.5	1.5 – 2
Area of inscribed rectangle				
Area of circumscribed rectangle				

(b) Hence determine an approximate value for the area of $R(x) = \dfrac{1}{\cos(x) + 1}$ where $0 \le x \le 2$.

(c) Compared to the actual area is your estimate larger or smaller? Explain.

70. *Calculator Assumed*

Buses depart at a bus rank every 15 minutes. A passenger's waiting time, T, in minutes is a uniformly distributed random variable.

(a) State the probability density function of T and hence sketch the graph.

(b) Calculate the probability that a passenger who arrives at the bus rank waits no more than 8 minutes for the bus to depart.

(c) What is the probability that less than three passengers from a random selection of 6 will have to wait at least 10 minutes for the bus to depart?

(d) If $P(T < 2t) = P(T > t)$ determine the value of t.

71. *Calculator Assumed*

A large household survey is conducted to determine the proportion, p, of people that service a car once a year. With a 95% confidence interval, the margin of error was no more than 5%.

(a) Determine the shape of the distribution.

(b) How many households were surveyed?

A further survey was conducted and from 200 households, 63 service their car once a year.

(c) For this sample, calculate a 95% confidence interval margin of error for the proportion, p, of households that service their car once a year.

(d) Determine the change in the number of households surveyed in part (b) above if the proportion, p, calculated in part (c) is used?

72. *Calculator Assumed*

The Richter scale, R, measures the magnitude of an earthquake and is given by $R = \log I,\ \ I > 1$ where I is the relative intensity of an earthquake.

The table below shows the magnitude of three typical earthquakes:

Earthquake	2012 Sumatra	2019 Pakistan	2021 Philippines
Magnitude	8.6	5.6	7.0

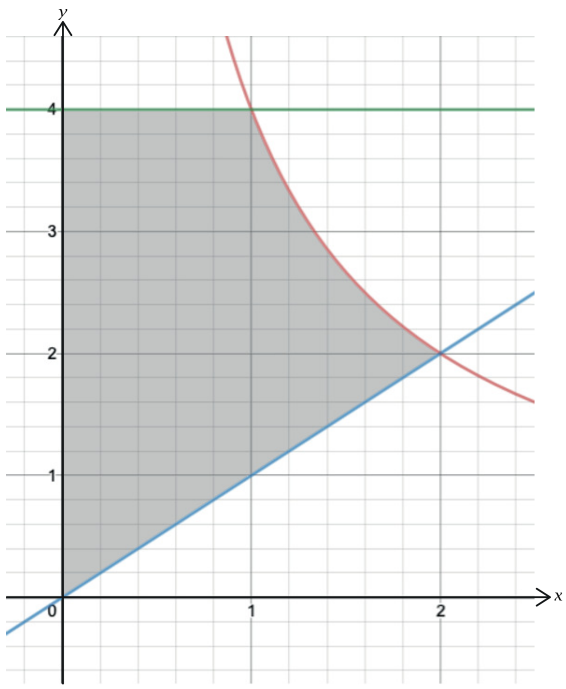
(a) The 2015 Queensland earthquake located in the Coral Sea had a relative intensity of 316 228. Determine its magnitude.

(b) Calculate the relative intensity of the 2021 Philippines earthquake.

(c) Compare the relative intensity of the Sumatran to Pakistan earthquakes by expressing in the form $k : 1$.

73. *Calculator Free*

Three functions: $y = x$, $y = \dfrac{4}{x}$ and $y = 4$ are shown on the graph below.



Determine the exact area of the shaded region.

74. *Calculator Free*

Calculate $\frac{d}{dp}\int_5^p 3\sqrt{3k+1}\ dk$

75. *Calculator Free*

(a) Evaluate $\int (4x-7)^6\ dx$

(b) Given $T(y) = \int_1^y (\sqrt{36-3^{2t}})\ dt$ determine the value(s) of y when $\frac{dT}{dy} = 3$

76. *Calculator Free*

(a) Evaluate $\frac{d}{dx}(x\sqrt{x+1})$

(b) Use your answer in part (a) to determine $\int \frac{x}{2\sqrt{x+1}}\ dx$.

77. *Calculator Free*

A small body moves from a fixed point *0* and has a displacement, x, in metres, given by $x(t) = \dfrac{t^2 + 12}{t + 2}$ where $t \geq 0$ and t is in seconds.

(a) State the velocity function $v(t)$ and acceleration function $a(t)$ of the small body simplifying the expressions.

(b) At the instant the body is stationary, calculate its displacement.

(c) Describe the acceleration of the body.

78. *Calculator Free*

Given $\int_3^6 f(p)\ dp = 24$ evaluate:

(a) $4\int_6^3 f(p)\ dp$

(b) $\int_3^4 [f(p) - 2]\ dp + \int_4^6 f(p)\ dp$

79. *Calculator Assumed*

A body moves in a straight line with acceleration $a(t) = 4t - 3k$ m/sec^2 where k is a constant and t is time in seconds. The body also has:

- a velocity of 4 m/s when $t = 3$
- a stationary velocity when $t = 2$
- a displacement of 6 metres when $t = 1$

(a) Determine the velocity equation of the body at time t.

(b) Hence determine the displacement equation of the body at time t.

80. *Calculator Free*

A particle moving in a straight line through a fixed point O has velocity given by $v(t) = t^3 - 5t^2 + 6t$ where v is in metres at time t seconds. Calculate:

(a) the value(s) of t when the particle is stationary.

(b) the acceleration when the particle is at rest.

81. *Calculator Assumed*

A body is moving such that its displacement, x, in metres at time t seconds from O is given by $x = (c + 4t)^2$ where $c \geq 0$, $t \geq 0$ and c is a constant.

(a) Show that $\frac{dx}{dt} = 8\sqrt{x}$.

(b) If the particle is 4 metres to the right of O, calculate its velocity.

(c) Describe the motion of the particles acceleration.

82. *Calculator Assumed*

A discrete random variable X has a probability function given by:

$$P(X = x) = \begin{cases} a(4 - x) & x = 0, 1, 2, 3 \\ b & x = 4, 5 \\ 0 & elsewhere \end{cases}$$

(a) Complete the table below.

x	0	1	2	3	4	5
$P(X = x)$		$3a$				

Given that $E(X) = 2.75$

(b) Determine the values of a and b.

Find:

(c) $P(0.5 < X < 3)$

(d) $E(2 - X)$

(e) $Var(X)$

(f) $Var(2 - X)$

83. *Calculator Free*

Given the function $f(x) = \dfrac{x^2 + 3}{x - 1}$ calculate the coordinates of the stationary points and their nature using the second derivative.

84. *Calculator Free*

Functions g and p are defined such that:

- $g(2) = 10$
- $g'(2) = 3$
- $p(2) = 25$
- $p'(2) = -4$

(a) If $f(x) = x.g(x)$ determine $f'(2)$

(b) If $h(x) = \dfrac{g(x)}{p(x)}$ determine $h'(2)$

(c) If $m(x) = \sqrt{p(x)}$ calculate $m'(2)$

85. *Calculator Free*

Determine:

(a) $f'(x)$ given $f(x) = \ln[(cos\pi x) + 3\pi^2]$

(b) $\int \frac{e^{2x} + 2}{e^x}\, dx$

(c) $\int_0^{\frac{\pi}{2}} \left[4 - \sin\left(\frac{x}{2}\right)\right] dx$

86. *Calculator Assumed*

When observation commenced the population was 220 000. It grows continuously according to $\frac{dP}{dt} = 0.08P$ where P is the size of the population (in thousands) and t is the number of years after the observation commenced.

(a) Determine an expression for the population, P, in terms of t.

(b) Calculate the length of time it will take for the population to triple in size.

(c) Calculate the rate of change of the population when $P = 500\ 000$.

87. *Calculator Free*

Use the increments formula to obtain an approximation for $g(2.01)$ given $g(2) = 3$ and $g'(x) = (2x - 1)^2$.

88. *Calculator Free*

$f(x) = x^3 + ax^2 + bx + 3$ where a and b are constants has the following features:

- A stationary point at (–2, 7).
- A point of inflection at (–1, 5).

(a) Determine the values of a and b.

(b) Use the second derivative to find the nature of both stationary points.

89. *Calculator Assumed*

The continuous random variable X has the following probability density function.

$$f(x) = \begin{cases} \frac{1}{3}, & 1 \le x \le 4 \\ 0 & \textit{elsewhere} \end{cases}$$

(a) Determine the cumulative distribution function $F(x)$.

(b) Hence calculate $P(X < 2.5)$

Another random variable Y has a cumulative distribution function as follows:

$$F(y) = P(Y \le y) = \begin{cases} 0, & y < 6 \\ \frac{1}{10}(y-6), & 6 \le y \le 16 \\ 1, & y > 16 \end{cases}$$

(c) Determine $f(y)$ the probability distribution function of Y.

(d) Hence determine the value of t where $P(Y \geq t) = \frac{3}{5}$.

90. *Calculator Assumed*

A supermarket chain decides to advertise on the self-serve computer screens as customers scan their items. To measure the effectiveness of these advertisements the company asks their marketing department to survey customers.

150 customers are surveyed as they left the self-serve counters in one supermarket at 5 pm and asked a question related to the advertisement. Of these 78 recalled the advertisement.

(a) State any bias or issues with the survey process.

(b) Calculate an approximate 95% confidence interval for the true proportion of customers who recalled the advertisement.

(c) If the company wants an approximate 95% confidence interval of a width no greater than 0.05, determine the minimum sample size.

91. *Calculator Free*

Given $h(x) = \frac{3}{2}x^4 - 6x^3 + 15$ determine:

(a) how many stationary points exist for $h(x)$ showing clearly your method?

(b) the interval(s) for which $h(x)$ is concave up.

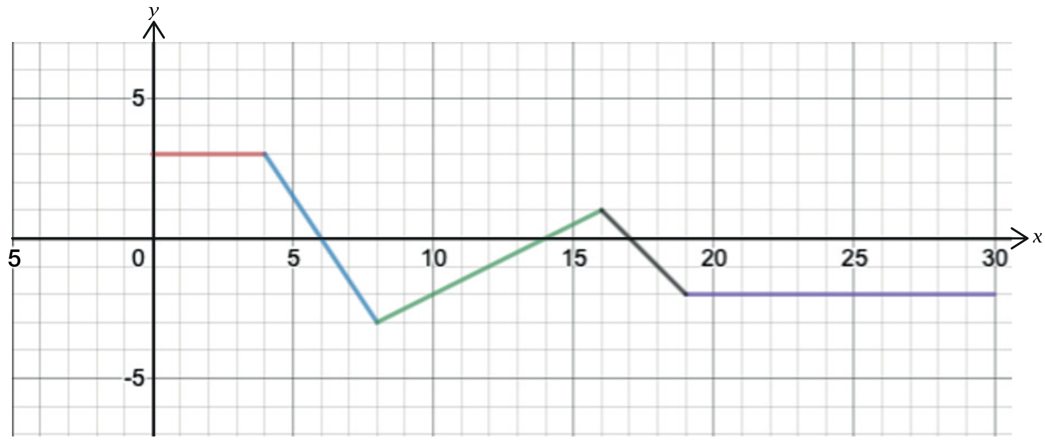
92. *Calculator Free*

Calculate the acceleration (in m/s) of a particle when $t = -1$ given the displacement from a fixed point *0* after t seconds is:

$$x(t) = \frac{2t^2 - 3}{t - 1}$$

93. *Calculator Assumed*

Given the graph of $y = f(x)$ and that $R(x) = \int_0^x f(p)\,dp$ determine the following:



(a) $\int_0^{14} f(x)\,dx$

(b) $\int_6^{14} f(x)\,dx$

(c) $R(8)$

(d) $R'(6)$

(e) Calculate the x intercept of the graph of $y = R(x)$.

94. *Calculator Assumed*

The random variable X has the discrete uniform distribution given as:

$P(X = x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$

(a) Calculate the value of $E(X)$ and $Var(X)$.

Find:

(b) $E(4 - 3X)$

(c) $Var(4 - 3X)$

95. *Calculator Free*

A cumulative distribution function $F(x)$ for a discrete random variable X is defined as follows:

$F(x) = \dfrac{(x+k)^2}{49}$ *for* $x = 1, 2, 3, 4$ where k is a positive integer.

(a) Determine the value of k.

(b) Find the probability distribution of X.

96. *Calculator Free*

A function $f(x)$ has the following criteria:

- $f(-4) = 4$
- $f(-3) = 0$
- $f(0) = -3$
- $f(1) = 0$
- $f(2) = 3$
- $f'(x) < 0$ *for* $-4 \le x < 0$
- $f'(0) = 0$
- $f'(x) > 0$ *for* $0 < x \le 2$

Determine:

(a) $\int_0^2 f'(x)\ dx.$

(b) $\int_{-4}^2 f'(x)\ dx$

(c) The area between the x axis and $f'(x)$ between $x = -4$ and $x = 2$

97. *Calculator Assumed*

The time taken, t, in minutes for 50 people to complete leg 2 of the 'Beach' race is shown in the table below.

Time t (minutes)	$20 \le t < 25$	$25 \le x < 30$	$30 \le x < 35$	$35 \le x < 40$	$40 \le x < 45$	$45 \le x < 50$
Relative frequency	0.02	0.08	0.56	0.22	0.11	0.01

Determine the probability that the next person to race in leg 2 will complete it in:

(a) more than 35 minutes.

(b) less than 45 minutes given they have already taken 30 minutes.

The completion time mean and standard deviation for leg 2 of the race are 34.25 and 4.44 minutes respectively.

(c) Due to unfavourable weather conditions for the race the time was adjusted with a reduction of 15% and an addition of a further 2.5 minutes. Calculate the new mean and standard deviation for completion of leg 2 of the race.

98. *Calculator Assumed*

Given Z is a standard normal distribution with $P(Z \le a) = 0.28$ and $P(Z > b) = 0.38$ determine:

(a) $P(a < Z < b)$

(b) $P(Z < a \mid Z < b)$

(c) $P(a \le Z < 0 \mid Z \le b)$

99. *Calculator Assumed*

In a random sample of 650 people, 125 are members of an AFL team.

(a) Determine the sample proportion of the number of people who are AFL members.

(b) If there are 253 000 people in a city suburb, determine an approximate number of people who are AFL members.

(c) For the proportion of people who are AFL members, construct an approximate margin of error at the 98% confidence interval level.

(d) Construct an approximate 98% confidence level for the proportion of people who are AFL members.

(e) Determine a new sample size if the approximate margin of error at the 98% confidence level is changed to 0.06.

100. *Calculator Assumed*

(a) Given the standard deviation of a Bernoulli distribution is 0.3, determine the mean.

(b) If $E(X) = 2$ and $Var(X) = \frac{4}{5}$ determine the value(s) of p and q given $E(pX + q) = 4$ and $Var(pX + q) = 20$.

101. *Calculator Assumed*

The probability of a bolt being faulty is 0.3. A random sample of 20 bolts are selected.

(a) Describe the distribution indicating all parameters.

Find the probability that there are:

(b) exactly 2 faulty bolts.

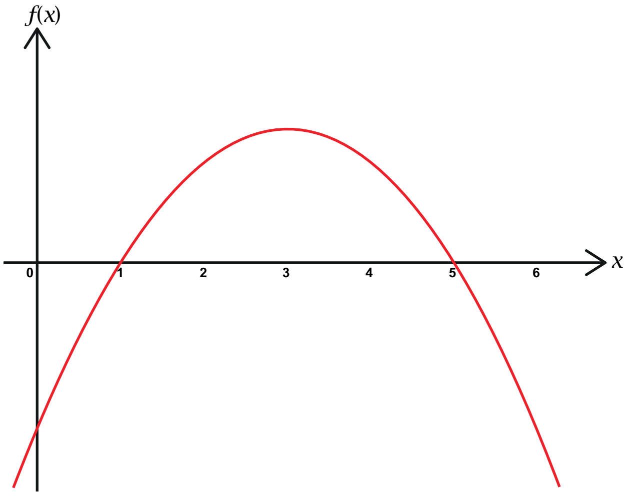
(c) more than 3 faulty bolts.

These bolts are sold in bags of 20. John buys 10 bags.

(d) Find the probability that exactly 6 of these bags contain more than 3 faulty bolts.

102. *Calculator Assumed*

The graph of $y = f(x)$ is shown below.

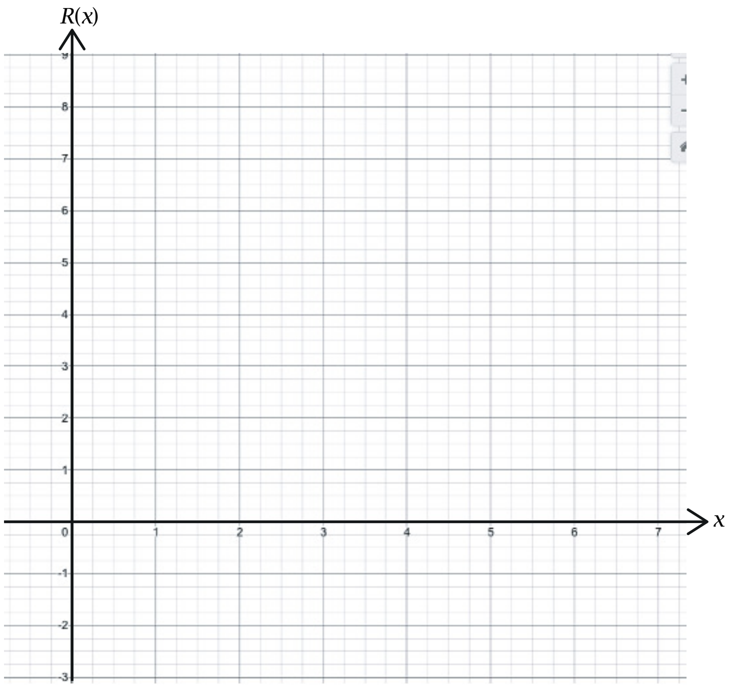


On the axes below graph the function $y = R(x) = \int_0^x f(p)\, dp$ over the domain $0 \le x \le 6$ and include the following features:

- turning point(s).
- x intercept(s)
- point(s) of inflection

$R(x)$ also has the following information:

- $R(1) = -2$
- $R(2) = 0$
- $R(5) = 8$
- $R(6) = 6$



103. *Calculator Assumed*

Michael's Chemistry test results over the past two weeks are recorded in the table below including the class average and variance.

	Michael's mark	Class Average	Class Variance
Test 1	81	83	36
Test 2	85	82	9

In which test did Michael perform better compared to the class? Explain your answer with appropriate calculations.

104. *Calculator Assumed*

Metal cans produced by a company have a capacity, X, mL with a mean of 270 mL and a standard deviation of 2.1 mL and follow a normal distribution model.

(a) State the parameters of this distribution.

(b) Determine:

(i) $P(X \geq 272)$

(ii) $P(X \leq 273 \mid X > 271)$

(iii) k where $P(X < k) = 0.72$

The capacity of the cans changes due to consumer demand. It increases by 33% and an addition of a further 20 *mL*

(c) Calculate the new $E(X)$ and the standard deviation of X.

105. *Calculator Assumed*

A body initially at the origin, O, moves with velocity $v(t) = 3\sin\left(\frac{t}{2} + \frac{\pi}{3}\right)$ *metres/s* where t is time in seconds and $0 \leq t \leq 22$.

(a) Determine the displacement equation of the body at time t.

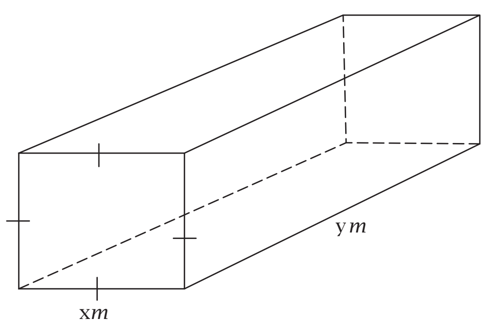
(b) Calculate the displacement of the body from the origin after 14 seconds.

(c) The body is returning to the origin for the final time and is decelerating at 0.8 m/s^2 at a **particular time**.

Calculate the total distance travelled by the body from its initial position to this particular time.

106. *Calculator Assumed*

A rectangular water trough is constructed from metal as follows excluding the top:



The volume of the tank is 2.5 m^3. The cost of the metal is as follows:

- Base costs \$3.60 *per* m^2
- Square ends cost \$2.50 *per* m^2
- Rectangular sides cost \$3.20 *per* m^2

(a) Show that the cost of manufacturing the water trough is:
$C(x) = 5x^2 + \frac{25}{x}$ where C is the cost in dollars.

(b) Use calculus to determine the minimum cost and dimensions of manufacturing the water trough.

107. *Calculator Free*

Given $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 5$ determine the interval(s) for which $f'(x) < 0$ and $f''(x) > 0$

108. *Calculator Assumed*

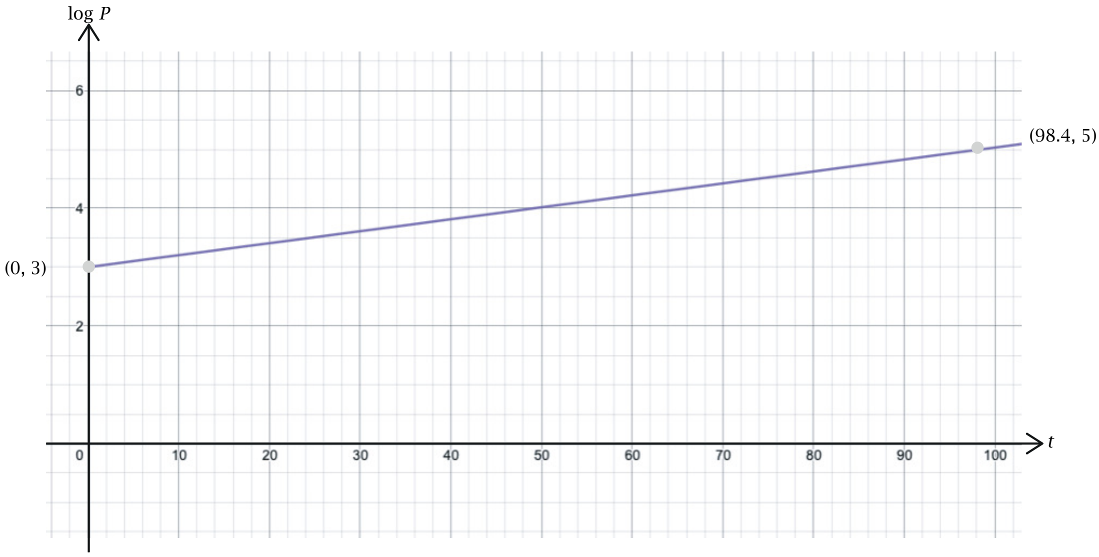
(a) In a random sample it was found that 146 out of 275 people prefer to catch public transport to and from work. The 99% confidence interval for the population proportion is calculated as (0.4534, 0.6084). Show clearly how this confidence interval was determined.

(b) A more recent random sample was taken on whether people prefer catching public transport to and from work. It was found that from another sample of 275 people, 82% prefer to catch public transport. From this sample construct a 99% confidence interval for the proportion of people who prefer to catch public transport to and from work.

(c) A further 10 samples were taken and for each a 99% confidence interval calculated. Determine the probability that at least 9 intervals will contain the true population proportion.

109. *Calculator Assumed*

The size of a growing population, P, after t years is according to the equation $P = ab^t$. The graph below shows the linear relationship between t and $\log P$ through the points (0, 3) and (98.4, 5)



(a) Using $P = ab^t$ **show** that the relationship between t and $\log P$ is linear.

(b) Determine the equation between t and $\log P$.

(c) Calculate the values of a and b in the equation $P = ab^t$ and interpret these values.

(d) When will the population reach 15 000?

(e) Calculate the population after 12 years.

110. *Calculator Assumed*

Statistical information is collected and reveals the number of TV's in each household. The results are shown in the table below:

Number of TV's per household	0	1	2	3 or more
Percentage of households	5	24	57	14

(a) A TV ratings company randomly and independently selects households. Calculate the probability that the first four households selected will contain at least 2 TV's.

The company decides to randomly sample 100 households. Determine the probability that:

(b) exactly 87 households will have at most 2 TV's.

(c) more than 80 households will have at least 2 TV's.

(d) If the random variable Z is the number of households with exactly 1 TV, determine the mean and standard deviation of Z.

111. *Calculator Assumed*

Calculate the area between the following two functions:

- $f(x) = (x - 1)(x + 2)(x + 3)$
- $g(x) = (x + 1)^2 - 4$

112. *Calculator Free*

The table below shows information regarding a polynomial $f(x)$.

No other roots of the function exist except those in the table below.

x	−4	−3	−2	−1	0	1	2	3
$f(x)$	−	0	+	+	+	+	0	+
$f'(x)$	+	0	+	+	0	−	0	+
$f''(x)$	−	0	+	0	−	0	+	+

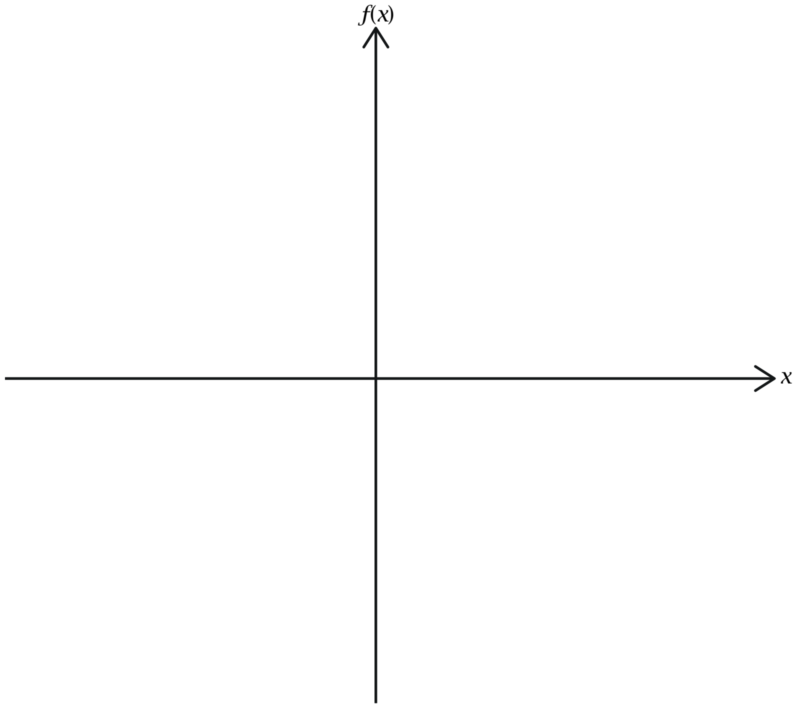
(a) State the x intercepts of $f(x)$.

(b) What value(s) of x does $f(x)$ have a turning point. Explain your reasoning.

(c) Explain what feature is associated at $x = -3$

(d) Determine the value(s) of x for which $f(x)$ is concave down.

(e) Draw a possible representation of $f(x)$ on the axes below.



113. *Calculator Assumed*

A call centre operator waits for phone calls to be answered by customers completing a phone survey. The time taken for the call to be answered, T, in seconds can be modelled by a continuous uniform distribution and occurs between 4 and 30 seconds.

(a) Sketch a graph which represents the probability density function for T.

(b) Determine:

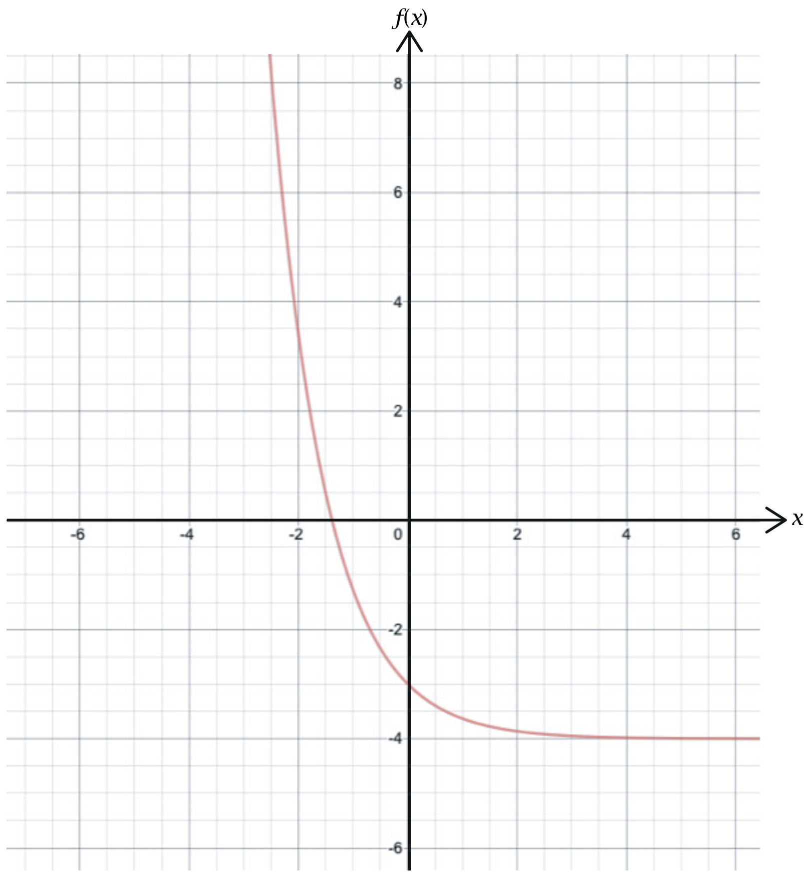
(i) $P(T > 15)$

(ii) $P(T < 25 \mid T \geq 10)$

(iii) k, such that $P(T > k) = 0.45$

114. *Calculator Assumed*

The graph of $f(x) = e^{-x} - 4$ is shown below:



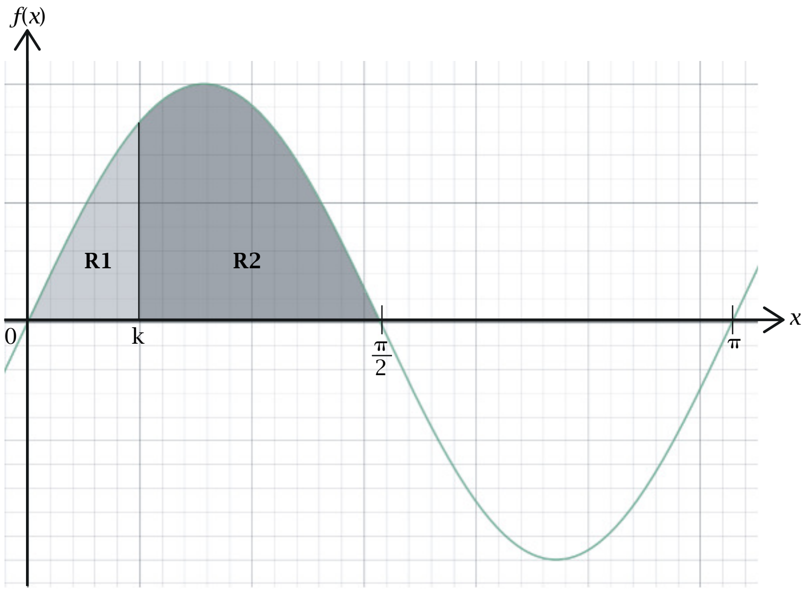
(a) Determine the equation of the tangent to the curve at the y intercept.

(b) Add the line to the graph above.

(c) Calculate the area enclosed between the curve, line and the x and y axes.

115. *Calculator Assumed*

The graph of $f(x) = \sin(2x)$ is shown below.



The area between the curve $y = \sin(2x)$ and the x axis between $x = 0$ and $x = \frac{\pi}{2}$ is divided into 2 regions. Calculate the value of k such that the areas $\frac{R_1}{R_2} = \frac{1}{3}$.

116. *Calculator Assumed*

A biased coin is tossed four times and the number of heads recorded. This is then repeated 140 times. Let X be the random variable of the number of heads in four tosses of the coin. The results are shown in the table below.

Number of heads	0	1	2	3	4
Frequency	5	19	34	51	31

(a) Describe the distribution.

(b) Calculate the average number of heads over the 140 tosses.

(c) Determine the probability of obtaining a head when the coin is tossed.

117. *Calculator Assumed*

The acceleration of a body moving in a straight line at time t seconds where $t \geq 0$ is given by:

$$a = -6t + 20 \ m/s^2$$

Given the body is initially at the origin and stationary determine:

(a) the velocity equation.

(b) the displacement equation.

(c) the maximum velocity and when this occurs.

(d) the change in the displacement of the body during the fourth second.

(e) the total distance travelled in the first 8 seconds.

(f) the velocity of the body when it's at the origin for the second time.

118. *Calculator Assumed*

A wildlife researcher in a protected forest area randomly sets traps to monitor the 'small mammal' population of which it is thought 56% are 'Western Quolls'. Traps are set each day that catch 12 small mammals to monitor the population.

Let Q be the random variable of the number of Western Quolls caught each day.

(a) Describe the distribution of Q and include all parameters.

(b) How many times in 3 weeks would the daily traps contain less 'Western Quolls' than any other small mammal?

(c) Determine the probability that over 2 consecutive days, the total number of 'Western Quolls' caught in the traps was at least 10.

119. *Calculator Assumed*

A function is defined as $f(x) = -\cos(2x)\ \ln(x + 2) \quad 0 \le x \le 3$

(a) Initially is the function increasing or decreasing. Justify your answer.

(b) Using the incremental formula find an approximate change in $f(x)$ when x changes from $\frac{11\pi}{12}$ to $\frac{19\pi}{20}$.

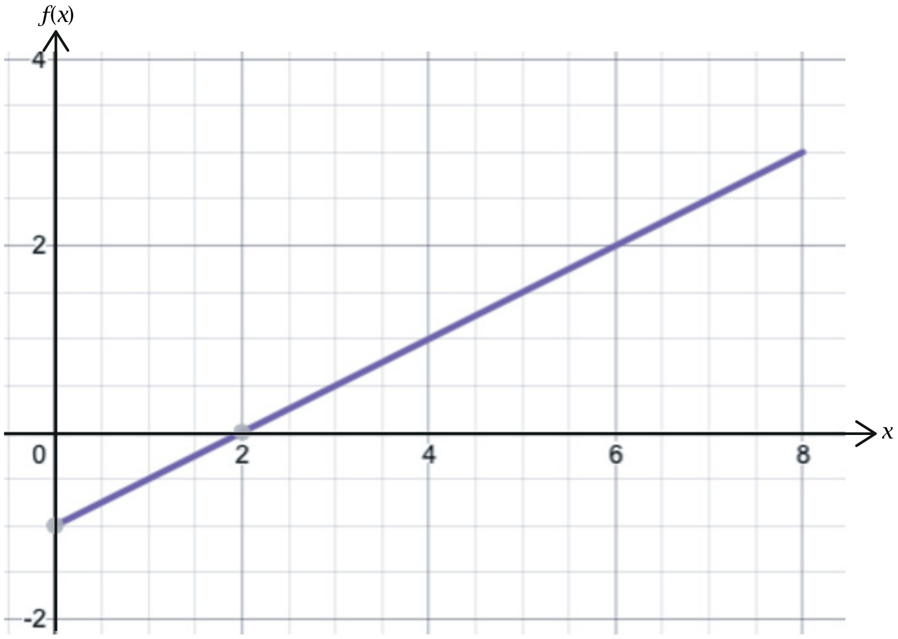
(c) Calculate when the instantaneous rate of change of $f(x)$ first starts to decrease.

120. *Calculator Free*

A curve with equation $y = 2x^3 - x^2 + 15$ passes through a point with an x value of -2. Use the increment formula to determine the approximate y coordinate of another point with an x value of -2.01.

121. *Calculator Free*

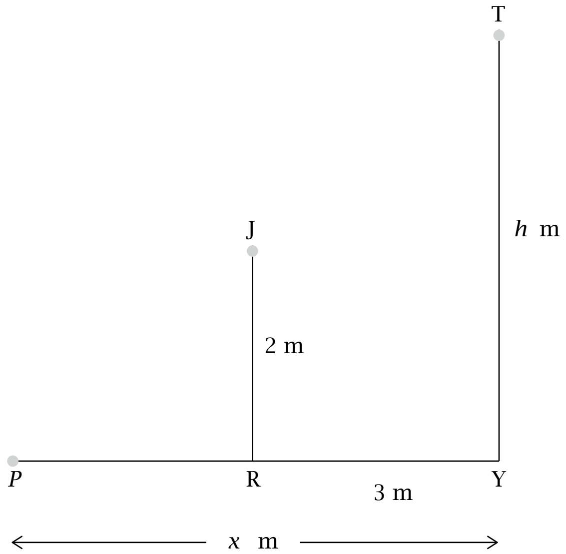
Given the graph of $y = f(x)$ over the domain $0 \le x \le 8$.



Draw the graph of $y = R(x) = \int_0^x f(p)\, dp$ over the interval $0 \le x \le 8$.

122. *Calculator Assumed*

A tree (T) h metres high is x metres from a peg (P) in the ground. Jack (J) 2 metres tall is positioned between the peg and 3 metres from the base of the tree. The diagram illustrates this information below.



(a) Show that the area of $\Delta\ PTY = \dfrac{x^2}{x-3}$

(b) Use calculus to find the minimum value of x that minimises the area of $\Delta\ PTY$ and hence find all dimensions and the area.

123. *Calculator Assumed*

At a school fete, attendees each receive 20 tokens that can be used to purchase food and play games.

One particular game is the money wheel. The circular wheel is divided into 30 equal sections numbered 1 to 30. The cost to play this game is two tokens and this is collected by a staff member in charge of this game.

The wheel is spun, and the indicator will point to a number when the wheel stops spinning. The outcome of this game is shown below:

- If the number is a 'multiple of 3' – **3 tokens are won**.
- If the number is a 'multiple of 5' – **5 tokens are won**.
- If the number is a 'multiple of 12' – **12 tokens are won**.
- If the number is <u>not</u> a 'multiple of 3, 5 or 12'– **no tokens are won**.

The random variable X is the number of tokens won playing the money wheel.

(a) Construct a probability distribution.

(b) Determine $E(X)$.

(c) The staff member spins the money wheel 300 times during the school fete. Will she win or lose tokens? Show working.

124. *Calculator Free*

Given $f(x) = \ln(ex) - kx$ where $k > 0$ determine:

(a) the coordinates of the stationary point(s) in terms of k.

(b) the nature of the stationary point(s) using the second derivative.

(c) $\int_1^3 \left(-k + \frac{1}{x}\right) dx$

125. *Calculator Assumed*

Adam represents his school in the 100 m freestyle swimming event. His training schedule finds that his times to complete the race are normally distributed with a mean of 58 seconds and a standard deviation of 7 seconds.

(a) Find the probability that Adam swims the 100 m freestyle in:

(i) exactly 55 seconds.

(ii) more than 60 seconds.

(iii) more than 56 seconds and less than 59 seconds.

(iv) at most 61 seconds given that he swims at least 55 seconds.

(b) Adam trains hard and swims five 100 m freestyle events. Determine the probability:

(i) the first two 100 m freestyle events were less than 55 seconds and the next three were more than 60 seconds.

(ii) that at least two of his swims were more than 60 seconds.

(c) What range of Adam's 100 m freestyle events would you expect that 95% of his times lie?

126. *Calculator Assumed*

$\hat{p}$ represents the sample proportion and p represents the true proportion. From a random sample of 110 people, 30 still own a video recorder.

(a) Calculate $\hat{p}$.

(b) Determine the standard error of the sample proportion.

(c) Calculate the approximate margin of error for a 90% confidence interval for p.

(d) Construct an approximate 90% confidence level for p.

127. *Calculator Assumed*

A card game consists of a pack of 52 cards each numbered either a 0 or 1. Twenty of the cards are numbered with a 0.

Let X be the random variable of the number on a card randomly selected from the pack of cards.

(a) Why is this a Bernoulli distribution? Give a reason for your answer.

(b) Calculate the mean and variance of the distribution X.

128. *Calculator Assumed*

The graph below illustrates the velocity of a body $v(t)$ m/s over the interval $0 \le t \le 15$ where t is the time in seconds.

(a) Determine the displacement of the body at

(i) $t = 7$

(ii) $t = 14$

(b) Determine the time(s) the body has a maximum displacement and state this displacement.

129. *Calculator Assumed*

The proportion, p, of young adults prefer cola as their choice of soft drink. If a 95% confidence interval is obtained with a margin of error of no more than 8%, determine an appropriate sample size.

130. *Calculator Assumed*

A company manufactures batteries with a lifetime which is normally distributed with a mean of 1450 hours and a standard deviation of 15 hours.

The company decides to improve the lifetime of the batteries. It is found that 45% of the batteries will last more than 1480 hours and 25% will last less than 1465 hours. Determine the new mean and standard deviation lifetime of the batteries.

131. *Calculator Assumed*

John operates a new game at the showground called 'Loto' consisting of 8 balls numbered as follows: 1 ,1, 2, 2, 3, 3, 4, 4 where two balls are drawn from the barrel, without replacement.

Each game costs $4 to play. Prizes are awarded according to the following criteria:

- A total of 5 from the two balls drawn wins $7.
- A total of 8 from the two balls drawn wins $14.

No other prizes are awarded and the daily fixed costs to John in running the game is $25.

How many games must be played per day for John to make a profit each day of $100?

132. *Calculator Assumed*

The rate at which a bottle is filled with water from a dripping tap is given by:
$\frac{dV}{dt} = \frac{1}{2}\sin\left(\frac{\pi t}{3}\right) + 4$, where $0 \le t \le 4$ where V is the volume in cm^3 and t the time in minutes .

(a) Determine the volume, V, of water in the bottle after t minutes.

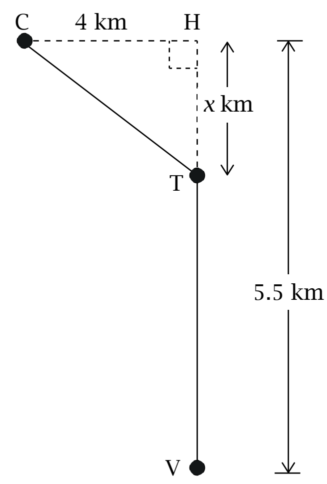
(b) Hence, calculate the amount of water in the bottle after 4 minutes.

(c) How long will it take to fill if the bottle has a capacity of 150 mL?

133. *Calculator Assumed*

Cabling is to be laid from offshore at point C to point T onshore. Point C is 4 km from point H located on the shore. The cabling is then to be laid from point T along the shore to point V located 5.5 km from H. The cost of the underwater cable is \$12 500 per km and along the shore is \$10 000 per km.

The diagram below illustrates this information.

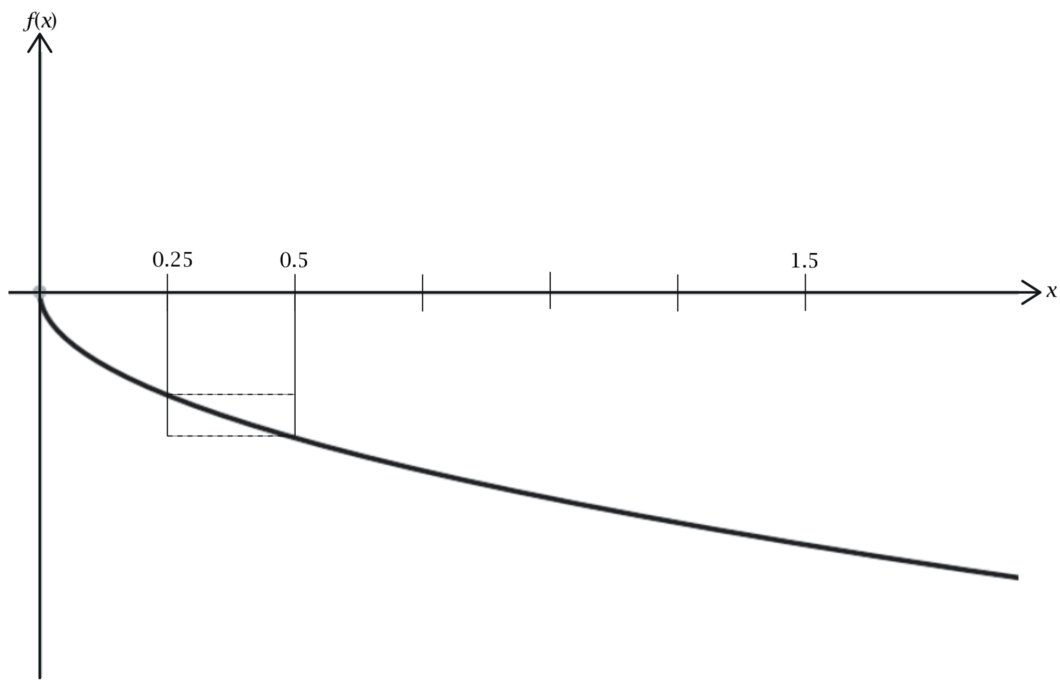


(a) Given point T is x km from point H, show that the cost of the cabling is $55\,000 - 10\,000x + 12\,500\sqrt{16 + x^2}$.

(b) Use calculus to determine the value of x which minimises the cost and hence state this minimum cost.

134. *Calculator Assumed*

The graph of $y = f(x)$ is shown below.



A table of values for the above graph is as follows:

x	0.25	0.5	0.75	1	1.25
$f(x)$	–0.5	–0.71	–0.87	–1	–1.12

Using the table of values and the sum of the rectangles based on $\Sigma_i f(x_i)\delta x_i$ where $\delta x = 0.25$ determine over the interval $0.25 \le x \le 1.25$.

(a) An overestimate for the area of the function.

(b) An underestimate for the area of the function.

(c) A numerical approximation for the area.

(d) Verify whether the approximation for the area is smaller or larger than the actual area. Give a reason for your answer.

(e) How can the approximation be improved? Explain.

135. *Calculator Free*

The volume of a particular container is given by $V = \dfrac{h^2 + 1}{h + 1}$ where V is the volume in cm^3 and h is the height in cm.

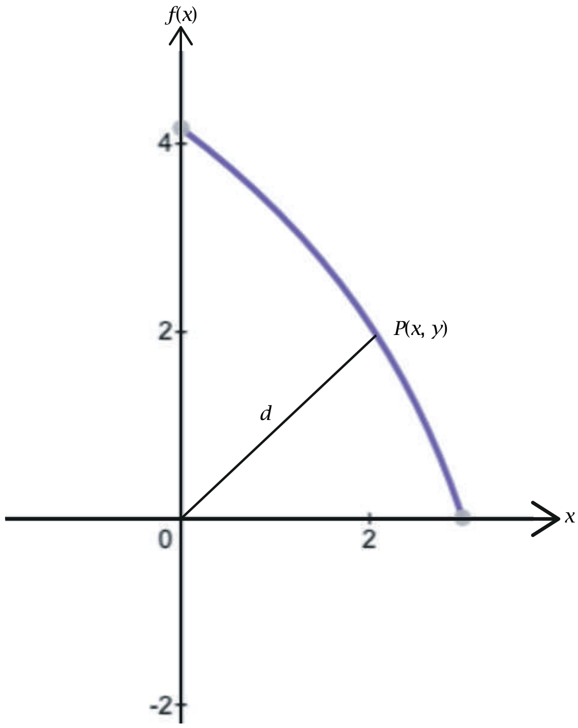
(a) Determine the volume given $h = 6$ cm.

(b) Determine $V(h + \delta h)$ given $h = 6$ cm and $\delta h = 1$.

(c) Use the increments formula to find the approximate small change in V between the $V = \frac{10}{4}\ cm^3$ and $V = \frac{17}{5}\ cm^3$.

136. *Calculator Assumed*

The graph of $f(x) = 3\ln(4 - x)$ is shown below.



(a) Show that the distance from 0 to P is given by:

$$d = \sqrt{(9[\ln(-x + 4)]^2 + x^2}$$

(b) Use calculus to find the minimum distance from *O* to *P*.

137. *Calculator Assumed*

A game is designed using a six numbered fair dice where a player will either win or lose money.

The amount won or lost is as follows:

- A one is thrown – the player loses $\$x$.
- A two or a three are thrown – the player wins \$5.
- A four is thrown – the player wins \$2.
- A five or six is thrown – the player wins \$10.

(a) The random variable X is the player winnings. Complete the probability distribution table below:

x				
$P(X = x)$				

(b) In terms of x, state the $E(X)$.

(c) Determine the value of x, for which the player will make a profit?

138. *Calculator Assumed*

A body has a velocity equation $v(t) = 5\cos\left(\frac{\pi t}{12}\right)$ *m/s* where $0 \le t \le 24$.

The body is at the origin when $t = 4$ seconds.

(a) State the equation for the displacement $x(t)$ of the body at t seconds.

(b) Calculate the acceleration when the body has a displacement of 2 m.

139. *Calculator Assumed*

An egg board classifies eggs according to weight W, in grams. These are:

Small size (s)

Medium size (m)

Large size (l)

Extra-large size (x)

Hens lay eggs in the ratio of $s : m : l : x$ as $1 : 2 : 4 : 3$.

Weights of eggs are normally distributed with a mean of 59.5 grams and a standard deviation of 2.7 grams.

(a) Find the probability that an egg randomly chosen is more than 62 grams and less than 65 grams.

(b) Determine the lower and upper bound weights of each size of egg.

(c) Calculate the probability that the first 3 eggs randomly chosen are medium size followed by an extra-large egg.

140. *Calculator Assumed*

The cost, C, in dollars to hire a car is directly proportional to the cube of the number of hired days, d. The cost is \$750 when the car is hired for 10 days with an additional once off fixed cost of \$1 093.50 for insurance.

(a) Show that the total cost T of hiring the car per day d is given by by $T = \frac{3d^2}{4} + \frac{1093.5}{d}$

(b) Determine by using calculus the minimum number of days to hire the car for the cost per day to be a minimum.

141. *Calculator Assumed*

It is known that 62% of a large population are in favour of a city bypass. After a significant large number of random samples of size 225, sample proportions are calculated.

(a) What type of distribution will be approximated by these sample proportions? Explain.

(b) State the expected mean and standard deviation for this distribution.

(c) Using this distribution, calculate the probability that the sample proportion is more than 60% in a randomly chosen sample.

142. *Calculator Assumed*

A game involves a bag containing 4 red marbles and 2 blue marbles. Two marbles are drawn from the bag without replacement. The random variable M is the number of blue marbles drawn from the bag.

(a) Complete a probability distribution table.

(b) Find the expected value and variance of M.

A game costs 50 cents to play with the following prizes awarded:

- Nothing if no blue marbles are drawn.
- \$1 for drawing 1 blue marble.
- \$10 for drawing both blue marbles.

If one game is played, P is the random variable on the profit of this game.

(c) Calculate $E(P)$.

143. *Calculator Free*

(a) If the confidence level remains the same and the sample size is decreased comment on the change to the width of the confidence interval.

(b) A random sample of 500 light switches found that 24 were defective. The 95% confidence interval is (0.0293, 0.0667). This was interpreted as:

"There is a 95% chance that the true proportion of defective light switches lies between 0.0293 and 0.0667."

Comment on the interpretation and correct if necessary.

A

ANSWERS

CHAPTER 1: *Differentiation*

1. (a) $(a, 0)$, $(c, 0)$

   (b) $(a, 0)$, (b, d)

   (c) i. $x < a$ and $x > b$

   ii. $a < x < b$

2. $f(x) = 2x^3$

$$f(x+h) = 2(x+h)^3$$
$$= 2(x^3 + 3x^2h + 3xh^2 + h^3)$$
$$= 2x^3 + 6x^2h + 6xh^2 + 2h^3$$
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 - \cancel{2x^3}}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2)}{\cancel{h}}$$
$$= 6x^2$$

3. (a) $y = 3x^2$
$$\frac{dy}{dx} = 6x$$

(b) $y = \frac{7}{x}$
$$y = 7x^{-1}$$
$$\frac{dy}{dx} = -7x^{-2}$$
$$\frac{dy}{dx} = \frac{-7}{x^2}$$

(c) $y = 4x$
$$\frac{dy}{dx} = 4$$

(d) $y = 2$
$$\frac{dy}{dx} = 0$$

(e) $y = -5x^3$
$$\frac{dy}{dx} = -15x^2$$

(f) $y = \frac{1}{\sqrt{x}}$
$$y = \frac{1}{x^{\frac{1}{2}}}$$
$$y = x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$
$$\frac{dy}{dx} = -\frac{1}{2x^{\frac{3}{2}}}$$

(g) $y = \frac{2}{5x^2}$
$$y = \frac{2x^{-2}}{5}$$
$$\frac{dy}{dx} = \frac{-4x^{-3}}{5}$$
$$\frac{dy}{dx} = \frac{-4}{5x^3}$$

(h) $y = -\frac{6}{\sqrt[3]{x}}$
$$y = \frac{-6}{x^{\frac{1}{3}}}$$
$$y = -6x^{-\frac{1}{3}}$$
$$\frac{dy}{dx} = 2x^{-\frac{4}{3}}$$
$$\frac{dy}{dx} = \frac{2}{x^{\frac{4}{3}}}$$

(i) $y = x^{-\frac{3}{4}}$
$$\frac{dy}{dx} = \frac{-3}{4}x^{-\frac{7}{4}}$$
$$\frac{dy}{dx} = \frac{-3}{4x^{\frac{7}{4}}}$$

(j) $y = 7\sqrt{x}$
$$y = 7x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{7}{2x^{\frac{1}{2}}}$$

4. (a) $y = 4x^3$
$$\frac{dy}{dx} = 12x^2$$
$$\left.\frac{dy}{dx}\right|_{x=-1} = 12$$

(b) $y = \frac{2}{x}$

$y = 2x^{-1}$

$\frac{dy}{dx} = -2x^{-2}$

$\left.\frac{dy}{dx}\right|_{x=2} = -\frac{1}{2}$

(c) $y = \sqrt{x}$

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2x^{\frac{1}{2}}}$

$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{4}$

(d) $y = \frac{-3}{\sqrt[3]{x}}$

$y = -3x^{-\frac{1}{3}}$

$\frac{dy}{dx} = x^{-\frac{4}{3}}$

$\frac{dy}{dx} = \frac{1}{x^{\frac{4}{3}}}$

$\left.\frac{dy}{dx}\right|_{x=27} = \frac{1}{81}$

5. (a) $y = 4x + 5$

$\frac{dy}{dx} = 4$

(b) $y = 3x^2 - 6x$

$\frac{dy}{dx} = 6x - 6$

(c) $y = 5x^3 + 9x^2 - 6$

$\frac{dy}{dx} = 15x^2 + 18x$

(d) $y = \frac{x^5}{2} + 7x^4 - x$

$\frac{dy}{dx} = \frac{5x^4}{2} + 28x^3 - 1$

(e) $y = \sqrt{x} + x$

$y = x^{\frac{1}{2}} + x$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 1$

$= \frac{1}{2\sqrt{x}} + 1$

(f) $y = x - \frac{3}{x}$

$y = x - 3x^{-1}$

$\frac{dy}{dx} = 1 + 3x^{-2}$

$\frac{dy}{dx} = 1 + \frac{3}{x^2}$

(g) $y = 3\sqrt{x} + \frac{1}{\sqrt{x}}$

$y = 3x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{3}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

(h) $y = 2x^{\frac{5}{2}} - 4x^{\frac{7}{2}} + 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = 5x^{\frac{3}{2}} - 14x^{\frac{5}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 5x^{\frac{3}{2}} - 14x^{\frac{5}{2}} + \frac{1}{\sqrt{x}}$

6. $y = x^2 - 3x + 2$

Gradient of tangent $y' = 2x - 3$

Where the curve cuts the x axis

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2,\ x = 1$

At $x = 1$ $\quad y' = -1$

$x = 2$ $\quad y' = 1$

7. $y = 9x - 3x^3$

$y = -1$ has a gradient of 0

$\therefore$ parallel tangent has $m = 0$ $\quad y' = 0$

$\therefore 9 - 9x^2 = 0$

$9x^2 = 9$

$x^2 = 1$

$x = \pm 1$

When $x = 1$ $\quad y = 6$

$x = -1$ $\quad y = -6$

Coordinates are (1, 6) and (–1, –6)

8. $y = kx^2 - 7x + 6$

$y' = 2kx - 7$

when $x = 3$ $\quad y' = 11$

$\therefore 11 = 2k(3) - 7$

$18 = 6k$

$3 = k$

9. (a) $v = 5t^3 + 4t$

$\frac{dv}{dt} = 15t^2 + 4$

(b) $c = 5m + 4m^{-2}$

$\frac{dc}{dm} = 5 - 8m^{-3}$

$\frac{dc}{dm} = 5 - \frac{8}{m^3}$

(c)
$$f = g^{-\frac{1}{4}}$$
$$\frac{df}{dg} = -\frac{1}{4} g^{-\frac{5}{4}}$$
$$\frac{df}{dg} = -\frac{1}{4g^{\frac{5}{4}}}$$

(d)
$$a = \frac{5b^{-4}}{12}$$
$$\frac{da}{db} = -\frac{20b^{-5}}{12}$$
$$\frac{da}{db} = -\frac{5}{3b^5}$$

10. $m = 5v \qquad v = 3h^2 - 2 \qquad h = 2x^3$
$$\frac{dm}{dv} = 5 \qquad \frac{dv}{dh} = 6h \qquad \frac{dh}{dx} = 6x^2$$
$$\therefore \frac{dm}{dx} = \frac{dm}{dv} \cdot \frac{dv}{dh} \cdot \frac{dh}{dx}$$
$$= (5)(6h)(6x^2)$$
$$= (5)(12x^3)(6x^2)$$
$$= 360x^5$$

11. (a)
$$y = (3x - 2)^2$$
$$\frac{dy}{dx} = 2(3x - 2)(3)$$
$$\frac{dy}{dx} = 6(3x - 2)$$

(b) $y = (4x - 5)^{\frac{1}{2}}$
$$= \frac{1}{2}(4x - 5)^{-\frac{1}{2}}(4)$$
$$= 2(4x - 5)^{-\frac{1}{2}}$$
$$= \frac{2}{\sqrt{(4x - 5)}}$$

(c)
$$y = (4x^2 + 2)^{-1}$$
$$\frac{dy}{dx} = -(4x^2 + 2)^{-2}(8x)$$
$$\frac{dy}{dx} = \frac{-8x}{(4x^2 + 2)^2}$$

(d)
$$y = 3(2x^2 - 5)^{-1}$$
$$\frac{dy}{dx} = -3(2x^2 - 5)^{-2}(4x)$$
$$\frac{dy}{dx} = \frac{-12x}{(2x^2 - 5)^2}$$

(e)
$$y = \frac{1}{2 + \sqrt{x}}$$
$$y = (2 + x^{\frac{1}{2}})^{-1}$$
$$\frac{dy}{dx} = -1(2 + x^{\frac{1}{2}})^{-2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}(2 + \sqrt{x})^2}$$

(f)
$$y = (x^2 + 1)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x)$$
$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}}$$

12. (a)
$$y = x^3 + 2x^2$$
$$\frac{dy}{dx} = 3x^2 + 4x$$
$$\left.\frac{dy}{dx}\right|_{x=2} = 20$$

(b)
$$y = \sqrt{1 - 3x}$$
$$y = (1 - 3x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(1 - 3x)^{-\frac{1}{2}}(-3)$$
$$\left.\frac{dy}{dx}\right|_{x=0} = -1.5$$

(c)
$$y = -\frac{4}{x}$$
$$y = -4x^{-1}$$
$$\frac{dy}{dx} = 4x^{-2}$$
$$\left.\frac{dy}{dx}\right|_{x=2} = 1$$

(d)
$$y = (x^2 + 2)^2$$
$$\frac{dy}{dx} = 2(x^2 + 2)(2x)$$
$$\left.\frac{dy}{dx}\right|_{x=1} = 12$$

(e)
$$y = \frac{1}{(3x + 1)^2}$$
$$y = (3x + 1)^{-2}$$
$$\frac{dy}{dx} = -2(3x + 1)^{-3}(3)$$
$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{3}{32}$$

13. (a)
$$y = (4x + 2)(2x - 6)$$
$$\frac{dy}{dx} = (4)(2x - 6) + (4x + 2)(2)$$
$$\frac{dy}{dx} = 8x - 24 + 8x + 4$$
$$\frac{dy}{dx} = 16x - 20$$

(b) $y = (3x^2 - 5)(5x + 1)$

$$\frac{dy}{dx} = (6x)(5x + 1) + (3x^2 - 5)(5)$$

$$\frac{dy}{dx} = 30x^2 + 6x + 15x^2 - 25$$

$$\frac{dy}{dx} = 45x^2 + 6x - 25$$

(c) $y = (2x^2 + x)(7x - 1)$

$$\frac{dy}{dx} = (4x + 1)(7x - 1) + (2x^2 + x)(7)$$

$$\frac{dy}{dx} = 28x^2 + 3x - 1 + 14x^2 + 7x$$

$$\frac{dy}{dx} = 42x^2 + 10x - 1$$

(d) $y = x(4x + 1)^2$

$$\frac{dy}{dx} = (1)(4x + 1)^2 + (x)(2)(4x + 1)(4)$$

$$\frac{dy}{dx} = 16x^2 + 8x + 1 + 32x^2 + 8x$$

$$\frac{dy}{dx} = 48x^2 + 16x + 1$$

(e) $y = (x - 1)^3 (3x + 2)^2$

$$\frac{dy}{dx} = 3(x - 1)^2 (3x + 2)^2 + (x - 1)^3 (2)(3x + 2)(3)$$

or

$$\frac{dy}{dx} = 3(x - 1)^2 (3x + 2)^2 + (18x + 12)(x - 1)^3$$

(f) $y = x^{\frac{1}{2}} (2x^2 - 1)^3$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} (2x^2 - 1)^3 + (x^{\frac{1}{2}})(3)(2x^2 - 1)^2 (4x)$$

or

$$\frac{dy}{dx} = \frac{(2x^2 - 1)^3}{2\sqrt{x}} + (12x)(\sqrt{x})(2x^2 - 1)^2$$

14. (a) $y = \dfrac{7x}{2x + 1}$

$$\frac{dy}{dx} = \frac{7(2x + 1) - (7x)(2)}{(2x + 1)^2}$$

(b) $y = \dfrac{x + 4}{x - 6}$

$$\frac{dy}{dx} = \frac{(x - 6) - (x + 4)}{(x - 6)^2}$$

(c) $y = \dfrac{3x^2}{7 - x}$

$$\frac{dy}{dx} = \frac{6x(7 - x) - 3x^2 (-1)}{(7 - x)^2}$$

(d) $y = \dfrac{5x^3}{(x^2 + 4)}$

$$\frac{dy}{dx} = \frac{15x^2 (x^2 + 4) - 5x^3 (2x)}{(x^2 + 4)^2}$$

(e) $y = \dfrac{(x + 3)^2}{(2x - 5)^3}$

$$\frac{dy}{dx} = \frac{2(x + 3)(2x - 5)^3 - (x + 3)^2 (3)(2x - 5)^2 (2)}{\left[(2x - 5)^3\right]^2}$$

(f) $y = \dfrac{x^{\frac{1}{2}}}{(3x - 5)}$

$$\frac{dy}{dx} = \frac{\frac{1}{2} x^{-\frac{1}{2}} (3x - 5) - x^{\frac{1}{2}} (3)}{(3x - 5)^2}$$

15. (a) $y = 4x^3 - 5x^2$

$$\frac{dy}{dx} = 12x^2 - 10x$$

$$\frac{d^2y}{dx^2} = 24x - 10$$

(b) $y = (2x - 9)^4$

$$\frac{dy}{dx} = 4(2x - 9)^3 (2)$$

$$\frac{dy}{dx} = 8(2x - 9)^3$$

$$\frac{d^2y}{dx^2} = 24(2x - 9)^2 (2)$$

$$\frac{d^2y}{dx^2} = 48(2x - 9)^2$$

(c) $y = 3x^2 - \sqrt{x}$

$$\frac{dy}{dx} = 6x - \frac{1}{2} x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 6 + \frac{1}{4} x^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = 6 + \frac{1}{4x^{\frac{3}{2}}}$$

(d) $y = \dfrac{x^4}{8} - \dfrac{x^3}{3} + 2x^2 - 6x$

$$\frac{dy}{dx} = \frac{{}^{1}\cancel{4}\, x^3}{\cancel{8}_{2}} - \frac{\cancel{3}\, x^2}{\cancel{3}} + 4x - 6$$

$$\frac{d^2y}{dx^2} = \frac{3x^2}{2} - 2x + 4$$

(e) $y = \dfrac{1}{\sqrt{x^2 - 9}}$

$$y = \frac{1}{\left(x^2 - 9\right)^{\frac{1}{2}}}$$

$$y = \left(x^2 - 9\right)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}\left(x^2 - 9\right)^{-\frac{3}{2}} \bullet 2x$$

$$\frac{dy}{dx} = -x\left(x^2 - 9\right)^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \left(-1\right)\left(x^2 - 9\right)^{-\frac{3}{2}} + \left(-x\right)\left(-\frac{3}{2}\right)\left(x^2 - 9\right)^{-\frac{5}{2}}\left(2x\right)$$

$$= -\frac{1}{\left(x^2 - 9\right)^{\frac{3}{2}}} + \frac{3x^2}{\left(x^2 - 9\right)^{\frac{5}{2}}}$$

(f) $y = \frac{x-3}{3}$

$$\frac{dy}{dx} = \frac{(1)(x)-(x-3)(1)}{x^2}$$
$$\frac{dy}{dx} = \frac{x-x+3}{x^2}$$
$$\frac{dy}{dx} = \frac{3}{x^2}$$
$$\frac{d^2y}{dx^2} = -\frac{6}{x^3}$$

16. $y = 3(2x-5)^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{9}{2}(2x-5)^{\frac{1}{2}} \bullet 2$$
$$\frac{dy}{dx} = 9(2x-5)^{\frac{1}{2}}$$
$$\left.\frac{dy}{dx}\right|_{x=3} = 9$$

17.
$$y = \left(\sqrt{(3x+1)^2} - \frac{12}{\sqrt[3]{(3x+1)^2}}\right)^2$$

y intercept when $x = 0$

$$\left.\frac{dy}{dx}\right|_{x=0} = -594$$

18. $t = 6v$ $\quad v = \frac{1}{p}$ $\quad p = 4x^3$

$\frac{dt}{dv} = 6$ $\quad \frac{dv}{dp} = -\frac{1}{p^2}$ $\quad \frac{dp}{dx} = 12x^2$

$$\frac{dt}{dx} = \frac{dt}{dv} \bullet \frac{dv}{dp} \bullet \frac{dp}{dx}$$
$$= (6)\left(-\frac{1}{p^2}\right)(12x^2)$$
$$= (6)\left(-\frac{1}{(4x^3)^2}\right)(12x^2)$$
$$= -\frac{72x^2}{16x^6}$$
$$= -\frac{9}{2x^4}$$

19. $f'(x) = 2(2x+k)(2)$

$= 4(2x+k)$

if $f'(1) = 16$

then $16 = 4(2(1)+k)$

$4 = 2 + k$

$2 = k$

20. $c = 5$

$\therefore y = x^3 - ax^2 + bx + 5$

$y' = 3x^2 - 2ax + b$

when $x = 0$, $y' = 6$

$6 = 3(0)^2 - 2a(0) + b$

$\therefore b = 6$

Sub in $(2, 13)$

$y = x^3 - ax^2 + 6x + 5$

$13 = 8 - 4a + 12 + 5$

$4a = 12$

$a = 3$

21. $y = (2x+1)^2(cx+d)$

$9 = (3)^2(c+d)$

$c + d = 1$ ①

$y = (2x+1)^2(cx+d)$

$y' = 2(2x+1)(2)(cx+d) + (2x+1)^2(c)$

$3 = 2(3)(2)(c+d) + 9c$

$3 = 12c + 12d + 9c$

$3 = 21c + 12d$ ②

Solve the equations ① and ② simultaneously

$c = -1$

$d = 2$

22. $y = \sqrt{x+5}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+5}}$$
$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{1}{4}$$

Coordinate $(-1, 2)$
Equation of tangent

$$y = \frac{1}{4}x + c$$
$$2 = \frac{1}{4}(-1) + c$$
$$c = \frac{9}{4}$$
$$y = \frac{1}{4}x + \frac{9}{4}$$

23. $$y=\frac{6}{\left(3x+1\right)^2}$$
$$y=6\left(3x+1\right)^{-2}$$
$$\frac{dy}{dx}=-12\left(3x+1\right)^{-3}\bullet 3$$
$$-2=-36\left(3x+1\right)^{-3}$$
$$x=0.5402$$
$$y=0.8737$$

(0.5402, 0.8737)

24. $y=3f$ $\quad f=2g^2+4$ $\quad g=6x^2$

$$\frac{dy}{df}=3 \qquad \frac{df}{dg}=4g \qquad \frac{dg}{dx}=12x$$
$$\frac{dy}{dx}=\frac{dy}{df}\bullet\frac{df}{dg}\bullet\frac{dg}{dx}$$
$$=\left(3\right)\left(4g\right)\left(12x\right)$$
$$=\left(3\right)\left(4\left(6x^2\right)\right)\left(12x\right)$$
$$=864x^3$$

25. $f(x)=x^3+ax^2+6x+b$
$f'(x)=3x^2+2ax+6$
$f'(0)=6$
$f'(3)=3(3)^2+2a(3)+6$
$\quad =27+6a+6$
$\therefore\ 6=27+6a+6$
$6a=-27$
$\underline{a=-4.5}$
$f(x)=x^3-4.5x^2+6x+b$
$4=(3)^3-4.5(3)^2+6(3)+b$
$4=27-40.5+18+b$
$\underline{b=-0.5}$

26. $$y=\frac{2x+3}{x-3}$$
$$\frac{dy}{dx}=-\frac{9}{\left(x-3\right)^2}$$
$$\frac{d^2y}{dx^2}=\frac{18}{\left(x-3\right)^3}$$

(a) $\left.\frac{dy}{dx}\right|_{x=4}=-9$

(b) $\left.\frac{d^2y}{dx^2}\right|_{x=4}=18$

27. $$y=\frac{a+4x}{3x-5}$$ (2, b)
gradient = 9
$$\frac{dy}{dx}=\frac{-3a-20}{\left(3x-5\right)^2}$$
$$9=\frac{-3a-20}{\left(3\left(2\right)-5\right)^2}$$
$$3a=-29$$
$$\underline{a=-\frac{29}{3}}$$
$$y=\frac{-\frac{29}{3}+4x}{3x-5}$$
when $x=2$
$$y=\frac{-\frac{29}{3}+8}{1}$$
$$y=-\frac{5}{3}$$
$$\underline{\therefore b=-\frac{5}{3}}$$

28. $$y=x+\sqrt{x^2-4}$$
$$\left(x^2-4\right)\left(\frac{d^2y}{dx^2}\right)+x\left(\frac{dy}{dx}\right)-y=0$$
$$\frac{dy}{dx}=1+x\left(x^2-4\right)^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2}=\left(x^2-4\right)^{-\frac{1}{2}}-x^2\left(x^2-4\right)^{-\frac{3}{2}}$$
$$\therefore\left(x^2-4\right)\left[\left(x^2-4\right)^{-\frac{1}{2}}-x^2\left(x^2-4\right)^{-\frac{3}{2}}\right]$$
$$+x\left[1+x\left(x^2-4\right)^{-\frac{1}{2}}\right]-x-\left(x^2-4\right)^{-\frac{1}{2}}$$
$$=\left(x^2-4\right)^{\frac{1}{2}}-x^2\left(x^2-4\right)^{-\frac{1}{2}}+x+x^2\left(x^2-4\right)^{\frac{1}{2}}$$
$$-x-\left(x^2-4\right)^{-\frac{1}{2}}$$
$$=0$$

29. (a) $f(x)=g(x)+x$
$f'(x)=g'(x)+1$
$f'(0)=g'(0)+1$
$f'(0)=16+1$
$f'(0)=17$

(b) $$h\left(x\right)=x\bullet g\left(x\right)-\left[g\left(x\right)\right]^{-1}$$
$$h'\left(x\right)=g\left(x\right)+x\bullet g'\left(x\right)+\left[g\left(x\right)\right]^{-2}\bullet g'\left(x\right)$$
$$h'\left(x\right)=g\left(x\right)+x\bullet g'\left(x\right)+\frac{g'\left(x\right)}{\left[g\left(x\right)\right]^2}$$
$$\therefore h'\left(0\right)=g\left(0\right)+0\bullet g'\left(0\right)+\frac{g'\left(0\right)}{\left[g\left(0\right)\right]^2}$$
$$=2+\frac{16}{4}$$
$$=6$$

30. $$y=8x^{\frac{1}{2}}+\frac{1}{2}x-4$$
$$\frac{dy}{dx}=4x^{-\frac{1}{2}}+\frac{1}{2}$$
$$\therefore\ \frac{3}{2}=4x^{-\frac{1}{2}}+\frac{1}{2}$$
$$x=16$$
$$y=36$$

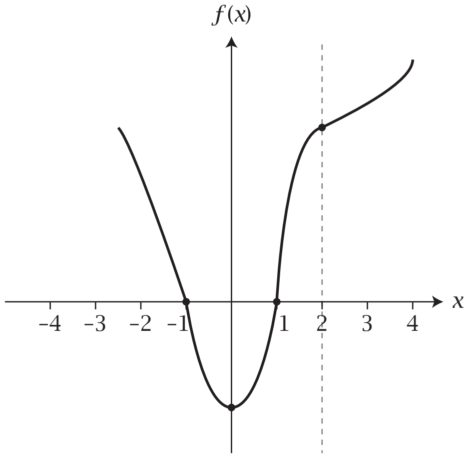
$-3x+2y=7$
$2y=3x+7$
$$y=\frac{3}{2}x+\frac{7}{2}$$
gradient of line $=\frac{3}{2}$

Equation of tangent line

$$y = mx + c$$
$$y = \frac{3}{2}x + c$$
$$36 = \frac{3}{2}(16) + c$$
$$36 = 24 + c$$
$$c = 12$$
$$\therefore y = \frac{3}{2}x + 12$$

CHAPTER 2: *Applications of Differentiation*

1.



2. $f(x) = x^3 - 9x^2 + 15x + 20$
$f(0) = 20$
$f(8) = 76$
$f'(x) = 3x^2 - 18x + 15$
$3(x^2 - 6x + 5) = 0$
$(x - 5)(x - 1) = 0$
$x = 5,\ x = 1$
$\therefore\ y = -5,\ y = 27$
(min) (max)
$f(0) = 20$ $\quad f(5) = -5$
$f(8) = 76$ $\quad f(1) = 27$
Maximum value of 76 when $x = 8$
Minimum value of -5 when $x = 5$

3.
$$y = \frac{x^2}{(3-x)}$$
$$\frac{dy}{dx} = \frac{2x(3-x)-(x^2)(-1)}{(3-x)^2}$$
when $\frac{dy}{dx} = 0$
$$\therefore\ 0 = 2x(3-x) + x^2$$
$$x^2 + 6x - 2x^2 = 0$$
$$-x^2 + 6x = 0$$
$$-x(x-6) = 0$$
$$\therefore\ x = 0,\ x = 6$$
Coordinates (0, 0), (6, −12)

4.
$$y = x(3-x)^2$$
$$\frac{dy}{dx} = (1)(3-x)^2 + (x)(-2)(3-x)$$
$$\frac{dy}{dx} = (3-x)^2 - 2x(3-x)$$
when $x = 2$
$$\frac{dy}{dx} = (3-2)^2 - 2(2)(3-2)$$
$$= 1 - 4$$
$$= -3$$
$$\therefore\ y = -3x + c$$
Sub in (2, 2)
$$2 = -3(2) + c$$
$$c = 8$$
$\therefore\ y = -3x + 8$ tangent line

Normal: $y = \frac{1}{3}x + c$

Sub in (2, 2)
$$2 = \frac{1}{3}(2) + c$$
$$c = \frac{4}{3}$$
$$\therefore\ y = \frac{1}{3}x + \frac{4}{3}$$

5. (a) $R(x) = 300x$

(b) $C'(x) = 8 + \frac{x}{150}$

(c) $\frac{C(300)}{300} = \frac{4700}{300}$
$= \$15.67$

(d) $P(x) = R(x) - C(x)$
$= 300x - \left(2000 + 8x + \frac{x^2}{300}\right)$
$= -2000 + 292x - \frac{x^2}{300}$

(e) $P(300) = \$85\,300$

(f) $P'(x) = 292 - \frac{x}{150} = 0$
$\therefore\ \frac{x}{150} = 292$
$x = 43\,800$
Max profit of \$6 392 800 when 43 800 items are produced and sold.

6.
$$f(x) = ax^3 + bx^2 + cx + d$$
$$f(0) = 2$$
$$\therefore d = 2$$

$$f'(x) = 3ax^2 + 2bx + c$$
$$f'(1) = 3a + 2b + c = 2$$
$$f''(x) = 6ax + 2b$$
$$f''\left(\frac{1}{2}\right) = 3a + 2b = 4$$
$$\therefore 4 + c = 2$$
$$c = -2$$

$$2b = 4 - 3a \qquad 3a = 4 - 2b$$
$$b = \frac{4-3a}{2} \qquad a = \frac{4-2b}{3}$$
$$\therefore f(x) = ax^3 + \left(2 - \frac{3a}{2}\right)x^2 - 2x + 2$$
or
$$f(x) = \left(\frac{4-2b}{3}\right)x^3 + bx^2 - 2x + 2$$

7. (a) $f(x) = (x-1)(x^2 - 3x + 1)$

x intercepts: when $y = f(x) = 0$

$\therefore x = 1,\ x \approx 2.62,\ x \approx 0.38$

(b) y intercept: when $x = 0$
$y = -1$

(c) Stationary point(s): when $f'(x) = 0$

$$f(x) = (x-1)(x^2 - 3x + 1) = x^3 - 4x^2 + 4x - 1$$
$$f'(x) = 3x^2 - 8x + 4$$
$$\therefore 3x^2 - 8x + 4 = 0$$
$$(3x - 2)(x - 2) = 0$$
$$\therefore x = \frac{2}{3},\ x = 2$$

Check for max/min using second derivative test.

$$f''(x) = 6x - 8$$
$f''\left(\frac{2}{3}\right) = -4 < 0 \therefore$ maximum

$f''(2) = 4 > 0 \quad \therefore$ minimum

Maximum turning point $\left(\frac{2}{3}, \frac{5}{27}\right)$

Minimum turning point $(2, -1)$

(d) Point of inflection when $f''(x) = 0$

$$\therefore 6x - 8 = 0$$
$$x = \frac{4}{3}$$

Point of inflection $\left(\frac{4}{3}, \frac{-11}{27}\right)$

8.
$$v = \frac{4}{3}\pi r^3$$
$$\frac{dv}{dr} = 4\pi r^2$$
$$\delta v = \frac{dv}{dr} \cdot \delta r$$
$$= 4\pi r^2 \cdot (\pm 0.1)$$
$$= 4\pi (4172)^2\ (\pm 0.1)$$
$$= \pm 21872501.93\ \text{km}^3$$

9. (a) $\dfrac{\delta L}{L} = -0.02 \qquad \dfrac{dT}{dL} = \dfrac{1}{2}k \cdot L^{-\frac{1}{2}}$

$$\frac{\delta T}{T} \approx \frac{dT}{dL} \cdot \frac{\delta L}{T}$$
$$\approx \frac{k}{2\sqrt{L}} \cdot \frac{-0.02L}{k\sqrt{L}}$$
$$\approx -0.01$$

Decrease of 1%.

(b)
$$\frac{\delta T}{T} \approx \frac{dT}{dL} \cdot \delta L \cdot \frac{100}{T}$$
$$\approx \frac{1}{\sqrt{L}}(-2) \cdot \frac{100}{2\sqrt{L}}$$
$$\approx \frac{-200}{2L}$$
$$\approx \frac{-200}{2(150)}$$
$$\approx \frac{-2}{3}\%$$

Approximate decrease of $\frac{2}{3}\%$ in T.

10. (a)
$$x = t^3 - 6t^2 + 9t$$
$$x(2) = 2^3 - 6(2)^2 + 9(2)$$
$$= 2 \text{ metres}$$

(b) $v(t) = 3t^2 - 12t + 9$

(c)
$$v(3) = 3(3)^2 - 12(3) + 9$$
$$= 0 \text{ m/s}$$

(d) Particle is at rest when $v(t) = 0$

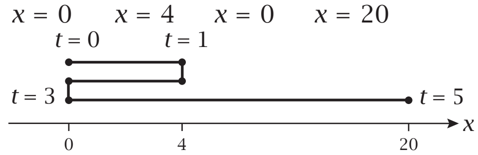
$$\therefore 3t^2 - 12t + 9 = 0$$
$$3(t-1)(t-3) = 0$$
$t = 1,\ t = 3$ seconds

(e) When $v(t) > 0$

$\therefore t < 1$ and $t > 3$

(f) $t = 0 \quad t = 1 \quad t = 3 \quad t = 5$

$x = 0 \quad x = 4 \quad x = 0 \quad x = 20$



Distance travelled $4 + 4 + 20 = 28$ m

11. *(a)* $v(t) = \begin{cases} 6t^2 - 6t & 0 \le t \le 2 \\ 20 - 6t & 2 < t \le 4 \end{cases}$

(b) When $v(t) = 0$

i.e. $6t^2 - 6t = 0$

$6t(t-1) = 0$

$\therefore t = 0,\ 1$

$\therefore x(t) = 2t^3 - 3t^2$

$x(1) = 2 - 3$

$\therefore x(1) = -1$

$v(t) = 0$

$20 - 6t = 0$

$6t = 20$

$t = \dfrac{10}{3}$

$x(t) = 20t - 3t^2 - 24$

$x\left(\dfrac{10}{3}\right) = 20\left(\dfrac{10}{3}\right) - 3\left(\dfrac{10}{3}\right)^2 - 24$

$x\left(\dfrac{10}{3}\right) = \dfrac{28}{3}$

(c) $t = 0 \quad t = 1 \quad t = 2 \quad t = 3$

$x = 0 \quad x = -1 \quad x = 4 \quad x = 9$

Distance travelled is 11 units.

(d) $v'(t) = \begin{cases} 12t - 6 \\ -6 \end{cases}$

when $v'(t) = 0$

$\therefore 12t - 6 = 0$

$t = \dfrac{1}{2}$

Minimum velocity $= 6\left(\dfrac{1}{2}\right)^2 - 6\left(\dfrac{1}{2}\right)$

$= -1.5$ units

12. *(a)* $V = \dfrac{\pi r^2 h}{3}$

$r : h = 5 : 3$

i.e. $\dfrac{r}{h} = \dfrac{5}{3}$

$r = \dfrac{5h}{3}$

$\therefore V = \pi \dfrac{\left(\dfrac{5h}{3}\right)^2}{3} \cdot h$

$V = \pi \cdot \dfrac{\dfrac{25h^2}{9} \cdot h}{3}$

$V = \dfrac{25\pi h^3}{27}$

(b) $\delta V \approx \dfrac{dV}{dh} \times \delta h$

$\delta V \approx \dfrac{75\pi h^2}{27} \times \delta h$

$\delta V \approx \dfrac{75\pi(5)^2}{27} \times 0.02$

$\delta V \approx \dfrac{25\pi}{18}$

$\underline{\delta V \approx 4.363 \text{ cm}^3}$

13. *(a)* $R(x) = x(40 - 0.02x)$

(b) $P(x) = x(40 - 0.02x) - (3000 + 5x)$

$= 40x - 0.02x^2 - 3000 - 5x$

$P(x) = -0.02x^2 + 35x - 3000$

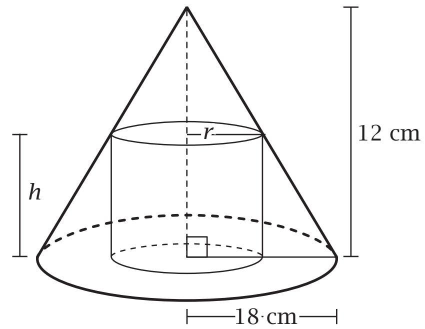
(c) Max profit: $12312.50
Number of items: 875

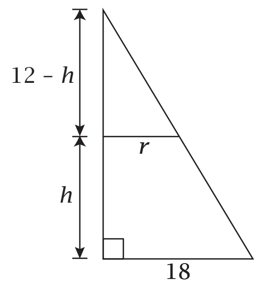
(d) A loss occurred when
$0 \le x \le 90, \quad x \ge 1660$

(e) $P'(x) = -0.04x + 35$

$P'(249) = \$25.04$

14.





$\dfrac{12 - h}{r} = \dfrac{12}{18}$

$12 - h = \dfrac{12r}{18}$

$h = -\dfrac{12r}{18} + 12$

Volume Cylinder $= \pi r^2 h$

$v = \pi r^2 \left(-\dfrac{12r}{18} + 12\right)$

$v = -\dfrac{12\pi r^3}{18} + 12\pi r^2$

$v' = -\dfrac{36\pi r^2}{18} + 24\pi r$

$v' = -2\pi r^2 + 24\pi r$

Max when $v' = 0$

$\therefore 0 = -2\pi r^2 + 24\pi r$

$\cancel{r = 0},\ r = 12$

$v'' = -48\pi + 24\pi$

$=$ negative $\therefore$ Maximum

$\therefore$ Maximum Volume $= -\dfrac{12\pi(12)^3}{18} + 12\pi(12)^2$

$= 576\pi\ cm^3$

15. $f(x) = x^3 - 6x^2 + 12x - 3$

Stationary points when $f'(x) = 0$

i.e. $3x^2 - 12x + 12 = 0$

$3(x^2 - 4x + 4) = 0$

$(x - 2)(x - 2) = 0$

$\therefore x = 2$

$y = 5$

Nature:

Sign test:

x	1	2	3
$f'(x)$	+ve /	0	+ve /

$\therefore$ (2, 5) – Horizontal point of inflection

16. (a) $A(x) = \frac{1}{2} b \times \text{p.h}$

$$\text{p.h} = \sqrt{8^2 - \left(\frac{1}{2}x\right)^2}$$

$$= \sqrt{64 - \frac{x^2}{4}}$$

$$= \sqrt{\frac{256 - x^2}{4}}$$

$$= \frac{\sqrt{256 - x^2}}{2}$$

$$A(x) = \frac{1}{2} b \times \text{p.h}$$

$$= \frac{1}{2} x \frac{\sqrt{256 - x^2}}{2}$$

$$= \frac{x}{4}\sqrt{256 - x^2}$$

(b)

$$\int A = \begin{bmatrix} \frac{1}{4}\sqrt{256 - x^2} \\ + \frac{x}{4}\left(\frac{1}{2}\right)\left(256 - x^2\right)^{-0.5}\left(-2x\right) \end{bmatrix} \bullet \delta x$$

$$= \begin{bmatrix} \frac{1}{4}\sqrt{256 - 5^2} \\ + \frac{5}{4}\left(\frac{1}{2}\right)\left(256 - 5^2\right)^{-0.5}\left(-2(5)\right) \end{bmatrix}(0.01)$$

$$= 0.0339 \text{ m}^2$$

17. $V = 500 \text{ cm}^3 \pm 10 \text{ cm}^3$

when $V = 500$

$\therefore \frac{4}{3}\pi r^3 = 500$

$\therefore r = 4.924$ cm

$\delta v \approx \frac{dV}{dr} \bullet \delta r$

$10 \approx 4\pi r^2 \bullet \delta r$

$\therefore \delta r \approx \frac{10}{4\pi r^2}$

$\delta r \approx \frac{10}{4\pi\left(4.924\right)^2}$

$\delta r \approx 0.0328$

$\therefore$ Radius of the sphere is:

4.924 ± 0.0328 cm

18. (a) $JY^2 = JB^2 + BY^2$

$JY^2 = 100 + k^2$

$JY = \sqrt{100 + k^2}$

(b) $YX = BX - BY$

$YX = 15 - k$

(c) $\text{Time}(t) = \frac{\sqrt{100 + k^2}}{5} + \frac{15 - k}{8}$

(d) For min time $\frac{dt}{dk} = 0$

$$\therefore 0 = \frac{k}{5(100 + k^2)^{\frac{1}{2}}} - \frac{1}{8}$$

$\therefore k \approx 8.006\, km$

$$\text{time} = \frac{\sqrt{100 + 8.006^2}}{5} + \frac{15 - 8.006}{8}$$

≈ 3.436

Least time for the journey is 3 hrs and 26 minutes.

19. (a) $\text{Volume (Cone)} = \frac{\pi r^2 h}{3}$

$$h = \frac{1}{2} r$$

$$\therefore \text{Volume} = \frac{\pi r^2 \left(\frac{1}{2} r\right)}{3}$$

$$V = \frac{\pi r^3}{6}$$

(b) $\frac{dV}{dr} = \frac{\pi r^2}{2}$ $\qquad \delta r = \frac{-10r}{100}$

$$\therefore \frac{\delta V}{V} = \frac{dv}{dr} \bullet \delta r \bullet \frac{1}{V}$$

$$\therefore \delta V = \frac{\pi r^2}{2} \bullet \frac{-10r}{100} \bullet \frac{6}{\pi r^3}$$

$$\delta V = \frac{-30}{100}$$

An approximate reduction of 30% in the volume.

20. $y = x^3(2 + x)$

$y = 2x^3 + x^4$

Roots: $(-2, 0)$, $(0, 0)$

For stationary points let $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 6x^2 + 4x^3$

$\therefore 6x^2 + 4x^3 = 0$

$2x^2(3 + 2x) = 0$

$\therefore x = 0, x = -\frac{3}{2}$

$y = 0, y = -\frac{27}{16}$

$(0, 0)$ $\quad \left(-\frac{3}{2}, -\frac{27}{16}\right)$

Nature: $\frac{d^2y}{dx^2} = 12x + 12x^2$

when $x = 0$ $\quad \frac{d^2y}{dx^2} = 0$

$\therefore$ (0, 0) is a horizontal point of inflection

When $x = -\frac{3}{2}$ $\quad \frac{d^2y}{dx^2} = 9$

$\therefore \left(-\frac{3}{2}, -\frac{27}{16}\right)$ is a minimum turning point.

Points of inflection: $\frac{d^2y}{dx^2} = 0$

	$12x + 12x^2 = 0$
(0, 0)	$12x(1 + x) = 0$
(–1, –1)	$x = 0,\ x = -1$
	$y = 0,\ y = -1$

21. $\delta V \approx \frac{dV}{dr} \cdot \delta r$

$V = \frac{4}{3}\pi r^3$

$34 = \frac{4}{3}\pi r^3$

$r = 2.00969$ cm

$\delta V \approx 4\pi r^2 \cdot \delta r$

$\delta V \approx 4\pi\left(2.9434\right)^2 \cdot \left(0.006\right)$

$\delta V \approx 0.3045\ \text{cm}^3$

22. *(a)* Area (square) = 400 cm^2

Area $\triangle SUP = 10x\ \text{cm}^2$

Area $\triangle PQT = \frac{1}{2}\left(20 - \frac{5x}{2}\right)\left(20\right)$

$= 200 - 25x\ \text{cm}^2$

Area $\triangle TRU = \frac{1}{2}\left(20 - x\right)\left(\frac{5x}{2}\right)$

$= 25x - \frac{5x^2}{4}\ \text{cm}^2$

$\therefore$ Area $\triangle PUT = 400 - 10x - \left(200 - 25x\right) - \left(25x - \frac{5x^2}{4}\right)$

$= 400 - 10x - 200 + 25x - 25x + \frac{5x^2}{4}$

$= 200 - 10x + \frac{5x^2}{4}\ \text{cm}^2$

(b) To minimise area: $\frac{dA}{dx} = 0$.

$A = 200 - 10x + \frac{5x^2}{4}$

$\frac{dA}{dx} = -10 + \frac{10x}{4}$

$-10 + \frac{10x}{4} = 0$

$\frac{10x}{4} = 10$

$10x = 40$

$\underline{x = 4}$

(c) $\therefore A = 200 - 10\left(4\right) + \frac{5\left(4\right)^2}{4}$

$= 200 - 40 + 20$

$\underline{= 180\ \text{cm}^2}$

23. *(a)* $V = \frac{4}{3}\pi\ ^{3}$ $\qquad \delta V \approx 3\ \text{cm}^3$

$\frac{dV}{dr} = 4\pi r^2$

$\delta V \approx \frac{dV}{dr} \cdot \delta r$

$3 \approx 4\pi\left(5\right)^2 \cdot \delta r$

$\frac{3}{4\pi\left(5\right)^2} \approx \delta r$

$\delta r = 0.00955$

(b) $S = 4\pi r^2$ $\qquad \frac{\delta S}{S} = 0.04$

$\frac{dS}{dr} = 8\pi r$

$\frac{\delta S}{S} \approx \frac{\frac{dS}{dr} \cdot \delta r}{S}$

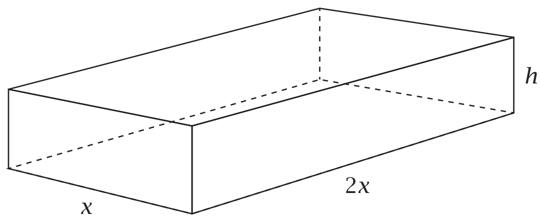
$\frac{\delta S}{S} \approx \frac{8\pi r \cdot \delta r}{4\pi r^2}$

$0.04 = \frac{2\,\delta r}{r}$

$0.02 = \frac{\delta r}{r}$

A 2% increase in the radius.

24.



(a) Surface area $= 2x^2 + 2\left(2xh\right) + 2\left(xh\right)$

$= 2x^2 + 4xh + 2xh$

$= 2x^2 + 6xh$

Cost $= \frac{5}{200}\left(2x^2 + 6xh\right)$

$= \frac{x^2 + 3xh}{20}$

(b) Cost $= \frac{x^2 + 3xh}{20}$

$15 = \frac{x^2 + 3xh}{20}$

$300 = x^2 + 3xh$

$h = \frac{100}{x} - \frac{x}{3}$

$V = 2x^2h$

$= 2x^2\left(\frac{100}{x} - \frac{x}{3}\right)$

$= 200x - \frac{2x^3}{3}$

(c) Max volume when $\frac{dV}{dx} = 0$

i.e. $V = 200x - \frac{2x^3}{3}$

$$\frac{dV}{dx} = 200 - 2x^2$$
$$-2x^2 + 200 = 0$$
$$2x^2 = 200$$
$$x^2 = 100$$
$$x = 10$$

Maximum volume of $\frac{4000}{3}$ cm^3

when width = 10 cm

length = 20 cm

height = $\frac{20}{3}$ cm

25. $\sqrt[3]{1003}$ $\quad y = \sqrt[3]{x}$

if $x = 1000$ $\quad \frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$

$y = 10$

$$\delta y \approx \frac{dy}{dx} \cdot \delta x$$
$$\delta y \approx \frac{1}{3\sqrt[3]{x^2}} \delta x$$
$$\delta y \approx \frac{1}{3\sqrt[3]{(1000)^2}} \cdot 3$$
$$\delta y \approx \frac{1}{100}$$
$$\sqrt[3]{1003} \approx 10 + \frac{1}{100}$$
$$\approx 10.01$$

26. $A = \frac{50}{1+t^2}$

(a) $A = \frac{50}{1+3^2}$

$A = \frac{50}{10}$

$A = 5$ mg

(b) $\frac{dA}{dt} = \frac{-100t}{(1+t^2)^2}$

$\left.\frac{dA}{dt}\right|_{t=3} = -3$

Rate of change of $A = -3$ mg/hour

27. $M = \frac{20t}{1+2t}$

(a) $t = 2 \quad M = 8$

$t = 3 \quad M = \frac{60}{7}$

Average rate of change $= \frac{\frac{60}{7} - 8}{3-2}$

$= 0.5714$ g/hour

(b) Instantaneous rate of change

$$\frac{dM}{dt} = \frac{20}{(2t+1)^2}$$
$$\left.\frac{dM}{dt}\right|_{t=2} = 0.8 \text{ g/hour}$$

CHAPTER 3: *Integration*

1. (a) $\frac{3x^5}{5} - 6x + c$

(b) $-\frac{1}{x^4} + x^2 + c$

(c) $\frac{2}{3}x^{\frac{3}{2}} + 3x + c$

(d) $4x + c$

(e) $\frac{x^3}{3} - 3x^2 + c$

(f) $2x^{\frac{1}{2}} + \frac{3}{4}x^{\frac{4}{3}} + c$

2. (a) $\frac{dy}{dx} = 4 - \frac{2}{x^2}$

$$\frac{dy}{dx} = 4 - 2x^{-2}$$
$$y = 4x - \frac{2x^{-1}}{-1} + c$$
$$y = 4x + \frac{2}{x} + c$$

when $x = -2 \quad y = 2$

$$\therefore 2 = 4(-2) + \frac{2}{-2} + c$$
$$2 = -8 - 1 + c$$
$$c = 11$$
$$\therefore y = 4x + \frac{2}{x} + 11$$

(b) $y = 4\left(\frac{1}{2}\right) + \frac{2}{\left(\frac{1}{2}\right)} + 11$

$y = 17$

(c) $-\frac{5}{3} = 4x + \frac{2}{x} + 11$

$x = -3$ or $x = -\frac{1}{6}$

3. (a) $\frac{(2x-3)^4}{8} + c$

(b) $\frac{(4-3x)^5}{-15} + c$

(c) $\frac{2(4x+1)^4}{16} + c$

$= \frac{(4x+1)^4}{8} + c$

(d) $3(2x+5)^{-2}$

$= \frac{3(2x+5)^{-1}}{-2} + c$

$= \frac{3}{-2(2x+5)} + c$

(e) $(3x-1)^{\frac{1}{2}}$

$= \dfrac{(3x-1)^{\frac{3}{2}}}{\frac{9}{2}} + c$

$= \dfrac{2(3x-1)^{\frac{3}{2}}}{9} + c$

(f) $2(5x-6)^{-\frac{1}{3}}$

$= \dfrac{3(5x-6)^{\frac{2}{3}}}{5} + c$

4. (a) $\int 10x\left(x^2-6\right)^4 dx$

$= 5\int 2x\left(x^2-6\right)^4 dx$

$= \dfrac{5\left(x^2-6\right)^5}{5} + c$

$= \left(x^2-6\right)^5 + c$

(b) $\int \left(2x-1\right)\left(x^2-x+4\right)^3 dx$

$= \dfrac{\left(x^2-x+4\right)^4}{4} + c$

(c) $\int 15x\left(2+x^2\right)^5 dx$

$= \dfrac{15}{2}\int 2x\left(2+x^2\right)^5 dx$

$= \dfrac{15\left(2+x^2\right)^6}{12} + c$

$= \dfrac{5\left(2+x^2\right)^6}{4} + c$

(d) $\int x^2\left(x^3+4\right)^9 + c$

$= \dfrac{1}{3}\int 3x^2\left(x^3+4\right)^9$

$= \dfrac{\left(x^3+4\right)^{10}}{30} + c$

(e) $\int \dfrac{x}{\left(5x^2+1\right)^3} dx$

$= \int x\left(5x^2+1\right)^{-3} dx$

$= \dfrac{1}{10}\int 10x\left(5x^2+1\right)^{-3} dx$

$= -\dfrac{\left(5x^2+1\right)^{-2}}{20} + c$

$= -\dfrac{1}{20\left(5x^2+1\right)^2} + c$

(f) $\int 2x\left(3+x\right)^2 dx$

$= \int 2x\left(9+6x+x^2\right)dx$

$= \int \left(18x+12x^2+2x^3\right)dx$

$= \dfrac{18x^2}{2} + \dfrac{12x^3}{3} + \dfrac{2x^4}{4} + c$

$= 9x^2 + 4x^3 + \dfrac{x^4}{2} + c$

5. (a) $\int (3x^2 + x)dx$

$= \dfrac{3x^3}{3} + \dfrac{x^2}{2} + c$

$= x^3 + \dfrac{x^2}{2} + c$

(b) $\int \dfrac{10}{(2x+3)^2} dx$

$= \int 10(2x+3)^{-2} dx$

$= \dfrac{10(2x+3)^{-1}}{-2} + c$

$= \dfrac{-5}{(2x+3)} + c$

(c) $\int \left(\dfrac{2-x^5}{x^2}\right)dx$

$= \int \left(\dfrac{2}{x^2} - \dfrac{x^5}{x^2}\right)dx$

$= \int (2x^{-2} - x^3)dx$

$= \dfrac{2x^{-1}}{-1} - \dfrac{x^4}{4} + c$

$= -\dfrac{2}{x} - \dfrac{x^4}{4} + c$

(d) $\int 2x(x^2-5x)dx$

$= \int (2x^3 - 10x^2)dx$

$= \dfrac{2x^4}{4} - \dfrac{10x^3}{3} + c$

$= \dfrac{x^4}{2} - \dfrac{10x^3}{3} + c$

(e) $\int (\sqrt{5x^3} + e^{-x})dx$

$= \int \left((5x^3)^{\frac{1}{2}} + e^{-x}\right)dx$

$= \int (\sqrt{5}\, x^{\frac{3}{2}} + e^{-x})dx$

$= \dfrac{\sqrt{5}\, x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} - \dfrac{1}{e^x} + c$

$= \dfrac{2x^{\frac{5}{2}}}{\sqrt{5}} - \dfrac{1}{e^x} + c$

6. (a) $\int_0^4 (2x-3)^3 dx$

$= \left[\dfrac{(2x-3)^4}{8}\right]_0^4$

$= 68$

(b) $\int_3^4 (3x - x^3)dx$

$= \left[\dfrac{3x^2}{2} - \dfrac{x^4}{4}\right]_3^4$

$= -33.25$

(c) $\int_0^7 (3x + 4)^{\frac{1}{2}} dx$

$= \left[\frac{2(3x + 4)^{\frac{3}{2}}}{9}\right]_0^7$

$= 26$

(d) $\int_1^3 (x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx$

$= \left[\frac{2x^{\frac{3}{2}}}{3} + 2x^{\frac{1}{2}}\right]_1^3$

≈ 4.2615

(e) $\int_0^2 (2x+1)(x^2+x-4)^2\, dx$

$= \left[\frac{(x^2+x-4)^3}{3}\right]_0^2$

$= \left(\frac{(2)^3}{3}\right) - \left(\frac{(-4)^3}{3}\right)$

$= \frac{8}{3} + \frac{64}{3}$

$= \frac{72}{3}$

$= 24$

7. (a) $\frac{d}{dx}\int_2^{x^2} \left(\frac{t^2}{4}\right) dt$

$= \frac{(x^2)^2}{4} \cdot 2x$

$= \frac{x^5}{2}$

(b) $\frac{d}{dx}\int_0^x \frac{3t}{(1-t^2)^2}\, dt$

$= \frac{3x}{(1-x^2)^2}$

(c) $\frac{d}{dx}\int_{3x^2}^0 \left(\frac{2+t}{t-3}\right) dt$

$= -\left(\frac{2+3x^2}{3x^2-3}\right) \cdot 6x$

$= -\left(\frac{4x+6x^3}{x^2-1}\right)$

(d) $\frac{d}{dx}\int_0^{2x^3} \sqrt{1+t^2}\, dt$

$= \sqrt{1+(2x^3)^2} \cdot 6x^2$

$= \sqrt{1+4x^6} \cdot 6x^2$

(e) $\int_1^2 \frac{d}{dx}\left[\frac{x^3}{x^2+1}\right] dx$

$= 1.1$

8. $\int_1^4 \sqrt{x}\,(x + 3)\, dx$

$= \int_1^4 (x^{\frac{3}{2}} + 3x^{\frac{1}{2}}) dx$

$= \left[\frac{2x^{\frac{5}{2}}}{5} + 2x^{\frac{3}{2}}\right]_1^4$

$= \left(\frac{2(4)^{\frac{5}{2}}}{5} + 2(4)^{\frac{3}{2}}\right) - \left(\frac{2(1)^{\frac{5}{2}}}{5} + 2(1)^{\frac{3}{2}}\right)$

$= \left(\frac{64}{5} + 16\right) - \left(\frac{2}{5} + 2\right)$

$= \frac{132}{5}$

9. (a) $\frac{dy}{dx} = 12(3x-2)^2$

$y = \frac{12(3x-2)^3}{9} + c$

$y = 3$ when $x = 0$

$3 = \frac{12(-2)^3}{9} + c$

$c = \frac{41}{3}$

$\therefore y = \frac{4(3x-2)^3}{3} + \frac{41}{3}$

(b) when $x = 2$

$y = 99$

10. Antiderivative of $5x - 8$

$y = \frac{5x^2}{2} - 8x + c$

when $x = 2,\ y = 1$

$1 = \frac{5(2)^2}{2} - 8(2) + c$

$c = 7$

$\therefore y = \frac{5x^2}{2} - 8x + 7$

11. $\int_{-\pi}^{\pi} (4x + 1) dx$

$= \left[\frac{4x^2}{2} + x\right]_{-\pi}^{\pi}$

$= [2x^2 + x]_{-\pi}^{\pi}$

$= \left(2(\pi)^2 + \pi\right) - \left(2(-\pi)^2 - \pi\right)$

$= 2\pi$

12. $\dfrac{dy}{dx} = (-3-2x)(x^2 + 3x-7)^5$

$y = \dfrac{(x^2 + 3x-7)^6}{-6} + c$

$y = -100$ when $x = 1$

$\therefore -100 = \dfrac{(1^2 + 3(1)-7)^6}{-6} + c$

$-100 = -121.5 + c$

$c = 21.5$

$\therefore y = \dfrac{(x^2 + 3x-7)^6}{-6} + 21.5$

13. (a) $f'(x) = 3x+1-\dfrac{1}{x^2}$

$f(x) = \dfrac{3x^2}{2}+x+\dfrac{1}{x}+c$

$f(1) = 4$

$\therefore 4 = \dfrac{3(1)^2}{2}+1+\dfrac{1}{1}+c$

$c = \dfrac{1}{2}$

$\therefore f(x) = \dfrac{3x^2}{2}+x+\dfrac{1}{x}+\dfrac{1}{2}$

(b) $f(3) = \dfrac{3(3)^2}{2}+3+\dfrac{1}{3}+\dfrac{1}{2}$

$= \dfrac{52}{3}$

14. $\int_t^1 (4x+5)\,dx = 4$

$\left[\dfrac{4x^2}{2}+5x\right]_t^1 = 4$

$\therefore \left(\dfrac{4(1)^2}{2}+5(1)\right)-\left(\dfrac{4t^2}{2}+5t\right) = 4$

$7-2t^2-5t = 4$

$2t^2+5t-3 = 0$

$(2t-1)(t+3) = 0$

$t = \dfrac{1}{2}$ or $\cancel{t=-3}$

$\therefore t = \dfrac{1}{2}$

15. $\int_2^5 f(x)\,dx = 15$

(a) $\int_5^2 3f(x)\,dx = -45$

(b) $\int_2^4 \left(f(x)+3\right)dx + \int_4^5 f(x)\,dx$

$= \int_2^4 f(x)\,dx + \int_2^4 3\,dx + \int_4^5 f(x)\,dx$

$= \int_2^5 f(x)\,dx + \int_2^4 3\,dx$

$= 15 + [3x]_2^4$

$= 15+12-6$

$= 21$

16. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-4x}}$

$y = \dfrac{-\sqrt{-4x+1}}{2}+c$

point $(-2,1)$

$\therefore 1 = \dfrac{-\sqrt{-4(-2)+1}}{2}+c$

$1 = \dfrac{-\sqrt{9}}{2}+c$

$1 = \dfrac{-3}{2}+c$

$c = \dfrac{5}{2}$

$\therefore$ Equation

$y = \dfrac{-\sqrt{-4x+1}}{2}+\dfrac{5}{2}$

17. (a) $\dfrac{dy}{dx} = k(x-a)(x-b)$

$\dfrac{dy}{dx} = k(x+2)(x-3)$

$\dfrac{dy}{dx} = kx^2-kx-6k$

$\therefore y = \dfrac{kx^3}{3}-\dfrac{kx^2}{2}-6kx+c$

$(-2,11)$ $\quad 11 = \dfrac{k(-2)^3}{3}-\dfrac{k(-2)^2}{2}+12k+c$

$(3,6)$ $\quad 6 = \dfrac{k(3)^3}{3}-\dfrac{k(3)^2}{2}-18k+c$

$\therefore k = \dfrac{6}{25} \quad c = \dfrac{231}{25}$

$\therefore a = -2$

$b = 3$

$k = \dfrac{6}{25}$

(b) Equation is

$y = \dfrac{6}{25}\left(\dfrac{x^3}{3}-\dfrac{x^2}{2}-6x\right)+\dfrac{231}{25}$

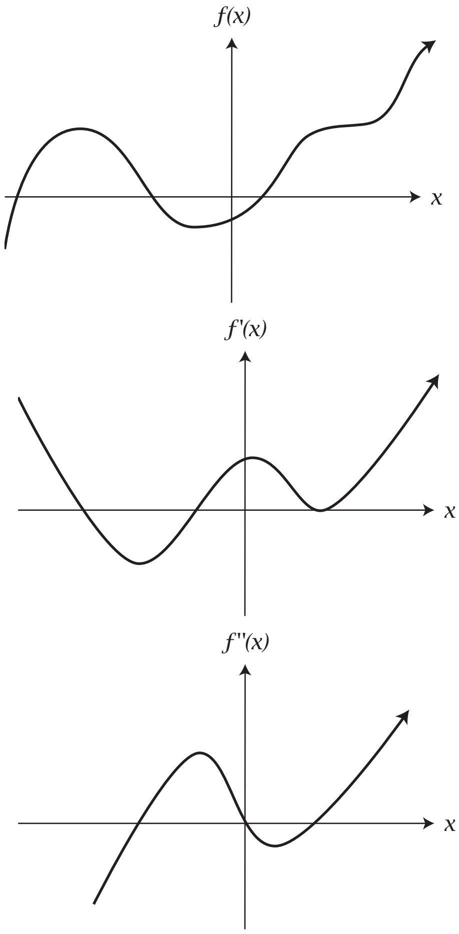
$y = \dfrac{6x^3}{75}-\dfrac{3x^2}{25}-\dfrac{36x}{25}+\dfrac{231}{25}$

CHAPTER 4: *Applications of Integration*

1. $$f(x) = x^3 + 1$$
$$g(x) = x + 1$$
$$\therefore x^3 + 1 = x + 1$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$\therefore x = 0,\ 1, -1$$
$$\text{Area} = \int_{-1}^{0} (x^3+1)-(x+1)\,dx + \int_{0}^{1} (x+1)-(x^3+1)\,dx$$
$$= 0.25 + 0.25$$
$$= 0.5 \text{ units}^2$$

2. $$\int_{-2}^{\frac{1}{2}} \left(2 + \frac{x}{2}\right) - (x^2 + 2x + 1)\,dx$$
$$\approx 2.604 \text{ units}^2$$

3.



4. (a) $$\frac{dv}{dt} = 0.2t^2 - 30$$
$$v = \int (0.2t^2 - 30)\,dt$$
$$= \frac{0.2t^3}{3} - 30t + c$$
when $t = 15 \quad v = 0$
$$0 = \frac{0.2(15)^3}{3} - 30(15) + c$$
$$0 = -225 + c$$
$$c = 225$$
Initially 225 kL

(b) $$\int_{7}^{8} (0.2t^2 - 30)\,dt$$
$= -18.73$ kL
$= 18.73$ kL drained

5. (a) $$x(t) = \int_{0}^{2} (t^3 - 7t^2 + 10t)\,dt$$
$$= \left[\frac{t^4}{4} - \frac{7t^3}{3} + \frac{10t^2}{2}\right]_0^2$$
$$= 5\frac{1}{3} \text{ m}$$

(b) $$\text{Distance} = \int_{0}^{3} \left|t(t-2)(t-5)\right|\,dt$$
$$= 8.41\dot{6} \text{ m}$$

6. (a) $$\frac{dC}{dx} = 60\sqrt{x}$$
$$\frac{dC}{dx} = 60x^{\frac{1}{2}}$$
$$C = \frac{60x^{\frac{3}{2}}}{1.5}$$
$$C = 40x^{\frac{3}{2}} + c$$
when $x = 0,\ C = 1500$
$$\therefore 1500 = 40(0)^{\frac{3}{2}} + c$$
$$C = 1500$$
$$\therefore C(x) = 40x^{\frac{3}{2}} + 1500$$

(b) when $x = 150$
$$C = \$74984.69$$

(c) Average cost $\dfrac{\$74984.69}{150}$
$$= \$499.90$$

(d) $$\text{Profit} = 600x - \left(40x^{\frac{3}{2}} + 1500\right)$$
$$= 600(150) - \$74984.69$$
$$= \$15015.31$$

7.

Rectangle	$0 \le x < 0.5$	$0.5 \le x < 1$	$1 \le x < 1.5$	$1.5 \le x < 2$	$2 \le x < 2.5$	$2.5 \le x \le 3$
Under-estimate	$A = ½ \times 8.75$ $= 4.375$	$A = ½ \times 8$ $= 4$	$A = ½ \times 6.75$ $= 3.375$	$A = ½ \times 5$ $= 2.5$	$A = ½ \times 2.75$ $= 1.375$	$A = ½ \times 0$ $= 0$
Over-estimate	$A = ½ \times 9$ $= 4.5$	$A = ½ \times 8.75$ $= 4.375$	$A = ½ \times 8$ $= 4$	$A = ½ \times 6.75$ $= 3.375$	$A = ½ \times 5$ $= 2.5$	$A = ½ \times 2.75$ $= 1.375$

Total Area Under-estimate = 15.625
Total Area Over-estimate = 20.125

$$\therefore \text{ Total Area} = \frac{15.625 + 20.125}{2}$$
$$= \underline{17.875}$$

8. (a)

Rectangle	$2 \le x < 2.5$	$2.5 \le x < 3$	$3 \le x < 3.5$	$3.5 \le x \le 4$
Under-estimate	$A = ½ \times 6$ $= 3$	$A = ½ \times 8.25$ $= 4.125$	$A = ½ \times 11$ $= 5.5$	$A = ½ \times 14.25$ $= 7.125$
Over-estimate	$A = ½ \times 8.25$ $= 4.125$	$A = ½ \times 11$ $= 5.5$	$A = ½ \times 14.25$ $= 7.125$	$A = ½ \times 18$ $= 9$

Total Area Under-estimate = 19.75
Total Area Over-estimate = 25.75

$$\therefore \text{ Total Area} = \frac{19.75 + 25.75}{2}$$
$$= 22.75$$

(b) Area can be improved by using *smaller* rectangles i.e. a larger number of rectangles.

9. (a)

$$v(t) = 3t^2 - 6t$$
$$x(t) = \frac{3t^3}{3} - \frac{6t^2}{2} + c$$
$$x(t) = t^3 - 3t^2 + c$$
$$\therefore x(t) = t^3 - 3t^2$$

(b)

$$x(t) = 0$$
$$t^3 - 3t^2 = 0$$
$$t^2(t-3) = 0$$
$$t = 0, \ t = 3$$

Object returns to starting point after 3 secs.

(c)

$$x(2) = -4m$$
$$x(5) = 50m$$
$$\therefore At \ t = 5\text{secs}$$

10. (a)

$$C(x) = \int (125 - 0.003x^2)\,dx$$
$$= 125x - \frac{0.003x^3}{3} + C$$
$$C(0) = 4000$$
$$\therefore C(x) = 125x - 0.001x^3 + 4000$$
$$C(40) = \$8936$$

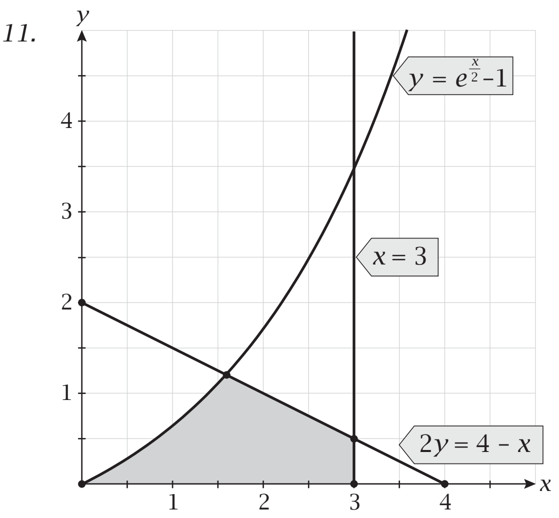
(b)

$$\frac{C(40)}{40} = \frac{\$8936}{40}$$
$$= \$223.40 \text{ per unit}$$

(c)

$$C(50) - C(40) = \int_{40}^{50} 125 - 0.003x^2\,dx$$
$$= \left(125(50) - 0.001(50)^3 + 4000\right) - 8936$$
$$= \$1189$$

11.



$$e^{\frac{x}{2}} - 1 = \frac{4-x}{2}$$
$$x \approx 1.584$$

$$\text{Area} \approx \int_0^{1.584} \left(e^{\frac{x}{2}} - 1\right) dx + \int_{1.584}^{3} \left(2 - \frac{x}{2}\right) dx$$
$$\approx 0.8316 + 1.2093$$
$$\approx 2.0409 \text{ units}^2$$

12. x intercepts: $x = 0$, $x = 3$

$$\text{Area} = \int_0^{-1} \left(3x - x^2\right) dx + \int_0^3 \left(3x - x^2\right) dx$$

$$= \frac{19}{3} \text{ units}^2$$

or

$$\text{Area} = \int_{-1}^{3} \left|3x - x^2\right| dx$$

$$= \frac{19}{3} \text{ units}^2$$

13. (a)
- Constant velocity for the first 6 seconds
- Moving to the right for the first 8 seconds and changes direction at $t = 8$
- Constant deceleration from $t = 6$ to $t = 10$.

(b) $v(3) = 5\,\text{m/s}$

(c) $v(t) = -2.5t + 20$

(d) $v(9) = -2.5\,\text{m/s}$

(e) i. $a(1) = 0\,\text{m/s}$

ii. $a(7) = -2.5\,\text{m/s}^{-2}$

(f) Distance travelled to right: 35 m
Distance travelled to left: 5 m
Change in displacement = 30 m

14. (a) i. When $t = 8$, $v = 10$ m/s

ii. When $t = 12$, $v = 5$ m/s

(b) i. $a = \dfrac{10-0}{8-4}$

$= 2.5\,\text{ms}^{-2}$

$\therefore$ acceleration

ii. $a = \dfrac{10-0}{8-16}$

$a = -1.25\,\text{ms}^{-2}$

$\therefore$ deceleration

(c) i. When $t = 4$, $x = 0$

ii. When $t = 8$, $x = 20$ m

iii. When $t = 12$, $x = 50$ m

(d) When $t = 12$ distance of particle from origin is 50 m, moving away. It is decelerating at $1.25\,\text{ms}^2$ until at $t = 16$ it is at instantaneous rest 60 m from the origin. It moves towards the origin at $1.25\,\text{m/s}^2$ until at $t = 20$ it is 50 m from the origin.

15. (a) $C'(x) = -3x + 0.03x^2$

$C'(110) = -3(110) + 0.03(110)^2$

$= \$33 / \text{printer}$

(b) $C(x) = \dfrac{-3x^2}{2} + \dfrac{0.03x^3}{3} + 250$

$= \dfrac{-3x^2}{2} + 0.01x^3 + 250$

(c) $P(x) = R(x) - C(x)$

$R(x) = 750x$

$P(x) = 750x - \left(\dfrac{-3x^2}{2} + 0.01x^3 + 250\right)$

(d) $P(x) = 750x + \dfrac{3x^2}{2} - 0.01x^3 - 250$

$P'(x) = 750 + 3x - 0.03x^2$

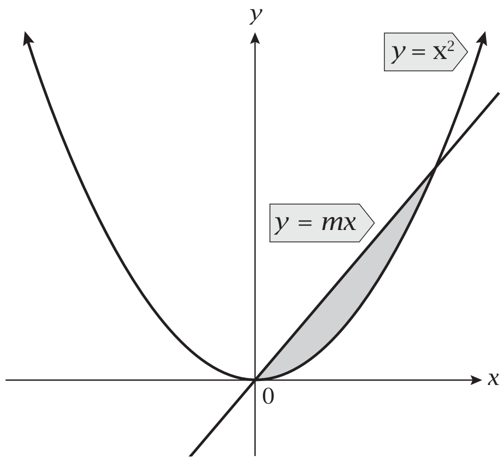
For max profit $P'(x) = 0$

$\therefore 750 + 3x - 0.03x^2 = 0$

$x = 215.83$

Maximum profit of \$130 957.04 when 216 printers are produced.

16.



$y = x^2$ and $y = mx$

$\therefore x^2 - mx = 0$

$x(x - m) = 0$

$\therefore x = 0, x = m$ *(points of intersection)*

$$\int_0^m (mx - x^2)\,dx = 24.813$$

$$\left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m = 24.813$$

$$\frac{m^3}{2} - \frac{m^3}{3} = 24.813$$

$$m^3 = 148.878$$

$$m = 5.3$$

17. (a) $\int_{-2}^{2} f(x)\,dx = -3$

(b) $\int_{0}^{2} f(-x)\,dx = -7$

(c) $\int_{0}^{-2} f(x)\,dx = 10$

(d) $\int_{-2}^{2} f(x)\,dx + \int_{-2}^{2} 3\,dx$

$= -3 + \left[3x\right]_{-2}^{2}$

$= -3 + 6 + 6$

$= 9$

18. $a = 10t - 30$

$v = \frac{10t^2}{2} - 30t + c$

$v = 5t^2 - 30t + 25$

Rest when $v = 0$

$\therefore 5t^2 - 30t + 25 = 0$

$5(t^2 - 6t + 5) = 0$

$5(t-1)(t-5) = 0$

$\therefore t = 1, t = 5$ secs

Comes to rest for 2nd time when $t = 5$ secs

Total distance

$\int_{0}^{5} \left|5t^2 - 30t + 25\right| dt$

$= 65$ m

19. (a) $\int_{400}^{500} \frac{14.5}{\sqrt[3]{x}} + 7\,dx$

(b) Average cost $= \frac{\$889.39}{100}$

$= \$8.89$ per magazine.

20. Area $= \int_{0}^{3} (-x^2 + 3x + 6)\,dx - \left(\frac{1}{2} \times 2 \times 6\right)$

$= \frac{33}{2}$ units^2

21. (a) $C = \frac{0.02x^2}{2} + 150x + d$

$C = 0.01x^2 + 150x + 2500$

$C(50) = 0.01(50)^2 + 150(50) + 2500$

$= \$10025$

(b) $\int_{100}^{200} (0.02x + 150)\,dx$

$= \$15300$

22. (a) $\int_{2}^{3} \left|(t^2 - 4t + 3)\right| dt$

$= \frac{2}{3}$ metres

(b) $\int_{1}^{3} \left|t^2 - 4t + 3\right| dt$

$= \frac{4}{3}$ metres

(c) $v = t^2 - 4t + 3$

$x = \frac{t^3}{3} - 2t^2 + 3t + c$

$-15 = c$

$\therefore x = \frac{t^3}{3} - 2t^2 + 3t - 15$

when $t = 3$

$x = -15$ metres

(d) when $t = 2$

$a = 2t - 4$

$a = 2(2) - 4$

$a = 0$ m^2/sec

23. (a) $B = \int_{0}^{4} \left[x - (x^2 - 3x)\right] dx$

$= 10\tfrac{2}{3}$ units^2

(b) $A = \int_{k}^{0} (x^2 - 3x - x)\,dx$

(c) $\int_{k}^{0} (x^2 - 3x - x)\,dx = 10\frac{2}{3}$

$k = -2$

Using CAS calculator

CHAPTER 5: *Trigonometric Functions*

1.

x	$\frac{1 - \cos x}{x}$
-0.1	-0.04996
-0.01	-0.00499996
-0.001	-0.00049999996
-0.0001	-0.00005
-0.00001	-0.000005
-0.000001	0
-0.0000001	0

x	
0.1	0.04996
0.01	0.00499996
0.001	0.00049999996
0.0001	0.00005
0.00001	0.000005
0.000001	0
0.0000001	0

$\therefore \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

2. (a) $y = \cos(x) + \sin(x)$

$\frac{dy}{dx} = \cos(x) - \sin(x)$

(b) $y = 3x - \sin(x)$

$\frac{dy}{dx} = 3 - \cos(x)$

(c) $y = \tan(x)$

$$y = \frac{\sin(x)}{\cos(x)}$$

$$\frac{dy}{dx} = \frac{\cos(x) \cdot (\cos x) - \sin(x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)}$$

(d) $f(x) = \sqrt{\cos(x)}$

$$f(x) = \left[\cos(x)\right]^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left[\cos(x)\right]^{-\frac{1}{2}} \cdot (-\sin(x))$$

$$= -\frac{\sin(x)}{2\sqrt{\cos(x)}}$$

(e) $f(x) = 2x \sin(3x)$

$$f'(x) = 2 \sin(3x) + (2x)(3 \cos(3x))$$

$$= 2 \sin(3x) + 6x \cos(3x)$$

3. (a) $y = 5 \sin(x)$

$$\frac{dy}{dx} = 5 \cos(x)$$

(b) $y = x \cos(x)$

$$\frac{dy}{dx} = (1)\cos(x) + x(-\sin x)$$

$$\cos(x) - x \sin(x)$$

(c) $y = x^3 \sin(x)$

$$\frac{dy}{dx} = (3x^2)(\sin(x)) + (x^3)(\cos x)$$

$$= 3x^2 \sin(x) + x^3 \cos(x)$$

4. (a) $y = \frac{\sin(x)}{x}$

$$\frac{dy}{dx} = \frac{(\cos x)(x) - \sin(x) \cdot (1)}{x^2}$$

$$= \frac{x \cos(x) - \sin(x)}{x^2}$$

(b) $y = \frac{x^2}{\cos(x)}$

$$\frac{dy}{dx} = \frac{2x \cos(x) - (x^2)(-\sin(x))}{\cos^2 x}$$

$$= \frac{2x \cos(x) + x^2 \sin(x)}{\cos^2 x}$$

(c) $y = \frac{3 \cos(x)}{\sin(2x)}$

$$\frac{dy}{dx} = \frac{(-3 \sin(x))(\sin(2x)) - (3 \cos(x))(2 \cos(2x))}{\left[\sin(2x)\right]^2}$$

5. (a) $f(x) = \sin(6x)$

$$f'(x) = 6 \cos(6x)$$

(b) $f(x) = 2 \cos(3x)$

$$f'(x) = -6 \sin(3x)$$

(c) $f(x) = 2x + 4 \cos(5x)$

$$f'(x) = 2 - 20 \sin(5x)$$

6. (a) $y = 2 \cos(x) + 3 \sin(2x)$

$$\frac{dy}{dx} = -2 \sin(x) + 6 \cos(2x)$$

(b) $y = \frac{3 + \sin(x)}{x^3}$

$$\frac{dy}{dx} = \frac{(\cos(x))(x^3) - (3x^2)(3 + \sin(x))}{(x^3)^2}$$

$$= \frac{x^3 \cos(x) - 9x^2 - 3x^2 \sin(x)}{x^6}$$

$$= \frac{x \cos(x) - 9 - 3 \sin(x)}{x^4}$$

(c) $y = 6 \sin^2(x)$

$$\frac{dy}{dx} = 12 \sin(x) \cdot \cos(x)$$

(d) $y = 3 \cos^3(x)$

$$\frac{dy}{dx} = 9 \cos^2(x) \cdot (-\sin x)$$

$$= -9 \cos^2(x) \cdot \sin(x)$$

(e) $y = \frac{4 \sin(2x)}{\cos(3x)}$

$$\frac{dy}{dx} = \frac{8 \cos(2x)\cos(3x) + 4 \sin(2x) \cdot 3 \sin(3x)}{\left[\cos(3x)\right]^2}$$

7. (a) $y = \sin(3x)$

$$\frac{dy}{dx} = 3 \cos(3x)$$

$$\frac{d^2y}{dx^2} = -9 \sin(3x)$$

(b) $y = x \cos(x)$

$$\frac{dy}{dx} = \cos(x) - x \sin(x)$$

$$\frac{d^2y}{dx^2} = -\sin(x) - \left[\sin(x) + x \cos(x)\right]$$

$$= -\sin(x) - \sin(x) - x \cos(x)$$

$$= -2 \sin(x) - x \cos(x)$$

(c) $y = \sin^2(x)$

$$\frac{dy}{dx} = 2\sin(x)\cos(x)$$

$$\frac{d^2y}{dx^2} = 2\cos(x)\cos(x) + 2\sin(x)(-\sin(x))$$

$$= 2\cos^2 x - 2\sin^2 x$$

8. $y = \sin(x)$

$$\frac{dy}{dx} = \cos(x)$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = \frac{1}{\sqrt{2}}$$

9. $y = \dfrac{4\sin(2x)}{1+\cos(x)}$

$$\frac{dy}{dx} = \frac{8\cos(x)\cos(2x) + 4\sin(x)\sin(2x) + 8\cos(2x)}{\left[1+\cos(x)\right]^2}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 0.9706\left(\frac{8}{4+3\sqrt{2}}\right)$$

10. $y = \sin(2x)$

$$\frac{dy}{dx} = 2\cos(2x)$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = -2$$

$\therefore y = mx + c$

$y = -2x + c$

$\left(\frac{\pi}{2}, 0\right)$ $\quad 0 = -2\left(\frac{\pi}{2}\right) + c$

$c = \pi$

$\therefore y = -2x + \pi$

11. $y = \sin^2(x)$

$$\frac{dy}{dx} = 2\sin(x)\cos(x)$$

$1 = 2\sin(x)\cos(x)$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$y = \frac{1}{2}, \frac{1}{2}$

$\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

12. $y = x\cos(3x)$

$$\frac{dy}{dx} = \cos(3x) - 3x(\sin 3(x))$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = -1$$

$\left(\frac{\pi}{3}, \frac{-\pi}{3}\right)$

$y = mx + c$

$y = -x + c$

$-\frac{\pi}{3} = -\frac{\pi}{3} + c$

$c = 0$

$\therefore y = -x$

13. (a) $\int\left[6\sin(x)\right]dx$

$= -6\cos(x) + c$

(b) $\int\left[-2\cos(3x)\right]dx$

$= -\frac{2}{3}\sin(3x) + c$

(c) $\int\left[15\sin(3x)\right]dx$

$= -5\cos(3x) + c$

(d) $\int\left[\sin\left(\frac{2x}{3}\right)\right]dx$

$$= \frac{-3\cos\left(\frac{2x}{3}\right)}{2} + c$$

(e) $\int\left[\cos(-x)\right]dx$

$= \sin(x) + c$

14. (a) $\int\left[5\sin(2x+3)\right]dx$

$$= \frac{-5\cos(2x+3)}{2} + c$$

(b) $\int\left[4\cos(4x-3)\right]dx$

$= \sin(4x-3) + c$

(c) $\int\left[\sin\left(2x + \frac{\pi}{3}\right)\right]dx$

$$= \frac{-\cos\left(2x + \frac{\pi}{3}\right)}{2} + c$$

(d) $\int\left[5\cos(2x) + 6\sin(3x)\right]dx$

$$= \frac{5\sin(2x)}{2} - 2\cos(3x) + c$$

(e) $\int\left[\sin(x)\cos^3(x)\right]dx$

$$= \frac{-\cos^4 x}{4} + c$$

(f) $\int\left[15\cos(x)\sin^2(x)\right]dx$

$= 5\sin^3(x) + c$

(g) $\int\left[\cos(2x)\cos(4x)-\sin(2x)\sin(4x)\right]dx$
$=\int\cos(6x)\,dx$
$=\dfrac{\sin(6x)}{6}+c$

15. (a) $\int_0^{\frac{\pi}{2}}\left[\cos(x)\right]dx$
$=\left[\sin x\right]_0^{\frac{\pi}{2}}$
$=\sin\left(\dfrac{\pi}{2}\right)-\sin(0)$
$=1$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\sin\left(\dfrac{x}{2}\right)\right]dx$
$=\left[-2\cos\left(\dfrac{x}{2}\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$
$=\left(-2\cos\left(\dfrac{\pi}{4}\right)\right)-\left(-2\cos\left(\dfrac{-\pi}{4}\right)\right)$
$=0$

16. (a) $y=3\sin(2t)+4$
$v=\dfrac{dy}{dt}=6\cos(2t)$
velocity: $v(t)=6\cos(2t)$

(b) Maximum velocity = 6 m/s

(c) Acceleration $a(t) = -12\sin(2t)$

(d) $v\left(\dfrac{\pi}{2}\right)=6\cos\left(2\times\dfrac{\pi}{2}\right)$
$=-6$ m/s

17. $x = 15 + 2\sin(3t)$

(a) Shortest length when $t=\dfrac{\pi}{2}$, $x = 13$ cm

(b) $\dfrac{dx}{dt}=6\cos(3x)$
$\left.\dfrac{dx}{dt}\right|_{x=\frac{\pi}{6}}=0$

18. (a) $y=\sin^2(x)$
Stationary points when $\dfrac{dy}{dx}=0$
i.e. $\dfrac{dy}{dx}=2\sin(x)\cos(x)$
$0=2\sin(x)\cos(x)$
$x=0,\ \dfrac{\pi}{2},\ \pi,\ \dfrac{3\pi}{2},\ 2\pi$
$y = 0, 1, 0, 1, 0$
Stationary points
$(0, 0)\ \left(\dfrac{\pi}{2}, 1\right)\ (\pi, 0)\ \left(\dfrac{3\pi}{2}, 1\right)\ (2\pi, 0)$

Points of inflection when
$\dfrac{d^2y}{dx^2}=0$
i.e. $-2\cos^2(x)-2\sin^2 x=0$
$x=\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{7\pi}{4}$
$y=\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2}$

Points of inflection
$\left(\dfrac{\pi}{4},\dfrac{1}{2}\right)\left(\dfrac{3\pi}{4},\dfrac{1}{2}\right)\left(\dfrac{5\pi}{4},\dfrac{1}{2}\right)\left(\dfrac{7\pi}{4},\dfrac{1}{2}\right)$

(b) $y = \sin(x) + \cos(x)$
Stationary points when $\dfrac{dy}{dx}=0$
i.e. $\dfrac{dy}{dx}=\cos(x)-\sin(x)$
$\cos(x)-\sin(x)=0$
$x=\dfrac{\pi}{4},\dfrac{5\pi}{4}$
$y=\sqrt{2},-\sqrt{2}$

Stationary points $\left(\dfrac{\pi}{4},\sqrt{2}\right),\left(\dfrac{5\pi}{4},-\sqrt{2}\right)$

Points of inflection when
$\dfrac{d^2y}{dx^2}=0$
i.e. $\dfrac{d^2y}{dx^2}=-\cos(x)-\sin(x)$
$-\cos(x)-\sin(x)=0$
$\therefore x=\dfrac{3\pi}{4},\dfrac{7\pi}{4}$
$y = 0, 0$

Points of inflection $\left(\dfrac{3\pi}{4}, 0\right)\left(\dfrac{7\pi}{4}, 0\right)$

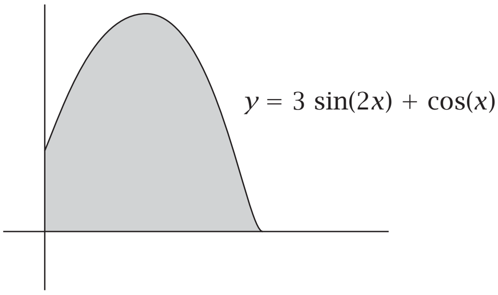
19. (a) Maximum height = 135 m

(b) Minimum height = 15 m

(c) One revolution is 25 minutes

(d) $t = 7.96$ and
$t = 17.04$ minutes

(e) $y=75-60\cos\left(\dfrac{2\pi t}{25}\right)$
$\dfrac{dy}{dt}=\dfrac{24\pi}{5}\sin\left(\dfrac{2\pi t}{25}\right)$
$\left.\dfrac{dy}{dt}\right|_{t=10.5}=7.2647$ m/min

20. 

$$\text{Area} = \int_0^{\frac{\pi}{2}} \left[3\sin(2x) + \cos(x)\right] dx$$
$$= 4 \text{ units}^2$$

21. (a) $(-\pi, -4)$

(b) $-\int_{-\pi}^{0} 4\sin\left(\frac{x}{2}\right) dx$

$A = 8 \text{ units}^2$

(c) Area Rectangle $= 4 \times \pi$
$= 4\pi$

$B = 4\pi - 8 \text{ units}^2$

22. $a(t) = \frac{-3\pi^2}{4}\cos\left(\frac{\pi t}{2}\right)$

(a) $v(t) = \int \left[\frac{-3\pi^2}{4}\cos\left(\frac{\pi t}{2}\right)\right] dt$

$v(t) = \frac{-3\pi}{2}\sin\left(\frac{\pi t}{2}\right)$

(b) $x(t) = 3\cos\left(\frac{\pi t}{2}\right) + c$

$3 = 3\cos\left(\frac{\pi(0)}{2}\right) + c$

$3 = 3 + c$

$c = 0$

$x(t) = 3\cos\left(\frac{\pi t}{2}\right)$

(c) $v(3) = \frac{-3\pi}{2}\sin\left(\frac{3\pi}{2}\right)$

$v(3) = \frac{3\pi}{2}$ m/s

$x(3) = 3\cos\left(\frac{3\pi}{2}\right)$

$x(3) = 0$ metres

23. (a) $y = e^x \sin(3x)$

$\frac{dy}{dx} = e^x \cdot \sin(3x) + e^x \cdot 3\cos(3x)$

(b) $y = \cos(x) e^{-2x}$

$\frac{dy}{dx} = -\sin(x) \cdot e^{-2x} + \cos(x) \cdot \left(-2e^{-2x}\right)$

(c) $y = x\, e^{\sin(x)}$

$\frac{dy}{dx} = e^{\sin(x)} + x\cos(x)\, e^{\sin(x)}$

(d) $y = \cos\left[\ln\left(\frac{1}{x}\right)\right]$

$\frac{dy}{dx} = -\sin\left[\ln\left(\frac{1}{x}\right)\right] \cdot \left(-\frac{1}{x}\right)$

$= \frac{\sin\left[\ln\left(\frac{1}{x}\right)\right]}{x}$

(e) $y = \left[\sin(x) + \ln(x)\right]^2$

$\frac{dy}{dx} = 2\left[\sin(x) + \ln(x)\right] \cdot \left(\cos x + \frac{1}{x}\right)$

CHAPTER 6: *Logarithms*

1. (a) $6^2 = 36$

(b) $4^{-2} = \frac{1}{16}$

(c) $a^c = b$

2. (a) $\log_2 32 = 5$

(b) $\log_7 \frac{1}{343} = -3$

(c) $\log_p q = r$

3. (a) $\log_3 81 = x$
$3^x = 81$
$3^x = 3^4$
$\therefore x = 4$
$\log_3 81 = 4$

(b) $\log_{10} 10^6$
$= 6\log_{10} 10$
$= 6$

(c) $\log_2\left(\frac{1}{32}\right) = x$

$2^x = \frac{1}{32}$

$2^x = \frac{1}{2^5}$

$2^x = 2^{-5}$

$x = -5$

$\therefore \log_2\left(\frac{1}{32}\right) = -5$

(d) $\log_6(1) = 0$

(e) $\log_5(0.04) = x$

$$5^x = 0.04$$
$$5^x = \frac{4}{100}$$
$$5^x = \frac{1}{5^2}$$
$$5^x = 5^{-2}$$
$$x = -2$$
$$\therefore \log_g(0.04) = -2$$

(f) $\log_e e^4$

$= 4 \log_e e$

$= 4$

(g) $\log_e \frac{1}{\sqrt{e}} = x$

$$e^x = \frac{1}{\sqrt{e}}$$
$$e^x = e^{-\frac{1}{2}}$$
$$x = -\frac{1}{2}$$
$$\therefore \ln \frac{1}{\sqrt{e}} = -\frac{1}{2}$$

(h) $\log_e \sqrt[5]{e} = x$

$$e^x = e^{\frac{1}{5}}$$
$$x = \frac{1}{5}$$
$$\therefore \ln \sqrt[5]{e} = \frac{1}{5}$$

4. (a) $\log_3 8 + \log_3\left(\frac{1}{8}\right)$

$= \log_3\left(8 \times \frac{1}{8}\right)$

$= \log_3 1$

$= 0$

(b) $2\log 5 + \log 8 - \frac{1}{2}\log 4$

$= \log 5^2 + \log 8 - \log 4^{\frac{1}{2}}$

$= \log(25 \times 8) - \log 2$

$= \log \frac{200}{2}$

$= \log 100$

$= 2$

(c) $\frac{\log 16}{\log 2}$

$= \frac{\log 2^4}{\log 2}$

$= \frac{4\cancel{\log 2}}{\cancel{\log 2}}$

$= 4$

(d) $\frac{2}{3}\log_2 8 + 6\log_2 \sqrt[3]{2} - \frac{1}{2}\log_2\left(\frac{1}{4}\right)$

$= \log_2 8^{\frac{2}{3}} + \log_2\left(2^{\frac{1}{3}}\right)^6 - \log_2\left(\frac{1}{4}\right)^{\frac{1}{2}}$

$= \log_2\left(2^3\right)^{\frac{2}{3}} + \log_2\left(2^{\frac{1}{3}}\right)^6 - \log_2\left(2^{-2}\right)^{\frac{1}{2}}$

$= \log_2 2^2 + \log_2 2^2 - \log_2 2^{-1}$

$= \log_2 2^4 - \log_2 2^{-1}$

$= \log_2\left(\frac{2^4}{2^{-1}}\right)$

$= \log_2 2^5$

$= 5\log_2 2$

$= 5$

(e) $\log_a\left(a^4\right) + \log_b\left(b^3\right) - \log_c\left(c^2\right)$

$= 4\log_a a + 3\log_b b - 2\log_c c$

$= 4 + 3 - 2$

$= 5$

5. (a) $3\log a - \log b - \log c$

$= \log a^3 - \log b - \log c$

$= \log \frac{a^3}{b} - \log c$

$= \log \frac{a^3}{bc}$

(b) $\log a + \frac{1}{2}\log c - 2\log b$

$= \log a + \log c^{\frac{1}{2}} - \log b^2$

$= \log \frac{ac^{\frac{1}{2}}}{b^2}$

(c) $2 + \log a^3 - 2\log a$

$= \log 100 + \log a^3 - \log a^2$

$= \log \frac{100a^3}{a^2}$

$= \log 100a$

6. $\log(pq) = 5 \quad \log\left(\frac{p}{q}\right) = 1$

$\therefore 10^5 = pq \quad 10' = \frac{p}{q}$

$\therefore p = 10q$

$$\therefore (10q)(q) = 10^5$$
$$10q^2 = 10^5$$
$$q^2 = 10^4$$
$$q = \sqrt{10^4}$$
$$\underline{q = 100}$$
$$p = 10(100)$$
$$\underline{p = 1000}$$

7. (a) $\log_c 6 = \log_c(2\times 3)$
$= \log_c 2 + \log_c 3$
$= q + p$

(b) $\log_c 1.5 = \log_c\left(\frac{3}{2}\right)$
$= \log_c 3 - \log_c 2$
$= p - q$

(c) $\log_c 1\frac{1}{8} = \log_c \frac{9}{8}$
$= \log_c 9 - \log_c 8$
$= \log_c 3^2 - \log_c 2^3$
$= 2\log_c 3 - 3\log_c 2$
$= 2p - 3q$

(d) $\log_c \sqrt[4]{36} = \log_c (36)^{\frac{1}{4}}$
$= \frac{1}{4}\log_c 36$
$= \frac{1}{4}\log_c\left(3^2\times 2^2\right)$
$= \frac{1}{4}\left[2\log_c 3 + 2\log_c 2\right]$
$= \frac{1}{4}\left[2p + 2q\right]$
$= \frac{1}{2}p + \frac{1}{2}q$

8. (a) $3\log_2 y = 2\log_2 x$
$\log_2 y^3 = \log_2 x^2$
$\therefore y^3 = x^2$
$y = \sqrt[3]{x^2}$

(b) $\log_2 y + 2 = \log_2 x^3$
$\log_2 y + \log_2 4 = \log_2 x^3$
$\log_2 4y = \log_2 x^3$
$\therefore 4y = x^3$
$y = \frac{x^3}{4}$

(c) $2\log_2(xy) = 5\log_2(x)$
$\log_2(xy)^2 = \log_2 x^5$
$\log_2 x^2y^2 = \log_2 x^5$
$\therefore x^2y^2 = x^5$
$y^2 = x^3$
$y = \sqrt{x^3}$

9. (a) $3^x = 8$
$\log 3^x = \log 8$
$x\log 3 = \log 8$
$x = \frac{\log 8}{\log 3}$

(b) $5^{2x+1} = 9$
$\log 5^{2x+1} = \log 9$
$(2x+1)\log 5 = \log 9$
$2x + 1 = \frac{\log 9}{\log 5}$
$2x = \frac{\log 9}{\log 5} - 1$
$x = \frac{\left(\frac{\log 9}{\log 5} - 1\right)}{2}$

(c) $6^{1-x} = 2^{3x+5}$
$\log 6^{1-x} = \log 2^{3x+5}$
$(1-x)\log 6 = (3x+5)\log 2$
$\log 6 - x\log 6 = 3x\log 2 + 5\log 2$
$3x\log 2 + x\log 6 = \log 6 - 5\log 2$
$x(3\log 2 + \log 6) = \log 6 - 5\log 2$
$x = \frac{\log 6 - 5\log 2}{3\log 2 + \log 6}$

(d) $2.2^{2x} = 11.2^x + 5 = 0$
Let $y = 2^x$ $\quad 2\bullet\left(2^x\right)^2 - 11\bullet\left(2^x\right) + 5 = 0$
$2y^2 - 11y + 5 = 0$
$(2y-1)(y-5) = 0$
$y = \frac{1}{2}, y = 5$
$\therefore 2^x = \frac{1}{2}, 2^x = 5$
$2^x = 2^{-1}, x\log 2 = \log 5$
$x = -1, x = \frac{\log 5}{\log 2}$

(e) $e^{5x+1} = 3$
$\ln e^{5x+1} = \ln 3$
$(5x+1)\ln e = \ln 3$
$5x + 1 = \ln 3$
$5x = \ln 3 - 1$
$x = \frac{\ln 3 - 1}{5}$

(f) $6e^{1-2x} = 360$
$e^{1-2x} = 60$
$\ln e^{1-2x} = \ln 60$
$(1-2x)\ln e = \ln 60$
$1 - 2x = \ln 60$
$-2x = \ln 60 - 1$
$x = \frac{\ln 60 - 1}{-2}$

10. (a) $y = \ln(2x+3)$
$\frac{dy}{dx} = \frac{2}{2x+3}$

(b) $y = \ln\left(x^4\right)$
$\frac{dy}{dx} = \frac{4}{x}$

(c) $y = x\ln(3x)$

$$\frac{dy}{dx} = \ln(3x) + x \bullet \frac{3}{3x}$$
$$= \ln(3x) + 1$$

(d) $y = \sqrt{x}\ln\left(\frac{x}{3}\right)$

$$\frac{dy}{dx} = \left(\frac{1}{2}x^{-\frac{1}{2}}\right)\left(\ln\frac{x}{3}\right) + \left(\sqrt{x}\right) \bullet \frac{1}{x}$$
$$= \frac{\ln\left(\frac{x}{3}\right)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

(e) $y = \frac{\ln(x)}{x}$

$$\frac{dy}{dx} = \frac{\left(\frac{1}{x} \bullet x\right) - (\ln x)(1)}{x^2}$$
$$= \frac{1 - \ln x}{x^2}$$

(f) $y = \frac{1}{10 - \ln x}$

$$y = (10 - \ln x)^{-1}$$
$$\frac{dy}{dx} = -1(10 - \ln x)^{-2} \bullet \left(-\frac{1}{x}\right)$$
$$= \frac{1}{x(10 - \ln x)^2}$$

(g) $y = (\ln x)^2$

$$\frac{dy}{dx} = 2(\ln x) \bullet \frac{1}{x}$$
$$= \frac{2\ln x}{x}$$

11. (a) $y = e^{2x}\ln(2x)$

$$\frac{dy}{dx} = \left(2e^{2x}\right)(\ln 2x) + \left(e^{2x}\right) \bullet \left(\frac{1}{x}\right)$$

(b) $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \bullet \cos x$$
$$\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

(c) $y = \ln\left(e^{-2x} + 4\right)$

$$\frac{dy}{dx} = \frac{1}{e^{-2x} + 4} \bullet \left(-2e^{-2x}\right)$$
$$= \frac{-2}{4e^{2x} + 1}$$

(d) $y = \frac{\cos^2 x}{\ln x}$

$$\frac{dy}{dx} = \frac{(-2\cos x \sin x)(\ln x) - \left(\cos^2 x\right)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

(e) $y = \ln\left[\frac{(x+4)^2}{(3x-1)}\right]$

$$y = \ln\left[(x+4)^2\right] - \ln(3x-1)$$
$$y = 2\ln(x+4) - \ln(3x-1)$$
$$\frac{dy}{dx} = \frac{2}{x+4} - \frac{3}{3x-1}$$

12. (a) $\int \frac{6}{x}\,dx = 6\ln(x) + c$

(b) $\int \frac{1}{6x+5}\,dx = \frac{1}{6}\ln(6x+5) + c$

(c) $\int \frac{4x}{2-x^2}\,dx = -2\int \frac{-2x}{2-x^2}\,dx$

$$= -2\ln\left(2 - x^2\right) + c$$

(d) $\int \frac{2x-1}{x^2-x} = \ln\left(x^2 - x\right) + c$

(e) $\int \frac{x-3}{x^2-6x+1}\,dx = \frac{1}{2}\int \frac{2x-6}{x^2-6x+1}\,dx$

$$= \frac{1}{2}\ln\left(x^2 - 6x + 1\right) + c$$

(f) $\int \frac{\cos x}{\sin x} = \ln(\sin x) + c$

(g) $\int \frac{\sin 2x}{1+\cos 2x}\,dx$

$$= -\frac{1}{2}\int \frac{-2\sin 2x}{1+\cos 2x}\,dx$$
$$= -\frac{1}{2}\ln(1 + \cos 2x) + c$$

(h) $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$

$$= -\ln(\sin x + \cos x) + c$$

13. (a) $\int_2^3 \left(\frac{5}{x}\right)dx$

$$= \left[5\ln x\right]_2^3$$
$$= 5\ln 3 - 5\ln 2$$

(b) $\int_1^4 \left(\frac{3}{2x-1}\right)dx$

$$= \frac{3}{2}\int_1^4 \left(\frac{2}{2x-1}\right)dx$$
$$= \frac{3}{2}\left[\ln(2x-1)\right]_1^4$$
$$= \frac{3}{2}\left[\ln(7) - \ln(1)\right]$$
$$= \frac{3}{2}\ln(7)$$

(c) $\int_2^4 \left(\frac{1}{x} + e^x\right)dx$

$$= \left[\ln(x) + e^x\right]_2^4$$
$$= \left(\ln(4) + e^4\right) - \left(\ln(2) + e^2\right)$$
$$= \ln(4) + e^4 - \ln(2) - e^2$$

14. $\int_1^p \frac{3}{2x-1}\,dx = 2$

$\frac{3}{2}\left[\ln(2x-1)\right]_1^p = 2$

CAS calculator $p \approx 2.4$

15. $y = x^2 \ln(x)$

$\frac{dy}{dx} = (2x)(\ln x) + x^2 \bullet \frac{1}{x}$

$\frac{dy}{dx} = 2x\ln x + x$

when $x = 1$ $\quad \frac{dy}{dx} = 1$

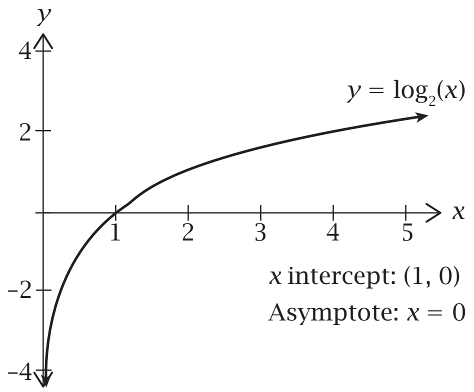
$\therefore y = x + c \quad (1,0)$

$0 = 1 + c$

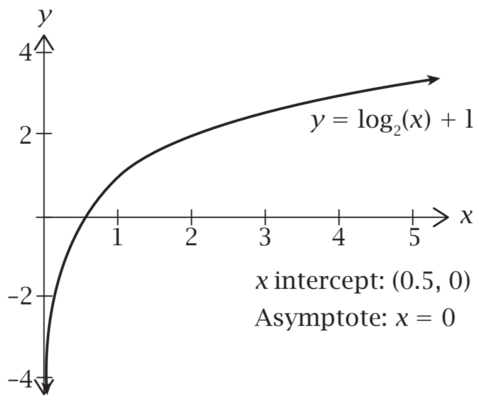
$c = -1$

Tangent $y = x - 1$

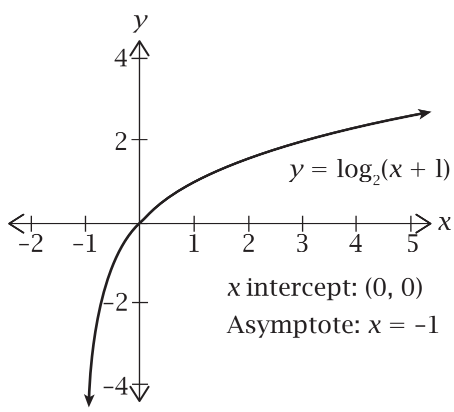
16. (a)



(b)



(c)



17. $y = \ln(kx-1)$

gradient = 1

when $x = 2$

$\frac{dy}{dx} = \frac{k}{kx-1}$

$1 = \frac{k}{2k-1}$

$2k - 1 = k$

$2k - k = 1$

$k = 1$

18. (a) $y = \log_2(x+3) + 2$

(b) $y = \log_3(x-2) - 1$

19. $y = x\ln x$

$\frac{dy}{dx} = \ln(x) + 1$

$1 = \ln(x) + 1$

$0 = \ln(x)$

$\therefore x = 1$

$y = 0$

$(1,0)$

20. $y = \ln x + \frac{1}{x}$

Minimum value when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2}$

$0 = \frac{x-1}{x^2}$

$\therefore x = 1$

$y = 1$

$(1,1)$

Minimum value is $y = 1$

21. $C(x) = 500 + 300\ln(x+1)$

$C'(x) = \frac{300}{x+1}$

when marginal cost = \$3

$3 = \frac{300}{x+1}$

$3x + 3 = 300$

$3x = 297$

$x = 99$

Cost $= 500 + 300\ln(1+99)$

$= \$1881.55$

Average cost $= \frac{\$1881.55}{99}$

$= \$19.01$

22. (a) $1 < a < e$

(b) $(1,0)$

23. $$y = x\ln x$$
$$\frac{dy}{dx} = \ln x + 1$$
when $x = e$
$$\frac{dy}{dx} = \ln e + 1$$
$$= 2$$
$$\therefore y = 2x + c$$
$$(e, e) \quad e = 2e + c$$
$$c = -e$$
$$\therefore y = 2x - e$$

24. $x = 15\ln(2t-5) - 5t$
$$v = \frac{30}{2t-5} - 5$$
$$a = -\frac{60}{(2t-5)^2}$$

(a) $$\frac{30}{2t-5} - 5 = -\frac{60}{(2t-5)^2}$$
$t \approx 1.7087$ and 6.2913 seconds

(b) Particle is at rest when
$$v = 0$$
$$\therefore \frac{30}{2t-5} - 5 = 0$$
$$\frac{30}{2t-5} = 5$$
$$30 = 10t - 25$$
$$55 = 10t$$
$t = 5.5$ secs

25. $$y = -\ln(x) + 2x^2$$
$$\frac{dy}{dx} = -\frac{1}{x} + 4x$$
$$0 = -\frac{1}{x} + 4x$$
$$\therefore x = \frac{1}{2} \text{ or } \cancel{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2} = 4 + \frac{1}{x^2}$$
when $x = \frac{1}{2}$
$$\frac{d^2y}{dx^2} = 8 > 0$$
$\therefore$ minimum value
Hence $\left(\frac{1}{2}, -\ln\left(\frac{1}{2}\right) + \frac{1}{2}\right)$
is a minimum turning point.

26. (a) $$\text{pH} = -\log\left[\text{H}^+\right]$$
$$6.89 = -\log\left[\text{H}^+\right]$$
$$10^{-6.89} = \text{H}^+$$
$$\text{H}^+ = 1.288 \times 10^{-7}$$

(b) $$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log\left[1.25 \times 10^{-8}\right]$$
$$\text{pH} = 7.903$$

27. (a) $$m - M = 5(\log d - 1)$$
$$0.25 - M = 5(\log 450 - 1)$$
$$M = -8.02$$

(b) $$7.4 - 0.9 = 5(\log d - 1)$$
$d = 199.53$ parsecs

28. $A = 400\, e^{-0.2t}$

(a) $A = 400\, e^{-0.2(3)}$
$A = 219.52$ grams

(b) $$200 = 400\, e^{-0.2t}$$
$$e^{-0.2t} = \frac{1}{2}$$
$$\ln e^{-0.2t} = \ln\left(\frac{1}{2}\right)$$
$$-0.2t \ln e = \ln\left(\frac{1}{2}\right)$$
$$-0.2t = \ln\left(\frac{1}{2}\right)$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.2}$$
$t = 3.466$ years

(c) $$100 = 400\, e^{-0.2t}$$
$$-0.2t \ln e = \ln\left(\frac{1}{4}\right)$$
$$-0.2t = \ln\left(\frac{1}{4}\right)$$
$$t = \frac{\ln\left(\frac{1}{4}\right)}{-0.2}$$
$t = 6.931$ years

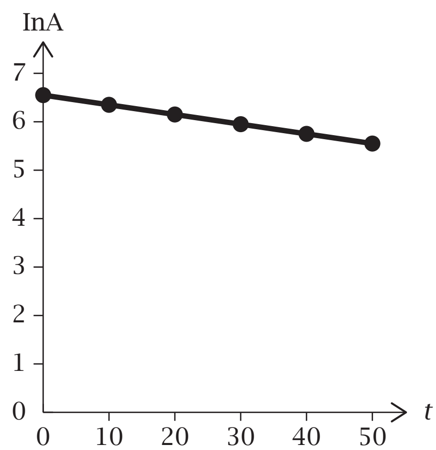
29. $p = -4$
$a = 4$
$q = 5$

30. (a) $A = 150\,000(1.06)^t$
$A = 150\,000(1.06)^4$
$A = \$189\,371.54$

(b) $$275000 = 150000(1.06)^t$$
$$\frac{275000}{150000} = 1.06^t$$
$$\frac{11}{6} = 1.06^t$$
$$\log\left(\frac{11}{6}\right) = \log(1.06)^t$$
$$\log\left(\frac{11}{6}\right) = t\log(1.06)$$
$$t = \frac{\log\left(\frac{11}{6}\right)}{\log(1.06)}$$
$$t = 10.402$$
10.402 years

(c) $$A = 150000(1.06)^t$$
$$450000 = 150000(x)^{10}$$
$$3 = x^{10}$$
$$x = \sqrt[10]{3}$$
$$x = 1.1161$$
Interest rate is 11.61% p.a.

31. (a)



(b) $$A = A_0e^{kt}$$
$$A_0 = 700$$
$$A = A_0e^{kt}$$
$$573 = 700e^{k(10)}$$
$$\frac{573}{700} = e^{10k}$$
$$\ln\left(e^{10k}\right) = \ln\left(\frac{573}{700}\right)$$
$$10k\cancel{\ln e} = \ln\left(\frac{573}{700}\right)$$
$$k = \frac{\ln\left(\frac{573}{700}\right)}{10}$$
$$k = -0.02$$

(c) $$A = 700e^{-0.02t}$$
$$150 = 700e^{-0.02t}$$
$$\frac{150}{700} = e^{-0.02t}$$
$$\ln\left(\frac{150}{700}\right) = -0.02t\ln e$$
$$t = \frac{\ln\left(\frac{150}{700}\right)}{-0.02}$$
$$t = 77.022 \text{ years}$$

CHAPTER 7: *Exponential Functions*

1. 'c' – translates c units right

   'b' – dilates parallel to x axis scale factor $\frac{1}{|b|}$ and if 'b' is negative a reflection in the y axis

   'a' – dilates parallel to y axis scale factor $|a|$ and if 'a' is negative a reflection in the x axis

   'd' – translates d units up

2. (a) i. Reflected in the y axis
   ii. Reflected in the x axis
   iii. Dilated parallel to the x axis scale factor of 0.5
   iv. Dilated parallel to the y axis scale factor of 2
   v. · Translated 4 units left
   · Dilated parallel to the x axis scale factor of $\frac{1}{3}$
   · Dilated parallel to the y axis scale factor of $\frac{1}{3}$
   · Translated 1 unit down

   (b) $y = -e^{\frac{1}{2}x - 3} + 4$

3. (a) $A = 8000e^{0.06(2)}$

   $A = \$9019.97$

   (b) $A = 8000e^{0.06(10)}$

   $= \$14576.95$

4. (a) $R = 2000e^{-0.02(10)}$

   $R \approx 1637.46$

   (b) $R = 2000e^{-0.02t}$

   $1000 = 2000e^{-0.02t}$

   $\therefore t \approx 34.66$ years

5. (a) $P = \dfrac{700}{1 + e^{-0.2(0)}}$

   $P = 350$

   (b) $P = \dfrac{700}{1 + e^{-0.2(5)}}$

   $P \approx 511.74$

(c) $650 = \dfrac{700}{1 + e^{-0.2t}}$

$t \approx 12.825$ days

6. (a) $y = 2e^{2x}$

$\dfrac{dy}{dx} = 4e^{2x}$

(b) $y = 5(e^x + 1)^{-1}$

$\dfrac{dy}{dx} = -5(e^x + 1)^{-2} e^x$

$\dfrac{dy}{dx} = \dfrac{-5e^x}{(e^x + 1)^2}$

(c) $y = \dfrac{1}{3e^{4x}}$

$y = \dfrac{e^{-4x}}{3}$

$\dfrac{dy}{dx} = \dfrac{-4e^{-4x}}{3}$

$= \dfrac{-4}{3e^{4x}}$

(d) $y = \sqrt{e^{4x}}$

$y = (e^{4x})^{\frac{1}{2}}$

$\dfrac{dy}{dx} = e^{2x}$

$= 2e^{2x}$

(e) $y = (e^{3x} + 1)^2$

$\dfrac{dy}{dx} = 2(e^{3x} + 1)3e^{3x}$

$\dfrac{dy}{dx} = 6e^{3x}(e^{3x} + 1)$

(f) $y = e^{\sqrt{x}}$

$y = e^{x^{\frac{1}{2}}}$

$\dfrac{dy}{dx} = \dfrac{e^{x^{\frac{1}{2}}}}{2x^{\frac{1}{2}}}$

$\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

7. (a) $y = xe^x$

$\dfrac{dy}{dx} = e^x + xe^x$

(b) $y = e^x(10 - x)^4$

$\dfrac{dy}{dx} = e^x(10 - x)^4 + (e^x)(4)(10 - x)^3(-1)$

$\dfrac{dy}{dx} = e^x(10 - x)^4 - 4e^x(10 - x)^3$

$\dfrac{dy}{dx} = e^x(10 - x)^3(6 - x)$

(c) $y = (3x + 1)^2 e^{-x}$

$\dfrac{dy}{dx} = 2(3x + 1)(3)e^{-x} + (3x + 1)^2(-1)e^{-x}$

$\dfrac{dy}{dx} = 6e^{-x}(3x + 1) - e^{-x}(3x + 1)^2$

$\dfrac{dy}{dx} = \dfrac{(3x + 1)(5 - 3x)}{e^x}$

(d) $y = \dfrac{e^{3x-2}}{e^{x-2}}$

$y = e^{2x}$

$\dfrac{dy}{dx} = 2e^{2x}$

(e) $y = \dfrac{x^3 - 4x^2 + 3x}{x^2}$

$y = x - 4 + \dfrac{3}{x}$

$\dfrac{dy}{dx} = 1 - \dfrac{3}{x^2}$

(f) $y = x^{\frac{1}{2}} e^x$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}e^x + x^{\frac{1}{2}}e^x$

$\dfrac{dy}{dx} = \dfrac{e^x}{2\sqrt{x}} + \sqrt{x}\,e^x$

8. $y = e^{2x+1}$

$\dfrac{dy}{dx} = 2e^{2x+1}$

$\left.\dfrac{dy}{dx}\right|_{x=-1} = \dfrac{2}{e}$

Point on the curve when $x = -1$ is $\left(-1, \dfrac{1}{e}\right)$

Equation of the tangent

$y = mx + c$

$\dfrac{1}{e} = \dfrac{2}{e}(-1) + c$

$c = \dfrac{3}{e}$

$\therefore y = \dfrac{2}{e}x + \dfrac{3}{e}$

9. $\dfrac{dy}{dx} = \dfrac{e^x(x) - e^x(1)}{x^2}$

$= \dfrac{e^x(x - 1)}{x^2}$

if $x = 1$ $\quad \dfrac{dy}{dx} = 0$

Equation of tangent line

$y = mx + c$

$e = c$

$\therefore y = e$

10. $\dfrac{dy}{dx} = 2xe^x + x^2 e^x$

$= xe^x (2 + x)$

Stationary when $\dfrac{dy}{dx} = 0$

$\therefore\ xe^x (2 + x) = 0$

$\therefore\ x = 0,\ x = -2$

$(0,\ 0)\quad \left(-2, \dfrac{4}{e^2}\right)$

Using first derivative test to classify

$(0,\ 0)\qquad \left.\dfrac{dy}{dx}\right|_{x=-0.5} = -0.45 < 0$

$\left.\dfrac{dy}{dx}\right|_{x=0.5} = 2.06 > 0$

$\therefore (0,\ 0)$ is a local minimum

$\left(-2, \dfrac{4}{e^2}\right)\qquad \left.\dfrac{dy}{dx}\right|_{x=-2.5} = 0.10 > 0$

$\left.\dfrac{dy}{dx}\right|_{x=-1.5} = -0.17 < 0$

$\therefore \left(-2,\ \dfrac{4}{e^2}\right)$ is a local maximum

11. (a) $y = ae^{-k^2x^2}$

$y' = -2ak^2 xe^{-k^2x^2}$

$\therefore$ Stationary points when $y' = 0$

$\therefore\ -2ak^2 xe^{-k^2x^2} = 0$

$\therefore\ x = 0$

when $x = 0$

$y = ae^0$

$y = a$

$\therefore (0,\ a)$ is a stationary point

· Nature of turning point

$y'' = -2ak^2 e^{-k^2x^2} + 4ak^4 x^2 e^{-k^2x^2}$

when $x = 0$

$y'' = -2ak^2 e^0 + 4ak^4 0 e^0$

$y'' = -2ak^2 < 0$

$\therefore (0,\ a)$ is a maximum turning point

(b) Point of inflection when $y''(x) = 0$

$\therefore -2ak^2 e^{-k^2x^2} + 4ak^4 x^2 e^{-k^2x^2} = 0$

$\therefore 2ak^2 e^{-k^2x^2} = 4ak^4 x^2 e^{-k^2x^2}$

$2ak^2 = 4ak^4 x^2$

$1 = 2k^2 x^2$

$x^2 = \dfrac{1}{2k^2}$

$x = \pm \dfrac{1}{\sqrt{2}\,k}$

$\therefore$ Points of inflection

$\left(\dfrac{1}{\sqrt{2}\,k}, \dfrac{a}{\sqrt{e}}\right)\left(\dfrac{-1}{\sqrt{2}\,k}, \dfrac{a}{\sqrt{e}}\right)$

12. (a) $\dfrac{e^{3x+1}}{3} + c$

(b) $\dfrac{4e^{2x}}{2} + c$

$= 2e^{2x} + c$

(c) $6e^{-5x}$

$= \dfrac{6e^{-5x}}{-5} + c$

$= \dfrac{6}{-5e^{5x}} + c$

(d) $3(e^{4x})^{\frac{1}{2}}$

$= 3e^{2x}$

$= \dfrac{3e^{2x}}{2} + c$

(e) $\dfrac{e^{5x}}{5} - \dfrac{e^{-5x}}{5} + c$

$= \dfrac{e^{5x}}{5} - \dfrac{1}{5e^{5x}} + c$

(f) $2xe^{x^2+1}$

$\dfrac{\cancel{2x}\, e^{x^2+1}}{\cancel{2x}} + c$

$= e^{x^2+1} + c$

13. (a) $\int 4e^{3x}\,dx$

$= \dfrac{4e^{3x}}{3} + c$

(b) $\int 3xe^{x^2-6}\,dx$

$= \dfrac{3xe^{x^2-6}}{2x} + c$

$= \dfrac{3e^{x^2-6}}{2} + c$

(c) $\int (\sqrt{5x^3} + e^{-x})\,dx$

$= \int \left((5x^3)^{\frac{1}{2}} + e^{-x}\right)dx$

$= \int (\sqrt{5}\ x^{\frac{3}{2}} + e^{-x})\,dx$

$= \dfrac{\sqrt{5}\ x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} - \dfrac{1}{e^x} + c$

$= \dfrac{2x^{\frac{5}{2}}}{\sqrt{5}} - \dfrac{1}{e^x} + c$

(d) $\int \dfrac{8e^{2x} + e^{-x}}{e^{-x}}\,dx$

$= \int \dfrac{8e^{2x}}{e^{-x}} + \dfrac{e^{-x}}{e^{-x}}\,dx$

$= \int (8e^{3x} + 1)\,dx$

$= \dfrac{8e^{3x}}{3} + x + c$

14. (a) $\int_{-1}^{4} e^{2x}\,dx$

$= \left[\frac{e^{2x}}{2}\right]_{-1}^{4}$

$= \left(\frac{e^{2(4)}}{2}\right) - \left(\frac{e^{-2}}{2}\right)$

$= \frac{e^8}{2} - \frac{1}{2e^2}$

(b) $\int_{0}^{1}\left(e^{3x} - 4(x+1)^{-2}\right)dx$

$= \left[\frac{e^{3x}}{3} + 4(x+1)^{-1}\right]_{0}^{1}$

≈ 4.3618

15. (a) $\frac{dP}{dt} = e^{4-2t}$

$P = \frac{e^{4-2t}}{-2} + c$

when $t = 1$ $P = \frac{e^2}{2}$

$\frac{e^2}{2} = \frac{e^{4-2(1)}}{-2} + c$

$\frac{e^2}{2} = \frac{e^2}{-2} + c$

$c = e^2$

$\therefore P = \frac{e^{4-2t}}{-2} + e^2$

(b) when $t = 2$

$P = \frac{e^0}{-2} + e^2$

$P = -\frac{1}{2} + e^2$

$P \approx 6.8891$

16. (a) $\frac{dC}{dt} = -kC$

i.e. $C = C_0\, e^{-kt}$

when $t = 0$ $C = 175$

$\therefore C = 175e^{-kt}$

when $t = 1$ $C = 125$

i.e. $125 = 175e^{-k(1)}$

$\frac{125}{175} = e^{-k}$

$k \approx 0.3365$

$\therefore C \approx 175e^{-0.3365t}$

$95 \approx 175e^{-0.3365t}$

$t \approx 1.815$ hours

(b) $C = 175e^{-0.3365t}$

when $t = 4.5$

$C = 175e^{-0.3365(4.5)}$

≈ 38.495 units

17. (a) $\frac{dP}{dt} = kP$

$\therefore P = P_0\, e^{kt}$

Third week increase of 150

i.e. $P_0\,(e^{3k} - e^{2k}) = 150$

Fourth week increase of 350

i.e. $P_0\,(e^{4k} - e^{3k}) = 350$

$\therefore \frac{e^{3k} - e^{2k}}{e^{4k} - e^{3k}} = \frac{150}{350}$

$k \approx 0.8473$

$\therefore P_0 \approx 21$ bacteria

(b) $P = 21e^{0.8473t}$

$P(6) = 21\,e^{0.8473(6)}$

≈ 3389 bacteria

18. (a) $y = e^{x^2}$

$\frac{dy}{dx} = 2xe^{x^2}$

(b) $y = x^5e^{-x}$

$\frac{dy}{dx} = 5x^4e^{-x} - x^5e^{-x}$

(c) $y = e^{-x}\sin x$

$\frac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$

(d) $y = \frac{e^{2x}}{\ln x}$

$\frac{dy}{dx} = \frac{\left(2e^{2x}\right)\left(\ln x\right) - \left(\frac{1}{x}\right)\left(e^{2x}\right)}{\left(\ln x\right)^2}$

(e) $y = e^{-2x}\cos x$

$\frac{dy}{dx} = \left(-2e^{-2x}\right)\left(\cos x\right) - \left(e^{-2x}\right)\left(\sin x\right)$

(f) $y = \left(e^x - e^{-x}\right)^2$

$y = \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right)$

$y = e^{2x} - e^0 - e^0 + e^{-2x}$

$y = e^{2x} - 2 + e^{-2x}$

$\frac{dy}{dx} = 2e^{2x} - 2e^{-2x}$

(g) $y = \frac{e^{2x}}{x^3 - x}$

$\frac{dy}{dx} = \frac{\left(2e^{2x}\right)\left(x^3 - x\right) - \left(e^{2x}\right)\left(3x^2 - 1\right)}{\left(x^3 - x\right)^2}$

19. (a) $\int\left(\frac{e^{5x}}{3}+\pi\right)dx$

$=\frac{e^{5x}}{15}+\pi x+c$

(b) $\int_0^2\left(2x\,e^{2x^2+3}\right)dx$

$=\frac{1}{2}\int_0^2\left(4x\,e^{2x^2+3}\right)dx$

$=\left[\frac{e^{2x^2+3}}{2}\right]_0^2$

$=\frac{e^{11}}{2}-\frac{e^3}{2}$

$=29927.03$

(c) $\int\left(e^{-4x}\pi-e\right)dx$

$=-\frac{\pi e^{-4x}}{4}-ex+c$

$=-\frac{\pi}{4e^{4x}}-ex+c$

(d) $\int_0^3\left(6\sqrt{e^x}+6x\right)dx$

$=\int_0^3\left(6e^{\frac{1}{2}x}+6x\right)dx$

$=\left[12e^{\frac{1}{2}x}+3x^2\right]_0^3$

$=\left(12e^{\frac{3}{2}}+27\right)-\left(12e^0+0\right)$

$=12e^{\frac{3}{2}}+27-12$

$=12e^{\frac{3}{2}}+15$

(e) $\int e^{x^2-5x+2}\left(2x-5\right)dx$

$=e^{x^2-5x+2}+c$

20. $f'\left(x\right)=\left(x-2\right)\left(e^{x^2-4x+3}\right)$

$f\left(x\right)=\frac{e^{x^2-4x+3}}{2}+c$

$2=\frac{e^{(1)^2-4(1)+3}}{2}+c$

$2=\frac{e^0}{2}+c$

$2=\frac{1}{2}+c$

$\frac{3}{2}=c$

$\therefore f\left(x\right)=\frac{e^{x^2-4x+3}}{2}+\frac{3}{2}$

21. e^{3x+2}

$\downarrow$ dilate parallel to the x axis scale factor of 3

e^{x+2}

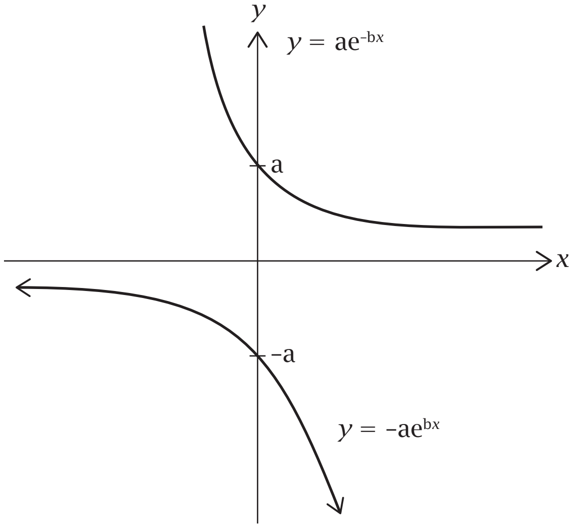
$\downarrow$ translate 1 unit right

$e^{(x-1)+2}$

$\downarrow$ reflect in the x axis

$-e^{x+1}$

22.



23. (a) $\frac{dP}{dt}=kP$

(b) $P=17.45e^{k(0)}$
$P=17.45$
Population = 17 450

(c) $P=17.45e^{kt}$
population increases to
17 450 + 125 = 17 575
$\therefore$ 17 575 = 17 450$e^{k(1)}$
$k=0.00714$

(d) 34 900 = 17 450$e^{0.00714t}$
$t=97.11$ years
$\therefore$ during the year 2111

24. (a) $H=H_0e^{kt}$
$H=H_0e^{0.065t}$
$3=e^{0.065t}$
$t=16.902$ years

(b) $\frac{dH}{dt}=0.065\left(550\,000\right)$
$=\$35750$ / year

(c) $t=12$ $\quad H=550\,000$
$\therefore 550\,000=H_0e^{0.065(12)}$
$H_0=\$252\,123.31$
$\therefore t=2$
$H=252\,123.31\ e^{0.065(2)}$
$H=\$287\,125.18$

(d) Average change $=\frac{550\,000-287\,125.18}{12-2}$
$=\$26\,287.48$ / year

CHAPTER 8: *Discrete Random Variables*

1. Bernoulli trials – (a) (c) (d)

2. (a) $P(\text{success})=\frac{1}{10}$

(b) $P(\text{success})=\frac{1}{5}$

(c) $P(\text{success})=\frac{3}{6}$

3. *(a)* Mean $= p = \dfrac{1}{10}$

$SD = \sqrt{p(1-p)}$

$= \sqrt{\dfrac{1}{10} \times \dfrac{9}{10}}$

$= 0.3$

(b) Mean $= p = \dfrac{1}{5}$

$SD = \sqrt{\dfrac{1}{5}\left(1-\dfrac{1}{5}\right)}$

$= 0.4$

(c) Mean $= p = \dfrac{1}{2}$

$SD = \sqrt{\dfrac{1}{2} \times \dfrac{1}{2}}$

$= \dfrac{1}{2}$

4. *(a)* Not a discrete random variable.
This is a continuous random variable.

(b) Sum of all probabilities is 1. $P(x) \geq 0$
$\therefore$ discrete random variable.

(c) $P(X = 6)$ is negative
$\therefore$ not a discrete random variable.

(d) $\sum P(X = x) = 1$

$P(X) \geq 0$

$\therefore$ a discrete random variable.

5. *(a)* $P(X = 0)$ is negative
$\therefore$ not a discrete random variable.

(b) $\sum P(X = x) = 1$

$P(X) \geq 0$
$\therefore$ a discrete random variable.

(c) $\left.\begin{matrix} P(X = 0) \\ P(X = 1) \\ P(X = 2) \end{matrix}\right\}$ are all negative
also $x \neq 3$
$\therefore$ not a discrete random variable

6. *(a)* For a p.d.f

$k + 2k + 3k + \dfrac{1}{6} = 1$

$k = \dfrac{5}{36}$

$P(X = x) \geq 0$ for all values

(b) $k = 0.3$

(c) For a p.d.f

$k + k + 2k + 1 - 8k + k + 3k = 1$

$1 = 1$

$0 \leq P(X = x) \leq 1$

$\therefore \quad 0 \leq k \leq \dfrac{1}{8}$

7. *(a)*

x	-3	-2	1	3
$P(X \leq x)$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{30}{36}$	$\frac{36}{36}$
$P(X \leq x)$	$\frac{5}{36}$	$\frac{5}{12}$	$\frac{5}{6}$	1

(b)

x	1	2	3	4	5	6
$P(X \leq x)$	0.1	0.3	0.4	0.7	0.8	1

8. *(a)* Mean $= \left(-3 \times \dfrac{5}{36}\right) + \left(-2 \times \dfrac{10}{36}\right) + \left(1 \times \dfrac{15}{36}\right) + \left(3 \times \dfrac{1}{6}\right)$

$= -\dfrac{1}{18}$

$$SD = \sqrt{\left(-3-\left(-\frac{1}{18}\right)\right)^2\left(\frac{5}{36}\right) + \left(-2-\left(-\frac{1}{18}\right)\right)^2\left(\frac{10}{36}\right) + \left(1-\left(-\frac{1}{18}\right)\right)^2\left(\frac{15}{36}\right) + \left(3-\left(-\frac{1}{18}\right)\right)^2\left(\frac{1}{6}\right)}$$

$= 2.068$

(b) Mean $= (1 \times 0.1) + (2 \times 0.2) + (3 \times 0.1) + (4 \times 0.3) + (5 \times 0.1) + (6 \times 0.2)$

$= 3.7$

$$SD = \sqrt{(1-3.7)^2(0.1) + (2-3.7)^2(0.2) + (3-3.7)^2(0.1) + (4-3.7)^2(0.3) + (5-3.7)^2(0.1) + (6-3.7)^2(0.2)}$$

$= 1.616$

9. As the distribution is skewed to the right
$\therefore$ mean $>$ median

10. *(a)* $P(X = 12) \approx 0.1797$

(b) $P(X \geq 15) \approx 0.1256$

(c) $P(X = 12 \mid X < 16) \approx 0.1894$

(d) $P(9 \leq X \leq 11) \approx 0.3479$

11. $X \sim B(6,\ 0.2)$

(a) $\mu = np$

$\mu = 6 \times 0.2$

$\mu = 1.2$

(b) $\sigma = \sqrt{np(1-p)}$

$\sigma = \sqrt{6(0.2)(0.8)}$

$\sigma = \sqrt{0.96}$

$\sigma \approx 0.9798$

12. $X \sim B(n,\ p)$

$$\begin{aligned} \mu &= 3.6 \\ \sigma &= 1.2 \\ \therefore \mu = np &= 3.6 \\ \sigma = \sqrt{np(1-p)} &= 1.2 \\ \therefore \sqrt{3.6(1-p)} &= 1.2 \\ 3.6(1-p) &= 1.44 \\ 1-p &= 0.4 \\ \therefore p &= 0.6 \\ \therefore n &= 6 \end{aligned}$$

13. (a)

$$\begin{aligned} P(X=0) &= 4k \\ P(X=1) &= 3k \\ P(X=2) &= 2k \\ P(X=3) &= k \\ \therefore 4k + 3k + 2k + k &= 1 \\ 10k &= 1 \\ k &= \frac{1}{10} \end{aligned}$$

(b) $P(X = \text{even}) = P(X=0) + P(X=2)$

$$= \frac{4}{10} + \frac{2}{10}$$
$$= \frac{3}{5}$$

(c) $P(X = \text{odd}) = P(X=1) + P(X=3)$

$$= \frac{3}{10} + \frac{1}{10}$$
$$= \frac{2}{5}$$

14. (a) i. $P(N < 3) = \frac{4}{10}$

$$= \frac{2}{5}$$

ii. $P(N = \text{odd}) = \frac{6}{10}$

$$= \frac{3}{5}$$

(b) $P(1,\ 2,\ 2) + P(1,\ 1,\ 3)$ in any order

$$= 3\left(\frac{1}{10} \times \frac{3}{10} \times \frac{3}{10}\right) + 3\left(\frac{1}{10} \times \frac{1}{10} \times \frac{2}{10}\right)$$
$$= \frac{27}{1000} + \frac{3}{500}$$
$$= \frac{33}{1000}$$

15. $X \sim B(10,\ 0.3)$

(a) $P(X \le 3) \approx 0.6496$

(b) $P(X \ge 6) \approx 0.04735$

16. $X \sim B(18,\ 0.9)$

(a) $P(X \ge 14) \approx 0.9718$

(b) $P(X = 18) \approx 0.1501$

(c) $P(12 \le X \le 15) \approx 0.2650$

17. (a)

x	0	1	2	3	4
$P(X=x)$	$\frac{4}{40}$	$\frac{16}{40}$	$\frac{12}{40}$	$\frac{6}{40}$	$\frac{2}{40}$

(b) Expected value

$$\mu = \sum (x \cdot P(X=x))$$
$$= \left(0 \times \frac{4}{40}\right) + \left(1 \times \frac{16}{40}\right) + \left(2 \times \frac{12}{40}\right) + \left(3 \times \frac{6}{40}\right) + \left(4 \times \frac{2}{40}\right)$$
$$\mu = 1.65$$

Standard Deviation

$$\sigma = \sqrt{\sum (x-\mu)^2 \cdot P(X=x)}$$

$$\sigma = \sqrt{\begin{aligned} &(0-1.65)^2\left(\frac{4}{40}\right) + (1-1.65)^2\left(\frac{16}{40}\right) \\ &+ (2-1.65)^2\left(\frac{12}{40}\right) + (3-1.65)^2\left(\frac{6}{40}\right) \\ &+ (4-1.65)^2\left(\frac{2}{40}\right) \end{aligned}}$$

$$= \sqrt{\begin{aligned} &\frac{1089}{4000} + \frac{169}{1000} + \frac{147}{4000} \\ &+ \frac{2187}{8000} + \frac{2209}{8000} \end{aligned}}$$

$$= \sqrt{1.0275}$$
$$\approx 1.0137$$

18. $X \sim B(15,\ 0.25)$

(a) $P(X \le 3) \approx 0.4613$

(b) $P(X \ge 10) \approx 0.00079$

(c) $P(X \le 6 \mid X \ge 3) \approx \frac{0.7073}{0.7639}$

$$\approx 0.9259$$

(d) $P(X = 7 \text{ or } X = 8) \approx 0.0393 + 0.0131$

$$\approx 0.0524$$

19.

	x	$P(X=x)$	$x \cdot P(X=x)$
Win	15	$\frac{1}{3}$	$\frac{15}{3}$
Lose	-10	$\frac{2}{3}$	$\frac{-20}{3}$

$$\mu = \frac{15}{3} + \left(-\frac{20}{3}\right)$$
$$= -\frac{5}{3}$$
$$= -1.67$$

Expected loss of \$1.67

20. For a p.d.f

$0.4 + 0.2 + a + b = 1$

$\therefore a + b = 0.4$

$\mu = \sum(x \cdot P(X = x))$

$2.3 = (1 \times 0.4) + (2 \times 0.2) + 3a + 4b$

$\therefore 3a + 4b = 1.5$

Solve Simultaneously

$a = 0.1$

$b = 0.3$

21. (a)

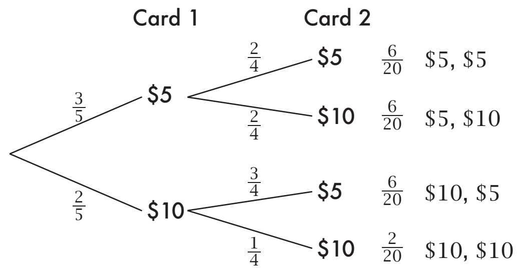
x	2	3	4
$P(X = x)$	0.5	0.25	0.25

(b) $\mu = (2 \times 0.5) + (3 \times 0.25) + (4 \times 0.25)$

$= 2.75$

Expected size is 2.75 (~3 children)

22.



Let x be the payout

x	$\$10 - d$	$\$15 - d$	$\$20 - d$
$P(X = x)$	0.3	0.6	0.1

To be fair $\sum(x \cdot P(x = x)) = 0$

$\therefore 0.3(10 - d) + 0.6(15 - d) + 0.1(20 - d) = 0$

$\therefore d = 14$

Player should pay \$14 per game.

23. (a) $P(X = 0) = \left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{3}{5}\right) = \frac{2}{7}$

$P(X = 1) = 3\left(\frac{2}{7}\right)\left(\frac{5}{6}\right)\left(\frac{4}{5}\right) = \frac{4}{7}$

$P(X = 2) = 3\left(\frac{2}{7}\right)\left(\frac{1}{6}\right)\left(\frac{5}{5}\right) = \frac{1}{7}$

x	0	1	2
$P(X = x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

(b) $P(X = 1)$ is the largest probability and the most likely number of black balls.

24. (a) $a_1 = 0.15$

$\therefore 0.15 = k(0.2)^1$

$k = 0.75$

(b)

a_0	a_1	a_2	a_3
0.75	0.15	0.03	0.006

(c) $P(a_n > 3) = 1 - P(a_n \leq 3)$

$= 1 - 0.936$

$= 0.064$

25. $X \sim B(20,\ 0.04)$

(a) $P(X \geq 1) \approx 0.558$

(b) i. $Y \sim B(5,\ 0.558)$

$P(Y = 5) \approx 0.0541$

ii. $X \sim B(20,\ 0.04)$

$P(X = 2) \approx 0.1458$

$Y \sim B(5,\ 0.1458)$

$P(Y = 5) \approx 0.00007$

(c) $X \sim B(20,\ 0.04)$

$P(X = 2) \approx 0.1458$

Expected number is 50×0.1458

$= 7$ boxes

26. $X \sim B(12,\ 0.4)$

(a) $P(X = 2) \approx 0.06385$

(b) $P(X \leq 5) \approx 0.6652$

(c) $P(X \geq 7) \approx 0.1582$

(d) $X \sim B(n,\ 0.6)$

$P(X \geq 8) = 0.8$

$n = 12 \quad P(X \geq 8) \approx 0.4382$

$n = 13 \quad P(X \geq 8) \approx 0.5744$

$n = 14 \quad P(X \geq 8) \approx 0.69245$

$n = 15 \quad P(X \geq 8) \approx 0.786$

$n = 16 \quad P(X \geq 8) \approx 0.8577$

Require 16 houses

27. $E(X) = 12$
$SD(X) = 2$

(a) $E(X + 3) = 15$

(b) $SD(X + 3) = 2$

(c) $E(2X - 4) = 20$

(d) $SD(2X - 4) = 4$

28. (a) $E(X) = (1 \times 0.3) + (2 \times 0.2) + (3 \times 0.4) + (4 \times 0.1)$
$= 2.3$

(b) $\text{Var}(X) = (1 - 2.3)^2(0.3) + (2 - 2.3)^2(0.2) + (3 - 2.3)^2(0.4) + (4 - 2.3)^2(0.1)$
$= 1.01$

(c) $E(Y) = 2.3 \times 4$
$= 9.2$

(d) $\text{Var}(Y) = 1.01 \times 4^2$
$= 16.16$

(e) $E(Z) = 2.3 \times 3 + 5$
$= 11.9$

(f) $\text{Var}(Z) = 1.01 \times 3^2$
$= 9.09$

CHAPTER 9: *Continuous Random Variables and Normal Distributions*

1. (a) $P(L < 35) = 0.19$

(b) $P(L > 40) = 0.52$

(c) $P(35 \leq L < 45) = 0.73$

(d) $P\left(L > 40 \mid L < 45\right) = \dfrac{0.44}{0.92}$
$= 0.4783$

2. (a) $\sum f(x) \neq 1$

$\therefore$ not a probability distribution

(b) $\int_{5}^{10} f(x)dx = 1$

$f(x) \geq 0$

$\therefore$ a continuous random variable

3. (a) $P(X < 5) = \dfrac{3}{8}$

(b) $P(3 < X < 8) = \dfrac{5}{8}$

(c) $P(X < 7 \mid X > 5) = \dfrac{2}{5}$

(d) $P(X = 5) = 0$

4. (a) $k = \dfrac{1}{9}$

(b) $P(X \geq 4) = \dfrac{8}{9}$

(c) $P(X \leq 7) = \dfrac{4}{9}$

(d) $P(X \leq 8 \mid X \leq 11) = \dfrac{5}{8}$

(e) $\mu = \dfrac{a + b}{2}$

$= \dfrac{3 + 12}{2}$

$= 7.5$

(f) Mean $= 7.5$

$$SD = \sqrt{\int_a^b \left[f\left(x\right) \bullet \left(x - \mu\right)^2\right] dx}$$

$$SD = \sqrt{\int_3^{12} \left(\frac{1}{9}\left(x - 7.5\right)^2\right) dx}$$

$= \sqrt{6.75}$
$= 2.598$

(g)
$$f\left(x\right) = \begin{cases} \dfrac{1}{9} \,, & 3 \leq x \leq 12 \\ 0 \,, & \text{elsewhere} \end{cases}$$

5. (a) $\int_{-0.25}^{0.25} f(x)dx \neq 1$

$\therefore$ not a continuous random variable

(b) $\int_{0.5}^{1} f(x) = 1$

$f(x) \geq 0$

$\therefore$ a continuous random variable

(c) These are discrete values of X. Hence not a continuous random variable.

(d) $\int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx = 1$

$f(x) \geq 0$

$\therefore$ a continuous random variable

(e) $\int_{3}^{5} f(x)\, dx \neq 1$

$\therefore$ not a continuous random variable

6. (a) $\int_{-1}^{1} \left[k\left(1 - x\right)\right] dx = 1$

$k = 0.5$

(b) $\int_{0}^{\infty} \left[ke^{-0.5x}\right] dx = 1$

$k = 0.5$

(c) $\int_{1}^{2} \left[k\left(x + 2\right)\right] dx = 1$

$k = \dfrac{2}{7}$

7. (a) Mean $= E\left(X\right)$

$$E\left(X\right) = \int_0^1 \left[x\left(3x^2\right)\right] dx$$

$= \dfrac{3}{4}$

$$SD\left(X\right) = \sqrt{\int_0^1 \left[\left(3x^2\right)\left(x - \frac{3}{4}\right)^2\right] dx}$$

$= \sqrt{\dfrac{3}{80}}$
$= 0.1936$

(b) $$E\left(X\right) = \int_0^{2.5} \left(0.4x\right) dx$$

$= 1.25$

$$SD\left(X\right) = \sqrt{\int_0^{2.5} \left[0.4\left(x - 1.25\right)^2\right] dx}$$

$= \sqrt{\dfrac{25}{48}}$
$= 0.7217$

(c) $E(X)=\int_0^{\infty}\left(0.25x\,e^{-0.25x}\right)dx$
$=4$
$SD(X)=\sqrt{\int_0^{\infty}(x-4)^2\left(0.25e^{-0.25x}\right)dx}$
$=\sqrt{16}$
$=4$

8. (a) $P(X\le k)=\int_0^k\left(3x^2\right)dx$
$=\left[x^3\right]_0^k$
$=k^3$
$$\therefore P(X\le x)=\begin{cases}0 & , \quad x<0\\ x^3 & , \quad 0\le x<1\\ 1 & , \quad x\ge 1\end{cases}$$

(b) $P(X\le k)=\int_0^k(0.4)\,dx$
$=\left[0.4x\right]_0^k$
$=0.4\,k$
$$\therefore P(X\le x)=\begin{cases}0 & , \quad x<0\\ 0.4x & , \quad 0\le x<2.5\\ 1 & , \quad x\ge 2.5\end{cases}$$

(c) $P(X\le k)=\int_0^k\left(0.25e^{-0.25x}\right)dx$
$=\left[-e^{-0.25x}\right]_0^k$
$=-e^{-0.25k}+1$
$$\therefore P(X\le x)=\begin{cases}0 & , \quad x<0\\ -e^{-0.25x}+1 & , \quad x\ge 0\end{cases}$$

9. $E(X)=10$
$SD(X)=5$

(a) $E(Y)=40$
$SD(Y)=20$

(b) $E(Y)=2(10)-5$
$=15$
$SD(Y)=2(5)$
$=10$

(c) $E(Y)=3(10)+1$
$=31$
$SD(Y)=3(5)$
$=15$

10. (a) $E(X)=\int_0^{140}\left(\frac{1}{140}x\right)dx$
$=70$

(b) $Var(X)=\int_0^{140}\left[\frac{1}{140}(x-70)^2\right]dx$
$=1633\frac{1}{3}$

(c) $SD=40.415$

(d) $E(Y)=3(70)+4$
$=214$

(e) $Var(Y)=\left(1633\frac{1}{3}\right)\times(3)^2$
$=14700$

(f) $SD(Y)=121.244$

11. (a) $P(X\ge 90)\approx 0.8413$

(b) $P(85\le X\le 101)\approx 0.4730$

(c) $P(X\le 80\,|\,X\le 96)\approx\frac{0.02275}{0.34458}$
≈ 0.0660

12. $X\sim N(200,\ 35^2)$

(a) $P(190\le X\le 210)\approx 0.2249$

(b) $P(X\ge 220)\approx 0.2839$

(c) $P(X\le 205)\approx 0.5568$

(d) $P(X\ge 210\,|\,X\le 220)\approx\frac{0.1037}{0.7162}$
≈ 0.1448

13. (a) $k\approx 12.0923$

(b) $k\approx 84.386$

(c) $k\approx 26.763$

14. (a) $P(X\le k)=0.24$
$k\approx 16.4685$

(b) $P(X\le k)=0.52$
$k\approx 20.2508$

15. (a) $\left(\frac{3+8}{2}\right)\times k=1$
$k=\frac{2}{11}$

(b)
$$f(x)=\begin{cases}\frac{2}{33}x & 0\le x\le 3\\ \frac{2}{11} & 3<x\le 6\\ \frac{-1}{11}x+\frac{8}{11} & 6<x\le 8\\ 0 & \text{otherwise}\end{cases}$$

(c) $P(1<X<5)=$
$P(1<X<3)=\left(\frac{2}{33}\times 2\right)+\left(\frac{4}{33}\times\frac{1}{2}\times 2\right)=\frac{8}{33}$
$+\ P(3<X<5)=2\times\frac{2}{11}=\frac{4}{11}$
$\therefore P(1<X<5)=\frac{4}{11}+\frac{8}{33}$
$=\frac{20}{33}$

16. (a) $X\sim$ travelling time in minutes
$$P(X=x)=\begin{cases}\frac{1}{30} & ,\ 90\le x\le 120\\ 0 & ,\ \text{otherwise}\end{cases}$$

(b) i. $P(X < 100) = \frac{1}{3}$

ii. $P(X = 110) = 0$

iii. $P(95 \le X \le 104) = \frac{9}{30} = \frac{3}{10}$

iv. $P(X \ge 92 \mid X \le 100) = \frac{\frac{8}{30}}{\frac{10}{30}}$

$= \frac{4}{5}$

(c) Expected travel time

$\mu = \frac{a+b}{2}$

$= \frac{90+120}{2}$

$= 105$ minutes

Expected standard deviation

$\sigma = \sqrt{\frac{(b-a)^2}{12}}$

$= \sqrt{\frac{(120-90)^2}{12}}$

$= 8.6603$

17. (a) 9.12 am and 30 seconds

(b) $P(X > 9.08) = \frac{12}{15}$

$= \frac{4}{5}$

(c) $P(X > 9.13) = \frac{7}{15}$

18. (a) $P(750 \le X \le 752) = \frac{1}{5}$

(b) $P(X \ge 748) = \frac{2}{5}$

(c) $P(748 \le X \le 752) = \frac{2}{5}$

$P(\text{Not meet Government regulation}) = \frac{3}{5}$

19. (a) $\int_0^4 (4kx - kx^2)\,dx = 1$

$\left[\frac{4kx^2}{2} - \frac{kx^3}{3}\right]_0^4 = 1$

$\left[2kx^2 - \frac{kx^3}{3}\right]_0^4 = 1$

$\left(32k - \frac{64k}{3}\right) = 1$

$k = \frac{3}{32}$

(b) $P(X > 1) = \int_1^4 f(x)\,dx$

$= 0.84375$

$P(\text{neither fails}) = \binom{2}{2}(0.84375)^2\,(1-0.84375)^0$

≈ 0.7119

(c) $P(0 < x < 2) = \int_0^2 f(x)\,dx$

$= 0.5$

$P(\text{exactly one fails}) = \binom{2}{1}(0.5)^1\,(0.5)^1$

$= 0.5$

(d) $\int_0^t \left(\frac{12}{32}x - \frac{3}{32}x^2\right)dx = 0.4$

$\left[\frac{12x^2}{64} - \frac{3x^3}{96}\right]_0^t = 0.4$

$\frac{12t^2}{64} - \frac{3t^3}{96} = 0.4$

$t \approx 1.732$ years

20. $X \sim N(17,\ 2.25^2)$

(a) $P(X > 20) \approx 0.0912$

(b) $P(14 \le X \le 20) \approx 0.8176$

(c) $P(X \le k) = 0.95$

$k \approx 20.701$

Maximum length is 20.701 mm

(d) $P(X < 12) \approx 0.0131$

$Y \sim B(8,\ 0.0131)$

$P(Y \ge 1) \approx 0.1001$

21. (a) Area $= \frac{\pi r^2}{2}$

$= \frac{\pi k^2}{2}$

$\therefore 1 = \frac{\pi k^2}{2}$

$k^2 = \frac{2}{\pi}$

$k = \sqrt{\frac{2}{\pi}}$

(b) $P(0 \le x \le 0.4) \approx 0.1963$

22. $X \sim N(1500,\ 70^2)$

(a) $P(X > 1480) \approx 0.61245$

(b) $X \sim N(\mu,\ 70^2)$

$P(X > 1480) = 0.98$

$$z = \frac{x-\mu}{\sigma}$$

$$-2.0537 \approx \frac{1480-\mu}{70}$$

$$\mu \approx 1623.759$$

New mean is 1623.759 hours

(c) $X \sim N(1500,\ \sigma^2)$

$P(X > 1480) = 0.98$

$$-2.0537 \approx \frac{1480-1500}{\sigma}$$

$$\sigma \approx 9.739$$

New standard deviation is 9.739 hours

23. $X \sim N(18,\ \sigma^2)$

(a) i. $P(11.981 \le X \le 22.723) = 0.9$

ii. $P(11.981 \le X \le 22.723 \mid X > 11.981)$

$$\approx \frac{0.9}{0.97}$$

$$\approx 0.9278$$

(b) $P(z < k) = 0.03$

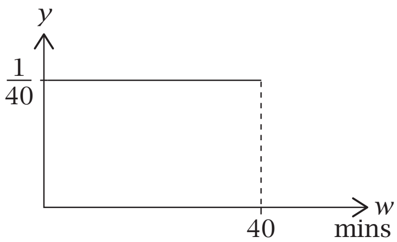
$$k \approx -1.8809$$

$$z = \frac{x-\mu}{\sigma}$$

$$-1.8809 \approx \frac{11.981-18}{\sigma}$$

$$\sigma \approx 3.2$$

24. (a)



p.d.f. $f(w) = \begin{cases} \dfrac{1}{40} & , \quad 0 \le w \le 40 \\ 0 & , \quad \text{elsewhere} \end{cases}$

(b) $P(0 \le w \le 15) = 15 \times \dfrac{1}{40}$

$= 0.375$

(c) $Y \sim B(8, 0.625)$

$P(Y < 3) = 0.036$

CHAPTER 10: *Random Sample, Proportion and Confidence Intervals*

1. (a) Random sampling

(b) Quota sampling

(c) Systematic sampling

2. (a) Biased – only one type of dealership (Holden) surveyed. Should survey a variety of car dealerships.

(b) Biased – only one floor surveyed. Should survey residents on every floor.

3. *Possible solutions* could include:

- when n is small: variation exists between the sample and population means.
- when n is small: variation exists between the sample means.
- when n is small: variation exists between the sample and population standard deviations.
- as n becomes larger the variation reduces.
- as n becomes larger the graph of each of the samples resembles a normal distribution.

4. $\hat{p} = \dfrac{543}{1234}$

5. (a) $\hat{p} = \dfrac{720}{900}$

$\hat{p} = 0.8$

(b) Estimated standard deviation of $\hat{p}$:

$$SD = \sqrt{\frac{p(1-p)}{n}}$$

$$= \sqrt{\frac{0.8(1-0.8)}{900}}$$

$$= \frac{1}{75}$$

6. (a) $p = 0.96$

(b) $\hat{p} = \dfrac{874}{950}$

$= 0.92$

(c) Mean $= p = 0.96$

$$SD = \sqrt{\frac{p(1-p)}{n}}$$

$$= \sqrt{\frac{0.96(1-0.96)}{950}}$$

$$= 0.006358$$

7. *Confidence Interval*

$$\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$\hat{p} = \frac{75}{250}$ $\quad n = 250$

$\hat{p} = 0.3$ $\quad k$ value for 95% CI = 1.96

Confidence Interval

$$0.3 - 1.96\sqrt{\frac{0.3(0.7)}{250}} \le p \le 0.3 + 1.96\sqrt{\frac{0.3(0.7)}{250}}$$

$0.2432 \le p \le 0.3568$

8. *Confidence Interval*

$\hat{p} = \frac{172}{400}$ $\quad n = 400$

$\hat{p} = 0.43$ $\quad k = 1.96$

$$0.43 - 1.96\sqrt{\frac{0.43(0.59)}{400}} \le p \le 0.43 + 1.96\sqrt{\frac{0.43(0.57)}{400}}$$

$0.3815 \le p \le 0.4785$

9. *Confidence Interval*

$\hat{p} = \frac{570}{1000}$ $\quad n = 1000$

$= 0.57$ $\quad k = 1.645$

$$0.57 - 1.645\sqrt{\frac{(0.57)(0.43)}{1000}} \le p \le 0.57 + 1.645\sqrt{\frac{(0.57)(0.43)}{1000}}$$

$0.5442 \le p \le 0.5958$

As the value of $p = 0.5$ is outside the interval the coin is biased.

10. (a) $\hat{p} = \frac{320}{500}$

$\hat{p} = 0.64$

(b) Confidence Interval

$n = 500$ $\quad k = 1.96$

$$0.64 - 1.96\sqrt{\frac{0.64 \times 0.36}{500}} \le p \le 0.64 + 1.96\sqrt{\frac{0.64 \times 0.36}{500}}$$

$0.5979 \le p \le 0.6821$

(c) Margin of error is

$$1.96\sqrt{\frac{0.64 \times 0.36}{500}} = 0.04207$$

11. $ME = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Sample size:

$$0.04 = 2.576\sqrt{\frac{(0.7)(0.3)}{n}}$$

$n = 870.95$

Sample size: 871

12. $\hat{p} = \frac{200}{250}$ $\quad p = 0.92$ success

$\hat{p} = 0.8$ $\quad n = 250$

CI: $\quad k = 1.96$

$$0.8 - 1.96\sqrt{\frac{(0.8)(0.2)}{250}} \le p \le 0.8 + 1.96\sqrt{\frac{(0.8)(0.2)}{250}}$$

$0.7504 \le p \le 0.8496$

The success rate of 92% is not within the confidence interval.

$\therefore$ the claim is not true.

13. $$\hat{p} \approx N\left[p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right]$$

$$\therefore \hat{p} \approx N\left[0.3, \left(\sqrt{\frac{(0.3)(0.7)}{200}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.3, (0.0324)^2\right]$$

$P(\hat{p} \le 0.27) = 0.1772$

14. (a) $$\hat{p} \approx N\left[p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.5, \left(\sqrt{\frac{0.5 \times 0.5}{200}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.5, (0.03536)^2\right]$$

$P(0.47 \le \hat{p} \le 0.51) = 0.4132$

(b) The probability reduces

$P(0.47 \le \hat{p} \le 0.51) = 0.305$

15. (a) $\hat{p} = \frac{280}{350}$

$= 0.8$

Percentage = 80%

(b) CI = $0.7648 \le p \le 0.8352$

(c) Margin of error:

$$1.645\sqrt{\frac{0.8 \times 0.2}{350}}$$

$= 0.03517$

16. (a) $\hat{p} = \frac{130}{520}$

$= 0.25$

(b) Standard deviation of $\hat{p}$:

$$\sqrt{\frac{0.25 \times 0.75}{520}}$$

$= 0.0190$

(c) $ME = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$0.03 = 1.96\sqrt{\frac{0.25\times0.75}{n}}$

$n = 800.\dot{3}$

Sample size is 801

(d) $\hat{p} = \frac{256}{1200}$

$= 0.21\dot{3}$

As this value of $0.21\dot{3}$ falls within the confidence interval the sample is representative of the population.

(e) $\hat{p} = \frac{205}{360}$

$= 0.5694$

As this value of 0.5694 falls outside of the confidence interval the sample is biased and not representative of the population.

17. $p = 0.25$

$n = 175$

$$\hat{p} \approx N\left[p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.25, \left(\sqrt{\frac{0.25\times0.75}{175}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.25, (0.0327)^2\right]$$

$P(\hat{p} \geq 0.28) = 0.1795$

18. (a) $\hat{p} = \frac{20}{320}$

$= 0.0625$

(b) $\hat{p} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$= \sqrt{\frac{0.0625\times(1-0.0625)}{320}}$

$= 0.01353$

(c) $$\hat{p} \approx N\left[p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.03, \left(\sqrt{\frac{0.03\times0.97}{360}}\right)^2\right]$$

$$\hat{p} \approx N\left[0.03, (0.00899)^2\right]$$

$P(\hat{p} \leq 0.04) = 0.867$

19. (a) $X \sim B(15, 0.07)$

(b) $p(X = 3) = 0.0653$

(c) $p(X > 3) = 0.0175$

(d) $\hat{p} = \frac{12}{150}$ $\quad n = 150$

$k = 1.645$

$\text{CI} = (0.0436, 0.1164)$

(e) $0.0436 \times 300 \approx 13$

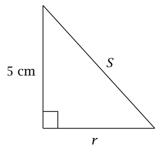
$0.1164 \times 300 \approx 35$

Between 13 and 35 people would not redeem their gift vouchers.

SOLUTIONS TO TRIAL TESTS

CALCULATOR ASSUMED: TRIAL TEST 1
Differentiation and Applications

1. $A = \pi rs$



$A = \pi r(\sqrt{25 + r^2})$

$\dfrac{dA}{dr} = \dfrac{2r^2\ \pi + 25\pi}{\sqrt{r^2 + 25}}$ ✓

$s = \sqrt{5^2 + r^2}$ ✓

$r = 3$ cm, $\quad \delta r = 0.03 \quad \delta A = ?$

$\delta A \approx \dfrac{dA}{dr} \times \delta r$

$\approx \dfrac{2(3)^2\pi + 25\pi}{\sqrt{(3)^2 + 25}} \times (0.03)$ ✓

≈ 0.695 cm^2 ✓

2. *(a)* $P(x) = R(x) - C(x)$

$= 250x^2 - 4x^3 - (50(x^2 - x) + 2000)$

$= 250x^2 - 4x^3 - 50x^2 + 50x - 2000$

$= -4x^3 + 200x^2 + 50x - 2000$ ✓

(b) $C(10) = \$6500$ ✓

(c) $C'(x) = 100x - 50$ ✓

$C'(6) = \$550$

The cost of producing the 7th computer desk. ✓

(d) $P(12) = \$20488$ ✓

Average profit $= \dfrac{20488}{12}$

$= \$1707.33$ ✓

(e) $P(x) > 0$

Number of computer desk

$4 \leq x \leq 50$

✓ ✓

(f) Maximum profit: \$73702 ✓ when 33 ✓ computer desks are produced and sold.

3. $\dfrac{dm}{dn} = 10n \qquad \dfrac{dg}{dm} = 3 \qquad \dfrac{df}{dg} = \dfrac{1}{2g^{\frac{1}{2}}}$ ✓

$\dfrac{df}{dn} = \dfrac{df}{dg} \times \dfrac{dg}{dm} \times \dfrac{dm}{dn}$

$\dfrac{df}{dn} = \dfrac{1}{2g^{\frac{1}{2}}} \times 3 \times 10n$ ✓

$\dfrac{df}{dn} = \dfrac{15n}{\sqrt{g}}$

$\dfrac{df}{dn} = \dfrac{15n}{\sqrt{15n^2 + 6}}$ ✓

4. $C = 40\sqrt{2 - 0.1t}$

(a) $C(0) = \$40\sqrt{2}$

$C(10) = \$40$ ✓

Average change $= \dfrac{40 - 40\sqrt{2}}{10}$

$= -1.657$

A decrease of \$1.66 per year. ✓

(b) $\dfrac{dC}{dt} = -\dfrac{2\sqrt{10}}{\sqrt{-t + 20}}$

$-\dfrac{2\sqrt{10}}{\sqrt{-t + 20}} < -2$ ✓

$t > 10$

$\therefore 10 < t \leq 15$ ✓

Between 2010 and 2015 ✓

5. *(a)* $v = 200\pi$ m^3

$v = \pi r^2 h + \dfrac{4}{3}\pi r^3 = 200\pi$ ✓

$h = \dfrac{200}{r^2} - \dfrac{4}{3}r$ ✓

$C = 2\pi rh \cdot p + 4\pi r^2 \cdot 2p$ ✓

$= 2\pi rp\left(\dfrac{200}{r^2} - \dfrac{4}{3}r\right) + 8\pi r^2 p$

$= \dfrac{400\pi p}{r} - \dfrac{8\pi r^2 p}{3} + 8\pi r^2 p$

$= \dfrac{400\pi p}{r} + \dfrac{16\pi r^2 p}{3}$ ✓

(b) $\dfrac{dc}{dr} = -\dfrac{400\pi p}{r^2} + \dfrac{32\pi rp}{3} = 0$ ✓

$32\pi r^3 p - 1200\pi p = 0$ ✓

$16\pi p(2r^3 - 75) = 0$

$2r^3 = 75$

$r^3 = \dfrac{75}{2}$

$r = \sqrt[3]{\dfrac{75}{2}}$ ✓

$\dfrac{d^2c}{dr^2} = \dfrac{32\pi p}{3} + \dfrac{800\pi p}{r^3}$

> 0 for $r = \sqrt[3]{\dfrac{75}{2}}$

$\therefore$ a minimum value ✓

(c) If $p = 10$

$$C = \frac{400\pi(10)}{r} + \frac{16\pi r^2\,(10)}{3} \checkmark$$

$$C = \frac{4000\pi}{r} + \frac{160\pi r^2}{3}$$

when $r = \sqrt[3]{\frac{75}{2}}$

$C = \$5631.50$ ✓

6. *(a)* $x(0) = 4$ m ✓

(b) $v(t) = \dfrac{6t^4 + 22t^3 - 24t^2 - 90t}{(t+3)^2}$ ✓

(c) At rest when $v(t) = 0$

$\dfrac{6t^4 + 22t^3 - 24t^2 - 90t}{(t+3)^2} = 0$ ✓

$t = 0, \quad t = 2.0146$ seconds ✓

(d) $a(t) = \dfrac{12t^4 + 94t^3 - 198t^2 - 54t - 270}{(t+3)^3}$ ✓

$a(4) = 34.315$ m/s^2 ✓

(e) Distance:

$x(1) = 0.5$ m

$x(2.0146) = -3.2012$ m

$x(4) = 33.7143$ ✓

Distance $= 0.5 + 3.2012 + 3.2012 + 33.7143$

$= 40.6167$ m ✓

Or by using integration –

Distance: $\displaystyle\int_1^4 |v(t)|\,dt = 40.6166$ m

CALCULATOR FREE: TRIAL TEST 1
Differentiation and Applications

1. *(a)* $f(x) = (6x^3 - 5)^4$

$f'(x) = (4)\,(6x^3 - 5)^3\,(18x^2)$

$f'(x) = 72x^2\,(6x^3 - 5)^3$ ✓

(b) $g(x) = (5 - 2x)(2 - 3x)^2$

$g'(x) = (-2)\,(2 - 3x)^2 + (5 - 2x)(2)(2 - 3x)(-3)$ ✓

$g'(-1) = (-2)(2 + 3)^2 + (5 + 2)(2)(2 + 3)(-3)$

$g'(-1) = -260$ ✓

(c) $\dfrac{d}{dx}\left(\dfrac{2x^3 - \pi}{\sqrt{4x+1}}\right)$

$$= \frac{(6x^2)(\sqrt{4x+1}) - (2x^3 - \pi)\left(\frac{1}{2}\right)(4x+1)^{-\frac{1}{2}}\,(4)}{(\sqrt{4x+1}^2)} \checkmark$$

$$= \frac{(6x^2)(\sqrt{4x+1}) - \dfrac{2(2x^3 - \pi)}{\sqrt{4x+1}}}{4x+1} \checkmark$$

2. *(a)* $f(x) = \dfrac{x^2 - 12}{x - 4}$

$f'(x) = \dfrac{(2x)(x-4) - (x^2 - 12)}{(x-4)^2}$ ✓

$f'(x) = \dfrac{x^2 - 8x + 12}{(x-4)^2}$ ✓

(b) When $f'(x) = 0$

$\dfrac{x^2 - 8x + 12}{(x-4)^2} = 0$

$(x - 6)(x - 2) = 0$ ✓

$x = 6, x = 2$

Coordinates $(6, 12)$ $(2, 4)$ ✓

(c) $f'(x) = \dfrac{(2x - 8)\,(x-4)^2 - (x^2 - 8x + 12)(2)(x - 4)}{(x-4)^4}$

$f''(x) = \dfrac{8}{(x-4)^3}$ ✓

$f''(2) = -1$ *Hence* $(2, 4)$ *is a maximum*

$f''(6) = 1$ *Hence* $(6, 12)$ *is a minimum* ✓

(d) Concave up when $f''(x) > 0$

$\dfrac{8}{(x-4)^3} > 0$ ✓

$\therefore (x - 4)^3 > 0$

Concave up when $x > 4$ ✓

3. *(a)* $g'(5) = -2$ ✓

(b) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$ when $x = 2.5$

$\dfrac{f'(2.5)g(2.5) - f(2.5)g'(2.5)}{(g(2.5))^2}$

$= \dfrac{f(0)(4) - (3)(4)}{4^2}$ ✓

$= -\dfrac{3}{4}$ ✓

(c) $\dfrac{d}{dx}\,[g(f(x))]$ at $x = 4$

$g'(f(4))f'(4) = g'(1.5)\,f'(4)$ ✓

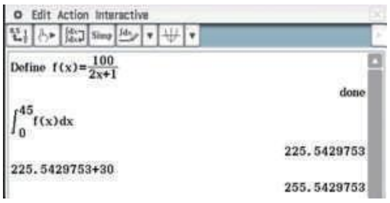
$= (0)(-1.5)$

$= 0$ ✓

CALCULATOR ASSUMED: TRIAL TEST 2

Integration and Applications

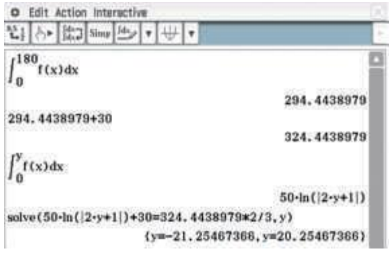
1. (a)



✓

Total amount = 255.54 L ✓

(b)



✓ ✓

Time = 20.25 minutes ✓

2. (a) -18 ✓

(b) 10 ✓

(c) 20 ✓

(d) $\int_{-2}^{2} 2\,dx + \int_{-2}^{2} f(x)dx$ ✓

$= [2x]_{-2}^{2} + (-7)$

$= [2(2) - 2(-2)] - 7$

$= 1$ ✓

(e) 38 ✓✓

3. (a) i. Displacement

$= [(\frac{1}{2}(1)(1) + (5)(1) + (\frac{1}{2}(2)(1)]$ ✓

$= 6.5$ m ✓

ii. Displacement

$= [(\frac{1}{2}(1)(1) + (5)(1) + \frac{1}{2}(2)(1) + \frac{1}{2}(2)(2)$

$+ \frac{1}{2}(2)(-2) + (6)(-2)]$ ✓

$= -5.5$ m ✓

(b) Max displacement when $v(t) = 0$ ✓

Time: $t = 8$ secs ✓ Displacement: 8.5 m ✓

4. (a) Overestimate

$= [0.25(1 + 1.22 + 1.41 + 1.58 + 1.73)]$ ✓

$= 1.735$ ✓

(b) Underestimate

$= [0.25(0.71 + 1 + 1.22 + 1.41 + 1.58)]$ ✓

$= 1.48$ ✓

(c) Approximation $= \frac{1.48 + 1.735}{2}$

$= 1.6075$ ✓

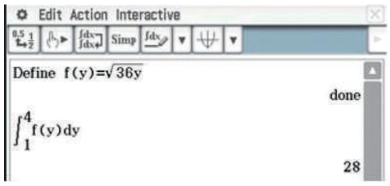
(d) Smaller as concave down hence an underestimate. ✓

(e) Smaller rectangles. ✓

5. Curve and lines intersect at (-6, 1) and (-12, 4) ✓

$y = 1, y = 4$ ✓

Curve: $x = \sqrt{36y}$ ✓



Area = 28 units^2 ✓✓

CALCULATOR FREE: TRIAL TEST 2

Integration and Applications

1. (a) $\int\left(1 - \frac{1}{\sqrt{x}}\right)dx$

$= x - 2\sqrt{x} + c$ ✓

(b) $\int_0^2 \left(\frac{2}{(2x+1)^3}\right)dx$

$= \int_0^2 (2(2x+1)^{-3})dx$

$= \left[\left(\frac{2(2x+1)^{-2}}{-4}\right)\right]_0^2$ ✓

$= \left[\left(-\frac{1}{2(2x+1)^2}\right)\right]_0^2$

$= \left(-\frac{1}{2(2(2)+1)^2}\right) - \left(-\frac{1}{2(2(0)+1)^2}\right)$ ✓

$= -\frac{1}{50} + \frac{1}{2}$

$= \frac{12}{25}$ ✓

(c) $\frac{d}{dx}\int_x^0 \left(\frac{2t}{(1-t^2)^3}\right) dt$

$= -\frac{2x}{(1-x^2)^3}$ ✓✓

2. $v(t) = \int -3(2t-1)^{-\frac{1}{2}}\ dt$

$= -\frac{3(2t-1)^{\frac{1}{2}}}{2 \times \frac{1}{2}} + c$

$= -3(2t-1)^{\frac{1}{2}} + c$ ✓

$x(t) = -\frac{3(2t-1)^{\frac{3}{2}}}{2 \times \frac{3}{2}} + ct + d$

$= -(2t-1)^{\frac{3}{2}} + ct + d$ ✓

$x(1) = 16$

$16 = -(2(1) - 1)^{\frac{3}{2}} + c + d$

$16 = -1 + c + d$

$17 = c + d$ ✓

$x(5) = 38$

$38 = -(2(5) - 1)^{\frac{3}{2}} + 5c + d$

$38 = -27 + 5c + d$

$65 = 5c + d$ ✓

Solving simultaneously

$c = 12$

$d = 5$ ✓

$v(t) = -3(2t - 1)^{\frac{1}{2}} + 12$

$x(t) = -(2t - 1)^{\frac{3}{2}} + 12t + 5$ ✓

3. (a) $\frac{d}{dx}(2x(x+1)^{\frac{1}{2}})$

$= 2(x+1)^{\frac{1}{2}} + \frac{1}{2}(x+1)^{-\frac{1}{2}} \times 2x$ ✓

$= 2(x+1)^{\frac{1}{2}} + \left(\frac{x}{(x+1)^{\frac{1}{2}}}\right)$ ✓

(b) $\frac{d}{dx}(2x\sqrt{x+1}) = 2\sqrt{x+1} + \left(\frac{x}{\sqrt{x+1}}\right)$ ✓

$\int \frac{d}{dx}(2x\sqrt{x+1}) = \int 2\sqrt{x+1}\,dx +$
$\int\left(\frac{x}{\sqrt{x+1}}\right)dx$

$\int\left(\frac{x}{\sqrt{x+1}}\right)dx = \int \frac{d}{dx}(2x\sqrt{x+1}) -$
$\int 2\sqrt{x+1}\,dx$ ✓

$\int\left(\frac{x}{\sqrt{x+1}}\right)dx = 2x\sqrt{x+1}) - \frac{4\sqrt{(x+1)^3}}{3} + c$ ✓

4. x values at points of intersection:

$2x = \sqrt{4x}$

$4x^2 - 4x = 0$

$4x(x - 1) = 0$

$\therefore x = 0,\ x = 1$ ✓

$\text{Area} = \int_0^1 (\sqrt{4x} - 2x)\ dx$

$= \left[\frac{4x^{\frac{3}{2}}}{3} - x^2\right]_0^1$ ✓

$= \left(\frac{4}{3} - 1\right) - (0)$

$= \frac{1}{3}u^2$ ✓

CALCULATOR ASSUMED: TRIAL TEST 3

Exponentials, Trigonometry, Binomial and Discrete Distributions –

1. (a) $B'(t) = 4.1e^{0.4t}$

$\therefore \int_2^3 4.1e^{0.4t}\,dt \approx 11.22$ ✓

$\therefore$ approximately 11 bats ✓

(b) $\int_6^x (4.1e^{0.4t})\,dt = 1750$ ✓

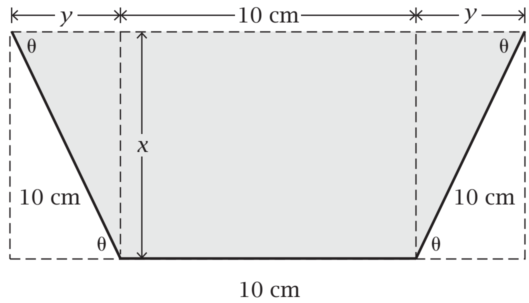
$[10.25e^{0.4t}]_6^x = 1750$

$10.25e^{0.4x} - 10.25e^{2.4} = 1750$

$t \approx 13.01$

$\therefore$ 13 weeks after the first study began or 7 weeks after the first study finished. ✓

2.



$x = 10\sin\theta$
$y = 10\cos\theta$
$\text{Area}\ \square = 10 \times 10\sin\theta$
$= 100\sin\theta$ ✓

$\text{Area}\ \triangle = 2 \times \frac{1}{2}(10\cos\theta \times 10\sin\theta)$

$= 100\cos\theta\sin\theta$ ✓

Hence total area =
$A = 100\sin\theta + 100\cos\theta\sin\theta$ ✓

For maximum area $\frac{dA}{d\theta} = 0$

$\frac{dA}{d\theta} = 100\cos\theta + 100(\cos\theta\cos\theta - \sin\theta\sin\theta)$ ✓

$0 = 100\cos\theta + 100(\cos^2\theta - \sin^2\theta)$ ✓

$0 = 100\cos\theta + 100[\cos^2\theta - [1 - \cos^2\theta]]$

$0 = 100\cos\theta + 100[\cos^2\theta - 1 + \cos^2\theta]$ ✓

$0 = 100\cos\theta + 200\cos^2\theta - 100$

$\therefore 100(2\cos^2\theta + \cos\theta - 1) = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$ ✓

$\theta = \frac{\pi}{3}$ ✓

3. (a) $v = -2\sin(2t) + 2$

$x = \int [2\sin(2t) + 2]\, dt$

$x = 2t + \cos(2t) + c$ ✓

when $t = 0 \quad x = 0$

$\therefore 0 = 2(0) + \cos(0) + c$

$0 = 1 + c$

$c = -1$

$x = 2t + \cos(2t) - 1$ ✓

(b) Distance travelled

$\int_0^3 (-2\sin(2t)) + 2$ ✓

$= 5.96$ m ✓

4. (a)

x	0	1	2
$P(X = x)$	$\frac{28}{45}$	$\frac{16}{45}$	$\frac{1}{45}$

✓✓

(b) i. $P(X = 0) = \frac{28}{45}$ ✓

ii. $P(X \geq 1) = \frac{17}{45}$ ✓

iii. $P(X = 0 \mid X \leq 1) = \dfrac{\frac{28}{45}}{\frac{44}{45}}$

$= \frac{28}{44}$

$= \frac{7}{11}$ ✓✓

5. $X \sim B(20,\ 0.08)$

(a) $P(X = 0) \approx 0.1887$ ✓

(b) $P(X = 5) \approx 0.0145$ ✓

(c) $P(X \leq 1) \approx 0.5169$ ✓

(d) $P(X = 2) \approx 0.2711$ ✓

$P(\text{accepted}) \approx 0.5169 + 0.2711 \times 0.1887$

≈ 0.5681 ✓

(e) Sample size:

$P(X \geq 1) \geq 0.688$

$n = 14$ ✓✓

6. (a)

Outcomes	Probability	Gain
H	$\frac{1}{2}$	5
TH	$\frac{1}{4}$	5
TTH	$\frac{1}{8}$	5
TTT	$\frac{1}{8}$	-35

✓✓

The probability of winning \$5 is

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ ✓

(b) Expected loss/gain

$\left(5 \times \frac{1}{2}\right) + \left(5 \times \frac{1}{4}\right) + \left(5 \times \frac{1}{8}\right) + \left(-35 \times \frac{1}{8}\right)$ ✓

$= 0$

No loss or gain. ✓

7. (a) $P = 0.8 + 0.2(0.2)$

$= 0.84$ ✓

$X \sim B(5,\ 0.84)$

$P(X = 3) = 0.1517$ ✓

(b) $E(X) = 5(0.84)$

$= 4.2$ ✓

(c) $\text{Var}(X) = np(1 - p)$

$= 5(0.84)(1 - 0.84)$

$= 0.672$ ✓✓

(d) $(0.2)^2 = 0.04$ ✓

(e) $Y = 2X - (5 - X)$

$= 3X - 5$ ✓

i. $E(Y) = 3E(X) - 5$

$= 7.6$ ✓

ii. $\text{Var}(Y) = 9\ \text{Var}(X)$

$= 6.048$ ✓✓

CALCULATOR FREE: TRIAL TEST 3

Exponentials, Trigonometry, Binomial and Discrete Distributions

1. (a) $f(x) = \sin^4 x$

$f'(x) = 4\cos(x)\sin^3(x)$ ✓✓

(b) $\int_0^1 \left(\frac{e^{4x}}{e^{2x}} - \frac{e^x}{e^{2x}}\right) dx$

$= \int_0^1 (e^{2x} - e^{-x})\, dx$

$= \left[\frac{e^{2x}}{2} + \frac{1}{e^x}\right]_0^1$

$= \left(\frac{e^2}{2} + \frac{1}{e}\right) - \left(\frac{e^0}{2} + \frac{1}{e^0}\right)$ ✓

$= \frac{e^2}{2} + \frac{1}{e} - \frac{1}{2} - 1$

$= \frac{e^2}{2} + \frac{1}{e} - \frac{3}{2}$ ✓

2. (a) $E(2X - 1) = 2(15) - 1$

$= 29$ ✓

(b) $SD(2X - 1) = 2(3)$

$= 6$ ✓

(c) $\text{Var}(2X - 1) = 36$ ✓

3. *(a)* Velocity function

$$v(t) = \frac{t\cos(t)e^{\sin(t)} + \cos(t)e^{\sin(t)} - e^{\sin(t)}}{(t+1)^2} \quad ✓✓$$

or

$$\frac{(t+1)e^{\sin t}\cos t - e^{\sin t}}{(t+1)^2}$$

(b) when $t = 0$

$v(t) = 0$ ✓

4. *(a)* $np = 10$ $\quad np(1 - p) = 9$

$$10(1 - p) = 9$$

$$(1 - p) = \frac{9}{10}$$

$$p = \frac{1}{10} \quad ✓$$

$$n = \frac{10}{\frac{1}{10}}$$

$$n = 100 \quad ✓$$

(b) Mean $Y = 3 - 2(10)$

$= -17$ ✓

Variance $Y = (9)(-2)^2$

$= 36$ ✓

5. *(a)* $\frac{d}{dx} = (-3k)(3x^2)e^{x^3}$ ✓

Stationary point when $\frac{dy}{dx} = 0$

$-9kx^2e^{x^3} = 0$

$x = 0$ ✓

$y = 2k - 3ke^0$

$y = -k$ ✓

Stationary point at $(0, -k)$

(b) $\frac{dy}{dx} = -9kx^2e^{x^3}$

$\frac{d^2y}{dx^2} = (-18kx)(e^{x^3}) + (-9kx^2)(3x^2e^{x^3})$ ✓

Sub in $x = 0$

$\frac{d^2y}{dx^2} = (-18kx)(e^{x^3}) + (-9kx^2)(3x^2e^{x^3})$

$= 0$ ✓

Stationary point is a horizontal point of inflection. ✓

6. *(a)* $y = -x\cos(x)$

$\frac{dy}{dx} = -\cos(x) + x\sin(x)$ ✓✓

(b) $\int\frac{dy}{dx} = \int(-\cos(x))dx + \int x\sin(x)dx$ ✓

$-x\cos(x) = -\sin(x) + \int x\sin(x)dx$ ✓

$\int x\sin(x)\,dx = \sin(x) - x\cos(x) + c$ ✓

CALCULATOR ASSUMED: TRIAL TEST 4

Exponentials and Logarithms

1. *(a)* $v(0) = \ln(1)$

$= 0$ ✓

Acceleration $a(t) = \frac{1}{t+1}$ ✓

Initial acceleration $= 1\text{ m/s}^2$ ✓

(b) $v(t) = \ln(t+1)$

$2 = \ln(t+1)$ ✓

$e^2 = t+1$

$t = e^2 - 1$ seconds ✓

Acceleration

$a(t) = \frac{1}{t+1}$

when $t = e^2 - 1$

$a(t) = \frac{1}{e^2 - 1 + 1}$ ✓

$= \frac{1}{e^2}\text{ m/s}^2$

(c) As $t \to \infty$

Acceleration = 0 m/s^2 ✓

2. $y = -e^{4x^2 - \ln(3 - 2x)} + 1$

$\frac{dy}{dx} = \frac{16x^2e^{4x^2} - 24xe^{4x^2} - 2e^{4x^2}}{(2x - 3)^2}$ ✓

When $x = 2$

$\frac{dy}{dx} = 14e^{16}$ ✓

Coordinate $(2, e^{16} + 1)$

Equation of tangent line

$y = mx + c$

$e^{16} + 1 = 14e^{16}(2) + c$

$c = 1 - 27e^{16}$ ✓

$\therefore y = 14e^{16}x - 27e^{16} + 1$ ✓

3. *(a)* $f(x) = 2x\ln(x) - 2x + 3$

$f'(x) = 2\ln(x)$ ✓

(b) $\int 2\ln(x)\;dx = 2x\ln(x) - 2x + c$

$\therefore \int\ln(x)\;dx = x\ln(x) - x + d$ ✓

(c) $\int\ln(x^3)dx = 3\int\ln(x)\;dx$

$= 3x\ln(x) - 3x + k$ ✓✓

4. *(a)* $10\log\left(\frac{I}{I_0}\right) = 90$ ✓

$\log\left(\frac{I}{I_0}\right) = 9$

$\frac{I}{I_0} = 10^9$

$I = 10^9 I_0$ ✓

(b) Lawn mower: $10^9\ I_0$

Car Horn: $10^{7.6}\ I_0$

Intensity $= 10^{2.6}$ ✓

$= 398.11$ times greater. ✓

(c) $S = 10 \log \dfrac{I}{I_0}$

$= 10 \log \dfrac{10^{7.6} I_0}{I_0}$ ✓

$= 10 \log 10^{7.6}$

$= 76$ dB ✓

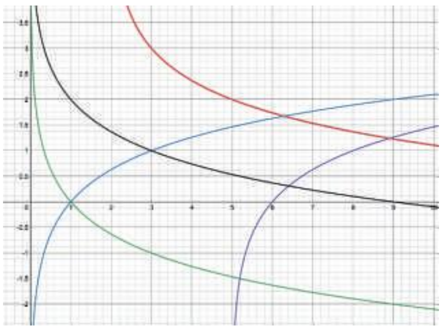
5. (a) $a = -1$ ✓

$b = -1$ ✓

$c = 2$ ✓

$d = -5$ ✓

$p = 3$ ✓

(b)  ✓

i. $x = 2$ ✓

ii. $(29, 0)$ ✓

6. (a) $P = ab^t$

$\log P = \log ab^t$ ✓

$\log P = \log a + \log b^t$

$\log P = \log a + t\log b$ ✓

Hence the relationship is linear.

(b) $\dfrac{4 - 1.3}{0 - 60} = -\dfrac{9}{200}$ ✓

$\log P = -\dfrac{9}{200}t + 4$ ✓

(c) $\log a = 4 \quad \therefore\ a = 10000$ initial population ✓

$\log b = -\dfrac{9}{200} \quad \therefore\ b = 0.9016$ rate of decrease ✓

(d) $P = 10000(0.9016)^t$

$2000 = 10000(0.9016)^t$

$t = 15.54$ years ✓

(e) $P = 10000(0.9016)^{30}$

$P = 447.11$

$P \approx 447$ ✓

7. (a) $P = 2200e^{0.05t}$

When $t = 6$

$P = 2200e^{0.05(6)}$

$P \approx 2970$ ✓

(b) $4000 = 2200e^{0.05t}$

$t = 11.96$ years ✓

(During 2023) ✓

(c) $\dfrac{dP}{dt} = 0.05P$

$\dfrac{dP}{dt} = 0.05(3000)$

$= 150$ marsupials/year ✓

(d) At the start of 2018

$P = 2970e^{-0.09t}$ ✓

$1300 = 2970e^{-0.09t}$

$t = 9.18$ years ✓

March 2027 ✓

CALCULATOR FREE: TRIAL TEST 4
Exponentials and Logarithms

1. $y = \ln(x)^2 e^{3x+1}$

$\dfrac{dy}{dx} = \ln(x^2) \bullet 3e^{3x+1} + \dfrac{2x}{x^2}e^{3x+1}$ ✓

$= e^{3x+1}\left(3\ln(x)^2 + \dfrac{2}{x}\right)$ ✓

(b) $\displaystyle\int \frac{8x}{1 + 3x^2}\,dx$

$= \dfrac{4}{3}\displaystyle\int \frac{6x}{1 + 3x^2}\,dx$ ✓

$= \dfrac{4}{3}\ln(1 + 3x^2) + c$ ✓

2. (a) $y = \dfrac{\ln(x)}{\frac{1}{2}x}$

$\dfrac{dy}{dx} = \dfrac{\frac{1}{x} \bullet \frac{x}{2} - \frac{1}{2}\ln(x)}{\frac{1}{4}x^2}$

$= \dfrac{\frac{1}{2} - \frac{1}{2}\ln(x)}{\frac{1}{4}x^2}$ ✓

$= \dfrac{2 - 2\ln(x)}{x^2}$

when $\dfrac{dy}{dx} = 0$

$\dfrac{2 - 2\ln(x)}{x^2} = 0$ ✓

$2 - 2\ln(x) = 0$

$2\ln(x) = 2$

$\ln(x) = 1$

$x = e$ ✓

$\left(e, \dfrac{2}{e}\right)$ ✓

(b)
$$y = mx + c$$
$$\frac{dy}{dx} = \frac{2 - 2\ln(x)}{x^2}$$
$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{2 - 2\ln(1)}{1^2}$$
$$= 2 \ ✓$$
$$(1, 0) \quad y = 2x + c$$
$$0 = 2(1) + c$$
$$c = -2$$
$$\therefore y = 2x - 2 \ ✓$$

3.
$$50e^{2x+1} = 3000$$
$$e^{2x+1} = 60 \ ✓$$
$$(2x+1)\ln e = \ln 60 \ ✓$$
$$2x = \ln 60 - 1$$
$$x = \frac{\ln 60 - 1}{2} \ ✓$$

4. (a)
$$2\log_2 3 + \log_2 4 - 1$$
$$= 2\log_2 3 + \log_2 4 - \log_2 2 \ ✓$$
$$= \log_2 3^2 + \log_2 4 - \log_2 2$$
$$= \log_2 \frac{36}{2}$$
$$= \log_2 18 \ ✓$$

(b)
$$\log_4 y = \log_4(4x) - 2$$
$$\log_4 y = \log_4(4x) - 2\log_4 4 \ ✓$$
$$\log_4 y = \log_4(4x) - \log_4 16$$
$$\log_4 y = \log_4\left(\frac{4x}{16}\right) \ ✓$$
$$\therefore y = \frac{x}{4} \ ✓$$

5. (a)
$$\log_p(tg) = \log_p t + \log_p g$$
$$= f + h \ ✓$$

(b)
$$\log_p\left(\frac{q}{t}\right) = \log_p q - \log_p t$$
$$= r - f \ ✓$$

(c)
$$\log_p \sqrt{qg} = \log_p (qg)^{\frac{1}{2}} \ ✓$$
$$= \frac{1}{2}\log_p(qg)$$
$$= \frac{1}{2}\left[\log_p q + \log_p g\right] \ ✓$$
$$= \frac{1}{2}(r + h)$$
$$= \frac{1}{2}r + \frac{1}{2}h \ ✓$$
$$= \frac{r + h}{2}$$

(d)
$$2\log_p\left(\frac{qt}{\sqrt{g}}\right) = 2\left[\log_p(qt) - \log_p\left(\sqrt{g}\right)\right] \ ✓$$
$$= 2\left[\log_p q + \log_p t - \frac{1}{2}\log_p g\right] \ ✓$$
$$= 2\left[r + f - \frac{1}{2}h\right]$$
$$= 2r + 2f - h \ ✓$$

CALCULATOR ASSUMED: TRIAL TEST 5

Continuous Distributions and Sample Proportions

1. (a)
$$\int_3^5 \left(\frac{k - x}{2}\right) dx \ ✓$$
$$1 = \left[\frac{kx}{2} - \frac{x^2}{4}\right]_3^5$$
$$1 = \left(\frac{5k}{2} - \frac{25}{4}\right) - \left(\frac{3k}{2} - \frac{9}{4}\right)$$
$$1 = k - 4$$
$$k = 5 \ ✓$$

(b)
$$\int_4^5 \left(\frac{5 - x}{2}\right) dx$$
$$= \left[\frac{5x}{2} - \frac{x^2}{4}\right]_4^5$$
$$= \left(\frac{25}{2} - \frac{25}{4}\right) - \left(\frac{20}{2} - \frac{16}{4}\right)$$
$$= \frac{5}{2} - \frac{9}{4}$$
$$= \frac{1}{4} \ ✓$$

2. (a)
$$P(Y = y) = \begin{cases} \frac{1}{12}, & 14 \le y \le 26 \\ 0, & \text{otherwise} \end{cases} \ ✓$$

(b) $P(Y < 18) = \frac{1}{3}$ ✓

(c) $P(16 \le Y \le 25) = \frac{3}{4}$ ✓

(d) Median travelling time = 20 minutes. ✓

(e) $P(Y \ge 15) = \frac{11}{12}$ ✓

(f) $P(\text{late on Thurs and Fri}) = \frac{11}{12} \times \frac{11}{12}$

$$= \frac{121}{144} \ ✓$$

(g) $X \sim B\left(5, \frac{11}{12}\right)$

$P(X \ge 3) \approx 0.9949$ ✓

3. $X \sim N(1.8,\ 0.28^2)$ ✓

(a) $P(X \geq 1.6 \mid X < 1.9) \approx \dfrac{0.40198}{0.6395}$

≈ 0.6286 ✓

(b) $P(X < a) = 0.2$ ✓

$\therefore a \approx 1.5644$ ✓

Actual weight is 1.5644 kg

4. (a) $\hat{p} = \dfrac{350}{500}$ $\quad n = 500$

$\hat{p} = 0.7$ $\quad k = 1.96$ ✓

CI: $0.6598 \leq p \leq 0.7402$ ✓

(b) $ME = k\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

$0.04 = 2.576\sqrt{\dfrac{(0.8)\times(0.2)}{n}}$ ✓

$n = 663.58$

Sample size 664 ✓

5. $X \sim N(30,\ 7.2^2)$

(a) $P(X < 25) \approx 0.2437$ ✓

(b) $P(27 \leq X \leq 32) \approx 0.27095$ ✓

(c) $P(X < 15) \approx 0.01861$

Number of patients $\approx 1000 \times 0.01861$

$= 18.61$

Approximately 19 patients. ✓

(d) $P(X > 25) = 0.9$ $\quad \mu = ?$

$z = \dfrac{x-\mu}{\sigma}$

$-1.28156 = \dfrac{25-\mu}{7.2}$ ✓

New mean: $\mu \approx 34.227$ kg ✓

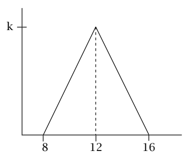
(e) $z = \dfrac{z-\mu}{\sigma}$ $\quad \sigma = ?$

$-1.28156 = \dfrac{25-30}{\sigma}$ ✓

New standard deviation:
$\sigma = 3.901$ kg ✓

6. Area: $\dfrac{1}{2}(8) \times k = 1$

$k = \dfrac{1}{4}$ ✓



Probability

$= \int_8^{11} (0.0625x - 0.5)\ dx$ ✓

$= 0.28125$ ✓

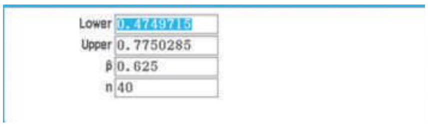
7. (a) $p = \hat{p} = \dfrac{25}{40} = \dfrac{5}{8}$ ✓

(b) Standard deviation $(\hat{p}) = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

$= \sqrt{\dfrac{\frac{5}{8}\left(\frac{3}{8}\right)}{40}}$ ✓

$= 0.0765$ ✓

(c)



✓

CI = [0.475, 0.775] ✓

(d) 95% of samples of the same size are likely to contain p in the interval 0.475 to 0.775. ✓✓

8. $np = 100 \times 0.58 = 58$ $\quad nq = 100 \times 0.42 = 42$ ✓

Thus, sample size is greater than 30: hence use a normal approximation or binomial.

$p = \hat{p} = 0.58$ ✓

Standard deviation $(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}}$

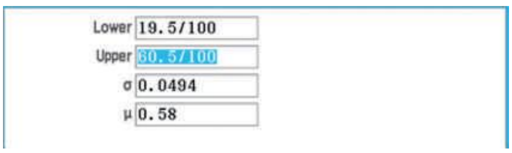
$= \sqrt{\dfrac{0.58 \times 0.42}{100}}$

$= 0.0494$ ✓

Using Binomial

✓✓

Or Using Normal Approximation



✓✓

EXAMINATION STYLE QUESTIONS – SOLUTIONS

1. (a) $\frac{dy}{dx} = 4(-2)^2 - 5$

$\frac{dy}{dx} = 11$

(b) $4x^2 - 5 = 1$

$4x^2 = 6$

$x^2 = \frac{3}{2}$

$x = \pm\sqrt{\frac{3}{2}}$

2. (a) $TP(3, -9)$

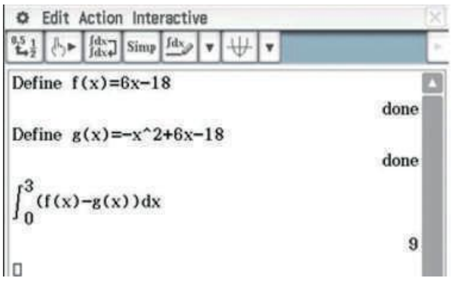
(b) $\frac{dy}{dx} = -2x + 6 \qquad (0, -18)$

$\frac{dy}{dx} = 6$

Eqn of tangent:

$y = 6x - 18$

(c) Eqn of line parallel to y axis is $x = 3$



Area = 9 units^2

3. (a) $\frac{dy}{dx} = 3x^2 - 12x + 25$

(b) $y = \frac{6x^7 - 14x^3}{x^3}$

$y = 6x^4 - 14$

$\frac{dy}{dx} = 24x^3$

(c) $y = x^2 - x^{\frac{2}{3}}$

$\frac{dy}{dx} = 2x - \frac{2}{3}x^{-\frac{1}{3}}$

$\frac{dy}{dx} = 2x - \frac{2}{3\sqrt[3]{x}}$

4. (a) i. 0.7

ii. $\frac{0.7}{0.8} = \frac{7}{8}$

(b) i. $E(X) = (-3 \times 0.1) + (-2 \times 0.2) + (-1 \times 0.1) + (0 \times 0.3) + (1 \times 0.1) + (2 \times 0.2)$

$= -0.3$

ii. $E(2X - 3) = 2(-0.3) - 3$

$= -3.6$

(c) i. $\text{Var}(X) = (0.1)(-2.7)^2 + (0.2)(-1.7)^2 + (0.1)(-0.7)^2 + (0.3)(0.3)^2 + (0.1)(1.3)^2 + (0.2)(2.3)^2$

$= 2.61$

ii. $\text{Var}(2X - 3) = 2^2 \times \text{Var}(X)$

$= 10.44$

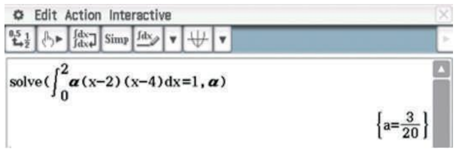
5. (a) $\frac{d}{dx}(x^3e^{2x}) = 3x^2e^{2x} + 2x^3e^{2x}$

(b) $\frac{d}{dx}(\cos(3x - 5)) = -3\sin(3x - 5)$

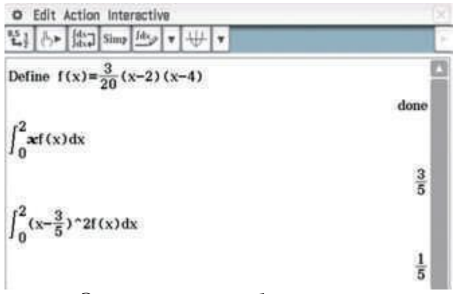
(c) $f(x) = \sin(2x)\ln(2x)$

$f'(x) = 2\cos(2x)\ln(2x) + \frac{\sin(2x)}{x}$

6. (a)



(b)



$E(X) = \frac{3}{5} \qquad \text{Var}(X) = \frac{1}{5}$

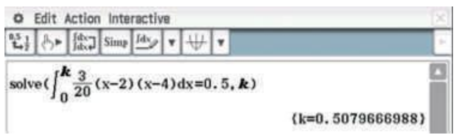
(c) $E(Y) = 3 - 2\left(\frac{3}{5}\right)$

$= \frac{9}{5}$

$\text{Var}(Y) = (-2)^2 \times \frac{1}{5}$

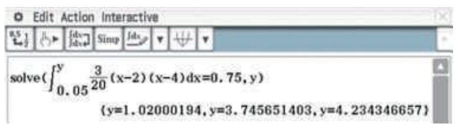
$= \frac{4}{5}$

(d)

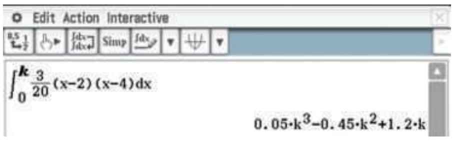


Median = 0.508 (3dp)

(e)

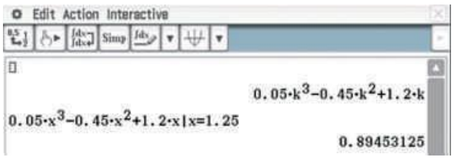


$k = 1.02$

(f) 

$F(x) = P(X \le x) =$

$$\begin{cases} 0 & x < 0 \\ 0.05x^3 - 0.45x^2 + 1.2x, & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

(g) 

$P(X \le 1.25) = 0.8945$

7\. (a) $y = x^3 - 4x^2$

$\frac{dy}{dx} = 3x^2 - 8x$

when $x = 3$

$\frac{dy}{dx} = 3$

(b) $21 = -p(-4)^2 + \frac{3}{4}(-4) + p$

$21 = -16p - 3 + p$

$24 = -15p$

$p = -\frac{8}{5}$

8\. (a) $np = 90.75$ and $np(1-p) = 8.25^2$

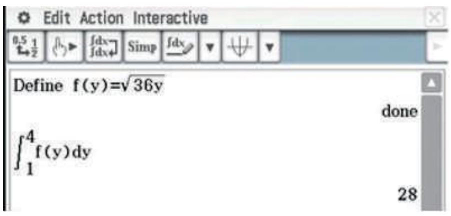
Using CAS: $n = 363$ and $p = 0.25$

(b) $P(T \ge 99) = 0.1734$

9\. Curve and lines intersect at (–6, 1) and (–12, 4)

$y = 1$, $y = 4$

Curve: $x = \sqrt{36y}$



Area = 28 units^2

10\. $y = x^2 - 8x + 15$

$\frac{dy}{dx} = 2x - 8$

when $x = 3$ $\frac{dy}{dx} = -2$

when $x = 5$ $\frac{dy}{dx} = 2$

$y = -2x + c$ (3, 0)
$0 = -2(3) + c$
$c = 6$
$\therefore\ y = -2x + 6$

$y = 2x + c$ (5, 0)
$0 = 2(5) + c$
$c = -10$
$\therefore\ y = 2x - 10$

11\. (a) $\int \frac{\sin(3t)}{2 + \cos(3t)}\, dt = \frac{-\ln(2 + \cos(3t))}{3} + c$

(b) $\int_1^4 \left(2 + \frac{3}{x}\right) dx = [2x + 3\ln(x)]_1^4$

$= (2(4) + 3\ln 4) - (2(1) + 3\ln 1)$

$= 6 + 3\ln 4$

(c) $\int \frac{-3}{e^{-5x+2}}\, dx = \frac{-3e^{5x-2}}{5} + c$

12\. (a) $\log(2^{3x-1}) = \log(3^{2x})$

$(3x - 1)\log 2 = (2x)\log 3$

$3x\log 2 - \log 2 = 2x\log 3$

$3x\log 2 - 2x\log 3 = \log 2$

$x(3\log 2 - 2\log 3) = \log 2$

$x = \frac{\log 2}{3\log 2 - 2\log 3}$

(b) $\log_3 27 \times \log_2 16^3 = \log_3 3^3 \times \log_2 (2^4)^3$

$= 3\log_3 3 \times 12\log_2 2$

$= 36$

(c) $2 - \log_2 25 + 2\log_2 10$

$= 2\log_2 2 - \log_2 25 + 2\log_2 10$

$= \log_2 2^2 - \log_2 25 + \log_2 10^2$

$= \log_2\left(\frac{4}{25} \times 100\right)$

$= \log_2 16$

$= 4$

13\. (a) $\int_0^{\frac{\pi}{6}} \frac{1}{2}\sin(x)\, dx = \left[-\frac{\cos(x)}{2}\right]_0^{\frac{\pi}{6}}$

$= \left(-\frac{\cos\frac{\pi}{6}}{2}\right) - \left(-\frac{\cos(0)}{2}\right)$

$= -\frac{\sqrt{3}}{4} + \frac{1}{2}$

(b) $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}\sin(x)\, dx = \left[-\frac{\cos(x)}{2}\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$

$= \left(-\frac{\cos\left(\frac{2\pi}{3}\right)}{2}\right) - \left(-\frac{\cos\left(\frac{\pi}{2}\right)}{2}\right)$

$= \frac{1}{4}$

14. (a) $\int 2x - \frac{6}{\sqrt{x}} + 3 \ dx$

$= x^2 - 12\sqrt{x} + 3x + c$

$f(x) = x^2 - 12\sqrt{x} + 3x + c \quad (1, 2)$

$2 = (1)^2 - 12\sqrt{1} + 3(1) + c$

$c = 10$

$\therefore f(x) = x^2 - 12\sqrt{x} + 3x + 10$

(b) $\frac{dy}{dx} = 2x - \frac{6}{\sqrt{x}} + 3 \quad (1, 2)$

$\frac{dy}{dx} = -1$

$y = -x + c$

$2 = -(1) + c$

$c = 3$

$\therefore y = -x + 3$

Eqn of tangent: $x + y - 3 = 0$

15. (a) $\int_0^1 ke^{3x} \ dx = 1$

$\left[\frac{ke^{3x}}{3}\right]_0^1 = 1$

$\left(\frac{ke^3}{3}\right) - \left(\frac{k}{3}\right) = 1$

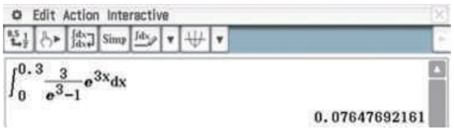
$\frac{ke^3 - k}{3} = 1$

$ke^3 - k = 3$

$k(e^3 - 1) = 3$

$k = \frac{3}{e^3 - 1}$

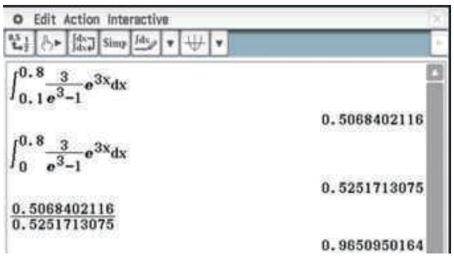
(b)



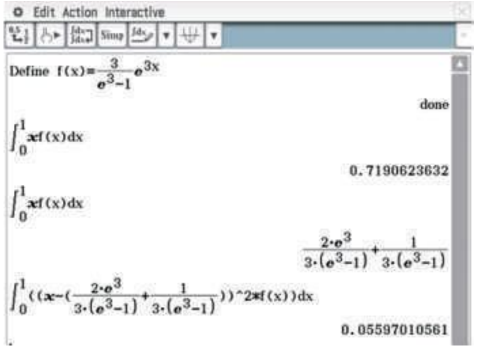
$P(X \leq 0.3) = 0.0765$

(c) $P(X > 0.1 \mid X \leq 0.8) = \frac{P(0.1 < X \leq 0.8)}{P(X \leq 0.8)}$

$= 0.9651$



(d)



$E(X) = 0.7191$

$\text{Var}(X) = 0.05597$

16. $\int 3x^{-\frac{1}{2}} + \frac{x}{\sqrt{x}} \ dx$

$y = 6\sqrt{x} + \frac{2x^{\frac{3}{2}}}{3} + c \quad (4, 27)$

$27 = 6\sqrt{4} + \frac{2(4)^{\frac{3}{2}}}{3} + c$

$c = \frac{29}{3}$

$y = 6\sqrt{x} + \frac{2x^{\frac{3}{2}}}{3} + \frac{29}{3}$

17. $\frac{dy}{dx} = 6x^2$

when $x = 2$ $\frac{dy}{dx} = 24$

$y = 24x + c \quad (2, 11)$

$11 = 24(2) + c$

$c = -37$

$\therefore y = 24x - 37$

y int curve $(0, -5)$ line $(0, -37)$

$\therefore$ Length of $RQ = 32$ units

18. (a) $M_0 = 6.25$ grams

(b) $2.75 = 6.25e^{-3k}$

$k = 0.2737$

(c) $M = 6.25e^{-0.2737t}$

$1.4 = 6.25e^{-0.2737t}$

$t = 5.47$ days

(d) $M = 6.25e^{-0.2737t}$

$\frac{dM}{dt} = (-0.2737)(6.25e^{-0.2737t})$

$-0.6 \ln(2) = (-0.2737)(6.25e^{-0.2737t})$

$t = 5.17$ days

(e) $\frac{dM}{dt} = -0.2737M$

$= -0.2737(5)$

$= -1.3685$

Decrease of 1.3685 grams/day

19. (a) $y = \sqrt{2}\sin(2x)$

$y = \sqrt{2}\cos(x)$

When $x = \dfrac{\pi}{6}$

$y = \sqrt{2}\sin\left(2\left(\dfrac{\pi}{6}\right)\right)$

$\therefore y = \dfrac{\sqrt{6}}{2}$

$y = \sqrt{2}\cos\left(\dfrac{\pi}{6}\right)$

$\therefore y = \dfrac{\sqrt{6}}{2}$

y coordinates identical

When $x = \dfrac{\pi}{2}$

$y = \sqrt{2}\sin\left(2\left(\dfrac{\pi}{2}\right)\right)$

$\therefore y = 0$

$y = \sqrt{2}\cos\left(\dfrac{\pi}{2}\right)$

$\therefore y = 0$

y coordinates identical

(b) Area $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sqrt{2}\sin(2x) - \sqrt{2}\cos(x))\, dx$

$= \left[\dfrac{-\sqrt{2}\cos(2x)}{2} - \sqrt{2}\sin(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(\dfrac{-\sqrt{2}\cos\left(\frac{2\pi}{2}\right)}{2} - \sqrt{2}\sin\left(\dfrac{\pi}{2}\right)\right)$

$- \left(\dfrac{-\sqrt{2}\cos\left(\frac{2\pi}{6}\right)}{2} - \sqrt{2}\sin\left(\dfrac{\pi}{6}\right)\right)$

$= \left(\dfrac{\sqrt{2}}{2} - \sqrt{2}\right) - \left(-\dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{2}}{2}\right)$

$= \dfrac{\sqrt{2}}{4}$ units^2

20. (a) $\int 6x^2 - 24x + 18dx$

$f(x) = 2x^3 - 12x^2 + 18x + c \quad (1, 8)$

$8 = 2(1)^3 - 12(1)^2 + 18(1) + c$

$c = 0$

$\therefore f(x) = 2x^3 - 12x^2 + 18x$

(b) y intercept $(0, 0)$

x intercepts $f(x) = 2x(x^2 - 6x + 9)$

$f(x) = 2x(x - 3)^2$

$\therefore\ (0, 0)\ (3, 0)$

Stationary points when

$f'(x) = 0$

$0 = 6x^2 - 24x + 18$

$0 = 6(x - 3)(x - 1)$

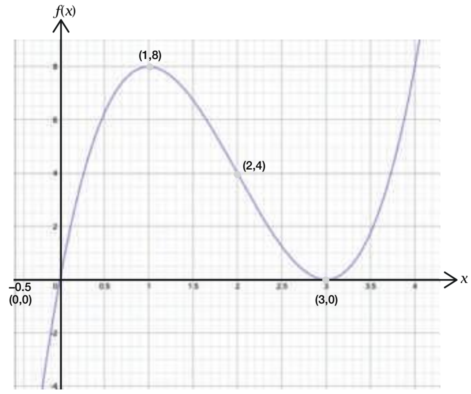
$\therefore\ (3, 0)$ min $\quad (1, 8)$ max

Points of inflection when

$f''(x) = 0$

$0 = 12x - 24$

$\therefore\ (2, 4)$



21. $\dfrac{dy}{dx} = 3x^2 - 12$

Stationary pts when $\dfrac{dy}{dx} = 0$

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

Coordinates $(2, -16)\ (-2, 16)$

Nature: $\dfrac{d^2y}{dx^2} = 6x$

At $x = 2$ $\dfrac{d^2y}{dx^2} = 12 \quad \therefore$ *min*

At $x = -2$ $\dfrac{d^2y}{dx^2} = -12 \quad \therefore$ *max*

$(2, -16)$ min T.P

$(-2, 16)$ max T.P

22. (a) $y = 2xe^{-0.5x}$

$\dfrac{dy}{dx} = 2e^{-0.5x} - xe^{-0.5x}$

Stationary point(s) when $\dfrac{dy}{dx} = 0$

$0 = 2e^{-0.5x} - xe^{-0.5x}$

$0 = e^{-0.5x}(2 - x)$

$\therefore e^{-0.5x} = 0,\ 2 - x = 0$

$\therefore x = 2$

Coordinate $= \left(2, \dfrac{4}{e}\right)$

(b) Point of inflection when $\dfrac{d^2y}{dx^2} = 0$

$\dfrac{d^2y}{dx^2} = -e^{-0.5x} - (e^{-0.5x} - 0.5xe^{-0.5x})$

$0 = -e^{-0.5x}(2 - 0.5x)$

$\therefore e^{-0.5x} = 0,\ 2 - 0.5x = 0$

$\therefore x = 4$

Coordinate $= \left(4, \dfrac{8}{e^2}\right)$

23. (a) Standard deviation $\hat{p} = \sqrt{\dfrac{p(1 - p)}{n}}$

$= \sqrt{\dfrac{0.6(1 - 0.6)}{42}}$

$$= 0.0756$$

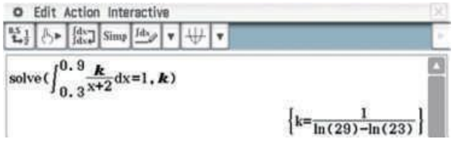
(b) Standard deviation $\hat{p} \leq 0.04$

$$\sqrt{\frac{p(1-p)}{n}} \leq 0.04$$

$$\sqrt{\frac{0.4(1-0.4)}{n}} \leq 0.04$$

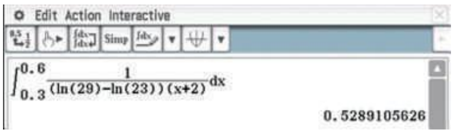
$$n \geq 150$$

24. (a)



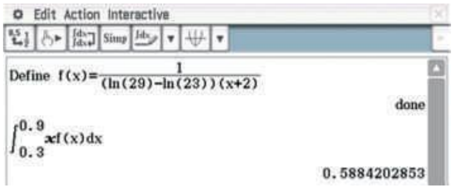
$$k = \frac{1}{\ln(29) - \ln(23)} \; (4.314)$$

(b)



$$P(X \leq 0.6) = 0.53$$

(c)



$$E(X) = 0.5884$$

(d) $E(Y) = 1 - 3(0.5884)$

$$= -0.7652$$

25. $\frac{dy}{dx} = x^2 - 2x - 3$

Stationary pts when $\frac{dy}{dx} = 0$

$\therefore x = 3, x = -1$

Minimum pt at (3, 0)

$$0 = \frac{1}{3}(3)^3 - (3)^2 - 3(3) + k$$

$$k = 9$$

26. (a) Systematic Sampling

(b) Main source of bias is only interviewing people who frequent fast food restaurants. Suggest a random sample should be taken where a selection of different people can be interviewed. Other possible solutions.

27. (a) $v = \frac{32}{(t+3)^2}$

$$x = \int_0^3 \frac{32}{(t+3)^2} dt$$

$$x = \int_0^3 32(t+3)^{-2} dt$$

$$x = \left[\frac{32(t+3)^{-1}}{-1}\right]_0^3$$

$$x = \left[-\frac{32}{t+3}\right]_0^3$$

$$x = \left(-\frac{32}{6}\right) - \left(-\frac{32}{3}\right)$$

$$x = \frac{16}{3} \text{ metres}$$

(b) $a = \frac{d}{dt}\left(\frac{32}{(t+3)^2}\right)$

$$a = -\frac{64}{(t+3)^3}$$

$$-1 = -\frac{64}{(t+3)^3}$$

$$(t+3)^3 = 64$$

$$t = 1 \text{ second}$$

At $t = 1$, $v = \frac{32}{(1+3)^2}$

$$v = 2 \text{ m/s}$$

28. (a) $(2\sqrt{x} + 3)^2 = (2\sqrt{x} + 3)(2\sqrt{x} + 3)$

$$= 4x + 6\sqrt{x} + 6\sqrt{x} + 9$$

$$= 4x + 12\sqrt{x} + 9$$

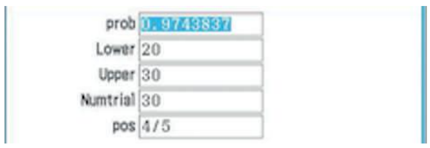
$\therefore k = 12$

(b) $\int (2\sqrt{x} + 3)^2 \; dx = \int 4x + 12\sqrt{x} + 9 \; dx$

$$= 2x^2 + 8x^{\frac{3}{2}} + 9x + c$$

29. (a) Binomial $X \sim B\left(30, \frac{4}{5}\right)$

(b) i.



$$P(X \geq 20) = 0.9744$$

ii.

$$P(X = 5) = 0.1723$$

(c)

$Y \sim B = (10, 0.9744)$

$P(Y \geq 9) = 0.2027$

(d) $P(X \geq 1) \geq 0.95$

$1 - P(X = 0) \geq 0.95$

$P(X = 0) \leq 0.05$

$0.92^n \leq 0.05$

$\therefore n \geq 35.93$

Therefore, require a sample size of 36.

(e) Mean $E(X) = (30)\left(\frac{4}{5}\right)$

$= 24$

St Deviation $= \sqrt{(30)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)}$

$= 2.191$

30. (a) $x = 2$ and $y = 0.25$

(b) Intersection: $f(x) = 2e^{x-2}$ and $y = 0.25$

$x = -0.0794$

Intersection: $f(x) = 2e^{x-2}$ and $g(x) = 3e^{-x}$

$x = 1.2027$

Area 1 $= \int_{-0.0794}^{1.2027} (2e^{x-2} - 0.25)\ dx$

$= 0.3306$

Area 2 $= \int_{1.2027}^{2} (3e^{-x} - 0.25)\ dx$

$= 0.2958$

Total area = 0.6264 units ^{2}

31. (a) i. $\frac{dy}{dx} = 9x^2 - 7$

ii. $\frac{d^2y}{dx^2} = 18x$

(b) $\int\left(3 + 2\sqrt{x} - \frac{1}{x^2}\right) dx = 3x + \frac{4x^{\frac{3}{2}}}{3} + \frac{1}{x} + c$

32. (a) $ke^{2(1)+1} + ke^{2(0)+1} = 1$

$ke^3 + ke^1 = 1$

$k(e^3 + e) = 1$

$k = \frac{1}{e^3 + e}$

(b) $E(X) = \frac{e^3}{e^3 + e}$

33. (a) $f(x)$ passes through (1, 0) and (2, 0.5)

$\therefore \quad 0.5 = \log_r 2$

$r^{0.5} = 2$

$r = 4$

$f(x + q)$ passes through (–1, 0) and (0, 0.5)

$\therefore \quad 0 = \log_4(-1 + q)$

$4^0 = -1 + q$

$q = 2$

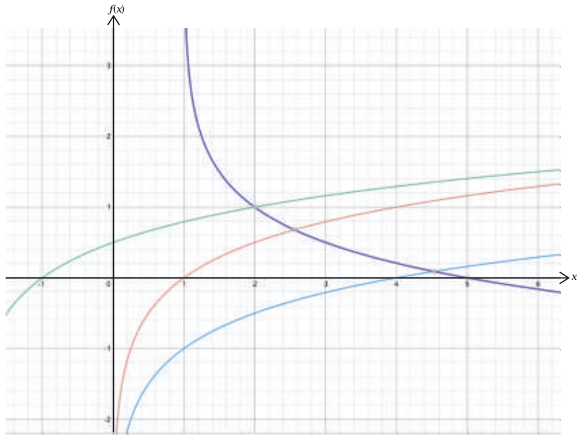
$f(x) + p$ passes through (4, 0)

$\therefore \quad 0 = (\log_4 4) + p$

$0 = 1 + p$

$p = -1$

(b)



i. Asymptote: $x = 1$

ii. Root: $x = 5$

$g(x) = -\log_r(x - 1) + 1$

$0 = -\log_4(x - 1) + 1$

$1 = \log_4(x - 1)$

$4 = x - 1$

$x = 5$

34. (a) $3k + 8k + 24k + 120k = 1$

$k = \frac{1}{155}$

(b)

x	2	3	5	11
$P(X = x)$	$\frac{3}{155}$	$\frac{8}{155}$	$\frac{24}{155}$	$\frac{120}{155}$

35. $\int_1^2\left(4 + 6x^2 - \frac{5}{x^2}\right) dx = \left[4x + 2x^3 + \frac{5}{x}\right]$

$= \left[4(2) + 2(2)^3 + \frac{5}{2}\right] -$

$[4 + 2 + 5]$

$= \frac{31}{2}$

36. $\frac{dy}{dx} = 3ax^2 - 6x + 8$

when $x = 2 \frac{dy}{dx} = 0$

$3a(2)^2 - 6(2) + 8 = 0$

$a = \frac{1}{3}$

$\therefore\ y = \frac{1}{3}x^3 - 3x^2 + 8x + b \quad (2, 10)$

$10 = \frac{1}{3}(2)^3 - 3(2)^2 + 8(2) + b$

$b = \frac{10}{3}$

37. (a)

t	0	1	2	3	4	6	9
$P(T = t)$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

(b) Y = \$profit for the player

y	-2.50	2.50	12.50
$P(Y = y)$	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{4}{16}$

$E(Y) = \dfrac{(-2.50 \times 7) + (2.50 \times 5) + (12.50 \times 4)}{16}$

$E(Y) = \$2.81$

Expected profit of \$2.81

38. $\dfrac{dy}{dx} = x^2 - x - 2$

Minimum point when $\dfrac{dy}{dx} = 0$

$x^2 - x - 2 = 0$

$x = -1, 2$

Minimum coordinate at $\left(2, -\dfrac{7}{3}\right)$

Equation of line $y = -\dfrac{25}{6}x + 6$

Area shaded region

$= \displaystyle\int_0^2 \left[\left(-\frac{25}{6}x + 6\right) - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x + 1\right)\right] dx$

$= \left[\left(-\dfrac{25x^2}{12} + 6x\right) - \left(\dfrac{x^4}{12} - \dfrac{x^3}{6} - x^2 + x\right)\right]$

$= \left[\left(-\dfrac{25(2)^2}{12} + 6(2)\right) - \left(\dfrac{(2)^4}{12} - \dfrac{(2)^3}{6} - (2)^2 + 2\right)\right]$

$= -\dfrac{100}{12} + 12 - \dfrac{16}{12} + \dfrac{8}{6} + 4 - 2$

$= \dfrac{68}{12}$ u^2

$= \dfrac{17}{3}$ units^2

39. *(a)* Max when $\dfrac{dy}{dx} = 0$ at when $x = 1$

$3(1)^2 - 7(1) + k = 0$

$k = 4$

(b) $\int 3x^2 - 7x + 4 \;\; dx$

$y = \dfrac{3x^3}{3} - \dfrac{7x^2}{2} + 4x + c \;\; (1, -3)$

$-3 = (1)^3 - \dfrac{7(1)^2}{2} + 4(1) + c$

$c = -\dfrac{9}{2}$

$\therefore y = x^3 - \dfrac{7}{2}x^2 + 4x - \dfrac{9}{2}$

40. *(a)* $\dfrac{dy}{dx} = -x^3 + x^4$

(b) $\int 28(2x - 5)^3 \, dx = \dfrac{28(2x - 5)^4}{2 \times 4} + c$

$= \dfrac{7(2x - 5)^4}{2} + c$

41. *(a)* -18

(b) -19

(c) -46

(d) $\displaystyle\int_{-1}^{2} 3 \;\, dx + \int_{-1}^{2} f(x) \;\, dx$

$= [3x]_{-1}^{2} + (-16)$

$= [3(2) - 3(-1)] - 16$

$= -7$

(e) -42

42. Number 4.

43. Increasing when $\dfrac{dy}{dx} > 0$

$\dfrac{dy}{dx} = 18 - 6x^2$

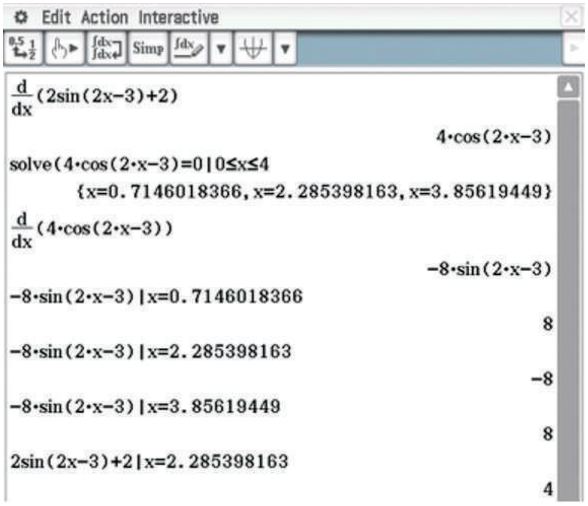
$18 - 6x^2 = 0$

$x = \pm\sqrt{3}$

Increasing function at $-\sqrt{3} < x < \sqrt{3}$

44. *(a)* $v = 4\cos(2t - 3)$

Max when $\dfrac{dv}{dt} = 0$



Max displacement of 4 metres at time $t = 2.2854$ secs.

Second derivative check:

$\dfrac{d^2v}{dt^2} = -8\sin(2t - 3)$

At $t = 2.2854$, $\dfrac{d^2v}{dt^2} < 0 \;\; \therefore$ Maximum

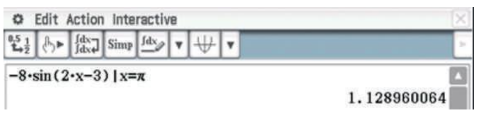
(b) Stationary when $\dfrac{dv}{dt} = 0$

i.e $t = 0.7146, 2.2854, 3.8562$ seconds

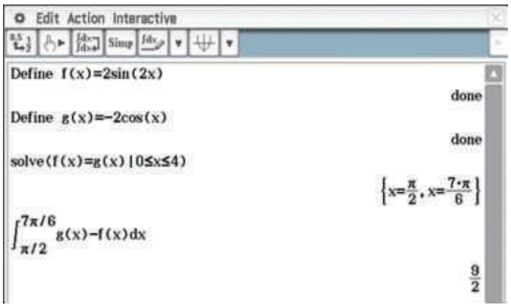
(c) Acceleration $a = -8\sin(2t - 3)$

At $t = \pi$

$a = 1.129$ m/s^2



45. Intersection when $f(x) = g(x)$



Area = 4.5 units^2

46. $v(t) = \int -3(2t+7)^{-\frac{1}{2}}\,dt$

$$= -\frac{3(2t+7)^{\frac{1}{2}}}{2\times\frac{1}{2}} + c$$

$$= -3(2t+7)^{\frac{1}{2}} + c$$

$$x(t) = -\frac{3(2t+7)^{\frac{3}{2}}}{2\times\frac{3}{2}} + ct + d$$

$$= -(2t+7)^{\frac{3}{2}} + ct + d$$

$x(1) = 6$
$6 = -(2(1)+7)^{\frac{3}{2}} + c + d$
$6 = -27 + c + d$
$33 = c + d$

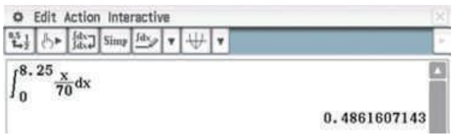
$x(9) = 28$
$28 = -(2(9)+7)^{\frac{3}{2}} + 9c + d$
$28 = -125 + 9c + d$
$153 = 9c + d$

Solving simultaneously
$c = 15$
$d = 18$
$v(t) = -3(2t+7)^{\frac{1}{2}} + 15$
$x(t) = -(2t+7)^{\frac{3}{2}} + 15t + 18$

47. (a) $\frac{1}{2}(14)(a) = 1$

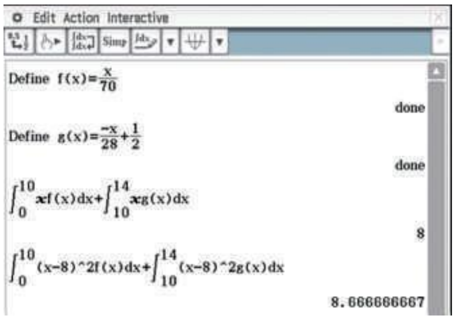
$a = \frac{1}{7}$

(b) $f(t) = \frac{t}{70} \quad 0 \le t \le 10$



$P(T < 8.25) = 0.4862$

(c)



$E(T) = 8$ hours and $Var(T) = 8$ hours 40 minutes

(d) $E(0.8T - 1.5) = 0.8 \times 8 - 1.5$
$= 4.9$ hours
$= 4$ hours and 54 minutes

$\text{Var}(0.8T) = (0.8)^2 \times 8\frac{2}{3}$
$= \frac{416}{75}$
$= 5$ hours 33 minutes

48. (a) $\hat{p} = \frac{97}{120} \approx 0.8083$

(b) $E \approx 1.96\sqrt{\frac{\frac{97}{120}\left(1-\frac{97}{120}\right)}{120}}$

≈ 0.0704

(c) Confidence Interval:

$\approx (0.7379, 0.8788)$

49. (a) $k\log_2 2 + k\log_2 4 + k\log_2 8 + k\log_2 16 = 1$

$1k + 2k + 3k + 4k = 1$

$k = \frac{1}{10}$

x	2	4	8	16
$P(X = x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

(b) $P(X < 8) = \frac{3}{10}$

(c) $P(x > 2 \mid X < 8) = \frac{2}{3}$

50. (a) $f(x) = x^2\ln(x)$
$f'(x) = (2x)(\ln(x)) + (x^2)\left(\frac{1}{x}\right)$
$f'(x) = (2x)(\ln(x)) + (x)$

$0 = x(2\ln(x) + 1)$
$\therefore\ 2\ln(x) + 1 = 0$
$\ln(x) = -\frac{1}{2}$
$x = e^{-\frac{1}{2}}$

(b) Nature of stationary points:

$f''(x) = 2\ln(x) + (2x)\left(\frac{1}{x}\right) + 1$

$f''(x) = 2\ln(x) + 3$

When $x = e^{-\frac{1}{2}}$ $f''(x) = 2$ $\therefore$ *Minimum*

$\left(e^{-\frac{1}{2}}, -\frac{1}{2e}\right)$ is a minimum stationary point.

51\. *(a)* $\frac{d}{dx}((\ln(x+1))(x+1) - x - 1)$

$= \frac{1}{x+1}(x+1) + \ln(x+1) \times 1 - 1$

$= 1 + \ln(x+1) - 1$

$= \ln(x+1)$

(b) Area $= \int_0^2 (\ln(x+1) - (x-1))\ dx$

$= \left[\ln(x+1)(x+1) - x - 1 - \left(\frac{x^2}{2} - x\right)\right]_0^2$

$= (\ln(3)(3) - 2 - 1 - 2 + 2) - (\ln(1)(1) - 1)$

$= 3\ln(3) - 2$

52\. *(a)* $\frac{0.5576 + 0.7224}{2} = 0.64$

$p = \frac{144}{n}$

$0.64 = \frac{144}{n}$

$n = 225$

(b) Standard error $= \sqrt{\frac{0.64(1 - 0.64)}{225}}$

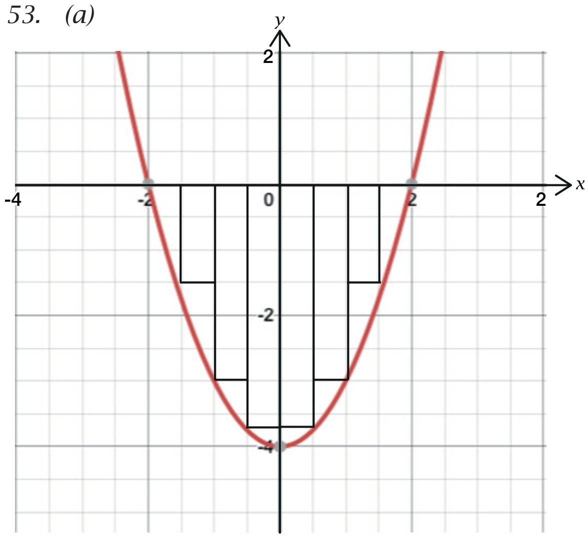
$= 0.032$

For Confidence interval: $0.64 + z(0.032) = 0.7224$

$z = 2.575$

Confidence interval is 99%

53\. *(a)*



Underestimate $= 2 \times 0.5 \times [(3.75) + (3) + (1.75)]$

$= 8.5$ units^2

(b) Actual area $= -\int_{-2}^{2} (x^2 - 4)\ dx$

$= -\left[\left(\frac{x^3}{3} - 4x\right)\right]$

$= -\left[\left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)\right]$

$= \frac{32}{3}$ units^2

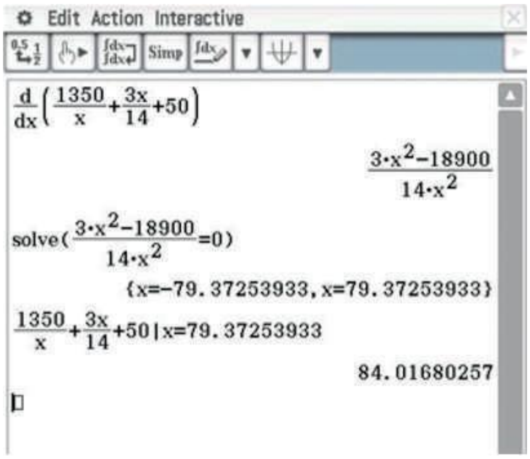
Difference is $\frac{13}{6}$ units^2.

54\. *(a)* $\frac{dy}{dx} = 2x - 3 - \frac{9}{\sqrt{x}}$

(b) $f'(x) = (-2x)(\sqrt{x} - 5x) + (2 - x^2)\left(\frac{1}{2\sqrt{x}} - 5\right)$

(c) $\frac{dy}{dx} = \dfrac{\frac{(3x^2 - 4)(5x^2 + 2)}{2\sqrt{x^3 - 4x}} - (10x)(\sqrt{x^3 - 4x})}{(5x^2 + 2)^2}$

55\.



Speed $v = 79.37$ km/h

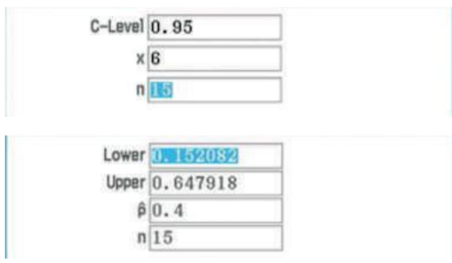
Cost is \$84.02

56\. *(a)* $\frac{40}{x} = \frac{6}{15}$

$x = 100$

Point estimate for the frog population is 100.

(b)



Confidence Interval $\approx (0.1521, 0.6479)$

(c) Upper *CI* for frog population $\approx \frac{40}{0.1521}$

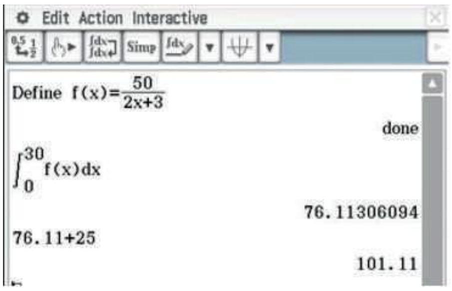
≈ 263

Lower *CI* for frog population $\approx \frac{40}{0.6479}$

≈ 62

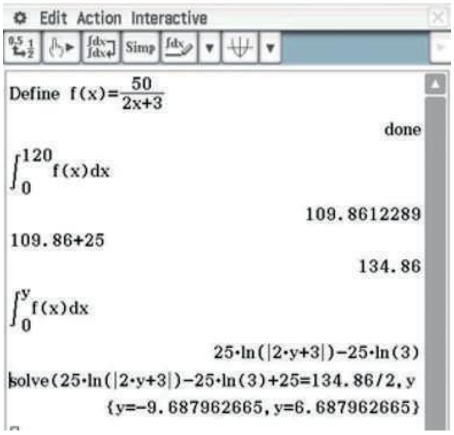
$\therefore$ *CI* $\approx (62, 263)$

57\. *(a)*



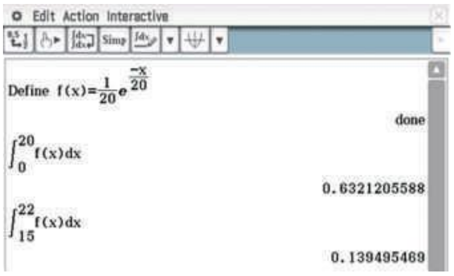
Total amount = 101.11 L

(b)



Time = 6.69 minutes

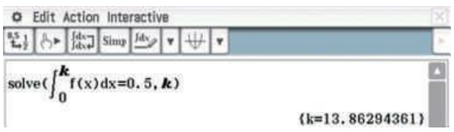
58\. *(a)*



i. $P(T \leq 20) = 0.6321$

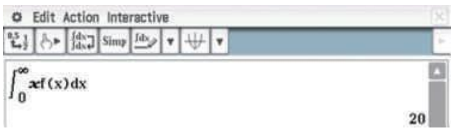
ii. $P(15 \leq T \leq 22) = 0.1395$

(b)



Median = 13.86 hours

(c)



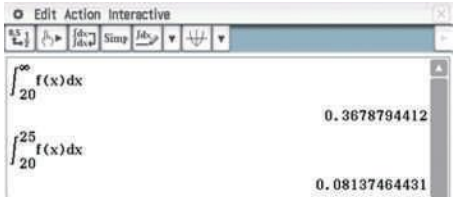
Mean = 20 hours

(d)

$$P(T < 25 \mid T > 20) = \frac{P(20 < T < 25)}{P(T > 20)}$$

$$= \frac{0.081375}{0.36788}$$

$$= 0.2212$$



(e) $Y \sim B(15, 0.6321)$

$P(Y \geq 7) = 0.9424$

59\. *(a)* *i.*

$$\log_k \frac{27}{8} = \log_k 27 - \log_k 8$$
$$= \log_k 3^3 - \log_k 2^3$$
$$= 3\log_k 3 - 3\log_k 2$$
$$= 3q - 3p$$

ii.

$$q = \log_k 3$$
$$k^q = 3$$
$$k^{2q} = (k^q)^2$$
$$= (3)^2$$
$$= 9$$

(b)

$$\log_t 5 + \log_t x = 0$$
$$\log_t (5x) = 0$$
$$5x = t^0$$
$$x = \frac{1}{5}$$

(c)

$$12^{1-x} + 1 = 12^x$$
$$\frac{12}{12^x} + 1 = 12^x$$
$$12 + 12^x = 12^{2x} \qquad \text{Let } y = 12^x$$
$$y^2 - y - 12 = 0$$
$$(y - 4)(y + 3) = 0$$
$$y = 4,\ y = -3$$
$$\therefore\ 12^x = 4$$
$$\log 12^x = \log 4$$
$$x \log 12 = \log 4$$
$$x = \frac{\log 4}{\log 12}$$

60\. *(a)* Bias exists.
- small sample size
- only customers from a single location and from the specific burger store.
- other answers possible.

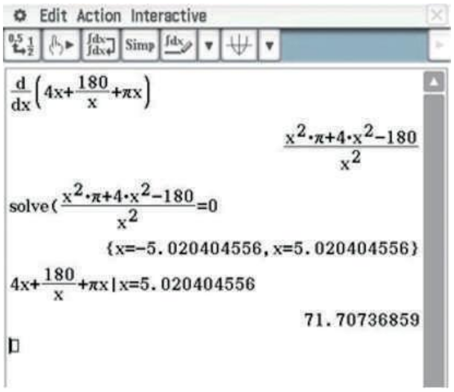
(b) Bias exists.
- only relevant to customers who visit the website and have an internet connection.
- volunteer sampling
- other answers possible.

61. (a) Area $= 4xy + \frac{\pi r^2}{2}$

$$360 = 4xy + \frac{\pi(2x)^2}{2}$$
$$360 = 4xy + \frac{4\pi x^2}{2}$$
$$360 = 4xy + 2\pi x^2$$
$$y = \frac{360}{4x} - \frac{2\pi x^2}{4x}$$
$$y = \frac{90}{x} - \frac{\pi x}{2}$$

Perimeter $= 4x + 2y + \pi r$

$$= 4x + 2\left(\frac{90}{x} - \frac{\pi x}{2}\right) + \pi(2x)$$
$$= 4x + \frac{180}{x} - \pi x + 2\pi x$$
$$= 4x + \frac{180}{x} + \pi x$$

(b) 

$x = 5.02$

Perimeter = 71.707 m

62. (a) Bernoulli with $p = \frac{22}{25}$

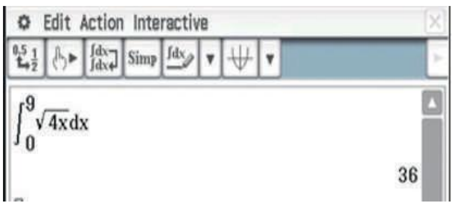
(b) $E(X) = \frac{22}{25}$

(c)

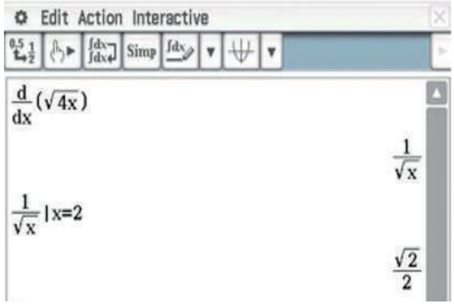
r	0	1	2	3
$P(R = r)$	$\frac{77}{115}$	$\frac{693}{2300}$	$\frac{33}{1150}$	$\frac{1}{2300}$

(d) $E(R) = \left(1 \times \frac{693}{2300}\right) + \left(2 \times \frac{33}{1150}\right) + \left(3 \times \frac{1}{2300}\right)$

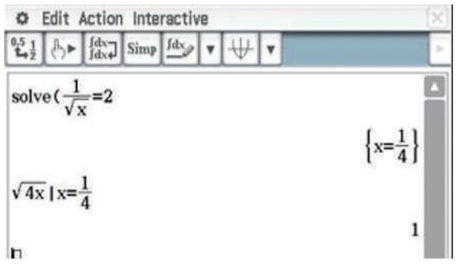
$$= \frac{9}{25}$$

63. (a) 

Distance = 36 m

(b) 

Acceleration $= \frac{\sqrt{2}}{2}$ m/s²

(c) 

Velocity = 1 m/s

64. $\frac{dy}{dx} = -\frac{2}{\sqrt{x}} + 3$

when $x = \frac{1}{4}$

$$\frac{dy}{dx} = -\frac{2}{\sqrt{\frac{1}{4}}} + 3$$
$$\frac{dy}{dx} = -\frac{2}{\frac{1}{2}} + 3$$
$$\frac{dy}{dx} = -1$$

$y = -x + c \qquad P\left(\frac{1}{4}, \frac{35}{4}\right)$

$$\frac{35}{4} = -\left(\frac{1}{4}\right) + c$$

$c = 9$

$\therefore y = -x + 9$

65. (a) $f(x) = (4 - x^2)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$
$$f'(x) = -\frac{x}{\sqrt{4 - x^2}}$$

(b) $f'(x) = \frac{5(x^2 + 3) - (2x)(5x + 1)}{(x^2 + 3)^2}$

$$f'(x) = \frac{5x^2 + 15 - 10x^2 - 2x}{(x^2 + 3)^2}$$
$$f'(x) = \frac{-5x^2 - 2x + 15}{(x^2 + 3)^2}$$

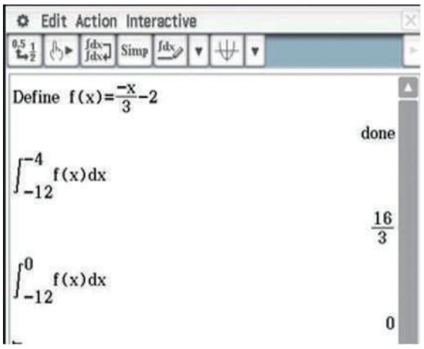
(c) $f'(x) = \frac{1}{3}(3x^2 - 1)^{-\frac{2}{3}}(6x)$

$$f'(x) = \frac{2x}{\sqrt[3]{(3x^2 - 1)^2}}$$

(d) $f'(x) = \sqrt{x + 3} + \frac{1}{2}(x + 3)^{-\frac{1}{2}}(x - 5)$

$$f'(x) = \sqrt{x + 3} + \frac{x - 5}{2\sqrt{x + 3}}$$

66. (a) i. ii.

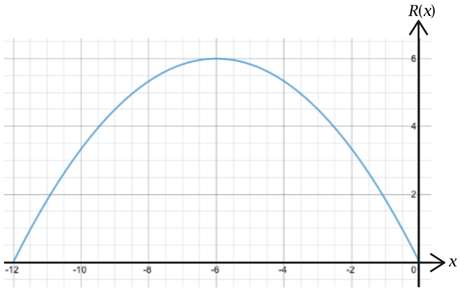


(b) $x > -6$

(c) $x = -6$

(d) $R(x) = -\frac{x^2}{6} - 2x$

(e)



67. (a) $g'(x) = 3x^2 + 2ax$

Stationary pt when $g'(x) = 0$.

$3x^2 + 2ax = 0$

$x(3x + 2a) = 0$

$\therefore x = 0,\ x = -\frac{2a}{3}$

Hence for all values of a and b,

Stationary pt at $x = 0$.

(b) Other point $(-4, 5)$ is when $x = -\frac{2a}{3}$

$-4 = -\frac{2a}{3}$

$a = 6$

$g(x) = x^3 + 6x^2 + b$

$5 = (-4)^3 + 6(-4)^2 + b$

$b = -27$

(c) i. $m = -\frac{2a}{3}$

$a = -\frac{3m}{2}$

ii. $n = m^3 + \left(-\frac{3m}{2}\right)(m)^2 + b$

$n = m^3 - \frac{3m^3}{2} + b$

$n = -\frac{1}{2}m^3 + b$

$b = n + \frac{m^3}{2}$

68. (a) $\frac{d}{dx}[3\ln(6 - 4x^3)] = \frac{18x^2}{2x^3 - 3}$

(b) $\frac{d}{dx}\left[\ln\left(\frac{6}{2 - x}\right)\right] = -\frac{1}{x - 2}$

(c) $\int \frac{x - 2}{x^2 - 4x - 5}\,dx = \frac{1}{2}\ln(x^2 - 4x - 5) + c$

(d) i. $\frac{1}{x - 1} - \frac{1}{x + 1} = \frac{(x + 1) - (x - 1)}{x^2 - 1}$

$= \frac{2}{x^2 - 1}$

ii. $\int \frac{2}{x^2 - 1}\,dx = \int \frac{1}{x - 1}\,dx - \int \frac{1}{x + 1}\,dx$

$= \ln(x - 1) - \ln(x + 1) + c$

(e) $\ln\frac{p^2}{\sqrt[3]{p}} = \ln p^2 - \ln p^{\frac{1}{3}}$

$= 2\ln p - \frac{1}{3}\ln p$

$= 2(0.45) - \frac{1}{3}(0.45)$

$= 0.75$

69. (a)

x interval	Area of inscribed rectangle	Area of circumscribed rectangle
0–0.5	0.2500	0.2663
0.5–1	0.2663	0.3246
1–1.5	0.3246	0.4670
1.5–2	0.4670	0.8564

(b) Area of inscribed rectangles = 1.3079 u²

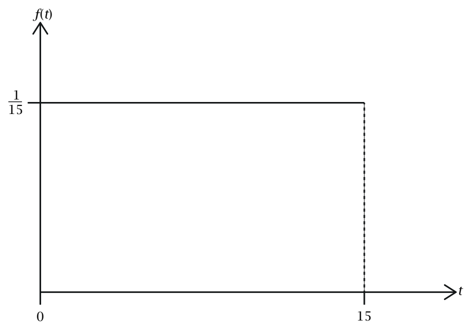
Area of circumscribed rectangles = 1.9143 u²

Area under the curve = $\frac{(1.3079 + 1.9143)}{2}$

= 1.611 u²

(c) Estimate is larger. Actual area is smaller. Curve is concave up.

70. (a) $f(t) = \begin{cases} \frac{1}{15} & 0 \le t \le 15 \\ 0 & \text{otherwise} \end{cases}$



(b) $P(T \le 8) = \frac{1}{15} \times 8$

$= \frac{8}{15}$

(c) $Y \sim B\left(6, \frac{1}{3}\right) \qquad P(T \ge 10) = 5 \times \frac{1}{15}$

$P(Y < 3) = 0.6804$

(d) $(2t - 0)\left(\frac{1}{15}\right) = (15 - t)\left(\frac{1}{15}\right)$

$2t = 15 - t$

$3t = 15$

$t = 5$

71\. (a) Sampling distribution is a close approximation to a normal distribution.

(b) Margin of error $0.05 = 1.96\sqrt{\frac{0.5(1-0.5)}{n}}$

$n = 385$

385 households

(c) Margin of error $E = 1.96\sqrt{\frac{\frac{63}{200}\left(1-\frac{63}{200}\right)}{200}}$

$= 0.0644$

(d) Margin of error

$0.05 = 1.96\sqrt{\frac{\frac{63}{200}\left(1-\frac{63}{200}\right)}{n}}$

$n = 332$

Reduction of 53 households.

72\. (a) $R = \log(316\ 228)$

$R = 5.5$

(b) $7 = \log(I)$

$I = 10\ 000\ 000$

(c) Sumatra : Pakistan $10^{8.6} : 10^{5.6}$

$= \frac{10^{8.6}}{10^{5.6}}$

$= 1000 : 1$

73\. Area

$= \int_0^1 (4 - x)\, dx + \int_1^2 \left(\frac{4}{x} - x\right) dx$

$= \left[4x - \frac{x^2}{2}\right]_0^1 + \left[4\ln(x) - \frac{x^2}{2}\right]_1^2$

$= \left(4(1) - \frac{1^2}{2}\right) - (0) + \left(4\ln(2) - \frac{2^2}{2}\right) - \left(4\ln(1) - \frac{1^2}{2}\right)$

$= \frac{7}{2} + 4\ln(2) - 2 - 4\ln(1) + \frac{1}{2}$

$= 2 + 4\ln(2)$ units^2

74\. $\sqrt[3]{3p + 1}$

75\. (a) $\frac{(4x - 7)^7}{4 \times 7} + c$

$\frac{(4x - 7)^7}{28} + c$

(b) $\frac{dT}{dy} = \sqrt{36 - 3^{2y}}$

$3 = \sqrt{36 - 3^{2y}}$

$9 = 36 - 3^{2y}$

$3^{2y} = 27$

$3^{2y} = 3^3$

$\therefore 2y = 3$

$y = \frac{3}{2}$

76\. (a) $\frac{d}{dx}\left(x(x + 1)^{\frac{1}{2}}\right)$

$= (x + 1)^{\frac{1}{2}} + \frac{1}{2}(x + 1)^{-\frac{1}{2}} \times x$

$= (x + 1)^{\frac{1}{2}} + \left(\frac{x}{2(x + 1)^{\frac{1}{2}}}\right)$

(b) $\frac{d}{dx}(x\sqrt{x + 1}) = \sqrt{x + 1} + \left(\frac{x}{2\sqrt{x + 1}}\right)$

$\int \frac{d}{dx}(x\sqrt{x + 1}) = \int \sqrt{x + 1}\, dx + \int\left(\frac{x}{2\sqrt{x + 1}}\right) dx$

$\int\left(\frac{x}{2\sqrt{x + 1}}\right) dx = \int \frac{d}{dx}(x\sqrt{x + 1}) - \int \sqrt{x + 1}\, dx$

$\int\left(\frac{x}{2\sqrt{x + 1}}\right) dx = x\sqrt{x + 1} - \frac{2\sqrt{(x + 1)^3}}{3} + c$

77\. (a) $v(t) = \frac{(2t)(t + 2) - (t^2 + 12)}{(t^2 + 2)^2}$

$v(t) = \frac{t^2 + 4t - 12}{(t + 2)^2}$

$a(t) = \frac{(2t + 4)(t + 2)^2 - 2(t + 2)(t^2 + 4t - 12)}{(t + 2)^4}$

$a(t) = \frac{(t + 2)[2(t + 2)^2 - 2(t^2 + 4t - 12)]}{(t + 2)^4}$

$a(t) = \frac{2t^2 + 8t + 8 - 2t^2 - 8t + 24}{(t + 2)^3}$

$a(t) = \frac{32}{(t + 2)^3}$

(b) Body is stationary $v(t) = 0$

$0 = \frac{t^2 + 4t - 12}{(t + 2)^2}$

$t^2 + 4t - 12 = 0$

$t = -6,\ t = 2$

as $t \geq 0$ solution is $t = 2$ seconds.

when $t = 2$

$x(2) = \frac{2^2 + 12}{2 + 2}$ m

Displacement = 4 m

(c) The body has an initial acceleration of 4 m/s^2.

As t increases the acceleration decreases but is never negative as the t axis is an asymptote.

Acceleration approaches 0 as $t \to \infty$

78. (a) -96

(b) $\int_3^4 f(p)\,dp - \int_3^4 2\,dp + \int_4^6 f(p)\,dp$

$= \int_3^6 f(p)\,dp - \int_3^4 2\,dp$

$= 24 - [2x]_3^4$

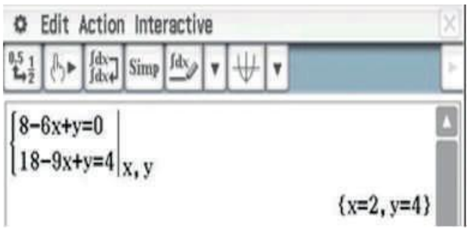
$= 24 - [2(4) - 2(3)]$

$= 22$

79. (a) $v(t) = 2t^2 - 3kt + c$

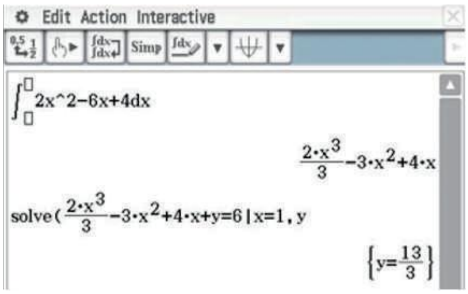
when $t = 2$ $\;2(2)^2 - 3k(2) + c = 0$

when $t = 3$ $\;2(3)^2 - 3k(3) + c = 4$



$\therefore\; v(t) = 2t^2 - 6t + 4$

(b)



$\therefore\; x(t) = \dfrac{2t^3}{3} - 3t^2 + 4t + \dfrac{13}{3}$

80. (a) Stationary when $v(t) = 0$.

$t^3 - 5t^2 + 6t = 0$

$t(t^2 - 5t + 6) = 0$

$t = 0,\ t = 2,\ t = 3$ seconds.

(b) $a(t) = 3t^2 - 10t + 6$

$a(0) = 6\text{ m/s}^2$

$a(2) = -2\text{ m/s}^2$

$a(3) = 3\text{ m/s}^2$

81. (a) $x = (c + 4t)^2$ *

$\dfrac{dx}{dt} = 2(c + 4t)(4)$

$\dfrac{dx}{dt} = 8(c + 4t)$

From * $c + 4t = \sqrt{x}$

$\therefore\; \dfrac{dx}{dt} = 8\sqrt{x}$

(b) $(c + 4t)^2 = 4$

$c + 4t = 2$

$\therefore\; \sqrt{x} = 2$

Velocity $\dfrac{dx}{dt} = 8\sqrt{x}$

$= 8(2)$

$= 16\text{ m/s}$

(c) Acceleration is $\dfrac{d^2x}{dt^2}$

$\dfrac{dx}{dt} = 8c + 32t$

$\dfrac{d^2x}{dt^2} = 32\text{ m/s}^2$ constant

82. (a)

x	0	1	2	3	4	5
$P(X = x)$	$4a$	$3a$	$2a$	a	b	b

(b) $10a + 2b = 1$ *and* $10a + 9b = 2.75$

$a = 0.05 \qquad b = 0.25$

(c) $P(0.5 < x < 3) = 0.25$

(d) $E(2 - 2.75) = -0.75$

(e) $\text{Var}(X) = (0.2)(-2.75)^2 + (0.15)(-1.75)^2 + (0.1)(-0.75)^2 + (0.05)(0.25)^2 + (0.25)(1.25)^2 + (0.25)(2.25)^2$

$= 3.6875$

(f) $\text{Var}(2 - x) = (-1)^2 \times 3.6875$

$= 3.6875$

83. $f'(x) = \dfrac{(2x)(x-1) - (x^2+3)}{(x-1)^2}$

$f'(x) = \dfrac{x^2 - 2x - 3}{(x-1)^2}$

Stationary pts when $f'(x) = 0$

$\therefore\; x^2 - 2x - 3 = 0$

$x = -1,\ x = 3$

Coordinates $(-1, -2)$ $\;(3, 6)$

Nature: $f''(x) = \dfrac{(2x-2)(x-1)^2 - 2(x-1)(x^2-2x-3)}{(x-1)^4}$

$f''(x) = \dfrac{8}{(x-1)^3}$

$f''(-1)$ = negative $\therefore$ $(-1, -2)$ is a max

$f''(3)$ = positive $\therefore$ $(3, 6)$ is a min

84. (a) $f'(x) = g(x) + x.g'(x)$

$f'(2) = g(2) + 2g'(2)$

$f'(2) = 10 + 2(3)$

$f'(2) = 16$

(b) $h'(x) = \dfrac{g'(x)p(x) - g(x)p'(x)}{(p(x))^2}$

$h'(2) = \dfrac{g'(2)p(2) - g(2)p'(2)}{(p(2)^2)}$

$h'(2) = \dfrac{(3)(25) - (10)(-4)}{(25)^2}$

$h'(2) = \dfrac{115}{625} \quad \left(\dfrac{23}{125}\right)$

(c) $m'(x) = \dfrac{p'(x)}{2\sqrt{p(x)}}$

$m'(2) = \dfrac{p'(2)}{2\sqrt{p(2)}}$

$m'(2) = \dfrac{-4}{2\sqrt{25}}$

$m'(2) = -\dfrac{2}{5}$

85. (a) $f(x) = \ln[(\cos \pi x) + 3\pi^2]$

$f'(x) = \dfrac{-\pi \sin(\pi x)}{\cos(\pi x) + 3\pi^2}$

(b) $\displaystyle\int \frac{e^{2x} + 2}{e^x}\, dx = e^x - \frac{2}{e^x} + c$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \left[4 - \sin\left(\frac{x}{2}\right)\right] dx = \left[4x + 2\cos\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$

$= \left(4\left(\dfrac{\pi}{2}\right) + 2\cos\left(\dfrac{\frac{\pi}{2}}{2}\right)\right) - \left(4(0) + 2\cos\left(\dfrac{0}{2}\right)\right)$

$= 2\pi + \sqrt{2} - 2$

86. (a) $P = 220e^{0.08t}$

(b) $3 = e^{0.08t}$

$t = 13.73$ years

(c) $\dfrac{dP}{dt} = 0.08P$ *when* $P = 500$

$\dfrac{dP}{dt} = 0.08(500)$

$\dfrac{dP}{dt} = 40$

40 000 people/year

87. $x = 2$

$\delta x = 0.01$

$\delta y \approx \dfrac{dy}{dx} \times \delta x$

$\delta y \approx (2x - 1)^2 \times \delta x$

$\delta y \approx (2 \times 2 - 1)^2 \times 0.01$

$\delta y \approx 0.09$

$g(2.01) \approx 3 + 0.09$

≈ 3.09

88. (a) $f'(x) = 3x^2 + 2ax + b$

$f''(x) = 6x + 2a$

For stationary point (–2, 7)

$0 = 3(-2)^2 + 2a(-2) + b$

$0 = 12 - 4a + b$ …(*eqn* 1)

For pt of inflection (–1, 5)

$0 = 6(-1) + 2a$

$\therefore\ a = 3$

Sub into (*eqn* 1)

$0 = 12 - 4(3) + b$

$\therefore\ b = 0$

(b) $f'(x) = 3x^2 + 6x$

$f''(x) = 6x + 6$

Stationary pts

$3x(x + 2) = 0$

$x = 0, x = -2$

(0, 3) *and* (–2, 7)

Nature

for (0, 3) $f''(0) = 6 \therefore$ *min pt*

for (–2, 7) $f''(-2) = -6 \therefore$ *max pt*

89. (a) $F(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \frac{1}{3}(x - 1) & 1 \le x \le 4 \\ 1 & x > 4 \end{cases}$

(b) $P(X < 2.5) = \dfrac{1}{3}(2.5 - 1)$

$= \dfrac{1}{2}$

(c) $f(y) = \begin{cases} \dfrac{1}{10}, & 6 \le y \le 16 \\ 0 & \textit{elsewhere} \end{cases}$

(d) $P(Y \ge t) = \dfrac{3}{5}$

$\dfrac{1}{10}(16 - t) = \dfrac{3}{5}$

$16 - t = 6$

$t = 10$

90. (a) Bias exists.

– Only customers surveyed at 5 pm rather than through the day.
– Only customers that use self-serve.
– Only responses from one supermarket.
– Other answers possible.

(b) $\hat{p} = \dfrac{78}{150} = 0.52$

$E \approx 1.96\sqrt{\dfrac{0.52(1 - 0.52)}{150}}$

≈ 0.07995

$CI = 0.52 \pm 0.07995$

$\approx (0.44005, 0.59995)$

(c) $E = \dfrac{0.05}{2} = 0.025$

$0.025 = 1.96\sqrt{\dfrac{0.52(1 - 0.52)}{n}}$

$n = 1534.18$

$n \approx 1535$

91. (a) $h'(x) = 6x^3 - 18x^2$

$0 = 6x^3 - 18x^2$

$0 = 6x^2(x - 3)$

$x = 0,\ x = 3\ \therefore$ *two stationary points.*

(b) concave up when $f''(x) > 0$.

Interval: $x < 0,\ x > 2$

92. $v(t) = \dfrac{(4t)(t-1) - (2t^2 - 3)}{(t-1)^2}$

$v(t) = \dfrac{2t^2 - 4t + 3}{(t-1)^2}$

$a(t) = \dfrac{(4t-4)(t-1)^2 - 2(t-1)(2t^2 - 4t + 3)}{(t-1)^4}$

$a(-1) = \dfrac{(-8)(-2)^2 - 2(-2)(9)}{(-2)^4}$

$a(-1) = \dfrac{1}{4}\ \text{m/s}^2$

93. (a) $\int_0^{14} f(x)\ dx = \left((3)(4) + \frac{1}{2}(2)(3) + \frac{1}{2}(8)(-3)\right)$

$= 3$

(b) $\int_6^{14} f(x)\ dx = -12$

(c) $R(8) = \left((3)(4) + \frac{1}{2}(2)(3) + \frac{1}{2}(2)(-3)\right)$

$= 12$

(d) $R'(6) = 0$

(e) Require $R(t) = 0$

$3 + \frac{1}{2}(3)(1) + \frac{1}{2}(2)(-2) + (t - 19)(-2) = 0$

$t = 20.25$

x intercept at $(20.25, 0)$

94. (a) $E(X) = 3.5$

$\text{Var}(X) = \frac{1}{6}[(-2.5)^2 + (-1.5)^2 + (-0.5)^2 + (0.5)^2 + (1.5)^2 + (2.5)^2]$

$= \frac{35}{12}\ (2.917)$

(b) $E(4 - 3X) = 4 - 3(3.5)$

$= -6.5$

(c) $\text{Var}(4 - 3X) = \frac{35}{12} \times (-3)^2$

$= \frac{105}{4}\ \ (26.25)$

95. (a) $\dfrac{(4+k)^2}{49} = 1$

$(4+k)^2 = 49$

$4 + k = 7$

$k = 3$

(b)

x	1	2	3	4
$P(X = x)$	$\frac{16}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

96. (a) $\int_0^2 f'(x)\ dx = f(2) - f(0)$

$= 3 - (-3)$

$= 6$

(b) $\int_{-4}^2 f'(x)\ dx = f(2) - f(-4)$

$= 3 - 4$

$= -1$

(c) Area $= \int_{-4}^0 |f'(x)|\ dx + \int_0^2 |f'(x)|\ dx$

$= (|f(0) - f(-4)|) + (|f(2) - f(0)|)$

$= |-3 - 4| + |3 - (-3)|$

$= 13\ \text{u}^2$

97. (a) $(0.22 + 0.11 + 0.01) = 0.34$

(b) $\dfrac{0.89}{0.90} = 0.9889$

(c) $E(T) = 34.25$

$St\ Dev\ (T) = 4.44$

$Y = 0.85T + 2.5$

$E(Y) = 0.85(34.25) + 2.5$

$= 31.6125$ mins

$St\ Dev\ (Y) = 4.44 \times 0.85$

$= 3.774$ mins

98. (a) $P(a < Z < b) = 1 - (0.28 + 0.38)$

$= 0.34$

(b) $P(Z < a\ |\ Z < b) = \dfrac{0.28}{1 - 0.38}$

$= \dfrac{0.28}{0.62}$

$= \dfrac{14}{31}$

(c) $P(a \le Z < 0\ |\ Z \le b) = \dfrac{0.22}{0.62}$

$= \dfrac{11}{31}$

99. (a) $\hat{p} = \dfrac{125}{650}\ \ (0.1923)$

(b) Approximate number $\approx \dfrac{125}{650} \times 253\ 000$

$\approx 48\ 654$

(c) Margin of error $E \approx 2.326\sqrt{\dfrac{\frac{125}{650}\left(1 - \frac{125}{650}\right)}{650}}$

≈ 0.03596

(d) $CI \approx 0.1923 \pm 0.03596$

$\approx (0.1563, 0.2283)$

(e) Margin of error

$$0.06 = 2.326\sqrt{\frac{\frac{125}{650}\left(1-\frac{125}{650}\right)}{n}}$$

$n = 234$

100. (a) $\sqrt{p(1-p)} = 0.3$

$p(1-p) = 0.09$

$p = 0.1, 0.9$

(b) Variance: $\frac{4}{5}p^2 = 20$

$p^2 = 25$

$p = \pm 5$

Mean: $pX + q = 4$

$5(2) + q = 4$

$q = -6$

$pX + q = 4$

$-5(2) + q = 4$

$q = 14$

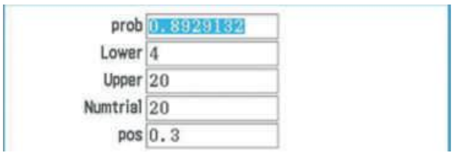
$\therefore p = 5 \quad q = -6 \quad or \quad p = -5 \quad q = 14$

101. (a) Binomial $X \sim B(20, 0.3)$

(b)

$P(X = 2) = 0.0278$

(c)

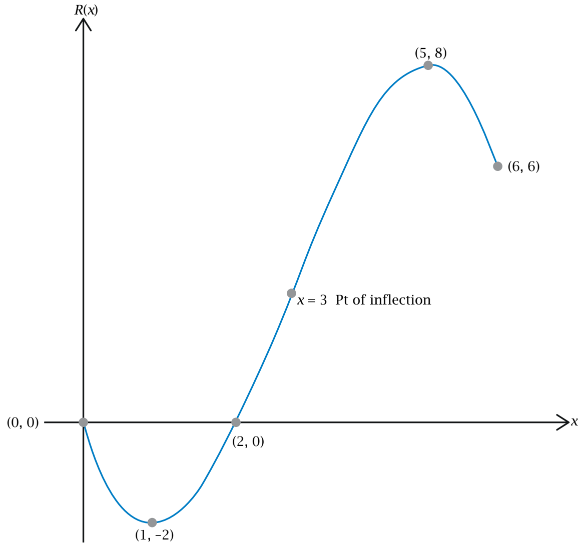


$P(X > 3) = 0.8929$

(d) $Y \sim B(10, 0.8929)$

$P(Y = 6) = 0.014$

102.



103. Standardising

Test 1: $z = \frac{81 - 83}{6}$

$z = -\frac{1}{3}$

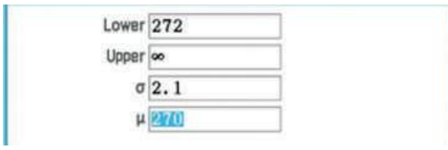
Test 2: $z = \frac{85 - 82}{3}$

$z = 1$

Standardised score shows Test 2 was better.

104. (a) $X \sim N(270, 2.1^2)$

(b)



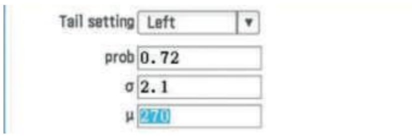
i. $P(X \geq 272) = 0.1705$

ii. $P(X \leq 273 \mid X > 271) = \frac{P(271 < X \leq 273)}{P(X > 271)}$

$= \frac{0.2404}{0.3170}$

$= 0.7584$

iii.



$P(X < k) = 0.72$

$k = 271.22$ mL

(c) $E(1.33X + 20) = 1.33(270) + 20$
$= 379.1$ mL

$St\ Dev = 1.33(2.1)$
$= 2.793$ mL

105. (a) $x(t) = -6\cos\left(\frac{t}{2} + \frac{\pi}{3}\right) + c$

Calculate c by letting $x(t) = 0$ when $t = 0$.

$0 = -6\cos\left(\frac{0}{2} + \frac{\pi}{3}\right) + c$

$c = 3$

$x(t) = -6\cos\left(\frac{t}{2} + \frac{\pi}{3}\right) + 3$

(b) $x(14) = -6\cos\left(\frac{14}{2} + \frac{\pi}{3}\right) + 3$
$= 4.152$ m

(c) $a(t) = \dfrac{3\cos\left(\frac{t}{2} + \frac{\pi}{3}\right)}{2}$

$-0.8 = \dfrac{3\cos\left(\frac{t}{2} + \frac{\pi}{3}\right)}{2}$

$t = 2.1723, 6.2053\ 14.7386, 18.7717$ seconds

Final time is 18.7717 seconds.

Distance travelled is:

$$\int_0^{18.7717} \left| 3\sin\left(\frac{t}{2} + \frac{\pi}{3}\right) \right| dt$$

$= 35.80$ metres (2dp)

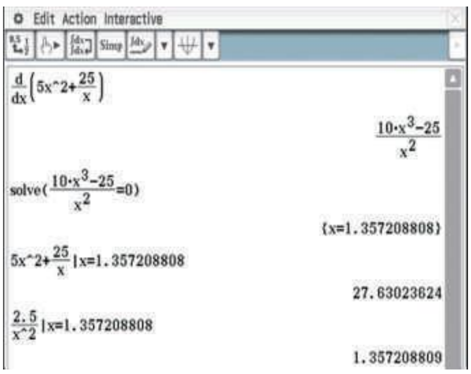
106. (a) Volume: $x^2y = 2.5$

$y = \dfrac{2.5}{x^2}$

Area: $2x^2 + xy + 2xy$

Cost: $= 2(2.5)x^2 + 3.6xy + 2(3.2)xy$
$= 5x^2 + 10xy$
$= 5x^2 + 10x\left(\frac{2.5}{x^2}\right)$
$= 5x^2 + \dfrac{25}{x^2}$

(b)



$x = 1.36$ m

$y = 1.36$ m

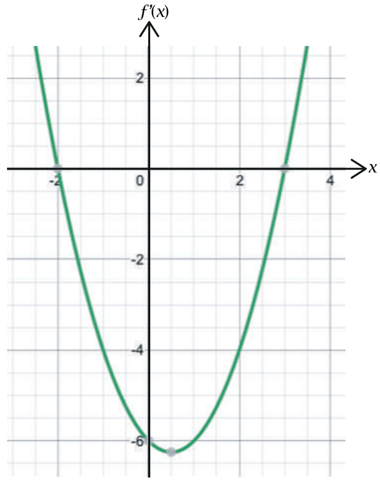
Cost: \$27.63

107. $f'(x) = x^2 - x - 6$
$f''(x) = 2x - 1$

x intercepts for $f'(x)$
$(x - 3)(x + 2) = 0$
$x = 3,\ x = -2$

TP at $\left(\frac{1}{2}, -\frac{25}{4}\right)$

Sketch a graph of $f'(x)$



Hence $f'(x) < 0$ when $(-2 < x < 3)$

$f''(x) > 0$ when $x > \frac{1}{2}$ as graph is increasing

Required interval: $\frac{1}{2} < x < 3$

108. (a) $\hat{p} = \dfrac{146}{275} = 0.5309$

$E = 2.576\sqrt{\dfrac{0.5309(1 - 0.5309)}{275}}$
≈ 0.07752

$CI = 0.5309 \pm 0.07752$
$\approx (0.4534, 0.6084)$

(b) $SE = \sqrt{\dfrac{0.82(1 - 0.82)}{275}}$
$= 0.023167$

$E = 2.576 \times 0.023167$
≈ 0.05968

$CI = 0.82 \pm 0.05968$
$\approx (0.7603, 0.8797)$

(c) $X \sim B(10, 0.99)$

$P(X \geq 9) = 0.9957$

109. (a) $P = ab^t$
$\log P = \log ab^t$
$\log P = \log a + \log b^t$
$\log P = \log a + t\log b$

Hence the relationship is linear.

(b) $\frac{5 - 3}{98.4 - 0} = \frac{5}{246}$

$\log P = \frac{5}{246}t + 3$

(c) $\log a = 3 \quad \therefore \quad a = 1000$ initial population

$\log b = \frac{5}{246} \quad \therefore \quad b = 1.0479$ growth rate

(d) $P = 1000(1.0479)^t$

$15000 = 1000(1.0479)^t$

$t = 57.88$ years

(e) $P = 1000(1.0479)^{12}$

$P = 1753.23$

$P \approx 1753$

110. (a) $p = 0.71$

$0.71^4 = 0.2541$

(b) $X \sim B(100, 0.86)$

$P(X = 87) = 0.1130$

(c) $Y \sim B(100, 0.71)$

$P(Y > 80) = 0.0154$

(d) $Z \sim B(100, 0.24)$

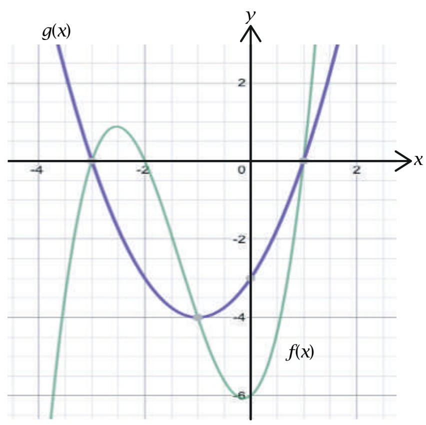
$E(Z) = 100 \times 0.24$

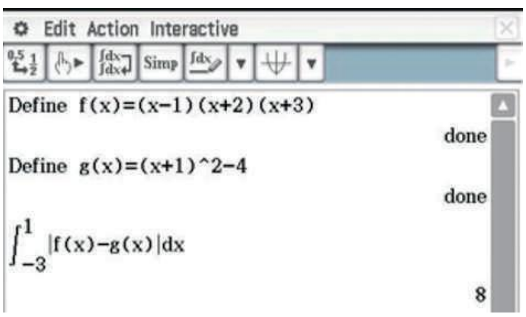
$= 24$

St Dev $Z = \sqrt{24(1 - 0.24)}$

$= 4.2708$

111.





Area = 8 units^2

112. (a) $(-3, 0)$ $(2, 0)$

(b) Turning points when $x = 0, 2$

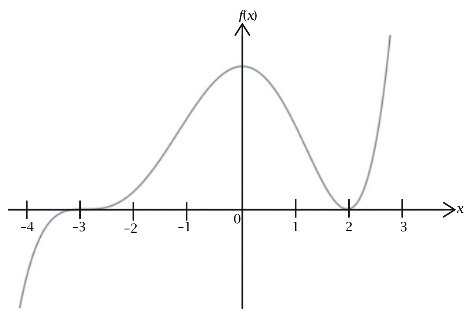
For $x = 0$ concave down as gradient is zero at $x = 0$ and when $x < 0$ positive and when $x > 0$ negative.

For $x = 2$ concave up as gradient is zero at $x = 2$ and when $x < 2$ negative and when $x > 2$ positive.

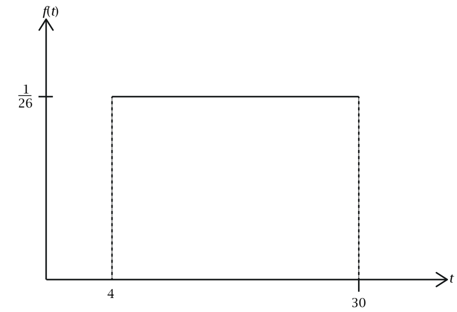
(c) Horizontal point of inflection

(d) $x < -3$ and $-1 < x < 1$

(e)



113. (a)



(b) i. $P(T > 15) = \frac{15}{26}$

ii. $P(T < 25 \mid T \geq 10) = \frac{P(10 \leq T < 25)}{P(T \geq 10)}$

$= \frac{\frac{15}{26}}{\frac{20}{26}}$

$= \frac{3}{4}$

iii. $P(T > k) = 0.45$

$\frac{30 - k}{26} = 0.45$

$k = 18.3$

114. (a) y intercept at $(0, -3)$

$\frac{dy}{dx} = -e^{-x}$ when $x = 0$ $\frac{dy}{dx} = -1$

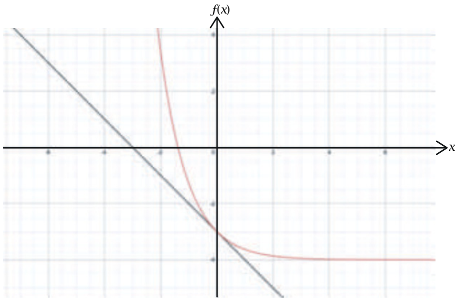
Equation of tangent line.

$y = -x + c \qquad (0, -3)$

$-3 = 0 + c$

$\therefore \; y = -x - 3$

(b)



(c) Area $= \frac{1}{2}(3)(3) + \int_{-2\ln(2)}^{0} (e^{-x} - 4)\, dx$

$= 4.5 + (-2.545)$

$= 1.955\ u^2$

115. Area $= \int_0^{\frac{\pi}{2}} (\sin 2x)\, dx = 1$

$R1 = \int_0^k \sin(2x)\, dx = 0.25$

$k = 0.5236$

116. (a) Binomial

(b) $\frac{(0\times 5)+(1\times 19)+(2\times 34)+(3\times 51)+(4\times 31)}{140} = 2.6$

(c) $E(X) = np \quad X \sim B(4, p)$

$2.6 = 4p$

$p = 0.65$

117. (a) $v(t) = -3t^2 + 20t$

(b) $x(t) = -t^3 + 10t^2$

(c) Max velocity when $a(t) = 0$

$-6t + 20 = 0$

$t = \frac{10}{3}$ sec

$v\left(\frac{10}{3}\right) = \frac{100}{3}$ m/s

(d) $\int_3^4 (-3t^2 + 20t)\, dt = 33$ m

(e) $\int_0^8 |-3t^2 + 20t|\, dt = \frac{4544}{27}$ m

(f) At origin when $x(t) = 0$

$-t^3 + 10t^2 = 0$

$t = 0,\ t = 10$ secs

At origin second time when $t = 10$

$v(10) = -100$ m/s

118. (a) Binomial $Q \sim B(12, 0.56)$

(b) $P(Q < 6) = 0.2380$

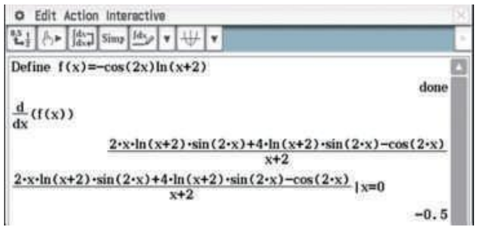
Number of times $= 0.2380 \times 21$

≈ 5 days

(c) $R \sim B(24, 0.56)$

$P(R \geq 10) = 0.9469$

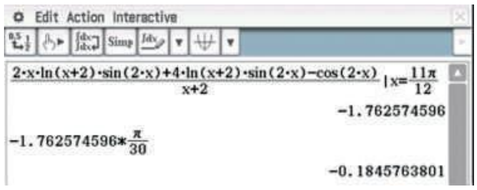
119. (a)



as $f'(0) < 0 \ \therefore$ function is decreasing.

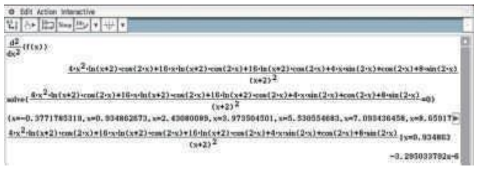
(b) Change in x from $\frac{11\pi}{12}$ to $\frac{19\pi}{20} = \frac{\pi}{30}$

$\delta f(x) = \frac{df(x)}{dx} \times \delta x$



$\delta f(x) = -0.1846$

(c)



When $f''(x) = 0 \quad x = 0.934863$

$\therefore f(0.934863) =$ *negative.*

This indicates that $f'(x)$ *is decreasing* when $x \approx 0.934863$

120. $\frac{dy}{dx} = 6x^2 - 2x$ Coordinate $(-2, -5)$

$\delta x = -0.01$

$\delta y \approx \frac{dy}{dx} \times \delta x$

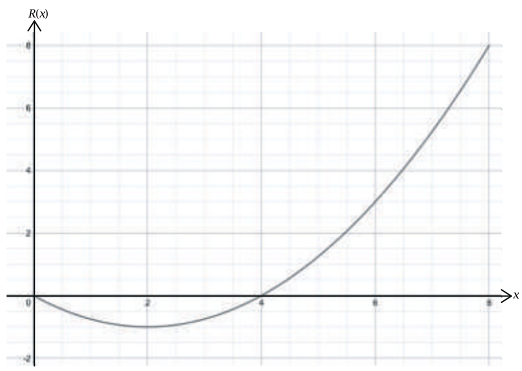
$\delta y \approx (6x^2 - 2x) \times \delta x$

$\delta y \approx [6(-2)^2 - 2(-2)] \times -0.01$

$\delta y \approx -0.28$

Approx y coordinate $= -5 - 0.28 = -5.28$

121.



122. (a) Area $\Delta\, PTY = \frac{1}{2} xh$

Using similar triangles:

$$\frac{2}{x-3} = \frac{h}{x}$$

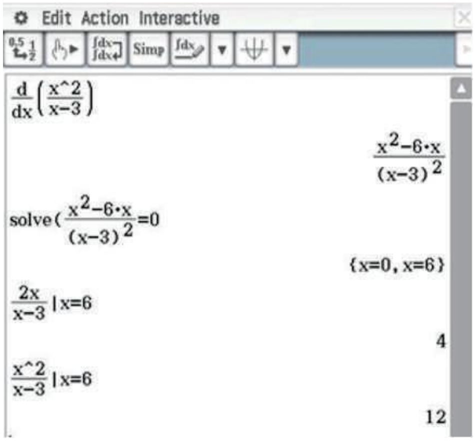
$$h = \frac{2x}{x-3}$$

$$\therefore \text{Area } \Delta\, PTY = \frac{1}{2} x \times \left(\frac{2x}{x-3}\right)$$

$$\text{Area } \Delta\, PTY = \frac{1}{2}\left(\frac{2x^2}{x-3}\right)$$

$$\text{Area } \Delta\, PTY = \frac{x^2}{x-3}$$

(b)



Dimensions: $x = 6$ m $h = 4$ m

Area $\Delta\, PTY = 12$ m^2

123. (a)

x	0	3	5	12
$P(X = x)$	$\frac{12}{30}$	$\frac{10}{30}$	$\frac{6}{30}$	$\frac{2}{30}$

(b) $E(X) = \left(0 \times \frac{12}{30}\right) + \left(3 \times \frac{10}{30}\right) + \left(5 \times \frac{6}{30}\right) + \left(12 \times \frac{2}{30}\right)$

$= 2.8$

(c) Number of tokens received by staff member: $300 \times 2 = 600$

Expected number won by players: $300 \times 2.8 = 840$

Staff member will **lose 240** tokens.

124. (a) $f'(x) = \frac{1}{x} - k$

Stationary point when $f'(x) = 0$

$$\frac{1}{x} - k = 0$$

$$x = \frac{1}{k}$$

$$\therefore f\left(\frac{1}{k}\right) = \ln\left(\frac{e}{k}\right) - 1$$

Coordinates are: $\left(\frac{1}{k},\ \ln\left(\frac{e}{k}\right) - 1\right)$

(b) Test using $f''(x) = \frac{-1}{x^2}$

- $f''\left(\frac{1}{k}\right) = -\frac{1}{\left(\frac{1}{k}\right)^2} < 0$

Stationary point is a maximum.

(c) $\int_1^3 \left(-k + \frac{1}{x}\right) dx = [-kx + \ln(x)]_1^3$

$= (-3k + \ln(3)) - (-k + \ln(1))$

$= -2k + \ln(3)$

125. (a) $X \sim N(58, 7^2)$

i. $P(X = 55) = 0$

ii. $P(X > 60) = 0.3875$

iii. $P(56 < X < 59) = 0.16925$

iv. $P(X \leq 61 \mid X \geq 55) = \frac{P(55 \leq X \leq 61)}{P(X \geq 55)}$

$= \frac{0.3318}{0.6659}$

$= 0.4983$

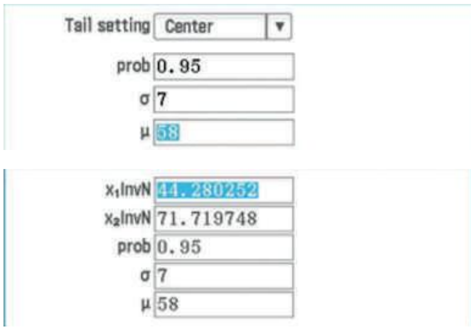
(b) i. $(0.3341)^2\,(0.3875)^3 = 0.006495$

ii. $Y \sim B(5, 0.3875)$

$P(Y \geq 2) = 0.6411$

(c) $P(-k \leq X \leq k) = 0.95$

$44.2803 \leq X \leq 71.7197$



126. (a) $\hat{p} = \frac{30}{110} = 0.2727$

(b) $SE = \sqrt{\frac{\frac{30}{110}\left(1 - \frac{30}{110}\right)}{110}}$

$= 0.04246$

(c) $E = 1.645\sqrt{\frac{\frac{30}{110}\left(1 - \frac{30}{110}\right)}{110}}$

≈ 0.06985

(d) $CI \approx 0.2727 \pm 0.06985$

$\approx (0.2029, 0.3426)$

127. (a) Bernoulli in a single trial the random variable X has only two values 0 or 1.

(b) Mean: $E(X) = \frac{32}{52}$

Variance: $\sigma^2 = \frac{32}{52}\left(1 - \frac{32}{52}\right)$

$= \frac{40}{169}$

128. *(a)* *i.* Displacement

$$= \left[\left(\frac{1}{2}(4)(2) + (2)(2) + \left(\frac{1+2}{2}\right) \times 1\right)\right]$$

$= 9.5$ m

ii. Displacement

$$= \left[\frac{1}{2}(4)(2) + (2)(2) + \frac{1}{2}(2)(2) + \frac{1}{2}(1)(-1) + (5)(-1)\right]$$

$= 4.5$ m

(b) Max displacement when $v(t) = 0$

Time: $t = 8$ secs Displacement: 10 m

129. Margin of error $0.08 = 1.96\sqrt{\dfrac{0.5(1-0.5)}{n}}$

$n = 151$

Sample size = 151

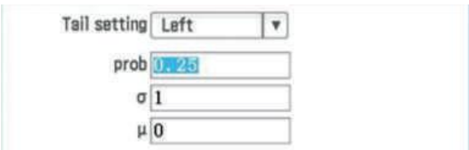
130. Use standardising formula:

$$z = \frac{x - \mu}{\sigma}$$

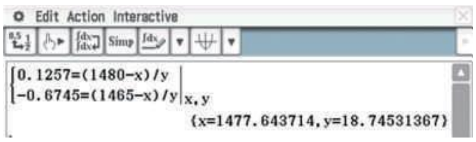
$$0.1257 = \frac{1480 - \mu}{\sigma} \quad \text{.....(1)}$$

$$z = \frac{x - \mu}{\sigma}$$

$$-0.6745 = \frac{1465 - \mu}{\sigma} \quad \text{.....(2)}$$



Solve 1 and 2 simultaneously



Mean = 1477.64 hours

Standard Deviation = 18.75 hours

131. Random variable X is the profit \$ per game for John

x	4	-3	-10
$P(X = x)$	$\frac{38}{56}$	$\frac{16}{56}$	$\frac{2}{56}$

$$E(X) = \left(4 \times \frac{38}{56}\right) + \left(-3 \times \frac{16}{56}\right) + \left(-10 \times \frac{2}{56}\right)$$

$= \$1.50$

Games to be played 'n'

$1.5n > 125$

$n > 83.3$

Must play at least 84 games.

132. *(a)* $V = \int_0^T \frac{1}{2}\sin\left(\frac{\pi t}{3}\right) + 4 \; dt$

(b) $V = \int_0^4 \frac{1}{2}\sin\left(\frac{\pi t}{3}\right) + 4 \; dt$

$= \frac{9}{4\pi} + 16$ (16.72 cm³)

(c) $150 = \int_0^T \frac{1}{2}\sin\left(\frac{\pi t}{3}\right) + 4 \; dt$

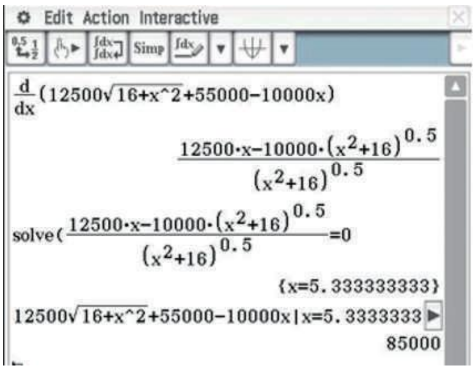
$T = 37.39$ minutes

133. *(a)* $CT = \sqrt{16 + x^2}$ $TV = 5.5 - x$

Cost: $12\,500\sqrt{16 + x^2} + 10\,000(5.5 - x)$

Cost: $12\,500\sqrt{16 + x^2} + 55\,000 - 10\,000x$

(b)



$x = \frac{16}{3}$ km Cost: \$85 000

134. *(a)* Overestimate

$= [0.25(0.71 + 0.87 + 1 + 1.12)]$

$= 0.925$ units²

(b) Underestimate

$= [0.25(0.5 + 0.71 + 0.87 + 1)]$

$= 0.77$ units²

(c) Approximation $= \dfrac{0.925 + 0.77}{2}$

$= 0.8475$ units²

(d) Smaller as concave up below the x axis hence an underestimate.

(e) Smaller rectangles

135. (a) $V = \frac{37}{7}\text{ cm}^3$

(b) $V(h + \delta h) = V(7) = \frac{50}{8}\text{ cm}^3$

(c) $\frac{dV}{dh} = \frac{2h(h+1) - (h^2+1)}{(h+1)^2}$

$\frac{dV}{dh} = \frac{h^2 + 2h - 1}{(h+1)^2}$

$\delta V \approx \frac{dV}{dh} \times \delta h$

when $h = 3$, $V = \frac{10}{4}$ and $h = 4$, $V = \frac{17}{5}$

$\therefore\ \delta h = 1$

$\delta V \approx \frac{h^2 + 2h - 1}{(h+1)^2} \times \delta h$

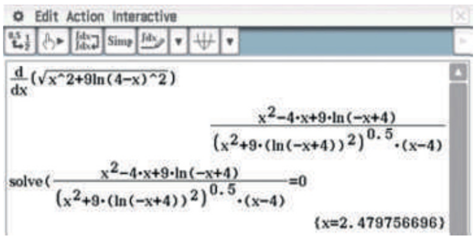
$\delta V \approx \frac{(3)^2 + 2(3) - 1}{(3+1)^2} \times 1$

$\delta V \approx \frac{7}{8}\text{ cm}^3$

136. (a) $d = \sqrt{x^2 + (3\ln(4-x))^2}$

$d = \sqrt{(x^2 + 9(\ln(4-x))^2}$

(b) For minimum $d' = 0$



$x = 2.4798$ units

$OP = d = 2.78$ units

137. (a)

x	$-x$	5	2	10
$P(X = x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

(b) $E(X) = \left(-x \times \frac{1}{6}\right) + \left(5 \times \frac{2}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(10 \times \frac{2}{6}\right)$

$= \frac{32 - x}{6}$

(c) When $E(X) \geq 0$

$\frac{32 - x}{6} \geq 0$

$x \leq 32$

$\therefore$ profit when $0 \leq x \leq 32$

138. (a) $x(t) = \int 5\cos\left(\frac{\pi t}{12}\right)\, dt$

$= \frac{60}{\pi}\sin\left(\frac{\pi t}{12}\right) + c$ when $t = 4$, $x(t) = 0$

$0 = \frac{60}{\pi}\sin\left(\frac{\pi(4)}{12}\right) + c$

$c = \frac{-30\sqrt{3}}{\pi}$

$\therefore\ x(t) = \frac{60}{\pi}\sin\left(\frac{\pi t}{12}\right) - \frac{30\sqrt{3}}{\pi}$

(b) $2 = \frac{60}{\pi}\sin\left(\frac{\pi t}{12}\right) - \frac{30\sqrt{3}}{\pi}$

$t = 5.0738$, $t = 6.9262$ seconds

$a(t) = -\frac{5\pi}{12}\sin\left(\frac{\pi t}{12}\right)$

$a(5.0738) = -\frac{5\pi}{12}\sin\left(\frac{\pi 5.0738}{12}\right)$

$= -1.2707\text{ m/s}^2$

$a(6.9262) = -\frac{5\pi}{12}\sin\left(\frac{\pi 6.9262}{12}\right)$

$= -1.2707\text{ m/s}^2$

139. $W \sim N(59.5, 2.7^2)$

(a) $P(62 < W < 65) = 0.1564$

(b) *Small* $W < 56.04$ grams

Medium $= 56.04 \leq W < 58.08$ grams

Large $= 58.08 \leq W < 60.92$ grams

Extra-large $= W \geq 60.92$ grams

(c) *Medium* $P = \frac{2}{10}$

Extra-large $P = \frac{3}{10}$

$P(3m + 1x) = \left(\frac{2}{10}\right)^3\left(\frac{3}{10}\right)$

$= \frac{3}{1250}$

140. (a) $C = kd^3$ when $d = 10$ $C = 750$

$750 = k(10)^3$

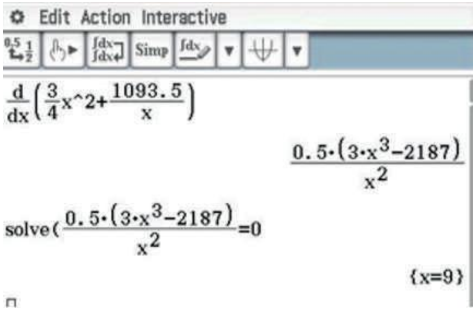
$k = \frac{3}{4}$

Cost $C = \frac{3}{4}d^3 + 1093.5$

Total cost per day:

$T = \frac{\frac{3}{4}d^3 + 1093.5}{d}$

$T = \frac{3}{4}d^2 + \frac{1093.5}{d}$

(b) 

Minimum number of days is 9.

141. (a) Normal distribution. As sample size is large, sample proportions will approach a normal distribution with mean p and standard deviation $\sqrt{\dfrac{p(1-p)}{n}}$.

(b) Mean = 0.62

Standard deviation = $\sqrt{\dfrac{0.62(1-0.62)}{225}}$

= 0.0324

(c) $P(X > 0.6) = 0.7315$

142. (a)

m	0	1	2
$P(M = m)$	$\frac{12}{30}$	$\frac{16}{30}$	$\frac{2}{30}$

(b) $E(M) = \left(0 \times \frac{12}{30}\right) + \left(1 \times \frac{16}{30}\right) + \left(2 \times \frac{2}{30}\right)$

$= \frac{2}{3}$

$\text{Var}(X) = \left(\frac{12}{30}\right)\left(0 - \frac{2}{3}\right)^2 + \left(\frac{16}{30}\right)\left(1 - \frac{2}{3}\right)^2 + \left(\frac{2}{30}\right)\left(2 - \frac{2}{3}\right)^2$

$= 0.3556$

(c)

p	-0.50	0.50	9.50
$P(P = p)$	$\frac{12}{30}$	$\frac{16}{30}$	$\frac{2}{30}$

$E(P) = \left(-0.5 \times \frac{12}{30}\right) + \left(0.50 \times \frac{16}{30}\right) + \left(9.5 \times \frac{2}{30}\right)$

= \$0.70

143. (a) The width of the confidence interval will increase.

(b) Should read –

"95% of the interval estimates include the true population proportion"

Other answers possible.

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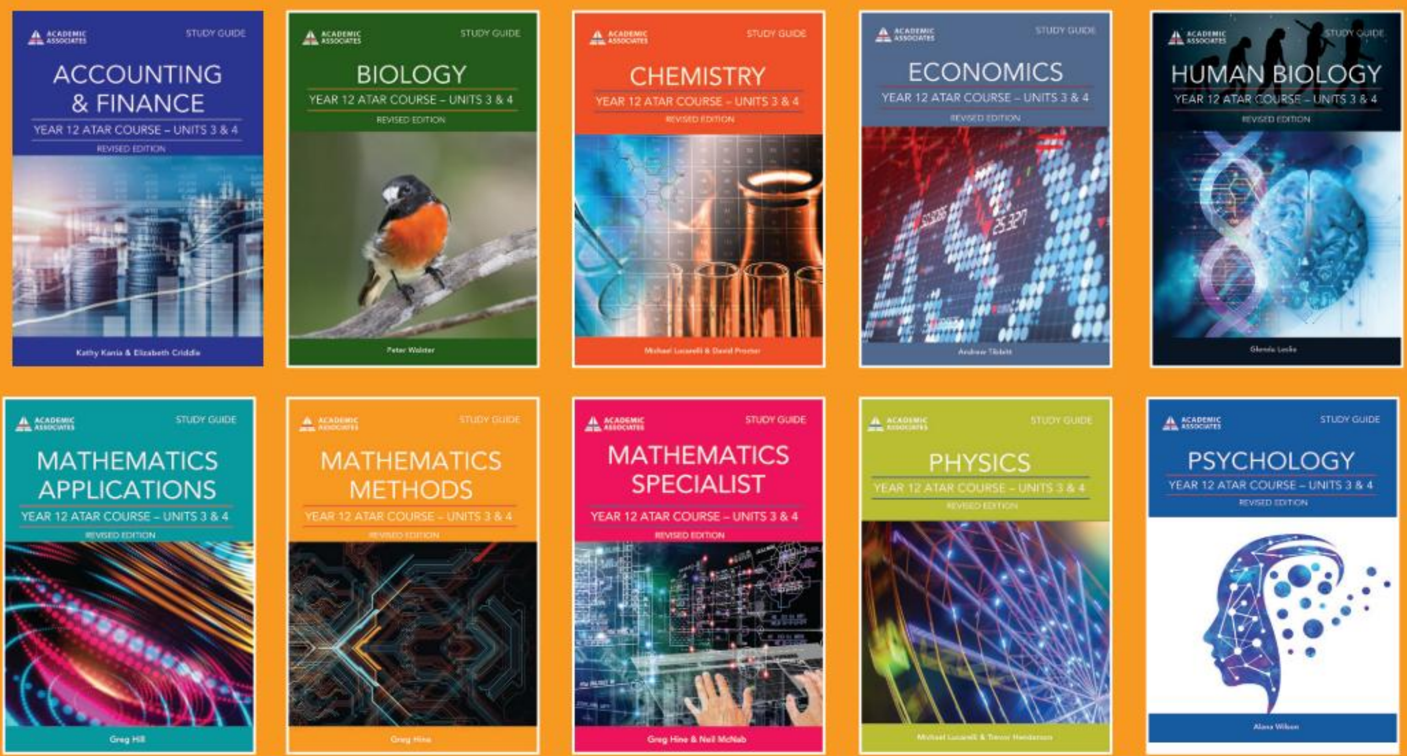

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