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11

MASTERING HSC MATHEMATICS

YEAR 11 EXTENSION 1 MATHEMATICS

NEW STAGE 6 HSC SYLLABUS

FOR STUDENTS AND TEACHERS

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Mathematical Association of NSW

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Errors:

This is a second print-run and many of the errors identified in the first print have been corrected. However, whilst the authors have taken utmost care with the editing and checking of this book, the authors cannot guarantee that all errors have been found and corrected. If you identify an error, please contact the authors directly at authors@masteringhscmathematics.com.au

Features of this book

This book is suitable for all students studying the HSC Mathematics Advanced and HSC Mathematics Extension 1 course. It has been designed in a thoroughly organised manner to help students master each syllabus topic in the new Stage 6 HSC Mathematics Advanced course. This book will teach, consolidate, test and challenge students. It is an essential resource for all students and teachers.

In flavour with the new course, this book has the following features:

- Technology-based questions.
- Interpretation questions.
- Modelling and application problems.
- Verification questions.

Within each chapter, there are subsections divided as follows.

Fundamentals

The carefully constructed *fundamentals* section appears before the main body of questions. The purpose of this section is to

- test all key formulae, definitions, concepts and theory.
- test essential mathematical terms and language through cloze-passages.
- ensure that the student has knowledge of the essential prerequisites.
- provide a summary of basic requirements for the topic.

Questions

This is the main body of questions with the following features.

- Step-by-step questions to assist the student with more difficult problems.
- Carefully graded exercises.
- "Show"-type questions, both guides the student, and offers good exam preparation.
- Proofs and explanations to strengthen understanding and develop problem-solving skills.
- Application questions to demonstrate future uses of learned theory.
- Technology-based questions to teach and reinforce concepts.

Challenge

These are more difficult questions that provide

- a challenge for students wishing to test their mastery of the topic.
- rigour and higher-order thinking skills.
- extension and more in-depth treatment of the unit of work.

Chapter Review

This section appears at the end of every chapter, and offers the following.

- Revision and consolidation of the previous exercises.
- Questions that require a combination of ideas from previous exercises.

Investigations

These tasks are potential assignments and research projects. Teachers may use and adapt these to cover the new NESA requirements on investigative assessment tasks. This section provides for the student

- application and modelling scenarios.
- research tasks involving data collection and analysis.
- scaffolding of learning tasks.
- open-ended style problems for discussions.
- opportunity to use appropriate technology effectively in a range of contexts.
- opportunity for students to demonstrate critical thinking.

Answers

- Quick answers to questions.
- "Show" and "prove" answers can be found in the full worked solutions.

Full worked Solutions

- Can be found online for free, or a full-colour hard copy purchased for convenience.
- Provide complete worked solutions to all questions, except investigative tasks to maintain the open-ended nature of the tasks.
- Includes several alternative solutions to problems, where possible.

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Table of Contents

Chapter 6:

Combinatorics and Binomial Expansions

Exercise

1

FURTHER FUNCTIONS

- Reciprocal and square root graphs
- Further reflections
- Adding graphs
- Multiplying graphs
- Inequalities
- Inverse functions
- Parametric forms

Exercise 1A

Reciprocal and square root graphs

Fundamentals

Fundamentals 1

Let $P(a, b)$ be a point on $y = f(x)$.

(a) The image of P under the transformation $y = \dfrac{1}{f(x)}$ is ______

(b) The image of P under the transformation $y = \sqrt{f(x)}$ is ______

Fundamentals 2

(a) As $f(x)$ increases, the graph of $\dfrac{1}{f(x)}$ increases/decreases (circle one).

(b) As $f(x)$ decreases, the graph of $\dfrac{1}{f(x)}$ increases/decreases (circle one).

(c) As $f(x) \to 0^+$, the graph of $\dfrac{1}{f(x)} \to$ ______.

(d) As $f(x) \to 0^-$, the graph of $\dfrac{1}{f(x)} \to$ ______.

(e) As $f(x) \to \infty$, the graph of $\dfrac{1}{f(x)} \to$ ______.

(f) As $f(x) \to -\infty$, the graph of $\dfrac{1}{f(x)} \to$ ______.

(g) All x-intercepts from $y = f(x)$ become v______ a________ on $y = \dfrac{1}{f(x)}$.

Fundamentals 3

Consider the equation of $y = \sqrt{f(x)}$.

(a) What happens if $f(x) < 0$?

(b) If $0 < f(x) < 1$, then $y = \sqrt{f(x)}$ is higher/lower (circle one) than $y = f(x)$.

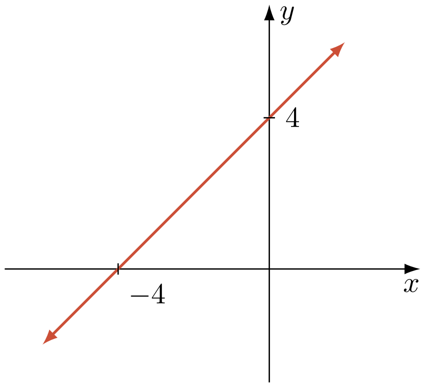
(c) If $f(x) > 1$, then $y = \sqrt{f(x)}$ is higher/lower (circle one) than $y = f(x)$.

(d) If $f(x)$ has a zero at $x = \alpha$, then $y = \sqrt{f(x)}$ also has a zero at $x = \alpha$. However, there will be a v________ tangent at $x = \alpha$, provided that $f'(\alpha) \neq 0$.

Fundamentals 4

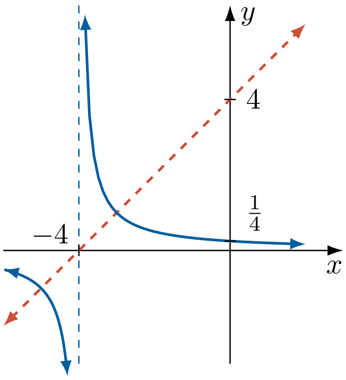
Explain the difference between the graphs of $y = \sqrt{f(x)}$ and $y^2 = f(x)$.

Question 1 The diagram below shows the graph of $y = f(x)$.

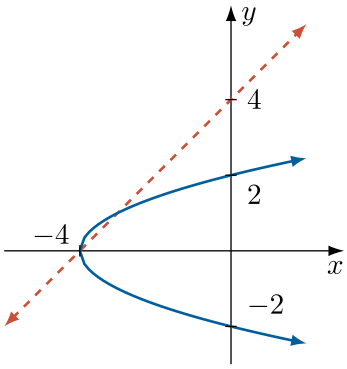


The following diagrams show the graphs of $y = \sqrt{f(x)}$, $y = \dfrac{1}{f(x)}$ and $y^2 = f(x)$ in a random order. Write down the transformation that matches each of the diagrams.

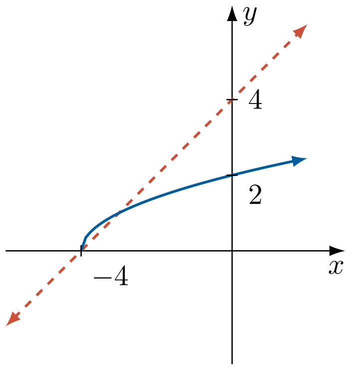
(a)



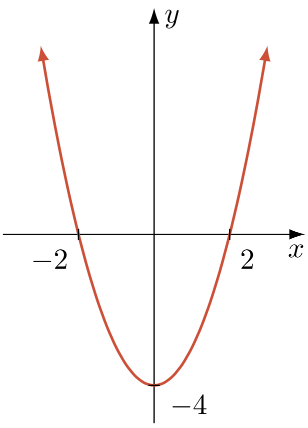
(b)



(c)

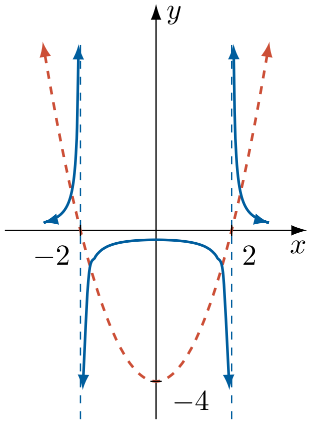


Question 2 The diagram below shows the graph of $y = f(x)$.

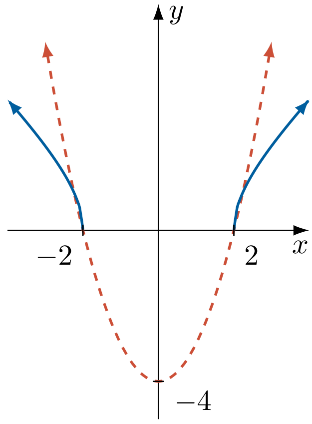


The following diagrams show the graphs of $y = \sqrt{f(x)}$, $y = \dfrac{1}{f(x)}$ and $y^2 = f(x)$ in a random order. Write down the transformation that matches each of the diagrams.

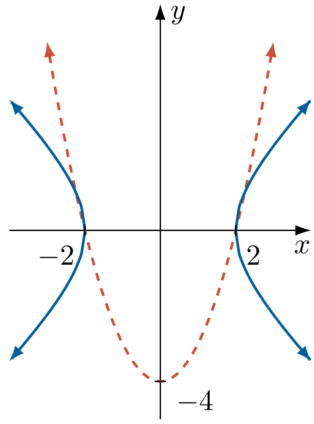
(a)



(b)



(c)



Question 3

(a) Draw the graph of $y = x^2 - 2x$, labelling all important features.

(b) Hence, draw a sketch of $y = \sqrt{x^2 - 2x}$ on the same set of axes.

Question 4

Use a similar technique to sketch the graph of the following.

(a) $y = \sqrt{2x - 1}$ (b) $y = \dfrac{1}{\sqrt{x}}$ (c) $y = \sqrt{x^3 - 4x}$

(d) $y = \sqrt{x^3 + 1}$ (e) $y = \sqrt{2 + x - x^2}$ (f) $y = \sqrt{4 + 2^{-x}}$

Question 5

By first drawing $y = f(x)$, sketch the following graphs of $y^2 = f(x)$.

(a) $y^2 = x + 2$ (b) $y^2 = x^2 - 4$ (c) $y^2 = 2^x - 1$

Question 6

(a) Draw the graph of $y = 4x - x^2$, labelling all important features.

(b) Hence, draw a sketch of $y = \dfrac{1}{4x - x^2}$ on the same set of axes.

Question 7

Use a similar technique to sketch the graph of the following. Draw the 'original' graph as a dashed curve, and draw the final answer on the same set of axes.

(a) $y = \dfrac{1}{2x - 1}$ (b) $y = \dfrac{1}{\sqrt{x}}$ (c) $y = \dfrac{1}{x^2}$

(d) $y = \dfrac{1}{x^2 + x - 2}$ (e) $y = \dfrac{1}{x^3 + 1}$ (f) $y = \dfrac{1}{1 + 2^x}$

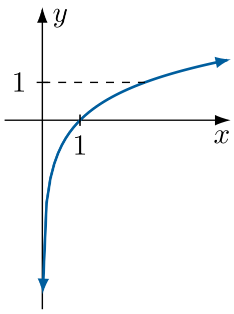
Question 8

(a) Sketch the graph of $y = x^3$.

(b) Hence, sketch the graph of the following.

(i) $y = \sqrt{x^3}$ (ii) $y^2 = x^3$ (iii) $y = \dfrac{1}{x^3}$

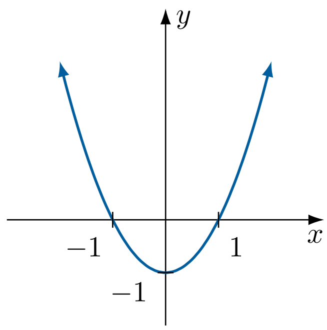
Question 9 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

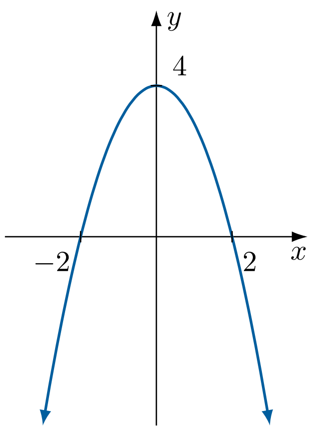
Question 10 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

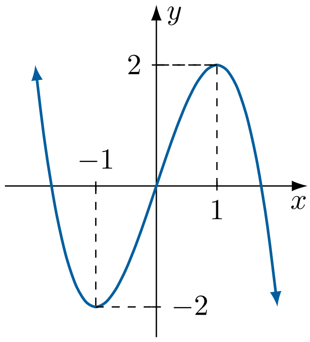
Question 11 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

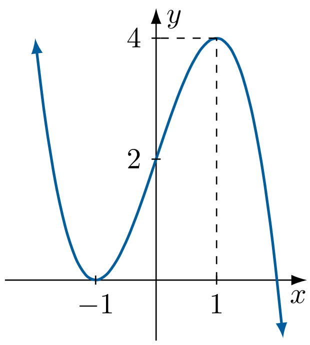
Question 12 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

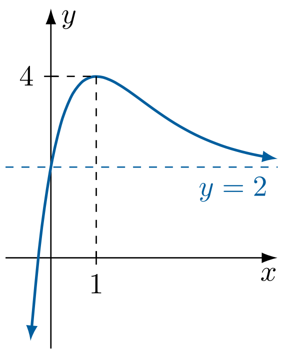
Question 13 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

Question 14 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

Challenge Problems

Problem 1 Sketch the following.

(a) $y = \dfrac{1}{\sqrt{x^2 - 1}}$

(b) $y = \dfrac{1}{\sqrt{1 - x^2}}$

Problem 2 Sketch the graph of $y = \dfrac{2^x}{1 + 2^x}$.

Hint: Divide the top and bottom by 2^x

Problem 3 [Kampyle of Eudoxus]

(a) Sketch the graph of $y = x^4 - x^2$.

(b) Hence, sketch the graph of $y^2 = x^4 - x^2$, which is called the *Kampyle of Eudoxus* named after the ancient Greek astronomer and mathematician Eudoxus of Cnidus (408 BC – 347 BC).

(c) Use graphing software to sketch $y^2 = x^4 - x^2$ and $y = x^2$ on the same set of axes. State what you observe.

(d) Prove your observation.

Problem 4 [Lemniscate]

Sketch the graph of $y^2 = x^2 - x^4$.

Problem 5 [Calculus required]

Prove the following statements about the reciprocal and square root graphs.

(a) If $f(x)$ has a zero at $x = \alpha$, and $f'(\alpha) \neq 0$, then $y = \sqrt{f(\alpha)}$ has a vertical tangent at $x = \alpha$.

(b) If $f(x)$ has a stationary point at $x = \alpha$ that is not also a zero, then $y = \dfrac{1}{f(x)}$ also has a stationary point at $x = \alpha$.

(c) If $f(x)$ has a turning point at $x = \alpha$ that is not also a zero, then $y = \dfrac{1}{f(x)}$ also has a turning point at $x = \alpha$, but with opposite concavity.

Exercise 1B

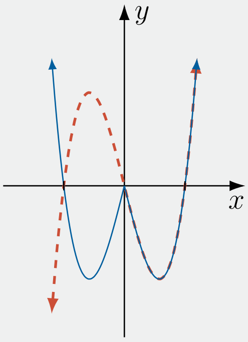
Further reflections

Fundamentals

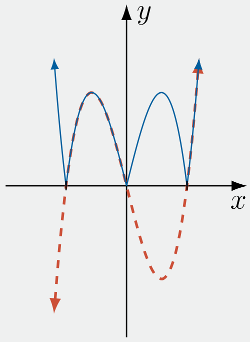
Fundamentals 1

The diagram below shows the graph of $y = f(x)$ (dashed red) and either $y = |f(x)|$ or $y = f(|x|)$ (blue). Determine which reflection the diagrams represent.

(a)



(b)



Fundamentals 2

Complete the following.

(a) $|x| = \begin{cases} \_\_\_, & \text{for} \quad x \geq 0 \\ \_\_\_, & \text{for} \quad x < 0 \end{cases}$

(b) $|f(x)| = \begin{cases} \_\_\_, & \text{for} \quad f(x) \geq 0 \\ \_\_\_, & \text{for} \quad f(x) < 0 \end{cases}$

Fundamentals 3

The below questions are about obtaining the graph of $y = |f(x)|$ from the graph of $y = f(x)$

(a) If $f(x) \geq 0$, then $|f(x)| =$ _____. Hence, when $f(x) \geq 0$, the graph of $y = |f(x)|$ is identical to the original.

(b) If $f(x) < 0$, then $|f(x)| =$ _____. Hence, the negative parts of $f(x)$ are reflected across the x/y (circle one) axis.

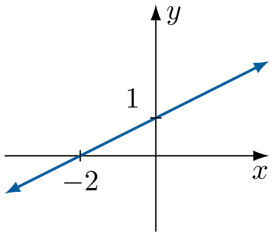
Fundamentals 4

The below questions are about obtaining the graph of $y = f(|x|)$ from the graph of $y = f(x)$.

(a) If $x \geq 0$, then $f(|x|) =$ _____. Hence, when $x \geq 0$, the graph of $y = f(|x|)$ is identical to the original.

(b) If $x < 0$, then $f(|x|) =$ _____. Hence, the left/right (circle one) half of the original graph is reflected across the x/y (circle one) axis.

Question 1 The diagram below shows the graph of $y = f(x)$.

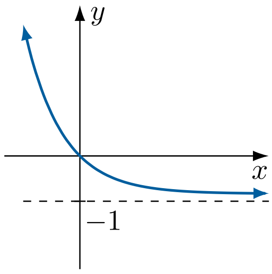


On separate axes, sketch the graph of

(a) $y = |f(x)|$

(b) $y = f(|x|)$

Question 2 The diagram below shows the graph of $y = f(x)$.

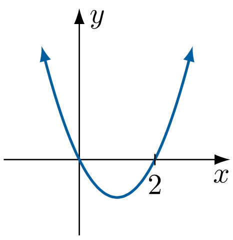


On separate axes, sketch the graph of

(a) $y = |f(x)|$

(b) $y = f(|x|)$

Question 3 The diagram below shows the graph of $y = f(x)$.

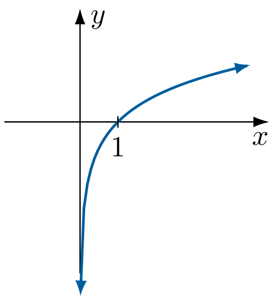


On separate axes, sketch the graph of

(a) $y = |f(x)|$

(b) $y = f(|x|)$

Question 4 The diagram below shows the graph of $y = f(x)$.

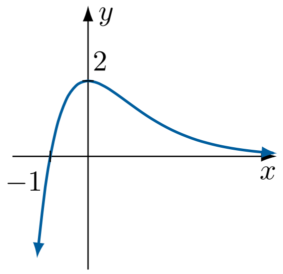


On separate axes, sketch the graph of

(a) $y = |f(x)|$

(b) $y = f(|x|)$

Question 5 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = |f(x)|$

(b) $y = f(|x|)$

Question 6

(a) Sketch the graph of $y = 4x - x^3$.

(b) Hence, sketch the graph of $y = |4x - x^3|$.

Question 7 By first sketching a base graph $y = f(x)$ and using it to sketch $y = |f(x)|$ on the same set of axes, draw a sketch of the following.

(a) $y = |2x + 1|$

(b) $y = \dfrac{1}{|x|}$

(c) $y = |2^x - 1|$

(d) $y = |1 - 2^{-x}|$

(e) $y = |x^2 - 4|$

(f) $y = |x^3 + 1|$

(g) $y = |x^3 - x|$

(h) $y = |x^2 + 2x - 3|$

(i) $y = \left|1 + \dfrac{1}{x}\right|$

Question 8

(a) Sketch the graph of $y = x^2 - 2x$

(b) Hence, sketch the graph of $y = |x|^2 - 2|x|$

Question 9 By first sketching a base graph $y = f(x)$ and using it to sketch $y = f(|x|)$ on the same set of axes, draw a sketch of the following.

(a) $y = 1 - 2|x|$

(b) $y = 2^{|x|}$

(c) $y = 2^{-|x|}$

(d) $y = \dfrac{1}{|x|}$

(e) $y = \sqrt{|x|}$

(f) $y = \sqrt{|x| + 1}$

(g) $y = 4|x| - |x|^2$

(h) $y = |x|^3 - 1$

(i) $y = |x|^3 - 16|x|$

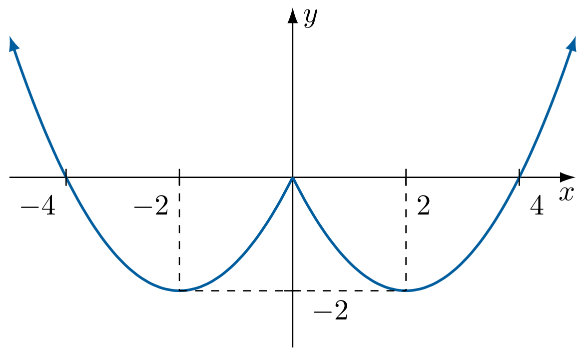
Question 10

(a) Let $f(x)$ be any function. Prove that $f(|x|)$ is an even function.

(b) Let $f(x)$ be any odd function. Prove that $|f(x)|$ is an even function.

Challenge Problems

Problem 1 A quadratic was transformed using a reflection to form the following graph. Find the equation of the original quadratic.



Problem 2

(a) Consider the graph of $y = f(x)$. Describe the effect on the graph if all x's in $y = f(x)$ were replaced with $|x|$.

(b) What do you think is the effect on the graph if the y was replaced with $|y|$.

(c) Hence, sketch the graph of $|x| + |y| = 1$.

Problem 3

(a) Show that $\dfrac{x-1}{x+1} = 1 - \dfrac{2}{x+1}$.

(b) Show that $\dfrac{x+1}{x-1} = 1 + \dfrac{2}{x+1}$.

(c) Sketch the graph of $y = \dfrac{|x|-1}{|x|+1}$.

(d) Sketch the graph of $y = \dfrac{|x|+1}{|x|-1}$.

Exercise 1C

Adding graphs

Fundamentals

Fundamentals 1

(a) If (a, b) is a point on $y = f(x)$ and (a, c) is a point on $y = g(x)$, then at $x = a$ the graph of $y = f(x) + g(x)$ has coordinates ________ .

(b) If $P(a, b)$ is a point of intersection of $y = f(x)$ and $y = g(x)$, then P has coordinates ______ on the graph of $y = f(x) + g(x)$.

Fundamentals 2

Instead of subtracting ordinates to sketch $y = f(x) - g(x)$ it can be easier to first sketch $y = f(x)$ and $y =$ ______, and then add the two curves.

Fundamentals 3

To sketch $y = f(x) + g(x)$ reasonably accurately, the base-graphs of $y = f(x)$ and $y = g(x)$ must be drawn to s______.

Fundamentals 4

Describe what happens to the graph of $y = f(x) + g(x)$ at $x = a$ if

(a) $f(x)$ has an x-intercept at $x = a$.

(b) $f(x)$ and $g(x)$ have the same value at $x = a$.

(c) $f(x)$ and $g(x)$ are negatives of each other at $x = a$.

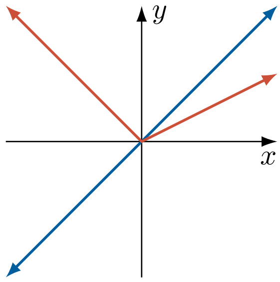
Fundamentals 5

(a) If $f(x)$ has domain $x \geq 0$ and $g(x)$ has domain all real x, then the graph of $y = f(x) + g(x)$ has domain ________ .

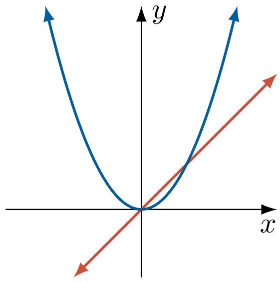
(b) In general, any domain restriction from either function still applies to their s___.

Question 1 The diagrams below show to-scale diagrams of $y = f(x)$ (in blue) and $y = g(x)$ (in red) on the same set of axes. Use the diagrams to sketch $y = f(x) + g(x)$ by addition of ordinates.

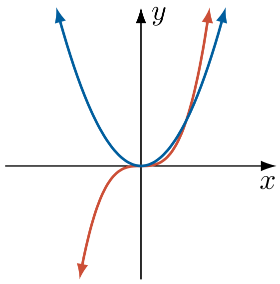
(a)



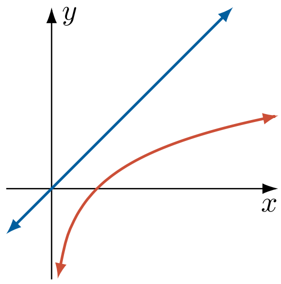
(b)



(c)



(d)



Question 2

(a) Sketch the graph of $y = x$ and $y = |x|$ on the same set of axes.

(b) Hence, sketch the graph of $y = x + |x|$ by addition of ordinates.

Question 3 Use a similar technique to the above question to sketch the following graphs.

(a) $y = x + x^2$ (b) $y = x + x^3$ (c) $y = |x - 1| + |x + 1|$

Question 4 [Alternative method to subtracting graphs]

(a) Sketch the graph of $y = |x|$ and $y = -x$ on the same set of axes.

(b) Hence, sketch the graph of $y = |x| - x$.

Question 5 Use a similar technique to the above question to sketch the following graphs.

(a) $y = x - x^3$ (b) $y = |x - 1| - |x + 1|$ (c) $y = \sqrt{x} - x$

Question 6 [Further absolute value graphs]

Sketch the following graphs.

(a) $y = |x + 1| + |x - 2|$ (b) $y = |x - 1| - |x + 2|$

(c) $y = |2x + 1| + |x - 1|$ (d) $y = |2x - 1| - |x + 1|$

Question 7 [Exponential functions]

(a) Use graphing software to sketch $y = x$, $y = 2^x$ and $y = x + 2^x$ on the same set of axes.

(b) What do you notice about the graph of $y = x$ and $y = x + 2^x$ as $x \to -\infty$?

(c) Explain your observation.

(d) Attempt to recreate the graph of $y = x + 2^x$ by addition of ordinates and compare your graph to the one generated using software.

Question 8 Sketch the following graphs.

(a) $y = x + 2^{-x}$ (b) $y = x - 2^{-x}$ (c) $y = x - 2^x$

Question 9

(a) Sketch the graph of $y = 2^x$ and $y = 2^{-x}$ on the same set of axes.

(b) Hence, sketch $y = 2^x + 2^{-x}$.

(c) Repeat part (a) and hence sketch $y = 2^x - 2^{-x}$

Question 10 [To do after radians are covered]

Sketch the graphs of the following curves over the domain $x \in [-\pi, \pi]$.

(a) $y = x + \sin x$ (b) $y = \sin x + \cos x$ (c) $y = \cos x - \sin x$

Challenge Problems

Problem 1 Sketch the graphs of the following using addition of ordinates.

(a) $y = x + \dfrac{1}{x}$ (b) $y = x - \dfrac{1}{x}$ (c) $y = x^2 + \dfrac{1}{x}$

(d) $y = \sqrt{x} + \dfrac{1}{x}$ (e) $y = 2^x + \dfrac{1}{x}$ (f) $y = 2^x - \dfrac{1}{x}$

Problem 2 Bob and Mary are both asked to sketch $y = f(x) + g(x)$, given that $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{1}{1-x}$. Bob draws $y = f(x)$ and $y = g(x)$ individually, then adds the ordinates. However, Mary has a more algebraic approach.

(a) Show that $\dfrac{x}{x-1} + \dfrac{1}{1-x} = 1$.

(b) Mary instead sketches $y = 1$ after following the steps of (a). Who produced the correct sketch, and why?

Exercise 1D

Multiplying graphs

Fundamentals

Fundamentals 1

(a) If (a,b) is a point on $y=f(x)$ and (a,c) is a point on $y=g(x)$, then at $x=a$ the graph of $y=f(x)g(x)$ has coordinates ________.

(b) If $P(a,b)$ is a point of intersection of $y=f(x)$ and $y=g(x)$, then P has coordinates ______ on the graph of $y=f(x)g(x)$.

Fundamentals 2

Instead of dividing ordinates to sketch $y=\dfrac{f(x)}{g(x)}$, it can be easier to first sketch $y=f(x)$ and $y=$ ______, and then multiply the two curves.

Fundamentals 3

Describe what happens to the graph of $y=f(x)g(x)$ at $x=a$ if

(a) $f(x)$ has an x-intercept at $x=a$.

(b) $f(x)$ is equal to one at $x=a$.

Fundamentals 4

(a) If $f(a)>0$ and $g(a)>0$, then $f(a)g(a)$ ______.

(b) If $f(a)>0$ and $g(a)<0$, then $f(a)g(a)$ ______.

(c) If $f(a)<0$ and $g(a)<0$, then $f(a)g(a)$ ______.

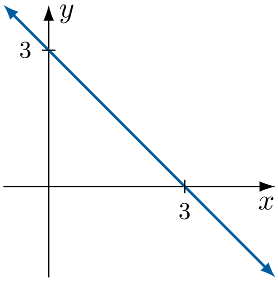
Fundamentals 5

(a) If $f(x)$ has the domain $x\geq 0$ and $g(x)$ has domain being all real x, then the graph of $y=f(x)g(x)$ has domain ________.

(b) In general, any domain restriction from either function still applies to their p________.

Question 1 The diagrams below show to-scale diagrams of $y = f(x)$. Use the diagrams to sketch $y = x\,f(x)$ by multiplication of ordinates.

(a)



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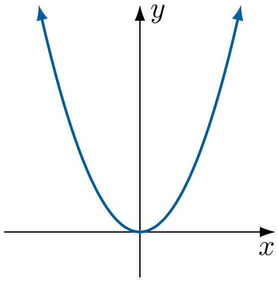

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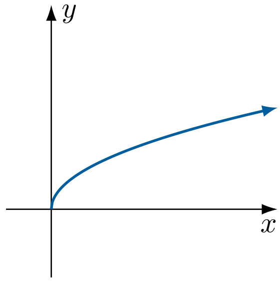

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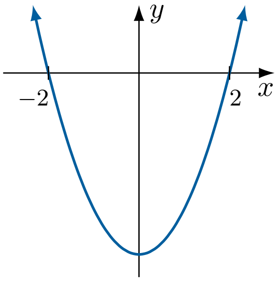

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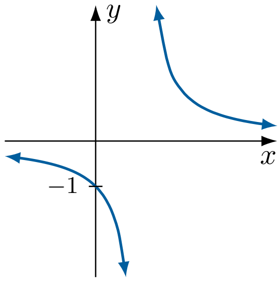

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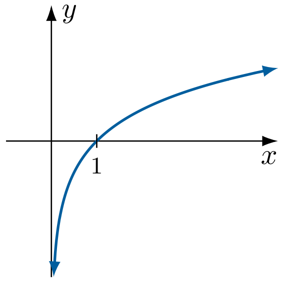


Question 2

(a) Sketch the graphs of $f(x) = x^2$ and $g(x) = x - 1$ on the same set of axes.

(b) Multiply their ordinates to obtain the graph of $y = x^2(x - 1)$.

(c) Sketch $y = x^2(x - 1)$ as you normally would and compare it with your graph from (b).

(d) Repeat the above steps for the following pairs of functions.

(i) $f(x) = x - 2,\ g(x) = -x + 4$

(ii) $f(x) = x,\ g(x) = x^2 - 1$

(iii) $f(x) = x^2,\ g(x) = x^2 - 9$

(iv) $f(x) = x^2 - 1,\ g(x) = x^2 - 4$

Question 3

(a) Sketch the graph of $y = x$ and $y = |x|$ on the same set of axes.

(b) Hence, sketch the graph of $y = x|x|$ by multiplication of ordinates.

Question 4 Use a similar technique to the above question to sketch the following graphs.

(a) $y = x|x - 1|$

(b) $y = \sqrt{x}(1 + x)$

(c) $y = |x|(x - 1)$

Question 5 [Dominating functions]

Consider the equation $y = x\,2^{-x}$.

(a) Complete the following table of values.

x	-10	-5	-1	0	1	5	10
$x\,2^{-x}$							

(b) Use the table to describe what happens to the curve as $x \to \infty$ and as $x \to -\infty$.

(c) Use graphing software to sketch $y = x$, $y = 2^{-x}$ and $y = x\,2^{-x}$ on the same set of axes.

(d) Draw the graph of $y = x\,2^{-x}$ by first sketching $y = x$ and $y = 2^{-x}$ and then multiplying ordinates. Compare your graph to the one generated using software.

Question 6 Sketch the following graphs. Use a calculator and a table of values for parts where the behaviour of the curve is not clear such as when $x \to \pm\infty$.

(a) $y = x\,2^{x}$

(b) $y = x^2\,2^{x}$

(c) $y = \dfrac{2^x}{x}$

(d) $y = \sqrt{x}\,2^{-x}$

Challenge Problems

Problem 1

(a) Sketch $y = x^2 - 4$ and hence sketch $y = \dfrac{1}{x^2 - 4}$

(b) Hence, sketch $y = \dfrac{x}{x^2 - 4}$

Problem 2

(a) Sketch $y = x\,3^x$.

(b) Explain why $y = x$ touches the curve $y = x\,3^x$ exactly once at the origin.

Problem 3 Sketch the following by drawing appropriate base-curves and multiplying their ordinates.

(a) $y = \dfrac{\sqrt{x}}{x - 1}$

(b) $y = \dfrac{\sqrt{x}}{x + 1}$

Problem 4 [To do after radians are covered]

Sketch the following by drawing appropriate base-curves and multiplying their ordinates.

(a) $y = x \sin x$ for $x \in [-\pi, \pi]$.

(b) $y = x \cos x$ for $x \in [-\pi, \pi]$.

(c) $y = x \tan x$ for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(d) $y = \sin x \cos x$ for $x \in [-\pi, \pi]$.

Problem 5 [To do after logarithms are covered]

Sketch the following graphs. Use a calculator and a table of values for parts where the behaviour of the curve is not clear such as when $x \to \infty$ or $x \to 0^+$.

(a) $y = x \ln x$

(b) $y = x^2 \ln x$

(c) $y = \dfrac{\ln x}{x}$

(d) $y = \sqrt{x} \ln x$

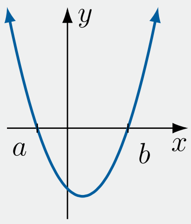
Exercise 1E

Inequalities

Fundamentals

Fundamentals 1

The diagram below shows the sketch of $y = (x - a)(x - b)$.



Write down the inequality that corresponds to

(a) $(x-a)(x-b) \geq 0$

(b) $(x-a)(x-b) < 0$

Fundamentals 2

(a) To solve a quadratic inequality in the form $ax^2 + bx + c \leq 0$ or $ax^2 + bx + c \geq 0$, first sketch the graph of $y =$ ________.

(b) Then, depending on the direction of the inequality, shade the region that is either a______ or b______ the ___-axis.

(c) The set of ___-values that are shaded is the solution set.

Fundamentals 3

(a) If an inequality involves a rational expression with an unknown in the denominator, we can multiply both sides by the s______ of the denominator.

(b) We cannot simply multiply both sides by the denominator itself to eliminate fractions because the denominator may be n________, which will flip the inequality sign around.

(c) Once this is done, we can then re-arrange and f________ the resultant polynomial and solve the inequality by shading the appropriate region.

Fundamentals 4

Suppose that $k > 0$.

(a) If $|x| \leq k$, then ___ $\leq x \leq$ ___.

(b) If $|x| \geq k$, then $x \geq$ ___ or $x \leq$ ___.

Fundamentals 5

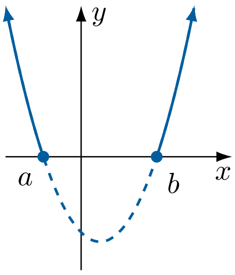
This question will extend the result from Fundamentals 1 further. Suppose that $k > 0$.

(a) If $|ax + b| \leq k$, then ___ $\leq ax + b \leq$ ___.

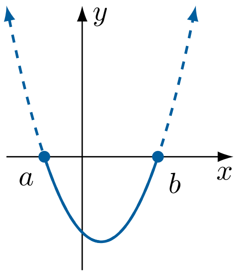
(b) If $|ax + b| \geq k$, then $ax + b \geq$ ___ or $ax + b \leq$ ___.

Question 1 The diagrams below show a highlighted section of the graph of $y = (x-a)(x-b)$, corresponding to either $(x-a)(x-b) \geq 0$ or $(x-a)(x-b) \leq 0$. State the appropriate inequality for each diagram.

(a)



(b)



Question 2 By first drawing a brief sketch of the quadratic polynomial and highlighting the appropriate section of the graph, solve the following inequalities.

(a) $(x-2)(x+2) \leq 0$

(b) $(x-5)(x+4) \geq 0$

(c) $x^2 - x - 2 < 0$

(d) $x^2 - 7x - 8 > 0$

(e) $x^2 \geq 4$

(f) $x^2 > 4x$

(g) $16 - x^2 \leq 0$

(h) $2x^2 - x - 6 < 0$

Question 3 By first drawing a brief sketch of the polynomial and highlighting the appropriate section of the graph, solve the following inequalities.

(a) $(x-1)(x+1)(x-4) \leq 0$

(b) $(x+2)(x-3)(x+6) > 0$

(c) $(x-3)^2(x^2-1) \geq 0$

(d) $x^4 - 13x^2 + 36 < 0$

Question 4 Consider the inequality $\dfrac{2}{x+1} \leq 4$

(a) Write down the restriction on the values of x.

(b) Multiply both sides by $(x+1)^2$ and hence, show that $4(x+1)^2 - 2(x+1) \geq 0$

(c) By factorising out $(x+1)$, show that this inequality simplifies to $(x+1)(2x+1) \geq 0$.

(d) Hence, find the solution of $\dfrac{2}{x+1} \leq 4$.

Hint: Make sure to check your answer against any restrictions.

Question 5 Solve the following inequalities. Be mindful to check that the endpoints of your domains work as well.

(a) $\dfrac{3}{x-2} < 1$

(b) $\dfrac{2x-3}{x+4} \leq 1$

(c) $\dfrac{x(x+3)}{x+1} < 2$

(d) $\dfrac{x^2-4}{x-2} > 0$

(e) $\dfrac{4}{x-3} \leq x$

(f) $\dfrac{1}{x} \leq \dfrac{1}{x+2}$

Question 6 [Using the distance definition of the absolute value]

Complete the following.

(a)

(i) $|x| = 4$ is read as "the distance of x from the o______ is ___".

(ii) $|x| < 4$ is read as "the distance of x from the o______ is l______ than 4".

(iii) Hence, the solution of $|x| < 4$ is ________________.

(iv) Likewise, the solution of $|x| > 4$ is __________ or __________.

(b)

(i) $|x - 2| = 5$ is read as "the distance of x from ___ is ___".

(ii) $|x - 2| < 5$ is read as "the distance of x from ___ is l______ than 5".

(iii) Hence, the solution of $|x - 2| < 5$ is ________________.

(iv) Likewise, the solution of $|x - 2| > 5$ is __________ or __________.

Question 7 [Proving the results in the Fundamentals section]

(a) Draw a graph of $y = |x|$ and $y = k$, for some $k > 0$, on the same set of axes.

(b) Find the x-coordinate where $y = |x|$ intersects $y = k$.

(c) Use your diagram to find the values of x such that $|x| \geq k$.

(d) Similarly, find the values of x such that $|x| \leq k$.

Question 8 Find the solution of the following, provided that $k > 0$.

(a) $|x - a| \leq k$

(b) $|x - a| \geq k$

Question 9 Solve the following absolute value inequalities either graphically, algebraically, or using the distance definition.

(a) $|x| \geq 4$

(b) $|x| < 2$

(c) $|x + 1| < 3$

(d) $|x - 3| \geq 2$

(e) $|4 - x| \leq 0$

(f) $|x + 5| \geq 0$

(g) $|3x - 4| > 0$

(h) $|4 - 3x| < 0$

(i) $|2x + 1| \geq 5$

(j) $|3 - 2x| \leq 7$

(k) $|3x - 4| > -1$

(l) $|2x + 5| < -2$

Challenge Problems

Problem 1 Sketch the graph of $|y| = x$ and $|y - 2| = x$ on the same set of axes. Comment on how the graphs differ.

Problem 2

(a) Draw a sketch of $y = |4x - 5|$ and $y = 2x + 3$ on the same set of axes.

(b) Hence, solve the inequality $|4x - 5| > 2x + 3$.

Problem 3 Solve the inequality $\dfrac{1}{|2x - 1|} < 2$.

Problem 4 Solve the inequality $x + \dfrac{1}{x} \geq 2$.

Problem 5 Explain why solving $x^3(x - 1) > 0$ is equivalent to solving $x(x - 1) > 0$.

Problem 6 By drawing a sketch, solve the inequality $|2x + 1| \leq |x - 4|$.

Exercise 1F

Inverse functions

Fundamentals

Fundamentals 1

(a) A one-to-one function has a single y-value for each ___-value, and vice versa.

(b) If $f(x)$ is a one-to-one function function in a particular domain, then we can find a new function called the inverse function, which is denoted by ___.

(c) The inverse function 'undoes' what a function f does to a value. For example, if $f(a) = b$, then $f^{-1}(b) =$ ___.

Fundamentals 2

(a) There are two types of inverses. One is an inverse f________ and another is an inverse r________.

(b) All functions are r________, but not necessary the other way around.

(c) An inverse r________ is obtained by simply swapping x and y. For example, the inverse relation of $y = x^2$ is ________.

(d) The inverse $f^{-1}(x)$ of a function $f(x)$ is itself only a function if $f(x)$ is o___-to-o___. The equation of $f^{-1}(x)$ is obtained by swapping x and y, and then making ___ the subject.

Fundamentals 3

(a) The inverse of a function $f(x)$ can be drawn by reflecting $f(x)$ across the line $y =$ ___.

(b) The resultant graph may or may not still be a function. If it is not a function, then we call it the i______ r______. However, if it is a function, then we we call it the i______ f______.

(c) To determine if the inverse is still a function, we can use the h________ line test on the original function $f(x)$.

(d) If $f(x)$ cuts the horizontal line more than once, then it passes/fails (circle one) the test and hence, the inverse is not a function.

(e) However, if $f(x)$ cuts the horizontal line at most once, then it passes/fails (circle one) the test and hence, the inverse is a function.

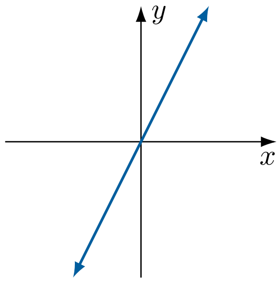
Fundamentals 4

(a) If the inverse of a function $f(x)$ is not a function, then we can restrict the d______ of $f(x)$ so that the function becomes o___-to-o___. This is done to ensure that the inverse also being a function.

(b) The d______ of the function becomes the r______ of the inverse function. Conversely, the r______ of the function becomes the d______ of the inverse function.

Question 1 For each of the following graphs, use the horizontal line test to determine whether the inverse is a relation or a function. Verify your findings by drawing the inverse.

(a)



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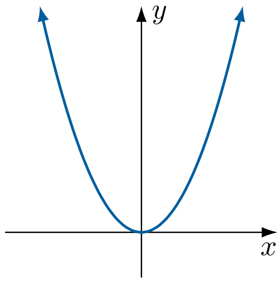

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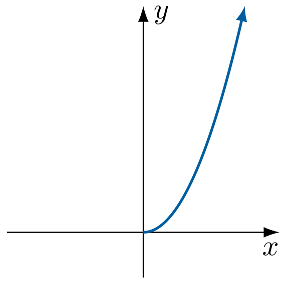

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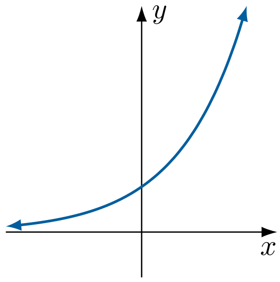

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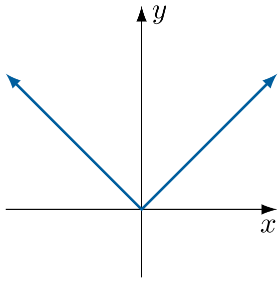

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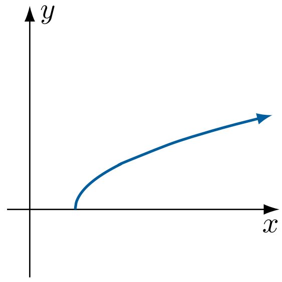

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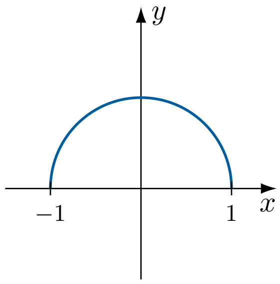

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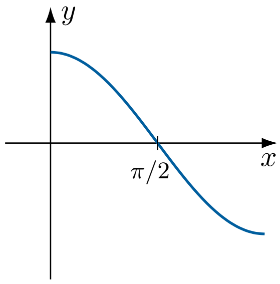
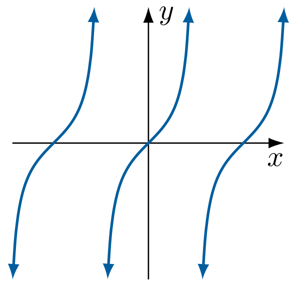


(i)



Question 2 By swapping x and y and then making y the subject, find the inverse of the following.

(a) $y = 2x$

(b) $y = 4x - 5$

(c) $2x + 3y + 5 = 0$

(d) $y = \dfrac{3x}{2} - 1$

Question 3 Let $f(x) = \dfrac{x-1}{x+2}$

(a) State the domain and range of $f(x)$.

(b) Sketch the graph of $y = f(x)$ and determine whether the inverse is a function or not.

(c) Find the equation of the inverse function.

Question 4 Find the inverse of the following.

(a) $y = \dfrac{1}{x+1}$

(b) $y = \dfrac{1}{4-x}$

(c) $y = \dfrac{2x-1}{x+2}$

Question 5 [Special case]

What is the inverse of the following?

(a) $y = 1$

(b) $x = -2$

Question 6 Explain how the horizontal line test works in determining whether the inverse is a relation or a function.

Question 7 By first finding the domain and range of $f(x)$, state the domain and range of the inverse function $f^{-1}(x)$.

(a) $f(x) = 3x + 4$

(b) $f(x) = \sqrt{x} - 3$

(c) $f(x) = x^2 + 1$, for $x \geq 0$

(d) $f(x) = \dfrac{3}{x}$

(e) $f(x) = 2^x$

(f) $f(x) = \dfrac{1}{\sqrt{x+2}}$

Question 8 [Intersecting the inverse]

Consider the function $f(x) = \sqrt{2x - 1}$, which has an inverse function.

(a) Find the equation of the inverse function $y = f^{-1}(x)$ and state the domain and range.

(b) Find the coordinates of the point where $y = f(x)$ intersects $y = x$.

(c) Explain why this point also represents the points where $y = f(x)$ intersects $y = f^{-1}(x)$.

(d) Hence, draw the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 9 For each of the functions in Question 2, sketch the function and inverse on the same set of axes.

Question 10

(a) Explain why $f(x) = x^2$ unrestricted does not have an inverse function.

(b) State the largest two possible domains so that $f(x) = x^2$ *does* have an inverse function.

(c) On a separate set of axes, sketch the two branches of $f(x) = x^2$ that have inverse functions.

(d) Find the equation of the two possible inverse functions, and match them up with the correct branch from the original function.

(e) Sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$, for both cases, on a separate set of axes.

Question 11 For each of the following quadratics, write down two ways the domain could be restricted so that the inverse function exists.

(a) $y = 1 - x^2$

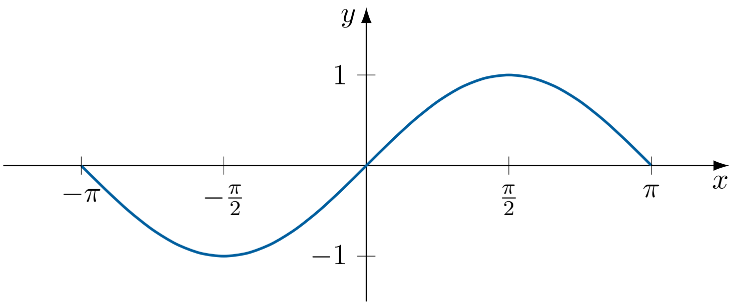
(b) $y = (x - 2)^2$

(c) $y = (x - 3)(x + 1)$

(d) $y = x^2 - 6x + 8$

Question 12 [Inverse Sine Graph]

The diagram below shows the graphs of $f(x) = \sin x$ in the domain $x \in [-\pi, \pi]$.



(a) Find the largest possible domain such that $f(x)$ is one-to-one, and so that the inverse contains the origin.

(b) Sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 13 Define $f(x) = (x-1)^2 - 3$.

(a) Find the largest domain containing $x = 3$ such that the inverse is defined.

(b) State the domain and range of $f^{-1}(x)$.

(c) Sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 14 Consider the function $f(x) = x^2 - 2x$.

(a) Find the largest domain containing $x = 2$ such that the inverse is a function.

(b) State the range of $f(x)$ in that domain.

(c) Show that the inverse relation is $y^2 - 2y - x = 0$.

(d) By either using the quadratic formula or completing the square, show that making y the subject yields the equations

$$y = 1 \pm \sqrt{1+x}$$

(e) State the range of the inverse function.

(f) Hence, determine whether the inverse function has equation $y = 1 + \sqrt{1+x}$ or $y = 1 - \sqrt{1+x}$.

(g) Sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 15 Define the function $f(x) = x^2 - 6x$ for the domain $x \geq 3$.

(a) Draw the graph of $y = f(x)$.

(b) Find where $f(x)$ intersects it's own inverse.

(c) Find the equation of the inverse function.

Challenge Problems

Problem 1 [Self-inverse]

Show that the function

$$f(x) = \frac{a-x}{1+bx}$$

where a and b are constants, is *self-inverse*. That is, the function is it's own inverse function.

Problem 2 Define the function $f(x) = x + \dfrac{1}{x}$.

(a) Show that $f\left(\dfrac{1}{a}\right) = f(a)$ for all $a \neq 0$.

(b) Hence, explain why $f(x)$ does not have an inverse function.

Problem 3 Define the function $f(x) = \dfrac{1}{1+x^2}$.

(a) Show that $f(x)$ is an even function.

(b) Hence, explain why $f(x)$ does not have an inverse function.

(c) What is the largest domain containing positive values such that $f^{-1}(x)$ exists?

(d) Use graphing software to sketch $y = f(x)$ in this domain. Hence, in your book, sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Problem 4 Define the function $f(x) = x - \dfrac{1}{x}$ in the domain $x > 0$.

(a) Use graphing software to sketch $y = f(x)$ and $y = x$. In your book, draw the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

(b) Show that the inverse has equation

$$f^{-1}(x) = \frac{x + \sqrt{x^2+4}}{2}.$$

Problem 5 Define the function $f(x) = x^2 + 4x + 5$ in the domain $x \geq -2$.

(a) Find the equation of the inverse function $f^{-1}(x)$.

(b) Show that

$$f^{-1}\left(f(x)\right) = f\left(f^{-1}(x)\right) = x$$

Exercise 1G

Parametric forms

Fundamentals

Fundamentals 1

(a) A C________ equation is an equation relating two variables x and y.

(b) These variables can be expressed as functions of a third variable called a p________.

(c) This p________ can be used to study the ___ or ___-coordinates individually, rather than studying them together all the time.

(d) Every point on the curve is now defined by only o___ number, which is the value of the p________.

(e) For a given Cartesian equation, the parametrisation is/is not (circle one) unique. In other words, a given Cartesian equation may/may not (circle one) have many parametric equations to represent it.

Fundamentals 2

(a) To obtain the C________ equation from the parametric equation, we need to e________ the parameter.

(b) This can often be done for most problems either by making the parameter the subject from one equation first and then s________ into the other, or by using a t________ identity.

Fundamentals 3

(a) The usual parametrisation for the circle $x^2 + y^2 =$ ___ is $x = r\cos\theta$ and $y =$ ________.

(b) It relies on the trigonometric identity ________________.

(c) If the circle is centred at (a, b), then a parametrisation is $x = a +$ ________ and $y =$ ___ $+ r\sin\theta$.

Question 1 Consider the curve defined parametrically by $x = t - 1$ and $y = t + 1$

(a) Complete the following table.

t	-2	-1	0	1	2
x					
y					

(b) Eliminate the parameter and hence find the Cartesian equation.

(c) What value of t yields the coordinate $(4, 6)$?

(d) Sketch the graph and plot the points corresponding to $t = 0$, 1, 2 on it.

Question 2 Consider the curve defined parametrically by $x = 3t$ and $y = t^2$

(a) Complete the following table.

t	-2	-1	0	1	2
x					
y					

(b) Eliminate the parameter and hence find the Cartesian equation.

(c) What value of t yields the coordinate $(6, 4)$?

(d) Sketch the graph and plot the points corresponding to $t = 0,\ 1,\ 2$ on it.

(e) Let T be the point on the parabola with parameter t. As t varies, the position of T will also vary. Describe what happens to T as $t \to \pm\infty$.

Question 3 For each of the following, eliminate the parameter and hence state the Cartesian equation.

(a) $x = 2t$
$y = 3t$

(b) $x = 3 + t$
$y = 2t$

(c) $x = 2 - 3t$
$y = 4 + 2t$

(d) $x = 4t$
$y = 16t^2$

(e) $x = 3t$
$y = 6t^2$

(f) $x = t - 3$
$y = 1 - t^2$

Question 4 For each of the following, show that the Cartesian equation is a circle and state the centre and radius.

(a) $x = \cos\theta$
$y = \sin\theta$

(b) $x = 2\cos\theta$
$y = 2\sin\theta$

(c) $x = -1 + \cos\theta$
$y = 2 - \sin\theta$

(d) $x = 4 + 3\cos\theta$
$y = -5 + 3\sin\theta$

Question 5 For each of the following circles, write down a suitable parametric equation.

(a) $x^2 + y^2 = 16$

(b) $(x - 2)^2 + (y + 5)^2 = 9$

(c) $x^2 + 6x + y^2 - 2y - 15 = 0$

Question 6 Sketch the following parametrically defined curves.

(a) $x = 3t - 5$
$y = 2t + 1$

(b) $x = 2t$
$y = t^2 - 1$

(c) $x = t - 2$
$y = 2t^2 + 1$

Question 7 [Trick question]

Find the Cartesian equation of the following.

(a) $x = 2t + 3$
$y = 5$

(b) $x = -2$
$y = t^3 + 1$

Question 8 [Importance of checking domain and range]

(a) Find the Cartesian equation of $(t^2, t^2 - 1)$.

(b) Bob claims that the graph is just the graph of $y = x - 1$ whereas Mary claims that it is only the right-hand side of the graph. By substituting a few values of t and plotting the resultant point, determine who is correct.

(c) Explain why they are not the same.

Question 9 Use a similar technique to Question 8 to find and sketch the Cartesian equation of the following. Remember to state any restrictions where necessary.

(a) $(3 - t^2,\ 2 + t^2)$ (b) $(\sqrt{t-1},\ t)$ (c) $(2,\ t^2 + 1)$

Challenge Problems

Problem 1 For the following Cartesian equations, find two possible parametric representations.

(a) $y = 2x + 3$ (b) $y = 4x^2 + 1$ (c) $x^2 + y^2 = 9$

Problem 2 [Folium of Descartes]

Consider the curve defined parametrically by

$$x = \frac{3t}{1+t^3}$$
$$y = \frac{3t^2}{1+t^3}$$

(a) Show that $\frac{y}{x} = t$.

(b) Deduce that $x^3 + y^3 = 3xy$.

(c) Use graphing software to produce a sketch of the *Folium of Descartes*.

Problem 3 [Parametrisation of the ellipse]

The ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are constants. Find a suitable parametrisation for the ellipse by modifying the standard parametrisation for the circle.

Problem 4 [More advanced algebraic parametrisations]

Eliminate the parameter in each of the following.

(a) $x = t + \frac{1}{t}$, $y = t^2 + \frac{1}{t^2}$

(b) $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$

Problem 5 [More advanced trigonometric parametrisations]

Eliminate the parameter in each of the following.

(a) $x = \sec\theta$, $y = \tan\theta$

(b) $x = \cos\theta + \sin\theta$, $y = \cos\theta - \sin\theta$

Chapter 1 Review

Further Functions

Review

Question 1 By first drawing a graph of $y = f(x)$, sketch a graph of $y = \dfrac{1}{f(x)}$.

(a) $f(x) = x + 1$

(b) $f(x) = x^2 + 2$

(c) $f(x) = x^2$

(d) $f(x) = x^2 - 4$

Question 2 By first drawing a graph of $y = f(x)$, sketch a graph of $y^2 = f(x)$.

(a) $f(x) = 2x - 4$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^2 - 16$

(d) $f(x) = 16 - x^2$

Question 3 By first drawing a graph of $y = f(x)$, sketch a graph of $y = |f(x)|$.

(a) $f(x) = 3x + 4$

(b) $f(x) = x^2 - 16$

(c) $f(x) = (x - 1)(x^2 - 4)$

(d) $f(x) = \sqrt{x} - 1$

Question 4 By first drawing a graph of $y = f(x)$, sketch a graph of $y = f(|x|)$.

(a) $f(x) = 6 - 2x$

(b) $f(x) = x^2 - 2x - 8$

(c) $f(x) = x^3 - 9x$

(d) $f(x) = \sqrt{x + 1}$

Question 5 By first drawing a graph of $y = f(x)$ and $y = g(x)$, sketch a graph of $y = f(x) + g(x)$.

(a) $f(x) = x,\ g(x) = -\sqrt{x}$

(b) $f(x) = x,\ g(x) = \sqrt{1 - x^2}$

(c) $f(x) = x^2,\ g(x) = \dfrac{1}{x}$

(d) $f(x) = \sqrt{x},\ g(x) = \dfrac{1}{x}$

Question 6 By first drawing a graph of $y = f(x)$ and $y = g(x)$, sketch a graph of $y = f(x)g(x)$.

(a) $f(x) = x,\ g(x) = x^2 + 1$

(b) $f(x) = x,\ g(x) = \sqrt{1 - x^2}$

(c) $f(x) = x^2,\ g(x) = \sqrt{1 - x^2}$

(d) $f(x) = x^2,\ g(x) = 4^{-x}$

Question 7

(a) Sketch the graph of $y = \sqrt{x} - 1$.

(b) Hence, sketch the graph of $y = \dfrac{1}{\sqrt{x} - 1}$.

Question 8 Solve the following inequalities.

(a) $x^2 \geq 25$

(b) $4x > x^2$

(c) $x^2 - x - 20 \leq 0$

(d) $12 - x - x^2 < 0$

Question 9 Solve the following inequalities.

(a) $\dfrac{2}{x-1} \geq 3$

(b) $\dfrac{3}{2x+1} \leq -1$

(c) $\dfrac{x}{x+1} \geq 2$

(d) $\dfrac{x-1}{x+1} \leq 4$

Question 10 Solve the following inequalities.

(a) $|x-2| \geq 5$

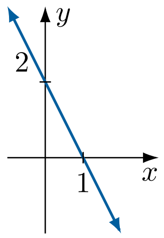
(b) $|2x+3| \leq 9$

(c) $|3-2x| > 7$

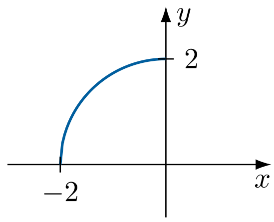
(d) $\left|\dfrac{3x+1}{2}\right| > 5$

Question 11 For each of the following graphs, sketch the inverse function.

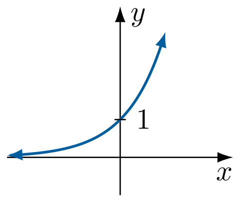
(a)



(b)



(c)



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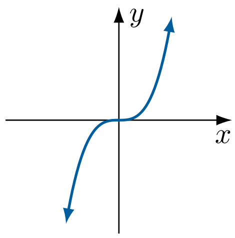
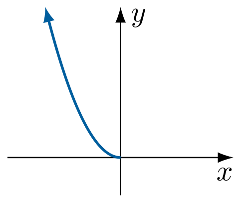
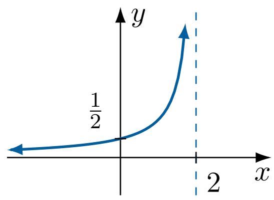


(e)



(f)



Question 12 For the following functions, find the equation of $f^{-1}(x)$ and hence show that

$$f^{-1}\left(f(x)\right) = f\left(f^{-1}(x)\right) = x$$

(a) $f(x) = 5 - x$

(b) $f(x) = 3x - 1$

(c) $f(x) = \sqrt{x}$

(d) $f(x) = \dfrac{1}{x-1}$

Question 13 Let $f(x) = x^2 - 8x$.

(a) Let $x \in [p, \infty)$ be a domain so that $f^{-1}(x)$ exists. Find the smallest value of p.

(b) Find the equation of the inverse and hence sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 14 Find the inverse of the following. If required, restrict the domain to contain only positive values of x.

(a) $y = x^2 - 6x + 14$

(b) $y = x^4 - 1$

(c) $y = \dfrac{2}{x^2}$

(d) $y = \sqrt{9 - x^2}$

Question 15

(a) Find the domain and range of $f(x) = \dfrac{3}{x-2}$.

(b) Find the equation of $f^{-1}(x)$.

(c) What are the x-coordinates of where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect?

(d) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 16 Eliminate the parameter t and hence find the Cartesian equation of the following.

(a) $x = 3t - 1$
$y = 2t + 3$

(b) $x = 2t$
$y = t^2 - 1$

(c) $x = 2at$
$y = at^2$

(d) $x = 2t - 1$
$y = t^2 - t$

(e) $x = 2\cos\theta$
$y = 2\sin\theta$

(f) $x = 2 + 2\cos\theta$
$y = -3 + 2\sin\theta$

Question 17 Write down the centre and radius of the circles defined parametrically by the following.

(a) $x = -2 + 5\cos\theta$
$y = 3 + 5\sin\theta$

(b) $x = 4 - 3\cos\theta$
$y = -1 + 3\sin\theta$

Question 18 Express the quadratic function $y = x^2 + 2x - 1$ in parametric form, given that $x = 2t - 1$.

Question 19 Show that the point $P\left(ap, \dfrac{a}{p}\right)$ is an appropriate parametrisation of $xy = a^2$.

Question 20 Draw the graph of the following by addition of ordinates.

(a) $y = |x| + |x - 2|$

(b) $y = |x| - |x - 2|$

Investigation Task

Further Reflections

So far, you have learned the following transformations (and their combinations), which require reflections.

$$y = -f(x), \quad y = |f(x)|$$
$$y = f(-x), \quad y = f(|x|)$$

This investigation task will take further the study of reflections.

Question 1 Create a function of your choice that lies both above and below the x-axis, and call it $f(x)$. Construct a graph of it using graphing software.

(a) Use graphing software to sketch $|y| = f(x)$ on the same set of axes as $y = f(x)$. Comment on your findings.

(b) Write down a set of instructions for a student on how to draw $|y| = f(x)$ for any given function.

(c) Bob makes the following argument.

> "Much like how $|x| = 5$ implies $x = \pm 5$, we can say that $|y| = f(x)$ implies $y = \pm f(x)$. So, the graph of $|y| = f(x)$ is just the positive and negative graphs on the same set of axes."

Is Bob's answer correct? If not, then is it partially correct or not-at-all correct? Give a detailed response and provide examples or counter-examples where necessary.

(d) Suppose $f(x) = -x^2 - 1$. Draw the graph of $|y| = f(x)$ and comment on your findings with justification. Repeat this for $f(x) = x^2 + 1$ and similarly comment on your findings.

Question 2 Create a function of your choice and call it $f(x)$. Construct a graph of it using graphing software.

(a) Use graphing software to sketch $y = f(4-x)$ on the same set of axes as $y = f(x)$. Comment on your findings.

(b) Write down a detailed set of instructions for a student on how to draw $y = f(a-x)$ for any given function and for various values of a.

(c) Explain why the graph of $y = f(a - x)$ has the effect that it does on the graph of $f(x)$.

Investigation Task

Parametric Curves

The following list contains well-known parametrically defined curves. Choose any four curves and research their parametric equations and use graphing software to plot them. Give a detailed description of the curve and how adjusting the constants affects the shape of the graph. Some graphs retain their general shape when the constants are adjusted whereas other graphs may change shape drastically (possibly more than once) based on what the constant is. For those examples, all cases are to be investigated.

Astroid	Cardioid	Catenary	Cayley's Sextic
Cochleoid	Conchoid	Epicycloid	Epitrochoid
Fermat's Spiral	Folium of Descartes	Hypocycloid	Hypotrochoid
Lemniscate	Lissajous Curves	Nephroid	Rhodonea Curves
Tractrix	Tricuspoid	Tschirnhaus's Cubic	Witch of Agnesi

2

POLYNOMIALS

- Remainder and factor theorem
- Odd and even polynomials
- Sum and product of roots (quadratic)
- Sum and product of roots (cubic and quartic)
- Roots of multiplicity
- Graphing polynomials

Exercise 2A

Remainder and factor theorem

Fundamentals

Fundamentals 1

(a) If a polynomial $P(x)$ is divided by a polynomial divisor $A(x)$ giving a quotient $Q(x)$ and remainder $R(x)$ then this division process can be expressed as the identity $P(x) =$ ____________. This is called the d________ transformation.

(b) The degree of $R(x)$ is always less/more/equal (circle one) than the degree of $A(x)$.

Fundamentals 2

(a) The value(s) of x that satisfy the equation $P(x) = 0$ are called the r______ of the polynomial.

(b) If we divide a polynomial $P(x)$ by $(x - \alpha)$ and the remainder is zero, then we say that $P(x)$ is d________ by $(x - \alpha)$. In other words, $(x - \alpha)$ is a f________ of $P(x)$.

(c) This also means that $x = \alpha$ is a r______ of $P(x) = 0$.

(d) If we divide $P(x)$ by $(x - \alpha)(x - \beta)$ and the remainder is zero, then we can conclude that both $(x - \alpha)$ and $(x - \beta)$ are f________ of $P(x)$ and hence $x =$ ___ and ___ are roots of $P(x) = 0$.

(e) If a cubic polynomial has roots $x = \alpha$, β and γ, then the polynomial can be expressed as $P(x) = a($_____$)($_____$)($_____$)$.

(f) In general, if an n-degree polynomial has roots $x = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then the polynomial has form $P(x) = a$____________________.

Fundamentals 3

(a) State the remainder theorem.

(b) State the factor theorem.

(c) If for a polynomial $P(x)$, $P\left(-\dfrac{b}{a}\right) = 0$, then ________ is a factor, and vice versa.

Fundamentals 4

If a polynomial $P(x)$ has all integer coefficients, then any integer roots *must* be factors of the c________ term of $P(x)$.

Question 1 Use polynomial long division to express the following in terms of the division transformation $P(x) = A(x)Q(x) + R(x)$.

(a) $(x^3 - x^2 + 2x + 1) \div (x + 1)$

(b) $(x^3 - 5x + 3) \div (x - 2)$

(c) $(x^4 - 3x^2 + x - 2) \div (x - 1)$

(d) $(x^4 - 2x^3 + 3x^2 + 3x + 1) \div (x^2 - x + 1)$

Question 2 Let $P(x) = x^3 - 3x - 2$.

(a) Show that $x + 1$ is a factor of $P(x)$.

(b) Complete the statement $P(x) = (x + 1)(\underline{\qquad\qquad})$.

(c) Hence, by factorising the resultant quadratic, express $P(x)$ as a product of linear factors.

Question 3 Let $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$.

(a) Show that $x + 4$ is a factor of $P(x)$.

(b) Show that $x - 1$ is also a factor of $P(x)$.

(c) Write down a quadratic factor of $P(x)$.

(d) Complete the statement $P(x) = (x^2 + 3x - 4)(\underline{\qquad\qquad})$.

(e) Hence, express $P(x)$ as a product of linear factors.

(f) Write down the solutions to $P(x) = 0$.

Question 4 Suppose $(x - 3)$ is a factor of $P(x) = 2x^3 - 9x + m$.

(a) Find the value of m.

(b) Use long division to find the polynomial $Q(x)$ such that $P(x) = (x - 3)Q(x)$.

(c) Hence, solve $P(x) = 0$.

Question 5 For what value of m is $P(x) = 2x^4 + mx^2 - 4$ divisible by $(x + 2)$?

Question 6 By first expressing the polynomial $P(x)$ as the product of linear factors and then expanding, find the equation of the *monic* polynomial with the following roots.

(a) $x = 0, \pm 2$

(b) $x = 0, \pm\sqrt{2}$

(c) $x = -3, -1, 2$

(d) $x = 1, 1 \pm \sqrt{2}$

Question 7 Suppose a polynomial $P(x)$ is divided by $A(x)$ and yields the remainder $R(x)$. Write down the most general form for the remainder $R(x)$ if the divisor $A(x)$ is

(a) linear of form $(x + \alpha)$

(b) linear of form $(\alpha x + \beta)$

(c) a quadratic

(d) a cubic

Question 8 If the divisor $A(x)$ is a quadratic, then what is the general form of the remainder $R(x)$? Does this imply necessarily that the remainder must be linear?

Question 9 Use polynomial long division to find the remainder when $P(x) = 2x^3 - 7x^2 - 2x + 4$ is divided by the following linear factors, and then verify your result using the remainder theorem.

(a) $(x + 2)$

(b) $(2x + 1)$

Question 10 Use the remainder theorem to find the remainder when

(a) $(x^4 - 5x^3 + 3x^2 - 7)$ is divided by $(x - 2)$

(b) $(2x^3 - 6x^2 + x - 4)$ is divided by $(2x - 1)$

Question 11 When $P(x)$ is divided by $(x^2 - 9)$, the remainder is $(2x + 5)$.

(a) Write the above in the form of the division transformation $P(x) = A(x)Q(x) + R(x)$.

(b) Hence, find the remainder when $P(x)$ is divided by $(x + 3)$.

Question 12

(a) Show that $P(x) = 2x^3 + x^2 - 13x + 6$ is divisible by $(2x - 1)$ and $(x + 3)$.

(b) Hence, find all the zeros of $P(x)$.

Question 13 Find the values of m and n if $(x - 1)$ and $(x - 2)$ are factors of $P(x) = x^4 + mx + n$.

Question 14

(a) Find the values of m if $P(x) = x^4 + 2x^3 - x^2 - 8x - m$ is divisible by $x^2 - 4$.

(b) Hence, find all the zeros of $P(x)$.

Question 15 Let $(x - 1)$ and $(x - 2)$ be factors of $P(x) = 2x^3 + px^2 + qx - 2$.

(a) Find the values of p and q.

(b) Hence, find the zeros of $P(x)$.

Question 16 Suppose $(x + 1)$ and $(x + 3)$ are factors of $P(x) = px^3 + qx^2 + 5x + 6$. Find the values of p and q, and hence find the zeros of $P(x)$.

Question 17

(a) Show that $(2x - 1)$ is a factor of $P(x) = 6x^3 + 23x^2 - 33x + 10$.

(b) Hence, fully factorise $P(x)$ and hence, state the zeros.

Question 18 [Faster factorising technique]

Suppose $6x^3 + ax^2 - 11x + 15 = (2x - 5)(px^2 + qx + r)$. Find the values of a, p, q and r.

Question 19 Let $P(x) = x^3 - 6x + 4$

(a) State all the possible integer zeros of $P(x)$.

(b) Find a root of $P(x)$ and hence state the corresponding linear factor of $P(x)$.

(c) Use polynomial long division, or otherwise, to express $P(x)$ as the product of a linear and quadratic factor.

(d) Hence, find all the zeros of $P(x)$.

Question 20 Use a similar technique to factorise and hence find all the zeros of the polynomial $P(x) = x^4 - x^3 - 5x^2 + 12$, given that it has an integer zero.

Question 21 Show that $P(x) = x^3 + x + 3$ cannot have an integer zero.
Hint: What are the only possible integer zeros?

Challenge Problems

Problem 1 An unknown polynomial gives a remainder of 1 when divided by $(x + 1)$, and a remainder of 5 when divided by $(x - 3)$. What is the remainder when the same polynomial is divided by $(x + 1)(x - 3)$?

Problem 2 When a polynomial $P(x)$ is divided by $(x - 2)(x + 5)$, the remainder is $x + 3$. Find the remainder when $P(x)$ is divided by $(x + 5)$.

Problem 3 The polynomial $P(x) = x^n - 1$ has zeros $x = 1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$.

(a) Expand $(x - 1)(1 + x + x^2 + \ldots + x^{n-1})$.

(b) Hence, show that $(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \ldots (1 - \alpha_{n-1}) = n$

Problem 4 [Proof of the integer zero theorem]

Define the polynomial

$$P(x) = c_n x^n + \ldots + c_2 x^2 + c_1 x + c_0$$

with integer coefficients and an integer zero α.

(a) Show that $c_0 = -\left(c_n \alpha^n + \ldots + c_2 \alpha^2 + c_1 \alpha\right)$.

(b) Deduce that α divides c_0. In other words, prove that α is a factor of c_0.

Exercise 2B

Odd and even polynomials

Fundamentals

Fundamentals 1

(a) If a polynomial $P(x) = ax^3 + bx^2 + cx + d$ is odd, what can you say about the coefficients a, b, c, d?

(b) If a polynomial $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, what can you say about the coefficients a, b, c, d and e?

(c) If a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ is even, what can you say about the powers of x.

(d) If a polynomial $P(x)$ is even it's axis of symmetry is __ axis

(e) If a polynomial $P(x)$ is even if α is a root so is __. If $-\beta$ is a root then so is __

(f) If a polynomial $P(x)$ is odd explain the symmetry one can observe with its graph.

(g) For an odd polynomial $P(x)$ explain what you notice about its zeros.

Fundamentals 2

A polynomial of odd degree always passes through the _____ and has at least one _____.

Question 1 Write down the form of every monic even quadratic polynomial.

Question 2 Write down the equation of the odd monic cubic polynomial that has a root at $x = -4$.

Question 3 Write down the equation of the even monic quartic polynomial that has zeroes at 3 and -3 and when divided by $x+2$, the remainder is -15. Write down the factors of this polynomial.

Question 4 $P(x)$ is an odd polynomial and $P(3) = -5$, find the value of

(a) $P(3) + P(-3)$ (b) $P(3) \times P(-3)$ (c) $P(3) \times P(-3) \times P(0)$

Question 5 $P(x)$ is an even polynomial and $P(3) = -5$, find the value of

(a) $P(3) + P(-3)$ (b) $P(3) \times P(-3)$

Question 6 A polynomial has zeros at -1 and 5. Write possible polynomials that satisfy the following conditions

(a) Monic and even.

(b) Monic and odd.

Challenge Problems

Problem 1 If a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is odd, what can you say about the powers of x and the constant a_0? Prove your statement using the definition of an odd function.

Problem 2 A monic even polynomial $P(x)$ of degree four has a zero at 2 and at only one other value of x

(a) Write down the value of the other zero.

(b) Write down the general form of all such polynomials $P(x)$.

(c) If $P(1) = 9$ determine the polynomial.

Exercise 2C

Sum and product of roots (quadratic)

Fundamentals

Fundamentals 1
Suppose the quadratic equation $ax^2 + bx + c = 0$ has roots α and β. Write down the formula for the

(a) sum of roots.

(b) product of roots.

Fundamentals 2

(a) Write down the equation of the monic quadratic equation that has roots α and β.

(b) Hence, the monic quadratic equation with roots α and β can also be written as

$$x^2 - \_\_\_\_\_ x + \_\_\_\_\_ = 0$$

Question 1 Let the roots of the following equations be α and β. Write down the value of $\alpha + \beta$ and $\alpha\beta$.

(a) $x^2 - 2x - 5 = 0$

(b) $4x^2 + 5x - 1 = 0$

Question 2 Recall that the equation of the monic quadratic polynomial with roots α and β is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Use this to find the quadratic equation, with integer coefficients, whose roots are

(a) ± 5

(b) 3 and -5

(c) $2 \pm \sqrt{3}$

(d) $\frac{2}{5}$ and $\frac{5}{2}$

Question 3 Let the roots of $4x^2 - 5x - 1 = 0$ be α and β. Find the value of the following, without explicitly solving for α and β.

(a) $\alpha + \beta$

(b) $\alpha\beta$

(c) $\frac{1}{\alpha} + \frac{1}{\beta}$

(d) $(\alpha - 3)(\beta - 3)$

(e) $\alpha^2 + \beta^2 + 2\alpha\beta$

(f) $\alpha^2\beta + \beta^2\alpha$

Question 4

(a) Expand $(\alpha + \beta)^2$.

(b) Hence, express $\alpha^2 + \beta^2$ in terms of $(\alpha + \beta)$ and $\alpha\beta$.

(c) Let the roots of $x^2 - 3x - 2 = 0$ be α and β. Find the value of $\alpha^2 + \beta^2$.

Question 5 Let the roots of $4x^2 - 5x - 1 = 0$ be α and β. Find the value of the following, without explicitly solving for α and β.

(a) $\alpha^2 + \beta^2$

(b) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

(c) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(d) $(\alpha - \beta)^2$

(e) $\left(\beta + \dfrac{1}{\alpha}\right)\left(\alpha + \dfrac{1}{\beta}\right)$

(f) $\left(\alpha + \dfrac{1}{\alpha}\right)\left(\beta + \dfrac{1}{\beta}\right)$

Question 6 Suppose that the quadratic equation $x^2 - kx - 4x - 35 = 0$ has the sum of roots being equal to 2. Find the value of k.

Question 7 The line $y = 2x + 3$ is tangential to $y = x^2 - kx$ at $x = 2$. Find the value of k.

Question 8 Write down the quadratic equation whose roots sum to -3, and multiply to -40.

Question 9 Consider the quadratic equation $px^2 + qx + r = 0$. Show that

(a) if the roots are opposite in sign but equal in magnitude, then $q = 0$.

(b) if the sum of roots is equal to the product of roots, then $q + r = 0$.

(c) if the roots are reciprocals of each other, then $p = r$.

Question 10 Let $3 - \sqrt{2}$ be a root of the quadratic equation $x^2 + mx + 7 = 0$, where m is a rational number. Find the

(a) second root.

(b) value of m.

Question 11

(a) Suppose the quadratic equation $4x^2 - px - 5 = 0$ has two roots opposite in sign but equal in magnitude. Find the value of p.

(b) Suppose the quadratic equation $x^2 + 5x + 3p - 1 = 0$ has two roots which are reciprocals of each other. Find the value of p.

Question 12 Let α be one of the roots of a general quadratic equation that has two roots that are reciprocals of each other. Write down the general equation of this quadratic.

Question 13 [Proof of the root-coefficient relationship]

Let α and β be the roots of the general quadratic equation $a^2 + bx + c = 0$.

(a) Complete the following.

$$ax^2 + bx + c = a(x - \_\_)(x - \_\_)$$

(b) Hence, prove the results $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Question 14 [Alternative proof of the root-coefficient relationship]

Let α and β be the roots of the general quadratic $a^2 + bx + c = 0$. Suppose $\alpha \geq \beta$.

(a) Use the quadratic formula to write down the value of α and β in terms of a, b and c.

(b) Hence, prove the results $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Question 15 Find the value(s) of m for which the quadratic equation

(a) $x^2 - (m - 5)x + 4 = 0$ has equal roots, using two different methods.

(b) $8x^2 - 3x + 4m + 9 = 0$ has one root equal to zero.

Question 16 Find the value of m if

(a) $x^2 - mx + 18 = 0$ has one root double the other.

(b) $x^2 + mx + 12 = 0$ has one root triple the other.

(c) $x^2 + (m - 3)x + (m + 9) = 0$ has the product of roots being twice the sum of the roots.

Question 17 One root of the equation $x^2 + px + q = 0$ is three times the other. Show that $3p^2 = 16q$.

Question 18 Let the roots of $x^2 - 3x + 10 = 0$ be α and β. Find the equation of the quadratic with roots $\alpha + 2$ and $\beta + 2$.

Question 19 If α and β are the roots of the quadratic equation $5x^2 + mx - 5 = 0$, then find the exact value of $(5\alpha + m)(5\beta + m)$.

Question 20 The two roots of $x^2 - px + q = 0$ differ by 2, and the two roots are both positive. Find p and q if $4p = 3q$.

Question 21 Let α and β be the roots of $x^2 - x + 4 = 0$.

(a) Find the value of $\alpha^2 + \beta^2$.

(b) Explain why this implies that the roots are not real.

Challenge Problems

Problem 1 Let α and β be the roots of $2x^2 - 8x + 1 = 0$. Form the quadratic equation whose roots are

(a) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$

(b) α^2 and β^2

Problem 2 Let the quadratic equation $ax^2 + bx + c = 0$ have roots α and β.

(a) Find an expression for $(\alpha - \beta)^2$ in terms of the sum and product of roots.

(b) Hence, show that $\alpha - \beta = \pm\dfrac{\sqrt{\Delta}}{a}$, where Δ is the discriminant.

(c) Verify your result in (b) using the quadratic formula.

Problem 3 Show that if the roots of $ax^2 + bx + c = 0$ differ by 1, then $b^2 - a^2 = 4ac$.

Exercise 2D

Sum and product of roots (cubic and quartic)

Fundamentals

Fundamentals 1

If α, β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

(a) $\alpha + \beta + \gamma = \_\_\_\_\_$ (b) $\alpha\beta + \alpha\gamma + \beta\gamma = \_\_\_\_\_$ (c) $\alpha\beta\gamma = \_\_\_\_\_$

Fundamentals 2

If α, β, γ and δ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

(a) $\alpha + \beta + \gamma + \delta = \_\_\_\_\_$ (b) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \_\_\_\_\_$

(c) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \_\_\_\_\_$ (d) $\alpha\beta\gamma\delta = \_\_\_\_\_$

Fundamentals 3

Useful identities to remember

(a) $\alpha^2 + \beta^2 + \gamma^2 = (\_\_\_\_\_\_\_\_\_)^2 - 2(\_\_\_\_\_\_\_\_\_\_\_)$

(b) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\_\_\_\_\_\_\_\_\_\_\_)^2 - 2(\_\_\_\_\_\_\_\_\_\_\_\_\_)$

(c) Generalising the above, we have

$$\text{Sum of squares} = (\text{Sum})^2 - 2(\_\_\_\_\_\_\_\_\_\_\_\_\_)$$

Question 1 Let α, β and γ be the roots of $2x^3 - 6x^2 + x - 4 = 0$. Find the values of

(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \alpha\gamma + \beta\gamma$ (c) $\alpha\beta\gamma$

Question 2 Let α, β and γ be the roots of $x^3 - 6x^2 + 10x - 4 = 0$. Find the values of

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ (b) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ (c) $\alpha^2 + \beta^2 + \gamma^2$

(d) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ (e) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$ (f) $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\alpha^2\gamma^2} + \dfrac{1}{\beta^2\gamma^2}$

Question 3 Two of the roots of $x^3 - 2x^2 - 9x + 18 = 0$ sum to zero.

(a) Since two of the roots sum to zero, we can let the roots be ___, ___, and β.

(b) Use the sum of roots to show that $\beta = 2$.

(c) Write down the corresponding linear factor.

(d) Use polynomial long division, or otherwise, to complete the following.

$$x^3 - 2x^2 - 9x + 18 = (x-2)(\underline{\hspace{4cm}})$$

(e) Hence, find the roots of $x^3 - 2x^2 - 9x + 18 = 0$.

Question 4 A set of numbers is said to be in *arithmetic progression* if the difference between successive terms is constant. For example, the set $\{1, 4, 7\}$ is an arithmetic progression with common difference 3. The polynomial equation $x^3 - 3x^2 - x + 3 = 0$ has its roots in arithmetic progression.

(a) Let the roots be $\alpha - d$, α and $\alpha + d$, where d is the common difference. Use the sum of roots to find one of them.

(b) Write down the corresponding linear factor.

(c) Use polynomial long division, or otherwise, to complete the following.

$$x^3 - 3x^2 - x + 3 = (x-1)(\underline{\hspace{4cm}})$$

(d) Hence, find the roots of $x^3 - 3x^2 - x + 3 = 0$.

Question 5 A set of numbers is said to be in *geometric progression* if the ratio between successive terms is constant. For example, the set $\{2, 6, 18\}$ is a geometric progression with common ratio 3. The polynomial equation $x^3 - 7x^2 + 14x - 8 = 0$ has its roots in geometric progression.

(a) Let the roots be $\frac{\alpha}{r}$, α and αr, where r is the common ratio. Use the product of roots to find one of them.

(b) Write down the corresponding linear factor.

(c) Use polynomial long division, or otherwise, to complete the following.

$$x^3 - 7x^2 + 14x - 8 = (x-2)(\underline{\hspace{4cm}})$$

(d) Hence, find the roots of $x^3 - 7x^2 + 14x - 8 = 0$.

Question 6 Find the roots of the following polynomial equations, given the fact that

(a) the roots of $x^3 - 6x^2 + 3x + 10 = 0$ are in arithmetic progression.

(b) the roots of $3x^3 - 14x^2 - 28x + 24 = 0$ are in geometric progression.

(c) the roots of $x^3 - x^2 - 24x - 36 = 0$ have one root being the product of the other two.

(d) the roots of $2x^3 + x^2 - 13x + 6 = 0$ have one root being the reciprocal of another.

Question 7 Two of the roots of $ax^3 + bx^2 + cx + d = 0$ are equal in magnitude but opposite in sign.

(a) Show that one of the roots is $x = -\dfrac{b}{a}$.

(b) By substituting this back into the polynomial, show that $ad = bc$.

Question 8

(a) The polynomial $P(x) = x^3 + px^2 + q$ has its zeros in an arithmetic progression. Show that $2p^3 + 27q = 0$.

(b) The polynomial $P(x) = x^3 + px^2 + qx + r$ has its zeros in a geometric progression. Show that $p^3 r = q^3$.

(c) The polynomial $P(x) = x^3 + px + q$ has its two of its zeros being reciprocals of each other, and $q \neq 0$.
Show that $p = 1 - q^2$.

(d) The polynomial $P(x) = x^3 + px^2 + qx + r$ has two zeros being negatives of each other. Show that $pq = r$.

Question 9

(a) Let α, β and γ be the roots of $2x^3 - 4x^2 + 3x - 1 = 0$. Find the cubic equation whose roots are 2α, 2β and 2γ.

(b) Let $x^3 - 3x^2 + 5x + 2 = 0$ have zeros α, β and γ. Find the equation of the polynomial equation with roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$.

Question 10 The polynomial equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ and δ.

(a) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = a^2 - 2b$.

(b) Hence, show that $x^4 - 3x^3 + 5x^2 + 4x - 1 = 0$ cannot have four real roots.

Question 11 The polynomial equation $x^4 + px^3 + qx^2 + rx + s = 0$ has two roots that sum to zero, and another two roots being reciprocals of each other. Show that

(a) $q = 1 + s$

(b) $r = ps$

Question 12 Let $x^3 + 5x - 1 = 0$ have roots α, β and γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

(b) Hence, state the number of x-intercepts of the graph of $y = x^3 + 5x - 1$.

Challenge Problems

Problem 1 Let $x^3 + px + q = 0$ have roots α, β and γ. Find the following in terms of p and q.

(a) $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$

(b) $(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)$

Problem 2 Let $x^3 - 4x + 2 = 0$ have roots α, β and γ.

(a) Show that $\alpha^3 = 4\alpha - 2$.

(b) Hence, find the value of $\alpha^3 + \beta^3 + \gamma^3$.

(c) Use part (a) to find the value of $\alpha^4 + \beta^4 + \gamma^4$.

Problem 3 Let the polynomial equation $x^3 + px^2 + qx + r = 0$ have roots α, β and γ.

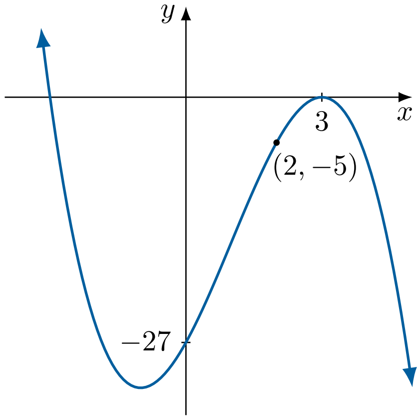
(a) Find in terms of p, q and r the values of $\alpha + \beta + \gamma$, $\alpha^2 + \beta^2 + \gamma^2$.

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = -p^3 + 3pq - 3r$.
Hint: Use a similar technique to Problem 2.

(c) Hence, find the solution to the system of equations

$$\begin{aligned} a + b + c &= -1 \\ a^2 + b^2 + c^2 &= 5 \\ a^3 + b^3 + c^3 &= -7 \end{aligned}$$

Problem 4 The diagram below shows a sketch of $y = -x^3 + 3x^2 + 9x - 27$.



(a) At $(0, -27)$, a tangent ℓ_1 is drawn to the curve. Find the equation of the tangent and find where the tangent cuts the curve again.

(b) Another tangent ℓ_2 is drawn at $(2, -5)$. Show that ℓ_1 and ℓ_2 are parallel, and hence find the equation of ℓ_2.

(c) Find where ℓ_2 intersects the curve again.

Problem 5 [Discriminant of the depressed cubic]

Let the polynomial equation $x^3 + px + q = 0$ have roots α, β and γ. Show that

(a) $(\alpha - \beta)^2 = -3\gamma^2 - 4p$

(b) $(\alpha - \beta)^2(\beta - \gamma)^2(\alpha - \gamma)^2 = -4p^3 - 27q^2$

(c) Hence, or otherwise, show that if $p > 0$, then the polynomial equation cannot have three real roots.

Exercise 2E

Roots of multiplicity

Fundamentals

Fundamentals 1
If a polynomial has equation $P(x) = (x-\alpha)^2(x-\beta)^n$ and $\alpha \neq \beta$, then we say that α is a root of multiplicity ___ but β is a root of multiplicity ___.

Fundamentals 2
If a polynomial $P(x)$ has α being a root of multiplicity n, then $P'(x)$ will have α being a root of multiplicity ____

Note: This exercise is to be attempted after covering Differentiation in the Year 11 Advanced Mathematics course.

Question 1 Find the equation of the quartic polynomial equation with a triple root at $x = 3$, single root at $x = -2$ and y-intercept -27.

Question 2 The polynomial $P(x) = px^4 + qx^3 - 2x^2 + 4x + 6$ has a double root at $x = -1$. Find the value of p and q.

Question 3 The polynomial equation $x^3 - x^2 - 8x + m = 0$ has an integer double root.

(a) Find the value of m.

(b) Hence, find the roots of the polynomial equation.

Question 4 Let $P(x) = 4x^3 - 8x^2 - 11x - 3$.

(a) Show that $x = -\dfrac{1}{2}$ is a double root of $P(x)$.

(b) Hence, find the third root.

Question 5 Find all the roots of $P(x) = x^4 - 6x^2 + 8x - 3$ given that it has a triple root.

Question 6 The polynomial $P(x) = x^4 + px^3 + qx^2 - 5x + 1$ has $x = 1$ being a double root. Find the value of p and q.

Question 7 Show that if $ax^3 + bx^2 + cx + d = 0$ has a triple root, then it must occur at $x = -\dfrac{b}{3a}$.

Question 8 The polynomial $P(x) = x^3 + 6x^2 + 9x + p$ has a double root. Find the value of p.

Question 9 The polynomial $P(x) = x^4 + px^2 + q$ has a double root at $x = \alpha$.

(a) Show that $x = -\alpha$ is also a double root of $P(x)$.

(b) Hence, show that $p^2 = 4q$.

Challenge Problems

Problem 1 The quartic polynomial $P(x) = x^4 + mx^3 + nx^2 + mx + 1$ has $x = \alpha$ being a double root.

(a) Show that $x = \dfrac{1}{\alpha}$ is also a root of $P(x)$.

(b) Find the fourth root.

(c) Hence, show that $m^2 = 4n - 8$.

Problem 2 [Proof of the multiple root theorem]

Let $P(x)$ be a polynomial with a zero $x = \alpha$ of multiplicity n.

(a) Complete the following.

$$P(x) = (\_\_\_\_\_\_\_\_)^{\_\_} \times Q(x),$$

where $Q(x)$ is some polynomial.

(b) Hence, show that $P'(x)$ has a zero $x = \alpha$ of multiplicity $n - 1$.

Problem 3 Let $ax^3 + bx^2 + c = 0$ have a non-zero double root. Prove that $27a^2c + 4b^3 = 0$.

Problem 4 [Discriminant of the depressed cubic]

Let $x^3 + px + q = 0$, where $p \neq 0$ and $q \neq 0$ have a double root at $x = \alpha$.

(a) Show that $\alpha^2 = -\dfrac{p}{3}$.

(b) Explain briefly why $\alpha^3 + p\alpha + q = 0$.

(c) Hence, show that $\alpha = -\dfrac{3q}{2p}$

(d) Deduce that $4p^3 + 27q^2 = 0$.

Problem 5 [Proof by contradiction]

Define the polynomial

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},$$

where $k!$ (read as "k factorial") is defined to be equal to $k \times (k-1) \times \ldots \times 3 \times 2 \times 1$.

Show that $P(x)$ cannot have a multiple root for any integer $n \geq 2$.

Exercise 2F

Graphing polynomials

Fundamentals

Fundamentals 1

The graph of a polynomial $y = P(x)$

(a) intersects the x-axis at the z_____ of the polynomial.

(b) crosses the x-axis at the roots that have o___ multiplicity.

(c) bounces off the x-axis at the roots that have e________ multiplicity.

Fundamentals 2

Consider the graph of $y = P(x)$, where $P(x)$ is a polynomial of even degree.

(a) The extremeties of the graph are either both p________ or n________ infinity.

(b) If the leading coefficient is positive, then the graph begins and ends at positive/negative (circle one) infinity.

(c) If the leading coefficient is negative, then the graph begins and ends at positive/negative (circle one) infinity.

Fundamentals 3

Consider the graph of $y = P(x)$, where $P(x)$ is a polynomial of odd degree.

(a) The extremeties of the graph begin and end at o__________ sides of each other.

(b) If the leading coefficient is positive, then the graph begins positive/negative (circle one) and ends positive/negative (circle one).

(c) If the leading coefficient is negative, then the graph begins positive/negative (circle one) and ends positive/negative (circle one).

Fundamentals 4

Name all the features that may be useful to consider when graphing a polynomial.

Question 1 Sketch the following polynomials.

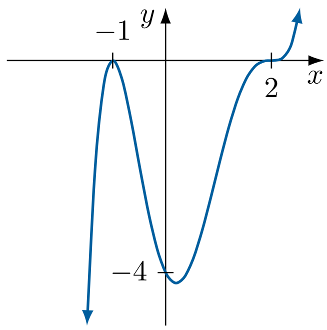
(a) $y = (x - 1)^2(x + 2)$

(b) $y = (x + 3)^2(x - 2)^3$

(c) $y = (x + 1)(1 - x)(x - 2)^2$

(d) $y = x(x + 1)^3(x - 2)^2$

Question 2 The diagram below shows the graph of a polynomial $y = P(x)$ with roots at $x = -1$ and $x = 2$.



(a) Complete the following general form of the polynomial.

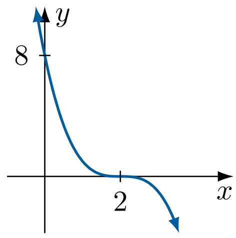
$$y = a(x + \underline{\quad})^{\underline{\quad}}(x - \underline{\quad})^{\underline{\quad}}$$

(b) Find the value of a by using the y-intercept.

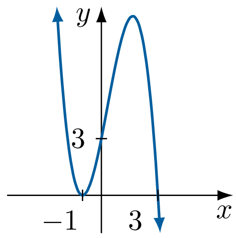
(c) Hence, state the equation of the polynomial graph.

Question 3 Use a similar technique to Question 2 to find the equation of the following polynomial graphs.

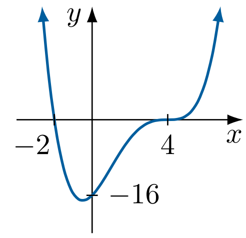
(a)



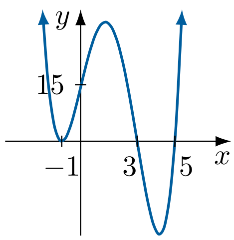
(b)



(c)



(d)



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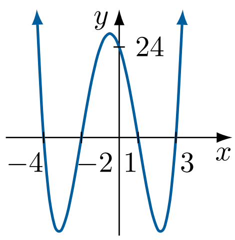

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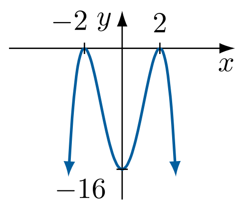

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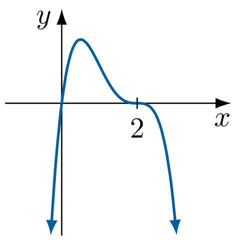

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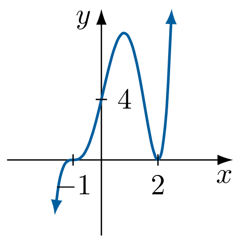

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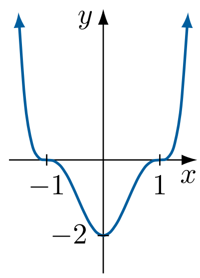


Question 4 For each of the graphs in Question 3, state the values of x where $y \geq 0$.

Question 5 Sketch the graphs of the following polynomials.

(a) $y = x(x-1)^2$

(b) $y = x(x-1)^3$

(c) $y = x(x-1)^4$

(d) $y = x(x-1)^5$

Question 6

(a) Use graphing software to sketch $y = x$, $y = x^2$ and $y = x^3$ for $x \geq 0$.

(b) In the domain $0 < x < 1$, determine whether increasing the power makes the curve go closer or further from the x-axis.

(c) In the domain $x > 1$, determine whether increasing the power makes the curve go closer or further from the x-axis.

(d) Hence, for the following pairs of functions in the given domains below, determine which curve is higher.

(i) $y = x^2$, $y = x^4$, $x > 1$

(ii) $y = x^3$, $y = x^5$, $0 < x < 1$

Challenge Problems

Problem 1 [Applications to inequalities]

Solve the following inequalities by first drawing a sketch.

(a) $(x+1)(1-x)(x-2)^2 \geq 0$

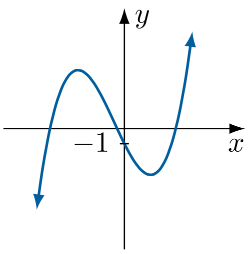
(b) $x(x+1)^3(x-2)^2 < 0$

(c) $x^3 - 4x > 0$

(d) $x^4 - 5x^2 + 4 < 0$

Problem 2 Consider the polynomial $P(x) = ax^3 + bx^2 + cx - 1$, where $a, b > 0$. Which of the following is a potential graph of $y = P(x)$?

(a)



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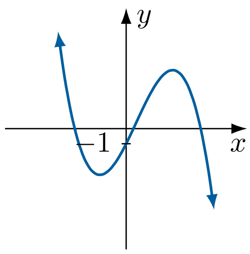

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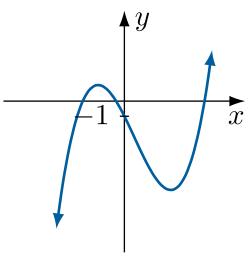

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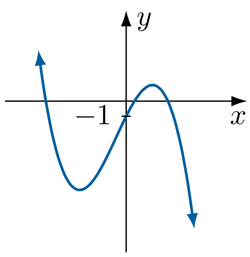


Chapter 2 Review

Polynomials

Review

Question 1 Write down the equation of the quadratic equation whose roots are

(a) 1 and -7

(b) $3+\sqrt{5}$ and $3-\sqrt{5}$

Question 2

(a) Find the value of m if $(m+2)x^2 - 8x + 4m = 0$ has -2 being a root.

(b) Hence, find the roots of the equation.

Question 3 Let α and β be the roots of $3x^2 - 2x + 9 = 0$. Find the value of

(a) $\alpha^2 + \beta^2$

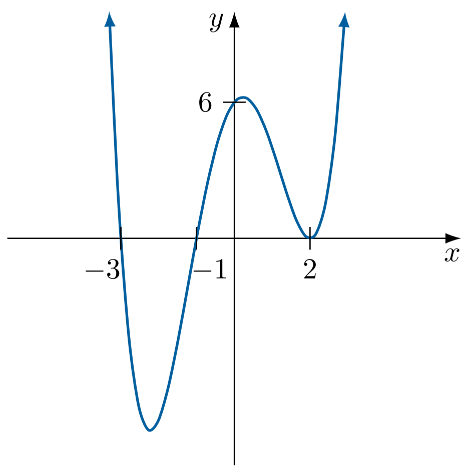
(b) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

Question 4 The equation $x^2 - (m+1)x + 2m + 2 = 0$ has two non-zero roots, with one being twice the other. Find the roots.

Question 5 Factorise $x^4 + x^2 + 1$, given that one factor is $x^2 + x + 1$.

Question 6 Solve the polynomial equation $x^3 + 2x^2 - 5x - 6 = 0$.

Question 7 The diagram shows the graph of a polynomial with two single roots and a double root. Find the equation of the polynomial, in factorised form.



Question 8 Solve $24x^3 - 14x^2 - 63x + 45 = 0$ if $x = \dfrac{3}{2}$ is a root.

Question 9 Find the value of m in $x^3 - 5x^2 + 3x + m = 0$, given that it has an integer double root.

Question 10 Let $x^3 + px^2 - qx + 1 = 0$ have two of its roots being 2 and $\frac{1}{2}$.

(a) Find the third root.

(b) Hence, find the value of p and q.

Question 11 Let two of the roots of $x^3 + px^2 + q = 0$, where $p, q \neq 0$, be reciprocals of each other. Show that $p = q - \frac{1}{q}$.

Question 12 The roots of $x^3 - 12x^2 + 12x + k = 0$ form an arithmetic progression.

(a) Find the value of k.

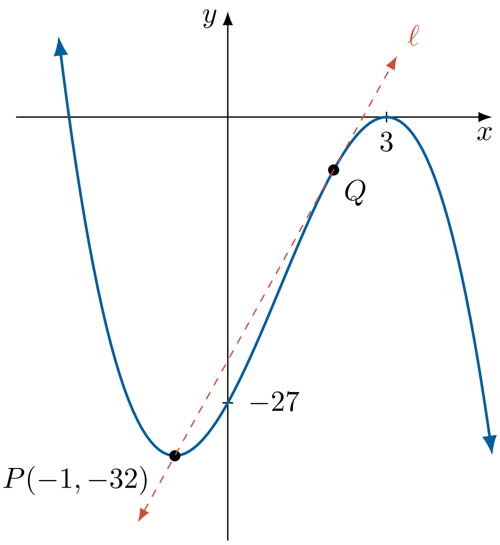
(b) Hence, find the roots of the equation.

Question 13 Consider the equation $x^3 + px^2 + qx + r = 0$. Show that

(a) if one of the roots is equal to the sum of the other two, then $p^3 = 4(pq - 2r)$.

(b) if the roots form a geometric progression, then $q^3 = p^3 r$.

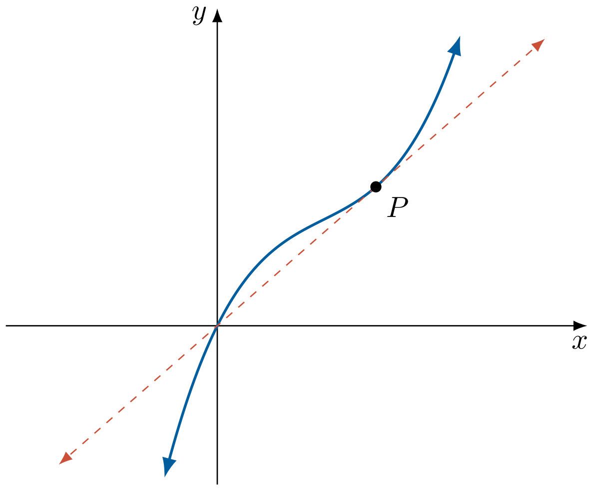
Question 14 The point $P(-1, -32)$ is a point on the curve $y = -x^3 + 3x^2 + 9x - 27$. A line ℓ from P touches the curve at Q.



(a) If the gradient of ℓ is m, then show that the equation of ℓ is $y = mx + m - 32$.

(b) Find the coordinates of Q.

Question 15 The line $y = mx$ intersects the curve $y = x^3 - 3x^2 + 4x$ at the origin. The line also touches the curve at P, as shown in the diagram below.



Find the equation of the straight line and the coordinates of P.

Investigation Task

Rational Root Theorem

In this course, the *integer root theorem* is covered. It states the following

Let $P(x)$ be a polynomial with integer coefficients. If $P(x) = 0$ has an integer root α, then α must be a factor of the constant term of $P(x)$.

This investigation task aims to allow students to explore the *rational root theorem*, which is an extension of the integer root theorem. We begin with a proof of the integer root theorem.

Question 1 [Integer root theorem]

Define the polynomial

$$P(x) = c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0$$

with integer coefficients and an integer zero α.

(a) Show that $c_0 = -\alpha \left(c_n \alpha^{n-1} + \cdots + c_2 x + c_1 \right)$

(b) Explain why this implies that α is a factor of c_0.

(c) Explain why it is so important that $P(x)$ has only integer coefficients.

(d) Explain why if the constant term is a prime number, then it makes it very easy to use the integer root theorem.

(e) Write down the worst-possible constant term that is still less than 20, and explain why it is the worst-case scenario.

Question 2

(a) Research and write down the statement of the rational root theorem.

(b) Prove the rational root theorem. Be sure to define all variables fully and to justify every step carefully.

(c) Explain which part of your proof relies on the fact that $P(x)$ has only integer coefficients.

(d) There are certain integer coefficients that may be problematic for the integer root theorem. Explain what they are, and why they are problematic.

Question 3

(a) Explain how the rational root theorem can be used to prove that a polynomial equation has only irrational or non-real roots.

(b) Create three random polynomials in the form $P(x) = px^3 + qx + r$, where p, q, r are non-zero integers, and use the rational root theorem to determine whether it has rational roots or not.

Investigation Task

Transforming Polynomials

Consider the polynomial equation $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ. Suppose we wish to find the equation of the polynomial equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. Normally you would need to solve for α, β and γ, and then re-construct the new polynomial from the transformed roots.

However, it is possible to obtain the new polynomial without explicitly finding the roots of the old polynomial equation. This investigation task will show how this is possible.

Question 1 Consider the polynomial $P(x) = x^3 - x^2 - 4x + 4$, which has zeroes α, β and γ. Suppose we wish to find the equation of the polynomial with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

(a) Let $u = \alpha + 1$. Explain why $P(u - 1) = 0$.

(b) Expand $P(u - 1) = 0$ and simplify to obtain a new polynomial equation in terms of u. Explain why this polynomial equation has roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. Call this polynomial $P_{new}(u)$.

(c) Calculate the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$ by using the product of roots of the new polynomial equation.

(d) Verify this result by expanding $(\alpha + 1)(\beta + 1)(\gamma + 1)$ and then using the root-coefficient relationships of the original polynomial.

(e) Find the value of $\dfrac{1}{\alpha + 1} + \dfrac{1}{\beta + 1} + \dfrac{1}{\gamma + 1}$.
Hint: Let the roots of the new polynomial be A, B and C.

(f) Use graphing software to sketch $y = P(x)$ and $y = P_{new}(u)$ on the same set of axes. Verify that indeed the roots are $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

(g) Explain why this result should have been obvious from the start.
Hint: Think of the graph of $y = x^2$ and $y = (x - 1)^2$.

Question 2 Repeat the set of steps above, with the same original polynomial, to find the equation of the new polynomial with roots

(a) 2α, 2β, 2γ (b) $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ (c) α^2, β^2, γ^2

3

FURTHER TRIGONOMETRY

- Compound angles
- t-formula
- Radian measure
- Trigonometric products to sums

Exercise 3A

Compound angles

Fundamentals

Fundamentals 1
Write down the expansion of the following.

(a) $\sin(A + B)$ (b) $\sin(A - B)$ (c) $\cos(A + B)$

(d) $\cos(A - B)$ (e) $\tan(A + B)$ (f) $\tan(A - B)$

Fundamentals 2
Write down the expansion of the following.

(a) $\sin(2A)$ (b) $\cos(2A)$ (c) $\tan(2A)$

Question 1 Use the compound angle results to prove the following identities.

(a) $\sin(180° - A) = \sin A$ (b) $\cos(180° + A) = -\cos A$

(c) $\tan(360° - A) = -\tan A$ (d) $\sin(90° - A) = \cos A$

Question 2 Use the compound angle formulae to expand and then simplify the following.

(a) $\sin(60° - A)$ (b) $\tan(A + 45°)$ (c) $\cos(150° + A)$

(d) $\sin(210° + A)$ (e) $\sin(90° + A)$ (f) $\cos(A - 60°)$

(g) $\cos(270° - A)$ (h) $\tan(270° + A)$ (i) $\sin(45° - A)\cos(45° - A)$

Question 3 Use the compound angle formulae to express the following in exact form.

(a) $\sin 15°$ (b) $\tan 15°$ (c) $\cos 105°$

(d) $\tan 105°$ (e) $\sin 75°$ (f) $\cos 75°$

Question 4 Simplify the following without using a calculator, and leave your answer in exact form.

(a) $2\sin 15° \cos 15°$ (b) $\cos 55° \cos 125° - \sin 55° \sin 125°$

(c) $\sin 65° \cos 70° + \cos 65° \sin 70°$ (d) $\dfrac{\cos^2 15° - \sin^2 15°}{1 - 2\sin^2 22\frac{1}{2}°}$

(e) $\dfrac{\tan 70° - \tan 10°}{1 + \tan 70° \tan 10°}$ (f) $\dfrac{2\tan 75°}{1 - \tan^2 75°}$

Question 5 Express the following as a single trigonometric function.

(a) $2\sin 3A\cos 3A$

(b) $\sin 2A\cos 2A$

(c) $\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

(d) $\cos^2 4A - \sin^2 4A$

(e) $2\cos^2 5A - 1$

(f) $2\cos^2\dfrac{A}{2} - 1$

(g) $\dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

(h) $\dfrac{2\tan 3A}{1-\tan^2 3A}$

(i) $\sqrt{1+\cos 2A}$

(j) $8\sin A\cos A\cos 2A$

(k) $\cos^4 A - \sin^4 A$

(l) $\sin 5A\cos 3A - \cos 5A\sin 3A$

Question 6 [Proving other compound angle identities from a base identity]

If we start with the compound angle identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we can derive further results using the following substitutions.

(a) Replace B with $-B$ to obtain $\sin(A-B) =$ ________

(b) Replace A with $90° - A$ to obtain $\sin(90° - A + B) = \cos($ ______ $)$. Expand and simplify

(c) Using your result in (b), replace B with $-B$ to obtain the expansion of $\cos(A+B)$.

(d) To obtain the result for the expansion of $\tan(A+B)$, expand and simplify $\dfrac{\sin(A+B)}{\cos(A+B)}$.

(e) In the $\tan(A+B)$ expansion replace B with $-B$ to obtain the $\tan(A-B)$ expansion.

Question 7 Find an expansion for $\cot(A+B)$ in terms of $\cot A$ and $\cot B$.

Question 8

(a) Expand $(x+y)(x^2 - xy + y^2)$.

(b) Hence, simplify $\dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A}$.

Question 9 Prove the following identities.

(a) $\dfrac{\cos(A-B)}{\sin A\cos B} = \cot A + \tan B$

(b) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

(c) $\sin^3 A + \sin 3A = \cos A\sin 2A + \sin A\cos^2 A$

(d) $\tan\left(\dfrac{\pi}{4} + A\right) + \tan\left(\dfrac{\pi}{4} - A\right) = 2\sec 2A$

Question 10 [Important result for integration]

By expanding $\cos 2A$ and rearranging, express $\sin^2 A$ and $\cos^2 A$ in terms of $\cos 2A$.

Question 11

(a) Express $\sin 3A$ in terms of $\sin A$

(b) Express $\cos 3A$ in terms of $\cos A$

Question 12 Prove the following identities.

(a) $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

(b) $\dfrac{\sin x}{1+\cos x} = \tan \dfrac{x}{2}$

(c) $\dfrac{\sin 2x}{1-\cos 2x} = \cot x$

(d) $\tan^2 3x = \dfrac{1-\cos 6x}{1+\cos 6x}$

(e) $\sin 4x = 4(\sin x \cos^3 x - \cos x \sin^3 x)$

(f) $\sec 2x + \tan 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

Question 13 Given that $\sin A = \dfrac{4}{5}$ and $\cos B = \dfrac{5}{13}$, where A and B are acute angles, find

(a) $\sin(A+B)$ (b) $\cos(A+B)$ (c) $\tan(A-B)$

Question 14 If $\cos\theta = \dfrac{3}{5}$ and θ is acute, find the exact values of the following.

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\cos\dfrac{\theta}{2}$ (d) $\tan\dfrac{\theta}{2}$

Question 15 If $\tan 2\theta = -\sqrt{3}$ and θ is acute, find the exact values of the following.

(a) $\sin 4\theta$ (b) $\cos 4\theta$ (c) $\tan 4\theta$ (d) $\tan\theta$

Question 16 Given that $\sin\alpha = \dfrac{1}{2}$ where $0 < \alpha < 90°$ and $\cos\beta = \dfrac{3}{5}$ where $-90° < \beta < 0$, find

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha-\beta)$ (c) $\tan 2\alpha$ (d) $\tan(\alpha-\beta)$

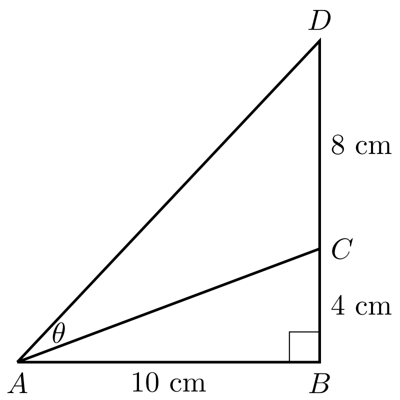
Question 17 If $\sin 2A = p$ and $\cos A = q$ write an expression in terms of p and q for

(a) $\sin A$ (b) $\cos 2A$ (c) $\tan A$

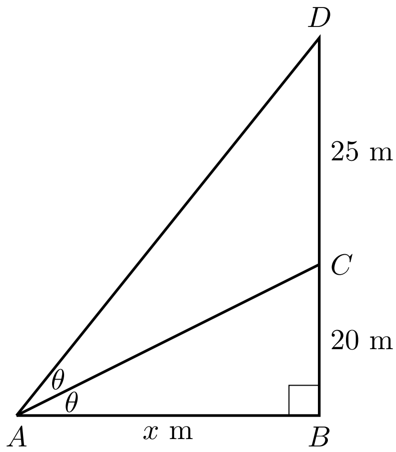
Question 18 Let $x = 2\sin A$. Simplify $\dfrac{x}{\sqrt{4-x^2}}$ and then evaluate for when $A = 135°$.

Question 19 Find the value of $\cos 5A\cos A + \sin 5A\sin A$ given that $\sin 2A = \dfrac{2}{5}$.

Question 20 Find the exact value of $\tan\theta$ in the diagram below.



Question 21 Daniel is standing at a bus stop and notices a sign on the building opposite 20m above the ground. When he doubles his angle of elevation to the building opposite he notices a person looking out the window 45m above the ground level. How far is the building from where Daniel is standing?



Question 22 If $\sin\left(x - \frac{\pi}{3}\right) = \cos\left(x - \frac{\pi}{3}\right)$, express $\tan x$ in the form $a + b\sqrt{3}$.

Question 23 Let $\cos x = \frac{7}{9}$ and $x = 2y$. Find $\sin y$.

Question 24 [t-formulae]

Let $t = \tan\left(\frac{x}{2}\right)$. By expanding and drawing an appropriate triangle, prove the following results.

(a) $\tan x = \frac{2t}{1 - t^2}$ (b) $\sin x = \frac{2t}{1 + t^2}$ (c) $\cos x = \frac{1 - t^2}{1 + t^2}$

Question 25 [Half-angle identities]

Prove the following identities.

(a) $\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$ (b) $\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$

(c) $\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A}$ (d) $\tan\frac{A}{2} = \frac{\sin A}{1 + \cos A}$

Question 26

(a) Show that $\dfrac{\sin(A+B)}{\cos(A-B)} = \dfrac{\tan A + \tan B}{1 + \tan A \tan B}$.

(b) Hence, or otherwise, show that

(i) $\dfrac{\sin(A+B)}{\cos(A-B)} = \dfrac{1}{\tan A + \cot B} + \dfrac{1}{\cot A + \tan B}$ (ii) $\dfrac{2}{\tan A + \cot A} = \sin 2A$

Challenge Problems

Problem 1

(a) Find a, b and c if $\cos 4\theta = a\cos^4\theta + b\cos^2\theta + c$.

(b) Hence, show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

Problem 2 Prove the following identities.

(a) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ (b) $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

Problem 3

(a) Complete the equation $x^4 + x^2y^2 + y^4 = (\_\_ + \_\_)^2 - \_\_\_\_$

(b) Factorise $x^6 - y^6$.

(c) Hence, show that

$$\cos^6 A - \sin^6 A = \cos 2A\left(1 - \frac{1}{4}\sin^2 2A\right)$$

Problem 4 Let $\triangle ABC$ be a non-right-angled triangle.

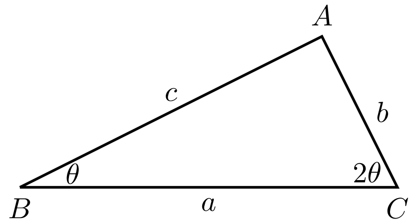
(a) $\sin C = \sin A\cos B + \cos A\sin B$.

(b) $\tan(A+B) = -\tan C$.

(c) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

Problem 5 In $\triangle ABC$, $B = 2A$ and $\tan A = \dfrac{1}{2}$. Show that $\cos C = -\dfrac{2\sqrt{5}}{25}$

Problem 6 In $\triangle ABC$, $\angle ACB = 2\theta$ and $\angle ABC = \theta$.



(a) Use the sine rule to show that $\cos\theta = \dfrac{c}{2b}$.

(b) State the sine rule again using angle $\angle BAC$ instead.

(c) Hence, show that $a = \dfrac{c^2 - b^2}{b}$.

Problem 7 [Geometric proof of the double-angle expansions]

In isosceles $\triangle ABC$, where $AB = BC = 1$, $\angle ABC$ is bisected by the line BD. Let $\angle ABD = \theta$.

(a) Write down the size of $\angle BDC$ and $\angle BDA$.

(b) Express CD and AD in terms of θ.

(c) Use the sine rule in $\triangle ABC$ to prove that $\sin 2\theta = 2\sin\theta\cos\theta$.

(d) Use the cosine rule in $\triangle ABC$ to prove that $\cos 2\theta = 1 - 2\cos^2\theta$.

Problem 8 [Application in conics]

(a) Show that the point $P(a\cos\theta,\, b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(b) The chord joining the points $P(a\cos\theta,\, b\sin\theta)$ and $Q(a\cos\phi,\, b\sin\phi)$ subtends a right angle at the point $A(a,\, 0)$.

Show that $\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2} = -\dfrac{b^2}{a^2}$

Problem 9 [Area of a regular n-sided polygon]

(a) Show that $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$.

(b) A regular n-sided polygon has side-length a. Show that the distance r from the centre to a vertex is given by

$$r^2 = \frac{a^2}{2\left(1-\cos\dfrac{360^\circ}{n}\right)}$$

(c) Show that the area of the polygon is given by $\dfrac{1}{2}nr^2\sin\dfrac{360^\circ}{n}$.

(d) Use (a) to write down an expression for $1-\cos\dfrac{360^\circ}{n}$.

(e) Hence, show that the area of the polygon can also be expressed as

$$\frac{1}{4}a^2 n\cot\frac{180^\circ}{n}$$

(f) Show that a 6-sided polygon of side-length 6 cm has area $54\sqrt{3}$ cm^2.

Exercise 3B

t-formula

Fundamentals

Fundamentals 1

If $t = \tan\left(\frac{A}{2}\right)$, write down the t-formula expansion of the following.

(a) $\sin A$ (b) $\cos A$ (c) $\tan A$

Fundamentals 2

The formula is not exclusive to $t = \tan\left(\frac{A}{2}\right)$. For example, if $t = \tan A$, then $\sin 2A =$ ________ and similarly, if $t = \tan($______$)$, then $\cos 4A = \frac{1-t^2}{1+t^2}$.

Question 1 If $t = \tan\left(\frac{A}{2}\right)$ express each of the following in terms of t.

(a) $\cot A$

(b) $\operatorname{cosec} A + \cot A$

(c) $\tan A \tan\left(\frac{A}{2}\right) + 1$

(d) $\frac{\sin A}{1 - \cos A}$

Question 2 Find the exact values of

(a) $\frac{1 - \tan^2 22.5°}{1 + \tan^2 22.5°}$ (b) $\frac{\tan 75°}{1 - \tan^2 75°}$ (c) $\frac{\tan 75°}{1 + \tan^2 75°}$

Question 3

(a) Express $\sin 2\theta$ and $\cos 2\theta$ in terms of t if $t = \tan\theta$.

(b) Hence, or otherwise, show that if $0 < 2\theta < \frac{\pi}{2}$ then

$$\frac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta} = \tan\theta$$

Question 4 Prove the following results using an appropriate t-formula substitution.

(a) $\cot\theta = \frac{1}{2}\left(\cot\frac{\theta}{2} - \tan\frac{\theta}{2}\right)$

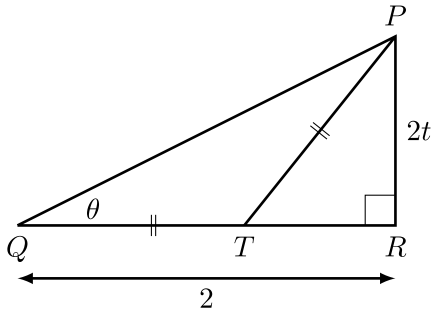
(b) $\frac{1}{1 + \sin 2\theta + \cos 2\theta} = \frac{1 + \tan^2\theta}{2(1 + \tan\theta)}$

(c) $\sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$

(d) $\frac{\sin 6\theta}{1 + \cos 6\theta} = \tan 3\theta$

Question 5 If $t = \tan A$ and $p\cos 2A - \sin 2A = 1$, show that $\dfrac{p-1}{p+1} = t$, where $0 \leq A \leq \dfrac{\pi}{2}$.

Question 6 Consider the diagram below. Let $\angle PQT = \theta$ and $QT = y$



(a) Write down the size of $\angle PTR$ in terms of θ.

(b) Express PT and TR in terms of y.

(c) Use Pythagoras theorem in $\triangle PTR$ to express PT and TR in terms of t.

(d) Hence prove the double angle formulae for $\tan 2\theta$, $\sin 2\theta$ and $\cos 2\theta$ where $t = \tan\theta$.

Question 7 [Using t-formulae to find exact values]

(a) Let $t = \tan 22\frac{1}{2}^\circ$. Use the formula for $\tan 2t$ to show that $t^2 + 2t - 1 = 0$.

(b) Hence, show that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

(c) Deduce that $\tan 112\frac{1}{2}^\circ = -\sqrt{2} - 1$.

(d) Use a similar method to above to find the exact value of

(i) $\tan 67\frac{1}{2}^\circ$ (ii) $\tan 157\frac{1}{2}^\circ$

Challenge Problems

Problem 1 Let a, b and c be the side-lengths of $\triangle ABC$. Show that

$$\frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{b^2 + c^2 - a^2}{2bc}$$

Problem 2 If $\tan \dfrac{A}{2} = \dfrac{1}{3}$ show that $\sin A = \dfrac{3}{5}$. Determine if the converse is true.

Problem 3 Express $\sec\theta + \tan\theta$ in terms of t, if $t = \tan\left(\dfrac{\theta}{2}\right)$.

Exercise 3C

Radian measure

Fundamentals

Fundamentals 1

Complete the following table.

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
$\sin\theta$								
$\cos\theta$								
$\tan\theta$								

Fundamentals 2

Fill in the following table.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin\theta$									
$\cos\theta$									
$\tan\theta$									

Fundamentals 3

Simplify the following.

(a) $\sin(\pi - \theta)$

(b) $\cos(\pi - \theta)$

(c) $\tan(\pi - \theta)$

(d) $\sin(\pi + \theta)$

(e) $\cos(\pi + \theta)$

(f) $\tan(\pi + \theta)$

(g) $\sin(2\pi - \theta)$

(h) $\cos(2\pi - \theta)$

(i) $\tan(2\pi - \theta)$

Question 1 Show that

(a) $\cos\frac{\pi}{12} = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$

(b) $\tan\frac{7\pi}{12} = -\sqrt{3} - 2$

(c) $\tan -\frac{\pi}{12} = \sqrt{3} - 2$

(d) $\sin\frac{13\pi}{12} = \frac{1 - \sqrt{3}}{2\sqrt{2}}$

Question 2 Simplify the following.

(a) $\cos\left(\frac{\pi}{2} - A\right) - \cos\left(\frac{\pi}{2} + A\right)$

(b) $\sin\left(A + \frac{\pi}{6}\right)\sin\left(A - \frac{\pi}{6}\right)$

Question 3

(a) If $\sin A = \frac{3}{5}$ and $\frac{\pi}{2} < A < \pi$, find the exact value of $\sin 2A$.

(b) If $\tan A = \frac{4}{3}$ and $\pi \leq A \leq \frac{3\pi}{2}$ find the value of $\cos 2A$.

Question 4 If $\sin A = \frac{2}{3}$ and $0 < A < \frac{\pi}{2}$ find the exact value of

(a) $\cos 2A$

(b) $\tan 2A$

Question 5 If $\sec A = -\frac{5}{3}$ and $A \in \left[\frac{\pi}{2}, \pi\right]$, find the exact value of

(a) $\sin 2A$

(b) $\cos 2A$

(c) $\tan 2A$

Question 6 Evaluate without using a calculator and leaving your answer in exact form.

(a) $2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

(b) $1 - 2\sin^2\frac{\pi}{8}$

(c) $2\cos^2\frac{\pi}{6} - 1$

Question 7 Find the exact value of $\tan x$ when $\sin\left(x + \frac{\pi}{4}\right) = 2\cos x$.

Question 8

(a) Show that if $0 \leq x \leq \pi$ then $\frac{1 - \cos x}{\sin x} = \tan\frac{x}{2}$

(b) Hence, show that $\tan\frac{\pi}{12} = \frac{1}{2 + \sqrt{3}}$

Question 9 Let $0 \leq A \leq \frac{\pi}{2}$.

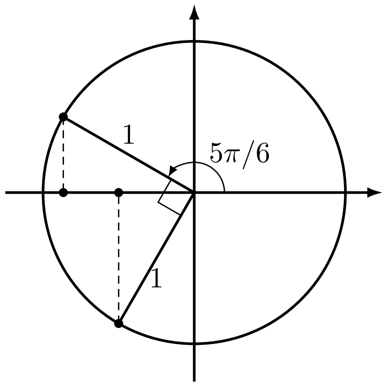
(a) Simplify $\sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$

(b) Hence, find the exact value of

(i) $\tan\frac{\pi}{8}$

(ii) $\tan\frac{\pi}{12}$

Question 10 Use the diagram below to simplify (but not evaluate) the following. Complete all missing angles and sides.



(a) $\tan \frac{5\pi}{6}$

(b) $\tan\left(\frac{\pi}{2} + \frac{5\pi}{6}\right)$

(c) $\tan\left(\pi + \frac{5\pi}{6}\right)$

Question 11 Using a similar method to above, simplify

(a) $\tan \frac{2\pi}{3}$

(b) $\tan\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)$

(c) $\tan\left(\pi + \frac{2\pi}{3}\right)$

Question 12

(a) Let x be a very small angle, in radians. Verify using your calculator that

$$x \approx \sin x \approx \tan x$$

and write down an approximate value for $\cos x$.

(b) Show that for small x, $\cos(\theta - x) \approx \cos\theta + x \sin\theta$

(c) Hence, show that $\cos 29°57' \approx \dfrac{3600\sqrt{3} + \pi}{7200}$

Question 13 Let $\tan A = -2$ for $0 \le A \le \pi$ and $\tan B = p$ for $\pi \le B \le 2\pi$.

(a) Find the value of p if $\tan(A + B) = -1$

(b) Hence, find the value of $A + B$ in terms of π

Challenge Problems

Problem 1 If $\sin A = \frac{3}{5}$ and $0 < A < \frac{\pi}{2}$ find the exact value of

(a) $\tan \frac{A}{2}$ (b) $\sin \frac{A}{2}$ (c) $\cos \frac{A}{2}$

Problem 2 Prove the following exact values

(a) $\cos \frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$ (b) $\sin \frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ (c) $\tan \frac{7\pi}{12} = -2-\sqrt{3}$

Problem 3 If $\tan(x+y) = -1$ and $\tan y = \frac{1}{3}$ where $\pi < y < 2\pi$

(a) Find $\tan x$ where $0 \leq x \leq \pi$ (b) Find $x + y$ in exact form

Exercise 3D

Trigonometric products to sums

Fundamentals

Fundamentals 1
Write down the expansions of the following.

(a) $\sin(A+B)$ (b) $\sin(A-B)$

(c) $\cos(A+B)$ (d) $\cos(A-B)$

Fundamentals 2
Write down the corresponding products-to-sums formulae.

(a) $\sin A\cos B$ (b) $\cos A\sin B$

(c) $\cos A\cos B$ (d) $\sin A\sin B$

Question 1 Evaluate the following.

(a) $\sin 75°\cos 15°$ (b) $\cos 285°\cos 15°$ (c) $\cos 105°\sin 15°$

(d) $\sin 105°\sin 15°$ (e) $\cos 105°\cos 75°$ (f) $\cos 165°\sin 75°$

Question 2 Evaluate each product using sum and difference of two functions, giving your answer in exact form.

(a) $\cos\frac{\pi}{12}\sin\frac{5\pi}{12}$ (b) $\sin\frac{\pi}{12}\sin\frac{5\pi}{12}$ (c) $\sin\frac{5\pi}{6}\cos\frac{\pi}{6}$

(d) $\cos\frac{5\pi}{6}\cos\frac{\pi}{6}$ (e) $\sin\frac{7\pi}{12}\sin\frac{\pi}{12}$ (f) $\cos\frac{5\pi}{24}\sin\frac{-\pi}{24}$

Question 3 Express the following as sum or difference of trigonometric functions.

(a) $\sin 4x\cos x$ (b) $\cos 5x\sin 3x$

(c) $\cos\frac{9x}{2}\cos\frac{3x}{2}$ (d) $\sin(x+y)\cos(x-y)$

Question 4 Prove the following identities.

(a) $\cos(A+B)\cos(A-B)=\cos^2 A+\cos^2 B-1$ (b) $\sin 4A\sin 2A=\cos^2 A-\cos^2 3A$

Question 5 Express $\tan 2x\tan x$ as a quotient using the sine and cosine ratio. Hence, show that

$$\tan 2x\tan x=\frac{\cos x-\cos 3x}{\cos x+\cos 3x}$$

Question 6 Show that

$$\cos 4x \cos 2x \cos x = \frac{1}{4}(\cos 7x + \cos 5x + \cos 3x + \cos x)$$

Question 7 [Sum-to-product formulae]

Use your product to sum formulae and the pair of substitutions $P = A + B$ and $Q = A - B$ to express the following as products of trigonometric functions.

(a) $\sin P + \sin Q$ (b) $\sin P - \sin Q$ (c) $\cos P + \cos Q$ (d) $\cos P - \cos Q$

Question 8 Express the following as products.

(a) $\sin 4x - \sin 2x$ (b) $\cos 4x + \cos 2x$

(c) $\sin 6x + \sin 4x$ (d) $\cos 3x + \cos 7x$

Question 9 Evaluate the sum by using the product formula and explain your results.

(a) $\sin \frac{7\pi}{12} - \sin \frac{5\pi}{12}$ (b) $\cos \frac{7\pi}{12} + \cos \frac{5\pi}{12}$

Challenge Problems

Problem 1 Express each of the following as the sum of basic trigonometric functions.

(a) $\cos x \cos 2x \sin 3x$ (b) $\sin x \sin 2x \cos 3x$

Problem 2 Find the exact value of

(a) $\sin 75° + \sin 15°$ (b) $\cos 75° - \cos 15°$ (c) $\sin 285° - \sin 15°$

Problem 3 Express $\sin 5A + 2\sin 3A + \sin A$ as a product of trigonometric functions in two different ways by grouping the terms differently. Verify that both of your results give you the same value for $A = \frac{\pi}{4}$ and then $A = \frac{\pi}{6}$.

Hint: You will need to reduce a double angle in one of the methods.

Problem 4 Prove the following identities

(a) $\cos A \cos B \cos C = \frac{1}{4}\big(\cos(A+B+C) + \cos(A+B-C) + \cos(B+C-A) + \cos(C+A-B)\big)$

(b) $\sin A \sin B \sin C = \frac{1}{4}\big(\sin(A+B-C) + \sin(A+C-B) + \sin(B+C-A) - \sin(A+B+C)\big)$

Chapter 3 Review

Further Trigonometry

Review

Question 1 Simplify the following.

(a) $\sin(45° - x)\cos(45° - x)$

(b) $\sin 6x \cos 3x - \cos 6x \sin 3x$

(c) $\cos 2x \cos x + \sin 2x \sin x$

(d) $\sin^2 x \cos^2 x$

(e) $\cos 4x - 2\cos^2 2x$

(f) $\sin 3x \cos 3x \cos 6x$

(g) $\dfrac{\tan 2x - \tan x}{1 + \tan 2x \tan x}$

(h) $\sin(x + y)\cos y - \cos(x + y)\sin y$

Question 2 Expand and simplify the following.

(a) $\sin\left(\dfrac{3\pi}{2} - A\right)$

(b) $\cos\left(A + \dfrac{\pi}{2}\right)$

(c) $\cos\left(\dfrac{5\pi}{3} + A\right)$

(d) $\sin\left(\dfrac{4\pi}{3} - A\right)$

Question 3 Find the exact values of

(a) $\dfrac{\tan 22.5°}{1 - \tan^2 22.5°}$

(b) $\dfrac{\tan 67.5°}{1 + \tan^2 67.5°}$

Question 4

(a) Express $\sin 195°$ in the form $a \sin\theta$ where $0 < \theta < 90°$

(b) Show that $\sin 195° = \dfrac{1 - \sqrt{3}}{2\sqrt{2}}$ using two different expansions.

Question 5 By expanding the following, show that

(a) $\sin(270° + \theta) = -\cos\theta$

(b) $\cos(270° - \theta) = -\sin\theta$

Question 6 Prove the following results.

(a) $\tan A + \tan B = \dfrac{\sin(A + B)}{\cos A \cos B}$

(b) $\tan\left(A + \dfrac{\pi}{4}\right)\tan\left(A - \dfrac{\pi}{4}\right) = -1$

Question 7 Prove that $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$ and hence find exact value of $\tan\dfrac{\pi}{12}$

Question 8 The angle between the lines $y = mx$ and $y = \dfrac{1}{3}x$ is $45°$, find the value of m.

Question 9 Express $\dfrac{1+\sin A-\cos A}{1+\sin A+\cos A}$ in terms of t.

Question 10 Let s be the perimeter of $\triangle ABC$ with side lengths a, b, and c. Show that

(a) Show that $\cos^2\left(\dfrac{A}{2}\right)=\dfrac{(b+c)^2-a^2}{4bc}$

(b) Deduce that $\cos^2\left(\dfrac{A}{2}\right)=\dfrac{s(s-2a)}{4bc}$

Question 11 Show that

$$\frac{1-\sec x+\tan x}{1+\sec x-\tan x}=\tan\left(\frac{x}{2}\right)$$

Question 12

(a) Show that $\sin^2\theta=\dfrac{1}{2}(1-\cos 2\theta)$.

(b) Hence, show that $\sin^4\theta=\dfrac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)$

(c) Deduce that $\sin^4\theta=\dfrac{1}{8}\left(\cos 4\theta-4\cos 2\theta+3\right)$

Question 13 If $\sin A=-\dfrac{8}{17}$ and $\pi<A<\dfrac{3\pi}{2}$ find the value of

(a) $\cot A$ (b) $\sin 2A$ (c) $\tan 2A$

Question 14 If $\sin A=\dfrac{3}{4}$, $\dfrac{\pi}{2}<A<\pi$ and $\sin B=\dfrac{2}{3}$, $0<B<\dfrac{\pi}{2}$ find the exact value of

(a) $\cos 2A$ (b) $\sin(A-B)$ (c) $\tan(A+B)$

Question 15 Write each product as a sum or difference of sines or cosines.

(a) $\cos 5A\sin 2A$ (b) $\cos 7A\cos 5A$ (c) $\sin A\sin 3A$ (d) $\sin 3A\cos A$

Question 16 Use the product-to-sum formula to evaluate

(a) $\cos 195°\sin 45°$ (b) $\cos 7.5°\cos 52.5°$ (c) $\cos\dfrac{11\pi}{12}\cos\dfrac{\pi}{12}$ (d) $\cos\dfrac{5\pi}{24}\cos\dfrac{\pi}{24}$

Question 17 Simplify the following.

(a) $\dfrac{\sin 7x+\sin x}{\sin 8x}$ (b) $\cos^2 4x-\sin^2 4x$ (c) $\dfrac{\sin 4x+\sin 6x}{\cos 4x-\cos 6x}$

Question 18 Simplify $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}$, *without* using the expansions for $\sin 3A$ and $\cos 3A$.

Question 19 Show that if $\tan\left(\dfrac{x}{2}\right)=\dfrac{p}{q}$, then $-p\cos x+q\sin x=p$.

Investigation Task

Tides and Simple Harmonic Motion

Simple harmonic motion is the study of periodic motion like the height of a mass bobbing up and down from a spring or in this case, the tides. Naturally, the sine and cosine functions become useful for modelling such oscillatory behaviour. The general equation for simple harmonic motion is

$$y = a\cos(nt + b) + c$$

where a, b, c and n are constants, and t is time.

Question 1 Suppose a student were to sketch the graph of

$$y = a\cos(nt + b) + c$$

Use graphing software to explain how each constant a, b, c and n affects the shape of the graph. Some key expressions to investigate include in your answer are *amplitude*, *period*, *phase-angle* and *centre of motion*.

Question 2 The tides are measured continuously at a particular location. It is found that high tide is 10 metres, low tide is 2 metres and the tide undergoes two complete cycles in a 24-hour time-span. Let t be time measured in hours from the first high-tide at 8am.

(a) From the above information, the values of a and c from $y = a\cos(nt + b) + c$ can be found instantly. Write down those values and explain how you found them. Your answers from Question 1 will be useful here.

(b) State the period.

(c) Hence, find the value of n.

(d) Explain why measuring t beginning from high-tide means that $b = 0$.

(e) Hence, show that $y = 6 + 4\cos\left(\frac{\pi}{6}t\right)$ is a suitable equation to model the above motion.

(f) Hence, find at what time(s) the tide drops to 3 metres within 24-hours from 8am that day.

Question 3 Go to `http://www.bom.gov.au/australia/tides/` and find the tidal data from a location of your choice. Use a similar technique to Question 2 to construct an equation that models the data provided. Use your equation to predict the next day's tide and compare it with the actual prediction provided by the Bureau of Meterology.

Investigation Task

t-formula substitutions

The t-formulae substitutions are extremely useful substitutions that allow us to convert trigonometric expressions into rational expressions, which we are generally better equipped to work with. As you have studied above, they state that if $t = \tan\left(\frac{\theta}{2}\right)$, then

$$\sin\theta = \frac{2t}{1+t^2}$$
$$\cos\theta = \frac{1-t^2}{1+t^2}$$
$$\tan\theta = \frac{2t}{1-t^2}$$

Research at least *two* different derivations of the t-formula substitutions, where at least one is an algebraic derivation and one is a geometric derivation. Your answer should include the full derivation as well as any necessary accompanying diagrams and constructions.

Investigation Task

Applications to Polynomials

In the topic *Polynomials*, polynomial equations were solved often by first finding an integer root and then factorising from there. However, the tools of trigonometry can also be used to help solve specific polynomials too. Furthermore, it will show how conversely, the tools of polynomials can then be used to help prove trigonometric identities. This investigation task will show how the branches of trigonometry and polynomials work together to develop some nice results.

Question 1 Let $P(x) = 8x^3 - 6x - 1$.

(a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) By letting $x = \cos\theta$, show that $P(x)$ is equivalent to $2\cos 3\theta - 1$

(c) Solve $2\cos 3\theta - 1 = 0$ and write down the first three positive solutions of θ.

(d) Hence, show that the zeroes of the polynomial $P(x)$ are

$$x = \cos\left(\frac{\pi}{9}\right), \cos\left(\frac{5\pi}{9}\right), \cos\left(\frac{7\pi}{9}\right)$$

(e) Hence, show that

$$\cos\left(\frac{\pi}{9}\right) + \cos\left(\frac{5\pi}{9}\right) + \cos\left(\frac{7\pi}{9}\right) = 0$$

and

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right) = \frac{1}{8}$$

Question 2 Suppose in Question 1 (c) you did not use the first three positive solutions of θ, and instead took the next three positive solutions. What do you notice about the resulting solutions in terms of x? Hence, explain why we take the first three positive solutions and not just any random three solutions.

Question 3 Use a similar technique to Question 1 to find the solutions of $8x^3 - 6x + 1 = 0$.

Question 4 [Using the tangent expansion]

By first proving that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

find all the solutions to $x^3 - 3x^2 - 3x + 1 = 0$.

4

INVERSE TRIGONOMETRY

- Exact values
- Graphs
- Applications with compound angle formulae

Exercise 4A

Exact values

Fundamentals

Fundamentals 1
Draw the graph of each of the following, and state the domain and range.

(a) $y = \sin^{-1}(x)$ (b) $y = \cos^{-1}(x)$ (c) $y = \tan^{-1}(x)$

Fundamentals 2
Write down an alternative notation for

(a) $\sin^{-1}(x)$ (b) $\cos^{-1}(x)$ (c) $\tan^{-1}(x)$

Fundamentals 3
Determine whether the following functions are odd, even or neither.

(a) $y = \sin^{-1}(x)$ (b) $y = \cos^{-1}(x)$ (c) $y = \tan^{-1}(x)$

Fundamentals 4
Complete the following identities.

(a) $\sin^{-1}(-x) =$ ________ (b) $\cos^{-1}(-x) =$ ________ (c) $\tan^{-1}(-x) =$ ________

Question 1 Find the exact value of the following.

(a) $\tan^{-1}(1)$ (b) $\cos^{-1}(1)$ (c) $\cos^{-1}(-1)$

(d) $\sin^{-1}(-1)$ (e) $\tan^{-1}(-1)$ (f) $\sin^{-1}\left(-\frac{1}{2}\right)$

(g) $\tan^{-1}\left(\sqrt{3}\right)$ (h) $\tan^{-1}\left(-\sqrt{3}\right)$ (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(j) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (k) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (l) $\cos^{-1}\left(-\frac{1}{2}\right)$

Question 2 Find the exact value of the following, where possible.

(a) $\arcsin(0.5)$ (b) $\arccos(0.5)$ (c) $\arctan(1)$

(d) $\arcsin(-1)$ (e) $\arcsin(2)$ (f) $\arccos(-3)$

Question 3 Use your calculator to find the following correct to 2 decimal places. Make sure that your calculator is in radian mode!

(a) $\sin^{-1}(0.32)$ (b) $\cos^{-1}(-0.27)$ (c) $\tan^{-1}(5.19)$

Question 4 Find the value of α, β and γ for each of the following values of k.

$$k = \sin^{-1}(\alpha) = \cos^{-1}(\beta) = \tan^{-1}(\gamma)$$

If no such value exists, then state that it is undefined.

(a) $k = 0$ (b) $k = \frac{\pi}{2}$ (c) $k = \frac{\pi}{3}$

(d) $k = -\frac{\pi}{4}$ (e) $k = \frac{2\pi}{3}$ (f) $k = -\frac{5\pi}{6}$

Question 5 Find the exact value of the following.

(a) $\sin\left(\cos^{-1}(0)\right)$ (b) $\cos\left(\sin^{-1}(1)\right)$

(c) $\cos\left(\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right)$ (d) $\sin\left(\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right)$

Question 6 [Review of trigonometric exact values]

Simplify the following.

(a) $\sin\left(\frac{5\pi}{3}\right)$ (b) $\tan\left(\frac{2\pi}{3}\right)$ (c) $\cos\left(\frac{5\pi}{4}\right)$

(d) $\sin\left(\frac{7\pi}{6}\right)$ (e) $\cos\left(\frac{5\pi}{6}\right)$ (f) $\sin\left(-\frac{\pi}{3}\right)$

(g) $\tan\left(-\frac{5\pi}{6}\right)$ (h) $\cos\left(-\frac{\pi}{4}\right)$ (i) $\sin\left(-\frac{7\pi}{6}\right)$

Question 7 Simplify the following.

(a) $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$ (b) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$ (c) $\cos^{-1}\left(\cos\frac{3\pi}{4}\right)$

Question 8 Consider the expression $\sin^{-1}(\sin\alpha)$.

(a) Explain why it is not always the case that $\sin^{-1}(\sin\alpha) = \alpha$.

(b) Hence, state the condition for α for which $\sin^{-1}(\sin\alpha) = \alpha$.

(c) Describe the steps to calculate $\sin^{-1}(\sin\alpha)$ if α does not satisfy the condition.

Question 9 Simplify the following.

(a) $\sin^{-1}\left(\sin\frac{\pi}{5}\right)$ (b) $\cos^{-1}\left(\cos\frac{3\pi}{5}\right)$ (c) $\tan^{-1}\left(\tan\frac{\pi}{7}\right)$

(d) $\sin^{-1}\left(\sin\frac{6\pi}{5}\right)$ (e) $\cos^{-1}\left(\cos -\frac{\pi}{7}\right)$ (f) $\tan^{-1}\left(\tan\frac{4\pi}{5}\right)$

(g) $\cos^{-1}\left(\cos\frac{6\pi}{5}\right)$ (h) $\sin^{-1}\left(\sin -\frac{4\pi}{7}\right)$ (i) $\tan^{-1}\left(\tan\frac{6\pi}{5}\right)$

Question 10 Find the exact value of the following.

(a) $\sin(\arcsin 1)$ (b) $\tan\left(\arctan\left(-\frac{3}{5}\right)\right)$ (c) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

Question 11 By drawing an appropriate right-angled triangle, find the exact value of the following.

(a) $\cos\left(\tan^{-1}\left(\frac{1}{3}\right)\right)$ (b) $\tan\left(-\sin^{-1}\left(\frac{1}{5}\right)\right)$

(c) $\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$ (d) $\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

Question 12 Find the exact value of the following.

(a) $\cos\left(\sin^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$ (b) $\tan\left(\sin^{-1}(1) + \sin^{-1}\left(\frac{1}{2}\right)\right)$

(c) $\cos\left(\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ (d) $\sin\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(-\sqrt{3}\right)\right)$

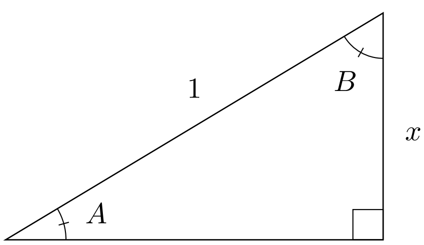
Question 13 Find the inverse of the following functions in the given domains.

(a) $y = 1 + \cos x$ for $x \in [0, \pi]$ (b) $y = 1 - 2\sin\left(\frac{x}{3}\right)$ for $x \in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

Question 14 If $\tan^{-1}(x) = \cos^{-1}(y)$, show that $y = \dfrac{1}{\sqrt{1+x^2}}$.

Question 15 Find the value of x such that $\sin^{-1}(x) = \cos^{-1}(x)$.

Question 16 Consider the following triangle



(a) Find an expression for A in terms of x.

(b) Find a similar expression for B in terms of x.

(c) Deduce that $\sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$.

(d) Verify this identity by substituting in $x = \dfrac{1}{2}$ and $x = 0.123$

Challenge Problems

Problem 1 Consider the function $f(x) = \operatorname{cosec} x$ in the domain $0 < x \leq \dfrac{\pi}{2}$.

(a) State the range of $f(x)$.

(b) State the domain of the inverse function $f^{-1}(x)$.

(c) Show that $f^{-1}(x) = \sin^{-1}\left(\dfrac{1}{x}\right)$.

(d) Sketch both $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Problem 2 Simplify the following expressions for $0 \leq x \leq \dfrac{\pi}{2}$.

(a) $\tan^{-1}(\cot x)$ (b) $\sin^{-1}(\cos x)$ (c) $\cos^{-1}(\sin x)$

Problem 3 Show the following results.

(a) $\cos\left(\tan^{-1}(x)\right) = \dfrac{1}{\sqrt{1+x^2}}$

(b) $\sin\left(\tan^{-1}(x)\right) = \dfrac{x}{\sqrt{1+x^2}}$

(c) $\tan\left(\cos^{-1}(x)\right) = \dfrac{\sqrt{1-x^2}}{x}$

(d) $\tan\left(\sin^{-1}(x)\right) = \dfrac{x}{\sqrt{1-x^2}}$

Exercise 4B

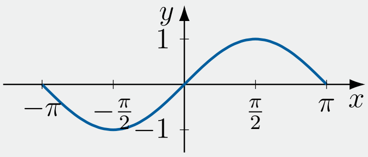
Graphs

Fundamentals

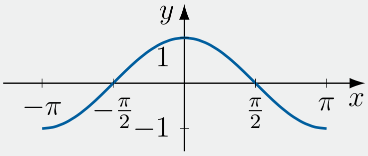
Fundamentals 1

Highlight the part of the curve that yields the corresponding inverse trigonometric graph.

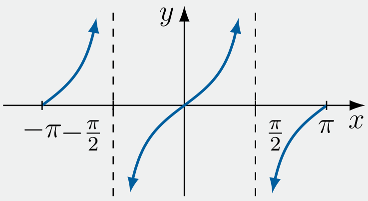
(a)



(b)



(c)



Fundamentals 2

Draw the graph of each of the following, and state the domain and range, and any symmetry.

(a) $y = \sin^{-1}(x)$ (b) $y = \cos^{-1}(x)$ (c) $y = \tan^{-1}(x)$

Fundamentals 3

State the value of $\tan^{-1}(x)$ as

(a) $x \to \infty$ (b) $x \to -\infty$

Fundamentals 4

Describe the effect of

(a) a on $y = a\sin^{-1}(x)$. (b) b on $y = \sin^{-1}(x - b)$. (c) c on $y = \sin^{-1}(cx)$.

Question 1 Draw the graph of the following by applying an appropriate reflection.

(a) $y = \sin^{-1}(-x)$ (b) $y = \cos^{-1}(-x)$ (c) $y = \tan^{-1}(-x)$

Question 2 State the domain and range for each of the following and hence, sketch the graph.

(a) $y = \sin^{-1}(x-1)$ (b) $y = \cos^{-1}(x-1)$

(c) $y = \cos^{-1}(x+2)$ (d) $y = \tan^{-1}(x-1)$

(e) $y = \sin^{-1}(x+1) + \dfrac{\pi}{2}$ (f) $y = \tan^{-1}(x) - \dfrac{\pi}{2}$

Question 3 State the domain and range for each of the following and hence, sketch the graph.

(a) $y = \sin^{-1}(2x)$ (b) $y = \sin^{-1}\left(\dfrac{x}{2}\right)$ (c) $y = \cos^{-1}(4x)$

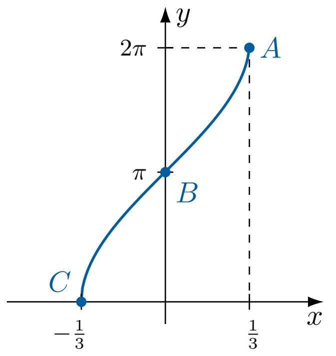
(d) $y = \cos^{-1}\left(\dfrac{x}{4}\right)$ (e) $y = \tan^{-1}(3x)$ (f) $y = \tan^{-1}\left(\dfrac{x}{3}\right)$

Question 4 State the domain and range for each of the following and hence, sketch the graph.

(a) $y = 2\sin^{-1}(3x)$ (b) $y = \dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{4}\right)$ (c) $y = \dfrac{1}{3}\cos^{-1}(2x)$

(d) $y = 3\cos^{-1}\left(\dfrac{x}{4}\right)$ (e) $y = 2\tan^{-1}(3x)$ (f) $y = \dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{3}\right)$

Question 5 The diagram below shows the graph of $y = \pi + 2\sin^{-1}(3x)$.



(a) Find the domain and range of $y = 2\sin^{-1}(3x)$ and hence, sketch the graph.

(b) Write down the coordinates of the points A, B and C from the diagram above.

(c) Draw the graph of $y = -\pi + 2\sin^{-1}(3x)$

(d) Draw the graph of $y = \pi + 2\sin^{-1}(-3x)$

Question 6 Consider the equation $y = 2\sin^{-1}(1-2x)$

(a) Find the domain.

(b) State the range.

(c) Find the domain and range of $y = 2\sin^{-1}(2x-1)$ and compare your answer to (a). What do you notice?

(d) Use graphing software to sketch the graphs of $y = 2\sin^{-1}(2x-1)$ and $y = 2\sin^{-1}(1-2x)$. Describe the difference between the two graphs.

(e) Sketch the graph of $y = 2\sin^{-1}(1-2x)$.

Question 7 Use a similar technique to Question 6 to sketch the following.

(a) $y = 3\cos^{-1}(2x+1)$ (b) $y = \dfrac{1}{2}\cos^{-1}(3-2x)$ (c) $y = 2\tan^{-1}(1-2x)$

Question 8

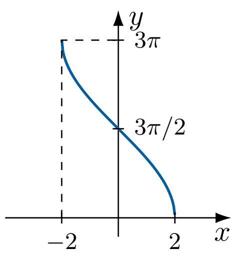
(a) Sketch $y = \sin^{-1}(x)$ and $y = \cos^{-1}(x)$ on the same set of axes.

(b) State the domain of $y = \sin^{-1}(x) + \cos^{-1}(x)$.

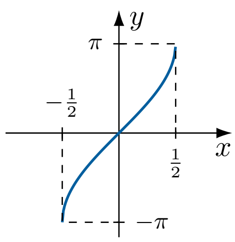
(c) Hence, sketch the graph of $y = \sin^{-1}(x) + \cos^{-1}(x)$ by adding the two graphs together.

Question 9 Find a possible equation for the following inverse trigonometric graphs.

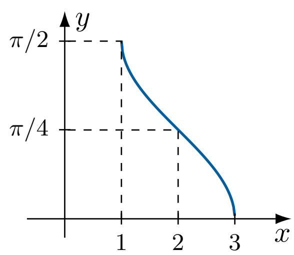
(a)



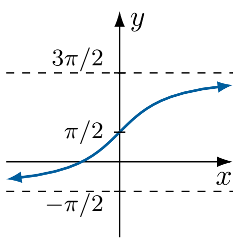
(b)



(c)



(d)

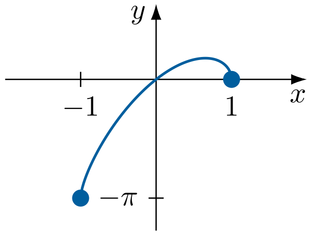


Question 10 State the domain and range of the following.

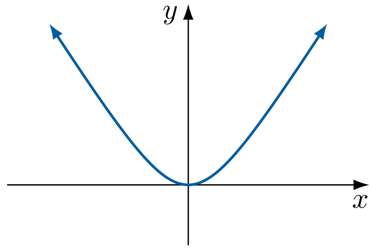
(a) $y = x\sin^{-1}(x)$ (b) $y = x\tan^{-1}(x)$

Question 11 The diagrams below show the graphs of $y = x\sin^{-1}(x)$, $y = x\cos^{-1}(x)$ and $y = x\tan^{-1}(x)$ in a random order. Match the equation to the graph by examining their features.

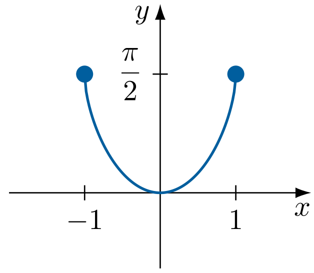
(a)



(b)



(c)



Question 12 Consider the function

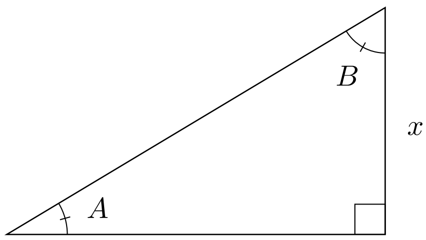
$$f(x) = \begin{cases} \sin^{-1}(x), & \text{for} \quad 0 < x \leq 1 \\ \cos^{-1}(x), & \text{for} \quad -1 \leq x \leq 0 \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

(b) Find the value of $f\left(-\frac{1}{2}\right) + f(0) - f\left(\frac{1}{2}\right)$.

Question 13

(a) Use the triangle below to find the exact value of $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ for $x > 0$.



(b) Let $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$. Show that $f(x)$ is an odd function.

(c) Hence, sketch the graph of $y = f(x)$.

Question 14 Find the domain and range of each of the following.

(a) $y = \sin\left(\sin^{-1}(x)\right)$

(b) $y = \cos\left(\cos^{-1}(x)\right)$

(c) $y = \tan\left(\tan^{-1}(x)\right)$

(d) $y = \sin^{-1}(\sin(x))$

(e) $y = \cos^{-1}(\cos(x))$

(f) $y = \tan^{-1}(\tan(x))$

Question 15 Consider the function $f(x) = \cos\left(\cos^{-1}(x)\right)$.

(a) Write down the domain and range of $f(x)$.

(b) Bob claims that the graph is just $y = x$ since the cosine and inverse cosine graphs ‘cancel’ each other out. Is he correct in saying this? Explain your answer.

(c) Complete the following table of values.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$					

(d) Hence, sketch the graph of $y = f(x)$.

Question 16 Use a similar technique to above to sketch the following graphs.

(a) $y = \sin\left(\sin^{-1}(x)\right)$

(b) $y = \tan\left(\tan^{-1}(x)\right)$

Challenge Problems

Problem 1

(a) Show that $\cos\left(\sin^{-1}(x)\right) = \sqrt{1 - x^2}$.

(b) Hence, sketch the graph of $y = \cos\left(\sin^{-1}(x)\right)$.

Problem 2 Draw a sketch of the following graphs.

(a) $y = \sin^{-1}(|x|)$

(b) $y = \cos^{-1}(|x|)$

(c) $y = \tan^{-1}(|x|)$

Problem 3 State the domain and range of the following, and hence draw a sketch of the following graphs.

(a) $y = \sin^{-1}(x^2)$

(b) $y = \cos^{-1}(x^2)$

(c) $y = \tan^{-1}(x^2)$

Problem 4 Find the domain and range of $y = \cos^{-1}\left(\frac{1}{x}\right)$.

Exercise 4C

Applications with compound angle formulae

Fundamentals

Fundamentals 1

To find the exact value of expressions like $\sin\left(2\cos^{-1}\left(\frac{x}{y}\right)\right)$

(a) Let $A =$ _____, so the expression becomes sin(___) = __________ when expanded.

(b) Since $A =$ _____, then $\cos A =$ ___ and hence a right-angled triangle can be drawn and all sides found using Pythagoras' Theorem.

(c) From the triangle, we can read off that $\sin A =$ _____.

(d) Hence, $\sin\left(2\cos^{-1}\left(\frac{x}{y}\right)\right) =$ _______

Fundamentals 2

To find the exact value of expressions like $\cos\left(\sin^{-1}(x) + \sin^{-1}(y)\right)$, let $A =$ _____ and $B =$ _____. The expression then becomes cos(_____), which can be expanded. After expanding, substitute the value of $\sin A$, $\sin B$, _____, _____ into the expansion.

Fundamentals 3

To prove expressions in the form

(a) $\sin^{-1}(p) + \sin^{-1}(q) = \sin^{-1}(r)$, begin by applying the sin/cos/tan (circle one) function to both sides, and prove that result instead by expanding the left hand side.

(b) $\cos^{-1}(p) + \cos^{-1}(q) = \cos^{-1}(r)$, begin by applying the sin/cos/tan (circle one) function to both sides, and prove that result instead by expanding the left hand side.

(c) $\tan^{-1}(p) + \tan^{-1}(q) = \tan^{-1}(r)$, begin by applying the sin/cos/tan (circle one) function to both sides, and prove that result instead by expanding the left hand side.

Question 1 Consider the expression $\cos\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$.

(a) Let $A = \sin^{-1}\left(\frac{4}{5}\right)$. Show that the expression is equivalent to $1 - 2\sin^2 A$.

(b) Write down the value of $\sin A$.

(c) Hence, find the exact value of $\cos\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$.

Question 2 Use a similar technique to Question 1 to find the exact value of the following.

(a) $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$

(b) $\sin\left(2\tan^{-1}\left(\sqrt{3}\right)\right)$

(c) $\cos\left(2\tan^{-1}\left(\frac{2}{3}\right)\right)$

(d) $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

(e) $\tan\left(2\sin^{-1}\left(\frac{1}{3}\right)\right)$

(f) $\tan\left(2\cos^{-1}\left(\frac{2}{5}\right)\right)$

Question 3 Consider the expression

$$\tan\left(\sin^{-1}\left(\frac{12}{13}\right)-\cos^{-1}\left(\frac{4}{5}\right)\right)$$

and let $A=\sin^{-1}\left(\frac{12}{13}\right)$ and $B=\cos^{-1}\left(\frac{4}{5}\right)$.

(a) Write down the expression in terms of A and B and expand it.

(b) By drawing appropriate triangles, find the value of $\tan A$ and $\tan B$.

(c) Use your values from (b) to find the exact value of $\tan\left(\sin^{-1}\left(\frac{12}{13}\right)-\cos^{-1}\left(\frac{4}{5}\right)\right)$.

(d) Hence, write down the exact value of $\sin^{-1}\left(\frac{12}{13}\right)-\cos^{-1}\left(\frac{4}{5}\right)$.

Question 4

(a) Simplify the expansion of

$$\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)+\cos^{-1}(x)\right)$$

by drawing appropriate triangles where needed.

(b) Deduce that

$$\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)+\cos^{-1}(x)=\frac{\pi}{2}$$

and state the values of x for which this is true.

(c) Hence, sketch the graph of $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)+\cos^{-1}(x)$.

Question 5 Prove the following identities.

(a) $\tan^{-1}\left(\frac{5}{3}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{2}$

(b) $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$

(c) $\tan^{-1}(2)+\tan^{-1}(3)=\frac{3\pi}{4}$

(d) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(-\frac{16}{65}\right)$

Question 6 Prove the following identities and state the values of x for which they hold. You may use graphing software to assist you in finding these values of x.

(a) $2\cos^{-1}(x) = \cos^{-1}(2x^2 - 1)$

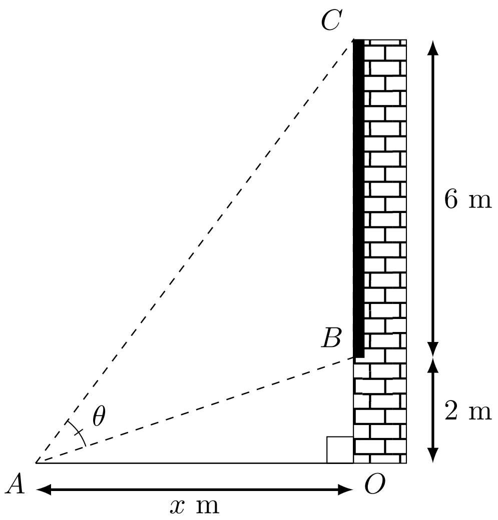
(b) $2\tan^{-1}(x) = \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)$

(c) $2\sin^{-1}(x) = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$

(d) $2\cos^{-1}(x) = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$

Question 7 [Application to billboard positioning]

An observer stands x metres away from a 6 metre tall billboard placed 2 metres above the ground as shown in the diagram below.



The *viewing angle* θ is the difference in angle of elevation from the observer to the top and the base of the billboard. Show that $\theta = \tan^{-1}\left(\dfrac{6x}{x^2 + 16}\right)$.

Question 8 Consider the expression

$$\cos^{-1}(x) - \sin^{-1}(x) = \frac{\pi}{4}.$$

(a) By considering the sine of both sides, then expanding and simplifying, show that

$$x = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$

(b) Use the identity $\sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$ to find another expression for x.

(c) Deduce that $\cos\left(\dfrac{3\pi}{8}\right) = \dfrac{\sqrt{2 - \sqrt{2}}}{2}$.

Challenge Problems

Problem 1 Show that $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$.

Problem 2

(a) Show that $\tan^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$.

(b) Draw a suitable right-angled triangle to justify this result.

(c) Suppose

$$\tan^{-1}(x) + \tan^{-1}(y) = \frac{\pi}{2}$$

where x, $y > 0$. Find the relationship between x and y.

Problem 3

(a) Show that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$.

(b) Suppose that

$$\sin^{-1}(x) + \sin^{-1}(y) = \frac{\pi}{2}$$

where x, $y > 0$. Find the relationship between x and y.

(c) Similarly, find the relationship between x and y if

$$\cos^{-1}(x) + \cos^{-1}(y) = \frac{\pi}{2}$$

where x, $y > 0$.

Chapter 4 Review

Inverse Trigonometry

Review

Question 1 Find the exact value of the following.

(a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (c) $\tan^{-1}(-1)$

(d) $\tan(\tan^{-1}(\sqrt{3}))$ (e) $\cos(\sin^{-1}(-1))$ (f) $\cos\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$

(g) $\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ (h) $\cos\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ (i) $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Question 2 Find the exact value of the following.

(a) $\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right)$ (b) $\tan^{-1}\left(\tan\left(\frac{6\pi}{5}\right)\right)$ (c) $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

(d) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$ (e) $\sin^{-1}\left(\sin\left(\frac{9\pi}{8}\right)\right)$ (f) $\cos^{-1}\left(\cos\left(\frac{9\pi}{8}\right)\right)$

Question 3 Find the exact value of the following.

(a) $\sin\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$ (b) $\cos\left(2\sin^{-1}\left(\frac{1}{3}\right)\right)$ (c) $\sin\left(2\tan^{-1}\left(\frac{1}{2}\right)\right)$

Question 4 Let $x = \sin^{-1}\left(\frac{2}{5}\right)$. Find the exact value of

(a) $\cos 2x$ (b) $\sin 2x$ (c) $\tan 2x$

Question 5 Show the following results.

(a) $\arctan\left(\frac{12}{5}\right) = 2\arctan\left(\frac{2}{3}\right)$ (b) $\arccos\left(\frac{3}{5}\right) - \arctan\left(\frac{3}{4}\right) = \arcsin\left(\frac{7}{25}\right)$

Question 6 Show the following results.

(a) $\sin^{-1}\left(\frac{2}{9}\right) + \cos^{-1}\left(\frac{2}{9}\right) = \frac{\pi}{2}$ (b) $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

(c) $\cos^{-1}\left(\frac{2}{5}\right) + \cos^{-1}\left(-\frac{2}{5}\right) = \pi$ (d) $\tan^{-1}\left(\frac{5}{12}\right) + \cos^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2}$

Question 7 Find the value of A in the following.

(a) $\sin^{-1}(A) = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{4}\right)$ (b) $\cos^{-1}(A) = \sin^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(\frac{1}{2}\right)$

Question 8 Show that

$$\sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

and state the values of x for which this is true.

Question 9 Show that if $A, B > 0$, then $\tan^{-1}(A) - \tan^{-1}(B) = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$.

Question 10

(a) Find the domain and range of $f(x) = 2\sin^{-1}(3x)$.

(b) Hence, sketch the graph of $y = f(x)$.

(c) Find the equation of $f^{-1}(x)$ and state its domain and range.

Question 11 Let $f(x) = 1 + 2\sin(3x)$ in the domain $-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$.

(a) Write down the range of $f(x)$.

(b) Find the equation of $f^{-1}(x)$.

(c) Write down the domain and range of $f^{-1}(x)$.

Question 12 Sketch the following graphs by first finding the domain and range. Make sure to label the end-points of the graph.

(a) $y = 2\sin^{-1}\left(\frac{x}{3}\right)$ (b) $y = -\cos^{-1}\left(\frac{x}{2}\right) + \pi$ (c) $y = \frac{\pi}{2} - 3\cos^{-1}(2x)$

(d) $y = \cos^{-1}(1-x)$ (e) $y = \sin^{-1}(2x-3)$ (f) $y = \sin^{-1}(3-2x)$

Question 13 Sketch the graphs of

(a) $y = |\sin^{-1}(x)|$ (b) $y = |\tan^{-1}(x)|$ (c) $y = \tan^{-1}(|x|)$

Question 14

(a) Sketch the graph of $y = \sin^{-1}(x) + \frac{\pi}{2}$

(b) Hence, sketch the graph of $y = \sin^{-1}(-x) + \frac{\pi}{2}$. What do you notice?

(c) Explain your findings.

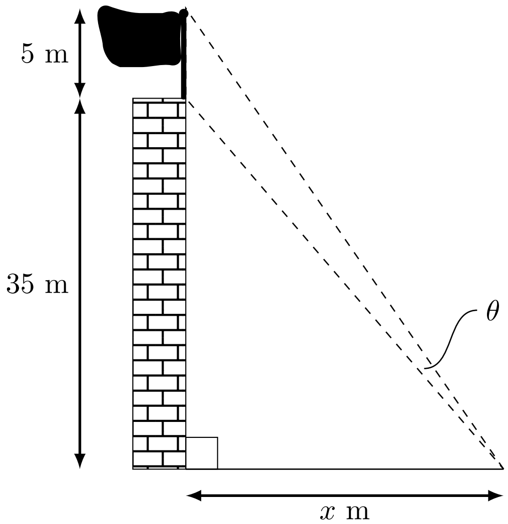
Question 15 Sketch the graph of the following by starting from the graph in (a) and applying the appropriate transformation. Make sure to label all appropriate endpoints.

(a) $y = 2\sin^{-1}(x) - \frac{\pi}{2}$ (b) $y = 2\sin^{-1}(|x|) - \frac{\pi}{2}$ (c) $y = \left|2\sin^{-1}(x) - \frac{\pi}{2}\right|$

Question 16 Suppose the line $y = ax + b$ bisects the angle between the two straight lines $y = px + q$ and $y = rx + s$. Show that

$$(a - p)(1 + ar) + (a - r)(1 + ap) = 0$$

Question 17 A 5 metre flag stands on top of a 35 metre-tall building as shown in the diagram below.



The flag subtends an angle of θ with the eye of an observer on the ground x metres away from the building. Show that

$$\theta = \arctan\left(\frac{5x}{x^2 + 1400}\right)$$

Investigation Task

Other Inverse Trigonometric Functions

So far you have learned the inverse trigonometric functions

$$y = \sin^{-1}(x)$$
$$y = \cos^{-1}(x)$$
$$y = \tan^{-1}(x)$$

But of course, there may be inverse functions for other trigonometric functions like $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$. Investigate their inverses, derive their equations and and express them in terms of familiar functions. Challenge Problem 1 of **Exercise 4A** may be useful for this task.

Investigation Task

Zig-zags

Earlier you learned how to sketch graphs like

$$y = \sin\left(\sin^{-1}(x)\right)$$
$$y = \cos\left(\cos^{-1}(x)\right)$$
$$y = \tan\left(\tan^{-1}(x)\right)$$

Composing the functions the other way around yields vastly differing results. Use graphing software to investigate the shapes of the following graphs, and use your knowledge of inverse trigonometric functions to explain their shape.

$$y = \sin^{-1}(\sin x)$$
$$y = \cos^{-1}(\cos x)$$
$$y = \tan^{-1}(\tan x)$$

5

PHYSICAL APPLICATIONS OF CALCULUS

- Rates of Change (Revision)
- Displacement, velocity and acceleration
- Exponential growth and decay
- Further growth and decay
- Related Rates of change

Exercise 5A

Rates of Change (Revision)

Fundamentals

Fundamentals 1
[Differentiation revision]

For each of the following functions of t, find the derivative with respect to t. Suppose that u, v are also functions of t, and a, b and k are constants.

(a) $Q(t) = (a + bt)^n$ (b) $Q(t) = uv$ (c) $Q(t) = \dfrac{u}{v}$

(d) $Q(t) = u^n$ (e) $Q(t) = e^{kt}$ (f) $Q(t) = e^{f(t)}$

Fundamentals 2

(a) To analyse the rate of change of a quantity Q at some time t_0, you need to analyse its i__________ rate of change at $t =$ ___. This is found by first d________ Q with respect to ___.

(b) If Q is some quantity, then the rate of change of Q is denoted by __.

Fundamentals 3
When finding the rate of change of a function $Q(t)$ at time $t = t_0$, you must d________ $Q(t)$ first, and then substitute $t =$ ___ afterwards.

Fundamentals 4

(a) A function $f(t)$ is increasing at $t = t_1$ if $f'(t_1)$ ___ 0.

(b) A function $f(t)$ is decreasing at $t = t_1$ if $f'(t_1)$ ___ 0.

Fundamentals 5

(a) If Q is increasing over an interval of t, then the rate of change of Q is p________ over that interval.

(b) If Q is decreasing over an interval of t, then the rate of change of Q is n________ over that interval.

(c) If Q remains the same over an interval of t, then the rate of change of Q is z______ over that interval.

Question 1 [Revision of differentiation techniques]

Differentiate the following.

(a) $N(t) = 4t^n + 3t + 2$ (b) $A(t) = \dfrac{t}{5} + \dfrac{5}{t} - \dfrac{2}{t^2}$ (c) $N(t) = \dfrac{3}{4t}$

(d) $P(t) = \dfrac{3t^2 - 6t + 8}{t}$ (e) $N(t) = t\sqrt{t}$ (f) $N(t) = (3t - 2)^3$

(g) $P(t) = (3t^2 - 5)^n$ (h) $Q(t) = t^2(2 - 3t)$ (i) $Q(t) = t^2(2 - 3t)^5$

(j) $X(t) = \dfrac{5t}{8 - t}$ (k) $S(t) = \dfrac{t^2}{t - 1}$ (l) $S(t) = \dfrac{2t}{(2t - 1)^2}$

(m) $X(t) = \dfrac{5}{8 - t^2}$ (n) $A(t) = \sqrt{2t^2 + 5}$ (o) $V(t) = \dfrac{1}{\sqrt{2t^2 + 5}}$

Question 2 [Differentiation techniques involving exponential functions]

Differentiate the following.

(a) $N(t) = 6e^{-0.5t}$ (b) $Q(t) = 120(1 - e^{-0.2t})$ (c) $Q(t) = 20(1 + 10e^{-0.2t})$

(d) $R(t) = e^{-t^2}$ (e) $V(t) = \dfrac{e^t + 1}{e^t}$ (f) $P(t) = \dfrac{3}{e^{3t} + 1}$

(g) $P(t) = \dfrac{t}{e^{3t} + 1}$ (h) $P(t) = \dfrac{e^{3t}}{e^{3t} + 1}$ (i) $P(t) = \dfrac{3}{e^{-t} + 1}$

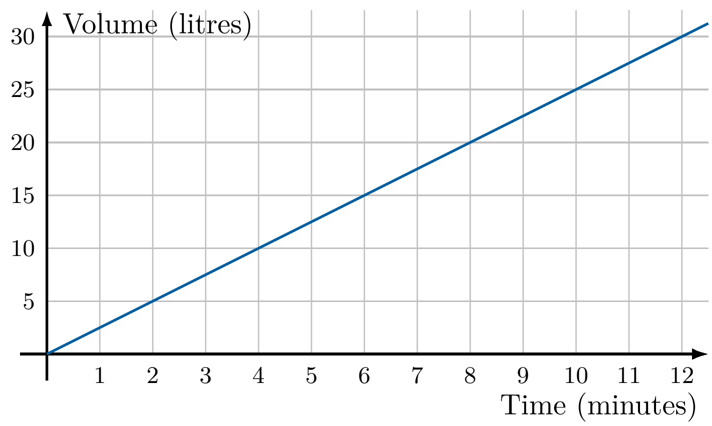
Question 3 Write down examples of suitable units of measurement you would be using in the following examples of rates of change with respect to time.

(a) Rate of pay. (b) Slowly draining water out of a tank.

(c) Cost of renting an apartment. (d) School fees.

Question 4 An empty bathtub is filled with water at a constant rate of 6 litres per minute. How long would it take to fill the bathtub with 42 litres of water?

Question 5 An empty bathtub is filled and the volume, in litres of water in the bath after t minutes is given by the linear function $V(t) = 2.5t$



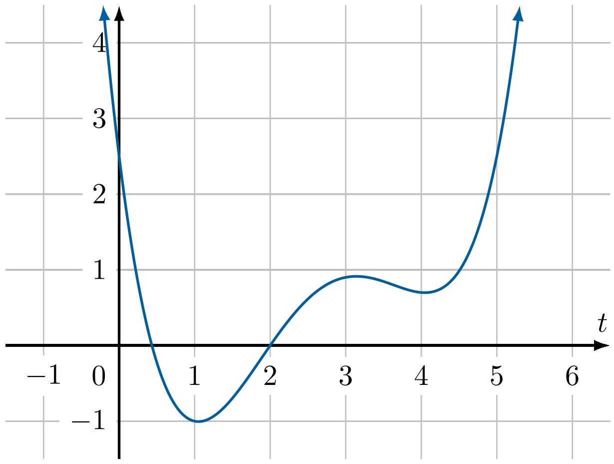
Find the rate of change of volume

Question 6 A circular disc expands as it is heated. The area, in cm^2 of the disc increases according to the formula $A = 3t^2 + t$. Find the rate of increase in the area after 4 minutes.

Question 7 In the domain $0 \leq t \leq 5$ write down the values of t for which the function below is

(a) increasing

(b) decreasing



Question 8 Let Q represents the population of a sports club and t represents time in years.

(a) What does $\dfrac{dQ}{dt}$ or Q' represent?

(b) What can we say about Q if $\dfrac{dQ}{dt} < 0$, $\dfrac{dQ}{dt} = 0$ or $\dfrac{dQ}{dt} > 0$?

(c) Sketch the graph $\dfrac{dQ}{dt} = t - 8$ and explain what happens to Q for different values of t.

Question 9 The number of bacteria in a culture is changing according to the formula

$$B = 2t^3 - 3t^2 + 800,$$

where t is time in hours. Find

(a) the number of bacteria initially.

(b) the rate at which the bacteria was increasing after 2 hours.

Question 10 A hose is being used to fill a small backyard pool and the volume function with respect to time is given by

$$V = 5t^2,$$

where V is litres after t minutes

(a) Sketch the graph of the volume of water in the pool against time.

(b) What do you notice about the rate of change of V?

(c) What does $V'(t)$ represent on the curve you have drawn?

(d) Find $V'(3)$ and explain its meaning in terms of the pool being filled.

Question 11 Joel is walking home from school. His distance from school, in meters, after t minutes is modelled by the function

$$D(t) = 20(t+2)(12-t)$$

(a) How far is his home from school?

(b) How long will Joel take to walk home?

(c) Graph this function

(d) Find $D'(t)$.

(e) What units should we use to express $D'(t)$?

(f) Find $D'(5)$ and explain your result.

(g) Does he go directly home? Explain your reasoning carefully.

Question 12 A water tank is being drained for a given amount of time and

$$V = 2000\left(1 - \frac{t}{25}\right)^2$$

represents volume in litres of the water remaining in the tank t minutes after the drain is opened.

(a) How much water is in the tank initially?

(b) When is the tank completely drained?

(c) Find the instantaneous rate of change of volume

(i) initially. (ii) when the tank is drained. (iii) at $t = 5$.

(d) Is the rate of decrease increasing or decreasing while the water is drained? Give reasons for your answer.

Question 13 Under ideal conditions, a certain bush grows so its height is increasing at a rate of $\frac{12}{3t+1}$ cm/month. A 24 cm bush is planted and grows under ideal conditions in a nursery.

(a) Express the information given as a derivative.

(b) Find the derivative of the above expression.

(c) How fast was the bush growing initially?

(d) How fast was the bush growing when $t = 12$?

(e) Explain the different growth rates using your answer in (b).

Question 14 Leah took a bottle which had 100 millilitres of water in it. She poured more water into it for thirty seconds until it was full. During this time the volume flow rate $\frac{dV}{dt}$ of water into the bottle, in millilitres per second, was given by $\frac{dV}{dt} = 2(30 - t)$.

(a) Show that $V = 100 + 60t - t^2$ for $t \leq 30$.

(b) How many millilitres of water were in the bottle when it was full?

(c) When is the flow rate the fastest? What is its value then?

Question 15 When a jet engine begins operating, the rate of fuel burn R kg per minute, is given by

$$R = 5 + \frac{5}{1+t}$$

where t is time in minutes.

(a) What is the rate of burn initially?

(b) Draw a sketch of R as a function of t.

(c) What is the rate of the burning after 5 minutes?

(d) What value does R approach as t becomes very large?

Question 16 Let

$$V = \frac{1}{3}t^3 - 4t^2 + 12t$$

represent volume of water in a container in litres over a 9 minute period

(a) Draw a graph of V during this period.

(b) How much water is in the container at the end of 9 minutes?

(c) Find $\frac{dV}{dt}$, graph this function and explain what it represents.

(d) Explain what is happening to the container initially and then for each case of $\frac{dV}{dt} > 0$, $\frac{dV}{dt} = 0$, $\frac{dV}{dt} < 0$.

Question 17 A container holds 100 mL of water, but has a leak. The volume of water remaining in the container after t seconds is

$$V(t) = 100\left(1 - \frac{t}{20}\right)^2$$

(a) How much water is in the container after four seconds?

(b) Find the rate of change of volume after t seconds.

(c) Find the rate of change of volume after 4 seconds.

(d) According to this model after how many seconds will the container be completely empty?

Challenge Problems

Problem 1 A flat circular disc is being heated so that the rate of increase of the area (A in cm^2), after t hours, is given by $\dfrac{dA}{dt} = \dfrac{1}{8}\pi t$.

(a) Verify that the area of the disk can be expressed as $A = \dfrac{1}{16}\pi t^2 + C$ for a constant C

(b) Initially the disc has a radius of 20 cm find the value of C.

(c) Find the rate of increase of the area when the area is 500π cm^2.

Problem 2 [Logistic equation]

It is assumed that the number N of termites in a certain mound at time t is given by

$$N = \frac{M}{9 + e^{-t}}$$

where M is a constant, and t is measured in months.

(a) At time $t = 0$, N is estimated at 1000 termites. Find the value of M.

(b) What is the value of N after three months, rounded to the nearest integer?

(c) What is the eventual size of the ants nest?

(d) Find an expression for the rate at which the number of termites changes at any time t.

(e) What is the rate of increase of the ants initially

(f) What happens to the rate of increase eventually?

(g) State whether the termite population is increasing or decreasing.

(h) Use graphing software to draw the graph of the number of ants in this nest and justify your answers above.

Exercise 5B

Displacement, velocity and acceleration

Fundamentals

Fundamentals 1

(a) Straight line motion can be modelled by giving the d________ as a function of time

(b) The rate of change of displacement is called the v________ of the particle.

(c) The rate of chance of velocity is called the a__________ of the particle.

Fundamentals 2

Write down at least two mathematical expressions used to represent

(a) velocity

(b) acceleration

Fundamentals 3

(a) When a particle moves to the right, the velocity is p________.

(b) When a particle moves to the left, the velocity is n________.

(c) When a particle is at rest, the velocity is z______.

Fundamentals 4

(a) Write down the value of v that means the particle is 'at rest'.

(b) Write down the value of t that means 'initially'

(c) Write down the value of t that means 'eventually'

Fundamentals 5

(a) If a particle is speeding up, then $\dot{x}$ and $\ddot{x}$ have the same/opposite (choose one) signs.

(b) If a particle is slowing down, then $\dot{x}$ and $\ddot{x}$ have the same/opposite (choose one) signs.

Question 1 A particle moves so that its displacement x metres at time t seconds is given by

$$x = 5 - 2t$$

(a) Find the position of the particle initially.

(b) Find the velocity of the particle.

(c) In what direction is the particle moving?

(d) What is the displacement of the particle when $t = 3$?

(e) When does the particle have displacement $x = -3$?

(f) When does the particle pass through the origin?

Question 2 The position of a particle x metres from O after t seconds is given by

$$x = t^2 - 2t - 8$$

Find the following.

(a) The initial position of the particle.

(b) The velocity-time function.

(c) The initial velocity of the particle.

(d) The direction the particle starts to move?

(e) When the particle is at rest.

(f) The acceleration-time function.

Question 3 [Changing direction]

Bob claims that to find when a particle changes direction, simply find when it is at rest. In other words, let $v = 0$. Mary claims that letting $v = 0$ does not guarantee that the particle changes direction. Their teacher gives them both the function $x = t^3 - 3t^2 + 3t$ to investigate.

(a) Find the velocity-time function.

(b) Find when the particle is at rest by letting $v = 0$.

(c) Complete the following table.

t	0	1	2
v			

(d) Who is correct?

(e) Hence, describe a series of steps to determine whether the particle changes direction or not.

Question 4 A projectile is fired into the air from level ground, and it's height y metres is

$$y = 6t - t^2$$

where t is time in seconds from the moment the projectile is launched.

(a) Find the velocity-time function.

(b) Show that the particle changes direction at $t = 3$.

(c) When does the particle land back on the ground?

(d) How many metres does the particle travel upwards before stopping?

(e) How many metres does the particle travel downwards before reaching the ground?

(f) Hence, how many metres does the particle travel overall?

(g) Bob claims that to find the distance the particle travels overall, just substitute the final value of $t = 6$ into the equation for y. Explain why this is incorrect, and state what this gives instead.

Question 5 Write down a series of steps describing how to determine the total distance that a particle travels over a given period of time.

Question 6 A projectile is fired into the air and its height y metres is given by

$$y = 9 + 8t - t^2$$

where t is time in seconds.

(a) Find the initial height.

(b) Find the velocity-time function.

(c) When does the particle reach maximum height?

(d) Find the maximum height reached

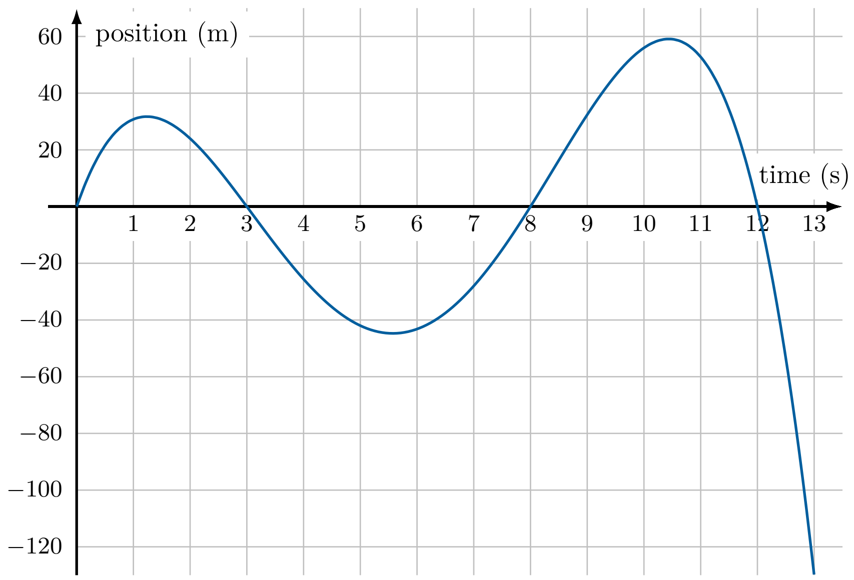
(e) Find when the projectile has height 21 metres.

(f) Find when the projectile reaches the ground

(g) Sketch the graph showing the height of the projectile over time t

(h) How far will the projectile travel in the first 9 seconds?

Question 7 A particle moves along a line. The graph below shows the particle's displacement over the first 13 seconds of movement.



(a) For the times indicated write down if the particle is moving to the right or left

(i) $t = 2$ (ii) $t = 7$ (iii) $t = 8$ (iv) $t = 9$ (v) $t = 11$

(b) Write down the approximate values of t when the particle is at rest.

Question 8 A particle moves so that its displacement is given by

$$x = t^3 - 3t$$

where x is in metres and t is time in seconds.

(a) In what direction is the particle moving initially?

(b) When does the particle first come to rest?

(c) Determine whether at this point, the particle changes direction or not.

(d) When and where does the particle change direction?

(e) Find the distance travelled in the first 3 secs

(f) Describe the motion of the particle

(g) Draw the displacement-time graph of the particle.

Question 9 A particle moves in a straight line so its displacement is given by

$$x = 3e^{2t}$$

cm at time t seconds.

(a) Find the initial position and velocity of the particle.

(b) Find the exact value of the velocity after 4 seconds.

(c) Find the exact time when the acceleration is 24 cm/s^2.

(d) Show that the particle is speeding up throughout it's entire motion.

(e) Sketch the displacement-time graph.

Question 10 The following are all FALSE statements, as they currently stand. Explain why each statement is false, and provide a counter-example.

(a) To find when a particle changes direction, let $v = 0$ and solve for t.

(b) If a particle is always moving to the right, then $x \to \infty$ as $t \to \infty$.

(c) If $v \to 0$ as $t \to \infty$, then the particle must approach some fixed value.

Question 11 A particle moves with displacement-time equation

$$x = \frac{1}{t+1}$$

where t is time in seconds and x is in metres.

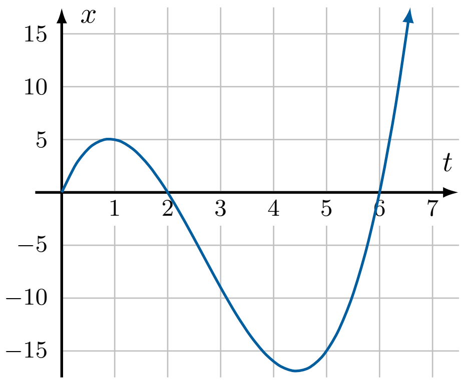
(a) Find the initial position and velocity of the particle.

(b) Hence, in which direction does the particle move initially?

(c) Show that the particle is always moving to the left.

(d) What is the limiting position and velocity of the particle?

Question 12 This graph shows the displacement of a particle as it moves on a straight line.



(a) When is the particle at the origin?

(b) In what direction does the particle start to move?

(c) Give an approximate value for when you think the particle is at rest.

(d) When is the particle moving to the right?

(e) Describe the motion of this particle eventually.

Question 13

(a) Suppose $a > 0$. Describe what happens to the particle's motion when

(i) $v > 0$ (ii) $v = 0$ (iii) $v < 0$

(b) Suppose $a = 0$. Describe what happens to the particle's motion if

(i) $v > 0$ (ii) $v = 0$ (iii) $v < 0$

(c) Suppose $a < 0$. Describe what happens to the particle's motion if

(i) $v > 0$ (ii) $v = 0$ (iii) $v < 0$

Question 14 A particle moves with displacement-time equation

$$x = 1 + e^{-t}$$

where t is time in seconds and x is in metres.

(a) Show that $\dot{x} < 0$.

(b) Show that $\ddot{x} > 0$.

(c) Find the initial velocity and displacement of the particle.

(d) Hence, describe the motion of the particle.

Question 15 A particle is moving along a straight line and the particle's velocity at time t seconds is given by

$$v = t^2 - 8t + 14$$

(a) What is the particle's velocity at $t = 4$?

(b) What is the particle's acceleration at $t = 4$?

(c) At $t = 4$ is the particle speeding up, slowing down or neither?

(d) Draw on the same diagram the velocity and acceleration of the particle.

(e) Highlight the values of t when the particle is slowing down.

Question 16 A particle moves in a straight line such that its displacement x cm after t seconds is given by

$$x = t^3 - 9t^2 + 15t + 2$$

(a) Find the velocity and acceleration of the particle.

(b) In what direction is the particle moving initially?

(c) When and where does the particle change direction?

(d) Draw on the same diagram the velocity and acceleration of the particle.

(e) Find the values of t where the particle is

(i) slowing down. (ii) speeding up.

(f) Find the total distance travelled by the particle for $0 \leq t \leq 6$.

Question 17 A particle moves in a straight line and at time t (seconds) its position x (cm) from the origin 0 is given by $x = 2 - e^{-t}$.

(a) Draw this function.

(b) Find the velocity v of this particle.

(c) What is the initial velocity?

(d) What information does the velocity give you about the motion of this particle?

(e) Describe what happens eventually to the position x and its velocity.

(f) Show that $x = \ddot{x} + 2$.

Question 18 Draw velocity-time graphs to represent the information below

(a) Leah runs at a constant speed for 10 seconds and has run 80 metres, she then accelerates at a constant rate so that 5 seconds later she reaches a speed which is twice her initial speed.

(b) Sam runs at a constant speed for 20 seconds and has run 120 metres and then slows down coming to rest 5 seconds later.

Question 19 A particle moves with velocity-time equation

$$v = 3t^2 - t^3$$

where t is time in seconds.

(a) When is the particle furthest from the origin?

(b) Sketch the velocity-time graph.

(c) Find the value of t when $a = 0$.

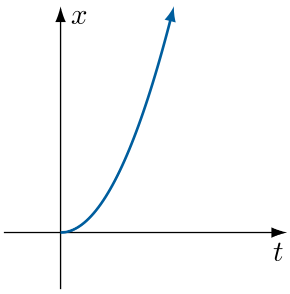
(d) Explain why this value of t also represents the highest point on the curve.
Hint: Remember that acceleration represents the rate of change of velocity.

(e) Find the maximum velocity of the particle.

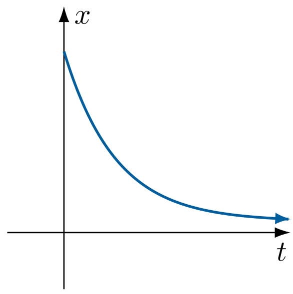
(f) Suppose the particle starts at the origin. Describe the motion of the particle over the domain $t \in [0, 3]$.

Question 20 The following diagrams show the displacement-time graphs of a particle with displacement that is either increasing at an increasing rate, increasing at a decreasing rate, decreasing at a decreasing rate or decreasing at an increasing rate. Match the correct description to the graph.

(a)



(b)



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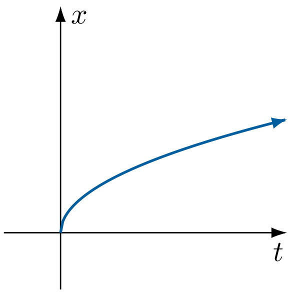
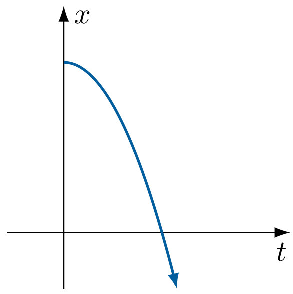
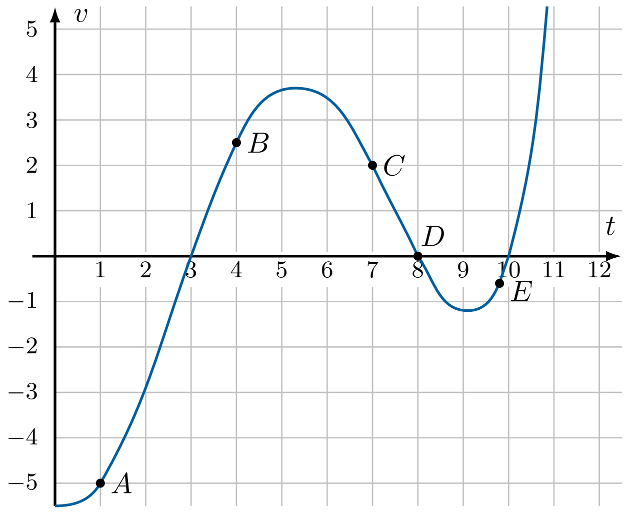


(d)



Question 21 A particle moves along a straight line. The graph drawn gives the particle's velocity over the first 11 seconds of motion.



(a) For each point indicated on the graph, state whether

(i) the particle has positive, negative or zero velocity.

(ii) the particle is moving to the right, to the left, or momentarily at rest.

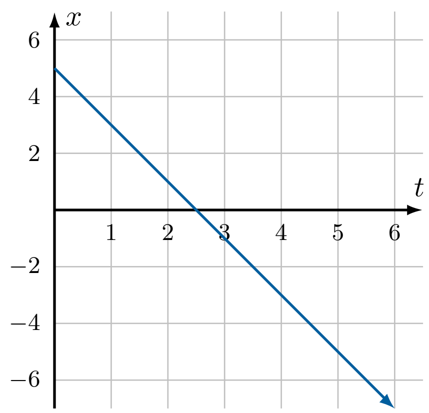
(iii) the particle is speeding up or slowing down.

(b) Explain what the gradients of the tangents mean on a velocity-time graph.

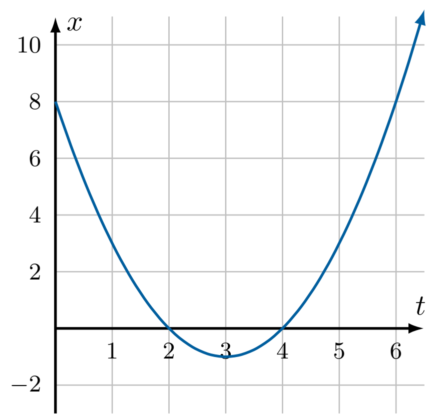
(c) Give the approximate times on this graph when the particle is neither speeding up nor slowing down, and give reasons for your answer.

Question 22 For each displacement-time graph, sketch the graphs for velocity and acceleration

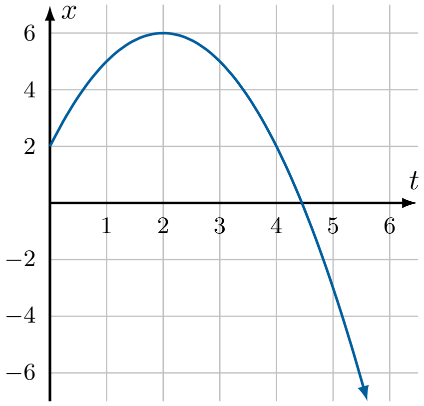
(a)



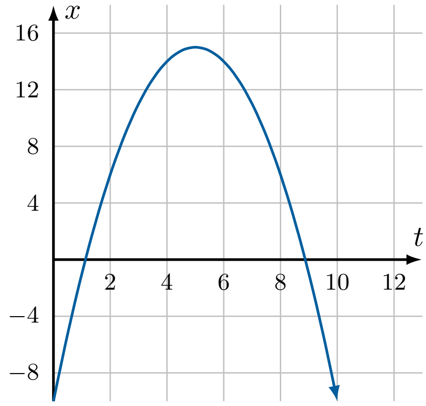
(b)



(c)

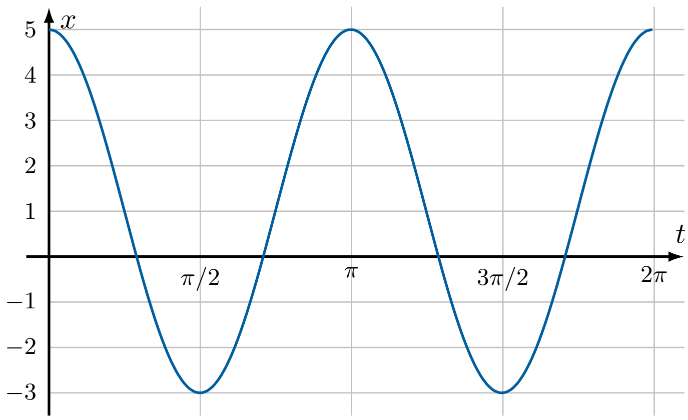


(d)



Question 23 [Simple harmonic motion]

The diagram shows the displacement, x metres, of a particle from the origin O after t seconds for $0 \leq t \leq 2\pi$.



(a) What is the initial displacement of the particle?

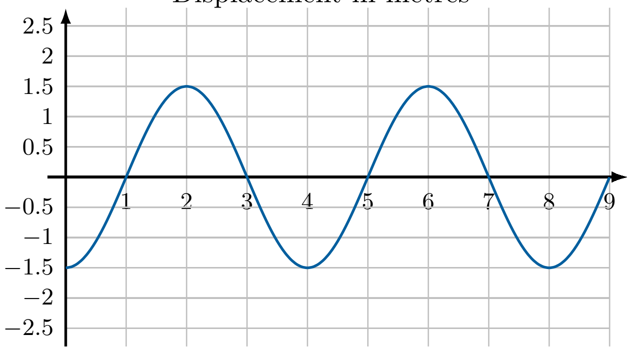
(b) When is the particle at rest during this motion?

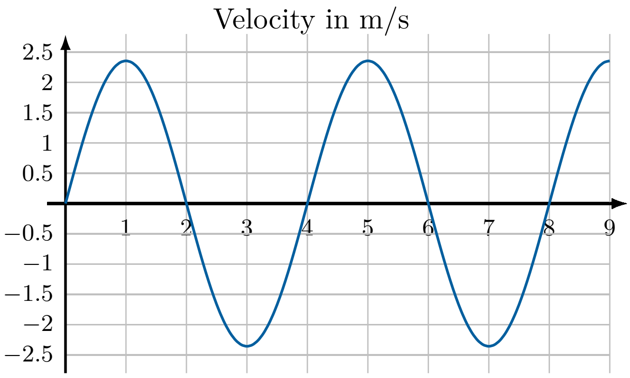
(c) What value of x does the particle appear to oscillate about?

(d) Describe the behaviour of the particle.

Question 24 A particle moves in a straight line. The diagrams below are the displacement-time and velocity-time graphs of this particle during the first 9 seconds of its motion.

Displacement in metres





(a) What is the initial position and velocity of the particle?

(b) After how long, does the particle return to its starting point?

(c) In what direction does the particle start to move?

(d) When does the particle change direction for the first time?

(e) Where does the particle reach its maximum velocity?

(f) Describe the motion of this particle.

Question 25 A particle moves with velocity-time equation

$$\dot{x} = \sqrt{16t - t^2}$$

where t is time in seconds and x is in metres.

(a) Show that this model is only valid for $t \in [0, 16]$.

(b) Find the initial velocity and the velocity when $t = 16$.

(c) Find the value of t for which $a = 0$.

(d) Hence, find the maximum speed of the particle.

Question 26 A particle moves with displacement-time equation

$$x = 2t^3 - 6t^2$$

By analysing the motion, find how far the particle travels in the first four seconds of motion.

Challenge Problems

Problem 1 [Resisted motion]

The velocity v metres per second of a body falling from rest after t seconds is given by

$$\frac{dx}{dt} = 50\left(1 - e^{-0.5t}\right)$$

Find

(a) the initial velocity.

(b) the acceleration at any time t.

(c) what happens to $\frac{dx}{dt}$ at $t \to \infty$.

(d) Draw the graph of $\frac{dx}{dt}$ against time.

Problem 2 The position x cm, of a particle moving along an x-axis is given by $x = 5t + e^{-2t}$ where t is the time in seconds.

(a) Find the initial position, velocity and acceleration of the particle.

(b) Draw a velocity-time graph.

(c) Will the particle ever come to rest? Explain.

(d) What is the eventual velocity and acceleration of this particle?

Problem 3 Describe the motion of the particle satisfying the following displacement-time equations. Your answer should include

- when the particle changes direction and where
- the displacement as $t \to \infty$
- the velocity as $t \to \infty$
- whether the particle speeds up or slows down as $t \to \infty$

(a) $x = te^{-t}$

(b) $x = \frac{2t}{1+t^2}$

(c) $x = \frac{1-t^2}{1+t^2}$

(d) $x = t^2e^{-t}$

Exercise 5C

Exponential growth and decay

Fundamentals

Fundamentals 1

(a) If the rate of change of some quantity Q is p________ to the quantity itself, then $\frac{dQ}{dt} = \_\_\_$, where k is a constant.

(b) This is called a d________ equation since it is a relationship between the rate and the quantity.

(c) If $k > 0$, then Q is i________.

(d) If $k < 0$, then Q is d________.

(e) If $k > 0$, then $Q \to \_\_\_$ as $t \to \infty$.

(f) If $k < 0$, then $Q \to \_\_\_$ as $t \to \infty$.

Fundamentals 2

Suppose that A and k are positive constants, and that t represents time.

(a) The equation $Q = Ae^{kt}$ represents exponential growth/decay (circle one).

(b) The equation $Q = Ae^{-kt}$ represents exponential growth/decay (circle one).

(c) In either case, when $t = 0$, $Q = \_\_\_$. This means that A is the i______ quantity.

(d) Hence, sometimes the equations are instead expressed as $Q = \_\_\_e^{kt}$ or $Q = \_\_\_e^{-kt}$ for exponential growth and decay respectively.

Fundamentals 3

(a) If $P = Ae^{kt}$, then $\frac{dP}{dt} = \_\_\_$.

(b) Conversely, if $\frac{dP}{dt} = kP$, then $P = \_\_\_\_\_\_\_\_$.

Fundamentals 4

Suppose M is the mass of a radioactive substance at time t.

(a) Half-life refers to the amount of time that it takes for a radioactive mass to decay to h______ of the original mass.

(b) In other words, when $t =$ half life, then $M = \_\_\_ \times$ original mass.

Question 1 Write down a mathematical equation that summarises each of the following statements. Assume that the growth constant k is positive.

(a) The rate of bacteria in a culture is proportional the the number of bacteria present.

(b) The rate of increase of the population of a country is proportional to the population present at any instant.

(c) The rate of decay of a radioactive mass is proportional to the mass of the substance at any given time.

Question 2 Sketch the following curves on the same set of axes.

(a) $P_1 = 10e^{0.5t}$ and $P_2 = 20e^{0.5t}$

(b) $P_1 = 10e^{-0.5t}$ and $P_2 = 20e^{-0.5t}$

Question 3 Use graphing software to draw the graph of the following equations, and comment how the graph changes as $k > 0$ varies. If each of the following represent the population of an animal species over time, comment on how k affects the growth or decay of the population.

(a) $P = 10e^{kt}$

(b) $P = 10e^{-kt}$

Question 4 [Revision of some algebra involving exponentials and logs]

Simplify the following.

(a) $e^{\ln 3}$

(b) $e^{2\ln 3}$

(c) $e^{-\frac{1}{2}\ln 9}$

(d) $e^{\ln A}$

(e) $e^{-\ln A}$

(f) $e^{k\ln A}$

Question 5 The number of people in a town is given by $P = 30000e^{0.03t}$, where t is the number of years since the year 2000.

(a) What is the initial population?

(b) What is the population projected to be in 2025?

(c) During which year will the population be double of what it is initially?

(d) Show that $\dfrac{dP}{dt} = 0.03P$.

(e) Hence, find the rate of increase of the population in the year 2000?

(f) The population of another town is $Q = 50000e^{-0.02t}$, where t is also the number of years since 2000. During which year will the two towns have the same population?

Question 6 The mass M, in kilograms, of a radioactive substance is given by $M = 12e^{-0.002t}$.

(a) What is the initial mass of the substance?

(b) After how many years, to the nearest year, will the mass be less than half of what it was originally?

Question 7 Ten years ago, the population of a certain town was 25000 and today, it is 30000. Let the rate of the population be given by $\frac{dP}{dt} = kP$, where k is a constant and t is the time since ten years ago.

(a) Show that $P = P_0 e^{kt}$ satisfies this equation.

(b) Find the value of k.

(c) Hence, find the rate at which the population was growing ten years ago, and the rate that it is growing today to the nearest person per year.

Question 8 The mass M of an organism growing in a pond after t days is given by $M = M_0 e^{kt}$, where k is a constant and M is in grams. It takes two days for an initial mass of 10 grams to grow to 14 grams.

(a) Find the value of M_0 and k.

(b) Find the mass after 10 days, correct to 1 decimal place.

(c) After how many days will the initial mass have at least doubled?

(d) Find the rate at which it is growing after ten days, correct to one decimal place?

Question 9 The population P of a town is falling at a rate given by $\frac{dP}{dt} = -kP$, where t is time in years and k is a positive constant.

(a) Show that $P = P_0 e^{-kt}$ satisfies this equation.

(b) Find the value of k if the population has halved after 20 years.

Question 10 The amount C of a carbon isotope in a dead tree trunk is given by $C = C_0 e^{-kt}$, where t is measured in years from the death of the tree, and k is a positive constant.

(a) Explain the significance of C_0.

(b) Find k, correct to four significant figures, if the half-life of the isotope is 6000 years.

(c) A particular tree trunk is found to have 12% of the amount of the isotope compared to what a typical living tree trunk would have. How many years ago at least did the tree die, correct to the nearest 1000 years?

Question 11 The mass M of a chemical in a system undergoing a reaction is given by the equation $M = M_0 e^{-kt}$. It is known that 200 kg of the chemical reduced to 160 kg after three hours.

(a) Show that there is exactly 128 kilograms of the chemical remaining after six hours.

(b) Find the rate at which the mass is reducing after six hours, correct to two decimal places

(c) Find how long it will take for 80% of the chemical to react, correct to two decimal places.

Question 12 The half-life of radium is approximately 1600 years. Find the percentage that a mass of radium will have decayed after

(a) 800 years.

(b) 400 years.

Challenge Problems

Problem 1 [Half-life is independent of mass]

Prove that the half-life of a radioactive substance depends only on the growth constant k.

Problem 2 The population of two identical bacterial colonies is compared after being given two different types of antibiotics. Their populations are given by

$$P_1 = Ae^{-kt}$$
$$P_2 = Ae^{-2kt}$$

(a) To which population was the antibiotic more effective?

(b) Draw a graph of the two populations.

(c) Let T be the time when the difference in population is a quarter of their initial population. Show that $T = \frac{1}{k}\ln 2$.

Problem 3 Let $P^{(n)}$ represent the n^{th} derivative of P with respect to time t. Consider a standard exponential growth population model $P = P_0 e^{kt}$.

(a) Show that $P^{(1)} = kP$.

(b) Hence, show that $P^{(n)} = k^n P$.

Problem 4 Consider two populations A and B sharing the same growth constant k, but with one population increasing and the other population decreasing. The populations are modelled respectively by

$$A = A_0 e^{kt}$$
$$B = B_0 e^{-kt}$$

(a) Show that the two populations will be the same when

$$t = \frac{1}{2k}\ln\left(\frac{B_0}{A_0}\right)$$

(b) Hence, or otherwise, explain why $A_0 < B_0$ for the populations to be the same at some point.

Exercise 5D

Further growth and decay

Fundamentals

Fundamentals 1

(a) The modified exponential model is $P = B + Ae^{kt}$, which has the differential equation $\frac{dP}{dt} = $ ______

(b) If $k > 0$, then the model can approach either ___ or ___ as $t \to \infty$.

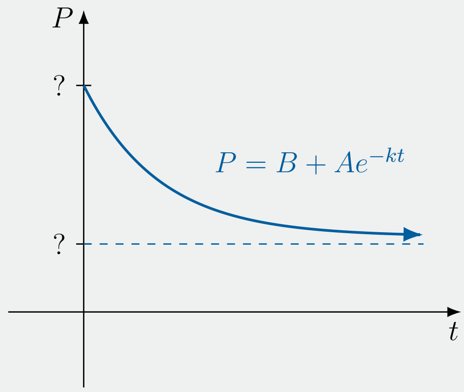
(c) If $k < 0$, then the model will always approach ___ as $t \to \infty$.

Fundamentals 2

Show that $P = B + Ae^{kt}$ satisfies $\frac{dP}{dt} = k(P - B)$.

Fundamentals 3

The diagram below shows the graph of $P = B + Ae^{-kt}$, where A, B, $k > 0$. Label the asymptote and y-intercept.



Fundamentals 4

The equation $T = B + Ae^{-kt}$ satisfying the differential equation $\frac{dT}{dt} = $ __________ is often used as the model for N________ Law of C__________.

Question 1 [Revision of sketching exponential curves]

Sketch the graphs of the following.

(a) $y = 2 + e^{-x}$

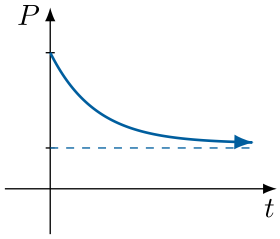
(b) $y = 2 - e^{-x}$

(c) $y = 2 - e^{x}$

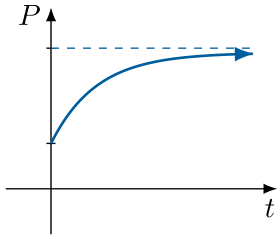
(d) $y = 2 + e^{x}$

Question 2 For each of the following graphs, determine whether it is more appropriate to use $P = B + Ae^{kt}$ or $P = B + Ae^{-kt}$ to model the graph, given that $k > 0$ and $P > 0$.

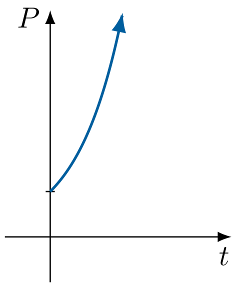
(a)



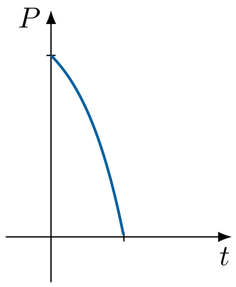
(b)



(c)



(d)



Question 3 Write down an appropriate differential equation, using appropriate variables, that models the following descriptions.

The rate of change of the temperature of an object is proportional to the difference between the environmental temperature and the temperature of the object.

Question 4

(a) Show that $P = 100 + 50e^{kt}$ satisfies the differential equation $\dfrac{dP}{dt} = k(P - 100)$.

(b) Show that $P = 100 - 50e^{kt}$ satisfies the differential equation $\dfrac{dP}{dt} = k(P - 100)$.

(c) Show that $P = 100 + 50e^{-kt}$ satisfies the differential equation $\dfrac{dP}{dt} = -k(P - 100)$.

(d) Show that $P = 100 - 50e^{-kt}$ satisfies the differential equation $\dfrac{dP}{dt} = -k(P - 100)$.

Question 5 A cup full of coffee initially has temperature 70°C, and it sits in a room of temperature 20°C. The temperature of the coffee cup can be modelled by the differential equation

$$\frac{dT}{dt} = k(B - T),$$

where B is the temperature of the room and T is the temperature of the coffee cup after t minutes.

(a) Show that $T = 20 + 50e^{-kt}$ satisfies the differential equation.

(b) After ten minutes, the temperature of the coffee cup reduces to 50°C. Find the value of k in exact form.

(c) What is the temperature of the coffee cup after five minutes?

(d) After how many minutes, to the nearest minute, will the temperature drop below 30°C?

(e) What is the limiting temperature of the coffee?

(f) Draw a graph of the temperature against time, labelling all important features.

Question 6 A frozen casserole dish with initial temperature $T = 0$°C is left in an oven set to 180°C. After ten minutes, the dish has temperature $T = 30$°C. The rate that the temperature of the dish increases is proportional to the difference in temperature between the oven and the dish.

(a) Is the growth constant positive or negative?

(b) Complete the equation $\dfrac{dT}{dt} =$ ____________.

(c) Show that $T = 180(1 - e^{-kt})$ satisfies the differential equation from (b).

(d) Find the value of k. Leave in exact form.

(e) Find the temperature of the dish after an hour.

(f) The dish is ready to serve when the interior temperature is at least 60°C. After how many minutes, to the nearest minute, will the dish be ready?

(g) Draw a graph of the temperature against time, labelling all important features.

Question 7 The population P of a bird species is modelled by the differential equation

$$\frac{dP}{dt} = -k(P - Q),$$

where t is time in years both k and Q are fixed constants.

(a) What is the significance of the value of Q with respect to the population of an animal species in this model?

(b) Show that the equation $P = Q + Ae^{-kt}$ satisfies the differential equation.

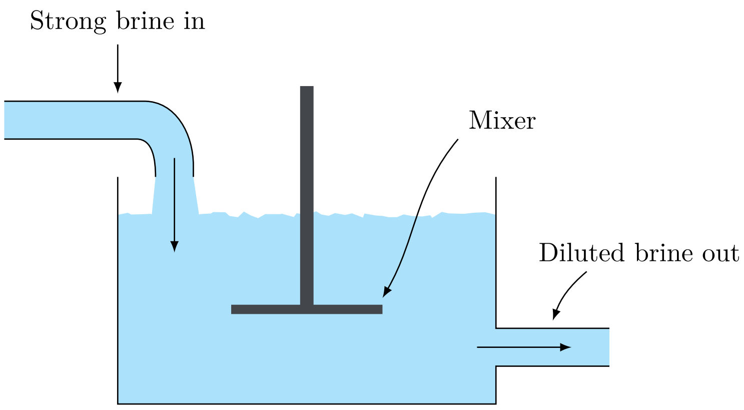
(c) Initially, there are 1000 birds. After ten years the population has doubled. After an extended period of time, the population of the bird species plateaus at 8000. Show that $k = \dfrac{1}{10}\ln\left(\dfrac{7}{6}\right)$.

(d) Hence, find how many years it took for the population to exceed half of its maximum.

Challenge Problems

Problem 1 [Brine problem]

The diagram below shows a simple model of a brine mixer.



Initially, the tank contains 500 litres of pure water. A strong brine with concentration 5 grams per litre is pumped into the tank continuously at a rate of R litres per minute, and then mixed instantaneously. At the same time, the mixture is being pumped out at a rate of R litres per minute so that the tank always has 500 litres of liquid at any given time. Let the amount of salt in the tank after t minutes be S.

(a) How many grams of salt are entering the system per minute?

(b) How many grams of salt are leaving the system after t minutes, in terms of S and R?

(c) Show that $\dfrac{dS}{dt} = -\dfrac{R}{500}(S - 2500)$.

(d) Show that

$$S = 2500 + Ae^{-\frac{Rt}{500}}$$

satisfies the differential equation in (c).

(e) How much salt will be in the tank, if the system is left to run for an extended period of time?

(f) After 1 hour, the salt concentration in the tank is 4 grams per litre. What was the value of R?

Problem 2 [Resisted motion]

A particle falls from rest with velocity v given by

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

where g is the constant due to gravity, k is also a constant and t is time in seconds.

(a) Show that v satisfies the acceleration equation $a = g - kv$.

(b) State the value of v as $t \to \infty$.

(c) Find the value of v when $a = 0$. What do you notice about this value of v and the answer from (b)? Can you think of a physical reasoning of this?

(d) Find the value of t when $v = \dfrac{g}{2k}$.

(e) Draw the graph of v against t.

(f) At some time $t = t_1$, the particle reaches half of its maximum speed. At another time $t = t_2$, the particle reaches three quarters of its maximum speed. Show that $t_2 = 2t_1$.

Exercise 5E

Related Rates of change

Fundamentals

Fundamentals 1

(a) A r________ rate of change is the rate of change of some variable, given the rate of change of another related variable.

(b) For example, the rate that the volume of a sphere increases can be found if we are given the rate that the r_____ increases. This is because the radius is r________ to the volume.

Fundamentals 2

Suppose we wish to find $\dfrac{dV}{dt}$ but we are given $\dfrac{dr}{dt}$. Then we can use the chain rule and split $\dfrac{dV}{dt}$.

$$\frac{dV}{dt} = \frac{dV}{\quad} \times \frac{\quad}{dt}$$

Fundamentals 3
For each of the statements below, complete the following.

$$\text{Required to find } \frac{d\,?}{d\,?} \text{ given that } \frac{d\,?}{d\,?} = ?$$

(a) Find the rate of change of the area A of a circle with respect to time, given that the radius r increases at a rate of 1 unit per second.

(b) Find the rate of change of the volume V of a sphere with respect to time, given that the area decreases at a rate of 4 units per second.

(c) Find the rate of change of the length l of the diagonal with respect to time, given that the side length x increases at a rate of 2 units per second.

Question 1 Write down the following chains to find the rate of the original quantity with respect to another quantity. For example if $A = 4x^2$, then

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = 8x \times \frac{dx}{dt}$$

(a) $V = \dfrac{1}{3}\pi r^3$ (b) $A = 6x^2$ (c) $y = \sqrt{25 - x^2}$

Question 2 The side length x of a square increases at a constant rate of 2 units per second.

(a) Write down an expression for the area A of the square.

(b) Find $\dfrac{dA}{dt}$ in terms of x.

(c) Hence, find the rate that the area is increasing when $x = 5$.

Question 3 The side length x of a cube increases at a constant rate of 3 units per second. Use a similar set of steps to Question 2 to find the rate of change of the following when $x = 2$.

(a) Surface area.

(b) Volume.

Question 4 A square metal sheet is being heated such that its length is expanding at a uniform rate of 0.05 cm/s. Find the rate of change of its area at the instant when the length is 4 cm.

Question 5 A circular plate of radius r is heated so that the area of the plate is increasing at a rate of 3 cm^2/min. At what rate is the radius increasing when $r = 5$ cm?

Question 6 A spherical balloon is being inflated by a pump at a constant rate of 8π cm^3/min.

(a) Find an expression in terms of r for the rate at which the radius of the balloon changes.

(b) Find the rate at which the radius of the balloon is changing when the radius is 4 cm.

(c) Find the rate at which the radius of the balloon is changing when the volume is 36π cm^3.

Question 7 A spherical snowball is melting in such a way that its radius is decreasing at a rate of 0.5 cm/min. At what rate is the volume of the snowball decreasing when the radius is 3 cm? Give your answer in exact form.

Question 8 The radius of the base of a cylinder is increasing at a rate of 6 cm/min. The height of the cylinder is fixed at 20 cm. Find the rate of change of the volume of the cylinder at the instant when the radius is 15 cm.

Question 9 The height of a cone is 12 cm and remains constant whilst the radius of the base is increasing at a rate of 6 cm/s. At what rate is the volume of the cone increasing at the instant when the radius is 10 cm?

Question 10 A cubic block of ice is melting so that the block remains cubic.

(a) If the edge is melting at 2 cm/min, find the rate at which the volume is melting when the edge is 12 cm.

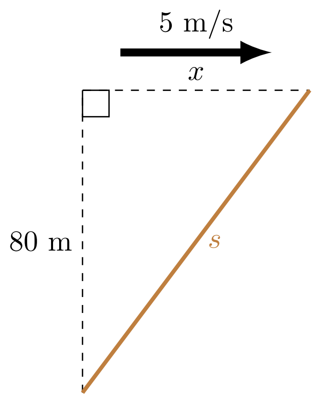
(b) If the volume is melting at 75 cm^3/min, find the rate at which the edges are melting when the volume is 64 cm^3.

Question 11 A cubic crystal is being grown in a laboratory so that rate of increase of its side length x is increasing at 0.5 cm/min. Find the rate of change of

(a) the surface area when the edge is 6 cm.

(b) the volume when the edge is 6 cm.

Question 12 A boy flies a kite 80 m above the ground and the wind is blows it on a horizontal course at 5 m/s. Find how fast the boy is paying out the string when the kite is 100 m from him.



Question 13 Gas escapes a spherical bubble so that the bubble always remains a sphere.

(a) Find the rate of decrease of the radius when the radius is 5 mm if the volume of the sphere is decreasing at a constant rate of 75 mm^3 per sec.

(b) Find the rate of decrease of the radius when the radius is 12 mm if the surface area of the sphere is decreasing at 24 mm^2 per sec.

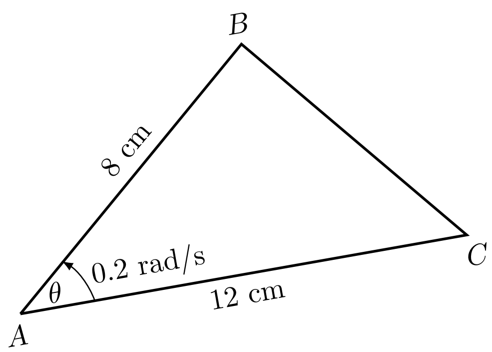
(c) Find the rate of decrease of the volume when the radius is 12 mm if the surface area of the sphere is decreasing at 24 mm^2 per sec.

Question 14 The radius of a sphere is increasing at a constant rate of k cm/s. If the radius is R cm, find the rate of increase of the following, in terms of π.

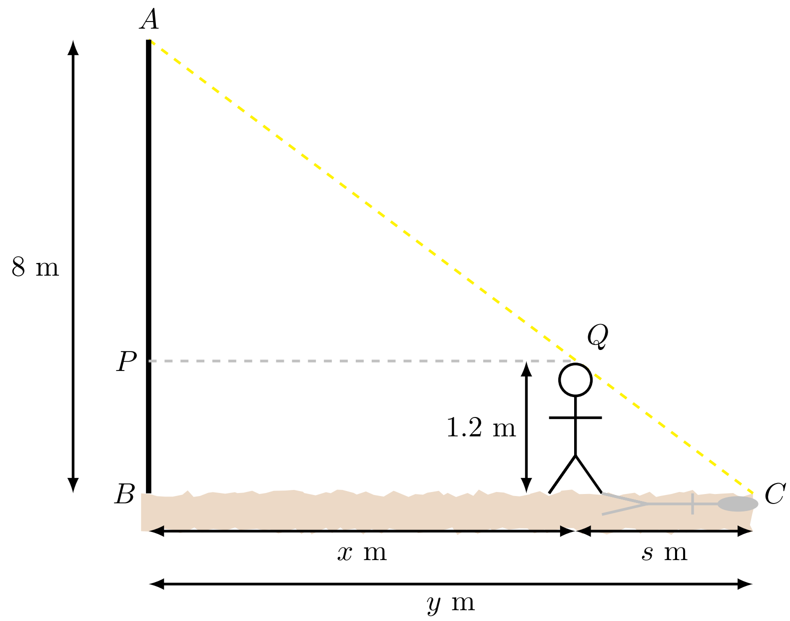
(a) Surface area.

(b) Volume.

Question 15 In this triangle ABC, $\angle BAC$ is increasing at a constant rate of 0.2 radians/s. Use the fact that $\dfrac{d}{d\theta}(\sin\theta) = \cos\theta$ to find the rate of change in the area of $\triangle ABC$ when $\theta = \dfrac{\pi}{3}$.



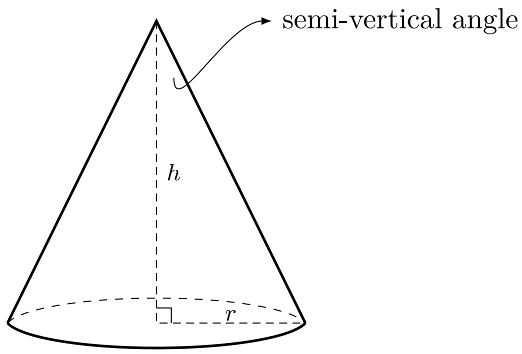
Question 16 A street light is is on top of a 8 metre pole. A person 1.2 m high walks away from the pole at 1 m/s. When the person is 5 m from the pole, find the rate at which



(a) the tip of the person's shadow is moving away from the pole.
Hint: This is $\dfrac{dy}{dt}$.

(b) the length of the shadow is changing.

Question 17 Water is poured into an inverted right conical vessel at a constant rate of 8 cm^3/sec.



(a) Find the relationship between h and r, given the semi vertical angle in each case below. Hence, express the volume of the cone in terms of h only.

(i) Semivertical angle is 45°

(ii) Semivertical angle is 30°

(b) Hence, find the rate the water level rises when the depth is 4 cm, given that

(i) the semi-vertical angle is 45°

(ii) the semi-vertical angle is 30°

Challenge Problems

Problem 1 [Double application of the chain rule]

Water is poured into an inverted cone with its height twice the radius.

(a) Show that when the water depth is h cm, the volume of water is given by

$$V = \frac{2\pi}{3} r^3$$

(b) The exposed surface area of the water decreases at a rate of 1 cm^2/s.
Show that $\dfrac{dV}{dt} = r$.

Problem 2 [Application of implicit differentiation]

Two cars are approaching the same intersection from two different roads at right angles to each other. Car A is travelling at 60 km/h and is x metres from the intersection while car B is travelling at 45 km/h and is y metres from the intersection. Let the distance between the two cars be l.

(a) Show that

$$l\frac{dl}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Hint: Differentiate both sides of $l^2 = x^2 + y^2$ with respect to t.

(b) Hence, find the rate of change of l when Car A is 800 metres from the intersection whilst Car B is 600 metres from the intersection.

Problem 3 One diagonal of a rhombus is decreasing at a rate of 5 centimetres per minute and the other diagonal of the rhombus is increasing at a rate of 9 centimetres per minute. At a certain instant, the decreasing diagonal is 6 centimetres and the increasing diagonal is 8 centimetres. What is the rate of change of the area of the rhombus at that instant?

Problem 4 The area A of a rectangle is increasing at a rate of 24 cm^2/s while its length is decreasing at a rate of 6 cm/s. Find the rate of increase of the breadth of the rectangle when the length is 96 cm and the breadth is 60 cm.

Problem 5 Find the rate of change of the area $A(t)$ of a triangle if we know that the base of the triangle is increasing at 10 cm/s and the height is decreasing at 6 cm/s, when the base is 8 cm and the height is 5 cm.

Chapter 5 Review

Physical Applications of Calculus

Review

Question 1 Let V represent the volume of water in a bath measured in litres and t represents time in minutes.

(a) What does $\dfrac{dV}{dt}$ or V' represent?

(b) What does it mean if $\dfrac{dV}{dt} < 0$?

(c) What does it mean if $\dfrac{dV}{dt} = 0$ and when $\dfrac{dV}{dt} > 0$?

(d) When would you expect $\dfrac{dV}{dt}$ to be constant?

(e) Do you think $\dfrac{dV}{dt} = -3$ is realistic? Explain and give reason for your answer.

(f) Sketch the graph $\dfrac{dV}{dt} = t - 10$ and explain what is happening to the volume of water in the bath for different values of t.

(g) Sketch the graph $\dfrac{dV}{dt} = t^2 - 7t + 10$ and explain what is happening to the volume of water in the bath for different values and ranges of t.

Question 2 The rate of water flow into a tank is given by $\dfrac{dV}{dt} = 10(2 - t)$, where $\dfrac{dV}{dt}$ is measured in litres per minute. There were 60 litres of water initially in the tank.

(a) Explain what is happening when $t = 1$ and $t = 4$.

(b) Show that the volume of water in the tank increases for a certain time only, and then decreases.

(c) Show that the volume of water in the tank is given by $V = 20t - 5t^2 + 60$.

(d) When does the rate of flow of water change from increasing to decreasing, and find the volume of water in the tank then.

(e) Find the time when the tank is empty.

Question 3 A block of ice of mass 6 kg is melting according to the formula $M = 6 - \dfrac{t^2}{54}$ where M is the mass of ice remaining in kg after t minutes.

(a) Find the time taken for all the ice to melt.

(b) Find the rate at which the ice is melting after 9 minutes.

(c) Find the rate at which the ice is melting when there is 3 kg of ice remaining.

(d) Draw the graph of mass of ice against time and explain from your graph when is the ice block melting fastest?

Question 4 A particle moves in a straight line so that the function $x(t) = (6 - t)^3$ gives the particle's position in metres at any time t seconds.

(a) Find particle's velocity-time equation, and describe the significance of the sign.

(b) When and where is the velocity zero?

(c) What is the particle's initial position and velocity?

(d) When is the particle at the origin?

(e) Find the distance travelled by the particle during the first 2 seconds.

(f) Write down the acceleration of the particle.

(g) Determine if the particle slows down during the first 6 seconds. Explain.

Question 5 A particle P moves along a straight line so that at time t secs its displacement from a fixed point O on that line is given by $x = \dfrac{3t}{4+t^2}$ metres.

(a) Find the velocity of the particle at time t.

(b) When is the particle momentarily at rest?

(c) Find the position of the particle when $t = 4$.

(d) Find the time when P is in exactly the same position as at $t = 4$.

(e) Use graphing software to sketch the displacement and velocity-time graphs.

(f) With the help of your graphs drawn, discuss the motion of this particle including as t increases without bound.

Question 6 A particle moves along a straight line so that it's position, x cm, at time, t mins, is given by $x = t^3 - 16t^2 + 64t$

(a) Find an expression for the velocity, v, and acceleration, a, of the particle at any time t.

(b) Draw the displacement, velocity and acceleration graphs with time.

(c) When is the particle at rest?

(d) When is the particle moving to the left?

(e) When is the acceleration zero?

(f) Discuss when the particle is speeding up and slowing down and in what direction it is travelling.

Question 7 A particle P moves so that its velocity is given by $v = \dfrac{1}{t+1}$ m/s.

(a) What is the initial velocity for this particle?

(b) What does the velocity approach for large t?

(c) What is the acceleration of P when $t = 1$?

Question 8 The height x metres of a ball thrown in the air is given by $x = 5t(8 - t)$ where t is in seconds.

(a) Find the velocity and acceleration equations of this motion.

(b) Describe what is happening to the ball initially.

(c) Explain why the acceleration is a constant.

(d) When does the ball change direction?

(e) Find the maximum height reached.

(f) When is the ball speeding up and when is it slowing down?

Question 9 Sketch the graphs of $Q = 5000e^{0.1t}$ and $Q = 5000e^{-0.1t}$ on the same set of axes.

Question 10 The rate of decay of a substance is proportional to the mass M of the substance present at any time t. In other words $\dfrac{dM}{dt} = -kM$, where k is a positive constant. The mass is initially 8 grams and after twelve years, the mass is now 7.5 grams.

(a) Show that $M = Ae^{-kt}$ satisfies this equation.

(b) Find the value of k.

(c) What was the decay rate initially?

(d) In how many years will the mass be less than 6 grams, rounded to the nearest year?

Question 11 In hot weather, the number of algae in a river satisfies the equation $N = N_0 e^{0.15t}$, where t is measured in days.

(a) Find how long it takes for the number of algae to increase by 25%, correct to two significant figures.

(b) After four days, the number of algae is estimated to be 1.8×10^9. Find N_0 correct to two significant figures and hence the rate of algae increase then.

(c) The number of algae doubles every T days. Find the value of T correct to two significant figures.

Question 12 The population P of a city is increasing at a rate given by $\dfrac{dP}{dt} = kP$, where k is a positive constant and t is time in years after the year 2000. The population of the city doubles every twelve years.

(a) Show that $P = P_0 e^{kt}$ satisfies this equation.

(b) Find the value of k.

(c) During which year will the city reach a population three times that it had at the beginning of the year 2000?

(d) Given that the population at the beginning of the year 2000 was 1.2 million, what will the population be at the beginning of the year 2030?

(e) Calculate the instantaneous rate at which the population will be increasing at the start of 2030.

Question 13 A vessel filled with liquid is being emptied and the volume V litres remaining after t minutes is given by $V = V_0 e^{-kt}$, where k is a positive constant.

(a) Show that $\dfrac{dV}{dt} = -kV$.

(b) If a quarter of the vessel is emptied in the first three minutes, find the exact value of k.

(c) What fraction of the original amount remains after six minutes?

(d) Show that the rate at which the liquid flowing out after six minutes is $\dfrac{3}{16} V_0 \ln\left(\dfrac{3}{4}\right)$ litres per minute.

(e) When full, the vessel holds 1200 litres of liquid. Find how much liquid is left in the vessel after an hour, correct to two decimal places.

Question 14 When a hot object is placed in a cool environment, the rate at which the temperature decreases is proportional to the difference between the temperature T of the object and the temperature E of the environment. The rate of change of the temperature satisfies the differential equation

$$\frac{dT}{dt} = -k(T - E)$$

where k is a positive constant and t is time in minutes.

(a) Show that $T = E + Ae^{-kt}$ is a solution of this equation, for any constant A.

(b) The temperature of the room is 20°C. A cup of coffee has a temperature of 90°C. Find the value of A and E.

(c) The coffee cools to 60°C in 5 minutes. Find the exact value of k.

(d) What is the temperature of the coffee after 15 minutes?

(e) Approximately how long will it take for the coffee to reach 40°C?

Question 15 A hot rod with temperature T placed in a room with a surrounding air temperature of 20°C and allowed to cool. It loses heat according to Newton's law of cooling

$$\frac{dT}{dt} = -k(T - A)$$

where t is time in minutes, A is the surrounding air temperature and k is a positive constant. After eight minutes, the temperature of the rod is 70°C, and after a further four minutes it is 45°C.

(a) Show that $T = A + Be^{-kt}$ satisfies the differential equation.

(b) What is the value of k?

(c) How long will it take for the rod to cool down to 25°C?

Question 16 In a community of 10000 residents, the rate at which people hear about a product is proportional to the number of people who have not heard about it. The growth model satisfies the differential equation

$$\frac{dR}{dt} = k(10000 - R)$$

where k is a constant and R is the number of residents who are aware of the product after t days. An advertising company is launching an advertising campaign to promote a new product. No one has heard of the product at the start of the campaign and ten days after the start of the campaign, 1000 people were aware of it.

(a) Show that $R = 10000 - Ae^{-kt}$ satisfies the differential equation.

(b) Find the value of A.

(c) Show that $k = \frac{1}{10} \ln\left(\frac{10}{9}\right)$.

(d) How many people were aware of the product twenty days after the launch of the advertising campaign?

(e) How many days did it take for at least half the residents to hear of the new product?

(f) Sketch the graph of $R = 10000 - Ae^{-kt}$ and state what happens to R eventually.

Question 17 A sky-diver opens his parachute when falling at 24ms^{-1}. The acceleration is given by

$$\ddot{x} = k(8 - \dot{x})$$

where k is a positive constant.

(a) Show that $\dot{x} = 8 + Be^{-kt}$ is a solution of the differential equation.

(b) Find the value of B.

(c) One second after opening the parachute, the velocity halves. Show that $k = 2\ln 2$.

(d) Find the value of $\dot{x}$ two seconds after opening the parachute.

(e) Find the acceleration two seconds after opening the parachute.

(f) What is the physical significance of 8ms^{-1} ?

(g) Sketch the graph of velocity against time.

Question 18 A circular plate of radius r is heated so that the radius of the plate is increasing at a rate of 5 cm/min. At what rate is the area increasing when $r = 10$ cm?

Question 19 The surface area of a cube is increasing at a rate of 6 cm^2/s. At the instant when the edge is 4 cm find

(a) the rate of increase of the edge length.

(b) the rate of increase of the volume.

Question 20 Sand is being dumped from a conveyer belt at a rate of 2.4 m^3/min. It forms a pile in the shape of a right circular cone whose height and radius are always equal. At what rate is the height of the pile increasing when the pile is 3 m high?

Question 21 A cylindrical storage tank of radius 0.5 m, stands on one of its circular ends.

(a) If water is pumped in at 2 m^3/min, find the rate at which the level of water is rising.

(b) If the water level is rising at 25 cm/min, find the rate of change of volume in m^3/min.

Investigation Task

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the environmental temperature. The differential equation, as studied earlier, is

$$\frac{dT}{dt} = -k(T - E)$$

where T is the temperature of the object, t is time, E is the environmental temperature and k is a constant.

Question 1 Measure the temperature of a room and place a cup of boiling water in that environment. Using a temperature probe, monitor the temperature of the water and log your data at 15-minute intervals over the span of three hours.

(a) Write down the value of the second logged temperature ($t = 15$).

(b) Use this temperature to create a model of the form $T = A + Be^{-kt}$, where t is measured in minutes.

(c) Plot your data and your model on the same set of axes.

(d) Comment and give possible explanations on any discrepancies in the comparison.

Question 2 Repeat the above experiment with a cup of cold water heating up in a room temperature environment and similarly comment on your findings.

6

COMBINATORICS AND BINOMIAL EXPANSIONS

- Multiplication principle
- Factorials
- Permutations
- Identical elements
- Combinations
- Arrangements in circles
- Applications to probability
- Pigeonhole principle
- Pascal's triangle and Binomial Expansions

Exercise 6A

Multiplication principle

Fundamentals

Fundamentals 1

(a) The multiplication principle says that if there are m ways to choose an object, and n options to choose another object, then there are ______ ways of choosing both objects.

(b) This answer does/does not (circle one) take order into consideration.

(c) In general, to find the number of ways of choosing objects, we m________ the number of options at each stage of selection.

Question 1 A dinner menu has 3 options for the entrée, 4 options for the main course, and 2 options for the dessert. How many possible meal selections can Jennifer have in total?

Question 2 A three-digit number is to be made from the digits $1, 2, 3, 4$ and 5. How many numbers can be made if

(a) digits can be repeated?

(b) digits cannot be repeated

Question 3 A race has three competitors. How many possible results can there be if

(a) the competitors cannot tie?

(b) the competitors can tie.

Question 4 Joshua has two hats, five shirts, three sets of jeans and three pairs of shoes. How many possible ways can Joshua be dressed if

(a) he does not wear a hat?

(b) he insists that he wears a hat?

Question 5 A car licence plate consists of three letters followed by three digits. How many possible licence plates can be formed if

(a) repetition is allowed?

(b) repetition is not allowed?

Question 6 A committee consists of a chairperson, treasurer and secretary. How many possible committees can be formed if

(a) there are three applicants?

(b) there are six applicants?

Question 7 A three-letter password is to be formed from the letters of the word DOCUMENT. How many possible passwords are there if

(a) there is no repetition?

(b) only vowels can be used and there is no repetition?

(c) only consonants can be used and there can be repetition?

Question 8 A three-digit number is to be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and repetition of digits is allowed. How many possible numbers can be made if

(a) there is no restriction?

(b) the number must end with 3?

(c) the number must be a multiple of 5?

(d) the number must be even?

(e) the number must consist only of even digits?

(f) repetition were not allowed?

Question 9 There are 10 contestants in a race, where there are medals for 1^{st}, 2^{nd} and 3^{rd} places. Suppose Andrew, Bob and Charles are contestants in this race. How many possible results of 1^{st}, 2^{nd} and 3^{rd} places are there if

(a) there are no restrictions?

(b) Andrew, Bob and Charles get medals?

(c) Bob comes first?

(d) Bob gets a medal?

Challenge Problems

Problem 1 There are five backpackers and four available rooms in a hotel. How many ways can the backpackers be accommodated in the four rooms if each room can hold at most four people?

Problem 2 How many ways can n objects be distributed amongst two people, if each person is to have at least one object?

Problem 3 [Bijections]

Consider the digits 1, 2, 3, 4 and 5.

(a) How many four-digit passwords can be formed if there is no repetition?

(b) How many five-digit passwords can be formed, if there is no repetition?

(c) Explain why the answers from (a) and (b) are the same, even though there are differing numbers of digits.

Exercise 6B

Factorials

Fundamentals

Fundamentals 1

(a) $n! = n \times \_\_\_\_\_ \times \_\_\_\_\_ \times \ldots \times \_\_$

(b) $\dfrac{n!}{k!} = n \times (n-1) \times (n-2) \times \ldots \times \_\_\_\_\_$

(c) $0! = \_\_$

Fundamentals 2

(a) $n! \times (n+1) = \_\_\_\_\_$

(b) $\dfrac{n!}{(n-1)!} = \_\_$

(c) $\dfrac{n!}{n} = \_\_\_\_\_$

Fundamentals 3

(a) ${}^nP_k = \_\_\_\_$

(b) ${}^nC_k = \dbinom{n}{k} = \_\_\_\_\_\_\_$

(c) ${}^nC_0 = \_\_$

(d) ${}^nC_n = \_\_$

(e) ${}^nC_1 = \_\_$

Question 1 Without using a calculator, find the value of the following.

(a) $4!$

(b) $5!$

(c) $\dfrac{6!}{4!}$

(d) $\dfrac{6!}{3!3!}$

Question 2 Simplify the following.

(a) $\dfrac{n!}{(n-2)!}$

(b) $\dfrac{(n+1)!}{(n-1)!}$

(c) $\dfrac{(n+2)!}{n!}$

Question 3 By factorising the appropriate factorial term, simplify the following and leave your answer in the form $k \times n!$.

(a) $5! + 4!$

(b) $9! - 8!$

(c) $6! + 5! + 4!$

Question 4 By factorising the appropriate factorial term, simplify the following.

(a) $(n+1)! - n!$

(b) $n! - (n-2)!$

(c) $(n+1)! - (n-1)!$

(d) $n! - (n-1)!$

Question 5 Let n and k be integers such that $n \geq k$. Show that

$$\frac{n!}{(n-k)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$$

Question 6 Show that ${}^nC_k = {}^nC_{n-k}$

Question 7 Show that ${}^nC_k + {}^nC_{k-1} = {}^{n+1}C_k$.

Question 8 Solve the following.

(a) ${}^{10}C_n = {}^{10}C_3$.

(b) ${}^nC_7 = {}^nC_3$

(c) ${}^nC_6 + {}^nC_5 = {}^{11}C_6$.

(d) ${}^8C_n + {}^8C_{n-1} = {}^9C_3$

Question 9 Show that ${}^{2n}P_{n+1} = 2n \times {}^{2n-1}P_n$

Question 10 Show that ${}^nC_k = \dfrac{n-k+1}{k}\,{}^nC_{k-1}$.

Question 11

(a) Show that ${}^mC_k = {}^{m+1}C_{k+1} - {}^mC_{k+1}$.

(b) Hence, show that

$${}^6C_6 + {}^7C_6 + {}^8C_6 + {}^9C_6 + {}^{10}C_6 + {}^{11}C_6 = {}^{12}C_7$$

(c) Generalising part (b), show that

$${}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + \cdots + {}^nC_k = {}^{n+1}C_{k+1}.$$

Challenge Problems

Problem 1

(a) Show that $(k+1)! - k! = k \times k!$

(b) Hence, simplify $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n!$.

Problem 2 Let $S(n) = \dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n+1)!}$

(a) Simplify $\dfrac{1}{k!} - \dfrac{1}{(k+1)!}$.

(b) Hence show that $S(n) = 1 - \dfrac{1}{(n+1)!}$.

(c) Write down the value of $\lim_{n\to\infty} S(n)$.

Problem 3 Show that ${}^{n+2}C_k = {}^{n}C_k + 2{}^{n}C_{k-1} + {}^{n}C_{k-2}$.

Problem 4 Show that ${}^{p+q+r}C_p{}^{q+r}C_q = {}^{p+q+r}C_q{}^{p+r}C_p$.

Problem 5 Show that ${}^{n+1}P_k = {}^{n}P_k + k{}^{n}P_{k-1}$.

Problem 6 A room contains $n+1$ people named P_1, P_2, P_3, ..., P_{n+1}, arranged in a line in that order. Person P_1 shakes hands with everybody else in the line and then leaves the room, having shaken everybody's hand. Afterwards, P_2 shakes hands with the remaining people in the room and similarly leaves. This is repeated until eventually, everybody has shaken hands with everybody else *exactly once*.

(a) How many hands did P_1 shake before leaving?

(b) When it was their turn, how many hands did P_2 shake before leaving?

(c) Continuing this pattern, how many hands did P_n and P_{n+1} shake?

(d) Explain why there were ${}^{n+1}C_2$ handshakes in total.

(e) Deduce that $1 + 2 + 3 + \cdots + n = \dfrac{n}{2}(n+1)$.

Problem 7

(a) Show that

$$k\frac{{}^{n}C_k}{{}^{n}C_{k-1}} = n - k + 1.$$

(b) Hence, use the identity in Problem 6 (e) to show that

$$\frac{{}^{n}C_1}{{}^{n}C_0} + 2\frac{{}^{n}C_2}{{}^{n}C_1} + 3\frac{{}^{n}C_3}{{}^{n}C_2} + \cdots + n\frac{{}^{n}C_n}{{}^{n}C_{n-1}} = \frac{n}{2}(n+1).$$

Exercise 6C

Permutations

Fundamentals

Fundamentals 1

(a) The number of ways of ordering n distinct objects is ______.

(b) This is because there are ___ ways of ordering the first object, then ______ ways of ordering the next object, and so on, until we finally have only ___ way of ordering the last object.

(c) The number of ways of making an ordered selection of k objects from a total of n is ______.

Fundamentals 2

(a) When answering counting problems, be sure to always deal with the r________ first.

(b) When you see the expression *'at least'* or *'at most'*, be mindful that it may be easier to consider the c__________ event.

Fundamentals 3

Consider the following problem.

Bob and Mary are in a line with four other people. Find the number of ways that they can be arranged if they are to be separated by at least one person.

A student attempts this problem by adding the cases when Bob and Mary are separated by one person, two people, three people and four people, and then adds them all up.

(a) In this case, it would be more practical to count the arrangements when Bob and Mary are together, and then subtract it from the total unrestricted arrangements. This is counting using the c________ event.

(b) Suppose Bob and Mary were to be separated by *at most one* person instead. Write down the cases that satisfy this condition, then write down the complementary event. Determine if it is more practical to count the cases directly, or to consider the complementary event.

(c) Repeat the above question for *'at most three'* and *'at least three'*.

Fundamentals 4

Write down ${}^nP_r$ in terms of factorials.

Fundamentals 5

Simplify the following.

(a) ${}^nP_0$ (b) ${}^nP_1$ (c) ${}^nP_n$

Question 1 Find the number of ways that

(a) three people can be ranked 1st, 2nd and 3rd.

(b) ten people can be arranged in a line.

(c) a four-letter code can be made from the letters A, B, C and D.

Question 2 Find the number of ways that seven people, including Bob and Mary, can be arranged in a straight line if

(a) there is no restriction?

(b) Bob must be in the left-most position?

(c) Bob must be in the middle?

(d) Bob must be next to Mary?

(e) Bob must *not* be next to Mary?

(f) Bob and Mary are on either ends of the line?

Question 3

(a) How many four-letter passwords can be made from the letters of the word PROBLEM?

(b) How many five-digit passwords can be made from the digits 1, 2, . . . , 9?

(c) How many ways can there be 1st, 2nd and 3rd in a race consisting of eight people?

Question 4 How many six-digit passwords can be formed from the digits 0, 1, 2, . . . , 9 if repetition is not allowed and the password

(a) has no other conditions?

(b) must be an even number?

(c) must be an odd number?

(d) cannot contain the number 5?

Question 5 A number with at most five digits is to be created from the digits 1 to 9 inclusive. How many possible numbers can be formed?

Question 6 Five boys and five girls are to be arranged in a line. How many arrangements are possible if the boys and girls are to alternate?

Question 7 Four boys, including Bob, and six girls, including Mary, are to be arranged in a line. How many ways can the ten people be arranged if

(a) there is no restriction?

(b) Bob must be next to Mary?

(c) Bob cannot be next to Mary?

(d) the boys must be grouped together?

(e) the boys and girls must be in separate groups?

(f) Bob and Mary are on the ends of the line?

Question 8 There are eight participants in a running race, including Bob and Mary. How many ways can the runners cross the line if

(a) Bob finishes immediately after Mary?

(b) Bob finishes after Mary?

Question 9 There are four boys and four girls, including Bob and Mary respectively, to sit in two rows of four seats so that there is a front and a back row. How many ways can they be arranged if

(a) there is no restriction?

(b) all boys are in the same row?

(c) Bob must be in the front row?

(d) Bob and Mary must be in the same row?

Challenge Problems

Problem 1 There are six backpackers and four available rooms for them to go in.

(a) Suppose that each room could accommodate an unrestricted number of people. How many ways can the backpackers be distributed?

(b) How many ways can the backpackers be distributed if one room is to have exactly five people, and another room is to have exactly one person?

(c) Now, suppose that each room could accommodate *at most* four people. Find the number of ways of distributing six backpackers amongst the four rooms.

Problem 2 A line contains three boys, including Adam, and seven girls. How many ways can they be arranged if Adam must have a girl on either side of him?

Problem 3 [Combinatoric proof]

A group of k people is to be selected from a total of $n+1$ people, and then placed on k seats in a line. Suppose Benny is one of the $n+1$ people.

(a) How many ways can this be done? Express your answer in terms of ${}^{a}P_{b}$ for appropriate values of a and b.

(b) How many ordered selections are possible if Benny is excluded from the selection?

(c) How many ordered selections are possible if Benny is included in the selection, and seated first before anybody else?

(d) Deduce that ${}^{n+1}P_k = {}^{n}P_k + k\,{}^{n}P_{k-1}$.

Exercise 6D

Identical elements

Fundamentals

Fundamentals 1

By manually listing out all possible permutations, find how many arrangements there are of the following letters.

(a) $XXXY$

(b) $XXYY$

Fundamentals 2

Consider the sequence of letters

$$XXXYYYYY.$$

(a) The number of ways of arranging the letters is ——.

(b) The naïve answer of 8! considers all the X's and Y's to be d________, when in actual fact they are i________.

(c) Explain why we must divide the naïve answer by 3!5! in order to obtain the correct result.

Fundamentals 3

In general, if there are n objects consisting of n_1 identical copies of one type of object, n_2 copies of another type etc... up to n_k such that $n = n_1 + n_2 + n_3 + \ldots + n_k$, then the number of ways of arranging the n objects is

$$\frac{\qquad\qquad}{\_\_!\,\_\_!\,\_\_!\ldots\_\_!}.$$

Question 1 How many distinct arrangements are there of the letters of the following words?

(a) PROOF

(b) BOTTLE

(c) COFFEE

(d) WOLLONGONG

(e) PARRAMATTA

(f) WOOLLOOMOOLOO

Question 2 Three identical black balls and three identical white balls are to be arranged in a line. How many possible ways can they be arranged?

Question 3 The digits of the number 122333 are used to form a 6-digit number. How many distinct numbers can be formed?

Question 4 Consider the letters of the word CLINICAL. How many ways can the letters be arranged if

(a) there are no restrictions?

(b) the C's are to be together?

(c) the L's must be on the ends?

(d) all the identical letters must be together?

Question 5 A survey has eight 'Yes/No'-type questions. How many ways can the survey be answered if there are

(a) no restrictions?

(b) exactly three 'No' answers?

(c) at least two 'No' answers?

(d) at least six 'Yes' answers?

(e) at most two 'No' answers?

(f) at most six 'Yes' answers?

Question 6 A three-letter code is to be made from the letters of the word MOON.

(a) Bob argues that the number of three-letter codes is ${}^4P_3$ because we want the number of ordered selections of three objects from a total of four. Explain why this is not correct.

(b) Find the number of codes that can be made if the code

(i) contains both O's.

(ii) has all distinct letters.

(c) Hence, how many possible three-letter codes are possible in total?

Question 7 A five-letter code is to be made from the letters of the word BOTTLE.

(a) How many codes can be made if the code contains both T's?

(b) How many codes can be made if the code contains exactly one T?

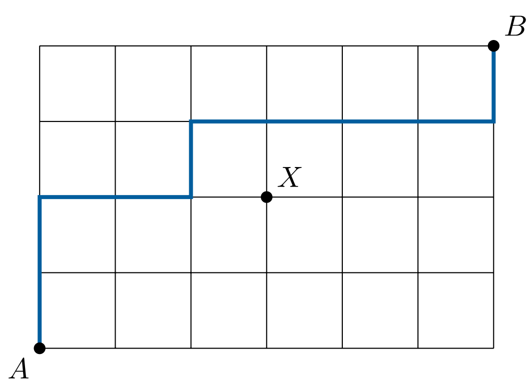
(c) Hence, in total how many five-letter codes can be made from the letters of the word BOTTLE?

Question 8 Use a similar technique to Question 7 to find how many five-letter codes can be formed from the letters of the word COFFEE.

Challenge Problems

Problem 1 [Application of identical elements]

The diagram below shows two points A and B on opposite corners of a 4×6 grid. Routes are to be created from A to B using sequences of only right movements and up movements. An example of a route has been highlighted.

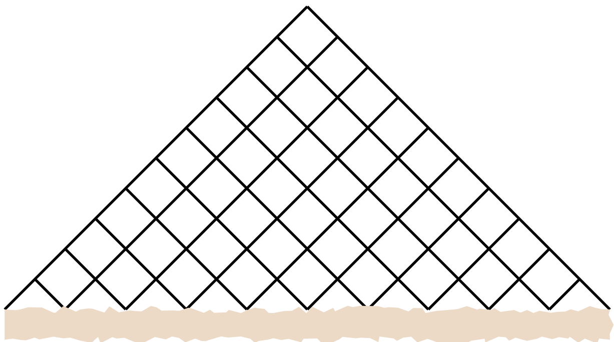


(a) If R represents moving to the right and U represents moving up, then what is the sequence that corresponds to the highlighted route above?

(b) Hence, find the number of possible routes from A to B.

(c) Find the number of possible routes from A to B, but passing through X.

Problem 2 A drop of water is placed at the top of the grid shown below.



If the drop must always run downwards along the edges of the grid, how many different ways can it reach the ground?

Problem 3 [Stars and bars]

Suppose there are six identical lollies and three students A, B and C. The lollies are to be distributed amongst the students. For example, they each can receive two lollies. However not all students need to be given lollies, so it is possible for A to have all six and the others none, or for A and B to have some but C has none.
Consider the diagram below.

$$\star\star\,|\,\star\,|\,\star\star\star$$

(a) Explain how this diagram can represent a possible distribution of the lollies to the students.

(b) Draw a similar diagram representing the scenario where A has two lollies, B has four lollies and C has no lollies.

(c) Hence, find the number of possible ways of distributing all six identical lollies amongst three students.

Exercise 6E

Combinations

Fundamentals

Fundamentals 1

(a) The number of ways of making an un-ordered selection of k objects from n is ______, which can also be written as ___.

(b) This is used when order does/does not (circle one) matter.

Fundamentals 2

(a) Expressed as a factorial, we have the formula

$$^{n}C_{k} = \binom{n}{k} = \_\_\_\_\_\_\_\_.$$

(b) Since $^{n}C_{k}$ is an un-ordered selection of k objects from n, we can make it an ordered selection by multiplying it by ___!

(c) This results in the identity

$$^{n}P_{k} = \_\_\_\_\_\_$$

Fundamentals 3

Simplify the following.

(a) $^{n}C_{0}$

(b) $^{n}C_{1}$

(c) $^{n}C_{n}$

Fundamentals 4

Explain briefly when it is appropriate to use $^{n}C_{r}$ instead of $^{n}P_{r}$.

Question 1

(a) How many ways can four people be selected for a committee if there are ten applicants?

(b) How many ways can three people be promoted if there are seven people on the team?

Question 2 There are six men and seven women in a room, including Bob and Mary respectively. How many ways can a committee of four people be formed if

(a) there is no restriction?

(b) the committee is to consist of only women?

(c) the committee is to have equal numbers of men and women?

(d) the committee is to have a majority of men?

(e) Bob and Mary are both to be on the committee?

(f) Either Bob or Mary, but not both, are on the committee?

Question 3 Five letters are to be chosen from the letters of the word KEYBOARD (without ordering). How many combinations are there if the selection

(a) has no restriction?

(b) must contain all three vowels?

(c) must contain the letter Y?

(d) must contain the letter K, but not D?

Question 4

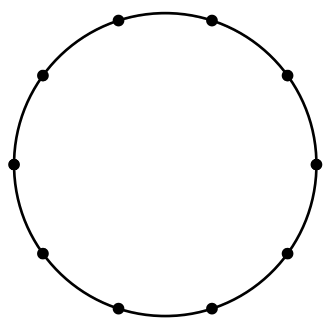
(a) How many ways can 3 people be hired from a list of 8 applicants?

(b) How many ways can 5 people be rejected from a list of 8 applicants?

(c) Explain why the answers from (a) and (b) are the same.

(d) Use similar logic to prove the identity ${}^nC_k = {}^nC_{n-k}$.

Question 5 Consider ten evenly-spaced points on a circle, as shown in the diagram below.



(a) How many triangles can be formed from the ten points?

(b) How many quadrilaterals can be formed from the ten points?

Question 6 At a fast-food restaurant, there is a promotion to create your own frozen drink from a combination of 8 flavours. The drink can be made by mixing at most two flavours, or it can consist of just the one flavour. How many possible unique drinks can be created?

Question 7 A group of four people A, B, C and D are to be split into two groups of two. Bob claims that the answer is simply ${}^4C_2 \times {}^2C_2$since we first choose two people out of four to go into one group, then the remaining two people become the other group, by default.

(a) By listing the possible combinations, how many possible ways can the four people be split into two groups of two?

(b) Explain why Bob's answer is exactly double the actual number of possible combinations?
Hint: Try filling out the following table.

Group 1	**Group 2**
A, B	C, D

Question 8

(a) How many ways can twelve people be split into three groups containing three people, four people and five people?

(b) How many ways can twelve people be split into two groups named A and B, each containing six people?

(c) Suppose the two groups did not have names. Does this change the number of ways of splitting twelve people into two groups of six?

(d) Explain your answer.

Question 9 [Distinguishing when to divide and when not to divide]

(a) How many ways can a group of six people be divided into two groups of three to play a game against each other?

(b) How many ways can six prizes be awarded to two people so that each person gets exactly three prizes?

Question 10 How many diagonals are there in a regular

(a) pentagon?

(b) hexagon?

(c) n-gon? (n-sided polygon)

Question 11 Consider a standard deck of 52 playing cards. How many hands of five cards can be formed if

(a) there is no restriction?

(b) all five are the same colour?

(c) all five are spades?

(d) all five are from the same suit?

(e) two are hearts and three are spades?

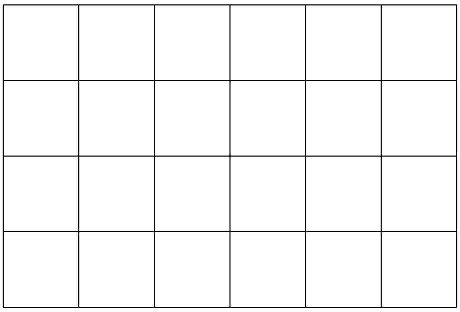
(f) there are only three aces, and two other cards?

(g) there is a pair and three of a kind?

(h) there are exactly two pairs and another random card?

Challenge Problems

Problem 1 The diagram below shows an 4×6 grid.
Hint: How many vertical lines and horizontal lines do we need to select in order to uniquely define a rectangle?



How many rectangles are there on the grid in total? Note that this includes all blocks of differing sizes as well as squares.

Problem 2 [Pascal's identity]

A team of k people is to be selected from a total of n applicants. Let Leon be one of these applicants.

(a) How many ways can the team be selected if Leon is on the team?

(b) How many ways can the team be selected if Leon is *not* on the team?

(c) Deduce the identity ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$.

Problem 3 [Binomial Coefficients]

(a) Consider a sequence of four A's and six B's arranged randomly. In terms of factorials, write down the number of ways of arranging the letters in the sequence.

(b) Express this as a binomial expression in the form ${}^nC_k$.

(c) Now consider a sequence of three A's and four B's arranged randomly. Write down the number of possible arrangements, in terms of ${}^nC_k$.

(d) Explain why (a) and (c), which are clearly permutation questions, can be answered using an expression normally used for combinations only.

(e) A coin is flipped twenty times. How many ways can there be exactly twelve tails?

Problem 4 Bob and Mary play a game where a coin is tossed repeatedly and the result is recorded. If the coin lands on heads, then Bob scores a point. Otherwise, if it lands on tails, then Mary scores a point. The first person to reach a score of five points wins the game. How many possible ways can Bob win the game?

Problem 5 [Classic binomial identity]

Consider a row of 10 light-switches, which can either be ON or OFF.

(a) How many ways can exactly five light switches be ON?

(b) How many ways can at least seven light switches be ON?

(c) By considering all the possible combinations of ON and OFF switches, deduce that

$$^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10} = 2^{10}.$$

Exercise 6F

Arrangements in circles

Fundamentals

Fundamentals 1

(a) If there are n distinct objects in a circle, then there are ______ possible arrangements of the n objects.

(b) This is because we first must fix the position of an object and then place the other ______ objects relative to it.

Fundamentals 2

How many ways can n people be arranged in a circle if

(a) two particular people must sit together?

(b) three particular people must sit together?

Question 1 There are five people sitting in a circle. How many ways can they be seated?

Question 2 How many ways can eight people be arranged

(a) in a straight line?

(b) in a circle?

Question 3 There are four boys and four girls sitting around a circular table, including Bob and Mary respectively. How many ways can they be seated if

(a) there is no restriction?

(b) the boys must be grouped together?

(c) the boys and girls must alternate?

(d) Bob must be next to Mary?

(e) Bob must not be next to Mary?

(f) Bob and Mary are separated by two people?

Question 4 There are three boys, including Anthony, and three girls, including Bethany, sitting around a circular table. How many ways can they be seated if

(a) the boys and girls are in distinct groups?

(b) Anthony is opposite Bethany?

(c) Anthony is not next to Bethany?

(d) Anthony wants to sit between two particular girls?

Question 5 Eight people, including Bob and Mary, are to be distributed amongst two different round tables of four. How many ways can the eight people be distributed and arranged if

(a) there is no restriction?

(b) Bob and Mary are on the same table?

(c) Bob and Mary are on different tables?

(d) Bob and Mary are on the same table, but are seated opposite each other?

Question 6 A group of eight people includes Carol and Josh. The eight people are to be split into two circular tables of three and five seats. How many ways can the people be seated if Carol and Josh are to be

(a) on separate tables?

(b) on the same table?

Question 7 How many ways can ten people be seated to two tables containing four and six seats in a circular arrangement?

Question 8 Bob and Mary are seated amongst six other people around a circular table. How many ways can they be seated if they are to be separated by at least two people?

Question 9 How many ways can six distinct beads be arranged on a bracelet?
Hint: A bracelet can be flipped.

Challenge Problems

Problem 1 There are four couples seated around a circular table. How many ways can they be arranged if

(a) everybody is to be sitting next to their partner?

(b) everybody is sitting opposite their partner?

Problem 2 There are n people sitting around a circular table. How many ways can they be arranged if

(a) there are no restrictions?

(b) two particular people sit next to each other?

(c) two particular people must be separated by a particular person?

Exercise 6G

Applications to probability

Fundamentals

Fundamentals 1

(a) The probability of a particular event with equally likely outcomes occurring is

$$\frac{\text{Number of outcomes in the event}}{\qquad\qquad}$$

(b) The total is usually just the unrestricted number of ways of arranging or c________ objects. Be mindful of whether r________ is allowed or not.

Fundamentals 2

In any permutations or combinations questions, always deal with the r________ first.

Fundamentals 3

(a) In some scenarios, it may be easier to find the probability using the c________ event, rather than finding the probability directly. Some key words or expressions that may indicate this are "at l______" and "not".

(b) For example, if we need to find the probability of an event E not occurring, it may be simpler to calculate $1 - P(\_\_)$.

Question 1 The letters of the word COMPUTER are arranged randomly.

(a) How many arrangements are there?

(b) Hence, find the probability that the vowels are grouped together.

Question 2 Bob and Mary are arranged randomly in a line with 6 other people. What is the probability that Bob and Mary are

(a) together?

(b) not together?

(c) at the ends of the line?

(d) separated by two people?

Question 3 Four boys and four girls are arranged randomly in a circle. Amongst the boys and girls are Andrew and Belinda respectively. What is the probability that

(a) the boys are grouped together?

(b) the boys and girls alternate?

(c) Andrew and Belinda are sitting next to each other?

(d) Andrew and Belinda are sitting opposite each other?

(e) Andrew and Belinda are sitting separately?

(f) Andrew and Belinda are separated by two people?

Question 4 A four-digit code is to be formed from the integers 1 to 9 inclusive. What is the probability that the digits are in increasing order?
Hint: Once we select the four digits, how many ways can they be arranged?

Question 5 The letters of the word OVERWATCH are arranged randomly. What is the probability that the 'word' formed

(a) has consonants at the ends?

(b) begins with a vowel and ends with a consonant?

Question 6 The letters of the word TEACHING are used to make a four-letter code. What is the probability that the code

(a) contains only consonants?

(b) contains all three vowels?

(c) begins and ends with a consonant?

(d) has two vowels and two consonants?

Question 7 Five men and five women are seated randomly around a circular table. Amongst the women is Kezia. What is the probability that Kezia has one man and one woman on either side of her?

Question 8 There are twelve students seated randomly around a table, including four from School A, three from school B and five from school C. What is the probability that students belonging to the same school are all seated together?

Question 9 In horse racing, a *Quinella* is a bet where the first two places are predicted, but not necessarily in the correct order. A *Trifecta* is a bet where the first three places are predicted in the correct order. A race has twelve horses competing. What is the probability of winning a

(a) Quinella?

(b) Trifecta?

Question 10 The letters of the word ADDITION are arranged randomly. What is the probability that the two D's are separated by two letters?

Question 11 Four students randomly guess all the answer to the last question in a multiple-choice exam, which has four possible options. What is the probability that exactly one of the options is not chosen by any of the students?

Question 12 Five cards are dealt randomly from a standard deck of 52 cards. How many ways are there of obtaining

(a) royal flush? (10, J, Q, K, A all same suit)

(b) a straight? (A,2,3,4,5) up to (10,J,Q,K,A).

(c) a straight flush? (a straight, but all five same suit)

(d) a flush? (five cards from the same suit, but not a straight)

(e) four-of-a-kind? (7,7,7,7,10)

(f) full house? (8,8,8,J,J)

(g) two-pair? (5,5,3,3,Q)

(h) one-pair? (Q,Q,3,5,8)

(i) three-of-a-kind? (K,K,K,2,7)

Challenge Problems

Problem 1 The letters of the word MONKEY are arranged randomly. What is the probability that

(a) the letters are in alphabetical order?

(b) the word 'KEY' appears, but the letters are not necessarily together? For example, the arrangement N**K**M**EY**O.

Problem 2 The letters of the word ARRANGEMENT are arranged randomly. What is the probability that

(a) the R's are grouped together?

(b) all consonants are grouped together?

Problem 3

(a) How many four-letter codes can be formed from the letters of the word COFFEE? **Hint:** You will have to consider cases here.

(b) A four-letter code is chosen randomly from the letters of the word COFFEE. What is the probability that it contains two E's?

Problem 4 Bob and Mary are arranged randomly in a queue with 8 other people.

(a) What is the probability that Bob is 3 places ahead of Mary?

(b) What is the probability that Bob is in front of Mary?

Exercise 6H

Pigeonhole principle

Fundamentals

Fundamentals 1

(a) If there are $n+1$ pigeons placed into n pigeonholes, then at least one pigeonhole must have at least ___ pigeons in it.

(b) In general, if there are n pigeones with k pigeonholes and $n > k$, then there is at least one pigeonhole with at least ____pigeons in it.

Fundamentals 2

If there are at least $mn+1$ pigeons placed into n pigeonholes, then at least one pigeonhole must have at least _____ pigeons in it.

Question 1 A sock drawer contains ten pairs of socks, where each pair is distinct from others. What is the minimum number of socks need to be taken to guarantee having at least a matching pair?

Question 2 Suppose there are 365 days in a year. What is the minimum number of people needed to guarantee that at least two people have the same birthday?

Question 3 Marbles are drawn without replacement from a bag that contains eight identical red marbles and eight identical blue marbles. What is the minimum number of marbles that must be drawn in order to guarantee that

(a) 4 marbles have the same colour.

(b) 6 marbles have the same colour.

Question 4 Suppose eight more black marbles are added to the same bag from Question 3. What is the minimum number of marbles that must be drawn in order to guarantee that at least 5 marbles have the same colour?

Question 5 What is the minimum number of times that a standard die needs to be rolled so that it is guaranteed that the same number is rolled at least

(a) two times?

(b) three times?

(c) n times?

Question 6 What is the minimum number of people needed to guarantee that at least two people have the same initials for their first name and surname?

Question 7 In a town, the first name of all the residents are placed in a box and names are drawn randomly without replacement. What is the minimum number of names that must be drawn to guarantee that at least

(a) three names begin with the same letter.

(b) two were born in the same day of the week.

(c) three were born on the same month.

(d) four people have the last digit of their year-of-birth being the same.

Question 8 A drawer contains an unknown number of pairs of different coloured socks. Socks are drawn randomly from the drawer in the dark. What is the minimum number of socks that must be drawn to guarantee a pair, if the sock drawer contains

(a) two different colours?

(b) three different colours?

(c) four different colours?

(d) n different colours?

Question 9 A box contains 4 red, 6 green, 8 blue, 10 yellow and 12 white balls. What is the minimum number of balls that must be chosen randomly from the box to guarantee obtaining 8 balls of the same colour?

Question 10 [Digit Sum]

A *digit sum* of a number n is the value obtained when the digits of n are added together. For example, the digit sum of 152 is $1+5+2=8$ and the digit sum of 230 is $2+3+0=5$.

(a) How many two-digit numbers must be selected to guarantee that at least two of them have the same digit sum?

(b) How many three-digit numbers must be selected to guarantee that at least two of them have the same digit sum?

Question 11 A conference has 600 attendees and all attendees are in one of eleven conference rooms. Show that at least one room must have at least 55 people.

Question 12 Ten people shake hands with each other randomly and there are 46 handshakes in total. Show that at least two people must have shaken hands twice.

Question 13 The human head can hold at most $120,000$ hairs. There are 24.8 million people living in Australia as of 2018. Show that there at least 207 people in Australia with exactly the same number of hairs on their head.

Question 14 Eight soccer teams play against each other and there were in total 29 soccer matches. Show that at least one pair of teams must have played against each other at least twice.

Question 15 A group of 98 children are enrolled in a holiday program. They can be enrolled in coding, jazz, drama, painting or cricket. If there are at least n people in one of the groups, find the value of n.

Question 16 There are 2958 people who have paid to attend Australia day fireworks in 8 different parks. At least one of the parks must have at least n people. Find the value of n.

Question 17 There are 293 students in a school. There must be at least one week during which at least n students in this school have their birthday. Find the value of n.

Question 18 A student is given 16 randomly chosen positive integers, and then instructed to find the remainder when divided by 5. The student finds that at least n of them have the same remainder. Find the value of n.

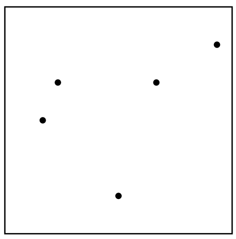
Challenge Problems

Problem 1 A university has an unknown number of students enrolled. However, it is known that at most four students have their phone numbers ending with the same two digits from 0 to 9 inclusive. What is the largest possible number of students enrolled?
Hint: Consider the cases for two students and three students.

Problem 2 A room has n people and some (but not necessarily all) people shake hands with each other randomly. Show that there must exist a pair of people who have shaken the exact same number of hands.
Hint: Consider the number of hands a person shakes to be the pigeonholes, and each person as the pigeon. Also, note that if two people shake no hands, then it is still counted as shaking the same number of hands, which is zero.

Problem 3 [Geometric application]

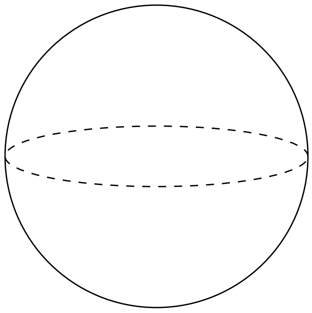
Consider five random points inside or on the border of a square of side-length 1 unit.



Show that there will always be a pair of points that are less than or equal to $\dfrac{1}{\sqrt{2}}$ units apart.
Hint: Divide the square into a 2×2 grid.

Problem 4 [Geometric application]

Consider five random points on the surface of a sphere.



Prove that there always exists a hemisphere (not necessarily with a horizontal cross section) that contains four of the points.
Hint: Construct an equator that passes through two of the points.

Problem 5 Consider the integers 1 to 2018 inclusive. What is the minimum number of integers that need to be selected to ensure that a pair of them sum to 2019?

Exercise 6I

Pascal's triangle and Binomial Expansions

Fundamentals

Fundamentals 1

Expand the following.

(a) $(x+y)^2 =$ __________

(b) $(x+y)^3 =$ _______________

Fundamentals 2

Draw Pascal's Triangle up to and including the sixth row. Note that the zero$^{\text{th}}$ row is just the number 1.

Fundamentals 3

(a) The expansion of $(x+y)^n$ has _____ terms.

(b) The numbers on the n^{th} row of Pascal's triangle are the coefficients of the expansion of ________.

Fundamentals 4

Fill in the following.

(a) $(1+x)^2 = \binom{2}{0} + \binom{2}{1}x + \_\_\,x^2$

(b) $(1+x)^3 = \binom{3}{0} + \_\_\,x + \_\_\,x^2 + \_\_\,x^3$

(c) $(1+x)^4 =$ _________________________

(d) $(1+x)^5 =$ _________________________

Fundamentals 5

Expand the following and express the coefficients in the form $\binom{n}{k}$.

(a) $(x+y)^3$

(b) $(x+y)^4$

(c) $(x+y)^5$

Fundamentals 6

In general

$$(1+x)^n = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Question 1 By first drawing Pascal's triangle up to the sixth row, complete the following.

(a) $(x+y)^4 = \_\_x^4 + \_\_x^3y + \_\_x^2y^2 + \_\_xy^3 + \_\_y^4$

(b) $(x+y)^5 = \_\_x^5 + \_\_x^4y + \_\_x^3y^2 + \_\_x^2y^3 + \_\_xy^4 + \_\_y^5$

(c) $(x+y)^6 = \_\_x^6 + \_\_x^5y + \_\_x^4y^2 + \_\_x^3y^3 + \_\_x^2y^4 + \_\_xy^5 + \_\_y^6$

Question 2 Write down the expansion of the following.

(a) $(1+x)^6$

(b) $(1+2x)^5$

(c) $(2x+3y)^5$

(d) $\left(2x+\frac{y}{2}\right)^4$

Question 3 Write down the expansion of the following.

(a) $(1-x)^5$

(b) $(2x-3)^4$

Question 4 Write down the expansion of the following.

(a) $(1+x^2)^5$

(b) $\left(x+\frac{1}{x}\right)^4$

(c) $\left(x^2-\frac{1}{x}\right)^5$

(d) $\left(x^3+\frac{2}{x^2}\right)^4$

Question 5 Find a and b such that

(a) $\left(1+\sqrt{2}\right)^5 = a+b\sqrt{2}$

(b) $\left(2-\sqrt{3}\right)^4 = a+b\sqrt{3}$

(c) $\left(\sqrt{2}+\sqrt{5}\right)^3 = a\sqrt{2}+b\sqrt{5}$

(d) $\left(\sqrt{3}-\sqrt{2}\right)^4 = a+b\sqrt{6}$

Question 6 Expand and simplify

(a) $\left(1+\sqrt{2}\right)^4+\left(1-\sqrt{2}\right)^4$

(b) $\left(1+\sqrt{2}\right)^4-\left(1-\sqrt{2}\right)^4$

Question 7

(a) By first expanding $(1+x)^4$, find the value of 1.1^4 without the use of a calculator.

(b) By first expanding $(1+2x)^5$, find the value of 1.02^5 correct to four decimal places.

Question 8 For what values of n does $\left(x+\frac{1}{x}\right)^n$ have a term independent of x when expanded?

Question 9 For each of the following binomial expressions, write down the term that contains x^2.

(a) $(1+x)^6$

(b) $(3+2x)^5$

(c) $(2-x)^4$

Question 10 [Reading off coefficients for simple expansions]

For the following questions, write down your answer in the form $\binom{n}{k}$ or ${}^nC_k$.

(a) By considering the expansion of $(1+x)^3$, write down the coefficient of x^2.

(b) By considering the expansion of $(1+x)^4$, write down the coefficient of x^2.

(c) By considering the expansion of $(1+x)^5$, write down the coefficient of x^2.

(d) What pattern do you notice?

(e) Hence, write down the coefficient of x^k in the expansion of $(1+x)^n$.

Question 11 [Binomial identity by equating coefficients]

Consider the expression

$$(1+x)^3(1+x)^4 = \left[\binom{3}{0}+\binom{3}{1}x+\binom{3}{2}x^2+\binom{3}{3}x^3\right]\left[\binom{4}{0}+\binom{4}{1}x+\binom{4}{2}x^2+\binom{4}{3}x^3+\binom{4}{4}x^4\right]$$

(a) There are four ways to obtain x^3 from the product. List them all.

(b) For each way you listed above, write down all the corresponding coefficients of x^3.
For example, one of them is $\binom{3}{0}\binom{4}{3}$.

(c) Deduce that

$$\binom{3}{0}\binom{4}{3}+\binom{3}{1}\binom{4}{2}+\binom{3}{2}\binom{4}{1}+\binom{3}{3}\binom{4}{0}=\binom{7}{3}$$

Question 12 Repeat a similar set of steps to above to prove the following results.

(a) $\binom{5}{0}\binom{3}{3}+\binom{5}{1}\binom{3}{2}+\binom{5}{2}\binom{3}{1}+\binom{5}{3}\binom{3}{0}=\binom{8}{3}$

(b) $\binom{5}{0}^2+\binom{5}{1}^2+\binom{5}{2}^2+\binom{5}{3}^2+\binom{5}{4}^2+\binom{5}{5}^2=\binom{10}{5}$

Hint: You will need to use the identity $\binom{n}{k}=\binom{n}{n-k}$

Question 13 For each of the following binomial expressions, find the coefficient of x^3.

(a) $(3+2x)(1+x)^3$ (b) $(2+x^2)(1-2x)^4$ (c) $(1+2x+3x^2)(4-5x)^3$

Question 14 Recall the expansion

$$(1+x)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5.$$

By substituting in a suitable value of x, simplify the following.

(a) $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

(b) $\binom{5}{0} + 2\binom{5}{1} + 4\binom{5}{2} + 8\binom{5}{3} + 16\binom{5}{4} + 32\binom{5}{5}$

(c) $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$

(d) $\binom{5}{0} - 2\binom{5}{1} + 4\binom{5}{2} - 8\binom{5}{3} + 16\binom{5}{4} - 32\binom{5}{5}$

Challenge Problems

Problem 1 In the expansion of

$$(3+2x+x^2)(a-3x)^4$$

the coefficient of x^4 is 81. Find the value of a.

Problem 2 [Binomial Identity]

(a) Differentiate $(1+x)^5$ using the chain rule.

(b) Differentiate $(1+x)^5$ by expanding and then differenting term-by-term.

(c) Hence, show that

$$\binom{5}{1} + 2\binom{5}{2} + 3\binom{5}{3} + 4\binom{5}{4} + 5\binom{5}{5} = 80$$

Problem 3 [Method of proving divisibility]

(a) Expand $(1+x)^5$.

(b) By substituting an appropriate value of x, show that $7^5 - 1$ is divisible by 6.

(c) Is $7^n - 1$, where n is any positive integer, divisible by 6? Justify your answer.

Problem 4 In the expansion of $(1+x+px^2)^n$, the coefficient of x^2 is zero. Find the value of p when

(a) $n = 2$ (b) $n = 3$ (c) $n = 5$

Chapter 6 Review

Combinatorics and Binomial Expansions

Review

Question 1 The following are assorted permutations problems.

(a) In how many ways can 5 people be arranged in line?

(b) How many different arrangements are possible using 3 letters of the word PALINDROME?

(c) In how many ways can 8 people be arranged in a circle?

(d) How many ways can 5 boys and 3 girls be arranged in a line so that the girls are together?

(e) How many ways can 4 boys and 4 girls be arranged in a line so that the boys and girls alternate?

(f) How many ways can 4 boys and 3 girls be arranged in a line so that the boys and girls alternate?

(g) How many 5 digit numbers can be formed by rolling a die 5 times.

(h) In how many ways can 6 friends sit at a round table if Adam and Ben do not sit next to each other?

(i) How many 10 digit mobile phone numbers can be made if all numbers start with 0 and the next 3 numbers cannot be 0?

Question 2 There are 4 men and 3 women to be seated in a line. Find the number of arrangements if

(a) there are no restrictions.

(b) men and women alternate.

(c) men and women are in separate groups.

(d) the three women want to sit together.

(e) each end of the line is occupied by a man.

Question 3 How many ways can 2 boys and 4 girls be arranged in a line so that

(a) the boys are at either end of the line?

(b) the two boys are not together?

(c) there are at least 3 girls separating the boys?

Question 4 Suppose 10 identical coins are tossed, and they show 6 heads and 4 tails.

(a) In how many ways can they be arranged in a straight line?

(b) In how many ways will the tails be together?

Question 5 Consider the digits 4, 5, 6, 7 and 8, where no repetition of digits is allowed.

(a) How many 3 digit numbers are possible?

(b) How many even 3 digit numbers are possible?

(c) How many numbers greater than 6000 can be formed?

Question 6 How many distinct permutations of the letters of the word 'COTTON' are possible in a straight line when

(a) the word begins and ends with the letter T?

(b) the two T's must be together?

Question 7

(a) How many ways can the letters of the word PAPER be arranged?

(b) There are five identical blue marbles and four identical yellow marbles arranged in a row. How many different arrangements are possible?

Question 8 There are 4 men and 3 women to be seated in a line. If there are 9 men and 4 women to select from, find the number of possible arrangements.

Question 9 Consider are 8 distinct keys. How many ways can they be arranged

(a) in a straight line? (b) in a circle? (c) on a key-ring?

Question 10 A committee of four is to be chosen from 5 men and 6 women. Find how many committees are possible if:

(a) all the members are to be male.

(b) there are 3 women and one man.

(c) a particular man must be included.

(d) a particular man must not be included.

(e) the committee consists of women only, but excluding a particular woman whilst including another particular woman.

Question 11

(a) How many committees of 5 people can be chosen from 9 people?

(b) How many committees consisting of 2 teachers and 4 students can be made up if there are 4 teachers and 6 students to choose from?

Question 12 Using the letters P, P, Q, R, S, T, T, find how many codes can be formed if they contain

(a) all 7 letters

(b) exactly four of the 7 letters
Hint: You will need to consider cases here!

Question 13 From the digits 0 to 9 inclusive, four digit numbers are formed. Find how many are possible if the digits are in

(a) ascending order.

(b) descending order.

Question 14 At a dinner party, the host, hostess and their 6 guests sit at a round table. In how many ways can they be arranged if the host and hostess are sitting

(a) together?

(b) opposite one another?

(c) separately?

Question 15 From a group of 9 people, we need to form two teams of four as well as a referee. Find the number of ways of forming this if

(a) there are no restrictions?

(b) the referee is already chosen from the 9 people?

(c) person X cannot be the referee?

(d) X and Y have to be on the same team?

(e) X and Y cannot be on the same team?

Question 16 Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set consisting of the single digit counting numbers. How many subsets of S are there if
Hint: Ordering does not matter for subsets.

(a) there is no restriction? Note that this can include the empty set.

(b) there are at least two numbers?

(c) there are at least two numbers, but excluding 3?

(d) there are at least two numbers, excluding 3 but including 1?

Question 17 The following are assorted probability problems.

(a) If a committee of 6 members is chosen from 4 teachers and 6 students, what is the probability that it contains 2 teachers and 4 students?

(b) At a football club, a team of 11 players is to be chosen from a pool of 30 players consisting of 18 local-born players and 12 players born in country towns. What is the probability that the team will consist of all local-born players?

(c) A tennis team of 4 players is to be selected at random from 9 players, of which two are twins. What is the probability that the team includes the twins.

(d) There are 4 boys and 3 girls arranged randomly in a line. What is the probability that the boys and girls alternate?

(e) A four-digit number is randomly formed using the digits $4, 5, 6, 7$ and 8, and repetition of digits is allowed. Find the probability that the 4 digit number ends with 7 or 8.
Hint: This can be done without any knowledge of permutations and combinations

Question 18 A bag contains 10 white marbles, 6 red marbles and 4 yellow marbles. They are all identical marbles, except for their colour. Three marbles are selected at random. What is the probability that all three marbles are

(a) different?

(b) the same colour?

Question 19 How many selections of at least one object can be made from n distinct objects?

Question 20 A sock drawer contains four pairs of socks. Each pair has a unique colour. How many socks need to be drawn randomly in order to guarantee exactly

(a) one matching pair?

(b) two matching pairs?

(c) three matching pairs?

Question 21 How many cards must be selected from a standard deck of 52 cards to guarantee that you have at least two cards with the same

(a) colour?

(b) suit?

(c) value?

Question 22 Show that a cohort of 55 students will have at least two students with first-names beginning with the same letter.

Question 23 In a season of soccer with twelve teams, there were 67 games played in total. Show that some pair of teams who must have played each other more than once.

Question 24 In a season of soccer with n teams, there were 16 games played in total. If there exists some pair of teams who played each other more than once, what is the largest value of n?

Question 25 Recall that a *digit sum* of a number n is the value obtained when the digits of n are added together. For example, the digit sum of 152 is $1 + 5 + 2 = 8$.

(a) How many four-digit numbers must be selected to guarantee that at least two of them have the same digit sum?

(b) 110 randomly-chosen three-digit numbers are selected. Show that at least five of them have the same digit sum.

Question 26 Seven points are randomly chosen on the number line in the interval $x \in [0, 1]$. Show that there exist two points that are less than 0.2 apart from each other.

Question 27 Four points are chosen randomly along the circumference of a circle. Show that three of these points lie on the same semi-circle.

Question 28 Show that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Question 29 Write down the expansion of

(a) $(x+y)^4$ (b) $(x+y)^5$

Question 30 Expand the following.

(a) $(2+x)^5$ (b) $(2-3x)^4$ (c) $(1+x^2)^4$

(d) $\left(x+\dfrac{2}{x}\right)^5$ (e) $\left(x^2+\dfrac{1}{x}\right)^4$ (f) $\left(2x^3-\dfrac{3}{x^2}\right)^4$

Question 31 Find the coefficient of x^3 in each of the following.

(a) $(2-x)(3+4x)^5$ (b) $(1+x+x^2)(3-2x)^3$ (c) $(3+4x)^3(1-2x)^4$

Question 32 In the expansion of

$$(2+3x+4x^2)(a-5x)^3$$

the coefficient of x^3 is -40. Find the value of a.

Investigation Task

Double Factorials

A double factorial is the product of every *second* consecutive integer leading down from the number until either 1 or 2. It is denoted by a double exclamation mark instead of a single one. For example

$$10!! = 10 \times 8 \times 6 \times 4 \times 2$$

$$9!! = 9 \times 7 \times 5 \times 3 \times 1$$

This investigation task aims to draw connections between the double factorial and the usual factorial.

Question 1

(a) Show that $8!! = 2^4 4!$

(b) Show that $9!! = \dfrac{9!}{8!!}$

(c) Deduce that $9!! = \dfrac{9!}{2^4 4!}$

Question 2 [Generalising]

(a) Show that if $n = 2k$, in other words if n is even, then

$$n!! = 2^k k!$$

(b) Show that if $n = 2k + 1$, in other words if n is odd, then

$$n!! = \frac{n!}{(n-1)!!} = \frac{n!}{2^k k!}$$

Question 3 Research some applications of the double factorial. Provide specific problems that require the use of it to answer, and then give solutions to those problems.

Investigation Task

Combinatoric Proofs

A combinatoric argument is a way of proving binomial identities by finding two logically different (but still valid) ways of counting the same scenario. For example, suppose there are ten applicants for a team of five people, which is to include a captain. The team of five, including a captain, can be selected in two different ways depending on how you like to think.

Method #1: Choose the five people first from the ten, then choose a captain from the group of five.

$$\binom{10}{5} \times 5$$

Method #2: Choose the captain first from the ten, then four more people from the remaining group of nine.

$$10 \times \binom{9}{4}$$

Since both techniques are valid ways of counting the same scenario, they must be equal to each other. Hence, we conclude that

$$\binom{10}{5} \times 5 = 10 \times \binom{9}{4}$$

This can be verified easily using a calculator.

Question 1 Research and explain in detail the combinatoric arguments for the following identities.

(a) $\binom{n}{k} = \binom{n}{n-k}$

(b) $\binom{n}{m}\binom{n-m}{k-m} = \binom{n}{k}\binom{k}{m}$

(c) ${}^nP_k = {}^nC_k \times k!$

(d) $\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$

Question 2 [Harder examples]

Research and explain in detail the combinatoric arguments for the following identities.

(a) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

(b) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

(c) $\binom{m}{k}\binom{n}{0} + \binom{m}{k-1}\binom{n}{1} + \binom{m}{k-2}\binom{n}{2} + \cdots + \binom{m}{0}\binom{n}{k} = \binom{m+n}{k}$

Investigation Task

Patterns in Pascal's Triangle

Pascal's Triangle is more than a tool used to find the coefficients in a binomial expansion. It also contains many nice properties and patterns when specific terms are grouped together. This investigation task aims to expose these nice properties to the student.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Question 1

(a) What do you notice about the result when you add all the terms in the same row?

(b) Write down the binomial identity associated with the pattern you notice.

Question 2 Research at least three more patterns in Pascal's Triangle and write down the corresponding binomial identity that characterises those patterns algebraically. Your answer should include discussion of Pascal's Identity as well as what happens when you sum across diagonals.

1. Further Functions

Exercise 1A

Reciprocal and square root graphs

F1

(a) $\left(a, \frac{1}{b}\right)$

(b) $\left(a, \sqrt{b}\right)$

F2

(a) decreases

(b) increases

(c) ∞

(d) $-\infty$

(e) 0^+

(f) 0^-

(g) vertical asymptotes

F3

(a) y is undefined so there is no graph.

(b) Higher

(c) Lower

(d) vertical

F4

$y^2 = f(x)$ is $y = \pm\sqrt{f(x)}$ whereas $y = \sqrt{f(x)}$ is only the part above the x-axis.

Q1

(a) $y = \dfrac{1}{f(x)}$

(b) $y^2 = f(x)$

(c) $y = \sqrt{f(x)}$

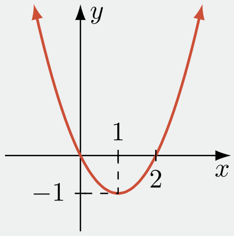
Q2

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$

(c) $y^2 = f(x)$

Q3

(a)



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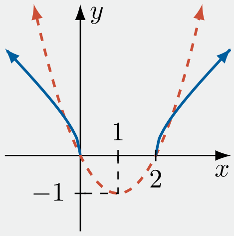
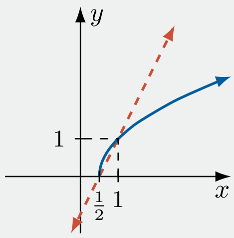


Q4

(a)



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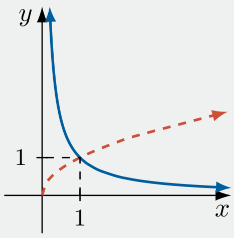

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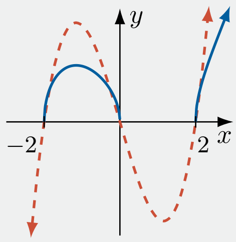

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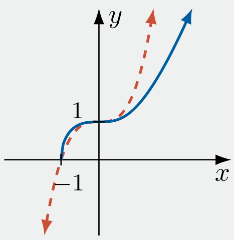

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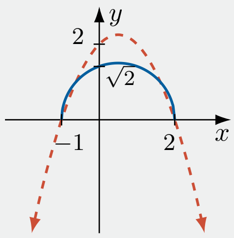

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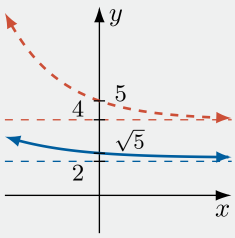
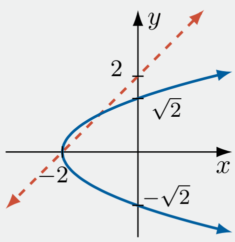


Q5

(a)



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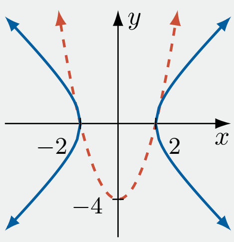

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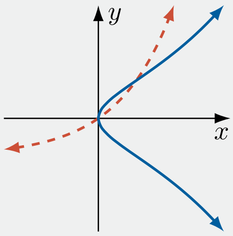
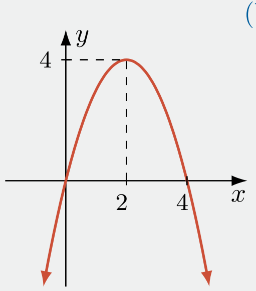


Q6

(a)



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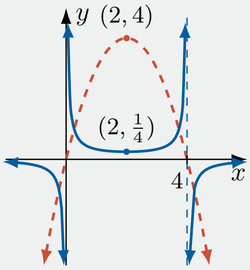
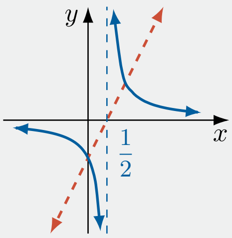


Q7

(a)



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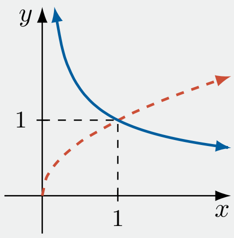

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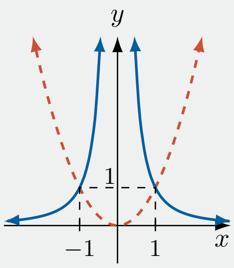

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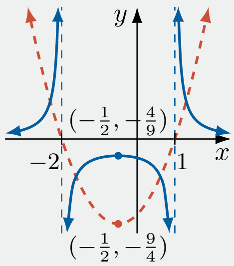

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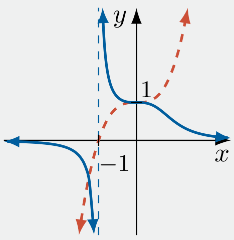

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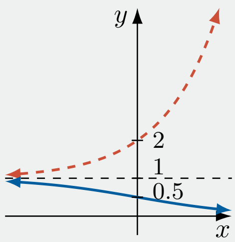

Q8

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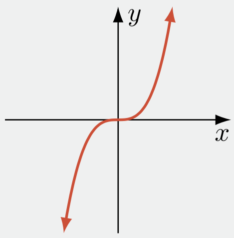
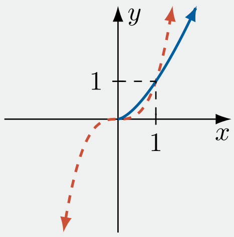


(b)

(i)



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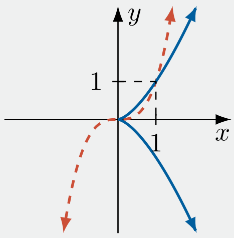

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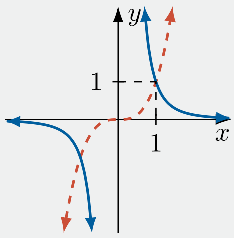
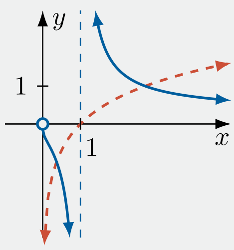


Q9

(a)



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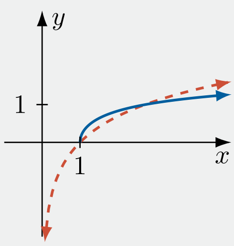

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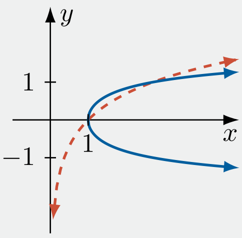

Q10

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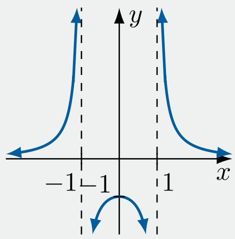

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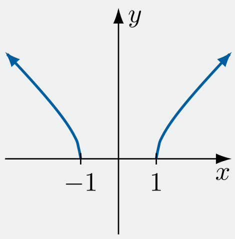

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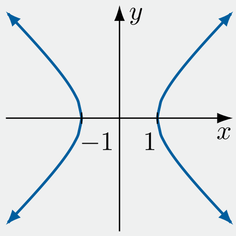

Q11

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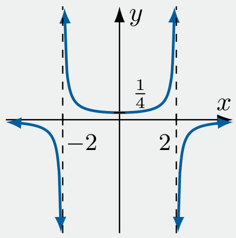

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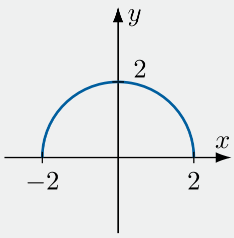

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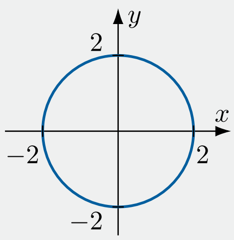

Q12

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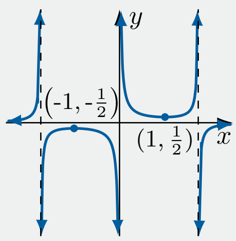

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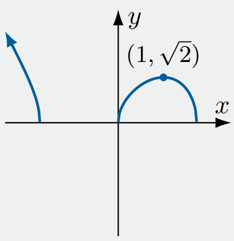

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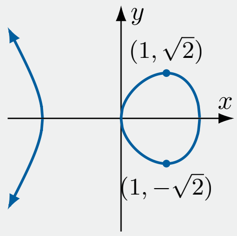

Q13

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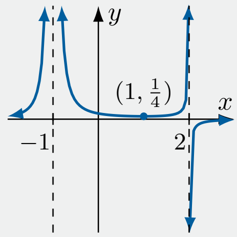

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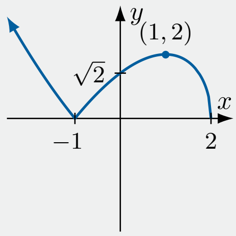

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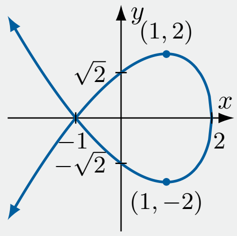

Q14

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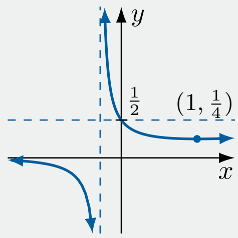
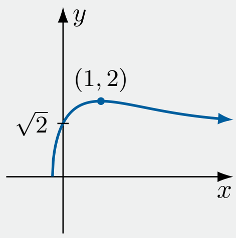
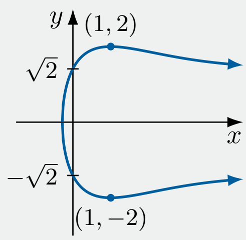


(b)

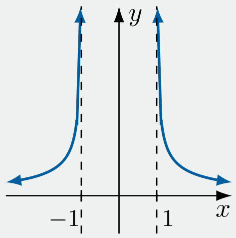


(c)

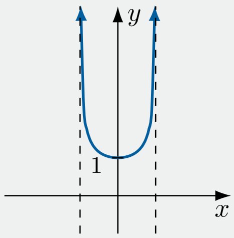


P1

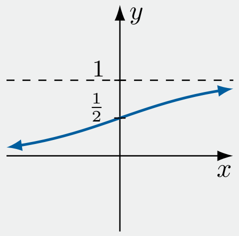
(a)



(b)

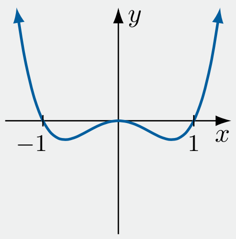


P2

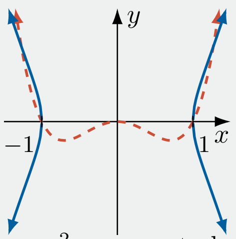


P3

(a)



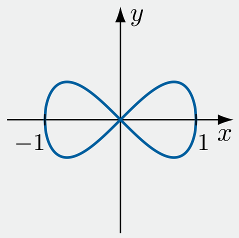
(b)



(c) $y = x^2$ appears to be an asymptote

(d) See full worked solutions.

P4



P5

See full worked solutions.

Exercise 1B

Further reflections

F1

(a) $y = f(|x|)$

(b) $y = |f(x)|$

F2

(a) $|x| = \begin{cases} x, & \text{for} \quad x \geq 0 \\ -x, & \text{for} \quad x < 0 \end{cases}$

(b) $|f(x)| = \begin{cases} f(x), & \text{for} \quad f(x) \geq 0 \\ -f(x), & \text{for} \quad f(x) < 0 \end{cases}$

F3

(a) $f(x)$

(b) $-f(x)$, x

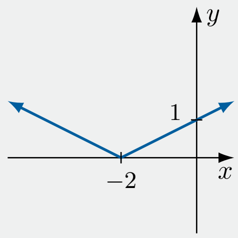
F4

(a) $f(x)$

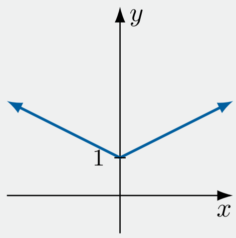
(b) $f(-x)$, left, y

Q1

(a)

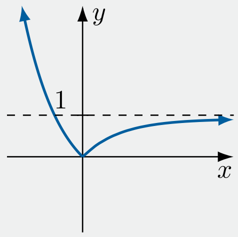


(b)

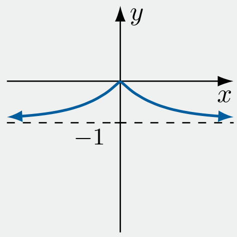


Q2

(a)

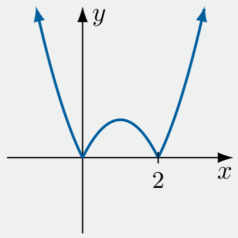


(b)

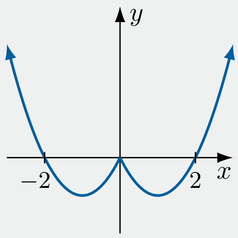


Q3

(a)

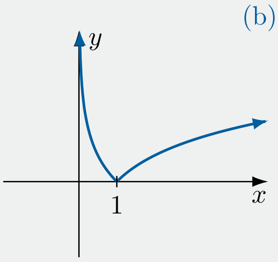


(b)

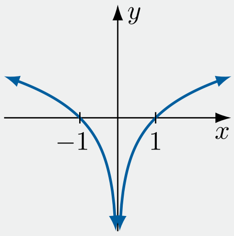


Q4

(a)

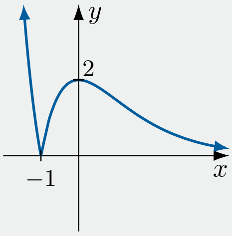


(b)

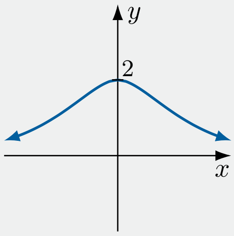


Q5

(a)

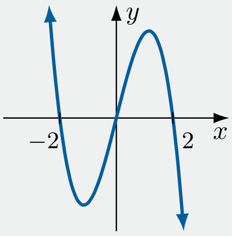


(b)

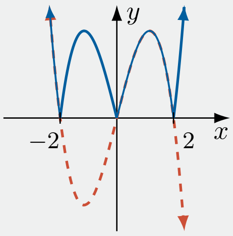


Q6

(a)

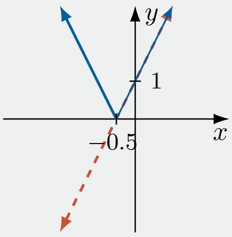


(b)

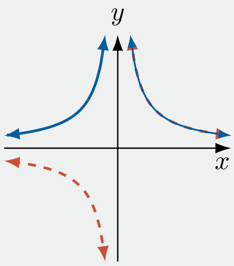


Q7

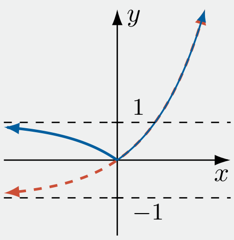
(a)



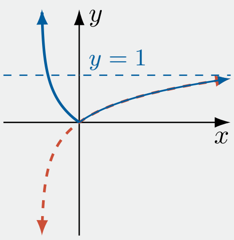
(b)



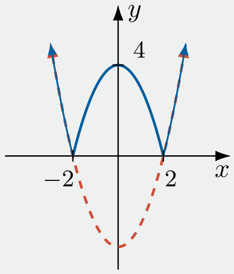
(c)



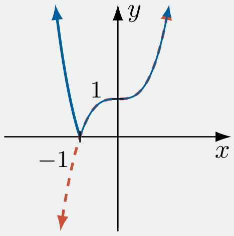
(d)



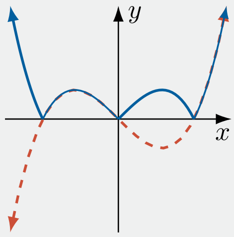
(e)



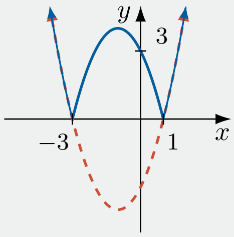
(f)



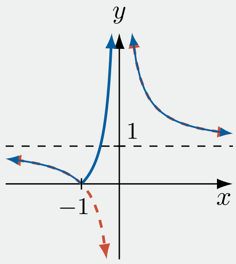
(g)



(h)

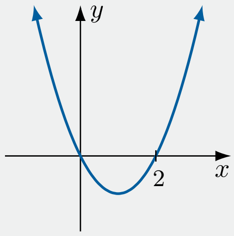


(i)

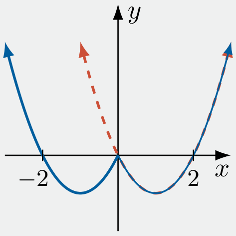


Q8

(a)

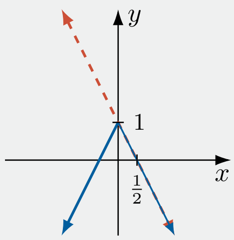


(b)

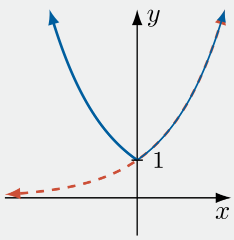


Q9

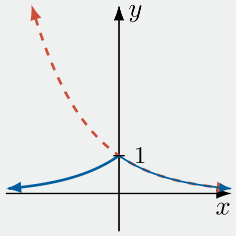
(a)



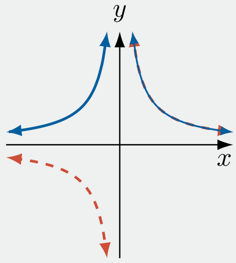
(b)



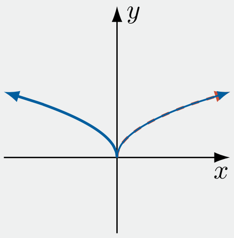
(c)



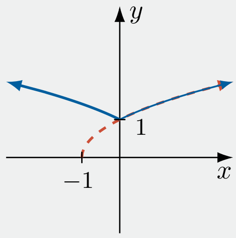
(d)



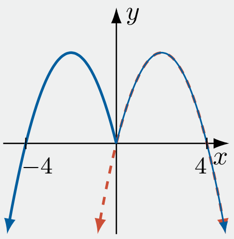
(e)



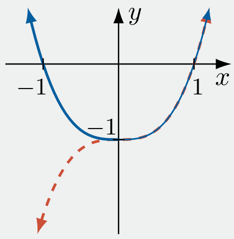
(f)



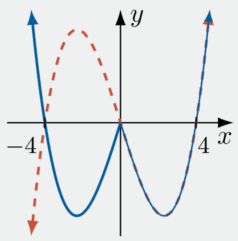
(g)



(h)



(i)



Q10

(a) See full worked solutions.

(b) See full worked solutions.

P1

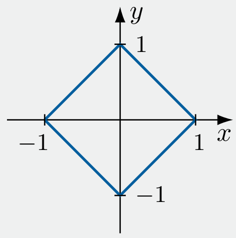
$y = \frac{1}{2}\left(x^2 - 4x\right)$

P2

(a) Delete all parts where $x < 0$ and reflect all parts where $x \geq 0$ across the y-axis.

(b) Reflect all parts where $y > 0$ across the x-axis.

(c)

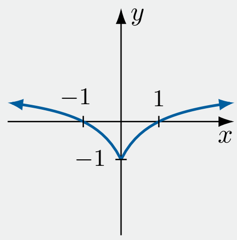


P3

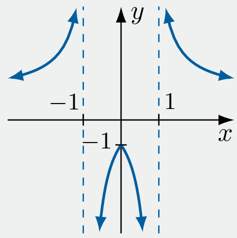
(a) See full worked solutions.

(b) See full worked solutions.

(c)



(d)



Exercise 1C

Adding graphs

F1

(a) $(a, b + c)$

(b) $(a, 2b)$

F2

$-g(x)$

F3

scale

F4

(a) $y = f(x) + g(x)$ takes on the value of $g(x)$ at $x = a$

(b) The y-coordinate of $y = f(x) + g(x)$ gets doubled.

(c) $y = f(x) + g(x)$ has an x-intercept.

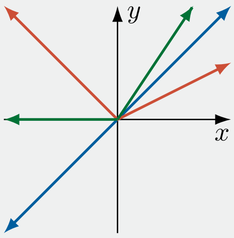
F5

(a) $x \geq 0$

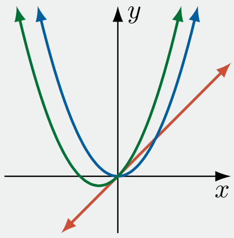
(b) sum

Q1

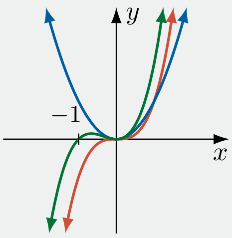
(a)



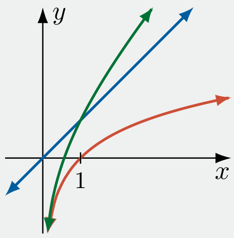
(b)



(c)

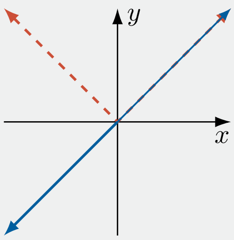


(d)

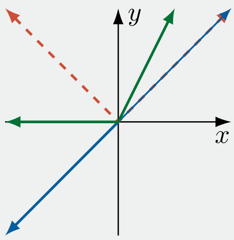


Q2

(a)

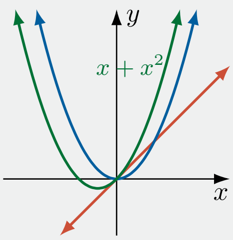


(b)

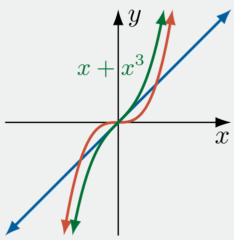


Q3

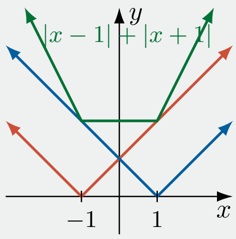
(a)



(b)

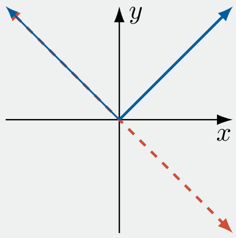


(c)

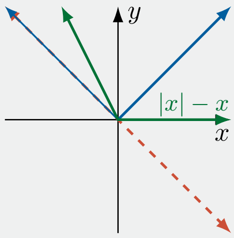


Q4

(a)

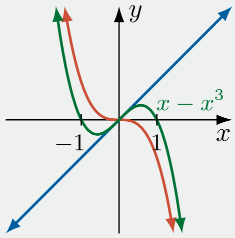


(b)

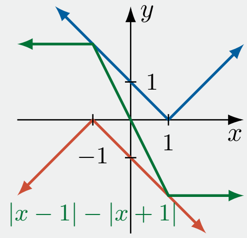


Q5

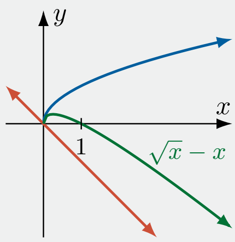
(a)



(b)

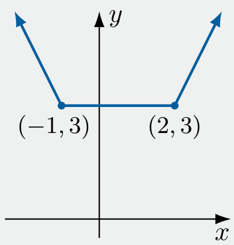


(c)

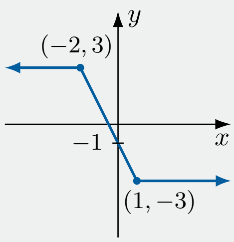


Q6

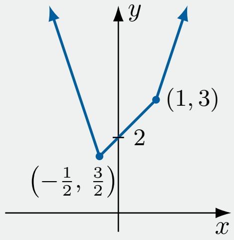
(a)



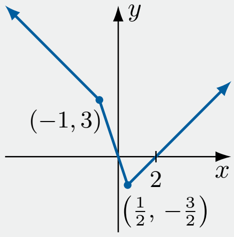
(b)



(c)

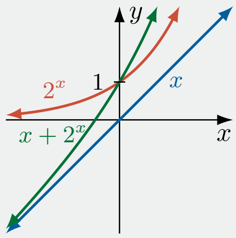


(d)



Q7

(a)



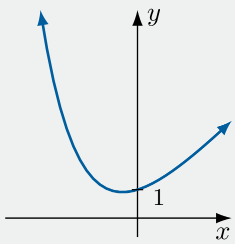
(b) $y = x$ is an asymptote.

(c) As $x \to -\infty$, $2^x \to 0$ so the curve basically is just $y = x$ around here.

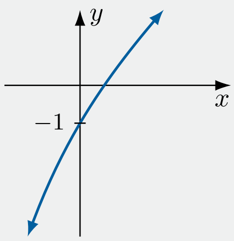
(d) Same as part (a)

Q8

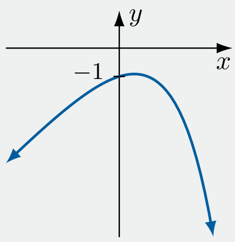
(a)



(b)

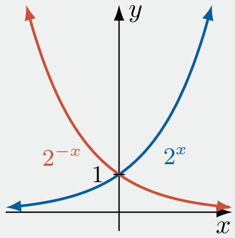


(c)

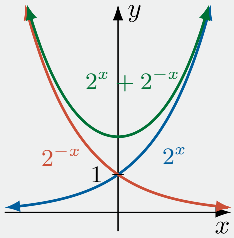


Q9

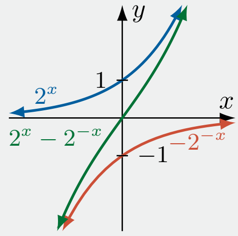
(a)



(b)

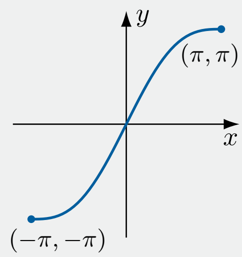


(c)

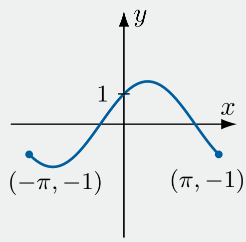


Q10

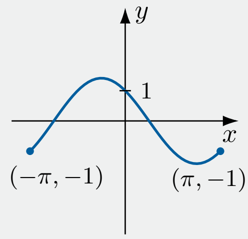
(a)



(b)

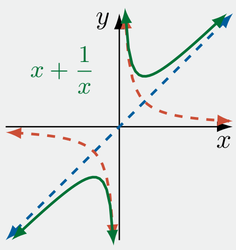


(c)

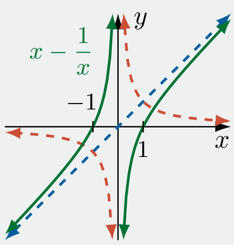


P1

(a)

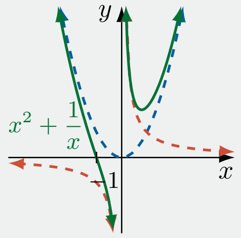


(b)

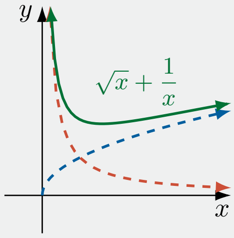


ANSWERS

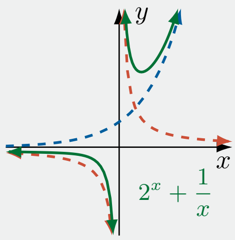
(c)



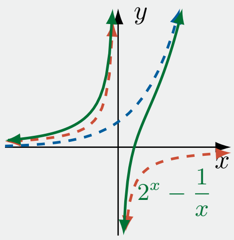
(d)



(e)



(f)



P2

(a) See full worked solutions.

(b) Bob has the correct one. Mary's answer disregards the condition that $x \neq 1$.

Exercise 1D

Multiplying graphs

F1

(a) (a, bc)

(b) (a, b^2)

F2

$\frac{1}{g(x)}$

F3

(a) $y = f(x)g(x)$ also has an x-intercept at $x = a$.

(b) $y = f(x)g(x)$ takes on the value of $g(x)$ at $x = a$

F4

(a) > 0

(b) < 0

(c) > 0

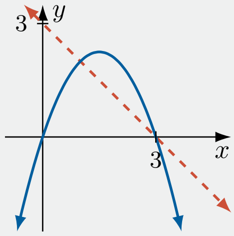
F5

(a) $x \geq 0$

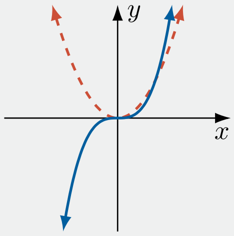
(b) product

Q1

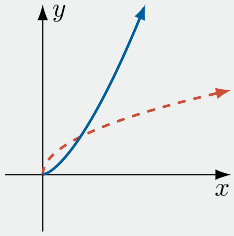
(a)



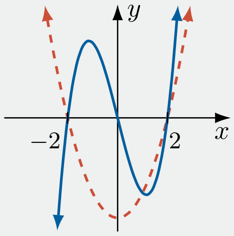
(b)



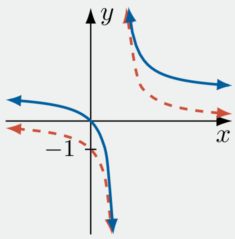
(c)



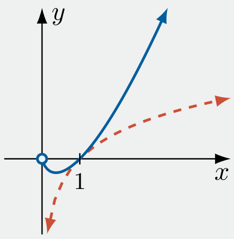
(d)



(e)

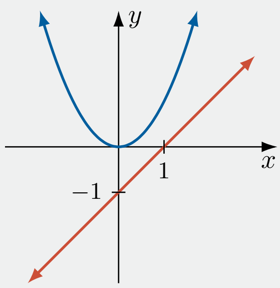


(f)

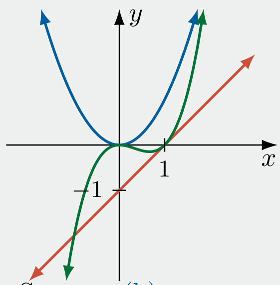


Q2

(a)



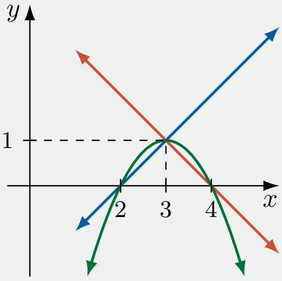
(b)



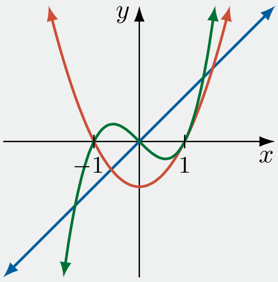
(c) Same as (b)

(d)

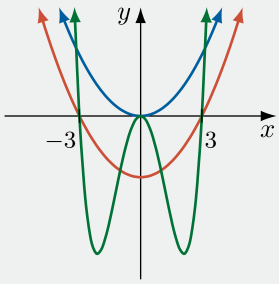
(i)



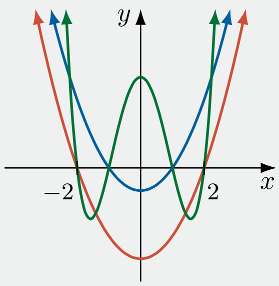
(ii)



(iii)

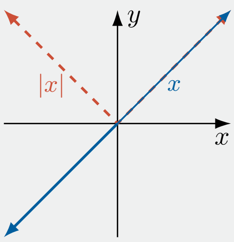


(iv)

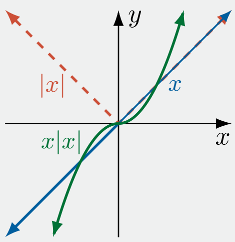


Q3

(a)

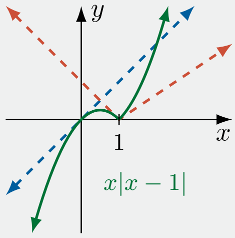


(b)

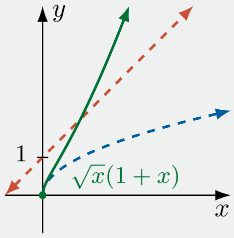


Q4

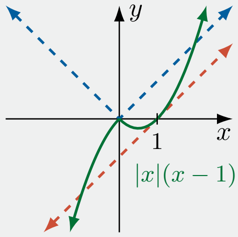
(a)



(b)



(c)



Q5

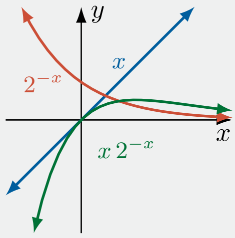
(a)

x	-10	-5	-1	0	1	5	10
$x\,2^{-x}$	-10240	-160	-2	0	0.5	0.1563	0.0098

(b) As $x \to \infty$, $y \to 0$.

As $x \to -\infty$, $y \to -\infty$.

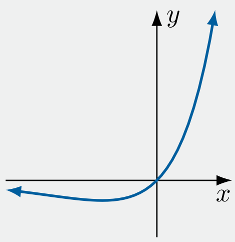
(c)



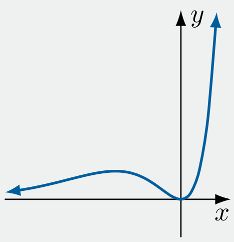
(d) Same as (c)

Q6

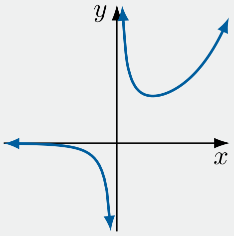
(a)



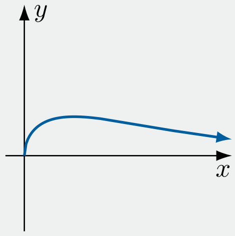
(b)



(c)

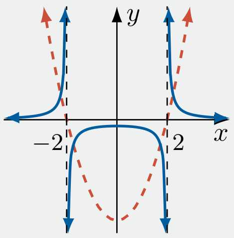


(d)

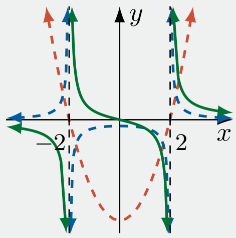


P1

(a)

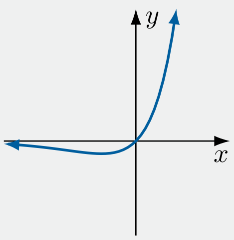


(b)



P2

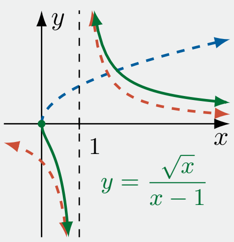
(a)



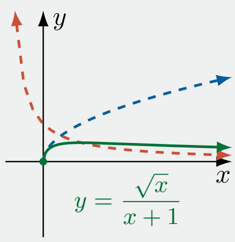
(b) When $x > 0$, $y = x\,3^x$ is above $y = x$. When $x < 0$, $y = x\,3^x$ is still above $y = x$. However, at $x = 0$, the two curves intersect. Hence, $y = x$ touches $y = y = x\,3^x$ at the origin.

P3

(a)

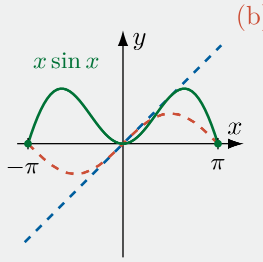


(b)

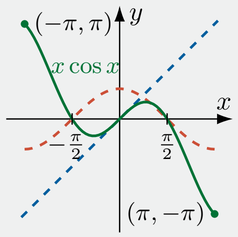


P4

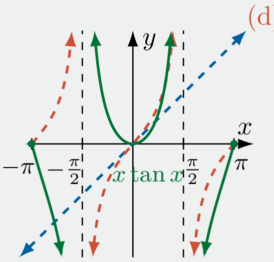
(a)



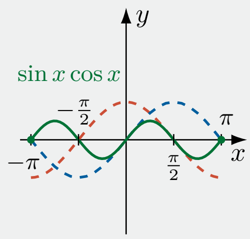
(b)



(c)

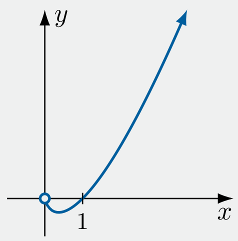


(d)

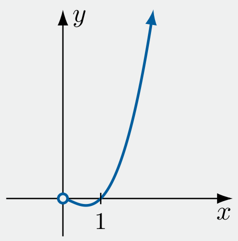


P5

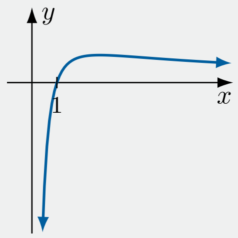
(a)



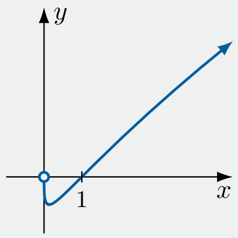
(b)



(c)



(d)



Exercise 1E
Inequalities

F1

(a) $x \geq a$ or $x \leq b$

(b) $a < x < b$

F2

(a) $ax^2 + bx + c$

(b) above, below, x

(c) x

F3

(a) square (b) negative

(c) factorise

F4

(a) $-k,\ k$ (b) $k,\ -k$

F5

(a) $-k,\ k$ (b) $k,\ -k$

Q1

(a) $(x-a)(x-b) \geq 0$

(b) $(x-a)(x-b) \leq 0$

Q2

(a) $-2 \leq x \leq 2$

(b) $x \geq 5$ or $x \leq -4$

(c) $-1 < x < 2$

(d) $x > 8$ or $x < -1$

(e) $x \leq -2$ or $x \geq 2$

(f) $x < 0$ or $x > 4$

(g) $x \leq -4$ or $x \geq 4$

(h) $-\frac{3}{2} < x < 2$

Q3

(a) $1 \leq x \leq 4$ or $x \leq -1$

(b) $-6 < x < -2$ or $x > 3$

(c) $x \geq 1$ or $x \leq -1$

(d) $-3 < x < -2$ or $2 < x < 3$

Q4

(a) $x \neq 1$

(b) See full worked solutions.

(c) See full worked solutions.

(d) $x < -1$ or $x \geq -\frac{1}{2}$

Q5

(a) $x > 5$ or $x < 2$

(b) $-4 < x \leq 7$

(c) $x < -2$ or $-1 < x < 1$

(d) $x > -2,\ x \neq 2$

(e) $-1 \leq x < 3$ or $x \geq 4$

(f) $-2 < x < 0$

Q6

(a)

(i) origin, 4

(ii) origin, less

(iii) $-4 < x < 4$

(iv) $x > 4,\ x < -4$

(b)

(i) 2, 5

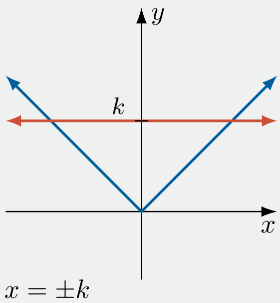
(ii) 2, less

(iii) $-3 < x < 7$

(iv) $x > 7,\ x < -3$

Q7

(a)



(b) $x = \pm k$

(c) $x \geq k$ or $x \leq k$

(d) $-k \leq x \leq k$

Q8

(a) $a-k \leq x \leq a+k$

(b) $x \geq a+k$ or $x \leq a-k$

Q9

(a) $x \geq 4$ or $x \leq -4$

(b) $-2 < x < 2$

(c) $-4 < x < 2$

(d) $x \geq 5$ or $x \leq 1$

(e) $x = 4$

(f) All real x

(g) All real x except $x = \frac{4}{3}$

(h) No solutions.

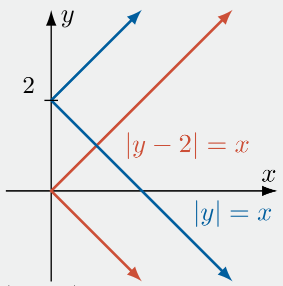
(i) $x \geq 2$ or $x \leq -3$

(j) $-2 \leq x \leq 5$

(k) All real x.

(l) No solutions.

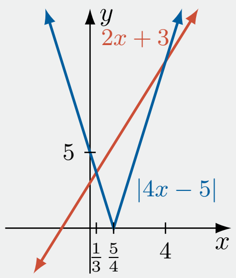
P1



$|y-2| = x$ is the same as $|y| = x$ but shifted up by 2 units.

P2

(a)



(b) $x > 4$ or $x < \frac{1}{3}$

P3

$x > \frac{3}{4}$ or $x < \frac{1}{4}$.

P4

$x > 0$

P5

x^2 is always positive anyway, so we can divide it out without affecting the inequality.

P6

$-5 \le x \le 1$

Exercise 1F

Inverse functions

F1

(a) x

(b) $f^{-1}(x)$

(c) a

F2

(a) function, relation

(b) relations

(c) relation, $y^2 = x$

(d) one, one, y

F3

(a) x

(b) inverse relation, inverse function

(c) horizontal

(d) fails

(e) passes

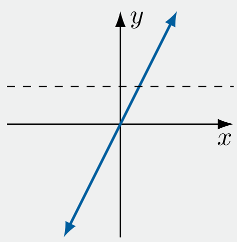
F4

(a) domain, one, one

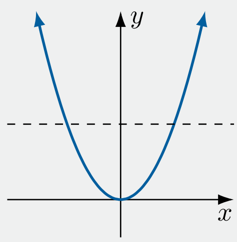
(b) domain, range, range, domain

Q1

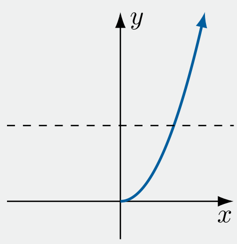
(a) Inverse is a function



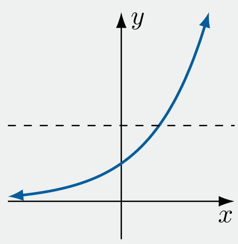
(b) Inverse is not a function



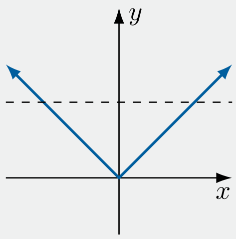
(c) Inverse is a function



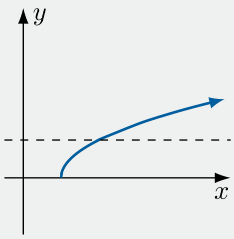
(d) Inverse is a function



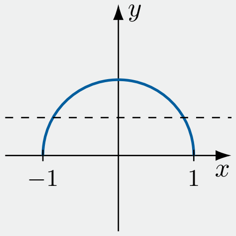
(e) Inverse is not a function



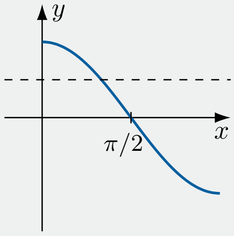
(f) Inverse is a function



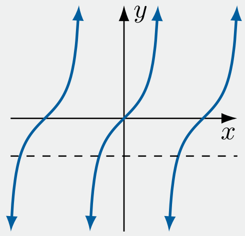
(g) Inverse is not a function



(h) Inverse is a function



(i) Inverse is not a function



Q2

(a) $y = \frac{1}{2}x$

(b) $y = \frac{1}{4}(x + 5)$

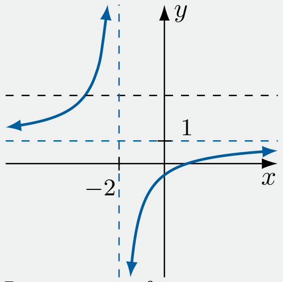
(c) $y = -\frac{1}{2}(3x + 5)$

(d) $y = \frac{2}{3}(x + 1)$

Q3

(a) Domain: All real x, $x \neq -2$
Range: $y \neq 1$

(b)



Inverse is a function since the graph passes the horizontal line test.

(c) $f^{-1}(x) = \frac{1 + 2x}{1 - x}$

Q4

(a) $y = \frac{1}{x} - 1$

(b) $y = 4 - \frac{1}{x}$

(c) $y = \frac{2x + 1}{2 - x}$

Q5

(a) $x = 1$

(b) $y = -2$

Q6

When we reflect across $y = x$ to obtain the inverse, the horizontal line becomes a vertical line, so it now becomes the vertical line test.

Q7

(a) Domain: All real x
Range: All real y

(b) Domain: $x \geq -3$
Range: $y \geq 0$

(c) Domain: $x \geq 1$
Range: $y \geq 0$

(d) Domain: All real x, $x \neq 0$
Range: All real y, $y \neq 0$

(e) Domain: $x > 0$
Range: All real y

(f) Domain: $x > 0$
Range: $y > -2$

Q8

(a) $y = \frac{1}{2}(x^2 + 1)$

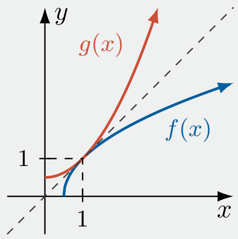
Domain: $x \geq 0$

Range: $y \geq \frac{1}{2}$

(b) $(1, 1)$

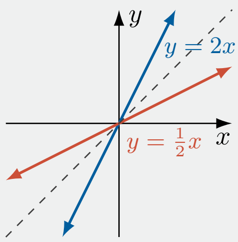
(c) See full worked solutions.

(d)

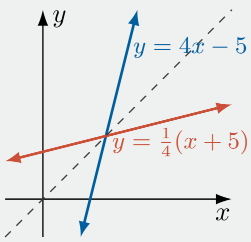


Q9

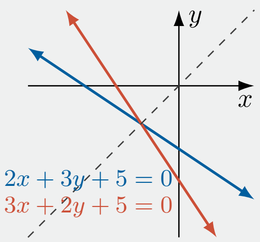
(a)



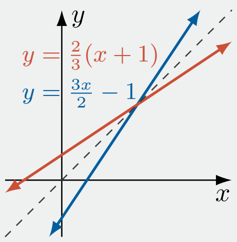
(b)



(c)



(d)

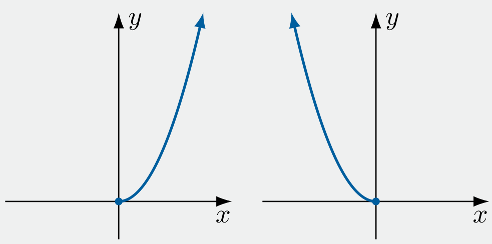


Q10

(a) It is not a one-to-one function since it does not pass the horizontal line test.

(b) $x \geq 0$ and $x \leq 0$

(c)



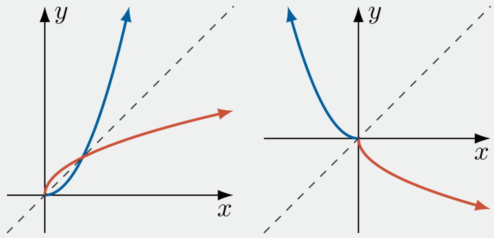
(d) $f^{-1}(x) = \sqrt{x}$ is the inverse of

$f(x) = x^2,\ x \geq 0.$

$f^{-1}(x) = -\sqrt{x}$ is the inverse of

$f(x) = x^2,\ x \leq 0.$

(e)



Q11

(a) $x \geq 0$ and $x \leq 0$

(b) $x \geq 2$ and $x \leq 2$

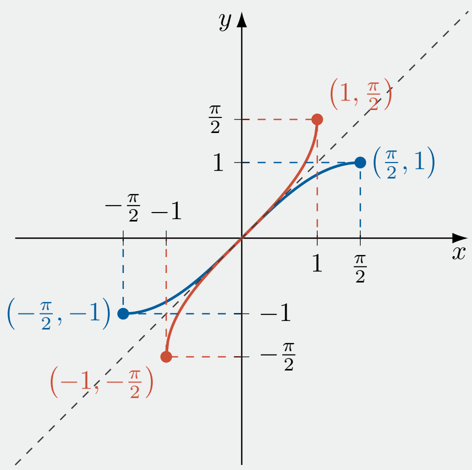
(c) $x \geq 1$ and $x \leq 1$

(d) $x \geq 3$ and $x \leq 3$

Q12

(a) $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b)

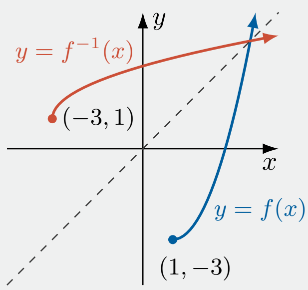


Q13

(a) $x \geq 1$

(b) Domain: $x \geq -3$
Range: $y \geq 1$

(c)



Q14

(a) $x \geq 1$

(b) $y \geq -1$

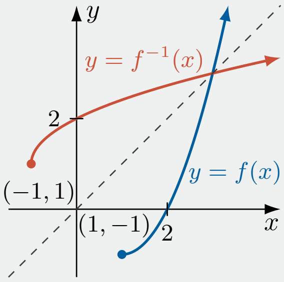
(c) See full worked solutions.

(d) See full worked solutions.

(e) $y \geq 1$

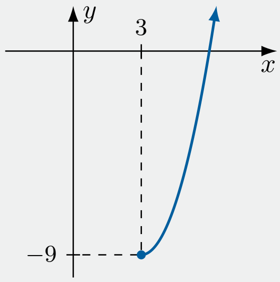
(f) $y = 1 + \sqrt{1 + x}$

(g)



Q15

(a)



(b) $(7, 7)$

(c) $y = 3 + \sqrt{x + 9}$

P1

See full worked solutions.

P2

(a) See full worked solutions.

(b) See full worked solutions.

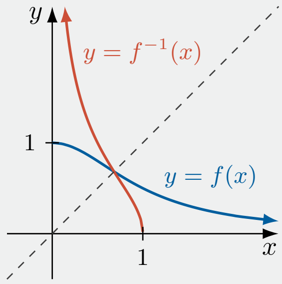
P3

(a) See full worked solutions.

(b) See full worked solutions.

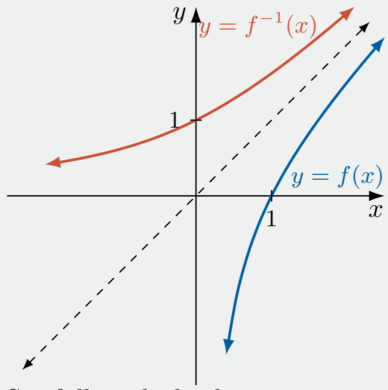
(c) $x \geq 0$

(d)



P4

(a)



(b) See full worked solutions.

P5

(a) $f^{-1}(x) = -2 + \sqrt{x - 1}$

(b) See full worked solutions.

Exercise 1G

Parametric forms

F1

(a) Cartesian

(b) parameter

(c) parameter, x, y

(d) one, parameter

(e) is not, may

F2

(a) Cartesian, eliminate

(b) substituting, trigonometric

F3

(a) r^2, $r \sin \theta$

(b) $\sin^2 \theta + \cos^2 \theta = 1$

(c) $r \cos \theta$, b

Q1

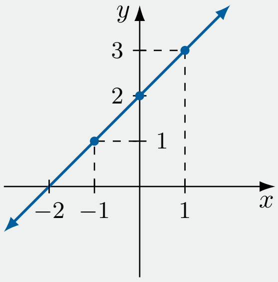
(a)

t	-2	-1	0	1	2
x	-3	-2	-1	0	1
y	-1	0	1	2	3

(b) $y = x + 2$

(c) $t = 5$

(d)



Q2

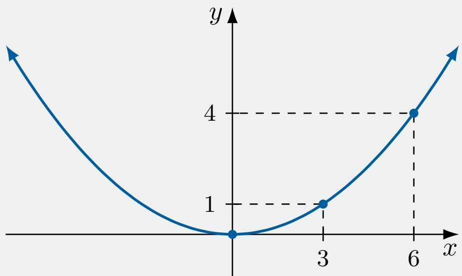
(a)

t	-2	-1	0	1	2
x	-6	-3	0	3	6
y	4	1	0	1	4

(b) $y = \dfrac{x^2}{9}$

(c) $t = 2$

(d)



(e) T moves further away from the origin.

Q3

(a) $y = \dfrac{3x}{2}$

(b) $y = 2x - 6$

(c) $2x + 3y = 16$

(d) $y = x^2$

(e) $y = \dfrac{2x^2}{3}$

(f) $y = -8 - 6x - x^2$

Q4

(a) $r = 1,\ C(0, 0)$

(b) $r = 2,\ C(0, 0)$

(c) $r = 1,\ C(-1, 2)$

(d) $r = 3,\ C(4, -5)$

Q5

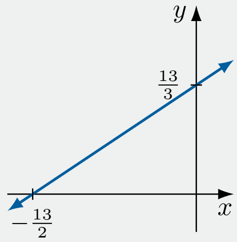
(a) $x = 4\cos\theta$
$y = 4\sin\theta$

(b) $x = 2 + 3\cos\theta$
$y = -5 + 3\sin\theta$

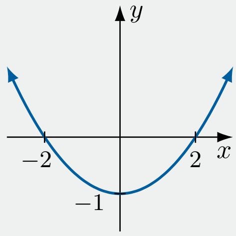
(c) $x = -3 + 5\cos\theta$
$y = 1 + 5\sin\theta$

Q6

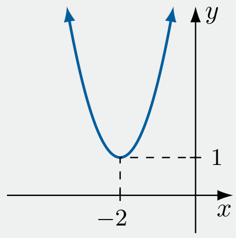
(a)



(b)



(c)



Q7

(a) $y = 5$

(b) $x = -2$

Q8

(a) $y = x - 1$

(b) Mary is correct.

(c) Mary's answer takes into account that $x = t^2 \geq 0$ whereas Bob's answer does not.

Q9

(a) $y = 5 - x$, where $x \leq 3$

(b) $y = x^2 + 1$, where $y \geq 1$ but this is clear from the Cartesian equation anyway.

(c) $x = 2$, where $y \geq 1$

P1

(a) $x = t,\ y = 2t + 3$ or $x = t - 3,\ y = 2t - 3$

(b) $x = t,\ y = 4t^2 + 1$ or $x = \dfrac{t}{2},\ y = t^2 + 1$

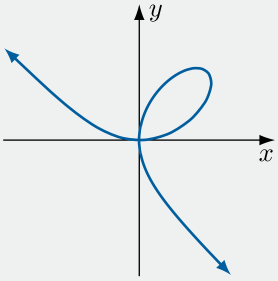
(c) $x = 3\cos\theta,\ y = 3\sin\theta$ or
$x = 3\sin\theta,\ y = 3\cos\theta$

P2

(a) See full worked solutions.

(b) See full worked solutions.

(c)



P3

$x = a\cos\theta$
$y = b\sin\theta$

P4

(a) $y = x^2 - 2$

(b) $x^2 - y^2 = 4$

P5

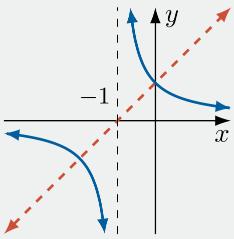
(a) $x^2 - y^2 = 1$

(b) $x^2 + y^2 = 2$

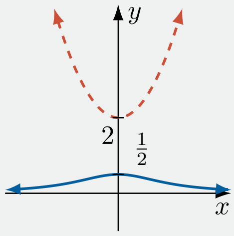
Chapter Review

R1

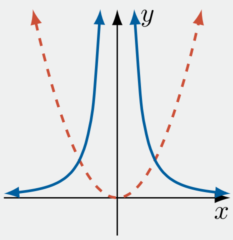
(a)



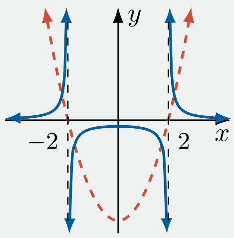
(b)



(c)

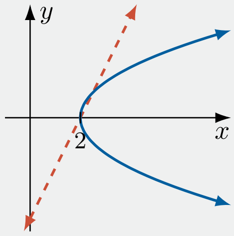


(d)



R2

(a)



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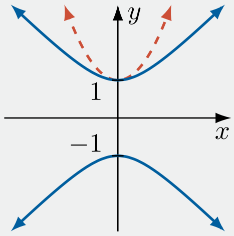

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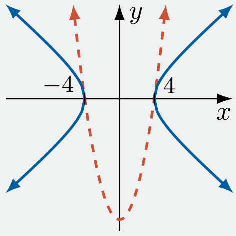

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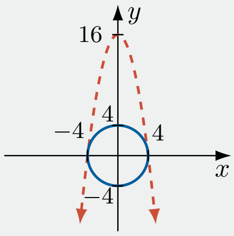
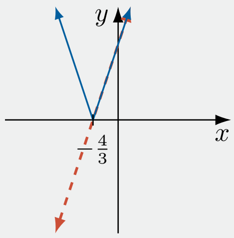
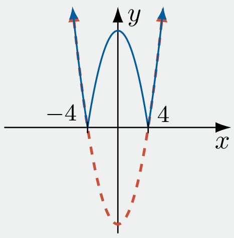


R3

(a)



(b)



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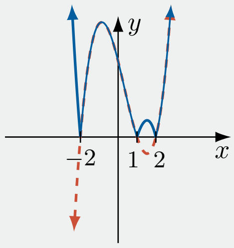

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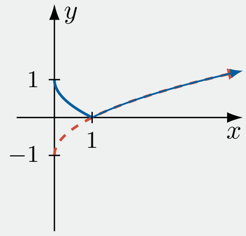
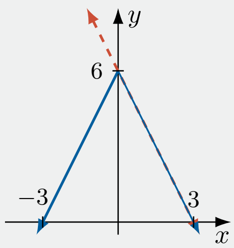


R4

(a)



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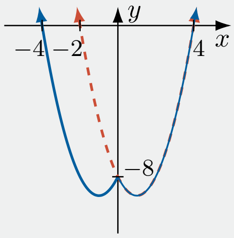

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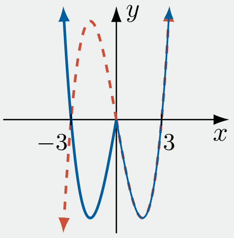

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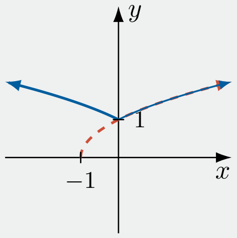
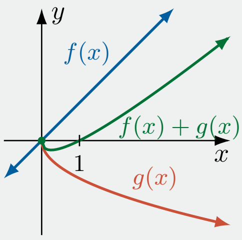


R5

(a)



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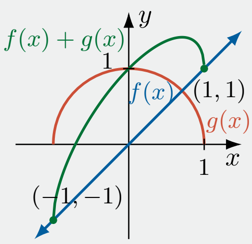
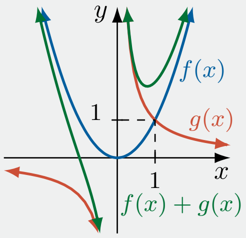


(c)



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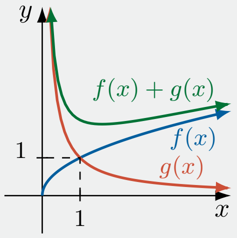
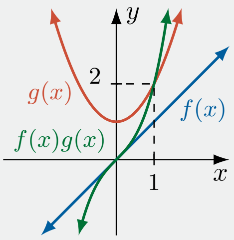


R6

(a)



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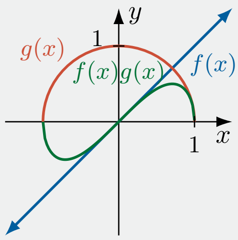

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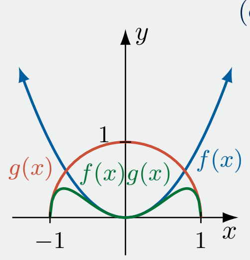

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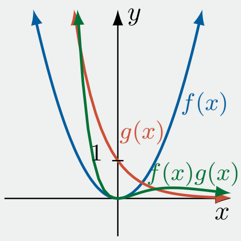
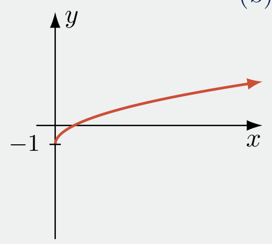


R7

(a)



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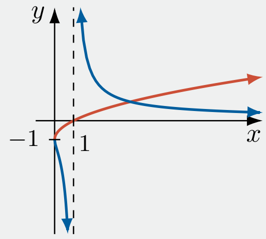


R8

(a) $x \leq -5$ or $x \geq 5$

(b) $0 < x < 4$

(c) $-4 \leq x \leq 5$

(d) $x < -4$ or $x > 3$

R9

(a) $1 < x \leq \frac{5}{3}$

(b) $-2 \leq x < -\frac{1}{2}$

(c) $-2 \leq x < -1$

(d) $x > -1$ or $x \leq -\frac{5}{3}$

R10

(a) $x \leq -3$ or $x \geq 7$

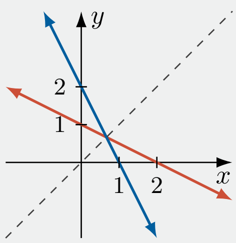
(b) $-6 \leq x \leq 3$

(c) $x < -2$ or $x > 5$

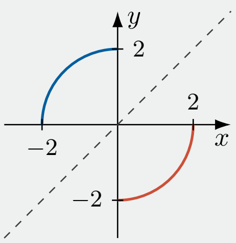
(d) $x < -\frac{11}{3}$ or $x > 3$

R11

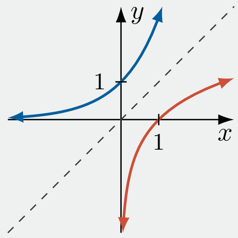
(a)



(b)



(c)



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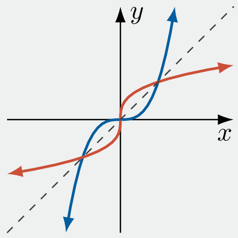

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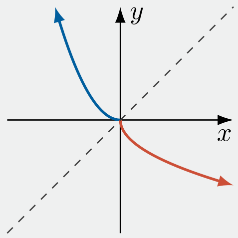
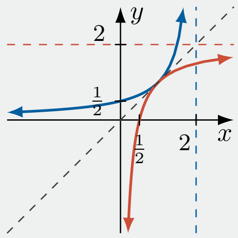


(f)



R12

(a) $f^{-1}(x) = 5 - x$

(b) $f^{-1}(x) = \dfrac{x+1}{3}$

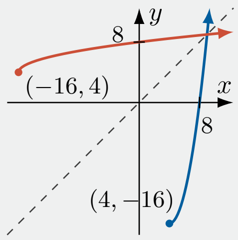
(c) $f^{-1}(x) = x^2,\ x \geq 0$

(d) $f^{-1}(x) = \dfrac{1}{x} + 1$

R13

(a) $p = 4$

(b) $y = 4 + \sqrt{x + 16}$



R14

(a) $x \geq 3,\ y = 3 + \sqrt{x - 5}$

(b) $x \geq 0,\ y = \sqrt[4]{x + 1}$

(c) $x \geq 0,\ y = \sqrt{\dfrac{2}{x}}$

(d) $0 \leq x \leq 3,\ y = \sqrt{9 - x^2}$

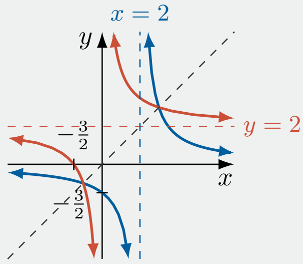
R15

(a) Domain: All real x, $x \neq 2$.
Range: All real y, $y \neq 0$.

(b) $f^{-1}(x) = \dfrac{2x + 3}{x}$

(c) $x = -1, 3$

(d)



R16

(a) $2x - 3y + 11 = 0$

(b) $y = \dfrac{x^2}{4} - 1$

(c) $x^2 = 4ay$

(d) $y = \dfrac{x^2 - 1}{4}$

(e) $x^2 + y^2 = 4$

(f) $(x - 2)^2 + (y + 3)^2 = 4$

R17

(a) $r = 5,\ C(-2, 3)$

(b) $r = 3,\ C(4, -1)$

R18

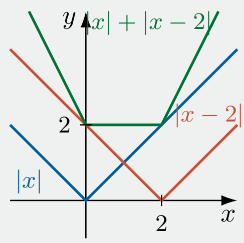
$x = 2t - 1$
$y = 4t^2 - 2$

R19

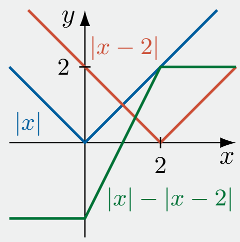
See full worked solutions.

R20

(a)



(b)



2. Polynomials

Exercise 2A

Remainder and factor theorem

F1

(a) $A(x)Q(x) + R(x)$, division

(b) less

F2

(a) roots

(b) divisible, factor

(c) root

(d) factors, α, β

(e) $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$

(f) $P(x) = a(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

F3

(a) When a polynomial $P(x)$ is divided by $(x-\alpha)$, the remainder is $P(\alpha)$.

(b) $P(\alpha) = 0$ if and only if $(x-\alpha)$ is a factor of $P(x)$.

(c) $ax + b$

F4

constant

Q1

(a) $x^3 - x^2 + 2x + 1 = (x+1)(x^2 - 2x + 4) - 3$

(b) $x^3 - 5x + 3 = (x-2)(x^2 + 2x - 1) + 1$

(c) $x^4 - 3x^2 + x - 2 = (x-1)(x^3 + x^2 - 2x - 1) - 3$

(d) $x^4 - 2x^3 + 3x^2 + 3x + 1 = (x^2 - x + 1)(x^2 - x + 1) + 5x$

Q2

(a) See full worked solutions.

(b) $x^2 - x - 2$

(c) $(x+1)^2(x-2)$

Q3

(a) See full worked solutions.

(b) See full worked solutions.

(c) $(x-1)(x+4)$

(d) $(x^2 - 2x + 1)$

(e) $(x-1)^3(x+4)$

(f) $x = 1, -4$

Q4

(a) $m = -27$

(b) $Q(x) = 2x^2 + 6x + 9$

(c) $x = 3$ is the only real solution.

Q5

$m = -7$

Q6

(a) $P(x) = x^3 - 4x$

(b) $P(x) = x^3 - 2x$

(c) $P(x) = x^3 + 2x^2 - 5x - 6$

(d) $P(x) = x^3 - 3x^2 + x + 1$

Q7

(a) $R(x) = c$ constant

(b) $R(x) = c$ constant

(c) $R(x) = ax + b$ linear

(d) $R(x) = ax^2 + bx + c$ quadratic

Q8

$R(x) = ax + b$. No it does not. The remainder can still be a constant. It just means that $a = 0$.

Q9

(a) -36 (b) 3

Q10

(a) -19 (b) $-\dfrac{19}{4}$

Q11

(a) $P(x) = (x^2 - 9)Q(x) + (2x + 5)$

(b) -1

Q12

(a) See full worked solutions. (b) $x = \dfrac{1}{2}, -3, 2$

Q13

$m = -15$, $n = 14$

Q14

(a) $m = 12$ (b) $x = \pm 2$

Q15

(a) $p = -7$, $q = 7$ (b) $x = \dfrac{1}{2}, 1, 2$

Q16

$p = -1$, $q = -2$, $x = -3, -1, 2$

Q17

(a) See full worked solutions.

(b) $P(x) = (2x - 1)(3x - 2)(x + 5)$,

$x = \frac{1}{2}, \frac{2}{3}, -5$

Q18

$p = 3$, $r = -3$, $q = 1$, $a = -13$

Q19

(a) $x = \pm 1, \pm 2, \pm 4$

(b) $x = 2$, $(x - 2)$

(c) $P(x) = (x - 2)(x^2 + 2x - 2)$

(d) $x = 2, -1 \pm \sqrt{3}$

Q20

$x = 2$ is the only (double) root.

Q21

See full worked solutions.

P1

$x + 2$

P2

-2

P3

(a) $x^n - 1$

(b) See full worked solutions.

P4

See full worked solutions.

Exercise 2B
Odd and even polynomials

F1

(a) $b = 0$ and $d = 0$

(b) $b = 0$ and $d = 0$

(c) The powers of x are all even + constant term.

(d) y

(e) $-\alpha, \beta$

(f) It has rotational symmetry through 180° about the origin.

(g) If α is a root so is $-\alpha$ and if $-\beta$ is a root then so is β as well and an odd polynomial always passes through $x = 0$

F2

origin, zero (or root)

Q1

$P(x) = x^2 + c$

Q2

$P(x) = x^3 - 16x$

Q3

$P(x) = (x + 3)(x - 3)(x + 1)(x - 1)$

Q4

(a) 0 (b) -25 (c) 0

Q5

(a) -10 (b) 25

Q6

(a) $P(x) = (x - 1)(x + 1)(x - 5)(x + 5)q(x)$ where $q(x)$ is an even function

(b) $P(x) = x(x - 1)(x + 1)(x - 5)(x + 5)q(x)$ where $q(x)$ is an even function

P1

The powers of x are all odd and $a_0 = 0$

P2

(a) $x = -2$

(b) $P(x) = (x + 2)(x - 2)(x^2 + a)$ where $a > 0$

(c) $P(x) = (x + 2)(x - 2)(x^2 + 2)$

Exercise 2C
Sum and product of roots (quadratic)

F1

(a) $-\frac{b}{a}$ (b) $\frac{c}{a}$

F2

(a) $(x - \alpha)(x - \beta) = 0$ (b) $(\alpha + \beta)$, $\alpha\beta$

Q1

(a) $\alpha + \beta = 2$, $\alpha\beta = -5$

(b) $\alpha + \beta = -\frac{5}{4}$, $\alpha\beta = -\frac{1}{4}$

Q2

(a) $x^2 - 25 = 0$

(b) $x^2 + 2x - 15 = 0$

(c) $x^2 - 4x + 1 = 0$

(d) $10x^2 - 29x + 10 = 0$

Q3

(a) $\frac{5}{4}$

(b) $-\frac{1}{4}$

(c) -5

(d) 5

(e) $\frac{25}{16}$

(f) $-\frac{5}{16}$

Q4

(a) $\alpha^2 + 2\alpha\beta + \beta^2$

(b) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(c) 13

Q5

(a) $\frac{33}{16}$

(b) 33

(c) $-\frac{33}{4}$

(d) $\frac{41}{16}$

(e) $-\frac{9}{4}$

(f) $-\frac{25}{2}$

Q6

$k = -2$

Q7

$k = -\frac{3}{2}$

Q8

$x^2 + 3x - 40 = 0$

Q9

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

Q10

(a) $3 + \sqrt{2}$

(b) -6

Q11

(a) $p = 0$

(b) $p = \frac{2}{3}$

Q12

$x^2 - \left(\alpha + \frac{1}{\alpha}\right)x + 1 = 0$

Q13

(a) $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

(b) See full worked solutions.

Q14

(a) $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

(b) See full worked solutions.

Q15

(a) $m = 1, 9$

(b) $m = -\frac{9}{4}$

Q16

(a) $m = \pm 9$

(b) $m = \pm 8$

(c) $m = -1$

Q17

See full worked solutions.

Q18

$x^2 - 7x + 20 = 0$

Q19

-25

Q20

$p = 6$, $q = 8$

Q21

(a) -7

(b) If the roots were real, then $\alpha^2 + \beta^2$ cannot be negative.

P1

(a) $x^2 - 8x + 2 = 0$

(b) $4x^2 - 60x + 1 = 0$

P2

(a) $(\alpha + \beta)^2 - 4\alpha\beta$

(b) See full worked solutions.

(c) See full worked solutions.

P3

See full worked solutions.

Exercise 2D

Sum and product of roots (cubic and quartic)

F1

(a) $-\frac{b}{a}$ (b) $\frac{c}{a}$ (c) $-\frac{d}{a}$

F2

(a) $-\frac{b}{a}$ (b) $\frac{c}{a}$ (c) $-\frac{d}{a}$ (d) $\frac{e}{a}$

F3

(a) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

(b) $\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

(c) Sum in pairs

Q1

(a) 3 (b) $\frac{1}{2}$ (c) 2

Q2

(a) $\frac{5}{2}$ (b) 24 (c) 16

(d) -1 (e) $\frac{3}{2}$ (f) 1

Q3

(a) $\alpha, -\alpha$ (b) See full worked solutions.

(c) $(x-2)$ (d) (x^2-9)

(e) $x=-3, 2, 3$

Q4

(a) $\alpha=1$ (b) $(x-1)$

(c) (x^2-2x-3) (d) $x=-1, 1, 3$

Q5

(a) $\alpha=2$ (b) $(x-2)$

(c) (x^2-5x+4) (d) $x=1, 2, 4$

Q6

(a) $x=-1, 2, 5$ (b) $x=\frac{2}{3}, -2, 6$

(c) $x=-3, -2, 6$ (d) $x=-3, \frac{1}{2}, 2$

Q7

See full worked solutions.

Q8

See full worked solutions.

Q9

(a) $x^3-4x^2+6x-4=0$

(b) $2x^3+5x^2-3x+1=0$

Q10

(a) See full worked solutions.

(b) $\alpha^2+\beta^2+\gamma^2+\delta^2=-1<0$

Q11

See full worked solutions.

Q12

(a) -10 (b) 1

P1

(a) p^2 (b) q

P2

(a) See full worked solutions. (b) -6

(c) 32

P3

(a) $\alpha+\beta+\gamma=-p$, $\alpha^2+\beta^2+\gamma^2=p^2-2q$

(b) See full worked solutions.

(c) $(a,b,c)=(-2,0,1)$ in any order.

P4

(a) $y=9x-27$, $(3,0)$ (b) $y=9x-23$

(c) $(-1,-32)$

P5

See full worked solutions.

Exercise 2E

Roots of multiplicity

F1

$2, n$

F2

$n-1$

Q1

$y=\frac{1}{2}(x-3)^3(x+2)$

ANSWERS

Q2

$p = q = 8$

Q3

(a) $m = 12$ (b) $x = 2, 2, -3$

Q4

(a) See full worked solutions. (b) $x = 3$

Q5

$x = 1, -3$

Q6

$p = -5, q = 8$

Q7

See full worked solutions.

Q8

$p = 0, 4$

Q9

See full worked solutions.

P1

(a) See full worked solutions.

(b) $x = \frac{1}{\alpha}$

(c) See full worked solutions.

P2

(a) $P(x) = (x - \alpha)^n \times Q(x)$

(b) See full worked solutions.

P3

See full worked solutions.

P4

(a) See full worked solutions.

(b) α is a zero, so substitute it in.

(c) See full worked solutions.

(d) See full worked solutions.

P5

See full worked solutions.

Exercise 2F

Graphing polynomials

F1

(a) zeros (b) odd (c) even

F2

(a) positive, negative (b) positive

(c) negative

F3

(a) opposite (b) negative, positive

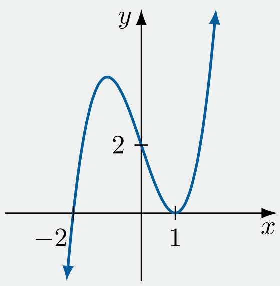
(c) positive, negative

F4

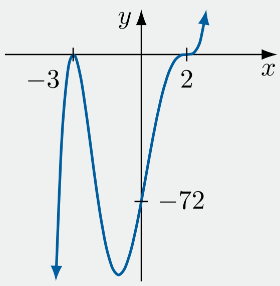
x and y-intercepts, multiplicity of the roots, even or odd degree, positive or negative polynomial

Q1

(a)



(b)



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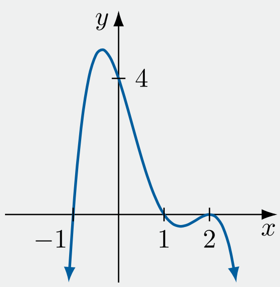

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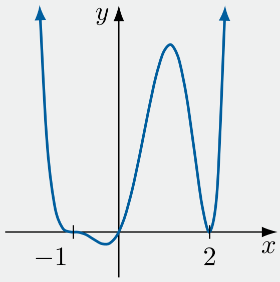


Q2

(a) $y = a(x+1)^2(x-2)^3$

(b) $a = \dfrac{1}{2}$

(c) $y = \dfrac{1}{2}(x+1)^2(x-2)^3$

Q3

(a) $y = -(x-2)^3$

(b) $y = (x+1)^2(3-x)$

(c) $y = \dfrac{1}{8}(x-4)^3(x+2)$

(d) $y = (x-3)(x-5)(x+1)^2$

(e) $y = (x-3)(x-1)(x+2)(x+4)$

(f) $y = -(x-2)^2(x+2)^2$

(g) $y = -x(x-2)^3$

(h) $y = (x+1)^3(x-2)^2$

(i) $y = 2(x-1)^3(x+1)^3$

Q4

(a) $x \le 2$

(b) $x \le 3$

(c) $x \le -2$ or $x \ge 4$

(d) $x \le 3$ or $x \ge 5$

(e) $x \le -4$ or $-2 \le x \le 1$ or $x \ge 3$

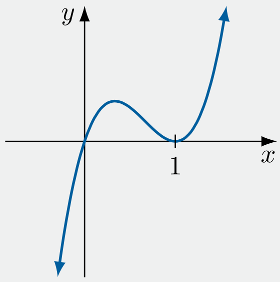
(f) $x = \pm 2$

(g) $0 \le x \le 2$

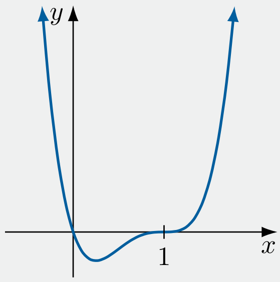
(h) $x \ge -1$

Q5

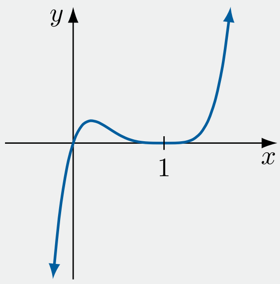
(a)



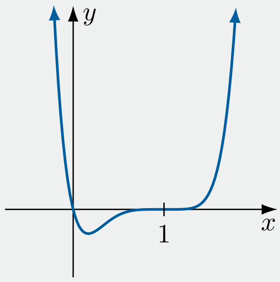
(b)



(c)

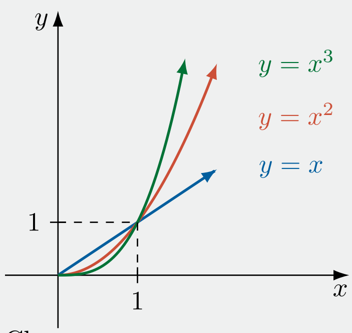


(d)



Q6

(a)



(b) Closer

(c) Further

(d)

(i) $y = x^4$ (ii) $y = x^3$

P1

(a) $-1 \le x \le 1,\ x = 2$

(b) $-1 < x < 0$

(c) $-2 < x < 0$ or $x > 2$

(d) $-2 < x < -1$ or $1 < x < 2$

P2
Option (a) is the correct graph.

Chapter Review

R1
(a) $x^2 + 6x - 7 = 0$ (b) $x^2 - 6x + 4 = 0$

R2
(a) $m = -3$ (b) $x = -2, -6$

R3
(a) $x = -\frac{50}{9}$ (b) $x = -\frac{50}{27}$

R4
$x = 3, 6$

R5
$(x^2 + x + 1)(x^2 - x + 1)$

R6
$x = -3, -1, 2$

R7
$y = \frac{1}{2}(x+1)(x+3)(x-2)^2$

R8
$x = -\frac{5}{3}, \frac{3}{2}, \frac{3}{4}$

R9
$m = 9$

R10
(a) $x = -1$ (b) $p = -\frac{3}{2}, q = \frac{3}{2}$

R11
See full worked solutions.

R12
(a) $k = 80$ (b) $x = -2, 4, 10$

R13
See full worked solutions.

R14
(a) See full worked solutions.
(b) $Q(2, -5)$

R15
$y = \frac{7}{4}x,\ P\left(\frac{3}{2}, \frac{21}{8}\right)$

3. Further Trigonometry

Exercise 3A
Compound angles

F1
(a) $\sin A \cos B + \cos A \sin B$
(b) $\sin A \cos B - \cos A \sin B$
(c) $\cos A \cos B - \sin A \sin B$
(d) $\cos A \cos B + \sin A \sin B$
(e) $\dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
(f) $\dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

F2
(a) $2 \sin A \cos A$
(b) $\cos^2 A - \sin^2 A,\ 2\cos^2 A - 1,\ 1 - 2\sin^2 A$
(c) $\dfrac{2 \tan A}{1 - \tan^2 A}$

Q1
See full worked solutions.

Q2
(a) $\frac{1}{2}(\sqrt{3}\cos A - \sin A)$
(b) $\dfrac{\tan A + 1}{1 - \tan A}$
(c) $-\frac{1}{2}(\sqrt{3}\cos A + \sin A)$
(d) $-\frac{1}{2}(\cos A + \sqrt{3}\sin A)$
(e) $\cos A$
(f) $\frac{1}{2}(\cos A + \sqrt{3}\sin A)$
(g) $-\sin A$
(h) $-\cot A$
(i) $\cos^2 A - \frac{1}{2}$

Q3

(a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ (b) $2-\sqrt{3}$

(c) $\dfrac{1}{4}(\sqrt{2}-\sqrt{6})$ (d) $-2-\sqrt{3})$

(e) $\dfrac{1}{4}(\sqrt{2}+\sqrt{6})$ (f) $\dfrac{1}{4}(\sqrt{6}-\sqrt{2})$

Q4

(a) $\dfrac{1}{2}$ (b) -1 (c) $\dfrac{1}{\sqrt{2}}$

(d) $\dfrac{\sqrt{6}}{2}$ (e) $\sqrt{3}$ (f) $-\dfrac{1}{\sqrt{3}}$

Q5

(a) $\sin 6A$ (b) $\dfrac{1}{2}\sin 4A$ (c) $\dfrac{1}{2}\sin A$

(d) $\cos 8A$ (e) $\cos 10A$ (f) $\cos A$

(g) $\tan A$ (h) $\tan 6A$ (i) $\sqrt{2}|\cos A|$

(j) $2\sin 4A$ (k) $\cos 2A$ (l) $\sin 2A$

Q6

(a) $\sin A\cos B - \cos A\sin B$

(b) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(c) $\cos A\cos B - \sin A\sin B$

(d) $\dfrac{\tan A + \tan B}{1-\tan A\tan B}$

(e) $\dfrac{\tan A - \tan B}{1+\tan A\tan B}$

Q7

$\dfrac{\cot A\cot B - 1}{\cot B + \cot A}$

Q8

(a) $x^3 + y^3$

(b) $2 - \sin 2A$ if $\sin A + \cos A \neq 0$

Q9

See full worked solutions.

Q10

$\sin^2 A = \dfrac{1}{2}(1-\cos 2A)$

$\cos^2 A = \dfrac{1}{2}(1+\cos 2A)$

Q11

(a) $\sin 3A = 3\sin A - 4\sin^3 A$

(b) $\cos 3A = 4\cos^3 A - 3\cos A$

Q12

See full worked solutions.

Q13

(a) $\dfrac{56}{65}$ (b) $-\dfrac{33}{65}$ (c) $-\dfrac{16}{63}$

Q14

(a) $\dfrac{24}{25}$ (b) $-\dfrac{7}{25}$ (c) $\dfrac{2}{\sqrt{5}}$ (d) $\dfrac{1}{2}$

Q15

(a) $-\dfrac{\sqrt{3}}{2}$ (b) $-\dfrac{1}{2}$ (c) $\sqrt{3}$ (d) $\sqrt{3}$

Q16

(a) $\dfrac{3-4\sqrt{3}}{10}$ (b) $\dfrac{3\sqrt{3}-4}{10}$

(c) $\sqrt{3}$ (d) $\dfrac{3-4\sqrt{3}}{4+3\sqrt{3}}$

Q17

(a) $\dfrac{p}{2q}$

(b) Any of the following are correct.

$$1-\frac{p^2}{2q^2} = 2q^2 - 1 = q^2 - \frac{p^2}{4q^2}$$

(c) $\dfrac{p}{2q^2}$

Q18

1

Q19

$\dfrac{17}{25}$

Q20

$\dfrac{20}{37}$

Q21

60 m

Q22

$-2-\sqrt{3}$

ANSWERS

Q23
See full worked solutions.

Q24
See full worked solutions.

Q25
See full worked solutions.

Q26
See full worked solutions.

P1
(a) $a = 8$, $b = -8$, $c = 1$
(b) See full worked solutions.

P2
See full worked solutions.

P3
(a) $(x^2 + y^2)^2 - x^2y^2$
(b) $x^6 - y^6 = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$
(c) See full worked solutions.

P4
See full worked solutions.

P5
See full worked solutions.

P6
(a) See full worked solutions.
(b) $\dfrac{\sin(\pi - 3\theta)}{a} = \dfrac{\sin\theta}{b}$
(c) See full worked solutions.
See full worked solutions.

P7
(a) $\angle BDC = 90°$ and $\angle BDA = 90°$
(b) $CD = \sin\theta$ and $DA = \sin\theta$
(c) See full worked solutions.
(d) See full worked solutions.

P8
See full worked solutions.

P9
See full worked solutions.

Exercise 3B
t-formula

F1
(a) $\dfrac{2t}{1+t^2}$ (b) $\dfrac{1-t^2}{1+t^2}$ (c) $\dfrac{2t}{1-t^2}$

F2
$\dfrac{2t}{1+t^2}$, $2A$

Q1
(a) $\dfrac{1-t^2}{2t}$ (b) $\dfrac{1}{t}$ (c) $\dfrac{1+t^2}{1-t^2}$ (d) $\dfrac{1}{t}$

Q2
(a) $\dfrac{1}{\sqrt{2}}$ (b) $-\dfrac{1}{2\sqrt{3}}$ (c) $\dfrac{1}{4}$

Q3
(a) $\sin 2\theta = \dfrac{2t}{1+t^2}$

$\cos 2\theta = \dfrac{1-t^2}{1+t^2}$
(b) See full worked solutions.

Q4
See full worked solutions.

Q5
See full worked solutions.

Q6
(a) $\angle PTR = 2\theta$
(b) $PT = y$ and $TR = 2 - y$
(c) $PT = 1 + t^2$ and $TR = 1 - t^2$
(d) $\tan 2\theta = \dfrac{2t}{1-t^2}$, $\sin 2\theta = \dfrac{2t}{1+t^2}$,

$\cos 2\theta = \dfrac{1-t^2}{1+t^2}$

Q7
(a) See full worked solutions.
(b) See full worked solutions.
(c) See full worked solutions.
(d)
(i) $1 + \sqrt{2}$ (ii) $1 - \sqrt{2}$

P1
See full worked solutions.

P2

See full worked solutions.

P3

$\dfrac{1+t}{1-t}$

Exercise 3C

Radian measure

F1

$\boldsymbol{\theta}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
sin $\boldsymbol{\theta}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
cos $\boldsymbol{\theta}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
tan $\boldsymbol{\theta}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

F2

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
sin $\boldsymbol{\theta}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0
cos $\boldsymbol{\theta}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
tan $\boldsymbol{\theta}$	0	1	Undef.	-1	0	1	Undef.	-1	0

F3

(a) $\sin\theta$ (b) $-\cos\theta$ (c) $-\tan\theta$

(d) $-\sin\theta$ (e) $-\cos\theta$ (f) $\tan\theta$

(g) $-\sin\theta$ (h) $\cos\theta$ (i) $-\tan\theta$

Q1

See full worked solutions.

Q2

(a) $2\sin A$ (b) $\sin^2 A - \frac{1}{4}$

Q3

(a) $-\frac{24}{25}$ (b) $-\frac{7}{25}$

Q4

(a) $\frac{1}{9}$ (b) $\frac{20}{\sqrt{5}}$

Q5

(a) $\frac{-24}{25}$ (b) $\frac{-7}{25}$ (c) $\frac{24}{7}$

Q6

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$

Q7

$2\sqrt{2}-1$

Q8

See full worked solutions.

Q9

(a) $\tan A$

(b)

(i) $\sqrt{2}-1$ (ii) $2-\sqrt{3}$

Q10

(a) $-\tan\frac{\pi}{6}$ (b) $\tan\frac{\pi}{3}$ (c) $-\tan\frac{\pi}{6}$

Q11

(a) $-\tan\frac{\pi}{3}$ (b) $\tan\frac{\pi}{6}$ (c) $-\tan\frac{\pi}{3}$

Q12

(a) $\cos x \approx 1$

(b) See full worked solutions.

(c) See full worked solutions.

Q13

(a) $p = \frac{1}{3}$ (b) $\frac{7\pi}{4}$

P1

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{10}}$ (c) $\frac{3}{\sqrt{10}}$

P2

See full worked solutions.

P3

(a) -2 (b) $\frac{7\pi}{4}$

ANSWERS

Exercise 3D

Trigonometric products to sums

F1

(a) $\sin A \cos B + \cos A \sin B$

(b) $\sin A \cos B - \cos A \sin B$

(c) $\cos A \cos B - \sin A \sin B$

(d) $\cos A \cos B + \sin A \sin B$

F2

(a) $\frac{1}{2}(\sin(A+B) + \sin(A-B))$

(b) $\frac{1}{2}(\sin(A+B) - \sin(A-B))$

(c) $\frac{1}{2}(\cos(A+B) + \cos(A-B))$

(d) $\frac{1}{2}(\cos(A-B) - \cos(A+B))$

Q1

(a) $\frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)$ (b) $\frac{1}{4}$

(c) $\frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$ (d) $\frac{1}{4}$

(e) $\frac{1}{2}\left(-1 + \frac{\sqrt{3}}{2}\right)$ (f) $-\frac{(2+\sqrt{3})}{4}$

Q2

(a) $\frac{2+\sqrt{3}}{4}$ (b) $\frac{1}{4}$ (c) $\frac{\sqrt{3}}{4}$

(d) $-\frac{3}{4}$ (e) $\frac{1}{4}$ (f) $\frac{1-\sqrt{2}}{4}$

Q3

(a) $\frac{1}{2}(\sin 5x + \sin 3x)$

(b) $\frac{1}{2}(\sin 8x - \sin 2x)$

(c) $\frac{1}{2}(\cos 6x + \cos 3x)$

(d) $\frac{1}{2}(\sin 2x + \sin 2y)$

Q4

See full worked solutions.

Q5

See full worked solutions.

Q6

See full worked solutions.

Q7

(a) $2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$

(b) $2 \cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$

(c) $2 \cos \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$

(d) $-2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$

Q8

(a) $2 \cos 3x \sin x$ (b) $2 \cos 3x \cos x$

(c) $2 \sin 5x \cos x$ (d) $2 \cos 5x \cos 2x$

Q9

(a) 0 (b) 0

P1

(a) $\frac{1}{4}(\sin 6x + \sin 4x + \sin 2x)$

(b) $\frac{1}{4}(\cos 2x + \cos 4x - \cos 6x - 1)$

P2

(a) $\frac{\sqrt{6}}{2}$ (b) $-\frac{1}{\sqrt{2}}$ (c) $-\frac{\sqrt{3}}{\sqrt{2}}$

P3

$4 \sin 3A \cos^2 A$ and $2 \sin 2A \cos A(2 \cos 2A + 1)$

P4

See full worked solutions.

Chapter Review

R1

(a) $\frac{1}{2}\cos 2x$ (b) $\sin 3x$ (c) $\cos x$

(d) $\frac{1}{4}\sin^2 2x$ (e) -1 (f) $\frac{1}{4}\sin 12x$

(g) $\tan x$ (h) $\sin x$

R2

(a) $-\cos A$ (b) $-\sin A$

(c) $\frac{1}{2}\cos A + \frac{\sqrt{3}}{2}\sin A$ (d) $\frac{1}{2}\sin A - \frac{\sqrt{3}}{2}\cos A$

R3

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{2}}{4}$

R4

(a) $-\sin 15°$

(b) See full worked solutions.

R5

(a) See full worked solutions.

(b) See full worked solutions.

R6

See full worked solutions.

R7

$2-\sqrt{3}$

R8

$m = 2$ and $m = -\frac{1}{2}$

R9

t

R10

See full worked solutions.

R11

See full worked solutions.

R12

See full worked solutions.

R13

(a) $\frac{15}{8}$ (b) $\frac{240}{289}$ (c) $\frac{120}{161}$

R14

(a) $-\frac{1}{8}$ (b) $\frac{3\sqrt{5}-2\sqrt{7}}{12}$

(c) $\frac{3\sqrt{5}+2\sqrt{7}}{\sqrt{35}-6}$

R15

(a) $\frac{1}{2}(\sin 7A - \sin 3A)$

(b) $\frac{1}{2}(\cos 12A + \cos 2A)$

(c) $\frac{1}{2}(\cos 2A - \cos 4A)$

(d) $\frac{1}{2}(\sin 4A + \sin 2A)$

R16

(a) $-\frac{1}{4}(\sqrt{3}+1)$ (b) $\frac{1}{4}(1+\sqrt{2})$

(c) $\frac{-(\sqrt{3}+2)}{4}$ (d) $\frac{1}{4}(\sqrt{3}+\sqrt{2})$

R17

(a) $\frac{\cos 3x}{\cos 4x}$ (b) $\cos 8x$ (c) $\cot x$

R18

2

R19

See full worked solutions.

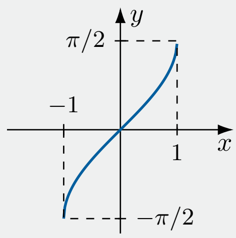
4. Inverse Trigonometry

Exercise 4A

Exact values

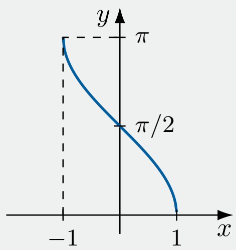
F1

(a)



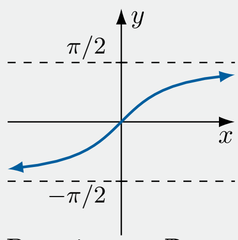
Domain:
$-1 \le x \le 1$
Range:
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(b)



Domain:
$-1 \le x \le 1$
Range: $0 \le y \le \pi$

(c)



Domain: $x \in \mathbb{R}$
Range:
$-\frac{\pi}{2} < y < \frac{\pi}{2}$

F2

(a) $\arcsin(x)$ (b) $\arccos(x)$ (c) $\arctan(x)$

F3

(a) Odd (b) Neither (c) Odd

F4

(a) $-\sin^{-1}(x)$ (b) $\pi - \cos^{-1}(x)$
(c) $-\tan^{-1}(x)$

Q1

(a) $\frac{\pi}{4}$ (b) 0 (c) π
(d) $-\frac{\pi}{2}$ (e) $-\frac{\pi}{4}$ (f) $-\frac{\pi}{6}$
(g) $\frac{\pi}{3}$ (h) $-\frac{\pi}{3}$ (i) $\frac{\pi}{4}$
(j) $\frac{\pi}{6}$ (k) $\frac{\pi}{6}$ (l) $\frac{2\pi}{3}$

Q2

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$
(d) $-\frac{\pi}{2}$ (e) Undefined. (f) Undefined.

Q3

(a) 0.33 (b) 1.84 (c) 1.38

Q4

(a) $\alpha = 0,\ \beta = 1,\ \gamma = 0$
(b) $\alpha = 1,\ \beta = 0,\ \gamma =$ Undefined
(c) $\alpha = \frac{\sqrt{3}}{2},\ \beta = \frac{\pi}{3},\ \gamma = \sqrt{3}$
(d) $\alpha = -\frac{1}{\sqrt{2}},\ \beta =$ Undefined, $\gamma = -1$
(e) $\alpha =$ Undefined, $\beta = -\frac{1}{2},\ \gamma =$ Undefined
(f) $\alpha =$ Undefined, $\beta =$ Undefined, $\gamma =$ Undefined

Q5

(a) 1 (b) 0 (c) $\frac{2\sqrt{2}}{3}$ (d) $\frac{1}{3}$

Q6

(a) $-\frac{\sqrt{3}}{2}$ (b) $-\sqrt{3}$ (c) $-\frac{1}{\sqrt{2}}$
(d) $-\frac{1}{2}$ (e) $-\frac{\sqrt{3}}{2}$ (f) $-\frac{\sqrt{3}}{2}$
(g) $\frac{1}{\sqrt{3}}$ (h) $\frac{1}{\sqrt{2}}$ (i) $\frac{1}{2}$

Q7

(a) $\frac{5\pi}{6}$ (b) $-\frac{\pi}{6}$ (c) $\frac{3\pi}{4}$

Q8

(a) $\sin^{-1}(x)$ can output only within the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If α exceeds this range, then $\sin^{-1}(\sin\alpha) \neq \alpha$.

(b) $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) Use supplementary identities so that $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and then the answer is just α.

Q9

(a) $\frac{\pi}{5}$ (b) $\frac{3\pi}{5}$ (c) $\frac{\pi}{7}$
(d) $-\frac{\pi}{5}$ (e) $\frac{\pi}{7}$ (f) $-\frac{\pi}{5}$
(g) $\frac{4\pi}{5}$ (h) $-\frac{3\pi}{7}$ (i) $\frac{\pi}{5}$

Q10

(a) 1 (b) $-\frac{3}{5}$ (c) $-\frac{1}{2}$

Q11

(a) $\frac{3}{\sqrt{10}}$ (b) $-\frac{1}{2\sqrt{6}}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{5}$

Q12

(a) $\frac{1}{2}$ (b) $-\sqrt{3}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

Q13

(a) $y = \cos^{-1}(x-1)$
(b) $y = 3\sin^{-1}\left(\frac{1-x}{2}\right)$

Q14

See full worked solutions.

Q15

$x = \frac{\pi}{4}$

ANSWERS

Q16

(a) $A = \sin^{-1}(x)$

(b) $B = \cos^{-1}(x)$

(c) See full worked solutions.

(d) They both work.

P1

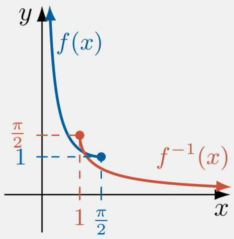
(a) $y \geq 1$

(b) Domain: $x \geq 1$

Range: $0 < y \leq \frac{\pi}{2}$

(c) See full worked solutions.

(d)



P2

(a) $\frac{\pi}{2} - x$

(b) $\frac{\pi}{2} - x$

(c) $\frac{\pi}{2} - x$

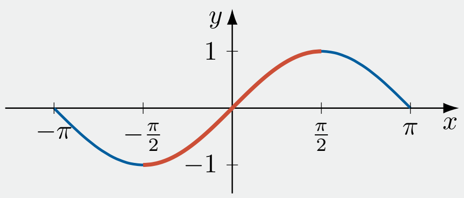
P3

See full worked solutions.

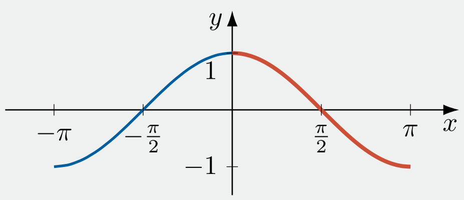
Exercise 4B
Graphs

F1

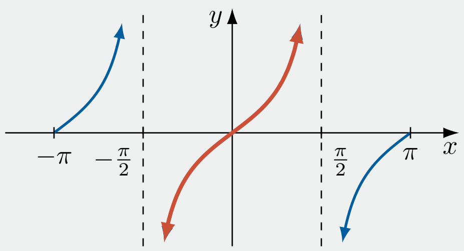
(a)



(b)

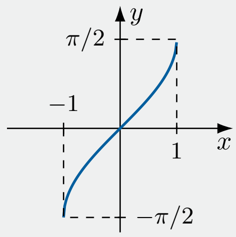


(c)



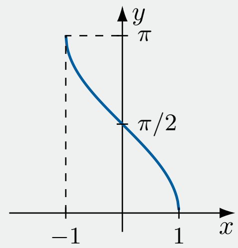
F2

(a)



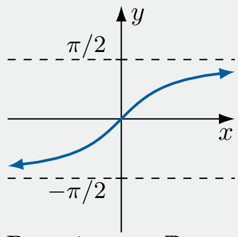
Domain:
$-1 \leq x \leq 1$
Range:
$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
Odd function

(b)



Domain:
$-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$
Neither

(c)



Domain: $x \in \mathbb{R}$
Range:
$-\frac{\pi}{2} < y < \frac{\pi}{2}$
Odd function

F3

(a) $\frac{\pi}{2}$

(b) $-\frac{\pi}{2}$

F4

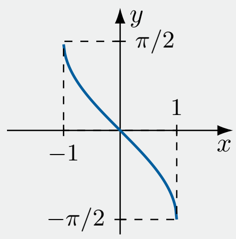
(a) Stretches or shrinks the curve vertically.

(b) Translates the curve horizontally.

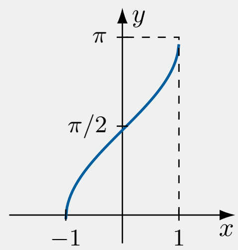
(c) Stretches or shrinks the curve horizontally.

Q1

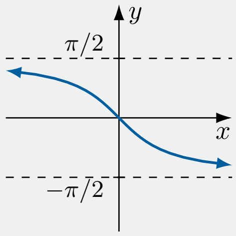
(a)



(b)



(c)



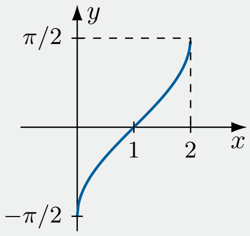
Q2

(a) Domain:

$0 \le x \le 2$

Range:

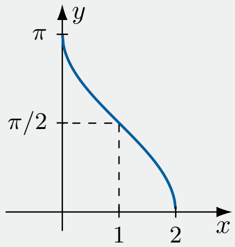
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



(b) Domain:

$0 \le x \le 2$

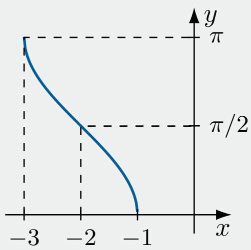
Range: $0 \le y \le \pi$



(c) Domain:

$-3 \le x \le -1$

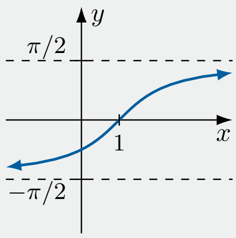
Range: $0 \le y \le \pi$



(d) Domain: $x \in \mathbb{R}$

Range:

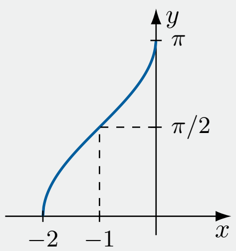
$-\frac{\pi}{2} < y < \frac{\pi}{2}$



(e) Domain:

$-2 \le x \le 0$

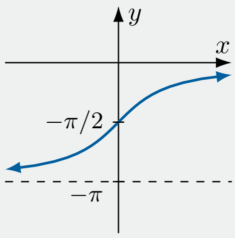
Range: $0 \le y \le \pi$



(f) Domain: $x \in \mathbb{R}$

Range:

$-\pi < y < 0$



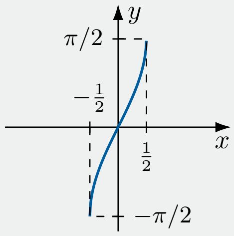
Q3

(a) Domain:

$-\frac{1}{2} \le x \le \frac{1}{2}$

Range:

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

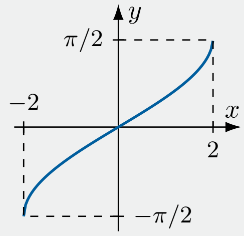


(b) Domain:

$-2 \le x \le 2$

Range:

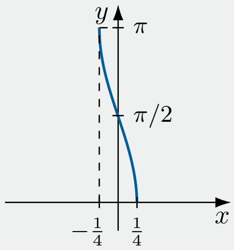
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



(c) Domain:

$-\frac{1}{4} \le x \le \frac{1}{4}$

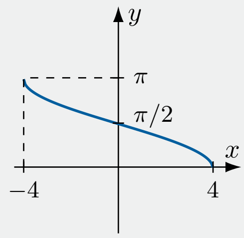
Range: $0 \le y \le \pi$



(d) Domain:

$-4 \le x \le 4$

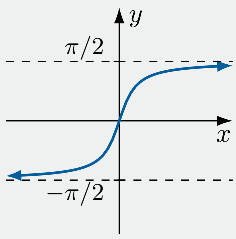
Range: $0 \le y \le \pi$



(e) Domain: $x \in \mathbb{R}$

Range:

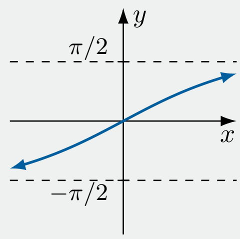
$-\frac{\pi}{2} < y < \frac{\pi}{2}$



(f) Domain: $x \in \mathbb{R}$

Range:

$-\frac{\pi}{2} < y < \frac{\pi}{2}$



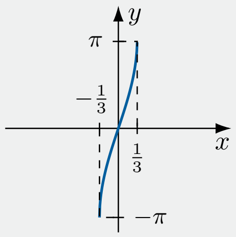
Q4

(a) Domain:

$-\frac{1}{3} \leq x \leq \frac{1}{3}$

Range:

$-\pi \leq y \leq \pi$

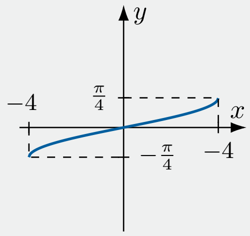


(b) Domain:

$-4 \leq x \leq 4$

Range:

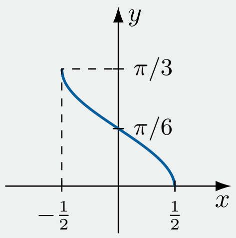
$-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$



(c) Domain:

$-\frac{1}{2} \leq x \leq \frac{1}{2}$

Range: $0 \leq y \leq \frac{\pi}{3}$

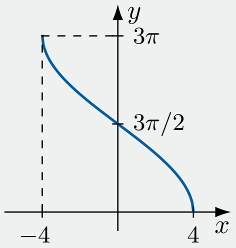


(d) Domain:

$-4 \leq x \leq 4$

Range:

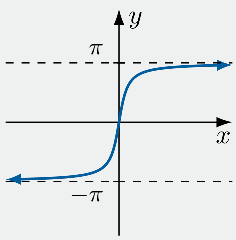
$0 \leq y \leq 3\pi$



(e) Domain: $x \in \mathbb{R}$

Range:

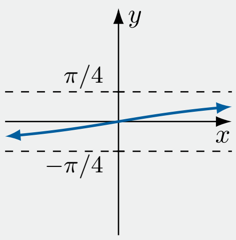
$-\pi < y < \pi$



(f) Domain: $x \in \mathbb{R}$

Range:

$-\frac{\pi}{4} < y < \frac{\pi}{4}$



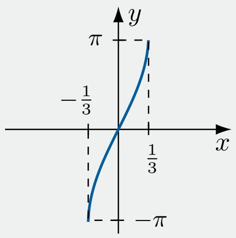
Q5

(a) Domain:

$-\frac{1}{3} \leq x \leq \frac{1}{3}$

Range:

$-\pi \leq y \leq \pi$

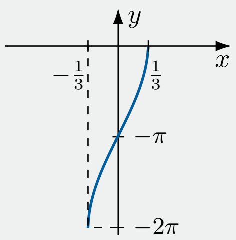


(b) $A\left(\frac{1}{3}, 2\pi\right)$,

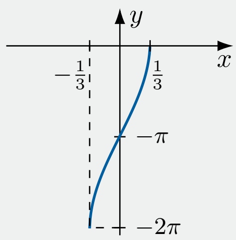
$B(0, \pi)$,

$C\left(-\frac{1}{3}, 0\right)$

(c)



(d)



Q6

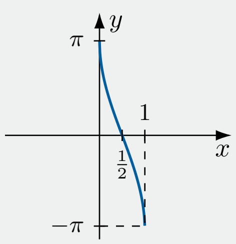
(a) $0 \leq x \leq 1$

(b) $-\pi \leq y \leq \pi$

(c) It has the exact same domain and range.

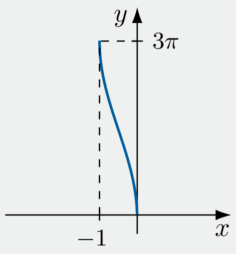
(d) The graph of $y = 2\sin^{-1}(1-2x)$ is the same as $y = 2\sin^{-1}(2x-1)$, but flipped across it's vertical axis of symmetry $x = \frac{1}{2}$.

(e)

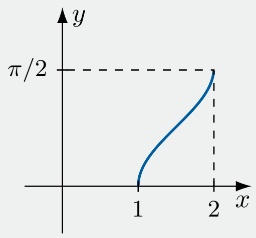


Q7

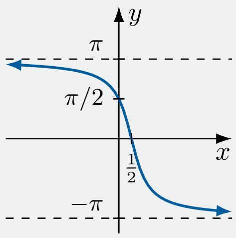
(a)



(b)

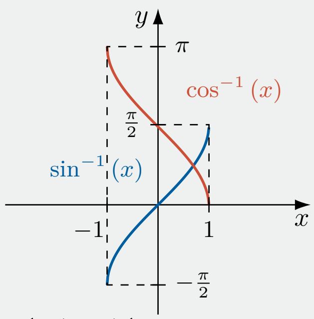


(c)



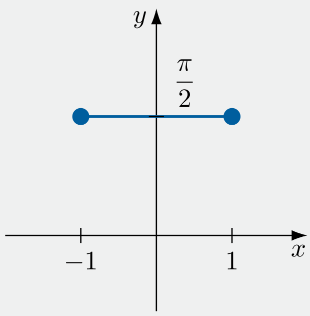
Q8

(a)



(b) $-1 \leq x \leq 1$

(c)



Q9

(a) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

(b) $y = 2\sin^{-1}(2x)$

(c) $y = \frac{1}{2}\cos^{-1}(x-2)$

(d) $y = 2\tan^{-1}(x) + \frac{\pi}{2}$

Q10

(a) Domain:

$-1 \leq x \leq 1$

Range: $0 \leq y \leq \frac{\pi}{2}$

(b) Domain: $x \in \mathbb{R}$

Range: $y \geq 0$

Q11

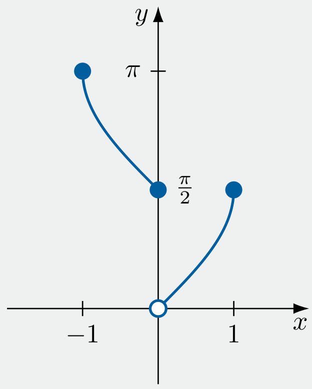
(a) $y = x\cos^{-1}(x)$

(b) $y = x\tan^{-1}(x)$

(c) $y = x\sin^{-1}(x)$

Q12

(a)



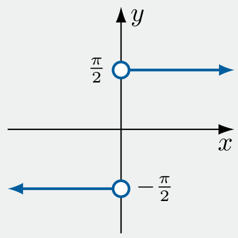
(b) π

Q13

(a) $\frac{\pi}{2}$

(b) See full worked solutions.

(c)



Q14

(a) Domain: $-1 \leq x \leq 1$

Range: $-1 \leq y \leq 1$

(b) Domain: $-1 \leq x \leq 1$

Range: $-1 \leq y \leq 1$

(c) Domain: $x \in \mathbb{R}$

Range: All real y

(d) Domain: $x \in \mathbb{R}$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(e) Domain: $x \in \mathbb{R}$

Range: $0 \leq y \leq \pi$

(f) Domain: $x \in \mathbb{R}$, $x \neq \dfrac{k\pi}{2}$ where $k \in \mathbb{Z}$

Range: $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

Q15

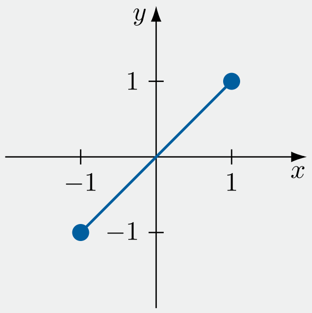
(a) Domain: $-1 \leq x \leq 1$
Range: $-1 \leq y \leq 1$

(b) He is partially correct. It will be the graph of $y = x$ but with a restricted domain and range.

(c)

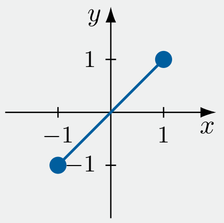
x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1

(d)

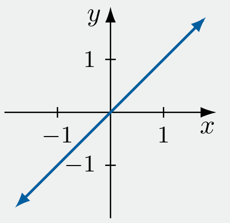


Q16

(a)



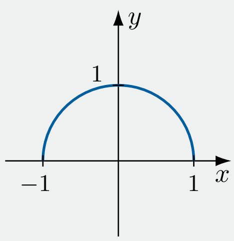
(b)



P1

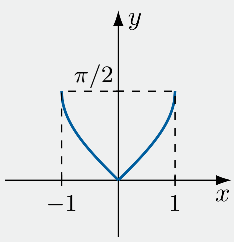
(a) See full worked solutions.

(b)

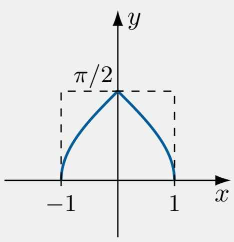


P2

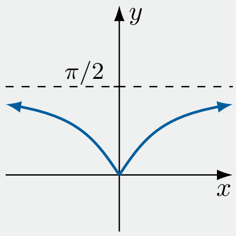
(a)



(b)

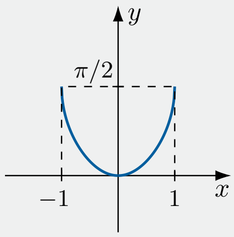


(c)

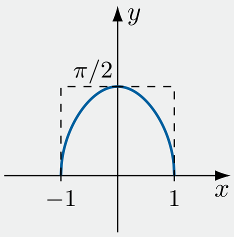


P3

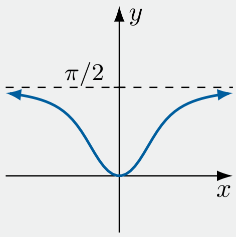
(a) Domain:
$-1 \leq x \leq 1$
Range: $0 \leq y \leq \dfrac{\pi}{2}$



(b) Domain:
$-1 \leq x \leq 1$
Range: $0 \leq y \leq \dfrac{\pi}{2}$



(c) Domain: $x \in \mathbb{R}$
Range: $0 \leq y < \dfrac{\pi}{2}$



P4

Domain: $x \geq 1$ or $x \leq -1$.

Range: $0 \leq y \leq \pi$, $y \neq \dfrac{\pi}{2}$

Exercise 4C

Applications with compound angle formulae

F1

(a) $\cos^{-1}\left(\dfrac{x}{y}\right)$, $2A$, $2\sin A\cos A$

(b) $\cos^{-1}\left(\dfrac{x}{y}\right)$, $\dfrac{x}{y}$

(c) $\dfrac{\sqrt{y^2 - x^2}}{y}$

(d) $2x\dfrac{\sqrt{y^2 - x^2}}{y^2}$

F2

$\sin^{-1}(x)$, $\sin^{-1}(y)$, $A+B$, $\cos A$, $\cos B$

F3

(a) sin (b) cos (c) tan

Q1

(a) See full worked solutions. (b) $\frac{4}{5}$

(c) $-\frac{7}{25}$

Q2

(a) $\frac{24}{25}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{5}{13}$

(d) $\frac{3}{5}$ (e) $\frac{4\sqrt{2}}{7}$ (f) $-\frac{4\sqrt{21}}{17}$

Q3

(a) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(b) $\tan A = \frac{12}{5}$, $\tan B = \frac{3}{4}$

(c) $\frac{33}{56}$

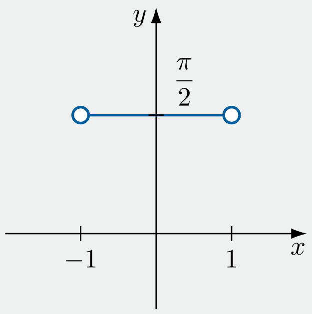
(d) $\tan^{-1}\left(\frac{33}{56}\right)$

Q4

(a) 0

(b) $-1 < x < 1$

(c)



Q5

See full worked solutions.

Q6

(a) $0 \le x \le 1$ (b) $-1 < x < 1$

(c) $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}} \le x \le 1$

Q7

See full worked solutions.

Q8

(a) See full worked solutions.

(b) $x = \cos\left(\frac{3\pi}{8}\right)$

(c) See full worked solutions.

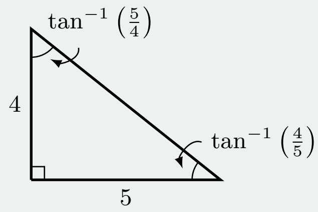
P1

See full worked solutions.

P2

(a) See full worked solutions.

(b)



(c) $xy = 1$ or in other words, they are reciprocals of each other.

P3

(a) See full worked solutions.

(b) $x^2 + y^2 = 1$

(c) $x^2 + y^2 = 1$

Chapter Review

R1

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $-\frac{\pi}{4}$

(d) $\sqrt{3}$ (e) 0 (f) $\frac{1}{\sqrt{2}}$

(g) $\frac{1}{2}$ (h) $-\frac{1}{2}$ (i) $-\frac{1}{2}$

R2

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{5}$ (c) $\frac{3\pi}{4}$

(d) $-\frac{\pi}{4}$ (e) $-\frac{\pi}{8}$ (f) $\frac{7\pi}{8}$

R3

(a) $\frac{6\sqrt{7}}{16}$ (b) $\frac{7}{9}$ (c) $\frac{4}{5}$

R4

(a) $\frac{17}{25}$ (b) $\frac{4\sqrt{21}}{25}$ (c) $\frac{4\sqrt{21}}{17}$

R5

See full worked solutions.

R6

See full worked solutions.

R7

(a) $A = \dfrac{\sqrt{15} - \sqrt{3}}{8}$ (b) $A = \dfrac{\sqrt{8} + \sqrt{3}}{6}$

R8

$0 \le x \le 1$

R9

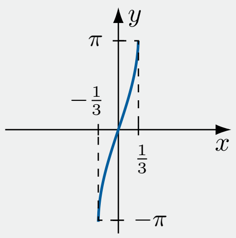
See full worked solutions.

R10

(a) Domain: $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$

Range: $-\pi \le y \le \pi$

(b)



(c) $f^{-1}(x) = \dfrac{1}{3}\sin\left(\dfrac{x}{2}\right)$

Domain: $-\pi \le x \le \pi$

Range: $-\dfrac{1}{3} \le y \le \dfrac{1}{3}$

R11

(a) $-1 \le y \le 3$

(b) $f^{-1}(x) = \dfrac{1}{3}\sin^{-1}\left(\dfrac{x-1}{2}\right)$

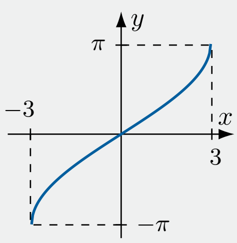
(c) Domain: $-1 \le x \le 3$

Range: $-\dfrac{\pi}{6} \le y \le \dfrac{\pi}{6}$

R12

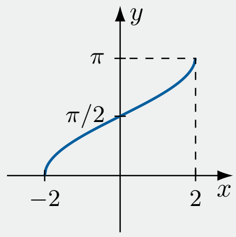
(a) Domain: $-3 \le x \le 3$

Range: $-\pi \le y \le \pi$



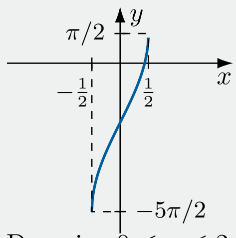
(b) Domain: $-2 \le x \le 2$

Range: $0 \le y \le \pi$



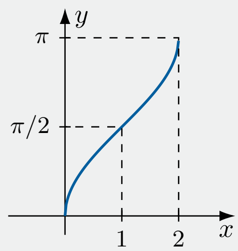
(c) Domain: $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$

Range: $-\dfrac{5\pi}{2} \le y \le \dfrac{\pi}{2}$



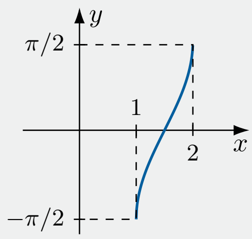
(d) Domain: $0 \le x \le 2$

Range: $0 \le y \le \pi$



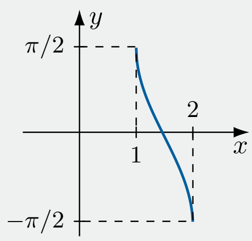
(e) Domain: $1 \le x \le 2$

Range: $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$



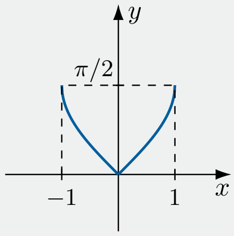
(f) Domain: $1 \le x \le 2$

Range: $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

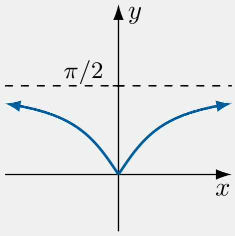


R13

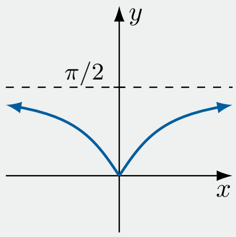
(a)



(b)

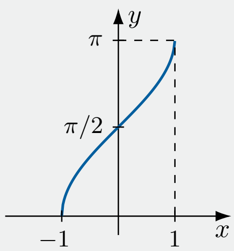


(c)

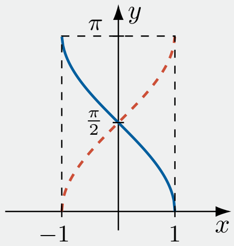


R14

(a)



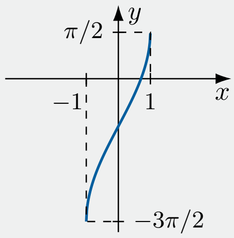
(b) It is just the graph of $y = \cos^{-1}(x)$.



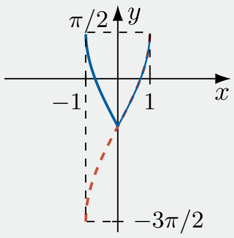
(c) See full worked solutions.

R15

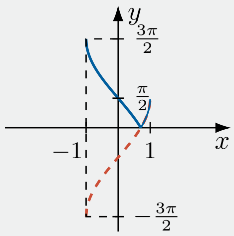
(a)



(b)



(c)



R16
See full worked solutions.

R17
See full worked solutions.

5. Physical Applications of Calculus

Exercise 5A

Rates of Change (Revision)

F1

(a) $bn(a+bt)^{n-1}$ (b) $u'v + v'u$

(c) $\dfrac{u'v - v'u}{v^2}$ (d) $nu^{n-1} \times u'$

(e) ke^{kt} (f) $f'(t)e^{f(t)}$

F2

(a) instantaneous, t_0, differentiating, t

(b) $\dfrac{dQ}{dt}$

F3
differentiate, t_0

F4

(a) $>$ (b) $<$

F5

(a) positive (b) negative (c) zero

Q1

(a) $4nt^{n-1} + 3$ (b) $\dfrac{4}{t^3} - \dfrac{5}{t^2} + \dfrac{1}{5}$

(c) $-\dfrac{3}{4t^2}$ (d) $3 - \dfrac{8}{t^2}$

(e) $\dfrac{3\sqrt{t}}{2}$ (f) $9(3t-2)^2$

(g) $6tn(3t^2-5)^{n-1}$ (h) $(4-9t)t$

(i) $-(2-3t)^4t(21t-4)$

(j) $\dfrac{40}{(8-t)^2}$

(k) $\dfrac{(t-2)t}{(t-1)^2}$

(l) $\dfrac{2(2t+1)}{(2t-1)^3}$

(m) $\dfrac{10t}{(8-t^2)^2}$

(n) $\dfrac{2t}{\sqrt{2t^2+5}}$

(o) $\dfrac{2t}{\sqrt{(2t^2+5)^3}}$

Q2

(a) $-3e^{-0.5t}$

(b) $24e^{-0.2t}$

(c) $-40e^{-0.2t}$

(d) $-2te^{-t^2}$

(e) $-e^{-t}$

(f) $\dfrac{9e^{3t}}{(e^{3t}+1)^2}$

(g) $\dfrac{e^{3t}(1-3t)+1}{(e^{3t}+1)^2}$

(h) $\dfrac{3e^{3t}}{(e^{3t}+1)^2}$

(i) $\dfrac{3e^{-t}}{(e^{-t}+1)^2}$

Q3

(a) \$/hour, \$/month, \$/year

(b) litres/sec or minutes or hours

(c) \$/week, \$/month

(d) \$/term, \$/year

Q4

7 minutes

Q5

2.5 litres per minute

Q6

25 cm^2/min

Q7

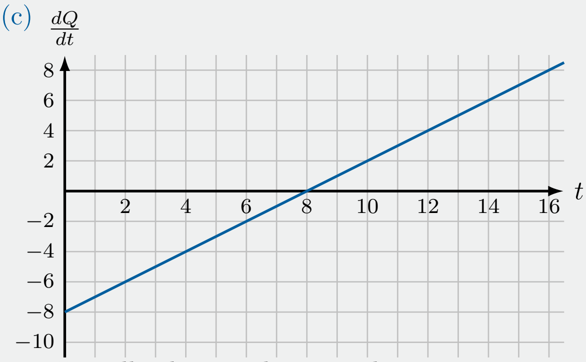
(a) $1 < t < 3$, $4 < t \leq 5$

(b) $0 \leq t < 1$, $3 < t < 4$

Q8

(a) Rate of change of population

(b) $\dfrac{dQ}{dt} < 0$, rate is negative, population is decreasing. $\dfrac{dQ}{dt} = 0$, rate is zero, population is not changing. $\dfrac{dQ}{dt} > 0$, rate is positive, population is increasing.

(c) 

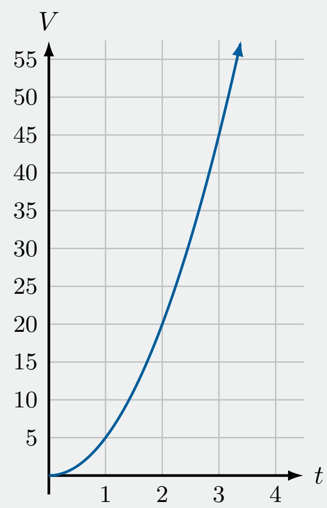
Initially the population is decreasing at 8 members per year, and after 8 years the number of members is increasing.

Q9

(a) 800

(b) $\dfrac{dB}{dt} = 6t^2 - 6t$ so $\dfrac{dB}{dt} = 12$ when $t = 2$

Q10

(a) 

(b) It is increasing.

(c) $V'(t)$ represents the slope or gradient of the graph at a particular point in t

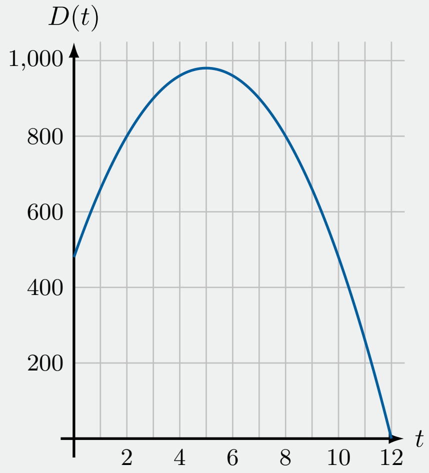
(d) 30 litres/min

Q11

(a) 480 m

(b) 12 minutes

(c)



(d) $D'(t) = 20(-2t + 10)$

(e) metres per minute

(f) 0

(g) See full worked solutions.

Q12

(a) 2000 litres

(b) At 25 minutes

(c)

(i) -160 litres/min

(ii) 0 litres/min

(iii) -128 litres/min

(d) See full worked solutions.

Q13

(a) $\frac{dh}{dt} = \frac{12}{3t+1}$

(b) $\frac{d^2h}{dt^2} = -\frac{36}{(3t+1)^2}$

(c) 12 cm/month

(d) $\frac{12}{37}$ cm/month

(e) See full worked solutions.

Q14

(a) See full worked solutions.

(b) 1000 mL

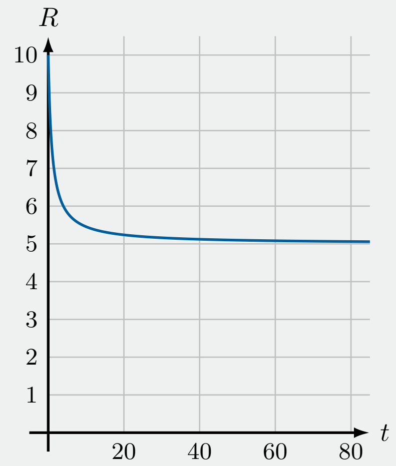
(c) Fastest when $t = 0$ and then

$$\frac{dV}{dt} = 60 \text{ ml/sec}$$

Q15

(a) 10 kg/min

(b)

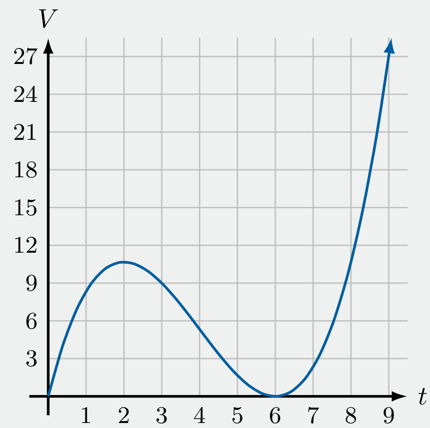


(c) $5\frac{5}{6}$ kg/min

(d) 5 kg/min

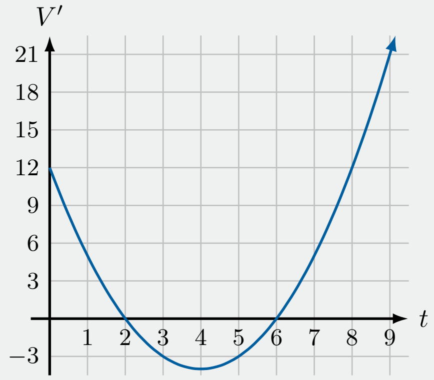
Q16

(a)



(b) $V = 27$ litres

(c)



(d) See full worked solutions.

Q17

(a) 64 mL (b) $\frac{t}{2} - 10$

(c) -8 mL/s (d) 20 s

P1

(a) See full worked solutions.

(b) 400π

(c) 5π

P2

(a) 10000

(b) 1105

(c) 1111

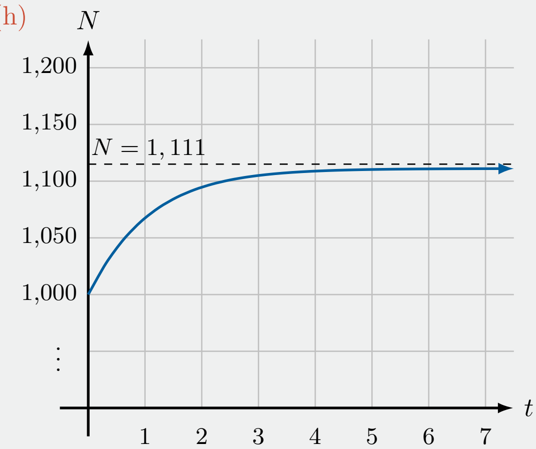
(d) $\dfrac{10000e^{-t}}{(9+e^{-t})^2}$

(e) 100

(f) Approaches zero.

(g) Increasing

(h)



Exercise 5B

Displacement, velocity and acceleration

F1

(a) displacement (b) velocity

(c) acceleration

F2

(a) v, $\dot{x}$, $\dfrac{dx}{dt}$ (b) a, $\ddot{x}$, $\dfrac{dv}{dt}$

F3

(a) positive (b) negative (c) zero

F4

(a) $v = 0$ (b) $t = 0$ (c) $t \to \infty$

F5

(a) same (b) opposite

Q1

(a) 5 m (b) $v = -2$ m/s

(c) To the left (d) 6 m

(e) $t = 4$ (f) $t = 2.5$

Q2

(a) -8 m (b) $2t-2$ m/s (c) -2 m/s

(d) Left (e) $t = 1$ (f) $a = 2$

Q3

(a) $v = 3t^2 - 6t + 3$

(b) $t = 1$

(c)

t	0	1	2
v	3	0	3

(d) Mary

(e) See full worked solutions.

Q4

(a) $v = 6 - 2t$ m/s

(b) See full worked solutions.

(c) $t = 6$

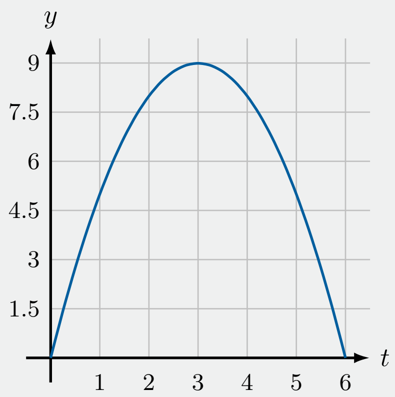
(d) 9 m

(e) 9 m

(f) 18 m

ANSWERS

(g) It gives the position at $t = 6$, but not the distance travelled. Since it lands on the ground again at $t = 6$, then Bob's answer gives zero, which is obviously not the distance travelled.



Q5

See full worked solutions.

Q6

(a) 9 m

(b) $v = 8 - 2t$ m/s

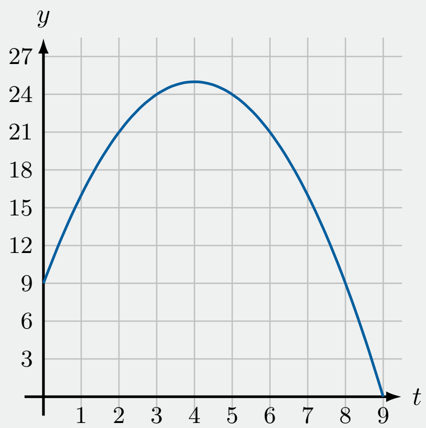
(c) $t = 4$

(d) 25 m

(e) $t = 2$, $t = 6$

(f) $t = 9$

(g)



(h) 41 m

Q7

(a)

(i) left

(ii) right

(iii) right

(iv) right

(v) left

(b) $t = 1.2,\ 5.5,\ 10.5$

Q8

(a) Left

(b) $t = 1$

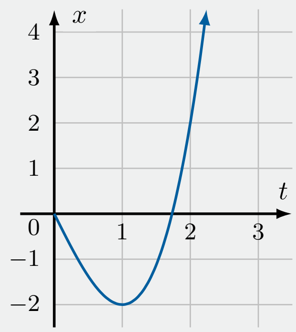
(c) Changes direction.

(d) When $t = 1$ and $x = -2$

(e) 22 metres

(f) See full worked solutions.

(g)



Q9

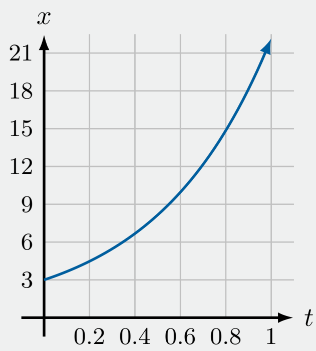
(a) $x = 3$ cm, $v = 6$ cm/s

(b) $6e^8$ cm/s

(c) $\frac{1}{2}\ln 2$ s

(d) See full worked solutions.

(e)



Q10

(a) Letting $v = 0$ gives the possible candidates for direction change, but it doesn't guarantee it. A table of values is needed to verify that the velocity changes sign.
$x = (t-1)^3$

(b) A particle could be approaching an asymptote.
$x = 1 - e^{-t}$

(c) A particle could slow down, but never actually reach zero velocity. This means it could still be moving to the right towards infnity, but just very slowly.
$x = \sqrt{t}$

Q11

(a) $x = 1$ m, $\dot{x} = -1$ m/s

(b) Left

(c) See full worked solutions.

(d) As $t \to \infty$, $x \to 0^+$, $\dot{x} \to 0^-$

Q12

(a) $t = 0$, 2 and 6

(b) To the right

(c) At 0.9 and 4.5 secs

(d) $0 < t < 1$, $t > 4.5$

(e) Particle moves indefinitely to the right with increasing speed.

Q13

(a)

(i) Particle moves right, speeding up

(ii) Particle stationary, but begins moving right

(iii) Particle moves right, slowing down

(b)

(i) Particle moves right with maximum or constant speed

(ii) Particle is stationary

(iii) Particle moves left with maximum or constant speed

(c)

(i) Particle moves right, slowing down

(ii) Particle stationary, but begins moving left

(iii) Particle moves left, speeding up

Q14

(a) $\dot{x} = -e^{-t} < 0$ for all $t \geq 0$

(b) $\ddot{x} = e^{-t} > 0$ for all $t \geq 0$

(c) $x = 2$ m, $\dot{x} = -1$ m/s

(d) See full worked solutions.

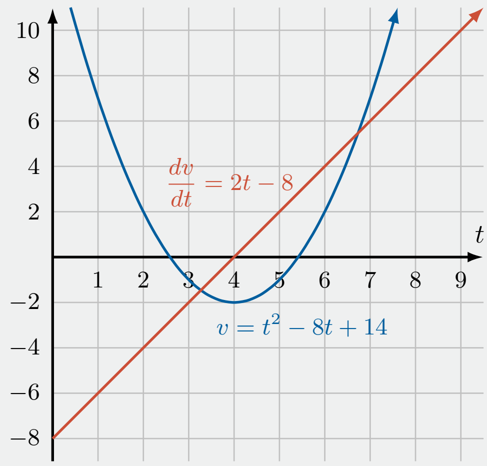
Q15

(a) -2 units/s

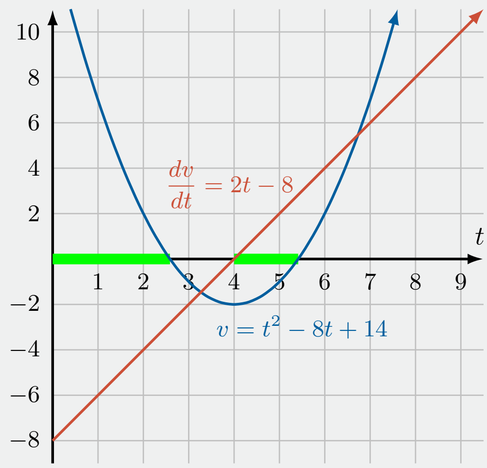
(b) 0 units/s^2

(c) Neither

(d)



(e)



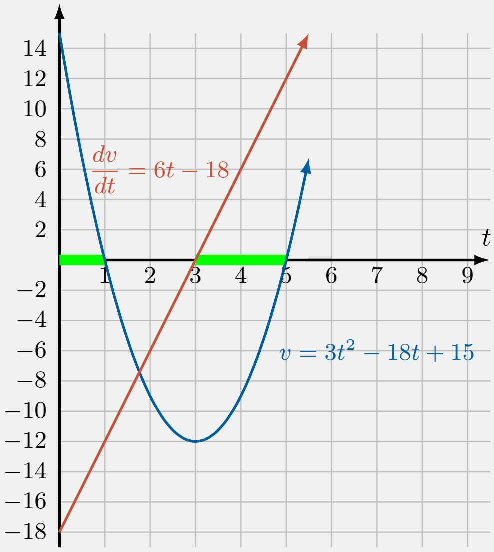
Q16

(a) $\dot{x} = 3t^2 - 18t + 15$ and $\ddot{x} = 6t - 18$

(b) When $t = 0$, $\dot{x} = 15$ so the particle is moving to the right

(c) $t = 1$, $x = 9$ and $t = 5$, $x = -23$

(d)



(e)

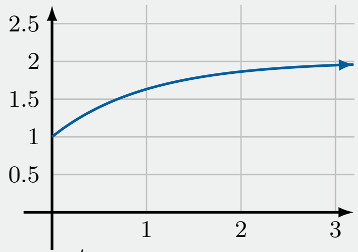
(i) When $0 \leq t < 1$ and $3 < t < 5$

(ii) When $1 < t < 3$ and $t > 5$

(f) 46 cm

Q17

(a)



(b) $v = e^{-t}$

(c) 1 cm/sec

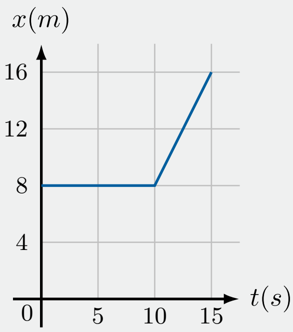
(d) Always moving to the right as $v > 0$

(e) As $t \to \infty$, $x \to 2$ and $v \to 0$

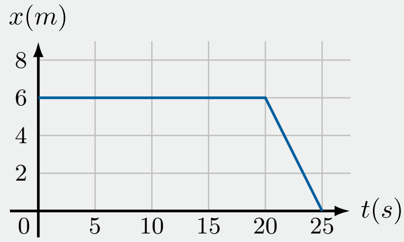
(f) See full worked solutions.

Q18

(a) $v = 8$ for $0 \leq t \leq 8$ and $v = \frac{8}{5}t - 8$



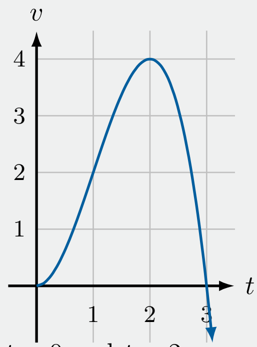
(b) $v = 6$ for $0 \leq t \leq 20$ and $v = \frac{6}{5}(t - 25)$



Q19

(a) $t = 3$

(b)



(c) $t = 0$ and $t = 2$

(d) See full worked solutions.

(e) $v = 4$ units/second

(f) See full worked solutions.

Q20

(a) Increasing at an increasing rate

(b) Decreasing at a decreasing rate

(c) Increasing at a decreasing rate

(d) Decreasing at an increasing rate

Q21

(a)

(i)

A	*B*	*C*	*D*	*E*
Negative	Positive	Positive	Zero	Negative

(ii)

A	*B*	*C*	*D*	*E*
Left	Right	Right	Stationary	Left

(iii)

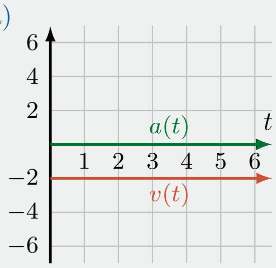
A	*B*	*C*	*D*	*E*
Slowing	Speeding	Slowing	At rest	Slowing

(b) The gradient of the tangent represents the acceleration of the particle at that point in time.

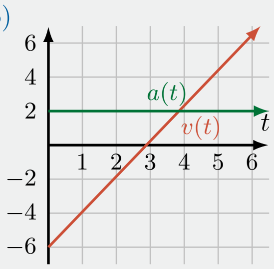
(c) $t \approx 5.2$ and $t \approx 9$

Q22

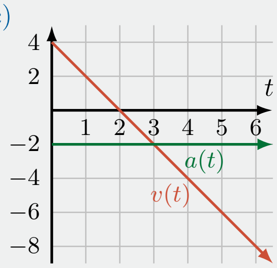
(a)



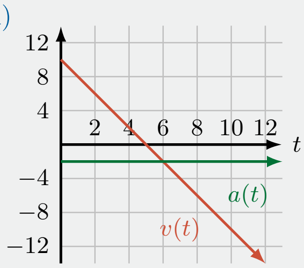
(b)



(c)



(d)



Q23

(a) $x = 5$ m

(b) $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(c) $x = 1$ m

(d) See full worked solutions.

Q24

(a) $x = -1.5,\ v = 0$

(b) 4 seconds

(c) right

(d) $t = 2$

(e) $x = 0$

(f) See full worked solutions.

Q25

(a) See full worked solutions.

(b) $v = 0$ at both.

(c) $t = 8$

(d) 8 m/s

Q26

48 metres

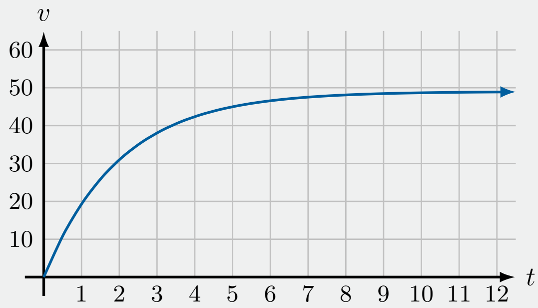
P1

(a) 0 m/s

(b) $25e^{-0.5t}$ m/s^2

(c) $\frac{dx}{dt} \to 50$

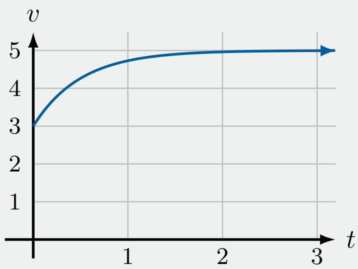
(d)



P2

(a) initially $x = 1$ cm, $v = 3$ cm/s, $a = 4$ cm/s^2

(b)



(c) No, velocity is never zero, $v \to 5$ cm/s as $t \to \infty$

(d) As $t \to \infty$, $v \to 5$ cm/s and $a \to 0$ cm/s^2

P3

See full worked solutions.

Exercise 5C

Exponential growth and decay

F1

(a) proportional, kQ

(b) differential

(c) increasing

(d) decreasing

(e) ∞

(f) 0

F2

(a) growth

(b) decay

(c) A, initial

(d) Q_0, Q_0

F3

(a) kP

(b) Ae^{kt}

F4

(a) half

(b) $\frac{1}{2}$

Q1

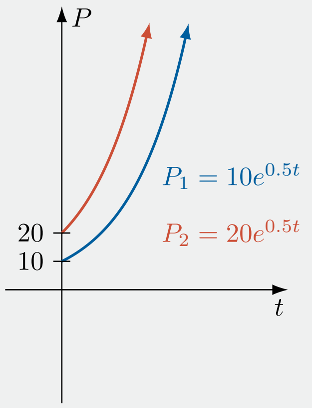
(a) $\frac{dN}{dt} = kN$

(b) $\frac{dP}{dt} = kP$

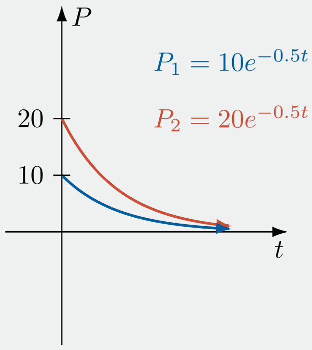
(c) $\frac{dM}{dt} = -kM$

Q2

(a)



(b)



Q3

(a) Larger k represents faster population growth (approaches infinity faster) whereas smaller k represents slower population growth (approaches infinity slower).

(b) Larger k represents faster population decline (approaches zero faster) whereas smaller k represents slower population decline (approaches zero slower).

Q4

(a) 3

(b) 9

(c) $\frac{1}{3}$

(d) A

(e) $\frac{1}{A}$

(f) A^k

Q5

(a) 30000

(b) 63510

(c) 2023

(d) See full worked solutions.

(e) 900 people per year

(f) 2010

Q6

(a) 12 kg.

(b) 347 years.

Q7

(a) See full worked solutions.

(b) $k = \frac{1}{10} \ln\left(\frac{6}{5}\right)$

(c) Ten years ago: 456 people per year
Today: 547 people per year

Q8

(a) $M_0 = 10$, $k = \frac{1}{2} \ln\left(\frac{7}{5}\right)$

(b) 53.8 grams

(c) 5 days

(d) 9.1 grams per day

Q9

(a) See full worked solutions.

(b) $k = \frac{1}{20} \ln 2$

Q10

(a) It is the amount of the carbon isotope from when the tree was alive.

(b) $k = 1.155 \times 10^{-4}$

(c) 18000 years ago.

Q11

(a) See full worked solutions.

(b) 9.52 kilograms per hour

(c) 21.64 hours

Q12

(a) 29.3%

(b) 15.9%

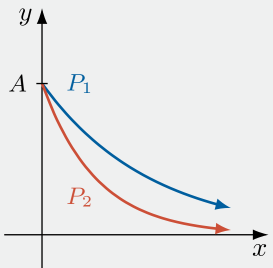
P1

See full worked solutions.

P2

(a) P_2

(b)



(c) See full worked solutions.

P3

See full worked solutions.

P4

See full worked solutions.

Exercise 5D

Further growth and decay

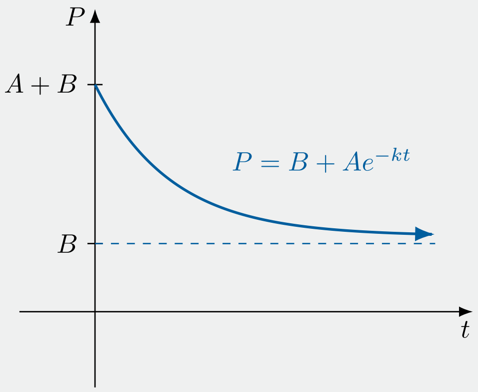
F1

(a) $k(P - B)$ (b) $\infty, -\infty$ (c) B

F2

See full worked solutions.

F3

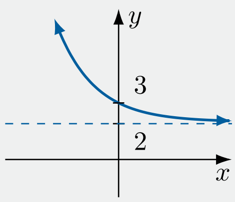


F4

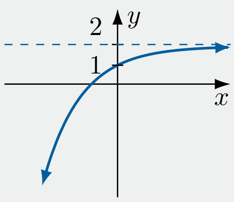
$-k(T - B)$, Newton's, Cooling

Q1

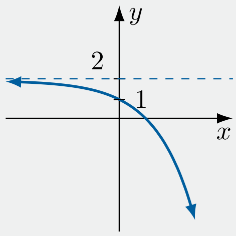
(a)



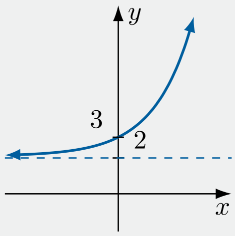
(b)



(c)



(d)



Q2

(a) $P = B + Ae^{-kt}$ (b) $P = B - Ae^{-kt}$

(c) $P = B + Ae^{kt}$ (d) $P = B - Ae^{kt}$

Q3

$\frac{dT}{dt} = k(T - T_e)$ where T is the temperature of the object, T_e is the environmental temperature and t is time.

Q4

See full worked solutions.

Q5

(a) See full worked solutions.

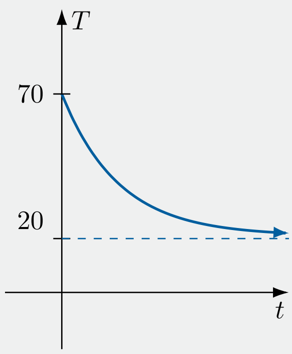
(b) $k = \frac{1}{10} \ln\left(\frac{5}{3}\right)$

(c) 59°C

(d) 32 minutes

(e) 20°C

(f)



Q6

(a) Positive

(b) $k(180 - T)$

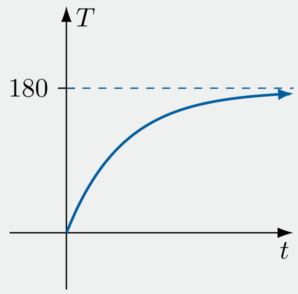
(c) See full worked solutions.

(d) $k = \frac{1}{10} \ln\left(\frac{6}{5}\right)$

(e) 120°C

(f) 23 minutes

(g)



Q7

(a) It is the limiting population of the species.

(b) See full worked solutions.

(c) See full worked solutions.

(d) 37 years

P1

(a) $5R$ grams per minute

(b) $\dfrac{RS}{500}$ grams per minute

(c) See full worked solutions.

(d) See full worked solutions.

(e) 2500 grams

(f) 13.4 litres per minute

P2

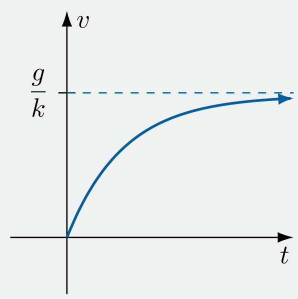
(a) See full worked solutions.

(b) $v \to \dfrac{g}{k}$

(c) It is the same $v = \dfrac{g}{k}$. This is because this value of v is the *terminal velocity* of the particle. In other words it is the limiting velocity and so the change in velocity gradually decreases as it approaches this constant. This means that the particle is no longer accelerating i.e. $a \to 0$.

(d) $t = \dfrac{1}{k} \ln 2$

(e)



(f) See full worked solutions.

Exercise 5E

Related Rates of change

F1

(a) related

(b) radius, related

F2

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

F3

(a) Required to find $\dfrac{dA}{dt}$ given that $\dfrac{dr}{dt} = 1$

(b) Required to find $\dfrac{dV}{dt}$ given that $\dfrac{dA}{dt} = -4$

(c) Required to find $\dfrac{dl}{dt}$ given that $\dfrac{dx}{dt} = 2$

Q1

(a) $\dfrac{dV}{dt} = \pi r^2 \times \dfrac{dr}{dt}$

(b) $\dfrac{dA}{dt} = 12x \times \dfrac{dx}{dt}$

(c) $\dfrac{dy}{dt} = \dfrac{-x}{\sqrt{25 - x^2}} \times \dfrac{dx}{dt}$

Q2

(a) $A = x^2$

(b) $\dfrac{dA}{dt} = 4x$

(c) 20 units^2/s

Q3

(a) 72 units^2/s

(b) 36 units^3/s

Q4

0.4 cm^2/s

Q5

$\dfrac{3}{10\pi}$ cm/s

Q6

(a) $\dfrac{2}{r^2}$ cm/min

(b) $\dfrac{1}{8}$ cm/min

(c) $\dfrac{2}{9}$ cm/min

Q7

18π cm^3/min

Q8

3600π cm^3/min

Q9

480π cm^3/s

Q10

(a) -864 cm^3/min (b) $-\frac{25}{16}$ cm/min

Q11

(a) 36 cm^2/min (b) 54 cm^3/min

Q12

3 m/s

Q13

(a) $\frac{3}{4\pi}$ mm/s (b) $\frac{1}{4\pi}$ mm/s (c) 144 mm^3/s

Q14

(a) $8\pi kR$ cm^2/s (b) $4\pi kR^2$ cm^3/s

Q15

4.8 cm^2/s

Q16

(a) $\frac{20}{17}$ m/s (b) $\frac{3}{17}$ m/s

Q17

(a)

(i) $V = \frac{1}{3}\pi h^3$ (ii) $V = \frac{\pi}{9}h^3$

(b)

(i) $\frac{1}{2\pi}$ cm/s (ii) $\frac{3}{2\pi}$ cm/s

P1

See full worked solutions.

P2

(a) See full worked solutions.

(b) 75 km/hr

P3

7 cm^2/min

P4

4 cm/s

P5

1 cm^2/s

Chapter Review

R1

(a) The rate at which the bath is being filled

(b) $\frac{dV}{dt} < 0$. Rate negative, water emptying

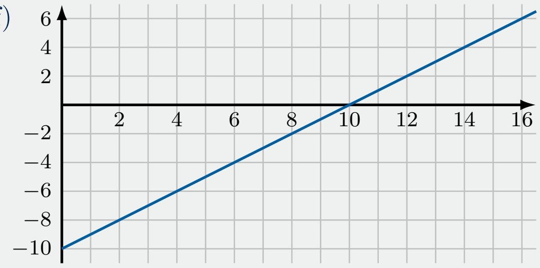
(c) $\frac{dV}{dt} = 0$ water level not changing and

$\frac{dV}{dt} > 0$ water level increasing

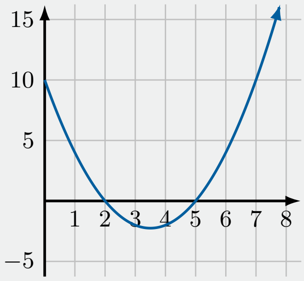
(d) See full worked solutions.

(e) See full worked solutions.

(f)



(g)



R2

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

(d) When $t = 2$, $V = 80$ litres

(e) At 6 minutes

R3

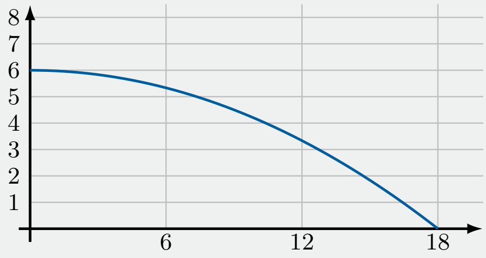
(a) 18 minutes

(b) $-\frac{1}{3}$ kg/m

(c) -0.471 kg/m

ANSWERS

(d)



R4

(a) $v = -3(6-t)^2$, since negative the particle is always moving to the left except at $t = 6$ when it is 0

(b) $t = 6$, $x = 0$

(c) $x = 216$, $v = -108$ m/s

(d) At $t = 6$

(e) 152 m

(f) $\dfrac{dv}{dt} = 6(6-t)$

(g) See full worked solutions.

R5

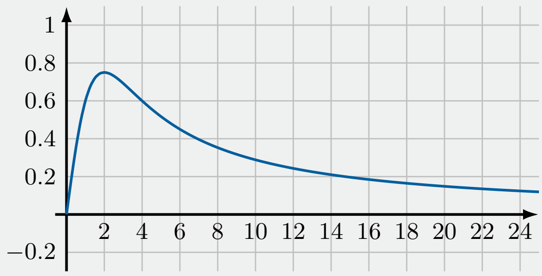
(a) $v = \dfrac{12-3t^2}{(4+t^2)^2}$

(b) At $t = 2$

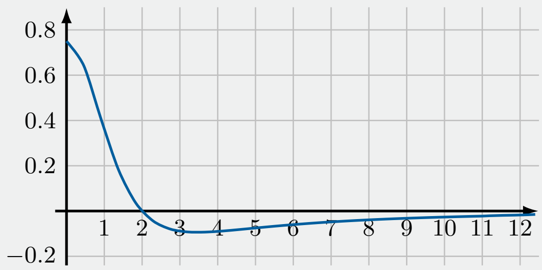
(c) $x = 0.6$

(d) $t = 1$

(e) Position



Velocity

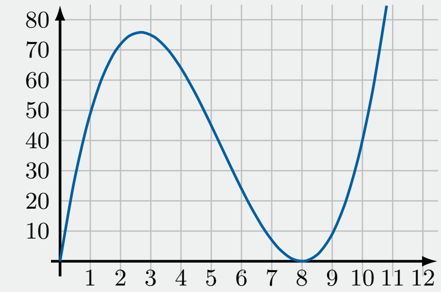


(f) See full worked solutions.

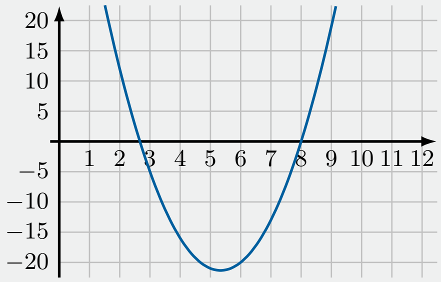
R6

(a) $v = 3t^2 - 32t + 64$, $a = 6t - 32$

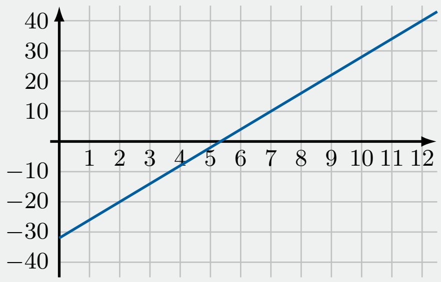
(b) Displacement



Velocity



Acceleration



(c) At $t = 8$ and $t = \dfrac{8}{3}$

(d) When $\dfrac{8}{3} < t < 8$

(e) At $t = 5\frac{1}{3}$

(f) See full worked solutions.

R7

(a) $v = 1$ m/s

(b) $v \to 0$

(c) $-\dfrac{1}{4}$ m/s^2

R8

(a) $v = 40 - 10t$, $\dfrac{dv}{dt} = \ddot{x} = -10$

(b) Initially when $t = 0$, $x = 0$, $v = 40$ m/s, $\ddot{x} = -10$ m/s^2. Since $v > 0$, ball is rising.

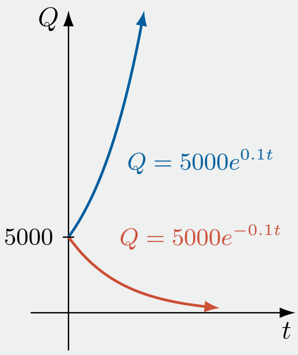
(c) Acceleration is a constant as it represents gravity

(d) At $v = 0$

(e) 80 m

(f) Speeding: $t > 4$. Slowing: $0 < t < 4$

R9



R10

(a) See full worked solutions.

(b) $k = \frac{1}{12}\ln\left(\frac{16}{15}\right)$

(c) -0.043 grams per year

(d) 54 years.

R11

(a) 1.5 days.

(b) 9.9×10^8 algae per day, 1.5×10^8 algae per day

(c) 4.6 days

R12

(a) See full worked solutions.

(b) $k = \frac{1}{12}\ln 2$

(c) 2019

(d) 6788225

(e) 392103 people per year

R13

(a) See full worked solutions.

(b) $\frac{1}{3}\ln\frac{4}{3}$

(c) $\frac{9}{16}$

(d) See full worked solutions.

(e) 3.8 litres

R14

(a) See full worked solutions.

(b) $A = 70,\ E = 20$

(c) $k = \frac{1}{5}\ln\left(\frac{7}{4}\right)$

(d) 33°C

(e) 11.2 minutes

R15

(a) See full worked solutions.

(b) $k = \frac{1}{4}\ln 2$

(c) 21.3 minutes

R16

(a) See full worked solutions.

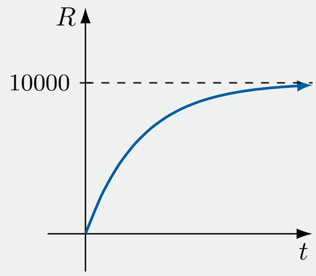
(b) $A = 10000$

(c) See full worked solutions.

(d) 1900

(e) 66 days

(f) Eventually $R \to 10000$



R17

(a) See full worked solutions.

(b) $B = 16$

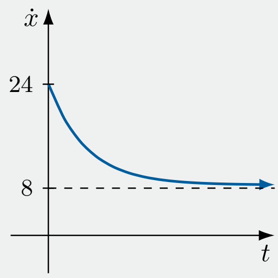
(c) See full worked solutions.

(d) 9ms^{-1}

(e) $-2\ln 2\text{ms}^{-2}$

(f) It is the limiting velocity of the sky-diver once the parachute is opened.

(g)



ANSWERS

R18

$100\pi\ \text{cm}^2/\text{s}$

R19

(a) $\frac{1}{8}$ cm/s (b) $6\ \text{cm}^3/\text{s}$

R20

$\frac{4}{15\pi}$ m/min

R21

(a) $\frac{8}{\pi}$ m/min (b) $\frac{\pi}{16}\ \text{m}^3/\text{min}$

6. Combinatorics and Binomial Expansions

Exercise 6A

Multiplication principle

F1

(a) mn (b) does (c) multiply

Q1

$3 \times 4 \times 2$

Q2

(a) $5 \times 5 \times 5$ (b) $5 \times 4 \times 3$

Q3

(a) $3 \times 2 \times 1$ (b) 3^3

Q4

(a) $5 \times 3 \times 3$ (b) $2 \times 5 \times 3 \times 3$

Q5

(a) $(26 \times 26 \times 26) \times (10 \times 10 \times 10)$

(b) $(26 \times 25 \times 24) \times (10 \times 9 \times 8)$

Q6

(a) $3 \times 2 \times 1$ (b) $6 \times 5 \times 4$

Q7

(a) $8 \times 7 \times 6$ (b) $3 \times 2 \times 1$ (c) 5^3

Q8

(a) 9^3 (b) 9^2 (c) 9^2

(d) $9 \times 9 \times 4$ (e) 4^3 (f) $9 \times 8 \times 7$

Q9

(a) $10 \times 9 \times 8$ (b) $3 \times 2 \times 1$

(c) $9 \times 8 = 72$ (d) 3×72

P1

$4^5 - 4$

P2

$2^n - 2$

P3

(a) $5 \times 4 \times 3 \times 2$

(b) $5 \times 4 \times 3 \times 2 \times 1$

(c) For every answer in (b), there exists a uniquely corresponding answer from (a). Just delete the last digit and the answer for (b) becomes a valid answer for (a).

Exercise 6B

Factorials

F1

(a) $n-1, n-2, 1$

(b) $k+1$

(c) 1

F2

(a) $(n+1)!$ (b) n (c) $(n-1)!$

F3

(a) $\frac{n!}{(n-k)!}$ (b) $\frac{n!}{k!(n-k)!}$

(c) 1 (d) 1

(e) n

Q1

(a) 24 (b) 120 (c) 30 (d) 20

Q2

(a) $n(n-1)$ (b) $n(n+1)$

(c) $(n+2)(n+1)$

Q3

(a) $6 \times 4!$ (b) $8 \times 8!$ (c) $36 \times 4!$

Q4

(a) $n \times n!$

(b) $(n^2 - n - 1) \times (n - 2)!$

(c) $(n^2 + n - 1) \times (n - 1)!$

(d) $(n - 1)(n - 1)!$

Q5

See full worked solutions.

Q6

See full worked solutions.

Q7

See full worked solutions.

Q8

(a) $n = 3, 7$ (b) $n = 10$

(c) $n = 10$ (d) $n = 3, 6$

Q9

See full worked solutions.

Q10

See full worked solutions.

Q11

See full worked solutions.

P1

(a) See full worked solutions.

(b) $(n + 1)! - 1$

P2

(a) $\dfrac{k}{(k + 1)!}$

(b) See full worked solutions.

(c) 1

P3

See full worked solutions.

P4

See full worked solutions.

P5

See full worked solutions.

P6

(a) n

(b) $n - 1$

(c) P_n shook 1 person's hand, and P_{n+1} did not need to shake anybody's hand since the room is now empty, so zero.

(d) There are $n + 1$ people in the room. To form a handshake, we need to select 2 people, so ${}^{n+1}C_2$.

(e) See full worked solutions.

P7

See full worked solutions.

Exercise 6C

Permutations

F1

(a) $n!$ (b) $n, n - 1, 1$

(c) ${}^{n}P_k$

F2

(a) restrictions (b) complementary

F3

(a) complementary

(b) Directly: Separated by 0 (together) and 1. Complementary: Separated by 2, 3 and 4. Easier to count directly.

(c) Easier to consider the complementary for 'at most three'. Easier to count directly for 'at least three'.

F4

$\dfrac{n!}{(n - r)!}$

F5

(a) 1 (b) n (c) $n!$

Q1

(a) 3! (b) 10! (c) 4!

Q2

(a) 7! (b) 6! (c) 6!

(d) 6!2! (e) $7! - 6!2!$ (f) $5! \times 2!$

ANSWERS

Q3

(a) $^7P_4$ (b) $^9P_5$ (c) $^8P_3$

Q4

(a) $^{10}P_6$ (b) $5 \times {}^9P_5$

(c) $^{10}P_6 - 5 \times {}^9P_5$ (d) $^9P_6$

Q5

$^9P_1 + {}^9P_2 + {}^9P_3 + {}^9P_4 + {}^9P_5$

Q6

$2 \times 5! \times 5!$

Q7

(a) $10!$ (b) $9!2!$

(c) $10! - 9!2!$ (d) $7!4!$

(e) $4!6! \times 2!$ (f) $8! \times 2!$

Q8

(a) $7!$ (b) $\frac{8!}{2}$

Q9

(a) $8!$ (b) $4!4! \times 2$

(c) $4 \times 7!$ (d) $^4P_2 \times 6! \times 2$

P1

(a) 4^6

(b) $^4P_2 \times 6$

(c) $4^6 - {}^4P_2 \times 6 - 4 = 4020$

P2

$8!{}^7P_2$

P3

(a) $^{n+1}P_k$

(b) ^nP_k

(c) $k{}^nP_{k-1}$

(d) See full worked solutions.

Exercise 6D
Identical elements

F1

(a) 4 (b) 6

F2

(a) $\frac{8!}{3!5!}$

(b) distinct, identical

(c) The naïve answer of 8! has a 3! and 5! inside it. Their jobs are to arrange the three X's and five Y's respectively. But it is incorrect to arrange them since they are identical anyway, so we must divide out 3! and 5! to 'undo' the permutations of the X's and Y's.

F3

$$\frac{n!}{n_1!n_2!n_3!\ldots n_k!}$$

Q1

(a) $\frac{5!}{2!}$ (b) $\frac{6!}{2!}$

(c) $\frac{6!}{2!2!}$ (d) $\frac{10!}{2!2!2!3!}$

(e) $\frac{10!}{2!2!4!}$ (f) $\frac{13!}{3!8!}$

Q2

$\frac{6!}{3!3!}$

Q3

$\frac{6!}{2!3!}$

Q4

(a) $\frac{8!}{2!2!2!}$ (b) $\frac{7!}{2!2!}$

(c) $\frac{6!}{2!2!}$ (d) $5!$

Q5

(a) 2^8 (b) $\frac{8!}{3!5!}$

(c) $2^8 - \frac{8!}{7!} - 1$ (d) $\frac{8!}{2!6!} + \frac{8!}{7!} + 1$

(e) $1 + \frac{8!}{7!} + \frac{8!}{2!6!}$ (f) $2^8 - 1 - \frac{8!}{7!}$

Q6

(a) The number of permutations of the code may differ, depending on whether the selection of letters contains both O's or not.

(b)

ANSWERS

(i) $2 \times \frac{3!}{2!}$ (ii) $3!$

(c) $2 \times \frac{3!}{2!} + 3!$

Q7

(a) $4 \times \frac{5!}{2!}$ (b) $5!$

(c) $4 \times \frac{5!}{2!} + 5! = 360$

Q8

$2 \times \frac{5!}{2!} + 2 \times \frac{5!}{2!2!}$

P1

(a) $UURRURRRRU$

(b) $\frac{10!}{4!6!}$

(c) $\frac{5!}{2!3!} \times \frac{5!}{2!3!}$

P2

2^{10}

P3

(a) See full worked solutions.

(b) $\star\star|\star\star\star\star|$

(c) $\frac{8!}{6!2!}$

Exercise 6E
Combinations

F1

(a) ${}^nC_k$, $\binom{n}{k}$ (b) does not

F2

(a) $\frac{n!}{k!(n-k)!}$ (b) $k!$ (c) $k! \times {}^nC_k$

F3

(a) 1 (b) n (c) 1

F4

Use ${}^nP_r$ if order matters, but use ${}^nC_r$ if order does not matter.

Q1

(a) ${}^{10}C_4$ (b) ${}^7C_3$

Q2

(a) ${}^{13}C_4$ (b) ${}^7C_4$

(c) ${}^6C_2{}^7C_2$ (d) ${}^6C_3{}^7C_1 + {}^6C_4$

(e) ${}^{11}C_2$ (f) $2 \times {}^{11}C_3$

Q3

(a) ${}^8C_5$ (b) ${}^5C_2$

(c) ${}^7C_4$ (d) ${}^6C_4$

Q4

(a) ${}^8C_3$

(b) ${}^8C_5$

(c) Selecting 3 to hire is equivalent to selecting 5 to reject.

(d) See full worked solutions.

Q5

(a) ${}^{10}C_3$ (b) ${}^{10}C_4$

Q6

36

Q7

(a) 3

(b) Bob's answer considers the two groups to be distinct from each other. In other words, his answer would be correct if the two groups had names like 'Group 1' and 'Group 2', where $\{A, B\}\&\{C, D\}$ is not the same as $\{C, D\}\&\{A, B\}$.

Group 1	Group 2
A, B	C, D
A, C	B, D
A, D	B, C

Q8

(a) ${}^{12}C_3{}^9C_4{}^5C_5$

(b) ${}^{12}C_6{}^6C_6$

(c) Yes.

(d) When the two groups have names, say A and B, then having six particular people in Group A and the rest in B is *not* the same as the other way around. But when the two groups are indistinguishable, then now they are the same.

ANSWERS

Q9

(a) $\frac{{}^6C_3}{2}$ (b) ${}^6C_3$

Q10

(a) ${}^5C_2 - 5$ (b) ${}^6C_2 - 6$ (c) ${}^nC_2 - n$

Q11

(a) ${}^{52}C_5$
(b) ${}^{26}C_5 \times 2$
(c) ${}^{13}C_5$
(d) ${}^4C_1{}^{13}C_5$
(e) ${}^{13}C_2{}^{13}C_3$
(f) ${}^4C_3{}^{48}C_2$
(g) ${}^{13}C_1{}^4C_2 \times {}^{12}C_1{}^4C_3 \times 2$
(h) ${}^{13}C_2{}^4C_2{}^4C_2{}^{44}C_1$

P1

${}^5C_2 \times {}^7C_2$

P2

(a) ${}^{n-1}C_{k-1}$
(b) ${}^{n-1}C_k$
(c) See full worked solutions.

P3

(a) $\frac{10!}{4!6!}$
(b) ${}^{10}C_4 = {}^{10}C_6$
(c) ${}^7C_3 = {}^7C_4$
(d) See full worked solutions.
(e) ${}^{20}C_{12} = {}^{20}C_8$

P4

${}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4 + {}^8C_4$

P5

(a) ${}^{10}C_5$
(b) ${}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$
(c) See full worked solutions.

Exercise 6F
Arrangements in circles

F1

(a) $(n-1)!$ (b) $n-1$

F2

(a) $2!(n-2)!$ (b) $3!(n-3)!$

Q1

4!

Q2

(a) 8! (b) 7!

Q3

(a) 7!
(b) 4!4!
(c) 4!3!
(d) 6!2!
(e) 7! – 6!2!
(f) $4! \times {}^6P_2 \times 2!$ or $2 \times 6!$

Q4

(a) 3!3! (b) 4!
(c) 5! – 4!2! (d) 3!2!

Q5

(a) ${}^8C_4{}^4C_4 3!3!$ (b) ${}^6C_2{}^4C_4 3!3! \times 2$
(c) ${}^6C_3{}^3C_3 3!3! \times 2$ (d) ${}^6C_2{}^4C_4 2!3! \times 2$

Q6

(a) ${}^6C_2{}^4C_4 2!4! \times 2$ (b) ${}^6C_1 2!4! + {}^6C_3 4!2!$

Q7

${}^{10}C_4 3! \times {}^6C_6 5!$

Q8

$7! - 6!2! - 5!{}^6C_1 2!$

Q9

$\frac{5!}{2}$

P1

(a) $3!2^4$ (b) $6 \times 4 \times 2 = 48$

ANSWERS

P2

(a) $(n-1)!$ (b) $(n-2)!2!$ (c) $(n-3)!2!$

Exercise 6G

Applications to probability

F1

(a) Total number of outcomes

(b) choosing, repetition

F2

restriction

F3

(a) complementary, least

(b) E

Q1

(a) $8!$ (b) $\frac{6!3!}{8!}$

Q2

(a) $\frac{7!2!}{8!}$

(b) $1-\frac{7!2!}{8!}$

(c) $\frac{6!2!}{8!}$

(d) $\frac{5\times 2!\times 6!}{8!}$ or $\frac{5!\,{}^6C_2 2!2!}{8!}$

Q3

(a) $\frac{4!4!}{7!}$

(b) $\frac{4!3!}{7!}$

(c) $\frac{6!2!}{7!}$

(d) $\frac{6!}{7!}$

(e) $1-\frac{6!2!}{7!}$

(f) $\frac{2\times 6!}{7!}$ or $\frac{4!\,{}^6C_2 2!2!}{7!}$

Q4

$\frac{1}{4!}$

Q5

(a) $\frac{{}^6P_2 7!}{9!}$ (b) $\frac{{}^3C_1\,{}^6C_1 7!}{9!}$

Q6

(a) $\frac{{}^5P_4}{{}^8P_4}$ (b) $\frac{{}^5C_1 4!}{{}^8P_4}$

(c) $\frac{{}^5P_2\,{}^6P_2}{{}^8P_4}$ (d) $\frac{{}^3C_2\,{}^5C_2 4!}{{}^8P_4}$

Q7

$\frac{{}^5C_1\,{}^4C_1 2!7!}{9!}$

Q8

$\frac{2!4!3!5!}{11!}$

Q9

(a) $\frac{1}{{}^{12}C_2}$ (b) $\frac{1}{{}^{12}P_3}$

Q10

$\frac{5\times\frac{6!}{2!}}{\frac{8!}{2!2!}}$

Q11

$\frac{4\times 3^4}{4^4}$

Q12

(a) $\frac{4}{{}^{52}C_5}$

(b) $\frac{10\times 4^5}{{}^{52}C_5}$

(c) $\frac{10\times {}^4C_1}{{}^{52}C_5}$

(d) $\frac{{}^4C_1\left({}^{13}C_5-10\right)}{{}^{52}C_5}$

(e) $\frac{{}^{13}C_1\,{}^{48}C_1}{{}^{52}C_5}$

(f) $\frac{{}^{13}C_1\,{}^4C_3\,{}^{12}C_1\,{}^4C_2}{{}^{52}C_5}$

(g) $\frac{{}^{13}C_2\,{}^4C_2\,{}^4C_2\,{}^{44}C_1}{{}^{52}C_5}$

(h) $\frac{{}^{13}C_1\,{}^4C_2\left({}^{48}C_3-{}^{12}C_1\,{}^4C_3-{}^{12}C_1\,{}^4C_2\,{}^{44}C_1\right)}{{}^{52}C_5}$

ANSWERS

(i) $\dfrac{{}^{13}C_1{}^4C_3{}^{48}C_2}{{}^{52}C_5} - \text{(e)}$

P1

(a) $\dfrac{1}{6!}$

(b) $\dfrac{\frac{6!}{3!}}{6!} = \dfrac{1}{3!}$

P2

(a) $\dfrac{\frac{10!}{2!2!2!}}{\frac{11!}{2!2!2!2!}}$

(b) $\dfrac{\frac{7!}{2!2!} \times \frac{5!}{2!2!}}{\frac{11!}{2!2!2!2!}}$

P3

(a) $\dfrac{4!}{2!2!} + 2 \times {}^3C_2\dfrac{4!}{2!} + 4!$

(b) $\dfrac{\frac{4!}{2!2!} + {}^3C_2\frac{4!}{2!}}{\frac{4!}{2!2!} + 2 \times {}^3C_2\frac{4!}{2!} + 4!}$

P4

(a) $\dfrac{7 \times 8!}{10!}$

(b) $\dfrac{1}{2}$

Exercise 6H

Pigeonhole principle

F1

(a) 2

(b) $\dfrac{n}{k}$

F2

$m + 1$

Q1

11

Q2

366

Q3

(a) 7

(b) 11

Q4

13

Q5

(a) 7

(b) 13

(c) $6n - 5$

Q6

$26^2 + 1$

Q7

(a) 53

(b) 8

(c) 25

(d) 31

Q8

(a) 3

(b) 4

(c) 5

(d) $n + 1$

Q9

$4 + 6 + 7 + 7 + 7 + 1 = 32$

Q10

(a) 19

(b) 28

Q11

See full worked solutions.

Q12

See full worked solutions.

Q13

See full worked solutions.

Q14

See full worked solutions.

Q15

20

Q16

370

Q17

6

Q18

4

P1

400

P2

See full worked solutions.

P3

See full worked solutions.

P4

Draw an equator through two points. Three points remaining. One of the hemispheres must have at least two points on it. That hemisphere now has four points in total.

P5

1010

Exercise 6I

Pascal's triangle and Binomial Expansions

F1

(a) $x^2+2xy+y^2$

(b) $x^3+3x^2y+3xy^2+y^3$

F2

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

F3

(a) $n+1$ (b) $(x+y)^n$

F4

(a) $\binom{2}{2}$

(b) $\binom{3}{1}, \binom{3}{2}, \binom{3}{3}$

(c) $\binom{4}{0}+\binom{4}{1}x+\binom{4}{2}x^2+\binom{4}{3}x^3+\binom{4}{4}x^4$

(d) $\binom{5}{0}+\binom{5}{1}x+\binom{5}{2}x^2+\binom{5}{3}x^3+\binom{5}{4}x^4+\binom{5}{5}x^5$

F5

(a) $\binom{3}{0}x^3+\binom{3}{1}x^2y+\binom{3}{2}xy^3+\binom{3}{3}y^3$

(b) $\binom{4}{0}x^4+\binom{4}{1}x^3y+\binom{4}{2}x^2y^2+\binom{4}{3}xy^3+\binom{4}{4}y^4$

(c) $\binom{5}{0}x^5+\binom{5}{1}x^4y+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}xy^4+\binom{5}{5}y^5$

F6

$\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n$

Q1

(a) $1, 4, 6, 4, 1$

(b) $1, 5, 10, 10, 5, 1$

(c) $1, 6, 15, 20, 15, 6, 1$

Q2

(a) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$

(b) $1+10x+40x^2+80x^3+80x^4+32x^5$

(c) $32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5$

(d) $16x^4+16x^3y+6x^2y^2+xy^3+\frac{y^4}{16}$

Q3

(a) $1-5x+10x^2-10x^3+5x^4-x^5$

(b) $16x^4-96x^3+216x^2-216x+81$

Q4

(a) $1+5x^2+10x^4+10x^6+5x^8+x^{10}$

(b) $x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

(c) $x^{10}-5x^7+10x^4-10x+\frac{5}{x^2}-\frac{1}{x^5}$

(d) $x^{12}+8x^7+24x^2+\frac{32}{x^3}+\frac{16}{x^8}$

Q5

(a) $a=41,\ b=29$ (b) $a=97,\ b=-56$

(c) $a=17,\ b=11$ (d) $a=49,\ b=-20$

Q6

(a) 34 (b) $24\sqrt{2}$

Q7

(a) 1.4641 (b) 1.1041

Q8

Even values of n

Q9

(a) $\binom{6}{2}x^2$ (b) $\binom{5}{2}2^2 3^3 x^2$ (c) $\binom{4}{2}2^2x^2$

Q10

(a) $\binom{3}{2}$ (b) $\binom{4}{2}$ (c) $\binom{5}{2}$

(d) The number on the top is the power of the expansion and the number on the bottom is the power of the term we want.

(e) $\binom{n}{k}$

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Q11

(a) $x^0 \times x^3$
$x^1 \times x^2$
$x^2 \times x^1$
$x^3 \times x^0$

(b) $x^0 \times x^3 \to \binom{3}{0}\binom{4}{3}$
$x^1 \times x^2 \to \binom{3}{1}\binom{4}{2}$
$x^2 \times x^1 \to \binom{3}{2}\binom{4}{1}$
$x^3 \times x^0 \to \binom{3}{3}\binom{4}{0}$

(c) See full worked solutions.

Q12

See full worked solutions.

Q13

(a) 9 (b) -72 (c) -245

Q14

(a) 32 (b) 243 (c) 0 (d) -1

P1

$a = 1$

P2

(a) $5(1+x)^4$

(b) $\binom{5}{1} + 2\binom{5}{2}x + 3\binom{5}{3}x^2 + 4\binom{5}{4}x^3 + 5\binom{5}{5}x^4$

(c) See full worked solutions.

P3

(a) $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

(b) Substitute $x = 6$.

(c) Yes, $7^n - 1$ is always divisible by 6 for positive integer n.

P4

(a) $p = -\frac{1}{2}$ (b) $p = -1$ (c) $p = -2$

Chapter Review

R1

(a) $5!$ (b) ${}^{10}P_3$ (c) $7!$

(d) $6!3!$ (e) $4!4! \times 2$ (f) $4!3!$

(g) 6^5 (h) $5! - 4!2!$ (i) $9^3 \times 10^6$

R2

(a) $7!$ (b) $4!3!$ (c) $4!3! \times 2$

(d) $5!3!$ (e) ${}^4P_2 \times 5!$

R3

(a) $4!2!$ (b) $6! - 5!2!$

(c) $2!2!4! + 4!2!$

R4

(a) $\frac{10!}{6!4!}$ (b) $\frac{7!}{6!}$

R5

(a) ${}^5P_3$ (b) $4 \times 3 \times 3$ (c) $3 \times {}^4P_3$

R6

(a) $\frac{4!}{2!}$ (b) $\frac{5!}{2!}$

R7

(a) $\frac{5!}{2!}$ (b) $\frac{9!}{5!4!}$

R8

${}^9C_4{}^4C_3 \times 7!$

R9

(a) $8!$ (b) $7!$ (c) $\frac{7!}{2}$

R10

(a) ${}^5C_4$ (b) ${}^6C_3{}^5C_1$ (c) ${}^{10}C_3$

(d) ${}^{10}C_4$ (e) ${}^4C_3$

R11

(a) ${}^9C_5$ (b) ${}^4C_2{}^6C_4$

R12

(a) $\frac{7!}{2!2!}$

(b) ${}^5P_4 + 2 \times {}^4C_2\frac{4!}{2!} + \frac{4!}{2!2!}$

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R13

(a) ${}^9C_4$ (b) ${}^{10}C_4$

R14

(a) $6!2!$ (b) $6!$ (c) $7! - 6!2!$

R15

(a) $\dfrac{{}^9C_1{}^8C_4{}^4C_4}{2}$ (b) $\dfrac{{}^8C_4{}^4C_4}{2}$

(c) $\dfrac{{}^8C_1{}^8C_4{}^4C_4}{2}$ (d) ${}^7C_1{}^6C_2$

(e) (a)–(d)

R16

(a) 2^9 (b) $2^9 - \left({}^9C_0 + {}^9C_1\right)$

(c) $2^8 - \left({}^8C_0 + {}^8C_1\right)$ (d) $2^7 - 1$

R17

(a) $\dfrac{{}^4C_2{}^6C_4}{{}^{10}C_6}$ (b) $\dfrac{{}^{18}C_{11}}{{}^{30}C_{11}}$ (c) $\dfrac{{}^7C_2}{{}^9C_4}$

(d) $\dfrac{4!3!}{7!}$ (e) $\dfrac{2}{5}$

R18

(a) $\dfrac{10 \times 6 \times 4}{{}^{20}C_3}$ (b) $\dfrac{{}^{10}C_3 + {}^6C_3 + {}^4C_3}{{}^{20}C_3}$

R19

$2^n - 1$

R20

(a) 5 (b) 6 (c) 7

R21

(a) 3 (b) 5 (c) 14

R22

See full worked solutions.

R23

See full worked solutions.

R24

6

R25

(a) 37

(b) See full worked solutions.

R26

See full worked solutions.

R27

See full worked solutions.

R28

See full worked solutions.

R29

(a) $\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

(b) $\binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$

R30

(a) $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

(b) $16 - 96x + 216x^2 - 216x^3 + 81x^4$

(c) $1 + 4x^2 + 6x^4 + 4x^6 + x^8$

(d) $x^5 + 10x^3 + 40x + \dfrac{80}{x} + \dfrac{80}{x^3} + \dfrac{32}{x^5}$

(e) $x^8 + 4x^5 + 6x^2 + \dfrac{4}{x} + \dfrac{1}{x^4}$

(f) $16x^12 - 96x^7 + 216x^2 - \dfrac{216}{x^3} + \dfrac{81}{x^8}$

R31

(a) 7200 (b) −26 (c) 640

R32

$a = 2, \dfrac{7}{4}$

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