Essential Insight Exam Guide

Mathematics Methods

Year 12 WACE

Western Australian Curriculum

2025 Edition

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Essential Insight Exam Guide
Mathematics Methods
Year 12 WACE

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Unit 3

Unit 3.1 – Further differentiation and applications

Section 1

2023 Section 1 Question 1 **Further differen-tiation and app-lications**	(a) Consider the function $f(x) = x^3 e^{2x}$. (i) Differentiate $f(x)$. (2 marks) (ii) Determine the value of x for any stationary points of $f(x)$. (3 marks) (b) Evaluate $\int_0^{\frac{\pi}{4}} \sin(2x + \pi)dx$. (3 marks)

2023 Section 1 Question 5

Further differentiation and applications

The table below contains values of the polynomial function $f(x)$, its first and second derivatives, and the function $F(x) = \int_0^x f(t)\,dt$ for $x = 0, 1, 2, 3, 4, 5, 6$.

$f(x)$ has no stationary points at non-integer values of x, and the letters a, b, c, d and e represent unspecified constants.

	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$f(x)$	a	b	4	c	0	d	e
$f'(x)$	16	0	–4	–2	0	–4	–20
$f''(x)$	–24	–9	0	3	0	–9	–24
$F(x)$	0	4.7	10.4	12.6	12.8	12.5	7.2

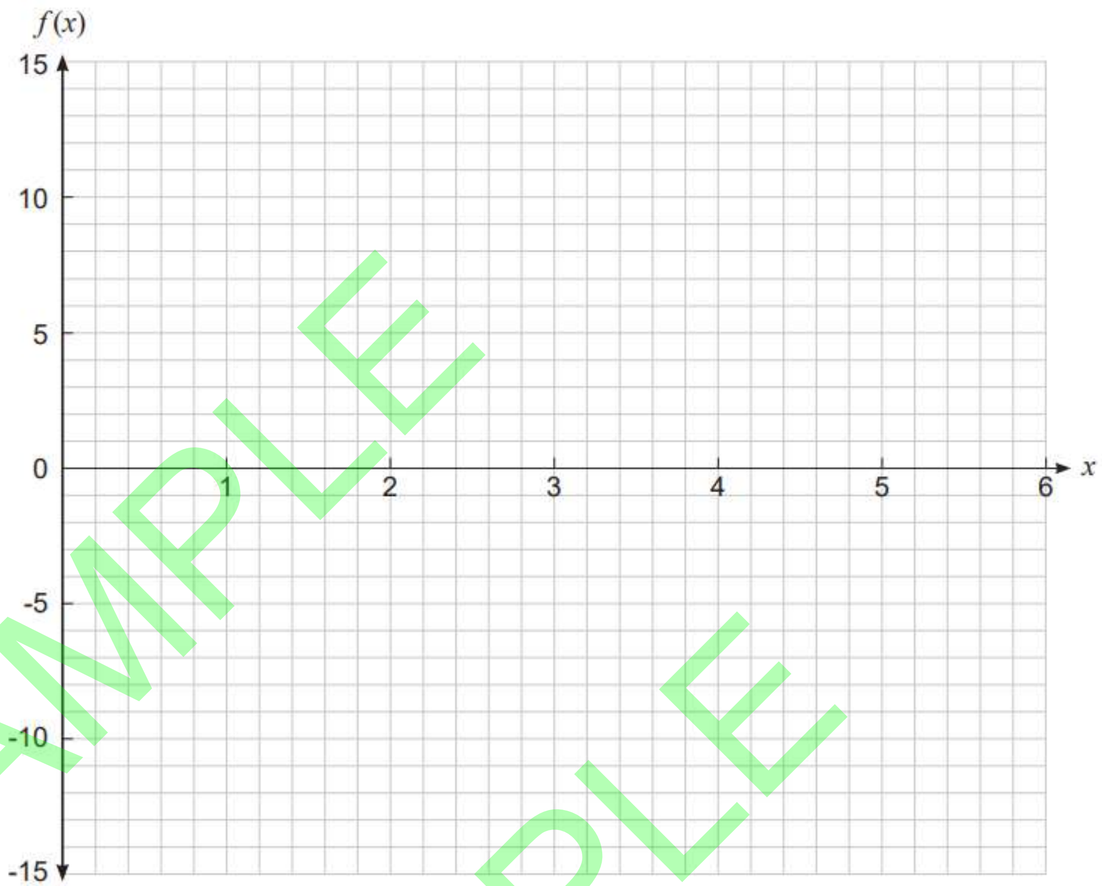
(a) Evaluate $\frac{d}{dx}\left(f(x)^2\right)$ when $x = 2$. (2 marks)

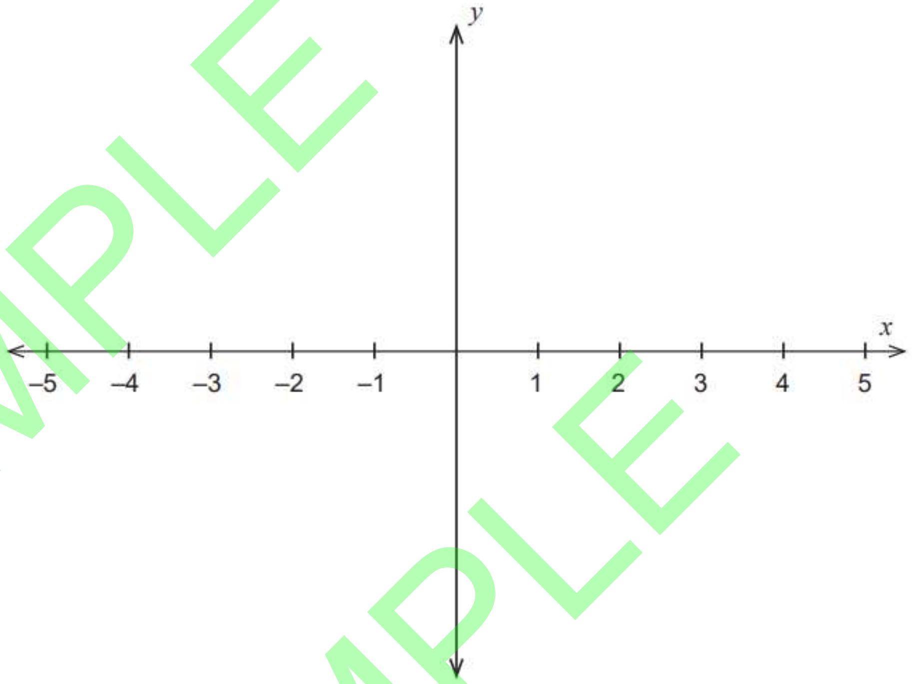
(b) Evaluate $\int_2^4 \left(f(x) + 2\right) dx$. (3 marks)

(c) Evaluate $\frac{d}{dx}\int_2^x f(t)\,dt$ when $x = 2$. (2 marks)

(d) Determine the x-coordinate of any stationary points and whether they are local maxima, local minima or inflection points. Justify your answer. (3 marks)

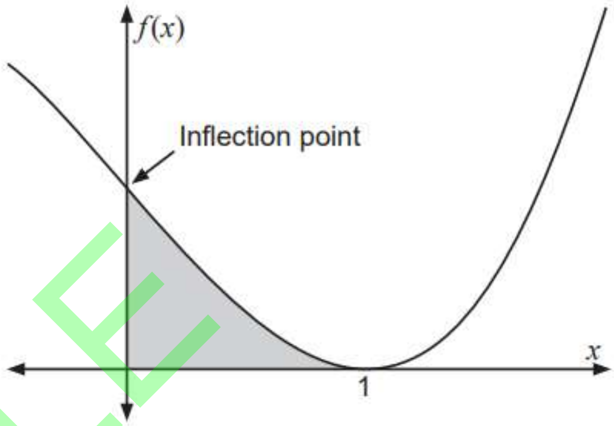
(e) Sketch a possible graph of $f(x)$ for $0 \leq x \leq 6$ on the axes below. (3 marks)

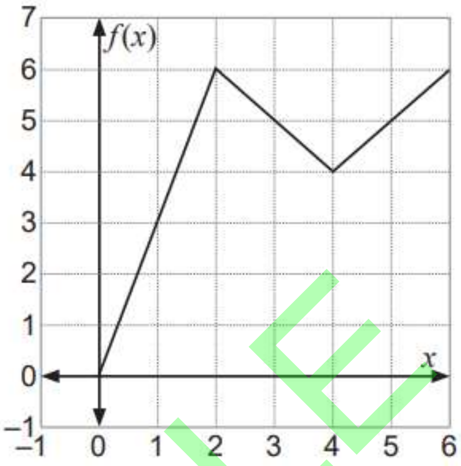
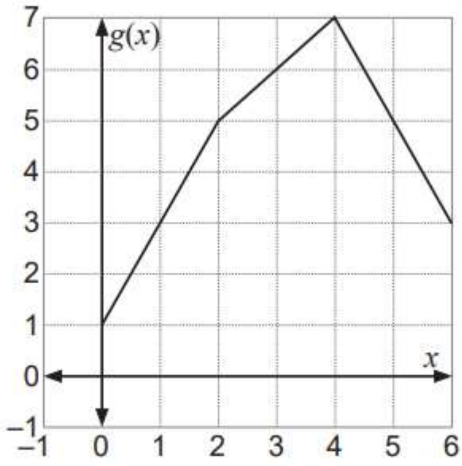


2022 Section 1 Question 5 Further differen-tiation and app-lications	A continuous function, f, satisfies the following conditions: • $f(2)=0$ • f has exactly 2 stationary points • $f'(-1)=0$ and $f'(1)=0$ • $f''(-1)=4$ • $f'(2)>0$. Sketch the function on the axes below. (5 marks) 

2021 Section 1 Question 1 **Further differen-tiation and app-lications**	(a) Differentiate $\frac{3x+1}{x^3}$ and simplify your answer. (3 marks) (b) Let $f'(x) = x\ln(2x)$. Determine a simplified expression for the rate of change of $f'(x)$. (3 marks) (c) Given that $g'(x) = 4e^{2x}$ and $g(1) = 0$, determine $g(5)$. (3 marks)

2020 Section 1 Question 2 **Further differen-tiation and app-lications**	If $h(x) = \dfrac{e^{-x}}{\cos x}$, then evaluate $h'(\pi)$. (4 marks)

2020 Section 1 Question 3 Further differen-tiation and app-lications	The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is shown below. A turning point is located at (1, 0) and the shaded region shown on the graph has an area of $\frac{3}{2}$ units^2.  Use the above information to determine the values of a, b, c and d. (7 marks)

2020 Section 1 Question 5 Further differen-tiation and app-lications	The graphs of the functions f and g are displayed below.   (a) Evaluate the derivative of $f(x)$ at $x = 3$. (1 mark) (b) Evaluate the derivative of $f(x)g(x)$ at $x = 5$. (2 marks) (c) Evaluate the derivative of $f(g(x))$ at $x = 1$. (2 marks)

2019 Section 1 Question 1 **Further differen-tiation and app-lications**	Consider the derivative function $f'(x) = xe^{x^2}$. (a) Determine $f''(1)$. (2 marks) (b) Explain the meaning of your answer to part (a). (1 mark) (c) Determine the expression for $y = f(x)$, given that it intersects the y-axis at the point (0,2). (3 marks)

2019 Section 1 Question 2 **Further differen-tiation and app-lications**	The values of the functions $g(x)$ and $h(x)$, and their derivatives $g'(x)$ and $h'(x)$ are provided in the table below for x = 1, x = 2 and x = 3.

	$x = 1$	$x = 2$	$x = 3$
$g(x)$	3	5	–3
$h(x)$	2	–2	6
$g'(x)$	–4	1	4
$h'(x)$	0	–6	–5

(a) Evaluate the derivative of $\frac{g(x)}{h(x)}$ at $x = 3$. (2 marks)

(b) Evaluate the derivative of $h(g(x))$ at $x = 1$. (2 marks)

(c) If $h''(1) = -1$, describe with justification, what the graph of $h(x)$ looks like at this point. (2 marks)

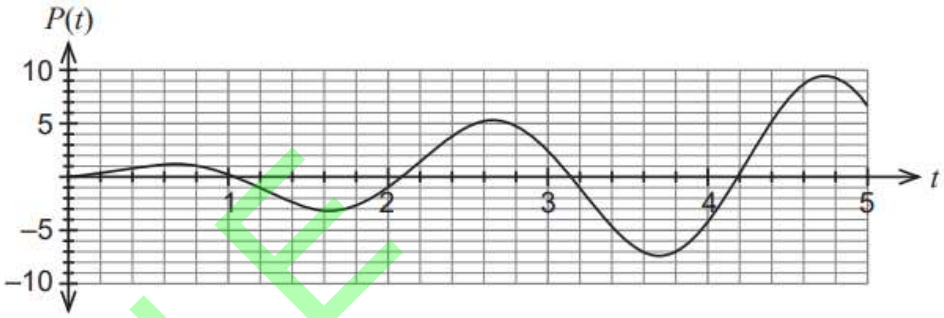
2019 Section 1 Question 7

Further differen-tiation and app-lications

A company's profit, in millions of dollars, over a five-year period can be modelled by the function:

$$P(t) = 2t\sin(3t) \quad 0 \le t \le 5 \text{ where } t \text{ is measured in years.}$$

The graph of $P(t)$ is shown below.



(a) Differentiate $P(t)$to determine the marginal profit function, $P'(t)$. (2 marks)

(b) Calculate the rate of change of the marginal profit function when $t = \frac{\pi}{18}$ years. (4 marks)

(c) Use the increments formula at $t = \frac{7\pi}{6}$ to estimate the change in profit for a one month change in time. (3 marks)

2023 Section 2 Question 6 Further differen-tiation and app-lications	A beekeeper is starting a new colony of bees. The population B of bees, in thousands, is given by $$B(t) = 4e^{1.4t}$$ where t is the number of years since the establishment of the colony. (a) Determine the initial population of the bee colony. (1 mark) (b) Determine the increase in the population of the bee colony in the first six months. (2 marks) (c) Determine the rate of population growth two years after the establishment of the colony. (2 marks) (d) After how many years will the rate of population growth be 65 000 bees/year? (2 marks)

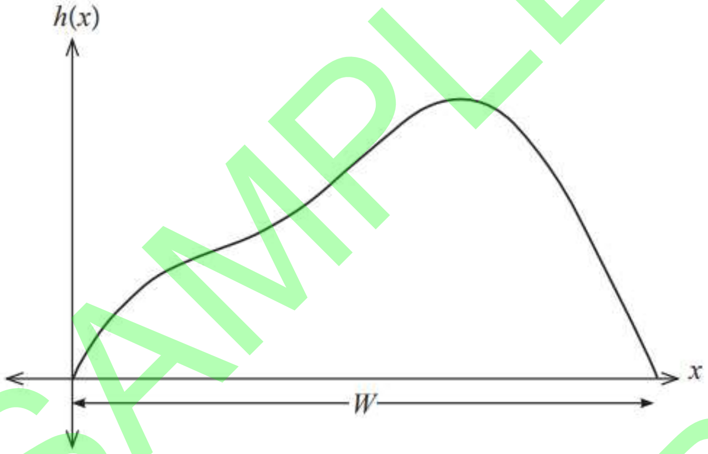
	After three years, the beekeeper notices that the number of bees begins to decline. The declining population, b, in thousands, has the form $b(t) = Ae^{rt}$ where t is the number of years since the start of the decline. (e) Determine A and r if one year after the start of the decline the bee population is 100 000. (4 marks)
2022 Section 2 Question 15 **Further differen-tiation and app-lications**	An object moves from the point $(0, 0)$ along the curve $y = \sqrt{3}\sin(x)$. The distance, D, travelled along the curve is given by $D(t) = \int_0^{\pi t} \sqrt{1 + 3\cos^2(x)}\, dx$ where D is measured in metres and t is measured in seconds. (a) Determine the speed $s = \frac{dD}{dt}$ of the object when $t = 1$. (3 marks)

(b) Use the increments formula to estimate the distance travelled by the object between $t = 1$ and $t = 1.02$. (2 marks)

2021 Section 2 Question 9

Further differentiation and applications

The Interesting Architecture company has designed a building with a constant cross-section shown in the figure below.



With reference to the figure, the height h(x) of the building at a point x along its width is given by

$$h(x) = 4\sin\left(x - \frac{3\pi}{2}\right) - x^2 + 3\pi x - 4 \text{, where } h \text{ and } 0 \le x \le W \text{ are measured in metres.}$$

(a) Determine the width W of the building to the nearest centimetre. (2 marks)

(b) Determine $h'(x)$. (1 mark)

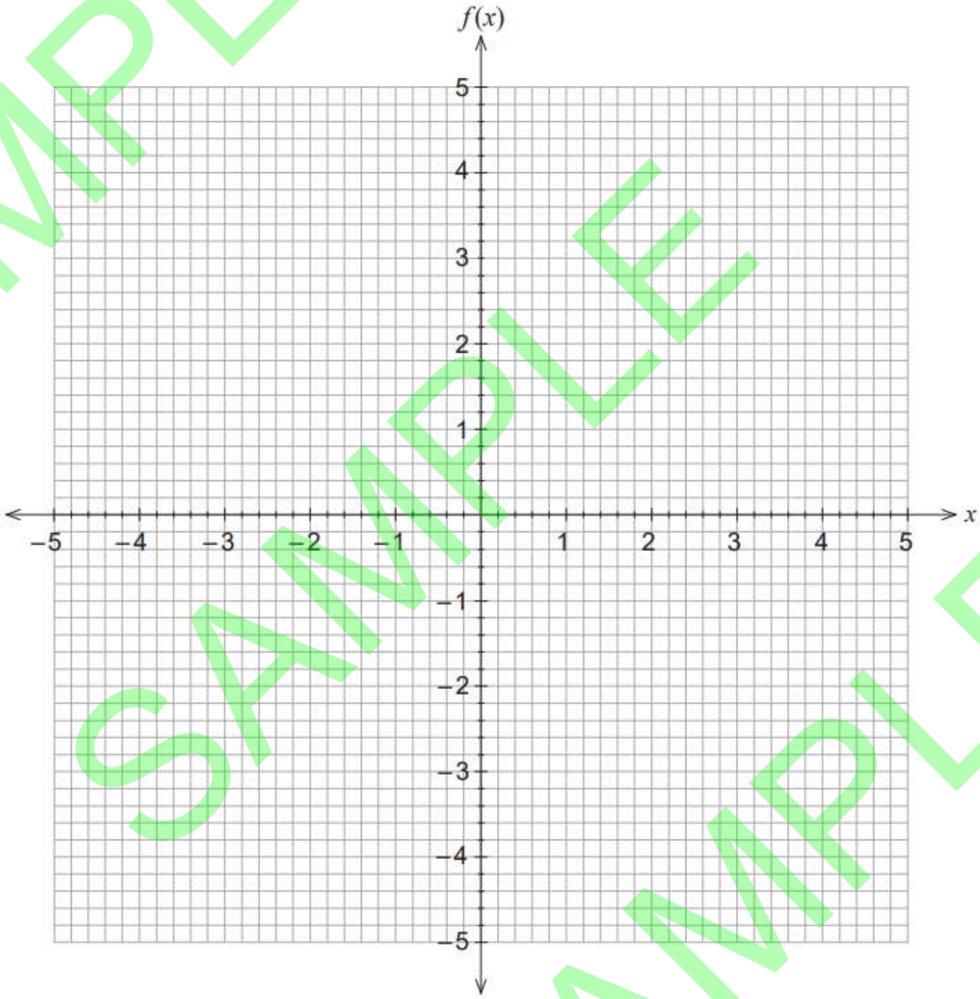
(c) Determine, to the nearest centimetre, the value of x at which the height of the building is maximum and state this maximum height. (2 marks)

(d) An adventure company allows tourists to climb from the ground on the left of the building, then along the outside of the building to the top. The company installs a platform that allows climbers to rest on their way up to the top. The platform is located on the second half of the climb, at the point where it is the steepest. How high off the ground, to the nearest centimetre, is it positioned? (3 marks)

2021 Section 2 Question 12 **Further differen-tiation and app-lications**	Let $f(x) = x^2e^x$. (a) Show that $f'(x) = xe^x(x+2)$. (2 marks) (b) Use calculus to determine all the stationary points of $f(x)$ and determine their nature. (7 marks)

(c) Determine the coordinates of any points of inflection. (2 marks)

(d) Hence sketch the graph of $f(x)$, clearly indicating the location of all stationary points and points of inflection. (4 marks)



2021 Section 2 Question 16 Further differen-tiation and app-lications	An analyst was hired by a large company at the beginning of 2021 to develop a model to predict profit. At that time, the company's profit was \$4 million. The model developed by the analyst was: $P(x) = \dfrac{20\ln(x+a)}{x+5}$, where $P(x)$ is the profit in millions of dollars after x weeks and a is a constant. (a) Show that $a = e$. (2 marks) (b) What does the model predict the profit will be after five weeks? (1 mark) (c) Showing use of the quotient rule, determine an equation that, when solved, will give the time when the model predicts the profit will be maximised. (3 marks) (d) What is this maximum profit and during which week will it occur? (2 marks)

(e) According to the model, during which week will the company's profit fall below its value at the beginning of 2021? (1 mark)

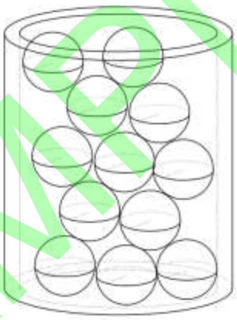
The model proved accurate and after 10 weeks the company implemented some changes. From this time the analyst used a new model to predict the profit:

$N(y) = 2e^{b(10+y)}$,
where $N(y)$ is the profit in millions of dollars y weeks from this point in time and b is a constant.

(f) The company is projecting its profit to exceed $5 million. During which week does the new model suggest this will happen? (3 marks)

2020 Section 2 Question 15 **Further differen-tiation and app-lications**	A chef needs to use an oven to boil 100 mL of water in five minutes for a new experimental recipe. The temperature of the water must reach 100 °C in order to boil. The temperature, T, of 100 mL of water t minutes after being placed in an oven set to T0 °C can be modelled by the equation below. $$T(t) = T_0 - 175e^{-0.07t}$$ In a preliminary experiment, the chef placed a 100 mL bowl of water into an oven that had been heated to $T_0 = 200$ °C. (a) What is the temperature of the water at the moment it is placed into the oven? (1 mark) (b) What is the temperature of the water five minutes after being placed in the oven? (1 mark) (c) What change could be made to the temperature at which the oven is set in order to achieve the five-minute boiling requirement? (2 marks) Assume that T_0 is still 200 °C. (d) Determine the rate of increase in temperature of the water five minutes after being placed in the oven. Give your answer rounded to two decimal places. (2 marks)

	(e) Explain what happens to the rate of change in the temperature of the water as time increases and how this relates to the temperature of the water. (3 marks)

2019 Section 2 Question 16 **Further differen-tiation and app-lications**	A cylindrical glass vase is filled with 20 spherical Christmas decorations as shown below (not all the decorations are visible). All the decorations have a diameter of one-third the internal diameter of the vase and they are completely contained within the vase. For design purposes the sum of the internal diameter of the base of the vase and the vase's internal height is to be 42 cm.  (a) Show that the volume of unused space in the vase, V, can be expressed as a function of the internal radius of the vase, r, and is given below as (3 marks) $$V(r) = 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$$

	(b) Use calculus to determine the dimensions of the vase that will maximise the unused space in it. Give your answers rounded to the nearest millimetre. (4 marks) (c) Can more than 20 of the spherical decorations fit inside the vase in part (b)? Use calculations to verify your answer. (3 marks)

2023 Section 1 Question 1

Further differentiation and applications

(a) Consider the function $f(x)=x^3e^{2x}$.

(i) Differentiate $f(x)$. (2 marks)

Solution
$f'(x)=\frac{d}{dx}(x^3)e^{2x}+x^3\frac{d}{dx}(e^{2x})$ $=3x^2e^{2x}+2x^3e^{2x}$
Specific behaviours
✓ demonstrates use of the product rule   ✓ obtains correct derivative

(ii) Determine the value of x for any stationary points of $f(x)$. (3 marks)

Solution
Setting $f'(x)=0$ gives $0=3x^2e^{2x}+2x^3e^{2x}$ $\Rightarrow 0=x^2e^{2x}(3+2x)$ $\Rightarrow x=0,\ -\frac{3}{2}$
Specific behaviours
✓ sets $f'(x)=0$ ✓ solves to obtain stationary point at $x=0$ ✓ solves to obtain stationary point at $x=-\frac{3}{2}$

(b) Evaluate $\int_0^{\frac{\pi}{4}}\sin(2x+\pi)dx$. (3 marks)

Solution
$\int_0^{\frac{\pi}{4}}\sin(2x+\pi)\,dx=\left[-\frac{\cos(2x+\pi)}{2}\right]_0^{\frac{\pi}{4}}$ $=-\frac{\cos\left(\frac{3\pi}{2}\right)}{2}-\left(-\frac{\cos(\pi)}{2}\right)$ $=0-\left(\frac{1}{2}\right)$ $=-\frac{1}{2}$
Specific behaviours
✓ antidifferentiates correctly   ✓ correctly substitutes integration bounds   ✓ evaluates to obtain correct answer

2023 Section 1 Question 5 Further differen-tiation and app-lications	The table below contains values of the polynomial function $f(x)$, its first and second derivatives, and the function $F(x)=\int_0^x f(t)\,dt$ for $x=0, 1, 2, 3, 4, 5, 6$.

$f(x)$ has no stationary points at non-integer values of x, and the letters a, b, c, d and e represent unspecified constants.

	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$f(x)$	a	b	4	c	0	d	e
$f'(x)$	16	0	–4	–2	0	–4	–20
$f''(x)$	–24	–9	0	3	0	–9	–24
$F(x)$	0	4.7	10.4	12.6	12.8	12.5	7.2

(a) Evaluate $\frac{d}{dx}\left(f(x)^2\right)$ when $x=2$.

(2 marks)

Solution

By the chain rule

$$\frac{d}{dx}\left(f(x)^2\right)=2f(x)f'(x)$$

Substituting $x=2$ gives

$$\left.\frac{d}{dx}\left(f(x)^2\right)\right|_{x=2}=2f(2)f'(2)$$
$$=2\times4\times-4$$
$$=-32$$

Or

By the product rule

$$\frac{d}{dx}\left(f(x)^2\right)=\frac{d}{dx}\left(f(x)f(x)\right)$$
$$=f(x)f'(x)+f(x)f'(x)$$

Substituting $x=2$ gives

$$\left.\frac{d}{dx}\left(f(x)^2\right)\right|_{x=2}=f(2)f'(2)+f(2)f'(2)$$
$$=4\times-4+4\times-4$$
$$=-16-16$$
$$=-32$$

Specific behaviours

✓ correctly applies the chain rule or product rule
✓ calculates correct derivative

(b) Evaluate $\int_2^4 (f(x)+2)\,dx$. (3 marks)

Solution
$\int_2^4 (f(x)+2)\,dx = \int_2^4 f(x)\,dx + \int_2^4 2\,dx$ $= F(4) - F(2) + [2x]_2^4$ $= 12.8 - 10.4 + (8-4)$ $= 6.4$
Specific behaviours
✓ correctly applies linearity of definite integrals   ✓ correctly applies fundamental theorem to first integral   ✓ correctly evaluates $\int_2^4 2\,dx$ and obtains correct answer

(c) Evaluate $\frac{d}{dx}\int_2^x f(t)\,dt$ when $x=2$. (2 marks)

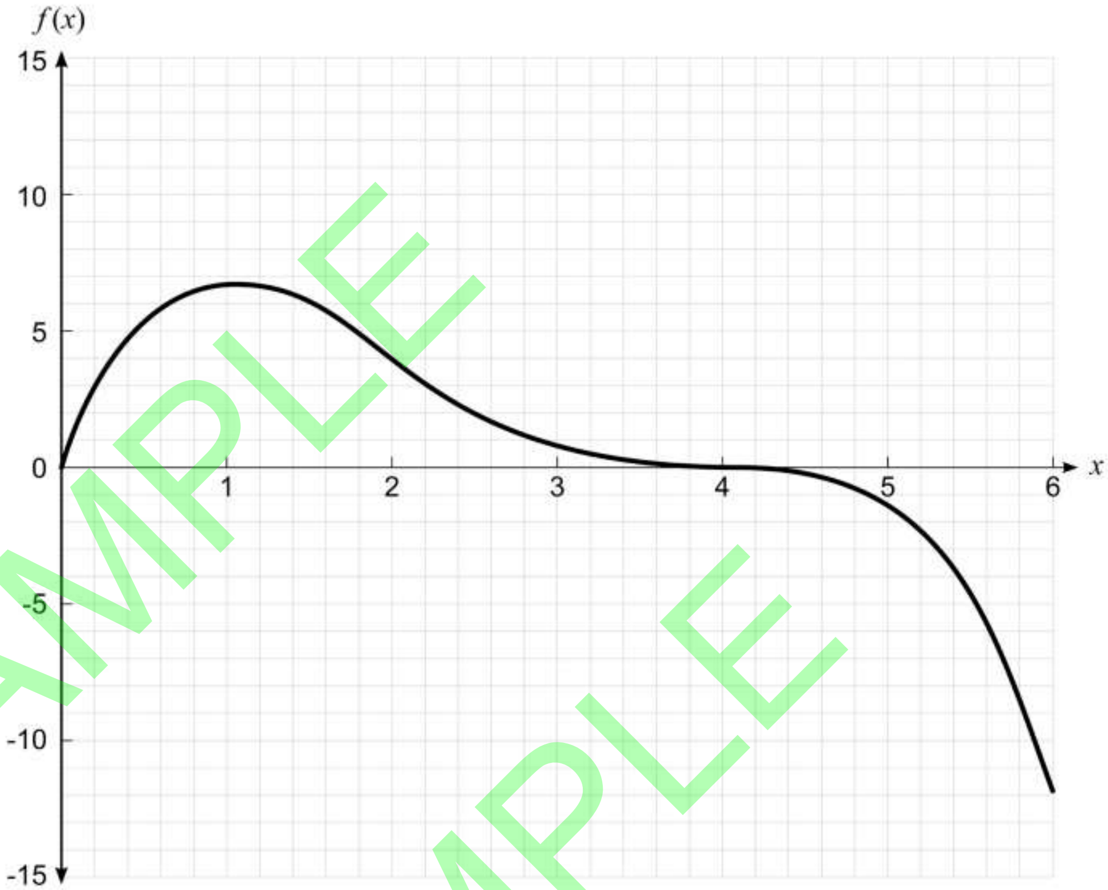
Solution
By the fundamental theorem of calculus $\frac{d}{dx}\int_2^x f(t)\,dt = f(x)$ Substituting $x=2$ gives $\left.\frac{d}{dx}\int_2^x f(t)\,dt\right\rvert_{x=2} = f(2)$ $= 4$
Specific behaviours
✓ correctly applies the fundamental theorem of calculus   ✓ correctly evaluates for $x=2$

(d) Determine the x-coordinate of any stationary points and whether they are local maxima, local minima or inflection points. Justify your answer. (3 marks)

Solution
Stationary points are when $f'(x)=0$, hence $x=1$ and $x=4$ are the stationary points.   Since $f''(1)=-9$ it follows that $x=1$ is a local maximum by the second derivative test.   Since $f''(4)=0$ the second derivative test fails. Since the gradient of f is negative on both sides of $x=4$ (i.e. $f'(3)=-2<0$, $f'(5)=-4<0$, and there are no stationary points for non-integer values of x) it follows that $x=4$ is a horizontal point of inflection.
Specific behaviours
✓ correctly identifies the coordinates $x=1$ and $x=4$ as stationary points   ✓ concludes that $x=1$ is a local maximum with correct justification   ✓ concludes that $x=4$ is an inflection point with correct justification

	(e) Sketch a possible graph of $f(x)$ for $0 \le x \le 6$ on the axes below. (3 marks)

Solution



Specific behaviours

✓ graph passes through the points $(2,4)$ and $(4,0)$
✓ graph includes local maximum at $x=1$ and a horizontal point of inflection at $x=4$
✓ concavity of graph is correct including inflection point at $x=2$

2023 Section 2 Question 6

Further differentiation and applications

A beekeeper is starting a new colony of bees. The population B of bees, in thousands, is given by

$$B(t) = 4e^{1.4t}$$

where *t* is the number of years since the establishment of the colony.

(a) Determine the initial population of the bee colony. (1 mark)

Solution
$B = 4000$ bees
Specific behaviours
✓ correctly determines initial population

(b) Determine the increase in the population of the bee colony in the first six months. (2 marks)

Solution
$B(0.5) = 4e^{1.4(0.5)}$ $\approx 8.055...$ Population increase ≈ 8055 – 4000 = 4055
Specific behaviours
✓ correctly calculates $B(0.5)$ ✓ correctly calculates increase in bee population

(c) Determine the rate of population growth two years after the establishment of the colony. (2 marks)

Solution
$\frac{dB}{dt} = 5.6e^{1.4t}$ $\left.\frac{dB}{dt}\right\|_{t=2} = 5.6e^{1.4(2)}$ $= 92.09...$ Hence the rate of population growth two years after the establishment of the colony is approximately 92 000 bees per year.
Specific behaviours
✓ correctly differentiates population equation   ✓ correctly determines rate of population growth

(d) After how many years will the rate of population growth be 65 000 bees/year? (2 marks)

Solution
$65 = 5.6e^{1.4t}$ $t = 1.751...$ Hence the rate of population growth will be 65 000 bees/year after 1.75 years
Specific behaviours
✓ correctly substitutes $B = 65$ into growth rate equation   ✓ correctly determines number of years

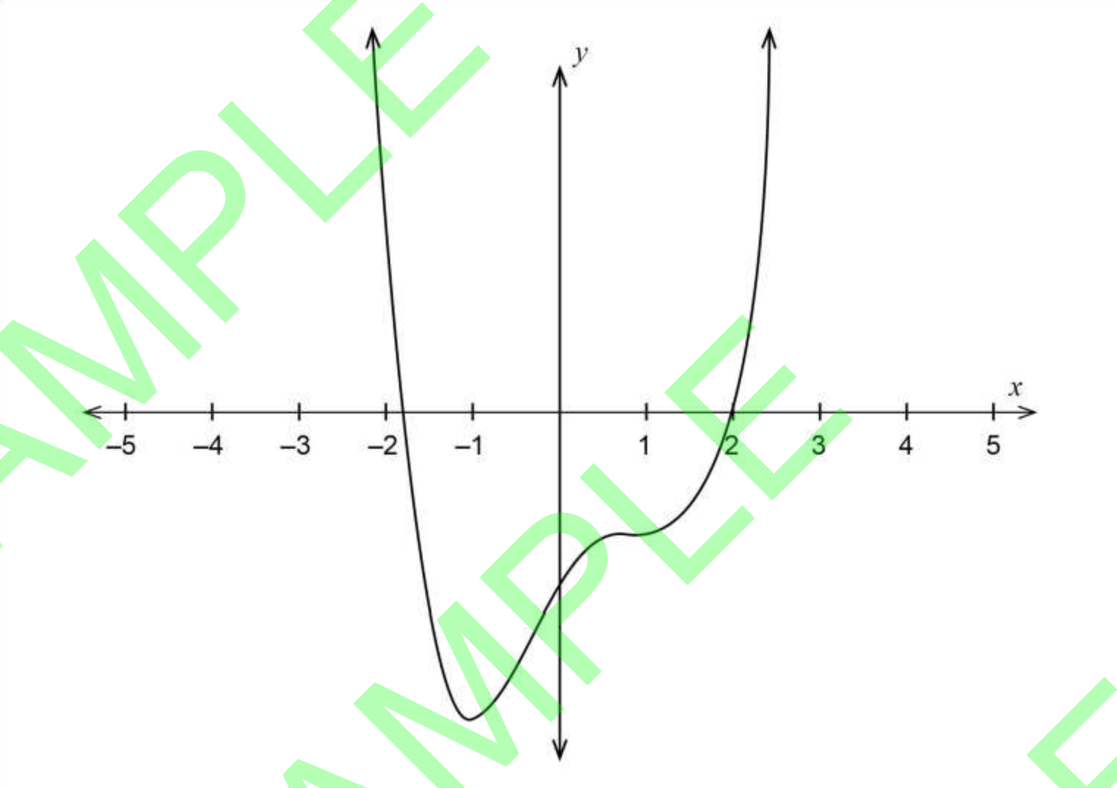
After three years, the beekeeper notices that the number of bees begins to decline. The declining population, b, in thousands, has the form $b(t) = Ae^{rt}$ where t is the number of years since the start of the decline.

	(e) Determine *A* and *r* if one year after the start of the decline the bee population is 100 000. (4 marks)

Solution
$B(3) = 4e^{1.4(3)}$ $= 266.745...$ $\therefore A = 267$ $100 = 267e^{r(1)}$ $\therefore r = -0.982...$ $\therefore b(t) = 267e^{-0.98t}$
Specific behaviours
✓ correctly determines population after 3 years of growth ✓ identifies value of A ✓ correctly substitutes value of A and $b = 100$ into equation to solve for r ✓ correctly solves for r

2022 Section 1 Question 5 **Further differen-tiation and app-lications**	A continuous function, f, satisfies the following conditions: • $f(2)=0$ • f has exactly 2 stationary points • $f'(-1)=0$ and $f'(1)=0$ • $f''(-1)=4$ • $f'(2)>0$. Sketch the function on the axes below. (5 marks)

Solution



Specific behaviours

✓ locates an intercept at $x = 2$
✓ indicates a positive gradient for $x > 1$
✓ indicates a local minimum at $x = -1$
✓ locates a horizontal point of inflection at $x = 1$
✓ sketches a continuous curve with correct shape

2021 Section 1 Question 1 Further differen-tiation and app-lications	

2021 Section 1 Question 1

Further differentiation and applications

(a) Differentiate $\frac{3x+1}{x^3}$ and simplify your answer.

(3 marks)

Solution
$\frac{d}{dx}\left(\frac{3x+1}{x^3}\right)=\frac{x^3(3)-3x^2(3x+1)}{x^6}$ $=\frac{3x^3-9x^3-3x^2}{x^6}$ $=\frac{-6x-3}{x^4}$
Specific behaviours
✓ recognises the need for the quotient rule ✓ correctly differentiate the expression ✓ simplifies the result

(b) Let $f'(x)=x\ln(2x)$. Determine a simplified expression for the rate of change of $f'(x)$.

(3 marks)

Solution
$f''(x)=x\times\frac{2}{2x}+1\times\ln(2x)$ $=1+\ln(2x)$
Specific behaviours
✓ identifies the rate of change as $f''(x)$ ✓ correctly determines $f''(x)$ ✓ simplifies the expression for $f''(x)$

(c) Given that $g'(x)=4e^{2x}$ and $g(1)=0$, determine $g(5)$. (3 marks)

Solution
$g(x)=2e^{2x}+c$ Since $g(1)=0$, $0=2e^2+c$ $\therefore c=-2e^2$ $\therefore g(x)=2e^{2x}-2e^2$ $\therefore g(5)=2e^{10}-2e^2$
Specific behaviours
✓ states an expression for $g(x)$, including the constant of integration ✓ correctly determines the constant ✓ correctly determines $g(5)$ as the final solution

2020 Section 1 Question 2 Further differen-tiation and app-lications	If $h(x) = \dfrac{e^{-x}}{\cos x}$, then evaluate $h'(\pi)$. (4 marks)

Solution

$$h'(x) = \frac{-e^{-x}(\cos x) - e^{-x} \times (-\sin x)}{(\cos x)^2}$$

$$h'(\pi) = \frac{-e^{-\pi}(\cos \pi) - e^{-\pi} \times (-\sin \pi)}{(\cos \pi)^2}$$

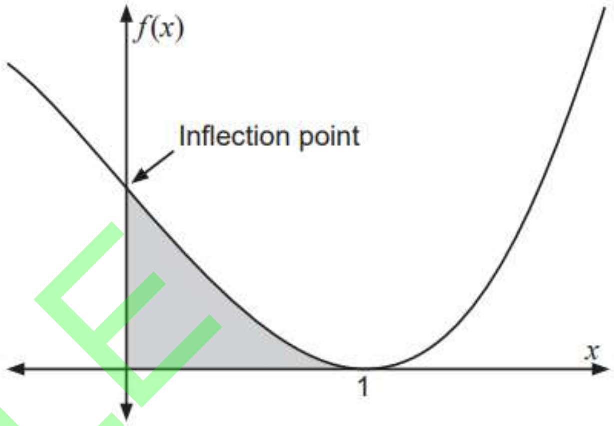
$$= \frac{-e^{-\pi} \times -1 - e^{-\pi} \times 0}{(-1)^2}$$

$$= e^{-\pi}$$

Specific behaviours

- ✓ demonstrates use of the quotient rule
- ✓ differentiates $\cos x$ and e^{-x} correctly
- ✓ substitutes $x = \pi$ correctly
- ✓ evaluates correctly

2020 Section 1 Question 3 Further differen-tiation and app-lications	The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is shown below. A turning point is located at (1, 0) and the shaded region shown on the graph has an area of $\frac{3}{2}$ units².



Use the above information to determine the values of a, b, c and d. (7 marks)

Solution

Firstly, note that

$$f'(x) = 3ax^2 + 2bx + c$$

and

$$f''(x) = 6ax + 2b$$

Given that there is an inflection point at $x = 0$ it follows that $f''(0) = 0$. Hence

$$0 = 2b$$
$$b = 0$$

Given that there is a turning point at $x = 1$ it follows that $f'(1) = 0$. Hence

$$0 = 3a + c$$
$$c = -3a$$

Given that there is an x-intercept at $x = 1$ it follows that $f(1) = 0$. Hence

$$0 = a - 3a + d$$
$$d = 2a$$

Finally, given that the area of the shaded region is $\frac{3}{2}$ it follows that $\int_0^1 f(x)dx = \frac{3}{2}$.

$$a\int_0^1 x^3 - 3x + 2\, dx = \frac{3}{2}$$

$$a\left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1 = \frac{3}{2}$$

$$\frac{3a}{4} = \frac{3}{2}$$

$$a = 2$$

Hence $a = 2$, $b = 0$, $c = -6$ and $d = 4$.

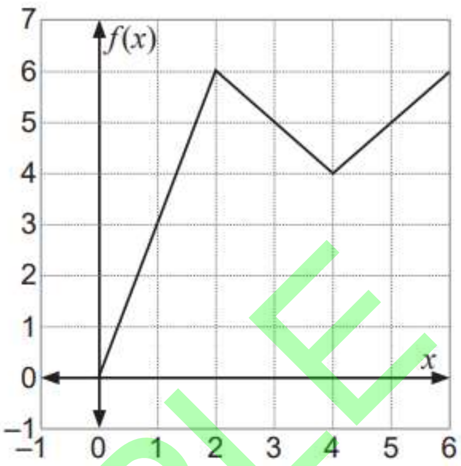
Specific behaviours

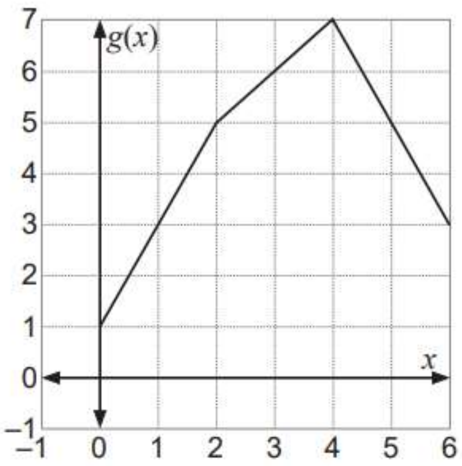
✓ states the first and second derivatives of f
✓ recognises that $f''(0) = 0$ and hence that $b = 0$
✓ recognises that $f'(1) = 0$ and hence that $c = -3a$
✓ recognises that $f(1) = 0$ and hence that $d = 2a$ (or $d = -a - c$)
✓ recognises that $\int_0^1 f(x)dx = \frac{3}{2}$
✓ evaluates definite integral to determine that $a = 2$
✓ solves for the values of c and d

2020 Section 1 Question 5

Further differentiation and applications

The graphs of the functions f and g are displayed below.





(a) Evaluate the derivative of $f(x)$ at $x = 3$. (1 mark)

Solution
$f'(3) = -1$
Specific behaviours
✓ states correct derivative

(b) Evaluate the derivative of $f(x)g(x)$ at $x = 5$. (2 marks)

Solution
$(fg)'(5) = f'(5)g(5) + g'(5)f(5)$ $= (1)(5) + (-2)(5)$ $= -5$
Specific behaviours
✓ uses product rule to express derivative   ✓ states correct derivative

(c) Evaluate the derivative of $f(g(x))$ at $x = 1$. (2 marks)

Solution
$f(g(x))'\|_1 = f'(g(1))g'(1)$ $= f'(3)2$ $= (-1)2$ $= -2$
Specific behaviours
✓ uses chain rule to express derivative   ✓ states correct derivative

2019 Section 1 Question 1 Further differen-tiation and app-lications	Consider the derivative function $f'(x) = xe^{x^2}$. (a) Determine $f''(1)$. (2 marks)

Solution
$f''(x) = x(2xe^{x^2}) + e^{x^2}$ $f''(1) = 3e$
Specific behaviours
✓ uses the chain rule to correctly differentiate $f'(x)$ ✓ evaluates at $x = 1$

(b) Explain the meaning of your answer to part (a). (1 mark)

Solution
$f''(1)$ is the rate of change of the derivative function when $x = 1$
Specific behaviours
✓ states it is the rate of change of the derivative AND includes when $x = 1$

(c) Determine the expression for $y = f(x)$, given that it intersects the y-axis at the point (0,2). (3 marks)

Solution
$\int xe^{x^2}\,dx = \frac{e^{x^2}}{2} + C$ $2 = \frac{e^0}{2} + C$ $C = \frac{3}{2}$ $\therefore f(x) = \frac{e^{x^2}}{2} + \frac{3}{2}$
Specific behaviours
✓ correctly integrates $f'(x)$ ✓ substitutes (0,2) into an expression involving C ✓ determines C and states the final expression for $y = f(x)$

2019 Section 1 Question 2 **Further differen-tiation and app-lications**	The values of the functions $g(x)$ and $h(x)$, and their derivatives $g'(x)$ and $h'(x)$ are provided in the table below for x = 1, x = 2 and x = 3.

	$x=1$	$x=2$	$x=3$
$g(x)$	3	5	–3
$h(x)$	2	–2	6
$g'(x)$	–4	1	4
$h'(x)$	0	–6	–5

(a) Evaluate the derivative of $\frac{g(x)}{h(x)}$ at $x = 3$. (2 marks)

Solution
$\left(\frac{g}{h}\right)'(3) = \frac{g'(3)h(3) - g(3)h'(3)}{h(3)^2}$ $= \frac{4(6)-(-3)(-5)}{6^2}$ $= \frac{1}{4}$
Specific behaviours
✓ expresses the derivative using the quotient rule ✓ evaluates the derivative

(b) Evaluate the derivative of $h(g(x))$ at $x = 1$. (2 marks)

Solution
$h(g(1))' = h'(g(1))g'(1)$ $= h'(3)(-4)$ $= (-5)(-4)$ $= 20$
Specific behaviours
✓ expresses the derivative using the chain rule ✓ evaluates the derivative

(c) If $h''(1) = -1$, describe with justification, what the graph of $h(x)$ looks like at this point. (2 marks)

Solution
Since $h'(1) = 0$ there is a stationary point at $x = 1$ Since 2nd derivative is negative the point is a maximum
Specific behaviours
✓ justifies stationary point ✓ determines point is a maximum

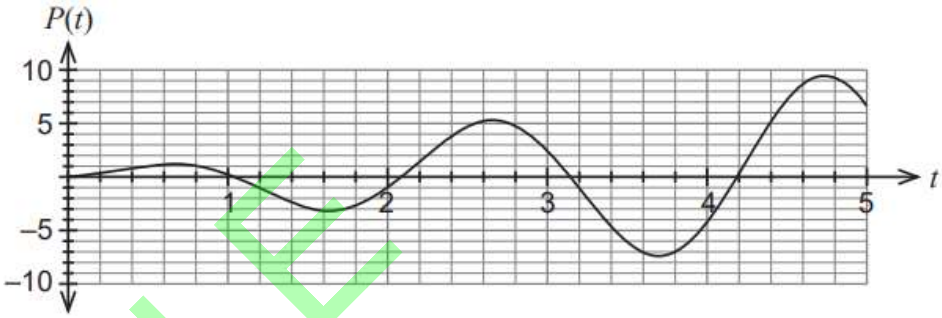
2019 Section 1 Question 7

Further differen-tiation and app-lications

A company's profit, in millions of dollars, over a five-year period can be modelled by the function:

$$P(t) = 2t\sin(3t) \quad 0 \le t \le 5 \text{ where } t \text{ is measured in years.}$$

The graph of $P(t)$ is shown below.



(a) Differentiate $P(t)$ to determine the marginal profit function, $P'(t)$. (2 marks)

Solution
$P'(t) = 2\sin(3t) + 6t\cos(3t) \quad \$/\,year$
Specific behaviours
✓ uses the product rule ✓ determines correct derivative

(b) Calculate the rate of change of the marginal profit function when $t = \frac{\pi}{18}$ years. (4 marks)

Solution
$P'(t) = 2\sin(3t) + 6t\cos(3t)$ $P''(t) = 6\cos(3t) + 6\cos(3t) - 18t\sin(3t)$ $= 12\cos(3t) - 18t\sin(3t)$ $P''\left(\frac{\pi}{18}\right) = 12\cos\left(\frac{\pi}{6}\right) - \pi\sin\left(\frac{\pi}{6}\right)$ $= 6\sqrt{3} - \frac{\pi}{2} \quad \$/\,year^2$
Specific behaviours
✓ uses product rule ✓ determines correct expression for the second derivative ✓ substitutes $\frac{\pi}{18}$ into second derivative expression ✓ calculates the exact rate of change

(c) Use the increments formula at $t=\frac{7\pi}{6}$ to estimate the change in profit for a one month change in time. (3 marks)

Solution
$P'\left(\frac{7\pi}{6}\right)=2\sin\left(\frac{7\pi}{2}\right)+6\left(\frac{7\pi}{6}\right)\cos\left(\frac{7\pi}{2}\right)$ $=-2$ $\delta P \approx \frac{dP}{dt}\times \delta t$ $\approx -2\times\frac{1}{12}$ $\approx -\frac{1}{6}$ The approximate change in profit is $-\frac{1}{6}$ million dollars.   [$\frac{1}{6}$ million dollar loss]
Specific behaviours
✓ calculates the correct value of P' when $t=\frac{7\pi}{6}$ ✓ states an appropriate approximation for the change in profit using the increments formula   ✓ substitutes and evaluates the change including units

2022 Section 2 Question 15

Further differentiation and applications

An object moves from the point $(0, 0)$ along the curve $y = \sqrt{3}\sin(x)$. The distance, D, travelled along the curve is given by

$$D(t) = \int_0^{\pi t} \sqrt{1 + 3\cos^2(x)}\, dx$$

where D is measured in metres and t is measured in seconds.

(a) Determine the speed $s = \frac{dD}{dt}$ of the object when $t = 1$. (3 marks)

Solution
Applying the fundamental theorem of calculus $$\frac{dD}{dt} = \pi\sqrt{1 + 3\cos^2(\pi t)}$$ When $t = 1$ $$\frac{dD}{dt}(1) = \pi\sqrt{1 + 3\cos^2(\pi)} = 6.283 \text{ m/s}$$
Specific behaviours
✓ applies fundamental theorem of calculus ✓ correctly applies chain rule ✓ evaluates $\frac{dD}{dt}$ at $t = 1$

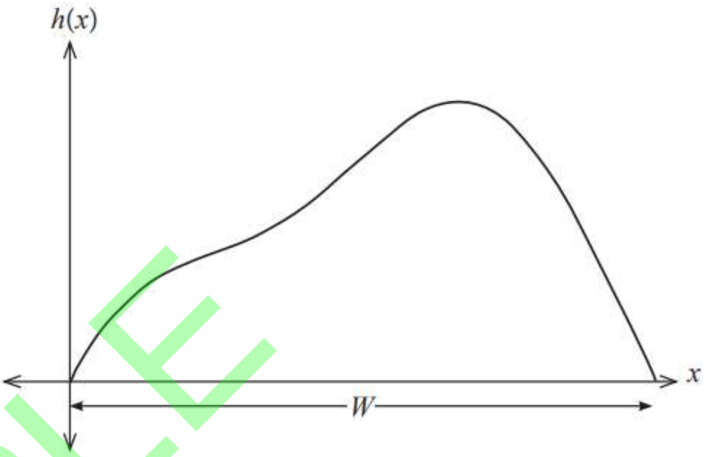
(b) Use the increments formula to estimate the distance travelled by the object between $t = 1$ and $t = 1.02$. (2 marks)

Solution
The increment in t is $\delta t = 1.02 - 1 = 0.02$. Hence $$\delta D \approx \frac{dD}{dt}(1) \times \delta t$$ $$= 6.283 \times 0.02$$ $$= 0.126 \text{ m}$$
Specific behaviours
✓ determines correct increment for t ✓ applies increments formula to obtain correct answer

2021 Section 2 Question 9

Further differentiation and applications

The Interesting Architecture company has designed a building with a constant cross-section shown in the figure below.



With reference to the figure, the height h(x) of the building at a point x along its width is given by

$$h(x) = 4\sin\left(x - \frac{3\pi}{2}\right) - x^2 + 3\pi x - 4$$, where h and $0 \le x \le W$ are measured in metres.

(a) Determine the width W of the building to the nearest centimetre. (2 marks)

Solution
$h(W) = 0$ $W = 8.64$ m (or 864 cm).
Specific behaviours
✓ sets $h(W) = 0$ ✓ solves for W

(b) Determine $h'(x)$. (1 mark)

Solution
$h'(x) = 4\cos\left(x - \frac{3\pi}{2}\right) - 2x + 3\pi$
Specific behaviours
✓ differentiates $h(x)$

(c) Determine, to the nearest centimetre, the value of x at which the height of the building is maximum and state this maximum height. (2 marks)

Solution
Setting $h'(x) = 0$ gives $x = 5.74$ m. Hence the maximum height $h(5.74) = 20.57$ m.
Specific behaviours
✓ sets $h'(x) = 0$ and solves it to obtain the value of x for maximum height ✓ states the maximum height

	(d) An adventure company allows tourists to climb from the ground on the left of the building, then along the outside of the building to the top. The company installs a platform that allows climbers to rest on their way up to the top. The platform is located on the second half of the climb, at the point where it is the steepest. How high off the ground, to the nearest centimetre, is it positioned? (3 marks)

Solution
The climb is steepest when the gradient is a maximum. The second derivative is given by $$h''(x) = -4\sin\left(x - \frac{3\pi}{2}\right) - 2$$ $h''(x) = 0$ at the points where $x = 2.09$ and $x = 4.19$ When $x = 4.19$ (second half of the climb), the platform is 15.93 metres off the ground.
Specific behaviours
✓ obtains the second derivative ✓ equates $h''(x)$ to zero and determines both x values ✓ identifies correct point and determines the height of the platform off the ground

2021 Section 2 Question 12 **Further differen-tiation and app-lications**	Let $f(x) = x^2e^x$. (a) Show that $f'(x) = xe^x(x+2)$. (2 marks)

Solution
$f'(x) = 2xe^x + x^2e^x = xe^x(x+2)$
Specific behaviours
✓ differentiates using product rule ✓ factorises correctly

(b) Use calculus to determine all the stationary points of $f(x)$ and determine their nature. (7 marks)

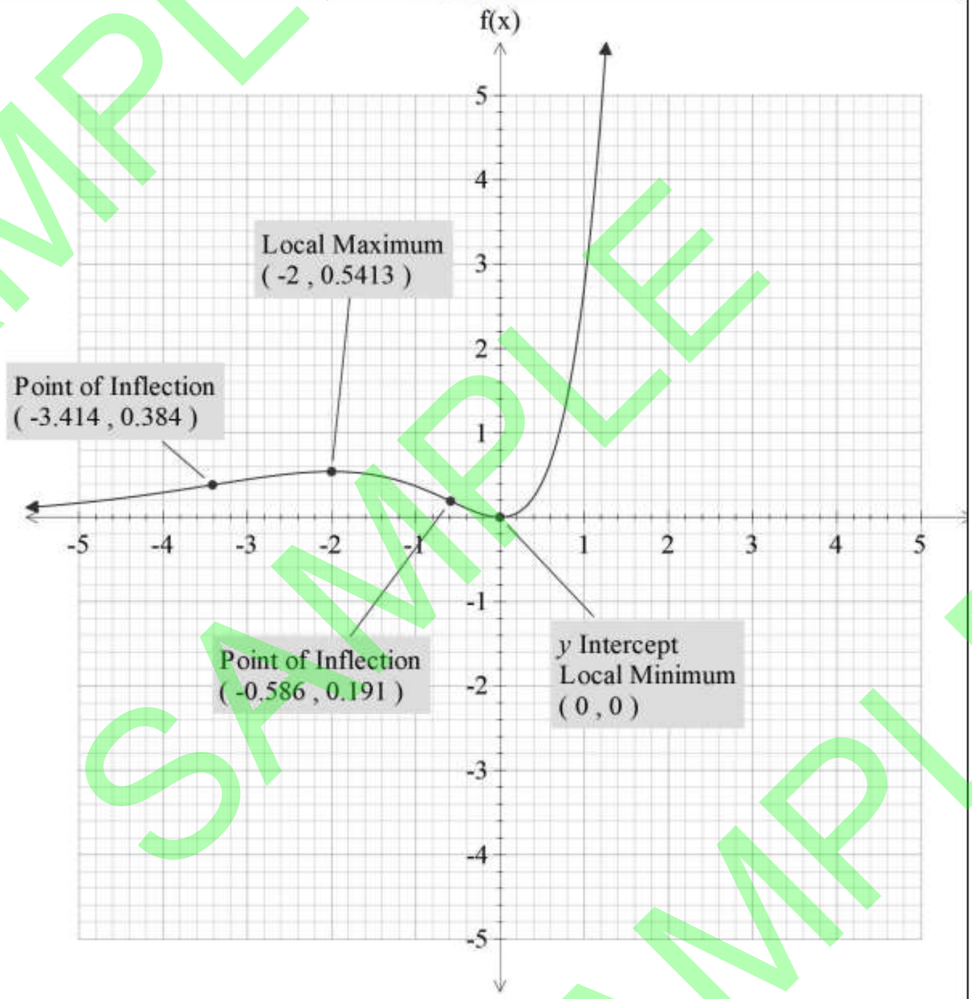
Solution
$f'(x) = 0$ $\Rightarrow xe^x(x+2) = 0$ $\Rightarrow x = 0, -2$ $f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2)$ $f''(0) = 2 > 0 \Rightarrow$ Local minima $f(0) = 0$ $f''(-2) = -2e^{-2} < 0 \Rightarrow$ Local maxima $f(-2) = 4e^{-2} \approx 0.54$
Specific behaviours
✓ sets first derivative equal to 0 ✓ obtains the two solutions ✓ finds the second derivative ✓ evaluates the second derivative at $x = 0$ and concludes it is a local minima ✓ obtains y coordinate at minima ✓ evaluates the second derivative at $x = -2$ and concludes it is a local maxima ✓ obtains y coordinate at maxima

	(c) Determine the coordinates of any points of inflection. (2 marks)

Solution
$f''(x) = 0$ $\Rightarrow e^x(2+4x+x^2) = 0$ $\Rightarrow x = -2 \pm \sqrt{2} \approx -3.4, -0.59$ $\Rightarrow$ points are $(-3.4, 0.38)$ and $(-0.59, 0.19)$
Specific behaviours
✓ sets second derivative equal to 0 ✓ obtains the two points

(d) Hence sketch the graph of $f(x)$, clearly indicating the location of all stationary points and points of inflection. (4 marks)

Solution



Specific behaviours
✓ indicates local minima
✓ indicates local maxima
✓ indicates points of inflections
✓ overall shape

2021 Section 2 Question 16

Further differen-tiation and app-lications

An analyst was hired by a large company at the beginning of 2021 to develop a model to predict profit. At that time, the company's profit was $4 million. The model developed by the analyst was:

$$P(x) = \frac{20\ln(x+a)}{x+5},$$

where $P(x)$ is the profit in millions of dollars after x weeks and a is a constant.

(a) Show that $a = e$. (2 marks)

Solution
Since $P(0) = 4$, we require $\ln(a) = 1$, giving $a = e$.
Specific behaviours
✓ recognises $P(0) = 4$ ✓ obtains $\ln(a) = 1$

(b) What does the model predict the profit will be after five weeks? (1 mark)

Solution
$P(5) = \frac{20\ln(5+e)}{5+5}$ $= 4.087$ (3 d.p.) Profit will be approximately $4 087 000
Specific behaviours
✓ states the profit

(c) Showing use of the quotient rule, determine an equation that, when solved, will give the time when the model predicts the profit will be maximised. (3 marks)

Solution
$P'(x) = \dfrac{(x+5)\frac{20}{(x+e)} - 20\ln(x+e)}{(x+5)^2}$ For maximum profit we require : $\dfrac{(x+5)\frac{20}{(x+e)} - 20\ln(x+e)}{(x+5)^2} = 0$
Specific behaviours
✓ demonstrates use of the quotient rule ✓ writes correct expression for $P'(x)$ ✓ equates $P'(x)$ to zero

(d) What is this maximum profit and during which week will it occur? (2 marks)

Solution
Maximum profit is approximately $4 436 000 This occurs when $x \approx 1.79$, so during the second week.
Specific behaviours
✓ states maximum profit ✓ states it occurs in the second week

(e) According to the model, during which week will the company's profit fall below its value at the beginning of 2021? (1 mark)

Solution
$4=\dfrac{20\ln(x+e)}{x+5}$ $x=0$ or 5.581 The model predicts during the 6th week
Specific behaviours
✓ determines when the profit falls below the 2021 value

The model proved accurate and after 10 weeks the company implemented some changes. From this time the analyst used a new model to predict the profit:

$N(y)=2e^{b(10+y)}$,
where $N(y)$ is the profit in millions of dollars y weeks from this point in time and b is a constant.

(f) The company is projecting its profit to exceed \$5 million. During which week does the new model suggest this will happen? (3 marks)

Solution
$P(10)=3.39072=N(0)$ $3.39072=2e^{10b}$ $b=0.05279$ $5=2e^{b(10+y)}$ $y\approx 7.36$ Profit should exceed \$5 million during the 8th week after the changes.
Specific behaviours
✓ determines $P(10)$ ✓ determines the value of the constant b ✓ determines the week when the profit exceeds \$5 million

2020 Section 2 Question 15

Further differentiation and applications

A chef needs to use an oven to boil 100 mL of water in five minutes for a new experimental recipe. The temperature of the water must reach 100 °C in order to boil. The temperature, T, of 100 mL of water t minutes after being placed in an oven set to T0 °C can be modelled by the equation below.

$$T(t) = T_0 - 175e^{-0.07t}$$

In a preliminary experiment, the chef placed a 100 mL bowl of water into an oven that had been heated to $T_0 = 200$ °C.

(a) What is the temperature of the water at the moment it is placed into the oven? (1 mark)

Solution
$T(0) = 200 - 175e^{-0.07(0)}$ $= 25$ °C
Specific behaviours
✓ states correct temperature

(b) What is the temperature of the water five minutes after being placed in the oven? (1 mark)

Solution
$T(5) = 200 - 175e^{-0.07(5)}$ $= 76.68$ °C
Specific behaviours
✓ states correct temperature

(c) What change could be made to the temperature at which the oven is set in order to achieve the five-minute boiling requirement? (2 marks)

Solution
$100 = T_0 - 175e^{-0.07(5)}$ $T_0 = 100 + 175e^{-0.07(5)}$ ≈ 223 °C
Specific behaviours
✓ states correct equation to be solved   ✓ solves for T_0, giving changed temperature

Assume that T_0 is still 200 °C.

(d) Determine the rate of increase in temperature of the water five minutes after being placed in the oven. Give your answer rounded to two decimal places. (2 marks)

Solution
$T'(t) = 12.25e^{-0.07t}$ $T'(5) = 12.25e^{-0.07(5)}$ $= 8.63$ °C/*min*
Specific behaviours
✓ states correct derivative of T with respect to t ✓ calculates correct rate

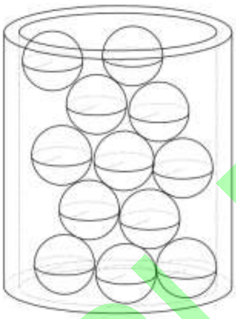
	(e) Explain what happens to the rate of change in the temperature of the water as time increases and how this relates to the temperature of the water. (3 marks)

Solution
As time increases, the rate of change in the temperature of the water $\rightarrow$ 0. The temperature of the water $\rightarrow$ the constant value of T_0.
Specific behaviours
✓ states that the rate of change in the temperature $\rightarrow 0$ ✓ states the water temperature approaches a constant ✓ states the water temperature approaches T_0

2019 Section 2 Question 16

Further differentiation and applications

A cylindrical glass vase is filled with 20 spherical Christmas decorations as shown below (not all the decorations are visible). All the decorations have a diameter of one-third the internal diameter of the vase and they are completely contained within the vase. For design purposes the sum of the internal diameter of the base of the vase and the vase's internal height is to be 42 cm.



(a) Show that the volume of unused space in the vase, V, can be expressed as a function of the internal radius of the vase, r, and is given below as (3 marks)

$$V(r) = 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$$

Solution
$2r + h = 42$ $h = 42 - 2r$ $V(r,h) = \pi r^2 h - 20\left(\frac{4}{3}\pi\left(\frac{r}{3}\right)^3\right)$ $V(r) = \pi r^2(42-2r) - \frac{80\pi}{81}r^3$ $= 2\pi\left(21r^2 - r^3 - \frac{40}{81}r^3\right)$ $= 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$
Specific behaviours
✓ determines an expression for h in terms of r ✓ states an expression for the volume of unused space in terms of h and r ✓ clearly shows that the expression for h in terms of r can substitute into V and simplifies to determine required result

(b) Use calculus to determine the dimensions of the vase that will maximise the unused space in it. Give your answers rounded to the nearest millimetre. (4 marks)

Solution
$V'(r) = 2\pi\left(42r - \frac{363r^2}{81}\right)$ $0 = 42r - \frac{363r^2}{81}$ $0 = r(42 - \frac{363r}{81})$ $r = 0$ or $\frac{1134}{121}\{=9.372(3dp)\}$ $V''(r) = 2\pi\left(42 - \frac{726r}{81}\right)$ $V''(9.372) = -ve\{=-84\pi\} \Rightarrow \text{max}$ Dimensions are: $r = 9.4$ cm and $h = 23.3$ cm
Specific behaviours
✓ determines first derivative of $V(r)$ ✓ equates to zero and determines 0 and 9.4 are solutions ✓ clearly shows the use of the second derivative or sign test to show that $r = 9.4$ is a maximum ✓ states the dimensions of the vase that maximise the unused space rounded to the nearest mm

(c) Can more than 20 of the spherical decorations fit inside the vase in part (b)? Use calculations to verify your answer. (3 marks)

Solution
$V(9.4) = 3863.1 \text{ cm}^3$ $V(\text{decoration}) = 127.7 \text{ cm}^3$ There is likely space for more decorations, but it is not certain as it would depend on the way the balls were packed into the vase.
Specific behaviours
✓ states the volume of unused space and the volume of one decoration ✓ infers likely to fit more ✓ states the limitation of packing

Unit 3.2 – Integrals

Section 1

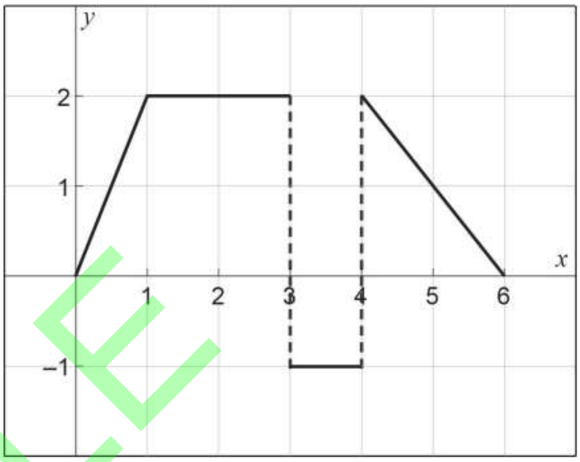
2022 Section 1 Question 1 **Integrals**	Consider the derivative function $f'(x) = \frac{4x}{x^2+3}$. (a) Determine the rate of change of $f'(x)$ when $x = 1$. (3 marks) (b) Determine $f(x)$ given that $f(1)$ = $\ln(32)$. (4 marks) (c) Determine $\frac{d}{dt}\int_t^3 f(x)\,dx$. (2 marks)

2022 Section 1 Question 2 **Integrals**	Consider the function $f(x)$ shown below.

2022
Section 1
Question 2

Integrals

Consider the function $f(x)$ shown below.



Evaluate the following integrals.

(a) $\int_0^6 f(x)\,dx$ (2 marks)

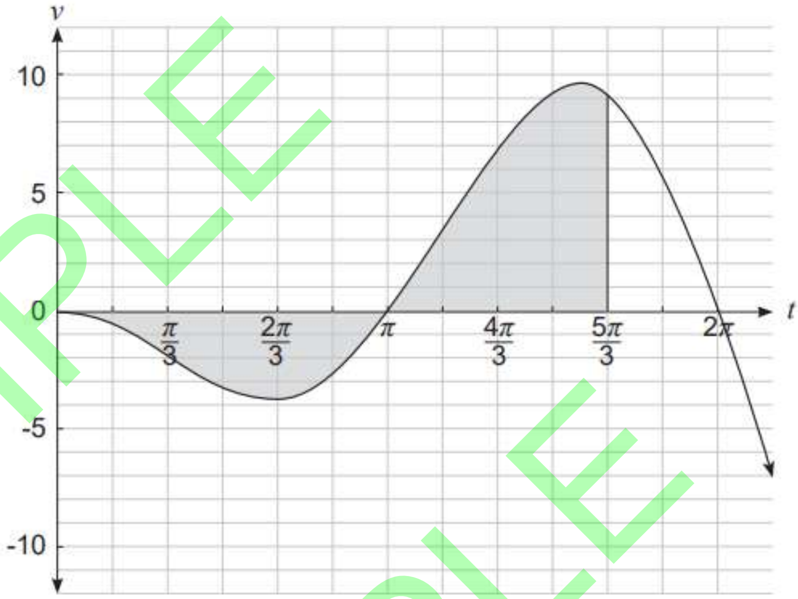
(b) $\int_0^4 f(x) - 2\,dx$ (2 marks)

(c) $\int_4^6 f'(x)\,dx$ (2 marks)

2021 Section 1 Question 4 Integrals	Determine the following: (a) $\int \left(2x^2 - x^3\right) dx$ (2 marks) (b) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3 - \cos(x)} dx$ (3 marks) (c) $\frac{d}{dy} \int_{-1}^{y} 3x^2 \cos(2x) \, dx$ (2 marks)

2021 Section 1 Question 5 **Integrals**	(a) Determine the area between the parabola $y = x2 - x + 3$ and the straight line $y = x + 3$. (4 marks) (b) The area between the parabola $y = x2 - x - 2$ and the straight line $y = x - 2$ is the same as the area determined in part (a). Explain why this is the case. (2 marks)

2019 Section 1 Question 5 **Integrals**	(a) Determine the area bound by the graph of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$. (3 marks) (b) Hence, determine the area bound by the graph of $f(x) = e^x$, the line $y = 2$ and the y-axis. (2 marks) (c) Determine the area bound by the graph of $f(x) = e^x$, the line $y = a$ and the y-axis, where a is a positive constant. (3 marks)

2023 Section 2 Question 8 **Integrals**	An oscillating mass has a velocity, v, given by $$v(t) = 2t\cos\left(t + \frac{\pi}{2}\right) \quad \text{for } t \geq 0.$$ The velocity is given in metres per second, and the time, t, is given in seconds. A graph of the velocity of the mass' motion is shown below.  (a) Determine the first two times, $t > 0$, at which the mass changes direction. State your answers exactly. (2 marks) (b) What does the signed area of the shaded region in the figure represent? (2 marks)

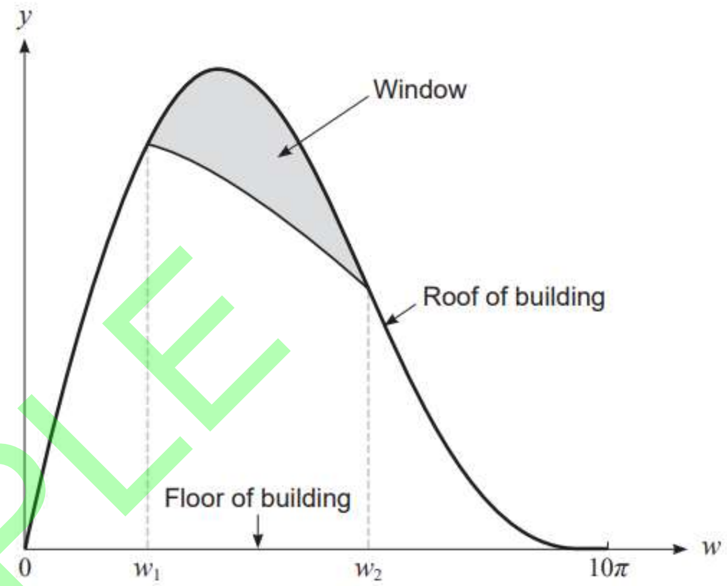
(c) Write an integral expression for the distance travelled from $t=\frac{\pi}{3}$ to $t=\frac{4\pi}{3}$. (3 marks)

(d) Determine the first time after $t=\pi$ that the acceleration of the object will be 0 m/s^2. (2 marks)

(e) The displacement of the mass is given by

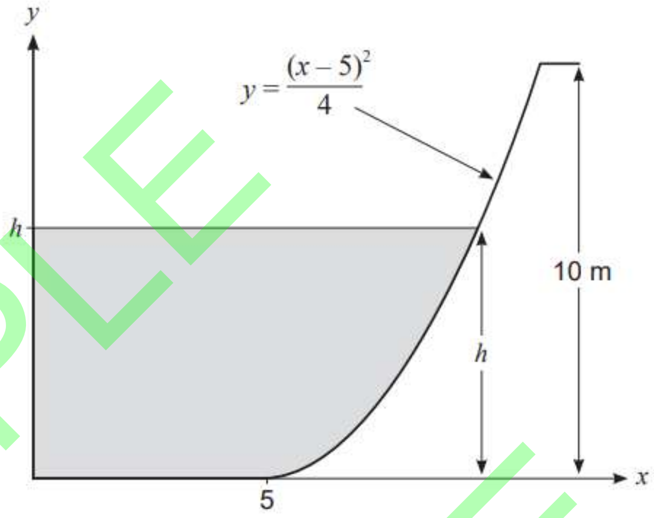
$$x(t)=A\sin\left(t+\frac{\pi}{2}\right)+B\cos\left(t+\frac{\pi}{2}\right)+2t\sin\left(t+\frac{\pi}{2}\right)$$

where *A* and *B* are constants. Determine the value of *A* and *B*. (3 marks)

2023 Section 2 Question 9 Integrals	A new entertainment venue is being proposed. The preliminary design has a constant cross-section, as shown in the figure below.  The roof height *h*(*w*) of the building at any point w along its width is given by $$h(w) = 6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right)$$ where h and $0 \le w \le 10\pi$ are measured in metres. (a) Determine the cross-sectional area of the building. (2 marks) The designer would like to place a window, as shown in the figure above, that is bounded above by the roof of the building and below by the formula $$g(w) = 7\cos\left(\frac{w}{20}\right).$$ (b) With reference to the figure (i) determine the values of w_1 and w_2. (2 marks)

	(ii) determine the area of the window. (2 marks) (c) Use calculus techniques to determine the maximum height of the building. (4 marks)

2023 Section 2 Question 14 **Integrals**	A small dam on an agricultural property has a length of 20 m, and a uniform cross-section shown below where x and y are in metres. The base of the dam is flat for $0 \le x \le 5$, and the right side is given by $y = \frac{(x-5)^2}{4}$ for $5 < x \le 11.325$. The shaded region on the graph below represents the cross-section of a volume of water V (m^3) in the dam with water level h (m).



(a) Using calculus, show that the volume of water in the dam is given by

$$V(h) = 100h + \frac{80}{3}h^{\frac{3}{2}}.$$

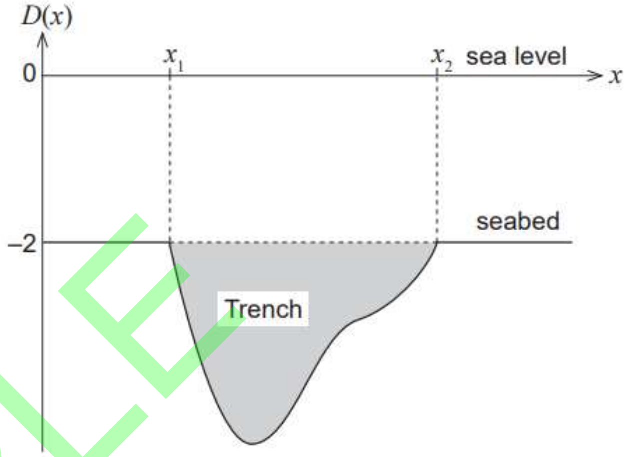
(5 marks)

(b) Use the increments formula to estimate the change in water volume if the water level rises from 6 m to 6.1 m. (3 marks)

Suppose the water volume at the start of winter is 1000 m^3. On the basis of rainfall data from previous years, the volume of water V_R (m^3) that will flow into the dam over winter is normally distributed with a mean of 600 m3 and a standard deviation of 200 m^3.

(c) Assuming that there are no other sources of water and no losses, determine the probability that the dam will reach full capacity (i.e. depth of 10 m) during winter.(3 marks)

2022 Section 2 Question 7 **Integrals**	A team of oceanographers surveyed the depth of the ocean in a region populated by a particular endangered fish species. They discovered a large trench extending below the otherwise flat seabed as shown in the figure below.



The displacement, in kilometres, from sea level to the ocean floor is given by

$$D(x) = \begin{cases} (x-4)^2 + \cos(2x - 3\pi) - 5, & x_1 \le x \le x_2 \\ -2, & \text{otherwise} \end{cases}$$

where x (measured in kilometres) is the east–west horizontal displacement relative to a reference marker at sea level.

(a) With reference to the figure above:

(i) determine the values of x_1 and x_2. (2 marks)

(ii) use calculus to determine the cross-sectional area of the trench shaded in the figure above. (3 marks)

	(b) Using calculus, determine the maximum distance of the trench below sea level. (5 marks)

2022 Section 2 Question 10 **Integrals**	The displacement, x, of a mass on the end of a damped spring is given by $$x(t) = 3e^{-t}\sin(t), \quad t \geq 0$$ where x is in centimetres and t is in seconds (a) Determine when the mass first returns to its starting position at x = 0. (2 marks) (b) Determine an expression for the velocity of the mass. (2 marks)

	(c) Determine the displacement of the mass when it first changes direction. (3 marks) (d) The mass is considered to have stopped oscillating when the oscillation amplitude $A(t) = 3e^{-t}$ drops to 0.01 cm. How long does it take for the spring to stop oscillating? (2 marks)

2022 Section 2 Question 11 **Integrals**	The 100 m sprint is a race run on a straight section of track. During a race the velocity, v, measured in metres per second, of an athlete is given by $v(t) = -10e^{-0.8t} - 0.05e^{0.2t} + 10.05$ where t is the time, in seconds, measured from the moment the athlete starts to move from the start line. (a) Determine the acceleration of the athlete three seconds after moving from the start line. (2 marks)

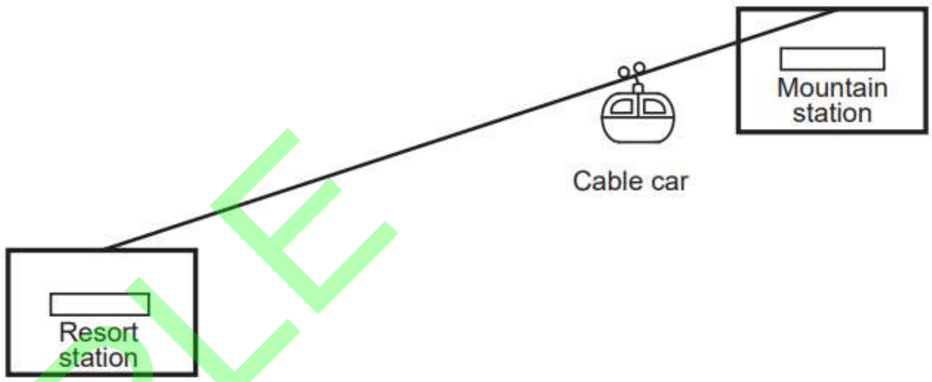
(b) Using calculus, determine the maximum velocity of the athlete during the race, and the time, t, at which it is achieved. (4 marks)

(c) The displacement, x, of the athlete is 0 m at the start of the race. Determine an expression for the displacement of the athlete during the race. (3 marks)

(d) Determine the time, t, at which the athlete finishes the 100 m race. (2 marks)

2021 Section 2 Question 14 **Integrals**	The displacement in metres, $x(t)$, of a power boat t seconds after it was launched is given by: $x(t) = \frac{5t(t^2 - 15t + 48)}{6}, \quad t \geq 0$ How far has the power boat travelled before its acceleration is zero? (5 marks)

2021 Section 2 Question 17 **Integrals**	A resort in the Swiss Alps features a cable car that travels from the resort station to the mountain station. Engineers are fixing a cable car that unexpectedly stopped shortly before it reached the mountain station. The engineers are ready to test the cable car. For the purposes of the test, the cable car will initially be at rest in its current position, will head up the mountain, stop at the mountain station and immediately return to the resort station where it will stop, and the test will be complete.



The test begins and engineers believe that the acceleration, $a(t)$, of the cable car during the test will be: $a(t) = kt^2 - 23t + 20k$, measured in m/min^2. The variable t is the number of minutes from the moment the cable car leaves its position and k is a constant. After two minutes, the engineers expect that the cable car will be travelling with velocity 18 m/min and will not yet have reached the mountain station.

(a) Determine the value of the constant k. (3 marks)

(b) Once the cable car leaves the mountain station, how long should it take to return to the resort station? (3 marks)

	(c) Unfortunately, 10 minutes into the test, the cable car breaks down again. According to the engineers' model, how far is the cable car from the mountain station at this time? (2 marks)

2020
Section 2
Question 11

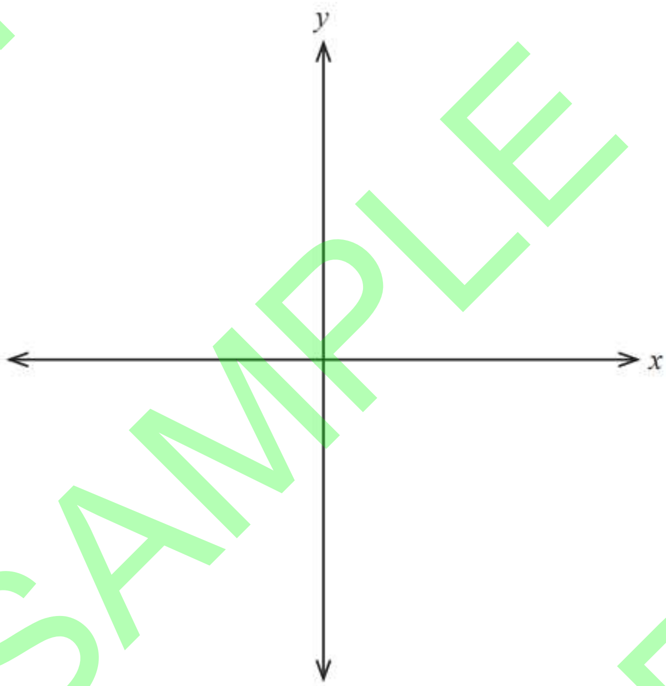
Integrals

The line $y = x + c$ is tangent to the graph of $f(x) = e^x$.

(a) Obtain the coordinates of the point of intersection of the tangent with the graph of $f(x)$. (2 marks)

(b) What is the value of c? (1 mark)

(c) Sketch the graph of $f(x)$ and the tangent on the axes below. (1 mark)



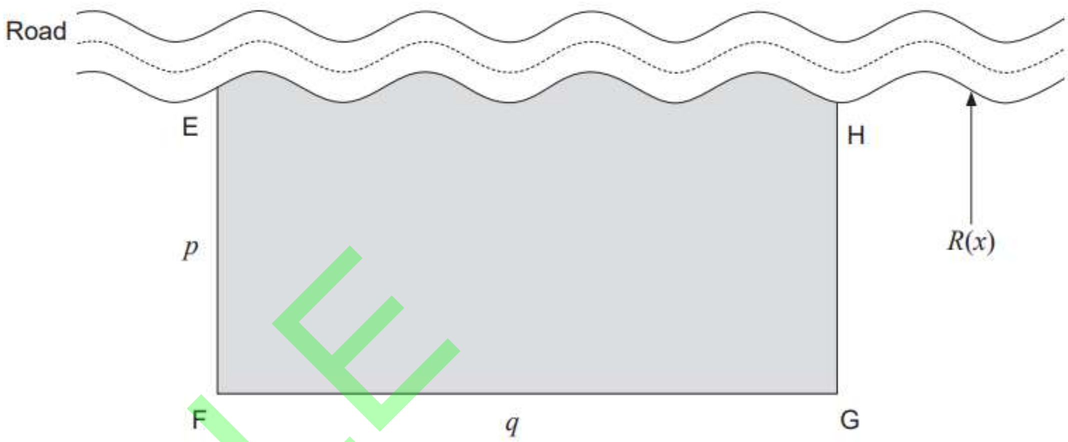
(d) Evaluate the exact area between the graph of $f(x)$, the tangent line, and the line $x = \ln 2$. (3 marks)

	(e) Given that g(x) is the inverse function of $f(x)$, write a definite integral that could be used to determine the area between the graph of $g(x)$, the x-axis, and the line $x = \ln 2$. (2 marks)

2020
Section 2
Question 17

Integrals

David and Katrina have a small farm and wish to fence off an area of their land so they can raise sheep. The area they have chosen has one border along a road as shown in the diagram below.



The enclosure is shown as the shaded area above and has right angles at points F and G. David and Katrina want the combined lengths of the fencing from E to F and F to G to equal 500 metres. Let the length of fence EF be equal to p metres and the length of fence FG be equal to q metres. If we locate the origin at point F and the x-axis along the line FG, the equation defining the fence along the road is given by:

$$R(x) = 10\sin\left(\frac{x}{15}\right) + p$$

(a) Show that the equation defining the area of the enclosure, $A(q)$, can be given in terms of q as follows:

$$A(q) = 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$

(4 marks)

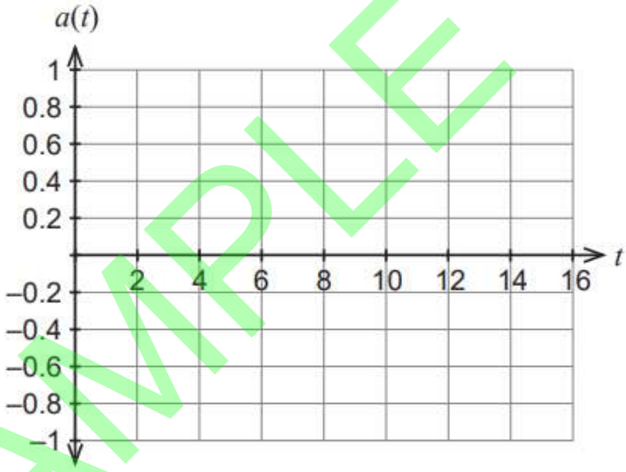
(b) Determine, to the nearest metre, the value of q that will allow the sheep to graze over the maximum area and state this maximum area. (4 marks)

The length of the fence from E to H is given by the equation:

$$L_{EH} = \int_0^q \sqrt{1 + (R'(x))^2}\, dx$$, where $R'(x)$ is the first derivative of $R(x)$.

(c) (i) Determine $R'(x)$. (1 mark)

(ii) Hence determine the total length of fencing required by David and Katrina to enclose their sheep with maximum area for grazing. (3 marks)

2019 Section 2 Question 9 **Integrals**	It takes an elevator 16 seconds to ascend from the ground floor of a building to the sixth floor. The velocity of the elevator during its ascent is given by $$v(t) = \frac{9\pi}{16}\sin\left(\frac{\pi t}{16}\right) \text{ m/s.}$$ The velocity, v, is measured in metres per second, while the time, t, is measured in seconds. (a) Determine the acceleration of the elevator during its ascent and provide a sketch of the acceleration function for $0 \le t \le 16$. (2 marks)  (b) With reference to your answer from part (a), explain what is happening to the velocity of the elevator in the interval $0 < t < 8$ and in the interval $8 < t < 16$. (3 marks) (c) Suppose that the ground floor has displacement $x = 0$ m. Determine the displacement function of the elevator and hence determine the height above the ground floor of the sixth floor. (3 marks)

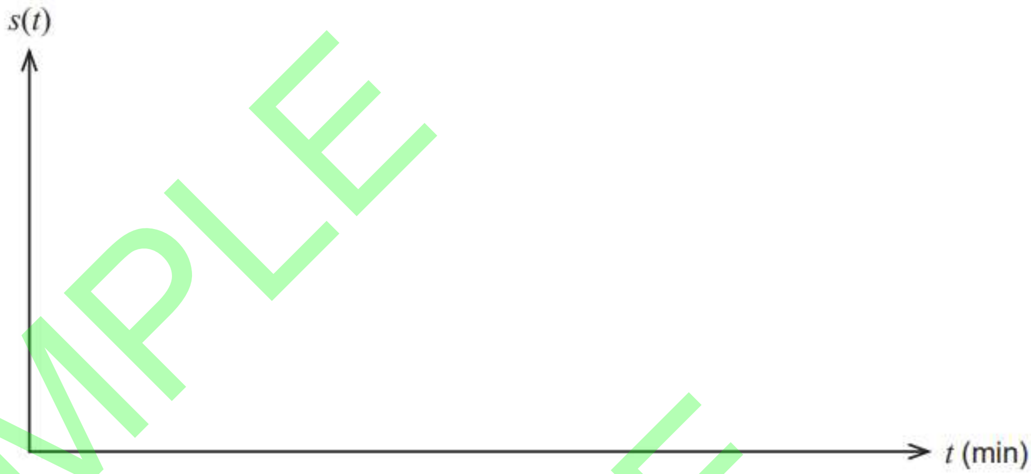
2019 Section 2 Question 12

Integrals

Part of Josie's workout at her gym involves a 10 minute run on a treadmill. The treadmill's program makes her run at a constant 12.3 km/h for the first 2 minutes and then her speed, $s(t)$, is determined by the equation below, where t is the time in minutes after she began running.

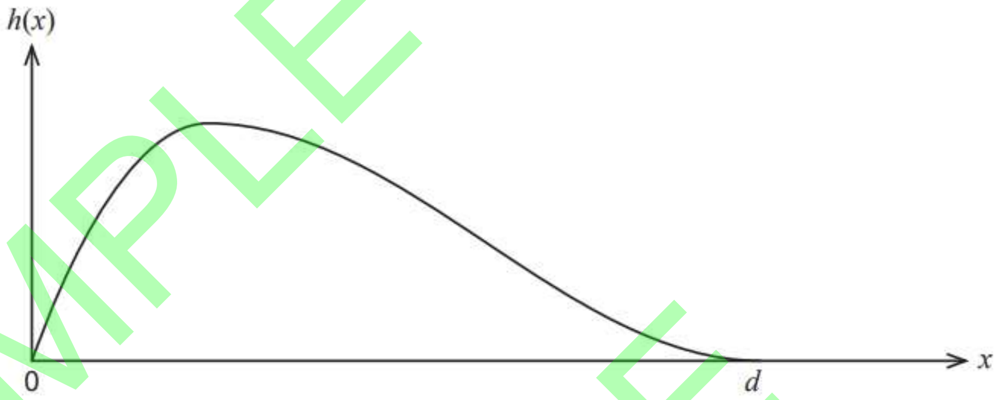
$$s(t) = 10 - \frac{\ln(t - 1.99)}{t} \text{ km/h}$$

(a) Sketch the graph of her speed during this run versus time on the axes below. (3 marks)



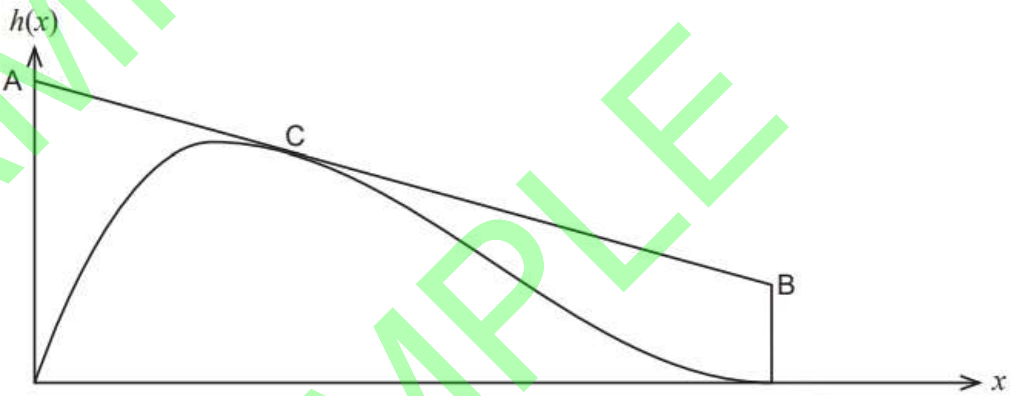
(b) At what time(s) is Josie's speed 10 km/h? (1 mark)

(c) At what time(s) during her run is Josie's acceleration zero? (2 marks)

2019 Section 2 Question 15 **Integrals**	A wall in a new Western Australian hotel is to feature a rolling, wave-shaped window. Engineers have modelled the top edge of the wave shape by joining together two functions, $h_1(x) = 4 - 4(x-1)^2, \quad 0 \le x \le 1$ and $h_2(x) = a(\cos(x-1)+1), \quad 1 < x \le d \quad a,d$ constants. The functions give the height, h, above ground level of the top edge of the window measured in metres. The origin is defined as the leftmost point of the window which is at ground level and x is the horizontal distance to the right of the origin measured in metres. The graph of the two functions is shown below.  (a) Determine the value of the constant a in the function $h2\ (x) = a(\cos(x-1)+1)$. (3 marks) (b) Determine the length of the bottom edge of the window. (2 marks)

(c) Determine the volume of glass required for the window if it has a uniform thickness of 3 cm. (5 marks)

The top edge of the wall, shown as the line AB below, is to just touch the window at the point C shown below. Point A is 1.39 m above the point B.



(d) How high is point C above the ground? (4 marks)

2022 Section 1 Question 1

Integrals

Consider the derivative function $f'(x) = \dfrac{4x}{x^2+3}$.

(a) Determine the rate of change of $f'(x)$ when $x = 1$. (3 marks)

Solution

The second derivative is

$$f''(x) = \frac{4(x^2+3) - 4x(2x)}{(x^2+3)^2}$$

Evaluating at $x = 1$ gives

$$f''(1) = \frac{(4)(4) - (4)(2)}{(4)^2}$$

$$= \frac{1}{2}$$

Specific behaviours

✓ correctly differentiates $f'(x)$
✓ indicates rate of change of $f'(x)$ when $x = 1$ is $f''(1)$
✓ correctly substitutes into $f''(x)$ and evaluates

(b) Determine $f(x)$ given that $f(1) = \ln(32)$. (4 marks)

Solution

$$f(x) = \int \frac{4x}{x^2+3}\,dx$$

$$= 2\int \frac{2x}{x^2+3}\,dx$$

$$= 2\ln(x^2+3) + c$$

Substituting $f(1) = \ln(32)$ gives

$$\ln(32) = 2\ln(1^2+3) + c$$

$$\Rightarrow c = \ln(32) - 2\ln(4)$$

$$= \ln(32) - \ln(16)$$

$$= \ln\left(\frac{32}{16}\right)$$

$$= \ln(2)$$

Hence

$$f(x) = 2\ln(x^2+3) + \ln(2)$$

Specific behaviours

✓ integrates correctly
✓ substitutes $f(1) = \ln(32)$
✓ correctly applies log laws to simplify
✓ evaluates c and states equation for $f(x)$

(c) Determine $\frac{d}{dt}\int_t^3 f(x)\,dx$.

(2 marks)

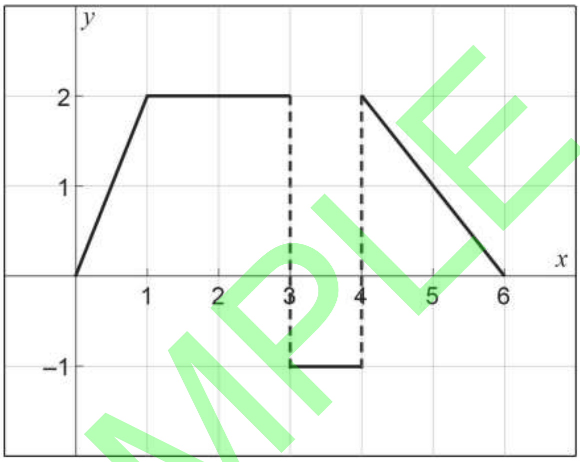
Solution
$\frac{d}{dt}\int_t^3 f(x)\,dx = -\frac{d}{dt}\int_3^t f(x)\,dx$ $= -f(t)$ $= -2\ln(t^2+3) - \ln(2)$

Specific behaviours
✓ uses properties of integrals to reverse integration bounds   ✓ applies the fundamental theorem of calculus to obtain answer

2022
Section 1
Question 2

Integrals

Consider the function $f(x)$ shown below.



Evaluate the following integrals.

(a) $\int_0^6 f(x)\,dx$

(2 marks)

Solution
Summing signed areas $\int_0^6 f(x)dx = 1 + 4 - 1 + 2 = 6$

Specific behaviours
✓ expresses integral in terms of signed areas   ✓ evaluates integral correctly

(b) $\int_0^4 f(x) - 2\,dx$ (2 marks)

Solution
$$\int_0^4 f(x) - 2\,dx = \int_0^4 f(x)\,dx - \int_0^4 2\,dx$$ $$= 4 - 2 \times 4$$ $$= -4$$
Specific behaviours
✓ expresses integral as the difference between $\int_0^4 f(x)\,dx$ and $\int_0^4 2\,dx$ ✓ evaluates integral correctly

(c) $\int_4^6 f'(x)\,dx$ (2 marks)

Solution
$$\int_4^6 f'(x)dx = f(6) - f(4) = 0 - 2 = -2$$
Specific behaviours
✓ applies fundamental theorem of calculus ✓ evaluates correctly
Note: Accept other answers that apply the fundamental theorem of calculus.

2021 Section 1 Question 4 Integrals	Determine the following:
	(a) $\int\left(2x^2 - x^3\right)dx$ (2 marks)

Solution
$\int\left(2x^2 - x^3\right)dx = \frac{2x^3}{3} - \frac{x^4}{4} + c$
Specific behaviours
✓ integrates correctly ✓ includes the constant of integration

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3 - \cos(x)}\,dx$ (3 marks)

Solution
$\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3-\cos(x)}\,dx = \left[\ln(3-\cos(x))\right]_0^{\frac{\pi}{2}}$ $= \ln(3-\cos(\frac{\pi}{2})) - \ln(3-\cos(0))$ $= \ln 3 - \ln 2$ $= \ln\left(\frac{3}{2}\right)$
Specific behaviours
✓ correctly integrates ✓ substitutes limits ✓ determines the correct simplified answer

(c) $\frac{d}{dy}\int_{-1}^{y} 3x^2\cos(2x)\,dx$ (2 marks)

Solution
$\frac{d}{dy}\int_{-1}^{y} 3x^2\cos(2x)\,dx = 3y^2\cos(2y)$
Specific behaviours
✓ identifies the need for the Fundamental Theorem of Calculus ✓ states the correct result

2021
Section 1
Question 5

Integrals

(a) Determine the area between the parabola $y = x2 - x + 3$ and the straight line $y = x + 3$. (4 marks)

Solution
Point of intersection:
$x^2 - x + 3 = x + 3$
$x^2 - 2x = 0$
$x(x-2) = 0$
$\therefore x = 0, 2$
$\int_0^2 \left[(x+3) - (x^2 - x + 3)\right] dx$
$= \int_0^2 \left[2x - x^2\right] dx$
$= \left[x^2 - \frac{x^3}{3}\right]_0^2$
$= 4 - \frac{8}{3}$
$= \frac{4}{3}$ units^2
Specific behaviours
✓ determines x coordinates of the points of intersection
✓ states correct integral for area
✓ evaluates integral
✓ determines correct area

(b) The area between the parabola $y = x2 - x - 2$ and the straight line $y = x - 2$ is the same as the area determined in part (a). Explain why this is the case. (2 marks)

Solution
Both graphs from part (a) have been vertically translated down 5 units. The shape of both graphs is unchanged. Therefore, the area between them remains unchanged.
Specific behaviours
✓ states both graphs have been translated in the same direction by the same amount
✓ states both graphs retain the same shape

2019 Section 1 Question 5 **Integrals**	(a) Determine the area bound by the graph of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$. (3 marks)

Solution

First we obtain the area under the graph of $f(x)$ between $x = 0$ and $x = \ln 2$. This is given by

$$A = \int_0^{ln2} e^x dx = e^x|_0^{ln2} = 2 - 1 = 1.$$

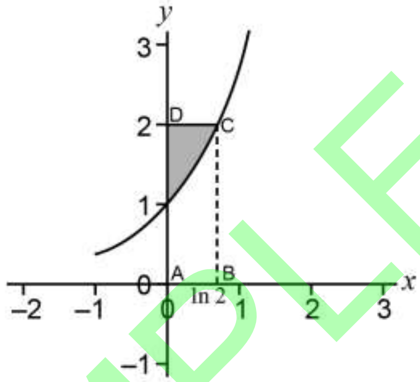
Specific behaviours

✓ writes down the correct integral
✓ integrates correctly
✓ simplifies to obtain final answer

(b) Hence, determine the area bound by the graph of $f(x) = e^x$, the line $y = 2$ and the y-axis. (2 marks)

Solution

This is given by the area shown below.



That is,

$$Area = 2\ln 2 - 1$$

Specific behaviours

✓ correctly defines the area
✓ calculates the area correctly

(c) Determine the area bound by the graph of $f(x) = e^x$, the line $y = a$ and the y-axis, where a is a positive constant. (3 marks)

Solution

$$\int_0^{lna} e^x dx = e^x|_0^{lna} = a - 1$$

$$A = \ln(a) \times a - (a - 1)$$

$$= a\ln(a) - a + 1$$

Specific behaviours

✓ writes down the correct integral
✓ integrates correctly and simplifies to obtain $a - 1$
✓ determines the correct expression for area

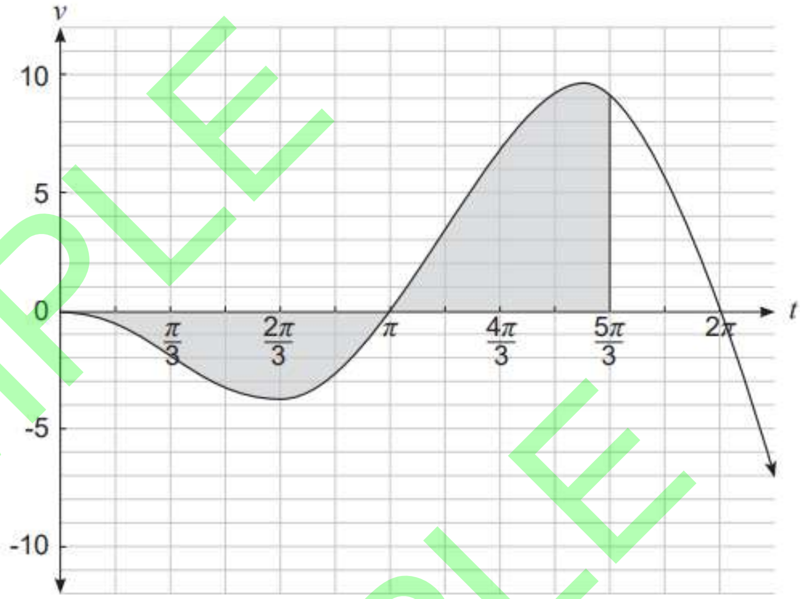
2023
Section 2
Question 8

Integrals

An oscillating mass has a velocity, *v*, given by

$$v(t) = 2t\cos\left(t + \frac{\pi}{2}\right) \quad \text{for } t \geq 0.$$

The velocity is given in metres per second, and the time, *t*, is given in seconds. A graph of the velocity of the mass' motion is shown below.



(a) Determine the first two times, $t > 0$, at which the mass changes direction. State your answers exactly. (2 marks)

Solution
The mass changes direction when $v(t) = 0$. From the graph it is clear that $v(t) = 0$ when $t = \pi$ seconds, and $t = 2\pi$ seconds.
Specific behaviours
✓ states that a change in direction occurs when $v(t) = 0$ ✓ correctly determines the first two times the mass changes direction

(b) What does the signed area of the shaded region in the figure represent? (2 marks)

Solution
The shaded region represents the change in the displacement of the mass from $t = 0$ to $t = \frac{5\pi}{3}$ seconds.
Specific behaviours
✓ states that the shaded region represents a change in displacement ✓ specifies the correct start and finish time

(c) Write an integral expression for the distance travelled from $t=\frac{\pi}{3}$ to $t=\frac{4\pi}{3}$. (3 marks)

Solution
distance $= \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left| 2t\cos\left(t+\frac{\pi}{2}\right) \right| dt$
Specific behaviours
✓ identifies distance as the integral of speed (absolute value of velocity) ✓ writes the correct integral (including bounds) ✓ dt included

Alternative solution one
distance $= -\int_{\frac{\pi}{3}}^{\pi} 2t\cos\left(t+\frac{\pi}{2}\right) dt + \int_{\pi}^{\frac{4\pi}{3}} 2t\cos\left(t+\frac{\pi}{2}\right) dt$
Specific behaviours
✓ split into two integrals with correct bounds and correct integrand ✓ correct sign (+/-) in front of integrals ✓ dt included in both integrals

Alternative solution two
distance $= \left| \int_{\frac{\pi}{3}}^{\pi} 2t\cos\left(t+\frac{\pi}{2}\right) dt \right| + \left| \int_{\pi}^{\frac{4\pi}{3}} 2t\cos\left(t+\frac{\pi}{2}\right) dt \right|$
Specific behaviours
✓ split into two integrals with correct bounds and correct integrand ✓ absolute value signs included (either inside the integral or outside) ✓ dt included in both integrals

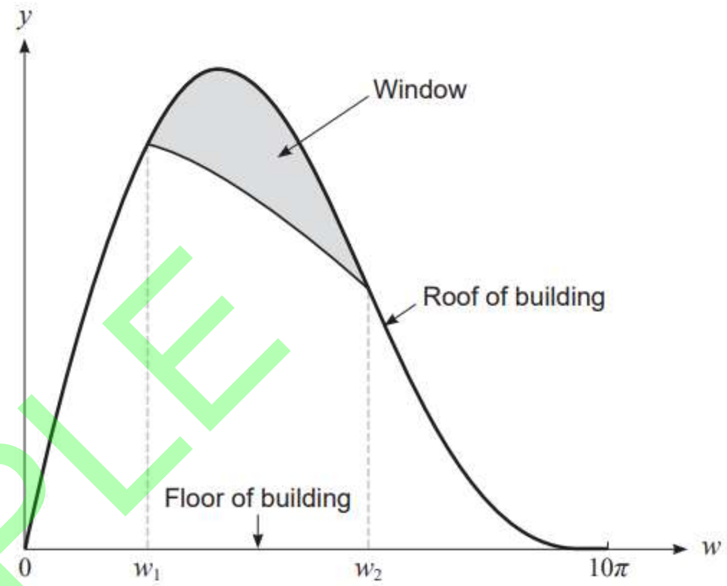
(d) Determine the first time after $t=\pi$ that the acceleration of the object will be 0 m/s^2. (2 marks)

Solution
The acceleration is given by $a(t)=v'(t)=2\cos\left(t+\frac{\pi}{2}\right)-2t\sin\left(t+\frac{\pi}{2}\right)$ or $a(t)=v'(t)=-2\sin(t)-2t\cos(t)$ Solving $v'(t)=0$ gives $t\approx 4.91$ seconds
Specific behaviours
✓ states correct expression for acceleration ✓ solves to obtain correct time

	(e) The displacement of the mass is given by $$x(t) = A\sin\left(t+\frac{\pi}{2}\right) + B\cos\left(t+\frac{\pi}{2}\right) + 2t\sin\left(t+\frac{\pi}{2}\right)$$ where *A* and *B* are constants. Determine the value of *A* and *B*. (3 marks)

Solution
$$v(t) = x'(t) = A\cos\left(t+\frac{\pi}{2}\right) - B\sin\left(t+\frac{\pi}{2}\right) + 2\sin\left(t+\frac{\pi}{2}\right) + 2t\cos\left(t+\frac{\pi}{2}\right)$$ Given that $v(t) = 2t\cos\left(t+\frac{\pi}{2}\right)$ it follows that $A = 0$ and $$-B + 2 = 0$$ $$\Rightarrow B = 2$$
Specific behaviours
✓ correctly differentiates $x(t)$ ✓ compares $x'(t)$ and $v(t)$ to determine that $A = 0$ ✓ compares $x'(t)$ and $v(t)$ to determine that $B = 2$

2023 Section 2 Question 9 **Integrals**	A new entertainment venue is being proposed. The preliminary design has a constant cross-section, as shown in the figure below.



The roof height $h(w)$ of the building at any point w along its width is given by

$$h(w) = 6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right)$$

where h and $0 \le w \le 10\pi$ are measured in metres.

(a) Determine the cross-sectional area of the building. (2 marks)

Solution
$\text{Area} = \int_0^{10\pi} \left(6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right)\right) dx$ $= 120 \text{ m}^2$
Specific behaviours
✓ states a correct integral expression for the cross-sectional area ✓ correctly determines the cross-sectional area including units

The designer would like to place a window, as shown in the figure above, that is bounded above by the roof of the building and below by the formula

$$g(w) = 7\cos\left(\frac{w}{20}\right).$$

(b) With reference to the figure

(i) determine the values of w_1 and w_2. (2 marks)

Solution

Solving $h(w)=g(w)$:

$$\Rightarrow 6\sin\left(\frac{w}{10}\right)+3\sin\left(\frac{w}{5}\right)=7\cos\left(\frac{w}{20}\right)$$

$$\Rightarrow w=6.6511,\ 18.4122$$

Hence $w_1=6.6511$ and $w_2=18.4122$.

Specific behaviours

✓ states correct equation to solve
✓ determines the correct values of w_1 and w_2

(ii) determine the area of the window. (2 marks)

Solution

$$\text{Area}=\int_{6.6511}^{18.4122}\left(6\sin\left(\frac{w}{10}\right)+3\sin\left(\frac{w}{5}\right)-7\cos\left(\frac{w}{20}\right)\right)dx$$

$$=13.94\text{ m}^2$$

Specific behaviours

✓ states a correct integral expression for the cross-sectional area
✓ correctly determines the cross-sectional area

(c) Use calculus techniques to determine the maximum height of the building. (4 marks)

Solution

The derivative of $h(w)$ is given by

$$h'(w)=\frac{3}{5}\cos\left(\frac{w}{10}\right)+\frac{3}{5}\cos\left(\frac{w}{5}\right)$$

Setting $h'(w)=0$ yields

$$w=\frac{10\pi}{3}\ (\approx 10.47)$$

The second derivative of $h(w)$ is

$$h''(w)=-\frac{3}{50}\sin\left(\frac{w}{10}\right)-\frac{3}{25}\sin\left(\frac{w}{5}\right)$$

Since $h''\left(\frac{10\pi}{3}\right)=-\frac{9\sqrt{3}}{100}\approx -0.156<0$ it follows that $w=\frac{10\pi}{3}$ is a local maximum.

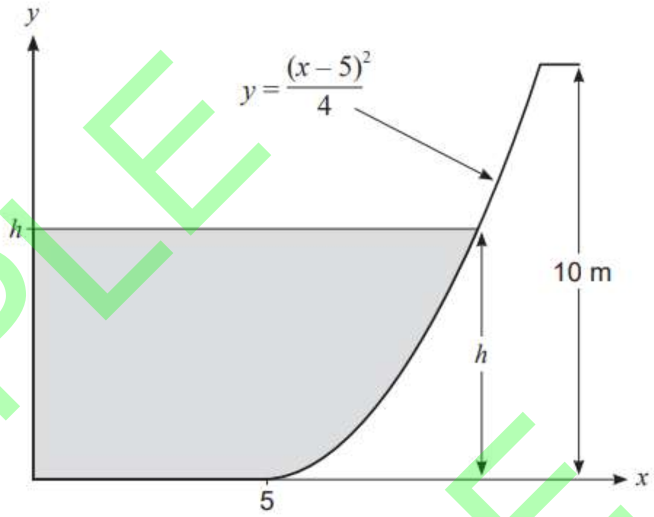
The maximum height of the building is

$$h\left(\frac{10\pi}{3}\right)=6\sin\left(\frac{\pi}{3}\right)+3\sin\left(\frac{2\pi}{3}\right)$$

$$=\frac{9\sqrt{3}}{2}\text{ m }(\approx 7.79\text{ m})$$

Specific behaviours

✓ states correct derivative for $h(w)$
✓ sets $h'(w)=0$ and obtains correct critical value
✓ calculates $h''\left(\frac{10\pi}{3}\right)$ and concludes local maximum
✓ calculates correct maximum height

2023 Section 2 Question 14 **Integrals**	A small dam on an agricultural property has a length of 20 m, and a uniform cross-section shown below where x and y are in metres. The base of the dam is flat for $0 \le x \le 5$, and the right side is given by $y = \frac{(x-5)^2}{4}$ for $5 < x \le 11.325$. The shaded region on the graph below represents the cross-section of a volume of water V (m^3) in the dam with water level h (m).  (a) Using calculus, show that the volume of water in the dam is given by $V(h) = 100h + \frac{80}{3}h^{\frac{3}{2}}$. (5 marks)

Solution
Determine an expression for x in terms of h by solving $$h=\frac{(x-5)^2}{4}$$ $$\Rightarrow 4h=(x-5)^2$$ $$\Rightarrow x=5+2\sqrt{h}$$ Hence the volume is given by $$V(h)=20\left(5h+\int_5^{5+2\sqrt{h}} h-\frac{(x-5)^2}{4}\,dx\right)$$ $$=20\left(5h+\left[hx-\frac{(x-5)^3}{12}\right]_5^{5+2\sqrt{h}}\right)$$ $$=20\left(5h+\left(h\left(5+2\sqrt{h}\right)-\frac{\left(2\sqrt{h}\right)^3}{12}\right)-5h\right)$$ $$=20\left(5h+\frac{4}{3}h^{\frac{3}{2}}\right)$$ $$=100h+\frac{80}{3}h^{\frac{3}{2}}$$
Specific behaviours
✓ determines an expression for the upper bound of the volume integral ✓ states a correct integral expression for the volume or cross-sectional area of water ✓ states correct antiderivative of integrand ✓ correctly applies fundamental theorem of calculus by substituting integration bounds ✓ simplifies to give desired result

(b) Use the increments formula to estimate the change in water volume if the water level rises from 6 m to 6.1 m. (3 marks)

Solution

The derivative of V with respect to h is

$$\frac{dV}{dh} = 100 + 40\sqrt{h}$$

The change in h is

$$\delta h = 6.1 - 6 = 0.1$$

Hence the change in V is approximately

$$\begin{aligned}\delta V &\approx \frac{dV}{dh}\delta h \\ &= \left(100 + 40\sqrt{h}\right) \times 0.1 \\ &= 10 + 4\sqrt{h}\end{aligned}$$

When $h = 6$ we have

$$\begin{aligned}\delta V &\approx 10 + 4\sqrt{6} \\ &\approx 19.80 \text{ m}^3\end{aligned}$$

Specific behaviours

✓ determines correct expression for the derivative of V with respect to h
✓ determine the increment in h
✓ obtains correct estimate for the change in V (exact or decimal)

Suppose the water volume at the start of winter is 1000 m^3. On the basis of rainfall data from previous years, the volume of water V_R (m^3) that will flow into the dam over winter is normally distributed with a mean of 600 m3 and a standard deviation of 200 m^3.

(c) Assuming that there are no other sources of water and no losses, determine the probability that the dam will reach full capacity (i.e. depth of 10 m) during winter.(3 marks)

Solution

The full capacity of the dam is

$$\begin{aligned}V(10) &= 100(10) + \frac{80}{3}(10)^{\frac{3}{2}} \\ &\approx 1843.27 \text{ m}^3\end{aligned}$$

Hence the dam will reach capacity if

$$\begin{aligned}V_R &\geq V(10) - 1000 \\ &= 1843.27 - 1000 \\ &= 843.27 \text{ m}^3\end{aligned}$$

Since $V_R \sim N\left(600,\ 200^2\right)$ it follows that

$$P\left(V_R \geq 843.27\right) \approx 0.1119$$

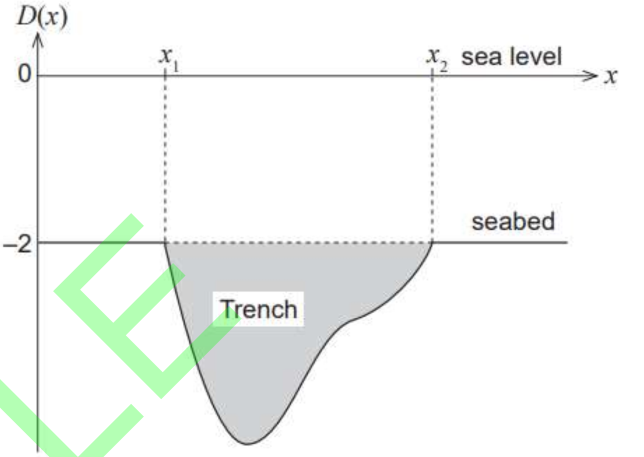
Specific behaviours

✓ determines the correct volume $V(10)$
✓ determines that we need $V_R \geq 843.27$
✓ obtains correct probability

2022
Section 2
Question 7

Integrals

A team of oceanographers surveyed the depth of the ocean in a region populated by a particular endangered fish species. They discovered a large trench extending below the otherwise flat seabed as shown in the figure below.



The displacement, in kilometres, from sea level to the ocean floor is given by

$$D(x) = \begin{cases} (x-4)^2 + \cos(2x - 3\pi) - 5, & x_1 \le x \le x_2 \\ -2, & \text{otherwise} \end{cases}$$

where x (measured in kilometres) is the east–west horizontal displacement relative to a reference marker at sea level.

(a) With reference to the figure above:

(i) determine the values of x_1 and x_2. (2 marks)

Solution
The values x_1 and x_2 are the solutions to the equation $(x-4)^2 + \cos(2x - 3\pi) - 5 = -2$ $x_1 = 2.3004$ and $x_2 = 5.9438$.
Specific behaviours
✓ states equation to solve for x_1 and x_2 ✓ obtains correct values for x_1 and x_2