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JACARANDA MATHS QUEST

FURTHER MATHEMATICS 12

VCE UNITS 3 AND 4 | SIXTH EDITION

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Sixth edition published 2019 by
John Wiley & Sons Australia, Ltd
42 McDougall Street, Milton, Qld 4064

Fifth edition published 2016
Fourth edition published 2013
Third edition published 2010
Second edition published 2006
First edition published 2000

Typeset in 11/14 pt Times Ltd Std

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ISBN: 978-0-7303-5553-3

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Illustrated by diacriTech and Wiley Composition Services

Typeset in India by diacriTech

A catalogue record for this
book is available from the
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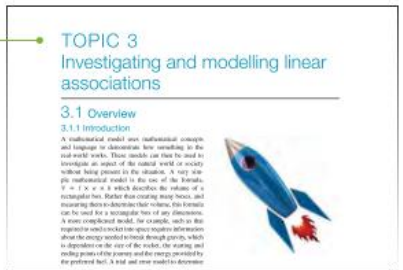
ABOUT THIS RESOURCE

Jacaranda Maths Quest 12 Further Mathematics VCE Units 3 & 4 Sixth Edition has been updated to enhance our in-depth coverage of the Study Design, with hundreds of new questions to expand students' understanding and improve learning outcomes. The *Jacaranda Maths Quest* series provides easy-to-follow text and is supported by a bank of resources for both teachers and students. At Jacaranda we believe that every student should experience success and build confidence, while those who want to be challenged are supported as they progress to more difficult concepts and questions.

Preparing students for exam success

Topic openers place mathematics in real-world contexts to drive engagement.



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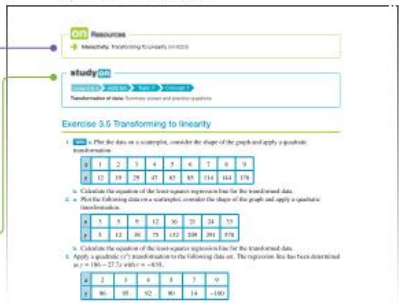
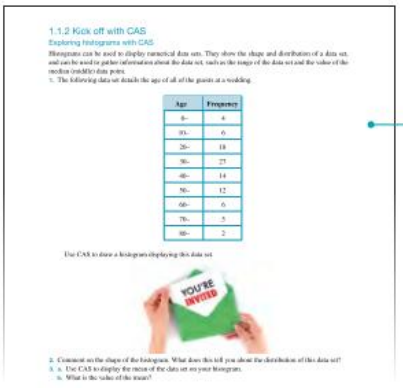
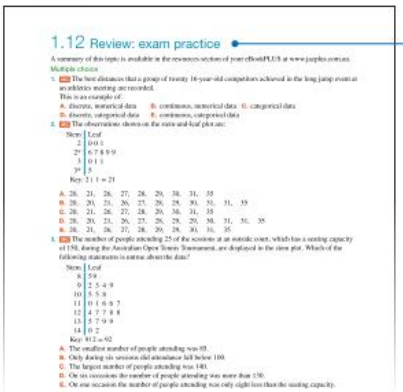


Each subtopic concludes with carefully graded technology free and technology active questions.

The *Manual for the TI-Nspire CAS calculator* and *Manual for the Casio ClassPad II calculator* in the Prelims of your eBookPLUS provide step-by-step instructions on how to use CAS technology.



Each topic opens with an engaging *Kick off with CAS* activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

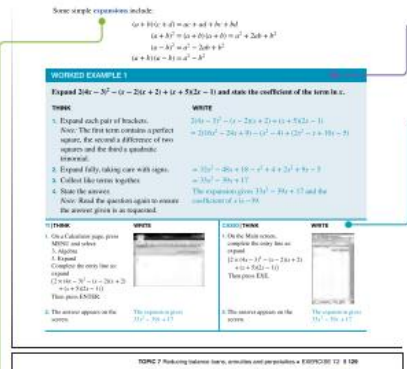


At the conclusion of each topic, Review questions allow you to revise the entire topic. Short-answer technology free questions allow you to consolidate your learning. Technology active questions are provided through multiple-choice and extended-response questions.

Features of the *Maths Quest* series

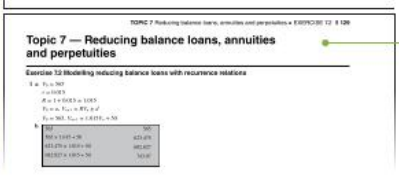
Questions and topics are sequenced from lower to higher levels of complexity; ideas and concepts are logically developed and questions are carefully graded, allowing every student to achieve success.

An extensive glossary of mathematical terms is provided in print and as a hover-over feature in the eBookPLUS.



Fully worked examples in the Think/Write format provide guidance and are linked to questions.

Selected worked examples demonstrate the use of CAS calculators.



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eBookPLUS features

Fully worked solutions for every question

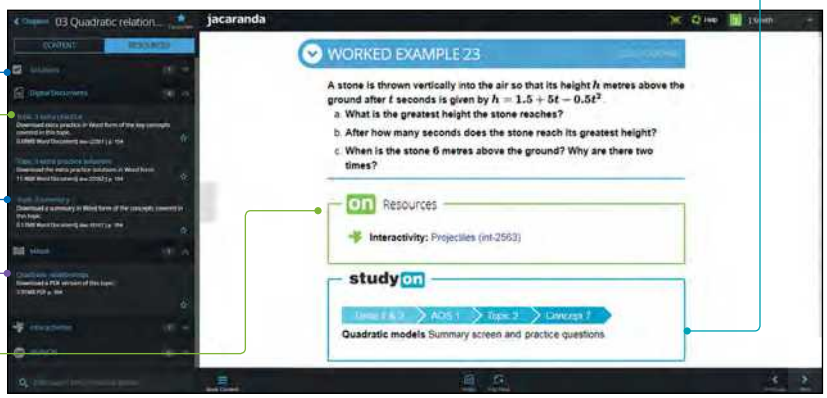
Digital documents: downloadable extra practice questions and fully worked solutions covering all key concepts in the topic

Topic summaries in downloadable format to assist in study and exam preparation

A downloadable PDF of the entire topic of the print text

Interactivities and video eLessons placed at the point of learning to enhance understanding and correct common misconceptions

Concept summary links to studyON for study, revision and exam practice



In the Prelims section of your eBookPLUS

A downloadable PDF of the entire solutions manual, containing fully worked solutions for every question in the text

FREE copies of the *Maths Quest Manual for the TI-Nspire CAS calculator* and the *Maths Quest Manual for the Casio ClassPad II calculator*

Downloadable fully worked solutions, with screenshots, to every question in the Casio manual's problem-solving chapter.

A downloadable PDF of the entire print text

Additional resources for teachers available in the eGuidePLUS

In the Resources tab of every topic there are two topic tests in downloadable, customisable Word format with worked solutions.

In the Prelims section of the eGuidePLUS

Additional resources for teachers include an application task and two problem solving and modelling tasks.

A work program is provided to assist with classroom planning.

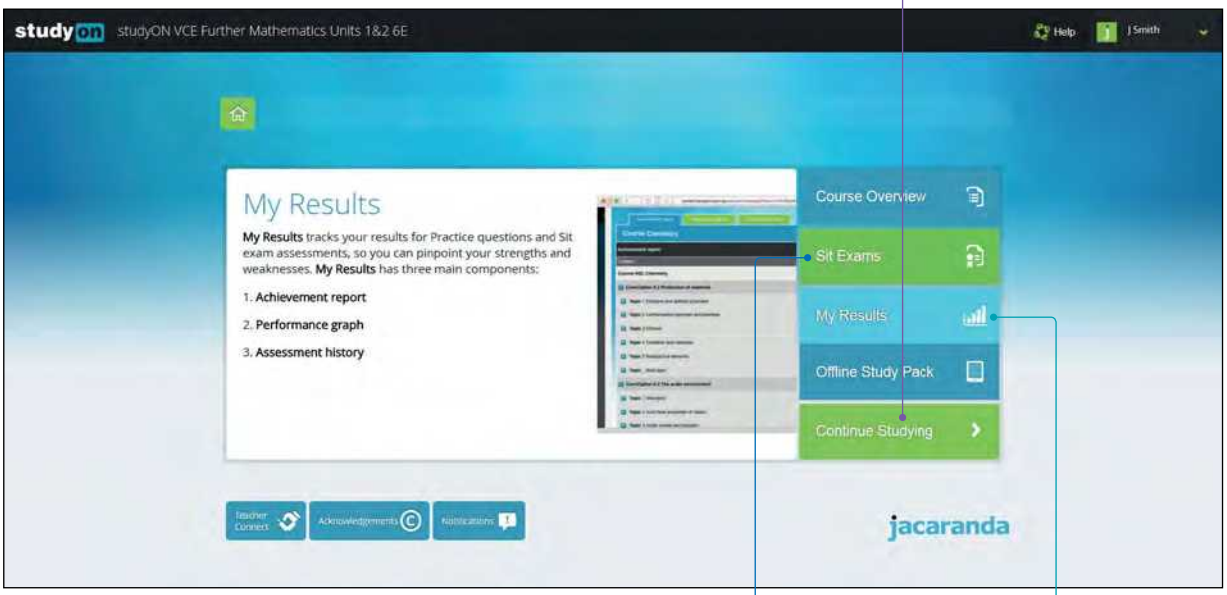
studyON — an invaluable exam preparation tool

studyON provides a complete study solution. An interactive and highly visual online study, revision and exam practice tool, it is designed to help students and teachers maximise exam results.

Concept summary screens and interactivities summarise key concepts and help prevent misconceptions.

Direct links from the eBookPLUS help scaffold students' understanding and study practices.

The studyON question hierarchy allows students in the *Continue Studying* feature to revise across the entire course, or to drill down to concept level for a more granular set of questions.



- The Sit Exams feature allows students to sit past VCAA exams (Units 3 & 4) or topic tests (Units 1 & 2) in timed, exam-like situations.
- Students have access to 1000+ past VCAA questions or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback for every question.

studyON's built-in progress tracker enables self-diagnosis of strengths and weaknesses at a topic and concept level, so students know exactly what needs extra revision and can sit their exams with confidence.

studyON Teacher edition is a powerful diagnostic tool

Enables teachers to assign activities for extra revision and practice, and track progress at an individual, group and classroom level

Allows teachers to monitor students' activities and results to pinpoint strengths and weaknesses. Armed with evidence-based insights, teachers can intervene at the right time.

Alignment with the Jacaranda text helps with planning, and instant feedback saves marking time. Built-in reporting functionality lets teachers easily schedule, export and print reports in Excel.

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Access your online Jacaranda resources anywhere, anytime, from any device in three easy steps:

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STEP 2 Enter your registration code.
STEP 3 Instant access!

eBookplus + studyon

eBookPLUS is an electronic version of the textbook, together with a targeted range of supporting multimedia resources.

eBookPLUS features:

- **eBook** — the entire textbook in electronic format
- **Digital documents** designed for easy customisation and editing
- **Interactivities** to reinforce and enhance students' learning
- **eLessons** — engaging video clips and supporting material
- **Weblinks** to relevant support material on the internet

eGuidePLUS features assessment and curriculum material to support teachers.

studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access 1000+ past VCAA questions** or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback.
- **Sit past VCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

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- **Contact** John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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TOPIC 1
Investigating data distributions

1.1 Overview

1.1.1 Introduction

Due to advances in technology, the collection, analysis and interpretation of data is at an unprecedented level in the world today. These advances have ensured that after the collection and analysis of large datasets, the results can be distributed very quickly. It is very important that future planning and decisions are based on current empirical data as change is fast-paced and costly in the modern world. In 1880 a census was conducted in the USA and it took 7 years for the information collected by the census to be published. The very detailed dataset collected by the Australian census in August 2016 was released in June 2017.

Data has the power to initiate change and business and governments ignore it at their peril. Meteorologists, urban planners, health scientists, traffic engineers, pharmaceutical companies, environmental scientists, educators and agriculturalists to name a few, use data to monitor change and make future predictions. These changes may be as small as choosing to block off a suburban street to stop too many people using it as a short cut and creating traffic problems, or as big as ensuring that there is enough food production to feed an ever-increasing world population. Decisions that require major societal change, practice and investment need to be justified. Empirical evidence can provide this justification and therefore it is important that everyone has some knowledge and understanding of the collection and interpretation of data.

LEARNING SEQUENCE

1.1 Overview
1.2 Types of data
1.3 Stem plots
1.4 Dot plots, frequency tables and histograms, and bar charts
1.5 Describing the shape of stem plots and histograms
1.6 The median, the interquartile range, the range and the mode
1.7 Boxplots
1.8 The mean of a sample
1.9 Standard deviation of a sample
1.10 Populations and simple random samples
1.11 The 68–95–99.7% rule and *z*-scores
1.12 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.1.2 Kick off with CAS

Exploring histograms with CAS

Histograms can be used to display numerical data sets. They show the shape and distribution of a data set, and can be used to gather information about the data set, such as the range of the data set and the value of the median (middle) data point.

1. The following data set details the age of all of the guests at a wedding.

Age	Frequency
0–	4
10–	6
20–	18
30–	27
40–	14
50–	12
60–	6
70–	3
80–	2

Use CAS to draw a histogram displaying this data set.

2. Comment on the shape of the histogram. What does this tell you about the distribution of this data set?

3. a. Use CAS to display the mean of the data set on your histogram.

b. What is the value of the mean?

4. a. Use CAS to display the median of the data set on your histogram.

b. What is the value of the median?

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

1.2 Types of data

Univariate data are data that contain one variable. That is, the information deals with only one quantity that changes. Therefore, the number of cars sold by a car salesman during one week is an example of *univariate* data. Sets of data that contain two variables are called **bivariate data** and those that contain more than two variables are called *multivariate* data.

Data can be classified as either numerical or categorical. The methods we use to display data depend on the type of information we are dealing with.

1.2.1 Types of data

Numerical data is data that has been assigned a numeric value. Numerical data can be:

- **discrete** — data that can be counted but that can have only a particular value, for example the number of pieces of fruit in a bowl
- **continuous** — data that is not restricted to any particular value, for example the temperature outside, which is measured on a continuous scale.

Categorical data is data split into two or more categories. Categorical data can be:

- **nominal** — data that can be arranged into categories but not ordered, for example arranging shoes by colour or athletes by gender
- **ordinal** — data that can be arranged into categories that have an order, for example levels of education from high school to post-graduate degrees.

1.2.2 Discrete and continuous data

Data are said to be *discrete* when a variable can take only certain fixed values. For example, if we counted the number of children per household in a particular suburb, the data obtained would always be whole numbers starting from zero. A value in between, such as 2.5, would clearly not be possible. If objects can be counted, then the data are discrete.

Continuous data are obtained when a variable takes any value between two values. If the heights of students in a school were obtained, then the data could consist of any values between the smallest and largest heights. The values recorded would be restricted only by the precision of the measuring instrument. If variables can be measured, then the data are continuous.

WORKED EXAMPLE 1

MC Which of the following is *not* numerical data?

A. Maths test results **B.** Ages **C.** Names of AFL football teams
D. Heights of students in a class **E.** Lengths of bacteria

THINK	WRITE
1. To be numerical data, it is to be measurable or countable.	
2. Look for the data that does not fit the criteria.	A: Measurable B: Countable C: Names, so not measurable or countable D: Measurable E: Measurable

3. Answer the question.	The data that is not numerical is names of AFL football teams. The correct option is C.

WORKED EXAMPLE 2

MC Which of the following is *not* discrete data?
A. Number of students older than 17.5 years old
B. Number of girls in a class
C. Number of questions correct in a multiple choice test
D. Number of students above 180 cm in a class
E. Height of the tallest student in a class

THINK	WRITE
1. To be discrete data, it is to be a whole number (countable).	
2. Look for the data that does not fit the criteria.	A: Whole number (countable) B: Whole number (countable) C: Whole number (countable) D: Whole number (countable) E: May not be a whole number (measurable)
3. Answer the question.	The data that is not discrete is the height of the tallest student in a class. The correct option is E.

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Units 3 & 4 > AOS DA > Topic 1 > Concepts 1 & 2

Classification of data Summary screen and practice questions
Defining univariate data Summary screen and practice questions

Exercise 1.2 Types of data

1. MC WE1 Which of the following is *not* numerical data?
   - **A.** Number of students in a class
   - **B.** The number of supporters at an AFL match
   - **C.** The amount of rainfall in a day
   - **D.** Finishing positions in the Melbourne Cup
   - **E.** The number of coconuts on a palm tree

2. MC Which of the following is *not* categorical data?
   - **A.** Preferred political party
   - **B.** Gender
   - **C.** Hair colour
   - **D.** Salaries
   - **E.** Religion
3. Write whether each of the following represents numerical or categorical data.
   - **a.** The heights, in centimetres, of a group of children
   - **b.** The diameters, in millimetres, of a collection of ball bearings
   - **c.** The numbers of visitors to an exhibit each day
   - **d.** The modes of transport that students in Year 12 take to school
   - **e.** The 10 most-watched television programs in a week
   - **f.** The occupations of a group of 30-year-olds
4. Which of the following represent categorical data?
   - **a.** The numbers of subjects offered to VCE students at various schools
   - **b.** Life expectancies
   - **c.** Species of fish
   - **d.** Blood groups
   - **e.** Years of birth
   - **f.** Countries of birth
   - **g.** Tax brackets
5. MC WE2 Which of the following is *not* discrete data?
   - **A.** Number of players in a netball team
   - **B.** Number of goals scored in a football match
   - **C.** The average temperature in March
   - **D.** The number of Melbourne Storm members
   - **E.** The number of twins in Year 12
6. MC Which of the following is *not* continuous data?
   - **A.** The weight of a person
   - **B.** The number of shots missed in a basketball game
   - **C.** The height of a sunflower in a garden
   - **D.** The length of a cricket pitch
   - **E.** The time taken to run 100 m
7. For each set of numerical data identified in question 3 above, state whether the data are discrete or continuous.
8. MC An example of a numerical variable is:
   - **A.** attitude to 4-yearly elections (for or against)
   - **B.** year level of students
   - **C.** the total attendance at Carlton football matches
   - **D.** position in a queue at the pie stall
   - **E.** television channel numbers shown on a dial

9. MC The weight of each truck-load of woodchips delivered to the wharf during a one-month period was recorded. This is an example of:
   - **A.** categorical and discrete data
   - **B.** discrete data
   - **C.** continuous and numerical data
   - **D.** continuous and categorical data
   - **E.** numerical and discrete data

10. When reading the menu at the local Chinese restaurant, you notice that the dishes are divided into sections. The sections are labelled chicken, beef, duck, vegetarian and seafood. What type of data is this?
11. MC NASA collects data on the distance to other stars in the universe. The distance is measured in light years. What type of data is being collected?
    A. Discrete **B.** Continuous **C.** Nominal **D.** Ordinal **E.** Bivariate
12. MC The number of blue, red, yellow and purple flowers in an award winning display is counted. What type of data is being collected?
    A. Nominal **B.** Ordinal **C.** Discrete **D.** Continuous **E.** Bivariate
13. Students in a performing arts class watch a piece of modern dance and are then asked to rate the quality of the dance as poor, average, above average or excellent. What type of data is being collected?
14. Given the set of data: 12, 6, 21, 15, 8, 2, describe what type of numerical data this data set is.
15. If a tennis tournament seeds the players to organise the draw, what type of categorical data is this?
16. MC The height of the players in the basketball team is measured. The data collected is best described as:
    A. categorical and nominal **B.** categorical and numerical
    C. numerical and discrete **D.** numerical and continuous
    E. none of the above

1.3 Stem plots

A *stem-and-leaf plot*, or **stem plot** for short, is a way of ordering and displaying a set of data, with the advantage that all of the raw data is kept. Since all the individual values of the data are being listed, it is only suitable for smaller data sets (up to about 50 observations).

The following stem plot shows the ages of people attending an advanced computer class.

Stem	Leaf
1	6
2	2 2 3
3	0 2 4 6
4	2 3 6 7
5	3 7
6	1

Key: 2|2 = 22 years old

The ages of the members of the class are 16, 22, 22, 23, 30, 32, 34, 36, 42, 43, 46, 47, 53, 57 and 61.

A stem plot is constructed by splitting the numerals of a record into two parts — the stem, which in this case is the first digit, and the leaf, which is always the last digit.

With your stem-and-leaf plot it is important to include a key so it is clear what the data values represent.

In cases where there are numerous leaves attached to one stem (meaning that the data is heavily concentrated in one area), the stem can be subdivided. Stems are commonly subdivided into halves or fifths. By splitting the stems, we get a clearer picture about the data variation.

WORKED EXAMPLE 3

The number of cars sold in a week at a large car dealership over a 20-week period is given as shown.

16	**12**	**8**	**7**	**26**	**32**	**15**	**51**	**29**	**45**
19	**11**	**6**	**15**	**32**	**18**	**43**	**31**	**23**	**23**

Construct a stem plot to display the number of cars sold in a week at the dealership.

THINK

1. In this example the observations are one- or two-digit numbers and so the stems will be the digits referring to the 'tens', and the leaves will be the digits referring to the units.
   Work out the lowest and highest numbers in the data in order to determine what the stems will be.
2. Before we construct an ordered stem plot, construct an unordered stem plot by listing the leaf digits in the order they appear in the data.
3. Now rearrange the leaf digits in numerical order to create an ordered stem plot.
   Include a key so that the data can be understood by anyone viewing the stem plot.

WRITE

Lowest number = 6
Highest number = 51
Use stems from 0 to 5.

Stem	Leaf
0	8 7 6
1	6 2 5 9 1 5 8
2	6 9 3 3
3	2 2 1
4	5 3
5	1

Stem	Leaf
0	6 7 8
1	1 2 5 5 6 8 9
2	3 3 6 9
3	1 2 2
4	3 5
5	1

Key: 2|3 = 23 cars

WORKED EXAMPLE 4

The masses (in kilograms) of the members of an Under-17 football squad are given as shown.

70.3	**65.1**	**72.9**	**66.9**	**68.6**	**69.6**	**70.8**
72.4	**74.1**	**75.3**	**75.6**	**69.7**	**66.2**	**71.2**
68.3	**69.7**	**71.3**	**68.3**	**70.5**	**72.4**	**71.8**

Display the data in a stem plot.

THINK

1. In this case the observations contain 3 digits. The last digit always becomes the leaf and so in this case the digit referring to the tenths becomes the leaf and the two preceding digits become the stem.
   Work out the lowest and highest numbers in the data in order to determine what the stems will be.
2. Construct an unordered stem plot. Note that the decimal points are omitted since we are aiming to present a quick visual summary of data.
3. Construct an ordered stem plot. Provide a key.

WRITE

Lowest number = 65.1
Highest number = 75.6
Use stems from 65 to 75.

Stem	Leaf
65	1
66	9 2
67	
68	6 3 3
69	6 7 7
70	3 8 5
71	2 3 8
72	9 4 7
73	
74	1
75	3 6

Stem	Leaf
65	1
66	2 9
67	
68	3 3 6
69	6 7 7
70	3 5 8
71	2 3 8
72	4 4 9
73	
74	1
75	3 6

Key: 74 | 1 = 74.1 kg

WORKED EXAMPLE 5

A set of golf scores for a group of professional golfers trialling a new 18-hole golf course is shown on the following stem plot.

Stem	Leaf
6	1 6 6 7 8 9 9 9
7	0 1 1 2 2 3 7

Key: 6 | 1 = 61

Produce another stem plot for these data by splitting the stems into:

a. halves **b. fifths.**

THINK

a. By splitting the stem 6 into halves, any leaf digits in the range 0–4 appear next to the 6, and any leaf digits in the range 5–9 appear next to the 6*. Likewise for the stem 7.

WRITE

a.

Stem	Leaf
6	1
6*	6 6 7 8 9 9 9
7	0 1 1 2 2 3
7*	7

Key: 6 | 1 = 61

b. Alternatively, to split the stems into fifths, each stem would appear 5 times. Any 0s or 1s are recorded next to the first 6. Any 2s or 3s are recorded next to the second 6. Any 4s or 5s are recorded next to the third 6. Any 6s or 7s are recorded next to the fourth 6 and, finally, any 8s or 9s are recorded next to the fifth 6.
This process would be repeated for those observations with a stem of 7.

b.

Stem	Leaf
6	1
6	
6	
6	6 6 7
6	8 9 9 9
7	0 1 1
7	2 2 3
7	
7	7
7	

Key: 6 | 1 = 61

on Resources

- **Interactivity:** Stem plots (int-6242)
- **Interactivity:** Create stem plots (int-6495)

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Units 3 & 4 > AOS DA > Topic 2 > Concept 4 **Stem plots** Summary screen and practice questions

Exercise 1.3 Stem plots

1. **WE3** The number of iPads sold in a month from a department store over 16 weeks is shown.

   28, 31, 18, 48, 38, 25, 21, 16, 33, 42, 35, 39, 49, 30, 29, 28

   Construct a stem plot to display the number of iPads sold over the 16 weeks.
2. The money (correct to the nearest dollar) earned each week by a busker over an 18-week period is shown below. Construct a stem plot for the busker's weekly earnings. What can you say about the busker's earnings?

   5 19 11 27 23 35 18 42 29

   31 52 43 37 41 39 45 32 36

3. **WE4** The test scores (as percentages) of a student in a Year 12 Further Maths class are shown.

   88.0 86.8 92.1 89.8 92.6 90.4 98.3 94.3 87.7

   94.9 98.9 92.0 90.2 97.0 90.9 98.5 92.2 90.8

   Display the data in a stem plot.
4. The heights of members of a squad of basketballers are given below in metres. Construct a stem plot for these data.

   1.96 1.85 2.03 2.21 2.17 1.89 1.99 1.87

   1.95 2.03 2.09 2.05 2.01 1.96 1.97 1.91

5. The ages of those attending an embroidery class are given below. Construct a stem plot for these data and draw a conclusion from it.

   39, 68, 51, 57, 63, 51, 37, 42, 63, 49, 52, 61, 58, 59, 49, 53

6. **MC** The observations shown on the stem plot at right are:
   - **A.** 4, 10, 27, 28, 29, 31, 34, 36, 41
   - **B.** 14, 10, 27, 28, 29, 29, 31, 34, 36, 41, 41
   - **C.** 4, 22, 27, 28, 29, 29, 30, 31, 34, 36, 41, 41
   - **D.** 14, 22, 27, 28, 29 30, 30, 31, 34, 36, 41, 41
   - **E.** 4, 2, 27, 28, 29, 29, 30, 31, 34, 36, 41

Stem	Leaf
0	4
1	
2	2 7 8 9 9
3	0 1 4 6
4	1 1

Key: 2|5 = 25

7. The ages of the mothers of a class of children attending an inner-city kindergarten are given below. Construct a stem plot for these data. Based on your display, comment on the statement 'Parents of kindergarten children are young' (less than 30 years old).

32	30	19	28	25	29	32	28	29	34
32	35	39	30	37	33	29	35	38	33

8. The number of hit outs made by each of the principal ruckmen in each of the AFL teams for Round 11 is recorded below. Construct a stem plot to display these data. Which teams had the three highest scoring ruckmen?

Team	Number of hit outs
Collingwood	19
Bulldogs	41
Kangaroos	29
Port Adelaide	24
Geelong	21
Sydney	31
Melbourne	40
Brisbane	25

Team	Number of hit outs
Adelaide	32
St Kilda	34
Essendon	31
Carlton	26
West Coast	29
Fremantle	22
Hawthorn	33
Richmond	28

9. The 2015 weekly median rental price for a 2-bedroom unit in a number of Melbourne suburbs is given in the following table. Construct a stem plot for these data and comment on it.

Suburb	Weekly rental ($)
Alphington	400
Box Hill	365
Brunswick	410
Burwood	390
Clayton	350
Essendon	350
Hampton	430
Ivanhoe	395
Kensington	406
Malvern	415

Suburb	Weekly rental ($)
Moonee Ponds	373
Newport	380
North Melbourne	421
Northcote	430
Preston	351
St Kilda	450
Surrey Hills	380
Williamstown	330
Windsor	423
Yarraville	390

10. **WE5** LeBron James's scores for his last 16 games are shown in the following stem plot.

Stem	Leaf
2	0 2 2 5 6 8 8
3	3 3 3 7 7 8 9 9 9

Key: 2|0 = 20

Produce another stem plot for these data by splitting the stems into:

a. halves

b. fifths.

11. The data below give the head circumference (correct to the nearest cm) of 16 four-year-old girls.

48 49 47 52 51 50 49 48

50 50 53 52 43 47 49 50

Construct a stem plot for head circumference, using:

a. the stems 4 and 5

b. the stems 4 and 5 split into halves

c. the stems 4 and 5 split into fifths.

12. For each of the following, write down all the pieces of data shown on the stem plot. The key used for each stem plot is 3|2 = 32.

a.

Stem	Leaf
0	1 2
0*	5 8
1	2 3 3
1*	6 6 7
2	1 3 4
2*	5 5 6 7
3	0 2

b.

Stem	Leaf
1	0 1
2	3 3
3	0 5 9
4	1 2 7
5	5
6	2

c.

Stem	Leaf
10	1 2
11	5 8
12	2 3 3
13	6 6 7
14	1 3 4
15	5 5 6 7

d.

Stem	Leaf
5	0 1
5	3 3
5	4 5 5
5	6 6 7
5	9

e.

Stem	Leaf
0	1 4
0*	5 8
1	0 2
1*	6 9 9
2	1 1
2*	5 9

13. A random sample of 20 screws is taken and the length of each is recorded to the nearest millimetre below.

23 15 18 17 17 19 22 19 20 16

20 21 19 23 17 19 21 23 20 21

Construct a stem plot for screw length using:

a. the stems 1 and 2

b. the stems 1 and 2 split into halves

c. the stems 1 and 2 split into fifths.

Use your plots to help you comment on the screw lengths.

14. The first 20 scores that came into the clubhouse in a local golf tournament are shown. Construct a stem plot for these data.

102 98 83 92 85 99 104 112 88 91

78 87 90 94 83 93 72 100 92 88

15. Golf handicaps are designed to even up golfers on their abilities. Their handicap is subtracted from their score to create a net score. Construct a stem plot on the golfer's net scores below and comment on how well the golfers are handicapped.

76, 76, 73, 74, 69, 72, 73, 86, 73, 72, 75, 74, 77, 73, 75, 75, 71, 71, 68, 67

16. The following data represents percentages for a recent Further Maths test.

63 71 70 89 88 69 76 83 93 80 73

77 91 75 81 84 87 78 97 89 98 60

Construct a stem plot for the test percentages, using:

a. the stems 6, 7, 8 and 9

b. the stems 6, 7, 8 and 9 split into halves.

17. The following data was collected from a company that compared the battery life (measured in minutes) of two different Ultrabook computers. To complete the test they ran a series of programs on the two computers and measured how long it took for the batteries to go from 100% to 0%.

Computer 1	358	376	392	345	381	405	363	380	352	391	410	366
Computer 2	348	355	361	342	355	362	353	358	340	346	357	352

a. Draw a back-to-back stem plot (using the same stem) of the battery life of the two Ultrabook computers.

b. Use the stem plot to compare and comment on the battery life of the two Ultrabook computers.

18. The heights of 20 Year 8 and Year 10 students (to the nearest centimetre), chosen at random, are measured. The data collected is shown in the table below.

Year 8	151	162	148	153	165	157	172	168	155	164	175	161	155	160	149	155	163	171	166	150
Year 10	167	164	172	158	169	159	174	177	165	156	154	160	178	176	182	152	167	185	173	178

a. Draw a back-to-back stem plot of the data.

b. Comment on what the stem plot tells you about the heights of Year 8 and Year 10 students.

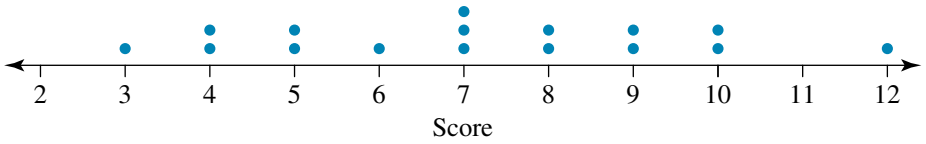
1.4 Dot plots, frequency tables and histograms, and bar charts

Dot plots, frequency **histograms** and **bar charts** display data in graphical form.

1.4.1 Dot plots

In picture graphs, a single picture represents each data value. Similarly, in dot plots, a single dot represents each data value. Dot plots are used to display discrete data where values are not spread out very much. They are also used to display categorical data.

When representing discrete data, dot plots have a scaled horizontal axis and each data value is indicated by a dot above this scale. The end result is a set of vertical 'lines' of evenly-spaced dots.



WORKED EXAMPLE 6

The number of hours per week spent on art by 18 students is given as shown.

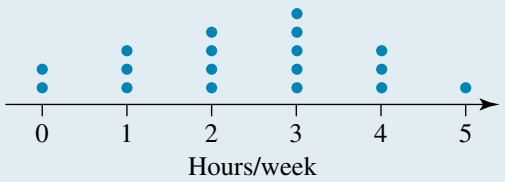
4	**0**	**3**	**1**	**3**	**4**	**2**	**2**	**3**
4	**1**	**3**	**2**	**5**	**3**	**2**	**1**	**0**

Display the data as a dot plot.

THINK

1. Determine the lowest and highest scores and then draw a suitable scale.
2. Represent each score by a dot on the scale.

DRAW



1.4.2 Frequency tables and histograms

A histogram is a useful way of displaying large data sets (say, over 50 observations). The vertical axis on the histogram displays the frequency and the horizontal axis displays class intervals of the variable (for example, height or income).

The vertical bars in a histogram are adjacent with no gaps between them, as we generally consider the numerical data scale along the horizontal axis as continuous. Note, however, that histograms can also represent discrete data. It is common practice to leave a small gap before the first bar of a histogram.

When data are given in raw form — that is, just as a list of figures in no particular order — it is helpful to first construct a **frequency table** before constructing a histogram.

WORKED EXAMPLE 7

The data below show the distribution of masses (in kilograms) of 60 students in Year 7 at Northwood Secondary College. Construct a frequency histogram to display the data more clearly.

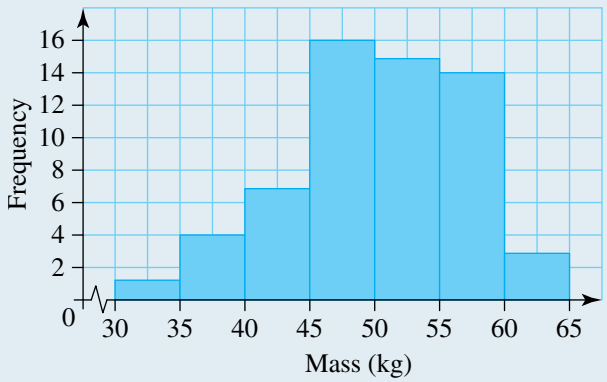
45.7	45.8	45.9	48.2	48.3	48.4	34.2	52.4	52.3	51.8	45.7	56.8	56.3	60.2	44.2
53.8	43.5	57.2	38.7	48.5	49.6	56.9	43.8	58.3	52.4	54.3	48.6	53.7	58.7	57.6
45.7	39.8	42.5	42.9	59.2	53.2	48.2	36.2	47.2	46.7	58.7	53.1	52.1	54.3	51.3
51.9	54.6	58.7	58.7	39.7	43.1	56.2	43.0	56.3	62.3	46.3	52.4	61.2	48.2	58.3

THINK

1. First construct a frequency table. The lowest data value is 34.2 and the highest is 62.3. Divide the data into class intervals. If we started the first class interval at, say, 30 kg and ended the last class interval at 65 kg, we would have a range of 35. If each interval was 5 kg, we would then have 7 intervals which is a reasonable number of class intervals. While there are no set rules about how many intervals there should be, somewhere between about 5 and 15 class intervals is usual. Complete a tally column using one mark for each value in the appropriate interval. Add up the tally marks and write them in the frequency column.
2. Check that the frequency column totals 60. The data are in a much clearer form now.
3. A histogram can be constructed.

WRITE/DRAW

Class interval	Tally	Frequency
30–	|	1
35–	||||	4
40–	𝍸 ||	7
45–	𝍸 𝍸 𝍸 |	16
50–	𝍸 𝍸 𝍸	15
55–	𝍸 𝍸 ||||	14
60–	|||	3
	Total	60



When constructing a histogram to represent continuous data, as in Worked example 7, the bars will sit between two values on the horizontal axis which represent the class intervals. When dealing with discrete data the bars should appear above the middle of the value they're representing.

WORKED EXAMPLE 8

The marks out of 20 received by 30 students for a book-review assignment are given in the frequency table.

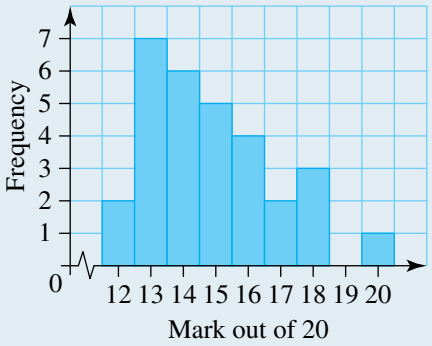
Mark	12	13	14	15	16	17	18	19	20
Frequency	2	7	6	5	4	2	3	0	1

Display these data on a histogram.

THINK

In this case we are dealing with integer values (discrete data). Since the horizontal axis should show a class interval, we extend the base of each of the columns on the histogram halfway either side of each score.

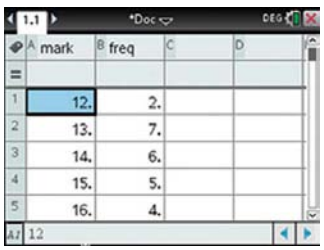
DRAW



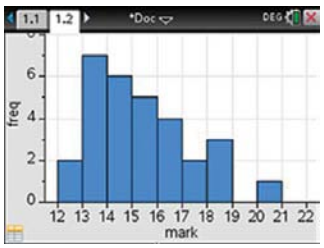
TI | THINK

1. On a Lists & Spreadsheet page, enter the data into the lists, named Mark and Freq.

WRITE



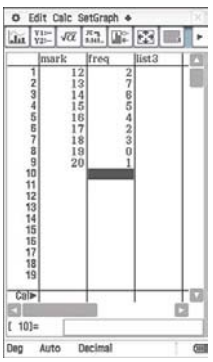
2. Press CTRL + I and select:
   5: Add Data & Statistics
   Use the tab key to move to the horizontal axis and press the click key or ENTER to place *mark* on the horizontal axis.
   To place *freq* on the vertical axis, move the cursor to the vertical axis and press MENU
   Select:
   1: Plot Type
   3: Histogram
   Press MENU:
   2: Plot Properties
   9: Add Y Summary List
   Select freq.



CASIO | THINK

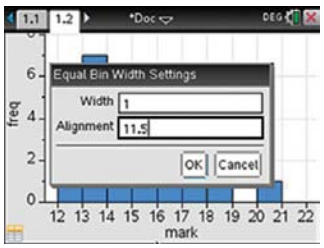
1. On a Statistics screen, enter the data into the lists, named Mark and Freq.

WRITE

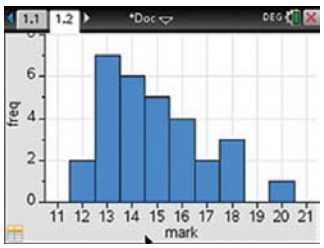


2. Tap:
   SetGraph
   Setting
   Complete the fields under Set StatGraphs as:
   Draw: On
   Type: Histogram
   XList: main\mark
   Freq: main\freq
   Tap Set to lock your selection.
   Tap graph icon.

3. As the data in this example is discrete data, to change the column width, press MENU and select:
   2: Plot Properties
   2: Histogram Properties
   2: Bin Settings
   1: Equal Bin Width
   Complete the fields as shown.
   Press ENTER.
   To Zoom out, press MENU and select:
   5: Window/Zoom
   2: Zoom-Data

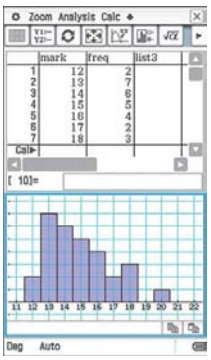


4. The histogram is shown on the screen.



3. Complete the Set Interval fields as:
   HStart: 11.5
   HStep: 1
   Tap OK.

4. The histogram is shown on the screen.

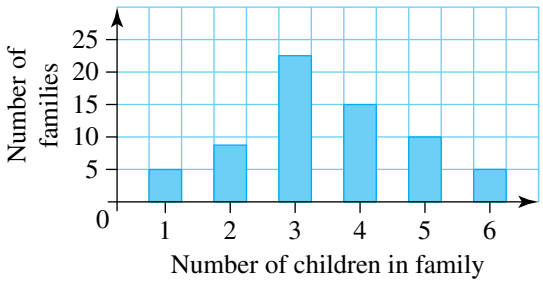


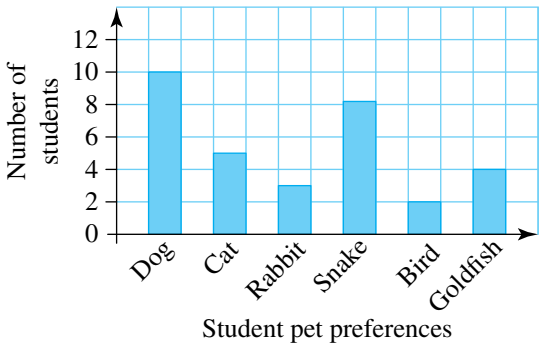
1.4.3 Bar charts

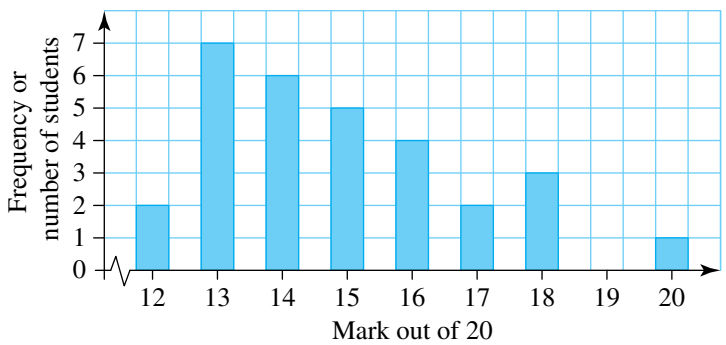
A *bar chart* consists of bars of equal width *separated* by small, equal spaces and may be arranged either *horizontally* or *vertically*. Bar charts are often used to display categorical data.

In bar charts, the frequency is graphed against a variable as shown in both figures.

The variable may or may not be numerical. However, if it is, the variable should represent discrete data because the scale is broken by the gaps between the bars.







The bar chart shown represents the data presented in Worked example 8. It could also have been drawn with horizontal bars (rows).

1.4.4 Segmented bar charts

A **segmented bar chart** is a single bar which is used to represent all the data being studied. It is divided into segments, each segment representing a particular group of the data. Generally, the information is presented as percentages and so the total bar length represents 100% of the data.

WORKED EXAMPLE 9

The table shown represents fatal road accidents for one year in Australia. Construct a segmented bar chart to represent this data.

	Accidents involving fatalities								
Year	NSW	Vic.	Qld	SA	WA	Tas.	NT	ACT	Aust.
2008	376	278	293	87	189	38	67	12	**1340**

Source: Australian Bureau of Statistics 2010, Year Book Australia 2009–10, cat. no. 1301.0. ABS. Canberra, table 24.20, p. 638.

THINK

1. To draw a segmented bar chart the data needs to be converted to percentages.

WRITE/DRAW

State	Number of accidents	Percentage
NSW	376	$376 \div 1340 \times 100\% = 28.1\%$
Vic.	278	$278 \div 1340 \times 100\% = 20.7\%$
Qld	293	$293 \div 1340 \times 100\% = 21.9\%$
SA	87	$87 \div 1340 \times 100\% = 6.5\%$
WA	189	$189 \div 1340 \times 100\% = 14.1\%$
Tas.	38	$38 \div 1340 \times 100\% = 2.8\%$
NT	67	$67 \div 1340 \times 100\% = 5.0\%$
ACT	14	$12 \div 1340 \times 100\% = 0.9\%$

2. To draw the segmented bar chart to scale decide on its overall length, let's say 100 mm.

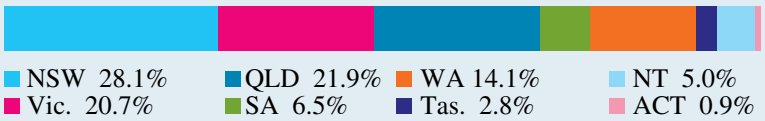
Measure a line 100 mm in length.

3. Therefore NSW = 28.1%, represented by 28.1 mm.
Vic = 20.7%, represented by 20.7 mm and so on.

Measure off each segment and check it adds to the set 100 mm.

4. Draw the answer and colour code it to represent each of the states and territories.

The segmented bar chart is drawn to scale. An appropriate scale would be constructed by drawing the total bar 100 mm long, so that 1 mm represents 1%. That is, accidents in NSW would be represented by a segment of 28.1 mm, those in Victoria by a segment of 20.7 mm and so on. Each segment is then labelled directly, or a key may be used.

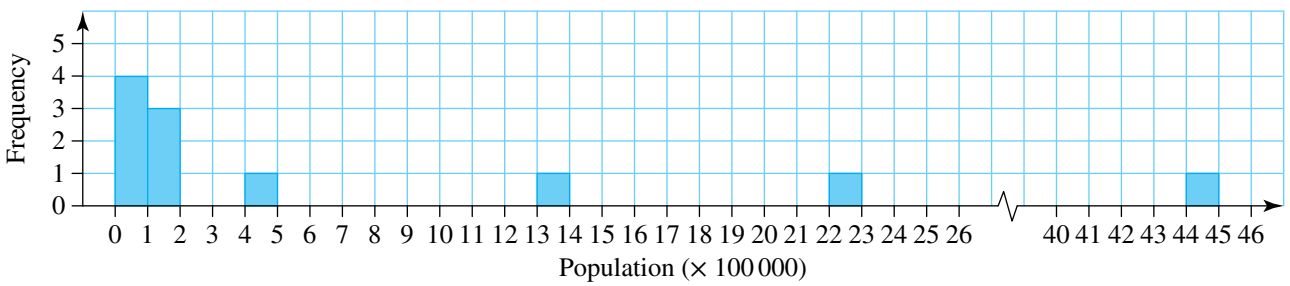


1.4.5 Using a log (base 10) scale

Sometimes a data set will contain data points that vary so much in size that plotting them using a traditional scale becomes very difficult. For example, if we are studying the population of different cities in Australia we might end up with the following data points:

City	Population
Adelaide	1 304 631
Ballarat	98 543
Brisbane	2 274 460
Cairns	146 778
Darwin	140 400
Geelong	184 182
Launceston	86 393
Melbourne	4 440 328
Newcastle	430 755
Shepparton	49 079
Wagga Wagga	55 364

A histogram splitting the data into class intervals of 100 000 would then appear as follows:



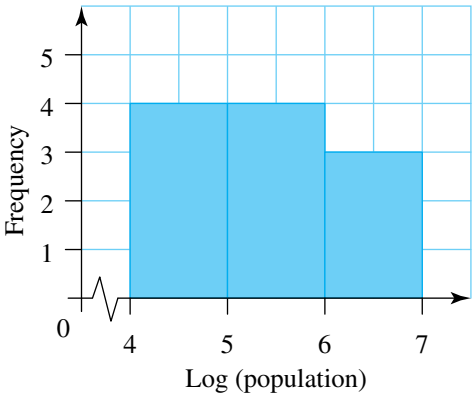
A way to overcome this is to write the numbers in **logarithmic (log) form**. The log of a number is the power of 10 which creates this number.

$$\log(10) = \log\left(10^1\right) = 1$$
$$\log(100) = \log\left(10^2\right) = 2$$
$$\log(1000) = \log\left(10^3\right) = 3$$
$$\vdots$$
$$\log(10^n) = n$$

Not all logarithmic values are integers, so use the log key on CAS to determine exact logarithmic values.
For example, from our previous example showing the population of different Australian cities:
log(4 440 328) = 6.65 (correct to 2 decimal places)
log(184 182) = 5.27 (correct to 2 decimal places)
log(49 079) = 4.69 (correct to 2 decimal places)
and so on …

We can then group our data using class intervals based on log values (from 4 to 7) to come up with the following frequency table and histogram.

Population	Log (population)	Frequency
10 000–	4–5	4
100 000–	5–6	4
1 000 000–	6–7	3



WORKED EXAMPLE 10

The following table shows the average weights of 10 different adult mammals.

Mammal	Weight (kg)
African elephant	4800
Black rhinoceros	1100
Blue whale	136 000
Giraffe	800
Gorilla	140
Humpback whale	30 000
Lynx	23
Orang-utan	64
Polar bear	475
Tasmanian devil	7

Display the data in a histogram using a log base 10 scale.

THINK

WRITE/DRAW

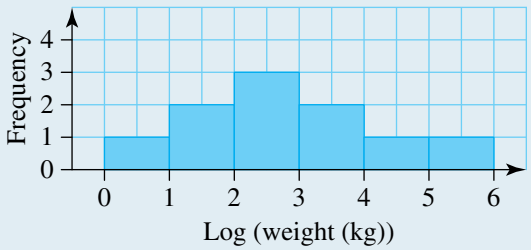
1. Using CAS, calculate the logarithmic values of all of the weights, e.g.:
   log(4800) = 3.68 (correct to 2 decimal places)

Weight	Log (weight (kg))
4800	3.68
1100	3.04
136 000	5.13
800	2.90
140	2.15
30 000	4.48
23	1.36
64	1.81
475	2.68
7	0.85

2. Group the logarithmic weights into class intervals and create a frequency table for the groupings.

Log (weight (kg))	Frequency
0–1	1
1–2	2
2–3	3
3–4	2
4–5	1
5–6	1

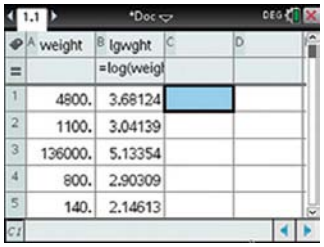
3. Construct a histogram of the data set.



TI | THINK

1. On a Lists & Spreadsheet page, enter the data into the list named weight.
   Label Column B as lgwght and in the formula cell for Column B, complete the entry line as:
   = log(*weight*)
   Press ENTER.

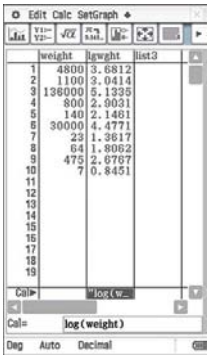
WRITE



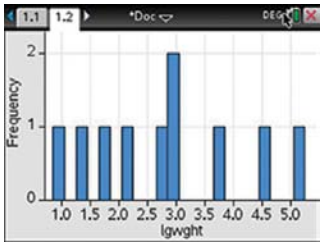
CASIO | THINK

1. On a Statistics screen, enter the data into List1 and rename the list weight.
   Label List2 as lgwght.
   Place the cursor in the Cal cell at the base of the List2 column and complete the entry line as:
   = log(*weight*)
   Tap EXE.

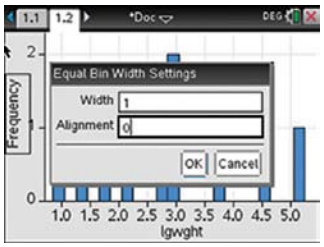
WRITE



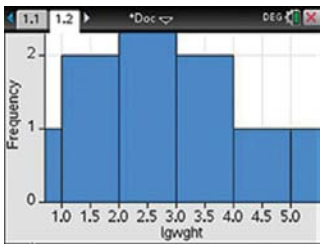
2. Press CTRL + I and select:
   5: Add Data & Statistics
   Use the tab key to move to the horizontal axis and press the click key or ENTER to place *lgwght* on the horizontal axis.
   Press MENU and select:
   1: Plot Type
   3: Histogram
   Press ENTER.



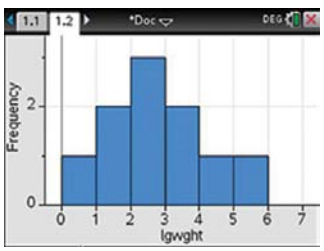
3. To change the column width, press MENU and select:
   2: Plot Properties
   2: Histogram Properties
   2: Bin Settings
   1: Equal Bin Width
   Complete the fields as shown.
   Press ENTER.



4. To Zoom out, press MENU and select:
   5: Window/Zoom
   2: Zoom-Data



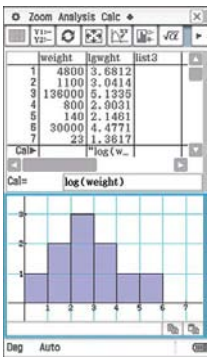
5. The histogram is shown on the screen.



2. Tap:
   SetGraph
   Setting
   Complete the fields under Set StatGraphs as:
   Draw: On
   Type: Histogram
   XList: main\lgwght
   Freq: main\1
   Tap Set to lock your selection.
   Tap graph icon.

3. Complete the Set Interval fields as:
   HStart: 11.5
   HStep: 1
   Tap OK.

4. The histogram is shown on the screen.



Interpreting log (base 10) values

If we are given values in logarithmic form, by raising 10 to the power of the logarithmic number we can determine the conventional number.

For example, the number 3467 in log (base 10) form is 3.54, and $10^{3.54} = 3467$.

We can use this fact to compare values in log (base 10) form, as shown in the following worked example.

WORKED EXAMPLE 11

The Richter Scale measures the magnitude of earthquakes using a log (base 10) scale. How many times stronger is an earthquake of magnitude 7.4 than one of magnitude 5.2? Give your answer correct to the nearest whole number.

THINK	WRITE
1. Calculate the difference between the magnitude of the two earthquakes.	$7.4 - 5.2 = 2.2$
2. Raise 10 to the power of the difference in magnitudes.	$10^{2.2} = 158.49$ (correct to 2 decimal places)
3. Express the answer in words.	The earthquake of magnitude 7.4 is 158 times stronger than the earthquake of magnitude 5.2.

on Resources

- **Interactivity:** Create histograms (int-6494)
- **Interactivity:** Dot plots, frequency tables and histograms, and bar charts (int-6243)
- **Interactivity:** Create bar charts (int-6493)

studyon

Units 3 & 4 > AOS DA > Topic 2 > Concepts 1, 2, 3, 5, 7

Dot plots Summary screen and practice questions
Frequency histograms Summary screen and practice questions
Bar charts Summary screen and practice questions
Segmented graphs Summary screen and practice questions
Log base 10 scale Summary screen and practice questions

Exercise 1.4 Dot plots, frequency tables and histograms, and bar charts

1. WE6 The number of questions completed for maths homework each night by 16 students is shown below.

5	6	5	9	10	10	6	8
10	9	8	5	7	8	7	9

Display the data as a dot plot.

2. The data below represent the number of hours each week that 40 teenagers spent on household chores. Display these data on a bar chart and a dot plot.

2 5 2 0 8 7 8 5 1 0 2 1 8 0 4 2 2 9 8 5

7 5 4 2 1 2 9 8 1 2 8 5 8 10 0 3 4 5 2 8

3. The number of hours spent on homework for a group of 20 Year 12 students each week is shown.

15 15 18 21 20 21 24 24 20 18

18 21 22 20 24 20 24 21 18 15

Display the data as a dot plot.

4. **WE7** The data shows the distribution of heights (in cm) of 40 students in Year 12. Construct a frequency histogram to display the data more clearly.

167 172 184 180 178 166 154 150 164 161

187 159 182 177 172 163 179 181 170 176

177 162 172 184 188 179 189 192 164 160

166 169 163 185 178 183 190 170 168 159

5. Construct a frequency table for each of the following sets of data.

a. 4.3, 4.5, 4.7, 4.9, 5.1, 5.3, 5.5, 5.6, 5.2, 3.6, 2.5, 4.3, 2.5, 3.7, 4.5, 6.3, 1.3

b. 11, 13, 15, 15, 16, 18, 20, 21, 22, 21, 18, 19, 20, 16, 18, 20, 16, 10, 23, 24, 25, 27, 28, 30, 35, 28, 27, 26, 29, 30, 31, 24, 28, 29, 20, 30, 32, 33, 29, 30, 31, 33, 34

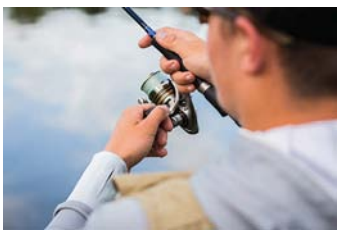
c. 0.4, 0.5, 0.7, 0.8, 0.8, 0.9, 1.0, 1.1, 1.2, 1.0, 1.3, 0.4, 0.3, 0.9, 0.6

Using the frequency tables above, construct a histogram by hand for each set of data.

6. The number of dogs at a RSPCA kennel each week is as shown. Construct a frequency table for these data.

7, 6, 2, 12, 7, 9, 12, 10, 5, 7, 9, 4, 5, 9, 3, 2, 10, 8, 9, 7, 9, 10, 9, 4, 3, 8, 9, 3, 7, 9

7. **WE8** The number of fish caught by 30 anglers in a fishing competition are given in the frequency table below.



Fish	0	1	2	3	4	5	7
Frequency	4	7	4	6	5	3	1

Display these data on a histogram.

8. The number of fatal car accidents in Victoria each week is given in the frequency table for a year.

Fatalities	0	1	2	4	6	7	9	10	13
Frequency	12	3	6	10	7	6	5	2	1

Display these data on a histogram.

9. Using the frequency table in question **6**, construct a histogram by hand.

10. Using CAS construct a histogram from the data in question **6**, and compare it to the histogram in question **9**.

11. WE9 The following table shows how many goals each of the 18 AFL team's leading goal kickers scored in the 2014 regular season. Construct a segmented bar chart to represent this data.

Club	Goals
Adelaide Crows	51
Brisbane Lions	33
Carlton	29
Collingwood	39
Essendon	27
Fremantle	49
Geelong Cats	62
Gold Coast Suns	46
GWS Giants	29

Club	Goals
Hawthorn	62
Melbourne	20
North Melbourne	41
Port Adelaide	62
Richmond	58
St Kilda	49
Sydney Swans	67
West Coast Eagles	61
Western Bulldogs	37

12. Information about adult participation in sport and physical activities in 2017–18 is shown in the following table. Draw a segmented bar graph to compare the participation of all persons from various age groups. Comment on the statement, 'Only young people participate in sport and physical activities'.

Participation in sport and physical activities — 2017–18

	Males		Females		Persons	
Age group (years)	**Number (×1000)**	**Participation rate (%)**	**Number (×1000)**	**Participation rate (%)**	**Number (×1000)**	**Participation rate (%)**
18–24	735.2	73.3	671.3	71.8	1406.4	72.6
25–34	1054.5	76.3	1033.9	74.0	2088.3	75.1
35–44	975.4	66.7	1035.9	69.1	2011.2	68.0
45–54	871.8	63.5	923.4	65.7	1795.2	64.6
55–64	670.1	60.4	716.3	64.6	1386.5	62.5
65 and over	591.0	50.8	652.9	48.2	1243.9	49.4
Total	**4898**	**64.6**	**5033.7**	**64.4**	**9931.5**	**64.5**

13. A group of students was surveyed, asking how many children were in their family. The data is shown in the table.

Number of children	1	2	3	4	5	6	9
Number of families	12	18	24	10	8	3	1

Construct a bar chart that displays the data.

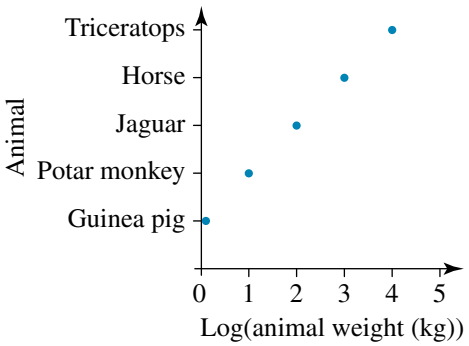
14. **WE10** The following table shows the average weights of 10 different adult mammals.

Mammal	**Weight (kg)**
Black wallaroo	18
Capybara	55
Cougar	63
Fin whale	70 000
Lion	175
Ocelot	9
Pygmy rabbit	0.4
Red deer	200
Quokka	4
Water buffalo	725

Display the data in a histogram using a log base 10 scale, using class intervals of width 1.

15. **MC** The following graph shows the weights of animals. If a gorilla has a weight of 207 kilograms then its weight is between that of:

A. Potar monkey and jaguar.
B. horse and triceratops.
C. guinea pig and Potar monkey.
D. jaguar and horse.
E. none of the above.



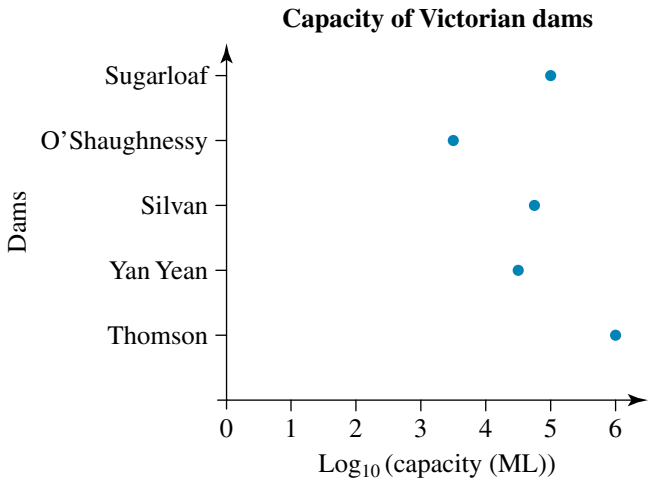
16. **MC** The following table shows a variety of top speeds.

F1 racing car	370 000 m/h
V8 supercar	300 000 m/h
Cheetah	64 000 m/h
Space shuttle	28 000 000 m/h
Usain Bolt	34 000 m/h

The correct value, to 2 decimal places, for a cheetah's top speed using a log (base 10) scale would be:

A. 5.57 **B.** 5.47 **C.** 7.45
D. 4.81 **E.** 11.07

17. MC The following graph represents the capacity of five Victorian dams.



a. The capacity of Thomson Dam is closest to:

A. 100 000 ML **B.** 1 000 000 ML **C.** 500 000 ML
D. 5000 ML **E.** 40 000 ML

b. The capacity of Sugarloaf Dam is closest to:

A. 100 000 ML **B.** 1 000 000 ML **C.** 500 000 ML
D. 5000 ML **E.** 40 000 ML

c. The capacity of Silvan Dam is between which range?

A. 1 and 10 ML **B.** 10 and 100 ML **C.** 1000 and 10 000 ML
D. 10 000 and 100 000 ML **E.** 100 000 and 1 000 000 ML

The table shows the flow rate of world famous waterfalls. Use this table to complete questions **18** and **19**.

Waterfalls	Flow rate
Victoria Falls	1 088 m^3/s
Niagara Falls	2 407 m^3/s
Celilo Falls	5 415 m^3/s
Khane Phapheng Falls	11 610 m^3/s
Boyoma Falls	17 000 m^3/s

18. MC The correct value, correct to 2 decimal places, to be plotted for Niagara Falls' flow rate using a log(base 10) scale would be:

A. 3.38 **B.** 3.04 **C.** 4.06
D. 4.23 **E.** 5.10

19. MC The correct value, correct to 2 decimal places, to be plotted for Victoria Falls' flow rate using a log (base 10) scale would be:

A. 3.38 **B.** 3.04 **C.** 4.06
D. 4.23 **E.** 5.10

20. WE11 The Richter scale measures the magnitude of earthquakes using a log (base 10) scale. How many times stronger is an earthquake of magnitude 8.1 than one of magnitude 6.9? Give your answer correct to the nearest whole number.

21. The pH scale measures acidity using a log (base 10) scale. For each decrease in pH of 1, the acidity of a substance increases by a factor of 10.

If a liquid's pH value decreases by 0.7, by how much has the acidity of the liquid increased?

22. Using the information provided in the table below:

a. calculate the proportion of residents who travelled in 2017 to each of the countries listed

b. draw a segmented bar graph showing the major destinations of Australians travelling abroad in 2014.

Short-term resident departures by major destinations					
	2013 (×1000)	**2014 (×1000)**	**2015 (×1000)**	**2016 (×1000)**	**2017 (×1000)**
New Zealand	815.8	835.4	864.7	902.1	921.1
United States of America	376.1	426.3	440.3	479.1	492.3
United Kingdom	375.1	404.2	412.8	428.5	420.3
Indonesia	335.1	319.7	194.9	282.6	380.7
China (excluding Special Administrative Regions (SARs))	182.0	235.1	251.0	284.3	277.3
Thailand	188.2	202.7	288.0	374.4	404.1
Fiji	175.4	196.9	202.4	200.3	236.2
Singapore	159.0	188.5	210.9	221.5	217.8
Hong Kong (SAR of China)	152.6	185.7	196.3	206.5	213.1
Malaysia	144.4	159.8	168.0	181.3	191.0

23. The number of people who attended the Melbourne Grand Prix are shown.

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Attendance 000s	360	359	301	301	303	287	305	298	313	323

a. Construct a bar chart by hand to display the data.

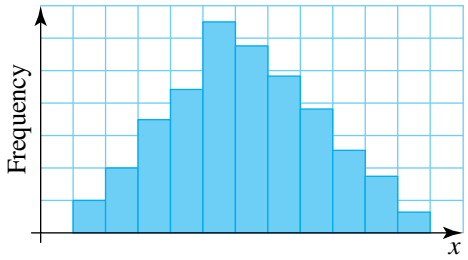
b. Use CAS to construct a bar chart and compare the two charts.

1.5 Describing the shape of stem plots and histograms

1.5.1 Symmetric distributions

The data shown in the histogram below can be described as **symmetric**.

There is a single peak and the data trail off on both sides of this peak in roughly the same fashion.



Similarly, in the stem plot at right, the distribution of the data could be described as symmetric.

The single peak for these data occur at the stem 3. On either side of the peak, the number of observations reduces in approximately matching fashion.

Stem	Leaf
0	7
1	2 3
2	2 4 5 7 9
3	0 2 3 6 8 8
4	7 8 9 9
5	2 7 8
6	1 3

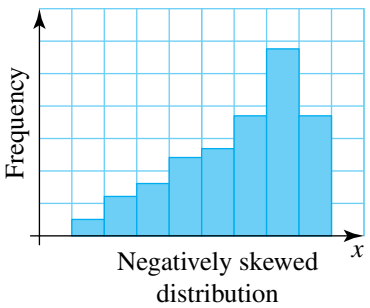
Key: 0|7 = 7

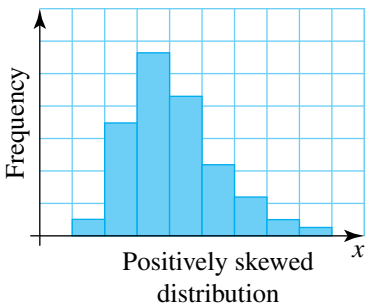
1.5.2 Skewed distributions

Each of the histograms shown on next page are examples of skewed distributions.

The figure on the left shows data which are **negatively skewed**. The data in this case peak to the right and trail off to the left.

The figure on the right shows **positively skewed** data. The data in this case peak to the left and trail off to the right.





WORKED EXAMPLE 12

The ages of a group of people who were taking out their first home loan is shown below.

Stem	Leaf
1	9 9
2	1 2 4 6 7 8 8 9
3	1 1 1 2 3 4 7
4	1 3 5 6
5	2 3
6	7

Key: 1 | 9 = 19 years old

Describe the shape of the distribution of these data.

THINK

Check whether the distribution is symmetric or skewed. The peak of the data occurs at the stem 2. The data trail off as the stems increase in value. This seems reasonable since most people would take out a home loan early in life to give themselves time to pay it off.

WRITE

The data are positively skewed.

studyon

Units 3 & 4 > AOS DA > Topic 2 > Concept 6 **Shape** Summary screen and practice questions

Exercise 1.5 Describing the shape of stem plots and histograms

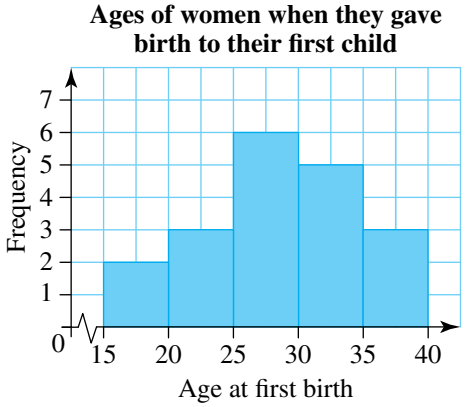
1. WE12 The ages of a group of people when they bought their first car are shown.

Stem	Leaf
1	7 7 8 8 8 8 9 9
2	0 0 1 2 3 6 7 8 9
3	1 4 7 9
4	4 8
5	3

Key: 1 | 7 = 17 years old

Describe the shape of the distribution of these data.

2. The ages of women when they gave birth to their first child is shown. Describe the shape of the distribution of the data.



3. For each of the following stem plots, describe the shape of the distribution of the data.

a.

Stem	Leaf
0	1 3
1	2 4 7
2	3 4 4 7 8
3	2 5 7 9 9 9 9
4	1 3 6 7
5	0 4
6	4 7
7	1

Key: 1|2 = 12

b.

Stem	Leaf
1	3
2	6
3	3 8
4	2 6 8 8 9
5	4 7 7 7 8 9 9
6	0 2 2 4 5

Key: 2|6 = 2.6

c.

Stem	Leaf
2	3 5 5 6 7 8 9 9
3	0 2 2 3 4 6 6 7 8 8
4	2 2 4 5 6 6 6 7 9
5	0 3 3 5 6
6	2 4
7	5 9
8	2
9	7
10	

Key: 10|4 = 04

d.

Stem	Leaf
1	
1*	5
2	1 4
2*	5 7 8 8 9
3	1 2 2 3 3 3 4 4
3*	5 5 5 6
4	3 4
4*	

Key: 2|4 = 24

e.

Stem	Leaf
3	
3	8 9
4	0 0 1 1 1
4	2 3 3 3 3 3 4
4	5 5 5
4	6 7
4	8

Key: 4|3 = 0.43

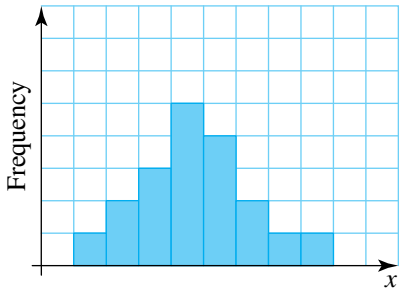
f.

Stem	Leaf
60	2 5 8
61	1 3 3 6 7 8 9
62	0 1 2 4 6 7 8 8 9
63	2 2 4 5 7 8
64	3 6 7
65	4 5 8
66	3 5
67	4

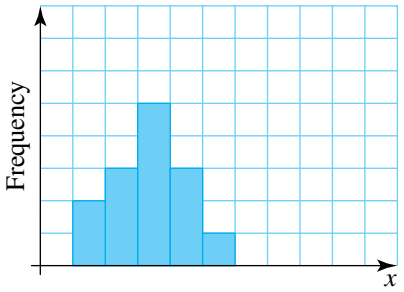
Key: 62|3 = 623

4. For each of the following histograms, describe the shape of the distribution of the data and comment on the existence of any outliers.

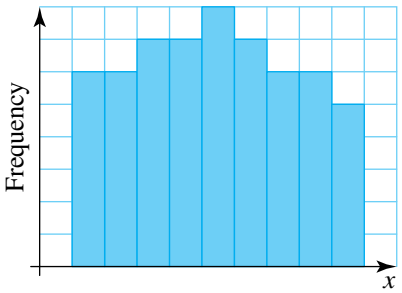
a.



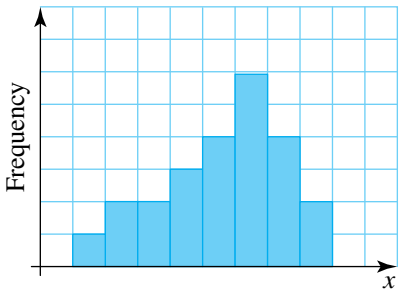
b.



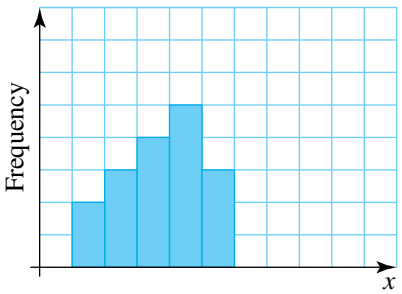
c.



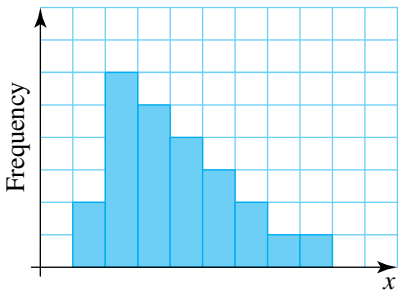
d.



e.



f.



5. **MC** The distribution of the data shown in this stem plot could be described as:

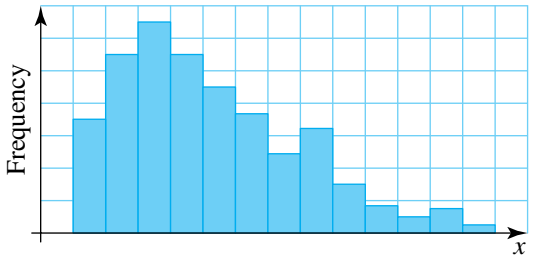
A. negatively skewed
B. negatively skewed and symmetric
C. positively skewed
D. positively skewed and symmetric
E. symmetric

Stem	Leaf
0	1
0	2
0	4 4 5
0	6 6 6 7
0	8 8 8 8 9 9
1	0 0 0 1 1 1 1
1	2 2 2 3 3 3
1	4 4 5 5
1	6 7 7
1	8 9

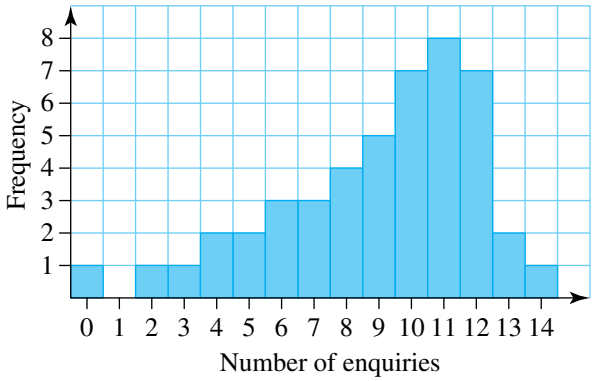
Key: 1|8 = 18

6. MC The distribution of the data shown in the histogram could be described as:

A. negatively skewed
B. negatively skewed and symmetric
C. positively skewed
D. positively skewed and symmetric
E. symmetric



7. The average number of product enquiries per day received by a group of small businesses who advertised in the Yellow Pages telephone directory is given below. Describe the shape of the distribution of these data.



8. The number of nights per month spent interstate by a group of flight attendants is shown in the stem plot. Describe the shape of distribution of these data and explain what this tells us about the number of nights per month spent interstate by this group of flight attendants.

Stem	Leaf
0	0 0 1 1
0	2 2 3 3 3 3 3 3 3 3
0	4 4 5 5 5 5 5
0	6 6 6 6 7
0	8 8 8 9
1	0 0 1
1	4 4
1	5 5
1	7

Key: 1|4 = 14 nights

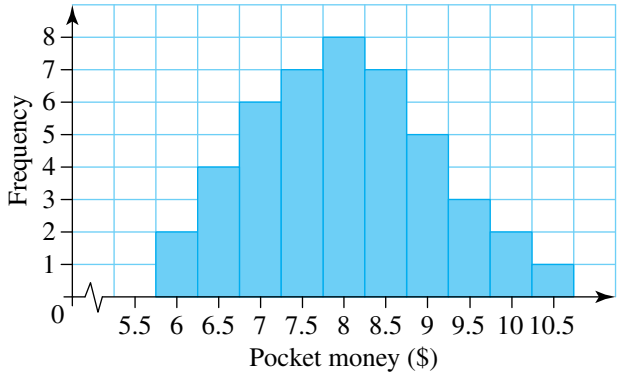
9. The mass (correct to the nearest kilogram) of each dog at a dog obedience school is shown in the stem plot.

a. Describe the shape of the distribution of these data.
b. What does this information tell us about this group of dogs?

Stem	Leaf
0	4
0*	5 7 9
1	1 2 4 4
1*	5 6 6 7 8 9
2	1 2 2 3
2*	6 7

Key: 0|4 = 4 kg

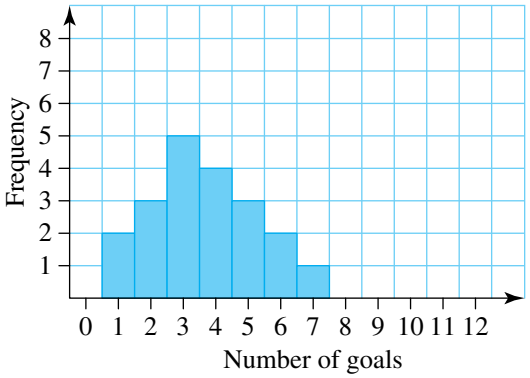
10. The amount of pocket money (correct to the nearest 50 cents) received each week by students in a Grade 6 class is illustrated in the histogram.



a. Describe the shape of the distribution of these data.

b. What conclusions can you reach about the amount of pocket money received weekly by this group of students?

11. Statistics were collected over 3 AFL games on the number of goals kicked by forwards over 3 weeks. This is displayed in the histogram.



a. Describe the shape of the histogram.

b. Use the histogram to determine:

i. the number of players who kicked 3 or more goals over the 3 weeks

ii. the percentage of players who kicked between 2 and 6 goals inclusive over the 3 weeks.

12. The number of hours a group of students exercise each week is shown in the stem plot.

a. Describe the shape of the distribution of these data.

b. What does this sample data tell us about this group of students?

Stem	Leaf
0	0 0 0 0 1 1 1
0	2 2 2 3
0	4 4
0	6
0	8 8 9
1	0 0 1
1	2 2 2 3

Key: 0|1 = 1

13. The stem plot shows the age of players in two bowling teams.

a. Describe the shape of the distribution of Club A and Club B.

b. What comments can you make about the make-up of Club A compared to Club B?

c. How many players are over the age of 70 from:

i. Club A

ii. Club B?

Club A	Stem	Club B
1	4	
	5	5 7 8 9
	6	0 2 3 5 7
6 5 4 3	7	0 1 2
8 6 5 4 3 2 1	8	8
	9	0

Key: 5|5 = 55

14. The following table shows the number of cars sold at a dealership over eight months.

Month	April	May	June	July	August	September	October	November
Cars sold	9	14	27	21	12	14	10	18

a. Display the data on a bar chart.
b. Describe the shape of the distribution of these data.
c. What does this sample data tell us about car sales over these months?
d. Explain why the most cars were sold in the month of June.

1.6 The median, the interquartile range, the range and the mode

After displaying data using a histogram or stem plot, we can make even more sense of the data by calculating what are called *summary statistics*. Summary statistics are used because they give us an idea about:

1. where the centre of the distribution is
2. how the distribution is spread out.

We will look first at four summary statistics — the **median**, the **interquartile range**, the **range** and the **mode** — which require that the data be in ordered form before they can be calculated.

1.6.1 The median

The *median* is the midpoint of an ordered set of data. Half the data are less than or equal to the median and half the data are greater than or equal to the median.

Consider the set of data: 2 5 6 8 11 12 15. These data are in ordered form (that is, from lowest to highest). There are 7 observations. The median in this case is the middle or fourth score; that is, 8.

Consider the set of data: 1 3 5 6 7 8 8 9 10 12. These data are in ordered form also; however, in this case there is an even number of scores. The median of this set lies halfway between the 5th score (7) and the 6th score (8). So the median is 7.5. $\left(\text{Alternatively, median} = \frac{7+8}{2} = 7.5.\right)$

When there are n records in a set of ordered data, the median is located at the $\left(\frac{n+1}{2}\right)$th position.

Checking this against our previous example, we have $n = 10$; that is, there were 10 observations in the set. The median was located at the $\left(\frac{10+1}{2}\right) = 5.5$th position; that is, halfway between the 5th and the 6th terms.

A stem plot provides a quick way of locating a median since the data in a stem plot are already ordered.

WORKED EXAMPLE 13

Consider the stem plot below which contains 22 observations. What is the median?

Stem	Leaf
2	3 3
2*	5 7 9
3	1 3 3 4 4
3*	5 8 9 9
4	0 2 2
4*	6 8 8 8 9

Key: 3|4 = 34

THINK	WRITE
1. Find the median position, where $n = 22$.	Median $= \left(\frac{n+1}{2}\right)$th position $= \left(\frac{22+1}{2}\right)$th position $= 11.5$th position
2. Find the 11th and 12th terms.	11th term = 35 12th term = 38
3. The median is halfway between the 11th and 12th terms.	Median = 36.5

1.6.2 The interquartile range

We have seen that the median divides a set of data in half. Similarly, **quartiles** divide a set of data in quarters. The symbols used to refer to these quartiles are Q_1, Q_2 and Q_3.

The middle quartile, Q_2, is the median.

The interquartile range IQR = $Q_3 - Q_1$.

The interquartile range gives us the range of the middle 50% of values in a set of data.

There are four steps to locating Q_1 and Q_3.

Step 1. Write down the data in ordered form from lowest to highest.

Step 2. Locate the median; that is, locate Q_2.

Step 3. Now consider just the lower half of the set of data. Find the middle score. This score is Q_1.

Step 4. Now consider just the upper half of the set of data. Find the middle score. This score is Q_3.

The four cases given below illustrate this method.

Case 1

Consider data containing the 6 observations: 3, 6, 10, 12, 15, 21.

The data are already ordered. The median is 11.

Consider the lower half of the set, which is 3, 6, 1, 0. The middle score is 6, so $Q_1 = 6$.

Consider the upper half of the set, which is 12, 15, 21. The middle score is 15, so $Q_3 = 15$.
So the IQR is $15 - 6 = 9$.

Case 2

Consider a set of data containing the 7 observations: 11, 13, 30, 4, 17, 9, 23.
Order the data. 4, 9, 11, 13, 17, 23, 30
Locate the median. The median is 13.
Consider the lower half of the set, which is 4, 9, 11. The middle score is 9, so $Q_1 = 9$.
Consider the upper half of the set, which is 17, 23, 30. The middle score is 23, so $Q_3 = 23$.
So the IQR is $23 - 9 = 14$.

Case 3

Consider a set of data containing the 8 observations: 1, 3, 9, 10, 15, 17, 21, 26.
The data are already ordered. The median is 12.5.
Consider the lower half of the set, which is 1, 3, 9, 10. The middle score is 6, so $Q_1 = 6$.
Consider the upper half of the set, which is 15, 17, 21, 26. The middle score is 19, so $Q_3 = 19$.
So the IQR is $19 - 6 = 13$.

Case 4

Consider a set of data containing the 9 observations: 2, 7, 13, 14, 17, 19, 21, 25, 29.
The data are already ordered. The median is 17.
Consider the lower half of the set, which is 2, 7, 13, 14. The middle score is 10, so $Q_1 = 10$.
Consider the upper half of the set, which is 19, 21, 25, 29. The middle score is 23, so $Q_3 = 23$.
So the IQR is $23 - 10 = 13$.

WORKED EXAMPLE 14

The ages of the patients who attended the casualty department of an inner-suburban hospital on one particular afternoon are shown below.

14	**3**	**27**	**42**	**19**	**17**	**73**
60	**62**	**21**	**23**	**2**	**5**	**58**
33	**19**	**81**	**59**	**25**	**17**	**69**

Find the interquartile range of these data.

THINK

1. Order the data.
2. Find the median.
3. Find the middle score of the lower half of the data.

WRITE

2, 3, 5, 14, 17, 17, 19, 19, 21, 23, 25, 27, 33, 42, 58, 59, 60, 62, 69, 73, 81

The median is 25 since ten scores lie below it and ten lie above it.

For the scores
2, 3, 5, 14, 17, 17, 19, 19, 21, 23,
the middle score is 17.
So, $Q_1 = 17$.

4. Find the middle score of the upper half of the data.

For the scores
27, 33, 42, 58, 59, 60, 62, 69, 73, 81,
the middle score is 59.5.
So, $Q_3 = 59.5$.

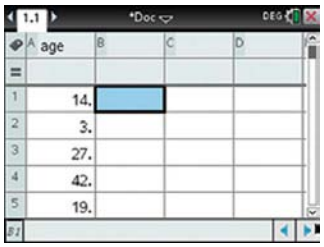
5. Calculate the interquartile range.

$\text{IQR} = Q_3 - Q_1$
$= 59.5 - 17$
$= 42.5$

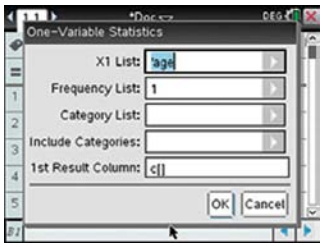
TI | THINK

1. On a Lists & Spreadsheet page, enter the data into Column A and rename the column *age*.

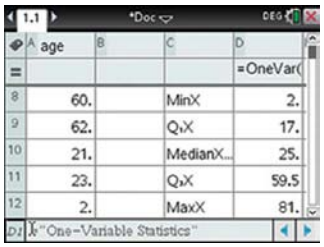
WRITE



2. Press MENU and select:
4: Statistics
1: Stat Calculations
1: One-Variable Statistics
Complete the field:
Num of Lists: 1
Press ENTER.
Complete the fields under One-Variable Statistics as shown.
Press ENTER or Click OK.



3. The One-Variable Statistics are displayed in Column D.



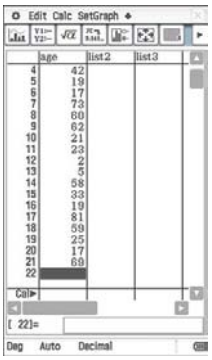
4. The answer is displayed on the screen.

$Q_1 = 17$
$Q_3 = 59.5$
$\text{IQR} = 59.5 - 17$
$= 42.5$

CASIO | THINK

1. On a Statistics screen, enter the data into List 1 and rename the list *age*.

WRITE



2. Tap:
Calc
One-variable
Complete the fields under Set Calculation as:
XList: main\age
Freq: 1
Tap OK.

3. The One-Variable Statistics are displayed.

4. The answer is displayed on the screen.

$Q_1 = 17$
$Q_3 = 59.5$
$\text{IQR} = 59.5 - 17$
$= 42.5$

WORKED EXAMPLE 15

Parents are often shocked at the amount of money their children spend. The data below give the amount spent (correct to the nearest whole dollar) by each child in a group that was taken on an excursion to the Royal Melbourne Show.

15	12	17	23	21	19	16
11	17	18	23	24	25	21
20	37	17	25	22	21	19

Calculate the interquartile range for these data.

THINK

1. Enter the data into CAS to generate one-variable statistics. Copy down the values of the first and third quartiles.
2. Calculate the interquartile range.

WRITE

$Q_1 = 17$ and $Q_3 = 23$

So, IQR $= Q_3 - Q_1$
$= 23 - 17$
$= 6$

1.6.3 The range

The *range* of a set of data is the difference between the highest and lowest values in that set.

It is usually not too difficult to locate the highest and lowest values in a set of data. Only when there is a very large number of observations might the job be made more difficult. In Worked example 15, the minimum and maximum values were 11 and 37, respectively. The range, therefore, can be calculated as follows.

$$\begin{aligned}\text{Range} &= \max_x - \min_x \\ &= 37 - 11 \\ &= 26\end{aligned}$$

While the range gives us some idea about the spread of the data, it is not very informative since it gives us no idea of how the data are distributed between the highest and lowest values.

Now let us look at another measure of the centre of a set of data: the mode.

1.6.4 The mode

The mode is the score that occurs most often; that is, it is the score with the highest frequency. If there is more than one score with the highest frequency, then all scores with that frequency are the modes.

The mode is a weak measure of the centre of data because it may be a value that is close to the extremes of the data. If we consider the set of data in Worked example 13, the mode is 48 since it occurs three times and hence is the score with the highest frequency. In Worked example 14 there are two modes, 17 and 19, because they equally occur most frequently.

Resources

Interactivity: The median, the interquartile range, the range and the mode (int-6244)
Interactivity: Mean, median, mode and quartiles (int-6496)

studyon

Units 3 & 4 > AOS DA > Topic 3 > Concepts 2 & 3

Measures of centre—median and mode Summary screen and practice questions

Measures of spread—range and interquartile range Summary screen and practice questions

Exercise 1.6 The median, the interquartile range, the range and the mode

1. **WE13** The stem plot shows 30 observations. What is the median value?

Stem	Leaf
2	1 1 3 4 4 4
2*	5 5 7 8 9
3	0 0 1 3 3 3 3
3*	6 6 7 9
4	0 1 1
4*	6 7 9 9 9

Key: 2|1 = 21

2. The following data represents the number of goals scored by a netball team over the course of a 16 game season. What was the team's median number of goals for the season?

28, 36, 24, 46, 37, 21, 49, 32, 33, 41, 47, 29, 45, 52, 37, 24

3. **WE14** From the following data find the interquartile range.

33 21 39 45 31 28 15 13 16

21 49 26 29 30 21 37 27 19

12 15 24 33 37 10 23 28 39

4. The ages of a sample of people surveyed at a concert are shown.

21 25 24 18 19 16 19 27 32 24

15 20 31 24 29 33 27 18 19 21

Find the interquartile range of these data.

5. **WE15** The data shows the amount of money spent (to the nearest dollar) at the school canteen by a group of students in a week.

3 5 7 12 15 10 8 9 21 5

7 9 13 15 7 3 4 2 11 8

Calculate the interquartile range for the data set.

6. The amount of money, in millions, changing hands through a large stocks company, in one-minute intervals, was recorded as follows.

45.8 48.9 46.4 45.7 43.8 49.1 42.7 43.1 45.3 48.6

41.9 40.0 45.9 44.7 43.9 45.1 47.1 49.7 42.9 45.1

Calculate the interquartile range for these data.

7. Write the median, the range and the mode of the sets of data shown in the following stem plots. The key for each stem plot is 3|4 = 34.

a.

Stem	Leaf
0	7
1	2 3
2	2 4 5 7 9
3	0 2 3 6 8 8
4	4 7 8 9 9
5	2 7 8
6	1 3

b.

Stem	Leaf
0	0 0 1 1
0	2 2 3 3
0	4 4 5 5 5 5 5 5 5 5
0	6 6 6 6 7
0	8 8 8 9
1	0 0 1
1	3 3
1	5 5
1	7
1	

c.

Stem	Leaf
0	1
0	2
0	4 4 5
0	6 6 6 7
0	8 8 8 8 9 9
1	0 0 0 1 1 1 1
1	2 2 2 3 3 3
1	4 4 5 5
1	6 7 7
1	8 9

d.

Stem	Leaf
3	1
3	
3	
3	6
3	8 9
4	0 0 1 1 1
4	2 2 3 3 3 3
4	4 5 5 5
4	6 7
4	9

e.

Stem	Leaf
60	2 5 8
61	1 3 3 6 7 8 9
62	0 1 2 4 6 7 8 8 9
63	2 2 4 5 7 8
64	3 6 7
65	4 5 8
66	3 5
67	4

8. For each of the following sets of data, determine the median and the range.

a. 2, 4, 6, 7, 9

b. 12, 15, 17, 19, 21

c. 3, 4, 5, 6, 7, 8, 9

d. 3, 5, 7, 8, 12, 13, 15, 16

e. 12, 13, 15, 16, 18, 19, 21, 23, 24, 26

f. 3, 8, 4, 2, 1, 6, 5

g. 16, 21, 14, 28, 23, 15, 11, 19, 25

h. 7, 4, 3, 4, 9, 5, 10, 4, 2, 11

9. a. The number of cars that used the drive-in at a McBurger restaurant during each hour, from 7.00 am until 10.00 pm on a particular day, is shown below.

14, 18, 8, 9, 12, 24, 25, 15, 18, 25, 24,
21, 25, 24, 14

Find the interquartile range of this set of data.

b. On the same day, the number of cars stopping during each hour that the nearby Kenny's Fried Chicken restaurant was open is shown below.

7, 9, 13, 16, 19, 12, 11, 18, 20, 19, 21, 20, 18, 10, 14

Find the interquartile range of these data.

c. What do these values suggest about the two restaurants?

10. Write down a set of data for which $n = 5$, the median is 6 and the range is 7. Is this the only set of data with these parameters?

11. Give an example of a data set where:

a. the lower quartile equals the lowest score

b. the IQR is zero.

12. MC The quartiles for a set of data are calculated and found to be $Q_1 = 13$, $Q_2 = 18$ and $Q_3 = 25$. Which of the following statements is true?

A. The interquartile range of the data is 5.
B. The interquartile range of the data is 7.
C. The interquartile range of the data is 12.
D. The median is 12.
E. The median is 19.

13. For each of the following sets of data find the median, the interquartile range, the range and the mode.

a. 16 12 8 7 26 32 15 51 29 45
19 11 6 15 32 18 43 31 23 23

b. 22 25 27 36 31 32 39 29 20 30
23 25 21 19 29 28 31 27 22 29

c. 1.2 2.3 4.1 2.4 1.5 3.7 6.1 2.4 3.6 1.2
6.1 3.7 5.4 3.7 5.2 3.8 6.3 7.1 4.9

14. For each set of data shown in the stem plots, find the median, the interquartile range, the range and the mode. Compare these values for both data sets.

a.

Stem	Leaf
2	3 5 5 6 7 8 9 9
3	0 2 2 3 4 6 6 7 8 8
4	2 2 4 5 6 6 6 7 9
5	0 3 3 5 6
6	2 4
7	5 9
8	2
9	7
10	
11	4

Key: 4|2 = 42

b.

Stem	Leaf
1	4
1*	
2	1 4
2*	5 7 8 8 9
3	1 2 2 2 4 4 4 4
3*	5 5 5 6
4	3 4
4*	

Key: 2|1 = 21

15. For the data in the stem plots shown, find the range, median, mode and interquartile range.

a.

Stem	Leaf
0	1
1	1 4 7
2	3 4 6 7
3	4 6 7 8 9
4	2 3 6 6
5	2 3 5
6	7
7	3

Key: 0|1 = 1

b.

Stem	Leaf
40	3 5 7
41	1 1 1 3 4 6 7 8 9
42	0 2 3 6 7 8 9
43	2 3 3 6 8
44	1 2
45	0

Key: 40|3 = 403

16. From the following set of data, find the median and mode.

4, 7, 9, 12, 15, 2, 3, 7, 9, 4, 7, 9, 2, 8, 13, 5, 3, 7, 5, 9, 7, 10

17. The following data represents distances (in metres) thrown during a javelin-throwing competition. Use CAS to calculate the interquartile range and median.

40.3	42.8	41.0	50.3	52.2	46.1	44.5
41.6	44.3	47.4	45.1	48.8	46.1	44.5
45.3	42.9	41.1	49.0	47.5	40.8	51.1

18. The data below shows the distribution of golf scores for one day of an amateur tournament.

a. Use CAS to calculate the median, interquartile range, mode and range.

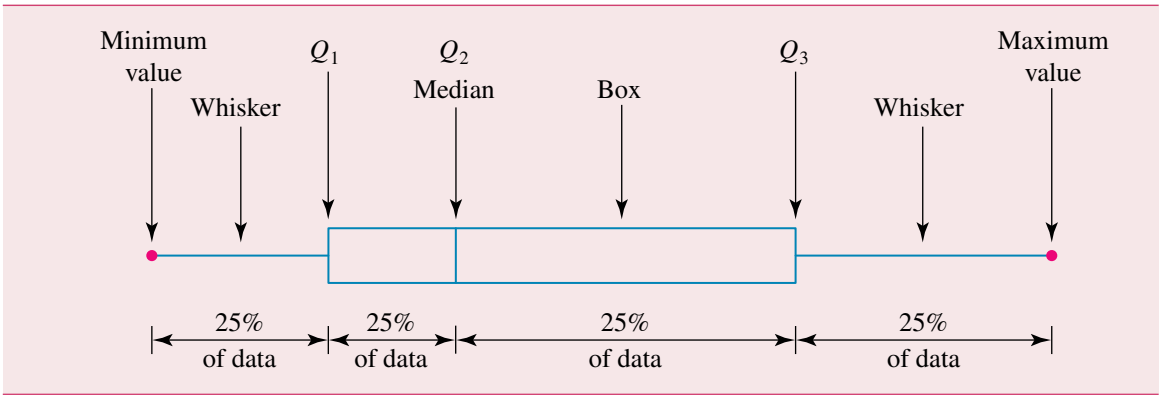
b. Comment on what you suggest the average handicap of the players listed should be if par for the course is 72.

111	93	103	85	81	90	101	95	84
93	101	85	87	85	93	100	86	91
93	95	93	99	95	93	92	96	93
97	93	93	97	96	92	100	95	104

1.7 Boxplots

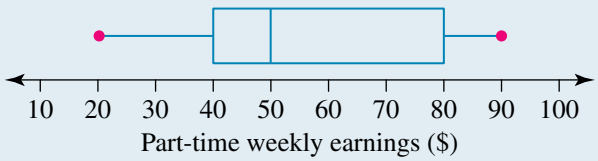
The five number summary statistics that we looked at in the previous section ($\min_x$, Q_1, Q_2, Q_3, $\max_x$) can be illustrated very neatly in a special diagram known as a boxplot (or box-and-whisker diagram). The diagram is made up of a box with straight lines (whiskers) extending from opposite sides of the box.

A boxplot displays the minimum and maximum values of the data together with the quartiles and is drawn with a labelled scale. The length of the box is given by the interquartile range. A boxplot gives us a very clear visual display of how the data are spread out.



WORKED EXAMPLE 16

The boxplot below shows the distribution of the part-time weekly earnings of a group of Year 12 students. Write down the range, the median and the interquartile range for these data.



THINK

1. Range = maximum value − minimum value
   The minimum value is 20 and the maximum value is 90.

WRITE

Range = 90 − 20
= 70

2. The median is located at the bar inside the box.

Median = 50

3. The ends of the box are at 40 and 80.
   IQR = $Q_3 - Q_1$

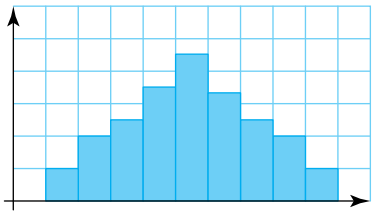
$Q_1 = 40$ and $Q_3 = 40$
IQR = 80 − 40
= 40

Boxplots can be drawn horizontally or vertically.

Horizontal boxplot

Vertical boxplot

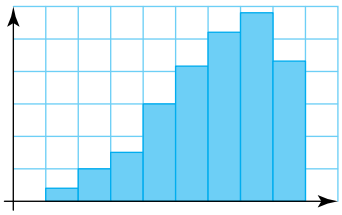
Earlier, we noted three general types of shape for histograms and stem plots: symmetric, negatively skewed and positively skewed. It is useful to compare the corresponding boxplots of distributions with such shapes.



Symmetric histogram

Symmetric boxplot

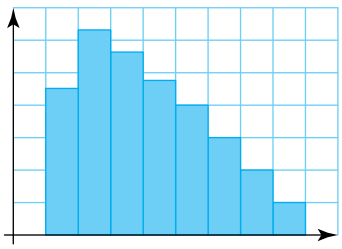
The median is in the centre of the boxplot.



Negatively skewed histogram

Negatively skewed boxplot

The median is to the right of the centre of the boxplot.



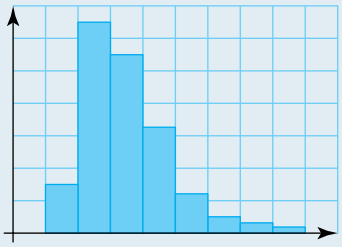
Positively skewed histogram

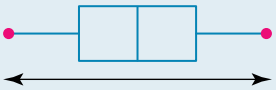
Positively skewed boxplot

The median is to the left of the centre of the boxplot.

WORKED EXAMPLE 17

Explain whether or not the histogram and the boxplot shown below could represent the same data.





THINK	WRITE
The histogram shows a distribution which is positively skewed. The boxplot shows a distribution which is approximately symmetric.	The histogram and the boxplot could not represent the same data since the histogram shows a distribution that is positively skewed and the boxplot shows a distribution that is approximately symmetric.

WORKED EXAMPLE 18

The results (out of 20) of oral tests in a Year 12 Indonesian class are:

15, 12, 17, 8, 13, 18, 14, 16, 17, 13, 11, 12

Display these data using a boxplot and discuss the shape obtained.

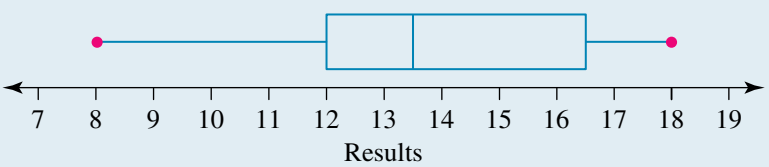
THINK

1. Find the lowest and highest scores, Q_1, the median (Q_2) and Q_3 by first ordering the data.

WRITE/DRAW

8, 11, 12, 12, 13, 13, 14, 15, 16, 17, 17, 18
The median score is 13.5.
The lower half of the scores are
8, 11, 12, 12, 13, 13.
So, $Q_1 = 12$
The upper half of the scores are
14, 15, 16, 17, 17, 18.
So, $Q_3 = 16.5$
The lowest score is 8.
The highest score is 18.

2. Using these five number summary statistics, draw the boxplot.

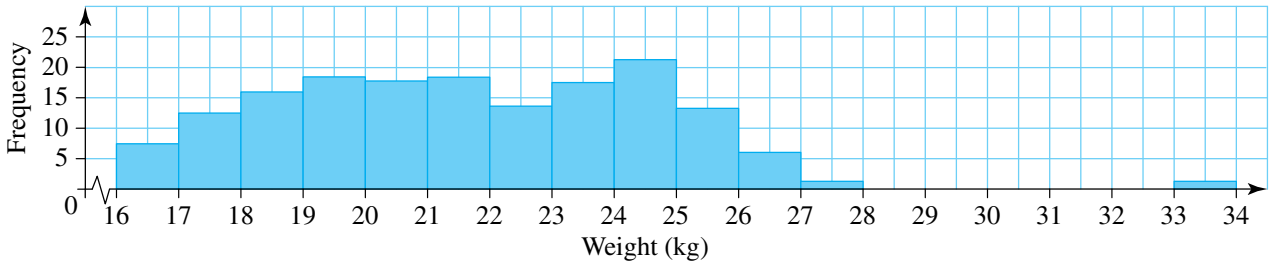


3. Consider the spread of each quarter of the data.

The scores are grouped around 12 and 13, as well as around 17 and 18 with 25% of the data in each section. The scores are more spread elsewhere.

1.7.1 Outliers

When one observation lies well away from other observations in a set, we call it an outlier. Sometimes an outlier occurs because data have been incorrectly obtained or misread. For example, here we see a histogram showing the weights of a group of 5-year-old boys.



The outlier, 33, may have occurred because a weight was incorrectly recorded as 33 rather than 23 or perhaps there was a boy in this group who, for some medical reason, weighed a lot more than his counterparts. When an outlier occurs, the reasons for its occurrence should be checked.

1.7.2 The lower and upper fences

We can identify outliers by calculating the values of the lower and upper fences in a data set. Values which lie either below the lower fence or above the upper fence are outliers.

The lower fence $= Q_1 - 1.5 \times \text{IQR}$

The upper fence $= Q_3 + 1.5 \times \text{IQR}$

An outlier is not included in the boxplot, but should instead be plotted as a point beyond the end of the whisker.

WORKED EXAMPLE 19

The times (in seconds) achieved by the 12 fastest runners in the 100 – *m* sprint at a school athletics meeting are listed below.

11.2	**12.3**	**11.5**	**11.0**	**11.6**	**11.4**
11.9	**11.2**	**12.7**	**11.3**	**11.2**	**11.3**

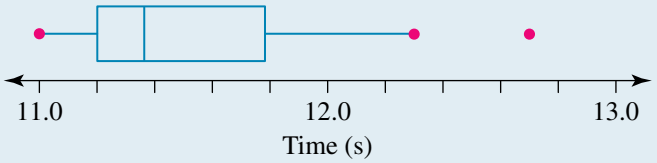
Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.

THINK	WRITE/DRAW
1. Determine the five number summary statistics by first ordering the data and obtaining the interquartile range.	11.0, 11.2, 11.2, 11.2, 11.3, 11.3, 11.4, 11.5, 11.6, 11.9, 12.3, 12.7 Lowest score = 11.0 Highest score = 12.7 Median = Q_2 = 11.35 Q_1 = 11.2 Q_3 = 11.75 IQR = 11.75 – 11.2 = 0.55

2. Identify any outliers by calculating the values of the lower and upper fences.

$Q_1 - 1.5 \times \text{IQR} = 11.2 - 1.5 \times 0.55$
$= 10.375$
The lowest score lies above the lower fence of 10.375, so there is no outlier below.
$Q_3 + 1.5 \times \text{IQR} = 11.75 + 1.5 \times 0.55$
$= 12.575$
The score 12.7 lies above the upper fence of 12.575, so it is an outlier and 12.3 becomes the end of the upper whisker.

3. Draw the boxplot with the outlier.



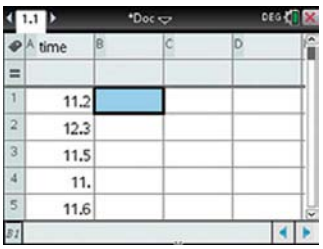
4. Describe the shape of the distribution. Data peak to the left and trail off to the right with one outlier.

The data are positively skewed with 12.7 seconds being an outlier. This may be due to incorrect timing or recording but more likely the top eleven runners were significantly faster than the other competitors in the event.

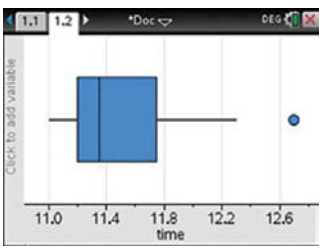
TI | THINK

1. On a Lists & Spreadsheet page, enter the data into Column A and rename the column *time*.

WRITE



2. Press Ctrl + I and select:
5: Add Data and Statistics
Move the cursor to select *time* on the horizontal axis.
Press MENU and select:
1: Plot Type
2: Box Plot



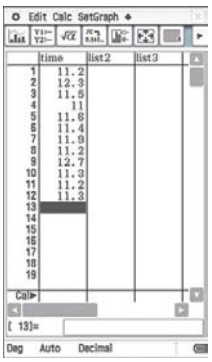
3. The answer is displayed in the screen.

The data is positively skewed with 12.7 seconds being an outlier.

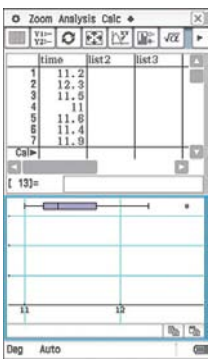
CASIO | THINK

1. On a Statistics screen, enter the data into List1 and rename the list *time*.

WRITE



2. Tap:
SetGraph
Settings
Complete the fields under Set StatGraphs as:
Draw: On
Type: MedBox
XList: main\time
Freq: 1
Tick Show Outliers.
Tap Set.



3. The answer is displayed in the screen.

The data is positively skewed with 12.7 seconds being an outlier.

on Resources

Interactivity: Boxplots (int-6245)

studyon

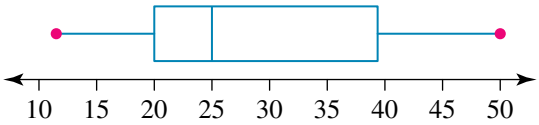
Units 3 & 4 > AOS DA > Topic 3 > Concepts 5 & 6

Boxplots Summary screen and practice questions

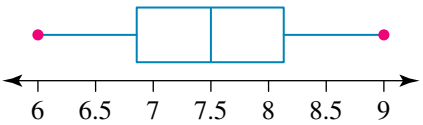
Boxplots with outlier(s) Summary screen and practice questions

Exercise 1.7 Boxplots

1. **WE16** Write down the range, median and interquartile range for the data in the boxplot shown.

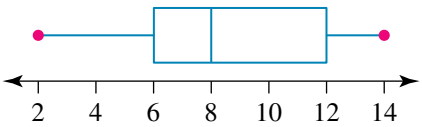


2. Find the median, range and interquartile range of the data displayed in the boxplot shown.

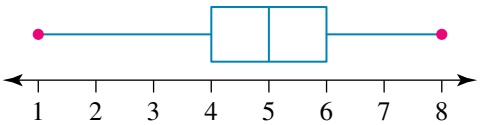


3. For the boxplots shown, write down the range, the interquartile range and the median of the distributions which each one represents.

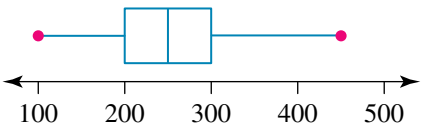
a.



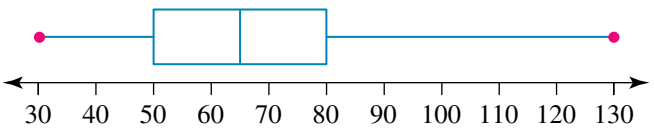
b.



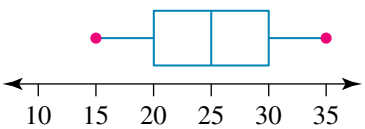
c.



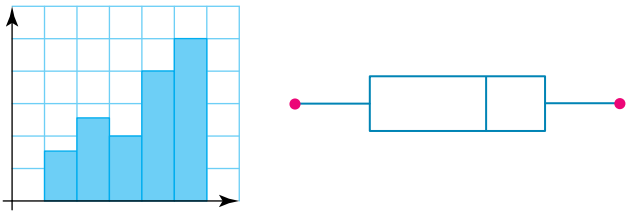
d.



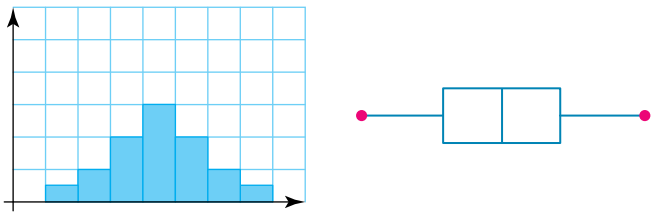
e.



4. **WE17** Explain whether or not the histogram and the boxplot shown could represent the same data.

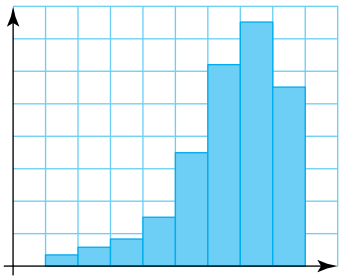


5. Do the histogram and boxplot represent the same data? Explain.

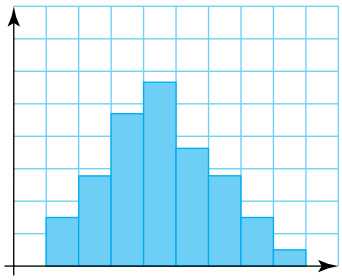


6. Match each histogram below with the boxplot which could show the same distribution.

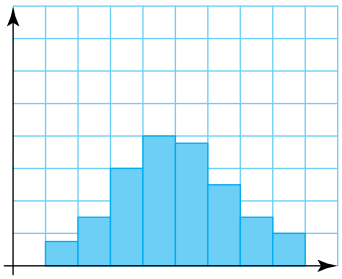
a.



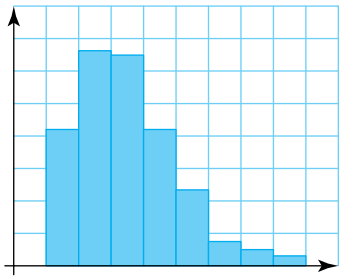
b.



c.



d.



i.

ii.

iii.

iv.

7. **WE18** The results for a Physics test (out of 50) are shown.

32, 38, 42, 40, 37, 26, 46, 36, 50, 41, 48, 50, 40, 38, 32, 35, 28, 30

Display the data using a boxplot and discuss the shape obtained.

8. The numbers of hours Year 12 students spend at their part-time job per week are shown.

4, 8, 6, 5, 12, 8, 16, 4, 7, 10, 8, 20, 12, 7, 6, 4, 8

Display the data using a boxplot and discuss what the boxplot shows in relation to part-time work of Year 12 students.

9. For each of the following sets of data, construct a boxplot.

a. 3, 5, 6, 8, 8, 9, 12, 14, 17, 18

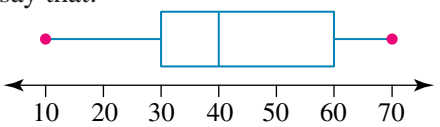
b. 3, 4, 4, 5, 5, 6, 7, 7, 7, 8, 8, 9, 9, 10, 10, 12

c. 4.3, 4.5, 4.7, 4.9, 5.1, 5.3, 5.5, 5.6

d. 11, 13, 15, 15, 16, 18, 20, 21, 22, 21, 18, 19, 20, 16, 18, 20

e. 0.4, 0.5, 0.7, 0.8, 0.8, 0.9, 1.0, 1.1, 1.2, 1.0, 1.3

10. MC For the distribution shown in the boxplot below, it is true to say that:



A. the median is 30

B. the median is 45

C. the interquartile range is 10

D. the interquartile range is 30

E. the interquartile range is 60

11. The number of clients seen each day over a 15-day period by a tax consultant is:

3, 5, 2, 7, 5, 6, 4, 3, 4, 5, 6, 6, 4, 3, 4

Represent these data on a boxplot.

12. The maximum daily temperatures (in °C) for the month of October in Melbourne are:

18, 26, 28, 23, 16, 19, 21, 27, 31, 23, 24, 26, 21, 18, 26, 27

23, 21, 24, 20, 19, 25, 27, 32, 29, 21, 16, 19, 23, 25, 27

Represent these data on a boxplot.

13. WE19 The heights jumped (in metres) at a school high jump competition are listed:

1.35, 1.30, 1.40, 1.38, 1.45, 1.48, 1.30, 1.36, 1.45, 1.75, 1.46, 1.40

a. Draw a boxplot to represent the data.

b. Describe the shape of the distribution and comment on the existence of any outliers by finding the lower and upper fences.

14. The amount of fuel (in litres) used at a petrol pump for 16 cars is listed:

48.5 55.1 61.2 58.5 46.9 49.2 57.3 49.9

51.6 30.3 45.9 50.2 52.6 47.0 55.5 60.3

Draw a boxplot to represent the data and label any outliers.

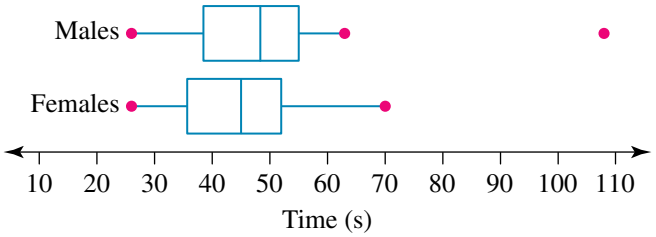
15. The number of rides that 16 children had at the annual show are listed below.

8, 5, 9, 4, 9, 0, 8, 7, 9, 2, 8, 7, 9, 6, 7, 8

a. Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.

b. Use CAS to draw a boxplot for these data.

16. A concentration test was carried out on 40 students in Year 12 across Australia. The test involved the use of a computer mouse and the ability to recognise multiple images. The less time required to complete the activity, the better the student's ability to concentrate.
The data are shown by the parallel boxplots.



a. Identify two similar properties of the concentration spans for boys and girls.
b. Find the interquartile range for boys and girls.
c. Comment on the existence of an outlier in the boys' data by finding the lower and upper fences.

17. You work in the marketing department of a perfume company. You completed a survey of people who purchased your perfume, asking them how many times a week they used it. Analyse the data by drawing a boxplot and comment on the existence of any outliers by finding the lower and upper fences.

7 2 5 4 7 5 7 2 5 4 3 5

7 7 9 8 5 6 5 3 15 8 7 5

18. For the data set shown:

11, 11, 14, 16, 19, 22, 24, 25, 25, 27, 28, 28, 36, 38, 38, 39

a. construct a boxplot by hand
b. comment on the presence of outliers by finding the lower and upper fences
c. construct a boxplot using CAS and compare the two boxplots.

19. From the stem plot shown construct a boxplot using CAS and comment on any outliers if they exist in the data.

Stem	Leaf
2	3
3	
4	
5	
6	1 5 8
7	2 5 5 7
8	1 2 3 3 6 7 7
9	1 1 3 7 7
10	6 7 9
11	1 3

Key: 2|3 = 23

1.8 The mean of a sample

When dealing with sets of data, we are always working with either the **population**, or a **sample** from the population. The means of a data set representing either the population or a sample are calculated in the same way; however, they are represented by different symbols. The **mean** of a population is represented by the Greek letter μ (mu) and the mean of a sample is represented by $\bar{x}$ (x-bar), a lower case x with a bar on top. In this section we will only use $\bar{x}$ to represent the mean.

The *mean* of a set of data is what is referred to in everyday language as the *average*.

For the set of data {4, 7, 9, 12, 18}:

$$\bar{x} = \frac{4 + 7 + 9 + 12 + 18}{5}$$
$$= 10$$

The formal definition of the mean is:

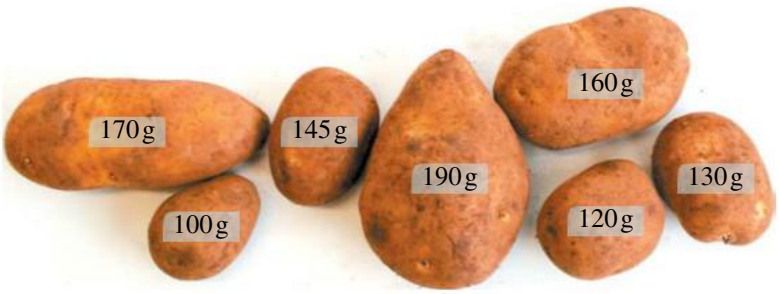
$$\bar{x} = \frac{\sum x}{n}$$

where $\sum x$ represents the sum of all of the observations in the data set and n represents the number of observations in the data set.

Note that the symbol, $\sum$ is the Greek letter, sigma, which represents 'the sum of'.

The mean is also referred to as a *summary statistic* and is a measure of the centre of a distribution. The mean is the point about which the distribution 'balances'.

Consider the masses of 7 potatoes, given in grams, in the photograph.



The mean is 145 g. The observations 130 and 160 'balance' each other since they are each 15 g from the mean. Similarly, the observations 120 and 170 'balance' each other since they are each 25 g from the mean, as do the observations 100 and 190. Note that the median is also 145 g. That is, for this set of data the mean and the median give the same value for the centre. This is because the distribution is symmetric.

Now consider two cases in which the distribution of data is not symmetric.

1.8.1 Case 1

Consider the masses of a different set of 7 potatoes, given in grams below.

100, 105, 110, 115, 120, 160, 200

The median of this distribution is 115 g and the mean is 130 g. There are 5 observations that are less than the mean and only 2 that are more. In other words, the mean does not give us a good indication of the *centre* of

the distribution. However, there is still a 'balance' between observations below the mean and those above, in terms of the spread of all the observations from the mean. Therefore, the mean is still useful to give a measure of the central tendency of the distribution but in cases where the distribution is skewed, the median gives a better indication of the centre. For a positively skewed distribution, as in the previous case, the mean will be greater than the median. For a negatively skewed distribution the mean will be less than the median.

1.8.2 Case 2

Consider the data below, showing the weekly income (to the nearest $10) of 10 families living in a suburban street.

$$\$600,\ \$1340,\ \$1360,\ \$1380,\ \$1400,\ \$1420,\ \$1420,\ \$1440,\ \$1460,\ \$1500$$

In this case, $\bar{x} = \frac{13\,320}{10} = \$1332$, and the median is $1410.

One of the values in this set, $600, is clearly an outlier. As a result, the value of the mean is below the weekly income of the other 9 households. In such a case the mean is not very useful in establishing the centre; however, the 'balance' still remains for this negatively skewed distribution.

The mean is calculated by using the values of the observations and because of this it becomes a less reliable measure of the centre of the distribution when the distribution is skewed or contains an outlier. Because the median is based on the order of the observations rather than their value, it is a better measure of the centre of such distributions.

WORKED EXAMPLE 20

Calculate the mean of the set of data shown.
10, 12, 15, 16, 18, 19, 22, 25, 27, 29

THINK

1. Write the formula for calculating the mean, where $\sum x$ is the sum of all scores; n is the number of scores in the set.
2. Substitute the values into the formula and evaluate.

WRITE

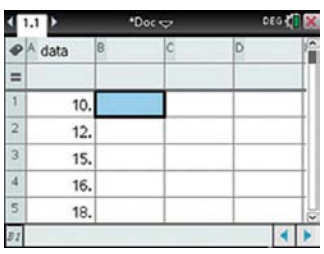
$$\bar{x} = \frac{\sum x}{n}$$
$$= \frac{10 + 12 + 15 + 16 + 18 + 19 + 22 + 25 + 27 + 29}{10}$$
$$\bar{x} = 19.3$$

The mean, $\bar{x}$, is 19.3.

TI | THINK

1. On a Lists & Spreadsheet page, enter the data into Column A and rename the column *data*.

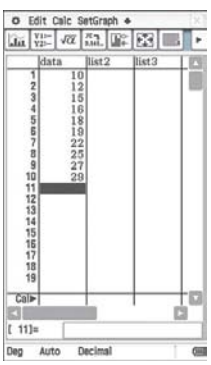
WRITE



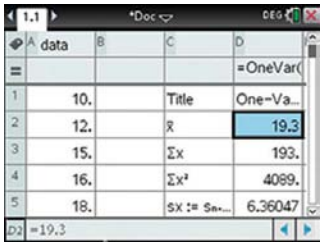
CASIO | THINK

1. On a Statistics screen, enter the data into List1 and rename the list *data*.

WRITE



2. Press MENU and select:
   4: Statistics
   1: Stat Calculations
   1: One-Variable Statistics
   Complete the field:
   Num of Lists: 1
   Press ENTER.
   Complete the fields under One-Variable Statistics as:
   X1List: data
   Frequency List: 1
   Leave other fields blank.
   Press ENTER or Click OK.
   The One-Variable Statistics are displayed in Column D.
4. The answer is displayed on the screen.



The mean, $\bar{x} = 19.3$

2. Tap:
   Calc
   One-variable
   Complete the fields under Set Calculation as:
   XList: main\data
   Freq: 1
   Tap OK.
4. The answer is displayed on the screen.

The mean, $\bar{x} = 19.3$

When calculating the mean of a data set, sometimes the answer you calculate will contain a long stream of digits after the decimal point.

For example, if we are calculating the mean of the data set

$$44,\ 38,\ 55,\ 61,\ 48,\ 32,\ 49$$

Then the mean would be:

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} \\ &= \frac{327}{7} \\ &= 46.714\,285\,71\ldots\end{aligned}$$

In this case it makes sense to round the answer to either a given number of decimal places, or a given number of **significant figures**.

1.8.3 Rounding to a given number of significant figures

When rounding to a given number of signified figures, we are rounding to the digits in a number which are regarded as 'significant'.

To determine which digits are significant, we can observe the following rules:

- All digits greater than zero are significant
- Leading zeros can be ignored (they are placeholders and are not significant)
- Zeros included between other digits are significant
- Zeros included after decimal digits are significant
- Trailing zeros for integers are not significant (unless specified otherwise)

The following examples show how these rules work:

0.003 561 — leading digits are ignored, so this has 4 significant figures

70.036 — zeros between other digits are significant, so this has 5 significant figures

5.320 — zeros included after decimal digits are significant, so this has 4 significant figures

450 000 — trailing zeros are not significant, so this has 2 significant figures

78 000.0 — the zero after the decimal point is considered significant, so the zeros between other numbers are also significant; this has 6 significant figures

As when rounding to a given number of decimal places, when rounding to a given number of significant figures consider the digit after the specified number of figures. If it is 5 or above, round the final digit up; if it is 4 or below, keep the final digit as is.

5067.37 — rounded to 2 significant figures is 5100
3199.01 — rounded to 4 significant figures is 3199
0.004 931 — rounded to 3 significant figures is 0.004 93
1 020 004 — rounded to 2 significant figures is 1 000 000

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Units 3 & 4 > AOS DA > Topic 3 > Concept 1

Measures of centre—mean Summary screen and practice questions

Exercise 1.8 The mean of a sample

1. WE20 Calculate the mean of the data set shown.
   9, 12, 14, 16, 18, 19, 20, 25, 29, 33, 35, 36, 39
2. Calculate the mean of the data set shown.
   5.5, 6.3, 7.7, 8.3, 9.7, 6.7, 12.9, 10.5, 9.9, 5.1
3. Find the mean of each of the following sets of data.
   a. 5, 6, 8, 8, 9 (correct to 2 significant figures)
   b. 3, 4, 4, 5, 5, 6, 7, 7, 7, 8, 8, 9, 9, 10, 10, 12 (correct to 4 significant figures)
   c. 4.3, 4.5, 4.7, 4.9, 5.1, 5.3, 5.5, 5.6 (correct to 5 significant figures)
   d. 11, 13, 15, 15, 16, 18, 20, 21, 22 (correct to 1 decimal place)
   e. 0.4, 0.5, 0.7, 0.8, 0.8, 0.9, 1.0, 1.1, 1.2, 1.0, 1.3 (correct to 4 decimal places)
4. Calculate the mean of each of the following and explain whether or not it gives us a good indication of the centre of the data.
   a. 0.7, 0.8, 0.85, 0.9, 0.92, 2.3
   b. 14, 16, 16, 17, 17, 17, 19, 20
   c. 23, 24, 28, 29, 33, 34, 37, 39
   d. 2, 15, 17, 18, 18, 19, 20
5. The number of people attending sculpture classes at the local TAFE college for each week during the first semester is given.
   15, 12, 15, 11, 14, 8, 14, 15, 11, 10, 7, 11, 12, 14, 15, 14, 15, 9, 10, 11
   What is the mean number of people attending each week? (Express your answer correct to 2 significant figures.)
6. MC The ages of a group of junior pilots joining an international airline are indicated in the stem plot below.

Stem	Leaf
2	1
2	2
2	4 5
2	6 6 7
2	8 8 8 9
3	0 1 1
3	2 3
3	4 4
3	6
3	8

Key: 2|1 = 21 years

The mean age of this group of pilots is:

A. 20 **B.** 28 **C.** 29 **D.** 29.15 **E.** 29.5

7. MC The number of people present each week at a 15-week horticultural course is given by the stem plot at right.
The mean number of people attending each week was closest to:

A. 17.7
B. 18
C. 19.5
D. 20
E. 21.2

Stem	Leaf
0	4
0*	7
1	2 4
1*	5 5 6 7 8
2	1 2 4
2*	7 7 7

Key: 2|4 = 24 people

8. For each of the following, write down whether the mean or the median would provide a better indication of the centre of the distribution.

a. A positively skewed distribution
b. A symmetric distribution
c. A distribution with an outlier
d. A negatively skewed distribution

1.9 Standard deviation of a sample

The **standard deviation** gives us a measure of how data are spread around the mean. For the set of data {8, 10, 11, 12, 12, 13}, the mean, $\bar{x} = 11$.

The amount that each observation 'deviates' (that is, differs) from the mean is calculated and shown in the table below.

Particular observation (x)	Deviation from the mean ($x - x$)
8	$8 - 11 = -3$
10	$10 - 11 = -1$
11	$11 - 11 = 0$
12	$12 - 11 = 1$
12	$12 - 11 = 1$
13	$13 - 11 = 2$

The deviations from the mean are either positive or negative depending on whether the particular observation is lower or higher in value than the mean. If we were to add all the deviations from the mean we would obtain zero.

If we square the deviations from the mean we will overcome the problem of positive and negative deviations cancelling each other out. With this in mind, a quantity known as sample **variance** (s^2) is defined:

$$s^2 = \frac{\sum(x - \bar{x})^2}{n - 1}.$$

Technically, this formula for variance is used when the data set is a sub-set of a larger population; that is, a sample of the population.

Variance gives the average of the squared deviations and is also a measure of spread. A far more useful measure of spread, however, is the standard deviation, which is the square root of variance (s). One reason for it being more useful is that it takes the same unit as the observations (for example, cm or number of people). Variance would square the units, for example, cm^2 or number of people squared, which is not very practical.

Other advantages of the standard deviation will be dealt with later in the topic.

In summary,

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

where s represents sample standard deviation
$\sum$ represents 'the sum of'
x represents an observation
$\bar{x}$ represents the mean
n represents the number of observations.

Note: This is the formula for the standard deviation of a sample. The standard deviation for a population is given by σ (sigma) and is calculated using a slightly different formula, which is outside the scope of this course.

While some of the theory or formulas associated with standard deviation may look complex, the calculation of this measure of spread is straightforward using CAS. Manual computation of standard deviation is therefore rarely necessary, however application of the formula is required knowledge.

WORKED EXAMPLE 21

The price (in cents) per litre of petrol at a service station was recorded each Friday over a 15-week period. The data are given below.

152.4	**160.2**	**159.6**	**168.6**	**161.4**	**156.6**	**164.8**	**162.6**
161.0	**156.4**	**159.0**	**160.2**	**162.6**	**168.4**	**166.8**	

Calculate the standard deviation for this set of data, correct to 2 decimal places.

THINK

1. Enter the data into CAS to determine the sample statistics.
2. Round the value correct to 2 decimal places.

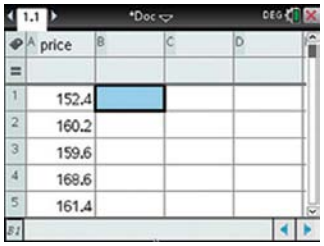
WRITE

$s = 4.515\,92$
$= 4.52$ cents/L

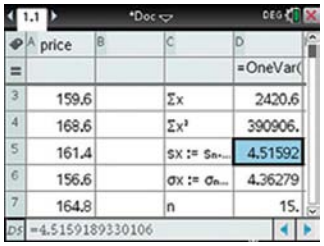
TI | THINK

WRITE

1. On a Lists & Spreadsheet page, enter the data into Column A and rename the column *price*.



2. Press MENU and select:
   4: Statistics
   1: Stat Calculations
   1: One-Variable Statistics
   Complete the field:
   Num of Lists: 1
   Press ENTER.
   Complete the fields under One-Variable Statistics as:
   X1List: price
   Frequency List: 1
   Leave other fields blank.
   Press ENTER or Click OK.
   The One-Variable Statistics are displayed in Column D.



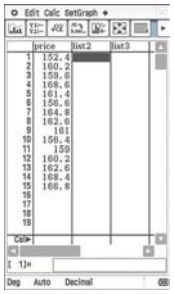
4. The answer is displayed on the screen.

The standard deviation, correct to 2 decimal places is, $s = 4.52$ cents/L.

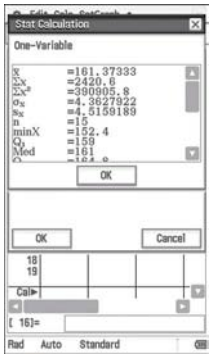
CASIO | THINK

WRITE

1. On a Statistics screen, enter the data into List1 and rename the list *price*.



2. Tap:
   Calc
   One-variable
   Complete the fields under Set Calculation as:
   XList: main\price
   Freq: 1
   Tap OK.



4. The answer is displayed on the screen.

The standard deviation, correct to 2 decimal places, is $s = 4.52$ cents/L.

WORKED EXAMPLE 22

The number of students attending SRC meetings during the term is given in the stem plot shown. Calculate the standard deviation for this set of data, correct to 4 significant figures, by using the formula $s = \sqrt{\dfrac{\sum(x - \bar{x})^2}{n - 1}}$.

Stem	Leaf
0	4
0*	8 8
1	1 3 4
1*	5 8
2	3
2*	5

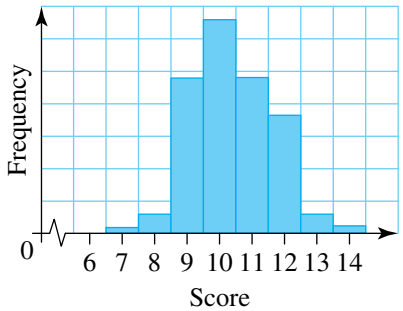
Key: 1|4 = 14 students

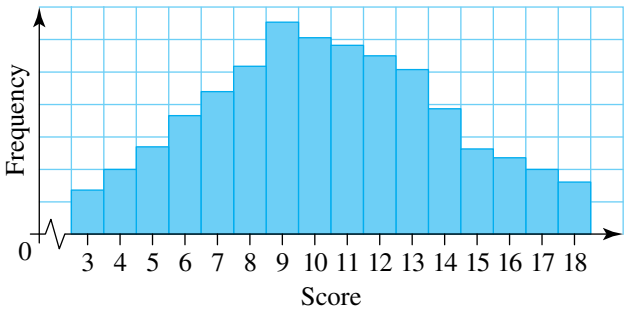
THINK	WRITE
1. Calculate the value of the mean ($\bar{x}$).	$\bar{x} = \frac{\sum x}{n}$ $= \frac{4 + 8 + 8 + 11 + 13 + 14 + 15 + 18 + 23 + 25}{10}$ $= 13.9$
2. Set up a table to calculate the values of $(x - \bar{x})^2$.	

x	$x - \bar{x}$	$(x - \bar{x})^2$
4	−9.9	98.01
8	−5.9	34.81
8	−5.9	34.81
11	−2.9	8.41
13	−0.9	0.81
14	0.1	0.01
15	1.1	1.21
18	4.1	16.81
23	9.1	82.81
25	11.1	123.21
$\sum (x - \bar{x})^2 = 400.9$		

3. Enter the values into the formula to calculate the standard deviation (s).	$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$ $= \sqrt{\frac{400.9}{9}}$ $= 6.6741 \ldots$
4. Round the value correct to 4 significant figures.	$s = 6.674$ (correct to 4 significant figures)

The standard deviation is a measure of the spread of data from the mean. Consider the two sets of data shown.





Each set of data has a mean of 10. The left set of data has a standard deviation of 1 and the right set of data has a standard deviation of 3.

As we can see, the larger the standard deviation, the more spread are the data from the mean.

Resources

Interactivity: The mean and the standard deviation (int-6246)

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Units 3 & 4 > AOS DA > Topic 3 > Concept 4

Measures of spread—standard deviation Summary screen and practice questions

Exercise 1.9 Standard deviation of a sample

1. **WE21** The Australian dollar is often compared to the US dollar. The value of the Australian dollar compared to the US dollar each week over a six-month period is shown.

97.2	96.8	98.0	98.3	97.5	95.9	96.1	95.6	95.0	94.8	94.6	94.9
93.9	93.3	92.9	90.6	90.4	89.6	88.1	88.3	89.5	88.8	88.5	88.1

Calculate the standard deviation for this set of data, correct to 2 decimal places.

2. The test results for a maths class are shown.

67	98	75	81	70	64	55	52	78	90	76
92	78	80	83	59	67	45	78	48	82	62

Calculate the standard deviation for the set of data, correct to 2 decimal places.

3. **WE22** Calculate the standard deviation of the data set shown, correct to 4 significant places, by using the formula $s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$.

Stem	Leaf
1	1 1
1*	6 9
2	4
2*	5 5 8
3	6 9

Key: 2|4 = 24

4. The number of students attending the service learning meetings in preparation for the year's fundraising activities is shown by the stem plot. Calculate the standard deviation for this set of data, correct to 3 decimal places, by using the formula $s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$.

Stem	Leaf
1	2 3 3
1*	6 7 8
2	1 4
2*	7
3	
3*	5

Key: 1|3 = 13

5. For each of the following sets of data, calculate the standard deviation correct to 2 decimal places.

a. 3, 4, 4.7, 5.1, 6, 6.2

b. 7, 9, 10, 10, 11, 13, 13, 14

c. 12.9, 17.2, 17.9, 20.2, 26.4, 28.9

d. 41, 43, 44, 45, 45, 46, 47, 49

e. 0.30, 0.32, 0.37, 0.39, 0.41, 0.43, 0.45

6. First-quarter profit increases for 8 leading companies are given below as percentages.

2.3, 0.8, 1.6, 2.1, 1.7, 1.3, 1.4, 1.9

Complete the table and hence calculate the standard deviation for this set of data. Express your answer correct to 2 decimal places.

x	$x - \bar{x}$	$(x - \bar{x})^2$
2.3	0.6625	□
0.8	□	0.7014
1.6	−.0.0375	□
2.1	□	0.2139
1.7	0.0625	□
1.3	−0.3375	0.1139
1.4	−0.2375	0.05 64
1.9	0.2625	1.5987

$$\frac{\sum (x - \bar{x})^2}{n - 1} = \frac{1.5987}{\square}$$

$$s = \square$$

7. The number of outgoing phone calls from an office each day over a 4-week period is shown in the stem plot below.

Stem	Leaf
0	8 9
1	3 4 7 9
2	0 1 3 7 7
3	3 4
4	1 5 6 7 8
5	3 8

Key: 2|1 = 21 calls

Calculate the standard deviation for this set of data and express your answer correct to 4 significant figures.

8. MC A new legal aid service has been operational for only 5 weeks. The number of people who have made use of the service each day during this period is set out at right.
The standard deviation (to 2 decimal places) of these data is:

A. 6.00
B. 6.34
C. 6.47
D. 15.44
E. 16.00

Stem	Leaf
0	2 4
0*	7 7 9
1	0 1 4 4 4 4
1*	5 6 6 7 8 8 9
2	1 2 2 3 3 3
2*	7

Key: 1|0 = 10 people

9. The speed of 20 cars (in km/h) is monitored along a stretch of road that is a designated 80 km/h zone. Calculate the standard deviation of the data, correct to 2 decimal places.

80, 82, 77, 75, 80, 80, 81, 78, 79, 78, 80, 80, 85, 70, 79, 81, 81, 80, 80, 80

10. Thirty pens are randomly selected off the conveyor belt at the factory and are tested to see how long they will last, in hours. Calculate the standard deviation of the data shown, correct to 3 decimal places.

20, 32, 38, 22, 25, 34, 47, 31, 26, 29, 30, 36, 28, 40, 31,
26, 37, 38, 32, 36, 35, 25, 29, 30, 40, 35, 38, 39, 37, 30

11. Calculate the standard deviation of the data shown, correct to 2 decimal places, representing the temperature of the soil around 25 germinating seedlings:

28.9	27.4	23.6	25.6	21.1	22.9	29.6	25.7	27.4
23.6	22.4	24.6	21.8	26.4	24.9	25.0	23.5	26.1
23.7	25.3	29.3	23.5	22.0	27.9	23.6		

12. Aptitude tests are often used by companies to help decide who to employ. An employer gave 30 potential employees an aptitude test with a total of 90 marks. The scores achieved are shown.

67, 67, 68, 68, 68, 69, 69, 72, 72, 73, 73, 74, 74, 75, 75,
77, 78, 78, 78, 79, 79, 79, 81, 81, 81, 82, 83, 83, 83, 86

Calculate the mean and standard deviation of the data, correct to 1 decimal place.

13. The number of players attending basketball try-out sessions is shown by the stem plot. Calculate the standard deviation for this set of data, correct to 4 significant figures.

Stem	Leaf
2	1
2	
2	
2	6
2	8 9
3	0 0 1 1 1
3	2 3 3 3 3 3
3	4 4 5 5
3	6 7
3	8

Key: 2|1 = 21

14. The scores obtained out of 20 at a dancing competition are shown. Calculate the standard deviation of the scores, correct to 3 decimal places.

18.5	16.5	18.0	12.5	13.0	18.0	15.5	17.5	18.5	19.0
17.0	12.5	16.5	13.5	19.0	20.0	17.5	19.5	16.0	15.5

1.10 Populations and simple random samples

1.10.1 Populations

A group of Year 12 students decide to base their statistical investigation for a maths project on what their contemporaries — that is, other Year 12 students — spend per year on Christmas and birthday presents for their family members. One of their early decisions is to decide what the population is going to be for their investigation. That is, are they looking at Year 12 students in Australia or in Victoria or in metropolitan Melbourne or in their suburb or just in their school? In practice, it is difficult to look at a large population unless, of course, you have a lot of resources available to you! The students decide that their population will be the Year 12 students at their school. This means that any conclusions they draw as a result of their investigation can be generalised to Year 12 students at their school but not beyond that.

1.10.2 Samples

Given that there are 95 students in Year 12 at the school, it would be too time-consuming to interview all of them. A smaller group known as a sample is therefore taken from the population. The way in which this smaller group is chosen is of paramount importance. For the investigation to have credibility, the sample should be a random selection from the population and every member of that population should have an equal chance of being chosen in the sample. Also, the selection of one person from the population should not affect whether or not another person is chosen; that is, the selections should be independent. A **simple random sample** provides such a sample.

The students conducting the investigation decide to choose a sample of 12 fellow students. While it would be simplest to choose 12 of their mates as the sample, this would introduce **bias** since they would not be representative of the population as a whole.

The students obtain a list of names of the 95 students in Year 12. They then write next to the name of each student a number from 1 to 95. Using a calculator, the students generate 12 random numbers between 1 and 95. Alternatively, the students could have used a table of random numbers. Any point on the table can be taken as the starting point. The students decide which direction to move through the table; for example, across the table to the right or to the left or down. Once a direction is chosen, they must stay with that movement and write down the 2-digit numbers as they go along.

The numbers chosen by the students are then matched to the numbers on the name list and the students in their sample can be identified.

These 12 students are then asked what they spent in the last year on family presents.

The students conducting the investigation can then record the data.

Random numbers can also be generated with the aid of CAS technology.

WORKED EXAMPLE 23

Generate 5 random numbers (integers) between 1 and 50.

THINK	WRITE
1. Find the appropriate menu in CAS to generate random integers.	
2. Generate 5 random numbers between 1 and 50.	
3. An example of a set of numbers is displayed.	{48, 46, 8, 26, 21}.

1.10.3 Displaying the data

The raw data for a sample of 12 students from our population of 95 are given below in dollars.

25, 30, 35, 38, 34, 22, 30, 40, 35, 25, 32, 40

Since there are not many responses, a stem plot is an appropriate way of displaying the data.

Stem	Leaf
2	2
2*	5 5
3*	0 0 2 4
3*	5 5 8
4	0 0

Key: 2|2 = 22 dollars

To summarise and comment further on the sample, it is useful to use some of the summary statistics covered earlier in this topic. The most efficient way to calculate these is to use CAS. Using the steps outlined in the previous sections, we obtain a list of summary statistics for these data.

$\bar{x} = 32.2$
$s = 6$
$Q_1 = 27.5$
Median = 33
$Q_3 = 36.5$

To measure the centre of the distribution, the median and the mean are used. Since there are no outliers and the distribution is approximately symmetric, the mean is quite a good measure of the centre of the distribution. Also, the mean and the median are quite close in value.

To measure the spread of the distribution, the standard deviation and the interquartile range are used.

The interquartile range is equal to $36.5 - 27.5 = 9$. This means that 50% of those in the sample spent within $9 of each other on family presents. Again, one might have expected a greater variation in what students spent. It would be interesting to know whether students confer about what they spend and therefore whether they tended to allocate about the same amount of money to spend.

At another school, the same investigation was undertaken and the results are shown in the following stem plot.

The summary statistics for these data are as follows:
$\bar{x} = 47.5$, $s = 16.3$, $Q_1 = 35$, median $= 50$, $Q_3 = 60$.

Stem	Leaf
2	0
2*	5 5
3	
3*	5
4	
4*	5 5
5	0 0
5*	5 5
6	0 0
6*	5
7	
7*	5

Key: 2|2 = 22 dollars

The distribution is approximately symmetric, albeit very spread out. The mean and the median are therefore reasonably close and give us an indication of the centre of the distribution. The mean value for this set of data is higher than for the data obtained at the other school. This indicates that students at this school in this year level, in general, spend more than their counterparts at the other school. Reasons for this might be that this school is in a higher socio-economic area and students receive greater allowances, or perhaps at this school there is a higher proportion of students from cultures where spending more money on family presents is usual.

The range of money spent on family presents at this school and at this particular year level is $55. This is certainly much higher than at the other school. The interquartile range at this school is $25. That is, the middle 50% of students spend within $25 of each other which is greater than the students at the other school.

studyon

Units 3 & 4 > AOS DA > Topic 5 > Concepts 1& 2

Population Summary screen and practice questions

Sample and simple random sample Summary screen and practice questions

Exercise 1.10 Populations and simple random samples

1. WE23 Generate 5 random numbers (integers) between 1 and 100.
2. Generate 10 random numbers (integers) between 1 and 250.
3. MC Students are selecting a sample of students at their school to complete an investigation. Which of the following are examples of choosing this sample randomly?
   - **A.** Choosing students queuing at the tuckshop
   - **B.** Assigning numbers to a list of student names and using a random number table to select random numbers
   - **C.** Calling for volunteers
   - **D.** Choosing the girls in an all-girls science class
   - **E.** Choosing students in a bus on the way home
4. Generate 10 random numbers (integers) between 1 and 100.
5. Generate 20 random numbers (integers) between 1 and 500.
6. Which is larger: A population or a sample? Explain why.

7. MC When selecting students for a simple random sample of a year level, the students selected should be:
   A. of similar age **B.** a group of mates **C.** independent
   D. female **E.** the tallest students
8. MC The students selected for a simple random sample of a year level should be selected by:
   A. a group of mates
   B. a group who all dance
   C. a selection of males
   D. the students with the best test results
   E. using random numbers
9. Would the mean be a good measure of the centre of the distribution shown? Explain.

Stem	Leaf
0	7
1	0 2
2	3 6 7 8 8
3	2 4 5 7 9 9
4	2 3 4 7 8
5	1 3 7
6	2 8

Key: 1|2 = 12

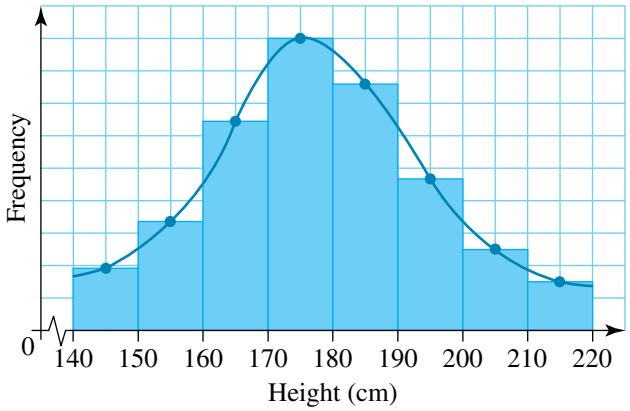
10. MC The mean is a good measure of the centre of a distribution if the data is:
    A. skewed left
    B. symmetric
    C. skewed right
    D. has outliers
    E. bimodal
11. MC The interquartile range is 12, since $Q_1 = 24$ and $Q_3 = 36$. The percentage of data that fit between 24 and 36 is:
    A. 12% **B.** 30% **C.** 50% **D.** 68% **E.** 95%
12. Conduct an investigation into how much money students in your year level earn per week (this might be an allowance or a wage). Write a report on your findings, ensuring you include:
    a. an explanation of the population for your investigation
    b. the manner in which your sample was selected
    c. the number in your sample
    d. your results as raw data
    e. your results in a stem plot or histogram
    f. the summary statistics for your data.

    Comment on your results based on the summary statistics.
13. Repeat question **12**, but this time investigate the following for students in your year level:
    a. the number of hours spent on homework each week
    b. the number of hours spent working in part-time jobs.
14. Conduct a similar investigation to that which you completed in questions **12** and **13**; however, this time sample students in another year group. Compare these data with those obtained for your year level.

1.11 The 68–95–99.7% rule and *z*-scores

1.11.1 The 68–95–99.7% rule

The heights of students at a graduation ceremony were recorded and are shown in the histogram at right.



This set of data is approximately symmetric and has what is termed a *bell* shape. Many sets of data fall into this category and are often referred to as **normal distributions**. Examples are birth weights and people's heights. Data which are normally distributed have their symmetrical, bell-shaped distribution centred on the mean value, $\bar{x}$.

A feature of this type of distribution is that we can predict what percentage of the data lie 1, 2 or 3 standard deviations (s, the standard deviation) either side of the mean using what is termed the 68–95–99.7% rule.

Note: For the purpose of this course, we will use s, to denote standard deviations, but for large populations the symbol for standard deviation is σ.

The 68–95–99.7% rule for a bell-shaped curve states that approximately:

1. **68% of data lie within 1 standard deviation either side of the mean**
2. **95% of data lie within 2 standard deviations either side of the mean**
3. **99.7% of data lie within 3 standard deviations either side of the mean.**

Figure 1

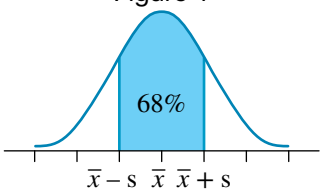


Figure 2

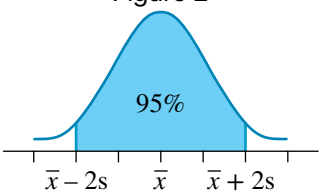
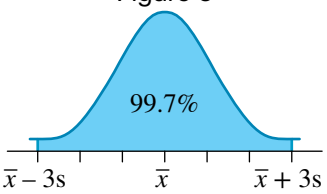


Figure 3



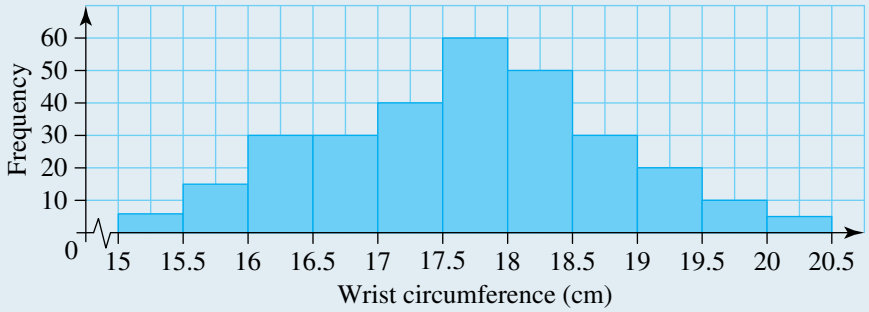
In Figure 1, 68% of the data shown lie between the value which is 1 standard deviation below the mean, that is $\bar{x} - s$, and the value which is 1 standard deviation above the mean, that is, $\bar{x} + s$.

In Figure 2, 95% of the data shown lie between the value which is 2 standard deviations below the mean, that is, $\bar{x} - 2s$, and the value which is 2 standard deviations above the mean, that is $\bar{x} + 2s$.

In Figure 3, 99.7% of the data shown lie between the value which is 3 standard deviations below the mean, that is, $\bar{x} - 3s$, and the value which is 3 standard deviations above the mean, that is, $\bar{x} + 3s$.

WORKED EXAMPLE 24

The wrist circumferences of a group of people were recorded and the results are shown in the histogram below. The mean of the set of data is 17.7 and the standard deviation is 0.9. Write down the wrist circumferences between which we would expect approximately:



a. **68% of the group to lie** **b.** **95% of the group to lie** **c.** **98.7% of the group to lie.**

THINK

a. The distribution can be described as approximately bell-shaped and therefore the 68–95–99.7% rule can be applied. Approximately 68% of the people have a wrist circumference between $\bar{x} - s$ and $\bar{x} + s$ (or one standard deviation either side of the mean).

b. Similarly, approximately 95% of the people have a wrist size between $\bar{x} - 2s$ and $\bar{x} + 2s$.

c. Similarly, approximately 99.7% of the people have a wrist size between $\bar{x} - 3s$ and $\bar{x} + 3s$.

WRITE

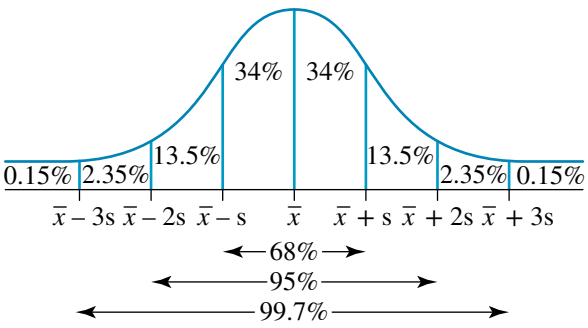
a. $\bar{x} - s = 17.7 - 0.9 = 16.8$
$\bar{x} + s = 17.7 + 0.9 = 18.6$
So approximately 68% of the people have a wrist size between 16.8 and 18.6 cm.

b. $\bar{x} - 2s = 17.7 - 1.8 = 15.9$
$\bar{x} + 2s = 17.7 + 1.8 = 19.5$
Approximately 95% of people have a wrist size between 15.9 cm and 19.5 cm.

c. $\bar{x} - 3s = 17.7 - 2.7 = 15.0$
$\bar{x} + 3s = 17.7 + 2.7 = 20.4$
Approximately 99.7% of people have a wrist size between 15.0 cm and 20.4 cm.

Using the 68–95–99.7% rule, we can work out the various percentages of the distribution which lie between the mean and 1 standard deviation from the mean and between the mean and 2 standard deviations from the mean and so on. The diagram at right summarises this.

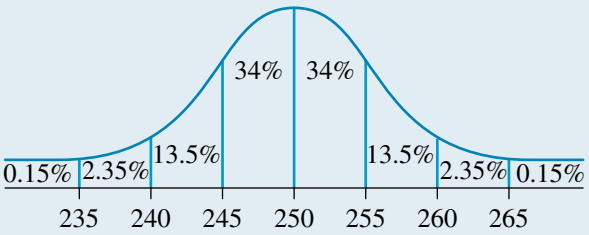
Note that 50% of the data lie below the mean and 50% lie above the mean due to the symmetry of the distribution about the mean.



WORKED EXAMPLE 25

The distribution of the masses of a number of packets of 'Fibre-fill' breakfast cereal is known to be bell-shaped with a mean of 250 g and a standard deviation of 5 g. Find the percentage of Fibre-fill packets with a mass which is:

a. less than 260 g
b. less than 245 g
c. more than 240 g
d. between 240 g and 255 g.

THINK	WRITE/DRAW
1. Draw the bell-shaped curve. Label the axis. $\bar{x} = 250$, $\bar{x} + s = 255$, $\bar{x} + 2s = 260$ etc.	
a. 260 g is 2 standard deviations above the mean. Using the summary diagram, we can find the percentage of data which is less than 260 g.	**a.** Mass of 260 g is 2 standard deviations above the mean. Percentage of distribution less than 260 g is: $13.5\% + 34\% + 34\% + 13.5\% + 2.35\% + 0.15\% = 97.5\%$ or $13.5\% + 34\% + 50\% = 97.5\%$
b. 245 g is 1 standard deviation below the mean.	**b.** Mass of 245 g is 1 standard deviation below the mean. Percentage of distribution less than 245 g is: $13.5\% + 2.35\% + 0.15\% = 16\%$ or $50\% - 34\% = 16\%$
c. 240 g is 2 standard deviations below the mean.	**c.** Mass of 240 g is 2 standard deviations below the mean. Percentage of distribution more than 240 g is: $13.5\% + 34\% + 34\% + 13.5\% + 2.35\% + 0.15\% = 97.5\%$ or $13.5\% + 34\% + 50\% = 97.5\%$
d. Now, 240 g is 2 standard deviations below the mean while 255 g is 1 standard deviation above the mean.	**d.** Mass of 240 g is 2 standard deviations below the mean. Mass of 255 g is 1 standard deviation above the mean. Percentage of distribution between 240 g and 255 g is: $13.5\% + 34\% + 34\% = 81.5\%$

WORKED EXAMPLE 26

The number of matches in a box is not always the same. When a sample of boxes was studied it was found that the number of matches in a box approximated a normal (bell-shaped) distribution with a mean number of matches of 50 and a standard deviation of 2. In a sample of 200 boxes, how many would be expected to have more than 48 matches?

THINK	WRITE
1. Find the percentage of boxes with more than 48 matches. Since $48 = 50 - 2$, the score of 48 is 1 standard deviation below the mean.	48 matches is 1 standard deviation below the mean. Percentage of boxes with more than 48 matches is: $34\% + 50\% = 84\%$
2. Find 84% of the total sample.	Number of boxes $= 84\%$ of 200 $= 168$ boxes

1.11.2 Standard *z*-scores

To find a comparison between scores in a particular distribution or in different distributions, we use the **z-score**. The z-score (also called the *standardised score*) indicates the position of a certain score in relation to the mean.

A z-score of 0 indicates that the score obtained is equal to the mean, a negative z-score indicates that the score is below the mean and a positive z-score indicates a score above the mean.

The z-score measures the distance from the mean in terms of the standard deviation. A score that is exactly one standard deviation above the mean has a z-score of 1. A score that is exactly one standard deviation below the mean has a z-score of -1.

To calculate a z-score we use the formula:

$$z = \frac{x - \bar{x}}{s}$$

where x = the score, $\bar{x}$ = the mean and s = the standard deviation.

WORKED EXAMPLE 27

In an IQ test, the mean IQ is 100 and the standard deviation is 15. Dale's test results give an IQ of 130. Calculate this as a z-score.

THINK	WRITE
1. Write the formula.	$z = \frac{x - \bar{x}}{s}$
2. Substitute for x, $\bar{x}$ and s.	$= \frac{130 - 100}{15}$
3. Calculate the z-score.	$= 2$

Dale's z-score is 2, meaning that his IQ is exactly two standard deviations above the mean.

Not all z-scores will be whole numbers; in fact most will not be. A whole number indicates only that the score is an exact number of standard deviations above or below the mean.

Using Worked example 28, an IQ of 88 would be represented by a z-score of -0.8, as shown below.

$$\begin{aligned} z &= \frac{x - \bar{x}}{s} \\ &= \frac{88 - 100}{15} \\ &= -0.8 \end{aligned}$$

The negative value indicates that the IQ of 88 is below the mean but by less than one standard deviation.

1.11.3 Comparing data

An important use of Z-scores is to compare scores from different data sets. Suppose that in your maths exam your result was 74 and in English your result was 63. In which subject did you achieve the better result?

At first glance, it may appear that the maths result is better, but this does not take into account the difficulty of the test. A mark of 63 on a difficult English test may in fact be a better result than 74 if it was an easy maths test.

The only way that we can fairly compare the results is by comparing each result with its mean and standard deviation. This is done by converting each result to a z-score.

If, for maths, $\bar{x} = 60$ and $s = 12$, then

$$\begin{aligned} z &= \frac{x - \bar{x}}{s} \\ &= \frac{74 - 60}{12} \\ &= 1.17 \end{aligned}$$

And if, for English, $\bar{x} = 50$ and $s = 8$, then

$$\begin{aligned} z &= \frac{x - \bar{x}}{s} \\ &= \frac{63 - 50}{8} \\ &= 1.625 \end{aligned}$$

The English result is better because the higher z-score shows that the 63 is higher in comparison to the mean of each subject.

WORKED EXAMPLE 28

Janine scored 82 in her Physics exam and 78 in her Chemistry exam. In Physics, $\bar{x} = 62$ and $s = 10$, while in Chemistry, $\bar{x} = 66$ and $s = 5$.

a. Write both results as a standardised score.

b. Which is the better result? Explain your answer.

THINK

a. 1. Write the formula for each subject.

2. Substitute for x, $\bar{x}$ and s.

3. Calculate each z-score.

b. Explain that the subject with the highest z-score is the better result.

WRITE

a. Physics: $z = \frac{x - \bar{x}}{s}$

$= \frac{82 - 62}{10}$

$= 2$

Chemistry: $z = \frac{x - \bar{x}}{s}$

$= \frac{78 - 66}{5}$

$= 2.4$

b. The chemistry result is better because of the higher z-score.

In each example the circumstances must be analysed carefully to see whether a higher or lower z-score is better. For example, if we were comparing times for runners over different distances, the lower z-score would be the better one.

on Resources

Interactivity: 68–95–99.7% rule in a normal distribution (int-6247)

studyon

Units 3 & 4 > AOS DA > Topic 4 > Concepts 1, 2, 3

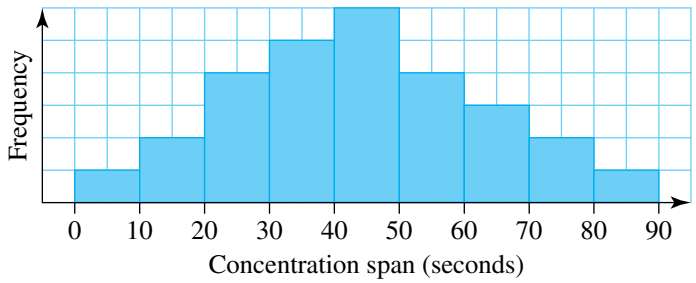
Normal distributions—the 68—95—99.7% rule Summary screen and practice questions

Standard z-scores Summary screen and practice questions

Comparing data values Summary screen and practice questions

Exercise 1.11 The 68–95–99.7% rule and z-scores

1. **WE24** The concentration ability of a group of adults is tested during a short task which they are asked to complete. The length of the concentration span of those involved during the task is shown.



The mean, $\bar{x}$, is 49 seconds and the standard deviation, s, is 14 seconds.
Write down the values between which we would expect approximately:

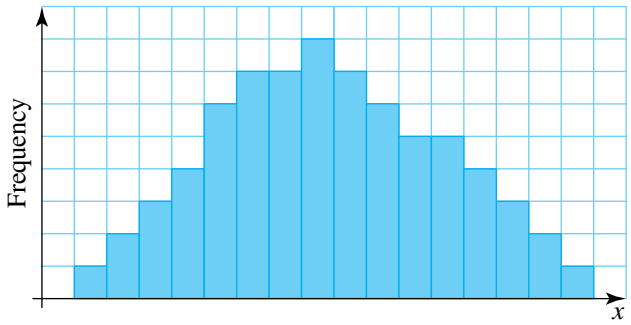
   a. 68% of the group's concentration spans to fall
   b. 95% of the group's concentration spans to fall
   c. 99.7% of the group's concentration spans to fall.

2. The monthly rainfall in Mathmania Island was found to follow a bell-shaped curve with a mean of 45 mm and a standard deviation of 1.7 mm. Write down the rainfall range which we would expect approximately:

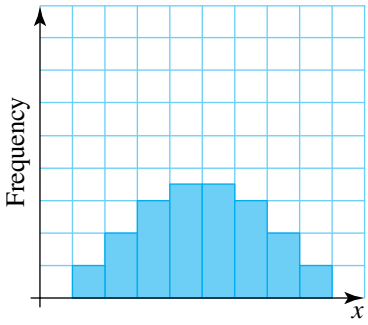
   a. 68% of the group to lie
   b. 95% of the group to lie
   c. 99.7% of the group to lie.

3. In each of the following, decide whether or not the distribution is approximately bell-shaped.

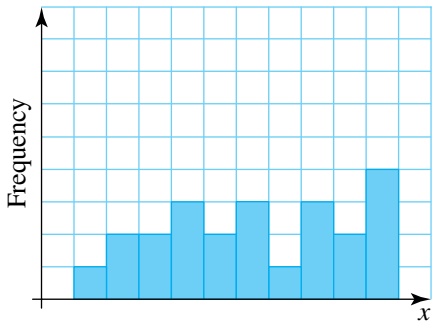
a.



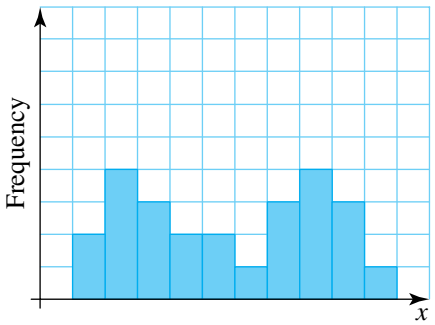
b.



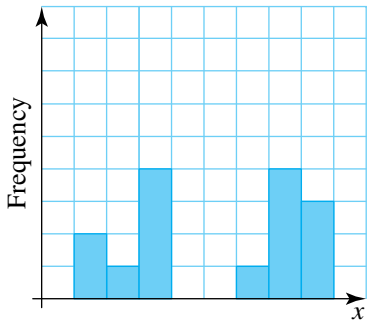
c.



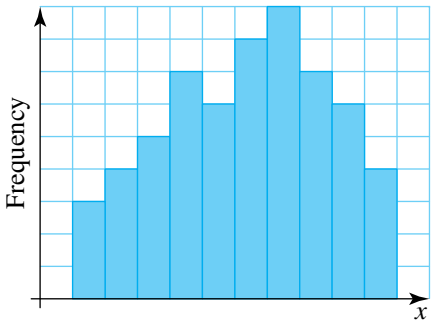
d.



e.

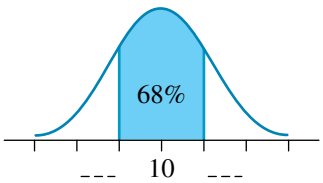


f.

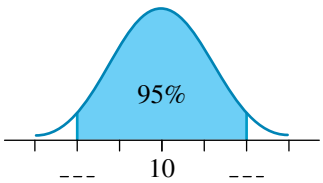


4. Copy and complete the entries on the horizontal scale of the following distributions, given that $\bar{x} = 10$ and $s = 2$.

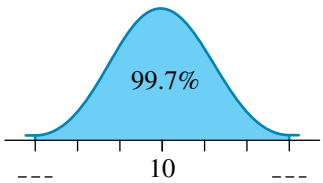
a.



b.

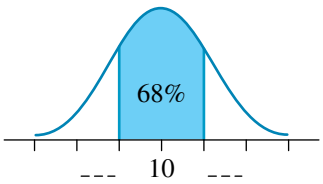


c.

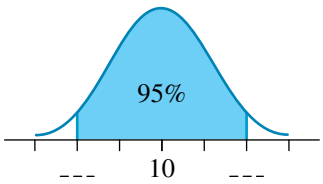


5. Copy and complete the entries on the horizontal scale of the following distributions, given that $\bar{x} = 5$ and $s = 1.3$.

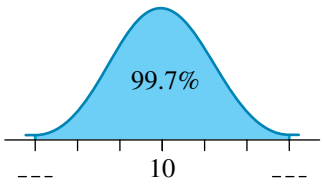
a.



b.



c.



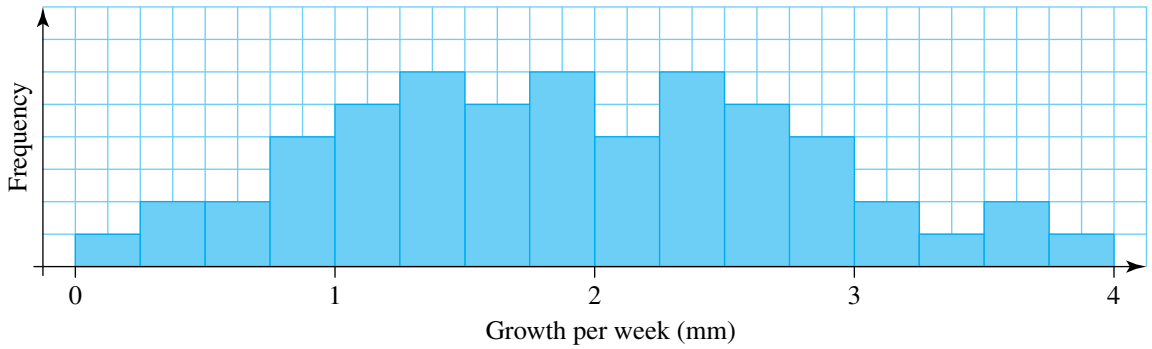
6. A research scientist measured the rate of hair growth in a group of hamsters. The findings are shown in the histogram.

The mean growth per week was 1.9 mm and the standard deviation was 0.6 mm. Write down the hair growth rates between which approximately:

a. 68% of the values fall

b. 95% of the values fall

c. 99.7% of the values fall.



7. The heights of the seedlings sold in a nursery have a bell-shaped distribution. The mean height is 7 cm and the standard deviation is 2.

Write down the values between which approximately:

a. 68% of seedling heights will lie

b. 95% of seedling heights will lie

c. 99.7% of seedling heights will lie.

8. MC A distribution of scores is bell-shaped and the mean score is 26. It is known that 95% of scores lie between 21 and 31.

It is true to say that:

A. 68% of the scores lie between 23 and 28

B. 97.5% of the scores lie between 23.5 and 28.5

C. the standard deviation is 2.5

D. 99.7% of the scores lie between 16 and 36

E. the standard deviation is 5

9. WE25 The distribution of masses of potato chips is known to follow a bell-shaped curve with a mean of 200 g and a standard deviation of 7 g. Find the percentage of the potato chips with a mass which is:

a. more than 214 g

b. more than 200 g

c. less than 193 g

d. between 193 g and 214 g.

10. The distribution of heights of a group of Melbourne-based employees who work for a large international company is bell-shaped. The data have a mean of 160 cm and a standard deviation of 10 cm.

Find the percentage of this group of employees who are:

a. less than 170 cm tall

b. less than 140 cm tall

c. greater than 150 cm tall

d. between 130 cm and 180 cm in height.

11. The number of days taken off in a year by employees of a large company has a distribution which is approximately bell-shaped. The mean and standard deviation of this data are shown below.
Mean = 9 days Standard deviation = 2 days
Find the percentage of employees of this company who, in a year, take off:
a. more than 15 days **b.** fewer than 5 days **c.** more than 7 days
d. between 3 and 11 days **e.** between 7 and 13 days.

12. **WE26** The number of marbles in a bag is not always the same. From a sample of boxes it was found that the number of marbles in a bag approximated a normal distribution with a mean number of 20 and a standard deviation of 1. In a sample of 500 bags, how many would be expected to have more than 19 marbles?

13. The volume of fruit juice in a certain type of container is not always the same. When a sample of these containers was studied it was found that the volume of juice they contained approximated a normal distribution with a mean of 250 mL and a standard deviation of 5 mL. In a sample of 400 containers, how many would be expected to have a volume of:
a. more than 245 mL **b.** less than 240 mL **c.** between 240 and 260 mL?

14. A particular bolt is manufactured such that the length is not always the same. The distribution of the lengths of the bolts is approximately bell-shaped with a mean length of 2.5 cm and a standard deviation of 1 mm.
a. In a sample of 2000 bolts, how many would be expected to have a length:
i. between 2.4 cm and 2.6 cm
ii. less than 2.7 cm
iii. between 2.6 cm and 2.8 cm?
b. The manufacturer rejects bolts which have a length of less than 2.3 cm or a length of greater than 2.7 cm. In a sample of 2000 bolts, how many would the manufacturer expect to reject?

15. **WE27** In a Physics test on electric power, the mean result for the class was 76% and a standard deviation of 9%. Drew's result was 97%. Calculate his mark as a z-score.

16. In a maths exam, the mean score is 60 and the standard deviation is 12. Chifune's mark is 96. Calculate her mark as a z-score.

17. **WE28** Bella's Specialist Maths mark was 83 and her English mark was 88. In Specialist Maths the mean was 67 with a standard deviation of 9, while in English the mean was 58 with a standard deviation of 14.
a. Convert the marks in each subject to a z-score.
b. Which subject is the better result for Bella? Explain.

18. Ken's English mark was 75 and his maths mark was 72. In English, the mean was 65 with a standard deviation of 8, while in maths the mean mark was 56 with a standard deviation of 12.
a. Convert the mark in each subject to a z-score.
b. In which subject did Ken perform better? Explain your answer.

19. In a major exam, every subject has a mean score of 60 and a standard deviation of 12.5. Clarissa obtains the following marks on her exams. Express each as a z-score.
a. English 54 **b.** Maths 78 **c.** Biology 61 **d.** Geography 32 **e.** Art 95

20. MC In a normal distribution the mean is 58. A score of 70 corresponds to a standardised score of 1.5. The standard deviation of the distribution is:

A. 6 **B.** 8 **C.** 10
D. 12 **E.** 9

21. In the first maths test of the year, the mean mark was 60 and the standard deviation was 12. In the second test, the mean was 55 and the standard deviation was 15. Barbara scored 54 in the first test and 50 in the second test. In which test did Barbara do better? Explain your answer.

22. The table below shows the average number of eggs laid per week by a random sample of chickens with 3 different types of living conditions.

Number of eggs per week		
Cage chickens	**Barn chickens**	**Free range chickens**
5.0	4.8	4.2
4.9	4.6	3.8
5.5	4.3	4.1
5.4	4.7	4.0
5.1	4.2	4.1
5.8	3.9	4.4
5.6	4.9	4.3
5.2	4.1	4.2
4.7	4.0	4.3
4.9	4.4	3.9
5.0	4.5	3.9
5.1	4.6	4.0
5.4	4.1	4.1
5.5	4.2	4.1

a. Copy and complete the following table by calculating the mean and standard deviation of barn chickens and free range chickens correct to 1 decimal place.

Living conditions	**Cage**	**Barn**	**Free range**
Mean	5.2		
Standard deviation	0.3		

b. A particular free range chicken lays an average of 4.3 eggs per week. Calculate the z-score relative to this sample correct to 3 significant figures.
The number of eggs laid by free range chickens is normally distributed. A free range chicken has a z-score of 1.

c. Approximately what percentage of chickens lay fewer eggs than this chicken?

d. Referring to the table showing the number of eggs per week, prepare five-number summaries for each set of data.

i. State the median of each set of data.

ii. What could be concluded about the egg-producing capabilities of chickens in different living conditions?

1.12 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC The best distances that a group of twenty 16-year-old competitors achieved in the long jump event at an athletics meeting are recorded.
This is an example of:

A. discrete, numerical data **B.** continuous, numerical data **C.** categorical data
D. discrete, categorical data **E.** continuous, categorical data

2. MC The observations shown on the stem-and-leaf plot are:

Stem	Leaf
2	0 0 1
2*	6 7 8 9 9
3	0 1 1
3*	5

Key: 2 | 1 = 21

A. 20, 21, 26, 27, 28, 29, 30, 31, 35
B. 20, 20, 21, 26, 27, 28, 29, 30, 31, 31, 35
C. 20, 21, 26, 27, 28, 29, 30, 31, 35
D. 20, 20, 21, 26, 27, 28, 29, 29, 30, 31, 31, 35
E. 20, 21, 26, 27, 28, 29, 29, 30, 31, 35

3. MC The number of people attending 25 of the sessions at an outside court, which has a seating capacity of 150, during the Australian Open Tennis Tournament, are displayed in the stem plot. Which of the following statements is untrue about the data?

Stem	Leaf
8	5 9
9	2 3 4 9
10	5 5 8
11	0 1 6 6 7
12	4 7 7 8 8
13	5 7 9 9
14	0 2

Key: 9 | 2 = 92

A. The smallest number of people attending was 85.
B. Only during six sessions did attendance fall below 100.
C. The largest number of people attending was 140.
D. On six occasions the number of people attending was more than 130.
E. On one occasion the number of people attending was only eight less than the seating capacity.

4. MC Which one of the frequency tables on the next page accurately summarises the scores shown below?

7 9 6 4 3 8 7 9 2

1 3 4 7 6 2 8 9 4

3 8 1 2 7 6 5 4 9

A.

Score	Frequency
1	2
2	3
3	2
4	5
5	2
6	4
7	6
8	3
9	2

B.

Score	Frequency
1	3
2	2
3	3
4	1
5	5
6	2
7	3
8	4
9	3

C.

Score	Frequency
1	3
2	2
3	1
4	3
5	1
6	5
7	3
8	2
9	4

D.

Score	Frequency
1	2
2	3
3	3
4	4
5	1
6	3
7	4
8	3
9	4

E.

Score	Frequency
1	3
2	4
3	3
4	2
5	2
6	1
7	3
8	2
9	1

5. MC The distribution of data shown in the stem plot could best be described as:

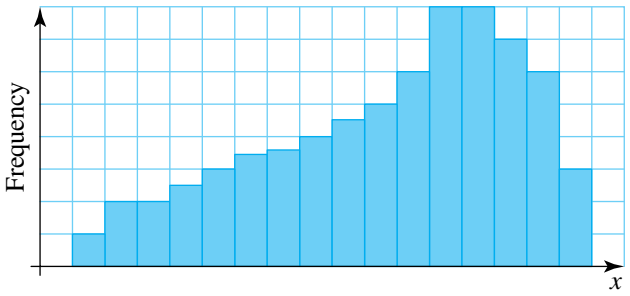
Stem	Leaf
2	3 4
2*	5 6 8
3	0 1 2 3 4 4
3*	5 5 7 9 9
4	0 1 3 3
4*	6 8 8
5	0 1
5*	6
6	
6*	9

Key: 3 | 1 = 31

A. negatively skewed
B. negatively skewed with one outlier
C. positively skewed
D. positively skewed with one outlier
E. symmetric

6. MC The distribution of the data shown in the histogram could best be described as:

A. negatively skewed
B. negatively skewed with one outlier
C. positively skewed
D. positively skewed with one outlier
E. symmetric



7. MC A set of data contains 7 observations and has a median of 5 and a range of 3. The set of data could be:

A. 4 4 5 6 7 **B.** 1 1 2 3 4 5 6

C. 4 5 5 5 6 7 7 **D.** 1 3 5 5 5 6 7

E. 3 5 7

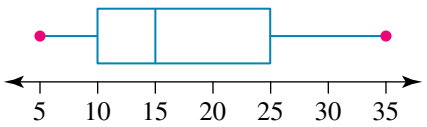
8. MC The median of the set of data shown in the stem plot below is:

Stem	Leaf
1	2 3
2	0 4 5 7
3	1 2 5 9
4	1 3 6 7
5	2 9 9
6	3

Key: 2|4 = 24

A. 5 **B.** 7 **C.** 9 **D.** 9.5 **E.** 37

9. MC For the distribution shown in this boxplot, it is true to say that:



A. the range is 35

B. the interquartile range is 10

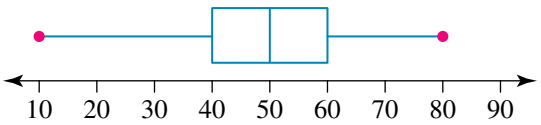
C. the median is 20

D. the interquartile range is 25

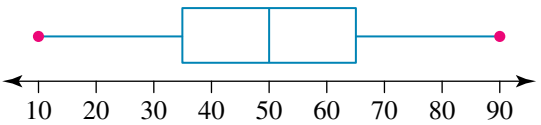
E. the median is equal to the interquartile range

10. MC A distribution has a range of 80, an interquartile range of 30 and a median of 50. Which one of the following boxplots could represent this distribution?

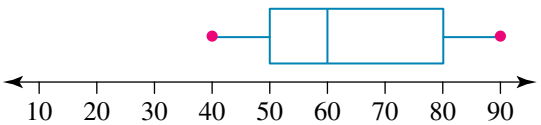
A.



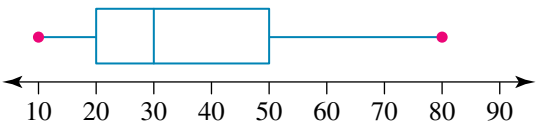
B.



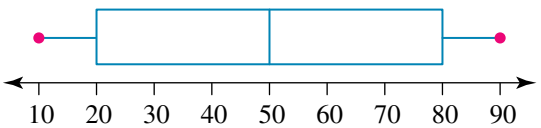
C.



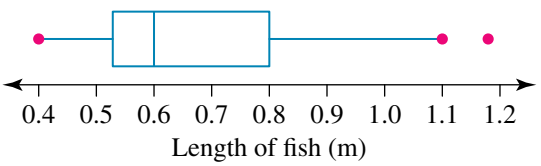
D.



E.



11. MC The boxplot represents the lengths of barracuda caught by fishing boats during one day. Which one of the following statements is not true about these data?



A. The data contain an outlier.

B. The shortest length is 0.4 m.

C. The median is 60 cm.

D. The interquartile range is 0.2 m.

E. The distribution is positively skewed.

12. MC For the following set of data: 14, 18, 20, 21, 23, 23, 24, 25, 29, 30, the mean is:

A. 10 **B.** 22.666 666 … **C.** 22.7

D. 23 **E.** 24.222 222 …

13. MC The ages of a group of students entering university for the first time is shown on the stem plot. What is the mean age?

Stem	Leaf
1	
1*	5 7 7 8 8 8 8 8 8 9
2	0 0 0 1 1
2*	6 8
3	1
3*	5

Key: 1 ∗ |5 = 15 years

A. 18

B. 18.9

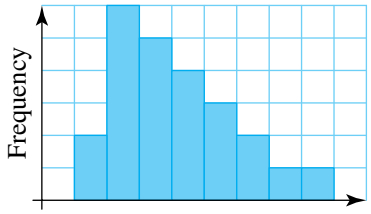
C. 19

D. 20.9

E. 21

14. MC In which case below would you expect the mean to be greater than the median?

A.



B.

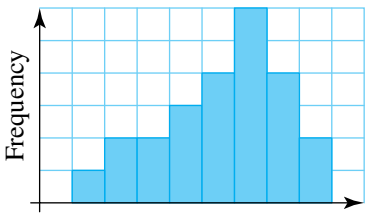
Stem	Leaf
1	
1*	5
2	1 4
2*	5 7 8 8 9
3	1 2 2 3 3 3 4 4
3*	5 5 5 6
4	3 4
4*	

Key: 2|4 = 24

C. The data:

11, 13, 16, 17, 18, 18, 19, 20

D.



E.

15. MC In which case in question **14** is the median not necessarily the better measure of the centre of the data?

16. MC The Millers obtained a number of quotes on the price of having their home painted. The quotes, to the nearest hundred dollars, were:

$4200, $4800, $5100, $5000, $4700, $4700, $4600, $4900

The standard deviation for this set of data, to the nearest whole dollar, is:

A. $277 **B.** $278 **C.** $324 **D.** $325 **E.** $4750

17. MC The number of Year 12 students who spent their spare periods studying in the resource centre during each week of terms 3 and 4 is shown on the stem plot.
The standard deviation for this set of data, to the nearest whole number is:

Stem	Leaf
0	8
1	
2	5 6 6 7
3	0 2 3 6 9
4	7 9
5	6
6	1

Key: 2|5 = 25 students

A. 10 **B.** 12 **C.** 14 **D.** 17 **E.** 35

18. MC The lifetime (in hours) of a particular type of battery is known to have a distribution which is bell-shaped. A large number of batteries of this type are sampled and are found to have a mean lifetime of 1200 hours and a standard deviation of 10 hours. We would expect that approximately 95% of the batteries in the sample would have a lifetime (in hours) between:

A. 10 and 1200 **B.** 1170 and 1230

C. 1200 and 1210 **D.** 1180 and 1220

E. 1190 and 1210

19. MC A set of marks from a maths test has a mean of 45 and a standard deviation of 5. The distribution of the marks is known to be bell-shaped. Which of the following statements is false?

A. Approximately 68% of the marks lie between 40 and 50.
B. The distribution is approximately symmetric.
C. Approximately 95% of the marks lie between 35 and 50.
D. A mark in the twenties would be most unusual.
E. Approximately 99.7%of the marks lie between 30 and 60.

20. MC The mean length of a large batch of broom handles is 120 cm. The data have a standard deviation of 3 cm. The percentage of broom handles, in this batch, which are shorter than 114 cm is:

A. 0.15% **B.** 2.5% **C.** 13.5% **D.** 16% **E.** 34%

21. MC The mean birth weight of babies at a hospital is 2.8 kg with a standard deviation of 0.4 kg. The standardised score for a weight of 3.3 kg would be:

A. 0.73
B. −1.25
C. 1.04
D. 1.25
E. −1.04.

22. MC When selecting a simple random sample of 20 teachers from all teachers in a particular high school, the teachers should be selected by:
   - **A.** choosing teachers from the Maths department
   - **B.** choosing teachers of similar age
   - **C.** assigning numbers to a list of teachers' names and using a random number generator to select random numbers
   - **D.** choosing from Year 12 teachers
   - **E.** choosing the first 20 teachers from the list of teachers' names

Short answer

1. Write an example of a variable which produces:
   - **a.** categorical data
   - **b.** numerical data that are:
     - **i.** discrete
     - **ii.** continuous
2. The money (rounded to the nearest whole dollar) raised by fifteen Year 12 students is shown below.

   78, 84, 61, 73, 71, 83, 87, 65, 60, 67, 71, 82, 84, 79, 78

   Construct a stem plot for the amount raised using:
   - **a.** the stems 6, 7 and 8
   - **b.** the stems 6, 7 and 8 split into halves
   - **c.** the stems 6, 7 and 8 split into fifths.

   Discuss your results.
3. **a.** The following frequency table shows the speeds of cars recorded by police. The cars were travelling through a 60 km/h zone. Construct a histogram to display the data.

Class interval	Frequency
50–51.9	3
52–53.9	5
54–55.9	6
56–57.9	7
58–59.9	9
60–61.9	10
62–63.9	9
64–65.9	10
66–67.9	8
68–69.9	5
70–71.9	3
72–73.9	4
74–75.9	2

   - **b.** Check your work using a CAS calculator.
   - **c.** Draw two conclusions about these data.

4. The money raised (correct to the nearest whole dollar) by each student in a Year 3 class on the school walkathon is shown in the stem plot.

Stem	Leaf
0	8 9
1	2 3 4 7
2	1 2 2 3 5 7 9
3	0 1 4 5 8
4	3 5 6 7
5	1 3 5
6	4 6
7	6

Key: 0|8 = $8

   a. Describe the shape of the distribution of these data.
   b. Describe how this distribution would need to change for it to become a symmetric one.

5. Find the range, the median, the mode and the interquartile range of this set of data.

Stem	Leaf
0	2
0∗	5 6 6 8 9
1	0 2 2 4 4 4
1∗	5 5 7 8 8 9
2	1 3
2∗	6

Key: 1|4 = 14

6. a. For the set of data below, construct a boxplot to display the distribution.

2 5 4 6 3 7 9 8 5 3

1 4 6 8 7 5 2 9 5 6

   b. Describe the shape of the distribution.

7. A local council monitored the use of the public swimming pool over a 12-day period. The attendance is recorded in the table below.

Attendance	275	180	201	345	480	150	500	155	302	650	362	410

   a. What is the range?
   b. For what percentage of days was the attendance greater than 300? Give your answer correct to one decimal place.

8. A chemical component is added to a filtering system on a weekly basis. The amount of chemical component required each week varies. The amounts required (in mL) over the past 20 weeks are shown in the stem plot.
Calculate, correct to 2 decimal places, the standard deviation of the amounts used.

Stem	Leaf
2	1
2	2 2
2	4 4 4 5
2	6 6
2	8 8 9 9
3	0
3	2 2
3	4 5
3	6
3	8

Key: 3|8 = 0.38 mL

9. The life spans of dogs of a particular breed follow a bell-shaped distribution. A group of this particular breed at a dog club was found to have a mean life span of 12 years with a standard deviation of 1.2 years.
   a. For this group, write down the expected values between which the life spans of approximately:
      i. 68% of the dogs would lie
      ii. 95% of the dogs would lie
      iii. 99.7% of the dogs would lie.
   b. What does this information suggest about this breed?
10. Ricardo scored 85 on an entrance test for a job. The test has a mean score of 78 and a standard deviation of 8. Kory sits a similar test and scores 27. In this test, the mean is 18 and the standard deviation is 6. Based on this test, who is the better candidate for the job? Explain your answer.
11. An experimental physicist recorded two measurements using log (base 10) scale. The measurements were 8.2 and 6.6. How many times greater was the first measurement compared to the second? Give your answer correct to the nearest whole number.
12. The quarterly incomes (in dollars) of the sample of 12 local businesses was recorded as follows:
$45 000, $20 000, $360 000, $750 000, $3 200, $1 048 500,
$34 590, $37 250, $65 710, $290 060, $59 070, $8 450
Complete the frequency table below for log (base 10) of incomes using class intervals of width 1.

$\log_{10}$(income)	Frequency

Extended response

1. Mr Fahey gives the same test to the two Year 10 classes that he teaches, 10C and 10E. This test is out of 20. The results in 10C are:

4 7 7 9 9 10 10 11 12

13 14 14 15 15 15 16 17 17

17 17 18 18 18 19 19

The results in 10E are:

8	9	10	11	11	12	12	12	13
13	13	13	13	14	14	14	14	14
14	15	15	15	16	16	19		

a. For each of these sets of data:
- **i.** display the data using a histogram or a stem plot. Give reasons for your choices. Also describe the shapes of the distribution
- **ii.** calculate the median, the interquartile range, the range and the mode
- **iii.** represent each set of data using a boxplot
- **iv.** calculate the mean and the standard deviation
- **v.** state whether the mean or the median is a better measure of the centre of each distribution
- **vi.** comment as to whether the 68–95–99.7% rule can be applied to either of the distributions.

b. Using the summary statistics you have calculated, comment on and compare the performances of 10C and 10E on the test.

2. A group of office workers and a group of sports instructors were asked to complete 5 minutes of exercise as part of a study of heart rates. Following the exercise, participants rested for 2 minutes before their pulse rates were measured. The results are set out in the stem plots.

Pulse rates for office workers (beats/min)

Stem	Leaf
7	6
8	
9	5
10	6 7
11	0 2
12	0 1 2 4 6 7 9
13	0 0 4

Pulse rates for sports instructors (beats/min)

Stem	Leaf
6	2 4 8 8 9
7	2 2 3 5 7 9
8	2 8
9	6
10	8

Key: 12|4 = 124 beats/min

a. Describe the shape of each distribution.
b. Calculate the median, the interquartile range, the mode and the range for both.
c. Represent each set of data using a boxplot.
d. Calculate the mean and the standard deviation for both sets of data.
e. Comment as to whether the 68–95–99.7% rule can be applied to either of the distributions.
f. Use the summary statistics that you have calculated to comment on the pulse rates of each group, noting any differences between the two.

3. A hatch of Atlantic salmon has been reared in a coastal environment over a period of 12 months. The lengths (to the nearest cm) of a sample of 20, out of the total number of 10 000 fish, are shown below.

13 16 17 14 16 19 15 17 16 15

16 18 16 13 17 14 18 15 19 16

a. Describe the type of data that the variable produces.
b. Construct an appropriate stem plot from these data and use it to describe the shape of the distribution.
c. Using your stem plot, calculate the five number summary statistics and then draw a boxplot.
d. Describe the shape of the distribution from the boxplot.
e. Does the stem plot or boxplot give a better indication of the distribution's shape?
f. For a symmetric distribution the mean is the same as the median. Is that the case here?
g. Given that the distribution is symmetric, the whole population of these salmon would form a normal or bell-shaped distribution.
Find the standard deviation (correct to 2 decimal places) for this sample and use it, along with the mean, to estimate the number of fish (out of 10 000) that would have lengths greater than 17.75 cm.
The same number of salmon was reared in a river environment over the same period of time. The lengths of 20 fish in a sample are shown below.

18 20 17 19 16 19 19 17 16 18

19 18 12 18 17 14 18 15 19 17

h. Use an appropriate method to help you describe the shape of this distribution.
i. Determine how many of this population of 10 000 salmon would have a length greater than 19.25 cm (calculate the standard deviation of the sample correct to 2 decimal places).
j. Comment on the growth of each hatch of salmon over the 12 months.

4. The birth weights (in kg) of 50 of the 220 babies that were born at a hospital during a one-month period are listed below.

2.9 2.7 3.1 2.5 2.4 2.6 2.9 2.6 3.2 4.1

2.3 2.8 2.4 3.2 2.7 2.5 2.6 2.9 3.0 2.2

3.4 3.1 3.3 2.9 3.2 2.9 3.4 3.1 2.3 3.5

3.1 3.0 2.9 3.6 3.1 2.7 2.6 1.8 1.9 3.6

2.0 3.4 3.5 2.4 3.5 3.0 2.2 2.8 3.5 3.1

a. Construct a frequency histogram for the data using class intervals of 1.5–1.9, 2.0–2.4, 2.5–2.9 and so on.

b. Comment on the shape of the distribution.

c. It has been said that the mean birth weight of babies is 3 kg. Using the given data, comment on this statement.

d. Using the mean and standard deviation (correct to 2 decimal places) for this sample of 50 birth weights, estimate how many of the 220 babies born at the hospital had weights:

- **i.** between 2.35 kg and 3.43 kg
- **ii.** between 3.43 kg and 3.97 kg
- **iii.** greater than 3.97 kg.

studyon

Units 3 & 4 Sit exam

Answers

Topic 1 Investigating Data Distributions

Exercise 1.2 Types of data

1. D
2. D
3. Numerical: **a**, **b**, **c**
4. Categorical: **c**, **d**, **e**, **f**, **g**
5. C
6. B
7. Discrete: **c**
   Continuous: **a**, **b**
8. C
9. C
10. Categorical
11. B
12. A
13. Categorical and ordinal
14. Discrete
15. Ordinal
16. D

Exercise 1.3 Stem plots

1.

Stem	Leaf
1	6 8
2	1 5 8 8 9
3	0 1 3 5 8 9
4	2 8 9

Key: 1|6 = 16

2.

Stem	Leaf
0	5
1	1 8 9
2	3 7 9
3	1 2 5 6 7 9
4	1 2 3 5
5	2

Key: 0|5 = $5

The busker's earnings are inconsistent.

3.

Stem	Leaf
86	8
87	7
88	0
89	8
90	2 4 8 9
91	
92	0 1 2 6
93	
94	3 9
95	
96	
97	0
98	3 5 9

Key: 86|8 = 86.8%

4.

Stem	Leaf
18	5 7 9
19	1 5 6 6 7 9
20	1 3 3 5 9
21	7
22	1

Key: 18|5 = 1.85 cm

5.

Stem	Leaf
3	7 9
4	2 9 9
5	1 1 2 3 7 8 9
6	1 3 3 8

Key: 3|7 = 37 years

It seems to be an activity for older people.

6. C

7.

Stem	Leaf
1*	9
2	
2*	5 8 8 9 9 9
3	0 0 2 2 2 3 3 4
3*	5 5 7 8 9

Key: 2|5 = 25 years

More than half of the parents are 30 or older with a considerable spread of ages, so this statement is not very accurate.

8.

Stem	Leaf
1*	9
2	1 2 4
2*	5 6 8 9 9
3	1 1 2 3 4
3*	
4	0 1

Key: 2|1 = 21 hit outs

Bulldogs, Melbourne, St Kilda

9.

Stem	Leaf
33	0
34	
35	0 0 1
36	5
37	3
38	0 0
39	0 0 5
40	0 6
41	0 5
42	1 3
43	0 0
44	
45	0

Key: 33|0 = $330

The stem plot shows a fairly even spread of rental prices with no obvious outliers.

10. a.

Stem	Leaf
2	0 2 2
2*	5 6 8 8
3	3 3 3
3*	7 7 8 9 9 9

Key: 2|0 = 20 points

b.

Stem	Leaf
2	0
2	2 2
2	5
2	6
2	8 8
3	
3	3 3 3
3	
3	7 7
3	8 9 9 9

Key: 2|0 = 20 cm

11. a.

Stem	Leaf
4	3 7 7 8 8 9 9 9
5	0 0 0 0 1 2 2 3

Key: 4|3 = 43 cm

b.

Stem	Leaf
4	3
4*	7 7 8 8 9 9 9
5	0 0 0 0 1 2 2 3
5*	

Key: 4|3 = 43 cm

c.

Stem	Leaf
4	
4	3
4	
4	7 7
4	8 8 9 9 9
5	0 0 0 0 1
5	2 2 3
5	
5	
5	

Key: 4|3 = 43 cm

12. a. 1, 2, 5, 8, 12, 13, 13, 16, 16, 17, 21, 23, 24, 25, 25, 26, 27, 30, 32

b. 10, 11, 23, 23, 30, 35, 39, 41, 42, 47, 55, 62

c. 101, 102, 115, 118, 122, 123, 123, 136, 136, 137, 141, 143, 144, 155, 155, 156, 157

d. 50, 51, 53, 53, 54, 55, 55, 56, 56, 57, 59

e. 1, 4, 5, 8, 10, 12, 16, 19, 19, 21, 21, 25, 29

13. a.

Stem	Leaf
1	5 6 7 7 7 8 9 9 9 9
2	0 0 0 1 1 1 2 3 3 3

Key: 1|5 = 15 mm

b.

Stem	Leaf
1	
1*	5 6 7 7 7 8 9 9 9 9
2	0 0 0 1 1 1 2 3 3 3
2*	

Key: 1|5 = 15 mm

c.

Stem	Leaf
1	
1	
1	5
1	6 7 7 7
1	8 9 9 9 9
2	0 0 0 1 1 1
2	2 3 3 3
2	
2	
2	

Key: 1|5 = 15 mm

Values are bunched together; they vary little.

14.

Stem	Leaf
7	2 8
8	3 3 5 7 8 8
9	0 1 2 2 3 4 8 9
10	0 2 4
11	2

Key: 7|2 = 72 shots

15.

Stem	Leaf
6	
6*	8 9
7	1 1 2 2 3 3 3 3 4 4
7*	5 5 5 6 6 7
8	
8*	6

Key: 7|1 = 71 net score

The handicapper has done a good job as most of the net scores are around the same scores; that is, in the 70s.

16. a.

Stem	Leaf
6	0 3 9
7	0 1 3 5 6 7 8
8	0 1 3 4 7 8 9 9
9	1 3 7 8

Key: 6|0 = 60%

b.

Stem	Leaf
6	0 3
6*	9
7	0 1 3
7*	5 6 7 8
8	0 1 3 4
8*	7 8 9 9
9	1 3
9*	7 8

Key: 6|0 = 60%

17. a.

Computer 1	Stem	Computer 2
5	34	0 2 6 8
8 2	35	2 3 5 5 7 8
6 3	36	1 2
6	37	
1 0	38	
2 1	39	
5	40	
0	41	

Key: 34|0 = 340 minutes

b. Computer 1 lasts longer but is not as consistent.
Computer 2 is more consistent but doesn't last as long.

18. a.

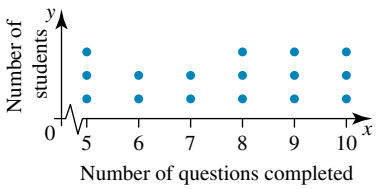
Year 8	Stem	Year 10
9 8	14	
7 5 5 5 3 1 0	15	2 4 6 8 9
8 6 5 4 3 2 1 0	16	0 4 5 7 7 9
5 2 1	17	2 3 4 6 7 8 8
	18	2 5

Key: 14|8 = 148 cm

b. As you would expect the Year 10 students are generally taller than the Year 8 students; however, there is a large overlap in the heights.

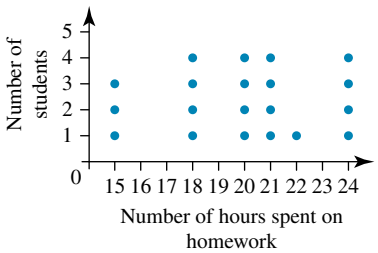
Exercise 1.4 Dot plots, frequency tables and histograms, and bar charts

1.

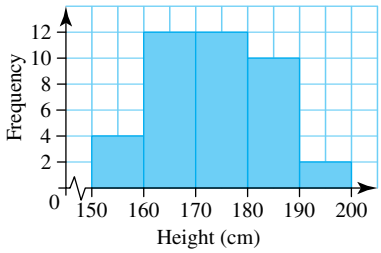


2. See graph at the bottom of the page

3.

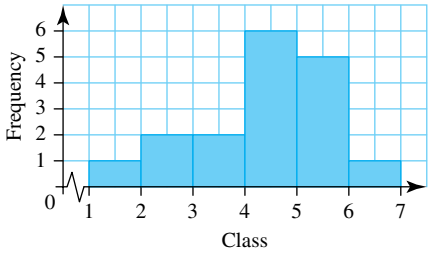


4.



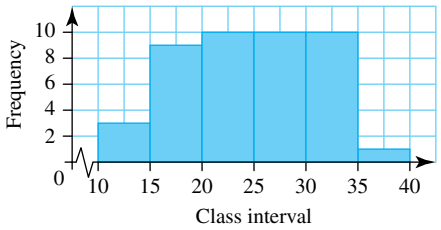
5. a.

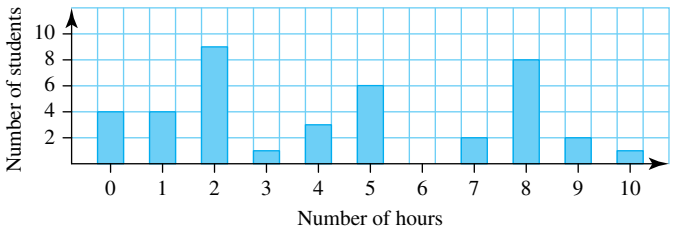
Class	Frequency
1–	1
2–	2
3–	2
4–	6
5–	5
6–	1

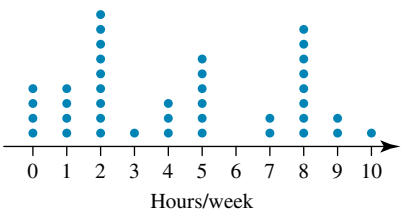


b.

Class interval	Frequency
10–	3
15–	9
20–	10
25–	10
30–	10
35–	1

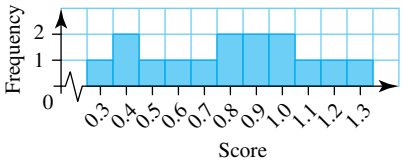






c.

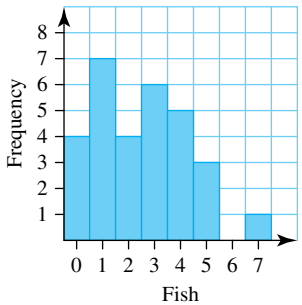
Score	Frequency
0.3	1
0.4	2
0.5	1
0.6	1
0.7	1
0.8	2
0.9	2
1.0	2
1.1	1
1.2	1
1.3	1



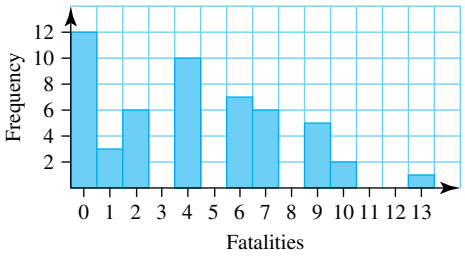
6.

Number of dogs	Tally	Frequency
2	II	2
3	III	3
4	II	2
5	II	2
6	I	1
7	IIII	5
8	II	2
9	IIII III	8
10	III	3
11		0
12	II	2
Total		30

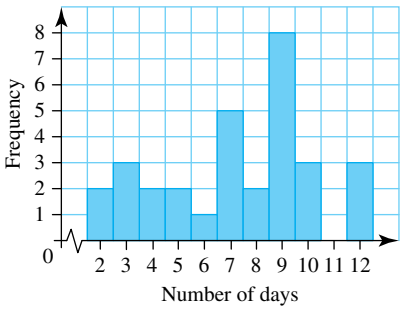
7.



8.

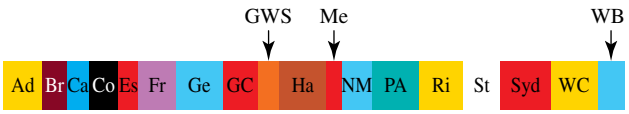


9.

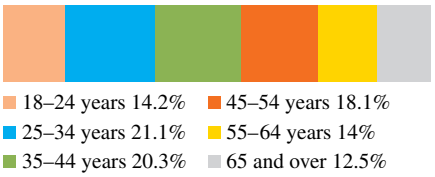


10. Check your histogram against that shown in the answer to question **9**.

11.

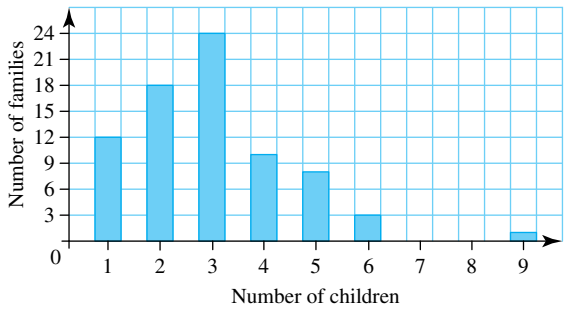


12. Participation in activities

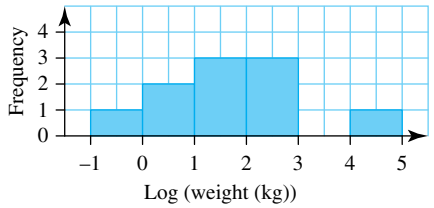


The statement seems untrue as there are similar participation rates for all ages. However, the data don't indicate types of activities.

13.



14.



15. D

16. D

17. a. B b. A c. D

18. A

19. B

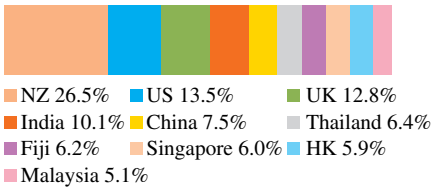
20. 16

21. 5 times

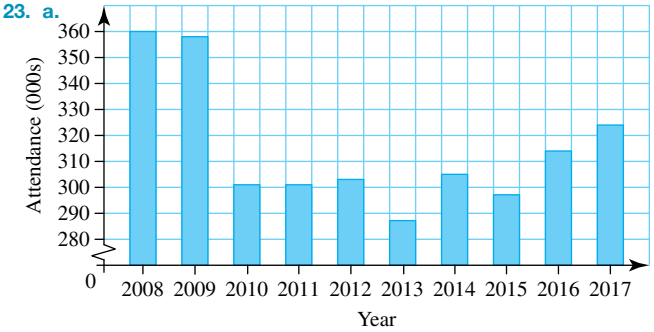
22. a.

NZ	26.5%
US	13.5%
UK	12.8%
India	10.1%
China	7.5%
Thailand	6.4%
Fiji	6.2%
Singapore	6.0%
HK	5.9%
Malaysia	5.1%

b.



23. a.



b. Check your bar chart against that shown in the answer to part a.

Exercise 1.5 Describing the shape of stem plots and histograms

1. Positively skewed

2. Negatively skewed

3. a. Symmetric b. Negatively skewed
   c. Positively skewed d. Symmetric
   e. Symmetric f. Positively skewed

4. a. Symmetric, no outliers
   b. Symmetric, no outliers
   c. Symmetric, no outliers
   d. Negatively skewed, no outliers
   e. Negatively skewed, no outliers
   f. Positively skewed, no outliers

5. E

6. C

7. Negatively skewed

8. Positively skewed. This tells us that most of the flight attendants in this group spend a similar number of nights (between 2 and 5) interstate per month. A few stay away more than this and a very few stay away a lot more.

9. a. Symmetric
   b. This tells us that there are few low-weight dogs and few heavy dogs but most dogs have a weight in the range of 10 to 19 kg.

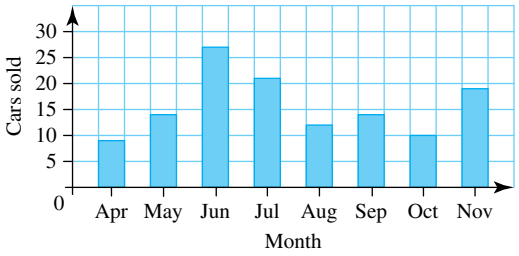
10. a. Symmetric
    b. Most students receive about $8 (give or take $2).

11. a. Positively skewed
    b. i. 15 ii. 85%

12. a. Positively skewed
    b. Since most of the data is linked to the lower stems, this suggests that some students do little exercise, but those students who exercise, do quite a bit each week. This could represent the students in teams or in training squads.

13. a. Club A: negatively skewed
    Club B: positively skewed
    b. Since Club A has more members of its bowling team at the higher stems as compared to Club B; you could say Club A has the older team as compared to Club B.
    c. i. Club A: 11 members over 70 years of age
    ii. Club B: 4 members over 70 years of age.

14. a.



    b. Positively skewed
    c. June, July and November represent the months with the highest number of sales.
    d. This is when the end of financial year sales occur.

Exercise 1.6 The median, the interquartile range, the range and the mode

1. Median = 33

2. Median = 36.5 goals

3. IQR = 14

4. IQR = 8

5. IQR = 6.5

6. IQR = 3.3

7.

	Median	Range	Mode
a	37	56	38, 49
b	5	17	5
c	11	18	8, 11
d	42.5	18	43
e	628	72	613, 628, 632

8.

	Median	Range
a	6	7
b	17	9
c	6	6
d	10	13
e	18.5	14
f	4	7
g	19	17
h	4.5	9

9. a. 10
   b. 8
   c. The IQRs (middle 50%) are similar for the two restaurants, but they don't give any indication about the number of cars in each data set.
10. An example is 2, 3, 6, 8, 9. There are many others.
11. a. The lowest score occurs several times. An example is 2, 2, 2, 3, 5, 6 .
    b. There are several data points that have the median value. An example is 3, 5, 5, 5, 5, 5, 7 .
12. C
13.

	Median	IQR	Range	Mode
a	21	18	45	15, 23, 32
b	27.5	8	20	29
c	3.7	3	5.9	3.7

14.

	Median	IQR	Range	Mode
a	42	21	91	46
b	32	7	30	34

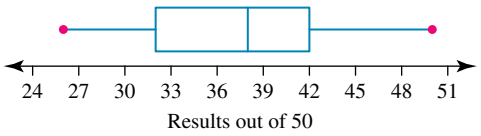
The data in set **a** have a greater spread than in set **b**, although the medians are similar. The spread of the middle 50% (IQR) of data for set **a** is bigger than for set **b** but the difference is not as great as the spread for all the data (range).

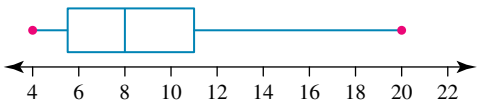
15. a. Range = 72, Median = 37.5, Mode = 46, IQR = 22
    b. Range = 47
    Median = 422
    Mode = 411
    IQR = 20
16. Median = 7, Mode = 7
17. $Q_1 = 42.2$, $Q_3 = 48.15$, IQR = 5.95, Median = 45.1
18. a. Median = 93, $Q_1 = 91.5$, $Q_3 = 97$, IQR = 5.5, Range = 30, Mode = 93
    b. The average handicap of the golfers should be around 21.

Exercise 1.7 Boxplots

1. Range = 39
   Median = 25
   IQR = 19
2. Range = 3
   Median = 7.5
   IQR = 1.4
3.

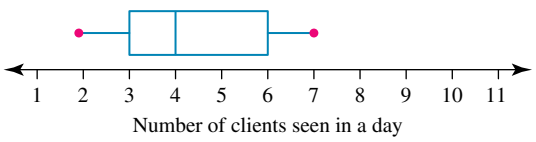
	Range	IQR	Median
a	12	6	8
b	7	2	5
c	350	100	250
d	100	30	65
e	20	10	25

4. They could represent the same data.
5. They could represent the same data.
6. a. iii b. iv c. i d. ii
7. 

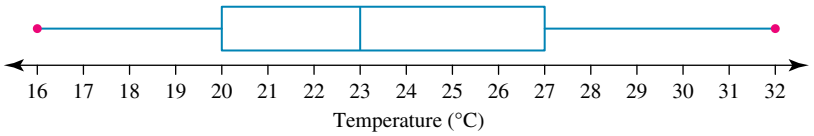
   Negatively skewed; 50% of results are between 32 and 42.
8. 

   Fairly symmetrical.
9. The boxplots should show the following:

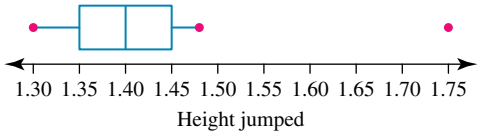
	Min	Q_1	Med	Q_3	Max
a	3	6	8.5	14	18
b	3	5	7	9	12
c	4.3	4.6	5	5.4	5.6
d	11	15.5	18	20	22
e	0.4	0.7	0.9	1.1	1.3

10. D
11. 

12.

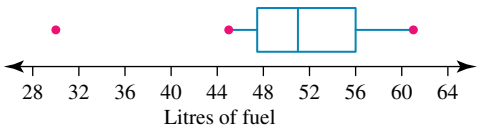


13. a.



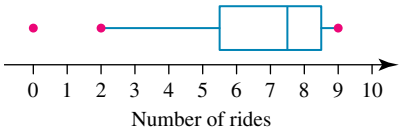
b. The data is symmetrical and 1.75 is an outlier.

14.



30.3 is an outlier.

15. a.



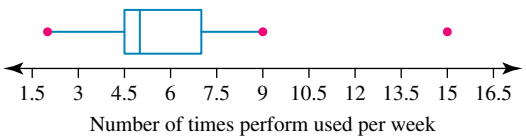
The data are negatively skewed with an outlier on the lower end. The reason for the outlier may be that the person wasn't at the show for long or possibly didn't like the rides.

16. a. Two similar properties: both sets of data have the same minimum value and similar IQR value.

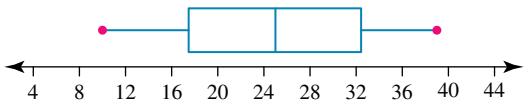
b. Boys IQR = 16
Girls IQR = 16.5

c. The reason for an outlier in the boys' data may be that the student did not understand how to do the test, or he stopped during the test rather than working continuously.

17. Median = 5
$Q_1 = 4.5$
$Q_3 = 7$
$\text{Min}_x = 2$, $\text{Max}_x = 15$
IQR = 2.5
15 is an outlier.



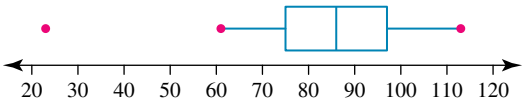
18. a. Median = 25
$Q_1 = 17.5$
$Q_3 = 32$
$\text{Min}_x = 11$, $\text{Max}_x = 39$
IQR = 14.5



b. No outliers

c. Check your boxplot against that shown in the answer to part **a**.

19. Median = 86
$Q_1 = 75$
$Q_3 = 97$
$\text{Min}_x = 23$, $\text{Max}_x = 113$
IQR = 22
23 is an outlier.



Exercise 1.8 The mean of a sample

1. 23.46

2. 8.26

3. a. 7.2 **b.** 7.125 **c.** 4.9875
d. 16.7 **e.** 0.8818

4. a. $1.078\dot{3}$ No, because of the outlier.
b. 17 Yes
c. 30.875 Yes
d. 15.57 No, because of the outlier.

5. 12

6. D

7. A

8. a. Median **b.** Mean **c.** Median **d.** Median

Exercise 1.9 Standard deviation of a sample

1. 3.54 cents

2. 14.27%

3. 9.489

4. 7.306

5. a. 1.21 **b.** 2.36 **c.** 6.01 **d.** 2.45 **e.** 0.06

6. 0.48%

7. 15.49

8. C

9. 2.96 km/h

10. 6.067 pens

11. 2.39 °C

12. $\bar{x} = 75.7$, $s = 5.6$

13. 3.786 players

14. 2.331

Exercise 1.10 Populations and simple random samples

1. Answers will vary.

2. Answers will vary.

3. B

4. Answers will vary.

5. Answers will vary.

6. Population is larger, since a sample is taken from the population.
7. C
8. E
9. Yes, because the distribution is reasonably symmetric with no outliers.
10. B
11. C
12. Answers will vary.
13. Answers will vary.
14. Answers will vary.

Exercise 1.11 The 68–95–99.7% rule and *z*-scores

1. a. 68% of group's concentration span falls between 35 secs and 63 secs.
   b. 95% of group's concentration span falls between 21 secs and 77 secs.
   c. 99.7% of group's concentration span falls between 7 secs and 91 secs.
2. a. 68% of the group to lie between 43.3 mm and 46.7 mm
   b. 95% of the group to lie between 41.6 mm and 48.4 mm
   c. 99.7% of the group to lie between 39.9 mm and 50.1 mm
3. a. Yes b. Yes c. No
   d. No e. No f. Yes
4. a. 8 and 12
   b. 6 and 14
   c. 4 and 16
5. a. 3.7 and 6.3
   b. 2.4 and 7.6
   c. 1.1 and 8.9
6. a. 1.3 mm and 2.5 mm
   b. 0.7 mm and 3.1 mm
   c. 0.1 mm and 3.7 mm
7. a. 5 and 9
   b. 3 and 11
   c. 1 and 13
8. C
9. a. 2.50% b. 50% c. 16% d. 81.5%
10. a. 84% b. 2.5% c. 84% d. 97.35%
11. a. 0.15% b. 2.5% c. 84% d. 83.85% e. 81.5%
12. 420 bags
13. a. 336 containers
    b. 10 containers
    c. 380 containers
14. a. i. 1360 ii. 1950 iii. 317
    b. 100
15. 2.33
16. 3
17. a. Specialist: $\bar{x} = 67, s = 9$
    English: $\bar{x} = 58, s = 14$
    $z_s = 1.78, z_e = 2.14$
    b. English has the higher result as it has the higher *z*-score.
18. a. English 1.25, Maths 1.33
    b. Maths mark is better as it has a higher *z*-score.
19. a. −0.48 b. 1.44 c. 0.08 d. −2.24 e. 2.8
20. B
21. Second test, Barbara's *z*-score was −0.33 compared to −0.5 in the first test.
22. a. Barn: $\bar{x} = 4.4$ $s = 0.3$
    FR: $\bar{x} = 4.1$ $s = 0.2$
    b. 1.18
    c. 84%
    d.

	Cage	Barn	Free range
Min_x	4.7	3.9	3.8
Q_1	5	4.1	4
med	5.15	4.35	4.1
Q_3	5.5	4.6	4.2
Max_x	5.8	4.9	4.4

    i. Cage: 5.15
    Barn: 4.35
    Free: 4.1
    ii. It could be concluded that the more space a chicken has, the fewer eggs it lays because the median is greatest for cage eggs.

1.12 Review: exam practice

Multiple choice

1. B 2. D 3. C 4. D 5. C 6. A
7. C 8. E 9. E 10. B 11. D 12. C
13. D 14. A 15. B 16. B 17. C 18. D
19. C 20. B 21. D 22. C

Short answer

1. Many answers possible
2. a.

Stem	Leaf
6	0 1 5 7
7	1 1 3 8 8 9
8	2 3 4 4 7

Key: 6|0 = $6

   b.

Stem	Leaf
6	0 1
6*	5 7
7	1 1 3
7*	8 8 9
8	2 3 4 4
8*	7

Key: 6|0 = $6

c.

Stem	Leaf
6	0 1
6	
6	5
6	7
6	
7	1 1
7	3
7	
7	
7	8 8 9
8	
8	2 3
8	4 4
8	7
8	

Key: 6|0 = $6

Stem plot **b** is probably an appropriate display. No real need for stems in fifths.

3. a. See first graph at the bottom of the page

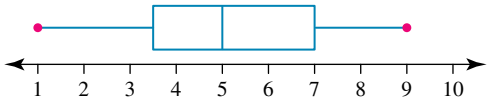
c. The data are approximately symmetrical. More than half the drivers exceeded the limit. The fastest drivers were about 15 km/h over the limit. Many other conclusions are possible.

4. a. Positively skewed

b. There would need to be a shift of some of the amounts in the twenties to the thirties and forties.

5. Range = 24, median = 14, mode = 14, interquartile range = 9.5

6. a.



b. Approximately symmetric

7. a. 500 **b.** 58.3%

8. 0.05 mL

9. a. i. 10.8 and 13.2 years

ii. 9.6 and 14.4 years

iii. 8.4 and 15.6 years

b. There is a large range of life spans for these dogs. The oldest dog is almost twice as old as the youngest.

10. Kory is the better candidate as he has a greater z-score (1.5 compared with 0.875).

11. 40 times

12.

$\log_{10}$(income)	Frequency
3 – <4	2
4 – <5	6
5 – <6	3
6 – <7	1

Extended response

1. a. i. A stem plot is more appropriate since there are only 25 observations in each set.

10C Stem	10C Leaf	10E Stem	10E Leaf
0		0	
0		0	
0	4	0	
0	7 7	0	
0	9 9	0	8 9
1	0 0 1	1	0 1 1
1	2 3	1	2 2 2 3 3 3 3 3
1	4 4 5 5 5	1	4 4 4 4 4 4 5 5 5
1	6 7 7 7 7	1	6 6
1	8 8 8 9 9	1	9

Key: 1|3 = 13

The distribution of 10C is negatively skewed with no outliers.

The distribution of 10E is symmetric with no outliers.

ii.

Class	Median	IQR	Range	Mode
10C	15	7	15	17
10E	13	2.5	11	14

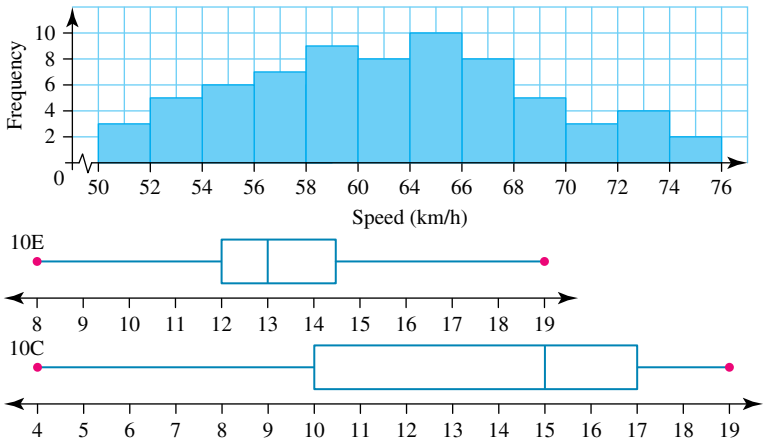
iii. See second graph at the bottom of the page

iv.

Class	Mean	Standard deviation
10C	13.64	4.24
10E	13.2	2.35

v. For 10C the median provides a better indication, whereas for 10E the mean and the median are close anyway.

vi. The 68 – 95 – 99.7% rule can be applied to the distribution of 10E's data since it is approximately bell-shaped.



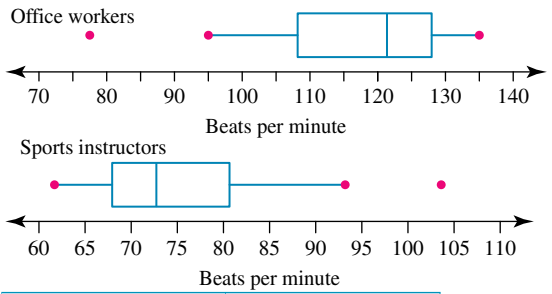
b. We use the median of the 10C scores to give us an indication of the centre of the distribution. The median of the 10C scores is 15. For 10E, the distribution is symmetric and hence we use the mean to give us an indication of the centre of the distribution. The mean of 10E scores is 13.2. The range of the 10C scores is 15 whereas for 10E the range is 11. Also, the standard deviation for 10C is 4.24 and for 10E it is 2.35. This means that the scores in 10C are more spread out than those in 10E, which are relatively bunched. So, while in 10C there are more students with higher marks than in 10E, the range of marks in 10C is greater and this would make it a more challenging class to teach.

2. a. Office workers: negatively skewed with outlier. Sports instructors: positively skewed with outlier.

b.

	Office workers	Sports instructors
Median	121.5 beats/min	73 beats/min
IQR	19.5 beats/min	14 beats/min
Range	58 beats/min	46 beats/min
Mode	130 beats/min	68, 72 beats/min

c.



d.

Office workers	Sports instructors
$\bar{x} = 116.8$ beats/min	$\bar{x} = 76.9$ beats/min
$s = 15.3$ beats/min	$s = 12.4$ beats/min

e. Not used since the distributions are not bell-shaped.

f. Office workers: Pulse rates are generally very high, clustered around 120 – 130 beats/min. Also, there is one person whose rate was much lower than the rest. This outlier (76) produces a large range and makes the mean slightly lower than the median. As a result the median is a more appropriate measure of the centre of the data rather than the mean.
Sports instructors: Pulse rates are generally low, clustered around 60 – 70 beats/min, although there are a few people with rates much higher, which makes the mean slightly higher than the median and also produces quite a large range. As a result of the skewed distribution the median is the more appropriate measure of the centre of the data rather than the mean, although there is little difference between these values.

3. a. Discrete, numerical data

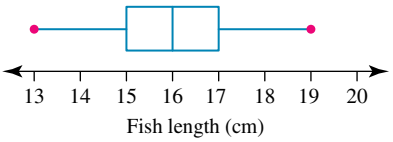
b.

Stem	Leaf
1	
1	3 3
1	4 4 5 5 5
1	6 6 6 6 6 7 7 7
1	8 8 9 9

Key: 1|3 = 13 cm

Looks to be slightly negatively skewed.

c. $Q_2 = 16$ cm
$Q_1 = 15$ cm
$Q_3 = 17$ cm
Lowest score = 13 cm
Highest score = 19 cm



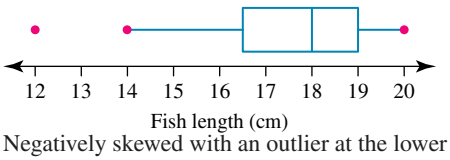
d. Symmetric

e. Given the data itself, the boxplot does.

f. Mean = 16 cm = median

g. $s = 1.75$ cm, 1600

h.



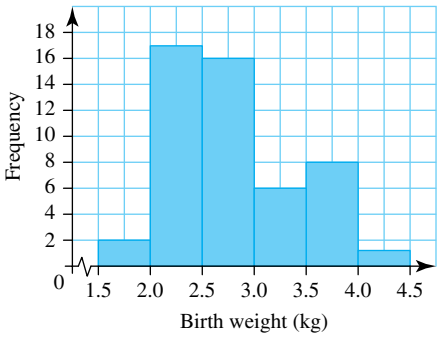
Negatively skewed with an outlier at the lower end

i. 1600

j. The river fish seem to be larger overall. Only 1600 of the coastal fish lie above 17.75 cm, whereas 1600 of the river fish lie above 19.25 cm. All the quartiles for the river fish are higher than those for the coastal ones. It would seem that the river fish have grown more than the coastal fish.

4. a.

Class interval	Tally	Frequency	Midpoint
1.5 – 1.9	II	2	1.7
2.0 – 2.4	𝍸 III	8	2.2
2.5 – 2.9	𝍸 𝍸 𝍸 II	17	2.7
3.0 – 3.4	𝍸 𝍸 𝍸 I	16	3.2
3.5 – 3.9	𝍸 I	6	3.7
4.0 – 4.4	I	1	4.2



b. Somewhat symmetrical or slightly positively skewed

c. Mean = 2.89 kg, $s = 0.54$ kg
The mean of the sample is only just less than the suggested mean; in fact, it is only about 0.2 of 1 standard deviation away from it. So the suggestion is probably right.

d. **i.** 149 **ii.** 29 **iii.** 5

TOPIC 2
Investigating associations between two variables

2.1 Overview

2.1.1 Introduction

Recent research by the University of Sydney using data from 80 000 adults over the age of 30 has determined that strength bearing exercise and cardio exercise are equally important in the prevention of premature death, but strength bearing exercise may be more effective when applied to premature death from cancer.

In this study, researchers would have considered the relationships between the many variables that pertained to the 80 000 people in their study. These variables would have included their age, diet, pre-existing health issues. After taking these variables into consideration, mortality rates could be compared based on the type of exercise undertaken by those in the study.

Detailed studies such as this require researchers with excellent skill sets in data science. Data scientists, engineers and statisticians are needed in every industry, university and government department to help set up data collection infrastructure and to provide in-depth analysis of the data to assist the decision makers within the company or organisation. Some of these roles require programming skills, statistical skills, technical skills, written and oral communication skills or any combination of these. Small and large businesses are also employing consultants to provide timely analysis of available data to grow and sustain their businesses. There is a vast range of careers within the data science field and it is an area that will continue to provide employment opportunities into the future.

LEARNING SEQUENCE

2.1 Overview
2.2 Response and explanatory variables
2.3 Two-way (contingency) frequency tables and segmented bar charts
2.4 Back-to-back stem plots
2.5 Parallel boxplots and dot plots
2.6 Scatterplots
2.7 Pearson's product–moment correlation coefficient
2.8 Calculating r and the coefficient of determination, r^2
2.9 Correlation and causation
2.10 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.1.2 Kick off with CAS

Exploring parallel boxplots with CAS

Parallel boxplots can be used to compare and contrast key information about two different numerical data sets.

1. Use CAS to draw parallel boxplots of the following two data sets, which detail the time it takes for two groups of individuals to complete an obstacle course (rounded to the nearest minute).
   Group A: 18, 22, 24, 17, 22, 27, 15, 20, 25, 19, 26, 19, 23, 26, 18, 20, 27, 24, 16
   Group B: 21, 22, 19, 21, 17, 21, 18, 24, 21, 20, 18, 24, 35, 22, 19, 17, 23, 20, 19

2. Use your parallel boxplots from question **1** to answer the following questions.
   a. Which group has the larger range of data?
   b. Which group has the larger interquartile range of data?
   c. Which group has the higher median value?
3. One of the data sets has an outlier.
   a. How is this marked on the parallel boxplot?
   b. State the value of the outlier.
4. a. Use CAS to draw a parallel boxplot of the following two data sets.
      Set A: 41, 46, 38, 44, 49, 39, 50, 47, 47, 42, 53, 44, 46, 35, 39
      Set B: 35, 31, 39, 41, 37, 43, 29, 40, 36, 38, 42, 33, 34, 30, 37
   b. Which data set has the largest range?
   c. Which data set has an interquartile range of 8?

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

2.2 Response and explanatory variables

A set of data involving two variables where one affects the other is called bivariate data. If the values of one variable 'respond' to the values of another variable, then the former variable is referred to as the **response** (dependent) **variable**. So an **explanatory** (independent) **variable** is a factor that influences the response (dependent) variable.

When a relationship between two sets of variables is being examined, it is important to know which one of the two variables responds to the other. Most often we can make a judgement about this, although sometimes it may not be possible.

Consider the case where a study compared the heights of company employees against their annual salaries. Common sense would suggest that the height of a company employee would not respond to the person's annual salary nor would the annual salary of a company employee respond to the person's height. In this case, it is not appropriate to designate one variable as explanatory and one as response.

In the case where the ages of company employees are compared with their annual salaries, you might reasonably expect that the annual salary of an employee would depend on the person's age. In this case, the age of the employee is the explanatory variable and the salary of the employee is the response variable.

The explanatory variable is a factor that influences the response variable.

WORKED EXAMPLE 1

For each of the following pairs of variables, identify the explanatory variable and the response variable. If it is not possible to identify this, then write 'not appropriate'.

a. The number of visitors at a local swimming pool and the daily temperature

b. The blood group of a person and his or her favourite TV channel

THINK

a. It is reasonable to expect that the number of visitors at the swimming pool on any day will respond to the temperature on that day (and not the other way around).

b. Common sense suggests that the blood type of a person does not respond to the person's TV channel preferences. Similarly, the choice of a TV channel does not respond to a person's blood type.

WRITE

a. Daily temperature is the explanatory variable; number of visitors at a local swimming pool is the response variable.

b. Not appropriate

studyon

Units 3 & 4 > AOS DA > Topic 6 > Concept 1

Defining two variable data Summary screen and practice questions

Exercise 2.2 Response and explanatory variables

1. WE1 For each of the following pairs of variables, identify the explanatory and the response variable. If it is not possible to identify this, then write 'not appropriate'.
   a. The number of air conditioners sold and the daily temperature
   b. The age of a person and their favourite colour

2. For each of the following pairs of variables, identify the explanatory and the response variable. If it is not possible to identify the variables, then write 'not appropriate'.
   a. The size of a crowd and the teams that are playing
   b. The net score of a round of golf and the golfer's handicap
3. For each of the following pairs of variables, identify the explanatory variable and the response variable. If it is not possible to identify this, then write 'not appropriate'.
   a. The age of an AFL footballer and his annual salary
   b. The growth of a plant and the amount of fertiliser it receives
   c. The number of books read in a week and the eye colour of the readers
   d. The voting intentions of a woman and her weekly consumption of red meat
   e. The number of members in a household and the size of the house
4. For each of the following pairs of variables, identify the explanatory variable and the response variable. If it is not possible to identify this, then write 'not appropriate'.
   a. The month of the year and the electricity bill for that month
   b. The mark obtained for a maths test and the number of hours spent preparing for the test
   c. The mark obtained for a maths test and the mark obtained for an English test
   d. The cost of grapes (in dollars per kilogram) and the season of the year
5. MC In a scientific experiment, the explanatory variable was the amount of sleep (in hours) a new mother got per night during the first month following the birth of her baby. The response variable would most likely have been:
   A. the number of times (per night) the baby woke up for a feed
   B. the blood pressure of the baby
   C. the mother's reaction time (in seconds) to a certain stimulus
   D. the level of alertness of the baby
   E. the amount of time (in hours) spent by the mother on reading
6. A paediatrician investigated the relationship between the amount of time children aged two to five spend outdoors and the vitamin D levels. The amount of time children spend outdoors is the explanatory variable. True or false?
7. Alex works as a personal trainer at the local gym. He wishes to analyse the relationship between the number of weekly training sessions and the weekly weight loss of his clients. Which one of the following statements is correct?

   The weekly weight loss of his clients is the explanatory variable. True or false?
8. If two variables investigated are the number of minutes on a basketball court and the number of points scored:
   a. which is the explanatory variable?
   b. which is the response variable?
9. Callum decorated his house with Christmas lights for everyone to enjoy. He investigated two variables, the number of Christmas lights he has and the size of his electricity bill.
   a. Which is the response variable?
   b. Which is the explanatory variable?
10. If the two variables investigated are the number of hours a child spends playing imaginary game with an adult and the vocabulary of the child
    a. Which is the explanatory variable?
    b. Which is the response variable?

2.3 Two-way (contingency) frequency tables and segmented bar charts

When we are examining the relationship between two categorical variables, **two-way (or contingency) tables** are an excellent tool. Consider the following example.

Once the two-way table is formed, **marginal distributions** and **conditional distributions** can both be found. Marginal distributions are the sums (totals) of the row or the column and are found in the margins of the table. The conditional distribution is the sub-population (sample) and this is found in the middle of the table.

If we were to look at mobile phone preference as shown in the table the marginal distributions are the totals, as shown by the green highlighted numbers.

	Apple	Samsung	Nokia	Total
Men	13	9	3	25
Women	17	7	1	25
Total	30	16	4	50

The conditional distribution is the sub-population, so if we are looking at people who prefer Samsung, the conditional distribution is shown by the purple highlighted numbers.

	Apple	Samsung	Nokia	Total
Men	13	9	3	25
Women	17	7	1	25
Total	30	16	4	50

WORKED EXAMPLE 2

At a local shopping centre, 34 females and 23 males were asked which of the two major political parties they preferred. Eighteen females and 12 males preferred Labor. Display these data in a two-way (contingency) table, and calculate the party preference for males and females.

THINK

1. Draw a table. Record the respondents' sex in the columns and party preference in the rows of the table.

WRITE

Party preference	Female	Male	Total
Labor			
Liberal			
Total			

2. We know that 34 females and 23 males were asked. Put this information into the table and fill in the total.

   We also know that 18 females and 12 males preferred Labor. Put this information in the

Party preference	Female	Male	Total
Labor	18	12	30
Liberal			
Total	34	23	57

table and find the total of people who preferred Labor.

3. Fill in the remaining cells. For example, to find the number of females who preferred the Liberals, subtract the number of females preferring Labor from the total number of females asked: $34 - 18 = 16$.

Party preference	Female	Male	Total
Labor	18	12	30
Liberal	16	11	27
Total	34	23	57

4. Marginal distributions for party preference for males and females refers to percentage (probability) of each party. For Labor there are 30 out of a total of 57.

Labor: $\frac{30}{57} = 0.53$

5. For Liberal there are 27 out of a total of 57.

Liberal: $\frac{27}{57} = 0.47$

In Worked example 2, we have a very clear breakdown of data. We know how many females preferred Labor, how many females preferred the Liberals, how many males preferred Labor and how many males preferred the Liberals.

If we wish to compare the number of females who prefer Labor with the number of males who prefer Labor, we must be careful. While 12 males preferred Labor compared to 18 females, there were fewer males than females being asked. That is, only 23 males were asked for their opinion, compared to 34 females.

To overcome this problem, we can express the figures in the table as percentages.

WORKED EXAMPLE 3

Fifty-seven people in a local shopping centre were asked whether they preferred the Australian Labor Party or the Liberal Party. The results are as shown.

Convert the numbers in this table to percentages.

Party preference	Female	Male	Total
Labor	18	12	30
Liberal	16	11	27
Total	34	23	57

THINK

Draw the table, omitting the 'total' column.
Fill in the table by expressing the number in each cell as a percentage of its column's total. For example, to obtain the percentage of males who prefer Labor, divide the number of males who prefer Labor by the total number of males and multiply by 100%.

$\frac{12}{23} \times 100\% = 52.2\%$ (correct to 1 decimal place)

WRITE

Party preference	Female	Male
Labor	52.9	52.2
Liberal	47.1	47.8
Total	100.0	100.0

We could have also calculated percentages from the table rows, rather than columns. To do that we would, for example, have divided the number of females who preferred Labor (18) by the total number of people who preferred Labor (30) and so on. The table shows this:

Party preference	Female	Male	Total
Labor	60.0	40.0	100
Liberal	59.3	40.7	100

By doing this we have obtained the percentage of people who were female and preferred Labor (60%), and the percentage of people who were male and preferred Labor (40%), and so on. This highlights facts different from those shown in the previous table. In other words, different results can be obtained by calculating percentages from a table in different ways.

In the above example, the respondent's gender is referred to as the explanatory variable, and the party preference as the response variable.

As a general rule, when the explanatory variable is placed in the columns of the table, the percentages should be calculated in columns.

Comparing percentages in each row of a two-way table allows us to establish whether a relationship exists between the two categorical variables that are being examined. As we can see from the table in Worked example 3, the percentage of females who preferred Labor is about the same as that of males. Likewise, the percentage of females and males preferring the Liberal Party are almost equal. This indicates that for the group of people participating in the survey, party preference is not related to gender.

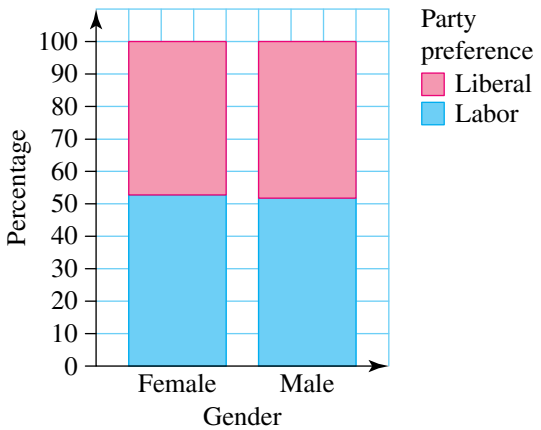
2.3.1 Segmented bar charts

When comparing two categorical variables, it can be useful to represent the results from a two-way table (in percentage form) graphically. We can do this using **segmented bar charts**.

A segmented bar chart consists of two or more columns, each of which matches one column in the two-way table. Each column is subdivided into segments, corresponding to each cell in that column.

For example, the data from Worked example 3 can be displayed using the segmented bar chart shown.

The segmented bar chart is a powerful visual aid for comparing and examining the relationship between two categorical variables.



WORKED EXAMPLE 4

Sixty-seven primary and 47 secondary school students were asked about their attitude to the number of school holidays which should be given. They were asked whether there should be fewer, the same number, or more school holidays. Five primary students and 2 secondary students wanted fewer holidays, 29 primary and 9 secondary students thought they had enough holidays (that is, they chose the same number) and the rest thought they needed to be given more holidays.

Present these data in percentage form in a two-way table and a segmented bar chart. Compare the opinions of the primary and the secondary students.

THINK

1. Put the data in a table. First, fill in the given information, then find the missing information by subtracting the appropriate numbers from the totals.

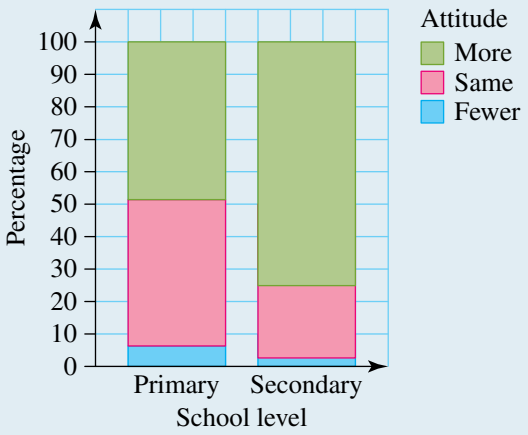
WRITE/DRAW

Attitude	Primary	Secondary	Total
Fewer	5	2	7
Same	29	9	38
More	33	36	69
Total	67	47	114

2. Calculate the percentages. Since the explanatory variable (the level of the student: primary or secondary) has been placed in the columns of the table, we calculate the percentages in columns. For example, to obtain the percentage of primary students who wanted fewer holidays, divide the number of such students by the total number of primary students and multiply by 100%.
   That is, $\frac{5}{67} \times 100\% = 7.5\%$.

Attitude	Primary	Secondary
Fewer	7.5%	4.3%
Same	43.3%	19.1%
More	49.2%	76.6%
Total	100%	100%

3. Rule out the set of axes. (The vertical axis shows percentages from 0 to 100, while the horizontal axis represents the categories from the columns of the table.) Draw two columns to represent each category — primary and secondary. Columns must be the same width and height (up to 100%). Divide each column into segments so that the height of each segment is equal to the percentage in the corresponding cell of the table. Add a legend to the graph.



4. Comment on the results.

Secondary students were much keener on having more holidays than were primary students.

on Resources

Interactivity: Two-way tables and segmented bar graphs (int-6249)

studyon

Units 3 & 4 > AOS DA > Topic 6 > Concept 4

Two-way frequency tables and segmented bar charts Summary screen and practice questions

Exercise 2.3 Two-way (contingency) frequency tables and segmented bar charts

1. WE2 A group of 60 people, 38 females and 22 males, were asked whether they prefer an Apple or Samsung phone. Twenty-three females and 15 males said they preferred an Apple phone. Display this data in a two-way (contingency) table and calculate the marginal distribution for phone preference for males and females.
2. A group of 387 females and 263 males were asked their preference from Coke and Pepsi. Two hundred and twenty-one females preferred Coke, whereas 108 males preferred Pepsi. Display this data in a two-way (contingency) table and calculate the conditional distribution of drink preference among females.
3. In a survey, 139 women and 102 men were asked whether they approved or disapproved of a proposed freeway. Thirty-seven women and 79 men approved of the freeway. Display these data in a two-way table (not as percentages), and calculate the approval or disapproval of the proposed freeway for men and women.
4. Students at a secondary school were asked whether the length of lessons should be 45 minutes or 1 hour. Ninety-three senior students (Years 10–12) were asked and it was found 60 preferred 1-hour lessons, whereas of the 86 junior students (Years 7–9), 36 preferred 1-hour lessons. Display these data in a two-way table (not as percentages), and calculate the conditional distribution on length of lessons and senior students.
5. WE3 A group of 60 people were asked their preferences on phones. The results are shown. Convert the numbers in this table to percentages.

Phone	Females	Males	Total
Apple	23	15	38
Samsung	15	7	22
Total	38	22	60

6. A group of 650 people were asked their preferences on soft drink. The results are shown. Convert the numbers in this table to percentages.

Drink	Females	Males	Total
Pepsi	221	155	376
Coke	166	108	274
Total	387	263	650

7. For each of the following two-way frequency tables, complete the missing entries.

a.

Attitude	Female	Male	Total
For	25	i	47
Against	ii	iii	iv
Total	51	v	92

b.

Party preference	Female	Male
Labor	i	42%
Liberal	53%	ii
Total	iii	iv

8. Sixty single men and women were asked whether they prefer to rent by themselves, or to share accommodation with friends. The results are shown below.

Preference	Men	Women	Total
Rent by themselves	12	23	35
Share with friends	9	16	25
Total	21	39	60

Convert the numbers in this table to percentages.

The information in the following two-way frequency table relates to questions 9 and 10.
The data show the reactions of administrative staff and technical staff to an upgrade of the computer systems at a large corporation.

Attitude	Administrative staff	Technical staff	Total
For	53	98	151
Against	37	31	68
Total	90	129	219

9. MC From the previous table, we can conclude that:
   - A. 53% of administrative staff were for the upgrade
   - B. 37% of administrative staff were for the upgrade
   - C. 37% of administrative staff were against the upgrade
   - D. 59% of administrative staff were for the upgrade
   - E. 54% of administrative staff were against the upgrade

10. MC From the previous table, we can conclude that:
   - A. 98% of technical staff were for the upgrade
   - B. 65% of technical staff were for the upgrade
   - C. 76% of technical staff were for the upgrade
   - D. 31% of technical staff were against the upgrade
   - E. 14% of technical staff were against the upgrade

11. **WE4** Sixty-one females and 58 males were asked which they prefer off the menu: entrée, main or dessert. Eight males and 18 females preferred entrée, while 31 males and 16 females said they preferred the main course, with the remainder having dessert as their preferred preference.

Present these data in percentage form in a two-way table and a segmented bar chart. Compare the opinions of the males and females on their preferences.

12. Ninety-three people less than 40 years of age and 102 people aged 40 and over were asked where their priority financially is, given the three options 'mortgage', 'superannuation' or 'investing'. Eighteen people in the 40 and over category and 42 people in the less than 40 years category identified mortgage as their priority, whereas 21 people under 40 years of age and 33 people aged 40 and over said investment was most important. The rest suggested superannuation was their most important priority.

Present these data in percentage form in a two-way table and segmented bar chart. Compare the opinions of the under 40s to the people aged 40 and over on their priority to their finances.

13. Delegates at the respective Liberal Party and Australian Labor Party conferences were surveyed on whether or not they believed that marijuana should be legalised. Sixty-two Liberal delegates were surveyed and 40 of them were against legalisation. Seventy-one Labor delegates were surveyed and 43 were against legalisation.

Present the data in percentage form in a two-way frequency table. Comment on any differences between the reactions of the Liberal and Labor delegates.

14. Use the results in question **13** to draw a segmented bar chart.

The information in the following table relates to questions **15**–**18**.
The amount of waste recycled by 100 townships across Australia was rated as low, medium or high and the size of the town as small, mid-sized or large.
The results of the ratings are:

	Type of town		
Amount of waste recycled	**Small**	**Mid-sized**	**Large**
Low	6	7	4
Medium	8	31	5
High	5	16	18

15. **MC** The percentage of mid-sized towns rated as having a high level of waste recycling is closest to:
A. 41% **B.** 25% **C.** 30% **D.** 17% **E.** 50%

16. **MC** The variables, *Amount of waste recycled* and *Type of town*, as used in this rating are:
A. both categorical variables
B. both numerical variables
C. numerical and categorical respectively
D. categorical and numerical respectively
E. neither categorical nor numerical variables

17. Calculate the conditional distribution for amount of waste and large towns.

18. Calculate the percentage of small towns rated as having a high level of waste recycling.

2.4 Back-to-back stem plots

In topic 1, we saw how to construct a stem plot for a set of univariate data. We can also extend a stem plot so that it compares two sets of univariate data. Specifically, we shall create a stem plot that displays the relationship between a numerical variable and a categorical variable. We shall limit ourselves in this section to categorical variables with just two categories, for example, gender. The two categories are used to provide two back-to-back leaves of a stem plot.

A back-to-back stem plot is used to display two sets of univariate data, involving a numerical variable and a categorical variable with 2 categories.

WORKED EXAMPLE 5

The girls and boys in Grade 4 at Kingston Primary School submitted projects on the Olympic Games. The marks they obtained out of 20 are as shown.

Girls' marks	16	17	19	15	12	16	17	19	19	16
Boys' marks	14	15	16	13	12	13	14	13	15	14

Display the data on a back-to-back stem plot.

THINK

1. Identify the highest and lowest scores in order to decide on the stems.

2. Create an unordered stem plot first. Put the boys' scores on the left, and the girls' scores on the right.

3. Now order the stem plot. The scores on the left should increase in value from right to left, while the scores on the right should increase in value from left to right.

WRITE

Highest score = 19
Lowest score = 12
Use a stem of 1, divided into fifths.

Leaf Boys	Stem	Leaf Girls
	1	
3 2 3 3	1	2
4 5 4 5 4	1	5
6	1	6 7 6 7 6
	1	9 9 9

Key: 1|2 = 12

Leaf Boys	Stem	Leaf Girls
3 3 3 2	1	2
5 5 4 4 4	1	5
6	1	6 6 6 7 7
	1	9 9 9

Key: 1|2 = 12

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Units 3 & 4 > AOS DA > Topic 6 > Concept 2

Back-to-back stem plots Summary screen and practice questions

The back-to-back stem plot allows us to make some visual comparisons of the two distributions. In Worked example 5, the centre of the distribution for the girls is higher than the centre of the distribution for the boys. The spread of each of the distributions seems to be about the same. For the boys, the scores are grouped around the 12 – 15 mark; for the girls, they are grouped around the 16 – 19 mark. On the whole, we can conclude that the girls obtained better scores than the boys did.

To get a more precise picture of the centre and spread of each of the distributions, we can use the summary statistics discussed in topic 1. Specifically, we are interested in:

1. the mean and the median (to measure the centre of the distributions), and
2. the interquartile range and the standard deviation (to measure the spread of the distributions).

We saw in topic 1 that the calculation of these summary statistics is very straightforward using CAS.

WORKED EXAMPLE 6

The number of 'how to vote' cards handed out by various Australian Labor Party and Liberal Party volunteers during the course of a polling day is as shown.

Labor	180	233	246	252	263	270	229	238	226	211
	193	202	210	222	257	247	234	226	214	204
Liberal	204	215	226	253	263	272	285	245	267	275
	287	273	266	233	244	250	261	272	280	279

Display the data using a back-to-back stem plot and use this, together with summary statistics, to compare the distributions of the number of cards handed out by the Labor and Liberal volunteers.

THINK

1. Construct the stem plot.

WRITE

Leaf Labor	Stem	Leaf Liberal
0	18	
3	19	
4 2	20	4
4 1 0	21	5
9 6 6 2	22	6
8 4 3	23	3
7 6	24	4 5
7 2	25	0 3
3	26	1 3 6 7
0	27	2 2 3 5 9
	28	0 5 7

Key: 18|0 = 180

2. Use CAS to obtain summary statistics for each party. Record the mean, median, IQR and standard deviation in the table. (IQR $= Q_3 - Q_1$)

	Labor	Liberal
Mean	227.9	257.5
Median	227.5	264.5
IQR	36	29.5
Standard deviation	23.9	23.4

3. Comment on the relationship.

From the stem plot we see that the Labor distribution is symmetric and therefore the mean and the median are very close, whereas the Liberal distribution is negatively skewed.
Since the distribution is skewed, the median is a better indicator of the centre of the distribution than the mean.
Comparing the medians therefore, we have the median number of cards handed out for Labor at 228 and for Liberal at 265, which is a big difference.
The standard deviations were similar, as were the interquartile ranges. There was not a lot of difference in the spread of the data.
In essence, the Liberal party volunteers handed out more 'how to vote' cards than the Labor party volunteers did.

on Resources

Interactivity: Back-to-back stem plots (int-6252)

Exercise 2.4 Back-to-back stem plots

1. **WE5** Boys and girls submitted an assignment on the history of the ANZACs. The results out of 40 are shown.

Girls' results	30	35	31	32	38	33	35	30
Boys' results	34	33	37	39	31	32	39	36

Display the data on a back-to-back stem plot.

2. The marks obtained out of 50 by students in Physics and Chemistry are shown. Display the data on a back-to-back stem plot.

Physics	32	45	48	32	24	30	41	29	44	45	36	34	28	49
Chemistry	46	31	38	28	45	49	34	45	47	33	30	21	32	28

3. The marks out of 50 obtained for the end-of-term test by the students in German and French classes are given as shown. Display the data on a back-to-back stem plot.

German	20	38	45	21	30	39	41	22	27	33	30	21	25	32	37	42	26	31	25	37
French	23	25	36	46	44	39	38	24	25	42	38	34	28	31	44	30	35	48	43	34

4. The birth masses of 10 boys and 10 girls (in kilograms, correct to the nearest 100 grams) are recorded in the table. Display the data on a back-to-back stem plot.

Boys	3.4	5.0	4.2	3.7	4.9	3.4	3.8	4.8	3.6	4.3
Girls	3.0	2.7	3.7	3.3	4.0	3.1	2.6	3.2	3.6	3.1

5. WE6 The number of promotional pamphlets handed out for company A and company B by a number of their reps is shown.

Company A	144	156	132	138	148	160	141	134	132	142	132	134	168	149
Company B	146	131	138	155	145	153	134	153	138	133	130	162	148	160

Display the data using a back-to-back stem plot and use this, together with summary statistics, to compare the number of pamphlets handed out by each company.

6. A comparison of student achievements (out of 100) in History and English was recorded and the results shown.

History	75	78	42	92	59	67	78	82	84	64	77	98
English	78	80	57	96	58	71	74	87	79	62	75	100

   a. Draw a back-to-back stem plot.
   b. Use summary statistics and the stem plot to comment on the two subjects.

7. The number of delivery trucks making deliveries to a supermarket each day over a 2-week period was recorded for two neighbouring supermarkets — supermarket A and supermarket B. The data are shown in the table.

A	11	15	20	25	12	16	21	27	16	17	17	22	23	24
B	10	15	20	25	30	35	16	31	32	21	23	26	28	29

   a. Display the data on a back-to-back stem plot.
   b. Use the stem plot, together with some summary statistics, to compare the distributions of the number of trucks delivering to supermarkets A and B.

8. The marks out of 20 obtained by males and females for a science test in a Year 10 class are given.

Females	12	13	14	14	15	15	16	17
Males	10	12	13	14	14	15	17	19

   a. Display the data on a back-to-back stem plot.
   b. Use the stem plot, together with some summary statistics, to compare the distributions of the marks of the males and the females.

9. The end-of-year English marks for 10 students in an English class were compared over 2 years. The marks for 2017 and for the same students in 2018 are as shown.

2017	30	31	35	37	39	41	41	42	43	46
2018	22	26	27	28	30	31	31	33	34	36

   a. Display the data on a back-to-back stem plot.
   b. Use the stem plot, together with some summary statistics, to compare the distributions of the marks obtained by the students in 2017 and 2018.

10. The age and gender of a group of people attending a fitness class are recorded.

Female	23	24	25	26	27	28	30	31
Male	22	25	30	31	36	37	42	46

   a. Display the data on a back-to-back stem plot.
   b. Use the stem plot, together with some summary statistics, to compare the distributions of the ages of the female members to the male members of the fitness class.

11. The scores on a board game for a group of kindergarten children and for a group of children in a preparatory school are given as shown.

Kindergarten	3	13	14	25	28	32	36	41	47	50
Prep. school	5	12	17	25	27	32	35	44	46	52

   a. Display the data on a back-to-back stem plot.
   b. Use the stem plot, together with some summary statistics, to compare the distributions of the scores of the kindergarten children compared to the preparatory school children.

12. **MC** A pair of variables that could be displayed on a back-to-back stem plot is:
   A. the height of a student and the number of people in the student's household
   B. the time put into completing an assignment and a pass or fail score on the assignment
   C. the weight of a businessman and his age
   D. the religion of an adult and the person's head circumference
   E. the income of an employee and the time the employee has worked for the company

13. **MC** A back-to-back stem plot is a useful way of displaying the relationship between:
   A. the proximity to markets in kilometres and the cost of fresh foods on average per kilogram
   B. height and head circumference
   C. age and attitude to gambling (for or against)
   D. weight and age
   E. the money spent during a day of shopping and the number of shops visited on that day

14. The score out of 100 a group of males and females received when going for their licence are shown. Construct a back-to-back stem plot of the data.

Male	86	92	100	90	94	82	72	90	88	94	76	80
Female	94	96	72	80	84	92	83	88	90	70	81	83

15. MC A back-to-back stem plot is used to display two sets of data, involving which two variables?
   - **A.** increasing variables
   - **B.** discrete and numerical variables
   - **C.** continuous and categorical variables
   - **D.** numerical and categorical variables
   - **E.** numerical and continuous variables

16. The study scores (out of 50) of students who studied both Mathematical Methods and Further Mathematics are shown.

Methods	28	34	41	36	33	39	44	40	39	42	36	31	29	44
Further	30	37	38	41	35	43	44	46	43	48	37	31	28	48

   a. Display the data in a back-to-back stem plot.
   b. Use the stem plot, together with some summary statistics, to compare the distributions of the scores for Mathematical Methods compared to Further Mathematics.

2.5 Parallel boxplots and dot plots

We saw in the previous subtopic that we could display relationships between a numerical variable and a categorical variable with just two categories, using a back-to-back stem plot.

When we want to display a relationship between a numerical variable and a categorical variable with two or *more* categories, **parallel boxplots** or **parallel dot plots** can be used.

Parallel boxplots are obtained by constructing individual boxplots for each distribution and positioning them on a common scale.

Parallel dot plots are obtained by constructing individual dot plots for each distribution and positioning them on a common scale.

Construction of individual boxplots was discussed in detail in topic 1. In this section we concentrate on comparing distributions represented by a number of boxplots (that is, on the interpretation of parallel boxplots).

WORKED EXAMPLE 7

The four Year 7 classes at Western Secondary College complete the same end-of-year maths test. The marks, expressed as percentages for the four classes, are given as shown.

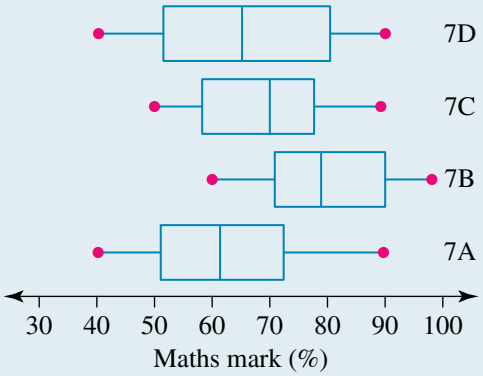
7A	40	43	45	47	50	52	53	54	57	60	69	63	63	68	70	75	80	85	89	90
7B	60	62	63	64	70	73	74	76	77	77	78	82	85	87	89	90	92	95	97	97
7C	50	51	53	55	57	60	63	65	67	69	70	72	73	74	76	80	82	82	85	89
7D	40	42	43	45	50	53	55	59	60	61	69	73	74	75	80	81	82	83	84	90

Display the data using parallel boxplots. Use this to describe any similarities or differences in the distributions of the marks between the four classes.

THINK	WRITE/DRAW
1. Use CAS to determine the five number summary for each data set.	

	7A	7B	7C	7D
Minimum	40	60	50	40
Q_1	51	71.5	58.5	51.5
Median $= Q_2$	61.5	77.5	69.5	65
Q_3	72.5	89.5	78	80.5
Maximum	90	97	89	90

2. Draw the boxplots, labelling each class. All four boxplots share a common scale.



3. Describe the similarities and differences between the four distributions.

Class 7B had the highest median mark and the range of the distribution was only 37. The lowest mark in 7B was 60.
We notice that the median of 7A's marks is 61.5. So, 50% of students in 7A received less than 61.5. This means that about half of 7A had scores that were less than the lowest score in 7B.
The range of marks in 7A was the same as that of 7D with the highest scores in each equal (90), and the lowest scores in each equal (40). However, the median mark in 7D (65) was slightly higher than the median mark in 7A (61.5) so, despite a similar range, more students in 7D received a higher mark than in 7A.
While 7D had a top score that was higher than that of 7C, the median score in 7C (69.5) was higher than that of 7D and almost 25% of scores in 7D were less than the lowest score in 7C. In summary, 7B did best, followed by 7C, then 7D and finally 7A.

Resources

Interactivity: Parallel boxplots (int-6248)

studyon

Units 3 & 4 > AOS DA > Topic 6 > Concept 3

Parallel boxplots Summary screen and practice questions

Exercise 2.5 Parallel boxplots and dot plots

1. **WE7** The times run for a 100 m race in grade 6 are shown for both boys and girls. The times are expressed in seconds.

Boys	15.5	16.1	14.5	16.9	18.1	14.3	13.8	15.9	16.4	17.3	18.8	17.9	16.1
Girls	16.7	18.4	19.4	20.1	16.3	14.8	17.3	20.3	19.6	18.4	16.5	17.2	16.0

   Display the data using parallel boxplots and use this to describe any similarities or differences between the boys' and girls' performances.

2. A teacher taught two Year 10 maths classes and wanted to see how they compared on the end of year examination. The marks are expressed as percentages.

10A	67	73	45	59	67	89	42	56	68	75	87	94	80	98
10D	76	82	62	58	40	55	69	71	89	95	100	84	70	66

   Display the data using parallel boxplots and parallel dot plots. Use this to describe any similarities or differences between the two classes.

3. The heights (in cm) of students in 9A, 10A and 11A were recorded and are shown in the table.

9A	120	126	131	138	140	143	146	147	150
10A	140	143	146	147	149	151	153	156	162
11A	151	153	154	158	160	163	164	166	167

9A	156	157	158	158	160	162	164	165	170
10A	164	165	167	168	170	173	175	176	180
11A	169	169	172	175	180	187	189	193	199

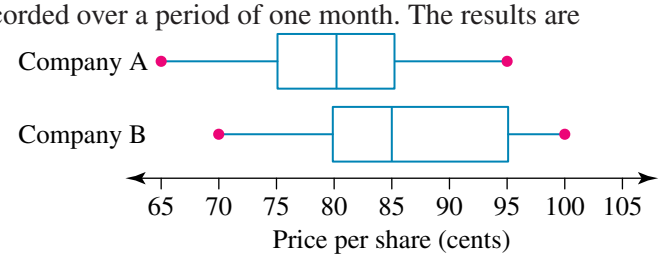
   a. Construct parallel boxplots to show the data.
   b. Use the boxplots to compare the distributions of height for the 3 classes.

4. The amounts of money contributed annually to superannuation schemes by people in 3 different age groups are as shown.

20–29	2000	3100	5000	5500	6200	6500	6700	7000	9200	10 000
30–39	4000	5200	6000	6300	6800	7000	8000	9000	10 300	12 000
40–49	10 000	11 200	12 000	13 300	13 500	13 700	13 900	14 000	14 300	15 000

   a. Construct parallel boxplots to show the data.
   b. Use the boxplots to comment on the distributions.

5. The daily share price of two companies was recorded over a period of one month. The results are presented as parallel boxplots.
   State whether each of the following statements is true or false.
   a. The distribution of share prices for Company A is symmetrical.

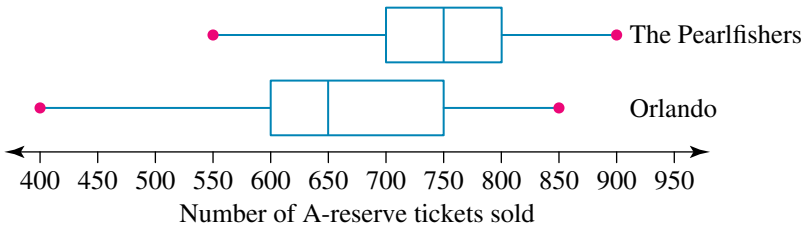


b. On 25% of all occasions, share prices for Company B equalled or exceeded the highest price recorded for Company A.

c. The spread of the share prices was the same for both companies.

d. 75% of share prices for Company B were at least as high as the median share price for Company A.

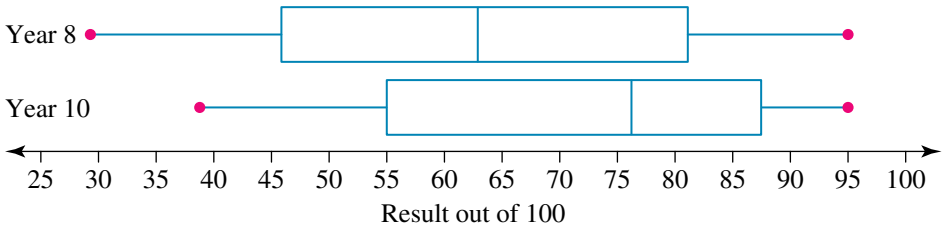
6. Last year, the spring season at the Sydney Opera House included two major productions: *The Pearlfishers* and *Orlando*. The number of A-reserve tickets sold for each performance of the two operas is shown as parallel boxplots.



a. Which of the two productions proved to be more popular with the public, assuming A-reserve ticket sales reflect total ticket sales? Explain your answer.

b. Which production had a larger variability in the number of patrons purchasing A-reserve tickets? Support your answer with the necessary calculations.

7. MC The exam results for two Maths classes, one in Year 8 and the other in Year 10, are given by the parallel boxplots.



The percentage of Year 10 students who obtained a mark greater than 87 was:

A. 2% **B.** 5% **C.** 20% **D.** 25% **E.** 75%

8. MC From the parallel boxplots in question **7**, it can be concluded that:

A. the Year 8 results were similar to the Year 10 results

B. the Year 8 results were lower than the Year 10 results and less variable

C. the Year 8 results were lower than the Year 10 results and more variable

D. the Year 8 results were higher than the Year 10 results and less variable

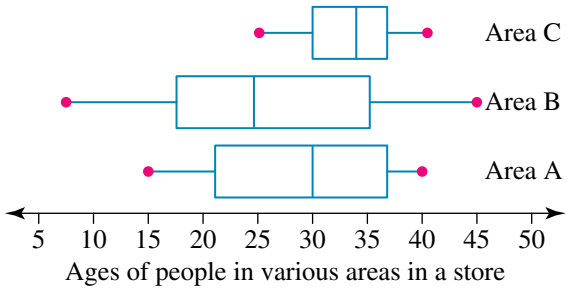
E. the Year 8 results were higher than the Year 10 results and more variable

9. The scores of 10 competitors on two consecutive days of a diving competition are recorded below:

Day 1	5.4	4.1	5.4	5.6	4.9	5.6	5.4	6.0	5.8	6.0
Day 2	4.9	5.1	5.3	5.8	5.7	5.2	5.8	5.4	5.5	6.0

Construct parallel dot plots to show the data and comment on the divers' results over the two days.

The following figure relates to questions **10** to **12**.
The ages of customers in different areas of a department store are as shown.



10. Which area has the largest range from Q_2 (the median) to Q_3: Area A, Area B or Area C?
11. Which area has the largest range: Area A, Area B or Area C?
12. Which area has the highest median age: Area A, Area B or Area C?
13. The numbers of jars of vitamin A, B, C and multi-vitamins sold per week by a local chemist are shown in the table.

Vitamin A	5	6	7	7	8	8	9	11	13	14
Vitamin B	10	10	11	12	14	15	15	15	17	19
Vitamin C	8	8	9	9	9	10	11	12	12	13
Multi-vitamins	12	13	13	15	16	16	17	19	19	20

Construct parallel boxplots to display the data and use it to compare the distributions of sales for the 4 types of vitamin.
14. Eleven golfers in a golf tournament play 18 holes each day. The scores for each of the golfers on the four days are given below. Display this data using parallel boxplots.

Thursday	Friday	Saturday	Sunday
70	77	81	70
71	78	83	81
75	81	84	81
79	82	84	88
80	83	86	88
81	83	86	89
83	85	87	90
83	85	87	90
84	85	87	91
85	88	88	93
90	89	89	94

2.6 Scatterplots

2.6.1 Fitting straight lines to bivariate data

The process of 'fitting' straight lines to bivariate data enables us to analyse relationships between the data and possibly make predictions based on the given data set.

We will consider the most common technique for fitting a straight line and determining its equation, namely least squares.

The **linear relationship** expressed as an equation is often referred to as the *linear regression equation* or line. Recall that when we display bivariate data as a **scatterplot**, the explanatory variable is placed on the horizontal axis and the response variable is placed on the vertical axis.

2.6.2 Scatterplots

We often want to know if there is a relationship between two numerical variables. A scatterplot, which gives a visual display of the relationship between two variables, provides a good starting point.

Consider the data obtained from last year's 12B class at Northbank Secondary College. Each student in this class of 29 students was asked to give an estimate of the average number of hours they studied per week during Year 12. They were also asked for the ATAR score they obtained.

Average hours of study	ATAR score
18	59
16	67
22	74
27	90
15	62
28	89
18	71
19	60
22	84
30	98
14	54
17	72
14	63
19	72
20	58

Average hours of study	ATAR score
10	47
28	85
25	75
18	63
19	61
17	59
16	76
14	59
29	89
30	93
30	96
23	82
26	35
22	78

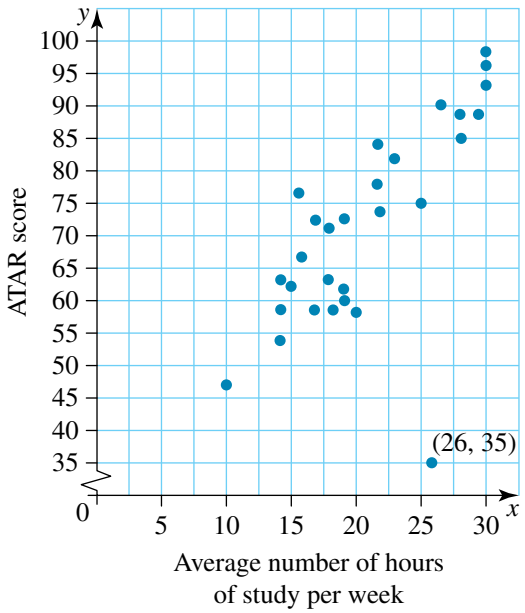
The figure below shows the data plotted on a scatterplot.

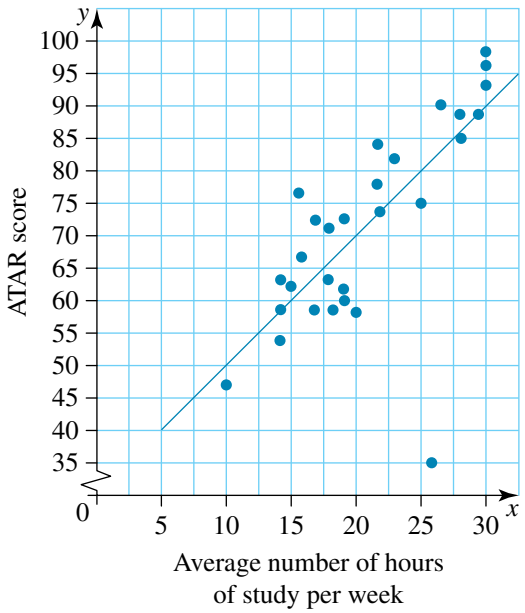
It is reasonable to think that the number of hours of study put in each week by students would affect their ATAR scores and so the number of hours of study per week is the explanatory (independent) variable and appears on the horizontal axis. The ATAR score is the response (dependent) variable and appears on the vertical axis.

There are 29 points on the scatterplot. Each point represents the number of hours of study and the ATAR score of one student.

In analysing the scatterplot we look for a pattern in the way the points lie. Certain patterns tell us that certain relationships exist between the two variables. This is referred to as **correlation**. We look at what type of correlation exists and how strong it is.

In the diagram we see some sort of pattern: the points are spread in a rough corridor from bottom left to top right. We refer to data following such a direction as having a *positive relationship*. This tells us that as the average number of hours studied per week increases, the ATAR score increases.





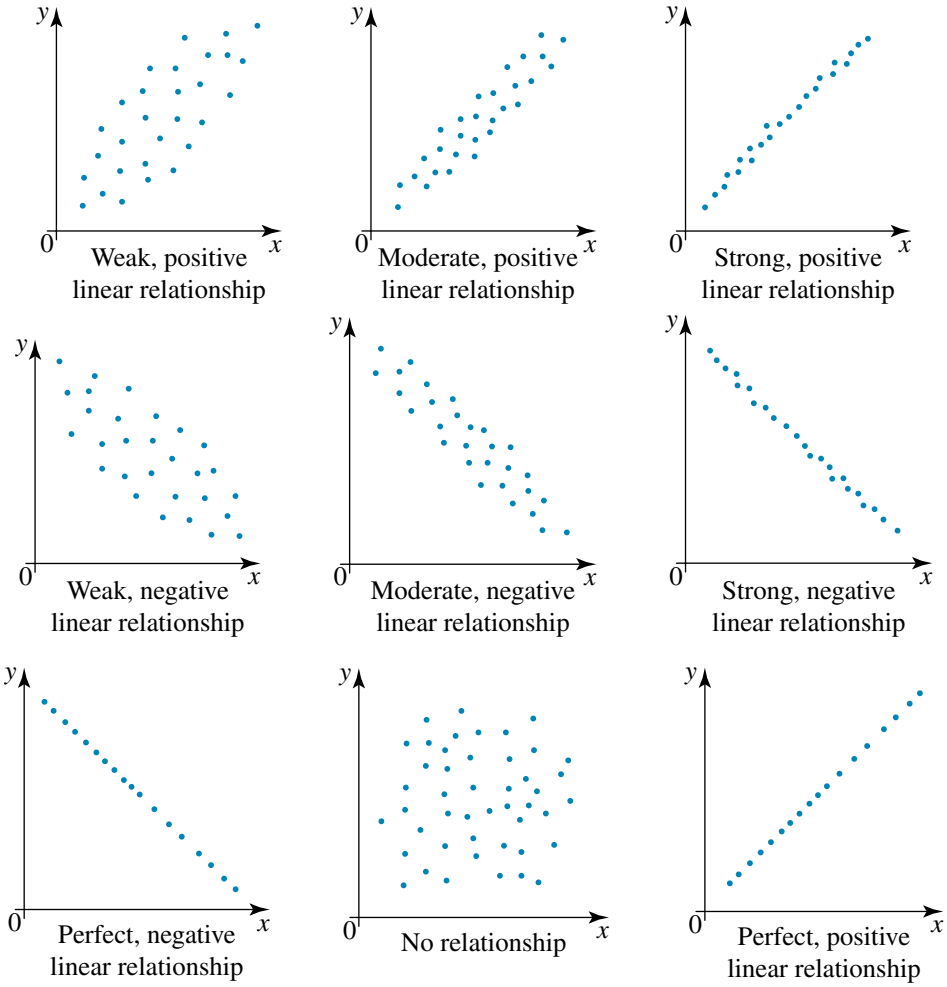
The point (26, 35) is an outlier. It stands out because it is well away from the other points and clearly is not part of the 'corridor' referred to previously. This outlier may have occurred because a student exaggerated the number of hours he or she worked in a week or perhaps there was a recording error. This needs to be checked.

We could describe the rest of the data as having a *linear* form as the straight line in the diagram indicates.

When describing the relationship between two variables displayed on a scatterplot, we need to comment on:

- **a.** the direction — whether it is positive or negative
- **b.** the form — whether it is linear or non-linear
- **c.** the strength — whether it is strong, moderate or weak
- **d.** possible outliers.

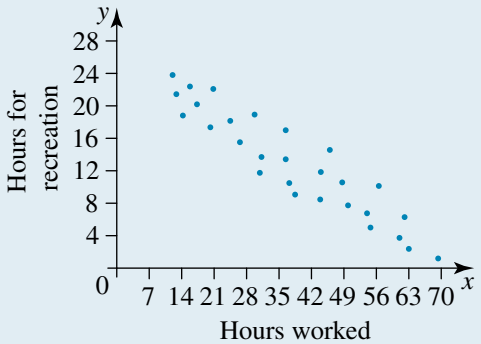
Here is a gallery of scatterplots showing the various patterns we look for.



WORKED EXAMPLE 8

The scatterplot shows the number of hours people spend at work each week and the number of hours people get to spend on recreational activities during the week.

Decide whether or not a relationship exists between the variables and, if it does, comment on whether it is positive or negative; weak, moderate or strong; and whether or not it has a linear form.



THINK

1. The points on the scatterplot are spread in a certain pattern, namely in a rough corridor from the top left to the bottom right corner. Interpret this pattern.
2. The corridor is straight (that is, it would be reasonable to fit a straight line into it).
3. The points are neither too tight nor too dispersed.

WRITE

As work hours increase, the recreation hours decrease.

4. The pattern resembles the central diagram in the gallery of scatterplots shown previously. Describe the relationship between the variables.	There is a moderate, negative linear relationship between the two variables.

WORKED EXAMPLE 9

Data showing the average weekly number of hours studied by each student in 12B at Northbank Secondary College and the corresponding height of each student (correct to the nearest tenth of a metre) are given in the table.

Average hours of study	18	16	22	27	15	28	18	20	10	28	25	18	19	17
Height (m)	1.5	1.9	1.7	2.0	1.9	1.8	2.1	1.9	1.9	1.5	1.7	1.8	1.8	2.1

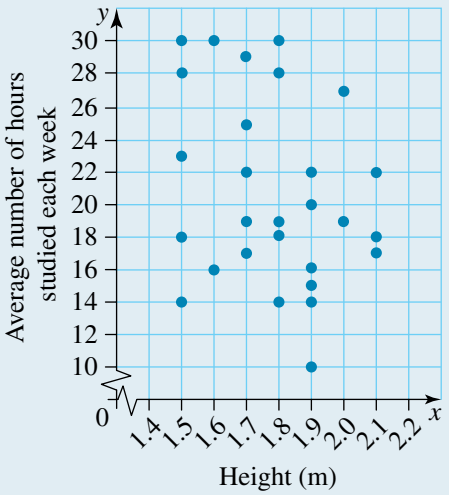
Average hours of study	19	22	30	14	17	14	19	16	14	29	30	30	23	22
Height (m)	2.0	1.9	1.6	1.5	1.7	1.8	1.7	1.6	1.9	1.7	1.8	1.5	1.5	2.1

Construct a scatterplot for the data and use it to comment on the direction, form and strength of any relationship between the number of hours studied and the height of the students.

THINK

1. CAS technology can be used to assist you in drawing a scatterplot.

WRITE/DRAW



2. Comment on the direction of any relationship.

There is no relationship; the points appear to be randomly placed.

3. Comment on the form of the relationship.

There is no form, no linear trend, no quadratic trend, just a random placement of points.

4. Comment on the strength of any relationship.

Since there is no relationship, strength is not relevant.

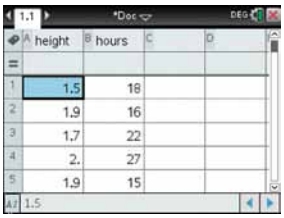
5. Draw a conclusion.

Clearly, from the graph, the number of hours spent studying for VCE has no relation to how tall you might be.

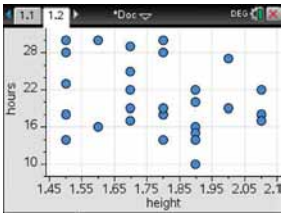
TI | THINK

1. On a Lists & Spreadsheet page, enter the data into the lists, named hours and height. Height is the explanatory variable and hours the response variable.

WRITE



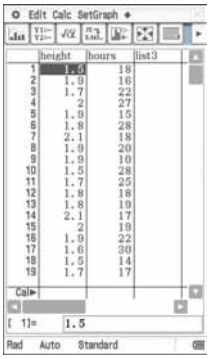
2. Press CTRL + I and select:
5: Add Data & Statistics
Use the tab key to move to each axis and press the click key or ENTER to place height on the horizontal axis and hours on the vertical axis.



CASIO | THINK

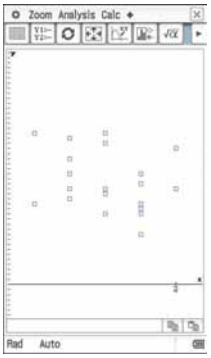
1. On a Statistics screen, enter the data into the lists, named hours and height. Height is the explanatory variable and hours the response variable.

WRITE



2. Tap:
SetGraph
Setting
Draw: On
Type: Scatter
XList: main\height
YList: main\hours
Freq: 1
Mark: square

Tap Set to lock your selection.
Tap Resize to enlarge the scatterplot.



Resources

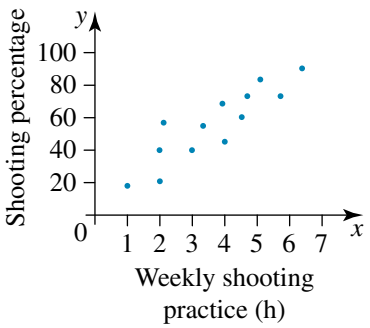
Interactivity: Scatterplots (int-6250)
Interactivity: Create scatter plots (int-6497)

studyon

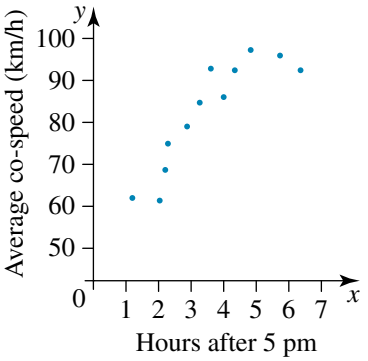
Units 3 & 4 > AOS DA > Topic 6 > Concept 5 **Scatterplots** Summary screen and practice questions

Exercise 2.6 Scatterplots

1. **WE8** The scatterplot shown represents the number of hours of basketball practice each week and a player's shooting percentage. Decide whether or not a relationship exists between the variables and, if it does, comment on whether it is positive or negative; weak, moderate or strong; and whether or not it is linear form.



2. The scatterplot shown shows the hours after 5 pm and the average speed of cars on a freeway. Explain the direction, form and strength of the relationship of the two variables.

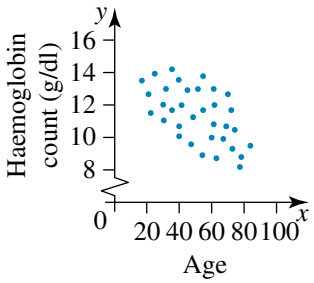


3. For each of the following pairs of variables, write down whether or not you would reasonably expect a relationship to exist between the pair and, if so, comment on whether it would be a positive or negative association.

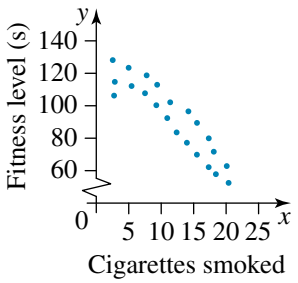
- **a.** Time spent in a supermarket and total money spent
- **b.** Income and value of car driven
- **c.** Number of children living in a house and time spent cleaning the house
- **d.** Age and number of hours of competitive sport played per week
- **e.** Amount spent on petrol each week and distance travelled by car each week
- **f.** Number of hours spent in front of a computer each week and time spent playing the piano each week
- **g.** Amount spent on weekly groceries and time spent gardening each week

4. For each of the scatterplots, describe whether or not a relationship exists between the variables and, if it does, comment on whether it is positive or negative, whether it is weak, moderate or strong and whether or not it has a linear form.

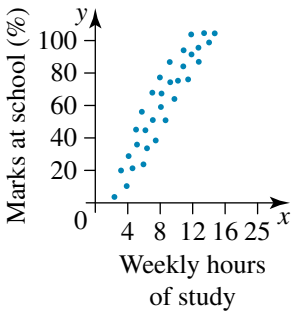
a.



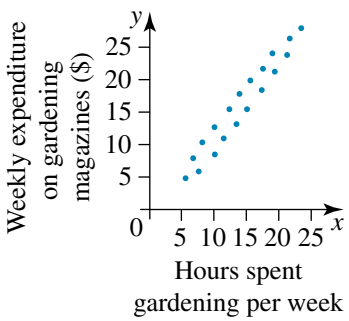
b.



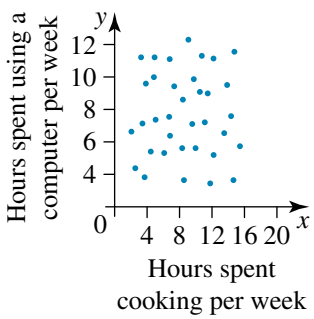
c.



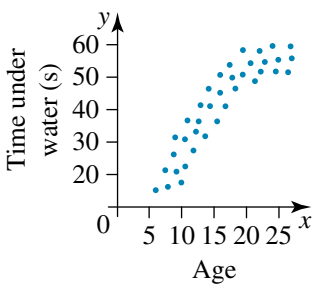
d.



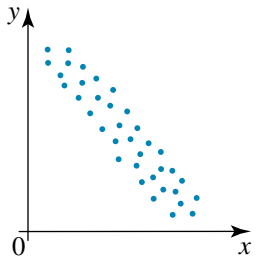
e.



f.



5. **MC** From the scatterplot shown, it would be reasonable to observe that:



   A. as the value of x increases, the value of y increases
   B. as the value of x increases, the value of y decreases
   C. as the value of x increases, the value of y remains the same
   D. as the value of x remains the same, the value of y increases
   E. there is no relationship between x and y

6. **WE9** Data on the height of a person and the length of their hair is shown. Construct a scatterplot for the data and use it to comment on the direction, form and strength of any relationship between the height of a person and the length of their hair.

Height (cm)	158	164	184	173	194	160	198	186	166
Hair length (cm)	18	12	5	10	7	3	10	6	14

7. The following table shows data on hours spent watching television per week and your age. Use the data to construct a scatterplot and use it to comment on the direction, form and strength of any relationship between the two variables.

Age (yr)	12	25	61	42	18	21	33	15	29
TV per week (h)	23	30	26	18	12	30	20	19	26

8. The population of a municipality (to the nearest ten thousand) together with the number of primary schools in that particular municipality is given below for 11 municipalities.

Population (×1000)	110	130	130	140	150	160	170	170	180	180	190
Number of primary schools	4	4	6	5	6	8	6	7	8	9	8

   Construct a scatterplot for the data and use it to comment on the direction, form and strength of any relationship between the population and the number of primary schools.

9. The table contains data for the time taken to do a paving job and the cost of the job. Construct a scatterplot for the data. Comment on whether a relationship exists between the time taken and the cost. If there is a relationship describe it.

Time taken (hours)	**Cost of job ($)**
5	1000
7	1000
5	1500
8	1200
10	2000
13	2500
15	2800
20	3200
18	2800
25	4000
23	3000

10. The table shows the time of booking (how many days in advance) of the tickets for a musical performance and the corresponding row number in A-reserve seating.

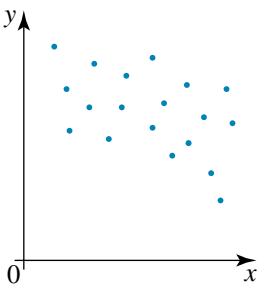
Time of booking	Row number
5	15
6	15
7	15
7	14
8	14
11	13
13	13

Time of booking	Row number
14	12
14	10
17	11
20	10
21	8
22	5
24	4

Time of booking	Row number
25	3
28	2
29	2
29	1
30	1
31	1

Construct a scatterplot for the data. Comment on whether a relationship exists between the time of booking and the number of the row and, if there is a relationship, describe it.

11. MC The correlation of this scatterplot is:



A. weak, positive, linear
B. no correlation
C. strong, positive linear
D. weak, negative, linear
E. strong, negative, linear

12. Draw a scatterplot to display the following data:
a. by hand **b.** using CAS.

Number of dry-cleaning items	1	2	3	4	5	6	7
Cost ($)	12	16	19	20	22	24	25

13. Draw a scatterplot to display the following data:
a. by hand **b.** using CAS.

Maximum daily temperature (°C)	26	28	19	17	32	36	33	23	24	18
Number of drinks sold	135	156	98	87	184	133	175	122	130	101

14. Describe the correlation between:
a. the number of dry-cleaning items and the cost in question **12**
b. the maximum daily temperature and the number of drinks sold in question **13**.

15. Draw a scatterplot and describe the correlation for the following data.

	NSW	VIC	QLD	SA	WA	TAS	NT	ACT
Population	7 500 600	5 821 000	4 708 000	1 682 000	2 565 000	514 000	243 000	385 000
Area of land (km^2)	800 628	227 010	1 723 936	978 810	2 526 786	64 519	1 335 742	2358

16. The table shown contains data giving the time taken to engineer a finished product from the raw recording (of a song, say) and the length of the finished product.
    a. Construct a scatterplot for these data.
    b. Comment on whether a relationship exists between the time spent engineering and the length of the finished recording.

Time spent engineering in studio (hours)	Finished length of recording (minutes)
1	3
2	4
3	10
4	12
5	20
6	16
7	18
8	25
9	30
10	28
11	35
12	36
13	39
14	42
15	45

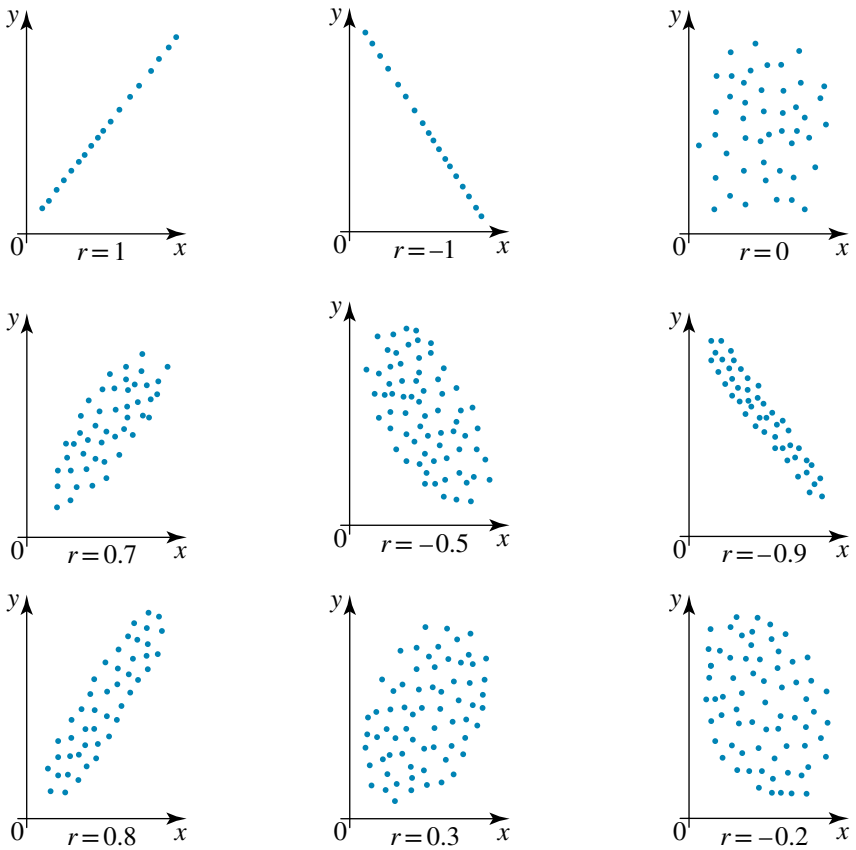
2.7 Pearson's product–moment correlation coefficient

In the previous section, we estimated the strength of association by looking at a scatterplot and forming a judgement about whether the correlation between the variables was positive or negative and whether the correlation was weak, moderate or strong.

A more precise tool for measuring correlation between two variables is **Pearson's product–moment correlation coefficient**. This coefficient is used to measure the strength of *linear relationships* between variables.

The symbol for Pearson's product–moment correlation coefficient is r. The value of r ranges from -1 to 1; that is, $-1 \leq r \leq 1$.

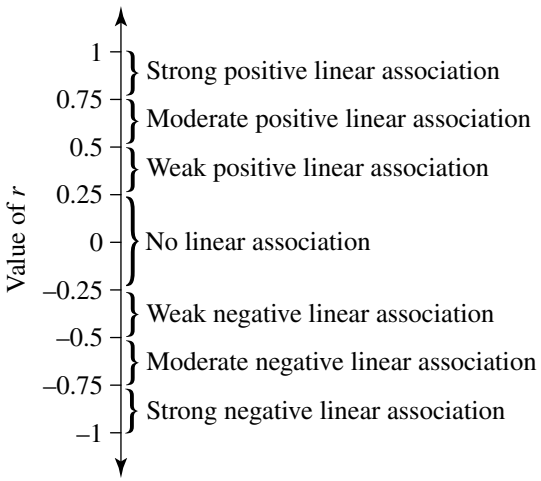
Following is a gallery of scatterplots with the corresponding value of r for each.



The two extreme values of r (1 and -1) are shown in the first two diagrams respectively.

From these diagrams we can see that a value of $r = 1$ or -1 means that there is perfect linear association between the variables.

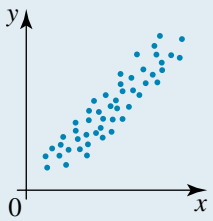
The value of the Pearson's product–moment correlation coefficient indicates the strength of the linear relationship between two variables. The diagram below gives a rough guide to the strength of the correlation based on the value of r.



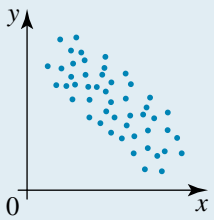
WORKED EXAMPLE 10

For each of the following:

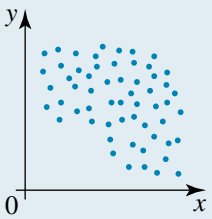
a.



b.



c.



i. Estimate the value of Pearson's product–moment correlation coefficient (r) from the scatterplot.

ii. Use this to comment on the strength and direction of the relationship between the two variables.

THINK	WRITE
a. i. Compare these scatterplots with those in the gallery of scatterplots shown previously and estimate the value of r.	**a. i.** $r \approx 0.9$
ii. Comment on the strength and direction of the relationship.	**ii.** The relationship can be described as a strong, positive, linear relationship.
b. Repeat parts **i** and **ii** as in **a**.	**b. i.** $r \approx -0.7$
	ii. The relationship can be described as a moderate, negative, linear relationship.
c. Repeat parts **i** and **ii** as in **a**.	**c. i.** $r \approx -0.1$
	ii. There is almost no linear relationship.

Note that the symbol $\approx$ means 'approximately equal to'. We use it instead of the $=$ sign to emphasise that the value (in this case r) is only an estimate.

In completing Worked example 10 above, we notice that estimating the value of r from a scatterplot is rather like making an informed guess. In the next section, we will see how to obtain the actual value of r.

on Resources

Interactivity: Pearson's product–moment correlation coefficient and the coefficient of determination (int-6251)

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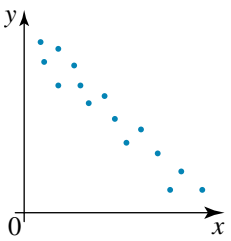
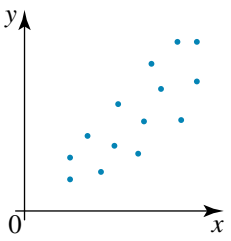
Units 3 & 4 > AOS DA > Topic 6 > Concept 6

Pearson's product–moment correlation coefficient (r) Summary screen and practice questions

Exercise 2.7 Pearson's product–moment correlation coefficient

1. **WE10** For each of the following:

   a.
   

   b.
   

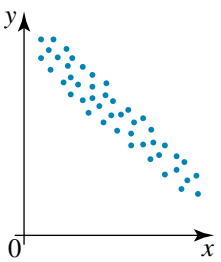
   i. estimate the Pearson's product–moment correlation coefficient (r) from the scatterplot.
   ii. use this to comment on the strength and direction of the relationship between the two variables.

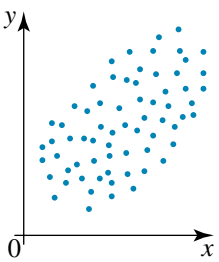
2. What type of linear relationship does each of the following values of r suggest?
   a. 0.85 **b.** −0.3

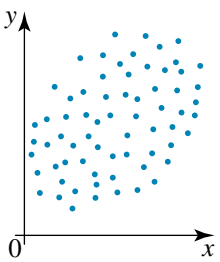
3. What type of linear relationship does each of the following values of r suggest?
   a. 0.21 **b.** 0.65 **c.** −1 **d.** −0.78

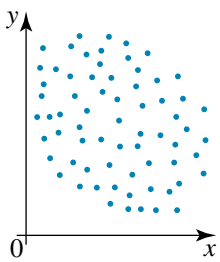
4. What type of linear relationship does each of the following values of r suggest?
   a. 1 **b.** 0.9 **c.** −0.34 **d.** −0.1

5. For each of the following:

   a.
   

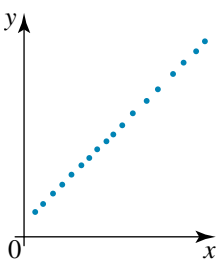
   b.
   

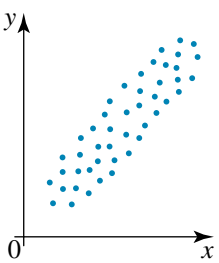
   c.
   

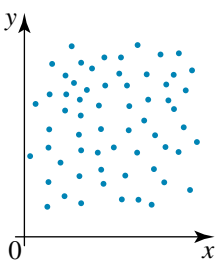
   d.
   

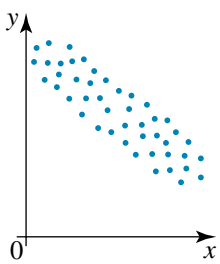
   i. estimate the value of Pearson's product–moment correlation coefficient (r), from the scatterplot.
   ii. use this to comment on the strength and direction of the relationship between the two variables.

6. For each of the following:

   a.
   

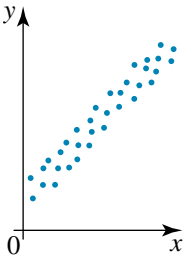
   b.
   

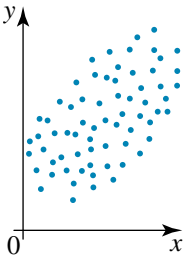
   c.
   

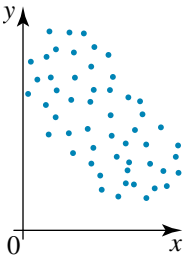
   d.
   

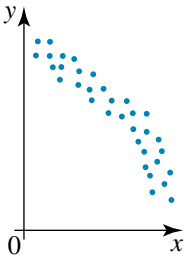
   i. estimate the value of Pearson's product–moment correlation coefficient (r), from the scatterplot.
   ii. use this to comment on the strength and direction of the relationship between the two variables.

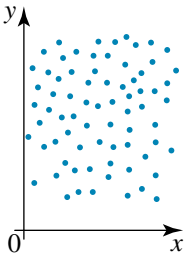
7. MC A set of data relating the variables x and y is found to have an r value of 0.62. The scatterplot that could represent the data is:

A. 

B. 

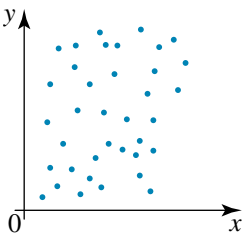
C. 

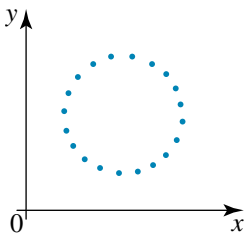
D. 

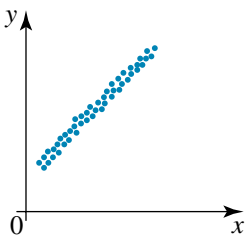
E. 

8. MC A set of data relating the variables x and y is found to have an r value of -0.45. A true statement about the relationship between x and y is:
   - **A.** There is a strong linear relationship between x and y and when the x-values increase, the y-values tend to increase also.
   - **B.** There is a moderate linear relationship between x and y and when the x-values increase, the y-values tend to increase also.
   - **C.** There is a moderate linear relationship between x and y and when the x-values increase, the y-values tend to decrease.
   - **D.** There is a weak linear relationship between x and y and when the x-values increase, the y-values tend to increase also.
   - **E.** There is a weak linear relationship between x and y and when the x-values increase, the y-values tend to decrease.

9. From the scatterplots shown estimate the value of r and comment on the strength and direction of the relationship between the two variables.

a. 

b. 

c. 

10. MC A weak, negative, linear association between two variables would have an r value closest to:
    A. -0.55 **B.** 0.55 **C.** -0.65 **D.** -0.45 **E.** 0.45

11. MC Which of the following is *not* a Pearson product–moment correlation coefficient?
    A. 1.0 **B.** 0.99 **C.** -1.1 **D.** -0.01 **E.** 0

12. Draw a scatterplot that has a Pearson product–moment correlation coefficient of approximately -0.7.

13. MC If two variables have an r value of 1, then they are said to have:
    - **A.** a strong positive linear relationship
    - **B.** a strong negative linear relationship
    - **C.** a perfect positive relationship
    - **D.** a perfect negative linear relationship
    - **E.** a perfect positive linear relationship

14. MC Which is the correct ascending order of positive values of r?
    - **A.** Strong, Moderate, Weak, No linear association
    - **B.** Weak, Strong, Moderate, No linear association
    - **C.** No linear association, Weak, Moderate, Strong
    - **D.** No linear association, Moderate, Weak, Strong
    - **E.** Strong, Weak, Moderate, No linear association

2.8 Calculating r and the coefficient of determination, r^2

2.8.1 Pearson's product–moment correlation coefficient (r)

The formula for calculating Pearson's correlation coefficient r is as follows:

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

where n is the number of pairs of data in the set
s_x is the standard deviation of the x-values
s_y is the standard deviation of the y-values
$\bar{x}$ is the mean of the x-values
$\bar{y}$ is the mean of the y-values.

The calculation of r is often done using CAS.

There are two important limitations on the use of r. First, since r measures the strength of a linear relationship, it would be inappropriate to calculate r for data which are not linear — for example, data which a scatterplot shows to be in a quadratic form.

Second, outliers can bias the value of r. Consequently, if a set of linear data contains an outlier, then r is not a reliable measure of the strength of that linear relationship.

The calculation of r is applicable to sets of bivariate data which are known to be linear in form and which do not have outliers.

With those two provisos, it is good practice to draw a scatterplot for a set of data to check for a linear form and an absence of outliers before r is calculated. Having a scatterplot in front of you is also useful because it enables you to estimate what the value of r might be — as you did in the previous exercise, and thus you can check that your workings are correct.

WORKED EXAMPLE 11

The heights (in centimetres) of 21 football players were recorded against the number of marks they took in a game of football. The data are shown in the following table.

a. Construct a scatterplot for the data.

b. Comment on the correlation between the heights of players and the number of marks that they take, and estimate the value of r.

c. Calculate r and use it to comment on the relationship between the heights of players and the number of marks they take in a game.

Height (cm)	Number of marks taken
184	6
194	11
185	3
175	2
186	7
183	5
174	4
200	10
188	9
184	7
188	6

Height (cm)	Number of marks taken
182	7
185	5
183	9
191	9
177	3
184	8
178	4
190	10
193	12
204	14

THINK

a. Height is the explanatory variable, so plot it on the x-axis; the number of marks is the response variable, so show it on the y-axis.

b. Comment on the correlation between the variables and estimate the value of r.

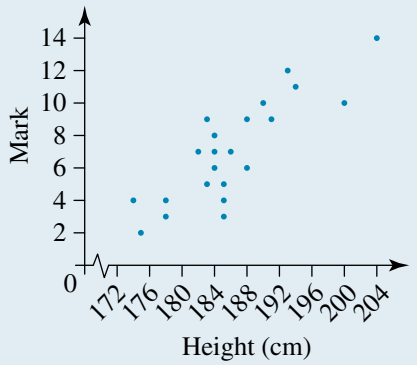
c. 1. Because there is a linear form and there are no outliers, the calculation of r is appropriate.

2. Use CAS to find the value of r. Round correct to 2 decimal places.

3. The value of $r = 0.86$ indicates a strong positive linear relationship.

WRITE/DRAW

a.

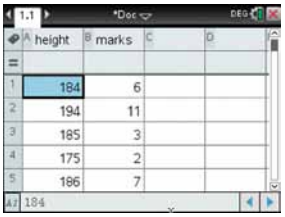
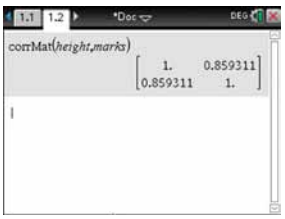


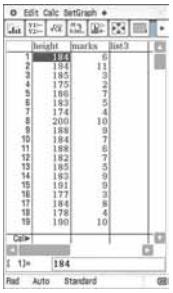
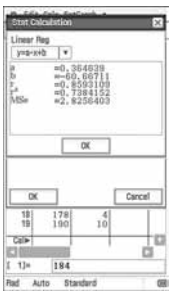
b. The data show what appears to be a linear form of moderate strength. We might expect $r \approx 0.8$.

c.

$r = 0.859\,311...$
≈ 0.86

$r = 0.86$. This indicates there is a strong positive linear association between the height of a player and the number of marks he takes in a game.

TI | THINK	WRITE
C. 1. On a Lists & Spreadsheet page, enter the data into the lists named height and marks.	
2. Press CTRL + I and select: 5: Add Calculator Complete the entry line as: corrMat(height, marks) Then press EXE.	
3. Read *r*, the correlation coefficient from the matrix on the screen and round correct to 2 decimal places.	$r = 0.859\,311...$ ≈ 0.86
The value of $r = 0.86$ indicates a strong positive linear relationship.	$r = 0.86$. This indicates there is a strong positive linear association between the height of a player and the number of marks he takes in a game.

CASIO | THINK	WRITE
1. On a Statistics screen, enter the data into the lists named height and marks.	
2. Tap: Calc Linear reg On the Set Calculation screen, select the following: XList: main\height YList: main\marks Freq: 1 Copy Formula: y1 OK	
Read *r*, the correlation coefficient from the screen and round correct to 2 decimal places.	$r = 0.859\,311...$ ≈ 0.86
The value of $r = 0.86$ indicates a strong positive linear relationship.	$r = 0.86$. This indicates there is a strong positive linear association between the height of a player and the number of marks he takes in a game.

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Units 3 & 4 › AOS DA › Topic 6 › Concept 7

Use and interpretation of *r* Summary screen and practice questions

The coefficient of determination (r^2)

The **coefficient of determination** is given by r^2. It is very easy to calculate, we merely square Pearson's product–moment correlation coefficient (r). The value of the coefficient of determination ranges from 0 to 1; that is, $0 \leq r^2 \leq 1$.

The coefficient of determination is useful when we have two variables which have a linear relationship. It tells us the proportion of variation in one variable which can be explained by the variation in the other variable.

The coefficient of determination provides a measure of how well the linear rule linking the two variables (x and y) predicts the value of y when we are given the value of x.

WORKED EXAMPLE 12

A set of data giving the number of police traffic patrols on duty and the number of drink-driving charges for the region was recorded and a correlation coefficient of $r = -0.8$ was found.

Calculate the coefficient of determination and interpret its value.

THINK	WRITE
1. Calculate the coefficient of determination by squaring the given value of r.	Coefficient of determination $= r^2$ $= (-0.8)^2$ $= 0.64$
2. Interpret your result.	We can conclude from this that 64% of the variation in the number of drink-driving charges can be explained by the variation in the number of police traffic patrols on duty. This means that the number of police traffic patrols on duty is a major factor in predicting the number of drink-driving charges.

Note: In Worked example 12, 64% of the variation in the number of drink-driving charges was due to the variation in the number of police cars on duty and 36% was due to other factors; for example, days of the week or hour of the day.

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Units 3 & 4 > AOS DA > Topic 7 > Concept 4

The coefficient of determination (r^2) Summary screen and pratice questions

Exercise 2.8 Calculating r and the coefficient of determination, r^2

1. WE11 The heights (cm) of basketball players were recorded against the number of points scored in a game. The data are shown in the following table.

Height (cm)	Points scored	Height (cm)	Points scored
194	6	201	13
203	4	196	10
208	18	205	20
198	22	215	14
195	2	203	3

a. Construct a scatterplot of the data.
b. Comment on the correlation between the heights of basketballers and the number of points scored, and estimate the value of r.
c. Calculate the r value and use it to comment on the relationship between heights of players and the number of points scored in a game.

2. The following table shows the gestation time and the birth mass of 10 babies.

Gestation time (weeks)	31	32	33	34	35	36	37	38	39	40
Birth mass (kg)	1.08	1.47	1.82	2.06	2.23	2.54	2.75	3.11	3.08	3.37

a. Construct a scatterplot of the data.
b. Comment on the correlation between the 'gestation time' and 'birth mass', and estimate the value of r.
c. Calculate the r value and use it to comment on the relationship between gestation time and birth mass.

3. The yearly salary ($\times$ \$1000) and the number of votes polled in the Brownlow medal count are given below for 10 footballers.

Yearly salary ($\times$ \$1000)	360	400	320	500	380	420	340	300	280	360
Number of votes	24	15	33	10	16	23	14	21	31	28

a. Construct a scatterplot for the data.
b. Comment on the correlation of salary and the number of votes and make an estimate of r.
c. Calculate r and use it to comment on the relationship between yearly salary and number of votes.

4. **WE12** Data on the number of booze buses in use and the number of drivers registering a blood alcohol reading over 0.05 was recorded and a correlation coefficient of $r = 0.77$ was found.
Calculate the coefficient of determination and interpret its value.

5. A set of data, obtained from 40 smokers, gives the number of cigarettes smoked per day and the number of visits per year to the doctor. The Pearson's correlation coefficient for these data was found to be 0.87. Calculate the coefficient of determination for the data and interpret its value.

6. Data giving the annual advertising budgets ($\times$ \$1000) and the yearly profit increases (%) of 8 companies are shown below.

Annual advertising budget ($\times$ \$1000)	11	14	15	17	20	25	25	27
Yearly profit increase (%)	2.2	2.2	3.2	4.6	5.7	6.9	7.9	9.3

a. Construct a scatterplot for these data.
b. Comment on the correlation of the advertising budget and profit increase and make an estimate of r.
c. Calculate r.
d. Calculate the coefficient of determination.
e. Write the proportion of the variation in the yearly profit increase that can be explained by the variation in the advertising budget.

7. Data showing the number of people in 9 households against weekly grocery costs are given below.

Number of people in household	2	5	6	3	4	5	2	6	3
Weekly grocery costs ($)	60	180	210	120	150	160	65	200	90

a. Construct a scatterplot for the data.
b. Comment on the correlation of the number of people in a household and the weekly grocery costs and give an estimate of r.
c. Calculate r.
d. Calculate the coefficient of determination.
e. Write the proportion of the variation in the weekly grocery costs that can be explained by the variation in the number of people in a household.

8. An investigation is undertaken with people following the Certain Slim diet to explore the link between weeks of dieting and total weight loss. The data are shown in the table.

Total weight loss (kg)	Number of weeks on the diet
1.5	1
4.5	5
9	8
3	3
6	6
8	9
3.5	4
3	2
6.5	7
8.5	10
4	4
6.5	6
10	9
2.5	2
6	5

a. Display the data on a scatterplot.
b. Describe the association between the two variables in terms of direction, form and strength.
c. Is it appropriate to use Pearson's correlation coefficient to explain the link between the number of weeks on the Certain Slim diet and total weight loss?
d. Estimate the value of Pearson's correlation coefficient from the scatterplot.
e. Calculate the value of this coefficient.
f. Is the total weight loss affected by the number of weeks staying on the diet?
g. Calculate the value of the coefficient of determination.
h. What does the coefficient of determination say about the relationship between total weight loss and the number of weeks on the Certain Slim diet?

2.9 Correlation and causation

2.9.1 Observation and experimentation

In an area known as Alsace on the eastern border of France there are a number of villages, many of them established in medieval times. It has long been noted that there is a strong correlation between the number of storks and the number of babies in these villages. This is an extreme example of why **correlation** does not necessary imply **causation**. The large number of storks do not cause the growth in births. Larger towns have more chimneys in which the storks make their nests and larger towns have a higher birth rate.

Establishing a high degree of correlation between two variables can provide a starting point for an investigation but establishing causation requires further observation and experimentation. In Worked example 11, there was a high positive correlation between the height of a footballer and the number of marks he takes. Does this mean that if a football team need players with greater marking ability then they just should recruit more tall players? An experienced recruiter knows that height is only one advantage in a marking contest; fitness levels, skill level, hand-eye coordination, ability to read the game are all necessary attributes of a player who can take marks. The recruiter would observe all of these attributes of potential new players and study the statistics of all the variables, before making a decision.

Therefore, a high degree of correlation between two variables is not a basis on which to state that one variable *causes* a change in the other variable, further investigation is necessary.

One way to establish causation is to conduct experiments where a control group is used. Scientists researching the effectiveness of a new drug designed to destroy a particular type of cancer cell, split the population (those participating in the study) into two groups. One group will be given the drug, the other group a placebo and all other variables are the same across both groups. A study of the correlation between the administered drug and a decrease in the presence of the cancer cells can then establish causation. In this example, the administered drug and placebo are the explanatory variables, and the amount of cancer cells present is the response variable.

2.9.2 Common response, confounding and coincidence explanations

In some cases, the correlation between two variables can be explained by a common response which provides the association. For example, a study may show that there is a strong correlation between house sizes and the life expectancy of home owners. While a bigger house will not directly lead to a longer life expectancy, a common response, the income of the house owner, provides a direct link to both variables and is more likely to be the underlying cause for the observed correlation.

In another example, a study of a group of people may provide a strong association between weight (response variable) and hours of exercise (explanatory variable). But there are other factors that can influence weight, such as age, sex, starting weight and diet. These are called **confounding variables** and a study that does not take these variables into account when setting up the study, means that the results cannot be used to show causality.

During a discussion over a community breakfast at a time-share resort in Queensland, it was established that of 6 people sitting together, 5 had a relative who lived on Phillip Island. Does this suggest a strong correlation between people who own time share and those who have a relative on Phillip Island? No, sometimes an association between variables can just be a **coincidence**.

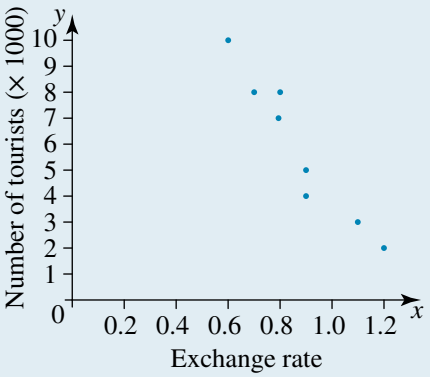
When observing an association between two variables it can never be stated that the one variable causes a response in the other variable. In professional research many controlled experiments must be carried out to determine and identify a causal relationship.

WORKED EXAMPLE 13

Data showing the number of tourists visiting a small country in a month and the corresponding average monthly exchange rate for the country's currency against the American dollar are as given.

Number of tourists (× 1000)	2	3	4	5	7	8	8	10
Exchange rate	1.2	1.1	0.9	0.9	0.8	0.8	0.7	0.6

a. Construct a scatterplot for the data.
b. Comment on the correlation between the number of tourists and the exchange rate
c. Calculate *r*.
d. Based on the value of *r* obtained in part c, would it be appropriate to conclude that the decrease in tourist numbers is caused by the increasing exchange rate?

THINK	WRITE
a. 1. Determine the explanatory and response variables.	Exchange rate (explanatory) Tourist numbers (response)
2. Sketch scatterplot.	
b. A negative association can be observed.	From the scatterplot there appears to be a strong negative correlation between the number of tourists visiting the country and the exchange rate.
c. Use CAS to find *r*.	$r = -0.96$
d. Correlation does not imply causation.	Just because $r = -0.96$, it cannot be stated that the increasing tourist numbers is caused by the decrasing exchasnge rate. There may be other factors that need to be considered. Further investigation is needed before any conclusions can be drawn.

studyon

Units 3 & 4 > AOS DA > Topic 6 > Concept 8

Non-causal explanations: Summary screen and practice questiuons

Exercise 2.9 Correlation and causation

1. WE13 Data showing the number of people on 8 fundraising committees and the annual funds raised are given in the table.

Number of people on committee	3	6	4	8	5	7	3	6
Annual funds raised ($)	4500	8500	6100	12500	7200	10000	4700	8800

   a. Construct a scatterplot for the data.
   b. Comment on the correlation between the number of people on the committee and the annual funds raised.
   c. Calculate r.
   d. Based on the value of r obtained in part c, would it be appropriate to conclude that the increase in funds raised is caused by an increase in the number of people on the committee?

2. Data was collected on the number of books read by each of 50 ten-year olds and their spelling skills. The correlation coefficient for this data was 0.90.
   What can be said about the association between the number of books read and spelling skills for this group of 10-year olds?

The following information applies to questions 3 and 4.

A set of data was collected from a large group of parents with children under 5 years of age. They were asked the number of hours they worked per week and the amount of money they spent on child care. The results were recorded, and the value of Pearson's correlation coefficient was found to be 0 92.

3. MC Which of the following is NOT true?
   A. There is a positive correlation between the number of working hours and the amount of money spent on child care.
   B. The correlation between the number of working hours and the amount of money spent on child care can be classified as strong.
   C. The relationship between the number of working hours and the amount spent on child care is linear.
   D. The linear relationship between the two variables suggests that as the number of working hours increase, so to does the amount spent on child care.
   E. The increase in the number of hours worked causes the increase in the amount of money spent on child care.

4. MC Which of the following is NOT true?
   A. The coefficient of determination is about 0.85.
   B. The number of working hours is the major factor in predicting the amount of money spent on child care.
   C. About 85% of the variation in the number of hours worked can be explained by the variation in the amount of money spent on child care.
   D. Other factors such as income levels can affect the amount of money spent on child care.
   E. About $\frac{17}{20}$ of the variation in the amount of money spent on child care can be explained by the variation in the number of hours worked.

5. MC Over a 10-week period a strong positive association ($r = 0.91$) was found between the number of baby penguins born on Phillip Island each week and the number of wildlife killed on the roads on Phillip Island each week. A strong positive association was also found both between the number of baby penguins born on the island ($r = 0.93$), the number of wildlife killed on the island ($r = 0.86$), and the number of visitors to the island.

Using this information, which of the following statements is true?

A. During peak breeding season, drivers get distracted looking at the baby penguins and don't notice the wildlife on the roads.
B. It is just a coincidence that there is a strong positive association between the number of baby penguins born on the island and the number of wildlife killed on the roads.
C. Breeding season attracts larger numbers of visitors and more visitors means more drivers on the roads which leads to more wildlife being killed.
D. Everyone knows that tourists are very poor drivers.
E. Penguins are very aggressive and cause other wildlife to run onto the road.

6. There is a strong positive correlation between rates of diabetes and the number of hip replacements. Does diabetes cause deterioration of the hip joint? What could be the common cause(s) that links these two variables?

7. It has been found that there is a strong positive association between weight gain and the length of holidays. Does this mean that holidays cause weight gain? What could be the common cause that links these variables?

8. A study of emergency service workers (police, fire and paramedics) showed a strong positive association between levels of stress and hours worked. Does this mean that the length of the shift causes stress? What other confounding variables could also contribute to this association?

2.10 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Mulitple choice

1. MC A back-to-back stem plot is a useful way of displaying the relationship between:

A. the number of children attending a day care centre and whether or not the centre has federal funding
B. height and wrist circumference
C. age and weekly income
D. weight and the number of takeaway meals eaten each week
E. the age of a car and amount spent each year on servicing it

The following information relates to questions **2** and **3**.

The salaries of people working at five different advertising companies are shown on the following parallel boxplots.

2. MC The company with the largest interquartile range is:

A. Company A **B.** Company B **C.** Company C **D.** Company D **E.** Company E

3. MC The company with the lowest median is:

A. Company A **B.** Company B **C.** Company C **D.** Company D **E.** Company E

Questions **4** and **5** relate to the following information.

Data showing reactions of junior staff and senior staff to a relocation of offices are given below in a contingency table.

Attitude	Junior staff	Senior staff	Total
For	23	14	37
Against	31	41	72
Total	54	55	109

4. MC From this table, we can conclude that:

A. 23% of junior staff were for the relocation

B. 42.6% of junior staff were for the relocation

C. 31% of junior staff were against the relocation

D. 62.1% of junior staff were for the relocation

E. 28.4% of junior staff were against the relocation

5. MC From this table, we can conclude that:

A. 14% of senior staff were for the relocation

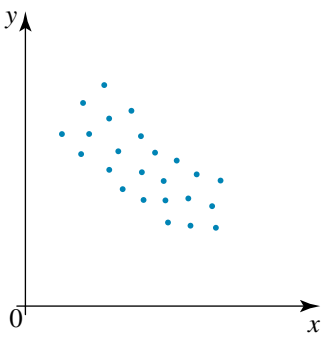
B. 37.8% of senior staff were for the relocation

C. 12.8% of senior staff were for the relocation

D. 72% of senior staff were against the relocation

E. 74.5% of senior staff were against the relocation

6. MC The relationship between the variables x and y is shown on the scatterplot.



The correlation between x and y would be best described as:

A. a weak positive association

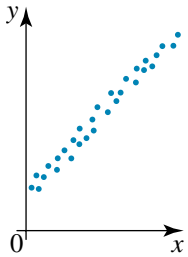
B. a weak negative association

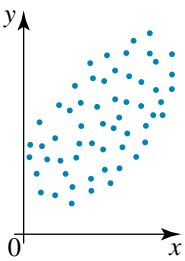
C. a strong positive association

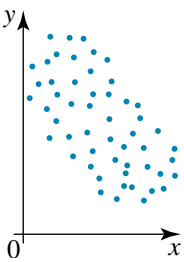
D. a strong negative association

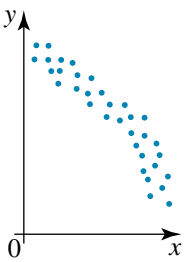
E. non-existent

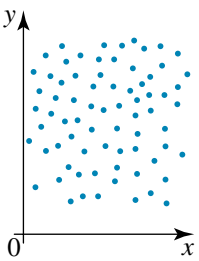
7. MC A set of data relating the variables x and y is found to have an r value of -0.83. The scatterplot that would best represent this data set is:

A.


B.


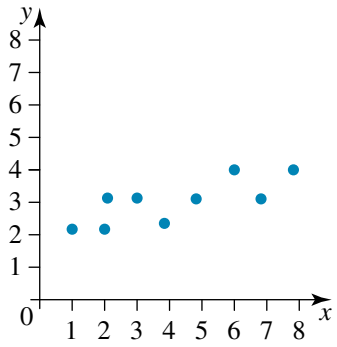
C.


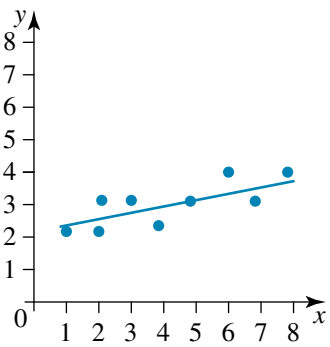
D.


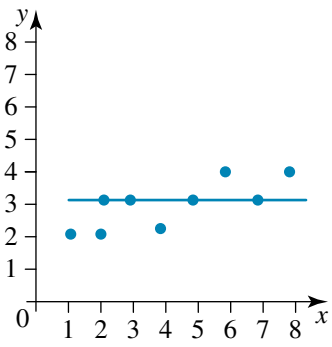
E.


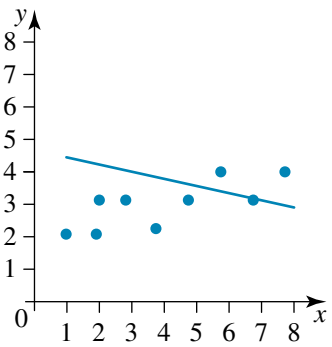
8. **MC** A set of data comparing age with blood pressure is found to have a Pearson's correlation coefficient of 0.86. The coefficient of determination for these data would be closest to:
   A. −0.86 **B.** −0.74 **C.** −0.43 **D.** 0.43 **E.** 0.74

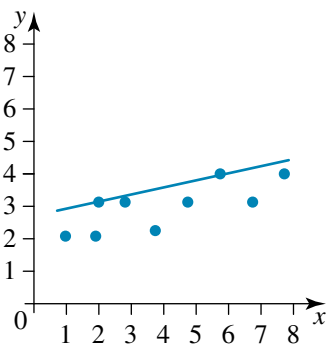
9. **MC** The most appropriate line of best fit for the figure is:

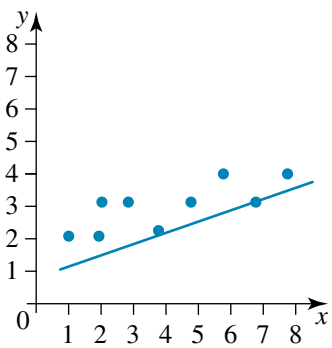


A.


B.


C.


D.


E.


10. **MC** The correlation between two variables x and y is −0.88. Which of the following statements is true?
   A. As y increases it causes x to increase.
   B. As y increases it causes x to decrease.
   C. There is a poor fit between x and y.
   D. As x increases, y tends to increase.
   E. As x increases, y tends to decrease.

11. **MC** When calculating a least-squares regression line, a correlation coefficient of −1 indicates that:
   A. the y-axis variable depends linearly on the x-axis variable
   B. the y-axis variable increases as the x-axis variable decreases
   C. the y-axis variable decreases as the x-axis variable decreases
   D. all the data lie on the same straight line
   E. the two variables depend upon each other

12. MC There is a strong negative association between the number of tourists in cities and the satisfaction levels of local residents.

From this information it can be concluded that:

A. increasing the number of tourists in a city will increases the satisfaction levels of local residents
B. decreasing the number of tourists in a city will increase the satisfaction levels of local residents
C. cities that have large numbers of tourists tend to have low satisfaction levels of local residents
D. cities that have large numbers of tourists tend to have high satisfaction levels of local residents
E. this association is a coincidence

Short answer

1. The number of hours of counselling received by a group of 9 full-time firefighters and 9 volunteer firefighters after a serious bushfire is given in the table.

Full-time	2	4	3	5	2	4	6	1	3
Volunteer	8	10	11	11	12	13	13	14	15

   a. Construct a back-to-back stem plot to display the data.
   b. Comment on the distributions of the number of hours of counselling of the full-time firefighters and the volunteers.

2. The IQ of 8 players in 3 different football teams were recorded and are shown below.

Team A	120	105	140	116	98	105	130	102
Team B	110	104	120	109	106	95	102	100
Team C	121	115	145	130	120	114	116	123

   Display the data in parallel boxplots.

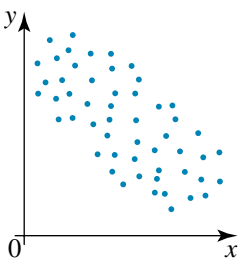
3. Delegates at the respective Liberal and Labor Party conferences were surveyed on whether or not they believed that uranium mining should continue. Forty-five Liberal delegates were surveyed and 15 were against continuation. Fifty-three Labor delegates were surveyed and 43 were against continuation.
   a. Present the data as percentages in a two-way frequency table and a segmented bar chart.
   b. Comment on any difference between the reactions of the Liberal and Labor delegates.

4. a. Construct a scatterplot for the data given in the table below.

Age	15	17	18	16	19	19	17	15	17
Pulse rate	79	74	75	85	82	76	77	72	70

   b. Use the scatterplot to comment on any relationship which exists between the variables.

5. For the variables shown on the scatterplot below, give an estimate of the value of r and use it to comment on the nature of the relationship between the two variables.



6. The table gives data relating the percentage of lectures attended by students in a semester and the corresponding mark for each student in the exam for that subject.
   a. Identify the explanatory and response variables.
   b. Construct a scatterplot for these data.
   c. Comment on the correlation between the lectures attended and the examination results and make an estimate of r.
   d. Calculate r.
   e. Calculate the coefficient of determination.
   f. Write the proportion of the variation in the examination results that can be explained by the variation in the number of lectures attended.

Lectures attended (%)	Exam result (%)
70	80
59	62
85	89
93	98
78	84
85	91
84	83
69	72
70	75
82	85

Extended response

1. For marketing purposes, the administration of the Arts Centre needs to compare the ages of people attending two different concerts: a symphony orchestra concert and a jazz concert. Twenty people were randomly selected from each audience and their ages were recorded as shown.

Event	Ages of people attending the event
Symphony orchestra concert	20, 23, 30, 35, 39, 42, 45, 45, 47, 48, 48, 49, 49, 50, 53, 54, 56, 58, 58, 60
Jazz concert	16, 18, 19, 19, 20, 23, 24, 27, 29, 30, 33, 34, 38, 39, 40, 42, 43, 45, 46, 62

   a. Display the data on a back-to-back stem plot.
   b. For each category calculate the following statistics:
      i. X_{min}
      ii. Q_1
      iii. median
      iv. Q_3
      v. X_{max}
      vi. mean
      vii. interquartile range (IQR)
      viii. standard deviation.

c. Use the stem plot together with some summary statistics to compare the distributions of the ages of patrons attending the two concerts.

One month later, at the beginning of the opera season, twenty people were again selected (this time from the opera audience) and their ages were recorded as shown.

Event	Ages of people attending the event
Opera	12, 18, 29, 30, 33, 35, 38, 39, 42, 46, 49, 50, 54, 56, 56, 57, 59, 63, 65, 68

The administration of the Arts Centre now wishes to compare all three distributions of the ages.

d. Explain why it is not possible to use a back-to-back stem plot for this task.

e. Calculate the eight summary statistics for the ages of the opera-goers (as in part **b** above).

f. Display the data for the three events using parallel boxplots.

g. Use the boxplots and some summary statistics to compare the three distributions.

2. In one study, 380 Year 12 students were asked how often they were engaged in any sporting activity outside school. Students were also asked to classify their stress level in relation to their VCE studies. The results were obtained, as shown in the table.

	Engaged in sporting activity outside school		
Level of stress	**Regularly**	**Sometimes**	**Never**
Low	16	32	36
Medium	12	40	56
High	6	52	130

a. How many students in this study reported a high level of stress?

b. How many students were engaged in sport activity outside of school?

c. Represent the data in a contingency table in percentage form.

d. Display the data from part **c** using a segmented bar chart.

e. Comment on any relationship between the stress level and the amount of sporting activity for this group of Year 12 students.

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Units 3 & 4 Sit exam

Answers

Topic 2 Investigating associations between two variable

Exercise 2.2 Response and explanatory variables

1. a. Daily temperature = explanatory variable, air conditioners sold = response variable
   b. Not appropriate
2. a. Size of the crowd = response variable, teams playing = explanatory variable
   b. Net score of a round of golf = response variable, golfer's handicap = explanatory variable
3. a. Explanatory — age, response — salary
   b. Explanatory — amount of fertiliser, response — growth
   c. Not appropriate
   d. Not appropriate
   e. Explanatory — number in household, response — size of house
4. a. Explanatory — month of the year, response — size of electricity bill
   b. Explanatory — number of hours, response — mark on the test
   c. Not appropriate
   d. Explanatory — season, response — cost
5. C
6. True
7. False
8. a. Minutes on the court: explanatory variable
   b. Points scored: response variable
9. a. Response variable: electricity bill
   b. Explanatory variable: number of Christmas lights
10. a. hours spent playing
    b. vocabulary

Exercise 2.3 Two-way (contingency) frequency tables and segmented bar charts

1. Note that black data is given in the question; red data are the answers.

Phone	Female	Male	Total
Apple	23	15	38
Samsung	15	7	22
Total	38	22	60

Marginal distribution:
Apple = 0.63 Samsung = 0.37

2. Note that black data is given in the question; red data are the answers.

Drink	Female	Male	Total
Coke	221	155	376
Pepsi	166	108	274
Total	387	263	650

Conditional distribution: Females who prefer
Coke = 0.57
Females who prefer Pepsi = 0.43

3.

Attitude	Female	Male	Total
For	37	79	116
Against	102	23	125
Total	139	102	241

Marginal distribution: For = 0.48, Against = 0.52

4.

Lesson length	Junior	Senior	Total
45 minutes	50	33	83
1 hour	36	60	96
Total	86	93	179

Conditional distribution:
Senior students prefer 45 mins = 0.35,
Senior students prefer an hour = 0.65

5.

Phone	Female	Male
Apple	60.5%	68.2%
Samsung	39.5%	31.8%
Total	100%	100%

6.

Drink	Female	Male
Coke	57.1%	58.9%
Pepsi	42.9%	41.1%
Total	100%	100%

7. a. i. 22 ii. 26 iii. 19 iv. 45 v. 41
   b. i. 47% ii. 58% iii. 100% iv. 100%

8.

Preference	Men	Women
Rent by themselves	57%	59%
Share with friends	43%	41%
Total	100%	100%

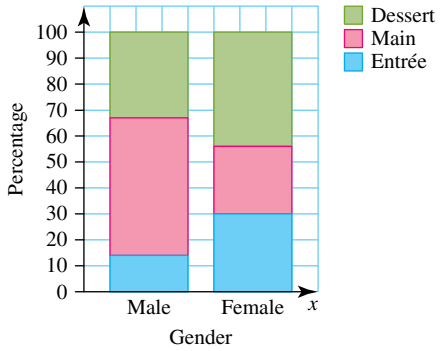
9. D

10. C

11.

Choice	Male	Female	Total
Entrée	8	18	26
Main	31	16	47
Dessert	19	27	46
Total	58	61	118

Choice	Male	Female
Entrée	14	30
Main	53	26
Dessert	33	44
Total	100	100



Males enjoy main meal the most compared to females who prefer their dessert the most.

12.

Choice	< 40	40+	Total
Mortgage	42	18	60
Superannuation	30	51	81
Investment	21	33	54
Total	93	102	195

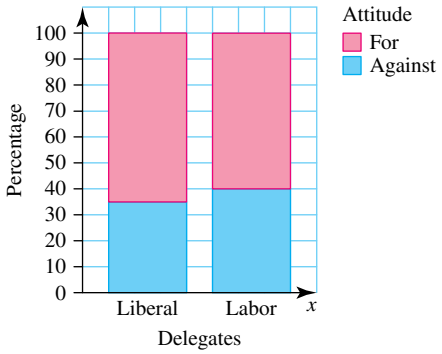
Choice	< 40	40+
Mortgage	45	18
Superannuation	32	50
Investment	23	32
Total	100	100

The under 40s have a focus on their mortgage, whereas the 40 and overs prioritise their superannuation.

13.

Attitude	Liberal	Labor
For	35.5	39.4
Against	64.5	60.6
Total	100.0	100.0

14.



There is not a lot of difference in the reactions.

15. C

16. A

17. Conditional distribution:
Large town and no waste = 0.15
Large town and medium waste = 0.19
Large town and high waste = 0.67
Note: rounding causes the total to be greater than 100%.

18. 26.32%

Exercise 2.4 Back-to-back stem plots

1. Key 3|1 = 31

Boys	Stem	Girls
1	3	0 0 1
3 2	3	2 3
4	3	5 5
7 6	3	
9 9	3	8

2. Key 2|4 = 24

Physics	Stem	Chemistry
4	2	1
9 8	2∗	8 8
4 2 2 0	3	0 1 2 3 4
6	3∗	8
4 1	4	
9 8 5 5	4∗	5 5 6 7 9

3. Key: 2|3 = 23

German	Stem	French
2 1 1 0	2	3 4
7 6 5 5	2∗	5 5 8
3 2 1 0 0	3	0 1 4 4
9 8 7 7	3∗	5 6 8 8 9
2 1	4	2 3 4 4
5	4∗	6 8

4. Key: 2 ∗ |7 = 2.7(kg)

Boys	Stem	Girls
	2∗	6 7
4 4	3	0 1 1 2 3
8 7 6	3∗	6 7
3 2	4	0
9 8	4∗	
0	5	

5. Key 13|0 = 130

Company A	Stem	Company B
4 4 2 2 2	13	0 1 3 4
8	13∗	8 8
4 2 1	14	
9 8	14∗	5 6 8
	15	3 3
6	15∗	5
0	16	0 2
8	16∗	

	Company A	Company B
Mean	143.57	144.71
Median	141.5	145.5
IQR	149 − 134 = 15	153 − 134 = 19
Standard deviation	11.42	10.87

They are both positively skewed. The median is a better indicator of the centre of the distribution than the mean. This shows Company B handing out more pamphlets, taking into account that the IQR and the standard deviations are quite similar.

6. a. Key 5|7 = 57

History	Stem	English
2	4	
9	5	7 8
7 4	6	2
8 8 7 5	7	1 4 5 8 9
4 2	8	0 7
8 2	9	6
	10	0

b.

	History	English
Mean	74.67	76.42
Median	77.5	76.5
IQR	83 − 65.5 = 17.5	83.5 − 66.5 = 17
Standard deviation	15.07	13.60

History has a slightly higher median; however, English has a slightly higher mean. Their standard deviations are similar, so overall the results are quite similar.

7. a. Key: 2 ∗ |5 = 25 trucks

Leaf A	Stem	Leaf B
2 1	1	0
7 7 6 6 5	1∗	5 6
4 3 2 1 0	2	0 1 3
7 5	2∗	5 6 8 9
	3	0 1 2
	3∗	5

b. For supermarket A the mean is 19, the median is 18.5, the standard deviation is 4.9 and the interquartile range is 7. The distribution is symmetric.
For supermarket B the mean is 24.4, the median is 25.5, the standard deviation is 7.2 and the interquartile range is 10. The distribution is symmetric.
The centre and spread of the distribution of supermarket B is higher than that of supermarket A.
There is greater variation in the number of trucks arriving at supermarket B.

8. a. Key: 1|2 = 12 marks

Females	Stem	Males
	1	0
3 2	1	2 3
5 5 4 4	1	4 4 5
7 6	1	7
	1	9

b. For the marks of the females, the mean is 14.5, the median is 14.5, the standard deviation is 1.6 and the interquartile range is 2. The distribution is symmetric.
For the marks of the males, the mean is 14.25, the median is 14, the standard deviation is 2.8 and the interquartile range is 3.5. The distribution is symmetric.
The centre of each distribution is about the same. The spread of marks for the boys is greater, however. This means that there is a wider variation in the abilities of the boys compared to the abilities of the girls.

9. a. Key: $2* | 6 = 26$ marks

2011	Stem	2012
	2	2
	2*	6 7 8
1 0	3	0 1 1 3 4
9 7 5	3*	6
3 2 1 1	4	
6	4*	

b. The distribution of marks for 2011 and for 2012 are each symmetric.
For the 2011 marks, the mean is 38.5, the median is 40, the standard deviation is 5.2 and the interquartile range is 7. The distribution is symmetric.
For the 2012 marks, the mean is 29.8, the median is 30.5, the standard deviation is 4.2 and the interquartile range is 6.
The spread of each of the distributions is much the same, but the centre of each distribution is quite different with the centre of the 2012 distribution lower. The work may have become a lot harder!

10. a. Key: $3* | 6 = 36$ years old

Female	Stem	Male
4 3	2	2
8 7 6 5	2*	5
1 0	3	0 1
	3*	6 7
	4	2
	4*	6

b. For the distribution of the females, the mean is 26.75, the median is 26.5, the standard deviation is 2.8 and the interquartile range is 4.5.
For the distribution of the males, the mean is 33.6, the median is 33.5, the standard deviation is 8.2 and the interquartile range is 12.
The centre of the distributions is very different: it is much higher for the males. The spread of the ages of the females who attend the fitness class is very small but very large for males.

11. a. Key: $5 | 0 = 50$ points

Kindergarten	Stem	Prep.
3	0	5
4 3	1	2 7
8 5	2	5 7
6 2	3	2 5
7 1	4	4 6
0	5	2

b. For the distribution of scores of the kindergarten children, the mean is 28.9, the median is 30, the standard deviation is 15.4 and the interquartile range is 27.
For the distribution of scores for the prep. children, the mean is 29.5, the median is 29.5, the standard deviation is 15.3 and the interquartile range is 27.
The distributions are very similar. There is not a lot of difference between the way the kindergarten children and the prep. children scored.

12. B

13. C

14. Key: $7 | 2 = 72$

Male	Stem	Female
2	7	0 2
6	7*	
2 0	8	0 1 3 3 4
8 6	8*	8
4 4 2 0 0	9*	0 2 4
	9*	6
0	10	

15. D

16. a. Key: $3 | 1 = 31$

Mathematical Method	Stem	Further Mathematics
9 8	2*	8
4 3 1	3	0 1
9 9 6 6	3*	5 7 7 8
4 4 2 1 0	4	1 3 3 4
	4*	6 8 8

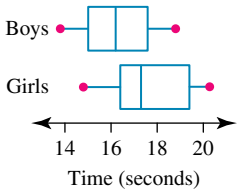
	Mathematical Methods	**Further Mathematics**
Mean	36.86	39.21
Median	37.5	39.5
IQR	$41 - 33 = 88$	$44 - 35 = 9$
Standard deviation	5.29	6.58

b. Mathematical Methods has a slightly lower IQR and standard deviation. It was found that Further Mathematics had a greater mean (39.21) as compared to Mathematical Methods (39.5), as well as a greater median; 39.5 as compared to 36.86. This suggests that students do better in Further Mathematics as compared to Mathematical Methods by an average of two study scores.

Exercise 2.5 Parallel boxplots and dot plots

Note: When comparing and contrasting data sets, answers will naturally vary. It is good practice to discuss your conclusions in a group to consider different viewpoints.

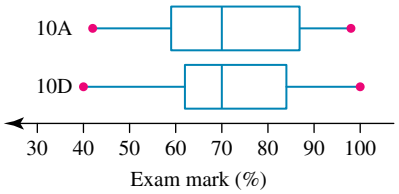
1.



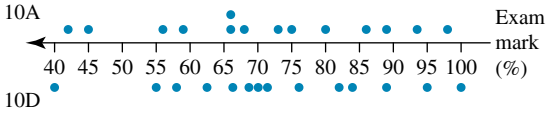
Boys: 13.8, 15, 16.1, 17.6, 18.8
Girls: 14.8, 16.4, 17.3, 19.5, 20.3

From the boxplots we can see that the boys have a significantly lower median. The boys' median is lower than Q_1 of the girls' time; that is; the lowest 25% of times for the girls is greater than the lowest 50% of times for the boys.

2.

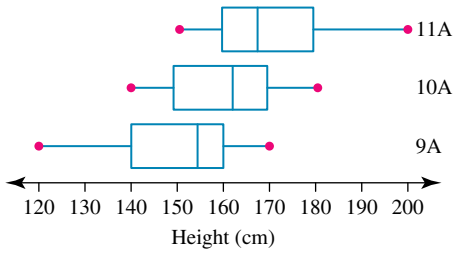


10A: 42, 59, 70.5, 87, 98
10D: 40, 62, 70.5, 84, 100



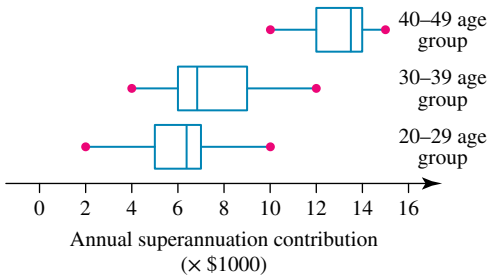
From the boxplots you can see the medians are the same but 10D has a higher mean. 10D also has the highest score of 100%, but 10D also has the lowest score. Since Q_1 and Q_3 are closer together for 10D their results are more consistent around the median. The parallel dot plot confirms this but doesn't give you any further information.

3. a.



b. Clearly, the median height increases from Year 9 to Year 11. There is greater variation in 9A's distribution than in 10A's. There is a wide range of heights in the lower 25% of the distribution of 9A's distribution. There is a greater variation in 11A's distribution than in 10A's, with a wide range of heights in the top 25% of the 11A distribution.

4. a.



b. Clearly, there is a great jump in contributions to superannuation for people in their 40s. The spread of contributions for that age group is smaller than for people in their 20s or 30s, suggesting that a high proportion of people in their 40s are conscious of superannuation. For people in their 20s and 30s, the range is greater, indicating a range of interest in contributing to super.

5. a. True **b.** True **c.** False **d.** True

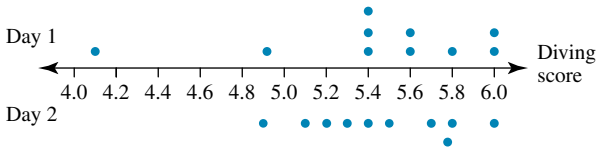
6. a. *The Pearlfishers*, which had a significantly higher medium number of A-reserve tickets sold, as well as a higher minimum and maximum number of A-reserve tickets sold.

b. *Orlando*, which had both a larger range and IQR of A-reserve tickets sold.

7. D

8. C

9.



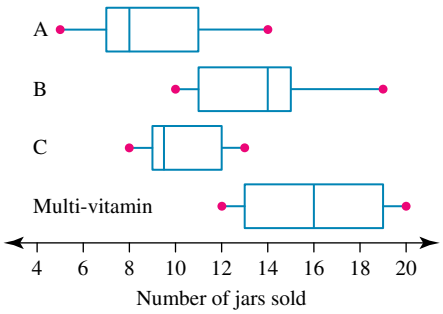
The dives on day 1 were more consistent than the dives on day 2 with most of the dives between 5.4 and 6.0 (inclusive), despite two lower dives. Day 2 was more spread with dives from 4.9 to 6.0 (inclusive). It must be noted that there were no very low scoring dives on the second day.

10. B

11. B

12. C

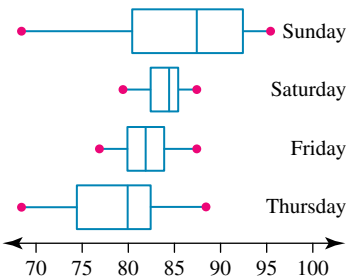
13.



Overall, the biggest sales were of multi-vitamins, followed by vitamin B, then C and finally vitamin A.

14. For all four days, the median is the 6th score.
For all four days, Q_1 is the 3rd score. For all four days, Q_3 is the 9th score.

Day	Min.	Max.	Range	Median
Thursday	70	90	20	81
Friday	77	89	12	83
Saturday	81	89	8	86
Sunday	70	94	24	89



Exercise 2.6 Scatterplots

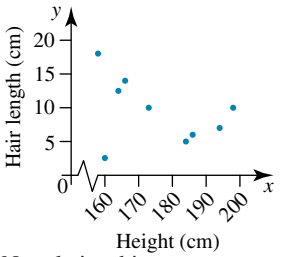
1. Moderate positive linear relationship

2. Moderate positive linear relationship

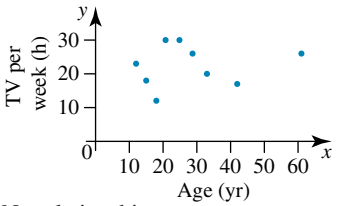
3. a. Yes — positive association
b. Yes — positive association
c. Yes — positive association
d. Yes — negative association
e. Yes — positive association
f. Yes — negative association
g. No — no association

4. **a.** Weak, negative association of linear form
b. Moderate, negative association of linear form
c. Moderate, positive association of linear form
d. Strong, positive association of linear form
e. No association
f. Non-linear association

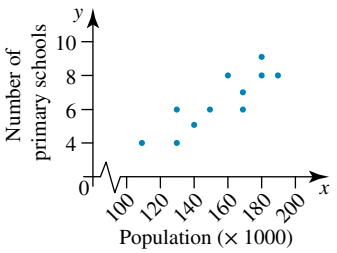
5. B

6.


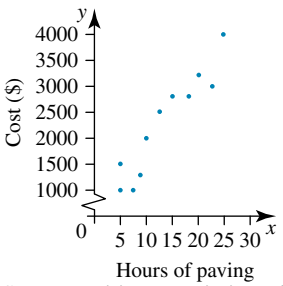
No relationship

7.


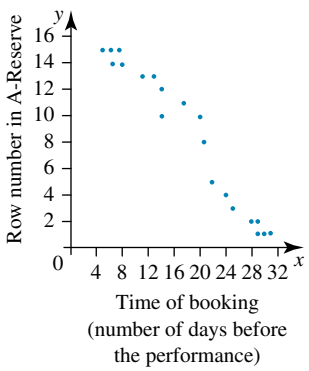
No relationship

8.


Moderate positive association of linear form, no outliers

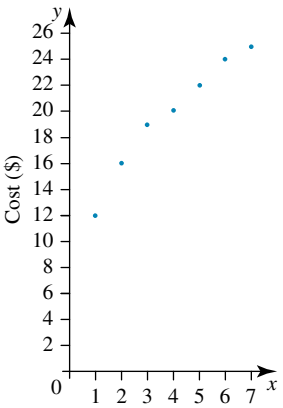
9.


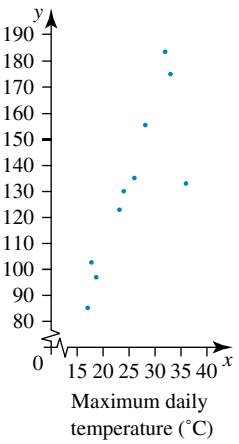
Strong positive association of linear form, no outliers

10.


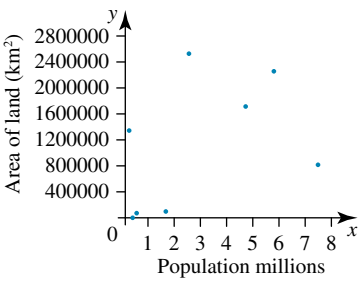
Strong negative association of linear form, no outliers

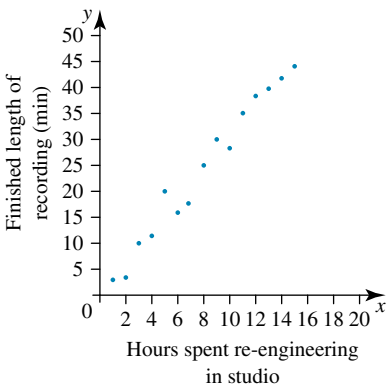
11. D

12. **a, b**


13. **a, b**


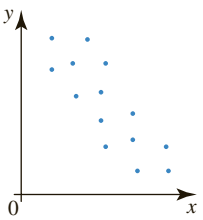
14. **a.** There is a strong positive correlation between the dry cleaning items and the cost.
b. There is a strong positive correlation between the maximum daily temperature and the number of drinks sold.

15. There appears to be no correlation between population and area of the various states and territories.


16. **a.**


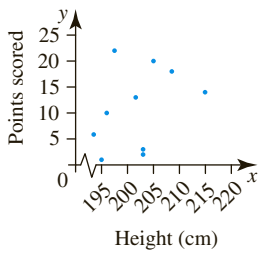
b. There is a relationship. It is strong, positive and linear.

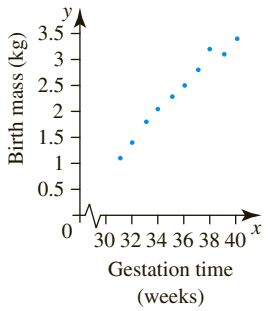
Exercise 2.7 Pearson's product–moment correlation coefficient

1. a. i. $r = -0.9$
   ii. Strong, negative, linear
   b. i. $r = 0.7$
   ii. Strong, positive, linear
2. a. Strong, positive, linear
   b. Weak, negative, linear
3. a. No association
   b. Moderate positive
   c. Strong negative
   d. Strong negative
4. a. Strong positive
   b. Strong positive
   c. Weak negative
   d. No association
5. a. i. $r \approx -0.8$
   ii. Strong, negative, linear association
   b. i. $r \approx 0.6$
   ii. Moderate, positive, linear association
   c. i. $r \approx 0.2$
   ii. No linear association
   d. i. $r \approx -0.2$
   ii. No linear association
6. a. i. $r \approx 1$
   ii. Perfect, positive, linear association
   b. i. $r \approx 0.8$
   ii. Strong, positive, linear association
   c. i. $r \approx 0$
   ii. No linear association
   d. i. $r \approx -0.7$
   ii. Moderate, negative, linear association
7. B
8. E
9. a. $r = 0.1$: no linear relationship
   b. $r = 0.2$: no linear relationship
   c. $r = 0.95$: strong, positive linear relationship
10. D
11. C
12. 

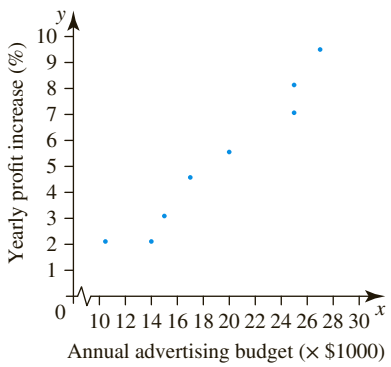
13. E
14. C

Exercise 2.8 Calculating r and the coefficient of determination, r^2

1. a. 

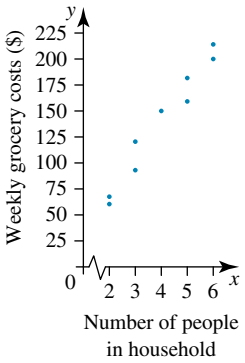
   b. No linear relationship, $r \approx 0.3$
   c. $r \approx 0.36$, weak, positive linear relationship
2. a. 

   b. Strong, positive, linear relationship. $r \approx 0.95$
   c. $r \approx 0.99$, very strong linear relationship
3. a.

   b. There is moderate, negative linear association. r is approximately -0.6.
   c. $r = -0.66$. There is a moderate negative linear association between the yearly salary and the number of votes. That is, the larger the yearly salary of the player, the fewer the number of votes we might expect to see.
4. a. Coefficient of determination = 0.59
   We can then conclude that 59% of the variation in the number of people found to have a blood alcohol reading over 0.05 can be explained by the variation in the number of booze buses in use. Thus the number of booze buses in use is a factor in predicting the number of drivers with a reading over 0.05.
5. Coefficient of determination is 0.7569. The portion of variation in the number of visits to the doctor that can be explained by the variation in the number of cigarettes smoked is about 76%.
6. a. 

   b. There is strong, positive linear association. r is approximately 0.8.
   c. $r = 0.98$
   d. Coefficient of determination is $r = 0.96$.
   e. The proportion of the variation in the yearly profit increase that can be explained by the variation in the advertising budget is 96%.

7. a.



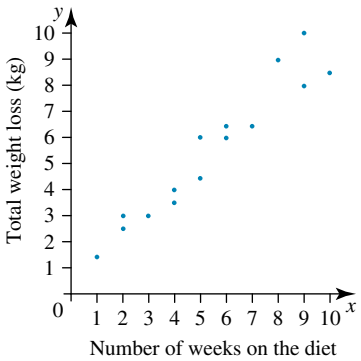
b. There is strong, positive association of a linear form and r is approximately 0.9.

c. $r = 0.98$

d. Coefficient of determination is 0.96.

e. The proportion of the variation in the weekly grocery costs that can be explained by the variation in the number of people in the household is 96%.

8. a.



b. The scatterplot shows strong, positive association of linear form.

c. It is appropriate since the scatterplot indicates association showing linear form and there are no outliers.

d. $r \approx 0.9$

e. $r = 0.96$

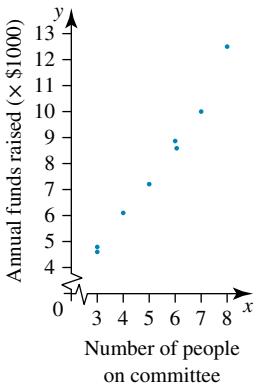
f. We cannot say whether total weight loss is affected by the number of weeks people stayed on the Certain Slim diet. We can only note the degree of correlation.

g. $r^2 = 0.92$

h. The coefficient of determination tells us that 92% of the variation in total weight loss can be explained by the variation in the number of weeks on the Certain Slim diet.

Exercise 2.9 Correlation and causation

1. a.



b. There is almost perfect positive correlation of a linear form and r is nearly 1.

c. $r = 0.99$

d. Causation can not be established on a strong association alone.

2. There is a strong positive association between the number of books read and the spelling levels for this group of students.

3. E

4. B

5. C

6. It cannot be concluded that diabetes causes deterioration of the hip joint. The common cause could be age.

7. It cannot be concluded that holidays cause weight gain. The common cause could be any or a combination of the following: amount of exercise, alcohol consumption, diet.

8. Confounding variables could be time away from family, amount of exercise, diet

2.10 Review: exam practice

Multiple choice

1. A	2. A	3. E	4. B	5. E	6. D
7. D	8. E	9. A	10. E	11. D	12. C

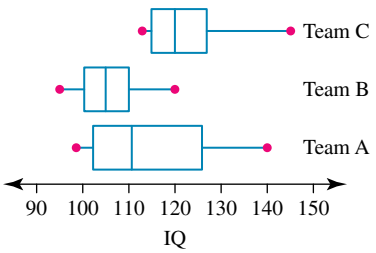
Short answer

1. a. Key: 0 | 3 = 3 hours

Full-time	Stem	Volunteer
1	0	
2 2	0	
4 4 3 3	0	
6 5	0	
	0	8
	1	0 1 1
	1	2 3 3
	1	4 5
	1	
	1	

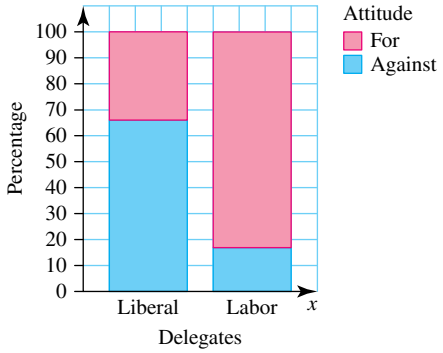
b. Both distributions are symmetric with the same spread. The centre of the volunteers' distribution is much higher than that of the full-time firefighters' distribution. Clearly, the volunteers needed more counselling.

2. a.

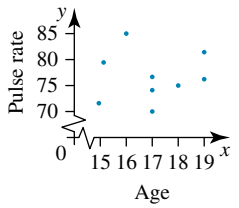


3. a.

Attitude	Liberal	Labor
For	66.7	18.9
Against	33.3	81.1
Total	100.0	100.0



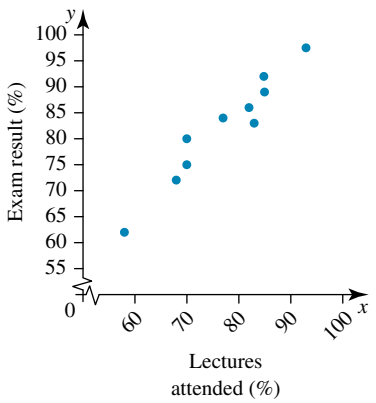
b. Clearly, the reaction to uranium mining is affected by political affiliation.

4. a.


b. There appears to be an extremely weak or no association between the variables.

5. r is approximately equal to -0.7. There is a moderate, negative linear association between the variables x and y.

6. a. The lectures attended is most likely to be the cause and the exam result is most likely to be the effect.

b.


c. There is strong, positive correlation of a linear form between the variables and r is approximately equal to 0 8.

d. $r = 0.96$

e. The coefficient of determination is 0.93.

f. The proportion of the variation in the exam results that can be explained by the variation in the number of lectures attended is 93%.

Extended response

1. a. Key 1|6 = 16 years old

Jazz concert	Stem	Symphony concert
9986	1	
97430	2	03
98430	3	059
65320	4	25578899
	5	034688
2	6	0

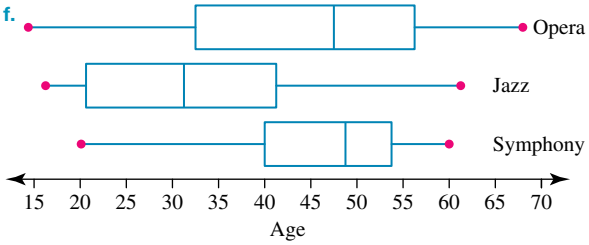
b.

	Summary	Symphony concert	Jazz concert
i.	X_{min}	20	16
ii.	Q_1	40.5	21.5
iii.	Median	48	31.5
iv.	Q_3	53.5	41
v.	X_{max}	60	62
vi.	Mean	45.45	32.35
vii.	IQR	13	19.5
viii.	Standard deviation	11.20	12.04

c. Overall, it appears that people who attended the symphony concert were older than those who attended the jazz concert. The spread of ages is nearly the same (slightly higher for the jazz audience).

d. Back-to-back stem plots can be used only for data with two categories. Since there are three events, parallel boxplots should be used.

e. $X_{min} = 12$, $Q_1 = 34$, median $= 47.5$, $Q_3 = 56.5$, $X_{max} = 68$, mean $= 44.95$, IQR $= 22.5$, standard deviation $= 15.55$

f.


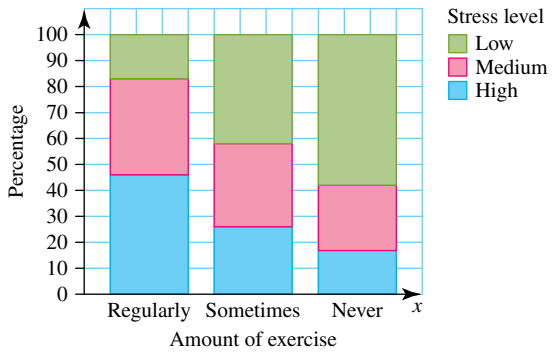
g. Overall, the people who went to the symphony concert and to the opera were of similar ages and older than those who went to the jazz concert. The ages of people who went to the opera are the most spread out, while the ages of people who attended the symphony concert are the least spread out.

2. a. 188

b. 158

c.

	Engaged in sport activity outside school		
Level of stress	**Regularly**	**Sometimes**	**Never**
Low	47.1%	25.8%	16.2%
Medium	35.3%	32.3%	25.2%
High	17.6%	41.9%	58.6%
Total	100%	100%	100%

d.


e. Overall, it appears that for this group of students, stress levels are related to the amount of physical activity they are engaged in outside of school.

TOPIC 3
Investigating and modelling linear associations

3.1 Overview

3.1.1 Introduction

A mathematical model uses mathematical concepts and language to demonstrate how something in the real-world works. These models can then be used to investigate an aspect of the natural world or society without being present in the situation. A very simple mathematical model is the use of the formula, $V = l \times w \times h$ which describes the volume of a rectangular box. Rather than creating many boxes, and measuring them to determine their volume, this formula can be used for a rectangular box of any dimensions. A more complicated model, for example, such as that required to send a rocket into space requires information about the energy needed to break through gravity, which is dependent on the size of the rocket, the starting and ending points of the journey and the energy provided by the preferred fuel. A trial and error model to determine the variables would compromise safety and waste valuable resources. Scientists set up mathematical models first and use these models to ensure the greatest chance of success.

A statistical model is a particular type of mathematical model. It is created by collecting data from a sample of the population that is under investigation. It then uses this data to create an idealised way of predicting how any future data will behave. The manufacturer of a new running shoe wants to determine the best tread thickness for maximum support. Rather than spending years measuring tread thicknesses and the extent of running injuries across the world, researchers take a sample and use the data from the sample to make predictions.

LEARNING SEQUENCE

3.1 Overview
3.2 Fitting a straight line – least-squares regression
3.3 Interpretation, interpolation and extrapolation
3.4 Residual analysis
3.5 Transforming to linearity
3.6 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.1.2 Kick off with CAS

Lines of best fit with CAS

Least-squares regression allows us to fit a line of best fit to a scatterplot. We can then use this line of best fit to make predictions about the data.

1. Using CAS, plot a scatterplot of the following data set, which indicates the temperature (x) and the number of visitors at a popular beach (y).

x	21	26	33	24	35	16	22	30	39	34	22	19
y	95	154	212	141	173	40	104	193	177	238	131	75

Are there any limitations on the data points?

2. If there appears to be a linear relationship between x and y, use CAS to add a least-squares regression line of best fit to the data set.
3. What does the line of best fit tell you about the relationship between the x- and y-values?
4. Use the line of best fit to predict y-values given the following x-values:
   a. 37 b. 24 c. 17.
5. The line of best fit can be extended beyond the limits of the original data set. Would you feel comfortable making predictions outside of the scope of the original data set?
6. Use CAS to plot scatterplots of the following sets of data, and if there appears to be a linear relationship, plot a least-squares regression line of best fit:

a.

x	62	74	59	77	91	104	79	85	55	74	90	83
y	108	83	127	90	62	55	86	70	141	92	59	77

b.

x	2.2	1.7	0.4	2.6	−0.3	1.5	3.1	1.1	0.8	2.9	0.7	−0.8
y	45	39	22	50	9	33	66	34	21	56	27	6

c.

x	40	66	38	55	47	61	34	49	53	69	43	58
y	89	112	93	90	75	106	101	77	86	120	81	99

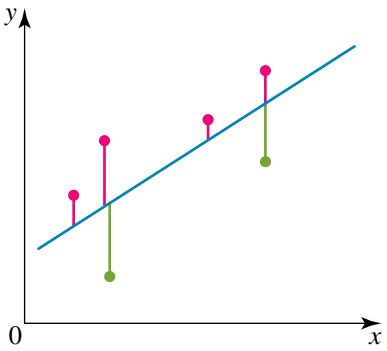
on Resources

Please refer to the Resources section in the Prelims of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

3.2 Fitting a straight line — least-squares regression

When data show a linear relationship and have no obvious outliers, a least squares line of best fit $y = a + bx$, can be found to model the data: x represents the explanatory variable and y represents the response variable.

To understand the underlying theory behind least-squares, consider the regression line shown.



We wish to minimise the total of the vertical lines, or 'errors' in some way. For example, balancing the errors above and below the line. This is reasonable, but for sophisticated mathematical reasons it is preferable to minimise the sum of the *squares* of each of these errors. This is the essential mathematics of least-squares regression.

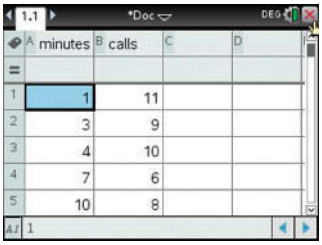
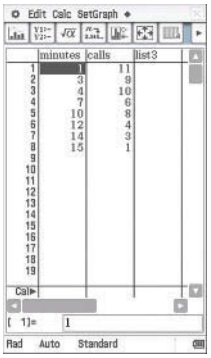
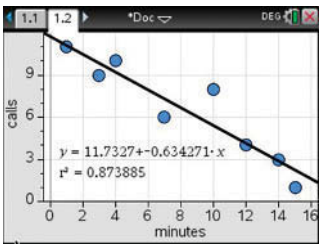
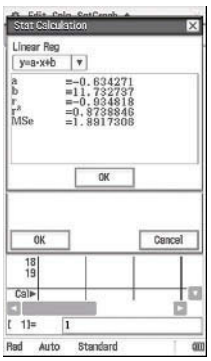
The calculation of the equation of a least-squares regression line is simple using CAS.

WORKED EXAMPLE 1

A study shows the more calls a teenager makes on their mobile phone, the less time they spend on each call. Find the equation of the linear regression line for the number of calls made plotted against call time in minutes using the least-squares method on CAS. Express coefficients correct to 2 decimal places, and calculate the coefficient of determination to assess the strength of the association.

Number of minutes	1	3	4	7	10	12	14	15
Number of calls	11	9	10	6	8	4	3	1

THINK	**WRITE**
1. Identify the explanatory and response variables.	Number of minutes – explanatory Number of calls – response
2. Enter the data into CAS to find the equation of least squares regression line.	$y = 11.7327 - 0.634\,271x$
3. Write the equation with coefficients expressed to 2 decimal places.	$y = 11.73 - 0.63x$
4. Write the equation in terms of the variable names. Replace x with number of minutes and y with number of calls.	Number of calls $= 11.73 - 0.63 \times$ no. of minutes
5. Read the r^2 value from your calculator.	$r^2 = 0.87$, so we can conclude that 87% of the variation in y can be explained by the variation in x. Therefore the strength of the linear association between y and x is strong.

TI | THINK	WRITE	CASIO | THINK	WRITE
1. Identify the explanatory and response variables.	Number of minutes: explanatory Number of calls: response	1. Identify the explanatory and response variables.	Number of minutes: explanatory Number of calls: response
2. On a Lists & Spreadsheet page, enter the data into the lists, named minutes and calls.		2. On a Statistics screen, enter the data into the lists, named minutes and calls.	
3. Press CTRL + I and select: 5: Add Data & Statistics Use the tab key to move to each axis and press the click key or ENTER to place minutes on the horizontal axis and calls on the vertical axis. Press MENU and select: 4: Analyze 6: Regression 2: Show Linear $(a + bx)$ *Note:* If the value of r^2 does not appear tick Diagnostics in Settings.		3. Tap: Calc Regression Linear reg Complete the Set Calculation fields: Linear Reg XList: main\minutes YList: main\calls Freq: 1 Copy Formula y1 Copy Residual Off Tap OK.	
4. The answer appears on the screen.	$y = 11.7327 - 0.634\,271x$	4. The answer appears on the screen.	$y = ax + b$ $y = 11.7327 - 0.634\,271x$
5. Write the equation with the coefficients expressed to 2 decimal places and replace x with the number of minutes and y with the number of calls.	No. of calls $= 11.73 - 0.63 \times$ no. of minutes	5. Write the equation with the coefficients expressed to two decimal places and replace x with the number of minutes and y with the number of calls.	No. of calls $= 11.73 - 0.63 \times$ no. of minutes
6. The answer appears on the screen.	$r^2 = 0.87$ (correct to 2 decimal places), so we can conclude that 87% of the variation in y can be explained by the variation in x. Therefore the strength of the linear association between y and x is strong.	6. The answer appears on the screen.	$r^2 = 0.87$ (correct to 2 decimal places), so we can conclude that 87% of the variation in y can be explained by the variation in x. Therefore the strength of the linear association between y and x is strong.

3.2.1 Calculating the least-squares regression line by hand

The least-squares regression equation minimises the average deviation of the points in the data set from the line of best fit. This can be shown using the following summary data and formulas to arithmetically determine the least-squares regression equation.

Summary data needed:

$\bar{x}$ the mean of the explana tory variable (x-variable)
$\bar{y}$ the mean of the response variable (y-variable)
s_x the standard deviation of the explanatory variable
s_y the standard deviation of the response variable
r Pearson's product–moment correlation coefficient.

Formula to use:

The general form of the least-squares regression line is

$$y = a + bx$$

where the slope of the regression line is $b = r\frac{s_y}{s_x}$

the y-intercept of the regression line is $a = \bar{y} - b\bar{x}$.

Alternatively, if the general form is given as $y = mx + c$, then $m = r\frac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$.

WORKED EXAMPLE 2

A study to find a relationship between the height of husbands and the height of their wives revealed the following details.
Mean height of the husbands: 180 cm
Mean height of the wives: 169 cm
Standard deviation of the height of the husbands: 5.3 cm
Standard deviation of the height of the wives: 4.8 cm
Correlation coefficient, $r = 0.85$
The form of the least-squares regression line is to be:
Height of wife $= a + b \times$ height of husband

a. Which variable is the response variable?
b. Calculate the value of b (correct to 2 significant figures).
c. Calculate the value of a (correct to 4 significant figures).
d. Use the equation of the regression line to predict the height of a wife whose husband is 195 cm tall (correct to the nearest cm).

THINK	WRITE
a. Recall that the response variable is the subject of the equation in $y = a + bx$ form; that is, y.	**a.** The response variable is the height of the wife.
b. 1. The value of b is the gradient of the regression line. Write the formula and state the required values.	**b.** $r = 0.85$, $S_y = 4.8$ and $S_x = 5.3$
2. Substitute the values into the formula and evaluate b.	$= 0.85 \times \frac{4.8}{5.3}$ $= 0.7698$ ≈ 0.77
c. 1. The value of a is the y-intercept of the regression line. Write the formula and state the required values.	**c.** $a = \bar{y} - b\bar{x}$ $\bar{y} = 169$, $\bar{x} = 180$ and $b = 0.7698$ (from part **b**)
2. Substitute the values into the formula and evaluate a.	$= 169 - 0.7698 \times 180$ $= 30.436$ ≈ 30.44
d. 1. State the equation of the regression line, using the values calculated from parts **b** and **c**. In this equation, y represents the height of the wife and x represents the height of the husband.	**d.** $y = 30.44 + 0.77x$ or height of wife $= 30.44 + 0.77 \times$ height of husband
2. The height of the husband is 195 cm, so substitute $x = 195$ into the equation and evaluate.	$= 30.44 + 0.77 \times 195$ $= 180.59$
3. Write a statement, rounding your answer correct to the nearest cm.	Using the equation of the regression line found, the wife's height would be 181 cm.

on Resources

Interactivity: Fitting a straight line using least squares regression (int-6254)

studyon

Units 3 & 4 > AOS DA > Topic 7 > Concept 1

Least-squares regression Summary screen and practice questions

Exercise 3.2 Fitting a straight line — least-squares regression

1. **WE1** A study shows that as the temperature increases the sales of air conditioners increase. Find the equation of the linear regression line for the number of air conditioners sold per week plotted against the temperature in °C using the least-squares method on CAS technology. Also find the coefficient of determination. Express the values correct to 2 decimal places and comment on the association between temperature and air conditioner sales.

Temperature °C	21	23	25	28	30	32	35	38
Air conditioner sales	3	7	8	14	17	23	25	37

2. Consider the following data set: x represents the month, y represents the number of dialysis patients treated.

x	1	2	3	4	5	6	7	8
y	5	9	7	14	14	19	21	23

Using CAS find the equation of the linear regression line and the coefficient of determination, with values correct to 2 decimal places.

3. Find the equation of the linear regression line for the following data set using the least-squares method, and comment on the strength of the association.

x	4	6	7	9	10	12	15	17
y	10	8	13	15	14	18	19	23

4. Find the equation of the linear regression line for the following data set using the least-squares method.

x	1	2	3	4	5	6	7	8	9
y	35	28	22	16	19	14	9	7	2

5. Find the equation of the linear regression line for the following data set using the least-squares method.

x	−4	−2	−1	0	1	2	4	5	5	7
y	6	7	3	10	16	9	12	16	11	21

6. **WE2** A study was conducted to find the relationship between the height of Year 12 boys and the height of Year 12 girls. The following details were found.
   Mean height of the boys: 182 cm
   Mean height of the girls: 166 cm
   Standard deviation of the height of boys: 6.1 cm
   Standard deviation of the height of girls: 5.2 cm
   Correlation coefficient, $r = 0.82$
   The form of the least squares regression line is to be:

$$\text{Height of Year 12 boy} = a + b \times \text{height of Year 12 girl}$$

   a. Which is the explanatory variable?
   b. Calculate the value of b (correct to 2 significant figures).
   c. Calculate the value of a (correct to 2 decimal places).

7. Given the summary details $\bar{x} = 4.4$, $s_x = 1.2$, $\bar{y} = 10.5$, $s_y = 1.4$ and $r = -0.67$, find the value of b and a for the equation of the regression line $y = a + bx$.

8. The following summary details were calculated from a study to find a relationship between mathematics exam marks and English exam marks from the results of 120 Year 12 students.

Mean mathematics exam mark = 64%
Mean English exam mark = 74%
Standard deviation of mathematics exam mark = 14.5%
Standard deviation of English exam mark = 9.8%
Correlation coefficient, $r = 0.64$

The form of the least-squares regression line is to be:

$$\text{Mathematics exam mark} = a + b \times \text{English exam mark}$$

a. Which variable is the response variable (y-variable)?
b. Calculate the value of b for the least-squares regression line (correct to 2 decimal places).
c. Calculate the value of a for the least-squares regression line (correct to 2 decimal places).
d. Use the regression line to predict the expected mathematics exam mark if a student scores 85% in an English exam (correct to the nearest percentage).

9. Find the least-squares regression equations, given the following summary data.

a. $\bar{x} = 5.6$ $s_x = 1.2$ $\bar{y} = 110.4$ $s_y = 5.7$ $r = 0.7$
b. $\bar{x} = 110.4$ $s_x = 5.7$ $\bar{y} = 5.6$ $s_y = 1.2$ $r = -0.7$
c. $\bar{x} = 25$ $s_x = 4.2$ $\bar{y} = 10200$ $s_y = 250$ $r = 0.88$
d. $\bar{x} = 10$ $s_x = 1$ $\bar{y} = 20$ $s_y = 2$ $r = -0.5$

10. Repeat questions **3**, **4** and **5**, collecting the values for $\bar{x}$, s_x, $\bar{y}$, s_y and r from CAS. Use these data to find the least-squares regression equation. Compare your answers to the ones obtained earlier from questions **3**, **4** and **5**. What do you notice?

11. A mathematician is interested in the behaviour patterns of her kitten, and collects the following data on two variables. Help her manipulate the data.

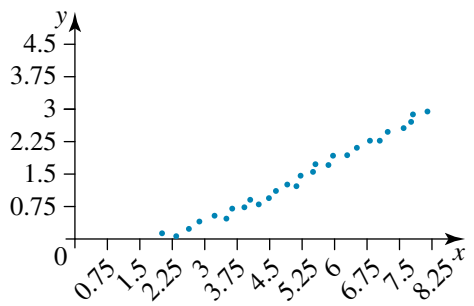
x	1	2	3	4	5	6	7	8	9	10
y	20	18	16	14	12	10	8	6	4	2

a. Fit a least-squares regression line.
b. Comment on any interesting features of this line.
c. Now fit the 'opposite regression line', namely:

x	20	18	16	14	12	10	8	6	4	2
y	1	2	3	4	5	6	7	8	9	10

12. MC The best estimate of the least-squares regression line for the scatterplot is:

A. $y = 2x$
B. $y = \frac{1}{2}x$
C. $y = 2 + \frac{1}{2}x$
D. $y = -2 + \frac{1}{2}x$
E. $y = -1 + \frac{1}{2}x$



13. The life span of adult males in a certain country over the last 220 years has been recorded.

Year	1780	1800	1820	1840	1860	1880	1900	1920	1940	1960	1980	2000
Life span (years)	51.2	52.4	51.7	53.2	53.1	54.7	59.9	62.7	63.2	66.8	72.7	79.2

a. Fit a least-squares regression line to these data.
b. Plot the data and the regression line on a scatterplot.
c. Do the data really look linear? Discuss.

14. The price of a long distance telephone call changes as the duration of the call increases. The cost of a sample of calls from Melbourne to Slovenia are summarised in the table.

Cost of call ($)	1.25	1.85	2.25	2.50	3.25	3.70	4.30	4.90	5.80
Duration of call (seconds)	30	110	250	260	300	350	420	500	600

Costof call ($)	7.50	8.00	9.25	10.00	12.00	13.00	14.00	16.00	18.00
Duration of call (seconds)	840	1000	1140	1200	1500	1860	2400	3600	7200

a. What is the explanatory variable likely to be?
b. Fit a least-squares regression line to the data.
c. View the data on a scatterplot and comment on the reliability of the regression line in predicting the cost of telephone calls. (That is, consider whether the regression line you found proves that costs of calls and duration of calls are related.)
d. Calculate the coefficient of determination and comment on the linear relationship between duration and cost of call.

15. MC In a study to find a relationship between the height of plants and the hours of daylight they were exposed to, the following summary details were obtained.
Mean height of plants = 40 cm
Mean hours of daylight = 8 hours
Standard deviation of plant height = 5 cm
Standard deviation of daylight hours = 3 hours
Pearson's correlation coefficient = 0.9
The most appropriate regression equation is:

A. height of plant (cm) $= -13.6 + 0.54 \times$ hours of daylight
B. height of plant (cm) $= -8.5 + 0.34 \times$ hours of daylight
C. height of plant (cm) $= 2.1 + 0.18 \times$ hours of daylight
D. height of plant (cm) $= 28.0 + 1.50 \times$ hours of daylight
E. height of plant (cm) $= 35.68 + 0.54 \times$ hours of daylight

16. Consider the following data set.

x	1	2	3	4	5	6
y	12	16	17	21	25	29

a. Perform a least-squares regression on the first two points only.
b. Now add the 3rd point and repeat.
c. Repeat for the 4th, 5th and 6th points.
d. Comment on your results.

3.3 Interpretation, interpolation and extrapolation

3.3.1 Interpreting slope and intercept (*b* and *a*)

Once you have a linear regression line, the **slope** and **intercept** can give important information about the data set.

The slope (b) indicates the change in the response variable as the explanatory variable increases by 1 unit.
The y-intercept indicates the value of the response variable when the explanatory variable = 0.

WORKED EXAMPLE 3

In the study of the growth of a species of bacterium, it is assumed that the growth is linear. However, it is very expensive to measure the number of bacteria in a sample. Given the data listed, find:

a. the equation, describing the relationship between the two variables

b. the rate at which bacteria are growing

c. the number of bacteria at the start of the experiment.

Day of experiment	1	4	5	9	11
Number of bacteria	500	1000	1100	2100	2500

THINK

a. 1. Find the equation of the least-squares regression line using CAS.

2. Replace x and y with the variables in question.

b. The rate at which bacteria are growing is given by the gradient of the least-squares regression.

c. The number of bacteria at the start of the experiment is given by the y-intercept of the least-squares regression line.

WRITE

a. $y = a + bx$

Number of bacteria = 202.5 + 206.25 × day of experiment

b. b is 206.25, hence on average, the number of bacteria increases by approximately 206 per day.

c. The y-intercept is 202.5, hence the initial number of bacteria present was approximately 203.

3.3.2 Interpolation and extrapolation

As we have already observed, any linear regression method produces a linear equation in the form:

$$y = a + bx$$

where b is the gradient and a is the y-intercept.

This equation can be used to 'predict' the y-value for a given value of x. These are only approximations, since the regression line itself is only an estimate of the 'true' relationship between the bivariate data. However, they can still be used, in some cases, to provide additional information about the data set (that is, make predictions).

There are two types of prediction: **interpolation** and **extrapolation**.

Interpolation

Interpolation is the use of the regression line to predict values within the range of data in a set, that is, the values that are *in between the values* already in the data set. If the data are highly linear (r near $+1$ or -1) then we can be confident that our interpolated value is quite accurate. If the data are not highly linear (r near 0) then our confidence is duly reduced. For example, medical information collected from a patient every third day would establish data for day 3, 6, 9, … and so on. After performing regression analysis, it is likely that an interpolation for day 4 would be accurate, given a high r value.

Extrapolation

Extrapolation is the use of the regression line to predict values outside the range of data in a set, that is, values that are *smaller than the smallest value* already in the data set or *larger than the largest value*.

Two problems may arise in attempting to extrapolate from a data set. Firstly, it may not be reasonable to extrapolate too far away from the given data values. For example, suppose there is a weather data set for 5 days. Even if it is highly linear (r near $+1$ or -1) a regression line used to predict the same data 15 days in the future is highly risky. Weather has a habit of randomly fluctuating and patterns rarely stay stable for very long.

Secondly, the data may be highly linear in a narrow band of the given data set. For example, there may be data on stopping distances for a train at speeds of between 30 and 60 km/h. Even if they are highly linear in this range, it is unlikely that things are similar at very low speeds (0–15 km/h) or high speeds (over 100 km/h).

Generally, one should feel *more confident about the accuracy of a prediction derived from interpolation* than one derived from extrapolation. Of course, it still depends upon the correlation coefficient (r). The closer to linearity the data are, the more confident our predictions in all cases.

Interpolation is the use of the regression line to predict values within the range of data in a set.

Extrapolation is the use of the regression line to predict values outside the range of data in a set.

WORKED EXAMPLE 4

Using interpolation and the following data set, predict the height of an 8-year-old girl.

Age (years)	1	3	5	7	9	11
Height (cm)	60	76	115	126	141	148

THINK

1. Find the equation of the least-squares regression line using your calculator. (Age is the explanatory variable and height is the response variable.)

WRITE

$y = 55.63 + 9.23x$

2. Replace x and y with the variables in question.	Height $= 55.63 + 9.23 \times$ age
3. Substitute 8 for age into the equation and evaluate.	When age $= 8$, Height $= 55.63 + 9.23 \times 8$ $= 129.5$ (cm)
4. Write the answer.	At age 8, the predicted height is 129.5 cm.

WORKED EXAMPLE 5

Use extrapolation and the data from Worked example 4 to predict the height of the girl when she turns 15. Discuss the reliability of this prediction.

THINK	WRITE
1. Use the regression equation to calculate the girl's height at age 15.	Height $= 55.63 + 9.23 \times$ age $= 55.63 + 9.23 \times 15$ $= 194.08$ cm
2. Analyse the result.	Since we have extrapolated the result (that is, since the greatest age in our data set is 11 and we are predicting outside the data set) we cannot claim that the prediction is reliable.

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Units 3 & 4 > AOS DA > Topic 7 > Concepts 2 and 3

Interpretation of slope and intercepts Summary screen and practice questions
Interpretation and extrapolation Summary screen and practice questions

Exercise 3.3 Interpretation, interpolation and extrapolation

1. WE3 A study on the growth in height of a monkey in its first six months is assumed to be linear. Given the data shown, find:
   a. the equation, describing the relationship between the two variables
   b. the rate at which the monkey is growing
   c. the height of the monkey at birth.

Month from birth	1	2	3	4	5	6
Height (cm)	15	19	23	27	30	32

2. The outside temperature is assumed to increase linearly with time after 6 am. Given the data shown, find:
   a. the equation, describing the relationship between the two variables
   b. the rate at which the temperature is increasing
   c. the temperature at 6 am.

Hours after 6 am	0.5	1.5	3	3.5	5
Temperature (°C)	15	18	22	23	28

3. A drug company wishes to test the effectiveness of a drug to increase red blood cell counts in people who have a low count. The following data are collected.

Day of experiment	4	5	6	7	8	9
Red blood cell count	210	240	230	260	260	290

   Find:
   a. the equation, describing the relationship between the variables in the form $y = a + bx$
   b. the rate at which the red blood cell count was changing
   c. the red blood cell count at the beginning of the experiment (that is, on day 0).

4. A wildlife exhibition is held over 6 weekends and features still and live displays. The number of live animals that are being exhibited varies each weekend. The number of animals participating, together with the number of visitors to the exhibition each weekend, is as shown.

Number of animals	6	4	8	5	7	6
Number of visitors	311	220	413	280	379	334

   Find:
   a. the rate of increase of visitors as the number of live animals is increased by 1
   b. the predicted number of visitors if there are no live animals.

5. An electrical goods warehouse produces the following data showing the selling price of electrical goods to retailers and the volume of those sales.

Selling price ($)	60	80	100	120	140	160	200	220	240	260
Sales volume (×1000)	400	300	275	250	210	190	150	100	50	0

   Perform a least-squares regression analysis and discuss the meaning of the gradient and y-intercept.

6. **WE4** Using interpolation and the following data set, predict the height of a 10-year-old boy.

Age (years)	1	3	4	8	11	12
Height (cm)	65	82	92	140	157	165

7. Using interpolation and the following data set, predict the length of Matt's pet snake when it is 15 months old.

Age (months)	1	3	5	8	12	18
Length (cm)	48	60	71	93	117	159

8. **WE5** Use extrapolation and the data from question **6** to predict the height of the boy when he turns 16. Discuss the reliability of this prediction.
9. Use extrapolation and the data from question **7** to predict the length of Matt's pet snake when it is 2 years old. Discuss the reliability of the prediction.

10. A study of the dining-out habits of various income groups in a particular suburb produces the results shown in the table.

Weekly income ($)	100	200	300	400	500	600	700	800	900	1000
Number of restaurant visits per year	5.8	2.6	1.4	1.2	6	4.8	11.6	4.4	12.2	9

Use the data to predict:
   a. the number of visits per year by a person on a weekly income of $680
   b. the number of visits per year by a person on a weekly income of $2000.

11. Fit a least-squares regression line to the following data.

x	0	1	2	4	5	6	8	10
y	2	3	7	12	17	21	27	35

Find:
   a. the regression equation
   b. y when $x = 3$
   c. y when $x = 12$
   d. x when $y = 7$
   e. x when $y = 25$.
   f. Which of **b** to **e** above are extrapolations?

12. The following table represents the costs for shipping a consignment of shoes from Melbourne factories. The cost is given in terms of distance from Melbourne. There are two factories that can be used. The data are summarised in the table.

Distance from Melbourne (km)	10	20	30	40	50	60	70	80
Factory 1 cost ($)	70	70	90	100	110	120	150	180
Factory 2 cost ($)	70	75	80	100	100	115	125	135

   a. Find the least-squares regression equation for each factory.
   b. Which factory is likely to have the lowest cost to ship to a shop in Melbourne (i.e. distance from Melbourne = 0 km)?
   c. Which factory is likely to have the lowest cost to ship to Mytown, 115 kilometres from Melbourne?
   d. Which factory has the most 'linear' shipping rates?

13. A factory produces calculators. The least-squares regression line for cost of production (C) as a function of numbers of calculators (n) produced is given by:

$$C = 600 + 7.76n$$

Furthermore, this function is deemed accurate when producing between 100 and 1000 calculators.
   a. Find the cost to produce 200 calculators.
   b. How many calculators can be produced for $2000?
   c. Find the cost to produce 10 000 calculators.
   d. What are the 'fixed' costs for this production?
   e. Which of **a** to **c** above is an interpolation?

14. A study of the relationship between IQ and results in a mathematics exam produced the following results. Unfortunately, some of the data were lost. Copy and complete the table by using the least-squares equation with the data that were supplied.
Note: Only use (x, y) pairs if both are in the table.

IQ	80		92	102	105		107	111	115	121
Test result (%)	56	60	68	65		74	71	73		92

15. The least-squares regression line for a starting salary (s) as a function of number of years of schooling (n) is given by the rule: $s = 37\,000 + 1800n$.
- **a.** Find the salary for a person who completed 10 years of schooling.
- **b.** Find the salary for a person who completed 12 years of schooling.
- **c.** Find the salary for a person who completed 15 years of schooling.
- **d.** Mary earned $60 800. What was her likely schooling experience?
- **e.** Discuss the reasonableness of predicting salary on the basis of years of schooling.

16. Fit a least-squares regression to the following data.

q	0	1	3	7	10	15
r	12	18	27	49	64	93

Find:
- **a.** the regression equation.
- **b.** r when $q = 4$
- **c.** r when $q = 18$
- **d.** q when $r = 100$.
- **e.** Which of **b** to **d** is extrapolation?

17. A plumbing company's charges follow the least-squares regression line:

$$C = 180 + 80n$$

where C is the total cost and n is the number of hours of work. This function is accurate for a single 8 hour day.

- **a.** Find the total cost if the plumber worked for 3 hours.
- **b.** If the total charge was $1250, how long did the plumber work, correct to 2 decimal places?
- **c.** Find the total cost if the plumber worked 9 hours and 30 minutes.
- **d.** What is the call-out fee (the cost to come out before they start any work)?
- **e.** Which of **a** to **c** is an extrapolation?

18. A comparison was investigated between AFL memberships sold and the amount of money spent on advertising by the club.

Advertising (in millions $)	2.3	1.8	1.2	0.8	1.6	0.6	1.0
Members	81 363	67 947	58 846	55 597	62 295	54 946	57 295

- **a.** Find the least-squares regression equation. Round the coefficient to the nearest whole number.
- **b.** Using the least-squares equation if $2 million was spent on advertising, how many members would you expect to have? Is this extrapolation or interpolation?
- **c.** If you wanted 70 000 members, how much would you expect to have to pay on advertising?
- **d.** Calculate the coefficient of determination and use it to explain the association between membership numbers and the amount of money spent on advertising.

3.4 Residual analysis

There are situations where the mere fitting of a regression line to some data is not enough to convince us that the data set is *truly* linear. Even if the correlation is close to +1 or −1 it still may not be convincing enough.

A residual plot is a way to test the assumption that there is a linear relationship between the variables. A residual is the vertical difference between each data point and the regression line.

3.4.1 Calculating residuals

A sociologist gathers data on the heights of brothers and sisters in families from different cultural backgrounds. He enters his records in the table shown.

x	2	3	5	8	9	9
y	3	7	12	10	12	16

He then plots each point, and fits a regression line as shown in Figure 1, which follows. He then decides to calculate the residuals.

The *residuals* are simply the vertical distances from the line to each point. These lines are shown as green and pink bars in Figure 2.

Figure 1

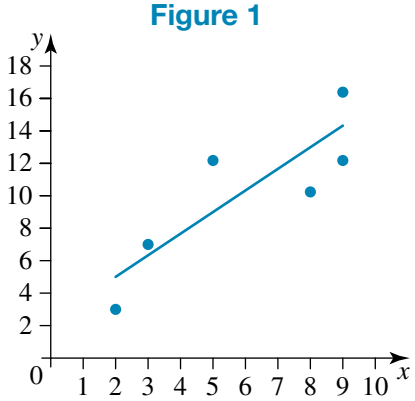
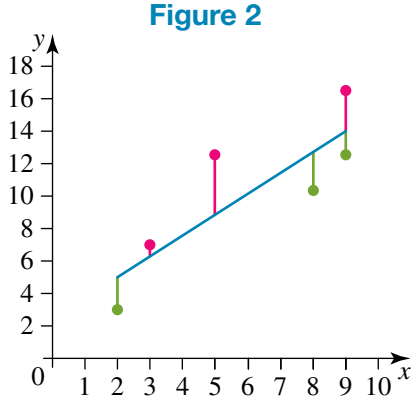


Figure 2



Finally, he calculates the residuals for each data point. This is done in two steps.

Step 1. He calculates the *predicted* value of y by using the regression equation.

Step 2. He calculates the *difference* between this predicted value and the original value.

Residual value = actual y-value − predicted y-value

WORKED EXAMPLE 6

Consider the data set shown. Find the equation of the least-squares regression line and calculate the residuals.

x	1	2	3	4	5	6	7	8	9	10
y	5	6	8	15	24	47	77	112	187	309

THINK	WRITE
1. Find the equation of a least-squares regression line using a calculator.	$y = -78.7 + 28.7x$
2. Use the equation of the least-squares regression line to calculate the predicted y-values (these are labelled as y_{pred}) for every x-value in the table. That is, substitute each x-value into the equation and evaluate record results in the table.	
3. Calculate residuals for each point by subtracting predicted y-values from the actual y-value. (That is, residual = observed y-value − predicted y-value.) Record results in the table.	

x-values	1	2	3	4	5
y-values	5.0	6.0	8.0	15.0	24.0
Predicted y-values	−50.05	−21.38	7.3	35.98	64.66
Residuals $(y - y_{pred})$	55.05	27.38	0.7	−20.98	−40.66

x-values	6	7	8	9	10
y-values	47.0	77.0	112.0	187.0	309.0
Predicted y-values	93.34	122.02	150.7	179.38	208.06
Residuals $(y - y_{pred})$	−46.34	−45.02	−38.7	7.62	100.94

Notes

1. The residuals may be determined by $(y - y_{pred})$; that is, the actual values minus the predicted values.
2. The *sum* of all the residuals *always* adds to 0 (or very close to 0 after rounding), when least-squares regression is used. This can act as a check for our calculations.

3.4.2 Introduction to residual analysis

As we observed in Worked example 6, there is not really a good fit between the data and the least-squares regression line; however, there seems to be a pattern in the residuals. How can we observe this pattern in more detail?

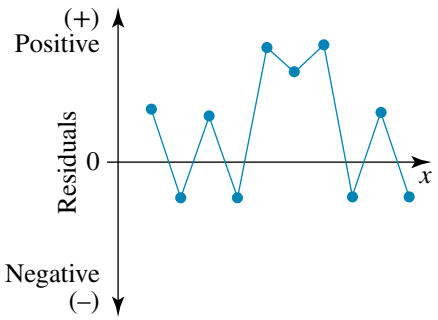
The answer is to plot the *residuals* themselves against the *original x-values*. If there is a pattern, it should become clearer after they are plotted.

Types of residual plots

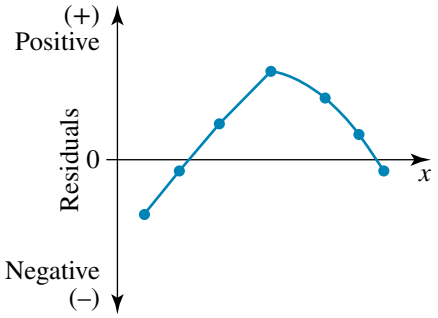
There are three basic types of **residual plots**. Each type indicates whether or not a linear relationship exists between the two variables under investigation.

Note: The points are joined together to see the patterns more clearly.

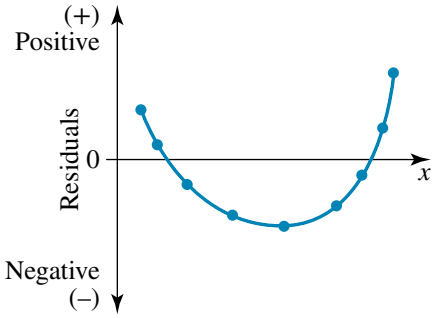
The points of the residuals are randomly scattered above and below the x-axis. The original data probably have a *linear* relationship.



The points of the residuals show a curved pattern (∩), with a series of negative, then positive and back to negative residuals along the x-axis. The original data probably have a **non-linear relationship**. Transformation of the data may be required.



The points of the residuals show a curved pattern (∪), with a series of positive, then negative and back to positive residuals along the x-axis. The original data probably have a *non-linear* relationship. Transformation of the data may be required.



The transformation of data suggested in the last two residual plots will be studied in more detail in the next section.

WORKED EXAMPLE 7

Using the same data as in Worked example 6, plot the residuals and discuss the features of the residual plot.

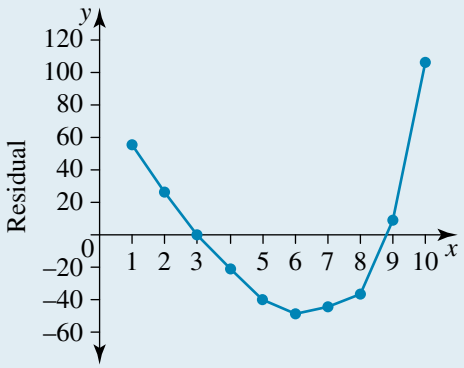
THINK

1. Generate a table of values of residuals against x.

WRITE/DRAW

x-values	1	2	3	4	5
Residuals ($y - y_{pred}$)	55.05	27.38	0.7	−20.98	−40.66
x-values	6	7	8	9	10
Residuals ($y - y_{pred}$)	−46.34	−45.02	−38.7	7.62	100.94

2. Plot the residuals against x. To see the pattern clearer, join the consecutive points with straight line segments.



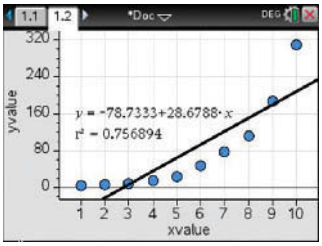
3. If the relationship was linear the residuals would be scattered randomly above and below the line. However, in this instance there is a pattern which looks somewhat like a parabola. This should indicate that the data were not really linear, but were more likely to be quadratic. Comment on the residual plot and its relevance.

The residual plot indicates a distinct pattern suggesting that a non-linear model could be more appropriate.

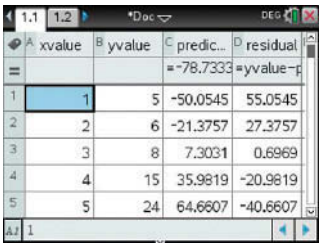
TI | THINK

1. On a Lists & Spreadsheet page, enter the data into the lists named *xvalue* and *yvalue*.
Construct a scatterplot and determine the equation of the least squares regression line.

WRITE



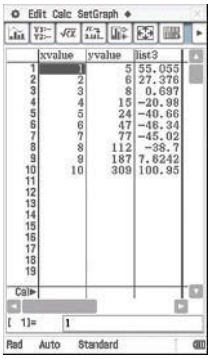
2. Return to the Lists & Spreadsheets page.
Name Column C predicted and Column D residual.
In the formula cell under the heading predicted, complete the entry line as:
$= -78.7333 + 28.6788 \times xvalue$
Press ENTER.
In the formula cell under the heading residual, complete the entry line as:
$= yvalue - predicted$
Press ENTER.



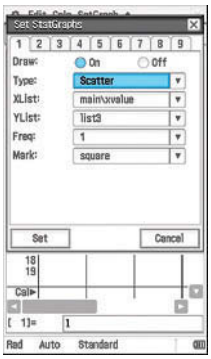
CASIO | THINK

1. On a Statistics screen, enter the data into the lists named *xvalue* and *yvalue*.
Tap:
Calc
Regression
Linear reg
Complete the Set Calculation fields:
Linear Reg
XList: main*xvalue*
YList: main*yvalue*
Freq: 1
Copy Formula y1
Copy Residual list3
Tap OK.

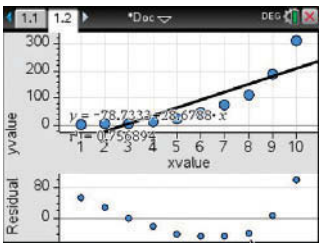
WRITE



2. Tap:
SetGraph
Setting
On the Set StatGraphs screen, select the following:
XList: main\xvalue
YList: list3
Freq: 1
Mark: square
Tap Set.



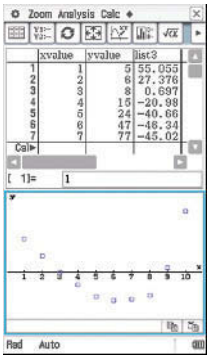
3. Return to the Data and Statistics page.
Press:
MENU
4: Analyze
7: Residuals
2: Show residual plot



4. If the relationship was linear the residuals would be scattered randomly above and below the line. However, in this instance there is a pattern which looks somewhat like a parabola. This should indicate that the data were not really linear, but were more likely to be quadratic. Comment on the residual plot and its relevance.

The residual plot indicates a distinct pattern suggesting that a non-linear model could be more appropriate.

3. Tap the graph icon.



4. If the relationship was linear the residuals would be scattered randomly above and below the line. However, in this instance there is a pattern which looks somewhat like a parabola. This should indicate that the data were not really linear, but were more likely to be quadratic. Comment on the residual plot and its relevance.

The residual plot indicates a distinct pattern suggesting that a non-linear model could be more appropriate.

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Units 3 & 4 > AOS DA > Topic 7 > Concepts 5 and 6

Residual analysis Summary screen and practice questions
Residual plots Summary screen and practice questions

Exercise 3.4 Residual analysis

1. WE6 Consider the data set shown. Find the equation of the least-squares regression line and calculate the residuals.

x	1	2	3	4	5	6	7	8	9
y	12	20	35	40	50	67	83	88	93

2. From the data shown, find the equation of the least-squares regression line and calculate the residuals.

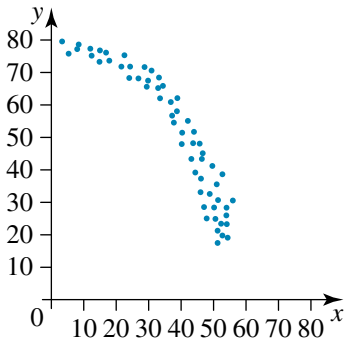
x	5	7	10	12	15	18	25	30	40
y	45	61	89	122	161	177	243	333	366

3. Find the residuals for the following data.

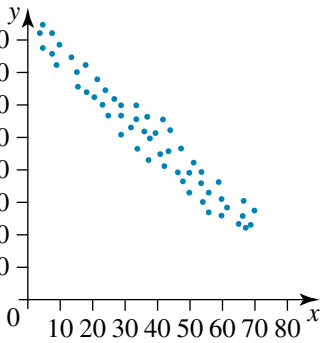
x	1	2	3	4	5	6
y	1	9.7	12.7	13.7	14.4	14.5

4. **WE7** Using the same data from question **1**, plot the residuals and discuss the features of the residual plot. Is your result consistent with the coefficient of determination?
5. Using the same data from question **2**, plot the residuals and discuss the features of the residual plot. Is your result consistent with the coefficient of determination?
6. For the results of question **5**, plot the residuals and discuss whether the relationship between x and y is linear.
7. **MC** Which of the following scatterplots shows linear relationship between the variables?

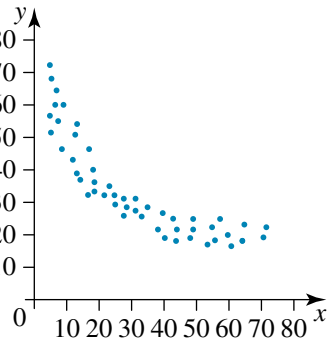
i.



ii.



iii.



A. All of them **B.** None of them **C.** **i** and **iii** only **D.** **ii** only **E.** **ii** and **iii** only

8. Consider the following table from a survey conducted at a new computer manufacturing factory. It shows the percentage of defective computers produced on 8 different days after the opening of the factory.

Day	2	4	5	7	8	9	10	11
Defective rate (%)	15	10	12	4	9	7	3	4

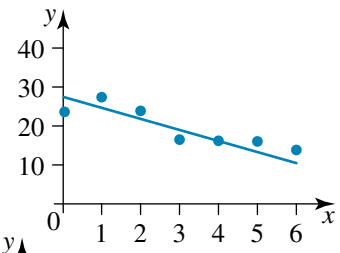
   a. The results of least-squares regression were: $b = -1.19$, $a = 16.34$, $r = -0.87$. Given $y = a + bx$, use the above information to calculate the predicted defective rates (y_{pred}).
   b. Find the residuals ($y - y_{\text{pred}}$).
   c. Plot the residuals and comment on the likely linearity of the data.
   d. Estimate the defective rate after the first day of the factory's operation.
   e. Estimate when the defective rate will be at zero. Comment on this result.
9. The following data represent the number of tourists booked into a hotel in central Queensland during the first week of a drought. (Assume Monday = 1.)

Day	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Bookings in hotel	158	124	74	56	31	35	22

The results of least-squares regression were:
$b = -22.5, a = 161.3, r = -0.94$, where $y = a + bx$.

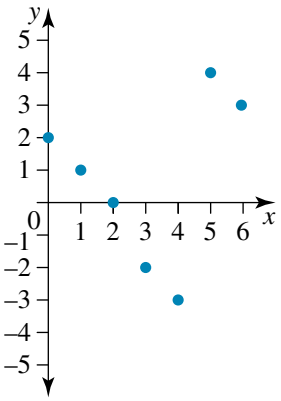
   a. Find the predicted hotel bookings (y_{pred}) for each day of the week.
   b. Find the residuals ($y - y_{\text{pred}}$).
   c. Plot the residuals and comment on the likely linearity of the data.
   d. Would this regression line be a typical one for this hotel?

10. MC A least-squares regression is fitted to the points shown in the scatterplot.

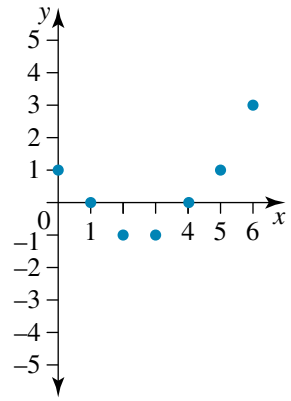


Which of the following looks most similar to the residual plot for the data?

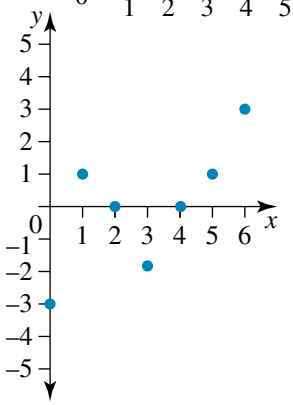
A.



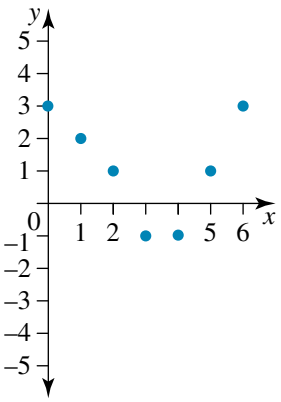
B.



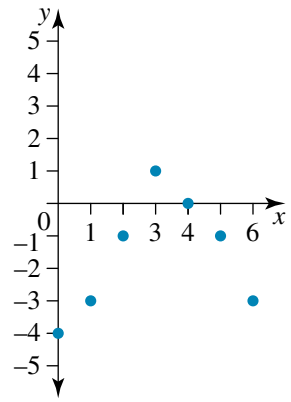
C.



D.



E.



11. From each table of residuals, decide whether or not the relationship between the variables is likely to be linear.

a.

x	y	Residuals
1	2	−1.34
2	4	−0.3
3	7	−0.1
4	11	0.2
5	21	0.97
6	20	2.3
7	19	1.2
8	15	−0.15
9	12	−0.9
10	6	−2.8

b.

x	y	Residuals
23	56	0.12
21	50	−0.56
19	43	1.30
16	41	0.20
14	37	−1.45
11	31	2.16
9	28	−0.22
6	22	−3.56
4	19	2.19
3	17	−1.05

c.

x	y	Residuals
1.2	23	0.045
1.6	25	0.003
1.8	24	−0.023
2.0	26	−0.089
2.2	28	−0.15
2.6	29	−0.98
2.7	34	−0.34
2.9	42	−0.01
3.0	56	0.45
3.1	64	1.23

12. Consider the following data set.

x	0	1	2	3	4	5	6	7	8	9	10
y	1	4	15	33	60	94	134	180	240	300	390

a. Plot the data and fit a least-squares regression line.
b. Find the correlation coefficient and interpret its value.
c. Calculate the coefficient of determination and explain its meaning.
d. Find the residuals.
e. Construct the residual plot and use it to comment on the appropriateness of the assumption that the relationship between the variables is linear.

13. Find the residuals for the following data set.

m	12	37	35	41	55	69	77	90
P	2.5	21.6	52.3	89.1	100.7	110.3	112.4	113.7

14. For the data in question **13**, plot the residuals and comment whether the relationship between x and y is linear.

15. Calculate the residuals of the following data.

k	1.6	2.5	5.9	7.7	8.1	9.7	10.3	15.4
D	22.5	37.8	41.5	66.9	82.5	88.7	91.6	120.4

16. For the data in question **15**:
a. plot the residuals and comment whether the relationship between x and y is linear
b. calculate the coefficient of determination and explain its meaning.

3.5 Transforming to linearity

Although linear regression might produce a 'good' fit (high r value) to a set of data, the data set may still be non-linear. To remove (as much as is possible) such *non-linearity*, the data can be transformed.

Either the x-values, y-values, or both may be transformed in some way so that the transformed data are more linear. This enables more accurate predictions (extrapolations and interpolations) from the regression equation. In Further Mathematics, six transformations are studied:

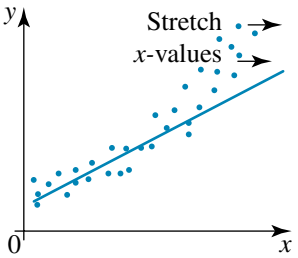
Logarithmic transformations:	y **versus** $\log_{10}(x)$	$\log_{10}(y)$ **versus** x
Quadratic transformations:	y **versus** x^2	y^2 **versus** x
Reciprocal transformations:	y **versus** $\frac{1}{x}$	$\frac{1}{y}$ **versus** x

3.5.1 Choosing the correct transformations

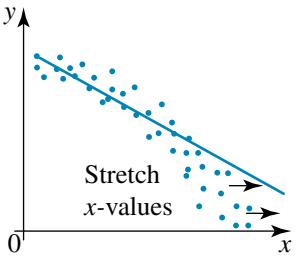
To decide on an appropriate transformation, examine the points on a scatterplot with high values of x and/or y (that is, away from the origin) and decide for each axis whether it needs to be stretched or compressed to make the points line up. The best way to see which of the transformations to use is to look at a number of 'data patterns'.

3.5.2 Quadratic transformations

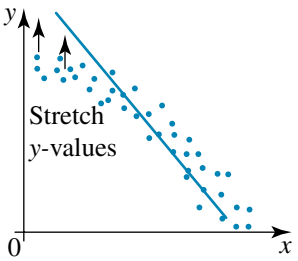
1. Use y versus x^2 transformation.



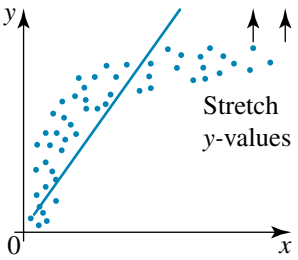
2. Use y versus x^2 transformation.



3. Use y^2 versus x transformation.

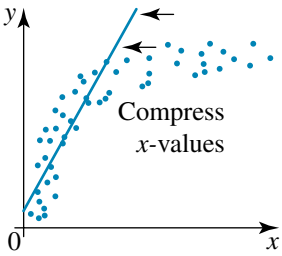


4. Use y^2 versus x transformation.

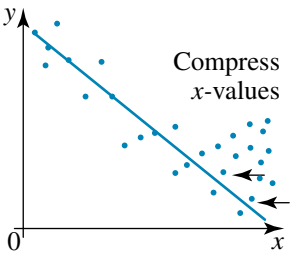


Logarithmic and reciprocal transformations

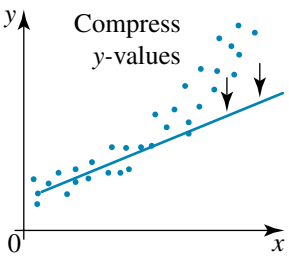
1. Use y versus $\log_{10}(x)$ or y versus $\frac{1}{x}$ transformation.



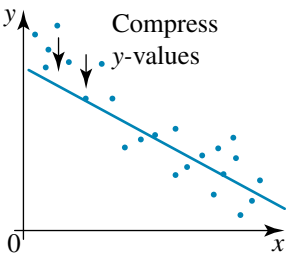
2. Use y versus $\log_{10}(x)$ or y versus $\frac{1}{x}$ transformation.



3. Use $\log_{10}(y)$ versus x or $\frac{1}{y}$ versus x transformation.



4. Use $\log_{10}(y)$ versus x or $\frac{1}{y}$ versus x transformation.



on Resources

Interactivity: Linearising data (int-6491)

3.5.3 Testing transformations

As there are at least two possible transformations for any given non-linear scatterplot, the decision as to which is the best comes from the coefficient of correlation. The least-squares regression equation that has a Pearson correlation coefficient closest to 1 or −1 should be considered as the most appropriate. However, there may be very little difference so common sense needs to be applied. It is sometimes more useful to use a linear function rather than one of the six non-linear functions.

WORKED EXAMPLE 8

a. Plot the following data on a scatterplot, consider the shape of the graph and apply a quadratic transformation.

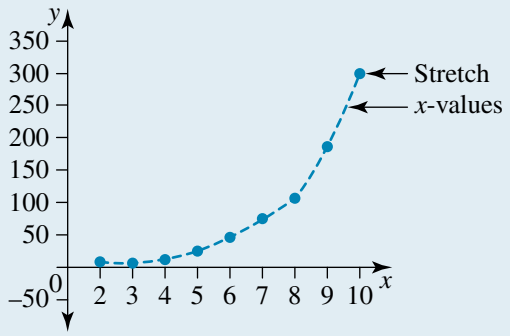
b. Calculate the equation of the least-squares regression line for the transformed data.

x	1	2	3	4	5	6	7	8	9	10
y	5	6	8	15	24	47	77	112	187	309

THINK

1. Plot the data to check that a quadratic transformation is suitable.
   Looking at the shape of the graph, the best option is to stretch the x-axis. This requires an x^2 transformation.

WRITE/DRAW

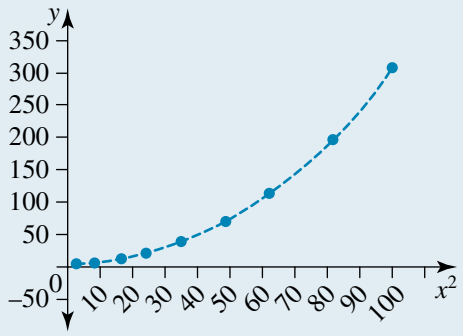


x^2	1	4	9	16	25	36	49	64	81	100
y	5	6	8	15	24	47	77	112	187	309

2. Square the x-values to give a transformed data set by using CAS.
3. Find the equation of the least-squares regression line for the transformed data.
   Using CAS: y-intercept $(a) = -28.0$
   gradient $(b) = 2.78$
   correlation $(r) = 0.95$.

$y = a + bx$
$y = -28.0 + 2.78x_T$ where $x_T = x^2$; that is,
$y = -28.0 + 2.78x^2$

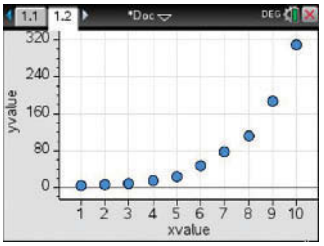
4. Plot the new transformed data.
   Note: These data are still not truly linear, but are 'less' parabolic. Perhaps another transformation would improve things even further. This could involve transforming the y-values, such as $\log_{10}(y)$, and applying another linear regression.



TI | THINK

a. 1. On a Lists & Spreadsheet page, enter the data into the lists named *xvalue* and *yvalue*.
Construct a scatterplot.

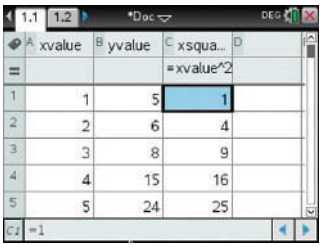
WRITE



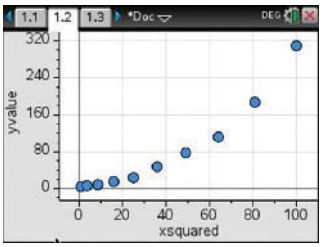
2. Looking at the shape of the graph, the best option is to stretch the *x*-axis. This requires an x^2 transformation.

Stretching the *x*-axis requires an x^2 transformation.

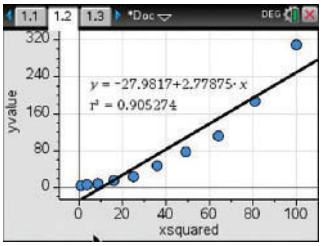
3. Return to the Lists & Spreadsheets page and in Column C, square the values on the horizontal axis.



4. Construct a scatterplot with the *xsquared* values on the horizontal axis and *yvalues* on the vertical axis.



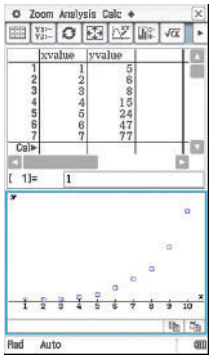
b. 1. Press MENU and select:
4: Analyze
6: Regression
2: Show Linear (a + bx)



CASIO | THINK

a. 1. On a Statistics screen, enter the data into the lists named *xvalue* and *yvalue*.

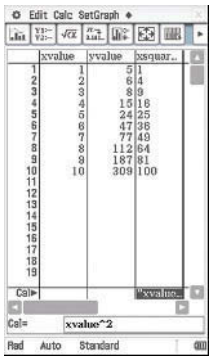
WRITE



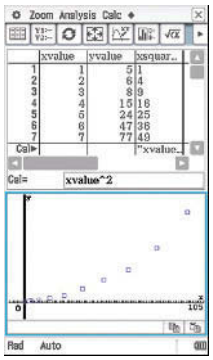
2. Looking at the shape of the graph, the best option is to stretch the *x*-axis. This requires an x^2 transformation.

Stretching the *x*-axis requires an x^2 transformation.

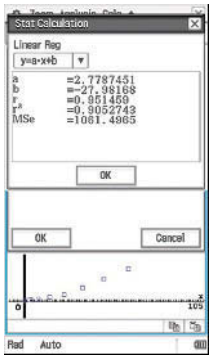
3. Rename list3 as xsquare.
Place the cursor in the calculation cell at the bottom of list3.
Type:
xvalue^2
Press EXE.



Construct a scatterplot with the *xsquared* values on the horizontal axis and *yvalues* on the vertical axis.



b. 1. Tap:
Calc
RegressionLinear reg
Complete the Set Calculation fields:
Linear Reg
XList: main\xsquared
YList: main\ yvalue
Freq: 1
Copy Formula: y1
Copy Residual: Off
Tap OK.



2. The answer appears on the screen. *Note:* These data are still not truly linear, but are 'less' parabolic. Perhaps another transformation would improve things even further. This could involve transforming the y-values, such as $\log_{10}(y)$, and applying another linear regression.

$y = -28.98 + 2.78x^2$ (correct to 2 decimal places)

2. The answer appears on the screen. *Note:* These data are still not truly linear, but are 'less' parabolic. Perhaps another transformation would improve things even further. This could involve transforming the y-values, such as $\log_{10}(y)$, and applying another linear regression.

$y = -28.98 + 2.78x^2$ (correct to 2 decimal places)

WORKED EXAMPLE 9

a. Tranform the data by applying a logarithmic transformation to the *y*-variable.
b. Calculate the equation of the least squares regression line for the transformed data.
c. Comment on the value of *r*.

Time after operation (h)	***x***	1	2	3	4	5	6	7	8
Heart rate (beats/min)	***y***	100	80	65	55	50	51	48	46

THINK

a. Transform the y data by calculating the log of y-values or, in this problem, the logof heart rate.

b. 1. Use a calculator to find the equation of least-squares regression line for x and log y.

2. Rewrite the equation in terms of the variables in question.

c. State the value of r and comment on the result.

WRITE

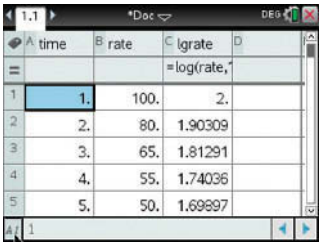
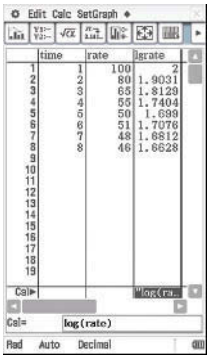
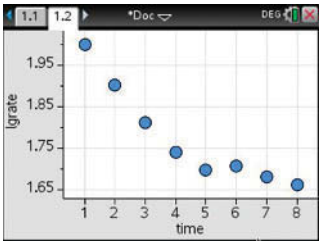
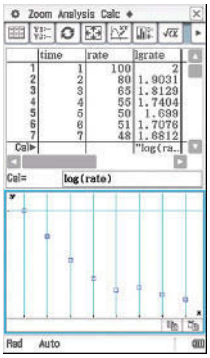
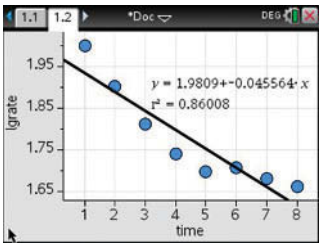
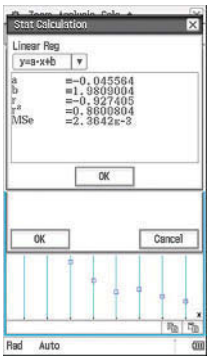
a.

Time	***x***	1	2	3	4	5	6	7	8
$\log_{10}$ (heart rate)	**$\log_{10}y$**	2	1.903	1.813	1.740	1.694	1.708	1.681	1.663

b. $\log_{10}(y) = 1.98 - 0.05x$

$\log_{10}(\text{heart rate}) = 1.98 - 0.05 \times \text{time}$
(i.e. time = number of hours after the operation.)

c. $r = -0.93$
There is a slight improvement of the correlation coefficient that resulted from applying logarithmic transformation.

TI | THINK	WRITE	CASIO | THINK	WRITE
a. 1. On a Lists & Spreadsheets page, enter the data into Columns A and B, renaming the columns as time, rate and lgrate. In the formula cell under lgrate, complete the entry line as = log(*rate*) Press ENTER.		**a. 1.** On a Statistics screen, enter the data into list1 and list2, renaming the lists as time, rate and lgrate. Place the cursor in the calculation cell at the bottom of list3 (lgrate). Type: log(rate) Press EXE.	
2. Press CTRL + I and select: 5: Add Data & Statistics Select time for the horizontal axis and lgrate for the vertical axis.		**2.** Construct a scatterplot with the time values on the horizontal axis and lgrate on the vertical axis.	
b. 1. Press MENU and then select: 4: Analyze 6: Regression 2: Show Linear (a + bx)		**b. 1.** Tap: Calc Regression Linear Reg Complete the Set Calculation fields: Linear Reg XList: main\time YList: main\lgrate Freq: 1 Copy Formula: y1 Copy Residual: Off Tap OK.	
2. The answer is on the screen.	$\log_{10}$(heart rate) = 1.98 − 0.05 × time (correct to 2 decimal places) Time = number of hours after the operation	**2.** The answer is on the screen.	$\log_{10}$(heart rate) = 1.98 − 0.05 × time (correct to 2 decimal places) Time = number of hours after the operation
c. The value of r^2 is on the screen. *Note:* The value of *r* is negative as the regression line has a negative gradient.	$r^2 = 0.86008$ $r = -0.93$ There is a slight improvement of the correlation coefficient that resulted from applying a logarithmic transformation.	**c.** The value of *r* is on the screen.	$r = -0.93$ There is a slight improvement of the correlation coefficient that resulted from applying a logarithmic transformation.

Further investigation

Often all appropriate transformations need to be performed to choose the best one. Extend Worked example 9 by compressing the *y* data using the reciprocals of the *y* data or even compress the *x* data. Go back to the steps for transforming the data. Did you get a better *r* value and thus a more reliable line of best fit? (*Hint:* The best transformation gives $r = -0.98$.)

3.5.4 Using the transformed line for predictions

Once the appropriate model has been established and the equation of least-squares regression line has been found, the equation can be used for predictions.

WORKED EXAMPLE 10

a. Using CAS, apply a reciprocal transformation to the following data.
b. Use the transformed regression equation to predict the number of students wearing a jumper when the temperature is 12°C.

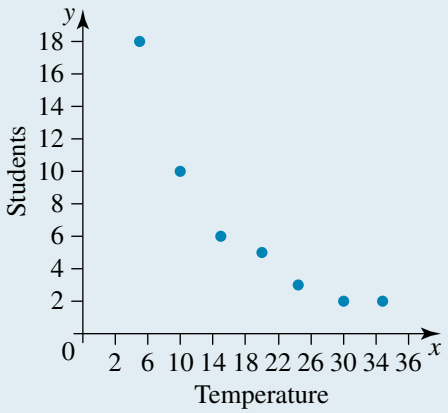
Temperature (°C)	x	5	10	15	20	25	30	35
Number of students in a class wearing jumpers	y	18	10	6	5	3	2	2

THINK

a. 1. Construct the scatterplot.
Temperature is the explanatory variable, while the number of students wearing jumpers is the response one.
Therefore, put *temperature* on the horizontal axis and *students* on the vertical axis.

WRITE/DRAW

a.



2. The x-values should be compressed, so it may be appropriate to transform the x-data by calculating the reciprocal of temperature.
Reciprocate each x-value $\left(\frac{1}{x}\right)$.

$\frac{1}{\text{Temperature}}$	$\frac{1}{x}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{30}$	$\frac{1}{35}$
Number of students wearing jumpers	y	18	10	6	5	3	2	2

3. Use CAS to find the equation of the least-squares regression line for $\frac{1}{x}$ and y.

$y = -0.4354 + 94.583x_T$, where $x_T = \frac{1}{x}$ or

$y = -0.4354 + \frac{94.583}{x}$

4. Replace x and y with the variables in question.

The number of students in class wearing jumpers $= -0.4354 + \frac{94.583}{\text{Temperature}}$

b. 1. Substitute 12 for x into the equation of the regression line and evaluate.

b. Number of students wearing jumpers

$= -0.4354 + \frac{94.583}{\text{Temperature}}$

$= -0.4354 + \frac{94.583}{12}$

$= 7.447$

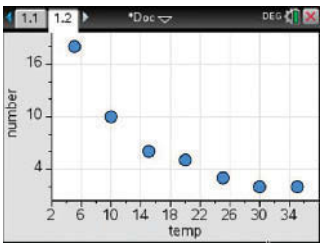
2. Write your answer to the nearest whole number.

7 students are predicted to wear jumpers when the temperature is 12 °C.

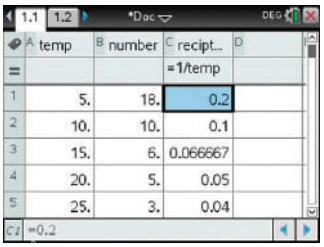
TI | THINK

WRITE

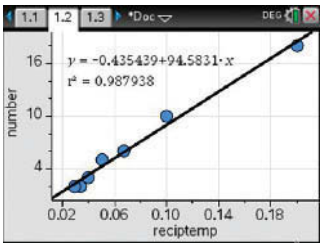
a. 1. Construct the scatterplot. Temperature is the explanatory variable, while the number of students wearing jumpers is the response one. Therefore, put temperature on the horizontal axis and number of students on the vertical axis.



2. The values should be compressed, so transform the data on the horizontal axis by calculating the reciprocal of the temperature.
Return to the Lists & Spreadsheets page and label Column C as reciptemp.
In the formula cell of Column C, complete the entry line as:
= 1 ÷ temp
Press ENTER.



3. Press CTRL + I and select:
5: Add Data & Statistics
Select reciptemp for the horizontal axis and number for the vertical axis. Press MENU and then select:
4: Analyze
6: Regression
2: Show Linear (a + bx)



4. The answer is on the screen.

$y = -0.44 + 94.58 \times \frac{1}{x}$

(correct to 2 decimal places)

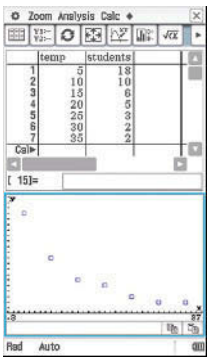
5. Replace x and y with the variables in question.

No. of students $= -0.44 + \frac{94.58}{\text{Temperature}}$

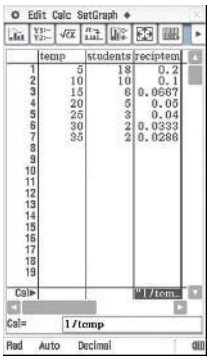
CASIO | THINK

WRITE

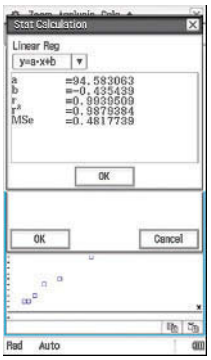
a. 1. Construct the scatterplot. Temperature is the explanatory variable, while the number of students wearing jumpers is the response one. Therefore, put temperature on the horizontal axis and number of students on the vertical axis.



2. The values should be compressed, so transform the data on the horizontal axis by calculating the reciprocal of the temperature.
Rename list3 as reciptem.
Place the cursor in the calculation cell at the bottom of list3 (reciptem).
Type:
1 ÷ temp
Press EXE.



3. Construct a scatterplot with the reciptem values on the horizontal axis and students on the vertical axis.
Tap:
Calc
Regression
Linear Reg
Complete the Set Calculation fields:
Linear Reg
XList: main\reciptem
YList: main\students
Freq: 1
Copy Formula: y1
Copy Residual: Off
Tap OK.



4. The answer is on the screen.

$y = -0.44 + 94.58 \times \frac{1}{x}$

(correct to 2 decimal places)

5. Replace x and y with the variables in question.

No. of students $= -0.44 + \frac{94.58}{\text{Temperature}}$

Resources

Interactivity: Transforming To Linearity (int-6253)

studyon

Units 3 & 4 > AOS DA > Topic 7 > Concept 7

Transformation of data: Summary screen and practice questions

Exercise 3.5 Transforming to linearity

1. WE8 **a.** Plot the data on a scatterplot, consider the shape of the graph and apply a quadratic transformation.

x	1	2	3	4	5	6	7	8	9
y	12	19	29	47	63	85	114	144	178

   b. Calculate the equation of the least-squares regression line for the transformed data.

2. **a.** Plot the following data on a scatterplot, consider the shape of the graph and apply a quadratic transformation.

x	3	5	9	12	16	21	24	33
y	5	12	38	75	132	209	291	578

   b. Calculate the equation of the least-squares regression line for the transformed data.

3. Apply a quadratic (x^2) transformation to the following data set. The regression line has been determined as $y = 186 - 27.7x$ with $r = -0.91$.

x	2	3	4	5	7	9
y	96	95	92	90	14	−100

4. WE9 Apply a logarithmic transformation to the following data, which represents a speed of a car as a function of time, by transforming the y-variable.

Time (s)	1	2	3	4	5	6	7	9
Speed (ms^{-1})	90	71	55	45	39	35	32	30

5. Apply a logarithmic transformation to the following data by transforming the y-variable.

x	5	10	15	20	25	30	35	40
y	1000	500	225	147	99	70	59	56

6. The *average* heights of 50 girls of various ages were measured as follows.

Age group (years)	9	10	11	12	13	14	15	16	17	18
Average height (cm)	128	144	148	154	158	161	165	164	166	167

The original linear regression yielded:

$$\text{Height} = 104.7 + 3.76 \times \text{age}, \text{ with } r = 0.92.$$

a. Plot the original data and regression line.

b. Apply a $\log_{10}(x)$ transformation.

c. Perform regression analysis on the transformed data and comment on your results.

7. a. Use the transformed data from question **6** to predict the heights of girls of the following ages:

i. 7 years old **ii.** 10.5 years old **iii.** 20 years old.

b. Which of the predictions in part **a** were obtained by interpolating?

8. Comment on the suitability of transforming the data of question **6** in order to improve predictions of heights for girls under 8 years old or over 18.

9. a. **WE10** Using CAS, apply a reciprocal transformation to the x-variable of the following data.

Time after 6 pm (h)	1	2	3	4	5	6	7	8
Temperature (°C)	32	22	16	11	9	8	7	7

b. Use the transformed regression equation to predict the temperature at 10.30 pm.

10. a. Using CAS, apply a reciprocal transformation to the x-variable of the following data.

x	2	5	7	9	10	13	15	18
y	120	50	33	15	9	5	2	1

b. Use the transformed regression equation to predict y when $x = 12$.

c. Use the transformed regression equation to predict x when $y = 20$.

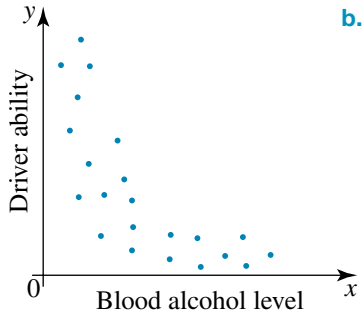
11. a. Apply a reciprocal transformation to the following data obtained by a physics student studying light intensity.

Distance from light source (meters)	1	2	3	4	5	10
Intensity (candlepower)	90	60	28	22	20	12

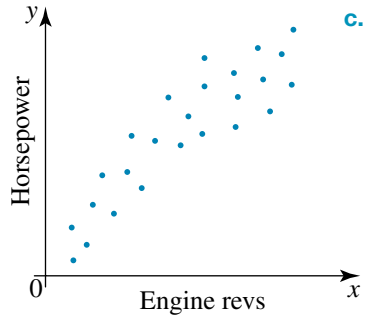
b. Use the transformed regression equation to predict the intensity at a distance of 20 metres.

12. For each of the following scatterplots suggest an appropriate transformation(s).

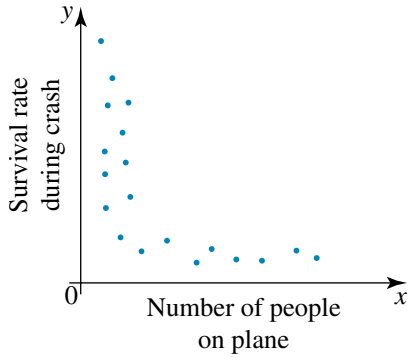
a.



b.



c.



13. Use the equation $y = -12.5 + 0.2x^2$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a. $x = 2.5$ **b.** $x = -2.5$.

14. Use the equation $y = -25 + 1.12\log_{10}(x)$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a. $x = 2.5$ **b.** $x = -2.5$ **c.** $x = 0$.

15. Use the equation $\log_{10}(y) = 0.03 + 0.2x$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a. $x = 2.5$ **b.** $x = -2.5$.

16. Use the equation $\frac{1}{y} = 12.5 + 0.2x$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a. $x = 2.5$ **b.** $x = -2.5$.

17. The seeds in the sunflower are arranged in spirals for a compact head. Counting the number of seeds in the successive circles starting from the centre and moving outwards, the following number of seeds were counted.

Circle	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Number of seeds	3	5	8	13	21	34	55	89	144	233

a. Plot the data and fit a least-squares regression line.

b. Find the correlation coefficient and interpret its value.

c. Using the equation of the regression line, predict the number of seeds in the 11th circle.

d. Find the residuals.

e. Construct the residual plot. Is the relation between the number of the circle and the number of seeds linear?

f. What type of transformation could be applied to:

i. the x-values? Explain why.

ii. the y-values? Explain why.

18. Apply a $\log_{10}(y)$ transformation to the data used in question **17**.

a. Fit a least-squares regression line to the transformed data and plot it with the data.

b. Find the correlation coefficient. Is there an improvement? Why?

c. Find the equation of the least-squares regression line for the transformed data.

d. Calculate the coefficient of determination and interpret its value.

e. Using the equation of the regression line for the transformed data, predict the number of seeds for the 11th circle.

f. How does this compare with the prediction from question **17**?

3.6 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC For the following data set, the least squares regression line (to 2 decimal places) is:

x	25	36	45	78	89	99	110
y	78	153	267	456	891	1020	1410

A. $y = 14.42 + 381.97x$ **B.** $y = 14.42 - 381.97x$ **C.** $y = -381.97 + 14.42x$

D. $y = -38 + 14x$ **E.** $y = 38 - 14x$

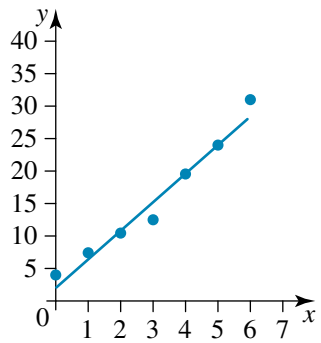
2. MC Given the following summary statistics

$$\bar{x} = 154.4 s_x = 5.8 \bar{y} = 172.5 s_y = 7.4 r = 0.9$$

the values of b and a,respectively, for the equation of the regression line $y = a + bx$ are:

A. 0.71 and 32.72 **B.** 1.15 and −4.79 **C.** 0.44 and 10.1
D. 0.04 and −0.16 **E.** −1.32 and 3.8

Use the following information to answer questions **3** to **5**.
A least-squares regression line is fitted to the 7 points as shown.



3. MC The equation of the least-squares line is closest to

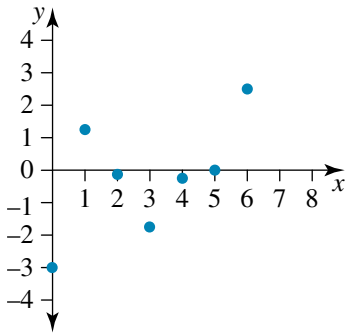
A. $y = x$ **B.** $y = \frac{1}{5}x$ **C.** $y = 2x$ **D.** $y = 5x$ **E.** $y = 20x$

4. MC From the data points $x = 6$, $y = 33$ if the least-squares line is used to predict the y value when $x = 6$, the residual will be

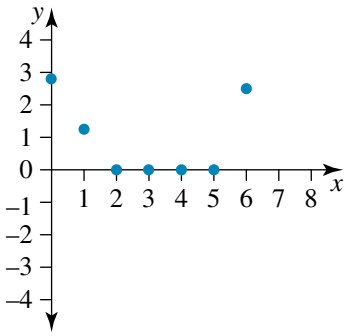
A. 0 **B.** 1 **C.** −1 **D.** 2 **E.** 3

5. MC The residual plot would look most similar to:

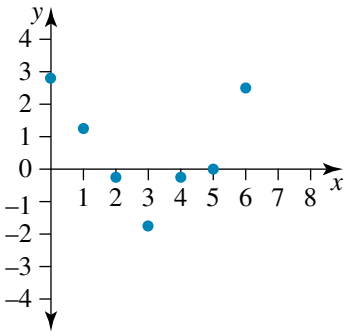
A.



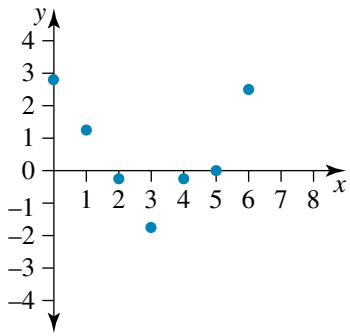
B.



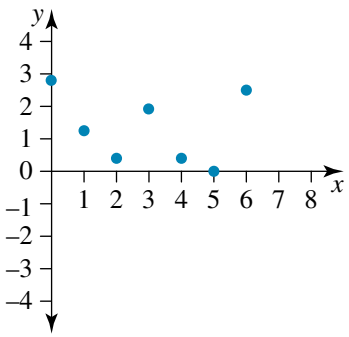
C.



D.



E.



6. MC After a transformation, a relationship was found to be $y = 0.4x^2 + 12.1$. The predicted value for y given that $x = 2.5$ is:

A. 6.25 **B.** 2.5 **C.** 14.6 **D.** 13.1 **E.** 12.5

Short answer

1. Use the following summary statistics to find:
   a. the slope, b, of the least-squares regression line
   b. the y-intercept, a, of the least-squares regression linewhere $y = a + bx$ is the equation of the regression line.

$$\bar{x} = 15 \quad s_x = 5 \quad \bar{y} = 10 \quad s_y = 2.5 \quad r = -0.9$$

2. a. Find the equation for the least-squares regression line for the following data.

x	1	2	4	8	9	10	12	15
y	23	21	20	14	16	9	12	5

   b. Using the equation you have found in part a, copy and complete the following table of predicted values.

x	3	5	7	9	11	13	15	17	20
y_{pred}									

   c. For the least-squares regression line from a, find the residuals.

Extended response

1. The data in the table show the number of hours spent by students learning to touch-type and their corresponding speed in words per minute (wpm).

Time (h)	20	33	22	39	40	37	46	44	24	36	50	48	29
Speed (wpm)	34	46	38	53	52	49	60	58	36	42	65	63	40

   a. State which variable is explanatory and which is response.
   b. Represent the data on a scatterplot.
   c. Use the scatterplot to comment on the relationship between the two variables.

2. Consider this data set which measures the sales figures for a new salesperson.

Day	1	2	3	4	5	6	7	8
Units sold	1	2	4	9	20	44	84	124

   The least-squares regression yielded the following equation:
   Units sold $= -39.1 + 16.7 \times$ day
   The correlation coefficient was 0.90.
   a. Use CAS to construct the scatterplot of the data. What kind of relationship between the variables does the scatterplot suggest?
   b. Comment on using the regression line to predict for small values of the explanatory variable.
   c. Use the equation of the regression line to predict the sales figures for the 10th day. Comment on the reliability of this prediction.
   d. Transform the data from part a using a quadratic (x^2) transformation.
   e. Perform a least-squares regression on the transformed data from part d.
   f. Use the regression line for the transformed data to predict the sales figures for the 10th day. Is this a better prediction than the one found in part c?

3. A mining company wishes to predict its gold production output. It collected the following data over a 9-month period.

Month (1 = January)	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
Production (tonnes)	3	8	10.8	12	11.6	14	15.5	15	18.1

   a. Find the equation of the least-squares regression line.
   b. Using the line from part **a**, predict the production after 12 months.
   c. Comment on the accuracy, usefulness and simplicity of the methods.
   d. Looking at the original data set, discuss whether linearity is a reasonable assertion.
   e. Research into goldmines has indicated that after about 10 months, production tends not to increase as rapidly as in earlier months. Given this information, a logarithmic transformation is suggested. Transform the original data using this method.
   f. Fit a straight line to this transformed data using least-squares regression.
   g. Discuss whether or not this transformation has removed any non-linearity.
   h. Predict the level of production of gold after 12 months using the equation obtained in part **f**. Compare the prediction from part **b** with the one obtained using the logarithmic transformation.

4. A number of students were asked to record the number of hours they spent studying for the test and the mark obtained on that test. The number of hours spent studying varied between 1.5 and 8 hours, while test results ranged from 28% to 92%. The equation of least squares regression line was found to be as follows:
   Test mark (%) $= 21.6 + 8.6 \times$ number of hours spent studying
   a. Interpret the value of the gradient in the context of this problem.
      Interpret the value of the y-intercept.
   b. Use equation of the least squares regression line to predict the test result for Nathan who spent 6 hours studying for the test. Give your answer correct to the nearest whole number. Is this an example of interpolation or extrapolation? Explain.
   c. Rachel was sick on the day of the test and will have to do it on another day. She claims spending 9 hours studying for the test. Use the equation of the least squares regression line to predict the test result for Rachel. Comment on reliability of this prediction.

studyon

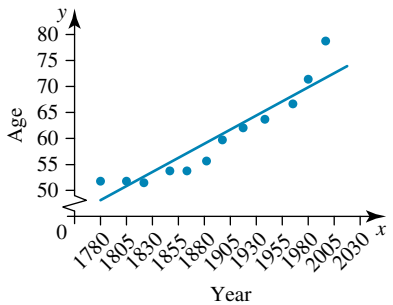
Units 3 & 4 Sit exam

Answers

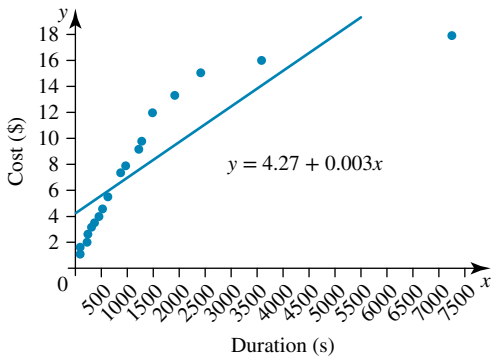
Topic 3 Investigating and modelling linear associations

Exercise 3.2 Fitting a straight line – least-squares regression

1. $y = -37.57 + 1.87x$ or air conditioner sales $= -37.57 + 1.87 \times$ (temperature), $r^2 = 0.97$, strong linear association
2. $y = 2.11 + 2.64x$ or number of dialysis patients $= 2.11 + 2.64 \times$ (month), $r^2 = 0.95$, strong linear association
3. $y = 4.57 + 1.04x$
4. $y = 35.47 - 3.72x$
5. $y = 9.06 + 1.20x$
6. **a.** Explanatory variable = 'height of Year 12 girl'.
   b. $b = 0.96$
   c. $a = 22.64$
7. $b = -0.78167$, $a = 13.94$, $y = 13.94 - 0.78x$
8. **a.** The mathematics exam mark
   b. 0.95
   c. 6.07
   d. 75%
9. **a.** $y = 91.78 + 3.33x$
   b. $y = 21.87 - 0.15x$
   c. $y = 8890.48 + 52.38x$
   d. $y = 30 - x$
10. The least-squares regression equations are exactly the same as obtained in questions 3, 4 and 5.
11. **a.** $y = 22 - 2x$
    b. A 'perfect' fit
    c. $y = 11 - 0.5x$
12. E
13. **a.** $y = -164.7 + 0.119x$
    b.



    c. The data definitely are not linear; there are big increases from 1880–1920, 1940–2000.
14. **a.** Duration of call is explanatory variable
    b. Cost of call $(\$)$ $= \$4.27 + \$0.002\,57 \times$ duration of call (sec)
    c.



    The line does not fit closely for all data points. The equation is not reliable due to outliers. If you eliminate the last two calls then there is a direct relationship.
    d. $r^2 = 0.73$. Therefore the linear association is only moderate.
15. D
16. **a.** $y = 8 + 4x$, perfect fit, but meaningless
    b. $y = 10 + 2.5x$, good fit, but almost meaningless
    c. $y = 9.5 + 2.8x$, $y = 8.9 + 3.1x$, $y = 8.4 + 3.3x$, good fit.
    d. The answers appear to be converging towards a 'correct' line.

Exercise 3.3 Interpretation, interpolation and extrapolation

1. **a.** $y = 12.13 + 3.49x$ or monkey height (cm) $= 12.13 + 3.49 \times$ (month from birth)
   b. 3.49 cm/month
   c. 12.13 cm
2. **a.** $y = 13.56 + 2.83x$ or temperature (°C) $= 13.56 + 2.83 \times$ (time after 6 a.m.)
   b. 2.83°C/hr
   c. 13.56°C
3. **a.** Red blood cell count $= 157.3 + 14 \times$ day of experiment
   b. 14 cell per day
   c. 157
4. **a.** 48.5, or 49 people per extra animal
   b. 31.8, or 32 visitors
5. Sale volume $= 464 - 1.72 \times$ selling price, $r = -0.98$. Gradient shows a drop of 1720 sales for every \$1 increase in the price of the item. Clearly, the y-intercept is nonsensical in this case since an item is not going to be sold for \$0! This is a case where extrapolation of the line makes no sense.
6. $y = 56.03 + 9.35x$ or height (cm) $= 56.03 + 9.35 \times$ (age)
   Height (cm) = 149.53 cm
7. $y = 40.10 + 6.54x$ or length (cm) $= 40.10 + 6.54 \times$ (months)
   Length (cm) = 138.20 cm
8. Height (cm) = 205.63 cm
   We cannot claim that the prediction is reliable, as it uses extrapolation.
9. Length (cm) = 197.06 cm
   We cannot claim that the prediction is reliable, as it uses extrapolation.

10. **a.** 7 **b.** 18

11. **a.** $y = 0.286 + 3.381x$ **b.** 10.4
c. 40.9 **d.** 1.99
e. 7.31 **f.** c

12. **a.** Factory 1: *cost* = 43.21 + 1.51 × *distance*
Factory 2: *cost* = 56.61 + 0.964 × *distance*
b. Factory 1 is cheaper at $43.21 (compared to Factory 2 at $56.61).
c. Factory 2 is cheaper at $167.47 (compared to Factory 1 at $216.86).
d. Factory 2 is marginally more linear (Factory 1: $r = 0.97$; Factory 2: $r = 0.99$).

13. **a.** $2152 **b.** 180 **c.** $78 200
d. $600 **e.** **a**, **b** only

14.

IQ	Test result (%)
80	56
87	60
92	68
102	65
105	73
106	74
107	71
111	73
115	80
121	92

15. **a.** $55 000 **b.** $58 600
c. $64 000 **d.** About 13 years
e. Various answers

16. **a.** $r = 11.73 + 5.35q$ **b.** $r = 33.13$
c. $r = 108.03$ **d.** $q = 16.50$
e. **c** and **d**

17. **a.** $C = \$420$ **b.** $n = 13.38$ hours
c. $C = \$940$ **d.** Call out fee = $180
e. **b** and **c**

18. **a.** $y = 42\,956 + 14.795 \times$ (millions spent on advertising) or
Members
$= 42\,956 + 14\,795 \times$ (millions spent on advertising)
b. 72 546 members
Interpolation
c. $1.83 million
d. $r^2 = 0.90$, strong linear relationship

Exercise 3.4 Residual analysis

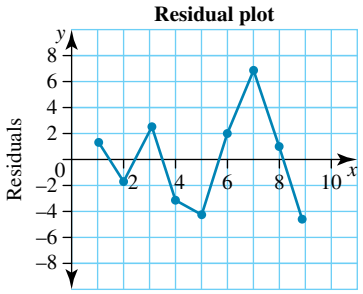
1. Using CAS: $y = -0.03 + 10.85x$
See table below bottom of the page*

2. Using CAS: $y = 0.69 + 9.82x$
See table below bottom of the page#

3. $r^2 = 0.98$, consistent with a linear relationship

x	y	y_{pred}	Residuals
1	1	5.1	−4.1
2	9.7	7.46	2.24
3	12.7	9.82	2.88
4	13.7	12.18	1.52
5	14.4	14.54	−0.14
6	14.5	16.9	−2.4

4.



$r^2 = 0.98$, consistent with a linear relationship

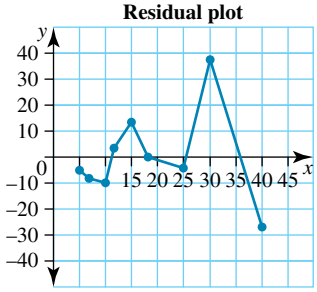
*1.

Predicted y	10.82	21.67	32.52	43.37	54.22	65.07	75.92	86.77	97.62
Residual ($y - y_{predicted}$)	1.18	−1.67	32.52	−3.37	−4.22	1.93	7.08	1.23	−4.62
x	1	2	3	4	5	6	7	8	9
y	12	20	35	40	50	67	83	88	93

#2.

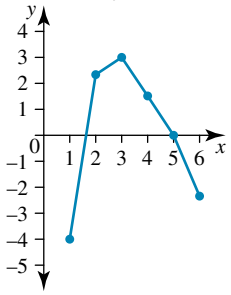
Predicted y	49.79	69.43	98.89	118.53	147.99	177.45	246.19	295.29	393.49
Residual ($y - y_{predicted}$)	−4.79	−8.43	−9.39	3.47	13.01	−0.45	−3.19	37.71	−27.49
x	5	7	10	12	15	18	25	30	40
y	45	61	89	122	161	177	243	333	366

5. Residual plot



$r^2 = 0.98$, consistent with a linear relationship

6. By examining the original scatterplot, and residual plot, data are clearly not linear.

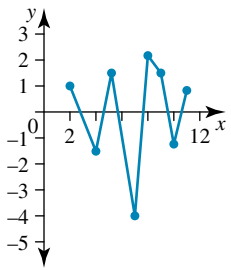


7. D

8. a, b

Day	Defective rate (%)	y_{pred}	Residuals
2	15	13.96	1.04
4	10	11.58	−1.58
5	12	10.39	1.61
7	4	8.01	−4.01
8	9	6.82	2.18
9	7	5.63	1.37
10	3	4.44	−1.44
11	4	3.25	0.75

c.



No apparent pattern in the residuals — likely to be linear

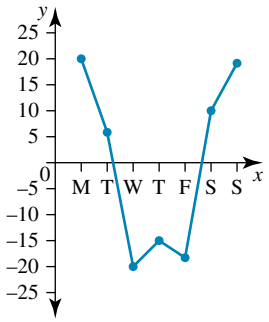
d. 15.15%

e. 13.7 days. Unlikely that extrapolation that far from data points is accurate. Unlikely that there would be 0% defectives.

9. a, b

Day	Bookings in hotel	y_{pred}	Residuals
1	158	138.8	19.2
2	124	116.3	7.7
3	74	93.8	−19.8
4	56	71.3	−15.3
5	31	48.8	−17.8
6	35	26.3	8.7
7	22	3.8	18.2

c.



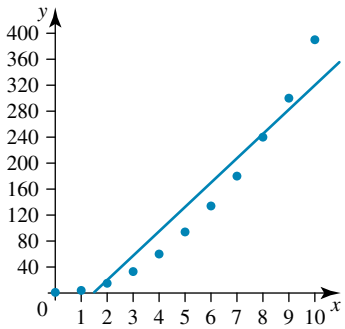
Slight pattern in residuals — may not be linear

d. Decline in occupancy likely due to drought — an atypical event.

10. C

11. a. Non-linear **b.** Linear **c.** Non-linear

12. a. $y = -57.73 + 37.93x$



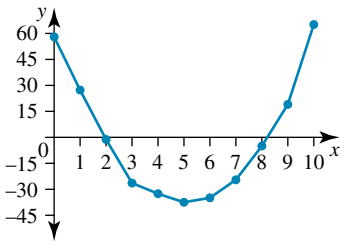
b. $r = 0.958$. This means that there is a strong positive relationship between variables x and y.

c. 0.9177, therefore 91.8% of the variation in y can be explained by the variation in x.

d.

x	Residuals
0	58.7
1	23.8
2	−3.1
3	−23.1
4	−34.0
5	−37.9
6	−35.8
7	−27.8
8	−5.7
9	16.4
10	68.5

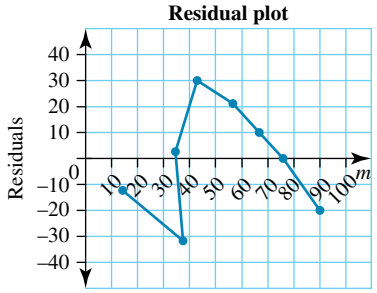
e.



There is a clear pattern; the relationship between the variables is non-linear.

13. $P = -3.94 + 1.52m$
See table below bottom of the page*

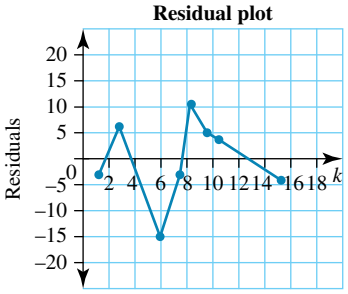
14.



Since the data shows a curved pattern, the original data probably have a non-linear relationship.

15. Using CAS: $D = 13.72 + 7.22k$
See table below bottom ofthe page#

16. a.

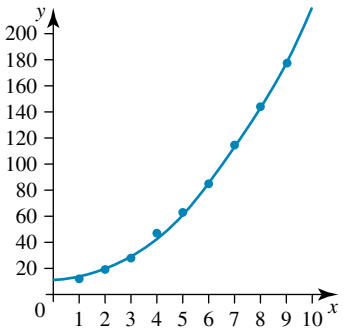


Since the data randomly jumps from above to below the k-axis; the data probably has a linear relationship.

b. $r^2 = 0.94$; 94% of variation in D can be explained by variation in k.

Exercise 3.5 Transforming to linearity

1. a.



Apply an x^2 transformation to stretch the x-axis.

b.

x^2	1	4	9	16	25	36	49	64	81
y	12	19	29	47	63	85	114	144	178

$y = 11.14 + 2.07x^2$ with $r = 0.999\,76$
This transformation has improved the correlation coefficient from 0.97 to 0.999 76; thus the transformed equation is a better fit of the data.

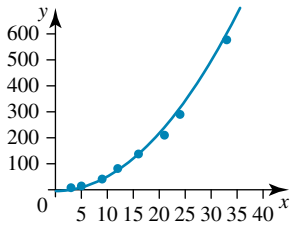
*13.

m	12	37	35	41	55	69	77	90
P	2.5	21.6	52.3	89.1	100.7	110.3	112.4	113.7
$P_{predicted}$	14.30	52.30	49.26	58.38	79.66	100.94	113.10	132.86
Residual $(P - P_{Predicted})$	−11.8	−30.7	3.04	30.72	21.04	9.36	−0.7	−19.16

#15.

k	1.6	5.9	178	7.7	8.1	9.7	10.3	15.4
D	22.5	37.8	41.5	66.9	82.5	88.7	91.6	120.4
$D_{predicted}$	25.27	31.77	56.32	69.31	72.20	83.75	88.09	124.91
Residual $(D - D_{predicted})$	−2.77	6.03	−14.82	−2.41	10.30	4.95	3.51	−4.51

2. a.



Apply an x^2 transformation to stretch the x-axis.

b.

x^2	9	25	81	144	256	441	576	1089
y	5	12	38	75	132	209	291	578

$y = -4.86 + 0.53x^2$ with $r = 0.9990$
This transformation has improved the correlation coefficient from 0.96 to 0.9990, thus the transformed equation is a better fit of the data.

3. $y = 128.15 - 2.62x_T$ where $x_T = x^2$, $r = -0.97$, which shows some improvement.

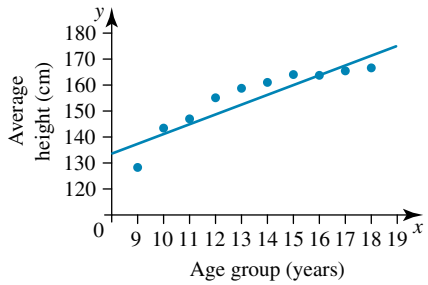
4. See table below bottom of the page*
$\log_{10}(\text{speed}) = 1.97 - 0.07 \times \text{time}$ with $r = -0.97$
Therefore there is an improvement of the correlation coefficient that resulted from applying a logarithmic transformation.

5.

x	5	10	15	20	25	30	35	40
$\log_{10} y$	3	2.70	2.35	2.17	2.00	1.85	1.77	1.75

$\log_{10} y = 3.01 - 0.04x$ with $r = -0.96$
Therefore there is a significant improvement of the correlation coefficient that resulted from applying a logarithmic transformation.

6. a.



b.

log (age group)	Average height (cm)
0.954	128
1	144
1.041	148
1.079	154
1.114	158
1.146	161
1.176	165
1.204	164
1.230	166
1.255	167

c. $y = 24.21 + 117.2x_T$ where $x_T = \log_{10}(x)$, $r = 0.95$, most non-linearity removed.

7. a. **i.** 123.3 cm **ii.** 143.9 cm **iii.** 176.7 cm
b. **a ii**

8. Normal growth is linear only within given range; eventually the girl stops growing. Thus logarithmic transformation is a big improvement over the original regression.

9. a.

$\frac{1}{\text{Time after 6 pm (hr)}}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
Temperature (°C)	32	22	16	11	9	8	7	7

$y = 3.85 + 29.87x_T$, where $x_T = \frac{1}{x}$
or
$$\text{Temperature} = 3.85 + \frac{29.87}{\text{Time after 6 pm}}$$

b. Temperature = 10.49 °C

10. a.

$\frac{1}{x}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{18}$
y	120	50	33	15	9	5	2	1

$y = -13.51 + 273.78x_T$, where $x_T = \frac{1}{x}$
or
$$y = -13.51 + \frac{273.78}{x}$$

b. $y = 9.31$
c. $y = 8.17$

11. a. $y = 2.572 + 90.867x_T$ where $x_T = \frac{1}{x} (r = 0.9788)$
b. The intensity is 7.1 candlepower.

12. a. Compress the y- or x-values using logs or reciprocals.
b. Stretch the y-values using y^2 or compress the x-values using logs or reciprocals.
c. Compress the y- or x-values using logs or reciprocals.

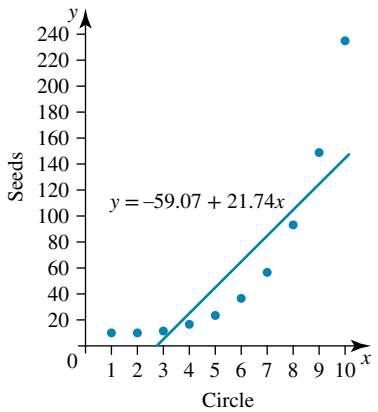
13. a. −11.25 **b.** −11.25

14. a. −24.55
b. Cannot take the log of a negative number.
c. Cannot take the log of zero.

15. a. 3.39 **b.** 0.34

16. a. −0.08 **b.** −0.08

17. a.



*4.

Time (sec)	1	2	3	4	5	6	7	8
Log_{10} Speed (ms^{-1})	1.95	1.85	1.74	1.65	1.59	1.54	1.51	1.48

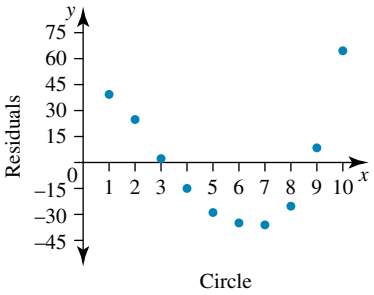
b. $r = 0.87$, which means it is a strong and positive relationship.

c. 180

d.

Circle	Seeds	Residual
1	3	40.33
2	5	20.59
3	8	1.85
4	13	−14.89
5	21	−28.63
6	34	−37.37
7	55	−38.11
8	89	−25.84
9	144	7.41
10	233	74.67

e.

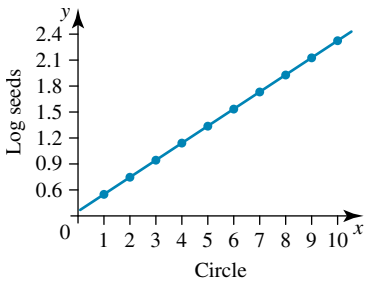


No, the relationship is not linear.

f. **i.** We can stretch the x-values towards linearity by using an x^2 transformation.

ii. We can compress the y-values towards linearity by using either a $\log_{10}(y)$ or a $\frac{1}{y}$ transformation.

18. a.



b. $r = 0.9999$, this is an almost perfect relation.

c. $\log_{10}(y) = 0.2721 + 0.2097x$
$\log_{10}(\text{number of seeds})$
$= 0.2721 + 0.2097 \times \text{circle number}$

d. 0.9999, 99.99%(100.0%) of variation in number of seeds is due to number of circles. This is a perfect relation, often found in nature (see the Golden Ratio).

e. 378

f. This is a much better prediction as it follows a steep upward trend.

3.6 Review: exam practice

Multiple choice

1. C **2.** B **3.** D **4.** E **5.** C **6.** C

Short answer

1. a. −0.45 **b.** 16.75

2. a. $y = 24.35 - 1.25x$, $r = -0.96$

b.

x	y_{pred}
3	20.6
5	18.1
7	15.6
9	13.1
11	10.6
13	8.1
15	5.6
17	3.2
20	−0.65

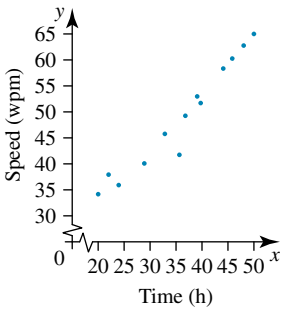
c.

x	y	y_{pred}	$y - y_{pred}$
1	23	23.1	−0.1
2	21	21.85	−0.85
4	20	19.35	0.65
8	14	14.35	−0.35
9	16	13.1	2.9
10	9	11.85	−2.85
11	12	10.6	1.4
15	5	5.6	−0.6

Extended response

1. a. Hours spent touch-typing — explanatory, speed of touch-typing — response.

b.



c. Strong, positive, linear relationship between the two variables

2. a. Likely to be a y versus x^2 relationship

b. A poor predictor for most values of x

c. 128; caution should always be taken when predicting outside the data set.

d.

Day	1	4	9	16	25	36	49	64
Units sold	1	2	4	9	20	44	84	124

e. $y = -13.86 + 1.96x_T$, where $x_T = x^2$

f. 182. The regression line is a better fit for the transformed data and hence is likely to be a better predictor of the sales figures.

3. a. $y = 4.27 + 1.55x$

b. 22.87.

c. Simplicity of eye fitting versus accuracy in this case is quite good. Little difference in the sum of squared errors. Least-squares regression gives quite a different answer from the other 2 methods, with consequent change in errors. (The 3-median method is subject to errors due to outliers, and computational errors.)

d. Not very linear, logarithmic transformation suggested

e.

$\log_{10}$ (month)	Production (tonnes)
0	3
0.301	8
0.477	10.8
0.602	12
0.699	11.6
0.778	14
0.845	15.5
0.903	15
0.954	18.1

f. $y = 3.30 + 14.08x_T$, where $x_T = \log_{10}(X)$

g. Square of residual error is reduced, correlation is closer to 1, graph looks more linear

h. Prediction of 18.49 against 22.87 using untransformed data. Given the nature of the data, likely to be more accurate.

4. a. On average, the test mark increases by 8.6% with every extra hour spent on studying for the test.
Any student who did not study for the test can expect to get 21.6%.

b. 73%; this prediction is an example of interpolation, as 6 hours is within the original set of data.

c. 99%; this prediction is an example of extrapolation, as 9 hours is outside the original set of data. Therefore it is probably not very reliable.

TOPIC 4
Investigating and modelling time series data

4.1 Overview

4.1.1 Introduction

Those who study time series data are interested in trends and patterns in the data. These may be short-term or long-term trends and patterns. Climate scientists at NASA have used time series data and other technologies to determine that the Earth's average surface temperature has increased by 1.1 degrees Celsius since the late 19th century with 16 of the 17 warmest years on record occurring since 2001.

Modelling using time series data can be used for forecasting and monitoring predicted sales figures, population growth, mortality rates and long-term health outcomes, endangered species, weather patterns, food consumption and production, economic growth, interest rate fluctuations and future consumer and voting trends. This information can be used to improve people's lives and ensure that our planet continues to thrive. It can also be used to influence choices and monitor behaviour. The more information that government, political lobby groups, business and other organisations and interest groups can obtain the more the information can be used in both positive and negative ways.

It is important that individuals are aware and informed regarding the mining of their own personal data and the ways in which this data can be used to manipulate choices and influence patterns of behaviour. Every action that connects one to a website leaves a digital footprint. This footprint can then be used to track an individual's political ideology, friends, consumer habits, credit rating, personal hobbies, location and interests. This may have a significant impact on an individual's ability to obtain employment, buy a house or a car, have a relationship and travel overseas.

LEARNING SEQUENCE

4.1 Overview
4.2 Time series and trend lines
4.3 Trend lines and forecasting
4.4 Smoothing time series
4.5 Smoothing with an even number of points
4.6 Median smoothing from a graph
4.7 Seasonal adjustment
4.8 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.1.2 Kick off with CAS

Time series graphs with CAS

A time series graph displays data where the x-variable is time. The value of the intervals in a time series can vary, from hours to days, weeks, months or years.

1. Use CAS to plot a time series graph for the following data set, which displays the average maximum quarterly temperature in Melbourne over a 36-month period starting from January 2011.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Temp. (°C)	24	17	15	21	25	17	16	22	25	18	16	20

2. Is there a pattern to your time series graph? If so, try to describe this pattern.
3. Use CAS to plot a comparative time series on a separate set of axes for the following data set, which displays the average maximum quarterly temperature in Auckland over the same 36-month period.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Temp. (°C)	23	18	15	19	22	17	16	20	23	17	15	19

4. Use CAS to plot a third time series on a separate set of axes for the following data set, which displays the average maximum quarterly temperature in New York over the same 36-month period.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Temp. (°C)	7	22	27	12	6	21	28	13	7	23	26	11

5. Are the patterns similar in all three time series? If not, what are the differences?

on Resources

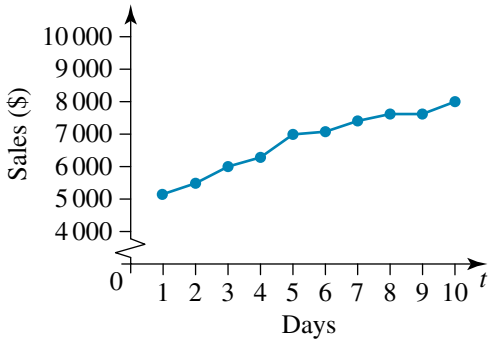
Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Time series and trend lines

In previous topics we looked at bivariate, or (x, y), data where both x and y could vary independently. In this topic we shall consider cases where the x-variable is time and, generally, where time goes up in even increments such as hours, days, weeks or years. In these cases we have what is called a **time series**. The main purpose of a time series is to see how some quantity varies with time. For example, a company may wish to record its daily sales figures over a 10-day period.

Time	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9	Day 10
Sales ($)	5200	5600	6100	6200	7000	7100	7500	7700	7700	8000

We could also make a graph of this time series as shown below.



As can be seen from this graph, there seems to be a trend upwards — clearly, this company is increasing its revenues.

studyon

Units 3 & 4 > AOS DA > Topic 8 > Concept 1

Nature of time series data Summary screen and practice questions

4.2.1 Types of time series

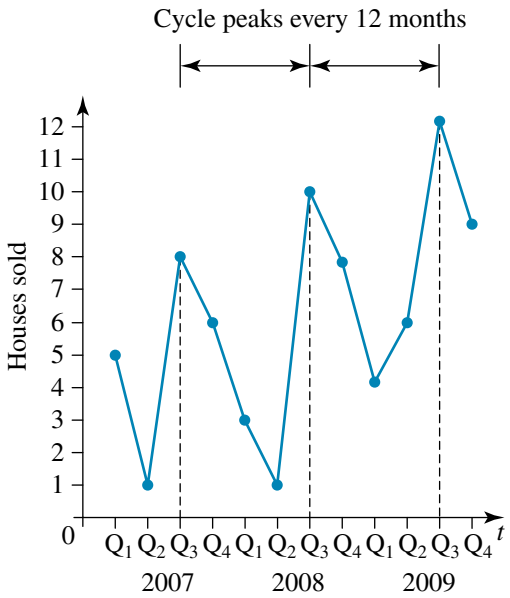
Seasonal fluctuations

Certain data seem to fluctuate during the year, as the seasons change. Consequently, this is termed a **seasonal time series**. The most obvious example of this would be total rainfall during summer, autumn, winter and spring in a year.

The name *seasonal* is not specific to the seasons of a year. It could also be related to other constant periods of highs and lows. For example, sales figures at a fast-food store could be consistently higher on Saturdays and Sundays and drop off during the weekdays. Here the seasons are days of the week and repeat once every week.

A key feature of seasonal fluctuations is that the seasons occur at the same time each cycle.

Here are some common seasonal periods.



	Seasons	Cycle	Example
Seasons	Winter, spring, summer, autumn	Four seasons in a *year*	Rainfall
Months	Jan., Feb., Mar., … , Nov., Dec.	12 months in a *year*	Grocery store monthly sales figures
Quarters	1st quarter (Q_1), 2nd quarter (Q_2), 3rd quarter (Q_3), 4th quarter (Q_4)	Four quarters in a *year*	Quarterly expenditure figures of a company
Week	Monday to Friday	Five days in a *week*	Daily sales for a store open from Monday to Friday only
Days	Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday	Seven days in a *week*	Number of hamburgers sold at a takeaway store daily

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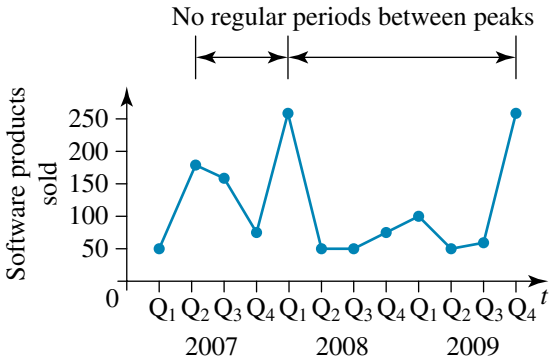
Units 3 & 4 › AOS DA › Topic 8 › Concepts 2 and 3

Long-term direction trends Summary screen and practice questions
Seasonal trends Summary screen and practice questions

Cyclic fluctuations

Like seasonal time series, **cyclic time series** show fluctuations upwards and downwards, but not according to season. Businesses often have cycles where at times profits increase, then decline, then increase again. A good example of this would be the sales of a new major software product. At first, sales are slow; then they pick up as the product becomes popular. When enough people have bought the product, sales may fall off until a new version of the product comes on the market, causing sales to increase again. This cycle can be repeated many times, which is why there are many versions of some software products.



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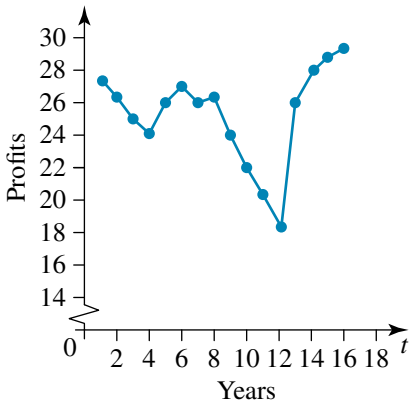
Cyclic trends Summary screen and practice questions

Irregular fluctuations

Fluctuations may seem to occur at random. This can be caused by external events such as floods, wars, new technologies or inventions, or anything else that results from random causes. There is no obvious way to predict the direction of the time series or even when it changes direction.

In the figure shown, there are a couple of minor fluctuations at $t = 4$ and $t = 8$, and a major one at $t = 12$. The major fluctuation could have been caused by a change in government which positively affected profits.



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Units 3 & 4 | AOS DA | Topic 8 | Concept 5

Irregular fluctuation Summary screen and practice questions

4.2.2 Plotting time series

To make better judgements about the type of time series, data in tabular form need to be plotted on a time series plot. This is similar to a scatterplot with some notable differences.

1. The explanatory variable is always *time*. This may be in days, days of the week, time of the day, weeks, months, quarters, years and so on. Thus, the *x*-axis variable is *time*.
2. As the periods are often labels and not numerical, the *x*-axis may be scaled using these period labels.
3. The points are connected. As they occur in chronological order, joining the points assists in identifying the type of time series and if a trend exists.

So that we can enter the data into CAS, time periods that are labels (and not numerical) need to be converted to numerals. For this, an **association table** is needed.

An association table summarises how the time periods are to be converted to numerical values. The first point is converted to 1, the second to 2 and so on, until the series is fully converted. Here are two examples.

Example 1

Week 1 Mon.	Week 1 Tues.	Week 1 Wed.	Week 1 Thurs.	Week 1 Fri.	Week 1 Sat.	Week 1 Sun.	Week 2 Mon.	Week 2 Tues.
1	2	3	4	5	6	7	8	9

Example 2

Jan. 2018	Feb. 2018	Mar. 2018	Apr. 2018	May 2018	June 2018	July 2018	Aug. 2018
1	2	3	4	5	6	7	8

4.2.3 Fitting trend lines

After we have plotted a time series graph, if there is a noticeable trend (upward, downward or flat) we can add a trend line, in the form of a straight line, to the data.

The trend lines we plot in this topic will be made using the least squares regression method we covered in topic 3. While these trend lines can be calculated manually, it is expected that you will use CAS to quicken this process.

WORKED EXAMPLE 1

The following table displays the school fees collected over a 10-week period. Plot the data and decide on the type of time series pattern. If there is a trend, fit a straight line.

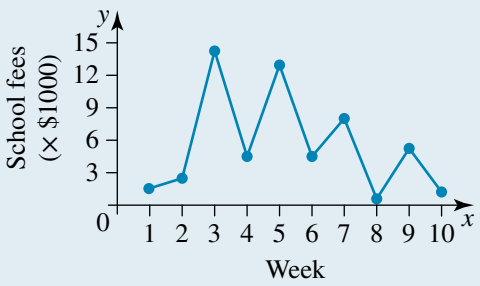
Week beginning	8 Jan.	15 Jan.	22 Jan.	29 Jan.	5 Feb.	22 Feb.	19 Feb.	26 Feb.	5 Mar.	12 Mar.
$ × 1000	1.5	2.5	14.0	4.5	13.0	4.5	8.5	0.5	5.0	1.0

THINK

1. Set up an association table. One method is to add another row and enter the numerical time code for each of the ten weeks starting at 1, 2, … , through to 10.
2. Construct a plot of the data. Place weeks on the horizontal axis and school fees on the vertical axis.
3. Identify the pattern as either seasonal, cyclical or irregular. If there is a trend, is it upwards or downwards?
4. Use CAS to fit a least-squares regression line.

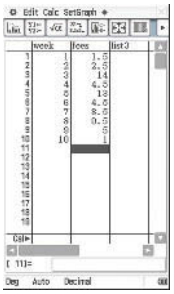
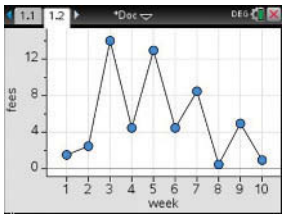
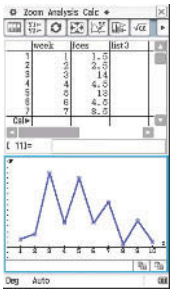
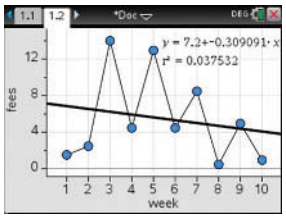
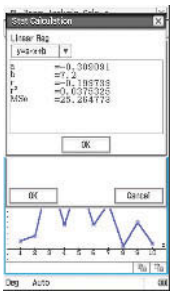
WRITE

Week beginning	8 Jan.	15 Jan.	22 Jan.	29 Jan.	5 Feb.	12 Feb.	19 Feb.	26 Feb.	5 Mar.	12 Mar.
$ × 1000	1 5	2.5	14.0	4.5	13.0	4.5	8.5	0.5	5.0	1.0
Time code	1	2	3	4	5	6	7	8	9	10



The school fees can be classified as cyclical or irregular with a downward trend. This is evident by the reducing totals in school fees collected.

$y = 7.2 - 0.31x$, where y represents school fees in thousands of dollars and $x = 1$ corresponds to week beginning on 8 January.

TI | THINK	WRITE	CASIO | THINK	WRITE
1. On a Lists & Spreadsheet page, enter the data into the lists, named *week* and *fees*.		1. On a Statistics screen, enter the data into the lists, named *week* and *fees*.	
2. Press CTRL + I and select: 5: Add Data & Statistics Use the tab key to move to each axis and press the click key or ENTER to place week on the horizontal axis and fees on the vertical axis. To join the dots, press MENU and then select: 2: Plot Properties 1: Connect Data Points		2. Tap: SetGraph Settings Complete the Set StatGraphs fields as: Draw: On Type: xy Line XList: main\week YList: main\fees Freq: 1 Mark: Square Tap Set and then the graph icon.	
3. The answer appears on the screen.	The school fees can be classified as cyclical or irregular with a downward trend. This is evident by the reducing totals in school fees collected.	3. The answer appears on the screen.	The school fees can be classified as cyclical or irregular with a downward trend. This is evident by the reducing totals in school fees collected.
4. To fit a least squares regression line, press MENU and select: 4: Analyze 6: Regression 2: Show Linear ($a + bx$)		4. To determine the least squares regression line, tap: Calc Regression Linear reg Complete the Set Calculation fields: Linear Reg XList: main\week YList: main\fees Freq: 1 Copy Formula y1 Copy Residual Off Tap OK.	
5. The answer appears on the screen.	$y = 7.2 - 0.31x$, where y represents school fees in thousands of dollars and $x = 1$ corresponds to week beginning on 8 January.	5. The answer appears on the screen.	$y = 7.2 - 0.31x$, where y represents school fees in thousands of dollars and $x = 1$ corresponds to week beginning on 8 January.

Exercise 4.2 Time series and trend lines

1. **WE1** The following table shows the sales for the first 8 years of a new business. Plot the data and decide on the type of time-series pattern. If there is a trend, fit a straight line.

Year	1	2	3	4	5	6	7	8
Sales ($)	1326	1438	1376	1398	1412	1445	1477	1464

2. Data was recorded about the number of families who moved from Melbourne to Ballarat over the last 10 years. Plot the data and decide on the type of time series pattern. If there is a secular trend, fit a straight line.

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Number moved	97	118	125	106	144	155	162	140	158	170

For questions **3** to **7**, identify whether the time series are likely to be seasonal, cyclic or irregular and if they are also displaying a trend:

3. the amount of rainfall, per month, in Western Victoria
4. the number of soldiers in the United States army, measured annually
5. the number of people living in Australia, measured annually
6. the share price of BHP Billiton, measured monthly
7. the number of seats held by the Liberal Party in Federal Parliament.

8. The following table shows the temperature in Victoria over a 10-day period.

Day	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	38	35	34	30	28	27	23	20	19	18

Fit a trend line to the data.

9. A wildlife park ranger is travelling on safari towards the centre of a wildlife park. Each day (t), he records the number of sightings (y) of zebra that he notes. He draws up the table shown.

t	1	2	3	4	5	6	7	8	9	10	11	12
y	6	9	13	8	9	14	15	17	14	11	15	19

Fit a trend line to the data.

10. The monthly share prices of a recently privatised telephone company were recorded as follows.

Date	January	February	March	April	May	June	July	August
Price ($)	2.50	2.70	3.00	3.20	3.60	3.70	3.90	4.20

Graph the data (let 1 = Jan., 2 = Feb., … and so on) and fit a trend line to the data. Comment on the feasibility of predicting share prices for the following year.

11. Plot the following monthly sales data for umbrellas. Discuss the type of time series reflected by the data and the limitations of a trend line.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Sales	5	10	15	40	70	95	100	90	60	35	20	10

12. Consider the data in the table shown, which represent the price of oranges over a 19-week period.

Week	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Price (cents)	40	45	53	46	40	45	62	58	67	60	72	60	64	78	74	66	78	81	80

a. Fit a straight trend line to the data.
b. Predict the price in week 25.

13. The following table represents the quarterly sales figures (in $000s) of a popular software product. Plot the data and fit a trend line. Discuss the type of time series best reflected by these data.

Quarter	Q_1-07	Q_2-07	Q_3-07	Q_4-07	Q_1-08	Q_2-08	Q_3-08	Q_4-08	Q_1-09	Q_2-09	Q_3-09	Q_4-09
Sales	120	135	150	145	140	120	100	110	120	140	190	220

14. The number of employees at the Comnatpac Bank was recorded over a 10-month period. Plot and fit a trend line to the data. What would you say about the trend?

Month	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Employees	6100	5700	5400	5200	4800	4400	4200	4000	3700	3300

4.3 Trend lines and forecasting

4.3.1 Association tables and forecasting

An association table is often required to convert period labels to a numerical value, so that a straight-line equation can be calculated. It is best to set up an extra row if data are in tabular form, or to change the labels shown on the axis of a time-series plot to numerical values. Here are three examples.

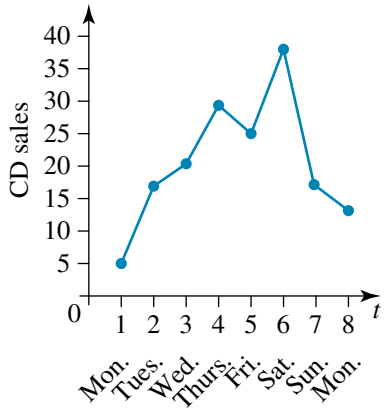
Example 1

Year	2014	2016	2017	2018
Time code	1	2	3	4

Example 2

1st quarter 2017	1
2nd quarter 2017	2
3rd quarter 2017	3
4th quarter 2017	4
1st quarter 2018	5
2nd quarter 2018	6

Example 3



For **forecasting**, use the association table to devise a time code for any period in the future. This time code will then be used in the straight-line equation.
From the three examples we can calculate that for:
Example 1: 2022 would have a time code of 8
Example 2: 1st quarter 2019 would have a time code of 9
Example 3: Monday week 4 would have a time code of 22.

WORKED EXAMPLE 2

A new tanning salon has opened in a shopping centre, with customer numbers for its first days shown in the following table. Fit a straight line to the data set using the least-squares regression method.

	Week 1							Week 2		
Period	**Mon.**	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**	**Sun.**	**Mon.**	**Tues.**	**Wed.**
Number of customers	9	9	11	13	16	18	19	20	23	27

Use the equation of the straight line to predict the number of customers for:
a. **Monday week 4**
b. **Thursday week 2.**

THINK

1. Complete an association table, where Monday week 1 is 1, Tuesday week 1 is 2, Wednesday week 2 is 10.

WRITE

	Week 1							Week 2		
Period	**Mon.**	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**	**Sun.**	**Mon.**	**Tues.**	**Wed.**
Number of customers	9	9	11	13	16	18	19	20	23	27
Time code	1	2	3	4	5	6	7	8	9	10

2. Use a calculator to find the equation of least-squares regression line.	$y = 5.67 + 1.97x$ Number of customers $= 5.67 + 1.97 \times$ time code, where time code 1 corresponds to Monday of week 1.
3. a. For Monday week 4, the time code is 22. Substitute $t = 22$ into the equation and evaluate. Round to the nearest integer.	**a.** Number of customers $= 5.67 + 1.97 \times$ time code $= 5.67 + 1.97 \times 22$ $= 49.01$ Number of customers $= 49$
b. For Thursday week 2, the time code is 11. Substitute $t = 11$ into the equation and evaluate. Round to the nearest whole number.	**b.** Number of customers $= 5.67 + 1.97 \times$ time code $= 5.67 + 1.97 \times 11$ $= 27.34$ Number of customers $= 27$

Note: Remember that forecasting is an extrapolation and if going too far into the future, the prediction is not reliable, as the trend may change.

Once an equation has been determined for a time series, it can be used to analyse the situation.

For the period given in Worked example 2, the equation is:

$$\text{Number of customers} = 5.67 + 1.97 \times \text{time code}.$$

The y-intercept (5.67) has no real meaning, as it represents the time code of zero, which is the day before the opening of the salon. The gradient or rate of change is of more importance. It indicates that the number of customers is changing; in this instance, growing by approximately 2 customers per day (gradient of 1.97).

WORKED EXAMPLE 3

The forecast equation for calculating share prices, y, in a sugar company was obtained from data of the share prices over the past 5 years . The equation is $y = 1.56 + 0.42t$, where $t = 1$ represents the year 2001, $t = 2$ represents the year 2002 and so on.

a. Rewrite the equation putting it in the context of the question.

b. Interpret the values of the gradient and y-intercept.

c. Predict the share price in 2013.

THINK	WRITE
a. The x-variable represents the time codes and the y-variable represents the share price in dollars.	**a.** Share price $= \$1.56 + \$0.42 \times$ time code Time code $t = 1$ is 2001, $t = 2$ is 2002 and so on.

b. The y-intercept of 1.56 represents the starting value; that is, when $t = 0$. The gradient of 0.42 represents the rate of change in share price with respect to time. That is, it will grow as it has a positive gradient.

c. If $t = 1$ is 2001, then for 2013, the time code will be $t = 13$. Substitute into the equation given.

b. The y-intercept of \$1.56 represents the approximate value of the shares in 2000. The gradient of \$0.42 means that on average the share price will grow by \$0.42 (42 cents) each year.

c. Share price $= \$1.56 + \$0.42 \times \text{time code}$
$= \$1.56 + \$0.42 \times 13$
$= \$1.56 + \$5.46$
$= \$7.02$

Before we fit a straight line to a data set using least-squares regression it is useful to draw a scatterplot. This is beneficial since it can:

1. Demonstrate how close the points are to a straight line, or if a curve is a better fit for the data.
2. Demonstrate if there is an outlier in the data set that could affect the least-squares regression.

If there is an outlier in the data and we are using the equation to make a prediction, then this can bias our prediction. If the outlier was removed from the data then this is more likely to give a better prediction since the least-squares regression will fit the data more closely.

WORKED EXAMPLE 4

The following table shows the sales for the first 8 years of the new business from question 1, Exercise 4.2.

a. Plot the data and decide if there are any outliers.
b. Re-plot the data after removing any outliers. Explain a possible reason for the outlier.
c. If there is a trend, fit a straight line to the data using a least-squares regression method.
d. Compare the gradient and y-intercept to that of question 1, Exercise 4.2.

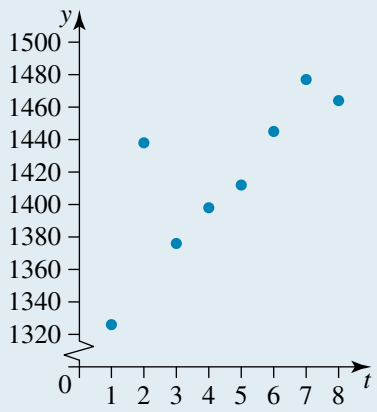
Year	1	2	3	4	5	6	7	8
Sales (\$)	1326	1438	1376	1398	1412	1445	1477	1464

THINK

a. Plot all the points on a scatterplot.

WRITE/DRAW

a.

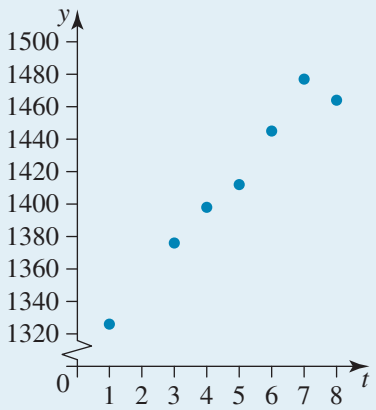


b. Looking at the scatterplot it appears that the second point is an outlier from a linear regression. Thus we remove this point and plot another scatterplot.

c. This looks like an upward trend, so use CAS to complete a least-squares regression on the reduced data set.

d. Compare the gradient and y-intercept to that of question **1**, Exercise 4.2, where the gradient was 16.45 and the y-intercept was 1342.96.

b.



Since we don't know the year in the second year of business or the type of business it is difficult to pinpoint a reason; however, they may have released a new line or developed something new to get this jump in sales. There could be many reasons for this outlier.

c. The linear trend line is $y = 1309.35 + 21.55x$, where y is the sales in $ and $x = 1$ corresponds to 1 year after the new business opening.

d. By removing the outlier the gradient has increased and the y-intercept has decreased. This would now be an improved trend line to make predictions from.

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Units 3 & 4 › AOS DA › Topic 9 › Concept 4

Time series trend lines and forecasting Summary screen and practice questions

Exercise 4.3 Trend lines and forecasting

1. **WE2** The number of people who watched the Channel 9 news over a fortnight is shown in the following table. Fit a straight line to the data set using a least-squares regression method.

Day	M	T	W	Th	F	S	Su
Viewers (1 000 000s)	1.20	1.18	1.16	1.18	0.9	0.75	1.0

Day	M	T	W	Th	F	S	Su
Viewers (1 000 000s)	1.21	1.23	1.19	1.16	0.95	0.68	0.98

Use the equation of the straight line to predict the number of viewers for:

a. Wednesday week 3

b. Monday week 4.

2. Data was recorded on the number of road fatalities 2009 in Australia in and 2010.

Month and year	01/09	02/09	03/09	04/09	05/09	06/09
Driver fatalities	115	117	132	153	130	131

Month and year	07/09	07/09	09/09	10/09	11/09	12/09
Driver fatalities	109	117	113	143	107	124

Month and year	01/10	02/10	03/10	04/10	05/10	06/10
Driver fatalities	126	98	104	115	134	111

Month and year	07/10	08/10	09/10	10/10	11/10	12/10
Driver fatalities	107	94	104	121	119	120

Source: Australian Government, Department of Infrastructure and Regional Development

Fit a straight line to the data set using the least-squares regression method and use the straight line to predict the number of fatalities in:

a. June 2015. **b.** April 2018.

3. The following table represents the number of cars remaining to be completed on an assembly line. Fit a straight line to the following data using the least-squares regression method.

Time (hours)	1	2	3	4	5	6	7	8	9
Cars remaining	32	26	27	23	16	17	13	10	9

a. Predict the number of cars remaining to be completed after 11 hours.

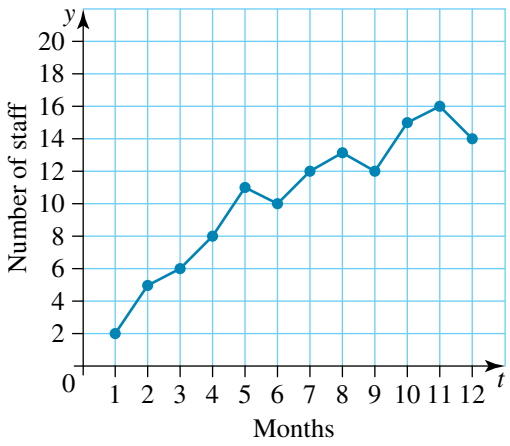
b. At what rate is the numbers of cars on the assembly line being reduced by?

4. From the equation of the trend line, it should be possible to predict when there are no cars left on the assembly line. This is done by finding the value of t which makes $y = 0$. Using the equation from question **3**, find the time when there will be no cars left on the assembly line.

5. When the MicroHard Company first started, it employed only one person. The company has consistently grown, so that after 12 months there are 14 people working there. The time-series data are shown by the graph.

a. Fit a straight line to the data using the least-squares regression method.

b. Predict the number of employees after a *further* 12 months.



6. The table below shows the share price of MicroHard during a volatile period in the stock market. Using CAS:

a. fit a least-squares regression line.

b. What type of time series is this?

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price ($)	2.75	3.30	3.15	2.25	2.10	1.80	1.50	2.70	4.10	4.20	3.55	1.65	2.60	2.95	3.25	3.70

7. The following time series shows the number of internet websites on a webring over a 9-month period. Plot the data and fit a line using the least-squares regression method. Comment on this line as a predictor of further growth.

Time (months)	1	2	3	4	5	6	7	8	9
Sites (millions)	2.00	2.20	2.50	3.10	3.60	4.70	6.10	7.20	8.50

8. **WE3** The forecast equation for calculating the share prices, y, of a bank company was obtained from the data of share prices over the past 7 years. The equation is $y = 22.74 + 0.28t$, where $t = 1$ represents the year 2007.
   a. Rewrite the equation putting it in the context of the question.
   b. Interpret the values of the gradient and y-intercept.
   c. Predict the share price in 2017.
9. The forecast equation for calculating the share prices, y, of a mining company was obtained from the data of share prices over the past 4 years. The equation is $y = 18.57 - 0.1t$, where $t = 1$ represents the year 2010.
   a. Rewrite the equation putting it in the context of the question.
   b. Interpret the values of the gradient and y-intercept.
   c. Predict the share price in 2019.
10. The forecast equation for calculating prices, y, of shares in a steel company was obtained from data of the share prices of the past 6 years. The equation is:

$$y = 2.56 + 0.72t$$

where $t = 1$ represents the year 2010, $t = 2$ represents the year 2011 and so on.
   a. Rewrite the equation putting it in the context of the question.
   b. Interpret the values of the gradient and the y-intercept.
   c. Predict the share price in 2020.
11. The Teeny-Tiny-Tot Company has started to make prams. Its sales figures for the first 8 months are given in the table shown.

Date	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.
Sales	65	95	130	115	145	170	190	220

   a. Using the sequence Jan. $= 1$, Feb. $= 2$, …, calculate the equation of the trend line using the least-squares regression method.
   b. Plot the data points and the trend line on the same set of axes.
   c. Use the trend line equation to predict the company's sales for December.
   d. Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.
12. The sales figures of Harold Courtenay's latest novel (in thousands of units) are given in the table shown. The book was released a week before the first figures were collected.

Time (weeks)	1	2	3	4	5	6	7	8	9
Sales (× 1000)	1	3	5	17	21	25	28	27	26

   a. Calculate the equation of the trend line for these data using the least-squares regression method.
   b. Plot the data points and the trend line on the same set of axes.
   c. Use the trend line equation to predict the sales for weeks 10, 12 and 14.
   d. Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

13. The average quarterly price of coffee (per 100 kg) has been recorded for 3 years.

Quarter	Q_1-16	Q_2-16	Q_3-16	Q_4-16	Q_1-17	Q_2-17	Q_3-17	Q_4-17	Q_1-18	Q_2-18	Q_3-18	Q_4-18
Price ($)	358	323	316	336	369	333	328	351	389	387	393	402

a. Calculate the equation of the trend line for these data using the least-squares regression method.
b. Plot the data points and the trend line on the same set of axes.
c. Use the trend line equation to predict the price for the next quarter.
d. Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

14. A mathematics teacher gives her students a test each month for 10 months, and the class average is recorded. The tests are carefully designed to be of similar difficulty.

Test	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.
Mark (%)	57	63	62	67	65	68	70	72	74	77

a. Calculate the equation of the trend line for these data using the least-squares regression method.
b. Plot the data points and the trend line on the same set of axes.
c. Use the trend line equation to predict the results for the last exam in December.
d. Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

15. WE4 The following table shows the All Ordinaries values from the Australian stock market over 8 years. The All Ordinaries consists of the 500 largest eligible companies.

Year	1995	1996	1997	1998	1999	2000	2001	2002
All Ords	1925	2140	2435	2500	2820	3200	2950	3050

a. Plot the data and decide if there are any outliers. Complete a least-squares regression including all data points.
b. Re-plot the data after removing any outliers.
c. If there is a trend, fit a straight line to the data using a least-squares regression method.
d. Compare the gradient and y-intercept of the two regressions.

16. The following table shows the stock price for Apple in 2009–10.

Month 2010	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
Stock price ($)	192.06	204.62	235.00	261.09	256.88	251.53	257.25	243.10	283.75

a. Plot the data and decide if there are any outliers. Complete a least-squares regression including all data points.
b. Re-plot the data after removing any outliers.
c. If there is a trend, fit a straight line to the data using a least-squares regression method.
d. Compare the gradient and y-intercept of the two regressions.

17. The average cost of a hotel room in Sydney in 2000 is shown in the table.

Month 2000	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
Hotel price ($)	250	240	235	237	239	230	228	237	332

a. Plot the data and decide if there are any outliers. Complete a least-squares regression including all data points.
b. Re-plot the data after removing any outliers.
c. If there is a trend, fit a straight line to the data using a least-squares regression method.
d. Compare the gradient and *y*-intercept of the two regressions and give a possible explanation for the outlier.

18. The percentage of homes that have a television is shown in the table.

Year	1970	1971	1972	1973	1974	1975	1976	1977	1978
Percentage	42	45	48	54	58	72	77	83	90

a. Plot the data and decide if there are any outliers.
b. Fit a straight line to the data using a least-squares regression method.
c. Explain any reason for the change in the data from its original trend.

4.4 Smoothing time series

When the data fluctuates a lot, it is often hard to see the underlying trend. In order to reveal the trend, we may need to try and remove some of these fluctuations before attempting to fit the trend line. This process is referred to as smoothing.

There are two basic techniques for smoothing *random* or *cyclical* variation: **median smoothing** and **moving-mean smoothing**.

Median smoothing is preferred where there are small data sets, as it can be done graphically on a time-series plot. Also, for data sets with many outliers due to the volatile random or cyclical trend, median smoothing is preferred.

We have seen earlier that the median is not affected by outliers, while the mean is.

Moving-mean smoothing is an option that is preferred for data sets with few random fluctuations.

4.4.1 Moving-mean smoothing

This technique relies on the principle that the means of data can be used to represent the original data. When applied to time series, a number of data points are averaged, then we move on to another group of data points in a systematic fashion and average them, and so on. It is generally quite simple. Consider the following example:

Notice how the third column in the table is computed from the first two columns.

1. Take the first three y-values (i.e. first, second and third) and find their average $\left(\frac{12 + 10 + 15}{3} = 12.3\right)$; place the result against $t = 2$.
2. Move down one line and again take three y-values (i.e. second, third and fourth). Find their average $\left(\frac{10 + 15 + 13}{3} = 12.7\right)$ and place the result against $t = 3$.
3. Continue moving down the table until you reach the last three points.

Time (t)	Data (y)	Moving mean
1	12	
2	10	$\frac{12 + 10 + 15}{3} = 12.3$
3	15	$\frac{10 + 15 + 13}{3} = 12.7$
4	13	$\frac{15 + 13 + 16}{3} = 14.7$
5	16	$\frac{13 + 16 + 13}{3} = 14.0$
6	13	$\frac{16 + 13 + 18}{3} = 15.7$
7	18	$\frac{13 + 18 + 21}{3} = 17.3$
8	21	$\frac{18 + 21 + 19}{3} = 19.3$
9	19	

As we use three points to average, moving down the table from top to bottom, the process is called a *3-point moving-mean smoothing*.

The number of points averaged at a time may vary: we could have a 4-point smoothing, a 5-point smoothing or even an 11-point smoothing. Although it is preferable to choose an odd number, such as 3 or 5, it is possible to choose even numbers as well, with a slight change in the method. Later in the topic, we will discuss how to choose the number of points for smoothing.

4.4.2 Moving-mean smoothing with odd numbers of points

As previously shown, the method for smoothing with an odd number (3, 5, ...) can be done in a vertical tabular form. The time values must be equally spaced, but they don't have to be in the sequence 1, 2, 3.

Note: There are fewer smoothed points than original ones. For a 3-point smooth, 1 point at either end is 'lost'; while for a 5-point smooth, 2 points at either end are 'lost'.

WORKED EXAMPLE 5

The temperature of a sick patient is measured every 2 hours and the results are recorded.

a. Use a 3-point moving-mean technique to smooth the data.

b. Plot both original and smoothed data on the same set of axes.

c. Predict the temperature for 18 hours using the last smoothed value.

Time (hours)	2	4	6	8	10	12	14	16
Temp. (°C)	36.5	37.2	36.9	37.1	37.3	37.2	37.5	37.8

THINK

a. 1. Put the data in a table.

2. Calculate a 3-point moving-mean for each data point.
Note: The 'lost' values are at $t = 2$ and $t = 16$. Therefore, the first point plotted is (4, 36.87).

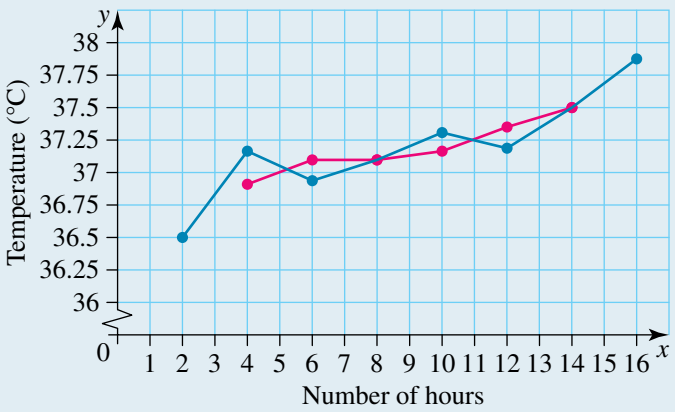
WRITE/DRAW

a.

Time (h)	Temp. (°C)	Smoothed temp. (°C)
2	36.5	
4	37.2	$\frac{1}{3}(36.5 + 37.2 + 36.9) = 36.87$
6	36.9	$\frac{1}{3}(37.2 + 36.9 + 37.1) = 37.07$
8	37.1	$\frac{1}{3}(36.9 + 37.1 + 37.3) = 37.1$
10	37.3	$\frac{1}{3}(37.1 + 37.3 + 37.2) = 37.2$
12	37.2	$\frac{1}{3}(37.3 + 37.2 + 37.5) = 37.33$
14	37.5	$\frac{1}{3}(37.2 + 37.5 + 37.8) = 37.5$
16	37.8	

b. 1. Plot the data. The smoothed line is the pink one.
Note: The smoothed data start at the 2nd time point and finish at the 7th point.

b.



2. Comment on the result.

The smoothed line has removed much of the fluctuation of the original time series and, in fact, clearly exposes the upwards trend in temperature.

c. Last smoothed data point is 37.5.

c. The temperature at 18 hours is predicted to be 37.5 °C.

4.4.3 Prediction using moving means

Because the moving mean does not generate a single linear equation, there are limited possibilities for using the resultant smoothed data for prediction. However, there are two things that can be done.

1. Predict the next value — use the last smoothed value to predict the next time point. In Worked example 5, our prediction for $t = 18$ would be temperature = 37.5. This is not necessarily an accurate prediction but it is the best we can do without a linear trend equation.
2. Fit a single straight line to the smoothed data — using the least-squares regression technique, one could find a single equation for the smoothed data points. This is often the preferred technique.

4.4.4 How many points should be in the moving mean?

When smoothing data, it is important to decide on the number of points to be used. Several factors that should be taken into account are discussed below. Here are some basic hints. (n = the number of points in the time series and p = the number of points taken at a time to be averaged.)

1. For small data sets, the value of p should be *considerably* smaller than n. For example, if $n = 7$, p should be no more than about 4.
2. The most common practice if there is a cyclic variation that we want to remove, is to let $p \approx$ length of cycle. If the data shows seasonal variation, p should be proportional to the number of seasons. For example, if there are quarterly data with seasonal variation, use $p = 4$.

Other preferred choices for number of points used in the smoothing procedures include:

Monthly sales figures	**use 12 point (centred)**
Daily sales for a store open each day of the week	**use 7 point**
Daily sales for a store open from Monday to Friday only	**use 5 point**
Quarterly electricity consumption figures	**use 4 point (centred)**

3. It is *always* preferable to use an odd value of p, regardless of whether n is even or odd.
4. The larger the value of p, the smoother the trend line of the resulting data becomes. More of the fluctuations will be removed. However, you can go too far as the bigger p is, the more data points are lost: when $p = 3$, we lose 2 points, if $p = 5$ we lose 4 points, if $p = 7$ we lose 6 points, etc. Therefore if the set is small, we may lose nearly all data by selecting a large value of p!

on Resources

Interactivity: Smoothing: moving mean method (int-6255)

studyon

Units 3 & 4 > AOS DA > Topic 9 > Concept 3

Moving-mean and moving-median smoothing Summary screen and practice questions

Exercise 4.4 Smoothing time series

1. WE5 The temperature of a pool was measured every hour over a summer's day and the results recorded.
   a. Use a 3-point moving-mean technique to smooth the data.
   b. Plot both the original and smoothed data on the same set of axes.
   c. Predict the temperature for 9 hours using the last smoothed value.

Time (h)	1	2	3	4	5	6	7	8
Temp (°C)	20.3	21.8	20.9	22.0	23.4	24.9	24.3	25.3

2. The membership numbers of the Collingwood Football Club over an eight year period are shown in the table.

Year (t)	2007	2008	2009	2010	2011	2012	2013	2014
Members (m)	38 587	42 498	45 972	57 408	71 271	72 688	78 427	72 170

   a. Use a 3-point moving-mean technique to smooth the data.
   b. Plot both the original and smoothed data on the same set of axes.
   c. Predict the number of members in 2016 using the last smoothed value.

3. The following table represents sales of a textbook.

Year (t)	2011	2012	2013	2014	2015	2016	2017	2018
Sales (y)	2250	2600	2400	2750	2900	2450	3100	3400

   a. Use a 3-point moving mean technique to smooth the data.
   b. Plot both the original and smoothed data.
   c. Predict the sales for 2019 using the last smoothed value.

4. The sales of a certain car seem to have been declining in recent months. The management wishes to find out if this is the case.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Sales	120	70	100	110	90	80	70	90	80	100	60	60

   a. Using a 3-point moving mean, smooth the data and comment on the result. Use Jan. = 1, Feb. = 2, ...
   b. Using the least-squares regression method, find the equation of the trend line for the smoothed data.
   c. Use the equation to predict the number of sales for March next year. Comment on the prediction.

5. Perform a 5-point moving mean smoothing on the data from question 4 and discuss the result.
6. Consider the quarterly rainfall data shown. Rainfall has been measured over a 3-years period. Perform a 3-point moving mean and comment on whether there is an underlying trend.

Time (t)	Spring 2015	Summer 2015	Autumn 2016	Winter 2016	Spring 2016	Summer 2016
Rainfall (mm)	100	50	65	120	90	50

Time (t)	Autumn 2017	Winter 2017	Spring 2017	Summer 2017	Autumn 2018	Winter 2018
Rainfall (mm)	60	110	85	40	50	100

7. The attendance at Bendigo Football Club games was recorded over 10 years. Management wishes to see if there is a trend.

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Attendance (×1000)	75	72	69	74	66	72	61	64	69	65

   a. Perform a 3-point moving mean smoothing on the data and comment on the result.
   b. Using least-squares repgression on the smoothed data, find the equation of the trend line.
   c. Use the equation from **b** to predict the attendance in 2020. Comment on the prediction.

8. Use technology to complete a 3-point moving mean smoothing on the following data which represent sales figures for a 21-week period.

Week	**Sales**	**Smoothed data**
1	34	
2	27	
3	31	
4	37	
5	41	
6	29	
7	32	
8	37	
9	47	
10	38	
11	41	

Week	**Sales**	**Smoothed data**
12	44	
13	47	
14	49	
15	41	
16	52	
17	48	
18	44	
19	49	
20	56	
21	54	

9. Coffee price data are shown in the table. Perform a 3-point moving average to smooth the data. Plot the smoothed and original data and comment on your result.

Quarter	Q_1-16	Q_2-16	Q_3-16	Q_4-16	Q_1-17	Q_2-17
Price ($)	358	323	316	336	369	333

Quarter	Q_3-17	Q_4-17	Q_1-18	Q_2-18	Q_3-18	Q_4-18
Price ($)	328	351	389	387	393	402

10. The sales of a new car can vary due to the effect of advertising and promotion. The sales figures for Nassin Motor Company's new sedan are shown in the table. Use 5-point moving means to smooth the data. Plot the data, and use the last smoothed value to predict sales for the next month.

Month	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Sales	141	270	234	357	267	387	288	303	367	465	398

11. A large building site requires varying numbers of workers. The weekly employment figures over the last 7 weeks have been recorded. By performing a 3-point moving mean smoothing, predict the number of people required for the next week.

Week	Number of employees
1	67
2	78
3	54
4	82
5	69
6	88
7	94

12. MC If you were to smooth quarterly gas bill data over a number of years, you would be best to use:
- **A.** 3-point
- **B.** 4-point (centred)
- **C.** 5-point
- **D.** 6-point (centred)
- **E.** 7-point

13. Use a spreadsheet or CAS to perform a 5-point moving mean smoothing on the data in question **8** and discuss your result.

14. Use a spreadsheet or CAS to perform a 5-point moving mean smoothing on the data in question **9** and discuss your result.

4.5 Smoothing with an even number of points

As mentioned in the previous section, it is usually preferable to use an odd number of points. However, there are situations when an even number of points should be used — that is, a 4-point, 6-point or even 12-point moving mean. When we used an odd number of points, the result was automatically centred; that is, the y-data had the same t-values as the original (except at the first and last 'lost' points). This does not occur with an even-point smoothing, as shown in the following example of a 4-point moving mean.

Time	*y*-value	4-point mean (smoothed value)	
		Calculation	Result
2012	6		
2013	10		
		(6 + 10 + 14 + 12) ÷ 4	10.5
2014	14		
		(10 + 14 + 12 + 11) ÷ 4	11.75
2015	12		
		(14 + 12 + 11 + 15) ÷ 4	13
2016	11		
		(12 + 11 + 15 + 16) ÷ 4	13.5
2017	15		
2018	16		

Observe that the first mean (10.5) is not aligned with any particular year — it is aligned with 2013.5! Also note that there are now three 'lost' values (the seven original records reduced to four). In other words, the moving mean is not centred properly. To align the data correctly, an additional step needs to be performed; this is called *centring*.

Use the following procedure to centre the data:

Step 1: Find the mean of the first two smoothed points and align it with the 3rd time point.

Step 2: Find the mean of the next two smoothed points and align it with the 4th time point.

Step 3: Repeat, leaving two blank entries at both top and bottom of the table.

This is demonstrated in the following table, using the data from the previous table.

Time	*y*-value	4-point mean (smoothed value)		4-point mean after centring	
		Calculation	Result	Calculation	Result
2012	6				
2013	10				
		(6 + 10 + 14 + 12) ÷ 4	10.5		
2014	14			(10.5 + 11.75) ÷ 2	11.125
		(10 + 14 + 12 + 11) ÷ 4	11.75		
2015	12			(11.75 + 13) ÷ 2	12.375
		(14 + 12 + 11 + 15) ÷ 4	13		
2016	11			(13 + 13.5) ÷ 2	13.25
		(12 + 11 + 15 + 16) ÷ 4	13.5		
2017	15				
2018	16				

The first mean (11.125) is now aligned with 2008, the second (12.375) aligned with 2015 and so on. This process not only introduces an extra step, but an *extra averaging* (or smoothing) as well. It is usually preferable to stick with an odd-point smoothing to reduce these difficulties.

WORKED EXAMPLE 6

The quarterly sales figures for a dress shop (in thousands of dollars) were recorded over a 2-years period. Perform a centred 4-point moving mean smoothing and plot the result. Comment on any trends that have been revealed.

Time	Summer	Autumn	Winter	Spring	Summer	Autumn	Winter	Spring
Sales (×$1000)	27	22	19	25	31	25	22	29

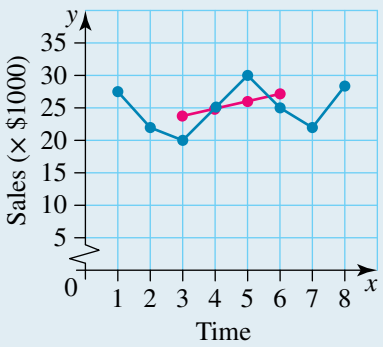
THINK

1. Arrange the data in a table. *Note:* Code the time column.
2. Calculate a 4-point moving mean in column 3.
3. Average the pairs of averages to find the 4-point centred data. This is done in column 4.

WRITE/DRAW

Time	Sales	4-point moving mean	4-point centred moving mean
1	27		
2	22		
		(27 + 22 + 19 + 25) ÷ 4 = 23.25	
3	19		(23.25 + 24.25) ÷ 2 = 23.75
		(22 + 19 + 25 + 31) ÷ 4 = 24.25	
4	25		(24.25 + 25.00) ÷ 2 = 24.63
		(19 + 25 + 31 + 25) ÷ 4 = 25.00	
5	31		(25.00 + 25.75) ÷ 2 = 25.38
		(25 + 31 + 25 + 22) ÷ 4 = 25.75	
6	25		(25.75 + 26.75) ÷ 2 = 26.25
		(31 + 25 + 22 + 29) ÷ 4 = 26.75	
7	22		
8	29		

4. Plot the data. The smoothed line is the red one. *Note:* The smoothed data start at the 3rd time point and finish at the 6th point.



5. Interpret the results.

Observe the steadily increasing trend (even with only four smoothed points) that was not obvious from the original data.

Exercise 4.5 Smoothing with an even number of points

1. WE6 The quarterly sales figures for a shoe shop (in thousands of dollars) were recorded over a 2-years period.
   a. Perform a centred 4-point moving mean smoothing and plot the results.
   b. Comment on any trend that has been revealed.

Time	Summer	Autumn	Winter	Spring	Summer	Autumn	Winter	Spring
Sales (×$1000)	59	48	43	50	63	52	47	61

2. The quarterly electricity cost figures for a shop were recorded over a 2-years period.
   a. Perform a centred 4-point moving mean smoothing and plot the results.
   b. Comment on any trend that has been revealed.

Time	Q_1-14	Q_2-14	Q_3-14	Q_4-14	Q_1-15	Q_2-15	Q_3-15	Q_4-15
Cost ($)	554	503	467	587	636	533	493	684

3. a. Perform a 4-point centred moving mean to smooth the following data and plot the result.
   b. Comment on any trends that you find.

t	1	2	3	4	5	6	7	8	9	10
y	75	54	62	60	70	45	54	59	62	64

4. The price of oranges fluctuates from season to season. Data have been recorded for 3 years. Perform a 4-point centred moving mean, plot the data and comment on any trends.

t	Autumn 2016	Winter 2016	Spring 2016	Summer 2016	Autumn 2017	Winter 2017
Price	45	67	51	44	52	76

t	Spring 2017	Summer 2017	Autumn 2018	Winter 2018	Spring 2018	Summer 2018
Price	63	48	58	80	66	52

5. **a.** Use technology to complete the following table. The time series represents the temperature of a hospital patient over 15 days.

Day	Temperature	4-point moving mean	4-point centred moving mean
1	36.6		
2	36.4	36.75	
3	36.8	36.825	
4	37.2	36.85	
5	36.9	36.95	
6	36.5	37	
7	37.2	37.05	
8	37.4	37.275	
9	37.1	37.375	
10	37.4	37.25	
11	37.6	37.275	
12	36.9	37.325	
13	37.2	37.15	
14	37.6		
15	36.9		

b. Using the smoothed data, find the equation of the least-squares regression line.

c. Use the trend line to predict the temperature of the patient on day 16.

6. The sales of summer clothing vary according to the season. The following table gives seasonal sales data (in thousands of dollars) for 3 years at a Darryl Jones department store.

Season	Q_3-15	Q_4-15	Q_1-16	Q_2-16	Q_3-16	Q_4-16
Sales	78	92	90	73	62	85

Season	Q_1-17	Q_2-17	Q_3-17	Q_4-17	Q_1-18	Q_2-18
Sales	83	70	61	78	74	59

a. Calculate a 4-point centred moving average.

b. Plot the original and smoothed data on the same set of axes.

c. Determine if there is an underlying upwards or downwards trend.

7. Calculate a 6-point centred moving mean on the data from question **5**.

8. An athlete wishes to measure her performance in running a 1 km race. She records her times over the last 10 days.

Day	1	2	3	4	5	6	7	8	9	10
Time (*s*)	188	179	183	180	173	171	182	168	171	166

a. Perform a 4-point centred moving mean smoothing.

b. Plot the original and smoothed data on the same set of axes.

c. Determine if there is a significant improvement in her times.

9. The following table shows the share price index of Industrial Companies during an unstable fortnight's trading. By calculating a 4-point centred moving mean, determine if there seems to be an upward or downward trend.

Day	1	2	3	4	5	6	7	8	9	10
Index	678	762	692	714	689	687	772	685	688	712

10. MC If data was given to you at the start of each season of the year, it would be best to smooth the data by using:

A. 3 point **B.** 4 point centred **C.** 5 point
D. 6 point centred **E.** 7 point

11. MC If you start with 12 points of data and carry out a 4-point centred moving average, the number of points you end up with is:

A. 8 **B.** 7 **C.** 6 **D.** 5 **E.** 4

12. MC You have data points for each day in the month of April and complete a 6-point centred moving average. The number of points you end up with is:

A. 30 **B.** 28 **C.** 26 **D.** 24 **E.** 22

13. Calculate a 6-point centred moving mean on the data from question **8**.

14. **a.** Use the smoothed data in question **9** to find the least-squares regression line.

b. Use the trend line to predict the price index of the Industrial Company after 15 days.

4.6 Median smoothing from a graph

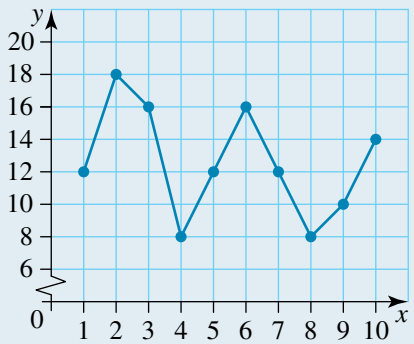
An alternative to moving-average smoothing is to replace the averaging of a group of points with the median of each group. It is a faster technique requiring no calculations and in the scope of this course, will only be addressed graphically.

Generally, the effect of median smoothing is to remove some random fluctuations. It performs poorly on cyclical or seasonal fluctuations — unless the size of the range being used (3, 5, 7, ... points) is chosen carefully.

Provided the graph has clearly marked data points, it is possible to find a median smooth directly from it.

WORKED EXAMPLE 7

Perform a 3-point median smoothing on the graph of a time series below.



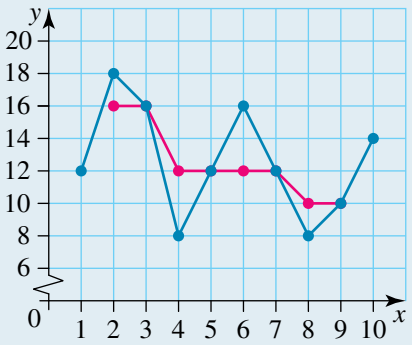
THINK

1. Read the data values and compute the median.

WRITE/DRAW

The 1st group of 3 points is: 12, 18, 16 — median = 16.
The 2nd group of 3 points is: 18, 16, 8 — median = 16.
The 3rd group of 3 points is: 16, 8, 12 — median = 12.
The 4th group of 3 points is: 8, 12, 16 — median = 12.
The 5th group of 3 points is: 12, 16, 12 — median = 12.
The 6th group of 3 points is: 16, 12, 8 — median = 12.
The 7th group of 3 points is: 12, 8, 10 — median = 10.
The 8th group of 3 points is: 8, 10, 14 — median = 10.

2. Plot the medians on the graph.
Note: Median smoothing has indicated a downward trend that is probably not in the real time series. This indicates that moving-average smoothing would be the preferred option.

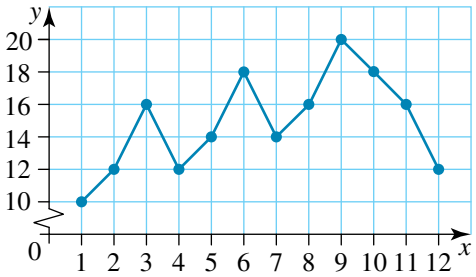


Resources

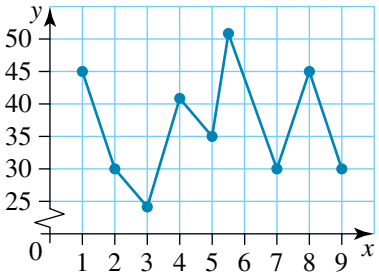
Interactivity: Smoothing: median method (int-6256)

Exercise 4.6 Median smoothing from a graph

1. **WE7** Perform a 3-point median smoothing on the graph of the time series shown.



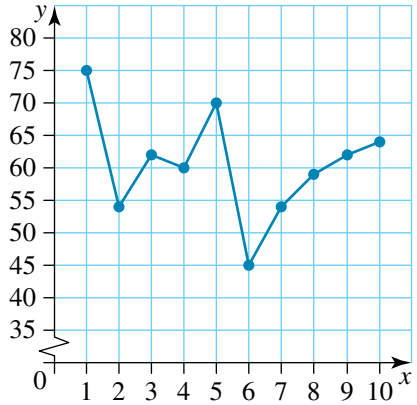
2. Perform a 3-point median smoothing on the graph of the time series shown.



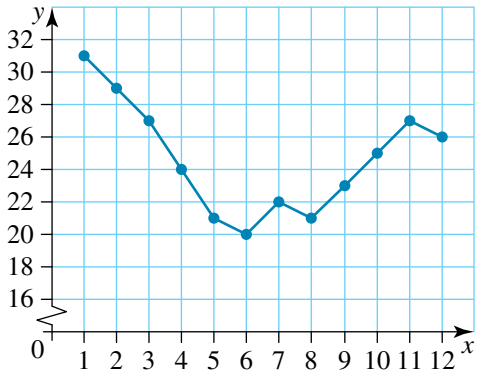
3. **a.** If you have a 12-point time series and perform a 3-point median smoothing on the data, how many data points would you be left with?

b. If you have an 11-point time series and perform a 5-point median smoothing on the data, how many data points would you be left with?

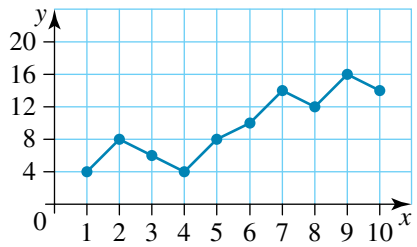
4. Perform a 3-point median smoothing on the graph below and plot the result. Comment on any trends that you find. These are the same data as in question 3, Exercise 4.5, so compare the graphs of the median smooth with the moving-mean smooth.



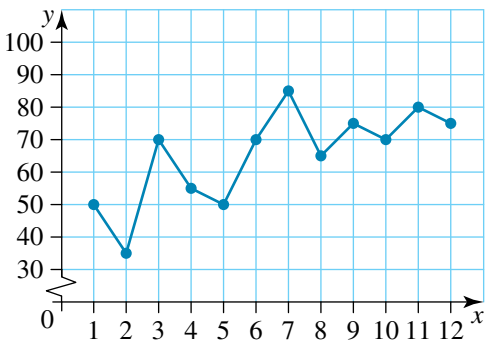
5. The maximum daily temperatures for a year were recorded as a monthly average. Perform a 3-point median smoothing on the following graph. Comment on your result.



6. Perform a 3-point median smoothing on the graphical time series shown below.

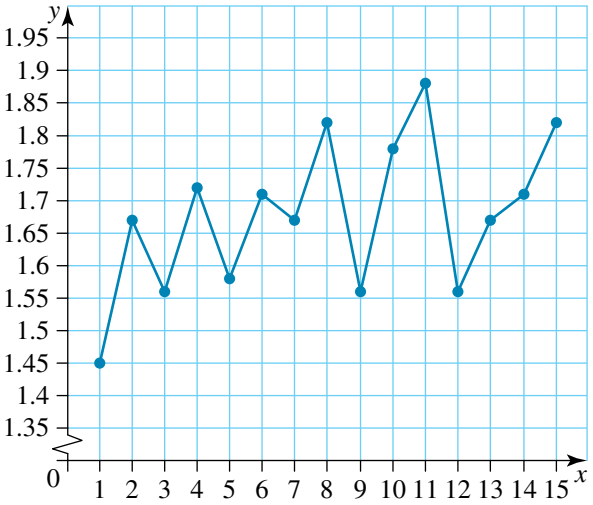


7. Comment on the effectiveness of the 3-point median smoothing in question 6.
8. Perform a 3-point median smoothing on the graphical time series shown below.

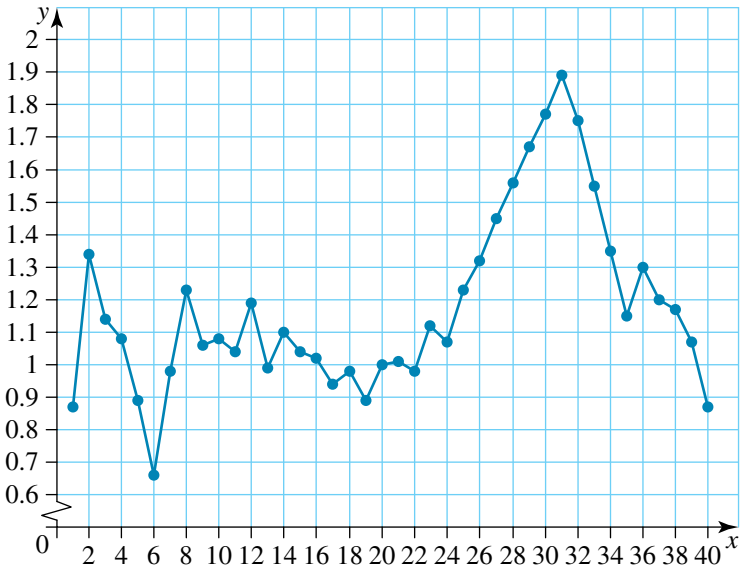


9. Comment on the effectiveness of the 3-point median smoothing in question 8.

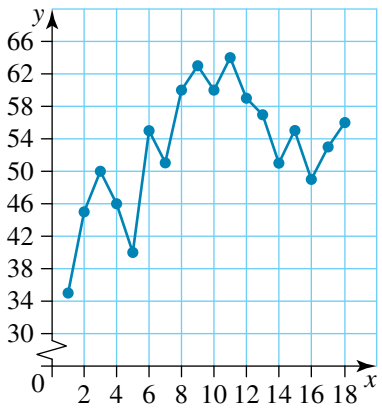
10. Perform a 3-point median smoothing on the graph shown, which represents the share price of the HAL computer company over the last 15 days.



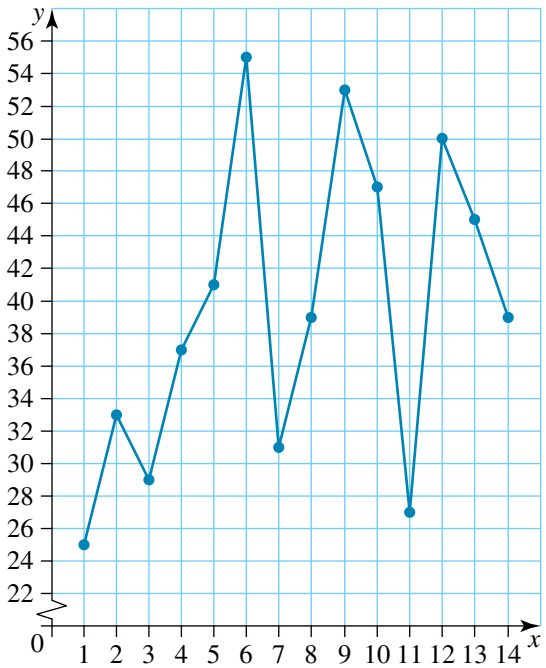
11. Perform a 5-point median smoothing on the graph, which represents the share price of the Pear-Shaped Computer Company over an 8-week trading period.



12. Perform a 5-point median smoothing on the following time series.



13. Perform a 3-point median smoothing on the following data.



14. Perform a 5-point median smoothing on the data from question **13**. Comment on the differences of the two smoothing techniques.

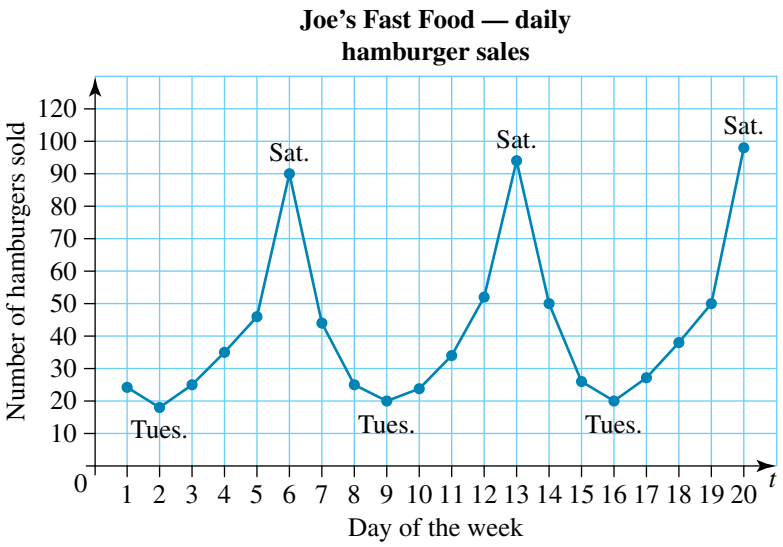
4.7 Seasonal adjustment

A seasonal time series is similar to a cyclical time series where there are defined peaks and troughs in the time-series data, except for one notable difference.

Seasonal time series have a fixed and regular period of time between one peak and the next peak in the data values. Similarly, there is a fixed and regular period of time between one trough and the next trough.

As we have seen in the sections on fitting a straight line to a time series, it is difficult to find an effective linear equation for such data. As well, the sections on smoothing indicated that seasonal data may not lend themselves to the techniques of moving-mean or median smoothing. We may just have to accept that the data vary from season to season and treat each record individually.

For example, the unemployment rate in Australia is often quoted as '6.8% — seasonally adjusted'. The Government has accepted that each season has its own time series, more or less independent of the other seasons. How can we



remove the effect of the season on our time series? The technique of seasonally adjusting, or **deseasonalising**, will modify the original time series, hopefully removing the seasonal variation, and exposing any other fluctuations (cyclic or irregular) which may be 'hidden' by seasonal variation.

4.7.1 Deseasonalising time series

The process of deseasonalising time series data involves calculating **seasonal indices**. A seasonal index compares a particular season to the average season.

That is, the seasonal index measures by what factor a particular season is above or below the average of all seasons for the cycle. For example:

Seasonal index = 1.3 means that season is 1.3 times the average season (that is, the figures for this season are 30% above the seasonal average). It is a peak or high season.

Seasonal index = 0.7 means that season is 0.7 times the average season (that is the figures for this season are 30% below the seasonal average). It is a trough or low season.

Seasonal index = 1.0 means that season is the same as the average season. It is neither a peak nor a trough.

To deseasonalise the data, we divide each value by the corresponding seasonal index. That is,

$$\textbf{Deseasonalised figure or value} = \frac{\textbf{actual original figure or value}}{\textbf{seasonal index}}.$$

The method of deseasonalising time series is best demonstrated with an example. Observe carefully the various steps, which must be performed in the order shown.

WORKED EXAMPLE 8

Unemployment figures have been collected over a 5-year period and presented in this table. It is difficult to see any trends.

Season	2014	2015	2016	2017	2018
Summer	6.2	6.5	6.4	6.7	6.9
Autumn	8.1	7.9	8.3	8.5	8.1
Winter	8.0	8.2	7.9	8.2	8.3
Spring	7.2	7.7	7.5	7.7	7.6

a. Calculate the seasonal indices.
b. Deseasonalise the data using the seasonal indices.
c. Plot the original and deseasonalised data.
d. Comment on your results, supporting your statements with mathematical evidence.

THINK

a. 1. Find the yearly averages over the four seasons for each year and put them in a table.

WRITE/DRAW

a. 2014: $(6.2 + 8.1 + 8.0 + 7.2) \div 4 = 7.375$
2015: $(6.5 + 7.9 + 8.2 + 7.7) \div 4 = 7.575$
2016: $(6.4 + 8.3 + 7.9 + 7.5) \div 4 = 7.525$
2017: $(6.7 + 8.5 + 8.2 + 7.7) \div 4 = 7.775$
2018: $(6.9 + 8.1 + 8.3 + 7.6) \div 4 = 7.725$

Year	2014	2015	2016	2017	2018
Average	7.375	7.575	7.525	7.775	7.725

2. Divide each term in the original time series by its yearly average. That is, divide each value for 2014 by the yearly average for 2014 (i.e. by 7.375); next divide each value for 2015 by the yearly average for 2015 (i.e. by 7.575) etc.

Summer 2014: $6.2 \div 7.375 = 0.8407$
Autumn 2014: $8.1 \div 7.375 = 1.0983$
Winter 2014: $8.0 \div 7.375 = 1.0847$
Spring 2014: $7.2 \div 7.375 = 0.9763$
Summer 2015: $6.5 \div 7.575 = 0.8581$
⋮
Spring 2018: $7.6 \div 7.725 = 0.9838$

Season	2014	2015	2016	2017	2018
Summer	0.8407	0.8581	0.8505	0.8617	0.8932
Autumn	1.0983	1.0429	1.1030	1.0932	1.0485
Winter	1.0847	1.0825	1.0498	1.0547	1.0744
Spring	0.9763	1.0165	0.9967	0.9904	0.9838

3. Determine the seasonal averages from this second table. That is, find the average of all five values for summer; next find the average of all values for autumn and so on. These are called *seasonal indices*.

Summer: $(0.8407 + 0.8581 + 0.8505 + 0.8617 + 0.8932) \div 5 = 0.8608$
Autumn: $(1.0983 + 1.0429 + 1.1030 + 1.0932 + 1.0485) \div 5 = 1.0772$
Winter: $(1.0847 + 1.0825 + 1.0498 + 1.0547 + 1.0744) \div 5 = 1.0692$
Spring: $(0.9763 + 1.0165 + 0.9967 + 0.9904 + 0.9838) \div 5 = 0.9927$

Season	Summer	Autumn	Winter	Spring
Seasonal index	0.8608	1.0772	1.0692	0.9927

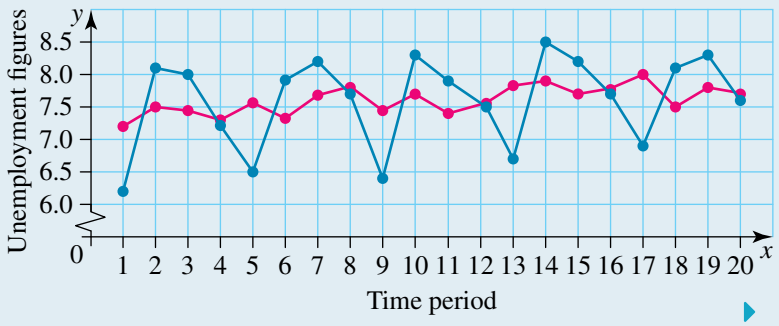
b. Divide each term in the original series by its seasonal index. That is, divide all summer figures by summer seasonal index (0.8608), all autumn figures by the autumn seasonal index (1.0772) and so on. This gives the seasonally adjusted or deseasonalised time series.
Note: Your answers may vary a little, depending upon how and when you rounded your calculations.

b. Summer 05: $6.2 \div 0.8608 = 7.2023$
Autumn 05: $8.1 \div 1.0772 = 7.5195$
⋮
Spring 09: $7.6 \div 0.9927 = 7.6557$

	2014	2015	2016	2017	2018
Summer	7.202	7.551	7.435	7.7483	8.015
Autumn	7.520	7.334	7.705	7.891	7.520
Winter	7.482	7.669	7.388	7.669	7.763
Spring	7.253	7.756	7.555	7.756	7.656

c. Graph the original and the seasonally adjusted (deseasonalised) time series.

c.



d. Note that most, but not all, of the seasonal variation has been removed. However, by using least-squares, we could more confidently fit a straight line to the deseasonalised data.

d. There appears to be a slight upward trend in unemployment figures.

4.7.2 Forecasting with seasonal time series

In the previous section we smoothed out the seasonal variation and are now able to see any trends more clearly. If there is an upward or downward secular trend, then a straight line equation can be calculated and used for making predictions into the future. Using the least-squares regression method on the *deseasonalised* data is always preferred.

Once the equation of the regression line for deseasonalised data has been obtained, it can be used for **forecasting**.

However, the prediction obtained using such an equation will also be deseasonalised or smoothed out to the average season. But as we have the relevant seasonal indices, we should be able to use it to remove the smoothing; that is, to **re-seasonalise** the predicted value.

The formula for re-seasonalising is:

Re-seasonalised figure or value = deseasonalised figure or value × seasonal index

WORKED EXAMPLE 9

Use the deseasonalised data from Worked example 8 to find the equation of the straight line for the deseasonalised data using the least-squares regression method. Predict the unemployment figure for summer in 2019. The deseasonalised data are reproduced in the table. (The seasonal index for summer is 0.8608.)

	2014	2015	2016	2017	2018
Summer	7.202	7.551	7.435	7.783	8.015
Autumn	7.520	7.334	7.705	7.891	7.520
Winter	7.482	7.669	7.388	7.669	7.763
Spring	7.253	7.756	7.555	7.756	7.656

THINK

1. Use CAS to find the equation of the least-squares regression line for the deseasonalised data.

WRITE

Deseasonalised unemployment (%) $= 7.357 + 0.0227 \times$ time code, where time code 1 represents summer 2014.

2. Using the association table, the summer of 2019 will be represented by $t = 21$. Substitute into the equation.

Summer of 2019:
Deseasonalised unemployment (%)
$= 7.357 + 0.0227 \times 21$
$= 7.834\%$

3. The predicted value is very high for summer. Re-seasonalise by using the seasonal index for summer, which was 0.8608. That is, this was a season or period of low unemployment.

Seasonalised value
= deseasonalised value × seasonal index
$= 7.834 \times 0.8608$
$= 6.74\%$

TI | THINK

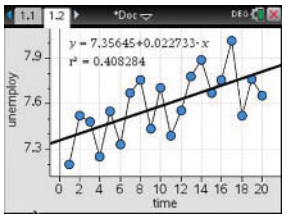
1. On a Lists & Spreadsheet page, enter the data into the lists, named *time* and *unemploy*.

WRITE

4. Press CTRL + I and select:
5: Add Data & Statistics

Use the tab key to move to each axis and press the click key or ENTER to place week on the horizontal axis and fees on the vertical axis.

To join the dots, press MENU and then select:
2: Plot Properties
1: Connect Data Points
To fit a least squares regression line, press MENU and select:
4: Analyze
6: Regression
2: Show Linear ($a + bx$)



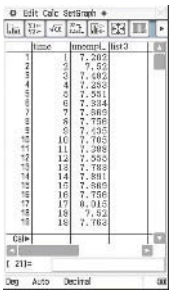
4. The answer appears on the screen.

Deseasonalised unemployment (%)
$= 7.36 + 0.0027 \times$ time code, where time code 1 represents summer 2014.

CASIO | THINK

1. On a Statistics screen, enter the data into the lists, named *time* and *unemploy*.

WRITE



4. To determine the least squares regression line, tap:
Calc
Regression
Linear reg

Complete the Set Calculation fields:
Linear Reg
XList: main\time
YList: main\unemploy
Freq: 1
Copy Formula y1
Copy Residual Off

Tap OK.

4. The answer appears on the screen.

Deseasonalised unemployment (%)
$= 7.36 + 0.0027 \times$ time code, where time code 1 represents summer 2014.

Note: Forecasting comes with limitations and cannot be used for an infinite time period. It should be used only for a small time period after your most recent recorded data.

WORKED EXAMPLE 10

Quarterly sales figures for a pool chemical supplier between 2013 and 2018 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Summer	1.8	1.2	0.2	0.8

Using the seasonal indices provided in the table, calculate the following.

a. Find the deseasonalised figure if the actual sales figure for the second quarter in 2017 was $4680.

b. Find the deseasonalised figure if the actual sales figure for the third quarter in 2017 was $800.

c. Find the predicted value if the deseasonalised predicted value for the first quarter in 2019 is expected to be $4000.

THINK

Use the formula for deseasonalising.

a. Use the 2nd quarter seasonal index.

b. Use the 3rd quarter seasonal index to obtain the deseasonalised figure.

c. Use the re-seasonalising formula and select the 1st quarter seasonal index.

WRITE

a. $$\text{Deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$
$$= \frac{4680}{1.2}$$
$$= \$3900$$

b. $$\text{Deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$
$$= \frac{800}{0.2}$$
$$= \$4000$$

c. Re-seasonalised figure
$= \text{deseasonalised figure} \times \text{seasonal index}$
$= 4000 \times 1.8$
$= \$7200$
The forecast sales figure for the first quarter in 2019 is $7200.

4.7.3 Seasonal indices

Finally, it should be noted that the sum of all the seasonal indices gives a specific result, which can be used to answer certain types of queries.

The sum of the seasonal indices is equal to the number of seasons.

This can be summarised as follows.

Type of data	Number of seasons	Cycle	Sum of all the seasonal indices
Monthly figures	12	A year	12
Quarterly figures	4	A year	4
Fortnightly figures	26	A year	26
Daily figures for data from Monday to Friday only	5	A week	5
Daily figures for data from Monday to Sunday	7	A week	7

WORKED EXAMPLE 11

A fast food store that is open seven days a week has the following seasonal indices.

Season	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Index	0.5	0.2	0.5	0.6		2.2	1.1

The index for Friday has not been recorded. Calculate the missing index.

THINK

1. The sum of the seasonal indices is equal to the number of seasons.

2. The missing index is the sum of all the other seasons subtracted from the total.

WRITE

There are 7 seasons (Monday to Sunday), therefore the sum of indices is 7.

Friday index
$= 7 - (\text{sum of all the other indices})$
$= 7 - (0.5 + 0.2 + 0.5 + 0.6 + 2.2 + 1.1)$
$= 7 - 5.1$
$= 1.9$

studyon

Units 3 & 4 > AOS DA > Topic 9 > Concepts 1 and 2

Seasonal indices Summary screen and practice questions
Deseasonalising data Summary screen and practice questions

on Resources

Interactivity: Seasonal adjustment (int-6257)

Exercise 4.7 Seasonal adjustment

Note: Your answers may vary slightly, depending upon rounding. Try to round correct to 4 decimal places for all intermediate calculations.

1. **WE8** The prices of 1 litre of milk have been collected over a 5-year period and are presented in the table.
   It is difficult to see any trends, other than seasonal ones.

Season	2014	2015	2016	2017	2018
Summer	1.55	1.60	1.50	1.60	1.45
Autumn	1.50	1.55	1.50	1.45	1.40
Winter	1.65	1.75	1.70	1.70	1.60
Spring	1.70	1.80	1.75	1.80	1.65

   a. Calculate the seasonal indices.
   b. Deseasonalise the data using the seasonal indices.
   c. Plot the original and deseasonalised data.
   d. Comment on your results, supporting your statements with mathematical evidence.

2. The average prices of 1 litre of petrol have been collected over a 5-year period and are presented in the table. It is difficult to see any trends, other than seasonal ones.

Season	2014	2015	2016	2017	2018
Summer	1.60	1.60	1.57	1.63	1.58
Autumn	1.48	1.50	1.46	1.31	1.27
Winter	1.40	1.35	1.39	1.28	1.19
Spring	1.65	1.60	1.63	1.69	1.59

   a. Calculate the seasonal indices.
   b. Deseasonalise the data using the seasonal indices.
   c. Plot the original and deseasonalised data.
   d. Comment on your results, supporting your statements with mathematical evidence.

3. **WE9** Use the deseasonalised data from question **1** to find the equation of the straight line for the deseasonalised data using the least-squares regression method. Predict the price of 1 litre of milk for summer 2019. The deseasonalised data are reproduced below. (The seasonal index for summer is 0.96.)

Season	2014	2015	2016	2017	2018
Summer	1.61	1.67	1.56	1.67	1.51
Autumn	1.63	1.68	1.63	1.58	1.52
Winter	1.57	1.67	1.62	1.62	1.52
Spring	1.57	1.67	1.62	1.67	1.53

4. Use the deseasonalised data from question 2 to find the equation of the straight line for the deseasonalised data using the least-squares regression method. Predict the price of 1 litre of petrol for summer 2019. The deseasonalised data are reproduced below. (The seasonal index for summer is 1.0731.)

Season	2014	2015	2016	2017	2018
Summer	1.49	1.49	1.46	1.52	1.47
Autumn	1.57	1.59	1.55	1.39	1.35
Winter	1.58	1.52	1.57	1.44	1.34
Spring	1.50	1.46	1.49	1.54	1.45

5. The price of sugar ($/kg) has been recorded over 3 years on a seasonal basis.

Season	2016	2017	2018
Summer	1.03	0.98	0.95
Autumn	1.26	1.25	1.21
Winter	1.36	1.34	1.29
Spring	1.14	1.07	1.04

   a. Compute the seasonal indices.
   b. Deseasonalise the data using the seasonal indices.
   c. Plot the original and deseasonalised data.
   d. Comment on your results, supporting your statements with mathematical evidence.

6. Data on the total seasonal rainfall (in mm) have been accumulated over a 6-year period.

Season	2013	2014	2015	2016	2017	2018
Summer	103	97	95	117	118	120
Autumn	93	84	82	100	99	98
Winter	143	124	121	156	155	151
Spring	123	109	107	125	122	124

   a. Compute the seasonal indices.
   b. Deseasonalise the original time series.
   c. Plot the original and deseasonalised time series.
   d. Comment on your result, supporting your statements with mathematical evidence.

7. It is known that young people (18–25) have problems in finding work; these problems are different from those facing older people. The youth unemployment statistics are recorded separately from the overall data. Using the youth unemployment figures for five years shown:

Season	2014	2015	2016	2017	2018
Summer	7.6	7.7	7.8	7.7	7.9
Autumn	10.9	11.3	11.9	12.6	13.1
Winter	11.7	12.4	12.8	13.5	13.9
Spring	9.9	10.5	10.8	11.4	11.9

a. compute the seasonal indices
b. deseasonalise the time series
c. plot the original and deseasonalised time series
d. comment on your result, supporting your statements with mathematical evidence.

8. The unemployment rate in a successful European economy is given in the table as a percentage.

Quarter	1	2	3	4
2016	5.8	4.9	3.5	6.7
2017	6.1	5.1	3.2	6.5
2018	5.7	4.5	4.1	7.1

a. Compute the seasonal indices.
b. Deseasonalise the time series.
c. Plot the original and deseasonalised time series.
d. Find the equation of the line-of-best-fit for the deseasonalised data using the least-squares method.
e. Use the equation of the line from part **d** to predict the unemployment rate for:
i. quarter 1 in 2019 **ii.** quarter 3 in 2023.

9. It is possible to seasonally adjust time series for other than the usual 4 seasons. Consider an expensive restaurant that wishes to study its customer patterns on a daily basis. In this case a 'season' is a single day and there are 7 seasons in a weekly cycle. Data are total revenue each day shown in the table which follows. Modify the spreadsheet solution to allow for these 7 seasons and deseasonalise the following data over a 5-week period. Comment on your result, supporting your statements with mathematical evidence.

Season	Week 1	Week 2	Week 3	Week 4	Week 5
Monday	1036	1089	1064	1134	1042
Tuesday	1103	1046	1085	1207	1156
Wednesday	1450	1324	1487	1378	1408
Thursday	1645	1734	1790	1804	1789
Friday	2078	2204	2215	2184	2167
Saturday	2467	2478	2504	2526	2589
Sunday	1895	1786	1824	1784	1755

10. MC A line-of-best-fit for deseasonalised data was given as:

$$\text{Deseasonalised monthly sales} = 10\,000 + 1500 \times \text{time code}$$

where June 2018 represents $t = 1$.

The predicted actual expected sales figure for June 2019, if the June seasonal index is 0.8, would be:

A. \$23 600 **B.** \$29 500 **C.** \$19 500 **D.** \$36 875 **E.** \$35 000

11. The following table gives the deseasonalised figures and corresponding seasonal indices for umbrella sales.

Season	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Number of umbrellas (deseasonalised)	24	24	25	26	25	27	27	28	30	31	33	34
Index	1.15	0.90	0.20	0.20	0.35	0.45	3.0	2.10	2.15	0.95	0.40	0.15

a. Find the equation of the straight line for the deseasonalised data using the least-squares regression method.

b. Predict the umbrella sales for January the following year.

12. WE10 Quarterly sales figures of a pet supplier between 2013 and 2018 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.4	1.2	0.6	0.8

Using the seasonal indices provided in the table, calculate the following.

a. Find the deseasonalised figure if the actual sales figure for the second quarter in 2017 was \$4345.

b. Find the deseasonalised figure if the actual sales figure for the third quarter in 2017 was \$950.

c. Find the predicted value if the deseasonalised predicted value for the first quarter in 2019 is expected to be \$5890.

13. Quarterly sales figures of a surf shop between 2013 and 2018 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.7	1.3	0.4	0.6

Using the seasonal indices provided in the table, calculate the following.

a. Find the deseasonalised figure if the actual sales figure for the second quarter in 2017 was $8945.
b. Find the deseasonalised figure if the actual sales figure for the third quarter in 2017 was $3250.
c. Find the predicted value if the deseasonalised predicted value for the first quarter in 2019 is expected to be $7950.

14. Quarterly sales figures for an ice-cream parlour between 2016 and 2018 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.50	1.00	0.25	1.25

Using the seasonal indices provided in the table, calculate the following.

a. Find the deseasonalised figure, if the actual sales figure for the second quarter in 2017 was $3000.
b. Find the deseasonalised figure, if the actual sales figure for the third quarter in 2017 was $800.
c. Find the predicted value, if the deseasonalised predicted value for the first quarter in 2019 is expected to be $3200.

15. WE11 A pizza shop that is open seven days a week has the following seasonal indices. The index for Friday has not been recorded. Calculate the missing index.

Season	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Index	0.4	0.3	0.6	0.8		1.8	1.3

16. A cinema open seven days a week has the following seasonal indices. Calculate the missing index.

Season	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Index	0.4	1.1	0.5	0.6	1.3		1.6

17. A newsagency store that is open seven days a week has the following seasonal indices.

Season	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Index	0.5	0.2		0.6	1.5	2.2	1.1

Find the value of the missing index.

18. Complete the following table of seasonal indices.

Season	Summer	Autumn	Winter	Spring
Index	1.23	0.89		1.45

Questions **19** and **20** relate to the following table, which contains the seasonal indices for the monthly sales of spring water in a particular supermarket.

Season	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Index	1.05		1.0	1.0	0.95	0.85	0.8	0.9	0.95	1.05	1.10	1.15

19. MC The seasonal index missing from the table is:

A. 1.0 **B.** 1.05 **C.** 1.10 **D.** 1.15 **E.** 1.20

20. MC If the actual sales figure for June 2012 was $102 000, then the deseasonalised figure would be:

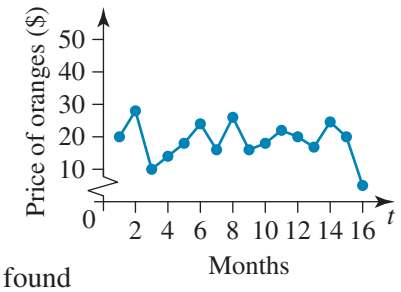
A. $96 900 **B.** $86 700 **C.** $107 368.42 **D.** $120 000 **E.** $102 000

4.8 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

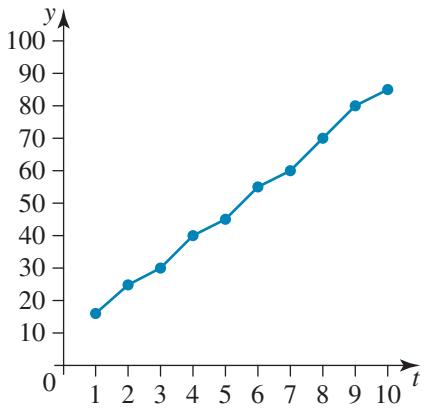
Multiple choice

1. **MC** The price of oranges over a 16-month period is recorded in the figure. The trend can be described as:
   - **A.** Cyclic
   - **B.** Seasonal
   - **C.** Irregular
   - **D.** Secular
   - **E.** There is no trend.



2. **MC** From another 16-month time series for the price of apples, it was found that the least-squares trend line was: price $= 8.45 + 0.415 \times$ month.
   A prediction for the price of apples in month 18 is:
   - **A.** 8.45
   - **B.** 0.42
   - **C.** 6.64
   - **D.** 15.92
   - **E.** unable to be determined with the above information

3. **MC** A least-squares trend line has been fitted to the time series in the figure below.



   Its equation is most likely to be:
   - **A.** $y = 10t$
   - **B.** $y = -8t + 10$
   - **C.** $y = 8t$
   - **D.** $y = 8t - 10$
   - **E.** $y = 8t + 10$

4. **MC** The following data represent the number of employees in a car manufacturing plant. The data are smoothed using a 3-point moving average.

Year	2011	2012	2013	2014	2015	2016	2017	2018
Number	350	320	300	310	270	240	200	160

   The first two points in the smoothed trend line are:
   - **A.** 320 and 300
   - **B.** 320 and 310
   - **C.** 323 and 310
   - **D.** 335 and 310
   - **E.** 323 and 273

5. **MC** How many points will the smoothed trend in question 4 contain?
   - **A.** 8
   - **B.** 7
   - **C.** 6
   - **D.** 5
   - **E.** 4

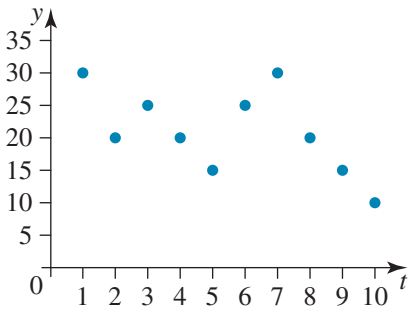
6. MC Consider the following data.

Time	2012	2013	2014	2015	2016	2017	2018
y-value	12	13	16	16	17	19	22

The value, after a 4-point moving average smoothing after centring, plotted against the year 2015 is:

A. 16.25 **B.** 14.25 **C.** 15.5 **D.** 17 **E.** 14.875

7. MC A 3-point median smoothing is performed on the data in the figure shown. The last smoothed value is:



A. 25 **B.** 21.7 **C.** 20 **D.** 15 **E.** 9

8. MC Seasonal indices and adjustment can be used when:

A. there are irregular variations in the data
B. there are seasonal variations along with a secular trend
C. there are seasonal variations only
D. there are seasonal or cyclic variations
E. there are at least 4 seasons' worth of data

9. MC The seasonal indices in the table were obtained from a time series.

Season	Spring	Summer	Autumn	Winter
Index	1.12	0.78	0.92	

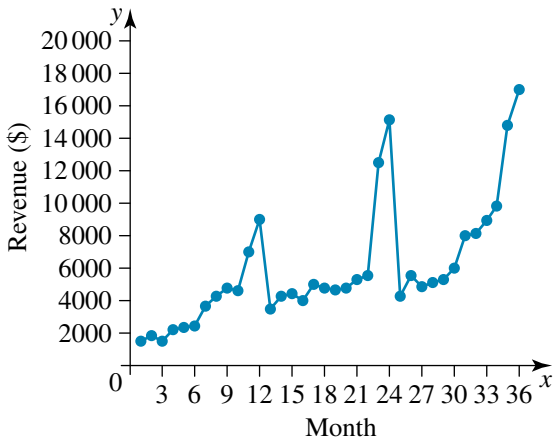
The value of the winter's seasonal index is:

A. 1.18
B. 0.94
C. 1.08
D. 1.06
E. unable to be determined from the given information

10. MC Using the data from question 9, a seasonally adjusted value for the summer of 2010, when the original value was 520, is closest to:

A. 406
B. 667
C. 464
D. 614
E. cannot be determined without additional information

11. MC The time series plot shows the revenue from sales (in dollars) each month made by a Queensland souvenir shop over a 3-year period.



This time series plot indicates that, over the 3-year period, revenue from sales each month showed:

A. no overall trend
B. no correlation
C. positive skew
D. an increasing trend only
E. an increasing trend with seasonal variation

Short answer

1. The number of uniforms sold in a school uniform shop is reported in the table.

Month	Number of uniforms sold
January	118
February	92
March	53
April	20
May	47
June	102
July	90
August	42
September	35
October	26
November	12
December	58

Fit a trend line to these data. What type of trend is best reflected by these data? Can you explain these trends?

2. Fit a least-squares regression line for the following data, which represent the sales at a snack bar during the recent Melbourne show. State the gradient and y-intercept.

Day	Sales
1	2300
2	2200
3	2600
4	3100
5	2900
6	3200
7	3300
8	3500

3. A hotel records the number of rooms booked over an 11-day period. Fit a trend line using the least-squares method.

Day	1	2	3	4	5	6	7	8	9	10	11
Rooms	12	8	15	20	22	20	25	24	26	28	30

   a. State the gradient and y-intercept, rounded to 2 decimal places.
   b. Predict the number of rooms booked for days 12 and 13.

4. Perform a 3-point moving average smoothing on the following rainfall data. Plot the original and smoothed data on the same set of axes. Give all answers rounded to 1 decimal place.

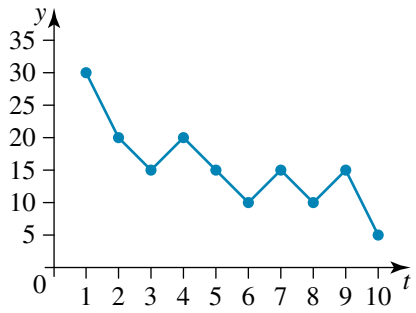
Day	1	2	3	4	5	6	7	8
Rooms	2	5	4	6	3	7	6	9

5. Apply a 5-point moving average smoothing to the following seasonal data of coat sales.

Season	Sales ($)
Winter 2017	690
Spring 2017	500
Summer 2017	400
Autumn 2017	720
Winter 2018	780
Spring 2018	660
Summer 2018	550
Autumn 2018	440

6. Apply a 4-point centred moving average smoothing to the data from question 5. Compare your results. What do you notice about the number of smoothed data points in each case?

7. Perform a 3-point median smoothing on the data shown below. Plot the smoothed points and join them with straight line segments.



8. The seasonal indices for the price of shares in CSP fruit canneries are:

Season	Index
Winter	1.7
Spring	0.6
Summer	0.5
Autumn	1.2

Use seasonal indices shown in the table above to deseasonalise the following data:

Season (2018)	Share price	
	Seasonalised	Deseasonalised
Spring	150	
Summer	100	
Autumn	300	
Winter	400	

Extended response

Task 1

1. Jazza's CD store has been opened for the past three weeks. The sales figures for the store were recorded and tabulated as follows.

Jazza's CD store daily sales figures — number of CDs sold							
	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Average daily sales for the week
Week 1	10	8	12	15	24	45	
Week 2	12	9	14	18	26	53	
Week 3	15	10	16	21	33	58	

a. Plot the data as a time series plot and comment on the type of trend that exists. Justify your choice.
b. How many seasons are there?
c. How many cycles are there?
d. Calculate the average daily sales for each of the weeks.
e. Complete the table of seasonal indices for each day.

Jazza's CD Store daily sales figures — number of CDs sold						
	Mon.	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**
Week 1	$\frac{10}{19} = 0.5263$	0.4211	$\frac{12}{19} =$	0.7895	1.2632	
Week 2	$\frac{12}{\quad} = 0.5455$	0.4091	$\frac{14}{\quad} = 0.6364$	0.8182		2.4091
Week 3	0.5882	0.3922			1.2941	2.2745

f. Complete the following table of seasonal indices.

	Mon.	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**
Seasonal indices	0.5533	$\frac{1.2224}{3} = 0.4075$	0.6318	$\frac{2.4312}{\quad} = 0.8104$	1.2464	

g. Interpret what an index of 2.3507 means.

2. Use the results from question **1** to answer question **2**.
a. Complete the table of deseasonalised CD sales figures.

Jazzas' CD store daily sales figures — number of CDs sold						
	Mon.	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**
Week 1	18.07	19.63	18.99	18.51	19.26	19.14
Week 2	21.69	$\frac{9}{0.4075} =$	22.16	22.21	20.86	
Week 3	27.11	$\frac{10}{0.4075} =$	25.32	25.91	26.48	

b. Find the equation of the trend line using the least-squares method and interpret the values of the gradient and the y-intercept.
c. Using the trend line, predict the deseasonalised sales figures for:
i. Monday week 4 **ii.** Saturday week 4 **iii.** Saturday week 6.
d. Using the deseasonalised values from part **c**, calculate the actual expected future sales for each. Comment on the reliability of the predictions.

Task 2

The next 7 questions relate to the following data, which represent seasonal rainfall (mm) in an Australian city.

Season	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall (mm)	43	75	41	13	47	78	50	19	51	83	55	25

1. Plot the data points and try to fit a trend line by eye. Comment on the ease of fitting the line to this plot.
2. Fit a trend line using the least-squares technique. Again, compare your result with the previous ones.
3. To smooth out the seasonal variation, 3-point and 5-point moving average smoothings are tried. Compare the results of these two methods with the results from questions **1** and **2** by plotting the smoothed data.
4. Upon observing the results with the 5-point smoothing, a trend appears. Take the data from the 5-point moving average smoothing and fit a straight line using the least-squares method. Put the first smoothed point at $t = 3$ and then centre the time data. State the y-intercept and gradient. Compare this trend line with that from question **2**.
5. Given the seasonal nature of the data, a 4-point moving average smoothing is tried. After calculating the 4-point moving average, fit a least-squares regression line, following the method of question **4**. Compare the results obtained with those from question **4**.
6. Finally, try seasonal adjustment. Take $t = 1$ to be summer and find the seasonal indices. Then, seasonally adjust the data.
7. Take the seasonally adjusted data from question **6** and fit a trend line using least-squares method. Comment on this result.

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Units 3 & 4 Sit exam

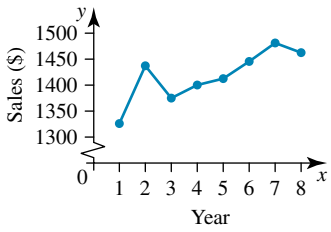
Answers

Topic 4 Investigating and modelling time series data

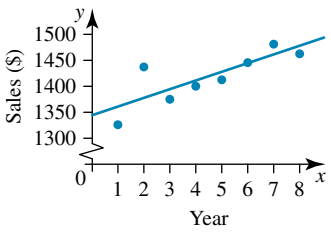
Exercise 4.2 Time series and trend lines

Note: Your answers may vary slightly due to using the 'by eye' method.

1.

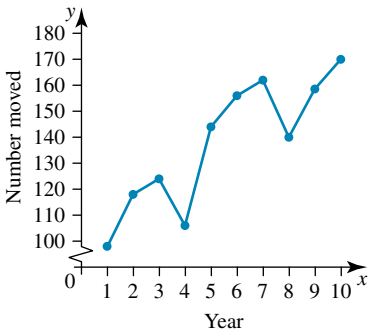


Upward trend with a possible outlier

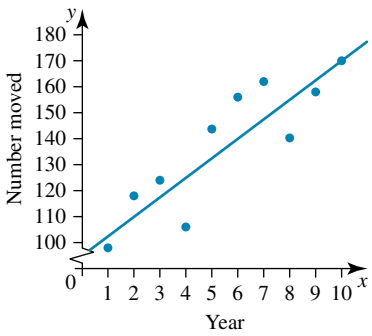


Sales = 1342.96 + 16.45 × year

2.



Irregular with an upward trend



Number moved = 97.8 + 7.22 × year

3. Seasonal

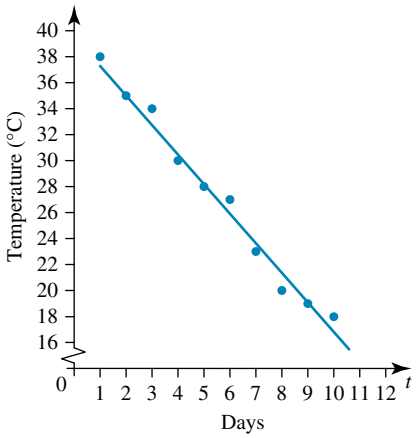
4. Irregular

5. Upwards trend

6. Irregular or cyclical with slight upward trend

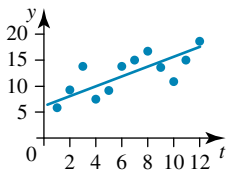
7. Cyclical

8. Definite downward trend



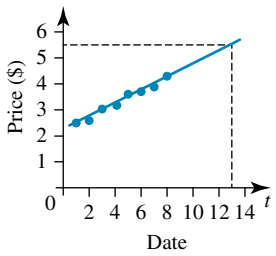
Temp = 40 − 2.33t

9. Although there are some random variations, the time series could also be cyclical, with an upward trend.



$y = 7.09 + 0.83t$

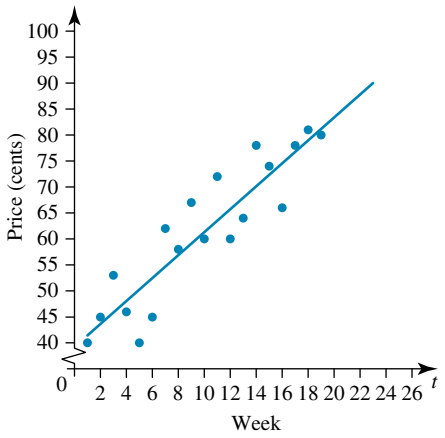
10. Predicting a share price in the following year is an extrapolated value (outside the plotted values) and can only be treated as an approximate value at best.



Price = 2.26 + 0.24t

11. The time series is cyclical, which means that it is difficult to fit an appropriate trend line to the data. There is no noticeable trend to the data.

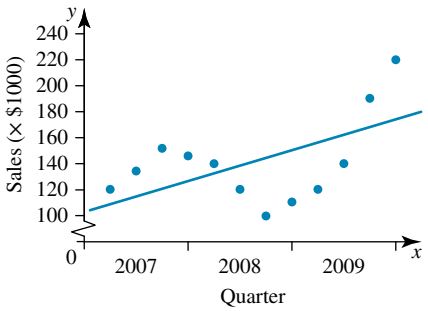
12. a.



Price = 39.42 + 2.21t

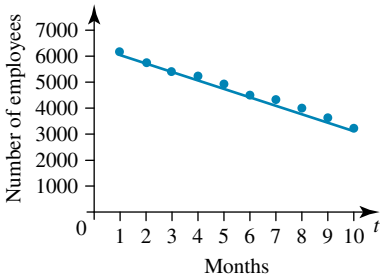
b. Prediction for $t = 25$ is about 95 cents.

13. Difficult to fit an accurate trend line, due to likely cyclical nature of software sales business.



Sales = 111.52 + 4.51 × quarter

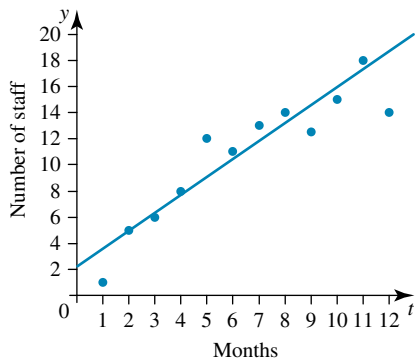
14. At current rate (about 300/month), bank will have no employees in another year! Although not likely, there is a clear downward trend.



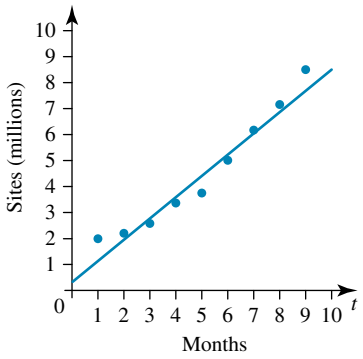
$y = 6333.33 - 300.61x$
Number of employees = 6333.33 − 300.61 × months

Exercise 4.3 Trend lines and forecasting

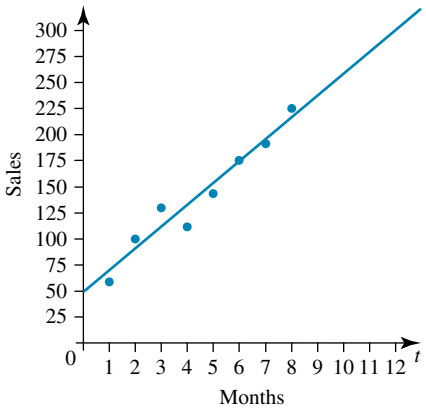
1. $y = 1.18 - 0.02x$ or millions of viewers = 1.18 − 0.02 × time code
   a. Number of viewers = 840 000
   b. Number of viewers = 740 000
2. $y = 128.1 - 0.77x$ or driver fatalities = 128.1 − 0.77 × time code
   a. Driver fatalities = 77
   b. Driver fatalities = 51
3. $y = 33.72 - 2.9t$, where y = cars remaining and t = time
   a. 2
   b. 2.9 per hour
4. 11.63 hours
5. a. $y = 3.02 + 1.13x$, where y = numbers of staff and x = months, where $x = 1$ is the first month of business



   b. $y = 30.14$, approximately 30 staff
6. a. Least-squares: $y = 2.48 + 0.044t$
   b. Probably a random (or cyclical) time series.
7. $y = 0.283 + 0.83x$; given the exponential nature of data, it is a very poor predictor.



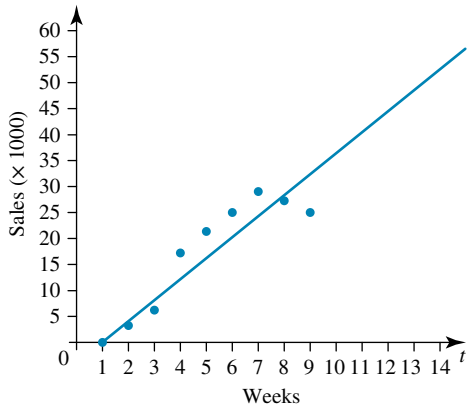
8. a. Share price ($) = $22.74 + $0.28 × time code
   Time code of $t = 1$ represents 2007
   b. The y-intercept of $22.74 represents the approximate value of the shares in 2006. The gradient of $0.28 means that on average the share price will grow by $0.28 each year.
   c. Share price ($) = $25.82
9. a. Share price ($) = $18.57 − $0.1 × time code
   Time code of $t = 1$ represents 2010
   b. The y-intercept of $18.57 represents the approximate value of the shares in 2009. The gradient of −$0.1 means that on average the share price will decline by $0.1 (10 cents) each year.
   c. Share price ($) = $17.57
10. a. Share price = $2.56 + $0.72 × time code
    Time code $t = 1$ represents 2010, $t = 2$ represents 2011 and so on.
    b. The y-intercept of $2.56 represents the approximate value of the shares in 2009. The gradient of +$0.72 means that the share value will grow by $0.72 (72 cents) each year.
    c. $10.48
11. a. $y = 49.64 + 20.36\,t$, where y = sales and t = months
    b.



    c. y(Dec.) = 293.96 (294)
    d. The trend line fits the data well, so it is a good predictor of future trends.

12. a. $y = -1.83 + 3.77t$, where y = sales and t = weeks

b.

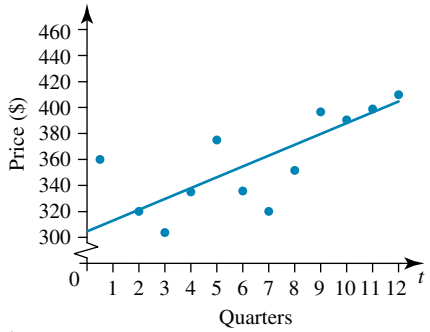


c. $y(10) = 35.87, y(12) = 43.41, y(14) = 50.95$

d. A poor predictor, given nature of data

13. a. $y = 315.8 + 6.35t$, where y = price and t = quarters

b.

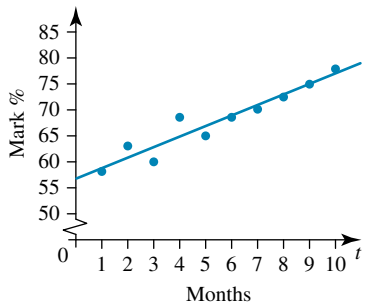


c. $398

d. The first 2 years displayed seasonal fluctuations, the last year an upward trend, so overall trend line would be a poor predictor.

14. a. $y = 56.9 + 1.93t$, where y = marks and t = months

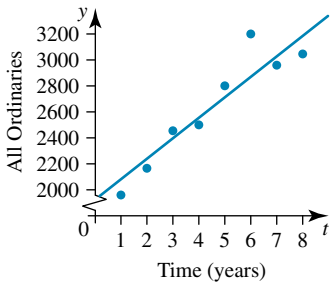
b.



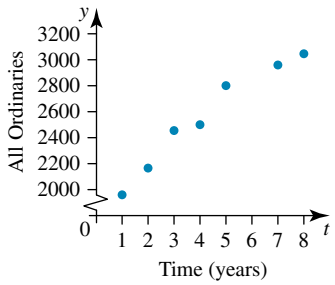
c. 78%

d. Given fairly even increase in averages, trend line is an excellent predictor.

15. a. $y = 1848.57 + 173.1x$, where y = All Ordinaries value and x = time, where $x = 1$ is 1995.



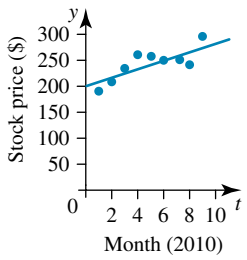
b. Taking out the 2000 data value:



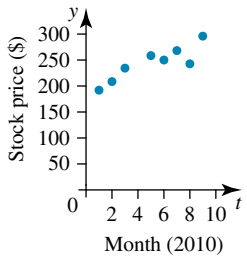
c. $y = 1862.17 + 159.49x$, where y = All Ordinaries and x = time

d. The gradient decreased and the y-intercept is higher when the outlier is taken out.

16. a. $y = 199.71 + 8.62x$, where y = stock price ($) and x = month, where $x = 1$ is January 2010.



b. The April value is a possible outlier.

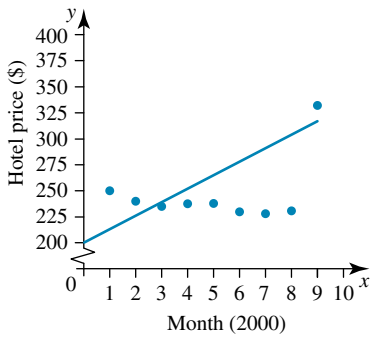


c. $y = 193.72 + 9.13x$, where y = stock price and x = month

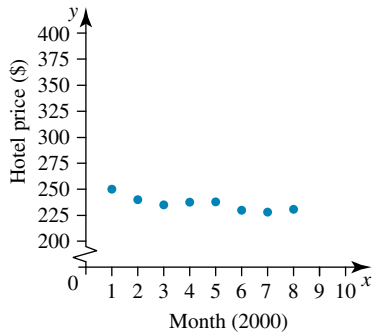
d. The gradient increased and the y-intercept was lower when the April value was taken out.

17. a. $y = 222.72 + 4.97x$, where y = hotel price ($) and x = months, where $x = 1$ is January 2000.

Month (2000)	1	2	3	4	5	6	7	8	9
Hotel price ($)	250	240	235	237	239	230	228	237	332



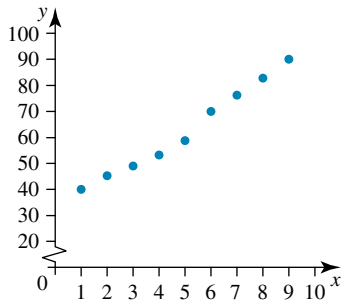
b. Outlier = September value



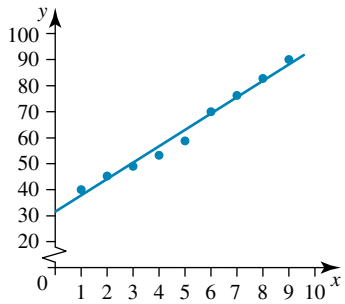
c. Hotel prices ($) = 245.79 − 1.95 × month

d. Gradient (positive to negative), *y*-intercept (increased)
The most likely reason for the steep jump in hotel price in the month of September 2000 is that this is when the Olympics were held in Sydney.

18. a. No obvious outliers



b. $y = 31.39 + 6.37x$, where y = percentage of homes with TV and x = year, where $x = 1$ is 1970.



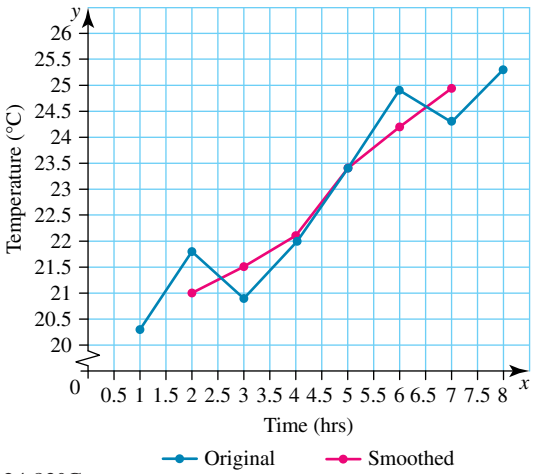
c. The data does jump significantly from 1974 going to 1975. The possible reason is that this is when colour TV was introduced into Australia.

Exercise 4.4 Smoothing time series

1. a.

Time (h)	Temp (°C)	Smoothed temperature (°C)
1	20.3	
2	21.8	$\frac{1}{3}(20.3 + 21.8 + 20.9) = 21$
3	20.9	$\frac{1}{3}(21.8 + 20.9 + 22.0) = 21.57$
4	22.0	$\frac{1}{3}(20.9 + 22.0 + 23.4) = 22.1$
5	23.4	$\frac{1}{3}(22.0 + 23.4 + 24.9) = 23.43$
6	24.9	$\frac{1}{3}(23.4 + 24.9 + 24.3) = 24.2$
7	24.3	$\frac{1}{3}(24.9 + 24.3 + 25.3) = 24.83$
8	25.3	

b.

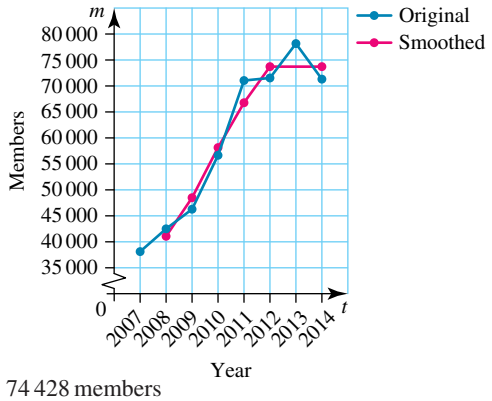


c. 24.83°C

2. a.

Year (t)	Members (m)	Smoothed members (m)
2007	38 587	
2008	42 498	$\frac{1}{3}(38\,587 + 42\,498 + 45\,972) = 42\,352$
2009	45 972	$\frac{1}{3}(42\,498 + 45\,972 + 57\,408) = 48\,626$
2010	57 408	$\frac{1}{3}(45\,972 + 57\,408 + 71\,271) = 58\,217$
2011	71 271	$\frac{1}{3}(57\,408 + 71\,271 + 72\,688) = 67\,122$
2012	72 688	$\frac{1}{3}(71\,271 + 72\,688 + 78\,427) = 74\,129$
2013	78 427	$\frac{1}{3}(72\,688 + 78\,427 + 72\,170) = 74\,428$
2014	72 170	

b.

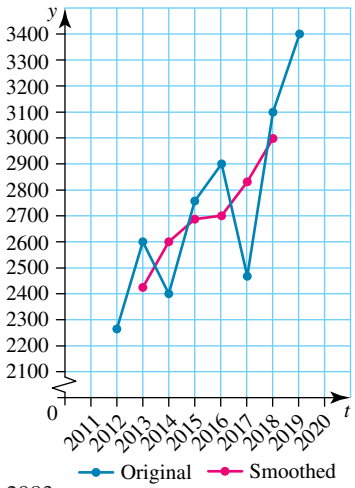


c. 74 428 members

3. a. Smoothed data:

2417	2583	2683	2700	2817	2983

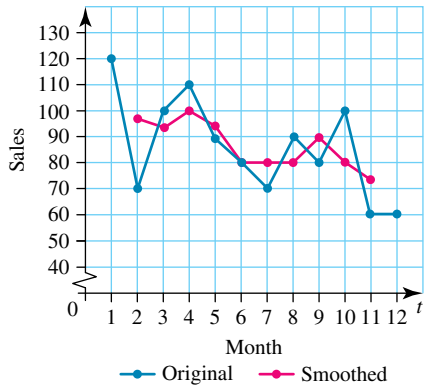
b.



c. 2983

4. a. Smoothed data: possible downward trend, but still fluctuations.

96.7	93.3	100	93.3	80	80	80	90	80	73.3

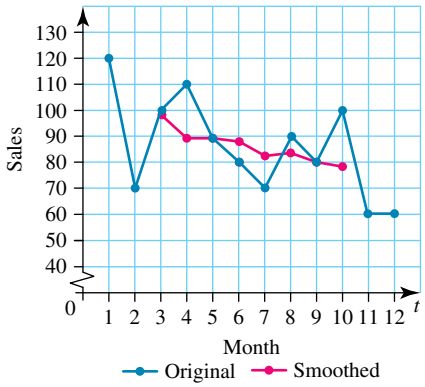


b. Price = $102.17 - 2.39t$

c. 66.32, the data are cyclical and the prediction is based on smoothed data that have removed this trend.

5.

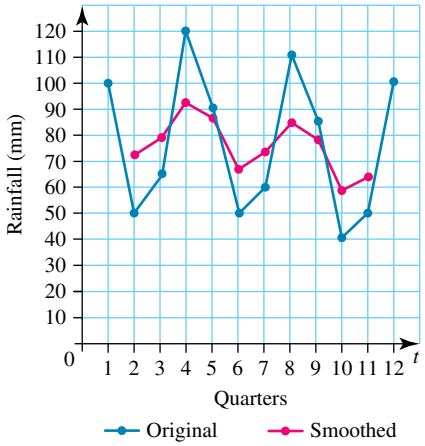
98	90	90	88	82	84	80	78



Smoothed data: definite downward trend is now apparent.

6.

71.7	63	52	86.7	66.7	73.3	85.0	78.3	58.3	63.3



Smoothed data did not remove seasonal fluctuation; from the figure, there may be a slight trend downward.

7. a.

72	71.7	69.7	70.7	66.3	65.7	64.7	66

Smoothed data: shows a clear downward trend.

b. Attendance = $74.47 - 1.11x$

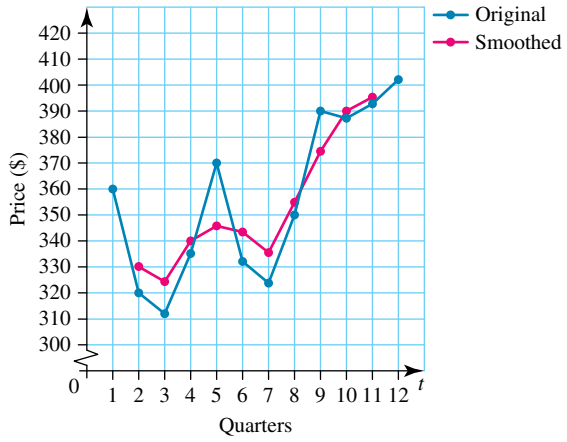
c. 61 150, this is a reasonable prediction as long as the trend continues to decline as given by the negative gradient.

8.

Week	Sales	Smoothed data
1	34	
2	27	30.67
3	31	31.67
4	37	36.33
5	41	35.67
6	29	34
7	32	32.67
8	37	38.67
9	47	40.67
10	38	42
11	41	41
12	44	44
13	47	46.67
14	49	45.67
15	41	47.33
16	52	47
17	48	48
18	44	47
19	49	49.67
20	56	53
21	54	

9.

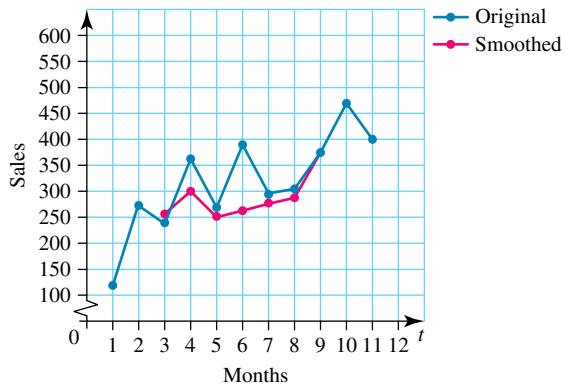
332.3	325	340.3	346	343.3	337.3	356	375.7	389.7	394



Smoothed data: Some but not all seasonal fluctuation removed.

10.

253.8	303	306.6	320.4	322.4	362	364.2



Smoothed data: most random variation smoothed, slight upward trend possible. Prediction for 12th month is 364.

11.

66.33	71.33	68.33	79.67	83.67

Smoothed data; prediction for week 7 is 84.

12. B

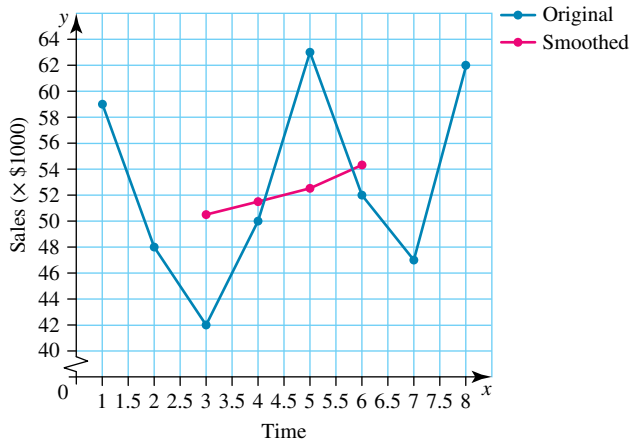
13. The 5-point moving mean smoothing has made it more evident that there is an upward trend in the data.

14. It has further smoothed out the seasonal fluctuations.

Exercise 4.5 Smoothing with an even number of points

1. a.

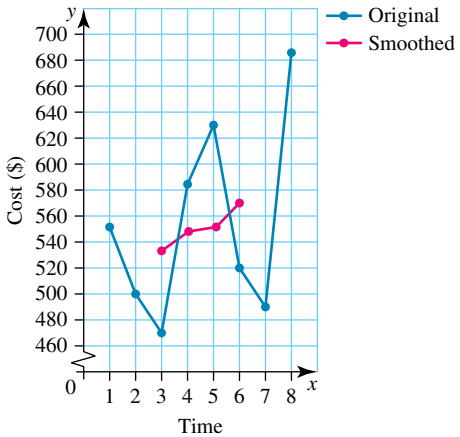
Time	Sale	4-point moving mean	4-point centred moving mean
1	59		
2	48		
		$\frac{(59+48+43+50)}{4} = 50$	
3	43		$\frac{(50+51)}{2} = 50.5$
		$\frac{(48+43+50+63)}{4} = 51$	
4	50		$\frac{(51+52)}{2} = 51.5$
		$\frac{(43+50+63+52)}{4} = 52$	
5	63		$\frac{(52+53)}{2} = 52.5$
		$\frac{(50+63+52+47)}{4} = 53$	
6	52		$\frac{(53+55.75)}{2} = 54.38$
		$\frac{(63+52+47+61)}{4} = 55.75$	
7	47		
8	61		



b. The data shows a steady increasing trend.

2. a.

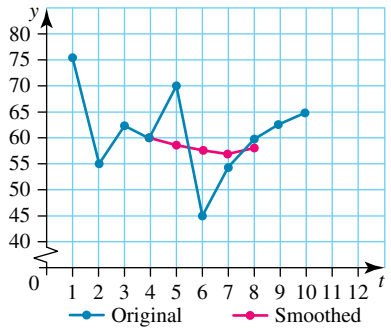
Time	Sale	4-point moving mean	4-point centred moving mean
1	554		
2	503		
		$\frac{(554 + 503 + 467 + 587)}{4} = 527.75$	
3	467		$\frac{(527.75 + 548.25)}{2} = 538$
		$\frac{(503 + 467 + 587 + 636)}{4} = 548.25$	
4	587		$\frac{(548.25 + 555.75)}{2} = 552$
		$\frac{(467 + 587 + 636 + 533)}{4} = 555.75$	
5	636		$\frac{(555.75 + 562.25)}{2} = 559$
		$\frac{(587 + 636 + 533 + 493)}{4} = 562.25$	
6	533		$\frac{(562.25 + 586.50)}{2} = 574.38$
		$\frac{(636 + 533 + 493 + 684)}{4} = 586.5$	
7	493		
8	684		



b. This data shows a steady increasing trend. This is not obvious with the original data.

3. a.

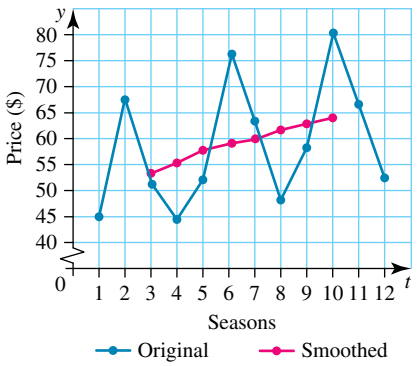
t	3	4	5	6	7	8
y	62.125	60.375	58.25	57.125	56	57.375



b. Smoothed data indicated a general downward trend, possibly with cyclic fluctuations in original data.

4.

t	Price ($)
Autumn '16	
Winter '16	
Spring '16	52.625
Summer '16	54.63
Autumn '17	57.25
Winter '17	59.25
Spring '17	60.50
Summer '17	61.75
Autumn '18	62.63
Winter '18	63.50
Spring '18	
Summer '18	



Smoothed data indicate a strong upward trend of almost 12 cents over 3 years.

5. a.

Day	Temperature	4-point moving mean	4-point centred moving mean
1	36.6		
2	36.4	36.75	
3	36.8	36.825	36.7875
4	37.2	36.85	36.8375
5	36.9	36.95	36.9
6	36.5	37	36.975
7	37.2	37.05	37.025
8	37.4	37.275	37.1625
9	37.1	37.375	37.325
10	37.4	37.25	37.3125
11	37.6	37.275	37.2625
12	36.9	37.325	37.3
13	37.2	37.15	37.2375
14	37.6		
15	36.9		

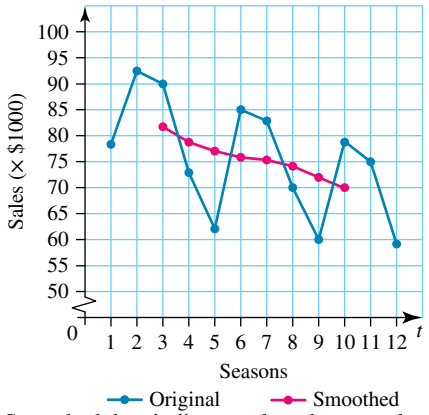
b. Temperature $= 36.654 + 0.056d$

c. 37.6 °C

6. a.

Season	Smoothed
Q_3-15	
Q_4-15	
Q_1-16	81.25
Q_2-16	78.375
Q_3-16	76.625
Q_4-16	75.375
Q_1-17	74.88
Q_2-17	73.875
Q_3-17	71.875
Q_4-17	69.375
Q_1-18	
Q_2-18	

b.



c. Smoothed data indicate a clear downward trend.

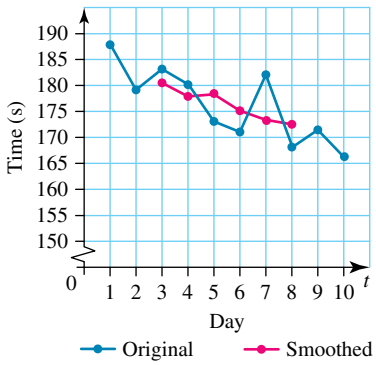
7.

Day	Smoothed
1	
2	
3	
4	36.78
5	36.92
6	37.03
7	37.07
8	37.14
9	37.24
10	37.27
11	37.29
12	37.29
13	
14	
15	

8. a.

Day	Smoothed
1	
2	
3	180.625
4	177.75
5	176.625
6	175
7	173.25
8	172.375
9	
10	

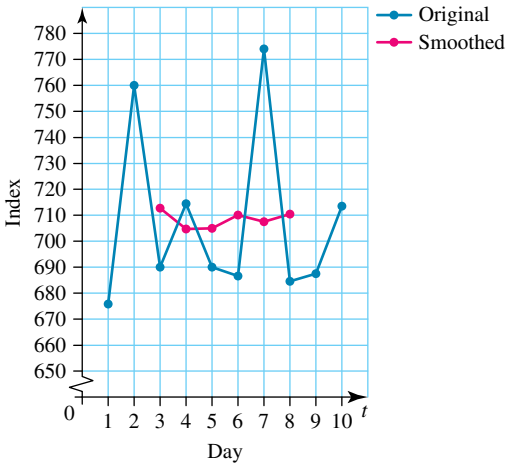
b.



c. Yes, there is a significant improvement in times.

9.

Day	Index
1	
2	
3	712.875
4	704.875
5	705.5
6	711.875
7	708.125
8	711.125
9	
10	



Smoothing indicates a flat trend (neither upward or downward)

10. B

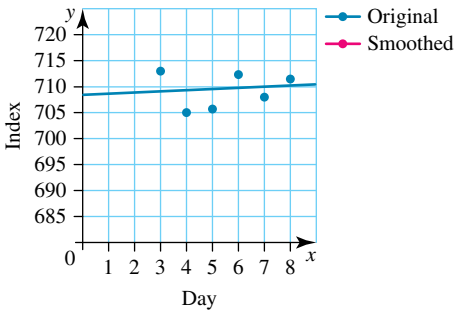
11. A

12. D

13.

Day	Time (s)	6-point moving average	6-point centred moving average
1	188		
2	179		
3	183	179	
4	180	178	178.5
5	173	176.17	177.08
6	171	174.17	175.17
7	182	171.83	173
8	168		
9	171		
10	166		

14. a.

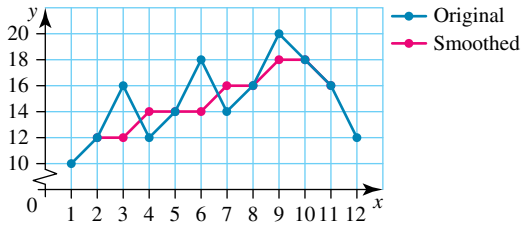


$y = 707.9 + 0.21x$, where y = price index and x = day number.

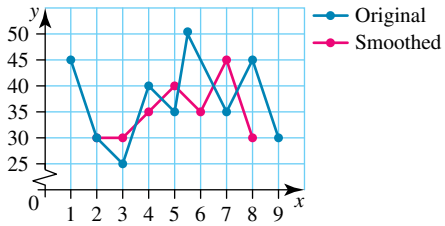
b. $y = 711.05$ therefore price index is approximately 711 after 15 days.

Exercise 4.6 Median smoothing from a graph

1.



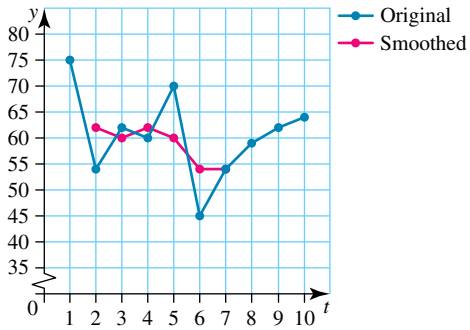
2.



3. a. 10

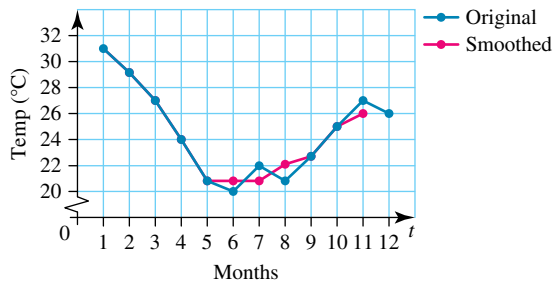
b. 7

4.



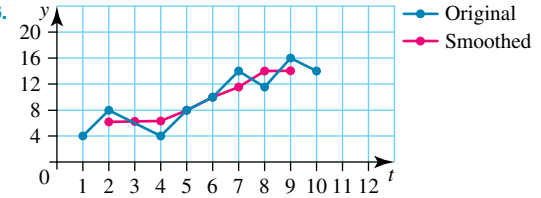
Not nearly as effective as moving mean smooth in demonstrating trend.

5.



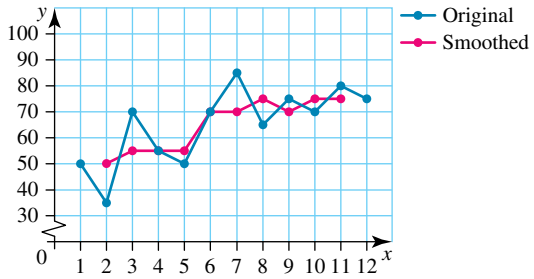
Smoothing had virtually no effect on data — only minor variations smoothed.

6.



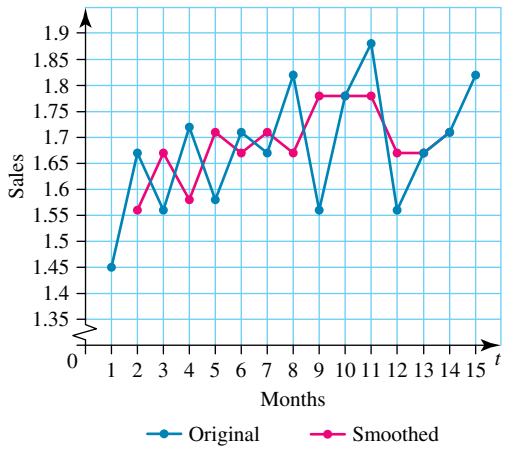
7. Effective at smoothing out small random variations.

8.

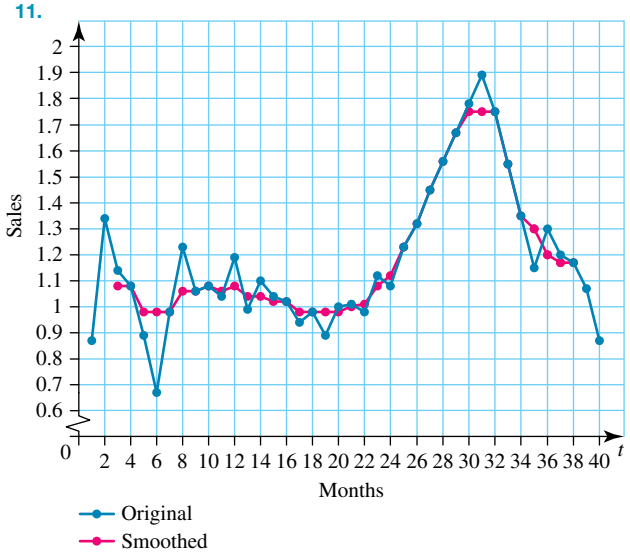


9. Smoothed out much of the variation, indicates a slight upward trend.

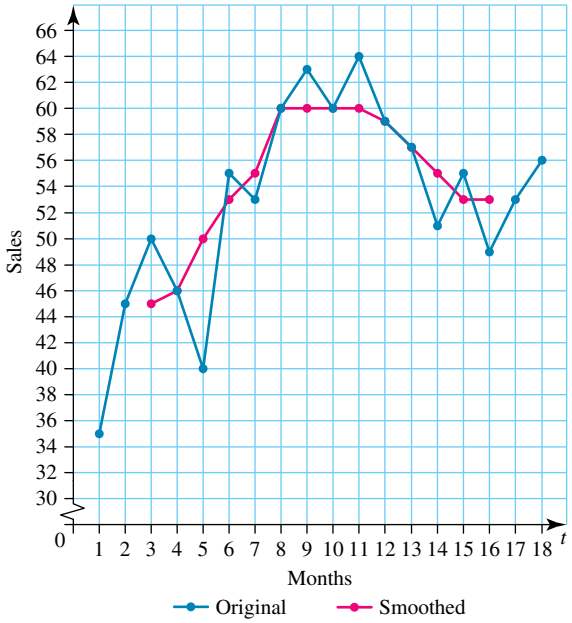
10.



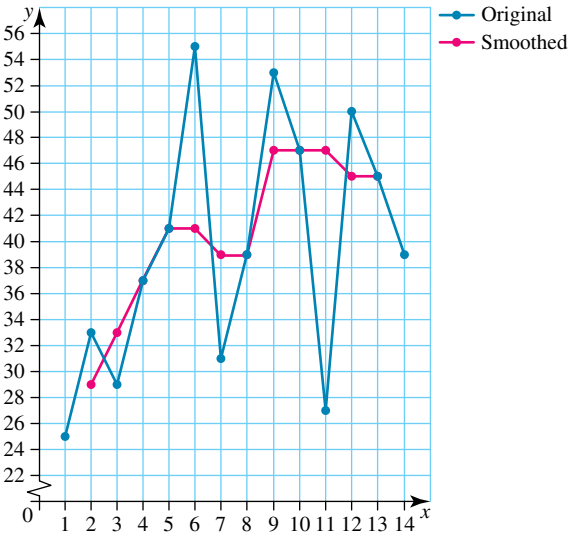
11.



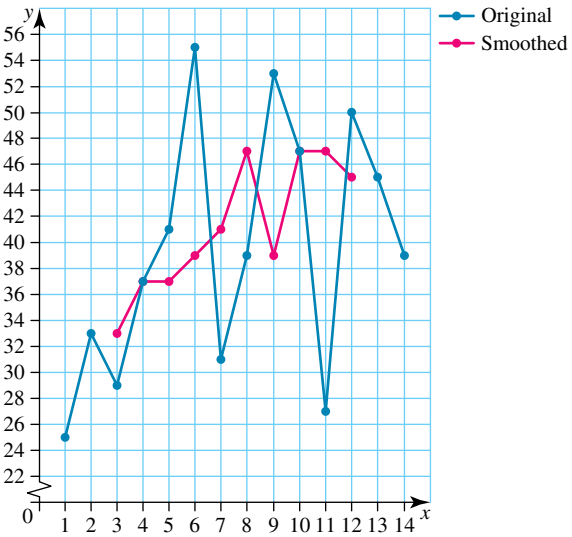
12.



13.



14.



The 3-point median smoothing in question **13** reduces the fluctuations more than the 5-point median smoothing in question **14**.

Exercise 4.7 Seasonal adjustment

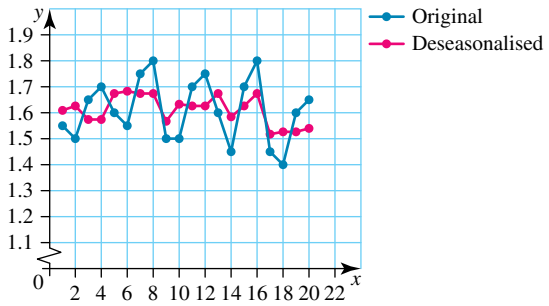
1. a.

Season	Summer	Autumn	Winter	Spring
Seasonal index	0.96	0.92	1.05	1.08

b.

Season	2014	2015	2016	2017	2018
Summer	1.61	1.67	1.56	1.67	1.51
Autumn	1.63	1.68	1.63	1.58	1.52
Winter	1.57	1.67	1.62	1.62	1.52
Spring	1.57	1.67	1.62	1.67	1.53

c.



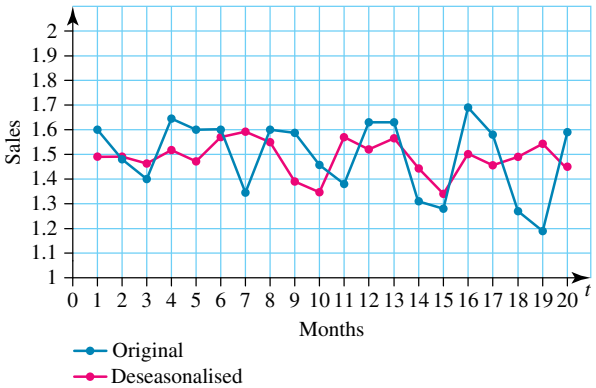
d. There seems to be a very slight downward trend in milk prices.

2. a. Seasonal indices: 1.0731, 0.9423, 0.8874, 1.0971

b.

Season	2014	2015	2016	2017	2018
Summer	1.4910	1.4910	1.4630	1.5189	1.4723
Autumn	1.5706	1.5918	1.5493	1.3902	1.3477
Winter	1.5777	1.5213	1.5664	1.4424	1.3410
Spring	1.5039	1.4583	1.4857	1.5404	1.4492

c.



d. There appears to be a slight downward trend in the price of 1 litre of petrol.

3. Deseasonalised 1 L of milk cost = 1.6527 − 0.004 45 × time code,
where time code 1 represents summer 2014
Predicted summer 2019 price = $1.50

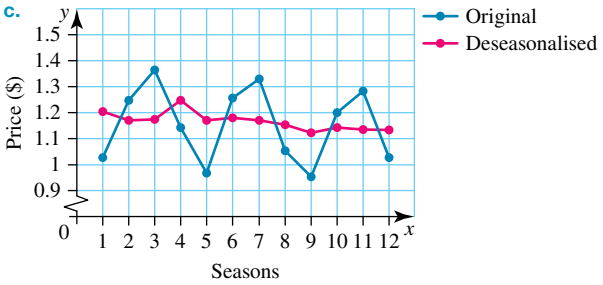
4. Deseasonalised 1 L of petrol cost = 1.5659 − 0.007 38 × time code,
where time code 1 represents summer 2014
Predicted summer 2019 price = $1.51

5. a. Seasonal indices: 0.8504, 1.0692, 1.1467, 0.9336

b.

Season	2016	2017	2018
Summer	1.211	1.152	1.117
Autumn	1.178	1.169	1.132
Winter	1.186	1.169	1.125
Spring	1.221	1.146	1.114

c.

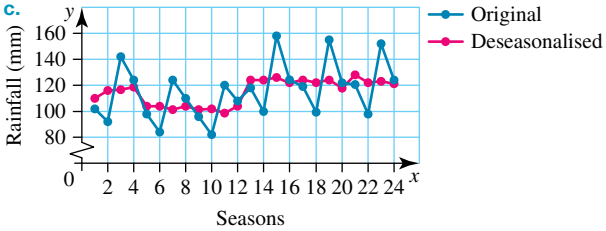


d. Slight trend downwards

6. a. Seasonal indices 0.9394, 0.8044, 1.2274, 1.0288

b. See table at the foot of the page*.

c.

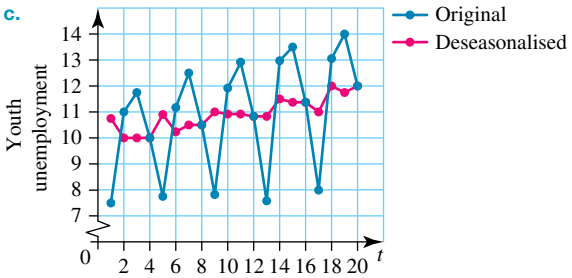


d. Probable drought in 2014–15

7. a. Seasonal indices: 0.7141, 1.100, 1.1832, 1.0027

b. See table at the foot of the page#.

c.



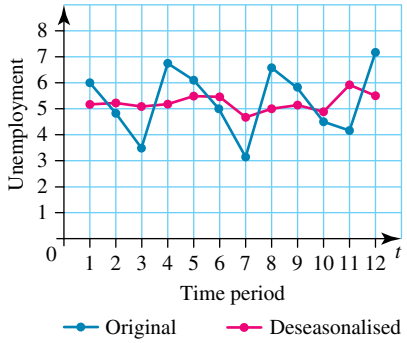
d. Youth unemployment increases in all seasons except in summer.

8. a. Seasonal indices: 1.1143, 0.9183, 0.6829, 1.2845

b.

Quarter	1	2	3	4
2016	5.205	5.336	5.125	5.216
2017	5.474	5.554	4.686	5.060
2018	5.115	4.900	6.004	5.528

c.

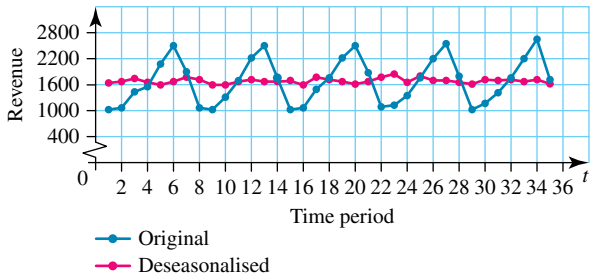


d. Deseasonalised unemployment rate
$= 5.1448 + 0.0188 \times t$

e. i. 6.0 **ii.** 3.9

9. Seasonal indices: 0.6341, 0.6613, 0.8329, 1.0354, 1.2822, 1.4850, 1.0692. Restaurant should probably close Mon.–Tues., steady sales over 5-week period — see following table.

Season	Week 1	Week 2	Week 3	Week 4	Week 5
Monday	1633.86	1717.45	1678.02	1788.42	1643.33
Tuesday	1667.88	1581.69	1640.66	1825.14	1748.02
Wednesday	1740.94	1589.66	1785.36	1654.49	1690.51
Thursday	1588.81	1674.77	1728.86	1742.38	1727.89
Friday	1620.72	1718.99	1727.57	1703.39	1690.13
Saturday	1661.32	1668.72	1686.23	1701.05	1743.47
Sunday	1772.29	1670.35	1705.89	1668.48	1641.36



***6. b.**

Season	2013	2014	2015	2016	2017	2018
Summer	109.643	103.257	101.128	124.548	125.612	127.741
Autumn	115.614	104.426	101.939	124.316	123.023	121.830
Winter	116.506	101.027	98.582	127.098	126.283	123.024
Spring	119.557	105.949	104.005	121.501	118.585	120.529

#7. b.

Season	2014	2015	2016	2017	2018
Summer	10.6431	10.7832	10.9232	10.7832	11.0633
Autumn	9.9090	10.2726	10.8181	11.4544	11.9090
Winter	9.8884	10.4800	10.8181	11.4097	11.7478
Spring	9.8732	10.4716	10.7708	11.3692	11.8678

10. A

11. a. Deseasonalised umbrella sales = $21.8788 + 0.9161 \times t$
    b. 39

12. a. \$3620.83 b. \$1583.33 c. \$8246

13. a. \$6880.77 b. \$8125 c. \$13 515

14. a. \$3000 b. \$3200 c. \$4800

15. 1.8 16. 1.5 17. 0.9

18. 0.43 19. E 20. D

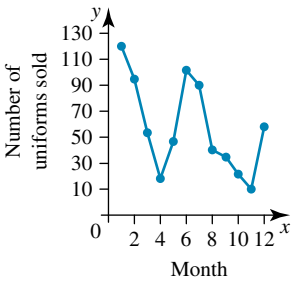
4.8 Review: exam practice

Multiple choice

1. C 2. D 3. E 4. C 5. C 6. A
7. D 8. B 9. A 10. B 11. E

Short answer

1.



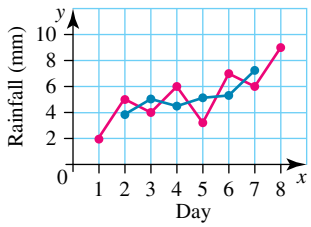
It's very difficult to fit a trend line as data is seasonal. Summer uniform is mostly bought at the end of the year and then the beginning of the year. Winter uniform is mostly bought near winter — these are peaks. The sales decrease throughout the other parts of the year.

2. Gradient = 184.52, y-intercept = 2057.14

3. a. Gradient = 1.58, y-intercept = 12.33
   b. $y(12) = 32$, $y(13) = 33$

4.

Day	2	3	4	5	6	7
Ave.	3.7	5.0	4.3	5.3	5.3	7.3



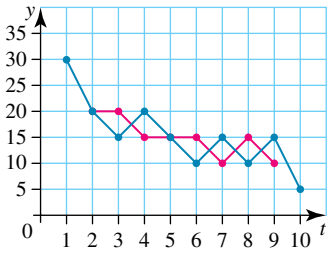
5.

Season	Sales (\$)
Winter '17	
Spring '17	
Summer '17	618
Autumn '17	612
Winter '18	622
Spring '18	630
Summer '18	
Autumn '18	

6. Same number of smoothed data points. (In fact a 4-point centred is a special case of a 5-point smoothing.)

Season	Sales (\$)
Winter '17	
Spring '17	
Summer '17	588.75
Autumn '17	620
Winter '18	658.75
Spring '18	642.5
Summer '18	
Autumn '18	

7.

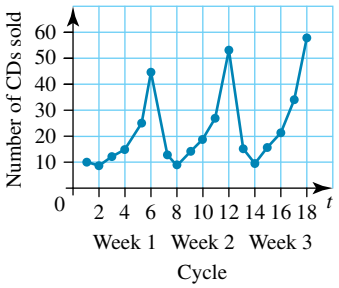


8. 250, 200, 250, 235

Extended response

Task 1

1. a. The time series is seasonal. There are peaks and troughs occurring on the same days of the week. There is also an upward secular trend. This can also be seen from the table as each day, week after week more CDs are sold.



   b. 6
   c. 3: there are 3 weeks of figures given
   d. Week1: 19, week 2: 22, week 3: 25.5
   e. See table at the foot of the page*.

*1. e.

Week 1	$\frac{10}{19} = 0.5263$	0.4211	$\frac{12}{19} = 0.6316$	0.7895	1.2632	$\frac{45}{19} = 2.3684$
Week 2	$\frac{12}{22} = 0.5455$	0.4091	$\frac{14}{22} = 0.6364$	0.8182	$\frac{26}{22} = 1.1512$	2.4091
Week 3	0.5882	0.3922	$\frac{16}{22} = 0.6349$	$\frac{21}{22} = 0.8235$	1.2941	2.2745

f. See table at the foot of the page*.

g. An index of 2.3507 means that Saturdays have about 2.35 times the sales compared to the average daily sales.

2. a. See table at the foot of the page*.

b. Deseasonalised CD sales = $17.43 + 0.4993t$
y-intercept of 17.4342 (17.43) means that these sales were expected the day before the data were calculated. The gradient of 0.4993 (0.5) means that as each day goes by Jazza can expect an increase in sales by half a CD each day (or more logically by 1 CD every two days).

c. i. Monday week 4, $t = 19$. Deseasonalised CD sales = 26.91

ii. Saturday week 4, $t = 24$. Deseasonalised CD sales = 29.40

iii. Saturday week 6, $t = 36$. Deseasonalised CD sales = 35.395

d. i. Monday week 4: 15 CDs approximately

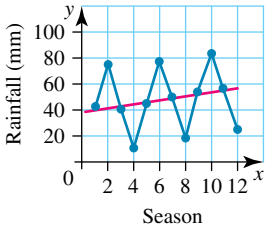
ii. Saturday week 4: 69 CDs approximately

iii. Saturday week 6: 83 CDs approximately

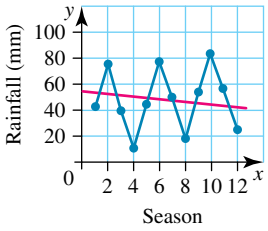
The first two predictions are reliable as they are only one week into the future. The Saturday week 6 prediction of 83 CDs is not so reliable as it is far into the future and the trend may change in the meantime, such as during holidays and so on.

Task 2

1. Very difficult to fit an accurate trend line. However, there seems to be an upward trend.



2. No improvement using this method.



3. See table and figure at the foot of the page*.
The 3-point average has only slightly reduced the variation, while the 5-point smooth seems more effective.

---

*1. f.

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Seasonal indices	0.5533	$\frac{1.2224}{3} = 0.4075$	0.6318	$\frac{2.4312}{3} = 0.8104$	1.2462	$\frac{7.052}{3} = 2.3507$

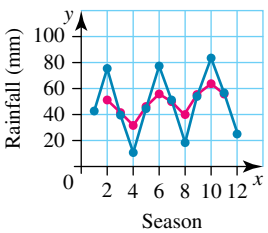
*2. a.

Jazza's CD store daily sales figures — Number of CDs sold						
	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Week 1	18.07	19.63	18.99	18.51	19.26	19.14
Week 2	21.69	$\frac{9}{0.4075} = 22.09$	22.16	22.21	20.86	$\frac{53}{2.3507} = 22.55$
Week 3	27.11	$\frac{10}{0.4075} = 24.54$	25.32	25.91	26.48	$\frac{58}{2.3507} = 24.67$

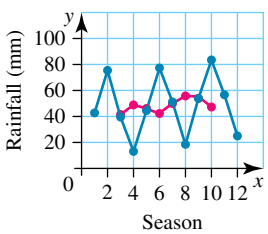
*3.

Season	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall	43	75	41	13	47	78	50	19	51	83	55	25
3-pt ave.		53	43	33.7	46	58.3	49	40	51	63	54.3	
5-pt ave.			43.8	50.8	45.8	41.4	49	56.2	51.6	46.6		

3-point smooth:



5-point smooth:



4. $y = 43.32 + 0.74t$
A positive trend. However, it is fairly weak ($m = 0.74$).

5. See table at the foot of the page*.
Least-squares: $y = 39.27 + 1.36t$. A stronger increasing trend shown.

6.

Season	Seasonal index
Summer	0.9741
Autumn	1.6346
Winter	1.0042
Spring	0.3871

Seasonally adjusted data:

Season	Rainfall	Seas. adj.
1	43	44.1
2	75	45.9
3	41	40.8
4	13	33.6
5	47	48.2
6	78	47.7
7	50	49.8
8	19	49.1
9	51	52.4
10	83	50.8
11	55	54.8
12	25	64.6

7. $y = 37.70 + 1.66t$
Strongest trend yet ($m = 1.66$). Also, all 12 points used rather than 8 or 10 as in moving average smoothing.

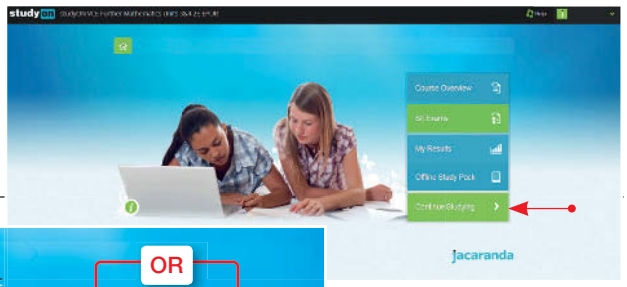
*5.

Season	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall	43	75	41	13	47	78	50	19	51	83	55	25
4-pt ave.			43.5	44.4	45.8	47.8	49.0	50.1	51.4	52.8		

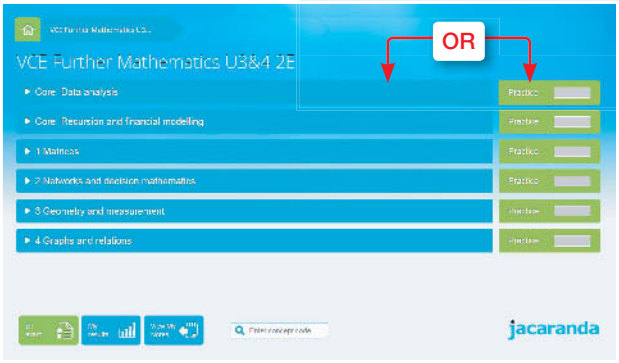
REVISION: AREA OF STUDY 1 Core: data analysis

TOPICS 1 to 4

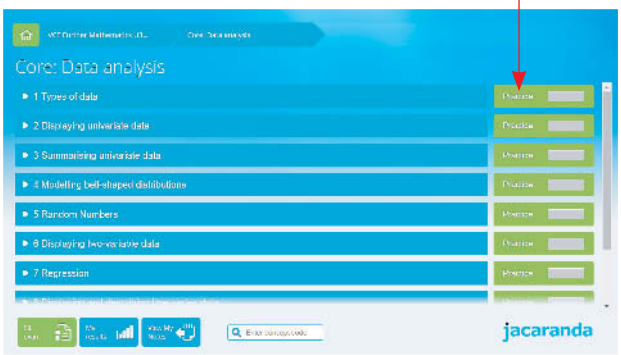
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



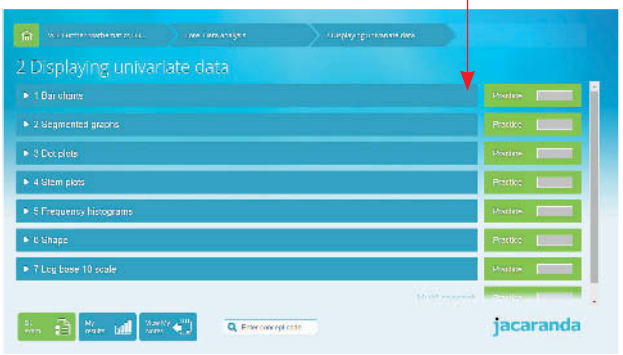
- Select your **course** *Further Mathematics VCE Units 3 & 4* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



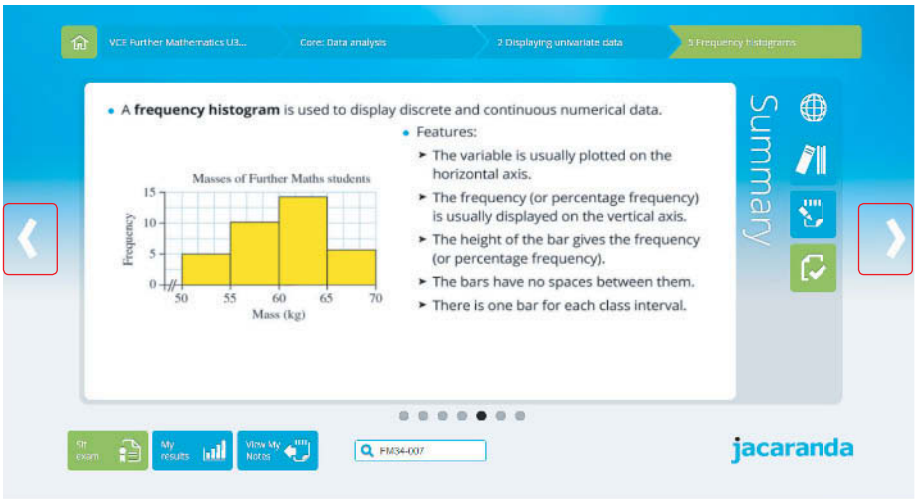
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 5
Depreciation of assets

5.1 Overview

5.1.1 Introduction

Many items such as antiques, jewellery or real estate increase in value (appreciate or increase in capital gains) with time. On the other hand, items such as computers, vehicles or machinery decrease in value (depreciate) with time or because of wear and tear, advances in technology or a lack of demand for those specific items. An example is farm machinery, similar to the image on the right.

The estimated loss in value of assets is called depreciation. Each financial year a business will set aside money equal to the depreciation of an item in order to cover the cost of the eventual replacement of that item. The estimated value of an item at any point in time is called its future value (or book value)

When the value becomes zero, the item is said to be 'written off'. At the end of an item's useful or effective life (as a contributor to a company's income) its future value is then called its scrap value.

There are three methods by which depreciation is calculated. They are:

1. Flat rate depreciation
2. Reducing balance depreciation
3. Unit cost depreciation

Currently, the Australian Taxation Office (ATO) allows depreciation of an asset as a tax deduction. This means that the annual depreciation reduces the amount of tax paid by a business in that year. The higher the depreciation, the greater the tax benefit. The ATO set the percentage depreciation rates, either flat rate or reducing balance, but once a method is applied to an article it cannot be changed for the life of the article.

LEARNING SEQUENCE

5.1 Overview
5.2 Generating the terms of a first order linear recurrence relation
5.3 Modelling flat rate depreciation with a recurrence relation
5.4 Modelling reducing balance depreciation with a recurrence relation
5.5 Modelling unit cost depreciation with a recurrence relation
5.6 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.1.2 Kick off with CAS

Generating terms of sequences with CAS

First-order recurrence relations relate a term in a sequence to the previous term in the same sequence. You can use recurrence relations to generate all of the terms in a sequence, given a starting (or initial) value.

1. Use CAS to generate a table of the first 10 terms of a sequence if the starting value is 7 and each term is formed by adding 3 to the previous term.

Term	Value
1	7
2	
3	
4	
5	
6	
7	
8	
9	
10	

2. Use CAS to generate the first 10 terms of a sequence if the starting value is 14 and each term is formed by subtracting 5 from the previous term.

Term	Value
1	14
2	
3	
4	
5	
6	
7	
8	
9	
10	

2. Use CAS to generate the first 10 terms of a sequence where the starting value is 1.5 and each term is formed by multiplying the previous term by 2.
3. Use CAS to generate the first 10 terms of a sequence where the starting value is 4374 and each term is formed by dividing the previous term by 3.

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

5.2 Generating the terms of a first-order recurrence relation

A **first-order recurrence relation** relates a term in a **sequence** to the previous term in the same sequence, which means that we only need an **initial value** to be able to generate all remaining terms of a sequence.

In a recurrence relation the nth term is represented by u_n, with the next term after u_n being represented by u_{n+1}, and the term directly before u_n being represented by u_{n-1}.

The initial value of the sequence is represented by u_0.

We can define the sequence $1, 5, 9, 13, 17, \ldots$ with the following equation:

$$u_{n+1} = u_n + 4 \qquad u_0 = 1$$

This expression is read as 'the next term is the previous term plus 4, starting at 1'.

Or, transposing the above equation, we get:

$$u_{n+1} - u_n = 4 \qquad u_0 = 1$$

A first-order recurrence relation defines a relationship between two successive terms of a sequence, for example, between:

u_n, the previous term and u_{n+1}, the next term.

The first term is represented by u_0.

A first order linear recurrence relation can be written as $u_0 = a$, $u_{n+1} = bu_n + c$.

WORKED EXAMPLE 1

Write the first 5 terms of the sequence that starts at 1 and each successive term increases by 2.

THINK	WRITE
1. This is a first-order recurrence relation.	1. This is a first-order recurrence relation. It has the pattern $u_{n+1} = u_n + 2$, $u_0 = 1$.
2. Write out the sequence of 5 numbers.	2. 1, 3, 5, 7, 9

Starting term

Earlier, it was stated that a starting term was required to fully define a sequence. As can be seen below, the same pattern with a different starting point gives a different set of numbers.

$$u_{n+1} = u_n + 2 \quad u_0 = 3 \qquad \text{gives } 3, 5, 7, 9, 11, \ldots$$

$$u_{n+1} = u_n + 2 \quad u_0 = 2 \qquad \text{gives } 2, 4, 6, 8, 10, \ldots$$

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Units 3 & 4 > AOS R&FM > Topic 1 > Concept 1

Generating a sequence Summary screen and practice questions

WORKED EXAMPLE 2

Write the first five terms of the sequence defined by the first-order recurrence relation:

$$u_{n+1} = 3u_n + 5, \quad u_0 = 2$$

THINK

1. Since we know the u_0 or starting term, we can generate the next term, u_1, using the pattern: The next term is $3\times$ the previous term $+ 5$.
   The 5 terms can be generated on the calculator (see instructions below).

2. Write your answer.

WRITE

$u_0 = 2$
$u_{n+1} = 3u_n + 5$

2	2
$2 \times 3 + 5$	11
$11 \times 3 + 5$	38
$38 \times 3 + 5$	119
$119 \times 3 + 5$	362

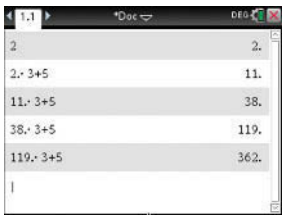
The sequence is
2, 11, 38, 119, 362.

TI | THINK

1. On a Calculator page, press 2 and then ENTER. Complete the entry line as: $\times 3 + 5$
   Then continue to press ENTER until the first 5 terms are displayed on the screen.

2. The answer appears on the screen.

WRITE



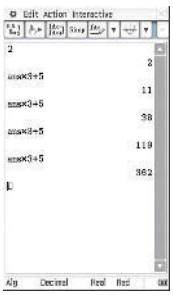
The sequence is
2, 11, 38, 119, 362.

CASIO | THINK

1. On the Main screen, press 2 and then EXE.
   Complete the entry line as: $\times 3 + 5$
   Then continue to press EXE until the first 5 terms are displayed on the screen.

2. The answer appears on the screen.

WRITE



The sequence is
2, 11, 38, 119, 362.

Resources

Interactivity: Initial values and first-order recurrence relations (int-6262)

Exercise 5.2 Generating the terms of a first-order recurrence relation

1. **WE1** Write down the first 5 terms of a sequence that starts at -2 and each successive term decreases by 3.
2. Write down the first 5 terms of a sequence that starts at 3 and each successive term is double the previous term.
3. **WE2** Write the first five terms of the sequence defined by the first-order recurrence relation:
   $$u_n = 4u_{n-1} + 3 \qquad u_0 = 5.$$
4. Write the first five terms of the sequence defined by the first-order recurrence relation:
   $$f_{n+1} = 5f_n - 6 \qquad f_0 = -2.$$

5. Which of the following equations are complete first-order recurrence relations?

a. $u_n = 2 + n$

b. $u_n = u_{n-1} - 1 \quad u_0 = 2$

c. $u_n = 1 - 3u_{n-1} \quad u_0 = 2$

d. $u_n - 4u_{n-1} = 5$

e. $u_n = -u_{n-1}$

f. $u_n = n + 1 \quad u_1 = 2$

g. $u_n = 1 - u_{n-1} \quad u_0 = 21$

h. $u_n = a^{n-1} \quad u_2 = 2$

i. $f_{n+1} = 3f_n - 1$

j. $p_n = p_{n-1} + 7 \quad u_0 = 7$

6. Write the first five terms of each of the following sequences.

a. $u_n = u_{n-1} + 2 \quad u_0 = 6$

b. $u_n = u_{n-1} - 3 \quad u_0 = 5$

c. $u_n = 1 + u_{n-1} \quad u_0 = 23$

d. $u_{n+1} = u_n - 10 \quad u_0 = 7$

7. Write the first five terms of each of the following sequences.

a. $u_n = 3u_{n-1} \quad u_0 = 1$

b. $u_n = 5u_{n-1} \quad u_0 = -2$

c. $u_n = -4u_{n-1} \quad u_0 = 1$

d. $u_{n+1} = 2u_n \quad u_0 = -1$

8. Write the first five terms of each of the following sequences.

a. $u_n = 2u_{n-1} + 1 \quad u_0 = 1$

b. $u_n = 3u_{n-1} - 2 \quad u_0 = 5$

9. Write the first seven terms of each of the following sequences.

a. $u_n = -u_{n-1} + 1 \quad u_0 = 6$

b. $u_{n+1} = 5u_{n-1} \quad u_0 = 1$

10. MC Which of the sequences is generated by the following first-order recurrence relation?

$$u_n = 3u_{n-1} + 4 \quad u_0 = 2$$

A. 2, 3, 4, 5, 6, …

B. 2, 6, 10, 14, 18, …

C. 2, 10, 34, 106, 322, …

D. 2, 11, 47, 191, 767, …

E. 6, 10, 14, 18, 22, …

11. MC Which of the sequences is generated by the following first-order recurrence relation?

$$u_{n+1} = 2u_n - 1 \quad u_0 = -3$$

A. $-3, 5, 9, 17, 33, \ldots$

B. $-3, -5, -9, -17, -33, \ldots$

C. $-3, 5, -3, 5, -3, \ldots$

D. $-3, -8, -14, -26, -54, \ldots$

E. $-3, -7, -15, -31, -63, \ldots$

12. Write the first-order recurrence relations for the following descriptions of a sequence and generate the first five terms of the sequence.

a. The next term is 3 times the previous term, starting at $\frac{1}{4}$.

b. Next year's attendance at a motor show is 2000 more than the previous year's attendance, with a first year attendance of 200 000.

c. The next term is the previous term less 7, starting at 100.

d. The next day's total sum is double the previous day's sum less 50, with a first day sum of $200.

5.3 Modelling flat rate depreciation with a recurrence relation

5.3.1 Flat rate (straight line) depreciation

If an item depreciates by the **flat rate method**, then its value decreases by a fixed amount each unit time interval, generally each year. This depreciation value may be expressed in dollars or as a percentage of the original cost price.

Since the depreciation is the same for each unit time interval, the flat rate method is an example of straight line (linear) decay. The relationship can be represented by the recurrence relation:

$$V_{n+1} = V_n - d$$

where V_n is the value of the asset after n depreciating periods and d is the depreciation each time period.

We can also look at the future value of an asset aftet n periods of depreciation, which can be calculated by:

$$V_n = V_0 - nd$$

We can use this relationship to generate a depreciation schedule (table) which can then be used to draw a graph of future value against time. The schedule displays the future value after each unit time interval, that is:

Time, n	Depreciation, d	Future value, V_n

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Units 3 & 4 > AOS R&FM > Topic 1 > Concept 2

Using recurrence relatives to find depreciation values Summary screen and practice questions

WORKED EXAMPLE 3

Fast Word Printing Company bought a new printing press for $15 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the prime cost price each year and its useful life was 5 years.

a. Find the annual depreciation.

b. Set up a recurrence relation to represent the depreciation.

c. Draw a depreciation schedule for the useful life of the press and use it to draw a graph of book value against time.

d. What is the scrap value?

THINK	WRITE/DRAW
a. 1. State the cost price.	a. $V_0 = 15\,000$

2. Find the depreciation rate as 15% of the prime cost price.

$$d = V_0 \times \frac{r}{100}$$
$$= 15\,000 \times \frac{15}{100}$$
$$= 2250$$

3. Write your answer.

Annual depreciation is $2250.

b. Write the recurrence relation for flat rate depreciation and substitute in the value for d.

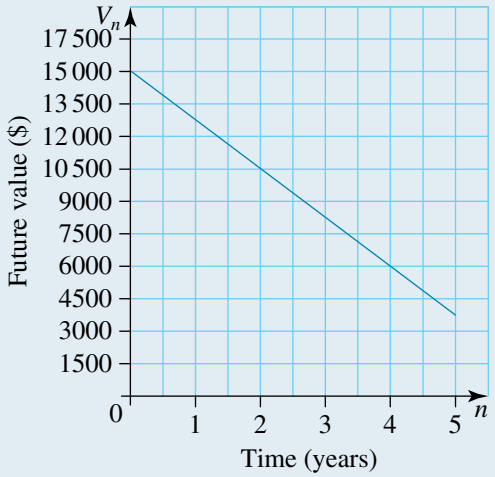
b. $V_{n+1} = V_n - d$
$V_{n+1} = V_n - 2250$

c. 1. Draw a depreciation schedule for 0–5 years, using depreciation of 2250 each year and a starting value of $15 000.
Using your calculator, in the entry type 15 000, press ENTER/EXE then –2250, continue to press ENTER/EXE five times to complete the schedule.

c.

Time, n (years)	Depreciation, d($)	Future value, V_n($)
0	—	15 000
1	2250	12 750
2	2250	10 500
3	2250	8250
4	2250	6000
5	2250	3750

2. Draw a graph of the tabled values for future value against time.



d. 1. The press is scrapped after 5 years so look at the schedule in part **b**.

d. When $n = 5$
$V_5 = 3750$

2. Write your answer.

The scrap value is $3750.

The depreciation schedule gives the scrap value, as can be seen in Worked example 3. So too does a graph of book value against time, since it is only drawn for the item's useful life and its end point is the scrap value.

Businesses need to keep records of depreciation for tax purposes on a year-to-year basis. What if an individual wants to investigate the rate at which an item has depreciated over many years? An example is the rate at which a private car has depreciated. If a straight line depreciation model is chosen, then the following example demonstrates its application.

WORKED EXAMPLE 4

Jarrod bought his car 5 years ago for $15 000. Its current market value is $7500. Assuming straight line depreciation, find:
a. the car's annual depreciation rate
b. the relationship between the future value and time and use your calculator to find when the car will have a value of $3000.

THINK	WRITE
a. 1. Find the total depreciation over the 5 years and thus the rate of depreciation.	a. Total depreciation = cost price − current value = $15 000 − $7500 = $7500 Rate of depreciation = $\frac{\text{total depreciation}}{\text{number of years}}$ = $\frac{\$7500}{5 \text{ years}}$ = $1500 per year
2. Write your answer.	The annual depreciation rate is $1500.
b. 1. Set up the recurrence relationship.	b. $V_{n+1} = V_n - 1500$ $V_0 = 15\,000$
2. Open a calculator screen and in the entry line type 15 000, then press ENTER/EXE. Type −1500 and press ENTER/EXE. Continue to press ENTER/EXE, counting the number of times that ENTER/EXE has been pressed until the value reaches 3000.	$V_n = 3000$ When $n = 8$
3. Write your answer.	The depreciation equation for the car is $V_n = 15\,000 - 1500n$. The future value will reach $3000 when the car is 8 years old.

on Resources

Interactivity: Depreciation: flat rate, reducing balance, unit cost (int-6266)

Exercise 5.3 Modelling flat rate depreciation with a recurrence relation

1. **WE3** A mining company bought a vehicle for $25 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the cost price each year and its useful life was 5 years.
   a. Find the annual depreciation.
   b. Set up a recurrence relation to represent the depreciation.
   c. Draw a depreciation schedule for the item's useful life and draw a graph of future value against time.
   d. What is the scrap value?

2. Write the recurrence relationships for the following items.
   a. A $30 000 car that is depreciated by 20% p.a. flat rate
   b. A $2000 display unit depreciated by 10% of its prime cost
   c. A $6000 piano depreciated by $500 p.a.

3. **WE4** For the situations described below, and using a straight line depreciation model, find:
   i. the annual rate of depreciation
   ii. the relationship between the future value and time, and use CAS to find when the item is written off (has a value of $0)
   a. A car purchased for $50 000 with a current value of $25 000; it is now 5 years old.
   b. A stereo unit bought for $850 seven years ago; it now has a current value of $150.
   c. A refrigerator with a current value of $285 bought 10 years ago for $1235.
4. A second-hand car is currently on sale for $22 500. Its value is expected to *depreciate* by $3200 per year.
   a. Write a recurrence relationship for this item.
   b. Use your calculator to work out the expected value of the car after 5 years.
5. All Clean carpet cleaners bought a cleaner for $10 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the cost price each year and its useful life was 5 years.
   a. find the annual depreciation
   b. using your calculator draw a depreciation schedule for the item's useful life and draw a graph of future value against time.

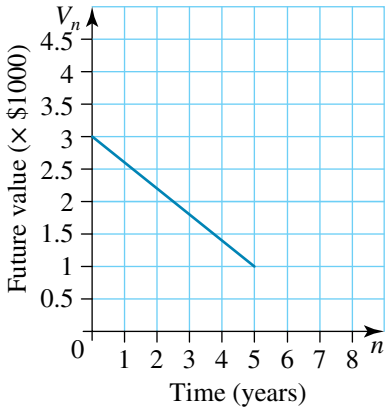
For the situations outlined in questions **6** and **7**:
a. draw a depreciation schedule for the item's first 5 years and draw a graph of future value against time
b. use your calculator to find the scrap value.

6. A farming company chose to depreciate a tractor by the prime cost method and the annual depreciation was $4000. The tractor was purchased for $45 000 and its useful life was 10 years.
7. A winery chose to depreciate a corking machine, that cost $13 500 when new, by the prime cost method. The annual depreciation was $2000 and its useful life was 6 years.

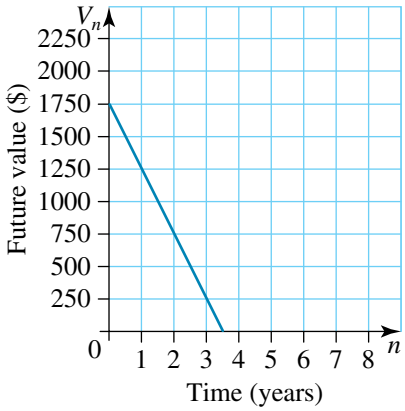
For the situations outlined in questions **8** and **9**:
a. find the annual depreciation
b. set up a recurrence relation to represent the depreciation
c. using your calculator draw a depreciation schedule for the item's useful life and draw a graph of future value against time
d. find how long it will take for the item to reach its scrap value.

8. Machinery is bought for $7750 and depreciated by the flat rate method. The depreciation is 20% of the cost price each year and its scrap value is $1550.
9. An excavation company buys a digger for $92 000 and depreciates it by the flat rate method. The depreciation is 15% of the cost price per year and its scrap value is $9200.
10. Each of the following graphs represents the flat rate depreciation of four particular items. In each case determine:
    i. the cost price of the item
    ii. the annual depreciation
    iii. the time taken for the item to reach its scrap value or to be written off.

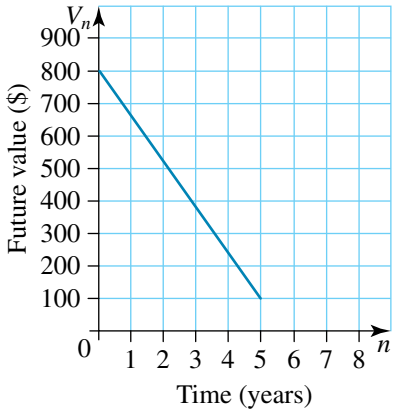
a.



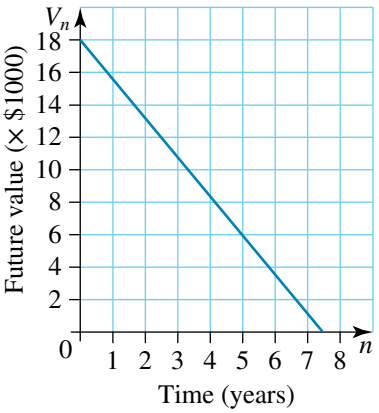
b.



c.



d.



11. MC A 1-tonne truck, bought for $31 000, was depreciated using the flat rate method. If the scrap value of $5000 was reached after 5 years, the annual depreciation would be:

A. $31 **B.** $1000 **C.** $5200 **D.** $6200 **E.** $26 000

12. A business buys two different photocopiers at the same time. One costs $2200 and is to be depreciated by $225 per annum. It also has a scrap value of $400. The other costs $3600 and is to be depreciated by $310 per annum. This one has a scrap value of $500.

a. Which machine would need to be replaced first?

b. How much later would the other machine need to be replaced?

13. MC A car valued at $20 000 was bought 5 years ago for $45 000. The straight line recurrence relation model is represented by:

A. $V_n = V_{n+1} - 5000,\ V_0 = 20\,000$ **B.** $V_{n+1} = V_n - 5000,\ V_0 = 20\,000$

C. $V_{n+1} = V_n + 5000,\ V_0 = 20\,000$ **D.** $V_{n+1} = V_n - 5000$

E. $V_{n+1} = 5000V_n,\ V_0 = 20\,000$

5.4 Modelling reducing balance depreciation with a recurrence relation

If an item depreciates by the **reducing balance depreciation** method then its value decreases by a fixed rate each unit time interval, generally each year. This rate is a percentage of the previous value of the item. Reducing balance depreciation is an example of geometric decay.

Reducing balance depreciation is also known as *diminishing value depreciation*.

Reducing balance depreciation can be expressed by the recurrence relation:

$$V_{n+1} = RV_n$$

where V_n is the value of the asset after n depreciating periods and $R = 1 - \frac{r}{100}$, where r is the depreciation rate.

WORKED EXAMPLE 5

Suppose the new $15 000 printing press considered in Worked example 3 was depreciated by the reducing balance method at a rate of 20% p.a. of the previous value.

a. Generate a depreciation schedule using a recurrence relation for the first 5 years of work for the press.

b. What is the future value after 5 years?

c. Draw a graph of future value against time.

THINK

a. 1. Calculate the value of R.

WRITE/DRAW

a. $R_n = 1 - \frac{r}{100}$

$= 1 - \frac{20}{100}$

$= 0.8$

2. Write the recurrence relation for reducing balance depreciation and substitute in the known information.

$V_{n+1} = R_n V$

$V_{n+1} = 0.8V_n,\ V_0 = 15\,000$

3. Use the recurrence relation and your calculator to calculate the future value for the first 5 years (up to $n = 5$).
On a calculator screen type 15 000, press ENTER/EXE. Type × 0.8, then press ENTER/EXE. Repeat this another 4 times.

15000	15000
15000 × 0.8	12000
12000 × 0.8	9600
96000 × 0.8	7680
7680 × 0.8	6144
6144 × 0.8	4915.20

4. Draw the depreciation schedule.

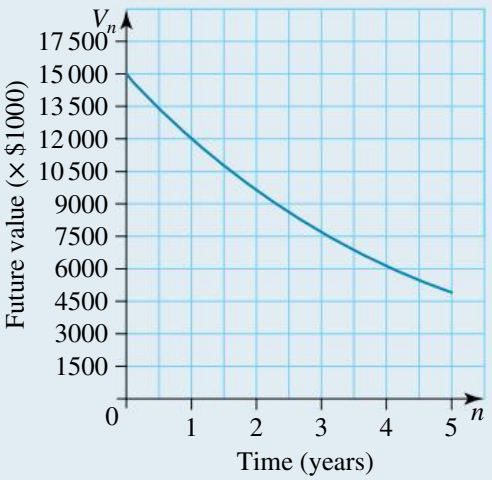
Time, n (years)	Future value, V_n($)
0	15 000
1	12 000
2	9600
3	7680
4	6144
5	4915.20

b. State the future value after 5 years from the depreciation schedule.

b. The future value of the press after 5 years will be $4915.20.

c. Draw a graph of the future value against time.

c.



It is clear from the graph and the schedule that the reducing balance depreciation results in greater depreciation during the early stages of the asset's life (the future value drops more quickly at the start since the annual depreciation falls from $3000 in year 1 to $1228.80 in year 4).

WORKED EXAMPLE 6

A transport business has bought a new bus for $60 000. The business has the choice of depreciating the bus by a flat rate of 20% of the cost price each year or by 30% of the previous value each year.

a. Generate depreciation schedules using both methods for a life of 5 years.

b. Draw graphs of the future value against time for both methods on the same set of axes.

c. After how many years does the reducing balance future value become greater than the flat rate future value?

THINK

a. 1. Calculate the flat rate depreciation per year.

2. Generate a flat rate depreciation schedule using your calculator for 0–5 years.

WRITE/DRAW

a. $d = 20\%$ of $60000
$= \$12\,000$ per year

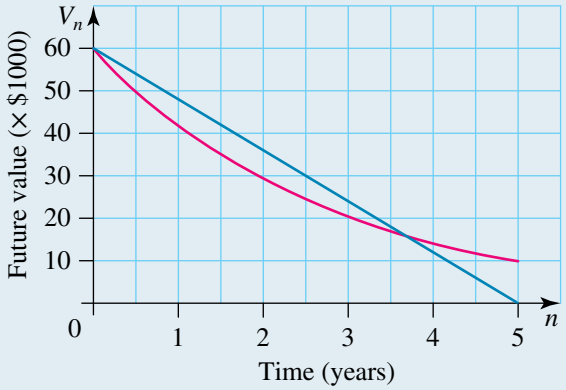
Time, n (years)	Depreciation, d ($)	Future value, V_n ($)
0	—	60 000
1	12 000	48 000
2	12 000	36 000
3	12 000	24 000
4	12 000	12 000
5	12 000	0

3. Generate a reducing balance depreciation schedule. Annual depreciation is 30% of the previous value, so to calculate the future value multiply the previous value by 0.7. Using your calculator continue to calculate the future value for a period of 5 years.

Time, n (years)	Future value, V_n ($)
0	60 000
1	$60\,000 \times 0.7 = 42\,000$
2	$42\,000 \times 0.7 = 29\,400$
3	$29\,400 \times 0.7 = 20\,580$
4	$20\,580 \times 0.7 = 14\,406$
5	$14\,406 \times 0.7 = 10\,084.20$

b. Draw graphs using values for V_n and n from the schedules. In this instance the blue line is the flat rate value and the pink curve is the reducing balance value.

b.



c. Look at the graph to see when the reducing balance curve lies above the flat rate line. State the first whole year after this point of intersection.

c. The future value for the reducing balance method is greater than that of the flat rate method after 4 years.

In the example above, for each consecutive year, the previous future value is multiplied by 0.7. So

$$\begin{aligned} V_5 &= 60\,000 \times 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.7 \\ &= 60\,000 \times 0.7^5 \end{aligned}$$

We can write a general formula for reducing balance depreciation.

The reducing balance depreciation formula is:

$$V_n = V_0 R^n$$

V_n = **book value after time,** n

R = **rate of depreciation** $\left(= 1 - \frac{r}{100}\right)$

V_0 = **cost price**

n = **time since purchase**

That is, given the cost price and depreciation rate we can find the future value (including scrap value) of an article at any time after purchase.

Let us now see how we can use this formula.

WORKED EXAMPLE 7

The printing press from Worked example 3 was depreciated by the reducing balance method at 20% p.a. What will be the future value and total depreciation of the press after 5 years if it cost $15 000 new?

THINK	WRITE
1. State V_0, r and n.	$V_0 = 15\,000$, $r = 20$, $n = 5$
2. Calculate the value of R.	$R = 1 - \frac{20}{100}$ $= 0.8$
3. Substitute into the depreciation formula and simplify.	$V_n = V_0 R$ $V_5 = 15\,000(0.8)^5$
4. Evaluate.	$= 4915.2$
5. Total depreciation is: cost price − future value.	Total depreciation $= V_0 - V_5$ $= 15\,000 - 4915.2$ $= 10084.8$
6. Write a summary statement.	The future value of the press after 5 years will be $4915.20 and its total depreciation will be $10 084.80.

Effective life

The situation may arise where the scrap value is known and we want to know how long it will be before an item reaches this value; that is, its useful or **effective life**.

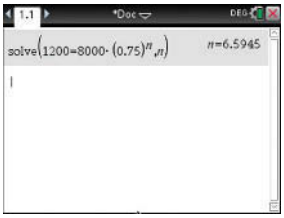
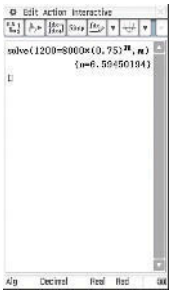
So, in the reducing balance formula $V_n = V_0 R^n$, n is needed.

WORKED EXAMPLE 8

A photocopier purchased for $8000 depreciates by 25% p.a. by the reducing balance method. If the photocopier has a scrap value of $1200, how long will it be before this value is reached? Use the CAS calculator to solve for the number of years.

THINK	WRITE
1. State the values of V_n, V_0 and r.	$V_n = \$1200$, $V_0 = \$8000$ and $r = 25\%$
2. Calculate the value of R.	$R = 1 - \frac{25}{100}$ $= 0.75$
3. Substitute the values of the pronumerals into the formula and simplify.	$V_n = V_0 R^n$ $1200 = 8000 \times (0.75)^n$
4. Use CAS to solve for n.	Solving $1200 = 8000 \times (0.75)^n$ for, $n = 6.59$ years
5. Interest is compounded annually, so n represents years. Raise n to the next whole year.	As the depreciation is calculated once a year, $n = 7$ years.

6. Write your answer.	It will take 7 years for the photocopier to reach its scrap value.

TI | THINK	**WRITE**	**CASIO | THINK**	**WRITE**
4. On a Calculator page, press MENU and select: 3. Algebra 1: Solve Complete the entry line as: solve$(1200 = 8000 \times (0.75)^n, n)$ Then press ENTER.		**4.** On the Main screen, complete the entry line as: solve$(1200 = 8000 \times (0.75)^n, n)$ Then press EXE.	
The answer appears on the screen.	$n = 6.59$ years It will take 7 years for the photocopier to reach its scrap value.	The answer appears on the screen.	$n = 6.59$ years It will take 7 years for the photocopier to reach its scrap value.

Exercise 5.4 Modelling reducing balance depreciation with a recurrence relation

1. **WE5** A laptop was bought new for $1500 and it depreciates by the reducing balance method at a rate of 17% p.a. of the previous book value.
- **a.** Generate a depreciation schedule using a recurrence relation for the first 5 years.
- **b.** What is the future value after 5 years?
- **c.** Draw a graph of future value against time.

2. A road bike was bought new for $3300 and it depreciates by the reducing balance method at a rate of 13% p.a. of the previous value.
- **a.** Generate a depreciation schedule using a recurrence relation for the first 5 years.
- **b.** What is the future value after 5 years?
- **c.** Draw a graph of future value against time.

3. A farming company chose to depreciate its new $60 000 bulldozer by the reducing balance method at a rate of 20% p.a. of the previous value.
- **a.** Write the recurrence relation that represents this depreciation.
- **b.** Draw a depreciation schedule for the first 5 years of the bulldozer's life.
- **c.** What is its future value after 5 years?
- **d.** Draw a graph of future value against time.

4. A retail store chose to depreciate its new $4000 computer by the reducing balance method at a rate of 40% p.a. of the previous value.
- **a.** Write the recurrence relation that represents this depreciation.
- **b.** Draw a depreciation schedule for the first 5 years of the computer's life.
- **c.** What is its future value after 5 years?
- **d.** Draw a graph of future value against time.

5. **WE6** A taxi company bought a new car for $29 990. The company has the choice of depreciating the car by a flat rate of 20% of the cost price each year or by 27% of the previous value each year.
   a. Generate depreciation schedules using both methods for a life of 5 years.
   b. Draw graphs of the future value against time for both methods on the same axis.
   c. After how many whole years does the reducing balance future value become greater than the flat rate future value?

6. A sailing rental company bought a new catamaran for $23 000. The company has the choice of depreciating the catamaran by a flat rate of 20% of the cost price each year or by 28% of the previous value each year.
   a. Generate depreciation schedules using both methods for a life of 5 years.
   b. Draw graphs of the future value against time for both methods on the same axis.
   c. After how many whole years does the reducing balance future value become greater than the flat rate future value?

7. A café buys a cash register for $550. The owner has the choice of depreciating the register by the flat rate method (at 20% of the cost price each year) or the reducing balance method (at 30% of the previous value each year).
   a. Draw depreciation schedules for both methods for a life of 5 years.
   b. Draw graphs of future value against time for both methods on the same set of axes.
   c. After how many whole years does the reducing balance future value become greater than the flat rate future value?

8. Speedy Cabs taxi service has bought a new taxi for $30 000. The company has the choice of depreciating the taxi by the flat rate method (at $33\frac{1}{3}$% of the cost price each year) or the diminishing value method (at 50% of the previous value each year).
   a. Draw depreciation schedules for both methods for 3 years.
   b. Draw graphs of future value against time for both methods on the same set of axes.
   c. After how many whole years does the reducing balance future value become greater than the flat rate future value?

9. **WE7** Using the reducing balance formula, find V_n (correct to 2 decimal places) given $V_0 = 45\,000$, $r = 15$, $n = 6$.

10. Using the reducing balance formula, find V_n (correct to 2 decimal places) given $V_0 = 2675$, $r = 22.5$, $n = 5$.

11. Using the reducing balance formula, find V_n (correct to 2 decimal places) given:
   a. $V_0 = 20\,000$, $r = 20$, $n = 4$
   b. $V_0 = 30\,000$, $r = 25$, $n = 4$.

12. **MC** A refrigerator costing $1200 new is depreciated by the reducing balance method at 20% a year. After 4 years its future value will be:
   A. $240 **B.** $491.52 **C.** $960 **D.** $1105 **E.** $2488.32

13. The items below are depreciated by the reducing balance method at 25% p.a. What will be the future value and total depreciation of:

a. a TV after 8 years, if it cost $1150 new

b. a photocopier after 4 years, if it cost $3740 new

c. carpets after 6 years, if they cost $7320 new?

14. The items below are depreciated at 30% p.a. by the reducing balance method. What will be the future value and total depreciation of:

a. a lawn mower after 5 years, if it cost $685 new

b. a truck after 4 years, if it cost $32 500 new

c. a washing machine after 3 years, if it cost $1075 new?

15. MC After 7 years, a new $3000 photocopier, which devalues by 25% of its value each year, will have depreciated by:

A. $400.45 **B.** $750 **C.** $2250
D. $2599.55 **E.** $2750

16. MC New office furniture valued at $17 500 is subjected to reducing balance depreciation of 20% p.a. and will reach its scrap value in 15 years. The scrap value will be:

A. less than $300 **B.** between $300 and $400 **C.** between $400 and $500
D. between $500 and $600 **E.** between $600 and $700

17. WE8 Use your CAS calculator to find how many years it takes an item of initial value $4500, depreciating at a reducing balance rate of 25% p.a., to reach a value of $900.

18. Use your CAS calculator to solve for n given reducing balance depreciation where $V_n = \$1500$, $V_0 = \$7600$, $r = 15\%$ p.a.

19. MC A new chainsaw bought for $1250 has a useful life of only 3 years. If it depreciates annually by a 60% reducing balance rate, its scrap value will be:

A. $0 **B.** $60 **C.** $80 **D.** $250 **E.** $270

20. An item is depreciated by using reducing balance depreciation. Use your CAS calculator to solve for n (correct to 2 decimal places), given:

a. $V_n = \$3000$, $V_0 = \$40\,000$, $r = 20$

b. $V_n = \$500$, $V_0 = \$3000$, $r = 30$

5.5 Modelling unit cost depreciation with a recurrence relation

The flat rate and reducing balance depreciations of an item are based on the age of the item. With the **unit cost method**, the depreciation is based on the possible maximum output (units) of the item. For instance, the useful life of a truck could be expressed in terms of the distance travelled rather than a fixed number of years; for example, 120 000 kilometres rather than 6 years. The actual depreciation of the truck for the financial year would be a measure of the number of kilometres travelled. (The value of the truck decreases by a certain amount for each kilometre travelled.)

The future value over time using unit cost depreciation can be expressed by the recurrence relation:

$$V_{n+1} = V_n - d$$

where V_n is the value of the asset after n outputs and d is the depreciation per output.

WORKED EXAMPLE 9

A motorbike purchased for $12 000 depreciates at a rate of $14 per 100 km driven.
a. Set up a recurrence relation to represent the depreciation.
b. Use the recurrence relation to generate a depreciation schedule for the future value of the bike after it has been driven for 100 km, 200 km, 300 km, 400 km and 500 km.

THINK

a. 1. Write the recurrence relation for unit cost depreciation as well as the known information.

2. Substitute the values into the recurrence relation.

b. 1. Use the recurrence relation and your calculator to calculate the future value for the first 5 outputs (up to 500 km). Open a calculator screen and type 12 000 press ENTER/EXE, then type −14, continue to press ENTER/EXE 5 more times.

2. Draw the depreciation schedule.

WRITE

a. $V_{n+1} = V_n - d$
$V_0 = 12\,000,\ d = 14$

$V_{n+1} = V_n - 14,\ V_0 = 12\,000$

b.

12000	12000
12000 – 14	11986
11986 – 14	11972
11972 – 14	11958
11958 – 14	11944
11944 – 14	11930

Distance driven (km)	Outputs (n)	Future value, V_n ($)
100	1	11 986
200	2	11 972
300	3	11 958
400	4	11 944
500	5	11 930

WORKED EXAMPLE 10

A taxi is bought for $31 000 and it depreciates by 28.4 cents per kilometre driven. In one year the car is driven 15 614 km. Find:
a. the annual depreciation for this particular year
b. its useful life if its scrap value is $12 000.

THINK

a. 1. Depreciation amount = distance travelled × rate

WRITE

a. Depreciation = 15 614 × $0.284
= $4434.38

2. Write a summary statement.

Annual depreciation for the year is $4434.38.

b. 1. Total depreciation = cost price − scrap value

$$\text{Distance travelled} = \frac{\text{total depreciation}}{\text{rate of depreciation}}$$

where rate of depreciation = 28.4 cents/km
= $0.284 per km

2. State your answer.

b. Total depreciation = 31 000 − 12 000
= $19 000

$$\text{Distance travelled} = \frac{19\,000}{0.284}$$
= 66 901 km

The taxi has a useful life of 66 901 km.

WORKED EXAMPLE 11

A photocopier purchased for $10 800 depreciates at a rate of 20 cents for every 100 copies made. In its first year of use 500 000 copies were made and in its second year, 550 000. Find:
a. the depreciation each year
b. the future value at the end of the second year.

THINK

a. To find the depreciation, identify the rate and number of copies made.
Express the rate of 20 cents per 100 copies in a simpler form of dollars per 100 copies, that is, $0.20 per 100 copies or $\frac{0.20}{100 \text{ copies}}$.

b. Future value
= cost price − total depreciation

WRITE

a. Depreciation = copies made × rate

$$\text{Depreciation 1st year} = 500\,000 \times \frac{0.20}{100 \text{ copies}}$$
= $1000

Depreciation in the first year is $1000.

$$\text{Depreciation 2nd year} = 550\,000 \times \frac{0.20}{100 \text{ copies}}$$
= $1100

Depreciation in the second year is $1100.

b. Total depreciation after 2 years
= 1000 + 1100
= $2100

Book value = 10 800 − 2100
= $8700

The future value after n outputs using unit cost depreciation can be expressed as:

$$V_n = V_0 - nd$$

where V_n is the value of the asset after n outputs and d is the depreciation per output.

WORKED EXAMPLE 12

The initial cost of a vehicle was $27 850 and its scrap value is $5050. If the vehicle needs to be replaced after travelling 80 000 km (useful life):
a. find the depreciation rate (depreciation ($) per km)
b. find the amount of depreciation in a year when 16 497 km were travelled
c. set up an equation to determine the value of the car after travelling n km
d. find the future value after it has been used for a total of 60 000 km
e. set up a schedule table listing future value for every 20 000 km.

THINK	WRITE
a. 1. To find the depreciation rate, first find the total depreciation. Total amount of depreciation = cost price − scrap value	**a.** Total amount of depreciation $= 27\,850 - 5050$ $= \$22\,800$
2. Find the rate of depreciation. It is common to express rates in cents per use if less than a dollar.	$\text{Depreciation rate} = \dfrac{\text{total depreciation}}{\text{total distance travelled}}$ $= \dfrac{22\,800}{80\,000}$ $= \$0.285 \text{ per km}$ $= 28.5 \text{ cents per km}$
b. 1. Find the amount of depreciation using the rate calculated. Amount of depreciation is always expressed in dollars.	**b.** Amount of depreciation = amount of use × rate of depreciation $= 16\,497 \times 28.5$ $= 470\,165 \text{ cents}$ $= \$4701.65$
c. 1. Write the equation for unit cost depreciation after n outputs as well as the known information.	**c.** $V_n = V_0 - nd$ $V_0 = 27\,850,\ d = 0.285$
2. Substitute the values into the equation.	$V_n = 27\,850 - 0.285d$
d. 1. Use the equation from part **c** to find the future value when $n = 60\,000$.	**d.** $V_{60\,000} = 27\,850 - 0.285 \times 60\,000$ $= 27\,850 - 17\,100$ $= 10\,750$
2. Write your answer.	The future value after the car has been used for 60 000 km is $10 750.
e. Calculate the future value for every 20 000 km of use and summarise in a table.	**e.**

Use, n (km)	Future value, V_n ($)
0	$27 850
20 000	$V_1 = 27\,850 - (20\,000 \times 0.285)$ $= \$22\,150$
40 000	$V_2 = 27\,850 - (20\,000 \times 0.285 \times 2)$ $= \$16\,450$
60 000	$V_3 = 27\,850 - (20\,000 \times 0.285 \times 3)$ $= \$10\,750$
80 000	$V_4 = 27\,850 - (20\,000 \times 0.285 \times 4)$ $= \$5\,050$

Exercise 5.5 Modelling unit cost depreciation with a recurrence relation

1. WE9 A washing machine purchased for $800 depreciates at a rate of 35 cents per wash.
   a. Set up a recurrence relation to represent the depreciation.
   b. Use the recurrence relation to generate a depreciation schedule for the future value of the washing machine after each of the first 5 washes.
2. An air conditioning unit purchased for $1400 depreciates at a rate of 65 cents per hour of use.
   a. Set up a recurrence relation to represent the depreciation.
   b. Use the recurrence relation to generate a depreciation schedule for the future value of the air conditioning unit at the end of each of the first 5 hours of use.
3. WE10 Below are the depreciation details for a vehicle. Find:
   a. the annual depreciation in the first year
   b. the useful life (km).

Purchase price ($)	Scrap value ($)	Rate of depreciation (cents/km)	Distance travelled in first year (km)
29 600	12 000	28.5	14 000

4. A taxi was purchased for $42 000 and depreciates by 25 cents per km driven. During its first year the taxi travelled 64 000 km; during its second year it travelled 56 000 km. Find:
   a. the depreciation in each of the first 2 years
   b. how far the car had travelled if its total depreciation was $20 000.
5. Below are depreciation details for 2 vehicles. In each case find:
   i. the annual depreciation in the first year
   ii. the useful life (km).

	Purchase price ($)	Scrap value ($)	Rate of depreciation (cents/km)	Distance travelled in first year (km)
a	25 000	10 000	26	12 600
b	21 400	8 000	21.6	13 700

In each situation in questions 6 to 8, find:
   a. the annual depreciation in the first year
   b. the item's useful life.

6. A company buys a $32 000 car which depreciates at a rate of 23 cents per km driven. It covers 15 340 km in the first year and has a scrap value of $9500.
7. A new taxi is worth $29 500 and it depreciates at 27.2 cents per km travelled. In its first year of use it travelled 28 461 km. Its scrap value was $8200.

8. A photocopier purchased for $7200 depreciates at a rate of $1.50 per 1000 copies made. In its first year of use, 620 000 copies were made and in its second year, 540 000 were made. Find:
   a. the depreciation for each year
   b. the future value at the end of the second year.

9. A photocopier bought for $11 300 depreciates at a rate of 2.5 cents for every 10 copies made.
   a. Set up a recurrence relation to represent the depreciation.
   b. Copy and complete the table below.

Time (years)	Outputs, n (10 copies)	Annual depreciation ($)	Future value at end of year, V_n ($)
1	35 000		
2	42 500		
3	37 620		
4	29 104		
5	38 562		

10. A corking machine bought for $14 750 depreciates at a rate of $2.50 for every 100 bottles corked.
   a. Set up a recurrence relation to represent the depreciation.
   b. Copy and complete the table below.

Time (years)	Outputs, n (100 bottles corked)	Depreciation ($)	Future value at end of year, V_n($)
1	400		
2	425		
3	467		
4	382.5		
5	430.6		

11. WE11 A photocopier is bought for $8600 and it depreciates at a rate of 22 cents for every 100 copies made. In its first year of use, 400 000 copies are made and in its second year, 480 000 copies are made. Find:
   a. the depreciation for each year
   b. the future value at the end of the second year.
12. A printing machine was purchased for $38 000 and depreciated at a rate of $1.50 per million pages printed. In its first year 385 million pages were printed and 496 million pages were printed in its second year. Find:
   a. the depreciation for each year
   b. the future value at the end of the second year.
13. MC A vehicle is bought for $25 900 and it depreciates at a rate of 21.6 cents per km driven. After its first year of use, in which it travels 13 690 km, the future value of the vehicle is closest to:
   A. $1000 **B.** $3000 **C.** $20 000 **D.** $23 000 **E.** $25 000

In each situation in questions 14 and 15, find:
   a. the depreciation for each year
   b. the future value at the end of the second year.

14. A company van is purchased for $32 600 and it depreciates at a rate of 24.8 cents per km driven. In its first year of use the van travels 15 620 km and it travels 16 045 km in its second year.

15. A taxi is bought for $35 099 and it depreciates at a rate of 29.2 cents per km driven. It travels 21 216 km in its first year of use and 19 950 km in its second year.

16. WE12 A delivery service purchases a van for $30 000 and it is expected that the van will be written off after travelling 200 000 km. It is estimated that the van will travel 1600 km each week.

- **a.** Find the depreciation rate (charge per km).
- **b.** Find how long it will take for the van to be written off.
- **c.** Set up an equation to determine the value of the van after travelling n kilometre.
- **d.** Find the distance travelled for the van to depreciate by $13 800.
- **e.** Find its future value after it has travelled 160 000 kilometres.
- **f.** Set up a schedule table for the value of the van for every 20 000 kilometres.

17. A car bought for $28 395 depreciates at a rate of 23.6 cents for every km travelled. Copy and complete the table below.

Time (years)	Distance travelled, n(km)	Depreciation ($)	Future value at end of year, V_n ($)
1	13 290		
2	15 650		
3	14 175		
4	9 674		
5	16 588		

18. A car is bought for $35 000 and a scrap value of $10 000 is set for it. The following three options for depreciating the car are available:

- **i.** flat rate of 10% of the purchase price each year
- **ii.** 20% p.a. of the reducing balance
- **iii.** 25 cents per km driven (the car travels an average of 10 000 km per year).
  - **a.** Which method will enable the car to reach its scrap value soonest?
  - **b.** If the car is used in a business the annual depreciation can be claimed as a tax deduction. What would the tax deduction be in the first year of use for each of the depreciation methods?
  - **c.** How would your answers to part **b** vary for the 5th year of use?

19. MC A machine which was bought for $8500 was depreciated at the rate of 2 cents per unit produced. By the time the book value had decreased to $2000, the number of units produced would be:

- **A.** 75 000
- **B.** 100 000
- **C.** 125 000
- **D.** 300 000
- **E.** 325 000

20. MC An $8500 machine depreciates by 2 cents per unit. By the time the machine had depreciated by $5000, it would have produced:

- **A.** 275 000 units
- **B.** 250 000 units
- **C.** 225 000 units
- **D.** 175 000 units
- **E.** 150 000 units

5.6 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC Which of the sequences is generated by the first order recurrence relation $u_{n+1} = u_n - 4, u_0 = -6$?
   A. $-4, -10, -16, -22, -28, \ldots$ **B.** $-6, -10, -14, -18, -22, -28, \ldots$
   C. $-6, -2, -6, -2, -6, \ldots$ **D.** $-6, -2, 2, 6, 10, \ldots$
   E. $-6, 2, -10, 4, -14, \ldots$
2. MC Given the recurrence relation
   $$V_{n+1} = 3V_n - 2,\ V_0 = 2$$
   The first 4 terms of the relation are
   A. $2, 6, 18, 54$ **B.** $1, 2, 3, 4$ **C.** $2, 4, 10, 28$
   D. $-2, -8, -26, -80$ **E.** $2, 4, 12, 34$
3. MC A recurrence relation is defined as
   $$V_{n+1} = 1.025\,V_n - 200, V_0 = 10\,000$$
   The fifth term in this recurrence relation would be closest to
   A. 11 038 **B.** 10 769 **C.** 10 506 **D.** 10 208 **E.** 10 110
4. MC The sequence $4, 2, 0, -2, -4, \ldots$ can be defined by the first order recurrence relation:
   A. $u_{n+1} = 2u_n \quad u_0 = 4$ **B.** $u_{n+1} = -2u_n \quad u_0 = 4$
   C. $u_{n+1} = u_n - 2 \quad u_0 = 4$ **D.** $u_{n+1} = 2u_n + 2 \quad u_0 = 4$
   E. $u_{n+1} = 4u_n + 2 \quad u_0 = 4$
5. MC The sequence $-3, 12, -48, 192, -768, \ldots$ can be defined by the first order recurrence relation:
   A. $A_{n+1} = -4A_n \quad A_0 = -3$ **B.** $A_{n+1} = -4A_n \quad A_0 = 3$
   C. $A_{n+1} = -3A_n \quad A_0 = -4$ **D.** $A_{n+1} = -3A_n \quad A_0 = 4$
   E. $A_{n+1} = 4A_n \quad A_0 = 3$
6. MC A yacht was purchased for $54 000. It depreciated at a rate of 12% each year. The recurrence relation that could be used to determine the value of the yacht, Y_n, after n years is
   A. $Y_{n+1} = -0.88Y_n \quad Y_0 = 54\,000$ **B.** $Y_{n+1} = Y_n - 6480 \quad Y_0 = 54\,000$
   C. $Y_{n+1} = Y_n - 54\,000 \quad Y_0 = 6480$ **D.** $Y_{n+1} = 1.12Y_n \quad Y_0 = 6480$
   E. $Y_{n+1} = 0.88Y_n \quad Y_0 = 54\,000$
7. MC A library adds 300 new books to its collection each year. The collection began with 4000 books and it is claimed that no book has ever been removed.
   A first order recurrence relation that reflects this situation is:
   A. $B_n = B_{n-1} + 300 \quad B_0 = 300$ **B.** $B_n = B_{n-1} + 300 \quad B_0 = 4000$
   C. $B_n = 1.03B_{n-1} + 100 \quad B_0 = 300$ **D.** $B_n = 1.03B_{n-1} + 4120 \quad B_0 = 4000$
   E. $B_n = 1.04B_{n-1} + 300 \quad B_0 = 300$
8. MC Consider the first order recurrence relations below.
   $V_{n+1} = V_n - 500,\ V_0 = 9500$
   $P_{n+1} = 0.7P_n,\ P_0 = 8000$
   V_{n+1} and P_{n+1} are examples of
   A. Linear decay and geometric decay respectively
   B. Geometric growth and linear decay respectively
   C. Linear growth and geometric growth respectively
   D. Geometric decay and linear decay respectively
   E. Linear decay

9. MC The reducing balance depreciation method is used to depreciate an air conditioner at a rate of 12.5% of the previous value. The purchase price of the air-conditioner was $8400.
Which recurrence relation below can be used to describe this situation?

A. $V_{n+1} = V_n - 1050, V_0 = 8400$
B. $V_{n+1} = 0.125V_n$
C. $V_{n+1} = 0.125V_n, V_0 = 8400$
D. $V_{n+1} = 0.875V_n, V_0 = 8400$
E. $V_{n+1} = 0.875V_n$

10. MC A coffee machine was purchased for $12 500. It has been depreciated using a flat rate model of $1000 per year. How many years before it reached the book value of $6500?

A. 2
B. 4
C. 6
D. 8
E. 10

Short answer

1. Write the first five terms of each of the following sequences.
   a. $u_{n+1} = 4u_n - 3 \qquad u_0 = -1$
   b. $u_{n+1} = 3 + 5u_n \qquad u_0 = 0$
2. A club loses 4% of its membership each year but adds 20 new members each year. The initial membership of the club was 300.
   a. Write a first-order recurrence relation to describe this situation, stating clearly the terms you use.
   b. What is the membership of the club at the end of 4 years?
3. A first-order recurrence relation is given as $C_{n+1} = 0.5C_n + 100$ and $C_0 = 400$. What is the value of C_4?
4. A security firm purchased a van for $35 000. The firm decided to use the reducing balance method of depreciation at 8% per year. Let P_n be the value of the van after n years.
   a. Write the recurrence relation for this situation.
   b. What will be the future value after 5 years?
5. Office equipment is bought for $18 000 and depreciated by the flat rate method. The depreciation is 15% of the cost price each year and its scrap value is $1800.
   Calculate:
   a. the annual depreciation.
   b. how long it will take for the equipment to reach its scrap value.
6. A 3D printer is purchased for $6 500 and it depreciates by $5.20 for every 20 items printed. In its first year of use, 2000 items were printed and in its second year 2500 copies were printed.
   Calculate:
   a. the depreciation each year.
   b. the future value at the end of the second year.

Extended response

1. A band has bought new electronic recording equipment for $150 000 and has a choice of depreciating the equipment by a flat rate of 15% of the cost price each year or by 25% of the previous value each year.
   a. Using your calculator, generate depreciation schedules using both methods for a life of 5 years
   b. After how many years does the reducing balance future value become greater than the flat rate future value?

2. Rachel bought a new limousine for her events management company. She paid $60 000 and is considering various methods of depreciating her new asset.

The limousine can be depreciated using flat rate method. The depreciation rate is 12% of the prime cost price per year.

a. Find annual depreciation of the limousine.
Hence, set up the recurrence relation to represent the depreciation.

b. Generate the relationship between the book value of the limousine and time.

c. Calculate the book value of the limousine and total amount of depreciation after 5 years.Another way is to use unit cost depreciation method. A limousine can be depreciated at a rate of 30 cents per kilometre driven.

d. Rachel estimates that her new limousine will be driven at least 25 000 km per year. According to this estimation, what would be the book value of the car after 5 years?
Rachel considers her final option – reducing balance depreciation method at 16% p.a.

e. Find the future value and the total depreciation of the limousine after 5 years.

f. Which of the three methods gives greatest total depreciation over the 5 year period?

studyon

Units 3 & 4 Sit exam

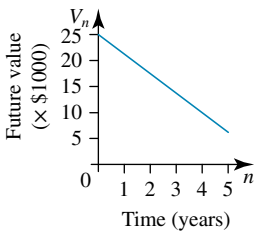
Answers

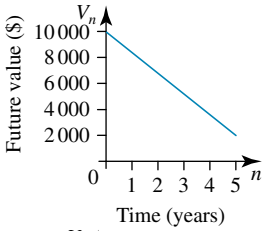
Topic 5 Depreciation of assets

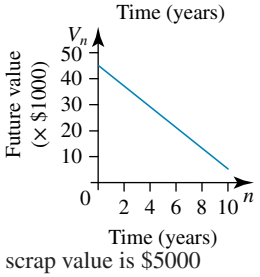
Exercise 5.2 Generating the terms of a first-order recurrence relation

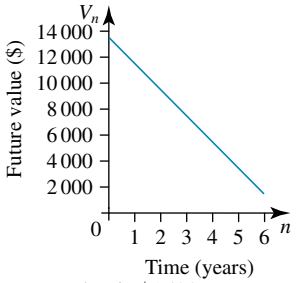
1. $-2, -5, -8, -11, -14$
2. $3, 6, 12, 24, 48$
3. $u_n = 4u_{n-1} + 3, u_0 = 5$
   The sequence is $5, 23, 95, 383, 1535$.
4. $f_{n+1} = 5f_n - 6, f_0 = -2$
   The sequence is $-2, -16, -86, -436, -2186$.
5. **b**, **c**, **g**, **j**
6. **a.** $6, 8, 10, 12, 14$
   b. $5, 2, -1, -4, -7$
   c. $23, 24, 25, 26, 27$
   d. $7, -3, -13, -23, -33$
7. **a.** $1, 3, 9, 27, 81$
   b. $-2, -10, -50, -250, -1250$
   c. $1, -4, 16, -64, 256$
   d. $-1, -2, -4, -8, -16$
8. **a.** $1, 3, 7, 15, 31$
   b. $5, 13, 37, 109, 325$
9. **a.** The first seven term are $6, -5, 6, -5, 6, -5, 6$.
   b. The first seven term are
   $1, 5, 25, 125, 625, 3125, 15\,625$.
10. C
11. E
12. **a.** $u_{n+1} = 3u_n, u_0 = \frac{1}{4}; \frac{1}{4}, \frac{3}{4}, 2\frac{1}{4}, 6\frac{3}{4}, 20\frac{1}{4}$
    b. $u_{n+1} = u_n + 2000, u_0 = 200\,000;$ $200\,000, 202\,000$
    $204\,000, 206\,000, 208\,000$
    c. $u_{n+1} = u_n - 7, u_0 = 100;$ $100, 93, 86, 79, 72$
    d. $u_{n+1} = 2u_n - 50,$
    $u_0 = \$200; \$200, \$350, \$650, \$1250, \$2450$

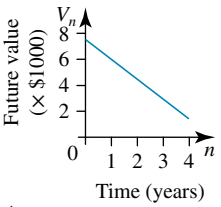
Exercise 5.3 Modelling flat rate depreciation with a recurrence relation

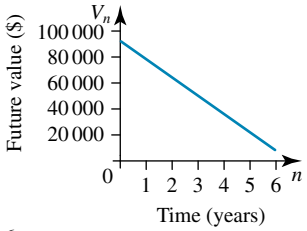
1. **a.** \$3750
   b. $V_{n+1} = V_n - 3750$
   c.


   d. \$6250
2. **a.** $V_0 = 30\,000, V_{n+1} = V_n - 6000$
   b. $V_0 = 2000, V_{n+1} = V_n - 200$
   c. $V_0 = 6000, V_{n+1} = V_n - 500$
3. **a.** **i.** \$5000 per year
   ii. 10 years old
   b. **i.** \$100 per year
   ii. 8.5 years old
   c. **i.** \$95 per year
   ii. 13 years old
4. **a.** $V_0 = 22,500$
   $V_{n+1} = V_n - 3200$
   b. \$6500
5. **a.** \$1500
   b.


6. **a.**


   b. scrap value is \$5000
7. **a.**


   b. scrap value is \$1500
8. **a.** \$1550
   b. $V_{n+1} = V_n - 1550, V_0 = 7750$
   c.


   d. 4 years
9. **a.** \$13 800
   b. $V_{n+1} = V_n - 13\,800, V_0 = 92\,000$
   c.


   d. 6 years
10. **a.** **i.** \$3000 **ii.** \$400 **iii.** 5 years
    b. **i.** \$1750 **ii.** \$500 **iii.** $3\frac{1}{2}$ years
    c. **i.** \$800 **ii.** \$140 **iii.** 5 years
    d. **i.** \$18 000 **ii.** \$2400 **iii.** $7\frac{1}{2}$ years
11. C
12. **a.** The cheaper machine **b.** 2 years
13. B

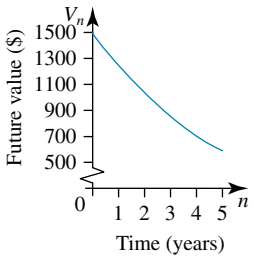
Exercise 5.4 Modelling reducing balance depreciation with a recurrence relation

1. a.

Time n (years)	Future value, V_n ($)
0	1500
1	1245.00
2	1033.35
3	857.68
4	711.87
5	590.86

b. 590.86

c.

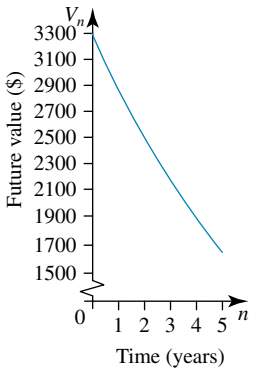


2. a.

Time n (years)	Future value, V_n ($)
0	3300.00
1	2871.00
2	2497.77
3	2173.06
4	1890.56
5	1644.79

b. 1644.79

c.



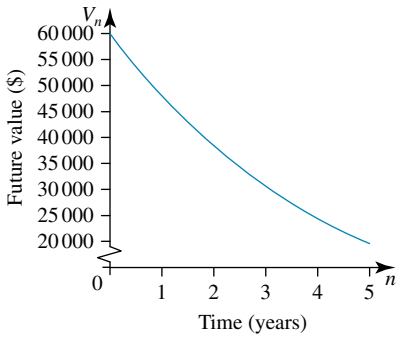
3. a. $V_{n+1} = 0.8V_n,\ V_0 = 60\,000$

b.

Time n (years)	Future value, V_n ($)
0	60 000.00
1	48 000.00
2	38 400.00
3	30 720.00
4	24 576.00
5	19 660.80

c. 19 660.80

d.



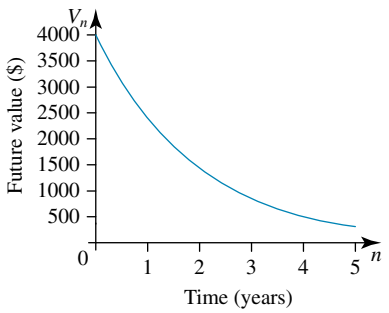
4. a. $V_n + 1 = 0.6V_n,\ V_0 = 4000$

b.

Time n (years)	Depreciation ($)	Future value, V_n ($)
0	–	4000.00
1	1600	2400.00
2	960	1440.00
3	576	864.00
4	345.60	518.40
5	207.36	311.04

c. 311.04

d.

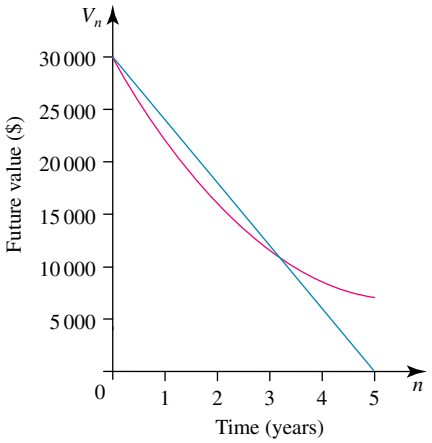


5. a.

Time n (years)	Depreciation ($)	Future value, V_n ($)
0	–	29 900
1	5998	23 992
2	5998	17 994
3	5998	11 996
4	5998	5 998
5	5998	0

Time n (years)	Future value, V_n ($)
0	29 900.00
1	21 892.70
2	15 981.67
3	11 666.62
4	8 516.63
5	6217.14

b.



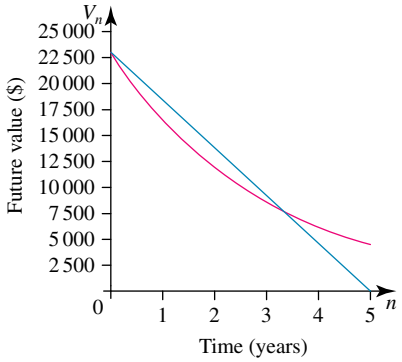
c. The future value for the reducing balance method is greater than the flat rate method after 4 years.

6. a.

Time n (years)	Depreciation ($)	Future value, V_n ($)
0	–	23 000
1	4600	18 400
2	4600	13 800
3	4600	9 200
4	4600	4 600
5	4600	0

Time n (years)	Future value, V_n ($)
0	23 000.00
1	16 560.00
2	11 923.20
3	8 584.70
4	6 180.98
5	4 450.31

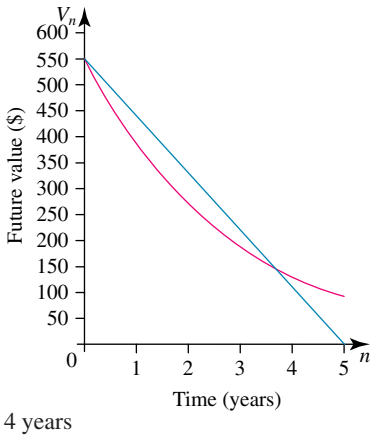
b.



c. The future value for the reducing balance method is greater than the flat rate method after 4 years.

7. a. See table at bottom of the page *

b.



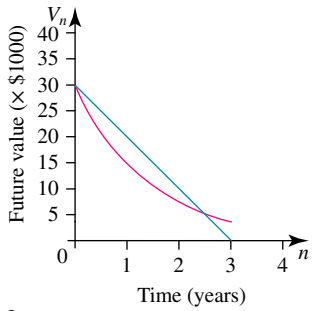
c. 4 years

*

Flat rate			Reducing balance		
Time (years)	Dep. ($)	Future value ($)	Time (years)	Dep. ($)	Future value ($)
0	–	550	0	–	550.00
1	110	440	1	165	385.00
2	110	330	2	115.50	269.50
3	110	220	3	80.85	188.65
4	110	110	4	54.60	132.05
5	110	0	5	39.62	92.43

8. a. See table at bottom of the page*

b.



c. 3 years

9. $16 971.73

10. $747.88

11. a. $8192 b. $9492.19

12. B

13. a. $115.13, $1034.87
b. $1183.36, $2556.64
c. $1302.80, $6017.20

14. a. $115.13, $569.87
b. $7803.25, $24 696.75
c. $368.73, $706.27

15. D

16. E

17. 5.59

18. 9.98

19. C

20. a. 11.61 b. 5.02

Exercise 5.5 Modelling unit cost depreciation with a recurrence relation

1. a. $V_{n+1} = V_n - 0.35n,\ V_0 = 800$

b.

Number of washes (n)	Future value, V_n ($)
1	799.65
2	799.30
3	798.95
4	798.60
5	798.25

2. a. $V_{n+1} = V_n - 0.65n,\ V_0 = 1400$

Hours of use (n)	Future value, V_n ($)
1	1399.35
2	1398.70
3	1398.05
4	1397.40
5	1396.75

3. a. $3990 b. 61 754 km

4. a. $16 000, $14 000 b. 80 000 km

5. a. i. $3276 ii. 57 692 km
b. i. $2959.20 ii. 62 037 km

6. a. $3528.20 b. 97 826 km

7. a. $7741.39 b. 78 309 km

8. a. $930, $810 b. $5460

9. a. $V_{n+1} = V_n - 0.0025n,\ V_0 = 11\,300$

b.

Annual depreciation ($)	Future value at end of year, V_n ($)
875	10 425
1062.50	9 362.50
940.50	8 422
727.60	7 694.40
964.05	6730.35

10. a. $V_{n+1} = V_n - 0.025n,\ V_0 = 14\,750$

b.

Depreciation ($)	Future value at end of year, V_n ($)
1000	13 750
1062.50	12 687.50
1167.50	11 520
956.25	10 563.75
1076.50	9487.25

11. a. $880, $1056 b. $6664

12. a. $577.50, $744 b. $36 678.50

*

Flat rate			Reducing balance		
Time (years)	Dep. ($)	Future value ($)	Time (years)	Dep. ($)	Future value ($)
0	–	30 000	0	–	30 000
1	10 000	20 000	1	15 000	15 000
2	10 000	10 000	2	7 500	7 500
3	10 000	0	3	3 750	3 750

13. D

14. a. $3873.76, $3979.16 b. $24 747.08

15. a. $6195.07, $5825.40 b. $23 078.53

16. a. 15 cents per km b. 2 years 21 weeks
c. $V_n = 30\,000 - 0.15n$ d. 92 000 km
e. $6000
f.

Distance n (km)	Value, ($)
0	30 000
20 000	27 000
40 000	24 000
60 000	21 000
80 000	18 000
100 000	15 000
120 000	12 000
140 000	9 000
160 000	6 000
180 000	3 000
200 000	0

17.

Depreciation ($)	Future value at end of years, V_n ($)
3136.44	25 258.56
3693.40	21 565.16
3345.30	18 219.86
2283.06	15 936.80
3914.77	12 022.03

18. a. Reducing balance depreciation (6 years compared to 7+ years and 10 years)
b. i. $3500 ii. $7000 iii. $2500
c. i. $3500 (same)
ii. $2867.20 ($4132.80 less)
iii. $2500 (same)

19. E

20. B

5.6 Review: exam practice

Multiple choice

1. B 2. C 3. D 4. C 5. A
6. E 7. B 8. A 9. D 10. C

Short answer

1. a. −1, −7, −31, −127, −384
b. 0, 3, 18, 93, 468

2. a. $M_{n+1} = 0.96M_n + 20, M_0 = 300$
b. 330 members

3. 212.5

4. a. $P_{n+1} = 0.92P_n$, $P_0 = 35\,000$
b. $21 222.43

5. a. $2700 b. 6 years

6. a. $520. $650 b. $5330

Extended response

1. a.

Time (n)	Future value (flat rate)	Future value (reducing balance)
0	150 000	150 000
1	127 500	112 500
2	105 000	84 375
3	82 500	63281.25
4	60 000	47 460.94
5	37 500	35 595.70
6	15 000	26 696.80

b. 6 years

2. a. $7200; $V_{n+1} = V_n - 7200, V_0 = 60\,000$
b. $V_n = 60\,000 - 7200n$
c. Book value after 5 years is $24 000; total depreciation is $36 000.
d. $22 500
e. Future value after 5 years is $25 092.70; total depreciation is $34 907.30.
f. Unit cost depreciation

TOPIC 6
Simple and compound interest — investments and loans

6.1 Overview

6.1.1 Introduction

Most people in the modern world will at some stage in their lives borrow and invest money. When a person enters the work force, they may want a car or a house or other goods and do not have the money to purchase these items outright. They then borrow the money believing that they have the means to service the loan and pay the money back over a specified time. In Australia, every employer must contribute to their employee's Superannuation Guarantee Fund which is set at 9.5% p.a. of their base salary. Many employees also make extra contributions to their superannuation fund. Self-employed people are encouraged to set up their own superannuation fund. The money in the superannuation funds is invested so that by the age of retirement, every person will have some money saved. Given the likelihood that every individual will need to grapple with decisions pertaining to borrowing and investing money throughout their adult life, it is essential that everyone has a competent understanding of financial mathematics. This will ensure that all their options have been thoroughly investigated and that their decision making is therefore informed.

LEARNING SEQUENCE

6.1 Overview
6.2 Simple interest
6.3 Compound interest as a geometric recurrence relation
6.4 Compound interest rule
6.5 Finding rate or time for compound interest
6.6 Effective annual interest rate
6.7 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

6.1.2 Kick off with CAS

Calculating interest with CAS

A recursion relation links one term in a series to the previous or next term in the same series.

We can use recursion equations to model financial situations, such as the amount of interest accrued in a bank account for a number of consecutive years.

1. $3000 was placed into a bank account earning simple interest at a rate of 5% per annum.

a. Using CAS, define the recurrence relation $V_{n+1} = V_n + 150,\ V_0 = 3000$ that represents this investment.

b. Use your recurrence relation defined in part **a** to complete the following table which represents the amount of money in the bank account after n years.

c. Use CAS to plot a graph of the amount of money in the bank after the first 5 years (using the figures in the table).

Years	Amount in bank account ($)
0	3000
1	
2	
3	
4	
5	

2. $3000 was placed into a bank account earning compound interest at a rate of 5% per annum.

a. Using CAS, define the recurrence relation $V_{n+1} = 1.05V_n,\ V_0 = 3000$ that represents this investment.

b. Use your recurrence relation defined in part **a** to complete the following table which represents the amount of money in the bank account after n years.

c. Use CAS to plot a graph of the amount of money in the bank after the first 5 years (using the figures in the table).

Years	Amount in bank account ($)
0	3000
1	
2	
3	
4	
5	

3. Comment on the shape of the two graphs drawn in questions **1** and **2**

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comphrehensive step-by-step guide on how to use your CAS technology.

6.2 Simple interest

6.2.1 Simple interest as a first-order linear recurrence relation

Simple interest can be represented by a first-order linear recurrence relation.

$$V_{n+1} = V_n + d,\ V_0 = \text{initial value;}$$

$$\text{where } d = \frac{V_0 \times r}{100}$$

where V_n represents the value of the investment after n time periods
d is the amount of interest earned per period
V_0 is the initial (or principal) amount and
r is the interest rate per period.

WORKED EXAMPLE 1

$325 is invested in a simple interest account for 5 years at 3% p.a. (per year).
a. Set up a recurrence relation to find the value of the investment after n years.
b. Use the recurrence relation from part a to find the value of the investment at the end of each of the first 5 years.

THINK

a. 1. Write the formula to calculate the amount of interest earned per period (d).
2. List the values of V_0 and r.
3. Substitute into the formula and evaluate the amount of interest earned per period.
4. Use the values of d and V_0 to set up your recurrence relation.

b. 1. Set up a table to find the value of the investment for up to $n = 5$.
2. Use the recurrence relation from part a to complete the table.

WRITE

a. $d = \dfrac{V_0 \times r}{100}$

$V_0 = 325, r = 3$

$d = \dfrac{325 \times 3}{100}$

$= 9.75$

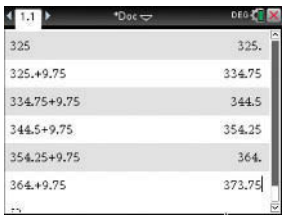
$V_{n+1} = V_n + 9.75, V_0 = \$325$

b.

$n + 1$	$V_n(\$)$	$V_{n+1}(\$)$
1	325	$325 + 9.75 = 334.75$
2	334.75	$334.75 + 9.75 = 344.50$
3	344.50	$344.50 + 9.75 = 354.25$
4	354.25	$354.25 + 9.75 = 364$
5	364	$364 + 9.75 = 373.75$

3. Write your answer.

The value of the investment at the end of each of the first 5 years is: $334.75, $344.50, $354.25, $364 and $373.75.

TI | THINK	WRITE
b. 1. On a Calculator page, type 325 and then ENTER. Complete the entry line as: +9.75 Then continue to press ENTER until the first 5 terms are displayed on the screen.	
2. The answer appears on the screen.	The value of the investment at the end of each of 5 years is: $354.25, $364, $373.75.

CASIO | THINK	WRITE
1. On the Main screen, type 325 and then EXE. Complete the entry line as: +9.75 Then continue to press EXE until the first 5 terms are displayed on the screen.	
2. The answer appears on the screen.	The value of the investment at the end of each of 5 years is: $354.25, $364, $373.75.

WORKED EXAMPLE 2

A bank offers 9% p.a. simple interest on an investment of $600.

a. Write the recurrence relation for this situation.

b. Using your calculator, complete a table to show the amount in the investment account at the end of 4 years.

c. What is the total interest earned after 4 years?

THINK	WRITE
a. 1. Write the formula to calculate the amount of interest earned per period (d).	**a.** $d = \frac{V_0 \times r}{100}$
2. List the values of V_0 and r.	$V_0 = 600, r = 9$
3. Substitute into the formula and evaluate the amount of interest earned per period.	$d = \frac{600 \times 9}{100}$ $= 54$
4. Use the values of d and V_0 to set up your recurrence relation.	$V_{n+1} = V_n + 54, V_0 = \$600$
b. 1. On a calculator screen, type 600, press ENTER/EXE, then type + 54, press ENTER/EXE 4 times to obtain the amount in the investment account at the end of each of 4 years.	**b.**

600	600
600 + 54	654
654 + 54	708
1308 + 54	762
1362 + 54	816

2. Write the answer.

At the end of 4 years there will be $816 in the account.

c. 1. The amount of interest earned is the total amount in the account at the end of the 4 years minus the principal (V_0, the initial amount of money in the account).

c. $I = 816 - 600$
$= 216$

2. Write the answer.

The interest earned in 4 years is $216.

6.2.2 Simple interest using a rule

You can also calculate the total amount of a simple interest loan or investment by using the following rule:

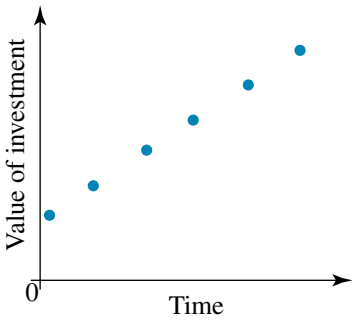
Total amount of loan or investment = initial amount (or principal) + $n \times d$

$$V_n = V_0 + nd$$

where V_n represents the value of the investment after n time periods
d is the amount of interest earned per period,
V_0 is the initial amount (or principal).

Simple interest is the percentage of the amount borrowed or invested multiplied by the number of time periods (usually years). The amount is added to the principal either as payment for the use of the money borrowed or as return on money invested.

In the case of simple interest, the total value of investment increases by the same amount per period. Therefore, if the values of the investment at the end of each time period are plotted, a straight-line graph is formed. Simple interest is an example of a linear model of growth.



WORKED EXAMPLE 3

Jan invested $210 with a building society in a fixed deposit account that paid 8% p.a. simple interest for 5 years.
a. Use the simple interest rule to determine how much was in her account after 5 years.
b. Represent the account balance for each of the 5 years graphically.

THINK

a. 1. Write the simple interest rule.

2. Write the formula to calculate the amount of interest earned per period.

WRITE/DRAW

a. $V_n = V_0 + nd$

$d = \dfrac{V_0 \times n}{100}$

3. List the values of V_0 and V, subtitute into the formula and evaluate.

$V_0 = 210,\ n = 8$

$$d = \frac{210 \times 8}{100}$$
$$= \$16.80$$

4. List the values of V_0, d and n to use in the simple interest rule.

$V_0 = 210,\ n = 5,\ d = \$16.80$

5. Substitute the values of the pronumerals into the rule and evaluate.

$$V_n = 210 + 5 \times 16.80$$
$$= 294$$

6. Write the answer.

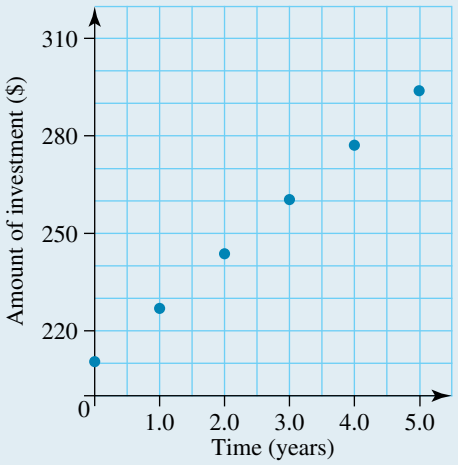
The amount in her account after 5 years is $294.

b. 1. As we are dealing with simple interest, the value of the investment increases each year by the same amount.

b. Increase per year = $16.80

2. Draw a set of axes. Put time (in years) on the horizontal axis and the amount of investment (in $) on the vertical axis. Plot the points: the initial value of investment is $210 and it grows by $16.80 each year.

Year 1 = 210 + 16.80
= $226.80
Year 2 = 226.80 + 16.80
= $243.60
Year 3 = 243.60 + 16.80
= $260.40
Year 4 = 260.40 + 16.80
= $277.20
Year 5 = $294 (from part a).



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Units 3 & 4 > AOS R&FM > Topic 2 > Concept 1

Simple interest Summary screen and practice questions

Exercise 6.2 Simple interest

1. WE1 $1020 is invested in a simple interest account for 5 years at 8.5% p.a.
   a. Set up a recurrence relation A_n to find the value of the investment after n years.
   b. Using your calculator and the recurrence relation from part a find the value of the investment at the end of each of the first 5 years.

2. $713 is invested in a simple interest account for 5 years at 6.75% p.a.
   a. Set up a recurrence relation B_n to find the value of the investment after n years.
   b. Using your calculator and the recurrence relation from part a to find the value of the investment at the end of each of the first 5 years.
3. **WE2** A bank offers $12\frac{1}{2}\%$ p.a. simple interest on an investment of $1500. The amount in the bank is A_n after n years.
   a. Write the recurrence relation for this situation.
   b. Using your calculator, complete a table to show the amount in the investment account at the end of 2 years.
   c. What is the total interest earned after 2 years?
4. An amount of $2400 is invested at $5\frac{1}{2}\%$ p.a. for 18 months.
   a. Write the recurrence relation for this situation.
   b. What is the amount in the investment account at the end of 18 months?
   c. What is the total interest earned after 18 months?
5. Find the value of the following investments at the end of each year of the investment by using a recurrence relation.
   a. $680 for 4 years at 5% p.a. simple interest
   b. $13 000 for 3 years at 7.5% p.a. simple interest
6. **WE3** $10 500 is invested in a simple interest account at 6.5% p.a.
   a. Use the simple interest rule to determine how much was in the account after 5 years.
   b. Represent the account balance for each of the 5 years graphically.
7. A building society offers an investment opportunity described using the following recurrence relation.

$$A_{n+1} = A_n + 565, A_0 = 6200$$

   a. Explain in words the building society's investment offer.
   b. Calculate the interest rate per annum correct to 2 decimal places.
8. Silvio invested the $1500 he won in Lotto with an insurance company bond that pays $12\frac{1}{4}\%$ p.a. simple interest provided he keeps the bond for 5 years.
   a. What is Silvio's total return from the bond at the end of the 5 years?
   b. Represent the balance at the end of each year graphically.

9. The value of a simple interest investment at the end of year 2 is $3377. At the end of year 3 the investment is worth $3530.50. Use a recurrence relation to work out how much was invested.
10. Carol has $3000 to invest. Her aim is to earn $450 in interest at a rate of 5% p.a. Over what term would she invest?
11. **MC** A loan of $1000 is taken over 5 years. The simple interest is calculated monthly. The total amount repaid for this loan is $1800. The simple interest rate per year on this loan is closest to:
    A. 8.9% **B.** 16% **C.** 36% **D.** 5% **E.** 11.1%
12. Find the interest charged or earned on the following loans and investments:
    a. $750 loaned at 12% p.a. simple interest for 4 years
    b. $7500 invested for 3 years at 1% per month simple interest
    c. $250 invested at $1\frac{3}{4}\%$ per month for $2\frac{1}{2}$ years.

6.3 Compound interest as a geometric recurrence relation

When interest is added to the initial amount (principal) at the end of an interest-bearing period, then both the interest and the principal earn further interest during the next period, which in turn is added to the balance. This process continues for the life of the investment. The interest is said to be compounded and is an example of geometric growth.

The result is that the balance of the account increases at regular intervals and so too does the interest earned.

Let the starting amount be V_n. Then the amount at the start of the next compounding period is V_{n+1}. The recurrence relation rule for geometric growth is:

$$V_{n+1} = RV_n$$

where V_n is the value of the loan after n compounding periods
V_{n+1} is the amount of the investment one time period after V_n
R is the growth or compounding factor; $R = 1 + \frac{r}{100}$
r is interest rate per period.

Consider \$1000 invested for 4 years at an interest rate of 12% p.a. with interest compounded annually (added on each year). Use the recurrence relation $A_{n+1} = 1.12A_n$, $A_0 = 1000$, to calculate the amount in the account at the end of each year.

V_n (\$)	Calculation	V_{n+1} (\$)
1000	1.12×1000	1120.00
1120	1.12×1120	1254.40
1254.40	1.12×1254.40	1404.93
1404.93	1.12×1404.93	1573.52

So the balance after 4 years is \$1573.52.

WORKED EXAMPLE 4

Laura invested \$2500 for 5 years at an interest rate of 8% p.a. with interest compounded annually.
a. Using A_0 to represent the initial investment, write Laura's investment as a recurrence relation.
b. Using your calculator, what is the amount invested at the end of 5 years?

THINK

a. 1. Write the recurrence relation rule.
Write the values of A_0 and r.

WRITE

a. $A_{n+1} = RA_n$
$A_0 = 2500$
$r = 8$

2. Calculate R, the compounding factor.

$$R = 1 + \frac{r}{100}$$
$$= 1 + \frac{8}{100}$$
$$= 1.08$$

3. Substitute into the recurrence relation rule.

$A_{n+1} = 1.08\,A_n$, $A_0 = \$2500$, where A is the amount invested and n is the number of years after the initial investment.

b. 1. Using your calculator type 2500, press ENTER/EXE, then type ×1.08.
Press ENTER/EXE 5 times to calculate the amount in the account for each of the 5 years.

b.

2500	2500
2500 × 1.08	2700
2700 × 1.08	2916
2916 × 1.08	3149.28
3149.28 × 1.08	3401.22
3401.22 × 1.08	3673.32

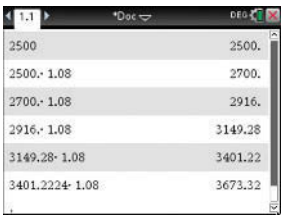
2. Write the answer.

At the end of 5 years, the amount invested is $3673.32.

TI | THINK

b. 1. On a Calculator page, type 2500 and then press ENTER.
Complete the entry line as: ×1.08
Then continue to press ENTER until the first 5 terms are displayed on the screen.

WRITE



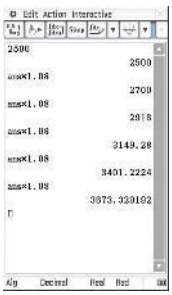
2. The answer appears on the screen.

At the end of 5 years, the amount invested is $3673.32.

CASIO | THINK

1. On the Main screen, type 2500 and then press EXE.
Complete the entry line as: ×1.08
Then continue to press EXE until the first 5 terms are displayed on the screen.

WRITE



2. The answer appears on the screen.

At the end of 5 years, the amount invested is $3673.32.

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Units 3 & 4 > AOS R&FM > Topic 2 > Concept 3

Recurrence relation for compound interest Summary screen and practice questions

Exercise 6.3 Compound interest as a geometric recurrence relation.

1. **WE4** Fraser invested $7500 for 4 years at an interest rate of 6% p.a. with interest compounded annually.
   a. Using A_0 to represent the initial investment, write Fraser's investment as a recurrence relation.
   b. Using your calculator, what is the amount invested at the end of 4 years?
2. Bobbi invested $3250 for 5 years at an interest rate of 7.5% p.a. with interest compounded annually.
   a. Using A_0 to represent the initial investment, write Bobbi's investment as a recurrence relation.
   b. Using your calculator, what is the amount invested at the end of 5 years?
3. Imani invested $600 in a short-term deposit for 6 months at an interest rate of 6% p.a. with interest compounded monthly.
   a. Using A_0 to represent the initial investment, write Imani's investment as a recurrence relation.
   b. Using your calculator, what is the amount invested at the end of the 6 months?
4. To join her friends on a holiday Siobhan borrowed $2500 for 2 months at an interest rate of 12% p.a. with interest compounded monthly.
   a. Using A_0 to represent the initial investment, write Siobhan's loan as a recurrence relation.
   b. What is the amount of interest owing on the loan at the end of the two months?

The following information relates to questions **5** to **9**.

Shannon invested money in a savings account. The amount in the account after n years, V_n, can be modelled by the recurrence relation below.

$$V_{n+1} = 1.05V_n,\ V_0 = 3000$$

5. **MC** The annual percentage compound interest rate for this account is:
   A. 1.05% **B.** 5% **C.** 50%
   D. 95% **E.** 105%
6. **MC** The interest earned in the first year is:
   A. $50 **B.** $100 **C.** $150
   D. $200 **E.** $300
7. **MC** The balance at the end of the first year is:
   A. $3000 **B.** $3050 **C.** $3100
   D. $3150 **E.** $3200
8. **MC** The principal at the start of the second year is:
   A. $3300 **B.** $3200 **C.** $3150
   D. $3250 **E.** $3100
9. **MC** The interest earned during the second year is:
   A. $100 **B.** $125 **C.** $150
   D. $157.50 **E.** $172.50
10. **MC** If $4500 is invested for 10 years at 12% p.a. with interest compounding annually and the interest earned in the third year was $677.38, then the interest earned in the fourth year is closest to:
    A. $600 **B.** $625 **C.** $650
    D. $677 **E.** $759

11. Declan invested $5750 for 4 years at an interest rate of 8% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

V_n ($)	Interest ($)	V_{n+1} ($)
5750	A% of 5750 = 460	6210
B	8% of C = 496.80	D
6706.80	8% of 6706.80 = 536.54	7243.34
7243.34	8% of 7243.34 = 579.47	E
F	8% of 7822.81 = 625.82	8448.63

12. Alex invested $12 000 for 4 years at an interest rate of 7.5% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

V_n ($)	Interest ($)	V_{n+1} ($)
12 000	7.5% of 12 000 = 900	12 900
12 900	7.5% of 12 900 = A	B
C	7.5% of 13 867.50 = 1040.06	14 907.56
14 907.56	7.5% of 14 907.56 = 1118.07	D
E	7.5% of 16 025.63 = 1201.92	F

13. Sarina invested $5500 for 3 years at an interest rate of 6.5% p.a. with interest compounded monthly. Complete the table shown to determine the value of the investment at the end of 5 months.

V_n ($)	V_{n+1} ($)
5500	$1.005\,42 \times 5500 = 5529.81$
5529.81	

14. Wai Keat runs his own architecture business. A client has requested to pay off an account of $15000 rather than pay the full amount by the due date. Wai Keat explains that he will need to charge interest on the money not paid by the due date at a rate of 1.2% per month.

a. Using A_0, A_n and A_{n+1}, write a recurrence relation that models this situation where n is the number of months after the due date.

b. How much interest will the client need to pay if he does not pay the full amount until 3 months after the due date?

15. MC Consider the following recurrence relations:

$P_{n+1} = 0.95P_n$, $P_0 = 3000$
$M_{n+1} = 1.15M_n$, $M_0 = 3000$
These are examples of

A. linear decay and geometric growth respectively
B. geometric decay and linear growth respectively
C. linear decay and linear growth respectively
D. geometric decay and geometric growth respectively
E. geometric growth

6.4 Compound interest rule

From the previous exercise we saw that we could write the value of a compounding investment as a recurrence relation.

This pattern can be expanded further to write the value of the investment in terms of the initial investment. This is known as the compound interest rule:

$$V_n = V_0 R^n$$

where V_n = final or total amount ($)
V_0 = principal ($)
R = growth or compounding factor; $R = 1 + \frac{r}{100}$
r = interest rate per period
n = number of interest-bearing periods

Note that the compound interest rule gives the *total amount* in an account, not just the interest earned as in the simple interest formula.

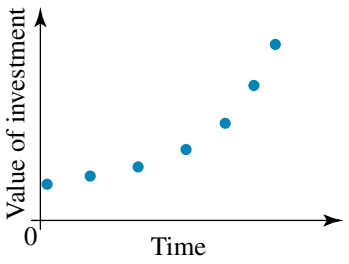
To find the total interest compounded, I:

$$I = V_n - V_0$$

where V_n = final or total amount ($)
V_0 = principal ($)

If compound interest is used, the value of the investment at the end of each period grows by an increasing amount. Therefore, when plotted, the values of the investment at the end of each period form an exponential curve.

Now let us consider how the rule is used.



WORKED EXAMPLE 5

$5000 is invested for 4 years at 6.5% p.a., interest compounded annually.
a. Generate the compound interest rule for this investment.
b. Find the amount in the balance after 4 years and the interest earned over this period.

THINK	WRITE
a. 1. Write the compound interest rule.	a. $V_n = V_0\left(1 + \frac{r}{100}\right)^n$
2. List the values of n, r and P.	$n = 4$ $r = 6.5$ $V_0 = 5000$
3. Substitute into the rule.	$V_n = 5000\left(1 + \frac{6.5}{100}\right)^n$
4. Simplify.	$V_n = 5000(1.065)^n$
b. 1. Substitute $n = 4$ into the rule.	b. $V_4 = 5000(1.065)^4$
2. Evaluate correct to 2 decimal places.	$= \$6432.33$
3. Subtract the principal from the balance.	$I = V_4 - V_0$ $= 6432.33 - 5000$ $= \$1432.33$
4. Write your answer.	The amount of interest earned is $1432.33 and the balance is $6432.33.

6.4.1 Non-annual compounding

In Worked example 5, interest was compounded annually. However as shown in section 6.3, in many cases the interest is compounded more often than once a year, for example, quarterly (every 3 months), weekly, or daily. In these situations n and r still have their usual meanings and we calculate them as follows.

Number of interest periods, n = number of years × number of interest periods per year

$$\textbf{Interest rate per period, } r = \frac{\textbf{nominal interest rate per annum}}{\textbf{number of interest periods per year}}$$

Note: Nominal interest rate per annum is simply the annual interest rate advertised by a financial institution.

WORKED EXAMPLE 6

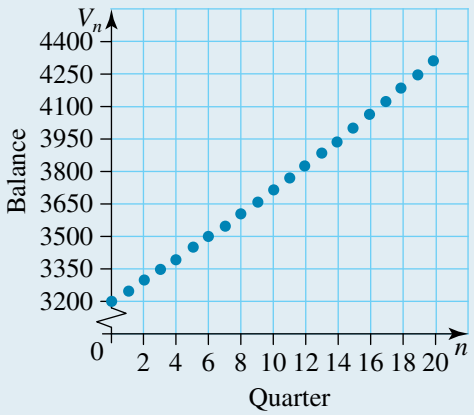
If $3200 is invested for 5 years at 6% p.a., interest compounded quarterly:

a. find the number of interest-bearing periods, *n*

b. find the interest rate per period, *r*

c. find the balance of the account after 5 years

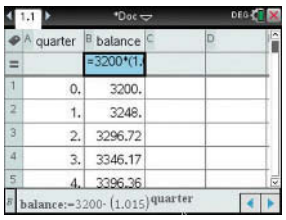
d. graphically represent the balance at the end of each quarter for 5 years. Describe the shape of the graph.

THINK	WRITE/DRAW
a. Recall the number of interest - bearing periods, n, is the product of the number of years multiplied by the number of interest periods per year.	**a.** $n = 5\text{ (years)} \times 4\text{(quarters)}$ $= 20$
b. Convert % p.a. to % per quarter to match the time over which the interest is calculated. Divide r% p.a. by the number of compounding periods per year, namely 4. Write as a decimal.	**b.** $r\% = \frac{6\%\text{ p.a.}}{4}$ $= 1.5\%\text{ per quarter}$ $r = 1.5$
c. 1. Write the compound interest formula.	**c.** $V_n = V_0R^n$, where $R = \left(1 + \frac{r}{100}\right)$
2. List the values of V_0, r and n.	$V_0 = 3200, r = 1.5, n = 20$
3. Substitute into the formula.	$V_{20} = 3200\left(1 + \frac{1.5}{100}\right)^{20}$
4. Simplify.	$= 3200\,(1.015)^{20}$
5. Evaluate correct to 2 decimal places.	$= \$4309.94$
6. Write your answer.	Balance of account after 5 years is $4309.94.
d. 1. Using CAS, find the balance at the end of each quarter and plot these values on the set of axes. (The first point is (0, 3200), which represents the principal.)	**d.** 
2. Comment on the shape of the graph.	The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the *new* balance.

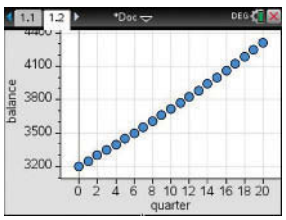
TI | THINK

WRITE

d. 1. On a Lists & Spreadsheets page, enter the quarters as consecutive whole numbers from 0 to 20 in Column A.
Rename Column A as quarter and Column B as balance.
In the formula cell for the column named balance, Complete the entry line as:
$= 3200 \times (1.015)^{\text{quarter}}$
Then press ENTER.



2. Press CTRL + I and select: 5: Add Data & Statistics
Construct scatterplot with quarter on the horizontal axis and balance on the vertical axis.



3. The answer appears on the screen.

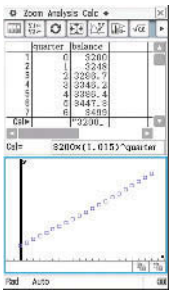
The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the new balance.

CASIO | THINK

WRITE

1. On a Statistics screen, enter the quarters as consecutive whole numbers from 0 to 20 in list1.
Rename list1 as quarter and list2 as balance.
In the formula cell for the column named balance, complete the entry line as:
$= 3200 \times (1.015)^{\text{quarter}}$
Then press EXE.

2. Tap:
SetGraph
Setting
Complete the Set StatGraph fields as:
Draw: On
Type: Scatter
XList: main\quarter
YList: main\balance
Freq: 1
Mark: Square
Tap graph icon.



3. The answer appears on the screen.

The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the new balance.

The situation often arises where we require a certain amount of money by a future date. It may be to pay for a holiday or to finance the purchase of a car. It is then necessary to know what principal should be invested now in order that it will increase in value to the desired final balance within the time available.

WORKED EXAMPLE 7

Find the principal that will grow to $4000 in 6 years, if interest is added quarterly at 6.5% p.a. (use CAS to solve the equation).

THINK

WRITE

1. Calculate n (there are 4 quarters in a year).

$n = 6 \times 4$
$= 24$

2. Calculate r, by converting % p.a. to % per quarter to match the time of which interest is calculated.

$r = \frac{6.5}{4}$
$= 1.625$

3. List the value of V_{24}.

$V_{24} = 4000$

4. Write the compound interest rule, substituting $R = 1 + \frac{r}{100}$. Use CAS to solve the equation.

$$V_{24} = V_0 R^n$$

$$V_{24} = V_0 \left(1 + \frac{r}{100}\right)^n$$

$$4000 = V_0 \left(1 + \frac{1.625}{100}\right)^{24}$$

Solving for V_0,
solve $(4000 = V_0(1.016\,25)^{24}, V_0)$

5. Evaluate correct to 2 decimal places.

$V_0 = 2716.73$

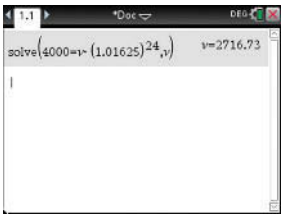
6. Write a summary statement.

$2716.73 would need to be invested.

TI | THINK

4. On a Calculator page, complete the entry line as:
solve(4000
$= v \times (1.016\,25)^{24}, v)$
Then press ENTER.

WRITE



2. The answer appears on the screen.

$2716.73 would need to be invested.

CASIO | THINK

4. On the Main screen, complete the entry line as:
solve(4000
$= v \times (1.016\,25)^{24}, v)$
Then press EXE.

WRITE

2. The answer appears on the screen.

$2716.73 would need to be invested.

Resources

Interactivity: Simple and compound interest (int-6265)
Interactivity: Non-annual compounding (int-6270)

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Units 3 & 4 > AOS R&FM > Topic 2 > Concept 2

Compound interest Summary screen and practice questions

Exercise 6.4 Compound interest rule

Note: Where it is the most efficient method CAS should be used to solve the following problems.

1. WE5 $2500 is invested for 5 years at 7.5% p.a., interest compounded annually.
   a. Generate the compound interest rule for this investment.
   b. Find the amount in the balance after 5 years and the interest earned over this period.
2. Find the amount in the account and interest earned after $6750 is invested for 7 years at 5.25% p.a., interest compounded annually.

3. WE6 If $4200 is invested for 3 years at 7% p.a., interest compounded quarterly:
   a. find the number of interest-bearing periods, n
   b. find the interest rate per period, r
   c. find the balance of the account after 3 years
   d. graphically represent the balance at the end of each quarter for 3 years. Describe the shape of the graph.
4. If $7500 is invested for 2 years at 5.5% p.a., interest compounded monthly:
   a. find the number of interest-bearing periods, n
   b. find the interest rate per period, r
   c. find the balance of the account after 2 years
   d. graphically represent the balance at the end of each month for 2 years. Describe the shape of the graph.
5. WE7 Find the principal that will grow to $5000 in 5 years, if interest is added quarterly at 7.5% p.a.
6. Find the principal that will grow to $6300 in 7 years, if interest is added monthly at 5.5% p.a.
7. Use the compound interest rule to find the amount, V_n, when:
   a. $V_0 = \$500, n = 2, r = 8$
   b. $V_0 = \$1000, n = 4, r = 13$
   c. $V_0 = \$3600, n = 3, r = 7.5$
   d. $V_0 = \$2915, n = 5, r = 5.25$.
8. Using a recurrence relation, find: i the balance, and ii the interest earned (interest compounded annually) after:
   a. $2000 is invested for 1 year at 7.5% p.a.
   b. $2000 is invested for 2 years at 7.5% p.a.
   c. $2000 is invested for 6 years at 7.5% p.a.
9. Find the number of interest-bearing periods, n, if interest is compounded:
   a. annually for 5 years
   b. quarterly for 5 years
   c. semi-annually for 4 years
   d. monthly for 6 years
   e. 6-monthly for $4\frac{1}{2}$ years
   f. quarterly for 3 years and 9 months.
10. Find the interest rate per period, r, if the annual rate is:
    a. 6% and interest is compounded quarterly
    b. 4% and interest is compounded half-yearly
    c. 18% and interest is compounded monthly
    d. 7% and interest is compounded quarterly.
11. $1500 is invested for 2 years into an account paying 8% p.a. Find the balance if:
    a. interest is compounded yearly
    b. interest is compounded quarterly
    c. interest is compounded monthly
    d. interest is compounded weekly.
    e. Compare your answers to parts a–d.
12. Use the recurrence relation $V_{n+1} = 1.045V_n$ to answer the following questions.
    a. If the balance in an account after 1 year is $2612.50, what will the balance be after 3 years?
    b. If the balance in an account after 2 years is $4368.10, what will the balance be after 5 years?
    c. If the balance in an account after 2 years is $6552.15, what was the initial investment?
13. Find the amount that accrues in an account which pays compound interest at a nominal rate of:
    a. 7% p.a. if $2600 is invested for 3 years (compounded monthly)
    b. 8% p.a. if $3500 is invested for 4 years (compounded monthly)
    c. 11% p.a. if $960 is invested for $5\frac{1}{2}$ years (compounded fortnightly)
    d. 7.3% p.a. if $2370 is invested for 5 years (compounded weekly)
    e. 15.25% p.a. if $4605 is invested for 2 years (compounded daily).
14. MC The greatest return is likely to be made if interest is compounded:
    A. annually B. semi-annually C. quarterly D. monthly E. fortnightly

15. MC If \$12 000 is invested for $4\frac{1}{2}$ years at 6.75% p.a., compounded fortnightly, the amount of interest that would accrue would be closest to:

A. \$3600 **B.** \$4200 **C.** \$5000
D. \$12 100 **E.** \$16 300

16. Use the compound interest rule to find the principal, V_0, when:

a. $V_n = \$5000,\ r = 9,\ n = 4$
b. $V_n = \$2600,\ r = 8.2,\ n = 3$
c. $V_n = \$3550,\ r = 1.5,\ n = 12$
d. $V_n = \$6661.15,\ r = 0.8,\ n = 36$

17. Find the principal that will grow to:

a. \$3000 in 4 years, if interest is compounded 6-monthly at 9.5% p.a.
b. \$2000 in 3 years, if interest is compounded quarterly at 9% p.a.
c. \$5600 in $5\frac{1}{4}$ years, if interest is compounded quarterly at 8.7% p.a.
d. \$10 000 in $4\frac{1}{4}$ years, if interest is compounded monthly at 15% p.a.

18. Find the interest accrued in each case in question **17**.

6.5 Finding rate or time for compound interest

6.5.1 Finding the rate in compound interest

Sometimes we know how much we can afford to invest as well as the amount we want to have at a future date. Using the compound interest rule we can calculate the interest rate that is needed to increase the value of our investment to the amount we desire. This allows us to 'shop around' various financial institutions for an account which provides the interest rate we want.

We must first find the interest rate per period, r, and convert this to the corresponding nominal rate per annum.

WORKED EXAMPLE 8

Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$3000 to grow to \$4000 over 2 years if interest is compounded quarterly.

THINK	WRITE
1. List the values of V_n, V_0 and n. For this example, n needs to represent quarters of a year and therefore r will be evaluated in % per quarter.	$V_n = \$4000$ $V_0 = \$3000$ $n = 2 \times 4$ $= 8$
2. Write the compound interest rule and substitute the known values.	$V_n = V_0 R^n$ $4000 = 3000R^8$
3. Replace R with $1 + \frac{r}{100}$.	$4000 = 3000\left(1 + \frac{r}{100}\right)^8$
4. Use CAS to solve the equation for r.	Solve $\left(4000 = 3000\left(1 + \frac{r}{100}\right)^8\right)$ for r.

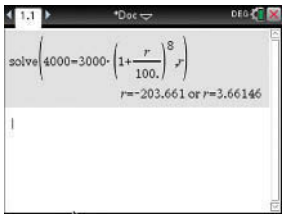
5. Write the solution for r.	$r = -203.66$ or $r = 3.661\,46$
6. $r > 1$	so $r = 3.661\,46$ $r\% = 3.661\,46\%$ per quarter
7. Multiply r by the number of interest periods per year to get the annual rate (correct to 2 decimal places).	Annual rate $= r\%$ per quarter $\times 4$ $= 3.661\,46\%$ per quarter $\times 4$ $= 14.65\%$ per annum
8. Write your answer.	Interest rate of 14.65% p.a. is required, correct to 2 decimal places.

TI | THINK

4. On a Calculator page, complete the entry line as: solve(4000 $= 3000 \times \left(1 + \frac{r}{100}\right)^8, r)$

Then press ENTER.

WRITE



5. The answer appears on the screen.

$r = -203.66$ or $r = 3.66$

CASIO | THINK

4. On the Main screen, complete the entry line as: solve(4000 $= 3000 \times \left(1 + \frac{r}{100}\right)^8, r)$

Then press EXE.

WRITE

5. The answer appears on the screen.

$r = -203.66$ or $r = 3.66$

6.5.2 Finding time in compound interest

We have seen how the compound interest formula, $V_n = V_0R^n$, where $R = 1 + \frac{r}{100}$ can be manipulated to solve situations where V_n, V_0 and r were unknown.

To find n, the number of interest-bearing periods, that is, to find the time period of an investment, use the compound interest rule and solve for n.

The value obtained for n may be a whole number, but it is more likely to be a decimal. That is, the time required will lie somewhere between two consecutive integers. The smaller of the two integers represents insufficient time for the investment to amount to the balance desired; the larger integer represents more than enough time.

WORKED EXAMPLE 9

How long it will take $2000 to amount to $3500 at 8% p.a. with interest compounded annually?

THINK	WRITE
1. State the values of V_n, V_0 and r.	$V_n = \$3500$, $V_0 = \$2000$ and $r = 8\%$ p.a.
2. Write the rule for compound interest and substitute in the values.	$V_n = V_oR^n$, $R = 1 + \frac{r}{100}$ $3500 = 2000\left(1 + \frac{8}{100}\right)^n$

3. Use CAS to solve for n.	solve$\left(3500 = 2000\left(1 + \frac{8}{100}\right)^n, n\right)$ $n = 7.27$ years
4. Interest is compounded annually, so n represents years. Raise n to the next whole year.	As the interest is compounded annually, $n = 8$ years.
5. Write your answer.	It will take 8 years for \$2000 to amount to \$3500.

WORKED EXAMPLE 10

Calculate the number of interest-bearing periods, n, required and hence the time it will take \$3600 to amount to \$5100 at a rate of 7% p.a., with interest compounded quarterly.

THINK	WRITE
1. State the values of V_n, V_0 and r.	$V_n = \$5100$, $V_0 = \$3600$ and $r = 7\%$ p.a.
2. Write the rule for compound interest and substitute in the values.	$V_n = V_0R^n$, $R = 1 + \frac{r}{100}$ $5100 = 3600 \times \left(1 + \frac{\frac{7}{4}}{100}\right)^n$
3. Use CAS to for n. Solve for n.	solve$\left(5100 = 3600 \times \left(1 + \frac{\frac{7}{4}}{100}\right)^n, n\right)$ $n = 20.08$ quarters
4. Write your answer using more meaningful units.	As the interest is compounded annually, $n = 21$ quarters.
5. Answer the question fully.	It will take $5\frac{1}{4}$ years for \$3600 to amount to \$5100.

Exercise 6.5 Finding rate or time for compound interest

Note: Where it is the most efficient method CAS should be used to solve the following problems.

1. **WE8** Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$4000 to grow to \$5000 over 2 years if interest is compounded quarterly.
2. Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$7500 to grow to \$10 000 over 3 years if interest is compounded monthly.

3. MC Lillian wishes to have $24 000 in a bank account after 6 years so that she can buy a new car. The account pays interest at 15.5% p.a. compounded quarterly. The amount (correct to the nearest dollar) that Lillian should deposit in the account now, if she is to reach her target, is:
   A. $3720 **B.** $9637 **C.** $10 109 **D.** $12 117 **E.** $22 320
4. Find the interest rates per annum (correct to 2 decimal places) that would enable investments of:
   a. $2000 to grow to $3000 over 3 years if interest is compounded 6-monthly
   b. $12 000 to grow to $15 000 over 4 years (interest compounded quarterly).

5. Find the interest rates per annum (correct to 2 decimal places) that would enable investments of:
   a. $25 000 to grow to $40 000 over $2\frac{1}{2}$ years (compounded monthly)
   b. $43 000 to grow to $60 000 over $4\frac{1}{2}$ years (compounded fortnightly)
   c. $1400 to grow to $1950 over 2 years (compounded weekly).
6. MC What is the minimum interest rate per annum (compounded quarterly) needed for $2300 to grow to at least $3200 in 4 years?
   A. 6% p.a.
   B. 7% p.a.
   C. 9% p.a.
   D. 8% p.a.
   E. 10% p.a.
7. WE9 How long will it take $3000 to amount to $4500 at 7% p.a. with interest compounded annually?
8. How long will it take $7300 to amount to $10 000 at 7.5% p.a. with interest compounded annually?
9. WE10 Calculate the number of interest-bearing periods, n, required and hence the time it will take $4700 to amount to $6100 at a rate of 9% p.a., with interest compounded quarterly.
10. Calculate the number of interest-bearing periods, n, required and hence the time it will take $3800 to amount to $6300 at a rate of 15% p.a., with interest compounded quarterly.
11. How long it will take (with interest compounded annually) for:
    a. $2000 to amount to $3173.75 at 8% p.a. b. $9250 to amount to $16 565.34 at 6% p.a.?
12. How long it will take (with interest compounded annually) for:
    a. $850 to amount to $1000 at 7% p.a. b. $12 000 to amount to $20 500 at 13.25% p.a.?
13. Calculate the number of interest-bearing periods, n, required, and hence the time in more meaningful terms when:
    a. $V_n = \$2100$, $V_0 = \$1200$, $r = 3\%$ per half-year
    b. $V_n = \$13\,500$, $V_0 = \$8300$, $r = 2.5\%$ per quarter
    c. $V_n = \$16\,900$, $V_0 = \$9600$, $r = 1\%$ per month.
14. Calculate the number of interest-bearing periods, n, required, and hence the time in more meaningful terms when:
    a. $V_n = \$24\,000$, $V_0 = \$16\,750$, $r = 0.25\%$ per fortnight
    b. $7800 is to amount to $10 000 at a rate of 8% p.a. (compounded quarterly)
    c. $800 is to amount to $1900 at a rate of 11% p.a. (compounded quarterly).
15. Wanda has invested $1600 in an account at a rate of 10.4% p.a., interest compounded quarterly. How long will it take to reach $2200?

16. MC What will be the least number of interest periods, n, required for \$6470 to grow to at least \$9000 in an account with interest paid at 6.5% p.a. and compounded half-yearly?
    - **A.** 10
    - **B.** 11
    - **C.** 12
    - **D.** 20
    - **E.** 22
17. About how long would it take for:
    - **a.** \$1400 to accrue \$300 interest at 8% p.a., interest compounded monthly
    - **b.** \$8000 to accrue \$4400 interest at 9.6% p.a., interest compounded fortnightly?
18. Jennifer and Dawn each want to save \$15 000 for a car. Jennifer has \$11 000 to invest in an account with her bank which pays 8% p.a., interest compounded quarterly. Dawn's credit union has offered her 11% p.a., interest compounded quarterly.
    - **a.** How long will it take Jennifer to reach her target?
    - **b.** How much will Dawn need to invest in order to reach her target at the same time as Jennifer? Assume their accounts were opened at the same time.

6.6 Effective annual interest rate

When money is borrowed or invested with compounding periods greater than one year then the value of the loan or investment increases with the number of compounding periods. The **effective annual interest rate** is used to compare the annual interest between loans or investments with these different compounding terms.

To calculate the effective annual interest rate, use the formula:

$$r = \left(1 + \frac{i}{n}\right)^n - 1$$

where r **= the effective annual interest rate**
i **= the nominal rate, as a decimal**
n **= the number of compounding periods per year**

For a loan of \$100 at 10% p.a. compounding quarterly over 2 years.
The effective annual interest rate:

$$\begin{aligned} r &= \left(1 + \frac{i}{n}\right)^n - 1 \\ &= \left(1 + \frac{0.10}{4}\right)^4 - 1 \\ &= 0.1038 \\ &= 10.38\% \end{aligned}$$

This means that the effective annual interest rate is actually 10.38% and not 10%. The comparison between the two can be shown in the following table.

Period	Amount owing ($)	Annual effective rate calculation ($)
1	$100\left(1+\frac{0.10}{4}\right)=102.50$	
2	$102.50\left(1+\frac{0.10}{4}\right)=105.06$	
3	$105.06\left(1+\frac{0.10}{4}\right)=107.69$	
4 (Year 1)	$107.69\left(1+\frac{0.10}{4}\right)=110.38$	$100\left(1+\frac{10.38}{100}\right)^{1}=110.38$
5	$110.38\left(1+\frac{0.10}{4}\right)=113.14$	
6	$113.14\left(1+\frac{0.10}{4}\right)=115.97$	
7	$115.97\left(1+\frac{0.10}{4}\right)=118.87$	
8 (Year 2)	$118.87\left(1+\frac{0.10}{4}\right)=121.84$	$100\left(1+\frac{10.38}{100}\right)^{2}=121.84$

WORKED EXAMPLE 11

Jason decides to borrow money for a holiday. If a personal loan is taken over 4 years with equal quarterly repayments compounding at 12% p.a., calculate the effective annual rate of interest (correct to 2 decimal places).

THINK	WRITE
1. Write the values for i and n.	$n = 4$ (since quarterly) $i = 0.12$ (12% as a decimal)
2. Write the formula for effective annual rate of interest.	$r=\left(1+\frac{i}{n}\right)^{n}-1$
3. Substitute $n = 4$ and $i = 0.12$.	$r=\left(1+\frac{0.12}{4}\right)^{4}-1$ $= 0.1255$ $= 12.55\%$
4. Write your answer.	The effective annual interest rate is 12.55% p.a. for a loan of 12%, correct to 2 decimal places.

TI | THINK

1. On a Calculator page, press MENU and then select:
   8: Finance
   5: Interest Conversion
   2: Effective Interest
   Rate: complete the entry line as:
   eff(12, 4)
   Then press ENTER.
   Note: The first entry in the brackets is the nominal interest rate, followed by the number of compounding periods per year.
2. The answer appears on the screen.

WRITE

The effective interest rate is 12.55% p.a., correct to 2 decimal places.

CASIO | THINK

1. On the Main screen, tap:
   Action
   Financial
   Interest Conversion
   convEff
   Complete the entry line as:
   convEff(4, 12)
   Then press EXE.
   Note: The first entry in the brackets is the number of compounding periods per year followed by the nominal interest rate.
2. The answer appears on the screen.

WRITE

The effective interest rate is 12.55% p.a., correct to 2 decimal places.

Resources

Interactivity: Effective annual interest rate (int-6288)

studyon

Units 3 & 4 > AOS R&FM > Topic 2 > Concept 6

Nominal and effective interest rates Summary screen and practice questions

Exercise 6.6 Effective annual interest rate

1. **WE11** Brad wants to take his family to Africa. He decides to take out a loan over 4 years with equal quarterly repayments compounding at 11% p.a. Calculate the effective annual rate of interest (correct to 1 decimal place).
2. Caitlin wants to landscape her garden. She decides to take out a loan over 3 years with equal quarterly repayments compounding at 14% p.a. Calculate the effective annual rate of interest (correct to 1 decimal place).
3. William is to purchase a new video recorder. If William pays $125 monthly instalments over 3 years at an interest rate of 11.5% p.a. compound interest, what effective annual interest rate is he paying? Give your answer correct to 2 decimal places.
4.

	Item	Cash price ($)	Deposit ($)	Monthly instalment ($)	Compound interest rate	Term of loan
a	Television	$875	$150		8% p.a.	2 years
b	New car	$23990	$2000		10% p.a.	5 years
c	Clothing	$550	$100		7.5% p.a.	1 year
d	Refrigerator	$1020	$50		$6\frac{3}{4}$% p.a.	2 years

For each of the items in the above table, calculate:

i. the monthly instalment on each item
ii. the total amount paid over the period given for each item
iii. the total amount of interest charged on each item
iv. the effective annual interest rate.

5. A camera valued at $1200 is purchased via a loan. A deposit of $200 is required and equal monthly instalments of $64 are paid over the 18-month agreed period. Calculate:
   a. the compound interest rate per annum
   b. the effective annual interest rate.

The following information relates to questions 6 and 7.

The bank approves a personal loan of $5000. An interest rate of 12.5% p.a. is charged, with repayments to be made over a 9-month period in equal weekly instalments.

6. Calculate the weekly instalment.
7. Calculate the effective annual interest rate.
8. Calculate the effective annual interest rate on a loan of $1000 if the monthly repayments are $45 and the loan is to be repaid over 2 years. (*Hint:* First calculate the compound interest rate.)
9. MC For a compound interest rate of 4.85% p.a. on an investment with monthly compounding periods for 2 years, the effective rate of interest is:
   A. 4.96% **B.** 4.85% **C.** 9.3% **D.** 5.12% **E.** 9.6%
10. MC A reducing balance loan was used to purchase a home theatre system valued at $2500. If the loan is paid off with quarterly repayments over 3 years at 9.6% p.a., then the effective annual interest rate is closest to:
    A. 10.03% **B.** 8.75% **C.** 5.2% **D.** 9.6% **E.** 9.95%
11. MC If the effective annual interest rate is 8.5% p.a. on a loan with monthly repayments over 4 years, then the compound interest rate is closest to:
    A. 9.23% **B.** 10% **C.** 4.3% **D.** 8.19% **E.** 7.95%
12. Carefully read the advertisement for the purchase of the phone at right and calculate:
    a. the compound interest rate
    b. the effective annual interest rate
    c. the total cost under the loan plan
    d. the increase in cost over a cash sale.

13. Sarah had two options to invest $10 000 for 2 years. The two options are:
    1. Compounding interest of 9% p.a. compounding monthly.
    2. An effective annual interest rate of 9.3%.

    With calculations, show which option she should take.
14. Calculate the effective annual interest rate that would require the same amount to be paid back as a $25 000 loan at 8.5% p.a. compounding monthly for 5 years.

6.7 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC A bank pays a flat rate of 6% p.a. on an investment of $5000. A recurrence relation, V_n that can be used to model this investment after n years is:
   A. $V_{n+1} = V_n + 5000$ **B.** $V_{n+1} = 300V_n$
   C. $V_{n+1} = V_n + 5000,\ V_0 = 5000$ **D.** $V_{n+1} = V_n + 5000,\ V_0 = 300$
   E. $V_{n+1} = V_n + 300,\ V_0 = 5000$
2. MC Clayton has invested $360 in a bank for 3 years at 8% simple interest each year. At the end of the 3 years, the total amount he will receive is:
   A. $86.40 **B.** $236.80 **C.** $28.80 **D.** $388.80 **E.** $446.40
3. MC A loan of $5000 is taken over 5 years. The simple interest is calculated monthly. The interest bill on this loan is $1125. The simple interest rate per year on this loan is:
   A. 3% **B.** 4.5% **C.** 3.75% **D.** 5% **E.** 3.5%
4. MC A loan of $10 000 is taken over 10 years. The total interest bill on this loan is $2000. The simple interest rate per year on this loan is:
   A. 3% **B.** 4.5% **C.** 2% **D.** 5% **E.** 2.5%
5. MC The following recurrence model was used to model an investment for n years.
   $V_0 = 2000,\ V_{n+1} = 1.05V_n$. The annual rate of interest for this investment is:
   A. 1% p.a. **B.** 5% p.a. **C.** 10.5% p.a. **D.** 50% p.a. **E.** 105% p.a.
6. MC An investment of $100 000 is compounded monthly at a rate of 12%. A recurrence relation for A_n, the value of the investment after n months is:
   A. $V_{n+1} = 1.01\,V_n,\ V_0 = 100\,000$ **B.** $V_{n+1} = 1.12\,V_n,\ V_0 = 100\,000$
   C. $V_{n+1} = V_n + 12,\ V_0 = 100\,000$ **D.** $V_{n+1} = 12\,V_n,\ V_0 = 100\,000$
   E. $V_n = 12\,n,\ V_0 = 100\,000$
7. MC An investment of $4500 earns compound interest at a rate of 6.4% p.a. and is made for 5 years. The balance in the account at the end of the investment period, if interest is compounded quarterly, is:
   A. $6181.40 **B.** $4871.71 **C.** $6136.50 **D.** $15561.27 **E.** $8592.20
8. MC After $4\frac{1}{2}$ years $1200 has grown to $1750 in an account where interest is compounded monthly. The annual interest rate is:
   A. 7.0% **B.** 0.7% **C.** 8.3% **D.** 3.2% **E.** 38%
9. MC A sum of $850 is invested at 8% p.a. compound interest, credited fortnightly. For the balance to grow to $1200 the investment should be left for a minimum of:
   A. 112 years **B.** 113 years **C.** 4 years 8 fortnights
   D. 4 years 9 fortnights **E.** 5 years
10. MC In an account which pays compound interest at 12% p.a., credited daily, the effective rate of interest is:
    A. 12% **B.** 12.75% **C.** 13.8% **D.** 15% **E.** 12.95%
11. MC $15 000 is invested in a simple interest account, which pays 4.2% per year. The recurrence relation that shows the value of the investment after n years is:
    A. $V_{n+1} = V_n + 630, V_0 = 15\,000$ **B.** $V_{n+1} = 630V_n, V_0 = 15\,000$
    C. $V_{n+1} = V_n + 630, V_0 = 15\,000$ **D.** $V_{n+1} = 630n, V_0 = 15\,000$
    E. $V_{n+1} = V_n + 630, V_0 = 15\,630$
12. MC Myvan wants to invest her savings of $6500 in an account that pays 5.5% p.a. compounded monthly. The effective rate of interest on this account is:
    A. 5.64% **B.** 5% **C.** 11% **D.** 6.5% **E.** 7%

Short answer

1. Cynthia invested $270 with a building society in a fixed deposit account that paid 8% p.a. simple interest for 4 years. How much did Cynthia receive at the end of the 4 years?
2. A bank offers 8.5% p.a. simple interest on an investment. At the end of 3 years the interest earned was $765. How much was invested?
3. If $725 is invested for 3 years and earns $206.65 interest, calculate the yearly interest rate.
4. Jack put some money away for $4\frac{1}{2}$ years in a bank account which is paying $3\frac{3}{4}$% p.a. interest. He found on his bank statement he had earned $67.50. How much did Jack invest?
5. Alex invested the $1800 he won at the races in an insurance company bond that pays $12\frac{1}{2}$% p.a. provided he keeps the bond for 4 years. What is Alex's total return from the bond at the end of the 4 years?
6. An investment bond is offered to the public at 10% per year. Louis buys a bond worth $4000 that will mature in $2\frac{1}{2}$ years. How much in total will Louis receive at the end of the $2\frac{1}{2}$ years?
7. If $5400 is to be invested for 5 years, which of the options below would be the most productive to use?
   a. 12% p.a. simple interest
   b. Compound interest at 11.8% p.a., credited quarterly
   c. Compound interest at 11.7% p.a., credited monthly
8. What amount must be invested at 9.25% p.a., interest compounded 6-monthly, if it is to grow to $5000 over 4 years?
9. How much interest would $950 earn if it was invested for 3 years at 12% p.a., interest credited daily?
10. How long would it take for $2000 to amount to $3450 by earning interest at 6.8% p.a., compounded monthly?

Extended response

1. Geoff wants to buy a windsurfer. Its retail price is $3995. He needs to save up until he has enough cash to pay for the windsurfer. Geoff's first option for financing the purchase is to place the balance of his savings account, $1983.50, into a term deposit offering 5.6% per annum for a 2-year term.
   a. Calculate the total value of his investment at the end of 2 years.
   b. Geoff uses the term deposit investment towards the purchase of the windsurfer. What extra fortnightly savings will be needed over the next 2 years to make up the balance of $3995?

   Another option for Geoff is to place his $1983.50 into a building society, which offers 5.4% interest, compounded monthly.

   c. How long will it take Geoff to accumulate enough funds for the windsurfer?

   Geoff decided to invest with the building society. Eight months later he received a $1000 bonus from his employer.

   d. How much money did Geoff have in his account at the time that he received his bonus?
   e. If Geoff immediately deposited the entire bonus into his building society account, how much longer would he need to wait to get enough funds for the windsurfer?

2. Bridie has invested her money in an account to save for a holiday. The amount of money in her account after n months, A_n can be modelled using the recurrence relation $A_{n+1} = 1.005A_n$, $A_0 = 6500$.
   a. What is the annual interest rate?
   b. How much money has Bridie saved before she opens the account?
   c. How much money does Bridie have in her account after two years?

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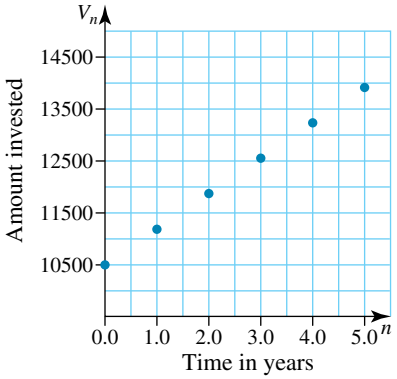
Units 3 & 4 Sit exam

Answers

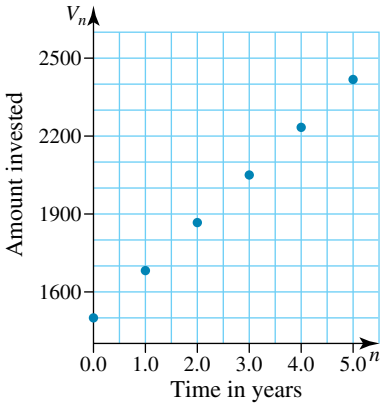
Topic 6 Simple and compound interest - investments and loan

Exercise 6.2 Simple interest

1. a. $A_{n+1} = A_n + 86.70$, $A_0 = 1020$
   b. $1453.50
2. a. $B_{n+1} = B_n + 48.13$, $B_0 = 713$
   b. $953.65
3. a. $A_{n+1} = A_n + 187.50$, $A_0 = 1500$
   b. $1875 c. $375
4. a. $A_{n+1} = A_n + 132$, $A_0 = 2400$
   b. $2730 c. $330
5. a. $816 b. $15 925
6. a. $13 912.50
   b.



7. a. For an initial investment of $6200 the building society is offering a flat rate of interest set at $565 per annum.
   b. 9.11%
8. a. $2418.50
   b.



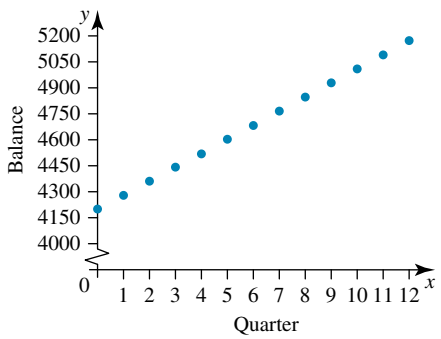
9. $3070
10. 3 years
11. B
12. a. $360 b. $2700 c. $131.25

Exercise 6.3 Compound interest as a geometric recurrence relation

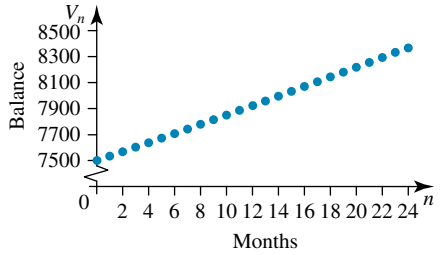
1. a. $A_{n+1} = 1.06A_n$, $A_0 = 7500$, where A is the amount invested and n is the number of years after the initial investment.
   b. $9468.58
2. a. $A_{n+1} = 1.075A_n$, $A_0 = 3250$, where A is the amount invested and n is the number of years after the initial investment.
   b. $4665.80
3. a. $A_{n+1} = 1.005A_n$, $A_0 = 600$, where A is the amount invested and n is the number of **months** after the initial investment.
   b. $618.23
4. a. $A_{n+1} = 1.01A_n$, $A_0 = 2500$, where A is the amount borrowed and n is the number of **months** after the initial loan.
   b. $50.25
5. B
6. C
7. D
8. C
9. D
10. E
11. $A = 8\%$
    $B = \$6210$
    $C = \$6210$
    $D = \$6706.80$
    $E = \$7822.81$
12. $A = \$967.50$
    $B = \$13\,867.50$
    $C = \$13\,867.50$
    $D = \$16\,025.63$
    $E = \$16\,025.63$
    $F = \$17\,227.55$
13. $5650.67
14. a. $A_{n+1} = 1.012A_n$, $A_0 = 15\,000$, where A is the amount owing and n is the number of months after the account due date.
    b. $546.51
15. D

Exercise 6.4 Compound interest rule

1. a. $V_s = 2500\left(1 + \frac{7.5}{100}\right)^5$
   b. $V_s = \$3589.07$; $I = \$1089.07$
2. $9657.36
3. a. 12 b. 1.75% c. $5172.05
   d.



Despite it being difficult to see on the graph, the graph is exponential.

4. a. 24
   b. 0.4583%
   c. $8369.92
   d. 

   Despite it being difficult to see on the graph, the graph is exponential.
5. $3448.40
6. $4290.73
7. a. $583.20 b. $1630.47
   c. $4472.27 d. $3764.86
8. a. i. $2150 ii. $150
   b. i. $2311.25 ii. $311.25
   c. i. $3086.60 ii. $1086.60
9. a. 5 b. 20 c. 8 d. 72 e. 9 f. 15
10. a. 1.5% b. 2%
    c. 1.5% d. 1.75%
11. a. $1749.60 b. $1757.49
    c. $1759.33 d. $1760.24
    e. The balance increases as the compounding periods become more frequent.
12. a. $2852.92 b. $4984.72 c. $6000
13. a. $605.60 b. $1314.84 c. $795.77
    d. $1043.10 e. $1641.82
14. E
15. B
16. a. $3542.13 b. $2052.54
    c. $2969.18 d. $5000
17. a. $2069.61 b. $1531.33
    c. $3564.10 d. $5307.05
18. a. $930.39 b. $468.67
    c. $2035.90 d. $4692.95

Exercise 6.5 Finding rate or time for compound interest

1. 11.31%
2. 9.63%
3. B
4. a. 13.98% b. 5.62%
5. a. 18.95% b. 7.41% c. 16.59%
6. 8.34%, C
7. Approx. 6 years
8. Approx. 5 years
9. Approx. 3 years
10. Approx. 3.5 years
11. a. 6 years b. 10 years
12. a. 3 years b. 5 years
13. a. 19, $9\frac{1}{2}$ years b. 20, 5 years
    c. 57, $4\frac{3}{4}$ years
14. a. 145, 5 years 15 fortnights
    b. 13, $3\frac{1}{4}$ years
    c. 32 quarters, 8 years
15. 13, $3\frac{1}{4}$ years
16. B
17. a. $n = 30$ months, $2\frac{1}{2}$ years
    b. $n = 119$ fortnights; 4 years 15 fortnights
18. a. 4 years b. $9718.11

Exercise 6.6 Effective annual interest rate

1. 11.5%
2. 14.8%
3. 12.13%
4. a. i. $32.79 ii. $936.96
      iii. $61.96 iv. 8.30%
   b. i. $467.22 ii. $30 033.20
      iii. $6043.20 iv. 10.47%
   c. i. $39.04 ii. $568.48
      iii. $18.48 iv. 7.76%
   d. i. $43.32 ii. $1089.68
      iii. $69.68 iv. 6.96%
5. a. 18.41% b. 20.05%
6. $134.46
7. 13.30%
8. 7.76%
9. A
10. E
11. D
12. a. 22.33% b. 24.99%
    c. $1029.60 d. $200.60
13. The effective annual interest rate of 9.3% is the better option (the compound interest rate has an effective annual interest rate of 9.38%).
14. 8.84%

6.7 Review: exam practice

Multiple choice

1. E 2. E 3. B 4. C 5. B 6. A
7. A 8. C 9. D 10. B 11. C 12. A

Short answer

1. $356.40 2. $3000
3. 9.5% 4. $400
5. $2700 6. $5000
7. **c** 8. $3482.46
9. $411.58 10. 8 years 1 month

Extended response

1. a. $2205.65 b. $34.41 c. 13 years
   d. $2056.04 e. 5 years
2. a. 6% p.a. b. $6500 c. $7326.54

TOPIC 7
Reducing balance loans, annuities and perpetuities

7.1 Overview

7.1.1 Introduction

When we invest money with a financial institution the institution pays us interest because it is using our money to lend to others. Conversely, when we borrow money from an institution we are using the institution's money and so it charges us interest.

In reducing balance loans**,** interest is usually charged every month by the financial institution and repayments are made by the borrower on a regular basis. These repayments nearly always amount to more than the interest for the same period of time and so the amount still owing is reduced. Since the amount still owing is continually decreasing and interest is calculated on the current balance but debited monthly, the amount of interest charged also decreases throughout the life of the loan.

This means that less of the amount borrowed is paid off in the early stages of the loan compared to the end. That is, the rate at which the loan is paid off increases as the loan progresses.

It is possible to have an 'interest only' loan account whereby the repayments only equal the interest added and so the balance doesn't reduce. This option is available to a borrower who wants to make the smallest repayment possible.

The theory behind reducing balance loans can also be applied to other situations such as superannuation payouts, bursaries and scholarships. These payouts are known as annuities. In Australia, all employers must pay a superannuation contribution to each employee, set at a minimum of 9.5% per annum. Many employees also make a voluntary contribution to their superannuation fund. Upon retirement, an employee may then take a pension from their superannuation fund. The larger their superannuation, the greater the interest generated from their fund.

LEARNING SEQUENCE

7.1 Overview
7.2 Modelling reducing balance loans with recurrence relations
7.3 Solving problems involving reducing balance loans using a Finance Solver
7.4 The effect of rate and repayment changes on reducing balance loans
7.5 Annuities and perpetuities
7.6 Annuity investments
7.7 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

7.1.2 Kick off with CAS

Finance Solver

As we saw in the previous topic, we can use CAS to solve finance problems. In this topic we will explore more complex financial problems, which involve many iterations. These can be solved using the CAS finance function called Finance Solver.

A reducing balance loan has regular repayments made during the life of the loan, with the interest being charged on the reducing balance of the loan. We can use the Finance Solver to find the amount still owing at any point during the life of a reducing balance loan.

1. Sylvia borrowed $15 000 exactly $2\frac{1}{2}$ years ago, with regular monthly repayments of $435. Interest is charged at 6.6% p.a. (adjusted monthly). Use CAS to calculate how much is still owing on Sylvia's loan.
2. Juliana is repaying a $3500 loan over 3 years with monthly instalments at 7.2% p.a. (adjusted monthly). Use CAS to determine how much Juliana's monthly repayments.
3. Calculate the answers to questions **1** and **2** using Finance Solver on your CAS.
4. Mohammed is repaying a $40 000 loan of 12 years with quarterly instalments at 6.3% p.a. (adjusted quarterly). How much does Mohammed still owe after 4 years?
5. Georgio took out a $22 000 loan with interest charged at 7.8% p.a. (adjusted monthly). He has made regular monthly repayments of $440.33. If he still owes $14 209.88, how long ago was the loan taken out?

on Resources

Please refer to the Resources section in the Prelims of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

7.2 Modelling reducing balance loans with recurrence relations

7.2.1 Recurrence relations of the form $V_0 = a$, $V_{n+1} = RV_n \pm d$

In Topic 5, we learned to write a sequence as a recurrence relation. These sequences were expressed as either linear sequences or geometric sequences. Many situations can be modelled using a combination of geometric and linear recurrence relations.

These can be expressed in the form of $V_0 = a$, $V_{n+1} = RV_n \pm d$, where V_0 is the initial term, V_n is the nth term of the sequence, r is the percentage increase or decrease, $R = 1 \pm \frac{r}{100}$ and d is the value added or subtracted.

WORKED EXAMPLE 1

The population of a small town is known to have a consistent increase in population of 3.5% each year. The population of the town at the beginning of 2018 is 23 500 people. A large factory in the town announces in 2018 that it needs to relocate 250 people in its workforce at the end of each year for the next 3 years.

a. Write a recurrence relation to describe this situation.

b. Use your calculator to determine the population of the town at the end of 2020.

c. Have the relocations at the factory in the three years caused the population to decrease from the 23 500 people at the beginning of 2018?

THINK	WRITE
a. 1. State the initial values for V_0, r, R and d.	**a.** $V_0 = 23500$ $r = 3.5\% = 0.035$ $R = 1 + 0.035 = 1.035$ $d = 250$
2. Write the general form of the recurrence relation. The percentage is increasing so $R = 1 + \frac{3.5}{100} = 1.035$. 250 is subtracted as this many people are being relocated.	$V_0 = a,\ V_{n+1} = RV_n \pm d$
3. Substitute the values for V_0, r, R and d.	$V_0 = 23\,500,\ V_{n+1} = 1.035V_n - 250$
b. 1. On a calculator screen, type 23 500, press ENTER/EXE, then type $\times 1.035 - 250$. Continue to press ENTER/EXE 3 times.	23500 — 23500 $23500 \times 1.035 - 250$ — 24072.5 $24072.5 \times 1.035 - 250$ — 24665.04 $24665.04 \times 1.035 - 250$ — 25278.31
2. Write the answer.	At the end of 2020 the population of the town is 25 278 people to the nearest whole number.
c. Write the answer.	The population has continued to increase from a population of 23 500 at the beginning of 2018 to a population of 25 278 at the end of 2020.

7.2.2 Modelling reducing balance loans with a recurrence relation

A recurrence relation can be used to model a reducing balance loan.

$V_0 = a$, $V_{n+1} = RV_n - d$
Where,
V_0 is the intial amount borrowed
V_n is the balance of the loan after n payments
$R = 1 + \frac{r}{100}$ (r is the interest rate per compounding period)
d is the payment made per compounding period.

WORKED EXAMPLE 2

A loan of $2500 is taken out at a rate of 7.5% p.a. (interest compounded monthly) and is to be paid back monthly with instalments of $509.42.
a. Write a recurrence relation to describe this situation.
b. Use your calculator to determine the number of monthly repayments needed to pay off this loan.

THINK

a. 1. State the initial values for V_0, r, R and d.

2. Write the general form of the recurrence relation.

3. Substitute in the values for V_0, r, R and d.

b. 1. On a calculator screen, type 2500, press ENTER/EXE and then type $\times 1.006\,25 - 509.42$.
Continue to press ENTER/EXE until the output is less than 0.
Count the number of iterations of the recurrence relation until the balance is less than 0.

2. Write the answer.

WRITE

a. $V_0 = 2500$

$r = \frac{7.5}{12} = 0.625$

$R = 1 + \frac{0.625}{100} = 1.006\,25$

$d = 509.42$

$V_0 = a,\ V_{n+1} = RV_n - d$

$V_0 = 2500,\ V_{n+1} = 1.006\,25V_n - 509.42$

2500	2500
$2500 \times 1.006\,25 - 509.42$	2006.21
$2006.21 \times 1.006\,25 - 509.42$	1509.32
$1509.32 \times 1.006\,25 - 509.42$	1009.34
$1009.34 \times 1.006\,25 - 509.42$	506.23
$506.23 \times 1.006\,25 - 509.42$	−0.03

After 5 months the account balance reads −0.03, so it would take 5 months to pay off the loan.
Note: As the above calculations have been rounded to 2 decimal places, this can impact on the final balance. If there was an overpayment of 3 cents, the final payment would be adjusted by 3 cents.

7.2.3 Amortisation of a reducing balance loan using a table

The process of paying off a loan by regular payments over a period of time is known as **amortisation**. The amortisation of a loan can be tracked on a step-by-step basis by following the payments made, the interest added and reduction in the principal. For each payment period, an amortisation table provides your loan balance, interest charges on your loan, and the amount of principal that you pay off.

Consider an amortisation table for Worked example 2 on the previous page.

- The *scheduled payment* of $509.42 remains fixed.
- The *interest rate* of 0.625% per month remains fixed.
- The *interest* is calculated on the previous month's balance.
- The *principal reduction* is the *scheduled payment – interest for that month*.
- The *balance of the loan* is the *previous month's balance – principal reduction*.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	2500.00
1	509.42	$0.006\,25 \times 2500 = 15.63$	$509.42 - 15.63 = 493.80$	$2500 - 493.80 = 2006.20$
2	509.42	$0.006\,25 \times 2006.20 = 12.54$	$509.42 - 12.54 = 496.88$	$2006.20 - 496.88 = 1509.32$
3	509.42	$0.006\,25 \times 1509.32 = 9.43$	$509.42 - 9.43 = 499.99$	$1509.32 - 499.99 = 1009.33$
4	509.42	$0.006\,25 \times 1009.33 = 6.31$	$509.42 - 6.31 = 503.11$	$1009.33 - 503.11 = 506.22$
5	509.42	$0.006\,25 \times 506.22 = 3.17$	$509.42 - 3.17 = 506.25$	$506.22 - 506.25 = -0.03$

WORKED EXAMPLE 3

Sharyn takes out a loan of $5500 at 8% p.a. to pay for solar heating for her pool. The loan is to be paid back half-yearly in instalments of $1515.20. The incomplete amortisation table for this loan is below.

a. What is the interest rate per payment period?
b. How much of the first payment of $1515.20 is interest?
c. What is the balance of the loan after two payments have been made?
d. What is the principal reduction after 3 payments have been made?
e. What is the total interest paid on this loan?
f. What should be the amount of the final payment to ensure a balance of zero at the end of the loan period?

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	5500.00
1	1515.20		1295.20	4204.80
2	1515.20	168.19	1347.01	
3	1515.20	114.31		1456.90
4	1515.20	58.28	1456.92	−0.02

THINK

a. 1. Write the value of the annual interest rate.

2. Calculate the half-yearly interest rate.

WRITE

a. Annual interest rate = 8%

Half-yearly interest rate $= \frac{8}{2} = 4\%$

	3. Write the answer.	The interest rate per payment period is 4%.
b.	1. Calculate 4% of $5500.	$0.04 \times 5500 = 220$
	2. Write the answer.	The interest component of the first payment is $220.
c.	1. The balance of the loan after two payments have been made is the previous balance less the balance reduction.	Previous balance – balance reduction $= 4204.80 - 1347.01$ $= 2857.79$
	2. Write the answer.	The balance of the loan after two payments have been made is $2857.79.
d.	1. Principal reduction = payment – interest	$1515.20 - 114.31 = 1400.89$
	2. Write the answer.	The principal reduction after 3 payments is $1400.89.
e.	1. The total interest paid is the sum of all interest payments.	$220+168.19+114.31+58.28 = 560.78$
	2. Write the answer.	The total interest paid is $560.78.
f.	1. The last payment results in an overpayment of 2 cents.	The last payment should be $1515.18.

studyon

Unit 3 > AOS R&FM > Topic 2 > Concept 8

Reducing balance loans Summary screen and practice questions

Exercise 7.2 Modelling reducing balance loans with recurrence relations

1. **WE1** The enrolment numbers at a school have been growing steadily at a rate of 1.5% per annum for a number of years. The school decides to accept a further 50 international students every year. The total number of enrolments at the beginning of 2018 is 565 students.
   a. Write a recurrence relation to describe this situation, where V_n is the enrolment number after n years.
   b. Use your calculator to determine the total number of enrolments at the end of 2020.
   c. How many new students enrol at the school in 2019?
2. At a child care centre, the sand in the sand pit decreases on average by 35% every year. The centre has a standing order with the local garden supplier to top up the sand by 1.5 m^3 once a year.The following recurrence relation can be used to model this situation, where A_n is the amount of sand in the sand pit after n years.

$$A_0 = 3.5,\ A_{n+1} = 0.65A_n + 1.5$$

   a. What is the initial amount of sand in the sand pit?
   b. How much sand will be in the sand pit after 4 years?
   c. How many years before the sand in the sand pit will exceed 4 m^3?

3. MC The recurrence relation, $P_0 = 10\,000$, $P_{n+1} = 1.03P_n - 250$ will generate a sequence that
   - **A.** is an example of linear decay.
   - **B.** is an example of geometric growth.
   - **C.** is an example of both geometric growth and linear growth.
   - **D.** is an example of both geometric growth and linear decay.
   - **E.** is an example of both geometric decay and linear decay.

4. WE2 A loan of \$15 000 is taken out at a rate of 12.5% p.a. (interest compounded half-yearly) and is to be paid back half-yearly with \$3074.44 instalments.
   - **a.** Write a recurrence relation to describe this situation.
   - **b.** Use your calculator to determine the number of half-yearly repayments needed to pay off this loan.

5. Nadjme invests \$1500 with interest paid at a rate of 5.5% p.a. (interest compounded monthly) and each month she adds another \$200 from her monthly pay to her investment.
   - **a.** Write a recurrence relation to describe this situation.
   - **b.** Use your calculator to determine the balance of her investment after 5 months.

6. WE3 Sourav takes out a loan of \$2400 at 9% p.a. to buy a new lap top. The loan is to be paid back monthly in instalments of \$489.64. The incomplete amortisation table for this loan is below.

Payment number (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	2400.00
1	489.64		471.64	1928.36
2	489.64	14.46	475.18	1453.18
3	489.64	10.90	478.74	
4	489.64	7.31		492.11
5	489.64	3.69	485.95	

   - **a.** What is the interest rate per payment period?
   - **b.** How much of the first payment of \$489.64 is interest?
   - **c.** What is the balance of the loan after three payments have been made?
   - **d.** What is the principal reduction after 4 payments have been made?
   - **e.** What is the total interest paid on this loan?
   - **f.** What is the balance left at the end of 5 months to pay out the loan?

7. A loan of \$25 000 is taken out over 10 years at a rate of 5.5% p.a. (interest debited monthly) and is to be paid back monthly with \$271.32 instalments. Complete the table below for the fourth payment.

Payment number (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	25 000.00
1	271.32	114.58	156.74	24 843.26
2	271.32	113.86	157.46	24 685.80
3	271.32	113.14	158.18	24 527.62
4	271.32			

8. Paul wanted to buy a new road bike so he took out a $10 000 loan over 6 years at a rate of 8% p.a. (interest debited quarterly) to be paid back quarterly with instalments of $528.71. Complete the table below for the first three payments and state how much interest he has paid and how much he owes at the end of this time.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	10 000.00
1	528.71			
2	528.71			
3	528.71			

9. Kali took out a loan for $23 000 to pay for her wedding over 7 years at a rate of 6.5% p.a. (interest debited monthly), which is to be paid back monthly with instalments of $341.54.
   a. Write down a recurrence relation to model this loan.
   b. Use your recurrence relation to find how much Kali owes after her third payment was made.

10. Noah took out a personal loan of $2000 to buy a new air conditioning unit, with the loan having an interest rate of 12% p.a. He will repay the loan with 8 equal monthly payments of $261.
    a. Write down a recurrence relation to model this loan.
    b. Use the recurrence relation to determine how much is still owed after 4 months (4 payments).
    c. Is the loan paid out exactly after 8 months? If not, how much will Noah need to add to the last payment to fully repay the loan?
11. Link borrows $18 000, taken out over 2 years and to be repaid in monthly instalments. (*Note:* As the interest rate increases, the monthly repayment increases if the loan period is to remain the same.) Find the amount still owing after 5 months if interest is debited monthly at a rate of:
    a. 6% p.a. and the repayment is $532.90
    b. 9% p.a. and the repayment is $608.04
    c. 12% p.a. and the repayment is $688.66.
12. MC Nithya wants to borrow $8000 for a second-hand car and her bank offers her a personal loan for that amount at an interest rate of 13% p.a., interest debited fortnightly, with fortnightly repayments of $124.11 over 3 years. After 8 weeks she still owes
    A. $7661.03 **B.** $7402.32 **C.** $7227.69
    D. $7488.99 **E.** $6899.45

13. MC Ben took out a loan for $20 000 for a small start-up business at 16.5% p.a.. The contract required that he repay the loan over 5 years with monthly instalments of $421.02. After 8 months Ben still owes
    A. $12 344.05 **B.** $14 7661.22 **C.** $9653.13
    D. $19 001.67 **E.** $18 774.05

7.3 Solving problems involving reducing balance loans using a Finance Solver

In the previous subtopic, recurrence relations were used to help solve reducing balance loan problems. The iterative process is a very simple way of generating the terms in a recurrence relation. There are limitations to this method as repeating the same process for a loan taken out monthly for 5 years, is equivalent to 60 iterations. The Finance Solver application on your CAS calculator can be used to solve problems where it is impractical to use the recurrence relation.

The Finance Solver application on the CAS calculator has a particular set of fields, similar to those in the following table.

N	The total number of repayment periods
I %	The interest rate per annum
PV	The initial value of the loan – this is positive as the money is coming to you
PMT (Pmt)	The periodic payment – this is negative if you are repaying a loan
FV	The future value of the loan – positive, negative or zero
P/Y (PpY)	The number of payment periods in a year
C/Y (CpY)	The number of compounding periods in a year (nearly always the same as P/Y)

Remember:

When entering values into the Finance Solver, pay particular attention to the sign of the values.

Cash flowing towards you is considered to be positive, cash flowing away from you is considered to be negative.

The values of N, I, P/Y and C/Y are always positive but particular attention must be paid to the signs of PV, PMT and FV.

If you have taken out a loan then that is a positive number. (Consider it as money coming to you.)

If you have to make a payment that is a negative number. (Consider this as money leaving you.)

If the FV is:

- negative then there is still money to be paid
- zero then the loan has been paid out
- positive then the borrower has overpaid.

Fill in all the fields except the unknown field, place your cursor in that field and press ENTER/EXE.

WORKED EXAMPLE 4

A loan of $50 000 is taken out over 20 years at a rate of 6% p.a. (interest debited monthly) and is to be repaid with monthly instalments of $358.22. Find the amount still owing after 10 years.

THINK

1. Using Finance Solver on your CAS calculator enter the appropriate values into the fields.

N	120	The number of payments for 10 years at 12 interest calculations per year.
I %	6	The annual interest rate
PV	50 000	The value of the loan
PMT (Pmt)	−358.22	The repayment per month
FV	?	This is the unknown value, **place the cursor in this cell**.
P/Y (PpY)	12	There are 12 payments per year.
C/Y (CpY)	12	There are 12 compounding periods per year.

Press ENTER / Tap Solve

2. Write the answer.

WRITE

N = 120 payments
I = 6% p.a.
Amount borrowed = $50 000
Repayment per month = $358.22

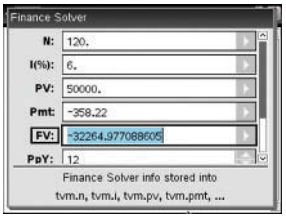
The balance of the loan after 10 years is $32 264.98.

TI | THINK

1. On a Calculator page, press MENU
   8: Finance
   1: Finance Solver
   Complete the fields as shown, place the cursor in the field marked FV and then press ENTER.

2. The answer appears on the screen.

WRITE



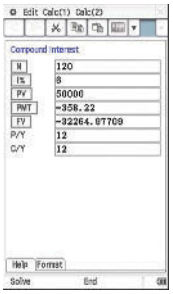
The balance of the loan after 10 years is $32 264.98.

CASIO | THINK

1. On the Financial screen, tap Compound Interest. Complete the fields as shown, place the cursor in the field marked FV and then tap Solve.

2. The answer appears on the screen.

WRITE



The balance of the loan after 10 years is $32 264.98.

Note that, even though 10 years is the halfway point of the term of the loan, more than half of the original $50 000 is still owing.

7.3.1 Number of repayments

The situation often arises in reducing balance loans when a potential borrower knows how much needs to be borrowed as well as the amount that can be repaid each month. The person then wants to know how long the loan needs to be to accommodate these conditions. The number of repayments, N, can be calculated using the Finance Solver on the CAS calculator.

WORKED EXAMPLE 5

Rob wants to borrow $2800 for a new sound system at 7.5% p.a., interest adjusted monthly.

a. What would be Rob's monthly repayment if the loan is fully repaid in $1\frac{1}{2}$ years?

b. What would be the total interest charged?

THINK

a. 1. Using Finance Solver on your CAS calculator enter the appropriate values into the fields.

N	18	The number of payments for $1\frac{1}{2}$ years at 12 interest calculations per year.
I %	7.5	The annual interest rate
PV	2800	The value of the loan
PMT (Pmt)	?	This is the unknown value, **place the cursor in this cell.**
FV	0	Fully paid means a future value of zero.
P/Y (PpY)	12	There are 12 payments per year.
C/Y (CpY)	12	There are 12 compounding periods per year.

Press ENTER/Tap Solve.

2. Write a statement.

b. 1. Total interest = total repayments − amount borrowed

2. Write a statement.

WRITE

a. N = 18 payments
I = 7.5% p.a.
Amount borrowed = $2800
Future value = 0

The monthly regular payment is $164.95 over 18 months.

b. Total interest = $164.95 \times 18 - 2800$
$= 2969.10 - 2800$
$= \$169.10$

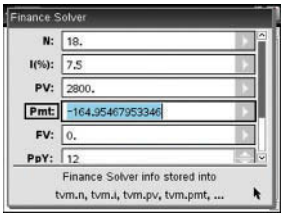
The total interest on the $2800 loan over 18 months is $169.10.

TI | THINK

a. 1. On a Calculator page, press MENU
8: Finance
1: Finance Solver
Complete the fields as shown, place the cursor in the field marked Pmt and then press ENTER.

2. The answer appears on the screen.

WRITE



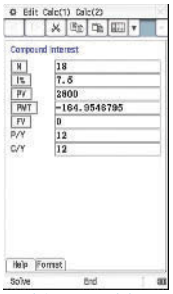
The monthly repayment is $164.95.

CASIO | THINK

a. 1. On the Financial screen, tap Compound Interest. Complete the fields as shown, place the cursor in the field marked PMT and then tap Solve.

2. The answer appears on the screen.

WRITE



Rob's monthly repayment would be $164.95.

WORKED EXAMPLE 6

Josh borrows $12 000 for some home office equipment. He agrees to repay the loan over 4 years with monthly instalments at 7.8% p.a. (adjusted monthly). Find:
a. the instalment value
b. the principal repaid and interest paid during the 10th repayment.

THINK

a. 1. Using Finance Solver on your CAS enter the appropriate values:

N	48	The number of payments for 10 years at 12 interest calculations per year.
I %	7.8	The annual interest rate
PV	12 000	The value of the loan
PMT (Pmt)	?	This is the unknown value, **place the cursor in this cell.**
FV	0	The loan is paid out so the future value is zero.
P/Y (PpY)	12	There are 12 payments per year.
C/Y (CpY)	12	There are 12 compounding periods per year.

2. Write a statement.

WRITE

a. N = 48 payments
I = 7.8 % p.a.
Amount borrowed = $12 000
Future value = 0

The monthly repayment over a 4-year period is $291.83.

b. 1. Find the amount owing after 9 months.

N	9
I %	7, 8
PV	12 000
PMT (Pmt)	−291.83
FV	**? place the cursor in this cell.**
P/Y (PpY)	12
C/Y (CpY)	12

Press ENTER/Tap Solve.

2. Write the amount owing after 9 months.

3. Find the amount owing after 10 months. Change $N = 9$ to $N = 10$ and press ENTER/Tap Solve.

4. Principal repaid $= V_9 - V_{10}$
Amount owing after 9 months less the amount owing after 10 months.

5. Interest paid = repayment − principal repaid

6. Write a statement.

b. 1. N = 9
I = 7.8% p.a.
Amount borrowed = $12 000
Payment per month = $291.83

Amount owing after 9 months is $10 024.73.

Amount owing after 10 months is $9798.06.

Principal repaid = 10 024.73 − 9798.06
= $226.67

Total interest = $291.83 − 226.67
= $65.16

In the 10th repayment, $226.67 principal is repaid and $65.16 interest is paid.

Sometimes we may want to find the time for only part of the loan term. The procedure that is followed is the same as in Worked example 5; however, V_n is zero only if we are calculating the time to repay the loan in full. Otherwise we should consider the amount still owing at that time.

WORKED EXAMPLE 7

Some time ago, Petra borrowed $14 000 to buy a car. Interest on this reducing balance loan has been charged at 9.2% p.a. (adjusted monthly) and she has been paying $446.50 each month to service the loan. Currently she still owes $9753.92. How long ago did Petra borrow the money?

THINK **WRITE**

1. Enter the following values into Finance Solver on your CAS:

N	**? place the cursor in this cell.**
I %	9.2
PV	14000
PMT (Pmt)	−446.50
FV	−9.753.92
P/Y (PpY)	12 (monthly payment)
C/Y (CpY)	12 (monthly compounds)

2. Press ENTER/Tap Solve.	N = 11.999 995 037 662
3. Interpret the results.	N = 12 months Time = 1 year
4. Write a statement.	Petra has had the loan for the past 12 months.

WORKED EXAMPLE 8

A loan of $11 000 is being repaid by monthly instalments of $362.74 with interest being charged at 11.5% p.a. (debited monthly). Currently, the amount owing is $7744.05. How much longer will it take to:

a. reduce the amount outstanding to $2105.11 **b. repay the loan in full?**

THINK

a. 1. Using the Finance Solver on your CAS, enter the following values:

N	**? place the cursor in this cell.**
I %	11.5
PV	7744.05
PMT (Pmt)	−362.74
FV	−2105.11
P/Y (PpY)	12
C/Y (CpY)	12

WRITE

2. Press Enter/Tap Solve.	N = 17.999 988 603 29
3. Interpret the results and write a statement.	N = 18 months It will take another $1\frac{1}{2}$ years to reduce the amount owing to $2105.11.
b. 1. Repeat part **a**, entering the appropriate values into Finance Solver. Enter FV: = 0 to represent the loan is fully repaid.	**b.** N = 23.999 534 856 457 = 24 months
2. Write a statement.	It will take another 2 years to repay the loan in full.

7.3.2 Effects of changing the repayment

Since most loans are taken over a long period of time it is probable that a borrower's financial situation will change during this time. For instance, a borrower may receive a pay rise and so their take-home pay is greater per week or fortnight. The person may then choose to increase the value of the repayments made to service the loan.

It may also be that a person's financial situation deteriorates, in which case he/she may request from their financial institution that the repayment value be decreased and the loan period subsequently extended.

In this section we will look at the effect that changing the repayment value has on the term of the loan and the total interest paid.

WORKED EXAMPLE 9

A reducing balance loan of $16 000 has a term of 5 years. It is to be repaid by monthly instalments at a rate of 8.4% p.a. (debited monthly).

a. Find the repayment value.

b. What will be the term of the loan if the repayment is increased to $393.62?

c. Calculate the total interest paid for repayments of $393.62.

d. By how much does the interest figure in c differ from that paid for the original offer?

THINK

a. 1. Using Finance Solver on your CAS, enter the following values:

N	60
I %	8.4
PV	16 000
PMT (Pmt)	**? place the cursor in this cell.**
FV	0
PpY	12
CpY	12

2. Press Enter/Tap Solve.

3. Write a statement.

WRITE

a.

Pmt = 327.49

$16 000 to be paid off in 5 years at 8.4% p.a. will need monthly repayments of $327.49.

b. 1. Using Finance Solver on your CAS, enter the following values:

N	**? place the cursor in this cell.**
I %	8.4
PV	16 000
PMT (Pmt)	−393.62
FV	0
P/Y (PpY)	12
C/Y (CpY)	12

2. Solve for N.

3. Interpret the results.

Write a statement.

b.

N = 47.999 695 088 867

N = 48 months

$$\text{Time} = \frac{48}{12} \text{ years}$$

$$= 4 \text{ years}$$

The new term of the loan would be 4 years.

c. Interest paid =
total repayments − principal repaid

d. 1. (a) Review the known quantities.
(b) Find the interest difference.

2. Write a statement.

c. When term = 4 years, $d = 393.62$.
Interest $= 48 \times 393.62 - 16\,000$
$= \$2893.76$

d. When term = 5 years, $d = 327.49$.
Interest $= 60 \times 327.49 - 16\,000$
$= \$3649.40$
Interest difference $= 3649.40 - 2893.76$
$= \$755.64$
If the repayment is increased from \$327.49 to \$393.62 per month then \$755.64 is saved in interest payments.

If a borrower does increase the value of each repayment and if all other variables remain the same, then the term of the loan is reduced. Conversely, a decrease in the repayment value increases the term of the loan. Consequently there are two stages to the loan, each with a different repayment.

WORKED EXAMPLE 10

Brad borrowed \$22 000 to start a business and agreed to repay the loan over 10 years with quarterly instalments of \$783.22 and interest debited at 7.4% p.a. However, after 6 years of the loan Brad was struggling to meet his repayments and decided to decrease the repayment value to \$648.68. Find:
a. the actual term of the loan
b. the total interest paid
c. the extra cost incurred by decreasing the repayment value.

THINK

a. 1. To find how much is left on the loan after 6 years:
Using Finance Solver on your CAS, enter the following values:

N	24
I %	7.4
PV	22000
PMT (Pmt)	−783.22
FV	? **place the cursor in this cell.**
P/Y (PpY)	4
C/Y (CpY)	4

WRITE

a. N = 24
Interest rate = 7.4% p.a.
The amount borrowed = \$22 000
The quarterly repayment = \$783.22
FV = \$10 761.83

2. Write a statement.

At the end of 6 years, the balance of the loan is $10 761.83.

3. Now find the N value to reduce $10 761.83 to zero; that is, the remaining part of the loan.

Balance is $10 761.88 and the repayments are $648.68.

4. Using Finance Solver, enter the appropriate values.

N	? place the cursor in this cell.
I %	7.4
PV	10 761.9
PMT (Pmt)	−648.68
FV	0
P/Y (PpY)	4
C/Y (CpY)	4

5. Solve for N.

N = 19.999 770 539 209

6. Interpret the results.

N = 20 quarters
Time = 5 years

7. Find the total term of the loan.

Total term = 6 + 5
= 11 years

b. For the two-repayment scenario:
Interest paid =
total repayments − principal repaid
In this case, in two stages.

b. For the two-repayment scenario:
Interest = 783.22 × 24 + 648.68 × 20 − 22 000
= $9770.88

c. 1. For the same repayment scenario: d = 783.22 for 10 years

c. For the same repayment scenario:
Interest = 783.22 × 40 − 22 000
= $9328.80

2. Find the difference between the two scenarios.

Interest difference = 9770.88 − 9328.80
= $447.08

3. Write a statement.

Brad has added $447.08 in interest by decreasing his repayment amount.

Exercise 7.3 Solving problems involving reducing balance loans using a Finance Solver

1. **WE4** A loan of $65 000 is taken out over 20 years at a rate of 12% p.a. (interest debited monthly) and is to be repaid with monthly instalments of $715.71. Find the amount still owing after:
   a. 5 years **b.** 10 years **c.** 15 years.
2. A loan of $52 000 is taken out over 15 years at a rate of 13% p.a. (interest debited fortnightly) and is to be repaid with fortnightly instalments of $303.37. Find the amount still owing after:
   a. 4 years **b.** 8 years **c.** 12 years.

3. WE5 Macca wants to borrow $3500 to buy a motorbike at 6.8% p.a., interest adjusted monthly.
   a. What would be Macca's monthly repayment if the loan is fully repaid in 2 years?
   b. What would be the total interest charged?
4. Willow wants to borrow $2400 for a new computer at 6.3% p.a., interest adjusted monthly.
   a. What would be Willow's monthly repayment if the loan is fully repaid in 2 years?
   b. What would be the total interest charged?
5. WE6 Grace has borrowed $18 000 to buy a car. She agrees to repay the reducing balance loan over 5 years with monthly instalments at 8.1% p.a. (adjusted monthly). Find:
   a. the instalment value
   b. the principal repaid and the interest paid during:
      i. the 10th repayment
      ii. the 50th repayment.
6. Sarah borrows $8000 for an overseas holiday. She agrees to repay the loan over 4 years with monthly instalments at 6.8% p.a. (adjusted monthly). Find:
   a. the instalment value.
   b. the principal repaid and interest paid during the:
      i. 10th repayment
      ii. 40th repayment.

7. Jim has a reducing balance loan of $3500 that he is using for a holiday and has agreed to repay it by monthly instalments of $206.35 at a rate of 7.6% p.a. (interest debited monthly). Find:
   a. the number of repayments needed to repay in full and this time in years
   b. the total interest charged.
8. WE7 Simon borrowed $8000. Interest on this reducing balance loan has been charged at 8.7% p.a. (adjusted monthly) and he has been paying $368.45 each month to service the loan. Currently he still owes $5489.56. How long ago did Simon borrow the money?
9. Aditya borrowed $11 000. Interest on this reducing balance loan has been charged at 6.5% p.a. (adjusted monthly) and he has been paying $409.50 each month to service the loan. Currently he still owes $5565.48. How long ago did Aditya borrow the money?
10. A loan of $2400 is taken out with a reducing balance interest rate of 4.5% per annum with interest debited monthly. The borrower wishes to pay instalments of $154.82 per month. What, correct to the nearest month, would be the term of such a loan?
11. WE8 A loan of $15 000 is being repaid by monthly instalments of $423.82 with interest being charged at 11.5% p.a. (debited monthly). Currently, the amount owing is $8357.65. How much longer will it take to:
    a. reduce the amount outstanding to $2450.15
    b. repay the loan in full?
12. A loan of $9000 is being repaid by monthly instalments of $273.56 with interest being charged at 8.8% p.a. (debited monthly). Currently, the amount owing is $6900.86. How much longer will it take to:
    a. reduce the amount outstanding to $1670.48
    b. repay the loan in full?
13. WE9 A reducing balance loan of $18 000 has a term of 5 years. It is to be repaid by monthly instalments at a rate of 7.8% p.a. (debited monthly).
    a. Find the repayment value.
    b. What will be the term of the loan if the repayment is increased to $390.50?
    c. Calculate the total interest paid for repayments of $390.50.
    d. By how much does the interest figure in **c** differ from that paid for the original offer?

14. A reducing balance loan of $25 000 has a term of 5 years. It is to be repaid by fortnightly instalments at a rate of 6.5% p.a. (debited fortnightly).
    a. Find the repayment value.
    b. What will be the term of the loan if the repayment is increased to $245?
    c. Calculate the total interest paid for repayments of $245.
    d. By how much does the interest figure in **c** differ from that paid for the original offer?
15. **WE10** James borrowed $21 000 for some home renovations and agreed to pay the loan over 7.5 years with quarterly instalments of $899.41 and interest debited at 6.8% p.a. However, after 6 years of the loan James decided to decrease the repayment value to $685.05. Find:
    a. the actual term of the loan
    b. the total interest paid
    c. the interest savings achieved by increasing the repayment value.
16. Gabriel borrowed $17 000 for some new furniture and agreed to pay the loan over 8 years with quarterly instalments of $670.29 and interest debited at 5.9% p.a. However, after 6 years of the loan Gabriel decided to increase the repayment value to $1724.02. Find:
    a. the actual term of the loan
    b. the total interest paid
    c. the extra interest charged by decreasing the repayment value.

17. Aimee has borrowed $5500 for some new outdoor furniture. She is to repay the reducing balance loan by quarterly instalments of $861.29 with interest debited quarterly at 9.4% p.a. Find:
    a. how long it will take Aimee to repay the loan in full
    b. the total interest charged.
18. Melpomeni's loan of $22 000 was taken out some time ago. Interest has been charged at 7.8% p.a. (adjusted monthly) and monthly repayments of $443.98 have serviced the loan. If the amount still owing is $14 209.88:
    a. how long ago was the loan taken out
    b. what was the term of the loan?
19. Some time ago, Elizabeth took out a loan of $25 000. Interest has been charged at 10.5% p.a. (adjusted monthly) and monthly repayments of $537.35 have serviced the loan. If the amount still owing is $11 586.64:
    a. how long ago was the loan taken out
    b. what was the term of the loan?
20. **MC** A reducing balance loan of $80 000 is taken out at 7.9% p.a. (adjusted monthly) to finance the purchase of a boat. It is to be repaid with monthly instalments of $639.84. The loan will be paid in full in:
    A. 10 years
    B. 15 years
    C. 20 years
    D. 22 years
    E. 25 years
21. Gila's reducing balance loan of $9000 is to be repaid by monthly instalments of $230.43 with interest charged at 10.5% p.a. (debited monthly).
    a. Currently, the amount owing is $8069.78. How much longer will it take to:
        i. reduce the amount owing to $3822.20
        ii. repay the loan in full?

b. Some time later the amount owing has fallen to $3226.06. How much longer will it take to:
   i. reduce the amount owing to $1341.23
   ii. repay the loan in full?

22. Megan wanted to borrow $50 000 and was offered a reducing balance loan over 20 years at 6.9% p.a. (adjusted monthly) with monthly instalments.
   a. What will be the monthly repayment value?
   b. What would be the term of the loan if instead the repayment was:
      i. increased to $577.97
      ii. increased to $486.33
      iii. decreased to $361.85
      iv. decreased to $352.90?
   c. In each case in b above, calculate the total interest paid.
   d. For each case above, calculate the interest difference from the original offer.

23. Jack borrowed $20 000 and agreed to repay the loan over 10 years with quarterly instalments of $750.48 with interest debited quarterly at 8.6% p.a. However, after 5 years he decided to increase the repayment value. Find:
   i. the actual term of the loan
   ii. the total interest paid
   iii. the interest saving achieved by increasing the repayment if the quarterly repayment was increased to:
      a. $901.48
      b. $1154.34.

24. MC Robin borrowed $25 000 and agreed to repay this reducing balance loan over 10 years with quarterly instalments of $975.06, interest being charged at 9.5% p.a. After 4 years Robin increased her repayment value to $1167.17. The term of her loan will be closest to:
   A. 6 years
   B. 7 years
   C. 8 years
   D. 9 years
   E. 10 years

25. A loan of $25 000 is repaid in 2 stages over $8\frac{3}{4}$ years with quarterly instalments. For the first 4 years the repayment was $975.06 and was increased to $1167.17 for the remaining time. The total amount of interest charged would be closest to:
   a. $9000
   b. $10 000
   c. $11 000
   d. $12 000
   e. $13 000

26. Anne is repaying a $26 000 loan over 8 years with monthly instalments of $383.61 at 9.2% p.a., debited monthly on the outstanding balance. She has made 2 years' worth of repayments but would like to repay the loan in full in the next 5 years. Find:
   a. the amount that she still owes
   b. the monthly repayment value needed to repay in full.

7.4 The effect of rate and repayment changes on reducing balance loans

7.4.1 Frequency of repayments

In this section we investigate the effect on the actual term of the loan and on the total amount of interest charged by making more frequent repayments. While the value of the repayment will change, the actual outlay will not. For example, a $3000 quarterly (every 3 months) repayment will be compared to a $1000 monthly repayment. That is, the same amount is repaid during the same period of time in each case. So the only variable will be how often repayments are made. In all cases in this section interest will be charged just before a repayment is made, although this may not be the case in practice.

WORKED EXAMPLE 11

Tessa wants to buy a dress shop. She borrows $15 000 at 8.5% p.a. (debited prior to each repayment) of the reducing balance. She can afford quarterly repayments of $928.45.
One-third of the quarterly repayment of $928.45 gives the equivalent monthly repayment of $309.48. The equivalent fortnightly repayment is $142.84.
Find:

a. the term of the loan and the amount still owing prior to the last payment if Tessa made payments monthly.

b. the term of the loan and the amount still owing prior to the last payment if Tessa made payments fortnightly.

THINK

a. 1. Identify the given values. Enter the appropriate values using Finance Solver. Remember that: PpY: = 12 and CpY: = 12 for monthly repayments.

2. Solve for N.

3. The value obtained for N is 59.58 which means that a 60th repayment is required. That is, N = 60

4. To find the amount still owing prior to the last payment, find V_n when N = 59. Enter the appropriate values using Finance Solver:

N	59
I %	8.5
PV	15 000
PMT (Pmt)	−309.48
FV	**? place the cursor in this cell.**
PpY:	12
CpY:	12

WRITE

a. i. Initial loan = 15 000
Monthly repayment = 309.48
Interest rate = 8.5%
N = ?
N = 59.582 518 723 273
N = 60 months
$\text{Term of loan} = \frac{60}{12}$
= 5 years

ii.

5. Press ENTER/Tap solve.
   FV = −179.273 603 537 66
6. State the amount still owing.
   The amount still owing prior to the last payment is $179.27.

b. 1. Enter the appropriate values using Finance Solver on your CAS.
Remember that:
PpY: = 26 and CpY: = 26
for fortnightly repayments.

b. i. For fortnightly repayments:
Initial loan = 15 000
For higthly repayment = 142.84
Interest rate = 8.5%

2. Solve for N.
   N =?
3. The value obtained for N is 128.85 which means that a 129th repayment is required. That is, N = 129.
   N = 129 fortnights
   Term of loan = 4 years, 25 fortnights
4. To find the amount still owing prior to the last payment, find F_V (or FV) when N = 128. Enter the appropriate values using Finance Solver:

   ii. FV = −120.636 212 821 95

N	128
I %	8.5
PV	15 000
PMT	−142.84
FV	**? place the cursor in this cell.**
PpY:	26
CpY:	26

5. State the amount still owing.
   The amount still owing prior to the last payment is $120.64.

It can be seen from Worked example 11 that while the same outlay is maintained there may be a slight decrease in the term of a loan when repayments are made more often. Let us now find what the savings are for such a loan. In this situation we should consider the final (partial) payment separately because the amount of interest that it attracts is less than a complete repayment, *d*.

The total interest paid is now calculated as usual:

Total interest = total repayments − principal repaid

WORKED EXAMPLE 12

In Worked example 11, Tessa's $15 000 loan at 8.5% p.a. gave the following three scenarios:

1. **quarterly repayments of $928.45 for 5 years**
2. **monthly repayments of $309.48 for 59 months with $179.27 still outstanding**
3. **fortnightly repayments of $142.84 for 128 fortnights with $120.64 still owing.**

Compare the total interest paid by Tessa if she repaid her loan:

a. quarterly **b. monthly** **c. fortnightly.**

THINK	WRITE
a. For quarterly repayments Total interest = total repayments – principal repaid	**a.** For quarterly repayments: Total interest $= 928.45 \times 20 - 15\,000$ $= \$3569$
b. 1. For monthly repayments find the interest rate and the amount owing after 59 months. (Refer to Worked example 11.)	**b.** For monthly repayments: $r = \frac{8.5}{12}\% = 0.7083\%$ $V_{59} = \$179.27$
2. Calculate the interest on the 59th repayment to find the final repayment.	Interest on the 59th repayment $= 0.7083\%$ of $\$179.27$ $= 0.007\,083 \times 179.27$ $= \$1.27$ Final repayment $= 179.27 + 1.27$ $= \$180.54$
3. Calculate the total interest paid.	Total interest $= 309.48 \times 59 + 180.54 - 15\,000$ $= \$3439.86$
c. 1. Fortnightly repayments Find the interest rate and the amount owing after 128 fortnights. (Refer to Worked example 11.)	**c.** For fortnightly repayments: $r = \frac{8.5}{26}\% = 0.3269\%$ The 128th repayment $= 120.64$
2. Calculate the interest on the 128th repayment to find the final repayment.	Interest on the 128th repayment $= 0.3269\%$ of $\$120.64$ $= 0.003\,269 \times 120.64$ $= \$0.39$ Final repayment $= 120.64 + 0.39$ $= \$121.03$
3. Calculate the total interest paid.	Total interest $= 142.84 \times 128 + 121.03 - 15\,000$ $= \$3404.55$
4. Calculate the interest saving with monthly repayments over quarterly repayments.	Monthly interest saving $= 3569 - 3439.86$ $= \$129.14$
5. Calculate the interest saving with fortnightly repayments over quarterly repayments.	Fortnightly interest saving $= 3569 - 3404.55$ $= \$164.45$
6. Write a comparison statement.	Tessa saves $\$164.45$ if she repays fortnightly rather than quarterly and $\$129.14$ if she repays monthly rather than quarterly.

The slight time savings calculated in Worked example 12 when repayments were made more often have now been transformed to money savings. The saving increases as the frequency of repayment increases. This is because the amount outstanding is reduced more often and so the amount of interest added is slightly less. A saving of $164 over 5 years, out of more than $18 000 repaid, might not seem much but the saving increases as the term of the loan increases and as the amount borrowed increases.

7.4.2 Changing the rate

Of all the variables associated with reducing balance loans, the one that is most likely to change during the term of a loan is the interest rate. These rates rarely stay the same for the life of a loan; for most loans the rate will change several times.

The Reserve Bank of Australia is the main monetary authority of the Federal Government and, as such, is the overall guiding influence on monetary factors in the Australian economy. Consequently, it indirectly controls the lending interest rates of financial institutions here.

There is usually some variation in rates between institutions; for example, a lower rate may be designed to attract more customers. Within each institution there are rate variations as well for different types of reducing balance loans. Banks advertise their loan rates to attract customers.

In this section we investigate the effect that changing the interest rate has on the term of the loan and on the total interest paid. It should be remembered that as the interest rate increases so too will the term (if d remains constant) of the loan since more interest needs to be paid.

First, let us simply compare loan situations by varying only the rate.

WORKED EXAMPLE 13

A reducing balance loan of $18 000 has been taken out over 5 years at 8% p.a. (adjusted monthly) with monthly repayments of $364.98.

a. What is the total interest paid?

b. If, instead, the rate was 9% p.a. (adjusted monthly) and the repayments remained the same, what would be:

i. the term of the loan

ii. the total amount of interest paid?

THINK

a. For 8% p.a.:

Total interest = total repayments − principal repaid

b. i. 1. Using the Finance Solver, enter the appropriate values.
Remember that PpY: and CpY: will both equal 12 for monthly repayments.

2. Solve for N.

3. N: = 61.81 means 61 full repayments plus a final lesser payment.

ii.1. Find the amount owing after 61 repayments to calculate the amount still owing.
Enter the following values:

N	61
I %	9
PV	18 000
FV	**? place the cursor here.**
PMT	−364.98
PpY:	12
CpY:	12

WRITE

a. For 8% p.a.:

Total interest = $364.98 \times 60 - 18\,000$

= $3898.80

b. i. For 9% p.a.:

$V_0 = 18\,000$, $d = 364.98$, I = 9, $V_n = 0$, N = ?

N = 61.810 665 384 123

N = 62 months

ii. Term = 5 years, 2 months

2. Solve for FV and interpret the result.	FV = −293.886 672 877 The amount still owing after 61 repayments is $293.88.
3. Calculate the interest on the 62nd repayment to find the final repayment. $r = \frac{9}{12} = 0.75\%$.	Interest on final repayment = 0.75% of $293.8 = $2.20 Final repayment = 293.88 + 2.20 = $296.08
4. Total interest = total repayments − principal repaid	Total interest paid = 364.98 × 61 + 296.08 − 18 000 = $4559.86

In Worked example 13 the rate was increased by only 1% p.a. on $18 000 for only 5 years, yet the amount of interest paid has increased from $3898.80 to $4559.86, a difference of $661.06. This difference takes on even more significant proportions over a longer period of time and with a larger principal.

Let us now consider varying the rate during the term of the loan.

WORKED EXAMPLE 14

Natsuko and Hymie took out a loan for home renovations. The loan of $42 000 was due to run for 10 years and attract interest at 7% p.a., debited quarterly on the outstanding balance. Repayments of $1468.83 were made each quarter. After 4 years the rate changed to 8% p.a. (debited quarterly). The repayment value didn't change.

a. Find the amount outstanding when the rate changed.

b. Find the actual term of the loan.

c. Compare the total interest paid to what it would have been if the rate had remained at 7% p.a. for the 10 years.

THINK	WRITE
a. 1. Rate changes after 4 years; that is, N = 16. State initial loan, repayments, interest rate and the number of payments.	**a.** $V_0 = 42\,000$, $d = 1468.83$, I = 7%, $n = 16$, $V_n = ?$ Initial loan = 42 000 Repayment = 1468.83 I = 7%, N = 16%
2. Find the amount owing after 16 payments using a CAS calculator.	FV = −285 84.356 811 602 The amount outstanding when the rate changed is $28 584.36.
b. 1. Find N to repay $28 584.36 in full at the new rate. (Use Finance Solver to enter the following values: I = 8, PV = 28 584.36, Pmt = −1468.83, FV = 0, PpY: = 4 and CpY: = 4)	**b.** New interest rate of 8%: N = −24.896 005 939 422 = 25 quarters Time = $6\frac{1}{4}$ years

2. Find the total term of the loan, that is, time at 7% plus time at 8%.

$$\text{Term} = 4 \text{ years} + 6\frac{1}{4} \text{ years}$$
$$= 10\frac{1}{4} \text{ years}$$

c. 1. Find the amount owing after 24 repayments to calculate the amount still owing. (That is, on your calculator change the value of N to 24 and solve for FV.)

c. FV = −1291.669 456 564 649
The amount still owing after the 24th repayment is $1291.61.

2. Calculate the interest on the outstanding amount to find the final repayment.
$r = \frac{8}{4} = 2\%$

$$\text{Interest on final payment} = 2\% \text{ of } \$1291.61$$
$$= \$25.83$$
$$\text{Final repayment} = 1291.61 + 25.83$$
$$= \$1317.44$$

3. Find the total interest for the rate change scenario.
The number of repayments is 40 at $1468.83 plus 1 at $1317.44.

$$\text{For the rate change scenario, total interest}$$
$$= 1468.83 \times 40 + 1317.44 - 42\,000$$
$$= \$18\,070.64$$

4. Calculate the total interest if the rate remained at 7%.

$$\text{For the rate at 7\% only, total interest}$$
$$= 1468.83 \times 40 - 42\,000$$
$$= \$16\,753.20$$

5. Find the interest difference between the two scenarios.

$$\text{Interest difference} = 18\,070.64 - 16\,753.20$$
$$= \$1317.44$$

6. Write a comparison statement.

An extra $1317.44 interest will be paid due to the interest rate change from 7% p.a. to 8% p.a.

In the situations studied so far the repayment value, remained the same, even though the rate varied. In practice, this is what happens if the rate decreases and so the term of the loan decreases. However, when the rate increases, financial institutions will generally increase the repayment value to maintain the original term of the loan. This was discussed in section 7.3.2. If this is not done the term of the loan can increase quite dramatically. In fact this may occur to such an extent that the repayments are insufficient to cover the interest added, so that the amount outstanding *increases*.

Consider a $44 000 loan over 15 years at 10% p.a. (monthly).
Monthly repayments = $472.83
After 5 years the amount owing is $35 779.02.
Suppose the interest rate rises dramatically to 16% p.a.
After a further 10 years under these conditions, the amount owing is $37 014.72.
That is, the amount owing has increased.
This situation is not beneficial to either the lender or the borrower.

7.4.3 Interest only loans

Interest only loans are loans where the borrower makes only the minimum repayment equal to the interest charged on the loan. As the initial principal and amount owing is the same for the period of this loan, we could use either the simple interest formula or CAS to solve problems of this type. When using Finance Solver, the present value (PV:) and future value (FV:) are entered as the same amount. Note that the future value is negative to indicate money owed to the bank.

This type of loan is used by two kinds of borrowers: investors in shares and/or property or families that are experiencing financial difficulties and seek short-term relief from high repayment schedules.

WORKED EXAMPLE 15

Jade wishes to borrow $40 000 to invest in shares. She uses an interest only loan to minimise her repayments and hopes to realise a capital gain if she can sell the shares at a higher value. The term of the loan is 6.9% p.a. compounded monthly with monthly repayments equal to the interest charged.

a. Calculate the monthly interest-only repayment.

b. If, in 3 years, she sells the shares for $50 000, calculate the profit she would make on this investment strategy.

THINK	WRITE
a. 1. Identify the initial loan, the interest rate and N, where N is equal to one payment period.	**a.** Initial loan = 40 000 Interest rate = 6.9% N = 1 month
2. Evaluate I using the simple interest formula.	$I = \frac{40\,000 \times 6.9 \times \frac{1}{12}}{100}$ = $230
3. Write your answer.	The monthly repayment to pay the interest only for the loan is $230.
b. 1. Find whether the capital gain on the shares exceeds the amount paid in interest.	**b.** Capital gain = selling price − purchase price = $50 000 − $40 000 = $10 000 Total interest charged = repayment × number of payments = $230 × 36 = $8280 Profit = capital gain − loan cost = $10 000 − $8280 = $1720
2. Write a statement.	Jade will make a profit of $1720.

Exercise 7.4 The effect of rate and repayment changes on reducing balance loans

1. **WE11** Bella wants to buy a shoe shop. She borrows $18 000 at 8.3% p.a. (debited prior to each repayment) of the reducing balance. She can afford quarterly repayments of $1108.80 and this will pay the loan in full in exactly 5 years.

One third of the quarterly repayment gives the equivalent monthly repayment of $369.60.

The equivalent fortnightly repayment is $170.58.

Find:

i. the term of the loan and

ii. the amount still owing prior to the last payment if Bella made repayments:

a. monthly **b.** fortnightly.

2. Simba wants to invest money in shares. He borrows $25 000 at 7.2% p.a. (debited prior to each repayment) of the reducing balance. He can afford quarterly repayments of $1292.02 and this will pay the loan in full in exactly 6 years.
   One third of the quarterly repayment gives the equivalent monthly repayment of $430.67.
   The equivalent fortnightly repayment is $198.77.
   Find:
   i. the term of the loan and
   ii. the amount still owing prior to the last payment if Simba made repayments:
      a. monthly b. fortnightly.

3. **WE12** In question 1, Bella's $18 000 loan at 8.3% p.a. gave the following scenarios:
   1. quarterly repayments of $1108.8 for 5 years.
   2. monthly repayments of $369.60 for 59 months with $ 217.37 outstanding.
   3. fortnightly repayments of $170.58 for 128 fortnights with $149.63 still owing.

   Compare the total interest paid by Bella if she repaid her loan:
   a. quarterly b. monthly c. fortnightly.

4. In question 2, Simba's $25 000 loan at 7.2% p.a. gave the following scenarios:
   a. quarterly repayments of $1292.02 for 6 years.
   b. monthly repayments of $430.67 for 71 months with $246.17 outstanding.
   c. fortnightly repayments of $198.77 for 154 fortnights with $164.36 still owing.

   Compare the total interest paid by Simba if he repaid his loan:
   i. quarterly ii. monthly iii. fortnightly.

5. **WE13** A reducing balance loan of $20 000 has been taken out over 5 years at 10% p.a. (adjusted monthly) with monthly repayments of $424.94.
   a. What is the total interest paid?
   b. If, instead, the rate was 11% p.a. (adjusted monthly) and the repayments remained the same, what would be:
      i. the term of the loan
      ii. the total amount of interest paid?

6. A reducing balance loan of $16 500 has been taken out over 4 years at 8% p.a. (adjusted monthly) with monthly repayments of $402.81.
   a. What is the total interest paid?
   b. If, instead, the rate was 9% p.a. (adjusted monthly) and the repayments remained the same, what would be:
      i. the term of the loan
      ii. the total amount of interest paid?

7. **WE14** Mi Lin took out a loan for a new kitchen. The loan of $32 000 was due to run for 10 years and attract interest at 8% p.a., debited quarterly on the outstanding balance. Repayments of $1169.78 were made each quarter. After 4 years the rate changed to 9% p.a. (debited quarterly). The repayment value didn't change.
   a. Find the amount outstanding when the rate changed.
   b. Find the actual term of the loan.
   c. Compare the total interest paid to what it would have been if the rate had remained at 8% p.a. for 10 years.

8. Roger took out a loan to get his large pine trees removed after they were affected after a massive storm. The loan of $28 000 was due to run for 8 years and attract interest at 6.5% p.a., debited monthly on the outstanding balance. Repayments of $374.81 were made each month. After 3 years the rate changed to 7.5% p.a. (debited monthly). The repayment value didn't change.
   a. Find the amount outstanding when the rate changed.
   b. Find the actual term of the loan.

c. Compare the total interest paid to what it would have been if the rate had remained at 6.5% p.a. for 8 years.

9. **WE15** Jodie wishes to borrow $30 000 to invest in shares. She uses an interest only loan minimising her repayments and hopes to realise a capital gain when she sells the shares at a higher value. The term of the loan is 5.8% p.a. compounded monthly with monthly repayments equal to the interest charged.
   a. Calculate the monthly interest only repayments.
   b. If, in 3 years, she sells the shares for $35 000, calculate the profit she would make on this investment strategy.

10. Max wishes to borrow $50 000 to invest in shares. He uses an interest only loan minimising his repayments and hopes to realise a capital gain when he sells the shares. The term of the loan is 7.5% p.a. compounded monthly with monthly repayments equal to the interest charged.
   a. Calculate the monthly interest only repayments.
   b. If, in 3 years, he sells the shares for $63 000, calculate the profit he would make on this investment.

11. Harper borrows $2000 to purchase new cricket gear. The loan is to be paid in full over 1 year with quarterly payments at an interest rate of 7.5% p.a.
   a. Calculate the quarterly payment required.
   b. Complete an amortisation table for the loan using a first-order recurrence relation for each payment.
   c. What is the principal outstanding after the third payment?
   d. Calculate the total amount paid on the loan.
   e. Calculate the total interest paid on the loan.

12. Grace takes a loan out for $3750 to purchase new curtains for her house. The loan is to be paid in full over 1 year with bi-monthly payments at a rate of 6% p.a.
   a. Calculate the bi-monthly payment required.
   b. Complete an amortisation table for the loan using a first-order recurrence relation.
   c. If the payments were monthly instead of bi-monthly what payments are required?
   d. Complete an amortisation table for the loan with monthly payments.
   e. Calculate the interest paid on the loan for both monthly and bi-monthly payments and comment on the answers.

13. Phul has a reducing balance loan of $40 000. The loan has interest charged at 8% p.a. (debited before each repayment) and can be repaid by quarterly instalments of $1462.23 over exactly 10 years. The equivalent monthly repayment is $487.41 and the equivalent fortnightly one is $224.96. Find the term of the loan and the amount still owing before the final repayment if repayments are made:
   a. monthly
   b. fortnightly.

14. A loan of $25 000 attracts interest at 8.25% p.a. on the outstanding balance and the following four scenarios are available:
   a. half-yearly repayments of $3101.48 for 5 years
   b. quarterly repayments of $1550.74 for $4\frac{3}{4}$ years with $1217.93 still owing
   c. monthly repayments of $516.91 for 58 months with $512.33 still owing
   d. fortnightly repayments of $238.58 for 127 fortnights with $140.27 still owing.
   Compare the total interest paid if the loan is repaid:
   i. half-yearly
   ii. quarterly
   iii. monthly
   iv. fortnightly.

The following information refers to questions **15** and **16**. Betty has borrowed $65 000 to finance her plant and flower nursery. Betty chooses to repay the loan, which attracts interest at 9.3% p.a. on the outstanding balance, by fortnightly repayments of $309.66 rather than the equivalent monthly repayment of $670.92.

15. MC The term of the loan will be:

A. 14 years 11 months **B.** 14 years 25 fortnights **C.** 15 years
D. 15 years 1 fortnight **E.** 15 years 1 month

16. MC The amount Betty will save is closest to:

A. $240 **B.** $260 **C.** $270
D. $300 **E.** $320

17. A reducing balance loan of $25 000 has been taken out over 5 years at 8% p.a. (adjusted monthly) with monthly repayments of $506.91.

a. What is the total interest paid?

b. If, instead, the rate was 9% p.a. (adjusted monthly) with the same repayments maintained, what would be:

i. the term of the loan now

ii. the total interest paid?

c. If, instead, the rate was 10% p.a. (adjusted monthly) with the same repayments maintained, what would be:

i. the term of the loan now

ii. the total interest paid?

The following information relates to questions **18** and **19**.

Clint's $28 000 loan for his house extensions has interest debited every month at 12% p.a. of the outstanding balance. The loan was due to run for 10 years and he was to make repayments of $401.72 per month to service the loan. After he had made 50 repayments his credit union reduced the interest rate to 10.75% p.a. (adjusted monthly) for the remainder of the loan.

18. MC If Clint maintained the monthly repayment, the term of the loan would be:

A. $9\frac{3}{4}$ years **B.** $10\frac{1}{4}$ years **C.** $10\frac{3}{4}$ years
D. 11 years **E.** 12 years

19. MC The total interest paid by Clint would lie between:

A. $18 700 and $18 800
B. $18 800 and $18 900
C. $18 900 and $19 000
D. $46 800 and $46 900
E. $46 900 and $47 000

20. The Risky brothers want to invest in $140 000 worth of shares. They use other people's money and take out an interest only loan from the bank. The loan is at 10.8% p.a. compounded quarterly with quarterly repayments.

a. Calculate the quarterly repayment amount.

b. If in 1 year they sell the shares for $152 000, calculate the amount of profit or loss they made on this investment strategy.

21. MC The Bigs have had a new addition to the family and John, the father, takes 12 months of leave from work to stay at home. To financially cope, they ask their bank manager for an interest only loan for this period on the outstanding amount on their home loan, which is currently $210 000. If the terms of the interest only loan are 6.79% p.a. compounded fortnightly, the fortnightly repayments will be closest to:

A. $548.42 **B.** $1188.25 **C.** $14 637 **D.** $4879 **E.** $21 032.18

7.5 Annuities and perpetuities

7.5.1 Annuities

An **annuity** is an investment account that at some future date will provide a regular stream of income for a set period of time. Annuities may be in the form of bursaries or scholarships, or retirement funds.

7.5.2 Using a recurrence relation to modelling annuities

A recurrence relation can be used to model an annuity.

$$V_0 = a,\ V_{n+1} = RV_n - d$$

Where,
V_0 is the intial amount borrowed
V_n is the balance of the loan after$n$ payments
$R = 1 + \frac{r}{100}$ (r is the interest rate per compounding period)
d is the payment made.

WORKED EXAMPLE 16

As part of her post graduate studies, Margaret has been awarded a 'writer in residence' scholarship of $8500 for 4 months at a local college. The money is invested in an annuity that pays 4.5% p.a., compounded monthly. Margaret is paid $2140 per month from the annuity.

a. Write a recurrence relation to describe this situation.
b. Use your calculator to determine how much money is left in the annuity at the end of 4 months.
c. What adjustment would need to be made to Margaret's last payment?
d. Complete the amortisation table below for this annuity.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	8500.00
1	2140	**A**	2108.13	6391.88
2	2140	23.97	**B**	4275.84
3	2140	16.03	2123.97	**C**
4	**D**	8.07	2131.93	0.00

THINK

a. 1. State the initial values for V_0, r, R and d.

WRITE

a. $V_0 = 8500$

$r = \frac{4.5}{12} = 0.375$

$R = 1 + \frac{0.375}{100} = 1.003\,75$

$d = 2000$

2. Write the general form of the recurrence relation.
$V_0 = a,\ V_{n+1} = RV_n - d$

3. Substitute in the values for V_0, r, R and d.
$V_0 = 8500,\ V_{n+1} = 1.003\,75V_n - 2140$

b. 1. On a calculator screen, type 8500, press ENTER/EXE and then type $\times 1.003\,75 - 2140$. Press ENTER/EXE 4 times.

8500	8500
$8500 \times 1.003\,75 - 2140$	6391.88
$6391.88 \times 1.003\,75 - 2140$	4275.84
$4275.84 \times 1.003\,75 - 2140$	2151.88
$2151.88 \times 1.003\,75 - 2140$	19.95

2. Write the answer.
After 4 months the balance of the annuity is $19.95.

c. If the balance is greater than zero then the last payment must be adjusted accordingly.
Margaret's last payment should be increased by $19.95.
Last payment = $2159.95

d. A is interest calculated on $8500.
B is principal reduction.
C is balance of the loan.
To calculate D: as there was $19.95 left after 4 payments, the final payment is adjusted by $19.95 to leave the balance of the investment at $0.
$A = 8500 \times 0.003\,75 = 31.88$
$B = 2140 - 23.97 = 2116.03$
$C = 4275.84 - 2123.97 = 2151.87$
$D = 2140 + 19.95 = 2159.95$

7.5.3 Solving annuity problems using the Finance Solver

The Finance Solver application on your CAS calculator can be used to solve annuity problems.

N	The total number of payments
I%	The interest rate per annum
PV	The present value of the investment or loan – this is negative for an annuity
PMT (Pmt)	The periodic payment – this is positive for an annuity providing periodic payments to you
FV	The future value of the loan or investment – this is positive or zero for an annuity providing periodic payments to you
P/Y (Ppy)	The number of payment periods in a year
C/Y (CpY)	The number of compounding periods in a year (nearly always the same as P/Y)

WORKED EXAMPLE 17

Jacob invests $120 000 in an annuity paying 7.5% per month. Using the Finance Solver on your calculator, find:
a. How long will this annuity last if the monthly payment is $4000?
b. How much should Jacob receive each month if he wishes the annuity to last for 20 years?

THINK

a. 1. Using Finance Solver on your CAS enter the appropriate values:

N	?	The number of payments for 10 years at 12 interest calculations per year. **place the cursor in this cell**.
I%	7.5	The annual interest rate
PV	−120 000	The value of the loan
PMT (Pmt)	4000	The value of the payment
FV	0	The investment is paid out so the future value is zero.
P/Y (PpY)	12	There are 12 payments per year.
C/Y (CpY)	12	There are 12compounding periods per year.

2. Press ENTER / Tap solve.

3. Write a statement.

b. 1. Find the payment each month if the annuity lasts for 20 years.

N	240
I%	7.5
PV	−120 000
PMT (PMT)	**? place the cursor in this cell.**
FV	0
P/V (PpY)	12
C/Y (CpY)	12

2. Using Finance Solver, Press ENTER / Tap solve.

3. Write the answer.

WRITE

a. I = 7.5 % p.a.
Amount invested = $120 000
Future value = 0

N = 33.32

The annuity will last 33 months, to the nearest month.

b. i. N = 20 × 12 = 240
I = 7.5% p.a.
Amount of annuity = $12 000

Payment = 966.71

The payment per month is $966.71.

7.5.4 Perpetuities

A **perpetuity** is an annuity where a permanently invested sum of money provides regular payments that continue forever.

Many *scholarships* or *grants* offered to students at universities are provided by funds known as perpetuities.

The funds last for an indefinite period of time as long as the amount paid out is no more than the interest earned on the initial lump sum deposited. The type of investment that is used to earn the interest is usually a bond, which offers a fixed interest amount, paid on a regular basis, over a long period of time. Wealthy people who wish to encourage and support a worthwhile cause usually set up these perpetuities.

The balance of the amount invested does not change and is the same for an indefinite period.

The perpetuity formula is:

$$d = \frac{V_0 r}{100}$$

where
d **= the amount of the regular payment per period (\$)**
V_0 **= the principal (\$)**
r **= the interest rate earned per period (%).**

Notes:

1. The period of the regular payment must be the same as the period of the given interest rate.
2. Finance Solver can be used in calculations involving perpetuities. As the principal does not change, the present value (PV: or negative cash flow) and the future value (FV: or positive cash flow) are entered as the same amount, but with opposite signs.

WORKED EXAMPLE 18

MC Robert wishes to use part of his wealth to set up a scholarship fund to help young students from his town further their education at university. Robert invests \$200 000 in a bond that offers a long-term guaranteed interest rate of 4% p.a. If the interest is calculated once a year, then the annual amount provided as scholarship will be:

A. \$188 000 **B.** \$288 000 **C.** \$666.67
D. \$8000 **E.** \$4000

THINK

1. Write the perpetuity formula.
2. List the values of V_0 and r.
3. Substitute the values into the formula and calculate the amount provided.
4. Select the appropriate answer.

WRITE

$$d = \frac{V_0 r}{100}$$

$V_0 = \$200\,000$ and $r = 4\%$ p.a.

$$d = \frac{\$200\,000 \times 4}{100}$$

$$= 8000$$

The annual amount provided for the scholarship is \$8000. Therefore **D** is the correct answer.

7.5.5 Finding V_0 and r

As was the case with earlier sections in this topic, there are calculations where we need to find the principal (V_0) or interest rate (r) needed to provide a certain regular payment (d). For example, how much needs to be invested at 3% p.a. interest to provide a $10 000 annual grant, or what interest rate is needed so that $100 000 will provide a $4000 yearly scholarship indefinitely? Other calculations involve finding what extra amount could be granted annually as a scholarship if the interest is compounded monthly in each year rather than once a year and if the scholarship is paid in two equal six-monthly instalments.

The perpetuity formula can be transposed to:

$$V_0 = \frac{100 \times d}{r} \quad \text{and} \quad r = \frac{100 \times d}{V_0}$$

If the frequency of the payments each year is not the same as the compounding period of the given interest rate, then Finance Solver is to be used with different values for PpY and CpY.

Notes:

1. The principal must be known to use Finance Solver.
2. Finance Solver gives the interest rate per annum.

WORKED EXAMPLE 19

A Rotary Club has $100 000 to set up a perpetuity as a grant for the local junior sporting clubs. The club invests in bonds that return 5.2% p.a. compounded annually.

a. Find the amount of the annual grant.

b. What interest rate (compounded annually) would be required if the perpetuity is to provide $6000 each year?
The Rotary Club wants to investigate other possible arrangements for the structure of the grant.

c. How much extra would the annual grant amount to if the original interest rate was compounded monthly?

d. What interest rate (compounded monthly) would be required to provide 4 equal payments of $1500 every 3 months? Give your answer correct to 2 decimal places.

THINK

a. 1. Write the perpetuity formula and list the values of V_0 and r.

2. Substitute the values into the formula and find the value of the annual grant.

3. Write a statement.

b. 1. Write the perpetuity formula and list the values of V_0 and d.

WRITE

a. $d = \frac{V_0 r}{100}$

$V_0 = \$100\,000$
$r = 5.2\%$ p.a.

$d = \frac{\$100\,000 \times 5.2}{100}$

$= 5200$

The amount of the annual grant is $5200.

b. $r = \frac{100 \times d}{V_0}$

$V_0 = \$100\,000$ and
$d = \$6000$

2. Substitute the values into the formula and find the interest rate.

$$r = \frac{100 \times 6000}{100\,000}$$
$$= 6$$

3. Write a statement.

For a $100 000 perpetuity to provide $6000 a year, the bond needs to offer an interest rate of 6% p.a.

c. 1. As the frequency of the payment is not the same as the compounding period, the perpetuity formula cannot be used. Use Finance Solver and enter the values as follows.

***n* (N)**	1
***r* (I%)**	5.2
V_0 (PV)	−100 000
***d* (Pmt)**	**? place the cursor in this cell.**
V_n (FV)	100 000
P_pY	1 (one payment per year)
C_pY	12 (there are 12 compound periods per year)

2. Solve for Pmt.

Pmt = 5325.741 057 054
If the interest was compounded monthly, the annual grant would amount to $5325.74.

3. Compare the sizes of the 2 grants and write a statement.

The extra amount is $5325.74 − $5200 = $125.74.
If the interest is compounded monthly, the annual grant would increase by $125.74.

d. 1. As the frequency of the payment is not the same as the compounding period, the perpetuity formula cannot be used. Use Finance Solver and enter the values as follows.

d.

***n* (N)**	1
***r* (I %)**	**? place the cursor in this cell.**
V_0 (PV)	−100 000
***d* (Pmt)**	1500
V_n (FV)	100 000
P_pY	4 (four payment per year)
C_pY	12 (there are 12 compound periods per year)

2. Solve for I to find the required interest rate.	$I = 5.970\,247\,527\,183$ $r = 5.97$
3. Write a statement.	An interest rate of 5.97% p.a. compounded annually is needed to provide four equal payments of $1500, correct to 2 decimal places.

WORKED EXAMPLE 20

A benefactor of a college has been approached to provide a Year 7 scholarship of $1000 per term. He is able to get a financial institution to offer a long-term interest rate of 8% per annum. What is the principal that needs to be invested?

THINK	WRITE
1. Write the perpetuity formula and list the values of d and r. Both d and r need to be expressed in the same period of time.	$V_0 = \frac{100 \times d}{r}$ $d = \$1000$ per term (4 terms per year) $R = 8\%$ p.a. $= \frac{8}{4}$ $= 2\%$ per term
2. Substitute the values into the formula and find the value of the annual grant.	$V_0 = \frac{100 \times 1000}{2}$ $= 50\,000$
3. Write a statement.	The principal that needs to be invested to provide a scholarship of $1000 per term at an annual interest rate of 8% is $50 000.

Note that Finance Solver cannot be used in Worked example 20 as the principal is not known. (Both PV and FV are unknowns.)

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Unit 3 > AOS R&FM > Topic 2 > Concept 8 **Perpetuity** Summary screen and practice questions

Exercise 7.5 Annuities and perpetuities

1. WE16 Abdo has been granted a small scholarship of $12 500 to attend a sporting institute for one year. The money is invested in an annuity that pays 6% p.a., compounded quarterly. Abdo is paid $3243 per quarter from the annuity.
   a. Write a recurrence relation to describe this situation.
   b. Use your calculator to determine how much money is left in the annuity at the end of the year.
   c. What adjustment would need to be made to Abdo's last payment?
   d. Complete the amortisation table below for this annuity.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	12 500
1	3243	**A**	3055.50	9444.50
2	3243	141.67	**B**	6343.17
3	3243	95.15	3147.85	**C**
4	**D**	47.93	3195.07	0

2. MC Maya has decided to study full time for 5 months. She has managed to save $20 000 and deposits this in an annuity fund, that pays 5.5% p.a. compounded monthly. Her payment from this fund each month is $3500. At the end of 5 months, the money left in her account will be:
   A. $5843.76 **B.** $4213.23 **C.** $3156.98
   D. $2801.38 **E.** $0
3. WE17 Alice invests $160 000 in an annuity paying 9.5% per month.. Using the Finance Solver on your calculator, find:
   a. how long this annuity will last if the monthly payment is $6000
   b. how much Alice should receive each month if she wishes the annuity to last for 30 years.
4. MC Genevieve wants to invest her inheritance from her grandmother of $100 000 in an annuity that offers an interest rate of 4.5% p.a. compounded quarterly. If she wishes to take a quarterly payment of $3000, this annuity will last for:
   A. 42 months **B.** 54 months **C.** 33 months
   D. 26 months **E.** 13 months
5. WE18 MC Chris wants to invest $150 000 in a bond that offers a long-term guaranteed interest rate of 5% p.a. If the interest is calculated once a year, then the annual interest earned is:
   A. $142 500 **B.** $7500 **C.** $130 000
   D. $6500 **E.** $8000
6. MC Freda chose to invest $95 000 in a bond that offers a long-term guaranteed interest rate of 3.8% p.a. If the interest is calculated once a year, then the annual interest earned is:
   A. $25 000 **B.** $4845 **C.** $91 390
   D. $3610 **E.** $5430
7. WE19 A charity has $75 000 to set up a perpetuity as a grant to help homeless people. The charity invests in bonds that return 4.8% p.a. compounded annually.
   a. Find the amount of the annual grant.
   b. What interest rate (compounded annually) would be required if the perpetuity is to provide $4800 each year?

The charity wants to investigate other possible arrangements for the structure of the grant.

c. How much extra would the annual grant amount to if the original interest rate was compounded monthly?

d. What interest rate (compounded monthly) would be required to provide 4 equal payments of $1200 every 3 months? Give your answer correct to 2 decimal places.

8. A family wants to use $125 000 to set up a perpetuity as a scholarship fund at their old school. The charity invests in bonds that return 5.7% p.a. compounded annually.

a. Find the amount of the annual grant.

b. What interest rate (compounded annually) would be required if the perpetuity is to provide $8000 each year?

The charity wants to investigate other possible arrangements for the structure of the grant.

c. How much extra would the annual grant amount to if the original interest rate was compounded monthly?

d. What interest rate (compounded monthly) would be required to provide 4 equal payments of $2000 every 3 months? Give your answer correct to 2 decimal places.

9. WE20 A benefactor of a school had been approached to provide a Year 12 scholarship of $5000 per term. She is able to get a financial institution to offer a long-term interest rate of 6.25% per annum. What is the principal that needs to be invested?

10. A benefactor of a college had been approached to provide a Year 11 scholarship of $2000 per term. He is able to get a financial institution to offer a long-term interest rate of 7.5% per annum. What is the principal that needs to be invested?

11. MC The owner of a technology company uses part of his wealth to set up research grants to help young Australian scientists with their endeavours. He invests $350 000 in a bond that offers a long-term guarantee of 5% p.a. If the interest is calculated once a year, then the annual amount provided as a research grant will be:

A. $70 000 **B.** $17 500 **C.** $1750
D. $7000 **E.** $3500

12. Use the perpetuity formula to calculate the annual payment as specified in each of the following situations.

a. $400 000 invested at 4% p.a., paid once a year

b. $300 000 invested at 1% per quarter, paid 4 times each year

13. Use the perpetuity formula to calculate the annual payment as specified in each of the following situations.

a. $100 000 invested at 12% p.a., calculated monthly, paid out monthly

b. $2 million invested at 6% p.a., compounded quarterly, paid out every 3 months

14. Check your answers to questions **12** and **13** using Finance Solver on your CAS.

15. An AFL club has $80 000 to set up a perpetuity as a grant for the local senior sporting clubs. The club invests in bonds that return 4% p.a. compounded annually.

a. Find the amount of the annual grant.

b. What interest rate (compounded annually) would be required if the perpetuity is to provide $5000 each year?

The club wishes to investigate other possible arrangements for the structure of the grant.

c. How much extra would the annual grant amount to if the original interest rate was compounded quarterly?

d. What interest rate (compounded monthly) would be required to provide 12 equal monthly payments of $400?

16. Prachi invested $15 000 in a fund that will earn on average 10.5% p.a. over a 1 year period with interest calculated monthly. On top of the initial investment Prachi contributes $250 at the start of each month after the initial investment. Complete the table to find the value of his investment at the end of the sixth month.

Time period (n)	Principal ($)	Interest earned ($)	Balance ($)
1	15 000		
2			
3			
4			
5			
6			

17. Use the perpetuity formula to calculate the interest rate (p.a.) required for each of the following perpetuities.

a. $400 000 provides $5000 per annum with interest compounded annually.

b. Half a million dollars provides $1000 each month with interest compounded monthly.

c. $800 000 provides $30 000 every six months with interest compounded biannually.

d. $100 000 provides $200 per fortnight with interest calculated fortnightly.

18. Check your answers to question **17** using Finance Solver on your CAS.

19. Use Finance Solver on your CAS to calculate the interest rate required for each of the following perpetuities. Give your answers correct to 2 decimal places.

a. $400 000 provides $5000 per annum with interest compounded monthly.

b. Half a million dollars provides $1000 each month with interest compounded annually.

c. $800 000 provides $30 000 every six months with interest compounded quarterly.

d. $100 000 provides $200 per fortnight with interest compounded monthly.

20. A benefactor of a college has been approached to provide a Year 9 scholarship of $200 per month. He is able to get a financial institution to offer a long-term interest rate of 3.6% per annum, compounded monthly. What is the principal that needs to be invested?

21. MC The total amount given by the perpetuity in question **20** over a 50-year period is:

A. $10 000 **B.** $2400 **C.** $57 000
D. $66 700 **E.** $120 000

22. Use the perpetuity formula to calculate the initial sum to be invested in a perpetuity specified as follows:

a. A $1200 per annum grant from a fund offering 6% p.a. compounded annually

b. A $10 000 per annum grant from a fund offering 4.5% p.a. compounded annually

c. A $300 per month scholarship from a fund offering 0.5% per month

d. A $120 per month grant from a fund offering 3% p.a. compounded monthly.

7.6 Annuity investments

A savings plan, like a Christmas Club account, is an investment where an initial sum as well as regular deposits are made. The interest earned is calculated regularly on the balance of the investment, which increases with each regular deposit (annuity). This is similar to reducing balance loans with the main difference being that it is an investment rather than a loan, so the principal amount is *growing*.

An **annuity investment** is an investment that has regular deposits made over a period of time. The annuity investment can be modelled using a recurrence relation.

$$V_0 = a,\ V_{n+1} = RV_n + d$$

Where:

V_0 = **initial investment**

V_{n+1} = **amount after** $n + 1$ **payments**

V_n = **amount at time** n

$R = 1 + \frac{r}{100}$

r = **interest rate per period**

d = **deposit amount**

Notice that the only difference to the recurrence relation for a reducing balance loan is that d is positive rather than negative.

WORKED EXAMPLE 21

An initial amount of $1000 was invested in an account for 5 years at a rate of 5.04% p.a. (interest calculated monthly), and an additional deposit of $100 was made each month. Use a recurrence relation to determine the balance of the investment at the end of 5 months.

THINK

1. State the initial values for V_0, r, and d.

WRITE

$V_0 = 1000$

$r = \frac{5.04}{12} = 0.42$

$R = 1 + \frac{0.42}{100} = 1.0042$

$d = 100$

2. Using the iterative process on your calculator, type 1000 and then press ENTER/EXE. Type $\times$ 1.0042 + 100 and press ENTER/EXE five times.

1000	1000
1000 $\times$ 1.0042 + 100	1104.20
1104.20 $\times$ 1.0042 + 100	1208.84
1208.84 $\times$ 1.0042 + 100	1313.91
1313.91 $\times$ 1.0042 + 100	1419.43
1419.43 $\times$ 1.0042 + 100	1525.39

3. Write the answer.

After 5 months the balance of the investment will be $1525.39.

An amortisation table can be completed for the annuity investment example in Worked example 21.

Time period (n)	Payments ($)	Interest ($)	Principal increase ($)	Balance of investment ($)
0	0	0	0	1000
1	100	$1000 \times 0.0042 = 4.2$	104.20	1104.20
2	100	$1104.20 \times 0.0042 = 4.64$	104.64	1208.84
3	100	$1208.84 \times 0.0042 = 5.08$	105.08	1313.91
4	100	$1313.92 \times 0.0042 = 5.52$	105.52	1419.43
5	100	$1419.43 \times 0.0042 = 5.96$	105.96	1525.39

The Finance Solver application on your CAS calculator can be used to solve annuity investment problems.

N	The total number of payments
I%	The interest rate per annum
PV	The present value of the investment or loan – this is negative for an annuity investment
PMT(Pmt)	The periodic payment to the bank – this is negative for an annuity investment
FV	The future value of the investment – this is positive for an annuity investment
P/Y(PpY)	The number of payment periods in a year
C/Y(CpY)	The number of compounding periods in a year (nearly always the same as P/Y)

WORKED EXAMPLE 22

Helen currently has $2000 in a savings account that is averaging an interest rate of 8% p.a. compounded annually. She wants to calculate the amount that she will receive in 5 years when she plans to go on an overseas trip.

a. If she deposits $6000 each year, find (correct to the nearest $1000) the amount available for her overseas trip.

b. If she places her $2000 and increases her deposits to $7000 each year into a different savings account that can offer 9% p.a. compounded annually, find (correct to the nearest $1000) the amount available for her overseas trip.

c. Calculate the extra amount saved by investing $7000 each year at 9% p.a. compared with $6000 each year at 8% p.a.

THINK	WRITE
a. 1. Use Finance Solver and enter the values as follows.	**a.** N = 5 r = 8% Money invested = $2000 Payment per year = $6000

N	5
I(%)	8
PV	−2000
Pmt	−6000
FV	? **place the cursor in this cell.**
PpY	1(one payment per year)
CpY	1

2. Solve for FV.	FV = 38 138.26
3. Write a statement, rounding the answer correctly.	In 5 years time the investment will be $38 000, correct to the nearest 1000.
b. 1. Use Finance Solver and enter the values as follows.	**b.** N = 5 r = 9% Money invested = $2000 Payment per year = $7000

N	5
I%	9
PV	−2000
Pmt	−7000
FV	? **place the cursor in this cell.**
PpY	1(one payment per year)
CpY	1

2. Solve for FV.	The future value of the investment is $44 970.22.
3. Write a statement, rounding the answer correctly.	The final balance of the investment after 5 years, if Helen deposits $7000 each year into an account offering 9% p.a., would be $45 000, correct to the nearest $1000.
c. The extra amount saved is the difference between the amounts found in parts **a** and **b**.	**c.** The extra amount is $45 000 − $38 000 = $7000.

7.6.1 Planning for retirement

When do you want to retire, and how many years will you want to spend in retirement? How much money will you need? An Australian male is now expected to live to 80 years of age and an Australian female to 84 years of age. So, most people will need to plan on 20 to 25 years in retirement. A common retirement income target is 60 to 65 per cent of your pre-retirement income, bearing in mind that you won't have to pay for expenses such as commuting, work clothes and, hopefully, mortgage repayments by then. Furthermore, you may be entitled to a part pension and/or tax concessions. So, if you earn $60 000 a year now, you may think of a retirement income of about $36 000 a year. Remember, these figures are in today's dollars.

Planning for retirement is an issue that you'll need to revise regularly, maybe with a financial planner. The annuities formula and Finance Solver can be used to calculate how much money is needed under different financial situations.

WORKED EXAMPLE 23

Andrew is aged 45 and is planning to retire at 65 years of age. He estimates that he needs $480 000 to provide for his retirement. His current superannuation fund has a balance of $60 000 and is delivering 7% p.a. compounded monthly.

a. Find the monthly contributions needed to meet the retirement lump sum target.

b. If in the final ten years before retirement, Andrew doubles his monthly contribution calculated from a, find the new lump sum amount available for retirement.

c. How much extra could Andrew expect if the interest rate from part b is increased to 9% p.a. (for the final 10 years) compounded monthly? Round the answer correct to the nearest $1000.

THINK

a. 1. Use Finance Solver and enter the values as follows.

N	240
I%	7
PV	−60 000
Pmt	? **place the cursor in this cell.**
FV	480 000
PpY	12
CpY	12

2. Solve for Pmt.

3. Write a statement.

b. 1. We need to find the balance after the first 10 years with $d = \$456.26$ and $N = 120$. Enter the values into the Finance Solver and solve for FV.

N	120
I(%)	9
PV	−2000
Pmt	−7000
FV	? **place the cursor in this cell.**
PpY	1(one payment per year)
CpY	1

WRITE

a. $N = 20 \times 12 = 240$
$r = 7\%$
Money invested = $60 000
Future value = $480 000

Payment = $456.26

The monthly contribution to achieve a retirement lump sum of $480 000 is $456.26.

b. $N = 10 \times 12 = 120$
$r = 7\%$
Money invested = $60 000
Monthy contributions = $456.26

2. Solve for FV.

After 10 years, the balance of the investment is $199 550.58.

3. We need to find the balance for the second ten years, with double monthly contributions. Enter the values into the Finance Solver and solve for FV.

N	120
I%	7
PV	−199 550.58
Pmt	−912.52
FV	? **place the cursor in this cell.**
PpY	12
CPY	12

Balance after 10 years is $199 550.58.
$N = 10 \times 12 = 120$
$r = 7\%$
Monthly contributions $= 456.26 \times 2 = \$912.52$
Money invested $= \$199\,550.58$

4. Solve for FV.

The future value $= \$558\,972.44$

5. Write a statement.

The new lump sum will be $= \$558\,972.44$

c. 1. Enter the values into the Finance solver and solve for FV.

N	120
I%	9
PV	−199 550.58
Pmt	−912.52
FV	**? place the cursor in this cell.**
PpY	12
CpY	12

c. Balance after 10 years is $199 550.58.
$N = 10 \times 12 = 120$
Money contributed $= 456.26 \times 2 = \$912.52$
The interest rate is 9%.

2. Solve for FV.

The future value $= \$665\,755.37$

3. Calculate the difference in answers.

The difference is:
$665\,755.37 - 558\,972.44 = 106\,782.93$

4. Write a statement, rounding the answer appropriately.

If the interest rate is increased to 9% for the final 10 years, Andrew could expect an extra $107 000, correct to the nearest $1000.

Exercise 7.6 Annuity investments

1. **WE21** Clifford invested $7000 in a managed fund that will earn on average 10% p.a. over a 3-year period with interest calculated monthly. Clifford contributes $150 monthly to his investment.
   a. Use a recurrence relation to find the value of his investment at the end of the fifth month.
   b. Complete the table on the next page by calculating the values of A, B, C and D to show the value of the investment at the end of each of the first 5 months.

Time period (n)	Payments ($)	Interest ($)	Principal increase ($)	Balance of investment ($)
0	0	0	0	7000
1	150	A	208.10	7208.10
2	150	59.83	B	7417.93
3	150	61.57	211.57	7629.50
4	150	C	213.32	7842.82
5	150	65.10	215.01	D

2. Pauline invested $10 000 in a fund that will earn on average 12% p.a. over a 1 year period with interest calculated monthly. If Pauline contributes $200 per month to her initial investment,
   a. Express this situation in the form of a recurrence relation.
   b. find the value of her investment at the end of the sixth month.
3. An initial deposit of $5000 was made on an investment taken out over 10 years at a rate of 7.5% p.a. (interest calculated monthly) and an additional deposit of $100 is made each month. Complete the table below for the first three deposits and hence calculate how much interest had been earned over this time.

period (n)	Payments ($)	Interest ($)	Principal increase ($)	Balance of investment ($)
0	0	0	0	5000
1	100			
2	100			
3	100			

4. An initial deposit of $10 000 was made on an investment taken out over 7 years at a rate of 8% p.a. (interest calculated monthly) and an additional deposit of $150 is made each month. Complete the last row of the table below and hence the value of the investment at the end of the fourth month.

Time period ($)	Payments ($)	Interest ($)	Principal increase ($)	Balance of investment ($)
0	0	0	0	10 000
1	150	66.67	216.67	10 216.67
2	150	68.04	218.04	10434.71
3	150	69.58	219.58	10654.27
4				

5. WE22 Sharyn has $2500 in a savings account that is averaging an interest rate of 6.5% p.a. compounded annually. She wants to calculate the amount she will receive in 5 years when she plans to buy a car.
   a. If she deposits $6500 each year, find (correct to the nearest $1000) the amount available for her car.
   b. If she places her $2500 and deposits $8000 each year into a different savings account that can offer 7% p.a. compounded annually, find (correct to the nearest $1000) the amount available for her car.

c. Calculate the extra amount saved by investing $8000 each year at 7.5% p.a. compared with $6500 each year at 6.5% p.a.

6. Rhonda has $4000 in a savings account that is averaging an interest rate of 5.5% p.a. compounded annually. She wants to calculate the amount she will receive in 5 years when she plans to go on a holiday.
   a. If she deposits $6000 each year, find (correct to the nearest $1000) the amount available for her holiday.
   b. If she places her $4000 and deposits $8000 each year into a different savings account that can offer 6.5% p.a. compounded annually, find (correct to the nearest $1000) the amount available for her holiday.
   c. Calculate the extra amount saved by investing $8000 each year at 6.5% p.a. compared with $6000 each year at 5.5% p.a.
7. An initial deposit of $20 000 was made on an investment taken out over 3 years at a rate of 6% p.a. (interest calculated monthly), and an additional deposit of $200 is made each month. Complete the table below for the first five deposits and calculate the amount of interest that was earned:
   a. in the second month
   b. in the fifth month
   c. over the first five months.

Time period (n)	Payments ($)	Interest ($)	Principal increase ($)	Balance of investment ($)
0	0	0	0	20 000
1	200			
2	200			
3	200			
4	200			
5	200			

8. Barbara currently has $60 000 in an investment account that is averaging an interest rate of 6% p.a., compounded annually. She wants to calculate the amount that she will receive after 20 years.
   a. If she deposits $9000 each year, find (correct to the nearest $1000) the amount available for her after 20 years.
   b. If she places her $60 000 and increases her deposits to $10 000 each year into a different savings account that can offer 8% p.a. compounded annually, find (correct to the nearest $1000) the amount available for her after 20 years.
   c. Calculate the extra amount saved by investing $10 000 each year at 8% p.a. compared with $9000 each year at 6% p.a.
9. WE23 Peter is 48 and is planning to retire at 65 years of age. He estimates that he needs $550 000 to retire at 65. His current superannuation fund has a balance of $55 000 and is delivering 8% p.a. compounded monthly.
   a. Find the monthly contributions needed to meet the retirement lump sum target.
   b. If in the final ten years before retirement Peter doubles his monthly contribution calculated from a, find the new lump sum amount available for retirement.
   c. How much extra could Peter expect if the interest rate from part b is increased to 10% p.a. (for the final 10 years) compounded monthly? Round the answer correct to the nearest $1000.
10. Patricia is 54 and is planning to retire at 65 years of age. She estimates that she needs $600 000 to retire at 65. Her current superannuation fund has a balance of $115 000 and is delivering 9.5% p.a. compounded monthly.
    a. Find the monthly contributions needed to meet the retirement lump sum target.

b. If in the final five years before retirement Patricia doubles her monthly contribution calculated from **a**, find the new lump sum amount available for retirement.

c. How much extra could Patricia expect if the interest rate from part **b** is increased to 11% p.a. (for the final 5 years) compounded monthly? Round the answer correct to the nearest $1000.

11. Justin is aged 38 and is planning to retire at 60 years of age. He estimates that he needs $680 000 to provide for his retirement. His current superannuation fund has a balance of $40 000 and is delivering 5% p.a. compounded monthly.

a. Find the monthly contributions needed to meet the retirement lump sum target.

b. If in the final ten years before retirement, Justin doubles his monthly contribution calculated from **a**, find the new lump sum amount target for his retirement.

c. How much extra could Justin expect if the interest rate from part **b** is increased to 8% p.a. compounded monthly (for the final 10 years)? Round the answer correct to the nearest $1000.

12. Mr Rookie is aged 25 and is planning to retire at 55 years of age. He estimates that he needs $880 000 to provide for his retirement. His current superannuation fund has a balance of $600 and is delivering 7% p.a. compounded monthly.

a. Find the monthly contributions needed to meet the retirement lump sum target.

b. If in the final ten years before retirement, Mr Rookie adds a further $300 to his monthly contribution calculated from **a**, calculate the new lump sum for retirement.

7.7 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC A loan of $14 000 is taken out over 4 years at 9.75% p.a. (debited fortnightly) on the outstanding balance. The fortnightly repayment needed to repay the loan in full, to the nearest dollar, is:
   A. $135 **B.** $145 **C.** $163 **D.** $170 **E.** $319
2. MC Rachel repaid a reducing balance loan of $22 000 in 5 years by quarterly repayments and with interest charged quarterly at 8.2% p.a. on the outstanding balance. The total amount of interest she paid at the end of the first quarter is closest to:
   A. $1804 **B.** $451 **C.** $150 **D.** $9000 **E.** $10 000
3. MC The number of monthly repayments required to repay a $41 000 reducing balance loan in full, if the repayments are $588.39 and interest is debited monthly at 10.5% p.a., will be closest to:
   A. 500 **B.** 600 **C.** 90 **D.** 100 **E.** 110
4. MC A reducing balance loan of $56 000 is repaid by quarterly instalments of $1332.24 over 15 years at an interest rate of 5% p.a. (adjusted quarterly). If, instead, repayments of $1500 per quarter were made throughout the loan (other variables remaining unchanged), the term of the loan would be:
   A. 15 years **B.** 14 years **C.** $12\frac{3}{4}$ years **D.** $12\frac{1}{2}$ years **E.** 12 years

Questions 5 to 7 refer to the following information. A reducing balance loan of $24 000 attracting interest at 6.5% p.a. can be repaid over 5 years by either quarterly repayments or fortnightly repayments.

5. MC The fortnightly repayment value is between:
   A. $120 and $140 **B.** $140 and $160 **C.** $160 and $180
   D. $180 and $200 **E.** $200 and $220
6. MC The quarterly repayment value is between:
   A. $1200 and $1220 **B.** $1250 and $1300 **C.** $1350 and $1360
   D. $1400 and $1420 **E.** $1460 and $1470
7. MC If, instead, a rival institution offered a rate of 5.5% p.a., the quarterly repayment value that would enable the loan to be repaid in full in the same time would be between:
   A. $1350 and $1375 **B.** $1375 and $1400 **C.** $1400 and $1425
   D. $1425 and $1450 **E.** $1450 and $1500
8. MC Elena has $2000 to invest in her new superannuation fund that is paying interest of 6.0% p.a. compounded monthly with $360 monthly contributions from her employer. Which of the following recurrence relations could be used to model this situation?
   A. $V_{n+1} = 1.5V_n + 360, V_0 = 2000$ **B.** $V_{n+1} = 1.005V_n - 360, V_0 = 2000$
   C. $V_{n+1} = 1.5V_n - 360, V_0 = 2000$ **D.** $V_{n+1} = 1.005V_n + 360, V_0 = 2000$
   E. $V_{n+1} = 1.05V_n + 360, V_0 = 2000$
9. MC Kerry Green borrows $305 000 to invest in an apartment. She wishes to use the bank's money to purchase the property. If the terms of an interest only loan are 6.47% p.a. compounded monthly, the monthly repayment is closest to:
   A. $1644.45 **B.** $19 733.50 **C.** $30 611.62
   D. $1644.46 **E.** none of the above
10. MC Claire is aged 48 and is planning to retire at 65. Her annual salary is $60 000 and her employer contributions are 10% of her gross monthly income. The superannuation fund has been returning an interest rate of 9.6% p.a. compounded monthly. Claire's current balance is $92 200 which she wants to grow to $800 000. The extra amount that Claire will have to contribute each month to ensure this final payout is achieved is closest to:
    A. $0 **B.** $1990 **C.** $650 **D.** $150 **E.** $240

Short answer

Questions 1 to 3 refer to the following information. Helmut and Su-Li want to buy a boat, so they borrow $70 000 at 7% p.a. (adjusted monthly) on the balance outstanding and agree to repay the loan over 20 years with instalments of $542.71 per month.

1. Find the total amount of interest to be paid.
2. Determine the amount still owing after 10 years.
3. If the repayment value is increased to $600 per month after 10 years, calculate the term of the loan.
4. Frieda is repaying a $55 000 housing loan with interest calculated quarterly at 7% p.a. of the amount outstanding. Quarterly repayments of $1487.93 are being made to service the loan. Twenty repayments have already been made. Frieda decides to change her repayments to $250 per fortnight. The bank responds by adjusting interest fortnightly.
   Calculate the difference that these changes would make to the overall term of the loan.
5. Bert and Ernie have just finished repaying a 5-year, $8000 reducing balance loan, which they required to re-carpet their home. During the first year interest was debited monthly at 8.5% p.a.; during the next 2 years the rate fell to 8% p.a. but rose to 8.3% p.a. for the remaining period of the loan. Before signing their contract Bert and Ernie were also given the option of fixing the interest rate at 8.25% p.a. (debited monthly) for the term of the loan. By considering the total amount of interest that they paid, determine whether Bert and Ernie made the right decision in choosing the *variable* interest rate for the loan rather than the *fixed* rate.
6. The actors' guild provides a donation of 1.2 million dollars to be set up as a perpetuity that earns 7.2% p.a. in interest, compounded annually.
   a. Find the amount of the annual grant.
   b. What interest rate compounded monthly would provide the same annual grant?
7. Use the Finance Solver to calculate the unknown value in each of the following annuity investments.
   a. A superannuation fund is to grow from $40 000 to $400 000 in 10 years where the interest rate is expected to be 7.7% p.a. compounded quarterly. What would the quarterly contributions need to be to achieve this growth?
   b. A superannuation fund's performance is often measured by the interest rate returned on superannuation investments. What is the required interest rate, compounded monthly, if a current balance of $40 000 is to mature to $400 000, if there is 20 years to go with monthly contributions of $200 per month?

Extended response

1. Alisha has saved $10 000 and has several options when she turns 18. She chooses to investigate these options. Answer the following recalling simple and compound interest formulas from your previous studies.
   a. Option 1 is for her to use the sum of $10 000 towards a new 4WD car valued at $27 000. The car dealership will accept $10 000 as a deposit. The remaining amount can be borrowed from the dealer at 6% p.a (simple interest) and is to be repaid in 4 years' time.
      Calculate the total cost of the car.

   b. Another option is to buy a second-hand car for $12 000. Alisha can borrow the extra $2000 at 7% p.a. compounded annually for 3 years with a single payment at the end of the term.
      i. Find the amount of interest Alisha will have to pay at the end of the 3 years for the $2000 loan.
      ii. What will be the total cost of the car?

c. The third option is to buy a reliable car for $5000 and use the other $5000 to buy a collectable painting, the value of which will increase with time.
   i. The value of the painting is expected to increase at a rate of 5% p.a. What will be the value of the painting in 4 years' time?
   ii. The $5000 car is expected to depreciate at 10% p.a. using the straight line depreciation model. What is its book value in 4 years? (Straight line depreciation means the item decreases in value at a fixed rate each year.)

d. The other 2 cars, being up-market models, depreciate at 15% p.a. on reducing value.
   i. What will be the value of the investment in 4 years for the $27 000 4WD car?
   ii. What will be the value of the investment in 4 years for the $12 000 car?
   iii. Find which of the three options loses the most money.

2. Complete question 1 and use the information to fill in a copy of the following table.

Description	$27 000 4WD car	$12 000 car	$5000 car and painting
Cost of loan (interest charged only)			
Depreciation of goods			
Total cost			
Less benefits (for example, increase in value)			
Total cost			

3. Nathan borrowed $6000 towards purchasing a motorbike. The reducing balance loan is to be repaid in full over 2 years with quarterly repayments. The interest rate is 4.5%.
   a. Calculate the quarterly repayments required.
   b. Use first-order recurrence relation to complete an amortisation table for the loan.
   c. Calculate the total amount paid on the loan.
   d. Calculate the total interest paid on the loan.
   e. Calculate the effective annual interest rate correct to 2 decimal places.

studyon

Unit 3&4 Sit exam

Answers

Topic 7 Reducing balance loans, annuities and perpetuities

Exercise 7.2 Modelling reducing balance loans with recurrence relations

1. a. $V_0 = 565,\ V_{n+1} = 1.015V_n + 50$
   b. 743 students
   c. 59 new students
2. a. $A_0 = 3.5\,\text{m}^3$
   b. $4.15\,\text{m}^3$
   c. 3 years
3. D
4. a. $V_0 = 15\,000,\ V_{n+1} = 1.0625V_n - 3074.44$
   b. 6
5. a. $V_0 = 1500,\ V_{n+1} = 1.004\,583V_n + 200$
   b. $2543.90
6. a. 0.75% b. $18 c. $974.44
   d. $482.33 e. $54.36 f. $6.16
7. For the fourth payment
   Interest for the fourth payment = $112.42
   Principal reduction = $158.90
   Balance of loan = $24 368.72
8. Interest = $580.15
   Balance at end of 3 payments $8994.02
9. a. $V_0 = 23\,000,\ V_{n+1} = 1.00\,542 \times V_n - 341.54$
   b. $22 345.80
10. a. $A_0 = 2000,\ A_{n+1} = 1.01A_n - 261$
    b. $1021.44
    c. $3.15
11. a. $15 763.24 b. $15 599.05 c. $15 405.32
12. A
13. E

Exercise 7.3 Solving problems involving reducing balance loans using a Finance Solver

1. a. $59 633.49 b. $49 884.16 c. $32 172.59
2. a. $46 102.98 b. $36 196.88 c. $19 556.12
3. a. $156.39 b. $253.36
4. a. $106.69 b. $160.56
5. a. $365.84
   b. i. $259.59, $106.25 ii. $339.75, $26.09
6. a. $190.83
   b. i. $153.09, $37.74 ii. $181.36, $9.47
7. a. $n = 18,\ 1\frac{1}{2}$ years b. $214.30
8. $n \approx 8$ months
9. $n \approx 15$ months
10. 16 months
11. a. $n \approx 16$ months b. $n \approx 22$ months
12. a. $n \approx 22$ months b. $n \approx 28$ months
13. a. $363.25 b. $n \approx 55$ months
    c. $3477.50 d. $317.50
14. a. $225.49 b. $n \approx 118$ fortnights
    c. $3910 d. $403.70
15. a. $n \approx 32$ quarters (8 years)
    b. $6066.24
    c. $83.94
16. a. $6\frac{3}{4}$ years b. $4259.02 c. $190.26
17. a. $n = 7,\ 1\frac{3}{4}$ years b. $529.03
18. a. $n = 24,\ 2$ years b. 5 years
19. a. $n = 36,\ 3$ years b. 5 years
20. D
21. a. i. 2 years ii. $3\frac{1}{2}$ years
    b. i. $\frac{3}{4}$ year ii. $1\frac{1}{4}$ years
22. a. $384.65
    b. i. 10 years
    ii. 13 years
    iii. 23 years
    iv. $24\frac{1}{2}$ years

	i	ii	iii	iv
c.	$19 356.40	$25 867.48	$49 870.60	$53 752.60
d.	$22 959.60 less	$16 448.52 less	$7554.60 more	$11436.60 more

23.

	i	ii	iii
a.	9 years	$9433.28	$585.92
b.	8 years	$8861.68	$1157.52

24. D
25. E
26. a. $21 164.60 b. $441.40

Exercise 7.4 The effect of rate and repayment changes on reducing balance loans

1. a. i. Term of loan = 5 years
   ii. $217.37
   b. i. Term of loan = 4 years 25 fortnights
   ii. $149.63
2. a. i. Term of loan = 6 years
   ii. $246.17
   b. i. Term of loan = 5 years 25 fortnights
   ii. $164.36
3. a. $4176 b. $4025.27 c. $3984.35
4. a. $6008.48 b. $5825.22 c. $5775.40
5. a. $5496.40
   b. i. Term = 5 years and 2 months
   ii. $6294.91
6. a. $2834.88
   b. i. Term = 4 years and 2 months
   ii. $3286.88
7. a. $22 125.28 b. $10\frac{1}{4}$ years c. $1070.37
8. a. $19 156.45
   b. Term = 8 years + 2 months
   c. $660.18 more
9. a. $145
   b. Jodie made a loss of $220.

10. **a.** $312.50 **b.** Profit = $1750

11. **a.** $523.66
b. See table at the foot of the page*.
c. $514.00
d. $2094.62
e. $94.62

12. **a.** $647.06
b. See table at the foot of the page*.
c. $322.75
d. See table at the foot of the page*.
e. Monthly: $122.99, Bi-monthly: $132.34
The interest paid on the monthly payments is almost $10 less than the interest paid on the bi-monthly payments.

13. **a.** 10 years, $A_{119} = \$102.61$
b. 9 years 24 fortnights, $A_{257} = \$185.59$

14. See table at the foot of the page*.

15. C

16. B

17. **a.** $5414.60
b. **i.** 5 years 2 months **ii.** $6333.26
c. **i.** 5 years 4 months **ii.** $7331.64

18. A

19. C

20. **a.** $3780 per quarter **b.** $3120 loss

21. A

---

*11. b.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0	0	0	2000
1	523.66	37.50	486.16	1513.84
2	523.66	28.38	495.28	1018.56
3	523.66	19.10	504.56	514.00
4	523.66	9.64	514.02	−0.02

*12. b.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0	0	0	3750
1	647.06	37.50	609.56	3140.44
2	647.06	31.40	615.66	2524.78
3	647.06	25.25	621.81	1902.97
4	647.06	19.03	628.03	1274.94
5	647.06	12.75	634.31	640.63
6	647.06	6.41	640.65	−0.02

*12. d.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0	0	0	3750
1	322.75	18.75	304	3446.00
2	322.75	17.23	305.52	3140.48
3	322.75	15.70	307.05	2833.43
4	322.75	14.17	308.58	2524.85
5	322.75	12.62	310.13	2214.71
6	322.75	11.07	311.68	1903.05
7	322.75	9.52	313.23	1589.81
8	322.75	7.95	314.80	1275.01
9	322.75	6.38	316.37	958.64
10	322.75	4.79	317.96	640.68
11	322.75	3.20	319.55	321.13
12	322.75	1.61	321.14	−0.01

*14.

	Frequency	Total interest ($)	Saving ($)
a	Half-yearly	6014.80	–
b	Quarterly	5707.11	$307.69 saving on half-yearly
c	Monthly	5496.63	$518.17 saving on half-yearly
d	Fornightly	5440.38	$574.42 saving on half-yearly

Exercise 7.5 Annuities and perpetuities

1. a. $V_0 = 12500$, $V_{n+1} = 1.015V_n - 3243$
b. 25 cents
c. 25 cents would need to be added to Aldo's last payment.
d. See table at the foot of the page*.

2. D

3. a. 30 months **b.** $1345.37

4. A

5. B

6. D

7. a. $3600 **b.** 6.4% p.a.
c. $3680.27 **d.** 6.37%

8. a. $7125 **b.** 6.4% p.a.
c. $7314.12 **d.** 6.37%

9. $320 000

10. $106 666.67

11. B

12. a. $16 000 per year
b. $3000 per quarter

13. a. $1000 per month
b. $30 000 per quarter

14. (Q12)

a.
N: = 1
I%: = 4
PV: = −400 000
Pmt: = 16 000
FV: = 400 000
PpY: = 1
CpY: = 1
PmtAt: = END

b.
N: = 1
I%: = 4
PV: = −300 000
Pmt: = 3000
FV: = 300 000
PpY: = 4
CpY: = 4
PmtAt: = END

(Q13)

a.
N: = 1
I%: = 12
PV: = −100 000
Pmt: = 1000
FV: = 100 000
PpY: = 12
CpY: = 12
PmtAt: = END

b.
N: = 1
I%: = 6
PV: = −2 000 000
Pmt: = 30 000
FV: = 2 000 000
PpY: = 4
CpY: = 4
PmtAt: = END

15. a. $3200 per year **b.** 6.25%
c. $48.32 **d.** 6%

16. $17 088.13

17. a. 1.25%
b. 2.40%
c. 7.50%
d. 5.20%

18. a.
N: = 1
I%: = 1.25
PV: = −400 000
Pmt: = 5000
FV: = 400 000
PpY: = 1
CpY: = 1
PmtAt: = END

b.
N: = 1
I%: = 2.4
PV: = −500 000
Pmt: = 1000
FV: = 500 000
PpY: = 12
CpY: = 12
PmtAt: = END

c.
N: = 1
I%: = 7.5
PV: = −800 000
Pmt: = 30 000
FV: = 800 000
PpY: = 2
CpY: = 2
PmtAt: = END

d.
N: = 1
I%: = 5.2
PV: = −100 000
Pmt: = 200
FV: = 100 000
PpY: = 26
CpY: = 26
PmtAt: = END

19. a.
N: = 1
I%: = 1.242 895 218
PV: = −400 000
Pmt: = 5000
FV: = 400 000
PpY: = 1
CpY: = 12
PmtAt: = END
$r = 1.24\%$ p.a.

b.
N: = 1
I%: = 2.426 576 795
PV: = −500 000
Pmt: = 1000
FV: = 500 000
PpY: = 12
CpY: = 1
PmtAt: = END
$r = 2.43\%$ p.a.

c.
N: = 1
I%: = 7.430 975 749
PV: = −800 000
Pmt: = 30 000
FV: = 800 000
PpY: = 2
CpY: = 4
PmtAt: = END
$r = 7.43\%$ p.a.

d.
N: = 1
I%: = 5.206 067 34
PV: = −100 000
Pmt: = 200
FV: = 100 000
PpY: = 26
CpY: = 12
PmtAt: = END
$r = 5.21\%$ p.a.

20. $66 666.67

21. E

22. a. $20 000
b. $222 222.22
c. $60 000
d. $48 000

---

*1. d.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	12500
1	3243	**A** = 187.50	3055.50	9444.50
2	3243	141.67	**B** = 3101.33	6343.17
3	3243	95.15	3147.85	**C** = 3195.32
4	**D** = 3243.25	47.93	3195.07	**0**

Exercise 7.6 Compound interest investments with periodic and equal additions to the principal

1. a. $8057.92
   b.
   A. = $58.10
   B. = $209.83
   C. = $63.92
   D. = $8057.83
2. a. $V_0 = 10\,000$, $V_{n+1} = 1.01V_n + 200$
   b. $11 845.60
3. $96.22
4. $10 875.37
5. a. $40 000 b. $50 000 c. $10 000
6. a. $39 000 b. $51 000 c. $12 000
7. a. $101.50 b. $106.05 c. $515.08
8. a. $523 000 b. $737 000 c. $214 000
9. a. $779.70 b. $692 647.66 c. $124 000
10. a. $d = \$1185.74$ b. $690 616.68
    c. $47 000
11. a. $1168.46 b. $861 442.14 c. $238 000
12. a. $717.34 b. $931 925.99
4. If changes not made, term = 15 years. With changes, term = 13 years 20 fortnights.
   So difference in term = 1 year 6 fortnights.
5. Fixed: interest = $1790.20
   Variable: interest = $1784.51 so correct decision is variable, $5.69 better off.
6. a. $86 400 b. 6.97%
7. a. $5287.59 b. 9.39% p.a.

Extended response

1. a. $31 080
   b. i. $450.09 ii. 12 450.09
   c. i. $6077.53 ii. $3000
   d. i. $14 094.17
   ii. $6264.08
   iii. The $27 000 car option
2. See table below bottom of the page*
3. a. $788.46
   b. See table below bottom of the page#
   c. $6307.68
   d. $307.68
   e. 4.58% (correct to 2 decimal places)

Exercise 7.7 Review: exam practice

Multiple choice

1. **C** 2. **B** 3. **E** 4. **C** 5. **E**
6. **D** 7. **B** 8. **D** 9. **D** 10. **D**

*

Description	$27 000 4WD car	$12 000 car	$5000 car and painting
Cost of loan (interest charged only)	4080.00	450.09	—
Depreciation of goods	12 905.83	5735.92	2000.00
Total cost	31 080.00	12 450.09	10 000.00
Less benefits (for example, increase in values)	—	—	1077.53
Total cost	31 080.00	12 450.09	8922.47

#

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0	0	0	6000
1	788.46	67.5	720.96	5279.04
2	788.46	59.39	729.07	4549.97
3	788.46	51.19	737.27	3812.70
4	788.46	42.89	745.57	3067.13
5	788.46	34.51	753.95	2313.17
6	788.46	26.02	762.44	1550.74
7	788.46	17.45	771.01	779.72
8	788.46	8.77	779.69	0.035

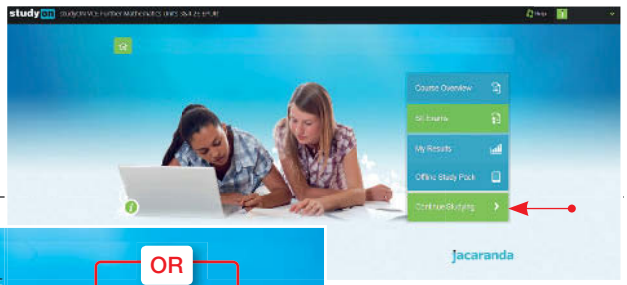
Short answer

1. $60 250.40
2. $46 741.44
3. 19 years (to the nearest whole year)

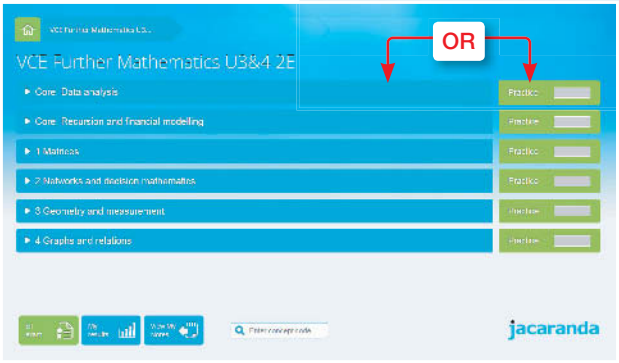
REVISION: AREA OF STUDY 1 Core: recursion and financial modelling

TOPICS 5 to 7

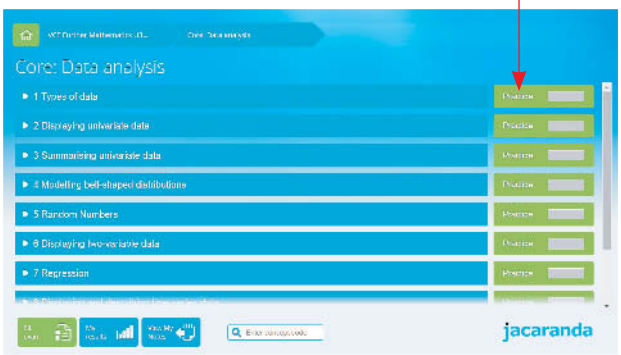
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



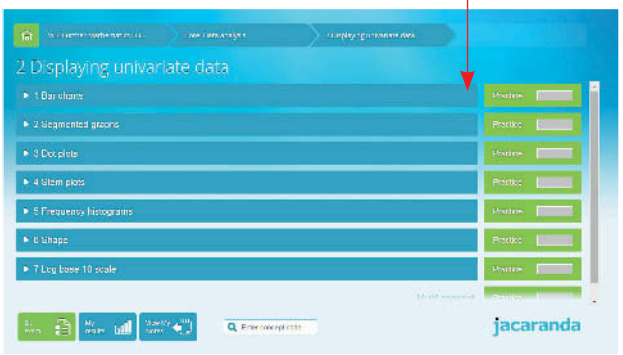
- Select your **course** *Further Mathematics VCE Units 3 & 4* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



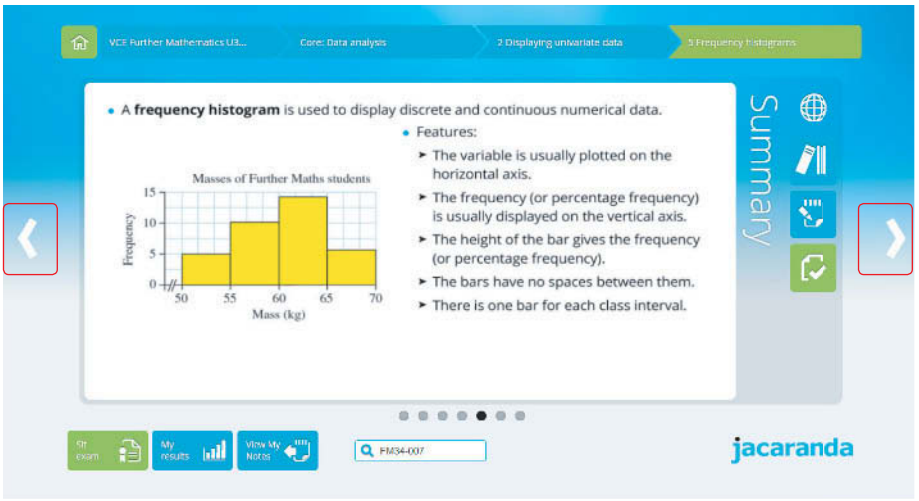
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 8
Matrices

8.1 Overview

8.1.1 Introduction

Matrices were being used by ancient civilisations as far back as 300 BC. There is evidence in the form of clay tablets and texts that both the Chinese and the Babylonians used matrix methods to study and solve simultaneous linear equations. For centuries, mathematicians used matrices to solve problems but it was not until the nineteenth century that the English mathematician Arthur Cayley published a formal definition of a matrix in his *Memoir on the Theory of Matrices* (1858), thus establishing matrices as a specific branch of mathematics. Matrices are used extensively in engineering, physics, geology, computer science, graphic design, economics, computer games and many other areas of applied mathematics. Computer programmers need an understanding of matrices to create the extraordinary 3D visual effects in movie block busters and realistic computers games that are now expected in this genre of movies and games. Matrices are used to encrypt code and hence play a vital role in the security of websites and data bases.

LEARNING SEQUENCE

8.1 Overview
8.2 Matrix representation
8.3 Addition, subtraction and scalar operations with matrices
8.4 Multiplying matrices
8.5 Multiplicative inverse and solving matrix equations
8.6 Dominance and communication matrices
8.7 Application of matrices to simultaneous equations
8.8 Transition matrices
8.9 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.1.2 Kick off with CAS

Dominance matrices

We can use dominance matrices to find which vertex in a matrix is the most dominant. This can have many purposes, for example determining the order of teams in a round robin tournament.

1. Five soccer teams took part in a round robin tournament with each team playing each other team once. The results were as follows:
   - B defeated C and E
   - A defeated B, C, and D
   - C defeated E
   - D defeated B, C, and E
   - E defeated A.

   Given the above results, try to rank the 5 teams in order.

2. The results can be represented in a 5×5 matrix. Complete the following matrix by placing a 1 in the elements where the team in the rows defeated the opposing teams in the columns. Place a 0 in all other elements.

$$D = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{array} \\ \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \end{array}$$

3. **a.** Using CAS, define the completed matrix from question **2**.
   b. Use CAS to determine the matrix D^2.
   c. Use CAS to determine the matrix $D + D^2$.
4. Sum each row of the matrix found in question **3c**.
5. Use your answer to question **4** to rank the teams in order (the team with the highest sum is the most dominant team, and so on).
6. Compare your answer to question **5** with your answer to question **1**.

on Resources

Please refer to the Resources section in the Prelims of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

8.2 Matrix representation

A **matrix** (plural *matrices*) is a rectangular array of numbers arranged in rows and columns. The numbers in a matrix are called the **elements** of the matrix.

The matrix shown here has 3 *rows* and 2 *columns*. We say that it is a 3×2 **rectangular matrix** and its **order** is 3 by 2.

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 4 \end{bmatrix}$$

Capital letters will be used to represent matrices.

In general, a matrix with m rows and n columns is known as a $m \times n$ matrix.

The elements in a matrix are referred to by the row and then by the column position.

The element in the second row and the first column of matrix A above is -1. This is represented as $a_{21} = -1$ or the 2, 1 element.

In general, the elements of matrix A are referred to as a_{ij} where i refers to the row position and j refers to the column position.

That is, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ . & . & . & . & a_{mn} \end{bmatrix}$ for the matrix of order $m \times n$.

The 2, 1 element is a_{21} and the 3, 2 element is a_{32}.

8.2.1 Types of matrices

A matrix with one row is called a **row matrix** or *row vector*.

$$B = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \text{ is a } 1 \times 3 \text{ row matrix.}$$

A matrix with one column is called a **column matrix** or *column vector*.

$$C = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ is a } 3 \times 1 \text{ column matrix.}$$

A matrix with an equal number of rows and columns is called a **square matrix**.

$$D = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & -1 \\ 3 & -2 & 0 \end{bmatrix} \text{ is a } 3 \times 3 \text{ square matrix}$$

$$E = \begin{bmatrix} 4 & 2 \\ -5 & 7 \end{bmatrix} \text{ is a } 2 \times 2 \text{ square matrix.}$$

A square matrix that has only non-zero elements on the main diagonal is called a **diagonal matrix**.

$$F = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ is a } 3 \times 3 \text{ diagonal matrix.}$$

A **zero matrix** consists only of elements that are 0.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a } 2 \times 2 \text{ zero matrix.}$$

Two matrices are *equal* if they are of the same order and all corresponding elements are equal.

$$\begin{matrix} & & A & & \\ \end{matrix} \quad \begin{matrix} & & B & & \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix}$$

These two 2×4 matrices are equal as the corresponding elements are equal.
That is, $A = B$ as $a_{11} = b_{11} = 1$, $a_{12} = b_{12} = 0$, …

A **symmetrical matrix** needs to be a square matrix.

Matrix A is symmetric if $A = A^T$, where A^T is the **transpose matrix**.

The elements with respect to the main diagonal are symmetric in a symmetric matrix, so $a_{ij} = a_{ji}$ for all indices i and j.

The following matrix is a 3×3 symmetric matrix, where the black numbers represent the main diagonal.

$$\begin{bmatrix} 1 & 3 & -7 \\ 3 & 5 & 9 \\ -7 & 9 & 2 \end{bmatrix}$$

A **triangular matrix** is a type of square matrix.

A triangular matrix can be labelled a **lower triangular matrix** if all elements above the main diagonal are zero. However, if all elements below the main diagonal are zero, then it is called an **upper triangular matrix**.

The following are both 3×3 triangular matrices.

$$\begin{bmatrix} 3 & 0 & 0 \\ 8 & 5 & 0 \\ 7 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & -1 \\ 0 & 5 & 8 \\ 0 & 0 & 2 \end{bmatrix}$$

Lower triangular matrix Upper triangular matrix.

A matrix that consists only of elements that are either 0 or 1 is known as a **binary matrix**. These are also referred to as (0, 1), Boolean or logical matrices.

WORKED EXAMPLE 1

For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2, 1 element and the position of the number 3 in x_{ij} form.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 9 & 7 & -5 \\ 7 & -1 & 4 \\ -5 & 4 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} -2 & 12 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

THINK	WRITE
1. A has 3 rows and 3 columns, with the numbered elements in the main diagonal. A zero is in the 2, 1 position and a 3 is in position 1, 1.	A is a 3×3 diagonal matrix. The element in the 2, 1 position is a 0 and the number 3 is represented by a_{11}.
2. B has 1 row and 3 columns. There is no element in the 2, 1 position and 3 is in position 1, 2.	B is a 1×3 row matrix. There is no element in the 2, 1 position and the number 3 is represented by b_{12}.
3. C has 2 rows and 2 columns. All elements are 0 or 1. A 0 is in the 2, 1 position and there is no number 3 in the matrix.	C is a 2×2 binary matrix. The element in the 2, 1 position is a 0 and there is no number 3 in the matrix.
4. D has 3 rows and 3 columns. The elements are symmetric about the main diagonal. A 7 is in the 2, 1 position and a 3 is in position 3, 3.	D is a 3×3 symmetrical matrix. The element in the 2, 1 position is a 7 and the number 3 is represented by d_{33}.
5. E has 3 rows and 3 columns; it is a square matrix. The elements have zeros below the main diagonal. A zero is in the 2, 1 position and a 3 is in position 1, 3.	E is a 3×3 upper triangular matrix. The element in the 2, 1 position is a 0 and the number 3 is represented by e_{13}.

8.2.2 Using matrices to store data

Matrices are useful for recording and storing bivariate data (data that depend on two categories). They can also keep track of the coefficients of systems of numbers in a simple two-dimensional format. For storing data, whether the data are organised in a row or column need not be specified; they are shown in the same manner as the original data format.

Distance between 3 local townships (kilometres)			
	Town A	**Town B**	**Town C**
Town A	0	23	17
Town B	23	0	43
Town C	17	43	0

$$\begin{bmatrix} 0 & 23 & 17 \\ 23 & 0 & 43 \\ 17 & 43 & 0 \end{bmatrix}$$

Labelling convention is not important as the matrix is simply being used to store data.

However, when performing mathematical processes using the data stored in matrices, the defining of the rows and columns must follow formal conventions. Two common examples where the columns and rows must follow a well-established convention are two-way frequency tables and simultaneous equations.

Two-way frequency tables follow the convention of organising explanatory variable headings in the columns and response variable headings in the rows.

Attitude	Primary	Secondary
Fewer	5	2
Same	29	9
More	33	36
Total	67	47

$$\begin{bmatrix} 5 & 2 \\ 29 & 9 \\ 33 & 36 \\ 67 & 47 \end{bmatrix}$$

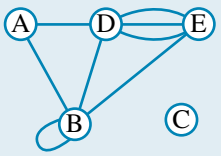
Simultaneous equations follow the convention of organising the variables into columns and the coefficients into rows.

$$\begin{matrix} 3x + 2y = 5 \\ 2x - 6y = 2 \end{matrix} = \begin{bmatrix} 3 & 2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Matrices are usually presented without headings or labels on the rows and columns.

WORKED EXAMPLE 2

a. Generate a matrix to show the number of major country roads between five nearby townships in the network shown.



b. Generate a matrix to represent the following contingency table for party preferences of females and males.

Party preference	Female	Male
Labor	18	12
Liberal	16	11
Total	34	23

c. Generate a 2 × 2 matrix to represent the information provided in the following scenario. Omit the totals from your matrix.
'In a survey, 139 women and 102 men were asked whether they approved or disapproved of a proposed freeway. Thirty-seven women and 79 men approved of the freeway.'

THINK

a. 1. Set up a blank 5 × 5 matrix. Thus, there are 25 entries inside the matrix. Label the rows and columns for accuracy.

WRITE

a.

$$\begin{matrix} \\ A \\ B \\ C \\ D \\ E \end{matrix} \begin{matrix} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{matrix}$$

2. Consider vertex A. It is connected to vertices B and D once each, so put a 1 in the corresponding columns of row 1 (shown in red) and in the corresponding rows of column 1 (shown in blue).

	A	B	C	D	E
A		1		1	
B	1				
C					
D	1				
E					

3. Consider vertex B. It is connected to itself once and vertices A, D and E once each. Put a 1 in the corresponding columns of row 2 (shown in red) and in the corresponding rows of column 2 (shown in blue).
Note: Some elements, such as m_{12} and m_{21}, will have been previously filled.

	A	B	C	D	E
A		1		1	
B	1	1		1	1
C					
D	1	1			
E		1			

4. Vertex C is not connected to any other vertex, so fill in column 3 and row 3 with zeros (shown in red).

	A	B	C	D	E
A		1	0	1	
B	1	1	0	1	1
C	0	0	0	0	0
D	1	1	0		
E		1	0		

5. Vertex D is connected to vertices A and B once. This was recorded in previous steps. Vertex 4 is also connected to vertex E three times, so put a 3 in elements m_{45} and m_{54} (shown in red).

	A	B	C	D	E
A		1	0	1	
B	1	1	0	1	1
C	0	0	0	0	0
D	1	1	0		3
E		1	0	3	

6. Vertex E has all its connections already entered, as they were considered in previous steps, so complete the matrix by placing a 0 in all unoccupied places, indicating no connections between the vertices.

	A	B	C	D	E
A	0	1	0	1	0
B	1	1	0	1	1
C	0	0	0	0	0
D	1	1	0	0	3
E	0	1	0	3	0

7. Remove the labels to complete the matrix. Check your result by comparing the entries in the matrix with the original network representation. This is best done on a vertex-by-vertex basis.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}$$

b. Set up a 3×2 matrix and record the data as shown with labels removed.

b. $\begin{bmatrix} 18 & 12 \\ 16 & 11 \\ 34 & 23 \end{bmatrix}$

c. 1. Set up a 2×2 matrix to record the data. For accuracy, you may label the rows and columns but remember to remove them in your final answer. Record the independent variable, the respondent's gender, in the columns and the dependent variable, party preference, in the rows.

c.

	m	f
approve		
disapprove		

2. We know that, of the 139 females, 37 approved of the freeway and the remainder (139 − 37 = 102) disapproved of the freeway. Enter this information into m_{12} and m_{22}.

3. We also know that, of the 102 males, 79 males approved of the freeway and the remainder (102 − 79 = 23) disapproved of the freeway. Put this information into m_{11} and m_{21}. Remove any labels on the rows or columns to complete the matrix.

$$\begin{array}{r} \\ \text{approve} \\ \text{disapprove} \end{array}\begin{array}{c}\begin{matrix}\text{m} & \text{f}\end{matrix}\\ \begin{bmatrix} & 37 \\ & 102 \end{bmatrix}\end{array}$$

$$\begin{bmatrix} 79 & 37 \\ 23 & 102 \end{bmatrix}$$

Matrices are not just useful for storing data. They can be added, subtracted, multiplied (but not divided) and generally manipulated to extract greater information from the data. For example, we can use matrices to quickly calculate percentages in the two-way frequency table used in Worked example 2. This will be examined more closely later in the topic.

studyon

Units 3 & 4 › AOS M1 › Topic 1 › Concepts 1, 2, 3

Defining a matrix Summary screen and practice questions

Matrix types Summary screen and practice questions

More matrix types Summary screen and practice questions

Exercise 8.2 Matrix representation

1. **WE1** For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2, 1 element and the position of the number 3 in x_{ij} form.

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 1 & -5 \\ 1 & -12 & 6 \\ -5 & 6 & 13 \end{bmatrix}, E = \begin{bmatrix} -2 & 0 & 0 \\ 5 & 4 & 0 \\ -10 & 3 & 6 \end{bmatrix}$$

2. For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 1, 2 element and the position of the number 3 in x_{ij} form.

$$A = \begin{bmatrix} 3 & -5 & 2 \\ 0 & 5 & -12 \\ 0 & 0 & 15 \end{bmatrix}, B = \begin{bmatrix} -1 & -3 & 1 \end{bmatrix}, C = \begin{bmatrix} 7 & 10 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 12 & 7 & -17 \\ 7 & -9 & 3 \\ -17 & 3 & 12 \end{bmatrix}$$

3. For each of the following:
   - i. state the order and type of the matrix
   - ii. where possible, write the 2, 1 element
   - iii. state the position of the number 3 in x_{ij} form.

a. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 2 & 0 \\ 0 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 7 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **c.** $\begin{bmatrix} & 2 & \\ 1 & 4 & 2 \\ & 3 & \end{bmatrix}$

d. $\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$ **e.** $\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$ **f.** $\begin{bmatrix} -1 & -4 & -12 \\ -4 & 15 & 3 \\ -12 & 3 & 4 \end{bmatrix}$

4. For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2, 2 element and the position of the number 7 in x_{ij} form.

$A = \begin{bmatrix} -13 & 0 & 0 \\ -11 & 4 & 0 \\ 10 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 7 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 & 5 \\ 0 & 0 & -9 \\ 5 & -9 & 7 \end{bmatrix}$

5. State the a_{21} element in each of the matrices given.

a. $\begin{bmatrix} 4 & -1 & 30 \\ 1 & -5 & 1 \\ 5 & 6 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} -9 & 2 & 5 \\ -5 & 4 & 2 \\ 12 & 3 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ **d.** $\begin{bmatrix} 0.5 \\ 2.1 \\ 3.7 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$ **f.** $\begin{bmatrix} 2 & -4 & -3 \end{bmatrix}$ **g.** $\begin{bmatrix} 2 & -1 \\ 3 & -3 \\ -5 & 0 \\ 6 & 3 \end{bmatrix}$ **h.** $\begin{bmatrix} 0.2 & 0.5 & 3.1 & 2.9 \\ 3.5 & 2.1 & 0.1 & 0.8 \end{bmatrix}$

6. For each of the matrices given in question **5**, state the ones in which x_{32} exists.

7. **MC** The value of a_{21} in the matrix is:

$\begin{bmatrix} 2.4 & 3.6 \\ -0.5 & 1.6 \\ 1.6 & 0 \\ -2.5 & 2.4 \end{bmatrix}$

A. 3.6
B. 1.6
C. −0.5
D. 0
E. 2.4

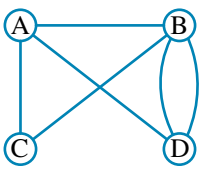
8. **MC** The number 3 in the following matrix can be represented using the notation:

$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 5 & 4 \end{bmatrix}$

A. a_{23}
B. a_{32}
C. a_{22}
D. a_{12}
E. none of these

9. Using the values 0 and 1 only, state the 4×4 diagonal matrix.

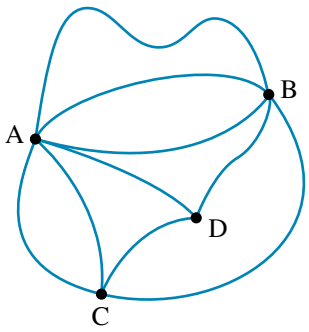
10. **WE2** Represent the following network as a matrix.



11. Represent the following contingency table for sport preference as a matrix.

Sport preference	Men	Women
AFL	26	22
Cricket	15	9
Total	41	31

12. The figure shown represents the number of routes between four towns.



Represent the possible routes as a 4×4 matrix in the following form.

$$\begin{array}{l} \\ \text{Number of routes from} \end{array} \begin{array}{c} \text{Number of routes to} \\ \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

13. Represent the following contingency table as a 3×3 matrix.

Rent preference	Men	Women	Total
Live independently	12	23	35
Share with friends	9	16	25
Total	21	39	60

14. Represent the final score at an AFL match as a matrix.

	Goals	Behinds	Points
Geelong	15	10	100
Carlton	12	15	87

15. The following information represents the goods to be delivered from a warehouse to individual suppliers.

To supplier A:	5 bicycles	12 helmets	6 tyres
To supplier B:	7 bicycles	2 helmets	15 tyres
To supplier C:	15 bicycles	7 helmets	0 tyres

Present the information as a matrix with the suppliers placed in the columns.

16. Represent the following coordinates of Cartesian points as a 4×2 matrix.
(3, 2), (−4, −1), (4, −1), (4, 1)

8.3 Addition, subtraction and scalar operations with matrices

8.3.1 Addition and subtraction

Matrices can be added or subtracted by applying the usual rules of arithmetic on corresponding elements of the matrices. It follows that:

1. **Addition and subtraction of matrices can be performed only if the matrices are of the same order.**
2. **Addition and subtraction of matrices is performed by adding or subtracting corresponding elements.**

WORKED EXAMPLE 3

Given the matrices

$A = \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 8 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 2 & 3 \\ 4 & -2 & -2 \end{bmatrix}$, **find, if possible:**

a. $A + B$ **b.** $B + A$ **c.** $B - C$ **d.** $B - A$ **e.** $D - C$.

THINK

a. 1. A and B are both of the same order 2×2, so they can be added.

2. Add the numbers in the corresponding positions of each matrix.

b. 1. B and A are both of the same order 2×2, so they can be added.

2. Add the numbers in the corresponding positions of each matrix.

WRITE

a. $A + B = \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 8 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2+1 & 4+6 \\ 3+8 & -2+2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 10 \\ 11 & 0 \end{bmatrix}$

b. $B + A = \begin{bmatrix} 1 & 6 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1+2 & 6+4 \\ 8+3 & 2+-2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 10 \\ 11 & 0 \end{bmatrix}$

c. The subtraction cannot be performed since the order of B is 2×2 and the order of C is 2×3.

c. $B - C$ cannot be calculated since B and C are of different orders.

d. Subtract the numbers in the corresponding positions of each matrix as B and A are of the same order.

d. $B - A = \begin{bmatrix} 1 & 6 \\ 8 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1-2 & 6-4 \\ 8-3 & 2--2 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix}$

e. 1. D and C are both of the same order 2×3, so they can be subtracted.

2. Subtract the numbers in the corresponding positions of each matrix.

e. $D - C = \begin{bmatrix} 5 & 2 & 3 \\ 4 & -2 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5-2 & 2-3 & 3--1 \\ 4-4 & -2--2 & -2-5 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 & 4 \\ 0 & 0 & -7 \end{bmatrix}$

Notice in parts **a** and **b** that the resulting matrices are both the same. In general, $A + B = B + A$ when A and B are of the same order; that is, matrix addition is **commutative**. Try this for other matrices. Using the matrices from above, try $A - B$.

What do you notice about $B - A$ and $A - B$?

8.3.2 Scalar multiplication

As we have seen in arithmetic, repeated addition can be more efficiently calculated using multiplication. For example, $2 + 2 + 2 = 3 \times 2$.

A similar approach applies to matrices. Consider the matrix $A = \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix}$.

To find the addition of $A + A + A$, the simplest approach is to find $3A$. This is achieved by multiplying each element of A by 3.

$$A + A + A = \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1+1+1 & 5+5+5 \\ 4+4+4 & -2+-2+-2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 15 \\ 12 & -6 \end{bmatrix}$$

$$3A = 3 \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \times 1 & 3 \times 5 \\ 3 \times 4 & 3 \times -2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 15 \\ 12 & -6 \end{bmatrix}$$

In the term $3A$, the number 3 is called a **scalar**, and the term $3A$ is an example of **scalar multiplication** of matrices.

Scalar multiplication applies to matrices of any order. The scalar quantity can be any number positive or negative, fraction or decimal, real or imaginary. This can be generalised as follows.

$$c \times \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ . & . & . & . & a_{mn} \end{bmatrix} = \begin{bmatrix} c \times a_{11} & c \times a_{12} & c \times a_{13} & c \times a_{14} & \dots \\ c \times a_{21} & c \times a_{22} & c \times a_{23} & c \times a_{24} & \dots \\ c \times a_{31} & c \times a_{32} & c \times a_{33} & c \times a_{34} & \dots \\ . & . & . & . & c \times a_{mn} \end{bmatrix}$$

When a matrix is multiplied by a scalar, each element in the matrix is multiplied by the scalar. The order of the matrix remains unchanged.

WORKED EXAMPLE 4

Given the matrices

$A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$,

calculate:

a. $2A$ **b.** $0.4B$ **c.** $3A + 4A$

d. $A + \frac{1}{3}B$ **e.** $3(A + B)$.

THINK

a. Multiply each element of A by 2.

b. Multiply each element of B by 0.4.

c. 1. $3A + 4A$ simplifies to $7A$.

2. Multiply each element of A by 7.

d. 1. Find $\frac{1}{3}B$ by multiplying each element of B by $\frac{1}{3}$.

WRITE

a. $2A = 2\begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 2 & 2 \times 6 \\ 2 \times -1 & 2 \times 3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 12 \\ -2 & 6 \end{bmatrix}$

b. $0.4B = 0.4\begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0.4 \times 3 & 0.4 \times 6 \\ 0.4 \times 1 & 0.4 \times 3 \end{bmatrix}$

$= \begin{bmatrix} 1.2 & 2.4 \\ 0.4 & 1.2 \end{bmatrix}$

c. $3A + 4A = 7A$

$7A = 7\begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 14 & 42 \\ -7 & 21 \end{bmatrix}$

d. $\frac{1}{3}B = \frac{1}{3}\begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{3} & \frac{6}{3} \\ \frac{1}{3} & \frac{3}{3} \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \end{bmatrix}$

2. Complete the addition by adding A to $\frac{1}{3}B$.

$$A + \frac{1}{3}B = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 8 \\ -\frac{2}{3} & 4 \end{bmatrix}$$

e. 1. Find the sum of A and B.

e.
$$A + B = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 12 \\ 0 & 6 \end{bmatrix}$$

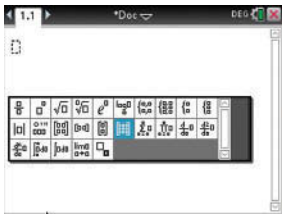
2. Multiply this matrix by 3.

$$3(A + B) = 3 \begin{bmatrix} 5 & 12 \\ 0 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 15 & 36 \\ 0 & 18 \end{bmatrix}$$

TI | THINK

1. To define Matrices A and B, on a Calculator page open the Maths expression palette and select the matrix template. Fill in the number of rows and columns as 2×2. Press ENTER.

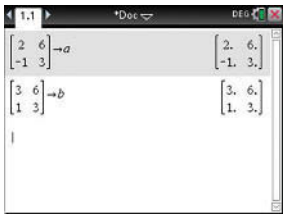
WRITE



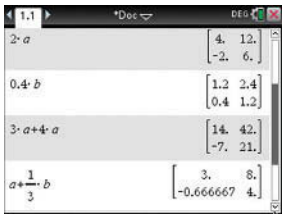
2. Use the tab key to enter the values into the matrix as shown, press tab to bring the cursor outside the matrix.
Press:
ctrl
var
a

Then press ENTER.

Repeat for Matrix B.



3. Complete the entry lines as:
$2 \times a$, then press ENTER.
$0.4 \times b$, then press ENTER.
$3 \times a + 4 \times a$, then press ENTER.
$a + \frac{1}{3} \times b$, then press ENTER.
$3 \times (a + b)$, then press ENTER.



CASIO | THINK

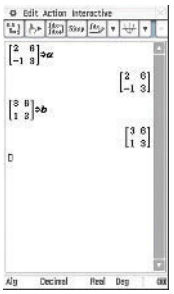
1. To define Matrices A and B, on a Main screen tap:
Keyboard
Math2

Tap the 2×2 template.

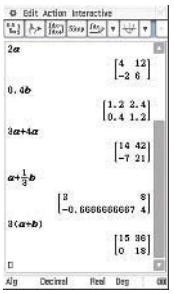
WRITE

2. Enter the values into the matrix, tap outside the matrix to bring the cursor outside.
Tap:
Store (arrow symbol)
a (Remember to use Var a)
Then tap EXE.

Repeat for Matrix B.



3. Complete the entry lines as:
$2 \times a$, then tap EXE.
$0.4 \times b$, then tap EXE.
$3 \times a + 4 \times a$, then tap EXE.
$a + \frac{1}{3} \times b$, then tap EXE.
$3 \times (a + b)$, then tap EXE.



4. The answers are shown on the screen.

a. $2A = \begin{bmatrix} 4 & 12 \\ -2 & 6 \end{bmatrix}$

b. $0.4B = \begin{bmatrix} 1.2 & 2.4 \\ 0.4 & 1.2 \end{bmatrix}$

c. $3A + 4B = \begin{bmatrix} 14 & 42 \\ -7 & 21 \end{bmatrix}$

d. $A + \frac{1}{3}B = \begin{bmatrix} 3 & 8 \\ -0.667 & 4 \end{bmatrix}$

e. $3(A + B) = \begin{bmatrix} 15 & 36 \\ 0 & 18 \end{bmatrix}$

4. The answers are shown on the screen.

a. $2A = \begin{bmatrix} 4 & 12 \\ -2 & 6 \end{bmatrix}$

b. $0.4B = \begin{bmatrix} 1.2 & 2.4 \\ 0.4 & 1.2 \end{bmatrix}$

c. $3A + 4B = \begin{bmatrix} 14 & 42 \\ -7 & 21 \end{bmatrix}$

d. $A + \frac{1}{3}B = \begin{bmatrix} 3 & 8 \\ -0.667 & 4 \end{bmatrix}$

e. $3(A + B) = \begin{bmatrix} 15 & 36 \\ 0 & 18 \end{bmatrix}$

8.3.3 Properties of addition of matrices

The following list is a summary of properties of addition of matrices. These properties hold true when A, B and C are $m \times n$ matrices, k and c are constants and O is a zero matrix (a matrix with all elements equal to zero).

Property	Example
Commutative (does not matter which order the matrices are operated on)	$A + B = B + A$
Associative (does not matter where the brackets are placed)	$(A + B) + C = A + (B + C)$ $(kc)A = k(cA)$
Identity	$A + O = A = O + A$
Inverse	$A + -A = O = -A + A$
Distributive	$kA + kB = k(A + B)$ $kA + cA = (k + c)A$

Note: For subtraction, matrices *do not* obey the **associative** or the commutative laws. For example, $A - B \neq B - A$ and $(A - B) - C \neq A - (B - C)$.

8.3.4 Simple matrix equations

To solve an algebraic equation such as $4x - 3 = 5$:

1. add 3 to both sides to obtain $4x - 3 + 3 = 5 + 3$ or $4x = 8$
2. divide both sides by 4 $\left(\text{or multiply by } \frac{1}{4}\right)$ to obtain $x = 2$.

Simple matrix equations that require the addition or subtraction of a matrix or multiplication of a scalar can be solved in a similar way.

WORKED EXAMPLE 5

Solve the following matrix equations.

a. $5E = \begin{bmatrix} 20 & 15 & -5 \\ 5 & 0 & 12 \end{bmatrix}$

b. $D + \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}$

c. If $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$, find C if $2C + A = 3B$.

THINK	WRITE
a. 1. To get E by itself, multiply both sides by $\frac{1}{5}$.	**a.** $5E = \begin{bmatrix} 20 & 15 & -5 \\ 5 & 0 & 12 \end{bmatrix}$ $E = \frac{1}{5}\begin{bmatrix} 20 & 15 & -5 \\ 5 & 0 & 12 \end{bmatrix}$
2. Simplify the matrix E.	$E = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 0 & 2.4 \end{bmatrix}$
b. 1. To get D by itself, subtract $\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$ from both sides.	**b.** $D + \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}$ $D = \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$
2. Simplify the matrix D.	$D = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}$
c. 1. First solve algebraically to get C by itself.	**c.** $C + A = 3B$ $2C = 3B - A$ $C = \frac{1}{2}(3B - A)$
2. Find the value of $3B - A$.	$3B - A = \begin{bmatrix} 9 & 18 \\ 3 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ $= \begin{bmatrix} 7 & 12 \\ 4 & 6 \end{bmatrix}$
3. Multiply this by $\frac{1}{2}$ to solve for C.	$C = \frac{1}{2}\begin{bmatrix} 7 & 12 \\ 4 & 6 \end{bmatrix}$ $= \begin{bmatrix} 3.5 & 6 \\ 2 & 3 \end{bmatrix}$

The matrix operations involving addition, subtraction and scalar multiplication can be applied to practical situations such as stock inventory, price discounting and marking up of store prices.

WORKED EXAMPLE 6

A retail chain of three stores has an inventory of three models each of televisions and DVD players, represented as matrices as follows. The first column represents the televisions and the second column represents the DVD players. Each row represents a different model.

$$\text{Store } A = \begin{bmatrix} 12 & 21 \\ 5 & 12 \\ 3 & 7 \end{bmatrix}, \text{Store } B = \begin{bmatrix} 23 & 32 \\ 8 & 15 \\ 1 & 11 \end{bmatrix}, \text{Store } C = \begin{bmatrix} 5 & 17 \\ 2 & 12 \\ 0 & 14 \end{bmatrix}$$

a. If the third row represents the most expensive models, which store has the most models of expensive televisions?

b. Give the matrix that would represent the total stock of televisions and DVD players for all three stores.

c. The wholesale price (in dollars) of each model of television and DVD player is presented in the following matrix.

$$\begin{bmatrix} 100 & 30 \\ 250 & 80 \\ 400 & 200 \end{bmatrix}$$

If the wholesale prices are marked up by 50%, show the recommended retail prices in a 3×2 matrix.

d. Store C wishes to have a sale. If it discounts all retail prices by 10%, represent the discounted prices as a 3×2 matrix.

THINK

a. The first column represents the televisions and the third row represents the most expensive model (a_{31}). Store A has 3 expensive televisions, Store B has 1 expensive television and Store C has 0 expensive televisions.

b. The total stock of televisions and DVD players for all three stores is given by the sum of the three matrices. Add the matrices and simplify.

c. 1. A mark-up of 50% represents 150% of the wholesale prices.

WRITE

a. Store A has the greatest number of expensive televisions.

b. Total inventory $= A + B + C$

$$= \begin{bmatrix} 12 & 21 \\ 5 & 12 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 23 & 32 \\ 8 & 15 \\ 1 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 17 \\ 2 & 12 \\ 0 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 12+23+5 & 21+32+17 \\ 5+8+2 & 12+15+12 \\ 3+1+0 & 7+11+14 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 70 \\ 15 & 39 \\ 4 & 32 \end{bmatrix}$$

c. $50\% + 100\% = 150\%$

2. Multiply each element of the wholesale price matrix by 1.5 (150%).

$$150\% \text{ of } \begin{bmatrix} 100 & 30 \\ 250 & 80 \\ 400 & 200 \end{bmatrix} = 1.5 \times \begin{bmatrix} 100 & 30 \\ 250 & 80 \\ 400 & 200 \end{bmatrix} = \begin{bmatrix} 150 & 45 \\ 375 & 120 \\ 600 & 300 \end{bmatrix}$$

d. 1. A discount of 10% represents 90% of the retail price.

d. $100\% - 10\% = 90\%$

2. Multiply each element of the retail price matrix by 0.9 (90%).

$$90\% \text{ of } \begin{bmatrix} 150 & 45 \\ 375 & 120 \\ 600 & 300 \end{bmatrix} = 0.9 \times \begin{bmatrix} 150 & 45 \\ 375 & 120 \\ 600 & 300 \end{bmatrix} = \begin{bmatrix} 135 & 40.5 \\ 337.5 & 108 \\ 540 & 270 \end{bmatrix}$$

on Resources

Interactivity: Adding and subtracting matrices (int-6463)

studyon

Units 3 & 4 > AOS M1 > Topic 1 > Concept 4

Addition, subtraction and scalar multiples Summary screen and practice questions

Exercise 8.3 Addition, subtraction and scalar operations with matrices

1. **WE3** Given the matrices

$$A = \begin{bmatrix} 4 & 7 \\ 3 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 & -3 \\ 5 & 2 \end{bmatrix}, C = \begin{bmatrix} 5 & 7 & -2 \\ 1 & 6 & -5 \end{bmatrix}, D = \begin{bmatrix} -7 & 7 & -2 \\ -5 & 6 & -8 \end{bmatrix}$$

find, if possible:

a. $A + B$ **b.** $B + A$ **c.** $B - C$ **d.** $B - A$ **e.** $D - C$.

2. Given the matrices

$$A = \begin{bmatrix} 12 & 9 \\ -8 & -10 \end{bmatrix}, B = \begin{bmatrix} 15 & 20 & -21 \\ 18 & 10 & -13 \end{bmatrix}, C = \begin{bmatrix} 8 & 17 & -4 \\ 1 & 12 & -9 \end{bmatrix}, D = \begin{bmatrix} -9 & 17 & -12 \\ -12 & 5 & -11 \end{bmatrix}$$

find, if possible:

a. $A + B$ **b.** $B + C$ **c.** $B - C$ **d.** $B - D$ **e.** $D - C$.

3. Given the matrices

$$A = \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 5 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 5 & 0 \\ 3 & -3 & 3 \end{bmatrix}, D = \begin{bmatrix} 4 & 2 & 2 \\ 3 & -2 & -1 \end{bmatrix}$$

calculate, if possible:

a. $A + B$ **b.** $A + A$ **c.** $B - C$ **d.** $B - A$ **e.** $D - C$.

4. MC The solution to $F - E$, given the matrices

$E = \begin{bmatrix} 1.2 & -0.5 \\ 3.6 & 5.0 \\ -3.5 & 2.2 \end{bmatrix}$ and $F = \begin{bmatrix} 0.2 & -0.5 \\ 2.4 & 2.5 \\ 0 & 1.1 \end{bmatrix}$ is:

A. $\begin{bmatrix} 1.0 & 0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$ **B.** $\begin{bmatrix} 1.4 & -1.0 \\ 6.0 & 7.5 \\ -3.5 & 3.3 \end{bmatrix}$ **C.** $\begin{bmatrix} 1.0 & -1.0 \\ 1.2 & 2.5 \\ 3.5 & 1.1 \end{bmatrix}$

D. $\begin{bmatrix} -1.0 & 0 \\ -1.2 & -2.5 \\ 3.5 & -1.1 \end{bmatrix}$ **E.** $\begin{bmatrix} 1.0 & -1.0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$

5. The cost price (C) and sale price (S) of four items at an electrical appliance store are given as follows.

$C = \begin{bmatrix} 23.50 & 45.00 & 87.50 & 140.00 \end{bmatrix}$ and $S = \begin{bmatrix} 49.50 & 135.00 & 169.00 & 299.95 \end{bmatrix}$

Calculate and show, as a matrix, the profit made on the sale of these four items.

6. WE4 Matrix $A = \begin{bmatrix} -6 & 8 \\ 6 & -4 \end{bmatrix}$.

Find:

a. $3A$ **b.** $-2A$ **c.** $\frac{1}{2}A$ **d.** $0.4A$.

7. Given $A = \begin{bmatrix} 2 & -6 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -8 & 0 \\ 7 & -3 \end{bmatrix}$, find:

a. $A + 2B$ **b.** $2A - B$.

8. Given the matrices

$$A = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix},$$

calculate:

a. $3A$ **b.** $0.1B$ **c.** $2A + 3A$ **d.** $A + \frac{1}{4}B$ **e.** $2(A + B)$ **f.** $\frac{1}{2}(3A - B)$.

9. Given the matrices

$$C = \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 2 & 2 & 4 & 1 & 3 \\ 1 & 0 & -2 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & 5 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 6 & 0 & 1 & -2 \\ 2 & 3 & -4 & 1 & 3 \\ 3 & 0 & 2 & 1 & 1 \\ 2 & 2 & -3 & 1 & 10 \\ 5 & 1 & 8 & 1 & 0 \end{bmatrix},$$

use CAS to evaluate the following:

a. $4C$ **b.** $-0.1D$ **c.** $3C - 3D$ **d.** $2(D + 3C)$ **e.** $\frac{1}{2}(C - 3D)$.

10. WE5 Solve the following matrix equations.

a. $4E = \begin{bmatrix} 24 & 40 & -36 \\ 48 & -8 & -80 \end{bmatrix}$ **b.** $D + \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 2 & -5 \end{bmatrix}$

c. If $A = \begin{bmatrix} 3 & 7 \\ -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix}$, find C if $2C + A = 4B$.

11. If $\begin{bmatrix} a & 2 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} -3 & b \\ c & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -2 & d \end{bmatrix}$, find the values of a, b, c and d.

12. Solve the following matrix equations.

a. $6E = \begin{bmatrix} 24 & 42 & -6 \\ 6 & 0 & 12 \end{bmatrix}$

b. $D + \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 5 \end{bmatrix}$

c. If $A = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix}$, calculate C if $2C + A = 4B$.

13. The order of five matrices is given as follows:

$$A = 3 \times 2,\ B = 2 \times 3,\ C = 3 \times 2,\ D = 3 \times 3 \text{ and } E = 3 \times 2.$$

Which of the following cannot be calculated?

a. $-0.3C$ **b.** $D + A$ **c.** $2C - 3A$

d. $2(E + 3C)$ **e.** $\frac{1}{2}(A - 3B)$ **f.** $A + C - E$

14. **WE6** A chain of three car yards has an inventory of front-wheel drive and 4WD cars. They have new and used vehicles of both types. The inventory is represented in matrix form, where the new vehicles are placed in the first column and the front-wheel drive vehicles are placed in the first row.

$$\text{Store } A = \begin{bmatrix} 6 & 4 \\ 4 & 7 \end{bmatrix}, \quad \text{Store } B = \begin{bmatrix} 5 & 6 \\ 2 & 5 \end{bmatrix}, \quad \text{Store } C = \begin{bmatrix} 3 & 3 \\ 3 & 0 \end{bmatrix}$$

a. Which store has the most new 4WDs? State the label of the element.

b. Give the matrix that would represent the total stock of each type of vehicle for all three stores combined.

15. A student has kept records of her test results in matrix form. In semester 1, for English tests she got 72%, 76% and 81% and for Further Maths tests she got 84%, 68% and 82%. In semester 2, for English tests she got 78%, 76% and 89% and for Further Maths tests she got 74%, 77% and 85%.

a. Write the semester results in two separate 3×2 matrices.

b. State the matrix equation that would give the average for each test of the two subjects.

c. Calculate the average result from the two semesters and present it as a 3×2 matrix.

16. A car dealership has three new car sales centres where they stock three models each of 4-wheel-drives and sedans, represented as matrices as follows. The first column represents the 4-wheel-drives and the second column represents the sedans. Each row represents a different model.

$$\text{Centre } A = \begin{bmatrix} 7 & 16 \\ 5 & 8 \\ 2 & 4 \end{bmatrix}, \quad \text{Centre } B = \begin{bmatrix} 13 & 12 \\ 8 & 15 \\ 3 & 1 \end{bmatrix}, \quad \text{Centre } C = \begin{bmatrix} 14 & 7 \\ 12 & 12 \\ 9 & 4 \end{bmatrix}$$

a. If the third row represents the most expensive models, which centre has the most models of expensive sedans?

b. Give the matrix that would represent the total stock of 4-wheel-drives and sedans for all three centres. The wholesale price (in dollars) of each model of 4-wheel-drives and sedans is presented in the following matrix.

$$\begin{bmatrix} 20\,000 & 13\,000 \\ 25\,000 & 18\,000 \\ 40\,000 & 28\,000 \end{bmatrix}$$

c. If the wholesale prices are marked up by 100% to get the selling price, calculate the recommended retail prices.

d. Centre B wishes to have a clearance sale. If it discounts all retail prices by 10%, represent the discounted prices as a matrix.

17. The percentages of households in a township that own no pets is 25%, households with one pet is 40%, those with two pets is 20% and more than two pets is 15%.

a. Set up a 1×4 matrix to represent the percentage ownership of pets.

b. Write an equation that will enable you to calculate the number of households for each category, given that there are 800 households in the town.

c. Evaluate the number of households for each category as a 1×4 matrix.

18. a. Place the following football team's total season score in a suitable matrix format.
Away games total score:
154 goals 214 behinds 1138 points
Home games total score:
207 goals 180 behinds 1422 points

b. Present the average game score for away games and home games as a suitable matrix. There are 11 away games and 11 home games.

19. The stock on hand (S) of items in a warehouse is summarised as a matrix as follows:

$$S = \begin{bmatrix} 23 & 150 & 112 & 5 & 1090 \\ 35 & 210 & 145 & 4 & 800 \\ 45 & 230 & 130 & 8 & 1200 \\ 32 & 700 & 230 & 4 & 400 \\ 77 & 420 & 160 & 7 & 850 \end{bmatrix}$$

a. For the company's plans to reduce costs, complete the following matrix equation if the new stock levels are to be set at 40% of the current stocks.

New stock level (N) = _____ of current stock levels (matrix _____)
= _____ × _____.

b. Evaluate the new stock levels and display as a matrix. (Round answers to the nearest whole number.)

20. Use CAS to store three 3×2 matrices as A, B and C (they can contain any numbers). Check to see whether the following are true.

a. $A + B = B + A$

b. $A - B = B - A$

c. $(A - B) - C = A - (B - C)$

d. $3A + 3B = 3(A + B)$

8.4 Multiplying matrices

In the previous section, we discussed the multiplication of a matrix by a scalar. In this situation, each element of the matrix is multiplied by the scalar, and the order of the matrix remains unchanged. Now we wish to investigate the multiplication of a matrix by another matrix.

8.4.1 The multiplication rule

Two matrices can be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. Consider the following matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 6 \\ 3 & -4 & 0 & 8 \end{bmatrix}$$

The number of columns in the first matrix is 3 and the number of rows in the second matrix is 3. It is possible for these two matrices to be multiplied. A way to remember this is to write the order of each matrix beside each other as shown.

$$2 \times 3 \times 3 \times 4$$

The order of the first matrix is 2×3 and the order of the second matrix is 3×4.

If the two inner numbers (shown in blue) are the same, it is possible to multiply the matrices. If they are not the same, then it will not be possible to multiply the matrices.

The resulting matrix will be of order 2×4 — the outer two numbers in the multiplication shown on the previous page.

If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then $A \times B$ exists and its order is $m \times p$.

The following matrix multiplication is not possible because the number of columns in the first matrix is not equal to the number of rows in the second matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & -4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$3 \times \mathbf{3} \times \mathbf{2} \times 3$$

The two inner numbers are not the same.

The following example shows how two 2×2 matrices are multiplied. (The resulting matrix will be of order 2×2.)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

Each row of the first matrix is multiplied by each column of the second matrix:

$$\begin{bmatrix} r_1c_1 & r_1c_2 \\ r_2c_1 & r_2c_2 \end{bmatrix} \text{ gives } \begin{bmatrix} 1 \times 5 + 2 \times 6 & 1 \times 7 + 2 \times 8 \\ 3 \times 5 + 4 \times 6 & 3 \times 7 + 4 \times 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 + 12 & 7 + 16 \\ 15 + 24 & 21 + 32 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 23 \\ 39 & 53 \end{bmatrix}$$

As long as the number of columns in the first matrix is equal to the number of rows in the second matrix, the method highlighted above can be used to multiply the matrices.

The procedure is repeated for two 3×3 matrices as shown.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ then

$$A \times B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31} & a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32} & a_{11} \times b_{13} + a_{12} \times b_{23} + a_{13} \times b_{33} \\ a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31} & a_{21} \times b_{12} + a_{22} \times b_{22} + a_{23} \times b_{32} & a_{21} \times b_{13} + a_{22} \times b_{23} + a_{23} \times b_{33} \\ a_{31} \times b_{11} + a_{32} \times b_{21} + a_{33} \times b_{31} & a_{31} \times b_{12} + a_{32} \times b_{22} + a_{33} \times b_{32} & a_{31} \times b_{13} + a_{32} \times b_{23} + a_{33} \times b_{33} \end{bmatrix}$$

The rows of the first matrix are multiplied by the columns of the second matrix.

When preparing to multiply two matrices, the order of the resultant matrix should be established first.

WORKED EXAMPLE 7

Given three matrices

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

a. write the order of the three matrices

b. find which of the following products exist

i. ***AB***

ii. ***AC***

iii. ***BA***

iv. ***CA***

c. write the order for the products that exist

d. calculate the products that exist.

THINK	WRITE
a. 1. Matrix A has 2 rows and 2 columns.	**a.** Matrix A is a 2×2 matrix.
2. Matrix B has 3 rows and 2 columns.	Matrix B is a 3×2 matrix.
3. Matrix C has 2 rows and 2 columns.	Matrix C is a 2×2 matrix.
b. i. The number of columns in matrix $A \neq$ the number of rows in matrix B, so AB does not exist.	**b. i.** AB does not exist.
ii. The number of columns in matrix $A =$ the number of rows in matrix C, so AC does exist.	**ii.** AC exists.
iii. The number of columns in matrix $B =$ the number of rows in matrix A, so BA exists.	**iii.** BA exists.
iv. The number of columns in matrix $C =$ the number of rows in matrix A, so CA exists.	**iv.** CA exists.

c. Identify the order of the products that exist by looking at the two outer numbers.

c. The order of AC is 2×2.
The order of BA is 3×2.
The order of CA is 2×2.

d. 1. Multiply the rows of matrix A by the columns of matrix C.

d. $AC = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 3 + 4 \times 4 & 1 \times 2 + 4 \times 1 \\ 3 \times 3 + 0 \times 4 & 3 \times 2 + 0 \times 1 \end{bmatrix}$

2. Simplify AC.

$= \begin{bmatrix} 19 & 6 \\ 9 & 6 \end{bmatrix}$

3. Multiply the rows of matrix B by the columns of matrix A.

$BA = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$

4. Simplify BA.

$= \begin{bmatrix} 2 \times 1 + 1 \times 3 & 2 \times 4 + 1 \times 0 \\ 1 \times 1 + -2 \times 3 & 1 \times 4 + -2 \times 0 \\ 3 \times 1 + 4 \times 3 & 3 \times 4 + 4 \times 0 \end{bmatrix}$

$= \begin{bmatrix} 5 & 8 \\ -5 & 4 \\ 15 & 12 \end{bmatrix}$

5. Multiply the rows of matrix C by the columns of matrix A.

$CA = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$

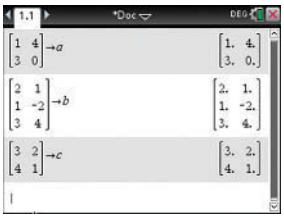
$= \begin{bmatrix} 3 \times 1 + 2 \times 3 & 3 \times 4 + 2 \times 0 \\ 4 \times 1 + 1 \times 3 & 4 \times 4 + 1 \times 0 \end{bmatrix}$

6. Simplify CA.

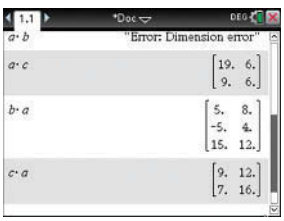
$= \begin{bmatrix} 9 & 12 \\ 7 & 16 \end{bmatrix}$

TI | THINK **WRITE**

b.1. On a Calculator page, define Matrices A, B and C.

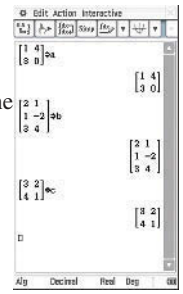


2. Complete the entry lines as:
$a \times b$, then press ENTER.
$a \times c$, then press ENTER.
$b \times a$, then press ENTER.
$c \times a$, then press ENTER.

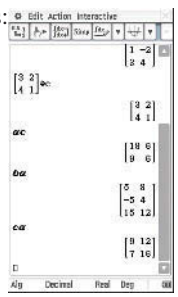


CASIO | THINK **WRITE**

1. On a Main screen, define Matrices A, B and C.
Note: To add an extra row to a 2×2 matrix, tap on the 1×2 icon.To add an extra column to a 2×2 matrix, tap on the 2×1 icon.



2. Complete the entry lines as:
$a \times b$, then tap EXE.
$a \times c$, then tap EXE.
$b \times a$, then tap EXE.
$c \times a$, then tap EXE.



3. The answers are shown on the screen.

i. AB – does not exist.

ii. $AC = \begin{bmatrix} 19 & 6 \\ 9 & 6 \end{bmatrix}$

i. $BA = \begin{bmatrix} 5 & 8 \\ -5 & 4 \\ 15 & 12 \end{bmatrix}$

ii. $CA = \begin{bmatrix} 9 & 12 \\ 7 & 16 \end{bmatrix}$

3. The answers are shown on the screen.

i. AB – The calculator states 'Invalid dimension' for this calculation.

ii. $AC = \begin{bmatrix} 19 & 6 \\ 9 & 6 \end{bmatrix}$

i. $BA = \begin{bmatrix} 5 & 8 \\ -5 & 4 \\ 15 & 12 \end{bmatrix}$

ii. $CA = \begin{bmatrix} 9 & 12 \\ 7 & 16 \end{bmatrix}$

8.4.2 The identity matrix

In Worked example 7, you would have noticed that the results of the multiplication of AC and CA did not produce the same answer. It can be said that matrix multiplication is not commutative.

In general, for two matrices A and B, $AB \neq BA$.

However, there is one exception where a matrix multiplication is commutative. The following situation demonstrates this.

$$\text{Let } A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 4 \times 0 & 1 \times 0 + 4 \times 1 \\ 3 \times 1 + 5 \times 0 & 3 \times 0 + 5 \times 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 0 \times 3 & 1 \times 4 + 0 \times 5 \\ 0 \times 1 + 1 \times 3 & 0 \times 4 + 1 \times 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

Look closely at matrix B. A square matrix with the number 1 for all the elements on the main diagonal and 0 for all the other elements is called an **identity matrix**. The identity matrix can only be defined for square matrices; that is, for matrices of order 1×1, 2×2, 3×3, 4×4 and so on. The identity matrix is commonly referred to as I. Note that the identity matrix is a specific type of diagonal matrix.

From this, it can be stated that $AI = IA = A$. The identity matrix I acts in a similar way to the number 1 when numbers are being multiplied (for example, $2 \times 1 = 1 \times 2 = 2$).

> **$AI = IA = A$ where A is a square matrix and I is the identity matrix of the same order as A.**

8.4.3 Permutation matrix

We mentioned at the start of the topic that a binary matrix is a matrix that consists of elements that are all either 0 or 1. Just like the identity matrix being a special type of a binary matrix, a **permutation matrix** is another special type. A permutation matrix is an $n \times n$ matrix that is a row or column permutation of the identity matrix. Each row and column of a permutation matrix contains the digit 1 exactly once. Permutation matrices reorder the rows or columns of another matrix via multiplication.

If you have a matrix Q and a permutation matrix P, then:

- QP is a column permutation of Q.
- PQ is a row permutation of Q.

Given $Q = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then:

- $QP = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 4 \\ 8 & 9 & 7 \end{bmatrix}$, which is a **column permutation** of Q.

  This is because the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has 1s in the positions a_{13}, a_{21} and a_{32}, thus column $1 \rightarrow 3$, column $2 \rightarrow 1$ and column $3 \rightarrow 2$.
- $PQ = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, which is a **row permutation** of Q.

  This is because the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has 1s in the position a_{13}, a_{21} and a_{32}, thus row $3 \rightarrow 1$, row $1 \rightarrow 2$ and row $2 \rightarrow 3$.

WORKED EXAMPLE 8

Given the matrix $R = \begin{bmatrix} 7 & 3 & -1 \\ 6 & 5 & 2 \\ -3 & 9 & 8 \end{bmatrix}$ and the permutation matrix, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, answer the following:

a. What type permutation is RP?

b. Which column does not change when performing a column permutation of R?

c. When performing a row permutation of R, to which row does row 2 go?

THINK	WRITE
a. Look at the order of the matrix multiplication.	a. Since the permutation matrix is the second matrix in the multiplication of RP, it is a column permutation of R.
b. 1. Look at the positions of the 1s in the permutation matrix to determine the change in columns.	b. The position of the 1s are a_{12}, a_{21} and a_{33}, therefore column $1 \rightarrow 2$, $2 \rightarrow 1$ and $3 \rightarrow 3$.
2. Write the answer.	The column that doesn't change is column 3.
c. 1. Look at the positions of the 1s in the permutation matrix to determine the change in rows.	c. The position of the 1s are a_{12}, a_{21} and a_{33}, therefore row $2 \rightarrow 1$, $1 \rightarrow 2$ and $3 \rightarrow 3$.
2. Write the answer.	Row 2 goes to row 1.

8.4.4 Properties of multiplication of matrices

The following list is a summary of properties of multiplication of matrices. These properties hold true when A, B and C are $m \times n$ matrices, I is an identity matrix and O is a zero matrix.

Property	Example
Associative	$(AB)C = A(BC)$
Identity	$AI = A = IA$
Distributive	$(A + B)C = AC + BC$ $C(A + B) = CA + CB$
Zero matrix	$AO = O = OA$

WORKED EXAMPLE 9

'Soundsmart' has three types of televisions priced at $350, $650 and $890 and three types of DVD players priced at $69, $120 and $250. The store owner wishes to mark up the prices of the televisions by 12% and mark down the prices of the DVD players by 10%.

a. Show the prices of the televisions and the DVD players as a suitable matrix.

b. Show the matrix obtained by marking up the prices of the televisions by 12% and marking down the prices of the DVD players by 10%.

c. Use matrix multiplication to calculate the new prices (correct to the nearest dollar).

THINK

a. Place the televisions and the DVD players in columns and the prices in the rows. This results in a 3×2 matrix.

b. 1. A 12% mark-up is equivalent to 112% or 1.12. A markdown of 10% is equivalent to 90% or 0.9.

2. We need to multiply the first column of the price matrix from **a** by 1.12 and multiply the second column by 0.9. This is a diagonal matrix.

c. 1. Multiply the two matrices to calculate the new prices. The order of the first matrix is 3×2 and the order of the second matrix is 2×2. The resulting matrix will be of order 3×2.

2. Round the answers to the nearest dollar.

WRITE

a. $\begin{bmatrix} 350 & 69 \\ 650 & 120 \\ 890 & 250 \end{bmatrix}$

b. $12\% + 100\% = 112\%, 100\% - 10\% = 90\%$

$\begin{bmatrix} 1.12 & 0 \\ 0 & 0.90 \end{bmatrix}$

c. New prices $= \begin{bmatrix} 350 & 69 \\ 650 & 120 \\ 890 & 250 \end{bmatrix} \times \begin{bmatrix} 1.12 & 0 \\ 0 & 0.90 \end{bmatrix}$

$= \begin{bmatrix} 350 \times 1.12 + 69 \times 0 & 350 \times 0 + 69 \times 0.9 \\ 650 \times 1.12 + 120 \times 0 & 650 \times 0 + 120 \times 0.9 \\ 890 \times 1.12 + 250 \times 0 & 890 \times 0 + 250 \times 0.9 \end{bmatrix}$

$= \begin{bmatrix} 392 & 62.1 \\ 728 & 108 \\ 996.8 & 225 \end{bmatrix}$

The marked-up prices for the three types of televisions will be $392, $728 and $997. The marked-down prices for the three types of DVD players will be $62, $108 and $225.

WORKED EXAMPLE 10

The number of desktop and notebook computers sold by four stores is given in the table.

	Desktop	Notebook
Store *A*	10	4
Store *B*	4	5
Store *C*	5	10
Store *D*	3	2

If the desktop computers were priced at $1500 each and the notebook computers at $2300 each, use matrix operations to find:

a. the total sales figures of each computer at each store

b. the total sales figures for each store

c. the store that had the highest sales figures for

i. desktop computers **ii. total sales.**

THINK

a. 1. Set up a 4×2 matrix to represent the sales figures and a 2×2 matrix to enable us to determine the total sales of each computer at each store.

2. Multiply the two matrices. The resultant matrix displays the total sales of each computer at each store.

b. 1. Use the 4×2 sales matrix from part **a** and a 2×1 column matrix to calculate the total sales figure for each store.

2. Multiply the matrices. The resultant matrix displays the total sales at each store.

Note: CAS can be used to perform the multiplication.

WRITE

a. $\text{Sales matrix} = \begin{bmatrix} 10 & 4 \\ 4 & 5 \\ 5 & 10 \\ 3 & 2 \end{bmatrix}$,

$\text{Price matrix} = \begin{bmatrix} 1500 & 0 \\ 0 & 2300 \end{bmatrix}$

Total sales figures at each store

$$= \begin{bmatrix} 10 & 4 \\ 4 & 5 \\ 5 & 10 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1500 & 0 \\ 0 & 2300 \end{bmatrix}$$

$$= \begin{bmatrix} 15\,000 & 9\,200 \\ 6\,000 & 11\,500 \\ 7\,500 & 23\,000 \\ 4\,500 & 4\,600 \end{bmatrix}$$

b. $\text{Matrix to determine the total sales} = \begin{bmatrix} 1500 \\ 2300 \end{bmatrix}$

$$\text{Total sales} = \begin{bmatrix} 10 & 4 \\ 4 & 5 \\ 5 & 10 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1500 \\ 2300 \end{bmatrix}$$

$$= \begin{bmatrix} 24\,200 \\ 17\,500 \\ 30\,500 \\ 9\,100 \end{bmatrix}$$

The total sales figure for store A was \$24 200, store B was \$17 500, store C was \$30 500 and store D was \$9100.

c. i. Examine the matrix from part **a**. The highest sales figure for desktop computers is the highest number in the first column.

c. i. Store A has the highest sales figure for desktop computers: \$15 000.

ii. Examine the matrix from part **b**. The store with the highest total sales figure is the row with the highest number.

ii. Store C has the highest total sales figure: \$30 500.

on Resources

Interactivity: Matrix multiplication (int-6464)

studyon

Units 3 & 4 > AOS M1 > Topic 1 > Concept 5

Product of matrices — matrix multiplication Summary screen and practice questions

Units 3 & 4 > AOS M1 > Topic 2 > Concept 4

Allocating tasks, binary and permutation matrices Summary screen and practice questions

Exercise 8.4 Multiplying matrices

1. WE7 Given $A = \begin{bmatrix} 8 & 2 \\ -6 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, find:

a. AB **b.** BA **c.** B^2.

2. For the three matrices

$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$:

a. write the order of the three matrices

b. find which of these products exist.

i. AB **ii.** AC **iii.** BA **iv.** CA **v.** BC

3. Given three matrices

$A = \begin{bmatrix} 2 & 4 \\ -3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}$

a. write the order of the three matrices

b. find which of the following products exist

i. AB **ii.** BA **iii.** AC **iv.** CA **v.** BC **vi.** CB

c. write the order of the products that exist

d. calculate the products that exist.

4. **a.** For $M = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, show all your working and find the products:

i. MN **ii.** NM.

b. For $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show all your working and find the products:

i. MN **ii.** NM.

c. Explain why $MN = NM$ (from part **b** above), but $MN \neq NM$ (from part **a**).

5. WE8 Given the matrix $R = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 5 & -7 \\ -4 & 9 & 6 \end{bmatrix}$ and the permutation matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, answer the following:

a. What type of permutation is RP?

b. Which column does not change when performing a column permutation of R?

c. When performing a row permutation of R, which row does row 1 go to?

6. Given the matrix $R = \begin{bmatrix} 2 & 3 & -7 \\ 4 & 6 & -2 \\ -1 & 9 & 8 \end{bmatrix}$ and the permutation matrix, $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, answer the following:

a. What type of permutation is PR?

b. With a row permutation of R, where does row 1 move to?

c. If a row permutation of R resulted in $R' = \begin{bmatrix} -1 & 9 & 8 \\ 2 & 3 & -7 \\ 4 & 6 & -2 \end{bmatrix}$, then find the permutation matrix that achieved this.

7. MC If $A = B \times C$, then the element a_{31} is the result of:

A. multiplying the third row by the first column **B.** multiplying the first row by the third column

C. multiplying the third column by the first row **D.** multiplying the first column by the third row

E. multiplying the third row by the first row

8. MC Given the matrices.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \\ 3 & 6 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

a. which one of the following products does not exist?

A. AB **B.** BC **C.** CA **D.** DE **E.** CE

b. the order of the matrix BE is:

A. 3×4 **B.** 2×2 **C.** 4×3 **D.** 3×2 **E.** 2×4

c. which of the following products gives a matrix of order 2×2?

A. BA **B.** BC **C.** CA **D.** AB **E.** BD

d. the matrix CE is:

A. $\begin{bmatrix} 1 & 5 & 0 & 16 \\ 5 & 12 & 13 & 28 \end{bmatrix}$ **B.** $\begin{bmatrix} 2 & 9 & 1 & 28 \\ 3 & 7 & 8 & 16 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 2 \\ 7 & 9 \\ 8 & 1 \\ 16 & 28 \end{bmatrix}$ **D.** $\begin{bmatrix} 2 & 3 \\ 9 & 7 \\ 1 & 8 \\ 28 & 16 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 7 & 8 & 16 \\ 2 & 9 & 1 & 28 \end{bmatrix}$

e. the matrix D^2 is:

A. $\begin{bmatrix} 4 & 30 & 12 \\ 6 & 37 & 16 \\ 12 & 60 & 24 \end{bmatrix}$ **B.** $\begin{bmatrix} 4 & 30 & 6 \\ 6 & 37 & 4 \\ -6 & 24 & 24 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 30 & 6 \\ 6 & 37 & 4 \\ 0 & 24 & 24 \end{bmatrix}$ **D.** $\begin{bmatrix} 4 & 30 & 6 \\ 6 & 37 & 4 \\ 6 & 24 & 24 \end{bmatrix}$

E. $\begin{bmatrix} 4 & 2 & 30 & 30 & 6 & 12 \\ 6 & 6 & 37 & 24 & 4 & 8 \\ -6 & 12 & 24 & 36 & 24 & 12 \end{bmatrix}$

9. Simplify the expressions below for the matrices

$A = \begin{bmatrix} 2 & 4 \\ -3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

a. $2A + AB$ **b.** $A(B + C)$ **c.** $AB + CD$ **d.** $BA + DC$ **e.** $2DB - D$.

10. A matrix of 3×3 order is the product of three matrices S, E and N in that order ($S \times E \times N$). If matrix $E = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ then the order of matrix N is:

A. 3×2 **B.** 2×3

C. 3×3 **D.** 2×2

E. Not enough information. Also need the order of matrix S.

11. **WE9** The 'Whiteguys Store' has three types of washing machines, priced at \$550, \$750 and \$990, and three types of dryers, priced at \$160, \$220 and \$350. The owner of the store wishes to mark up the prices of the washing machines by 8% and mark down the prices of the dryers by 8%.

a. Show the original prices of washing machines and dryers as a suitable matrix.

b. Show the diagonal matrix that would mark up washing machines by 8% and mark down the dryers by 8%.

c. Calculate the new prices (correct to the nearest dollar) using the matrices from a and b.

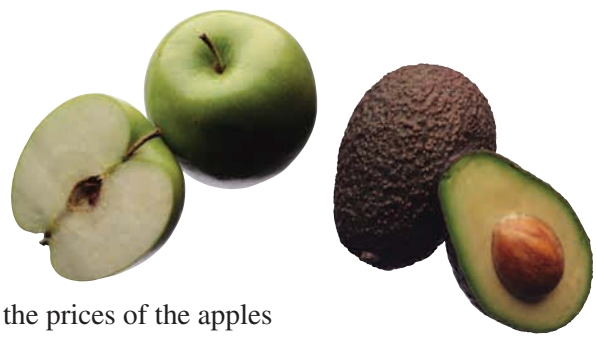
12. A supermarket has three types of lettuces priced at \$1.50, \$2.00 and \$2.75 each and three types of potatoes priced at \$2.50, \$3.00 and \$3.25 per kilogram.
The manager wants to mark up the prices of lettuces by 15% and mark down the prices of potatoes by 12%.

a. Show the original prices of the lettuces and potatoes as a suitable matrix.

b. Show the diagonal matrix that would mark down the prices of potatoes by 12% and mark up the prices of lettuces by 15%.

c. Use matrix multiplication to calculate the new prices (correct to the nearest cent).

13. A supermarket has three types of apples priced at \$2.50, \$3.50 and \$4.00 per kilogram and three types of avocados priced at \$0.90, \$1.90 and \$2.50. The manager wishes to mark up the prices of the avocados by 15% and mark down the prices of the apples by 15%.



a. Show the original prices of the apples and avocados as a suitable matrix.

b. Show the diagonal matrix that would mark down the prices of the apples by 15% and mark up the prices of the avocados by 15%.

c. Use matrix multiplication to calculate the new prices (correct to the nearest cent).

14. A sports store has four types of tennis racquets priced at $25.00, $35.00, $95.00 and $140.00 and four types of footballs priced at $9.90, $19.90, $75.00 and $128.00. The manager wishes to mark down the prices of all items by 20% in preparation for a sale.

a. Show the original prices of the tennis racquets and footballs as a suitable matrix.

b. Show the diagonal matrix that would mark down the prices by 20%.

c. Use matrix multiplication to calculate the sale prices (correct to the nearest 10 cents).

15. **WE10** The number of pies and cans of soft drinks sold to four year-level groups at the canteen on a particular day is given in the table below.

Year	Pies	Soft drinks
Year 12	10	26
Year 10/11	25	45
Year 8/9	22	30
Year 7	5	22

If the pies were priced at $2.50 each and the soft drinks at $1.00 each, use matrix operations to find:

a. the total sales figures of each food item for each year-level group

b. the total sales figures for each year-level group

c. the year-level group that had the highest sales figures for

i. pies **ii.** total sales.

16. The number of iPads and iPad minis sold by four stores is given in the table shown.

Store	iPad	iPad mini
Store A	14	5
Store B	9	7
Store C	10	8
Store D	7	6

If the iPads were priced at $550 each and the iPad minis at $320 each, use matrix operations to find:

a. the total sales figures of iPads at each store

b. the total sales figures for each store

c. the store that had the highest sales figures for:

i. iPads **ii.** total sales.

17. Four golf shops, A, B, C and D are supplied with boxes of different brands of golf balls (Longer, Higher, Further and Straighter) as shown in the table.

Shop	Longer	Higher	Further	Straighter
Shop A	12	10	10	12
Shop B	15	25	15	25
Shop C	10	10	10	10
Shop D	8	20	5	18

The cost of the golf balls, per box, is:

Longer: $15 Higher: $20 Further: $30 Straighter: $32

a. Present the stock of golf balls for the four shops as a 4×4 matrix.

b. Write the costs as a matrix.

c. Use matrix multiplication to determine the total value of each golf ball brand at each shop.

d. Calculate the total value of golf balls at each shop.

18. MC Given the matrix $D = \begin{bmatrix} 1 & 3 & -3 \\ 5 & 7 & 8 \\ -6 & 4 & 2 \end{bmatrix}$ and the permutation matrix, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, answer the following questions.

a. What type of permutation is PD?

b. With a row permutation of D, where does row 1 move to?

c. With a column permutation of D, where does column 3 move to?

d. Which of the following is not a permutation matrix?

A. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **D.** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. If a row permutation of D resulted in $D' = \begin{bmatrix} -6 & 4 & 2 \\ 1 & 3 & -3 \\ 5 & 7 & 8 \end{bmatrix}$, find the permutation matrix that achieved this.

19. The sales figures (in dollars) for three months, at three stores, of three brands of mobile phones are shown in the following tables.

February	Mobile A ($)	Mobile B ($)	Mobile C ($)
Store A	600	500	0
Store B	480	750	840
Store C	240	1000	0

March	Mobile A ($)	Mobile B ($)	Mobile C ($)
Store A	1200	1000	420
Store B	840	1500	1260
Store C	1200	1750	0

April	Mobile A ($)	Mobile B ($)	Mobile C ($)
Store A	1440	750	1680
Store B	600	1500	2100
Store C	1560	250	420

a. Write a matrix to represent the sales figures for each month.

b. Use a suitable matrix operation to show the total sales figures ($) for the three months by store and mobile phone brand.

c. If brand A mobile phones cost $120, brand B mobile phones cost $250 and brand C mobile phones cost $420, use an appropriate matrix operation to calculate how many mobile phones of each brand were sold in total by each store, for the three months.

d. Calculate the total sales for each store for the three months.

20. The Fibonacci numbers are a part of an interesting sequence of numbers that have been the focus of a great deal of study over the years. The first 12 numbers of the sequence are presented in the table.

Term	1	2	3	4	5	6	7	8	9	10	11	12
Number	1	1	2	3	5	8	13	21	34	55	89	144

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ can be used to find terms of the Fibonacci sequence.

a. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and complete the following calculations using CAS.

i. $A \times A$ (that is, A^2) **ii.** A^3 **iii.** A^4 **iv.** A^5

b. Study each of the answers found in part **a** carefully. How do the elements in each answer and the power in the question relate to the Fibonacci numbers?

c. Using your finding from part **b** (that is, without performing any calculations) what will be the elements in the matrix A^8?

d. Use CAS and matrix methods to find the 30th Fibonacci number.

8.5 Multiplicative inverse and solving matrix equations

Recall from arithmetic that any number multiplied by its reciprocal (multiplicative inverse) results in 1. For example, $8 \times \frac{1}{8} = 1$.

Now, consider the matrix multiplication below.

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 3 \times -3 & 2 \times -3 + 3 \times 2 \\ 3 \times 5 + 5 \times -3 & 3 \times -3 + 5 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that the answer is the identity matrix (I).

This means that one matrix is the **multiplicative inverse** of the other. In matrices, we use the symbol A^{-1} to denote the multiplicative inverse of A. A must be a square matrix.

If $AA^{-1} = A^{-1}A = I$, then A^{-1} is called the multiplicative inverse of A.

8.5.1 Finding the multiplicative inverse of a square 2 × 2 matrix

In matrices, we are often given a square 2×2 matrix and are asked to determine its multiplicative inverse. The following notes display how this is achieved.

Let A represent a matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. Swap the elements on the main diagonal and multiply the elements on the other diagonal by -1. This results in the matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2. Multiply this matrix by $\frac{1}{ad-bc}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Note: The value of $ad - bc$ is known as the **determinant** of matrix A. It is commonly written as det A or $|A|$.

WORKED EXAMPLE 11

Calculate the determinant of each of the following matrices.

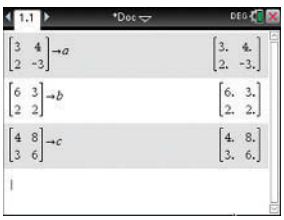
$A = \begin{bmatrix} 3 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}$

THINK	WRITE
1. For a matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is given by $ad - bc$.	det $A = ad - bc$
2. Calculate the determinant for each of the three matrices.	det $A = 3 \times -3 - 4 \times 2$ $= -17$ det $B = 6 \times 2 - 3 \times 2$ $= 6$ det $C = 4 \times 6 - 8 \times 3$ $= 0$

TI | THINK

1. On a Calculator page, define Matrices A, B and C.

WRITE



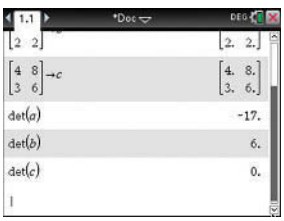
2. Press MENU and select:
7: Matrix & Vector
3: Determinant

Complete the entry line as:

det(a), then press ENTER.

Repeat for Matrices B and C.

WRITE



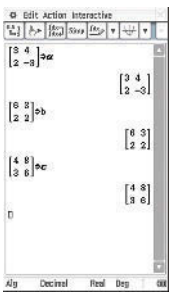
3. The answers are shown on the screen.

a. det(A) = −17
b. det(B) = 6
c. det(C) = 0

CASIO | THINK

1. On a Main screen, define Matrices A, B and C.

WRITE



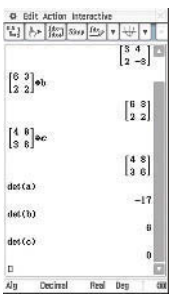
2. Tap:
Action
Matrix
Calculation
det

Complete the entry lines as:

det(a), then tap EXE.

Repeat for Matrices B and C.

WRITE



3. The answers are shown on the screen.

a. det(A) = −17
b. det(B) = 6
c. det(C) = 0

8.5.2 Singular and regular matrices

Notice that the determinant of matrix C in Worked example 11 was 0. For any matrix that has a determinant of 0, it is impossible for an inverse to exist. This is because $\frac{1}{0}$ is undefined. A matrix with a determinant equal to 0 is called a **singular matrix**.

If the determinant of a matrix is not 0, it is called a regular matrix.

WORKED EXAMPLE 12

If $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, find its inverse, A^{-1}.

THINK

1. Calculate the determinant of A. (If the determinant is equal to 0, the inverse will not exist.)
2. Use the rule to find the inverse. That is, $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Swap the elements on the main diagonal and multiply the elements on the other diagonal by -1. Multiply this matrix by $\frac{1}{ad - bc}$.

WRITE

$\det A = ad - bc$
$= 3 \times 4 - 5 \times 2$
$= 2$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$

Note: Fractional scalars can be left outside the matrix unless they give whole numbers when multiplied by each element.

TI | THINK

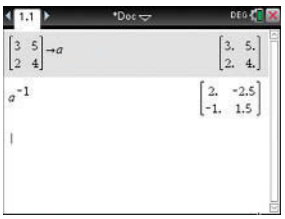
1. On a Calculator page, define the Matrix A.

   Complete the entry line as:

   a^{-1}, then press ENTER.

2. The answer is shown on the screen.

WRITE

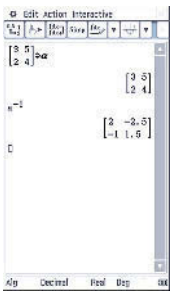


$A^{-1} = \begin{bmatrix} 2 & -2.5 \\ -1 & 1.5 \end{bmatrix}$

CASIO | THINK

1. On a Main screen, define the Matrix A.
2. The answer is shown on the screen.

WRITE



$A^{-1} = \begin{bmatrix} 2 & -2.5 \\ -1 & 1.5 \end{bmatrix}$

For matrices of a higher order than 2×2, for example, 3×3, 4×4, and so on, finding the inverse (and determinant) is more difficult and CAS is required.

8.5.3 Further matrix equations

Recall from algebra that to solve an equation in the form $4x = 9$, we need to divide both sides by 4 $\left(\text{or multiply both sides by } \frac{1}{4}\right)$ to obtain the solution $x = \frac{9}{4}$.

A matrix equation of the type $AX = B$ is solved in a similar manner. Both sides of the equation are multiplied by A^{-1}. Since the order of multiplying matrices is important, we must be careful of the position of the inverse (remember that the products of AX and XA are different).

Solving for X in the following situations:

1. For $AX = B$
   Pre-multiply by A^{-1}: $A^{-1}AX = A^{-1}B$
   $IX = A^{-1}B$ since $A^{-1}A = I$
   $X = A^{-1}B$ since $IX = X$
2. For $XA = B$
   Post-multiply by A^{-1}: $XAA^{-1} = BA^{-1}$
   $XI = BA^{-1}$ since $AA^{-1} = I$
   $X = BA^{-1}$ since $XI = X$

In summary:
1. **If $AX = B$, then $X = A^{-1}B$.**
2. **If $XA = B$, then $X = BA^{-1}$.**

WORKED EXAMPLE 13

For the given matrices
$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 16 & -3 \\ 3 & 7 \end{bmatrix}$
solve for the unknown matrix X if:

a. $AX = B$ **b.** $AX = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$.

THINK

a. 1. We are required to pre-multiply by A^{-1} to get matrix X by itself.

2. Find A^{-1}. First calculate the determinant. Then swap the elements on the leading diagonal of A and multiply the elements on the other diagonal by -1.

3. Write the equation to be solved and substitute in the matrices.

4. Calculate the product of A^{-1} and B. Multiply each element by the fractional scalar as they will all result in whole numbers.

WRITE

a. $AX = B$
$A^{-1}AX = A^{-1}B$
$X = A^{-1}B$
$\det A = ad - bc$
$= 2 \times 3 - 5 \times -1$
$= 11$
$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
$= \frac{1}{11}\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$
$X = A^{-1}B$
$= \frac{1}{11}\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 16 & -3 \\ 3 & 7 \end{bmatrix}$
$= \frac{1}{11}\begin{bmatrix} 33 & -44 \\ 22 & 11 \end{bmatrix}$
$= \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$

b. 1. We are required to pre-multiply by A^{-1} to get matrix X by itself.

2. Use A^{-1} from part **a**. Calculate the product of A^{-1} and $\begin{bmatrix} 13 \\ -1 \end{bmatrix}$. Multiply each element by the fractional scalar.

b
$$AX = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$$
$$A^{-1}AX = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$$
$$X = A^{-1}\begin{bmatrix} 13 \\ -1 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ -1 \end{bmatrix}$$
$$= \frac{1}{11}\begin{bmatrix} 44 \\ 11 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

on Resources

Interactivity: inverse matrices (int-6465)

studyon

Units 3 & 4 > AOS M1 > Topic 2 > Concepts 1 & 2

Determinants and inverses of matrices Summary screen and practice questions

Solving matrix equations Summary screen and practice questions

Exercise 8.5 Multiplicative inverse and solving matrix equations

1. WE 11 Calculate the determinants of the following matrices.
$A = \begin{bmatrix} 2 & 2 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 3 & -5 \end{bmatrix}$

2. Calculate the determinants of the following matrices.
$A = \begin{bmatrix} -6 & -8 \\ 15 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}$

3. Calculate the determinants of the following matrices.

a. $A = \begin{bmatrix} 9 & 8 \\ 7 & 5 \end{bmatrix}$ **b.** $B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ 1 & \frac{1}{3} \end{bmatrix}$ **c.** $C = \begin{bmatrix} 5 & 9 \\ 8 & 3 \end{bmatrix}$ **d.** $D = \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix}$

4. State which matrices from question **3** will not have an inverse. Explain your answer.

5. WE 12 If $A = \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix}$, find its inverse, A^{-1}.

6. For the matrix $C = \begin{bmatrix} -5 & -1 \\ 10 & 4 \end{bmatrix}$, find:

a. det C **b.** C^{-1}, the inverse of C.

7. Calculate the inverse matrix for each matrix (where possible) in question **3**.

8. If $C = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}$, find its inverse, C^{-1}.

9. Given the two matrices below, show that A and B are inverses of each other.

$A = \begin{bmatrix} 11 & 12 \\ 10 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & -12 \\ -10 & 11 \end{bmatrix}$

10. MC If $T = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix}$, then T^{-1} is equal to:

A. $-\frac{1}{4}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$ **B.** $\frac{1}{4}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$ **C.** $-\frac{1}{4}\begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}$

D. $-\frac{1}{4}\begin{bmatrix} 2 & -2 \\ 3 & -5 \end{bmatrix}$ **E.** $\frac{1}{4}\begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}$

11. MC If $P = \begin{bmatrix} 12 & 4 \\ -12 & -6 \end{bmatrix}$, then P^{-1} could be:

A. $\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$ **B.** $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **C.** $\begin{bmatrix} \frac{1}{20} & \frac{1}{30} \\ -\frac{1}{10} & -\frac{1}{10} \end{bmatrix}$

D. $\begin{bmatrix} 0.3 & 0.2 \\ -0.5 & -0.5 \end{bmatrix}$ **E.** $\begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

12. For the matrix $\begin{bmatrix} 0.1 & 0.2 \\ 0.25 & 0.45 \end{bmatrix}$:

a. calculate the determinant

b. state the inverse in the form $\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

13. Use CAS to find:

i. the determinant **ii.** the inverse matrixfor each of the following matrices.

a. $A = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} -3 & 4 \\ 3 & -6 \end{bmatrix}$

c. $C = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$ **d.** $D = \begin{bmatrix} 0.4 & 1.0 \\ 0.2 & 0.25 \end{bmatrix}$

e. $E = \begin{bmatrix} 5 & 2 \\ -10 & -4 \end{bmatrix}$ **f.** $F = \begin{bmatrix} 0.3 & 0.48 \\ 0.5 & 0.8 \end{bmatrix}$

g. $G = \begin{bmatrix} 4 & 6 & 8 \\ 6 & 4 & 6 \\ 8 & 6 & 4 \end{bmatrix}$ **h.** $H = \begin{bmatrix} 12 & 8 & 0 & 4 \\ 8 & 4 & 8 & 8 \\ 4 & 12 & 8 & 12 \\ 8 & 4 & 4 & 8 \end{bmatrix}$

i. $I = \begin{bmatrix} 0 & 1 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 2 & 1 & 0 & 4 \\ 2 & 1 & 3 & 0 \end{bmatrix}$ **j.** $J = \begin{bmatrix} 1 & 4 \\ 5 & 3 \\ 6 & 3 \end{bmatrix}$

14. WE13 For the matrices $A = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -3 \\ 3 & 7 \end{bmatrix}$, solve for the unknown matrix X, given:

a. $AX = B$ **b.** $AX = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$.

15. For the matrices $A = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 8 \\ -5 & 7 \end{bmatrix}$ solve for the unknown matrix X, given:

a. $AX = B$ **b.** $AX = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$.

16. For the given matrices $A = \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -3 \\ 3 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 0 \\ 3 & 6 \end{bmatrix}$ solve for the unknown matrix X if:

a. $AX = B$ **b.** $XA = B$ **c.** $XC = A$ **d.** $CX = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$.

17. If $F = \begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}$ and $G = \begin{bmatrix} 0.2 & -1.2 \\ 0.1 & 2.8 \end{bmatrix}$, solve the following matrix equations.

a. $FX = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ **b.** $GX = \begin{bmatrix} -1.0 \\ 2.9 \end{bmatrix}$

18. Solve the following matrix equation for X.

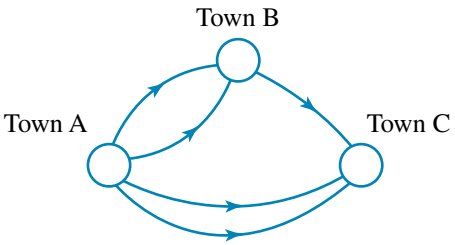
$$\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} X = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$$

8.6 Dominance and communication matrices

A **directed graph** (or digraph) is a graph or network where every edge has a direction. Directed graphs can be used to represent many situations, such as traffic flow, competitions between teams or the order of activities in a production line.

8.6.1 Reachability

As the name suggests, **reachability** is the concept of how it is possible to go from one vertex in a directed network to another. The different pathways that link the vertices are analysed.



Consider the directed network shown, representing possible pathways (routes) from town A to town C. By inspection, it can be seen that there are two pathways that go directly from A to C, without passing through B. That is, there are two **one-stage pathways** from A to C. A one-stage pathway is one that includes one edge only.

There are also two pathways that go from A to C via B. These are called **two-stage pathways**. A two-stage pathway is one that contains two edges only.

Notice that there are no routes entering town A but there are four leaving it. We say that the **indegree** of A is zero, while its **outdegree** is four. The indegree is the number of edges moving into a vertex and the outdegree is the number of edges moving away from a vertex. The indegree of B is two and its outdegree is one. The indegree of C is three and its outdegree is zero. A is the **source** and C is the **sink** of the network.

8.6.2 Matrix representation

The one-stage and two-stage pathways for a directed network can be represented in matrix form. The matrix here displays all of the possible one-stage pathways for the previous network. It is commonly known as the **adjacency matrix** and is denoted by A.

The matrix shows that there are two one-stage pathways from A to B and two one-stage pathways from A to C. There is also a one-stage pathway from B to C. Notice that the sum of each row is equal to the outdegree of each vertex and the sum of each column is equal to the indegree of each vertex. This can be a useful tip to ensure you have completed the adjacency matrix correctly.

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{From} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\text{A} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Adjacency matrix

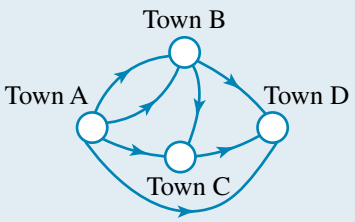
The matrix here displays all of the possible two-stage pathways of the network. This matrix is denoted by A^2. There are two two-stage pathways from A to C (via B).

Note that it is also possible to represent other stage pathways in matrix form; for example, three-stage is denoted by A^3, four-stage is denoted by A^4 and so on. This text will only concentrate on representing up to two-stage pathways in matrix form.

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{From} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{A}^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

WORKED EXAMPLE 14

Wendy is a businessperson working in Town A and wishes to meet with a colleague in Town D. She also needs to pick up some documents from Town B to take to the meeting.



a. Find the number of two-stage paths that Wendy could take and name them.

b. Find the number of three-stage paths that Wendy could take and name them.

c. Represent the one-stage and two-stage pathways of the directed network in matrix form.

THINK	WRITE
a. 1. Find the number of pathways from A to B (first stage).	**a.** There are two paths from A to B.
2. Find the number of pathways from B to D (second of the two stages).	There is only one path from B to D.
3. Write the pathways.	There are 2 two-stage paths. A–B–D and A–B–D
b. 1. What two-stage pathways are possible from B to D?	**b.** The only two-stage path from B to D is B–C–D.
2. Write the pathways.	The only three-stage paths are A–B–C–D and A–B–C–D. There are two possible routes.

c. 1. Draw a matrix to show all the possible one-stage pathways throughout the network.

c.

$$\text{From}\begin{matrix}\\ \text{A}\\ \text{B}\\ \text{C}\\ \text{D}\end{matrix}\begin{matrix}\begin{matrix}\text{To}\\ \text{A}\end{matrix} & \begin{matrix}\\ \text{B}\end{matrix} & \begin{matrix}\\ \text{C}\end{matrix} & \begin{matrix}\\ \text{D}\end{matrix}\\ \begin{bmatrix}0 & 2 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\end{matrix}$$

2. The matrix can be represented without the labels along the side and the top.

$$\text{A} = \begin{bmatrix}0 & 2 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

3. Repeat steps 1 and 2 and display all the two-stage pathways throughout the network.

$$\text{A}^2 = \begin{bmatrix}0 & 0 & 2 & 3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

8.6.3 Dominance

If an edge in a directed network moves from A to B, then it can be said that A is dominant, or has a greater influence, over B. If an edge moves from B to C, then B is dominant over C. However, we often wish to find the dominant vertex in a network; that is, the vertex that holds the most influence over all the other vertices. This may be clearly seen by inspection, by examining the pathways between the vertices. It may be the vertex that has the most edges moving away from it. Generally speaking, if there are more ways to go from A to B than there are to go from B to A, then A is the dominant vertex. In Worked example 14, town A is dominant over all the other vertices (towns) as it has edges moving to each of the other vertices. Similarly, B has edges moving to C and D, so B is dominant over C and D and C is dominant over D. Using this inspection technique, we can list the vertices in order of dominance from A then B then C and finally D.

However, the dominant vertex in a directed network may not be easily determined by inspection. There may be an edge moving from A to B and another one from B to A. Which is the more dominant then? A more formal approach to determine a dominant vertex can be taken using matrix representation. Using the matrices from Worked example 15, this approach is outlined below.

Take the matrices that represent the one-stage pathways (the adjacency matrix, A) and two-stage pathways (A^2) and add them together. (When adding matrices, simply add the numbers in the corresponding positions.)

$$\begin{matrix}A & & A^2\end{matrix}$$
$$\begin{bmatrix}0 & 2 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 2 & 3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 2 & 3 & 4\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The resulting matrix, which we will call the **dominance matrix**, consists of all the possible one- and two-stage pathways in the network. By taking the sum of each row in this matrix, we can determine the **dominant vertex**. The dominant vertex belongs to the row that has the highest sum.

The first row corresponds with vertex A and has a sum of 9. Row 2 (vertex B) has a sum of 3, row 3 (vertex C) has a sum of 1 and row 4 (vertex D) has a sum of 0. The highest sum is 9, so the dominant vertex is A. The order of dominance is the same as for the inspection technique described earlier.

This formal approach just described is not the only technique used to determine dominance in a network. Other approaches are possible, but this section will concentrate only on the inspection technique and summing the rows of the matrix that results from $A + A^2$.

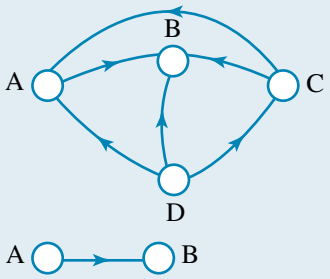
The concept of dominance can be applied to various situations such as transportation problems, competition problems and situations involving relative positions.

WORKED EXAMPLE 15

The results of a round robin (each competitor plays each other once) tennis competition are represented by the directed graph at right.

a. By inspection, determine the dominant vertex (dominant competitor); that is, the winner. Rank the competitors in finishing order.

b. Confirm your answer to part a by finding the matrix, $A + A^2$, and summing the rows of this matrix.



means A beats B

THINK

a. Vertex D has arrows moving to all other vertices, so D is the dominant vertex. Vertex C is dominant over B and A and vertex A is dominant over B. List the competitors (vertices) in finishing order. (It is clear that B is the loser as all the arrows lead into it.)

b. 1. Find the adjacency matrix, A (representing all of the one-stage pathways in the directed graph).

2. Find A^2, the matrix representing all the two-stage pathways in the matrix.

3. Find the resultant matrix when A and A^2 are added (add the numbers in the corresponding positions).

4. Determine the sum of each row of the resultant matrix.

5. Write a statement. The row with the highest sum is the dominant vertex and the row with the lowest sum is the least dominant.

WRITE

a. The finishing order from first to fourth is D, C, A and B.

b. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$

The resultant matrix is:

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix}$.

The sum of the first row (vertex A) is 1, the second row (vertex B) is 0, the third row (vertex C) is 3 and the final row (vertex D) is 6.

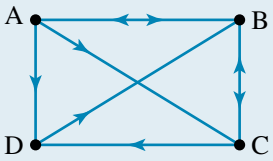
D is the winner, followed by C, then A and finally B.

8.6.4 Communication

In Worked example 15 we looked at results of a round robin tennis competition. This is where one competitor either wins or loses to another competitor. Thus, the network has arrows going only one way. A **communication network** has the ability of the arrows to travel both ways. A communication network contains a set of people where they can have a one-way or two-way communication link. A two-way communication link could be via phone. We can set up **communication matrices** from communication networks.

WORKED EXAMPLE 16

Find the communication matrix from the following communication network.



THINK

WRITE

1. Set up the communication matrix with 'send call' being the vertical and 'receive call' being the horizontal.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

2. Place 1s where A can communicate, that is with B, C and D.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

3. Place 1s where B can communicate, that is with A and C.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ & & & \\ & & & \end{bmatrix} \end{array}$$

4. Place 1s where C can communicate, that is with B and D.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & & & \end{bmatrix} \end{array}$$

5. Place 1s where D can communicate, that is B.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

6. Write the answer.

The communication matrix for the communication network shown is:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Resources

Interactivity: The adjacency matrix (int-6466)

studyon

Units 3 & 4 > AOS M2 > Topic 2 > Concepts 2, 5 and 6

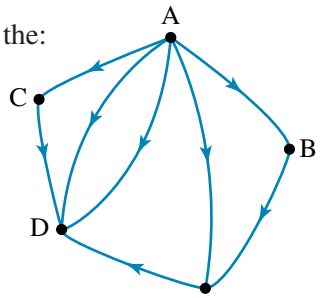
Matrix representation of directed graphs Summary screen and practice questions

Dominance matrix Summary screen and practice questions

Communication matrix Summary screen and practice questions

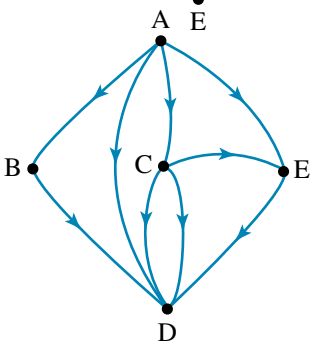
Exercise 8.6 Dominance and communication matrices

1. WE14 For the directed network at right determine the number and name of the:



- **a.** **i.** one-stage paths
  ii. two-stage paths
  iii. three-stage paths from A to D.
- **b.** Represent the one-stage and two-stage pathways of the directed network in matrix form.

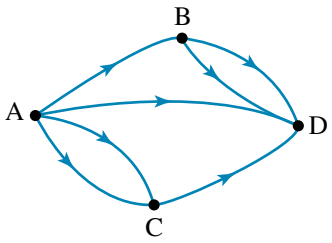
2. For the directed network at right determine the number and name of the:



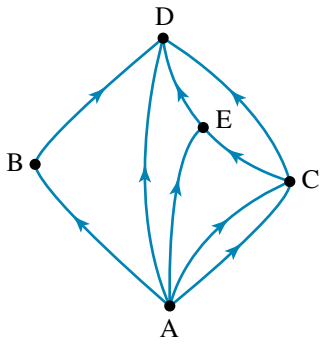
- **a.** **i.** one-stage paths
  ii. two-stage paths
  iii. three-stage paths from A to D.
- **b.** Represent the one-stage and two-stage pathways of the directed network in matrix form.

3. For each directed network shown determine the number and name of the:
   - i. one-stage paths
   - ii. two-stage paths
   - iii. three-stage paths from A to D.

a.

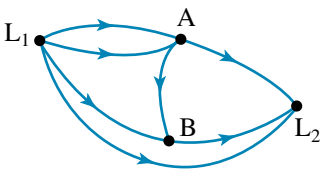


b.

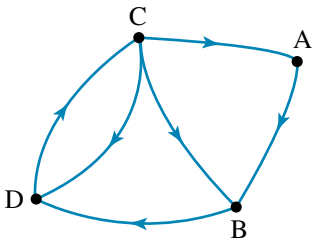


4. Represent the one-stage and two-stage pathways of the directed networks in question **3** in matrix form.

5. The directed graph represents part of a river system where the water flows from the lake, L_1, to another lake, L_2. If fish eggs flow from L_1 to L_2, via how many different routes is it possible for the eggs to go? Name all the routes.

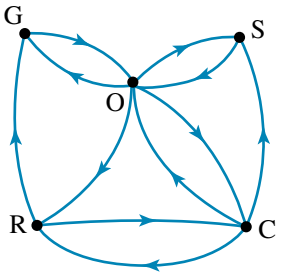


6. The bus routes between certain landmarks are shown in the diagram. Name all the different routes by which it is possible to reach:
   - **a.** D from A
   - **b.** A from D
   - **c.** B from D.



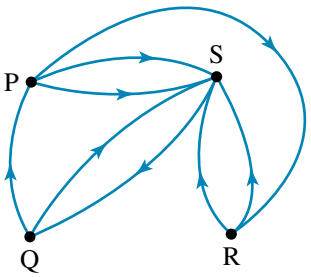
7. The directed network represents the pathways available to students as they move around their school. Name the different pathways by which the students can get from the:
   - **a.** office to the gym
   - **b.** common room to the science block
   - **c.** science block to the common room

   given that they wish to make a:
   - i. one-stage trip
   - ii. two-stage trip
   - iii. three-stage trip.

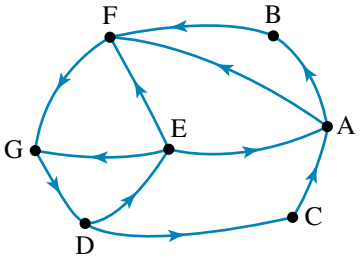


G = gym
S = science block
O = office
C = cafeteria
R = common room

8. **WE15** Determine the dominant vertex for the following directed graph.



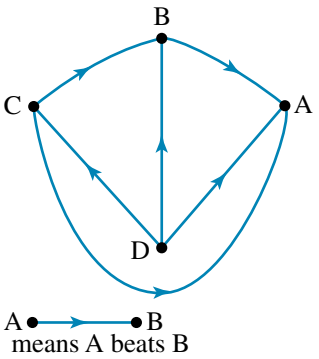
9. Determine the dominant vertex for the following directed graph.



10. The results of a round robin chess competition are represented by the directed graph.

a. By inspection, determine the dominant vertex (dominant competitor); that is, the winner. Rank the competitors in finishing order.

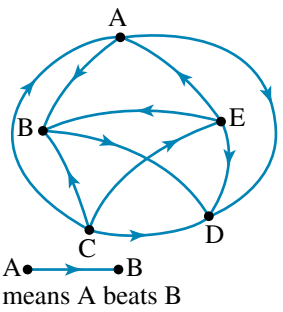
b. Confirm your answer to part **a** by finding the matrix, $A + A^2$, and summing the rows of this resultant matrix.



11. The results of a round robin basketball competition are represented by the directed graph.

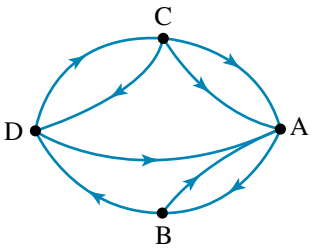
a. By inspection, determine the dominant vertex (dominant team); that is, the winner. Rank the teams in finishing order.

b. Confirm your answer to part **a** by finding the matrix, $A + A^2$, and summing the rows of this resultant matrix.

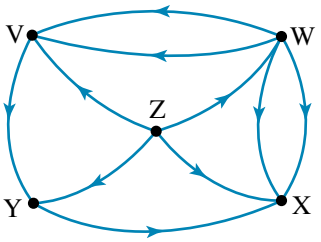


12. Determine the dominant vertex for each of the following directed graphs.

a.

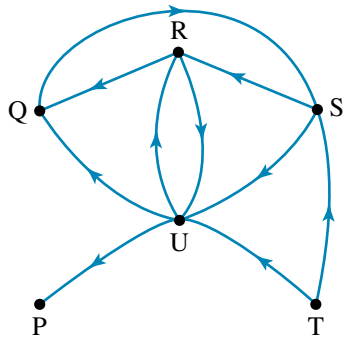


b.

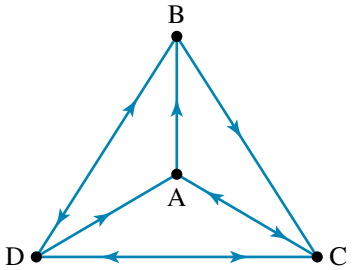


13. MC In the directed graph shown, the dominant vertex (by inspection) is:

A. Q **B.** R **C.** S **D.** T **E.** U



14. WE16 Find the communication matrix from the following communication network.



15. Given the following communication matrix, answer the following questions.

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

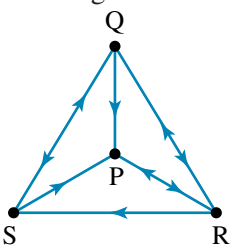
a. Who can C talk to?

b. Who can B receive calls from?

c. Why is the main diagonal all zeroes?

d. Who can D not call?

16. Find the communication matrix from the following communication network.



17. Given the following communication matrix, answer the following questions.

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a. Who can C talk to?

b. Who can B receive calls from?

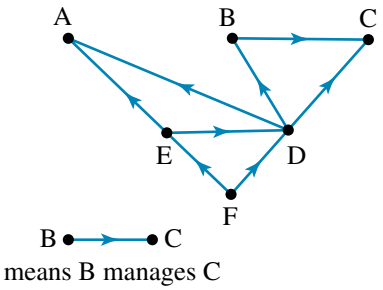
c. Who can call only one person?

d. Who can D not call?

e. Who can receive calls from everyone?

18. The personnel management roles of six employees are shown in the directed graph.

a. Which employee(s) exerts most influence in this group?
b. Which employee(s) exerts least influence?
c. Determine the order of influence of all six employees.



8.7 Application of matrices to simultaneous equations

When solving equations containing one unknown, only one equation is needed. The equation is transposed to find the value of the unknown. In the case where an equation contains two unknowns, two equations are required to solve the unknowns. These equations are known as **simultaneous equations**. You may recall the algebraic methods of substitution and elimination used in previous years to solve simultaneous equations.

Matrices may also be used to solve linear simultaneous equations. The following technique demonstrates how to use matrices to solve simultaneous equations involving two unknowns.

Consider a pair of simultaneous equations in the form:

$$ax + by = e$$
$$cx + dy = f$$

The equations can be expressed as a matrix equation in the form $AX = B$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the **coefficient matrix**, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$.

Notes

1. A is the matrix of the coefficients of x and y in the simultaneous equations.
2. X is the matrix of the pronumerals used in the simultaneous equations.
3. B is the matrix of the numbers on the right-hand side of the simultaneous equations.

As we have seen from subtopic 8.5, an equation in the form $AX = B$ can be solved by pre-multiplying both sides by A^{-1}.

$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

Simultaneous equations are not just limited to two equations and two unknowns. It is possible to have equations with three or more unknowns. To solve for these unknowns, one equation for each unknown is needed.

Simultaneous equations involving more than two unknowns can be converted to matrix equations in a similar manner to the methods described previously.

Let us consider an ancient Chinese problem that dates back to one of the oldest Chinese mathematics books, *The Nine Chapters on the Mathematical Art*.

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

This information can be converted to equations, using the pronumerals x, y and z to represent the three types of corn, as follows:

$$3x + 2y + 1z = 39$$
$$2x + 3y + 1z = 34$$
$$1x + 2y + 3z = 26$$

(Note the importance of lining up the pronumerals on the left side and the numbers on the right side.)

As was the case earlier with two simultaneous equations, this system of equations can also be written as a matrix equation in the form $AX = B$ as follows:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix}$$

X can be solved by pre-multiplying both sides of the equation by A^{-1}. As the order of A is greater than (2×2), CAS should be used to find the inverse (A^{-1}). Try to solve this problem for yourself after reading the following worked example.

WORKED EXAMPLE 17

a. Solve the two simultaneous linear equations by matrix methods.

$$2x + 3y = 13$$
$$5x + 2y = 16$$

b. Use matrix methods to solve the following system of equations.

$$x - 2y + z = -2$$
$$-2x + 3y = -3$$
$$2x - z = 4$$

THINK	WRITE
a. 1. Write the simultaneous equations as a matrix equation in the form $AX = B$. Matrix A is the matrix of the coefficients of x and y in the simultaneous equations, X is the matrix of the pronumerals and B is the matrix of the numbers on the right-hand side of the simultaneous equations.	**a.** $AX = B$ $\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$
2. Matrix X is found by pre-multiplying both sides by A^{-1}.	$X = A^{-1}B$
3. Calculate the inverse of A.	$\det A = 2 \times 2 - 3 \times 5$ $= -11$ The inverse (A^{-1}) is $-\frac{1}{11}\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}$.
4. Solve the matrix equation by calculating the product of A^{-1} and B and simplify.	$X = A^{-1}B$ $= -\frac{1}{11}\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$ $= -\frac{1}{11}\begin{bmatrix} -22 \\ -33 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5. Equate the two matrices and solve for x and y.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

6. Write the answers.

The solution to the simultaneous equations is $x = 2$ and $y = 3$.

b. 1. Use the information from the equations to construct a matrix equation. Insert a 0 in the coefficient matrix where the pronumeral is 'missing'.

b.
$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

2. To solve, pre-multiply both sides by the inverse of the coefficient matrix.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

3. Use CAS to evaluate the right-hand side.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

4. Interpret the results and answer the question. You can double-check your answer by substituting these values into the original equations.

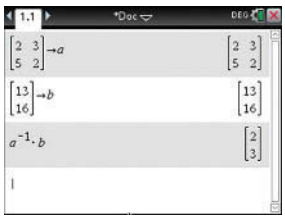
The values of the pronumerals are $x = 0$, $y = -1$ and $z = -4$.

TI | THINK

a. 1. On a Calculator page, define the Matrices A and B.

Complete the entry line as: $a^{-1} \times b$, then press ENTER.

WRITE



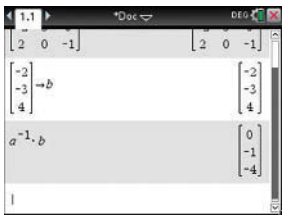
2. The answer is shown on the screen.

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The values of the pronumerals are $x = 2$, $y = 3$.

b. 1. On a Calculator page, define the Matrices A and B.

Complete the entry line as: $a^{-1} \times b$, then press ENTER.



2. The answer is shown on the screen.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

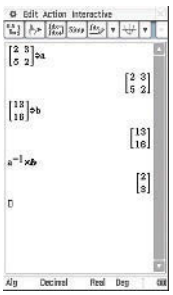
The values of the pronumerals are $x = 0$, $y = -1$ and $z = 4$.

CASIO | THINK

1. On a Main screen, define the Matrices A and B.

Complete the entry line as: $a^{-1} \times b$, then tap EXE.

WRITE



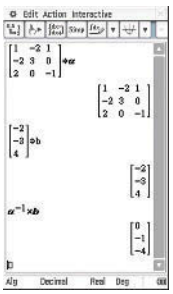
2. The answer is shown on the screen.

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The values of the pronumerals are $x = 2$, $y = 3$.

On a Main screen, define the Matrices A and B.

Complete the entry line as: $a^{-1} \times b$, then tap EXE.



2. The answer is shown on the screen.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

The values of the pronumerals are $x = 0$, $y = -1$ and $z = 4$.

Remember that if the determinant of a matrix equals zero, the inverse will not exist. When this happens there will not be a unique solution (i.e. one solution) for the system of simultaneous equations. It means there is either no solution (i.e. the graphs of the equations are parallel) or there are infinite solutions (i.e. the graphs of the equations draw the same line).

Dependent systems of equations

If the graphs of two or more equations coincide, that is if they are the same line, then there is no unique solution to the equations and we say that the equations are dependent.

If we try to solve **dependent equations** using matrix methods then we would find that there would be no solution as the determinant of the coefficient matrix is 0.

Inconsistent systems of equations

If the graphs of two or more equations do not meet, that is if the lines are parallel, then there is no solution to the equations and we say that the equations are inconsistent.

If we try to solve **inconsistent equations** using matrix methods then we would find that there would be no solution as the determinant of the coefficient matrix is 0.

WORKED EXAMPLE 18

Use a matrix method to decide if the simultaneous equations have a unique solution.

i. $2x - y = 8$
$x + y = 1$

ii. $2x + 4y = 12$
$3x + 6y = 8$

THINK	WRITE
i. 1. Write the simultaneous equations as a matrix equation in the form $AX = B$.	i. $AX = B$ $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$
2. Find the determinant of the matrix A.	$\det A = ad - bc$ $= (2 \times 1) - (-1 \times 1)$ $= 2 - -1$ $= 3$
3. Write the answer.	Since $\det A \neq 0$, a unique solution can be found.
ii. 1. Write the simultaneous equations as a matrix equation in the form $AX = B$.	ii. $AX = B$ $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$
2. Find the determinant of the matrix A.	$\det A = ad - bc$ $= (2 \times 6) - (3 \times 4)$ $= 12 - 12$ $= 0$
3. Write the answer.	Since $\det A = 0$, a unique solution cannot be found and the equations are either dependent or inconsistent.

Matrices are a very efficient tool for solving problems with two or more unknowns. As a result, matrices are used in many areas such as engineering, computer graphics and economics. Matrices may also be applied to solving problems from other modules of the Further Mathematics course, such as break-even analysis, finding the first term and the common difference in arithmetic sequences and linear programming.

When answering problems of this type, take care to follow these steps:

1. Read the problem several times to ensure you fully understand it.
2. Identify the unknowns and assign suitable pronumerals. (Remember that the number of equations needed is the same as the number of unknowns.)
3. Identify statements that define the equations and write the equations using the chosen pronumerals.
4. Use matrix methods to solve the equations. (Remember, for matrices of order 3×3 and higher, use CAS.)

WORKED EXAMPLE 19

A bakery produces two types of bread, wholemeal and rye. The respective processing times for each batch on the dough-making machine are 12 minutes and 15 minutes, while the oven baking times are 16 minutes and 12 minutes respectively. How many batches of each type of bread should be processed in an 8-hour shift so that both the dough-making machine and the oven are fully occupied?

THINK	WRITE
1. Identify the unknowns and choose a suitable pronumeral for each unknown.	We need to determine the number of batches of wholemeal bread and the number of batches of rye bread. Let x = the number of batches of wholemeal bread. Let y = the number of batches of rye bread.
2. Write two algebraic equations from the given statements. All times must be expressed in the same units. (8 hours = 480 minutes)	$12x + 15y = 480$ $16x + 12y = 480$
3. Write the simultaneous equations as a matrix equation in the form $AX = B$.	$AX = B$ $\begin{bmatrix} 12 & 15 \\ 16 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 480 \\ 480 \end{bmatrix}$
4. Solve the matrix equation to find the values for x and y.	$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{96} \begin{bmatrix} 12 & -15 \\ -16 & 12 \end{bmatrix} \begin{bmatrix} 480 \\ 480 \end{bmatrix}$ $= -\frac{1}{96} \begin{bmatrix} -1440 \\ -1920 \end{bmatrix}$ $= \begin{bmatrix} 15 \\ 20 \end{bmatrix}$
5. Write your answer, relating the pronumerals to the original problem.	$x = 15$ and $y = 20$. To fully utilise the dough-making machine and the oven during an 8-hour shift, 15 batches of wholemeal bread and 20 batches of rye bread should be processed.

on Resources

Interactivity: Using matrices to solve simultaneous equations (int-6291)

studyon

Units 3 & 4 > AOS M1 > Topic 2 > Concept 3

Simultaneous equations and matrices Summary screen and practice questions

Exercise 8.7 Application of matrices to simultaneous equations

1. **WE17** Solve these simultaneous linear equations by matrix methods.

$$x + y = 12$$
$$x + 4y = 36$$

2. Solve these simultaneous linear equations by matrix methods.

$$6x + 2y = 30$$
$$2x - y = 10$$

3. Solve each of the following matrix equations.

a. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 15 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 7 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -21 \\ 15 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 21 \end{bmatrix}$

4. Solve the following simultaneous linear equations by matrix methods.

a. $-3x + 7y = 65$
$-9x + 6y = -15$

b. $3x + 2y = 9$
$6x + 4y = 22$

c. $-x + 2y = 0$
$-6x + 14y = 2$

d. $4x - y = -3$
$3x - y = -1$

e. Use CAS and matrix methods to solve the following system of equations.

$$2x + y + 4z = 17$$
$$3x - y = -3$$
$$x + 4y + 5z = 7$$

5. **WE18** Use a matrix method to decide whether the simultaneous equations have a unique solution.

$$6x + 9y = 17$$
$$2x + 3y = 6$$

6. Use a matrix method to decide whether the simultaneous equations have a unique solution.

$$x + 2y = -5$$
$$-2x + y = 2$$

7. **a.** Use a matrix method to decide whether the simultaneous equations have a unique solution.

i. $2x + 2y = -4$
$-3x + y = 14$

ii. $x - 3y = 4$
$-2x + 6y = 12$

iii. $3x + 5y = -11$
$-6x - 10y = 22$

iv. $4x + y = 7$
$6x + 2y = 14$

b. If there is no unique solution, draw graphs to determine whether they are dependent or inconsistent systems of equations.

8. Consider the following two pairs of simultaneous linear equations.

i. $6x + 2y = 4$
$9x + 3y = 14$

ii. $\frac{1}{2}x - y = 3$
$\frac{3}{2}x - 3y = 9$

a. Write each pair of simultaneous equations as a matrix equation in the form $AX - B$.
b. Calculate the determinant for both coefficient matrices.
c. Find the solution for each pair of simultaneous equations. What do you notice? Suggest a reason for this.
d. Transpose the two equations in **i** into $y = mx + c$ form and graph them both using CAS. How do these graphs relate to your answer from part **c**?
e. Transpose the two equations in **ii** into $y = mx + c$ form and graph them both using CAS. How do these graphs relate to your answer from part **c**?

9. Consider the pair of simultaneous equations:

$$y = 2x + 3$$
$$y = x + 1$$

a. Transpose the equations so that they are in the form $ax + by = c$.
b. Write the simultaneous equations as a matrix equation in the form $AX = B$.
c. Solve the matrix equation, writing the solution in coordinate form.

10. Solve the following set of simultaneous equations using matrix methods and CAS.

$$2a - 3b + 6c + 2d = 16$$
$$2b + 4c - d = -3$$
$$-a - b - c - d = -4.5$$
$$0.1a + 0.4b - 0.6c + 1.2d = 3.1$$

11. **WE19** At a car spray-painting company, each car receives two coats of paint, which have to be completed within one day. There are two types of cars that this company spray paints — sedans and utilities.
The times are displayed in the following table.

	Stage of painting	
	1st coat	**2nd coat**
Sedan	5 minutes	9 minutes
Utility	7 minutes	8 minutes
Total time available for each stage	140 minutes	183 minutes

To fully utilise the company's time, how many sedans and utilities should be planned for in a day?

12. The cost of manufacturing jeans ($\$C$) is related to the number of pairs of jeans produced (n), by the formula $C = 1000 + 20n$.
The revenue ($\$R$) made from selling n pairs of jeans is $R = 45n - 800$.
The break-even point is when the cost and revenue are the same; that is, when $C = R$.
Using matrix methods, find out the number of pairs of jeans that need to be manufactured, and the amount that was invested in production if the break-even point was reached.

13. Consider the following problem studied by the Babylonians. (*Note:* We have substituted square metres, instead of square yards, as the units of area.)
There are two fields whose total area is 1800 square metres. One produces grain at the rate of $\frac{2}{3}$ of a bushel per square metre while the other produces grain at the rate of $\frac{1}{2}$ a bushel per square metre. If the total yield is 1100 bushels, what is the size of each field?

Use matrix methods to solve the problem.

14. The cost (in dollars) of manufacturing electronic components, d, is related to the number of components produced, n, by the formula $d = 6000 + 2.5n$. The revenue, d (in dollars), generated from selling n components is given by the formula $d = 4.5n - 8000$. Use matrix methods to calculate the number of components that need to be manufactured so that the manufacturing cost and revenue are equal.

15. The table shown displays the attendance numbers and the box-office takings for the first three shows of a new stage play.

Show	Adults	Children	Pensioners	Box-office takings ($)
First	40	20	5	945
Second	50	15	15	1165
Third	30	0	40	800

Use matrix methods and CAS to calculate the ticket prices for adults, children and pensioners.

16. Use matrix methods to find two numbers, where twice a number plus three times another number is 166 and the sum of the two numbers is 58.

17. A factory produces two different models of transistor radios. Each model requires two workers to assemble it. The time taken by each worker varies according to the following table.

	Worker 1	Worker 2
Model A	5 minutes	5 minutes
Model B	18 minutes	4 minutes
Maximum time available for each worker	360 minutes	150 minutes

a. Use matrix methods to calculate how many of each model should be produced so that each worker is used for the total time available.

b. If the company makes $2.50 on each model A sold and $4.00 on each model B sold, what is the maximum amount of revenue from the sales?

18. The sum of the first 15 terms in an arithmetic sequence is 633 and the 30th term is 187.4. To find the first term, a, of this sequence and the common difference, d, the following two equations can be used:

$$15a + 105d = 633$$
$$a + 29d = 187.4$$

Use matrix methods to find the first term and the common difference for this arithmetic sequence.

8.8 Transition matrices

Andrei Markov was a Russian mathematician whose name is given to a technique that calculates probability associated with the state of various transitions (which can be represented in matrix form). It answers questions such as, 'What is the probability that it will rain today given that it rained yesterday?' or 'What can be said about the long-term prospect of rainy days?'

Andrei Markov, 1856-1922

8.8.1 Powers of matrices

Throughout this section, it will be necessary to evaluate a matrix raised to the power of a particular number, for example M^3. Only square matrices can be raised to a power, as the order of a non-square matrix does not allow for repeated matrix multiplication. For example, a 2×3 matrix cannot be squared, because using the multiplication rule, we see the inner two numbers are not the same ($2 \times \mathbf{3} \times \mathbf{2} \times 3$).

WORKED EXAMPLE 20

For the given matrices

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}, B \begin{bmatrix} 0.81 & 0.6 \\ 0.19 & 0.4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

use CAS to:

a. evaluate

i. A^3 **ii.** B^2 **iii.** C^0

b. evaluate C^{40}, expressing the matrix in whole numbers multiplied by a fractional scalar.

THINK

a. i. Enter matrix A in CAS and raise it to the power of 3.

ii. Enter matrix B in CAS and square it.

iii. Enter matrix C in CAS and raise it to the power of 0.

WRITE

a. i. $A^3 = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}^3$

$= \begin{bmatrix} 0.583 & 0.556 \\ 0.417 & 0.444 \end{bmatrix}$

ii. $B^2 = \begin{bmatrix} 0.81 & 0.6 \\ 0.19 & 0.4 \end{bmatrix}^2$

$= \begin{bmatrix} 0.7701 & 0.726 \\ 0.2299 & 0.274 \end{bmatrix}$

iii. $C^0 = \begin{bmatrix} 5.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}^0$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. 1. Raise matrix C to the power of 40.

b. $C^{40} = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}^{40}$

$$= \begin{bmatrix} 0.333\,333 & 0.333\,333 & 0.333\,333 \\ 0.333\,333 & 0.333\,333 & 0.333\,333 \\ 0.333\,333 & 0.333\,333 & 0.333\,333 \end{bmatrix}$$

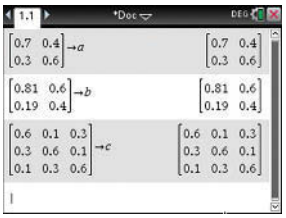
2. Identify the fractional scalar common to each element and place it outside the matrix to give your answer using whole number elements.

$$C^{40} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

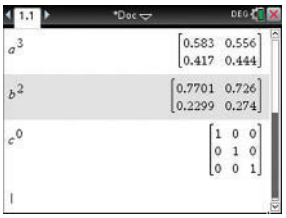
TI | THINK

WRITE

a. 1. On a Calculator page, define Matrices A, B and C.



2. Complete the entry lines as:
- **i.** a^3, then press ENTER.
- **ii.** b^2, then press ENTER.
- **iii.** c^0, then press ENTER.



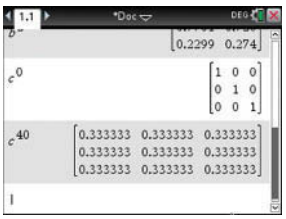
3. The answers are shown on the screen.

i. $a^3 = \begin{bmatrix} 0.583 & 0.556 \\ 0.417 & 0.444 \end{bmatrix}$

ii. $b^2 = \begin{bmatrix} 0.7701 & 0.726 \\ 0.2299 & 0.274 \end{bmatrix}$

iii. $c^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

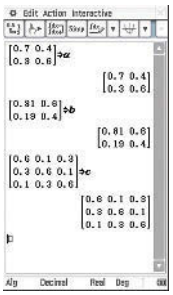
b. 1. Complete the entry line as:
c^{40}, then press ENTER.



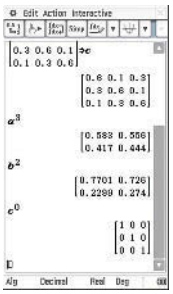
CASIO | THINK

WRITE

1. On a Main screen, define Matrices A, B and C.



2. Complete the entry lines as:
- **i.** a^3, then tap EXE.
- **ii.** b^2, then tap EXE.
- **iii.** c^0, then tap EXE.



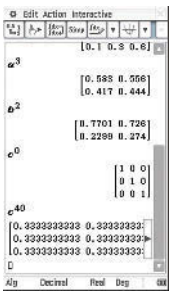
3. The answers are shown on the screen.

i. $a^3 = \begin{bmatrix} 0.583 & 0.556 \\ 0.417 & 0.444 \end{bmatrix}$

ii. $b^2 = \begin{bmatrix} 0.7701 & 0.726 \\ 0.2299 & 0.274 \end{bmatrix}$

iii. $c^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. 1. Complete the entry line as:
c^{40}, then tap EXE.



2. The answer is on the screen. Identify the fractional scalar common to each element and place it outside the matrix to give your answer using whole number elements.	$C^{40} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
2. The answer is on the screen. Identify the fractional scalar common to each element and place it outside the matrix to give your answer using whole number elements.	$C^{40} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

8.8.2 Markov systems and transition matrices

A **Markov system** (or Markov chain) is a system that investigates estimating the distribution of states (condition or location) of an event, given information about the current states. It also investigates the manner in which these states change from one state to the next, according to fixed probabilities. Matrices can be used to model such situations where:

- there are defined sets of conditions or *states*
- there is a *transition* from one state to the next, where the next state's probability is *conditional* on the result of the preceding outcome
- the conditional probabilities for each outcome are the same on each occasion; that is, the same matrix is used for each transition
- information about an **initial state** is given.

A Markov system can be illustrated by means of a state transition statement, a table or a diagram. The following transition statements describe the movement of delivery trucks between two locations.

A group of delivery trucks transfer goods between two warehouses A and B. They start the day at either warehouse and finish the day parked at one of them.

70% of the trucks that start at A will park at A that night and 30% will park at B.

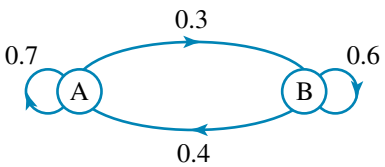
60% of the trucks that start at B will park at B that night and 40% will park at A.

This is how the statements are represented as a table and as a diagram. (In both, the values have been expressed as probabilities.)

Transition table

		Transition from	
		Warehouse A	**Warehouse B**
Transition to	**Warehouse A**	0.7	0.4
	Warehouse B	0.3	0.6

Transition diagram



The statements, the table and the diagram all represent the same information. They can all be summarised as a **transition matrix** as shown. Throughout this section, the transition matrix will be denoted as T.

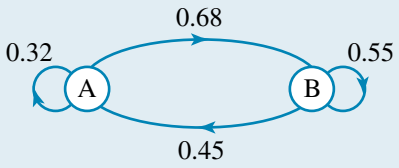
$$T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

Note that each of the columns of a transition matrix must add up to 1.

WORKED EXAMPLE 21

Represent each of the following as a transition matrix.

a.



b. There are a number of train carriages operating between two depots, North depot and South depot. At the end of each week, 40% of the carriages that started at North depot end up at South depot and 25% of the carriages that started at South depot end up at North depot.

THINK	WRITE
a. 1. Identify there are two states, A and B. Enter the values in the correct manner; that is, *from* the column *to* the row. 0.32 from A to A and 0.68 from A to B. 0.55 from B to B and 0.45 from B to A.	**a.** From A B To A $\begin{bmatrix} 0.32 & 0.45 \\ 0.68 & 0.55 \end{bmatrix}$ B
2. Check that each column adds up to 1 and remove any labels.	$T = \begin{bmatrix} 0.32 & 0.45 \\ 0.68 & 0.55 \end{bmatrix}$
b. 1. Identify there are two states, North depot and South depot. Enter the given percentage probabilities in decimal form.	**b.** From North South To North $\begin{bmatrix} & 0.25 \\ 0.40 & \end{bmatrix}$ South
2. The missing values can be calculated knowing the columns must add up to 1. Remove any labels.	$T = \begin{bmatrix} 0.60 & 0.25 \\ 0.40 & 0.75 \end{bmatrix}$

8.8.3 Distribution vector and powers of the transition matrix

A distribution vector is a column vector with an entry for each state of the system. It is often referred to as the **initial state matrix** and is denoted by S_0.

If S_0 is an $n \times 1$ initial distribution vector state matrix involving n components and T is the transition matrix, then the distribution vector after 1 transition is the matrix product $T \times S_0$.

Distribution after 1 transition: $S_1 = T \times S_0$

The distribution one stage later is given by:

Distribution after 2 transitions: $S_2 = T \times S_1$

$$= T \times (T \times S_0)$$
$$= (T \times T) \times S_0$$
$$= T^2 \times S_0$$

We can continue this pattern to create a matrix recurrence relation.

The matrix recurrence relation $S_{n+1} = TS_n$, where T is a transition matrix and S_n is a column state matrix, will generate a sequence of state matrices.

The distribution after n transitions can be obtained by premultiplying S_0 by T, n times or by multiplying T^n by S_0.

Distribution after n transitions: $S_n = T \times S_{n-1}$

$= T^n \times S_0$

The sequence of states, $S_0, S_1, S_2, ..., S_n$ is called a Markov chain.

8.8.4 Applications to marketing

One common use for the above approach is in marketing, where organisations can predict their share of the market at any given moment. Marketing records show that when consumers are able to purchase certain goods — for example, groceries — from competing stores A and B, we can associate conditional probabilities with the likelihood that they will purchase from a given store, or its competitor, depending on the store from which they had made their previous purchases over a set period, such as a month.

The following worked example highlights the application of transition matrices to marketing.

WORKED EXAMPLE 22

A survey shows that 75% of the time, customers will continue to purchase their groceries from store A if they purchased their groceries from store A in the previous month, while 25% of the time consumers will change to purchasing their groceries from store *B* if they purchased their groceries from store A in the previous month. Similarly, the records show that 80% of the time, consumers will continue to purchase their groceries from store B if they purchased their groceries from store B in the previous month.

a. Use a matrix recurrence relation to determine how many customers are still purchasing their groceries from A and B at the end of two months, if 300 customers started at A and 300 started at B?

b. What percentage (in whole numbers) of customers are purchasing their groceries at A and B at the end of 6 months, if 50% of the customers started at A?

THINK

a. 1. Set up the transition matrix using the correct methods. Complete any missing probabilities knowing the columns must add up to 1.

2. Set up the initial state matrix. The elements for this matrix are the initial number of customers for each store.

3. The number of transitions is 2. Calculate the number of customers still purchasing at A and B by using the recurrence relation $S_{n+1} = TS_n$ and evaluating S_1 and S_2.

WRITE

a.

$$T = \begin{matrix} \\ \text{A} \\ \text{B} \end{matrix} \begin{matrix} \begin{matrix} \text{A} & \text{B} \end{matrix} \\ \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix} \end{matrix}$$

$$T = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} 300 \\ 300 \end{bmatrix}$$

$$S_1 = TS_0$$

$$= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \begin{bmatrix} 300 \\ 300 \end{bmatrix}$$

$$= \begin{bmatrix} 285 \\ 315 \end{bmatrix}$$

$$S_2 = TS_1$$
$$= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \begin{bmatrix} 285 \\ 315 \end{bmatrix}$$
$$= \begin{bmatrix} 276.75 \\ 323.25 \end{bmatrix}$$

4. Interpret the answer represented by S_2.

After 2 months, 277 customers will be purchasing their groceries from store A and 323 will be purchasing their groceries from store B.

b. 1. Identify that after 6 months, $n = 6$. The initial state matrix contains the percentage of customers for each store at the start.

b. $S_0 = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$

2. Use the formula to set up the matrix calculation.

$$S_n = T^n \times S_0$$
$$S_6 = T^6 \times S_0$$
$$= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix}^6 \times \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

3. Use CAS to evaluate S_6.

$$S_6 = \begin{bmatrix} 44.5982 \\ 55.4018 \end{bmatrix}$$

4. Interpret the answer represented by S_6. (Round to the nearest per cent.)

After 6 months, 45% of the customers will be purchasing their groceries from store A and 55% of the customers will be purchasing their groceries from store B.

8.8.5 New state matrix with culling and restocking

The new state matrix $S_n = T^n S_0$ can be extended to include culling and restocking. This can be done by adding (restocking) or subtracting (culling) a matrix to our original new state matrix.

The matrix recurrence relation $S_{n+1} = TS_n + B$, where B is a matrix (usually a column matrix), represents a transition situation including culling and restocking.

WORKED EXAMPLE 23

Betta Health Centres run concurrent Lift and Cycle fitness classes at all 3 of their gyms in FitTown.
A study shows that 80% of the clients who attended a Lift class one week will attend the Lift class the next week, while the other 20% will move to the Cycle class the next week.
Similarly, 70% of the clients who attended a Cycle class one week will attend the Cycle class the next week, while the other 30% will move to the Lift class the next week.

The numbers are also affected by people joining and leaving the gym, with 2 additional people joining the Lift classes each week and 3 additional people joining the Cycle classes each week. In the first week 55 people attended the Lift classes and 62 people attended the Cycle classes. Set up the matrix recurrence relation that would be used to find how many people attended the Lift and Cycle classes in week 2.

THINK	WRITE
1. Set up a transition matrix to represent the changing numbers between the classes. Let L represent the Lift class and C represent the Cycle class.	$\begin{matrix} & \text{L} & \text{C} \\ \text{L} \\ \text{C} \end{matrix}\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$
2. Determine the value of matrix B (represented by the additional people joining the classes).	$B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{matrix} \text{L} \\ \text{C} \end{matrix}$
3. Set up the initial state matrix.	$S_1 = \begin{bmatrix} 55 \\ 62 \end{bmatrix}\begin{matrix} \text{L} \\ \text{C} \end{matrix}$
4. Enter the information into the matrix recurrence relation $S_2 = TS_1 + B$.	$S_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 55 \\ 62 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

WORKED EXAMPLE 24

A school was running extra maths and English classes each week, with students being able to choose which extra classes they would attend. A matrix equation used to determine the number of students expected to attend extra classes is given by

$$S_{n+1} = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} S_n - \begin{bmatrix} 7 \\ 9 \end{bmatrix},$$

where S_n is the column matrix that lists the number of students attending in week n.
The attendance matrix for the first week is given by

$$S_1 = \begin{bmatrix} 104 \\ 92 \end{bmatrix}\begin{matrix} \text{Maths} \\ \text{English} \end{matrix}$$

a. Calculate the number of students who are expected to attend extra English lessons in week 3.
b. Of the students who attended extra classes in week 3, how many are not expected to return for extra classes in week 4?

THINK	WRITE
a. 1. First calculate week 2 by using S_1.	a. $S_2 = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix}\begin{bmatrix} 104 \\ 92 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ $= \begin{bmatrix} 109 \\ 71 \end{bmatrix}$

2. Find week 3 by using S_2 and then round off.
Note: Round down when referring to discrete data.

$$S_3 = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} \begin{bmatrix} 109 \\ 71 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$
$$= \begin{bmatrix} 106.95 \\ 57.05 \end{bmatrix}$$
$$= \begin{bmatrix} 106 \\ 57 \end{bmatrix}$$

3. Write the answer.

In week 3 it is expected that 57 attend extra English classes.

b. 1. Find week 4 using S_3, not using rounded numbers, but round off at the end.

b. $$S_4 = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} \begin{bmatrix} 106.95 \\ 57.05 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$
$$= \begin{bmatrix} 101.0225 \\ 46.9775 \end{bmatrix}$$
$$= \begin{bmatrix} 101 \\ 46 \end{bmatrix}$$

2. Find total students doing extra classes in week 3 and subtract total student doing extra classes in week 4.

Total in week 3 = 106 + 57 = 163 students.
Total in week 4 = 101 + 46 = 147 students.
Difference = 163 − 147 = 16 students.

3. Write the answer.

It is estimated that there will be 16 students that attended extra classes in week 3 that don't attend in week 4.

8.8.6 Steady state

As higher and higher powers of T are taken, we find the values of the elements in the transition matrix show no noticeable difference, and approach a fixed matrix T^{∞}. We refer to T^{∞} as the **steady state** or equilibrium state matrix. To test for steady state, a suitable value of n to test is 50. Then test $n = 51$. If the elements in the matrix haven't changed, then a steady state has been reached.

When there is no noticeable change from one state matrix to the next, the system is said to have reached its steady state.

If a Markov system is regular, then its long-term transition matrix is given by the square matrix, whose columns are the same and equal to, the steady state probability vector. This occurs as long as the transition matrix squared, T^2, has no zeros.

If T^2 contains any zeros, then it is not possible to reach a steady state.

T	T^2	T^3
$\begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix}$	$\begin{bmatrix} 0.6125 & 0.31 \\ 0.3875 & 0.69 \end{bmatrix}$	$\begin{bmatrix} 0.536\,875 & 0.3705 \\ 0.463\,125 & 0.6295 \end{bmatrix}$
T^4	T^8	T^{16}
$\begin{bmatrix} 0.495\,281\,25 & 0.403\,775 \\ 0.504\,718\,75 & 0.596\,225 \end{bmatrix}$	$\begin{bmatrix} 0.449\,096\ldots & 0.440\,722\ldots \\ 0.550\,903\ldots & 0.559\,277\ldots \end{bmatrix}$	$\begin{bmatrix} 0.444\,48\ldots & 0.444\,41\ldots \\ 0.555\,51\ldots & 0.555\,58\ldots \end{bmatrix}$
T^{50}	T^{51}	
$\begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}$	$\begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}$	

These probabilities can be easily expressed as fractions if n is very large.

8.8.7 Applications to weather

Predicting the long-term weather forecast is important to insurance companies who insure event organisers against losses if the event is 'rained on'. To do this they need to predict the long-term probability of there being rain. Suppose that for a 'Melbourne spring', long run data suggest that there is a 65% chance that if today is dry, then the next day will also be dry. Conversely, if today is wet, there is an 82% chance that the next day will also be wet. What is the long-term probability for it being a wet day if the initial day was dry? This style of problem is highlighted in the following worked example.

WORKED EXAMPLE 25

An insurance company needs to measure its risk if it is to underwrite a policy for a major outdoor event planned. The company used the following information about the region.
Long run data gathered about the region's weather suggests that there is a 75% chance that if today is dry, then the next day will also be dry. Conversely, if today is wet, there is a 72% chance that the next day will also be wet. This information is given in the table.

	Today is dry	Today is wet
Next day dry	0.75	0.28
Next day wet	0.25	0.72

a. Find the probability it will rain in three days time if initially the day is dry.
b. Find the long-term probability of rain if initially the day is wet.
c. If the company insures only if they have the odds in their favour, will they insure this event?

THINK	WRITE
a. 1. Set up the transition matrix.	**a.** $T = \begin{matrix} \\ \text{dry} \\ \text{wet} \end{matrix} \begin{matrix} \begin{matrix} \text{dry} & \text{wet} \end{matrix} \\ \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix} \end{matrix}$ $= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}$
2. Set up the initial state matrix. For the initial day being dry, set dry as 1 and wet as 0.	$S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
3. Identify that in three days time, $n = 3$. Substitute the matrices and use CAS to evaluate S_3.	$S_3 = T^3 \times S_0$ $= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}^3 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.577\,275 \\ 0.422\,725 \end{bmatrix}$
4. Interpret the answer represented by S_3.	The probability of a dry day, three days after a dry day, is 57.7%.
b. 1. Identify that for a long-term steady state, n needs to be large, say $n = 50$. (Test that the same result is obtained when $n = 51$.)	**b.** $S_{50} = T^{50} \times S_0$
2. Set up the initial state matrix. For the initial day being wet, set dry as 0 and wet as 1.	$S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
3. Substitute the matrices and use CAS to evaluate S_{50} and S_{51}.	$S_{50} = T^{50} \times S_0$ $= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}^{50} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $= \begin{bmatrix} 0.528\,302 \\ 0.471\,698 \end{bmatrix}$ $S_{51} = T^{51} \times S_0$ $= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}^{51} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $= \begin{bmatrix} 0.528\,302 \\ 0.471\,698 \end{bmatrix}$
4. As there is no change from S_{50} to S_{51}, a steady state has been reached. Interpret the answer.	In the long run, there is a 47.2% chance it will be a wet day (or 25 in 53 chance) if the initial day is wet.

c. For it to be in the insurer's favour, there must be more than a 50% chance of it being dry.

c. In the long run, there is a 52.8% chance it will be a dry day. The odds are slightly in favour of the insurance company. Therefore, they will insure this event.

Other well-known examples of the application of transition matrices are to population studies, stock inventory and sport.

studyon

Units 3 & 4 › AOS M1 › Topic 3 › Concepts 1–5

Description of transition matrices Summary screen and practice questions

Initial state and transitions Summary screen and practice questions

Repeated transitions Summary screen and practise questions

Equilibrium or steady state Summary screen and practice questions

Modelling and applications Summary screen and practice questions

Exercise 8.8 Transition matrices

The following matrices are required to answer questions **1** and **2**. Define them in CAS to assist with your calculations.

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}, C = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$$

1. WE20 Use CAS to:

a. evaluate (correct to 2 decimal places)

i. A^3 **ii.** B^2 **iii.** C^4 **iv.** C^8

b. evaluate C^{46}, expressing the elements of the matrix as whole numbers.

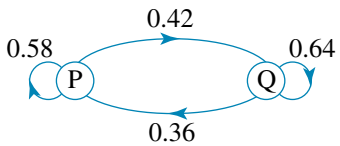
2. a. Use CAS to perform the following matrix operations.

i. A^0 **ii.** B^0 **iii.** C^0 **iv.** D^0

b. Describe the type of matrices produced in question **2a**.

3. WE21 Represent the following as a transition matrix.

a.



b. There are a number of delivery trucks operating between two warehouses, warehouse A and warehouse B. At the end of each week, 31% of the trucks that started at warehouse A end up at warehouse B and 27% of trucks that started at warehouse B end up at warehouse A.

4. MC The missing elements in the following transition matrix are:

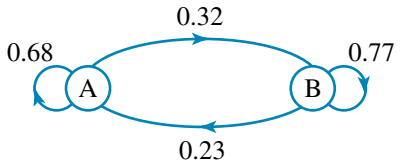
$$T = \begin{bmatrix} 0.3 & 0.2 & \\ 0.3 & 0.7 & 0.3 \\ & 0.1 & 0.6 \end{bmatrix}$$

A. $t_{31} = 0.3$ and $t_{13} = 0.5$ **B.** $t_{13} = 0.3$ and $t_{31} = 0.5$ **C.** $t_{13} = 0.5$ and $t_{31} = 0.4$

D. $t_{31} = 0.4$ and $t_{13} = 0.1$ **E.** $t_{13} = 0.4$ and $t_{13} = 0.1$

5. Represent each of the following as a transition matrix.

a.



b. There are a number of train carriages operating between two depots, North depot and South depot. At the end of each week, 20% of the carriages that started at North depot end up at South depot and 15% of the carriages that started at South depot end up at North depot.

6. **WE22** A railway knows that 250 goods wagons will be needed to carry goods from point A to point B. At the end of each week, it finds that 10% of the wagons that started the week at point A ended at point B, and 8% of the wagons that started at point B ended at point A.

a. Write a transition matrix to represent this situation.

b. Use a matrix recurrence relation to determine how many wagons are located at point A and point B at the end of two weeks, if 125 wagons started at point A and 125 wagons started at point B.

c. What percentage (in whole numbers) of wagons will be at point A and point B at the end of 6 weeks, if 40% of the wagons started at A?

7. **MC** A transition analysis of the movement of a population of people at the Tamworth Country Festival is given by $S_4 = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}^4 \begin{bmatrix} 10\,000 \\ 12\,000 \end{bmatrix}$. The most correct answer for S_4 is:

A. $\begin{bmatrix} 6\,300 \\ 15\,700 \end{bmatrix}$ B. $\begin{bmatrix} 6\,288 \\ 15\,711 \end{bmatrix}$ C. $\begin{bmatrix} 6\,320 \\ 15\,690 \end{bmatrix}$

D. $\begin{bmatrix} 6\,315.8 \\ 15\,684.2 \end{bmatrix}$ E. $\begin{bmatrix} 28.7 \\ 71.3 \end{bmatrix}$

8. **MC** Using the following table of events and their transition probabilities, find the correct transition matrix.

Percentage of customers and their choice of shopping the next time			
	Shop A	**Shop B**	**Shop C**
Shop A	85%		20%
Shop B	10%	60%	
Shop C	5%	20%	75%

A. $\begin{bmatrix} 0.85 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.2 & 0.75 \end{bmatrix}$ B. $\begin{bmatrix} 0.85 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.05 \\ 0.5 & 0.2 & 0.75 \end{bmatrix}$ C. $\begin{bmatrix} 0.85 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.05 \\ 0.05 & 0.2 & 0.75 \end{bmatrix}$

D. $\begin{bmatrix} 0.85 & -0.05 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.2 & 0.75 \end{bmatrix}$ E. $\begin{bmatrix} 85 & 20 & 20 \\ 10 & 60 & 5 \\ 5 & 20 & 75 \end{bmatrix}$

9. Write the transition matrices to represent the following tables of events and their transition probabilities. Complete any missing information.

a.

		From	
		Dry	**Wet**
To	**Dry**	85%	70%
	Wet	15%	30%

b.

		From		
	Shoe brand	**Gulius**	**Grasby**	**Warlow**
To	**Gulius**	0.65		0.03
	Grasby	0.15	0.68	
	Warlow		0.12	0.97

10. The following information relates to a survey conducted in October 2012 on supermarket shopping.

- 12% of store A customers will shop at store B the following month.
- 36% of store A customers will shop at store C the following month.
- 40% of store B customers will shop at store B the following month.
- 44% of store B customers will shop at store C the following month.
- 14% of store C customers will shop at store A the following month.
- 7% of store C customers will shop at store B the following month.

a. Represent this information as a transition matrix.

b. If the market share at the time of the survey showed that 1200 customers shopped at store A, 800 customers shopped at store B and 1000 customers shopped at store C, find the number of customers expected to be shopping at each store in March 2013.

c. Calculate the long-term share of the customers shopping at each store as a percentage (correct to 1 decimal place).

d. Express the answer for part **c** as a fraction of the customers shopping at each store.

11. If a train is late on one day, there is a 15% probability that the same train will be late the next day. If the train is on time one day, there is a 40% chance that it will be late the next day.

If, in a given week, a train arrives on time on Monday, calculate:

a. the probability that the train will be on time on the following Friday

b. the probability that the train will be late on the following Monday (assume that there are no trains operating on weekends).

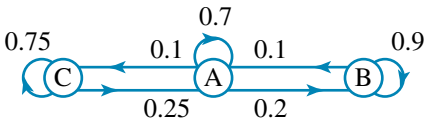
12. For the transition matrix

$$T = \begin{bmatrix} 0.6 & 0.1 & 0 & 0.05 \\ 0.2 & 0.8 & 0 & 0 \\ 0.1 & 0.1 & 1 & 0 \\ 0.1 & 0 & 0 & 0.95 \end{bmatrix}$$

a. State the matrix T^2.

b. State why this matrix (T) has no steady state.

c. MC The transition matrix that matches the following transition diagram is:



A. $\begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.2 & 0.9 & 0 \\ 0.1 & 0 & 0.75 \end{bmatrix}$

B. $\begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.2 & 0.9 & 0.45 \\ 0.1 & 0.2 & 0.75 \end{bmatrix}$

C. $\begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.1 & 0.9 & 0 \\ 0.2 & 0 & 0.75 \end{bmatrix}$

D. $\begin{bmatrix} 0.7 & 0 & 0.25 \\ 0.2 & 0.9 & 0 \\ 0.1 & 0.1 & 0.75 \end{bmatrix}$

E. $\begin{bmatrix} 0.7 & 0.1 & 0.75 \\ 0.2 & 0.9 & 0 \\ 0.1 & 0 & 0.25 \end{bmatrix}$

13. The following transition table describes the proportion of customers that purchase fuel from four nearby petrol stations each week.

	Petro	Texcal	Oilmart	CP
Petro	0.6	0.1	0	0.05
Texcal	0.2	0.8	0.05	0
Oilmart	0.1	0.1	0.9	0
CP	0.1	0	0.05	0.95

a. Represent the information as a transition matrix.

b. If the initial customer base is represented by the matrix $\begin{bmatrix} 4500 \\ 4000 \\ 2000 \\ 1500 \end{bmatrix}$, which petrol station has the largest customer base (include how many):

i. after 2 weeks

ii. after 8 weeks

iii. in the long term? (*Hint:* Investigate large values of n.)

14. A farmer has two types of cows that he milks each day, the Holstein Friesian and the Jersey. He also breeds them to increase his herd. A matrix recurrence relation used to determine the expected number of cows to be milked is given by

$$S_{n+1} = \begin{bmatrix} 0.60 & 0.35 \\ 0.40 & 0.65 \end{bmatrix} S_n + \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where S_n is the column matrix that lists the number of cows on day n.

The number of cows milked on the first day is given by the matrix

$$S_1 = \begin{bmatrix} 106 \\ 72 \end{bmatrix} \begin{matrix} \text{Holstein Friesian} \\ \text{Jersey} \end{matrix}$$

a. Calculate the number of Jersey cows that are expected to be milked on day 4.

b. How many more cows does the farmer have to milk on day 4 compared to the first day?

15. **WE23** Top Climbers run concurrent Agility and Climb classes at all 3 of their centres in FitTown.

A study shows that 65% of the clients who attended an Agility class one week will attend the Agility class the next week, while the other 35% will move to the Climb class the next week.

Similarly, 77% of the clients who attended a Climb class one week will attend the Climb class the next week, while the other 23% will move to the Agility class the next week.

The numbers are also affected by people joining and leaving the gym, with 5 additional people joining the Agility classes each week and 2 people leaving the Climb classes each week.

In the first week 48 people attended the Agility classes and 41 people attended the Climb classes.

Set up the matrix recurrence relation that would be used to find how many people attended the Agility and Climb classes in week 2.

16. A matrix recurrence relation representing the changing number of people who attend two different supermarkets on a weekly basis is given by

$$S_{n+1} = \begin{bmatrix} 0.61 & 0.18 \\ 0.39 & 0.82 \end{bmatrix} S_n + \begin{bmatrix} 16 \\ 7 \end{bmatrix}.$$

The first row in the state matrices represents Supermarket A and the second row in the state matrices represents Supermarket B.

a. What percentage of people who attend Supermarket A in week 1 go back to Supermarket A in week 2?

b. What percentage of people who attend Supermarket B in week 1 go back to Supermarket B in week 2?

c. How many additional people go to each supermarket each week?

17. The Fine Drink Co. manufacture two drinks, Sweet Cola and Tangy Lemon. The manufacturing of the two products is determined by the matrix recurrence relation:

$$S_{n+1} = \begin{bmatrix} 0.84 & 0.22 \\ 0.16 & 0.78 \end{bmatrix} S_n + \begin{bmatrix} 110 \\ 71 \end{bmatrix},$$

where S_n is the column matrix that lists the number of drinks produced in week n.The number of drinks produced in the first week is given by the matrix

$$S_1 = \begin{bmatrix} 950 \\ 1050 \end{bmatrix} \begin{matrix} \text{Sweet Cola} \\ \text{Tangy Lemon} \end{matrix}$$

a. Calculate the number of each type of drink produced in weeks 2, 3 and 4. Round all answers correct to whole numbers.

b. How many more drinks in total were produced in week 4 than week 1?

18. **WE24** A school was running extra music lessons for the piano and guitar each week, with students being able to choose which extra lessons they would attend. A matrix recurrence relation used to determine the number of students expected to attend extra lessons is given by

$$S_{n+1} = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix} S_n - \begin{bmatrix} 5 \\ 3 \end{bmatrix},$$

where S_n is the column matrix that lists the number of students attending in week n.
The attendance matrix for the first week is given by

$$S_1 = \begin{bmatrix} 34 \\ 26 \end{bmatrix} \begin{matrix} \text{Piano} \\ \text{Guitar} \end{matrix}$$

a. Calculate the number of students who are expected to attend extra guitar lessons in week 2.
b. Of the students who attended extra lessons in week 2, how many are not expected to return for extra lessons in week 3?

19. A farmer has his sheep moving between two paddocks, paddock A and paddock B, each week. A matrix recurrence relation used to determine the number of sheep expected to be in each of the paddocks (including the loss of sheep) is given by

$$S_{n+1} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} S_n - \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

where S_n is the column matrix that lists the number of sheep in week n.
The number of sheep in each paddock in the first week is given by the matrix

$$S_1 = \begin{bmatrix} 136 \\ 108 \end{bmatrix} \begin{matrix} \text{Paddock A} \\ \text{Paddock B} \end{matrix}$$

a. Calculate the number of sheep who are expected to be in paddock B in week 4.
b. How many sheep have been lost between week 4 and week 5?

20. The population of the two main cities on an island is determined by the matrix recurrence relation

$$S_{n+1} = \begin{bmatrix} 0.95 & 0.08 \\ 0.05 & 0.92 \end{bmatrix} S_n + \begin{bmatrix} 1200 \\ 955 \end{bmatrix}$$

where S_n is the column matrix that lists the population produced in year n.
The population in 2012 is given by the matrix

$$S_1 = \begin{bmatrix} 55\,000 \\ 72\,000 \end{bmatrix} \begin{matrix} \text{City A} \\ \text{City B} \end{matrix}$$

Calculate the population of each city in 2013, 2014 and 2015. Round all answers correct to whole numbers.

21. **WE25** An insurance company needs to measure its risk if it is to underwrite a policy for a major outdoor event planned. The company used the following information about the region where the event was to take place.

Long run data gathered about the region's weather suggests that there is a 95% chance that if today is dry, then the next day will also be dry. Conversely, if today is wet, there is a 45% chance that the next day will also be wet. This information is given in the table below.

	Today is dry	**Today is wet**
Next day is dry	0.95	0.55
Next day is wet	0.05	0.45

a. Find the probability it will rain in three days time if initially the day is dry.

b. Find the long-term probability if initially the day is wet.

c. If the company insures only if the odds are in its favour, will the company insure this event?

22. MC For $T = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 100 \\ 50 \\ 80 \end{bmatrix}$, the steady state distribution vector is:

A. $\begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}$ **B.** $\begin{bmatrix} 69 \\ 138 \\ 23 \end{bmatrix}$ **C.** $\begin{bmatrix} 30 \\ 60 \\ 100 \end{bmatrix}$

D. $\begin{bmatrix} 70 \\ 135 \\ 25 \end{bmatrix}$ **E.** $\begin{bmatrix} 100 \\ 50 \\ 80 \end{bmatrix}$

23. There are two supermarkets to choose from in Matrixtown, and as the town grows more shoppers shop at the supermarkets each week. A matrix recurrence relation used to determine the number of shoppers expected to shop in each of the supermarkets is given by

$$S_{n+1} = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix} S_n + \begin{bmatrix} 6 \\ 4 \end{bmatrix},$$

where S_n is the column matrix that lists the number of shoppers in week n.

The number of shoppers who shopped at each supermarket in the first week is given by the matrix

$$S_1 = \begin{bmatrix} 256 \\ 194 \end{bmatrix} \begin{matrix} \text{Supermarket A} \\ \text{Supermarket B} \end{matrix}$$

a. Calculate the number of shoppers who are expected to shop in supermarket A in week 3.

b. What is the difference in shoppers at supermarket A compared to supermarket B after 4 weeks?

24. Many Victorians have been retiring and moving to Queensland, yet only some have come from Queensland to retire in Victoria. The government of Victoria wants to investigate the expected long-term transition using the following information.

- 6000 Victorians moved to Queensland in 2009.
- 400 Queenslanders moved to Victoria in 2009.
- 73% of Victorians who moved to Queensland stayed.
- 27% of Victorians who moved to Queensland eventually returned to Victoria.
- 10% of Queenslanders who moved to Victoria stayed.
- 90% of Queenslanders who moved to Victoria returned to Queensland.

The government wants to budget for its proposed aged care program and needs to find out the long-term impact on numbers the 2009 transition will have. Calculate the net gain or loss of retirees to the state of Victoria.

8.9 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC The a_{12} element in the following matrix is:

$$A = \begin{bmatrix} 2.4 & -0.5 \\ 3.6 & 1.6 \\ 1.6 & 0 \\ -2.5 & 2.4 \end{bmatrix}$$

A. 3.6 **B.** 1.6 **C.** −0.5 **D.** 0 **E.** 2.4

2. MC The number 0 in the following matrix can be represented using the notation:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

A. a_{23} **B.** a_{32} **C.** a_{22} **D.** a_{12} **E.** none of these

3. MC The solution to $E - F$ given the matrices

$E = \begin{bmatrix} 1.2 & -0.5 \\ 3.6 & 5.0 \\ -3.5 & 2.2 \end{bmatrix}$ and $F = \begin{bmatrix} 0.2 & -0.5 \\ 2.4 & 2.5 \\ 0 & 1.1 \end{bmatrix}$ is:

A. $\begin{bmatrix} 1.0 & 0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$ **B.** $\begin{bmatrix} 1.4 & -1.0 \\ 6.0 & 7.5 \\ -3.5 & 3.3 \end{bmatrix}$ **C.** $\begin{bmatrix} 1.0 & -1.0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$

D. $\begin{bmatrix} -1.0 & 0 \\ -1.2 & -2.5 \\ -3.5 & -11 \end{bmatrix}$ **E.** $\begin{bmatrix} 1.0 & -1.0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$

4. MC If $A = B \times C$, then the element a_{13} is the result of:

A. multiplying the third row by the first column
B. multiplying the third column by the first row
C. adding the first row to the third column
D. multiplying the first row by the third column
E. multiplying the first column by the third row

Use the following matrices to answer questions **5** to **8**.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \\ 3 & 6 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 4 \end{bmatrix}, E = \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}$$

5. MC Which of the following products does not exist?

A. AB **B.** BC **C.** CA **D.** ED **E.** CE

6. MC The order of the matrix BD is:

A. 2×2 **B.** 4×2 **C.** 4×4 **D.** 3×4 **E.** 2×4

7. MC Which of the following products gives a matrix of order 2×2?

A. BA **B.** EC **C.** CA **D.** AC **E.** CE

8. MC The matrix ED is:

A. $\begin{bmatrix} 4 & 9 & 11 & 20 \\ 3 & 6 & 9 & 12 \\ 2 & 3 & 6 & 4 \end{bmatrix}$ **B.** $\begin{bmatrix} 4 & 9 & 11 & 20 \\ 3 & 6 & 9 & 12 \\ 2 & -3 & -7 & 4 \end{bmatrix}$ **C.** $\begin{bmatrix} 4 & 9 & 11 & 20 \\ 3 & 6 & 9 & 12 \\ 2 & 3 & 7 & 12 \end{bmatrix}$ **D.** $\begin{bmatrix} 4 & 9 & 11 & 20 \\ 3 & 6 & 9 & 12 \\ 2 & 3 & 7 & 4 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 7 & 8 & 16 \\ 2 & 9 & 1 & 28 \end{bmatrix}$

9. MC The determinant of the matrix $\begin{bmatrix} 9 & 8 \\ 5 & 6 \end{bmatrix}$ is:

A. $\frac{1}{14}$ **B.** $\frac{-1}{14}$ **C.** 14 **D.** 104 **E.** -8

10. MC If $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, its inverse P^{-1} is equal to:

A. $\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ **C.** $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ **D.** $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

E. cannot be found.

Use the pair of simultaneous equations below to answer questions **11** and **12**.

$$y = 3x - 2$$
$$y = -2x$$

11. MC The correct matrix equation to solve this system of linear equations is:

A. $\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ **B.** $\begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

C. $\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ **D.** $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

E. $\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

12. MC The solution, written in coordinate form, is:

A. (0.4, 0.8) **B.** $\left(\frac{2}{5}, \frac{-4}{5}\right)$ **C.** $(-2, 4)$ **D.** $\left(\frac{-2}{5}, \frac{4}{5}\right)$ **E.** $\left(\frac{-4}{5}, \frac{2}{5}\right)$

13. MC For the system of simultaneous equations

$$3x - 2y + 1 = 1$$
$$-x + 4y - 3z = 2$$
$$3x + z = 6$$

the coefficient matrix is:

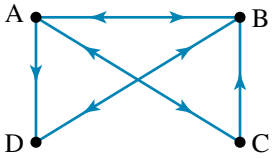
A. $\begin{bmatrix} 3 & -2 & 1 \\ -1 & 4 & -3 \\ 3 & 0 & 1 \end{bmatrix}$ **B.** $\begin{bmatrix} 3 & -2 & 1 \\ -1 & 4 & -3 \\ 3 & 1 & 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 3 & -2 & 1 \\ -1 & 4 & -3 \\ 3 & 1 & 6 \end{bmatrix}$

D. $\begin{bmatrix} 3 & -2 & 1 \\ -1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$ **E.** $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 3 \\ 3 & 0 & 1 \end{bmatrix}$

14. MC A railway goods distribution company has 300 wagons that transport coal from the coalfields at station A to the electricity generation plant at station B. At the end of each week, it is known that 20% of the wagons that started the week at station A end the week at station B and 30% of the wagons that started the week at station B end the week at station A. The transition matrix that describes the distribution of wagons is:

A. $\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$ **B.** $\begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix}$ **C.** $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ **D.** $\begin{bmatrix} 0.7 & 0.2 \\ 0.8 & 0.3 \end{bmatrix}$ **E.** $\begin{bmatrix} 120 \\ 180 \end{bmatrix}$

15. MC The communication matrix for the following communication network is:



A.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

B.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

C.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

D.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

E.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Use the following information for questions **16** and **17**.

Four players, Alex (A), Lena (L), Nathan (N) and Rachel (R) took part in a round robin badminton competition. The results of the competition are summarised by adjacency matrix A and A^2 below.

$$A = \begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad A^2 = \begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

16. MC Which of the following shows the dominance matrix?

A.
$$\begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

B.
$$\begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

C.
$$\begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

D.
$$\begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

E.
$$\begin{array}{c} \\ A \\ L \\ N \\ R \end{array}\begin{array}{c} \begin{array}{cccc} A & L & N & R \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

17. MC The order (starting from the winner) in which competitors finished is:

A. Alex, Lena, Rachel, Nathan
B. Lena, Nathan, Rachel, Alex
C. Nathan, Rachel, Lena, Alex
D. Nathan, Lena, Rachel, Alex
E. Alex, Rachel, Lena, Nathan

Short answer

1. For the following two matrices

$$A = \begin{bmatrix} 10 & 7 \\ 5 & -5 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix}$$

a. state the order and type of matrix
b. where possible, write the 3, 2 element
c. state the position of the number 5 in x_{ij} form.

2. Given the following matrices

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -5 & 0 \\ 3 & -3 & 3 \end{bmatrix}$$

find, if possible:

a. $A + B$ **b.** $B - C$ **c.** $2C$.

3. The matrix shown represents the prices (in dollars) of some mobile phone options. The first column displays the costs of two types of pre-paid mobile phones and the second column represents two types of 12-month-plan mobile phones.

$$\begin{bmatrix} 249 & 29 \\ 680 & 49 \end{bmatrix}$$

The company wants to increase the price of the pre-paid mobile phones by 12% and decrease the cost of the 12-month-plan mobile phones by 5%.

a. Show the matrix used to represent the price changes as a 2×2 matrix.

b. Use matrix multiplication to calculate the new prices.

4. In the matrix multiplication $A \times B = C$, the order of B is 2×3 and the order of C is 4×3. What is the order of matrix A?

5. For the matrix equation $RS = T$ where $S = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, find matrix R.

6. The food-vending machine at an office building had three varieties of potato chips that were restocked at the end of each day. The cash box was emptied at the same time. The number of packets sold and the cash takings for five days are shown in the table.

Day	Plain	Salt and vinegar	Jumbo pack	Cash takings
Monday	30	20	10	$68.00
Tuesday	22	17	0	$38.00
Wednesday	20	15	5	$44.00
Thursday	15	5	1	$20.00
Friday	32	20	7	$63.60

a. Use the data from Monday to Wednesday to find the price of each variety of chips.

b. Select any other combination of three days and show that the answer obtained is the same as in part **a**.

7. A survey shows that 85% of the time, customers will continue to purchase their groceries from store A if they purchased their groceries from store A in the previous month, while 15% of the time consumers will change to purchasing their groceries from store B if they purchased their groceries from store A in the previous month. Similarly, the records show that 60% of the time, consumers will continue to purchase their groceries from store B if they purchased their groceries from store B in the previous month.

a. How many customers are still purchasing their groceries at A and B at the end of three months, if 200 customers started purchasing at A and 200 started at B?

b. What percentage (in whole numbers) of customers are purchasing their groceries at A and B at the end of 6 months, if 40% of the customers started at A?

8. For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2, 2 element and the position of the number 7 in x_{ij} form.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & 5 & 9 \end{bmatrix}, B = \begin{bmatrix} 7 \\ -2 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} -3 & 2 & -1 \\ 2 & 0 & -4 \\ -1 & -4 & 7 \end{bmatrix}$$

9. Given the matrix $R = \begin{bmatrix} 6 & -2 & -8 \\ 3 & 5 & -4 \\ -1 & 10 & 12 \end{bmatrix}$ and the permutation matrix, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, answer the following:
   a. What type of permutation is PR?
   b. With a row permutation of R, where does row 1 move to?
   c. If a row permutation of R resulted in $R' = \begin{bmatrix} 3 & 5 & -4 \\ 6 & -2 & -8 \\ -1 & 10 & 12 \end{bmatrix}$, then find the permutation matrix that achieved this.

10. Given the following communication matrix, answer the following:

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

   a. Who can C talk to?
   b. Who can B receive calls from?
   c. Who can call the most people?
   d. Who can C not call?

Extended response

1. The owner of a chain of three local video stores is doing the bookkeeping for the months of January, February and March. The number of hires of video games, weekly DVDs and overnight DVDs for the three stores are given in the table.

	January			February			March		
	Video games	**Weekly DVDs**	**Overnight DVDs**	**Video games**	**Weekly DVDs**	**Overnight DVDs**	**Video games**	**Weekly DVDs**	**Overnight DVDs**
Store A	450	1200	1500	320	600	900	540	1300	1400
Store B	310	1000	1200	158	580	920	212	1080	1569
Store C	250	750	600	130	345	700	278	850	900

   a. Represent the number of hires from each store as a 3×3 matrix for each month.
   b. Write a matrix equation to calculate the total number of each hire option for the three stores during this three-month period.

      The owner has a profit margin for each type of hire as follows: Video games at \$0.50 each, weekly DVDs at \$1.50 each and overnight DVDs at \$4.50.
   c. Show the hire profit margins as a 3×1 matrix and use it to calculate the total profit for each of the stores at the end of this three-month period.

d. Calculate the total profit made by the owner of the chain, during this three-month period.
The owner has just bought 120 copies of a new-release blockbuster on DVD. He delivers an equal quantity to each outlet. However, over the years he has determined that after a period of a week, 80% of hires from store A are returned to store A while 15% are returned to store B and the remainder to store C. For hires from store B, 70% are returned to store B, while 25% are returned to store C and the remainder to store A. Similarly for store C, 60% are returned to store C, while 30% are returned to store B and 10% are returned to store A.

e. Represent this information as a transition matrix.

f. Calculate the number of copies of the new-release blockbuster at each store after 2 weeks.

g. Use appropriate calculations to suggest how the 120 copies could have been distributed among the stores initially, and thus avoid the slow transition of copies to a long-term steady state.

2. Another video store surveys its customers to establish the type of movie each customer prefers. The results have been summarised in the following two-way frequency table.

	Men	Women	Children
Romance	14	76	14
Sci-fi	43	16	86
Comedy	32	24	152
Dramas	51	44	28
Total	140	160	280

a. Use matrix methods to represent the figures in the table as percentages. That is, the percentage of men who prefer romance movies, the percentage of men who prefer sci-fi movies, and so on, through to the percentage of children who prefer dramas.

b. From the table calculated in **a**, what percentage of women prefer romance movies?
The computer system crashed for three days and the owner had only the following figures recorded by his sales assistants. Unfortunately, they forgot to label the type of hire as shown in the table.

	Men	Women	Children	Takings
Day 1	210	36	340	$2835.00
Day 2	110	110	110	$1457.50
Day 3	350	140	420	$4375.00

The hire charge fees for video games are $2.50, overnight DVDs are $7.00 and weekly DVDs are $3.75.

c. Use this information to determine the type of hire that each of the columns in the table represent.

d. How many overnight DVDs were hired during this three-day period?
The owner needs to increase all hire charges by 14.5%.

e. Using a 1×3 matrix to represent the hire charges, calculate the new hire charges to the nearest 10 cents. Show your working.

On average, each hire cost, costs the owner in overheads and staff wages: $0.75 for video games, $2.25 for weekly DVDs and $5.00 for overnight DVDs.

f. Use matrix methods to calculate the new profit margins for each type of hire

3. There are two petrol stations to choose from in the town of Naralleville, and as the town grows more people purchase fuel at the petrol stations each week. A matrix equation used to determine the number of customers expected to purchase fuel in each of the petrol stations is given by

$$P_{n+1} = \begin{bmatrix} 0.60 & 0.30 \\ 0.40 & 0.70 \end{bmatrix} S_n + \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

where P_n is the column matrix that lists the number of customers in week n.The number of customers who purchased petrol at each petrol station in the first week is given by the matrix

$$P_1 = \begin{bmatrix} 184 \\ 168 \end{bmatrix} \begin{matrix} \text{Station A} \\ \text{Station B} \end{matrix}$$

a. Calculate the number of customers who are expected to purchase fuel at station A in week 3.
b. What is the difference in the number of customers for station A after 4 weeks compared to 2 weeks?

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Answers

Topic 8 Matrices

Ezercise 8.2 Matrix representation

1. A is a 3×3 diagonal matrix; 2, 1 is 0; 3 is represented by a_{33}.
   B is a 2×2 square matrix; 2, 1 is 3; 3 is represented by b_{21}.
   C is a 2×2 binary matrix; 2, 1 is 0; 3 is not in the matrix.
   D is a 3×3 symmetrical matrix; 2, 1 is 1; 3 is represented by d_{11}.
   E is a 3×3 lower triangular matrix; 2, 1 position is 5; 3 is represented by e_{32}.
2. A is a 3×3 upper triangular matrix; 1, 2 is -5; 3 is represented by a_{11}.
   B is a 1×3 row matrix; 1, 2 is -3; 3 is not in the matrix.
   C is a 2×2 square matrix; 1, 2 is 10; 3 is represented by c_{12}.
   D is a 3×3 symmetrical matrix; 1, 2 is 7; 3 is represented by d_{32} and d_{23}.
3. a. i. 4×2 rectangular matrix
      ii. 2
      iii. a_{22}
   b. i. 3×3 square matrix
      ii. 0
      iii. b_{33}
   c. Not a matrix
   d. i. 3×1 column matrix
      ii. 3
      iii. d_{21}
   e. i. 1×3 row matrix
      ii. Does not exist
      iii. e_{11}
   f. i. 3×3 symmetrical matrix
      ii. -4
      iii. f_{32} and f_{23}
4. A is a (3×3) lower triangular matrix. The element in the 2, 2 position is 4 and the number 7 is represented by a_{33}.
   B is a (1×3) row matrix. The element in the 2, 2 position is 7 and the number 7 is represented by b_{12}.
   C is a (2×2) identity matrix. The element in the 2, 2 position is 1 and the number 7 is not in the matrix.
   D is a (3×3) symmetrical matrix. The element in the 2, 2 position is 0 and the number 7 is represented by d_{33}.
5. a. 1 b. -5
   c. 2 d. 2.1
   e. Does not exist f. Does not exist
   g. 3 h. 3.5
6. **a**, **b** and **g**
7. C
8. B
9. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
10. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$

    Vertex A is connected once to vertices B, C and D.
    Vertex B is connected once to vertices A and C and connected twice to vertex D.
    Vertex C is connected once to vertices A and B.
    Vertex D is connected once to vertex A and twice to vertex B.
11. $\begin{bmatrix} 26 & 22 \\ 15 & 9 \\ 41 & 31 \end{bmatrix}$
12. $\begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$
13. $\begin{bmatrix} 12 & 23 & 35 \\ 9 & 16 & 25 \\ 21 & 39 & 60 \end{bmatrix}$
14. $\begin{bmatrix} 15 & 10 & 100 \\ 12 & 15 & 87 \end{bmatrix}$
15. $\begin{bmatrix} 5 & 7 & 15 \\ 12 & 2 & 7 \\ 6 & 15 & 0 \end{bmatrix}$
16. $\begin{bmatrix} 3 & 2 \\ -4 & -1 \\ 4 & -1 \\ 4 & 1 \end{bmatrix}$

Exercise 8.3 Addition, subtraction and scalar operations with matrices

1. a. $\begin{bmatrix} 4 & 4 \\ 8 & -2 \end{bmatrix}$ b. $\begin{bmatrix} 4 & 4 \\ 8 & -2 \end{bmatrix}$
   c. This cannot be done. d. $\begin{bmatrix} -4 & -10 \\ 2 & 6 \end{bmatrix}$
   e. $\begin{bmatrix} -12 & 0 & 0 \\ -6 & 0 & -3 \end{bmatrix}$
2. a. This cannot be done. b. $\begin{bmatrix} 23 & 37 & -25 \\ 19 & 22 & -22 \end{bmatrix}$
   c. $\begin{bmatrix} 7 & 3 & -17 \\ 17 & -2 & -4 \end{bmatrix}$ d. $\begin{bmatrix} 24 & 3 & -9 \\ 30 & 5 & -2 \end{bmatrix}$
   e. $\begin{bmatrix} -17 & 0 & -8 \\ -13 & -7 & -2 \end{bmatrix}$
3. a. $\begin{bmatrix} 4 & 5 \\ 2 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 10 \\ -6 & 4 \end{bmatrix}$ c. Not possible
   d. $\begin{bmatrix} 2 & -5 \\ 8 & -4 \end{bmatrix}$ e. $\begin{bmatrix} 3 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$
4. D
5. $\begin{bmatrix} 26.00 & 90.00 & 81.50 & 159.95 \end{bmatrix}$
6. a. $\begin{bmatrix} -18 & 24 \\ 18 & -12 \end{bmatrix}$ b. $\begin{bmatrix} 12 & -16 \\ -12 & 8 \end{bmatrix}$
   c. $\begin{bmatrix} -3 & 4 \\ 3 & -2 \end{bmatrix}$ d. $\begin{bmatrix} -2.4 & 3.2 \\ 2.4 & -1.6 \end{bmatrix}$
7. a. $\begin{bmatrix} -14 & -6 \\ 19 & -6 \end{bmatrix}$ b. $\begin{bmatrix} 12 & -12 \\ 3 & 3 \end{bmatrix}$
8. a. $\begin{bmatrix} 12 & 15 \\ 3 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 0.4 & 0.8 \\ 0.2 & 0.6 \end{bmatrix}$ c. $\begin{bmatrix} 20 & 25 \\ 5 & 10 \end{bmatrix}$
   d. $\begin{bmatrix} 5 & 7 \\ 1.5 & 3.5 \end{bmatrix}$ e. $\begin{bmatrix} 16 & 26 \\ 6 & 16 \end{bmatrix}$ f. $\begin{bmatrix} 4 & 3.5 \\ 0.5 & 0 \end{bmatrix}$

9. a. $\begin{bmatrix} 4 & 12 & 0 & 8 & 8 \\ 8 & 8 & 16 & 4 & 12 \\ 4 & 0 & -8 & 4 & -4 \\ 8 & 4 & 12 & 4 & 0 \\ 4 & 4 & 20 & 4 & 4 \end{bmatrix}$

b. $\begin{bmatrix} -0.4 & -0.6 & 0 & -0.1 & 0.2 \\ -0.2 & -0.3 & 0.4 & -0.1 & -0.3 \\ -0.3 & 0 & -0.2 & -0.1 & -0.1 \\ -0.2 & -0.2 & 0.3 & -0.1 & -1 \\ -0.5 & -0.1 & -0.8 & -0.1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} -9 & -9 & 0 & 3 & 12 \\ 0 & -3 & 24 & 0 & 0 \\ -6 & 0 & -12 & 0 & -6 \\ 0 & -3 & 18 & 0 & -30 \\ -12 & 0 & -9 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 14 & 30 & 0 & 14 & 8 \\ 16 & 18 & 16 & 8 & 24 \\ 12 & 0 & -8 & 8 & -4 \\ 16 & 10 & 12 & 8 & 20 \\ 16 & 8 & 46 & 8 & 6 \end{bmatrix}$

e. $\begin{bmatrix} -5.5 & -7.5 & 0 & -0.5 & 4 \\ -2 & -3.5 & 8 & -1 & -3 \\ -4 & 0 & -4 & -1 & -2 \\ -2 & -2.5 & 6 & -1 & -15 \\ -7 & -1 & -9.5 & -1 & -0.5 \end{bmatrix}$

10. a. $E = \begin{bmatrix} 6 & 10 & -9 \\ 12 & -2 & -20 \end{bmatrix}$

b. $D = \begin{bmatrix} 6 & 0 \\ 5 & 2 \end{bmatrix}$

c. $C = \begin{bmatrix} \frac{13}{2} & \frac{13}{2} \\ 5 & 11 \end{bmatrix}$

11. $a = 9, b = -5, c = -8, d = -4$

12. a. $E = \begin{bmatrix} 4 & 7 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

b. $D = \begin{bmatrix} 5 & -3 \\ 4 & 3 \end{bmatrix}$

c. $C = \begin{bmatrix} 6 & 13.5 \\ 3.5 & 11 \end{bmatrix}$

13. **b**, **e**

14. a. Store A; $a_{21} = 4$

b. $\begin{bmatrix} 14 & 13 \\ 9 & 12 \end{bmatrix}$

15. a. Semester 1 $\begin{bmatrix} 72 & 84 \\ 76 & 68 \\ 81 & 82 \end{bmatrix}$, Semester 2 $\begin{bmatrix} 78 & 74 \\ 76 & 77 \\ 89 & 85 \end{bmatrix}$

b. $C = \frac{1}{2}(A + B)$

c. $\begin{bmatrix} 75 & 79 \\ 76 & 72.5 \\ 85 & 83\,5 \end{bmatrix}$

16. a. Centre A and Centre C

b. $\begin{bmatrix} 34 & 35 \\ 25 & 35 \\ 14 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 40\,000 & 26\,000 \\ 50\,000 & 36\,000 \\ 80\,000 & 56\,000 \end{bmatrix}$

d. $\begin{bmatrix} 36\,000 & 23\,400 \\ 45\,000 & 32\,400 \\ 72\,000 & 50\,400 \end{bmatrix}$

17. a. $\begin{bmatrix} 0.25 & 0.40 & 0.20 & 0.15 \end{bmatrix}$ **b.** $A = 800B$

c. $\begin{bmatrix} 200 & 320 & 160 & 120 \end{bmatrix}$

18. a. $\begin{bmatrix} 154 & 214 & 1138 \\ 207 & 180 & 1422 \end{bmatrix}$

b. $\begin{bmatrix} 14 & 19\frac{5}{11} & 103\frac{5}{11} \\ 18\frac{9}{11} & 16\frac{4}{11} & 129\frac{3}{11} \end{bmatrix}$

19. a. New stock level (N) = 40% of current stock levels (matrix S)
$= 0.4 \times S$

b. $\begin{bmatrix} 9 & 60 & 45 & 2 & 436 \\ 14 & 84 & 58 & 2 & 320 \\ 18 & 92 & 52 & 3 & 480 \\ 13 & 280 & 92 & 2 & 160 \\ 31 & 180 & 64 & 3 & 340 \end{bmatrix}$

20. a. True **b.** Not true

c. Not true **d.** True

Exercise 8.4 Multiplying matrices

1. a. $\begin{bmatrix} 14 & 14 \\ 12 & -18 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 & 14 \\ 30 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

2. a. A is a 2×3 matrix.
B is a 3×2 matrix.
C is a 2×2 matrix.

b. **i.** AB does exist. **ii.** AC does not exist.

iii. BA does exist. **iv.** CA does exist.

v. BC does exist.

3. a. $A = 2 \times 2$ $B = 2 \times 3$ $C = 2 \times 2$

b. **i**, **iii**, **iv** and **vi**

c. $AB = 2 \times 3$ $AC = 2 \times 2$ $CA = 2 \times 2$ $CB = 2 \times 3$

d. $AB = \begin{bmatrix} 16 & 0 & 16 \\ 3 & 9 & 12 \end{bmatrix}$, $AC = \begin{bmatrix} 12 & 16 \\ 9 & 21 \end{bmatrix}$,

$CA = \begin{bmatrix} 6 & -6 \\ -9 & 27 \end{bmatrix}$, $CB = \begin{bmatrix} -6 & -2 & -8 \\ 21 & -1 & 20 \end{bmatrix}$

4. a. **i.** $\begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$ **ii.** $\begin{bmatrix} 13 & 11 \\ 5 & 8 \end{bmatrix}$

b. **i.** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **ii.** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

c. N in part **b** is the identity matrix.

5. a. Column permutation of R

b. They all change.

c. Row 1 is now Row 3.

6. a. Row permutation of R

b. Row 1 moves to Row 3.

c. $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

7. A

8. a. D **b.** A **c.** D

d. E **e.** B

9. a. $\begin{bmatrix} 24 & 26 \\ -9 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 32 & 34 \\ 6 & 12 \end{bmatrix}$

c. $\begin{bmatrix} 16 & 20 \\ 10 & -5 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 44 \\ -3 & 9 \end{bmatrix}$

e. $\begin{bmatrix} 25 & 19 \\ 8 & 17 \end{bmatrix}$

10. C

11. a. $\begin{bmatrix} 550 & 160 \\ 750 & 220 \\ 990 & 350 \end{bmatrix}$

b. $\begin{bmatrix} 1.08 & 0 \\ 0 & 0.92 \end{bmatrix}$

c. The washing machines are priced at \$594, \$810 and \$1069, and the dryers are priced at \$147, \$202 and \$322.

12. a. $\begin{bmatrix} 1.50 & 2.50 \\ 2.00 & 3.00 \\ 2\,75 & 3\,75 \end{bmatrix}$

b. $\begin{bmatrix} 1.15 & 0 \\ 0 & 0\,88 \end{bmatrix}$

c. New price for lettuce is \$1.73, \$2.30 and \$3.16. New price for potatoes (per kg) is \$2.20, \$2.64 and \$2.86.

13. a. $\begin{bmatrix} 2.50 & 0.90 \\ 3.50 & 1.90 \\ 4.00 & 2.50 \end{bmatrix}$ **b.** $\begin{bmatrix} 0.85 & 0 \\ 0 & 1.15 \end{bmatrix}$

c. $\begin{bmatrix} 2.13 & 1.04 \\ 2.98 & 2.19 \\ 3.40 & 2.88 \end{bmatrix}$

14. a. $\begin{bmatrix} 25.00 & 9.90 \\ 35.00 & 19.90 \\ 95.00 & 75.00 \\ 140.00 & 128.00 \end{bmatrix}$ **b.** $\begin{bmatrix} 0.80 & 0 \\ 0 & 0.80 \end{bmatrix}$

c. $\begin{bmatrix} 20.00 & 7.90 \\ 28.00 & 15.90 \\ 76.00 & 60.00 \\ 112.00 & 102.40 \end{bmatrix}$

15. a. $\begin{bmatrix} 25.00 & 26.00 \\ 62.50 & 45.00 \\ 55.00 & 30.00 \\ 12.50 & 22.00 \end{bmatrix}$ **b.** $\begin{bmatrix} 51.00 \\ 107.50 \\ 85.00 \\ 34.50 \end{bmatrix}$

c. i. Year 10/11 **ii.** Year 10/11

16. a. Total sales at each store $= \begin{bmatrix} 7700 & 1600 \\ 4950 & 2240 \\ 5500 & 2560 \\ 3850 & 1920 \end{bmatrix}$

b. Store A = \$9300, Store B = \$7190, Store C = \$8060 and Store D = \$5770

c. i. Store A has the highest sales for iPads at \$7700.

ii. Store A has the highest total sales at \$9300.

17. a. $\begin{bmatrix} 12 & 10 & 10 & 12 \\ 15 & 25 & 15 & 25 \\ 10 & 10 & 10 & 10 \\ 8 & 20 & 5 & 18 \end{bmatrix}$ **b.** $\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$

c. $\begin{bmatrix} 180 & 200 & 300 & 384 \\ 225 & 500 & 450 & 800 \\ 150 & 200 & 300 & 320 \\ 120 & 400 & 150 & 576 \end{bmatrix}$ **d.** $\begin{bmatrix} 1064 \\ 1975 \\ 970 \\ 1246 \end{bmatrix}$

18. a. A row permutation of D.

b. Row 1 is moved to row 2.

c. Column 3 is moved to column 2.

d. D is not a permutation matrix since it can only have a single 1 in each column and row.

e. $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

19. a. February $= \begin{bmatrix} 600 & 500 & 0 \\ 480 & 750 & 840 \\ 240 & 1000 & 0 \end{bmatrix}$

March $= \begin{bmatrix} 1200 & 1000 & 420 \\ 840 & 1500 & 1260 \\ 1200 & 1750 & 0 \end{bmatrix}$

April $= \begin{bmatrix} 1440 & 750 & 1680 \\ 600 & 1500 & 2100 \\ 1560 & 250 & 420 \end{bmatrix}$

b. $\begin{bmatrix} 3240 & 2250 & 2100 \\ 1920 & 3750 & 4200 \\ 3000 & 3000 & 420 \end{bmatrix}$

c. $\begin{bmatrix} 27 & 9 & 5 \\ 16 & 15 & 10 \\ 25 & 12 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 7590 \\ 9870 \\ 6420 \end{bmatrix}$ Store A sold \$7590. Store B sold \$9870. Store C sold \$6420.

20. a. i. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **ii.** $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

iii. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ **iv.** $\begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$

b. The numbers produced in the solution matrix are Fibonacci numbers. If we let F_n represent the nth Fibonacci number, then the situation can be represented as follows.

For a matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then A^n can be written as $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$.

c. $A^8 = \begin{bmatrix} F_9 & F_8 \\ F_8 & F_7 \end{bmatrix}$

$= \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix}$

d. $A^{30} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{30}$

$= \begin{bmatrix} 1\,346\,269 & 832\,040 \\ 832\,040 & 514\,229 \end{bmatrix}$

Given that $A_{30} = \begin{bmatrix} F_{31} & F_{30} \\ F_{30} & F_{29} \end{bmatrix}$, the 30th Fibonacci number, F_{30} is 832 040.

Exercise 8.5 Multiplicative inverse and solving matrix equations

1. $\det A = 6$
$\det B = -42$

2. $\det A = 48$
$\det B = 0$

3. a. -11 **b.** $\frac{1}{3}$ **c.** 13 **d.** 0

4. a, **b** and **c** will have an inverse; **d** has no inverse because it is a singular matrix, that is, its determinant is equal to 0.

5. $A^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -6 \\ -6 & 5 \end{bmatrix}$

6. a. $|C| = -10$

b. $C^{-1} = \frac{1}{-10}\begin{bmatrix} 4 & 1 \\ -10 & -5 \end{bmatrix}$ or $\begin{bmatrix} -0.4 & -0.1 \\ 1 & 0.5 \end{bmatrix}$

7. a. $-\frac{1}{11}\begin{bmatrix} 5 & -8 \\ -7 & 9 \end{bmatrix}$ **b.** $3\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ -1 & \frac{1}{2} \end{bmatrix}$

c. $\frac{1}{13}\begin{bmatrix} 8 & -9 \\ -3 & 5 \end{bmatrix}$ **d.** No inverse exists.

8. $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$

9. $AB = BA = I$

10. A

11. E

12. a. -0.005 **b.** $\begin{bmatrix} -90 & 40 \\ 50 & -20 \end{bmatrix}$

13. a. i. -28 **ii.** $\begin{bmatrix} \frac{1}{14} & \frac{3}{28} \\ \frac{3}{14} & -\frac{5}{28} \end{bmatrix}$

b. i. 6 **ii.** $\begin{bmatrix} -1 & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

c. i. -1 **ii.** $\begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

d. i. -0.1 **ii.** $\begin{bmatrix} -2.5 & 10 \\ 2 & -4 \end{bmatrix}$

e. i. 0 **ii.** Does not exist — singular matrix

f. i. 0 **ii.** Does not exist — singular matrix

g. i. 96 **ii.** $\begin{bmatrix} -\frac{5}{24} & \frac{1}{4} & \frac{1}{24} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{24} & \frac{1}{4} & -\frac{5}{24} \end{bmatrix}$

h. i. -3072 **ii.** $\begin{bmatrix} \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 \\ \frac{1}{12} & 0 & \frac{1}{12} & -\frac{1}{6} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{5}{48} & -\frac{3}{16} & \frac{1}{48} & \frac{1}{3} \end{bmatrix}$

i. i. -72 **ii.** $\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \end{bmatrix}$

j. **i** and **ii** Not possible — J is not a square matrix.

14. a. $X = \begin{bmatrix} 3 & -44 \\ -3 & 61 \end{bmatrix}$ **b.** $X = \begin{bmatrix} 44 \\ -59 \end{bmatrix}$

15. a. $X = \begin{bmatrix} 9 & -3 \\ -17 & 7 \end{bmatrix}$ **b.** $X = \begin{bmatrix} 58 \\ -102 \end{bmatrix}$

16. a. $X = \frac{1}{30}\begin{bmatrix} 21 & -33 \\ 15 & 15 \end{bmatrix} = \begin{bmatrix} 0.7 & -1.1 \\ 0.5 & 0.5 \end{bmatrix}$

b. $X = \frac{1}{30}\begin{bmatrix} 36 & -45 \\ 18 & 0 \end{bmatrix} = \begin{bmatrix} 1.2 & -1.5 \\ 0.6 & 0 \end{bmatrix}$

c. $X = \frac{1}{30}\begin{bmatrix} 15 & 25 \\ -18 & 30 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.33 \\ -0.6 & 1 \end{bmatrix}$

d. $X = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

17. a. $X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **b.** $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

18. $X = \begin{bmatrix} 5 & 6 \\ -3 & -\frac{8}{3} \end{bmatrix}$

Exercise 8.6 Dominance and communication matrices

1. a. i. A–D, A–D (2)

ii. A–C–D, A–E–D (2)

iii. A–B–E–D (1)

b. $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2. a. i. A–D (1)

ii. A–B–D, A–C–D, A–C–D, A–E–D (4)

iii. A–C–E–D (1)

b. $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. a. i. A–D (1)

ii. A–B–D, A–B–D, A–C–D, A–C–D (4)

iii. none possible

b. i. A–D (1)

ii. A–B–D, A–E–D, A–C–D, A–C–D (4)

iii. A – C – E – D, A – C – E – D (2)

4. a. $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5. One-stage routes: L_1–L_2 (1)
Two-stage routes: L_1–B–L_2, L_1–A–L_2, L_1–A–L_2 (3)
Three-stage routes: L_1–A–B–L_2, L_1–A–B–L_2 (2)
6 routes overall

6. a. A–B–D **b.** D–C–A

c. D–C–B, D–C–A–B

7. a. i. O – G
ii. O – R – G
iii. O – C – R – G, O – C – O – G, O – S – O – G
b. i. None possible
ii. R – C – S
iii. R – G – O – S, R – C – O – S
c. i. None possible ii. S – O – R
iii. S – O – C – R

8. P

9. E

10. a. First: D, second: C, third: B and fourth: A

b. $A + A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

Sum of the first row is 0, A is placed fourth.
Sum of the second row is 1, B is placed third.
Sum of the third row is 3, C is placed second.
Sum of the fourth row is 6, D is placed first (as it has the highest sum).

11. a. First: C, second: E, third: A, fourth: B and fifth: D

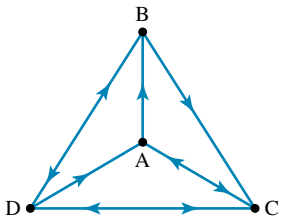
b. $A + A^2 = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 & 0 \end{bmatrix}$

Sum of the first row is 3, A is placed third.
Sum of the second row is 1, B is placed fourth.
Sum of the third row is 10, C is placed first (as it has the highest sum).
Sum of the fourth row is 0, D is placed fifth.
Sum of the fifth row is 6, E is placed second.

12. a. C **b.** Z

13. D

14.



Communication matrix $= \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$

15. a. 'Talk to' (vertical) so C talks to A, B and D.
b. 'Receive calls' (horizontal) so B receives calls from A, C and D.
c. Since they cannot call themselves.
d. 'Call' or 'talk to' (vertical) so D cannot call C.

16.

Communication matrix $= \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$

17. a. 'Talk to' (vertical) so C talks to A and D.
b. 'Receive calls' (horizontal) so B receives calls from D.
c. 'Call/talk to' (vertical) B can only call D.
d. D cannot call A.
e. D can receive calls from A, B and C.

18. a. F **b.** A and C
c. F, E, D, B and A/C

Exercise 8.7 Application of matrices to simultaneous equations

1. $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

2. $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

3. a. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ **b.** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$
c. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ **d.** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

4. a. $\begin{bmatrix} 11 \\ 14 \end{bmatrix}$ **b.** Cannot be solved
c. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **d.** $\begin{bmatrix} -2 \\ -5 \end{bmatrix}$
e. $\begin{bmatrix} 3\frac{1}{3} \\ -7 \\ 7\frac{2}{3} \end{bmatrix}$

5. No unique solution

6. Unique solution

7. a. i. Unique solution
ii. No unique solution
iii. No unique solution
iv. Unique solution
b. System **i** is inconsistent and System **ii** is dependent.

8. a. $\begin{bmatrix} 6 & 2 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{3}{2} & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$
b. The determinant for both coefficient matrices is 0.
c. No solutions. As the determinant is 0, (singular matrix), the inverse does not exist.
d. The two graphs are parallel and do not intersect. Therefore, there is no solution.
e. The two graphs are the same line.

9. a. $2x - y = -3$ or $-2x + y = 3$
$x - y = -1$ $-x + y = 1$
b. $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
c. $(-2, -1)$

10. $a = 2$, $b = -1$, $c = 0.5$ and $d = 3$

11. 7 sedans and 15 utilities

12. 72 jeans, $2440

13. Field 1: 1200 square metres; field 2: 600 square metres

14. The costs and revenue are equal ($23 500) when 7000 components are manufactured.

15. Adults: $20, Children: $6, Pensioners: $5

16. 8 and 50

17. a. Model A: 18, Model B: 15
b. $105

18. $a = -4$ and $d = 6.6$

Exercise 8.8 Transition matrices

1. a. i. $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ **ii.** $\begin{bmatrix} 0.63 & 0.38 \\ 0.38 & 0.63 \end{bmatrix}$

iii. $\begin{bmatrix} 0.56 & 0.44 \\ 0.44 & 0.56 \end{bmatrix}$ **iv.** $\begin{bmatrix} 0.51 & 0.49 \\ 0.49 & 0.51 \end{bmatrix}$

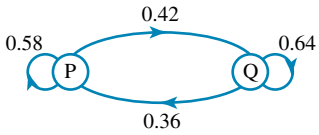
b. $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

2. a. i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iii. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **iv.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. The identity matrix

3. a.



To / From

	P	Q
P	0.58	0.36
Q	0.42	0.64

b. $\begin{bmatrix} 0.69 & 0.27 \\ 0.31 & 0.73 \end{bmatrix}$

4. D

5. a.

	A	B
A	0.68	0.23
B	0.32	0.77

b.

	North	South
North	0.80	0.15
South	0.20	0.85

6. a. $\begin{bmatrix} 0.9 & 0.08 \\ 0.1 & 0.92 \end{bmatrix}$

b. 120 wagons at point A and 130 wagons at point B

c. 43% of the wagons at point A and 57% of the wagons at point B

7. A

8. C

9. a. $\begin{bmatrix} 0.85 & 0.7 \\ 0.15 & 0.3 \end{bmatrix}$ **b.** $\begin{bmatrix} 0.65 & 0.20 & 0.03 \\ 0.15 & 0.68 & 0 \\ 0.20 & 0.12 & 0.97 \end{bmatrix}$

10. a. $T = \begin{bmatrix} 0.52 & 0.16 & 0.14 \\ 0.12 & 0.40 & 0.07 \\ 0.36 & 0.44 & 0.79 \end{bmatrix}$

b. Store A will have 694 customers, store B will have 369 customers and store C will have 1937 customers.

c. Store A: 23.0%, store B: 12.2%, store C: 64.9%

d. Store A: $\frac{17}{74}$, store B: $\frac{9}{74}$, store C: $\frac{24}{37}$

11. a. 68.1% **b.** 32.0%

12. a. $\begin{bmatrix} 0.385 & 0.14 & 0 & 0.0775 \\ 0.28 & 0.66 & 0 & 0.01 \\ 0.18 & 0.19 & 1 & 0.005 \\ 0.155 & 0.01 & 0 & 0.9075 \end{bmatrix}$

b. T has no steady state because the matrix T^2 contains zero elements.

c. A

13. a. $\begin{bmatrix} 0.6 & 0.1 & 0 & 0.05 \\ 0.2 & 0.8 & 0.05 & 0 \\ 0.1 & 0.1 & 0.9 & 0 \\ 0.1 & 0 & 0.05 & 0.95 \end{bmatrix}$

b. i. Texcal, 4128 customers

ii. Oilmart, 4116 customers

iii. CP, 5600 customers

14. a. There are 100 jersey cows expected to be milked on day 4.

b. Nine more cows were milked on day 4 compared to day 1.

15. $S_2 = \begin{bmatrix} 0.65 & 0.23 \\ 0.35 & 0.77 \end{bmatrix}\begin{bmatrix} 48 \\ 41 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

16. a. 61%

b. 82%

c. 16 additional people go to Supermarket A and 7 additional people go to Supermarket B.

17. a. Week 2: Sweet Cola: 1139, Tangy Lemon: 1042
Week 3: Sweet Cola: 1296, Tangy Lemon: 1066
Week 4: Sweet Cola: 1433, Tangy Lemon: 1110

b. 543

18. a. 21 students **b.** 8 students

19. a. 86 sheep **b.** Difference = 220 − 212 = 8 sheep

20. 2013: City A: 59 210, City B: 69 945
2014: City A: 63 045, City B: 68 265
2015: City A: 66 554, City B: 66 911

21. a. 0.078

b. Dry: $\frac{11}{12}$, wet: $\frac{1}{12}$

c. They should insure the event as there is a very good chance that the day will be dry.

22. B

23. a. 264 shoppers **b.** 56 shoppers

24. Loss of 5229 (6000 − 771) Victorians to Queensland

Review: Exam practice

Multiple choice

1. C	**2.** C	**3.** A	**4.** D
5. A	**6.** E	**7.** B	**8.** D
9. C	**10.** B	**11.** E	**12.** B
13. A	**14.** C	**15.** B	**16.** D
17. D			

Short answer

1. a. A: is a 3×2 rectangular matrix
B: is a 3×3 square matrix

b. a_{32} is 2 , b_{32} is 0

c. a_{21}, b_{31}

2. a. $\begin{bmatrix} 4 & 5 \\ 8 & 4 \end{bmatrix}$ **b.** Not possible **c.** $\begin{bmatrix} 2 & -10 & 0 \\ 6 & -6 & 6 \end{bmatrix}$

3. a. $\begin{bmatrix} 1.12 & 0 \\ 0 & 0.95 \end{bmatrix}$ **b.** $\begin{bmatrix} 278.88 & 27.55 \\ 761.60 & 46.55 \end{bmatrix}$

4. 4×2

5. $R = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$

6. a. Plain: \$0.80, Salt and vinegar: \$1.20, Jumbo: \$2.00

b. Plain: \$0.80, Salt and vinegar: \$1.20, Jumbo: \$2.00

7. a. Store A: 283, store B: 117

b. Store A: 72.5%, store B: 27.5%

8. A: Order (3×3); lower triangular matrix; $2, 2 = 4$; 7 is in x_{31}.
B: Order (3×1); column matrix; $2, 2$ doesn't exist; 7 is in x_{11}.
C: Order (2×2); identity matrix; $2, 2 = 1$; 7 is not in the matrix.
D: Order (3×3); symmetrical matrix; $2, 2 = 0$; 7 is in x_{33}.

9. a. PR is a row permutation of R.

b. Row 1 moves to row 3.

c. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. a. C can 'talk to' (vertical) B and D.

b. B can 'receive calls' (horizontal) from C and D.

c. D can talk to all three; therefore, D can talk to the most people.

d. C cannot call A or themselves; therefore, A and C.

Extended response

1. a. $January{:}A = \begin{bmatrix} 450 & 1200 & 1500 \\ 310 & 1000 & 1200 \\ 250 & 750 & 600 \end{bmatrix}$,

$February{:}B = \begin{bmatrix} 320 & 600 & 900 \\ 158 & 580 & 920 \\ 130 & 345 & 700 \end{bmatrix}$,

$March{:}C = \begin{bmatrix} 540 & 1300 & 1400 \\ 212 & 1080 & 1569 \\ 278 & 850 & 900 \end{bmatrix}$

b. $A + B + C = \begin{bmatrix} 1310 & 3100 & 3800 \\ 680 & 2660 & 3689 \\ 658 & 1945 & 2200 \end{bmatrix}$

c. $\begin{bmatrix} 0.50 \\ 1.50 \\ 4\ 50 \end{bmatrix}$

Store A: \$22 405, store B: \$20 930.50, store C: \$13 146.50

d. \$56 482

e. $T = \begin{bmatrix} 0.80 & 0.05 & 0.10 \\ 0.15 & 0.70 & 0.30 \\ 0.05 & 0.25 & 0.60 \end{bmatrix}$

f. Store A: 36, store B: 49, store C: 35

g. Store A: 31, store B: 52, store C: 37

2. a. $\begin{bmatrix} 10 & 47.5 & 5 \\ 30.7 & 10 & 30.7 \\ 22.9 & 15 & 54.3 \\ 36.4 & 27.5 & 10 \end{bmatrix}$

b. 47.5%

c. The first column represents the overnight DVDs, the second column represents the videos and the final column represents the weekly DVDs.

d. 670

e. Videos: \$2.90, overnight DVDs: \$8.00, weekly DVDs: \$4.30

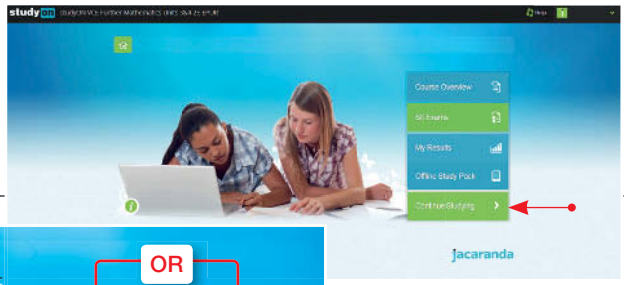
f. Videos: \$2.15, overnight DVDs: \$3.00, weekly DVDs: \$2.05

3. a. 160 customers **b.** 3 customers

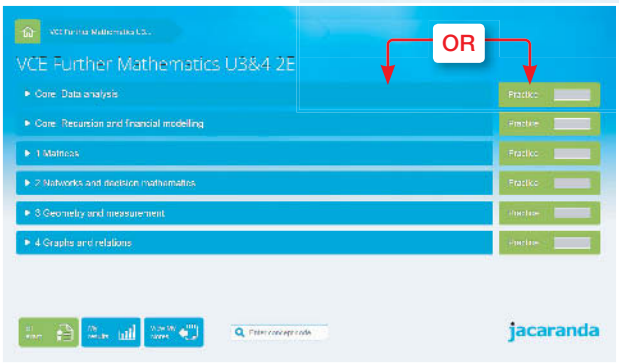
REVISION: AREA OF STUDY 2 Matrices

TOPIC 8

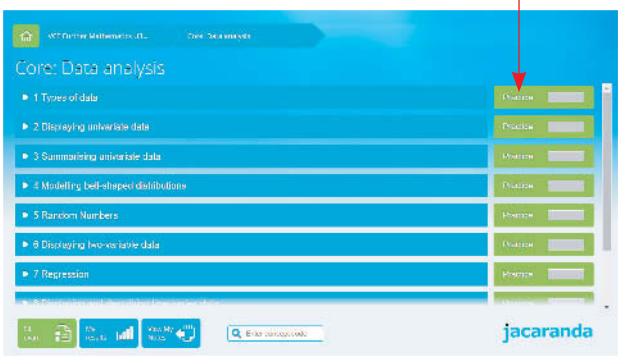
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



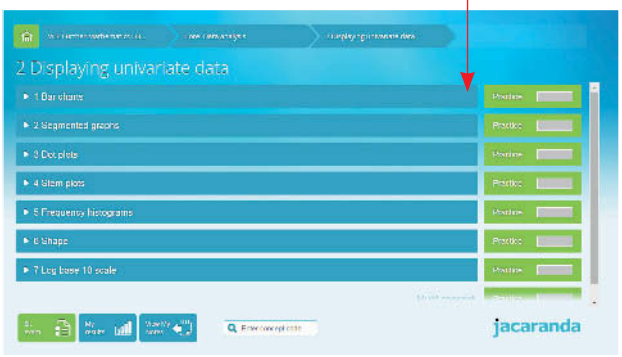
- Select your **course** *Further Mathematics VCE Units 3 & 4* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



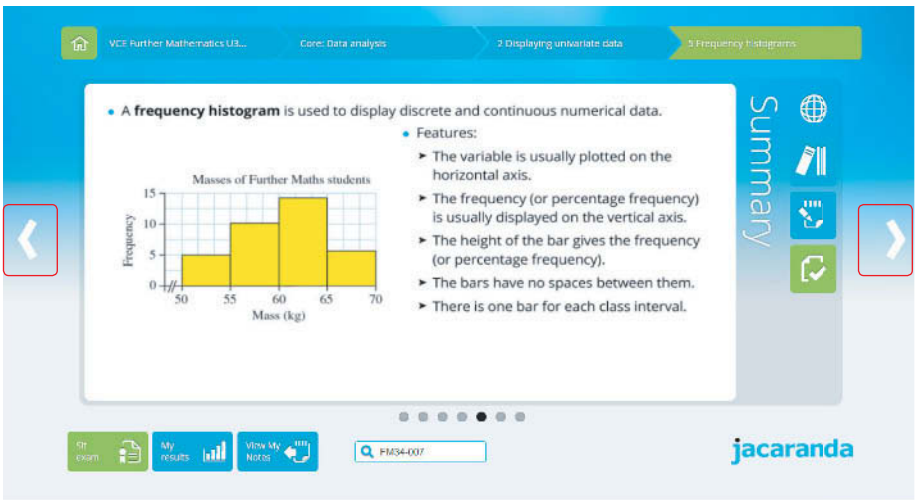
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 9
Undirected graphs and networks

9.1 Overview

9.1.1 Introduction

The study of networks and decision mathematics is a branch of graph theory. The graphs referred to in this area of study are not graphs such as bar graphs or graphs associated with the cartesian plane but rather refer to geometric representations of situations where the graphs are defined as sets of vertices and sets of edges.

The simplest form of networks are undirected graphs. These graphs show two-way connections between vertices and are not concerned with the direction of the path between vertices. The social connections between a group of friends could be illustrated using an undirected graph as a friendship would imply a reciprocal relationship. The network of airports in Australia would be an undirected graph as it would be safe to assume that if a plane flew into an airport from a city it would fly out of that airport and back to the first city.

Early studies of graph theory were in response to difficult problems. Francis Guthrie, a mathematician, first posed the Four Colour Theorem in 1852. This theorem states that if a map is drawn on a plane using only 4 colours, where each region touches the other three, then regions sharing a common boundary do not share the same colour. It took nearly 150 years for this theorem to be proved. In 1976, mathematicians at the University of Illinois were able to harness their knowledge of graph theory and the power of computer algorithms and test all possible combinations to prove the theorem.

LEARNING SEQUENCE

9.1 Overview
9.2 Basic concepts of a network
9.3 Planar graphs and Euler's formula
9.4 Walks, trails, paths, cycles and circuits
9.5 Trees and their applications
9.6 Review: exam practice

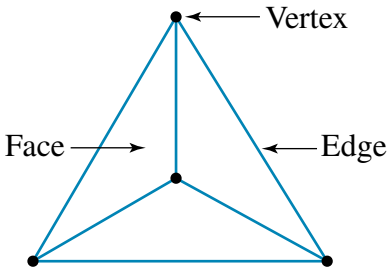
Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

9.1.2 Kick off with CAS

Planar graphs and Euler's formula

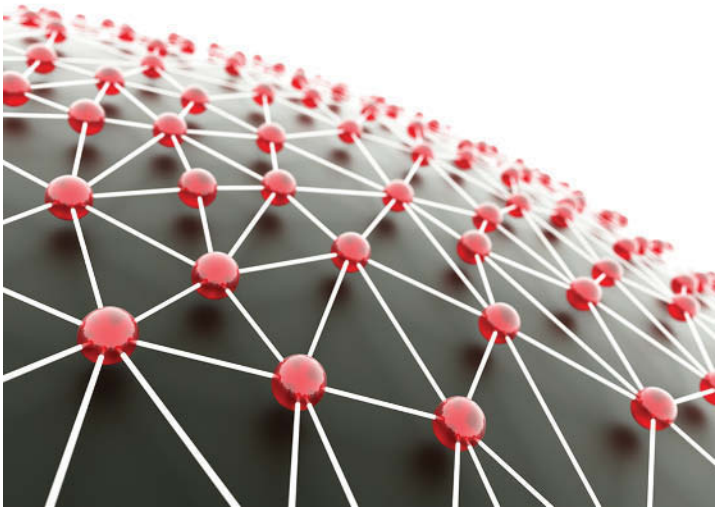
In graph theory, graphs are made up of vertices, with edges connecting the vertices.

Planar graphs are graphs in which there are no intersecting edges. The edges and vertices in a planar graph divide the graph into a number of faces, as shown in the following diagram. When counting the number of faces remember to include the infinite face — the region surrounding the graph.



Euler's formula links the number of vertices (v), edges (e) and faces (f) in a planar graph with the rule $v - e + f = 2$.

1. Using CAS, define and save Euler's formula for planar graphs.
2. Use CAS to solve Euler's formula for
   - **a.** the number of faces (f)
   - **b.** the number of edges (e)
   - **c.** the number of vertices (v).
3. Use your answer from **2a** to calculate the number of faces in a graph with:
   - **a.** 6 vertices and 8 edges
   - **b.** 5 vertices and 5 edges.
4. Use your answer from **2b** to calculate the number of edges in a graph with:
   - **a.** 5 vertices and 4 faces
   - **b.** 7 vertices and 5 faces.
5. Use your answer from **2c** to calculate the number of vertices in a graph with:
   - **a.** 6 edges and 4 faces
   - **b.** 9 edges and 5 faces.



on Resources

Please refer to the Resources section in the Prelims of your **eBookPlus** for a comprehensive step-by-step guide on how to use your CAS technology.

9.2 Basic concepts of a network

9.2.1 Definition of a network

What do the telephone system, the Australian Army, your family tree and the internet have in common? The answer is that they can all be considered **networks**.

The simplest possible definition of a network, which will suit our purposes throughout this topic, is:

> **A network is a collection of objects connected to each other in some specific way.**

In the case of the telephone system, the objects are telephones (and exchanges, satellites, ...). In the case of the Australian Army, the objects are units (platoons, companies, regiments, divisions, ...). In the case of the internet, the objects are computers; while your family tree is made up of parents, grandparents, cousins, aunts, ...

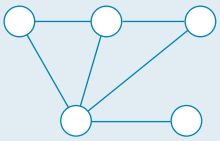
The mathematical term for these objects is a vertex. Consider the network represented in the diagram. This is perhaps the simplest possible network. It consists of two vertices (circles labelled 1 and 2) and one *connection* between them. This connection is called an **edge**.

In the case of the telephone system, the edges are the cables connecting homes and exchanges; in the Australian Army they are the commanding officers of various ranks; while in the family tree the links between the generations and between husband and wife can be considered as edges.

The first distinguishing features of a network are the total number of vertices and total number of edges.

WORKED EXAMPLE 1

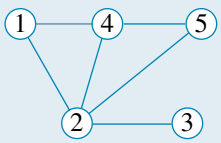
Count the number of vertices and edges in the network shown.



THINK

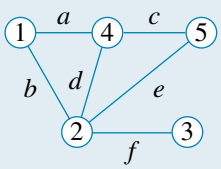
1. Count the vertices by labelling them with numbers.

WRITE/DRAW



Thus, there are 5 vertices.

2. Count the edges by labelling them with letters.



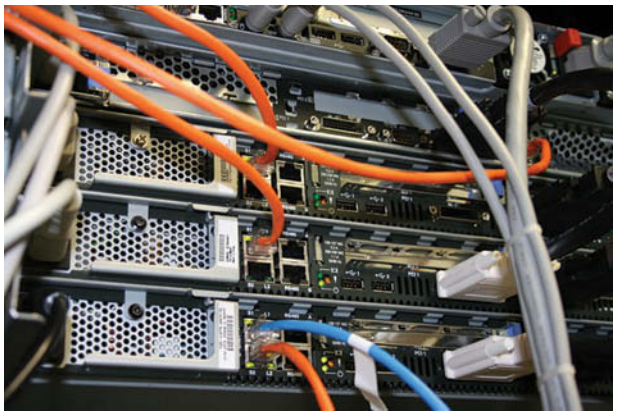
Thus, there are 6 edges.

There are two things worth noting about this classification of a network:

1. The vertices and edges can be labelled in any order, using any suitable labelling system.
2. Vertices may have different numbers of edges connected to them. How many edges are connected to vertex 2 in Worked example 1?

9.2.2 The degree of a vertex

Each vertex may have a number of edges connecting it with the rest of the network. This number is called the **degree**. To determine the degree of a vertex, simply count its edges. The following table shows the degree of each vertex in Worked example 1.



Vertex	1	2	3	4	5
Degree	2	4	1	3	2

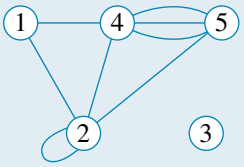
A vertex with degree 0 is *not connected* to any other vertex, and is called an **isolated vertex**.

An edge which connects a vertex to itself is called a **loop** and contributes 2 towards the degree.

If two (or more) edges connect the *same pair* of vertices they are called **parallel edges** (or *multiple edges*) and all count towards the degree. Otherwise, if there is only *one* connection between two vertices, the connection is called a *simple*, or *single*, connection.

WORKED EXAMPLE 2

Determine the degree of each vertex in the figure shown.



THINK	WRITE
1. Node 1: Has 2 simple edges.	Degree of node 1 = 2
2. Node 2: Has 3 simple edges and 1 loop.	Degree of node 2 = 3 + 2 = 5
3. Node 3: Has no edges — an isolated node.	Degree of node 3 = 0
4. Node 4: Has 2 simple edges and 3 parallel edges.	Degree of node 4 = 2 + 3 = 5
5. Node 5: Has 1 simple edge and 3 parallel edges.	Degree of node 5 = 1 + 3 = 5

9.2.3 Representations of networks

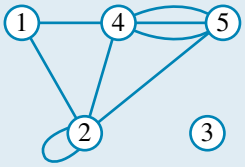
So far we have seen the graphical representation of a network as a two-dimensional collection of vertices and edges. Hence, networks are sometimes called *graphs*. There are other ways to represent the network without losing any of its essential features:

1. labelling vertices and labelling edges according to their vertices
2. matrix representation.

To label vertices, simply list them. If there are three vertices labelled A, B, and C write $V = \{A, B, C\}$. To label edges according to their vertices, identify the vertices that the edge connects. If an edge connects vertex 1 with vertex 3, we represent the edge as (1, 3). If there is a loop at vertex 4, its edge is (4, 4). If there are 2 parallel edges between vertices 2 and 4, we write (2, 4), (2, 4).

WORKED EXAMPLE 3

Label the vertices and edges for the figure shown, as in Worked example 2.



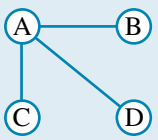
THINK	WRITE
1. Label the vertices.	$V = \{1, 2, 3, 4, 5\}$
2. Examine each edge, in turn.	
Vertex 1–vertex 4	(1, 4)
Vertex 1–vertex 2	(1, 2)
Vertex 2–vertex 2 (loop)	(2, 2)
Vertex 2–vertex 4	(2, 4)
Vertex 2–vertex 5	(2, 5)
Vertex 4–vertex 5 (3 parallel edges)	(4, 5), (4, 5), (4, 5)
3. Combine vertices and edges into a list.	$V = \{1, 2, 3, 4, 5\}$ $E = \{(1, 4), (1, 2), (2, 2), (2, 4), (2, 5), (4, 5), (4, 5), (4, 5)\}$

There are several points to note about this representation:

1. There is no '3' in the list of edges (E). This implies it is an isolated vertex.
2. The number of pairs in $E\{(1, 4), (1, 2), ...\} = 8$ which must be the same as the number of edges.
3. The number of times a vertex appears anywhere inside E equals the degree of the vertex. For example, the digit 4 appears 5 times, so the degree of vertex 4 = 5.
4. From this representation of V and E we can construct (or reconstruct) the original graph.

WORKED EXAMPLE 4

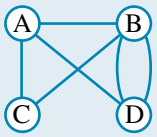
Construct a graph (network) from the following list of vertices and edges.
$V = \{A, B, C, D, E\}$
$E = \{(A, B), (A, C), (A, D), (B, C), (B, D), (C, E), (D, E), (E, E)\}$

THINK	WRITE/DRAW
1. Start with a single vertex, say vertex A, and list the vertices to which it is connected.	Vertex A is connected to B, C and D.
2. Construct a graph showing these connections.	

3. Take the next vertex, say B, and list the vertices to which it is connected.

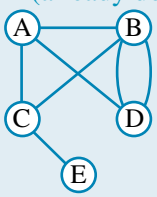
Vertex B is connected to A (already done), C and D (twice: parallel edge).

4. Add the edges from step 3.



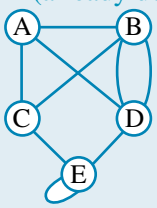
5. Repeat steps 3 and 4 for vertex C.

Vertex C is connected to A (already done), B (already done) and E.



6. Repeat steps 3 and 4 for vertex D and, finally, add the loop (E, E). As a check, count the edges in the list E (9) and compare it with the number of edges in your final graph.

Vertex D is connected to A (already done), B (already done) and E.
Vertex E is connected to C (already done), D (already done) and E (loop).



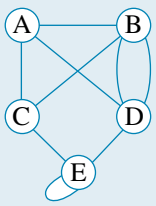
There may be other geometric configurations which can be drawn from the same vertex and edge lists, but they would be **isomorphic** (or equivalent) to this one.

9.2.4 Matrix representation of networks

A method of representing a network in concise form is through the use of a *matrix*. Recall that a matrix is a rectangular collection, or 'grid' of numbers. To represent the network, write the names of the vertices above the columns of the matrix and to the left side of the rows of the matrix. The number of edges connecting vertices is placed at the intersection of the corresponding row and column. This is best shown with an example.

WORKED EXAMPLE 5

Represent the network shown (from Worked example 4) as a matrix.



THINK

1. Set up a blank matrix, putting the vertex names across the top and down the side. Thus there are 25 possible entries inside the matrix.

WRITE

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

2. Consider vertex A. It is connected to vertices B, C, and D once each, so put 1 in the corresponding columns of row 1 and in the corresponding row of column 1.

	A	B	C	D	E
A		1	1	1	
B	1				
C	1				
D	1				
E					

3. Consider vertex B. It is connected to C once and D twice. Put 1 and 2 in the corresponding columns of row 2 and in the corresponding rows of column 2.

	A	B	C	D	E
A		1	1	1	
B	1		1	2	
C	1	1			
D	1	2			
E					

4. Repeat for vertices C, D and E.Vertex C is connected to vertex E once, so put 1 in the corresponding column of row 3 and in the corresponding row of column 3 (shown in red).
Vertex D is connected to vertex E once, so put 1 in the corresponding column of row 4 and in the corresponding row of column 4 (shown in black).
Vertex E is connected to itself once (loop), so put 1 in the corresponding column 5, row 5. *Note:* Only one entry is needed for loops (shown in green). A value of 1 in the leading diagonal denotes a loop in the network, connecting a vertex to itself. This is important to understand when calculating the degree.

	A	B	C	D	E
A		1	1	1	
B	1		1	2	
C	1	1			1
D	1	2			1
E			1	1	1

5. Complete the matrix by placing 0 in all unoccupied places.

6. Check your result by comparing the entries in the matrix with the original network representation. This is best done on a vertex-by-vertex basis.

	A	B	C	D	E
A	0	1	1	1	0
B	1	0	1	2	0
C	1	1	0	0	1
D	1	2	0	0	1
E	0	0	1	1	1

In matrix representation:

	A	B	C	D	E	
A	0	1	1	1	0	
B	1	0	1	2	0	$= 4$
C	1	1	0	0	1	
D	1	2	0	0	1	
E	0	0	1	1	1	$3 + 1 = 4$
		4			$3 + 1 = 4$	

1. the sum of a row (or a column) gives a degree of that vertex, except where a loop is present. Where a loop is present (denoted by 1 in the leading diagonal), add 1 to the sum of the row or column.
2. if an entire row and its corresponding entire column has only 0s then that vertex is isolated
3. the matrix is diagonally symmetric.

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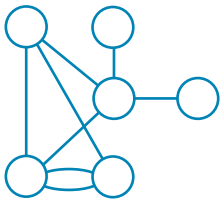
Units 3 & 4 > AOS M2 > Topic 1 > Concept 2

Matrix representation of networks Summary screen and practice questions

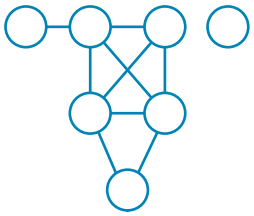
Exercise 9.2 Basic concepts of a network

1. **WE1** Count the number of vertices and edges in the following networks.

a.

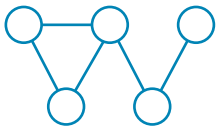


b.

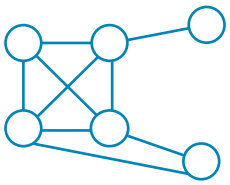


2. Count the number of vertices and edges in the following networks.

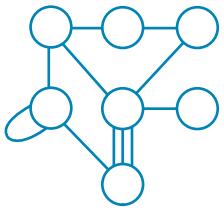
a.



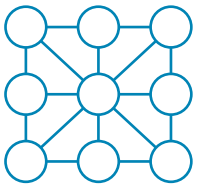
b.



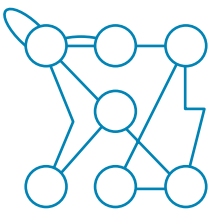
c.



d.



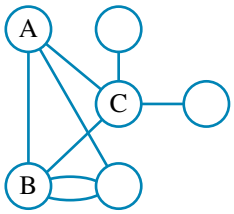
3. **MC** The number of vertices and edges in the figure is:



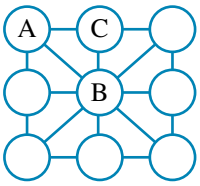
A. Vertices = 7, edges = 7
B. Vertices = 7, edges = 10
C. Vertices = 7, edges = 11
D. Vertices = 11, edges = 11
E. Vertices = 11, edges = 7

4. WE2 Determine the degree of the labelled vertices in each diagram.

a.

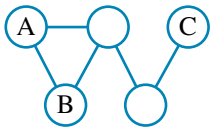


b.

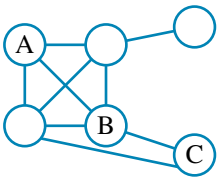


5. Determine the degree of the labelled vertices in each diagram.

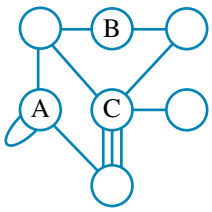
a.



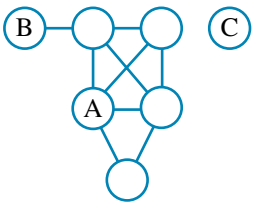
b.



c.

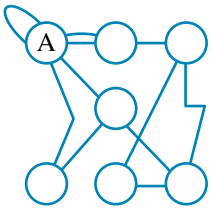


d.



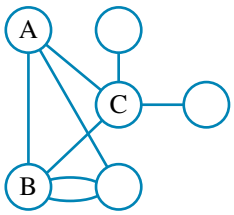
6. MC The degree of vertex A in the figure is:

A. 3
B. 4
C. 5
D. 6
E. 7

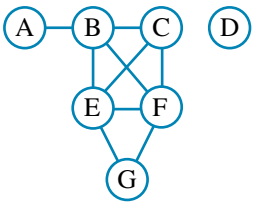


7. WE3 List the vertices and label *all* the edges, according to their vertices, in each of these diagrams.

a.

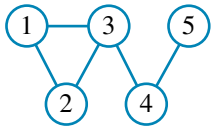


b.

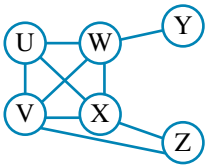


8. List the vertices and label *all* the edges, according to their vertices, in each of these diagrams.

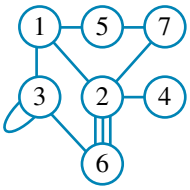
a.



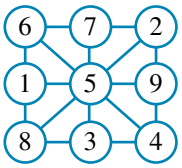
b.



c.

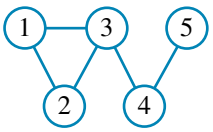


d.



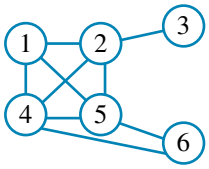
9. **WE4** Construct a network from the following list of vertices and edges.
   $V = \{1, 2, 3, 4, 5, 6\}$ $E = \{(1, 2), (1, 4), (1, 6), (2, 3), (2, 6), (3, 4), (4, 6)\}$
10. Construct a network from the following list of vertices and edges.
    $V = \{A, B, C, D, E, F, G\}$
    $E = \{(A, B), (A, C), (A, F), (B, F), (D, E), (D, F), (E, F), (F, G)\}$
11. Construct a network from the following list of vertices and edges.
    a. $V = \{1, 2, 3, 4\}$ $E = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$
    b. $V = \{A, B, C, D, E\}$ $E = \{(A, B), (A, C), (A, C), (B, B), (B, C), (B, D), (C, D)\}$
12. Consider a network of 4 vertices, where each vertex is connected to each of the other 3 vertices with a single edge (no loops, isolated vertices or parallel edges).
    a. List the vertices and edges.
    b. Construct a diagram of the network.
    c. List the degree of each vertex.
13. Repeat question **12** for:
    i. a network of 5 vertices
    ii. a network of 8 vertices.
14. Using the results from questions **12** and **13**, predict the number of edges for a similar network of:
    a. 10 vertices **b.** 20 vertices **c.** 100 vertices.
    Note that the increase in the number of edges is one of the problems that had to be overcome in the design of computer networks.
15. Construct a network representing the following family tree. Use a single node to represent each married couple.
    Allan and Betty had 3 children: Charles, Doris and Earl.
    Charles married Frances and had 2 children, George and Harriet.
    Doris married Ian and had 1 child, John.
    Earl married Karen and had 3 children, Louise, Mary and Neil.
16. **WE5** Represent the network from question **9** as a matrix.
17. Copy and complete the matrix representation of the network at right. The first few entries are shown.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & & & & \\ 1 & & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \end{array}$$

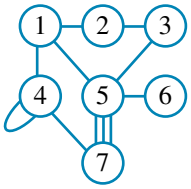


18. Represent the following networks by matrices.

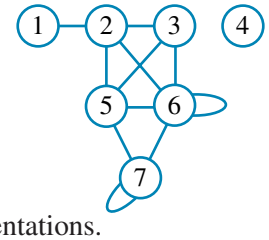
a.



b.



c.



19. Construct networks from the following matrix representations.
Note that the number of rows = number of columns = number of vertices. Watch out for loops.

a. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$

9.3 Planar graphs and Euler's formula

9.3.1 Planar graphs

A **planar graph** is a special kind of network or graph. The additional properties of planar graphs will allow us to map two-dimensional and even three-dimensional objects into graphs.

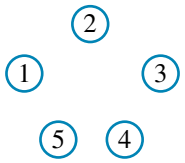
Degenerate graph

A graph with *no* edges is called a **degenerate graph** (or *null* graph).

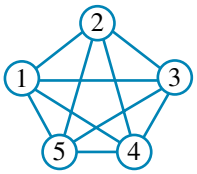
Complete graph

A graph where all vertices are connected directly to all other vertices without parallel edges or loops is called a **complete graph**.

The figure on the left is degenerate; the one on the right is complete. How many edges would there be in a complete graph of 6 vertices?



A degenerate graph



A complete graph

For a complete graph, if E = number of edges and V = number of vertices then $E = \frac{V(V-1)}{2}$.

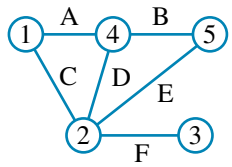
Planar graphs

A planar graph can be defined as follows:

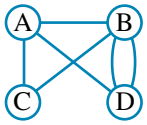
If a graph has no edges which cross, then it is a planar graph.

Consider the following graphs.

a.



b.



c.

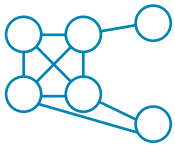
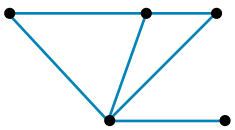


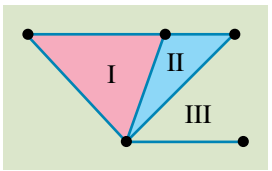
Figure **a** is a planar graph because none of the paths {A, B, C, D, E, F} cross each other. Figure **b** is apparently not a planar graph because the path (A, D) crosses the path (B, C). Is figure **c** a planar graph?

9.3.2 The regions of a planar graph



Consider a simplified version of the graph in figure **a**, as shown at right. Note that the large circular vertices have been replaced by small black circles. Otherwise this is the identical network to figure **a**.

Now, observe how this planar graph can be divided into 3 *regions*: region I, region II and region III.



Note also that one of the regions (III) will always be *infinite*, because it continues beyond the bounds of the diagram. All the other regions have a *finite* area. These regions are also called **faces**, for a reason which will soon become apparent.

The reason one region becomes infinite can be seen by considering the fact that when you look at three-dimensional objects, you can't see all the faces at the same time, no matter from which angle you look.

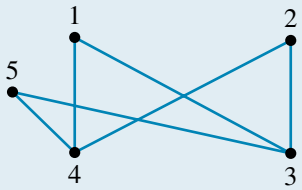
9.3.3 Converting non-planar graphs

Although it may *appear* that a graph is not planar, by modifying the graph it may become clearly planar.

There is no specific method, but by trial and error it may be possible to remove all the crossing paths. (It may also help to imagine the nodes as nails in a board and the edges as flexible rubber bands.) Alternatively, it may be possible to move the vertices so that the connecting edges don't cross. If there are no crossings left, the graph is planar.

WORKED EXAMPLE 6

Convert the graph below to a planar graph. Indicate the faces (regions) of the planar graph.

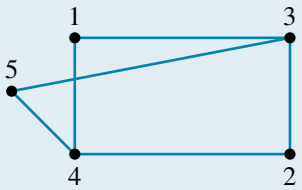


THINK

1. Confirm that the graph is non-planar.
   $E = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)\}$
2. Two crossings could be eliminated if vertex 2 were exchanged with vertex 3. Redraw the modified graph.
   Check that all the edges are connected to the same vertices.

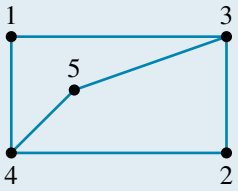
WRITE/DRAW

Edge (1, 3) crosses (2, 4).
Edge (3, 5) crosses (1, 4) and (2, 4).

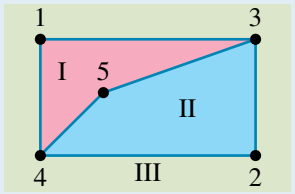


$E = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)\}$

3. Placing node 5 inside the rectangle is one way of eliminating all crossings. Note that this planar graph is only one of several possible answers.



4. Define the faces (regions).



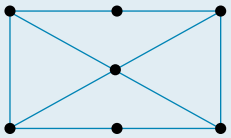
The *degree* of each face is the number of edges defining that region. Consider the last figure in Worked example 6.

Face I is defined by edges (1, 3), (1, 4), (4, 5) and (5, 3), so its degree = 4.
Face II is defined by edges (3, 5), (5, 4), (4, 2) and (2, 3), so its degree = 4.
Face III is defined by edges (1, 3), (1, 4), (4, 2) and (2, 3), so its degree = 4.

In almost all cases, each region will have a degree of *at least* 3. Why? Can you think of exceptions?

WORKED EXAMPLE 7

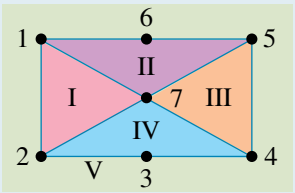
Find the degree of each face of the graph shown in the figure.



THINK

Define the edges and faces of the graph.

WRITE/DRAW



$E = \{(1,2),(2,3),(3,4),(4,5),(5,6),$
$(6,1),(1,7),(5,7),(4,7),(2,7)\}$

Count the edges for each face.
For example:
Face I — edges (1, 2), (2, 7), (7, 1)
Face II — edges (1, 6), (6, 5), (5, 7), (7, 1).

Face I — degree = 3
Face II — degree = 4
Face III — degree = 3
Face IV — degree = 4
Face V — degree = 6

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Units 3 & 4 > AOS M2 > Topic 1 > Concept 3

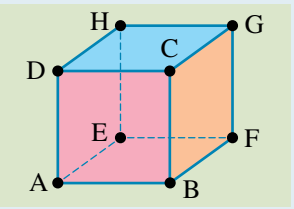
Types of graphs (networks) Summary screen and practice questions

9.3.4 Converting three-dimensional solids to planar graphs

Another application of planar graphs is the conversion of the graph representing a three-dimensional solid (with flat faces) to a planar graph.

WORKED EXAMPLE 8

The figure at right shows a cube with vertices, $V = \{A, B, C, D, E, F, G, H\}$.
Convert this to a planar graph.



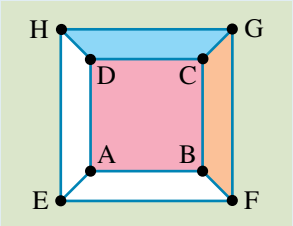
THINK

1. List the edges (12 in all).

WRITE/DRAW

$E = \{(A, B), (A, D), (A, E), (B, C), (B, F), (C, D), (C, G), (D, H), (E, F), (E, H), (F, G), (G, H)\}$

2. Imagine the three-dimensional cube 'collapsing' to a two-dimensional graph.
   Try collapsing the face A–B–C–D into the face E–F–G–H .



3. Check the edges to see that they are the same as in step 1. Note also the edges (A, E), (B, F), (C, G) and (D, H) which link the 'collapsed' faces.

There are some other interesting features of this planar graph:

1. The planar graph is, in a sense, a two-dimensional 'projection' of the original cube.
2. The original 'base' of the cube (A–B–F–E) has become the *infinite* region of the planar graph.

9.3.5 Euler's formula

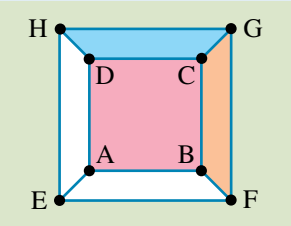
By now it may be clear that there is a mathematical relationship between the vertices, edges and faces of planar graphs.

In fact, it is the same relationship, known as **Euler's formula**, that you may have learned when studying solid geometry:

$$\text{Vertices} = \text{edges} - \text{faces} + 2$$
$$V = E - F + 2$$

WORKED EXAMPLE 9

Verify Euler's formula for the 'cube' of the last figure in Worked example 8.

THINK	WRITE/DRAW
1. List the vertices.	$V = \{A, B, C, D, E, F, G, H\}$
2. Count the vertices.	$V = 8$
3. List the edges.	$E = \{(A, B), (A, D), (A, E), (B, C), (B, F), (C, D), (C, G), (D, H), (E, F), (E, H), (F, G), (G, H)\}$
4. Count the edges.	$E = 12$
5. Define the faces (regions).	 There are 6 faces in all: {I, II, III, IV, V, VI} $F = 6$
6. Confirm Euler's formula by substitution.	$V = E - F + 2$ $8 = 12 - 6 + 2$ $8 = 8$ Therefore, Euler's formula is verified.

Note that the cube is a form of prism (an object with a uniform cross-section), and all prisms can be converted to planar graphs using the above technique of one face 'collapsing' into another.

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Units 3 & 4 > AOS M2 > Topic 1 > Concept 4

Planar graphs and Euler's formula Summary screen and practice questions

on Resources

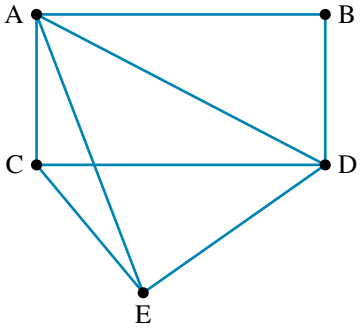
Interactivity: Planer graphs (int-6467)

Interactivity: Euler's formula (int-6468)

Exercise 9.3 Planar graphs and Euler's formula

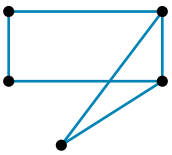
1. **WE6** Convert the following graph to a planar graph.

2. Redraw the following network diagram so that it is a planar graph.

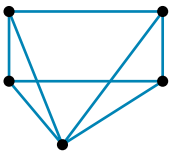


3. Modify the following graphs so that their representations are planar.

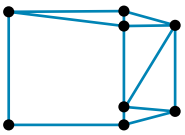
a.



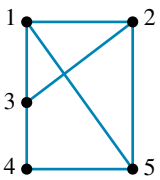
b.



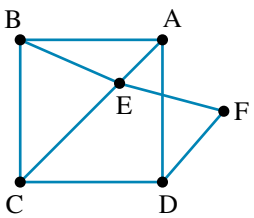
c.



d.

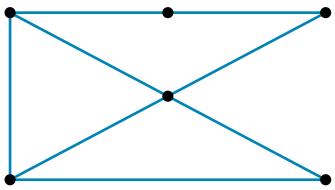


4. **MC** The graph represented in the figure is apparently not planar because:

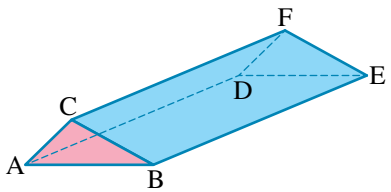


   A. edge (A, C) crosses edge (B, E)
   B. edge (A, D) crosses edge (E, F)
   C. edges (A, E), (F, E), (C, E) and (B, E) intersect
   D. vertex EE has a degree of 4
   E. none of the above

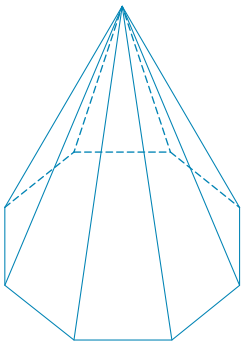
5. **WE7** Find the degree of each vertex of the graph in question **2**.
6. Find the degree of each vertex of the graph shown.



7. **MC** A complete graph with 7 vertices would have:
   A. 7 edges **B.** 14 edges **C.** 21 edges **D.** 28 edges **E.** 42 edges
8. **WE8** Convert the three-dimensional triangular prism to a planar graph.



9. Consider a 'pyramid' with an octagon for a base.



Convert the representation to a planar graph.

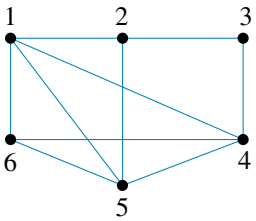
10. **a.** By moving vertex F only, modify the graph in question **4** so that it is clearly planar.

b. How many faces are there in your planar graph?

c. Find the degree of each vertex.

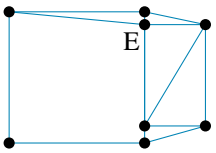
11. **a.** By moving vertex 5 only, modify the graph so that it is clearly planar.

b. How many faces are there in your planar graph?
(*Hint:* You may have to use curved edges to connect all the vertices.)



12. **MC** The degree of vertex E in the figure is:

A. 1 **B.** 2 **C.** 3
D. 4 **E.** 5



13. **WE9** Verify Euler's formula for the graph in question **9**.

14. Convert a triangular pyramid to a planar graph. Verify Euler's formula for your graph.

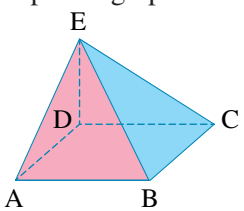
15. Verify Euler's formula for the figure in question **12**.

16. Show that the sum of the degrees of *all* the vertices of *any* planar graph is always an even number. Also show that if S = sum of the degrees, and E = number of edges, that $S = 2E$. (This is known as the handshaking lemma.)

17. **MC** In a planar graph, the number of edges = 5, the number of vertices = 4, therefore the number of faces is:

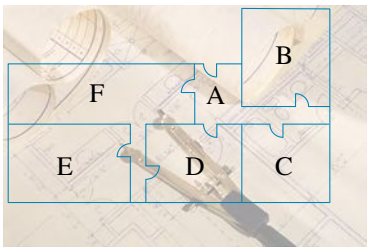
A. 1 **B.** 3 **C.** 9 **D.** 11
E. unable to be determined from the given information

18. Convert the rectangular pyramid shown into a planar graph.



19. The diagram shown is a crude floor plan for a small house with 6 rooms, labelled A, B, ... , F.

Convert this plan to a planar graph where rooms are considered as vertices.
(*Hint:* What should the edges be?)



20. Eight people in a room shake hands with each other once.

a. How many handshakes are there?

b. Represent the handshakes as a complete graph.

c. Represent the handshakes as a matrix.

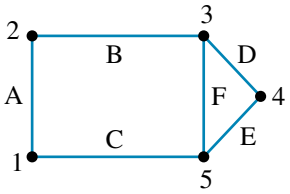
9.4 Walks, trails, paths, cycles and circuits

9.4.1 Walks and trails and paths

In planar graphs we can define a **walk** as a sequence of edges and look at various sequences or pathways through the network. Sometimes you may wish to have a walk that goes through *all* nodes only once, for example, for a travelling salesperson who wishes to visit each town once. Sometimes you may wish to use all edges only once, such as for a road repair gang repairing all the roads in a shire.

9.4.2 Walks

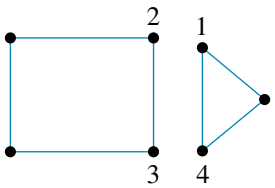
There are different ways of naming a walk. For example, consider travelling from node 1 to node 3 in the figure. A walk could be specified via node 2, namely A–B, or by specifying the vertices, 1–2–3. Alternatively one could take the walk C–E–D, or C–F. Each of these routes is a walk.

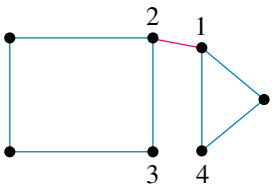


9.4.3 Connected graphs and bridges

If there is a walk between all possible pairs of vertices, then it is a connected graph.

For example, in the figure on the left, there is no walk between vertices 1 and 2, nor between vertices 3 and 4, so it is not a connected graph. However, if we add a single edge, as in the figure on the right, between vertices 1 and 2, the entire graph becomes connected.





A bridge is an edge in a connected graph whose deletion will no longer cause the graph to be connected.

In the above example on the right, the edge between vertices 1 and 2 is a bridge. When this edge is removed, the graph is no longer connected.

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Units 3 & 4 > AOS M2 > Topic 1 > Concept 5

Routes through a network Summary screen and practice questions

9.4.4 Euler trails

Consider a walk where no routes are repeated.

A trail is a walk in which no edges are repeated.

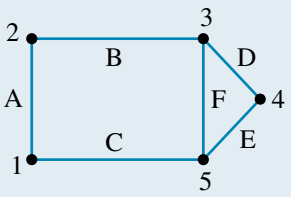
Consider a trail where every edge is used only once, as in our road repair gang example.

An Euler trail is one which uses every edge exactly once.

1. For an Euler trail to exist, all vertices must be of an even degree *or* there must be exactly two vertices of odd degree.
2. If the degrees of *all* the vertices are even numbers, start with *any* vertex. In this case the starting vertex and ending vertex are the same.
3. If there are two vertices whose degree is an odd number use either as a starting point. The other vertex of odd degree must be the ending point.

WORKED EXAMPLE 10

Using the following figure, identify an Euler trail.



THINK

1. Determine a starting vertex. Since there are vertices (3 and 5) whose degree is an odd number, use one of these to start.
2. Attempt to find a path which uses each edge exactly once. You are allowed to visit a vertex more than once; it is only edges that are restricted to one use.

WRITE

Use vertex 3 as the start.

B–A–C–F–D–E or B–A–C–E–D–F or D–E–F–B–A–C are all Euler trails. There are several other possible Euler trails.

Note that the starting vertex and ending vertex are not the same, but we started and ended on the vertices with odd degree (vertices 3 and 5).

9.4.5 Circuits

Euler circuits

A circuit in mathematical terms has much the same meaning as any circuit.

A circuit is a trail beginning and ending at the same vertex.

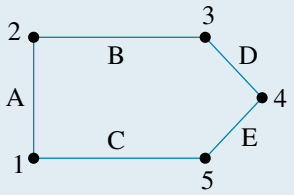
With our road repair gang example, it would be desirable that the Euler trail started and finished at the same point. This kind of Euler trail is called an Euler circuit.

An Euler circuit is an Euler trail where the starting and ending vertices are the same.

It is important to note that an Euler circuit *cannot* exist for planar graphs that have *any* vertices whose degree is odd. In such graphs there is no Euler circuit. Therefore, the planar graph of Worked example 10 does not contain an Euler circuit because vertices 3 and 5 were of odd degree.

WORKED EXAMPLE 11

Find an Euler circuit for the planar graph shown.



THINK	WRITE
1. Confirm that all vertices have *even* degree.	Vertex 1 — degree = 2 Vertex 2 — degree = 2 Vertex 3 — degree = 2 Vertex 4 — degree = 2 Vertex 5 — degree = 2
2. Pick any vertex to start and determine a trail that uses each edge and ends at the same vertex.	Start with vertex 1. Then the Euler circuit could be: A–B–D–E–C or C–E–D–B–A

9.4.6 An Euler circuit algorithm

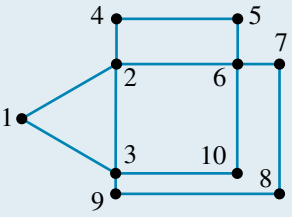
For some networks, it may be difficult to determine an Euler circuit, even after determining that all vertices have even degree. Here is an algorithm that 'guarantees' an Euler circuit.

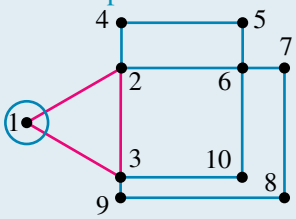
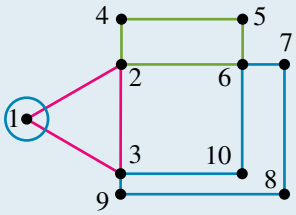
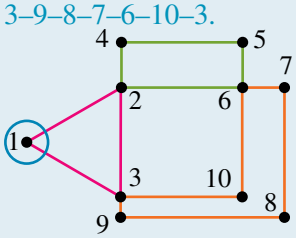
Consider a network where all vertices are of even degree. Let $V = \{1, 2, 3, ...\}$ be the list of vertices.

Step 1. Choose a starting vertex from the list V. Call this vertex A.
Step 2. From vertex A, find the *smallest possible* path which returns to vertex A. This is a '*subcircuit*' of the original network. Let S be the list of vertices in this subcircuit.
Step 3. For each vertex in S, choose a single vertex in turn as the starting vertex of a different subcircuit. It should also be as small as possible, and not use any previously used edge.
Step 4. For each of these new subcircuits (if there are any), add any new vertices to the list in S.
Step 5. Repeat steps 3 and 4 until there are no more new vertices, edges or subcircuits left; that is, the lists S and V are the same.
Step 6. Join the subcircuits at their intersection points.

WORKED EXAMPLE 12

Find one possible Euler circuit for the network shown using the Euler circuit algorithm.



THINK	WRITE/DRAW
1. Choose a starting vertex, and find its smallest subcircuit. The subcircuit is marked in pink.	Vertex 1 is selected. The path 1–2–3–1 is the smallest possible. 
2. Create the list S from the first subcircuit. Find new subcircuits, not using any edges already used (apply step 3 of the algorithm). The new subcircuit is marked in green. Note that this subcircuit does not use any previously used edges.	$S = \{1, 2, 3\}$ From vertex 2, there is a subcircuit 2–4–5–6–2. 
3. Add to the list S (apply step 4 of the algorithm).	$S = \{1, 2, 3, 4, 5, 6\}$
4. Find the new subcircuits (re-apply step 3). The new subcircuit is marked in orange.	From vertex 3, there is a subcircuit 3–9–8–7–6–10–3. 
5. Add to list S (re-apply step 4). Check that all vertices are in the list (step 5).	$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $= V$, so stop.
6. Apply step 6. Form the Euler trail, starting with the first subcircuit, and proceeding through all the other subcircuits at their intersections. Note that the second subcircuit is in the 1st set of square brackets [ ] and the next subcircuit is in the 2nd set of square brackets [ ].	The red circuit and the blue circuit are connected at vertex **2**. The red circuit and the green circuit are connected at vertex **3**. Euler circuit = 1–[**2**–4–5–6–**2**]–[**3**–9–8–7–6–10–**3**]-1
7. List the Euler circuit.	1–2–4–5–6–2–3–9–8–7–6–10–3–1

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Units 3 & 4 › AOS M2 › Topic 1 › Concept 6

Eulerian trails and circuits Summary screen and practice questions

9.4.7 Paths and cycles

A path is a walk in which no vertices are repeated (except possibly the start and finish).
A cycle is a path beginning and ending at the same vertex.

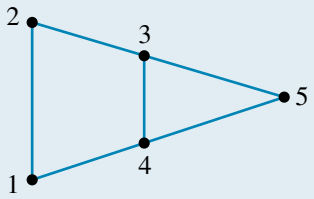
In Euler trails and circuits each edge was used *exactly once*, while vertices could be re-used. Now, consider the case where it is desirable to use each vertex *exactly once*, as in our travelling salesman problem mentioned at the start of this section.

A Hamiltonian path uses every vertex exactly once.

It is important to note that not all edges need to be used. Furthermore, there can be only up to 2 vertices with degree 1 (dead ends). In this case these would be the start and/or the finishing vertices.

WORKED EXAMPLE 13

Determine a Hamiltonian path in the planar graph shown.



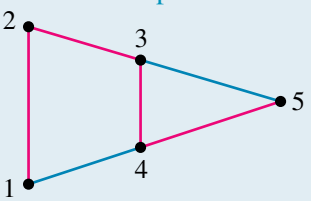
THINK

1. Choose a starting node. If there is a vertex with degree = 1, then use it to start.
2. Attempt to visit each vertex. This will work for all feasible planar graphs.
   The Hamiltonian path found is shown in red.

WRITE

Since there are no vertices with degree 1, choose any node to start. Choose vertex 1.

The path connecting nodes 1–2–3–4–5 was chosen as one of the Hamiltonian paths.



Note that there are several possible Hamiltonian paths for the planar graph in Worked example 13 and there are several paths which will not result in a Hamiltonian path. Can you find such a path?

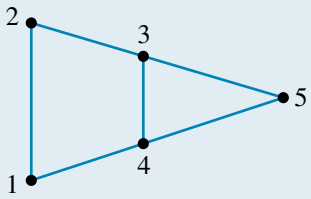
9.4.8 Hamiltonian cycles

When determining a Hamiltonian path, sometimes it is desirable to start and finish with the same vertex. For example, our travelling salesperson may live in one of the towns (vertices) she visits and would like to start and finish at her home town after visiting all the other towns once. This is similar to the concept of an Euler circuit.

A Hamiltonian cycle is a Hamiltonian path which starts and finishes at the same vertex.

WORKED EXAMPLE 14

Determine a Hamiltonian cycle in the planar graph shown. (This is the same graph used in Worked example 13.)

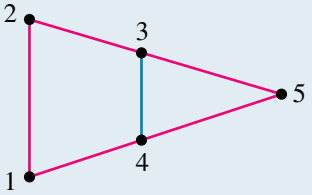


THINK

1. Choose a starting (and finishing) vertex.
2. Attempt to visit each vertex and return to vertex 1. The Hamiltonian cycle found is shown in pink.

WRITE

Choose vertex 1.



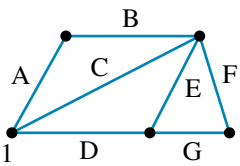
The path connecting nodes 1–2–3–4–5–4–1 was chosen as one of the Hamiltonian cycles.

on Resources

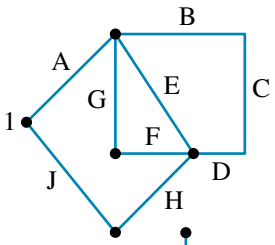
- **Interactivity:** Traversing connected graphs (int-6469)
- **Interactivity:** Euler trails and Hamiltonian paths (int-6479)

Exercise 9.4 Walks, trails, paths, cycles and circuits

1. **WE10** Using the figure shown at right and starting at vertex 1, identify an Euler trail.
2. Choosing the other vertex of degree 3 in the figure used for question **1**, identify another Euler trail.

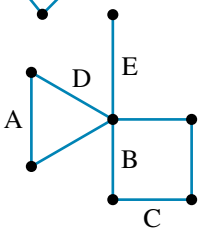


3. Using the figure shown at right and starting at vertex 1, identify an Euler trail.

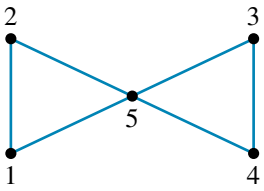


4. Starting with a vertex of degree 4 from the figure in question **3**, identify another Euler trail.

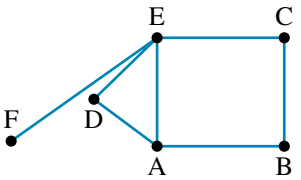
5. **MC** Which edge in the following connected graph is a bridge?
   - **A.** A
   - **B.** B
   - **C.** C
   - **D.** D



6. **WE 11** Using the figure shown and starting at vertex 1, identify an Euler circuit.



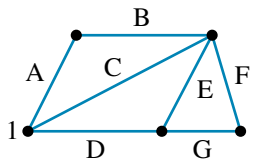
7. **a.** **MC** What additional edge should be added to the planar graph at right so that it could be possible to define an Euler circuit?
   - **A.** CA
   - **B.** FA
   - **C.** FD
   - **D.** EB
   - **E.** None of the above



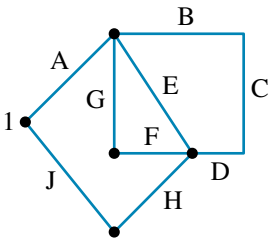
   b. Which edge in the original planar graph is a bridge?

8. **MC** Considering the networks as shown, which have Euler circuits?
   - **A.** Both
   - **B.** Neither
   - **C.** Figure **a** only
   - **D.** Figure **b** only
   - **E.** None of the above

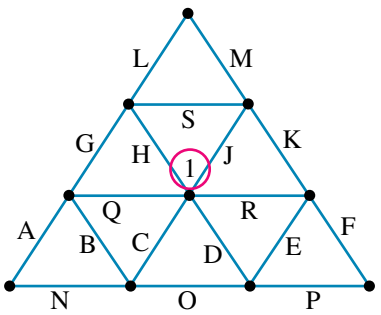
a.



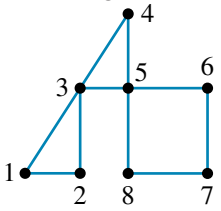
b.



9. Using the graph shown at right, determine:
   a. an Euler trail
   b. an Euler circuit starting at vertex 1 (circled in pink).

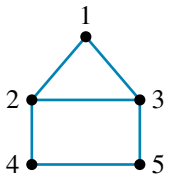


10. WE12 Find an Euler circuit for the graph shown, using the Euler circuit algorithm.

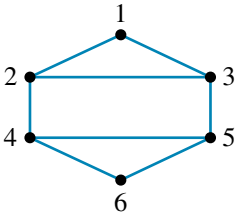


11. MC In which of the following does an Euler circuit exist?

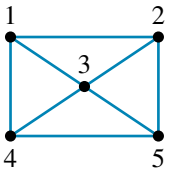
A.



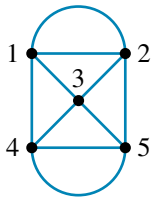
B.



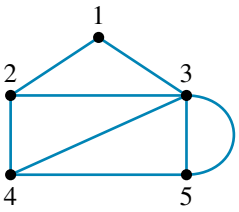
C.



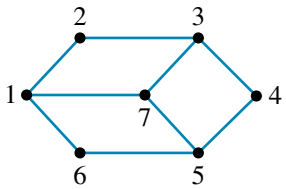
D.



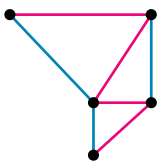
E.



12. WE13 Starting at vertex 2, determine a Hamiltonian path for the graph shown.

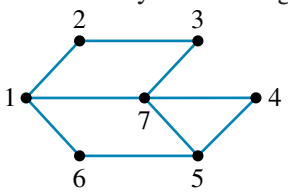


13. MC The path shown in pink in the figure is:

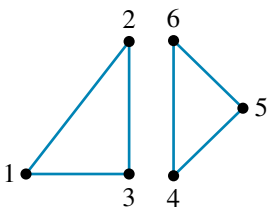


A. an Euler trail
B. an Euler circuit
C. a Hamiltonian path
D. a Hamiltonian cycle
E. none of the above

14. WE14 Starting at vertex 2, determine a Hamiltonian cycle for the graph shown.

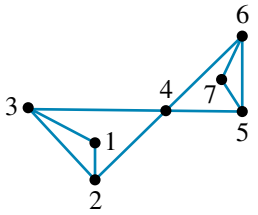


15. MC The network shown has:

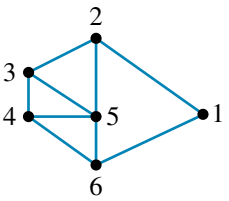


A. an Euler circuit and a Hamiltonian path
B. a Hamiltonian path and cycle
C. an Euler trail and a Hamiltonian cycle
D. an Euler trail only
E. none of the above

16. Starting at vertex 7, determine a Hamiltonian path for the graph shown.

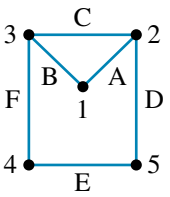


17. MC Which of the following paths is a Hamiltonian cycle for the figure at right?



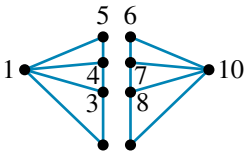
A. 2–3–4–5–6–1–2
B. 2–3–4–5–2
C. 2–5–3–4–5–6–1–2
D. 2–5–6–4–3–2
E. 2–1–6–4–5–2

18. Using the network shown at right:



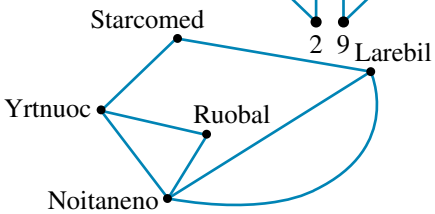
a. determine an Euler trail
b. determine an Euler circuit
c. determine a Hamiltonian path
d. determine a Hamiltonian cycle.

19. a. Using the network shown at right, what two edges should be added to the network so that it has both an Euler circuit and a Hamiltonian cycle?



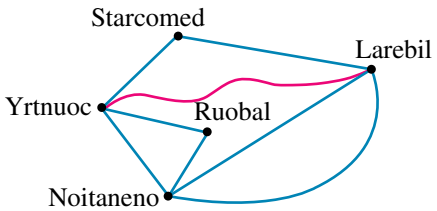
b. Determine the Euler circuit and Hamiltonian cycle.

20. The Police Commissioner wishes to give the impression of an increased police presence on the roads. The roads that the commissioner has to cover are depicted in the network diagram. A speed camera is set up once during the day on each of the roads.



a. Determine a walk that the police officer could follow so that she does not travel more than once on any road.
b. What type of walk is this?

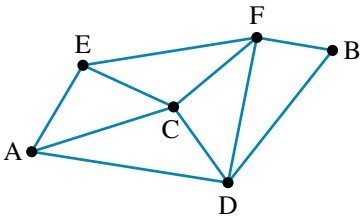
21. The police officer knows that there is a dirt track linking the towns of Larebil and Yrtnuoc. She wishes to meet the commissioner's instructions from question **20** but also wishes to start and finish in her home town, Larebil.



a. Determine a walk that would meet both the police officer's and the commissioner's requirements.
b. What type of walk is this?

22. A security company, 'Wotchemclose', is responsible for patrolling stores in the towns from question **20**. The company wants a patrol car to visit each town once each night without resorting to using the dirt road.
a. If the security guard starts at Ruobal, determine a walk that will meet the company's requirements.
b. What type of walk is this?

23. A physical education teacher, I. M. Grate, wishes to plan an orienteering course through a forest following marked tracks. She has placed checkpoints at the points shown in the diagram. The object of any orienteer is to visit each of the checkpoints once to collect a mark.



a. What walk could an efficient orienteer follow if the course starts at C and finishes at B?

b. What walk should the orienteer follow if starting and finishing at C?

c. What type of walks are these?

9.5 Trees and their applications

There are many applications where only part of the network is required as a solution to a problem. This section will look at such problems involving **subgraphs** and **trees**. To begin with we need a few more definitions.

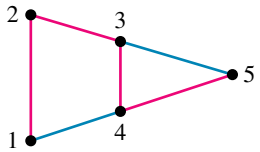
9.5.1 Graphs and subgraphs

Until now we have used the term network to refer to a collection of vertices and edges. This network can also be called a graph. In practice, a graph should have at least 2 vertices and 1 edge. All or part of this graph can be considered as a *subgraph*.

For example, in the figure, the entire network can be considered as a graph, while the path in pink can be considered a subgraph.

Another subgraph could be defined by the path 1–2–3–4–1.

A 'minimum' subgraph could be defined by the path 1–2.



Often the edges in a graph are not just simply connectors, but could be assigned some quantity, such as distance, time or cost. For example, in the figure the distance between vertex 1 and vertex 2 could be assigned a distance of 40 metres. If the graph contains such quantities, then it is called a **weighted graph**.

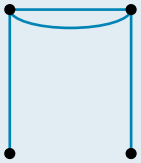
9.5.2 Trees

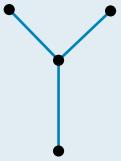
A *tree* is a connected subgraph which cannot contain any:

1. **loops**
2. **parallel (or multiple) edges**
3. **cycles.**

WORKED EXAMPLE 15

Determine whether each of the figures is a tree, and if not, explain why not.

a. 

b. 

c.

d.

THINK	WRITE
1. Examine each figure in turn, looking for loops, parallel edges or cycles.	
2. Examine figure **a**.	Figure **a** has parallel edges (at the top) so it is not a tree.
3. Examine figure **b**.	Figure **b** has no loops, parallel edges or cycles so it is a tree.
4. Examine figure **c**.	Figure **c** has a cycle (at the top) so it is not a tree.
5. Examine figure **d**.	Figure **d** has a loop (at the bottom) so it is not a tree.

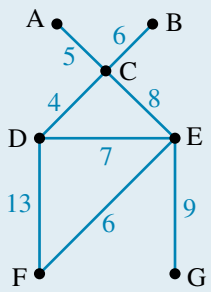
The advantage of trees within a network is that the tree could determine an 'efficient' connection between vertices in the sense that there is a minimum distance, cost or time.

9.5.3 Shortest paths

Sometimes it may be useful to determine the **shortest path** between 2 selected vertices of a graph. For example, when going shopping, a person may leave his home and travel east via the playground, or north via the parking lot and still end up at the same shop. In one case the distance travelled may be the minimum.

WORKED EXAMPLE 16

Determine the shortest path between nodes A and F in the figure shown. Nodes are labelled A, ... , G and distances (in metres) between them are labelled in blue.



THINK	WRITE
1. From the starting node, by inspection, determine the possible trees between A and F.	(1) A–C–E–D–F (2) A–C–D–F (3) A–C–D–E–F (4) A–C–E–F
2. For each tree, calculate the total distance travelled.	(1) $5 + 8 + 7 + 13 = 33$ (2) $5 + 4 + 13 = 22$ (3) $5 + 4 + 7 + 6 = 22$ (4) $5 + 8 + 6 = 19$
3. Choose the path with the shortest distance.	Path A–C–E–F is the shortest path with a distance of 19 m.

When choosing the possible paths in step 1, there is no point in finding paths that are *not* trees. There will *always* be a tree which covers the same vertices in less distance. Non-tree paths will include cycles and loops, which only add to the total distance.

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Weighted graphs Summary screen and practice questions

9.5.4 A shortest path algorithm

Sometimes it can be difficult to list all the paths between the starting and ending vertex. Here is an algorithm which 'guarantees' the shortest path — assuming that the starting vertex is already chosen.

Step 1. From the starting vertex, find the *shortest* path to all other *directly connected* vertices. Include all such vertices, including the starting one in the list $S = \{A, B, ...\}$.

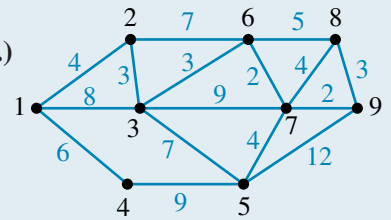
Step 2. Choose a vertex (V) directly connected to those in S and find the shortest path to the starting vertex. Generally, there is one possible path for each degree of V, although some obvious paths can be eliminated immediately.

Step 3. Add the new vertex, V, to the list S.

Step 4. Repeat steps 2 and 3 until all vertices are in S. Find the shortest path to the vertex you want.

WORKED EXAMPLE 17

Find the shortest path from vertex 1 to vertex 9. Vertices are labelled in black, distances in blue. (*Note:* Lines are not to scale.)



THINK

1. From the starting vertex (1), find the shortest path to each of the vertices directly connected to it.

WRITE

From	To	Via	Distance	
1	2	—	4	shortest path to 2
1	3	—	8	
1	3	2	$4 + 3 = 7$	shortest path to 3
1	4	—	6	shortest path to 4

2. Determine the set of vertices, S.

$S = \{1, 2, 3, 4\}$

3. Apply step 2 of the algorithm for a vertex connected directly to one in S.

Select vertex 5.

From	To	Via	Distance	
1	5	3	$7 + 7 = 14$	shortest path to 5
1	5	4	$6 + 9 = 15$	

4. Apply step 3 of the algorithm and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5\}$

5. Re-apply step 2 of the algorithm for another vertex directly connected to one in S.

Select vertex 6.

From	To	Via	Distance	
1	6	2	$4 + 7 = 11$	
1	6	3	$7 + 3 = 10$	shortest path to 6
1	6	7	$7 + 9 + 2 = 18$	

6. Re-apply step 3 of the algorithm and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6\}$

7. Re-apply step 2 for another vertex directly connected to one in S.

Select vertex 7.

From	To	Via	Distance
1	7	6	$10 + 2 = 12$ shortest path to 7
1	7	3	$7 + 9 = 16$
1	7	5	$14 + 4 = 18$

8. Re-apply step 3 and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6, 7\}$

9. Re-apply step 2 for another vertex directly connected to one in S.

Select vertex 8.

From	To	Via	Distance
1	8	6	$10 + 5 = 15$ shortest path to 8
1	8	7	$12 + 4 = 16$

10. Re-apply step 3 and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

11. Re-apply step 2 for another vertex directly connected to one in S.

Select vertex 9.

From	To	Via	Distance
1	9	8	$15 + 3 = 18$
1	9	7	$12 + 2 = 14$ shortest path to 9
1	9	5	$14 + 12 = 26$

12. Re-apply step 3 and add to the set of vertices, S.

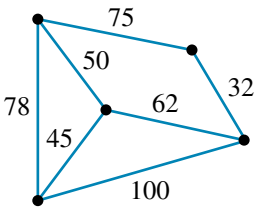
$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Stop, because all vertices are in the list.

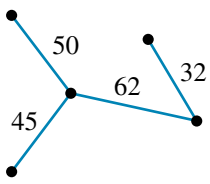
13. Apply step 4 and determine the shortest path.

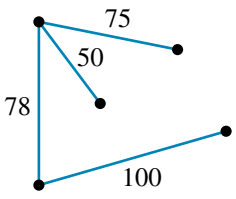
1 to 9 via 7 (see step 11)
1 to 7 via 6 (see step 7)
1 to 6 via 3 (see step 5)
1 to 3 via 2 (see step 1)
1 to 2 (see step 1)
Path = 1–2–3–6–7–9
Distance = $4 + 3 + 3 + 2 + 2 = 14$

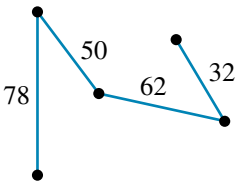
9.5.5 Spanning trees

In the network shown, the vertices represent school buildings and the edges represent footpaths. The numbers represent the distance, in metres, between the buildings. The school council has decided to cover some of the footpaths so that the students can access any building during rainy weather without getting wet. Three possible trees which would accomplish this are shown in the figures.









Note that each of these trees included all the vertices of the original network. These trees are called **spanning trees** because of this property. In practice, the school council would like to make the total distance of covered footpaths as small as possible, in order to minimise cost. In this case, they would have the **minimum spanning tree**. Can you determine which of the figures is the minimum spanning tree?

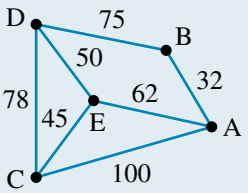
9.5.6 Minimum spanning trees and Prim's algorithm

One method of determining the minimum spanning tree is called **Prim's algorithm**. The steps are as follows:

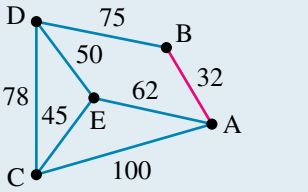
Step 1. Choose the edge in the network which has the smallest value. If 2 or more edges are the smallest, choose any of these.
Step 2. Inspect the 2 vertices included so far and select the smallest edge leading from either vertex. Again, if there is a 'tie', arbitrarily choose any one.
Step 3. Inspect all vertices included so far and select the smallest edge leading from any included vertex. If there is a 'tie', choose any, arbitrarily.
Step 4. Repeat step 3 until all vertices in the graph are included in the tree.

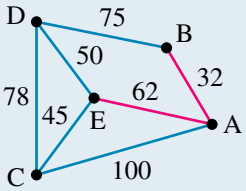
WORKED EXAMPLE 18

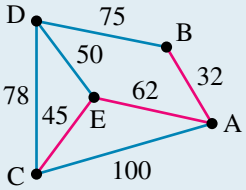
Determine the minimum spanning tree for the network representing footpaths in a school campus.

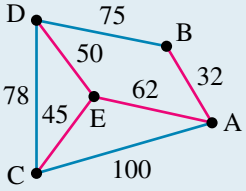


THINK	WRITE
1. Find the edge with the smallest distance. This can be done by listing all the edges and choosing the smallest.	A–B = 32, by inspection
2. Inspect A and B and find the shortest edge connecting one of these to a third vertex.	A–E = 62 — choose this A–C = 100 B–D = 75
3. Inspect A, B and E and find the shortest edge connecting one of these to another vertex.	A–C = 100 E–C = 45 — choose this E–D = 50 B–D = 75
4. Continue until all vertices have been connected. In this case only vertex D remains.	B–D = 75 E–D = 50 — choose this C–D = 78



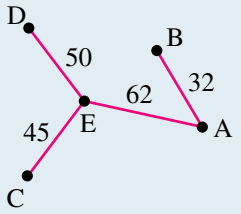






5. Since all vertices have been connected, this is the minimum spanning tree. Calculate the total distance of the minimum spanning tree.

Total distance
$= 32 + 62 + 45 + 50$
$= 189$ m



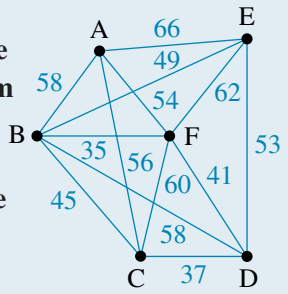
Maximum spanning tree

In some cases you may be required to find the **maximum spanning tree** instead of the minimum spanning tree. In this case, Prim's algorithm works by finding the largest edges at each stage instead of the smallest edges.

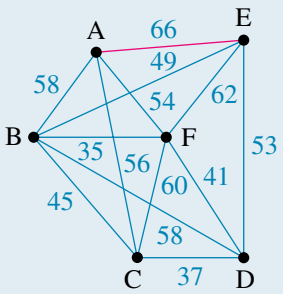
WORKED EXAMPLE 19

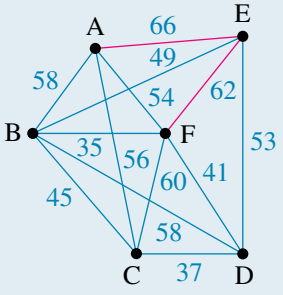
The figure shown represents a telephone network connecting 6 towns, A, B, … , F. The numbered edges represent the 'capacity' of the telephone connection between the towns connected, that is, the maximum number of calls that can be made at the same time along that edge.

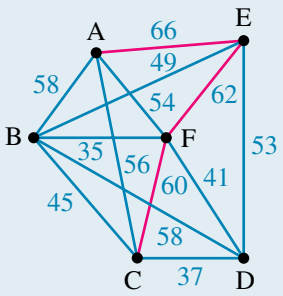
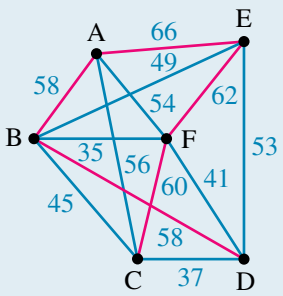
A telephone engineer wishes to determine the maximum capacity of the system in terms of a tree connecting all the towns so that calls can be routed along that tree.



THINK	WRITE
1. Because this is a maximum spanning tree, find the edge with the largest capacity.	This is edge A–E (66).
2. Inspect edges from vertices A and E, find the edge with the largest capacity.	This is edge E–F (62).





3. Inspecting edges from vertices, A, E and F, find the edge with the largest capacity.	This is edge F–C (60).	
4. Repeat until all towns are connected.	A–B (58) and B–D (58) are the edges selected.	
5. Determine maximum capacity.	Maximum capacity $= 66 + 62 + 60 + 58 + 58$ $= 304$ telephone calls at the same time.	

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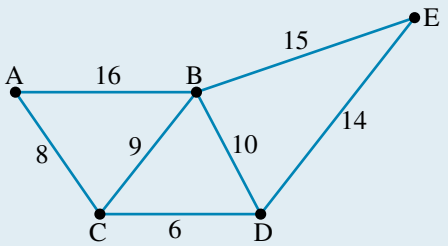
Minimum spanning trees and Prim's algorithm Summary screen and practice questions

9.5.7 Dijkstra's algorithm

Another method for determining the shortest path between a given vertex and each of the other vertices is by using **Dijkstra's algorithm**. An effective way to use this algorithm is using a tabular method, as shown in Worked example 20.

WORKED EXAMPLE 20

Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra's algorithm in tabular form.



THINK

1. Construct a table representing the network. Where there is no value, you are unable to take that path.

WRITE

	A	B	C	D	E
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

2. Starting at E, place a 0 above it. The option are B (15 km) and D (14 km). Write these above the letters.

		15		14	0
	A	B	C	D	E
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

3. Go to the shortest distance – D and go down vertically.
   B: Add 14 and 10 (= 24); this is greater than the 15 already above B so not a choice.
   C: Add 14 and 6 (= 20); since nothing above C write this above C.
   E: We started at E so don't want to go back there.

		15	20	14	0
	A	B	C	~~D~~	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

4. Now E and D are done, the next lowest is B; again work vertically.
   A: Add 15 and 16 (= 31); since nothing above A write this above.
   C: Add 15 and 9 (= 34); this is greater than 20 so leave it as 20.
   D: Already been to D.
   E: Already been to E.

	31	15	20	14	0
	A	~~B~~	C	~~D~~	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

5. Now E, D and B are done, the next lowest is C, again work vertically.
   A: Add 20 and 8 (= 28); since this is lower than 31 that is currently there, replace it with 28.
   B: Already been to B.
   D: Already been to D.

	28				
	~~31~~	15	20	14	0
	A	~~B~~	~~C~~	~~D~~	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

6. The shortest distance from A to E is then the number above E, in this case 28.

The shortest distance from A to E is 28 km.

Resources

Interactivity: A shortest path algorithm (int-6284)
Interactivity: Minimum spanning trees and Prim's algorithm (int-6285)

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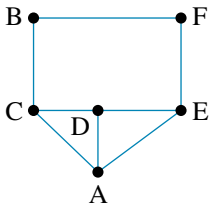
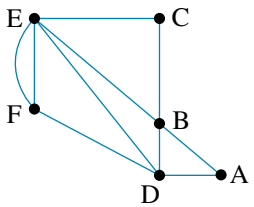
Shortest path and Dijkstra's algorithm Summary screen and practice questions

Exercise 9.5 Trees and their applications

1. **WE15** Determine *all* the trees connecting vertices A and B, without going through vertex F.



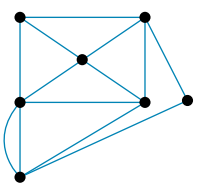
2. **MC** Consider the following paths:
   - i. A–B–C–E–F–D–A
   - ii. A–D–F–E–C–B
   - iii. A–B–D–A
   - iv. A–B–D–F–E–C



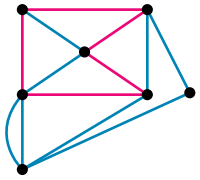
Which (if any) are trees?

A. All are trees
B. i and iv only
C. ii and iv only
D. iv only
E. None are trees

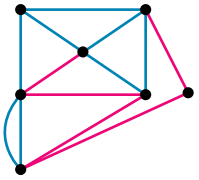
3. Consider the figure shown at right. Which of the paths marked in pink on the figures are trees?



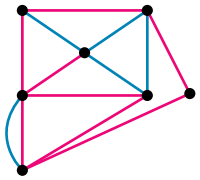
a.



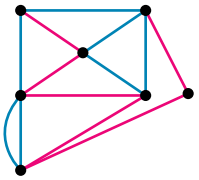
b.



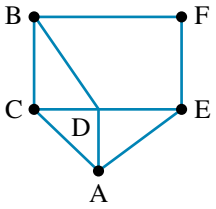
c.



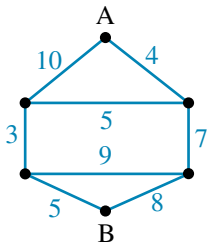
d.



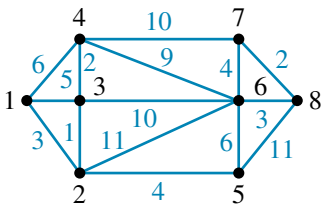
4. Determine *all* the trees connecting vertices A and B, without going through vertex F.



5. **WE16** Determine the shortest path from A to B, where the distances (in blue) are in kilometres.



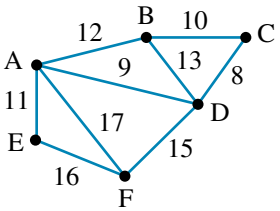
6. The network shown at right represents the time (blue numerals in minutes) that it takes to walk along pathways connecting 8 features in a botanical garden. Vertices 1 and 8 are entrances. Find the minimum time to walk between the entrances, along pathways only.



7. **WE17** Referring to the network shown at right, where distances are in km, find:

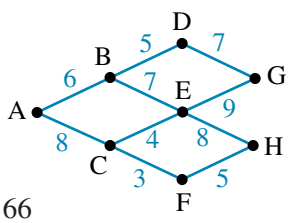
a. the shortest path from B to F

b. the shortest path from A to C.



8. Referring to the network in question **7**, where distances are in km, find the shortest path from E to C.

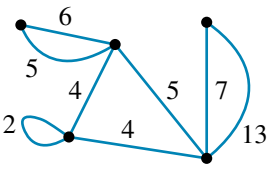
9. **WE18** Find the minimum spanning tree for the network shown at right.



10. **MC** The total length of the minimum spanning tree in question **7** is:

A. 20 **B.** 37 **C.** 40 **D.** 51 **E.** 66

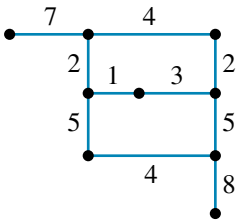
11. MC WE19 In the figure shown, the calculations using Prim's algorithm for the minimum spanning tree would be:



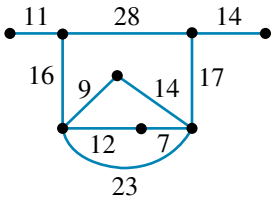
A. $2+4+4+5+7$ **B.** $4+4+5+7$
C. $4+4+6+7$ **D.** $4+5+7+4$
E. 20

12. Draw the minimum spanning tree in each of the following graphs and calculate the total length.

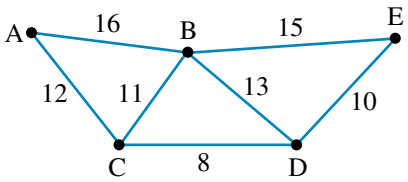
a.



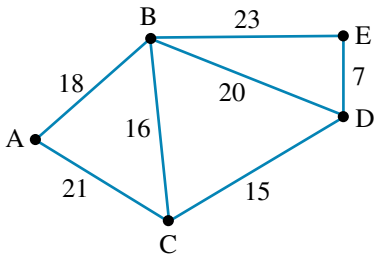
b.



13. WE20 Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra's algorithm in tabular form.



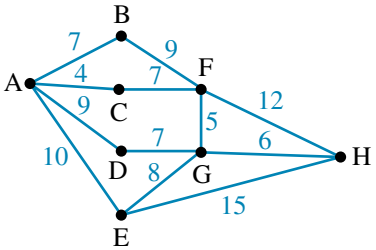
14. Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra's algorithm in tabular form.



15. MC Which of the following statements is true?

A. A Hamiltonian cycle is a tree.
B. A tree can contain multiple edges or loops.
C. A Hamiltonian path is not a tree.
D. A tree can visit the same vertex more than once.
E. A tree can have one edge.

Questions **16** to **18** refer to the network shown in the figure. Vertices are labelled, A, B, …, H and the *time* it takes to travel between them, in minutes, is given by the numbers in blue.



16. **a.** List all possible trees connecting A and H, passing through B.
b. Use Dijkstra's algorithm to determine the shortest time for A–H.

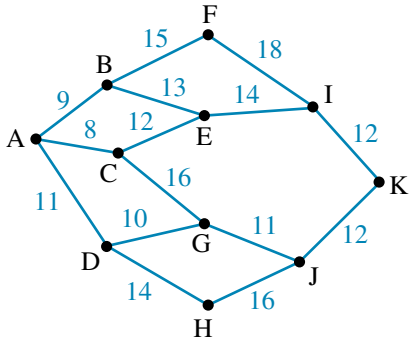
17. MC The total number of possible trees connecting A and H is given by:

A. 8 **B.** 10 **C.** 12
D. 15 **E.** 28

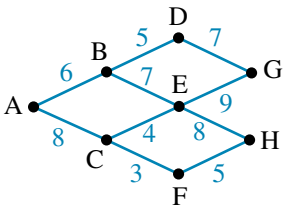
18. MC The shortest time it would take to travel between A and H is given by the tree:

A. A–B–F–H **B.** A–B–F–G–H **C.** A–C–F–H
D. A–C–F–G–H **E.** A–E–H

19. Use Dijkstra's algorithm to determine the shortest path from A to K, where the distances are in km.



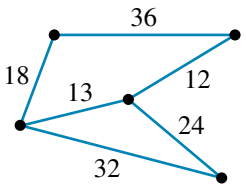
20. The figure shows a network connecting vertices A, …, H



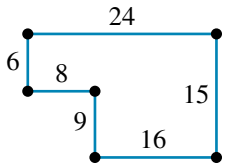
a. How many different trees are there connecting A to G?
b. Find the shortest path connecting A to G.
c. How many different trees are there connecting D to F?
d. Find the shortest path connecting D to F.

21. Using Prim's algorithm, determine the length of the minimum spanning tree in each of the graphs shown.

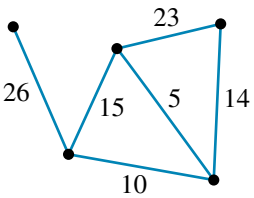
a.



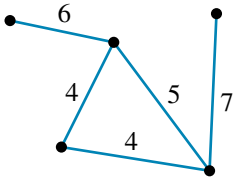
b.



c.

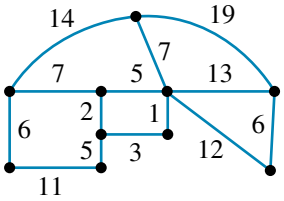


d.

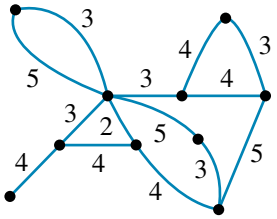


22. Draw the minimum spanning tree in each of the following graphs and calculate the total length.

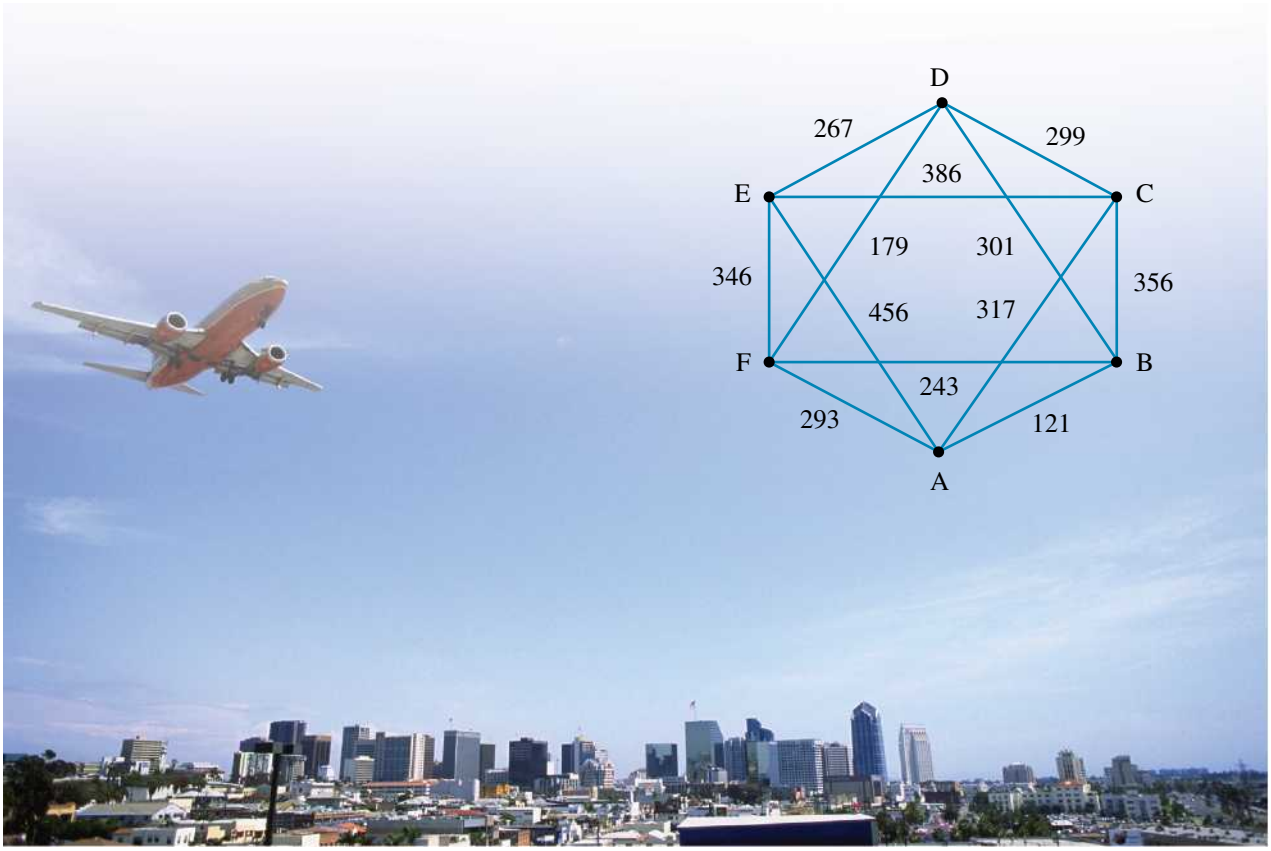
a.



b.



23. Flyemsafe Airlines wish to service six cities. The directors have decided that it is too costly to have direct flights between all the cities. The airline needs to minimise the number of routes which they open yet maximise the total number of passengers that they can carry.



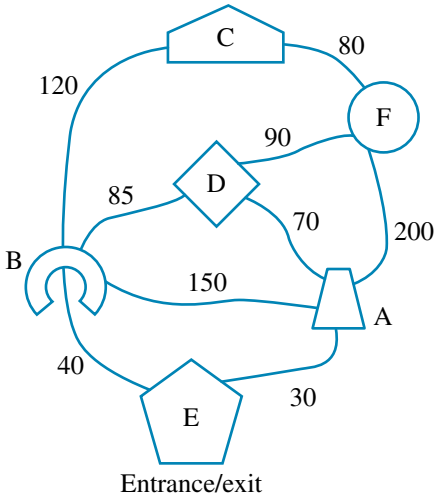
The network diagram shown has edges representing routes and vertices representing cities. The numbers on the edges are projected capacities.

Find:

a. the maximum spanning tree that will meet the airline's requirements

b. the total carrying capacity of this tree.

24. A fairground has 5 main attractions which are joined by paths to the entrance/exit gate. The numbers show the distance along the paths in metres.



a. Draw an undirected graph to represent the fairground and then write down:
  i. the number of edges
  ii. the number of vertices
  iii. the degree of each vertex.

b. What is the minimum distance a person would have to walk to visit every attraction, beginning and ending at the entrance/exit?

c. If each attraction needs to be able to communicate via a phone line, draw the minimum possible tree to represent this.

d. Complete a matrix for the graph shown.

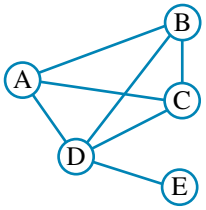
e. Following a Hamiltonian cycle would be an efficient way to visit every attraction in the fairground. Suggest a route a visitor could follow in order to create a Hamiltonian cycle, beginning and ending at the entrance/exit.

9.6 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

Questions **1** and **2** refer to the network shown.



1. MC The sum of the degrees of all the vertices in the network is:

A. 5 **B.** 10 **C.** 12 **D.** 13 **E.** 14

2. MC The list of edges can be written as:

A. {A, B, C, D, E}

B. {(A, A), (B, B), (C, C), (D, D), (E, E)}

C. {(A, B), (A, C), (A, D), (B, C), (B, D), (C, D), (D, E)}

D. {(A, B), (A, C), (A, D), (B, C), (C, D), (D, E)}

E. {(A, B, C, D), (B, A, C, D), (C, A, B, D), (D, A, B, C, E), (E, D)}

3. MC From the following matrix representation of a network with vertices A, B, C, D, E, the degree of vertex A is:

A. 1

B. 2

C. 5

D. 6

E. 10

	A	B	C	D	E
A	1	2	1	0	1
B	2	0	1	0	1
C	1	1	0	0	1
D	0	0	0	0	0
E	1	0	1	0	0

4. MC Using the matrix in question **3**, the isolated vertex is:

A. A

B. B

C. C

D. D

E. E

5. MC A certain planar graph with 6 vertices can be divided into 4 regions (or faces). How many edges does it have?

A. 2

B. 6

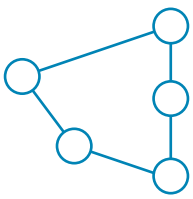
C. 8

D. 10

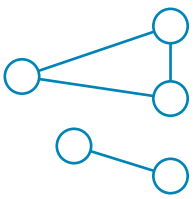
E. Cannot be determined from the given information

6. MC Consider the networks **i**, **ii**, **iii** and **iv**. Which are connected graphs?

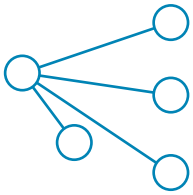
i.



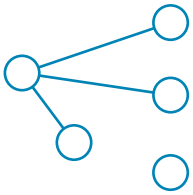
ii.



iii.

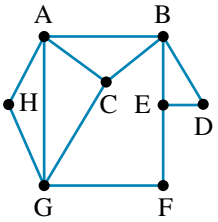


iv.



A. **i** only **B.** **i**, **ii** and **iii** **C.** **i** and **iii** **D.** **ii** and **iv** **E.** All are connected.

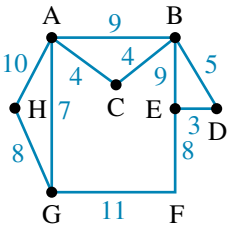
7. MC For the graph shown, which is a Hamiltonian path?



A. C–G–A–H–G–F–E–D–B–A–C–B–E
B. C–G–H–A–B–D–E–B–C
C. C–G–H–A–B–D–E–F
D. A–B–C–D–E–F–G–H
E. There is no Hamiltonian path for this graph.

8. MC For the graph in question **7**, an Euler trail:
A. does not exist because there are 2 vertices whose degree is 3
B. exists because there are exactly 2 vertices whose degree is 3
C. exists because there is an even number of vertices
D. must not start on a vertex with odd degree
E. requires the starting vertex and the ending vertex to be the same

9. MC The shortest path from H to E is:



A. 24
B. 25
C. 26
D. 27
E. 28

10. MC The length of the minimum spanning tree for the graph in question **9** is:
A. 39
B. 40
C. 43
D. 49
E. dependent upon the starting vertex

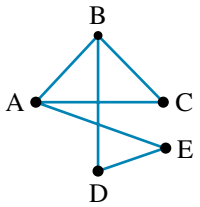
11. MC A connected planar graph has 12 edges. This graph could have:
A. 5 vertices and 6 faces
B. 5 vertices and 8 faces
C. 6 vertices and 8 faces
D. 6 vertices and 9 faces
E. 7 vertices and 9 faces

Short answer

1. For the network in Multiple choice question **9**:
a. label the vertices and edges in terms of the vertices
b. write a matrix representation of the network.

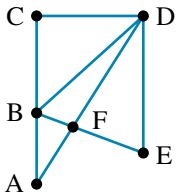
2. Draw a diagram of a network represented by $V = \{A, B, C, D, E, F\}$ and $E = \{(A, B), (A, C), (B, C), (B, D), (B, E), (C, F), (D, E), (E, F)\}$. Calculate the sum of the degrees of all the vertices.

3. **a.** Convert the network in the figure to a planar graph.



b. Confirm Euler's formula for the result.

4. Starting at vertex A, determine an Euler circuit in the figure shown.



5. Starting at vertex A, determine two different Hamiltonian cycles for the graph in question **4**.
6. In a town there are five friends: Paul, Ben, Kevin, Matt and David. The friends' houses are linked by the number of footpaths as given in the matrix.

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Due to an oversight, the names of all the friends were not listed with the rows and columns; however, the following information is known:

- Ben and Paul have two footpaths between their houses.
- There is only one path between Kevin and each of his friends.
- Paul and David have one path linking their houses to each other.
- The second column in the matrix representation above represents Paul's edges.

a. Redraw the matrix with the names in their correct places above the columns and beside the rows.

b. Using the answer to part **a**, draw the network of footpaths and houses.

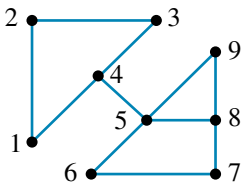
7. Use the information in question **6** to complete the following.

a. Redraw the network and add a path to it that will enable an Euler circuit to be travelled by the young men.

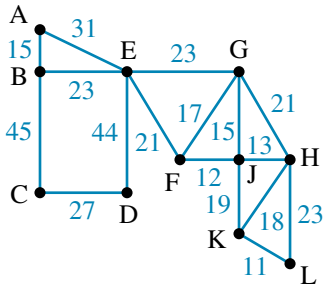
b. Why would the friends wish there to be the extra footpath in their network of footpaths?

c. Using the new network diagram, state an Euler circuit.

8. Determine a Hamiltonian path for the network drawn.

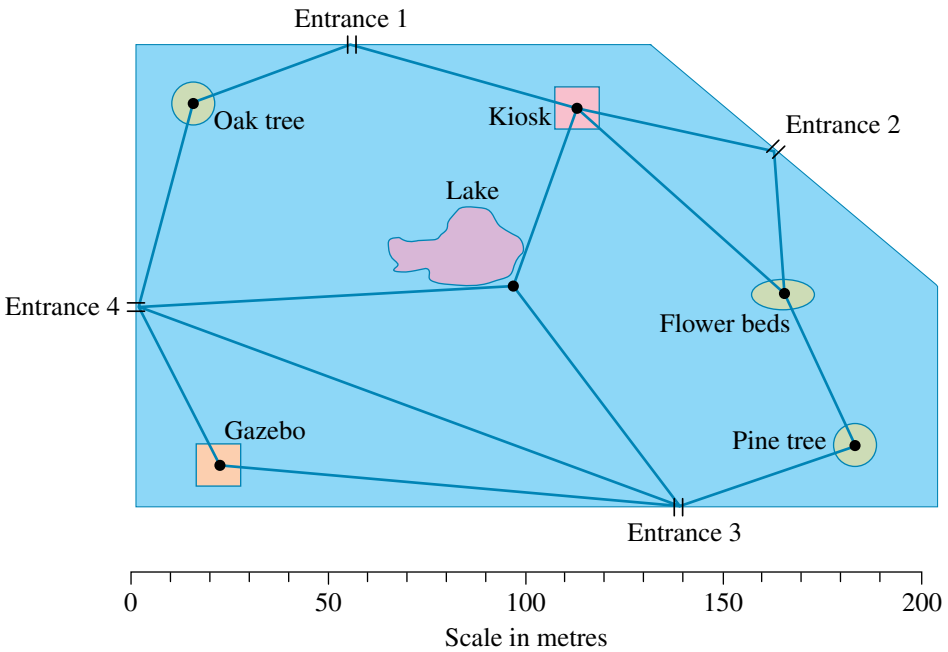


9. The figure represents rail connections between 11 towns.
A telegraph system is to be set up connecting all the towns. Find the shortest total distance for this system. Note that messages can be relayed from one station to another via a third station.



Extended response

1. The plan represents a botanical garden in the town of Lovely Banks. There are 6 features (Oak tree, Kiosk, Flower beds, Pine tree, Gazebo and Lake) and entrances. Paths connecting various features are drawn as blue lines and can be taken as straight lines connecting the dots for each feature.



   a. Draw this 'network' of 10 vertices as a planar graph.
   b. A visitor arriving at Entrance 2 wishes to travel along all paths exactly once. This is not possible, but, by adding a new path between two features, it will become possible. Add this new path and determine the visitor's path. What kind of path is this?
   c. Measure the distance along the paths using a ruler and the scale at the bottom of the plan. Quote distances to the nearest 5 metres. (Do not include the path from part **b**.)
   d. Find the shortest path that connects all the vertices (including entrances). What kind of path is this? Make a drawing of this path. (Do not include the path from part **b**.)
   e. Find the shortest path between Entrance 4 and Entrance 2. (Do not include the path from part **b**.)

2. In a state forest there are six camp sites: Angel Valley, Booming Falls, Crater Face, Dewey Seat, Echidna Spike and Fearful Drop. The camp sites have walking tracks linking them to various other camp sites. The matrix representing the walks and camp sites is given as shown.

	A	B	C	D	E	F
A	1	0	1	3	1	0
B	0	0	1	1	0	0
C	1	1	0	1	0	1
D	3	1	1	0	0	1
E	1	0	0	0	0	1
F	0	0	1	1	1	0

a. In the matrix how many loops are there?
b. In the matrix how many multiple edges are there?
c. State the degree of vertex A.
d. State the degree of vertex D.
e. Represent the matrix as a planar graph.
f. From the graph draw any tree.

Questions **3** to **5** are based upon information obtained from answers to question **2**, specifically the drawing of the graph in **2e**.

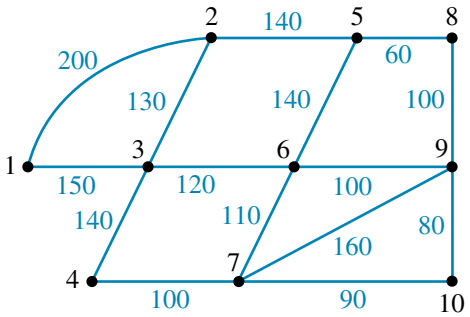
3. Two of the paths from A to D have been closed due to wash-outs, and the loop out from A has been closed due to the next week being the breeding season for the rare albino ground-dwelling earthworm. The rangers have provided the following (incomplete) matrix of distances (km) between the camp sites.

	A	B	C	D	E	F
A	0	–	24	21	6	–
B		0	15	8	–	–
C			0	26	–	16
D				0	–	10
E					0	7
F						0

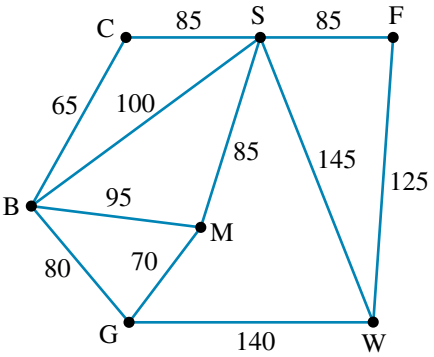
a. Why are the zeros running through the diagonal?
b. What do the dashes (–) indicate in terms of pathways between camp sites?
c. Fill in the blank spaces in the matrix.
d. Was it necessary to complete the matrix? Why or why not?
e. Redraw the network with the closed paths removed, but include the distances given in the matrix.

4. A group of members from TLTHTB Fitness Club wish to complete a run which includes each of the paths over three days.
a. Starting at A, plan a course for the runners so that each path is covered exactly once.
b. What type of path is this?
c. Is it possible to meet the requirements of the group and also have the start and finish at A? Why or why not?
d. If your answer to part **c** was no:
  i. could you suggest one path which the rangers must add to enable the circuit to be completed?
  ii. would this still enable the network to be planar?
e. If the graph was no longer planar, what would this mean about the actual paths?

5. Another group, The Walkers Club, is planning to camp at camp site F. They wish to visit each camp site during their 3-day stay and return to their base camp each night by bus. The total distance that they plan to cover over the three days is 47 km. Assuming that the walks get progressively longer and that no walk is over 20 km:
   a. draw and clearly label each of the daily walks
   b. draw the minimum spanning tree and calculate the total distance of this tree
   c. determine what requirement of the walkers prevents the minimum spanning tree being utilised.
6. Recyclable materials are to be collected from households in a part of a particular suburb. The network represents this area, where the numbered dots are street intersections and the edges are the streets. The numbers (in blue) indicate the lengths (in metres) of the streets between intersections.



   a. Does this network represent a planar graph? Explain your answer.
   b. Verify Euler's formula for this network.
   c. Represent the network as a matrix.
   d. The collectors wish to travel along each street once only, in order to keep travel distance to a minimum. What sort of path is needed? Determine this path.
   e. As well as the criterion mentioned in part **d** above, they would also like to start and finish at the same intersection. What sort of path is this? If this is possible, determine this path. If it is not, what would need to happen for it to be possible?
   f. The Transport Department has placed traffic density monitors at each intersection. Determine a route that a departmental officer could take in order for her to collect all the monitors without retracing a street. What would this path be called?
   g. A taxi driver wants to transport a customer from 1 to 10. Determine the shortest distance for him to travel between intersections 1 and 10 and indicate this path.
7. An airfreight company operates out of Melbourne (M) to Geelong (G), Ballarat (B), Castlemaine (C), Seymour (S), Mansfield (F) and Warragul (W). A planar graph of the company's flight network is outlined, with distances shown in kilometres. Note that some flight paths, such as Melbourne to Warragul, are prohibited so they do not appear on the network.



   a. Verify Euler's formula for this network.
   b. Represent the network as a matrix.

c. The company wishes to fly to all six of these satellite towns and return to Melbourne. Mathematically, what is this type of path called? Determine the two possible flight paths for this.
d. Draw a spanning tree for this network.
e. Now determine the minimum spanning tree and the total distance represented. Comment on the usefulness of this for the company.
f. Use Dijkstra's algorithm to determine the shortest distance between Geelong and Mansfield.

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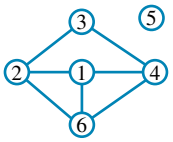
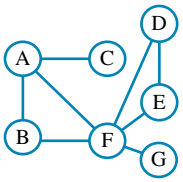
Units 3 & 4 Sit exam

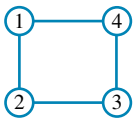
Answers

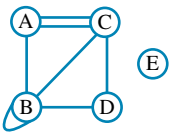
Topic 9 Undirected graphs and networks

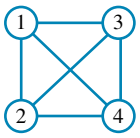
Exercise 9.2 Basic concepts of a network

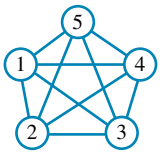
1. a. Vertices = 6, edges = 8
   b. Vertices = 7, edges = 9
2. a. Vertices = 5, edges = 5
   b. Vertices = 6, edges = 9
   c. Vertices = 7, edges = 11
   d. Vertices = 9, edges = 16
3. C
4. a. deg (A) = 3, deg (B) = 4, deg (C) = 4
   b. deg (A) = 3, deg (B) = 8, deg (C) = 3
5. a. deg (A) = 2, deg (B) = 2, deg (C) = 1
   b. deg (A) = 3, deg (B) = 4, deg (C) = 2
   c. deg (A) = 4, deg (B) = 2, deg (C) = 6
   d. deg (A) = 4, deg (B) = 1, deg (C) = 0
6. D
7. a. $V = \{A, B, C, D, E, F\}$
   $E = \{(A, B), (A, D), (A, E), (B, D), (B, E), (B, E), (C, D), (D, F)\}$
   b. $V = \{A, B, C, D, E, F, G\}$
   $E = \{(A, B), (B, C), (B, E), (B, F), (C, E), (C, F), (E, F), (E, G), (F, G)\}$
8. a. $E = \{(1, 2), (1, 3), (2, 3), (3, 4), (4, 5)\}$
   b. $V = \{U, V, W, X, Y, Z\}$
   $E = \{(U, V), (U, W), (U, X), (V, W), (V, X), (V, Z), (W, X), (W, Y), (X, Z)\}$
   c. $V = \{1, 2, 3, 4, 5, 6, 7\}$
   $E = \{(1, 2), (1, 3), (1, 5), (2, 4), (2, 6), (2, 6), (2, 6), (2, 7), (3, 3), (3, 6), (5, 7)\}$
   d. $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9$
   $E = \{(1, 5), (1, 6), (1, 8), (2, 5), (2, 7), (2, 9), (3, 4), (3, 5), (3, 8), (4, 5), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7)\}$
9. 

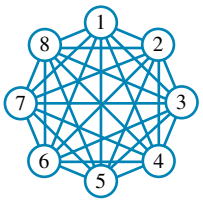
10. 

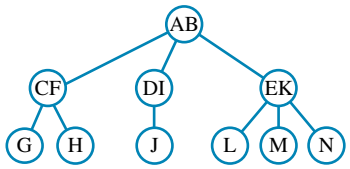
11. a. 

    b. 

12. a. $V = \{1, 2, 3, 4\}$
    $E = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
    b. 

    c. Degree of each vertex = 3
13. i. a. $V = \{1, 2, 3, 4, 5\}$
    $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$
    b. 

    c. Degree of each vertex = 4
    ii. a. $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
    $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8)\}$
    b. 

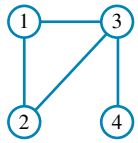
    c. Degree of each vertex = 7
14. a. 45 b. 190 c. 4950
15. 

16. $\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$
17. $\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$
18. a. $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

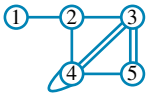
b.
$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

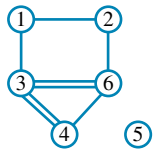
19. a.



b.



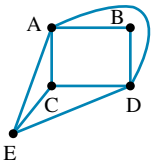
c.



Exercise 9.3 Planer graphs and Euler's formula

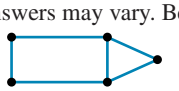
1.

2.

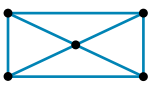


3. Answers may vary. Below are suggested solutions.

a.

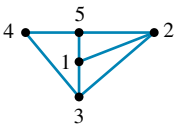


b.



c. Already planar

d.



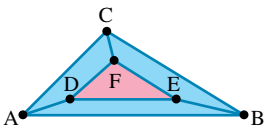
4. B

5. deg(A) = 4
deg(B) = 2
deg(C) = 3
deg(D) = 4
deg(E) = 3

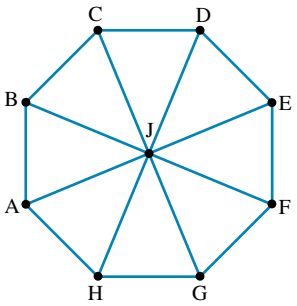
6. deg(A) = 3
deg(B) = 2
deg(C) = 2
deg(D) = 2
deg(E) = 3
deg(F) = 4

7. C

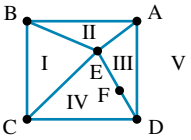
8.



9.



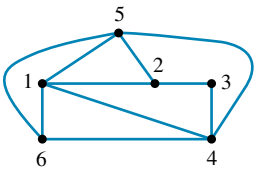
10. a.



b. 5

c. deg(A) = 3
deg(B) = 3
deg(C) = 3
deg(D) = 3
deg(E) = 4
deg(F) = 2

11. a.

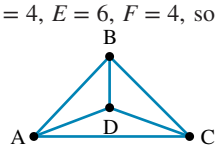


b. 6

12. D

13. $V = 9$, $E = 16$, $F = 9$, so $9 = 6 - 9 + 2$

14. $V = 4$, $E = 6$, $F = 4$, so $4 = 6 - 4 + 2$

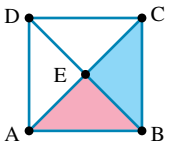


15. $V = 8$, $E = 13$, $F = 7$, so $8 = 13 - 7 + 2$

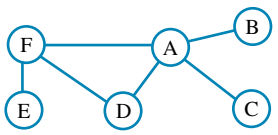
16. Since the degree of a single node is determined by the number of edges (E) 'leaving' it, and each such edge must be the 'entering' edge of another node, each edge is counted twice in the sum of degrees (S). Thus the sum must be an even number. And since each edge is counted twice $S = 2E$.

17. B

18.



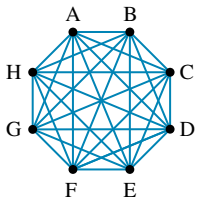
19.



The edges should represent the doorways between rooms. (*Note:* The hall space between rooms E and D belongs to room F. Similarly, the hall space between rooms B and C belongs to room A.)

20. a. 28

b.



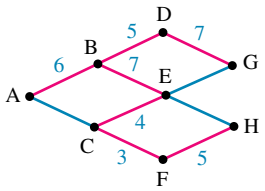
c.

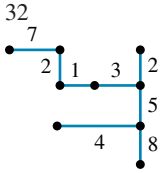
	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	1	1
B	1	0	1	1	1	1	1	1
C	1	1	0	1	1	1	1	1
D	1	1	1	0	1	1	1	1
E	1	1	1	1	0	1	1	1
F	1	1	1	1	1	0	1	1
G	1	1	1	1	1	1	0	1
H	1	1	1	1	1	1	1	0

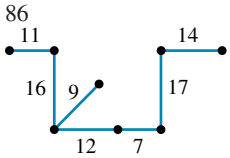
Exercise 9.4 Walks, trails, paths, cycles and circuits

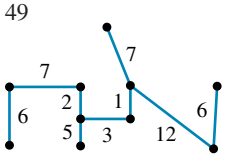
1. A–B–C–D–E–F–G
2. E–F–G–D–C–B–A
3. A–B–C–D–E–G–F–H–J
4. A–J–H–E–G–F–D–C–B
5. E
6. 1–2–5–3–4–5–1
7. **a.** B **b.** FE
8. D
9. **a.** A–G–L–M–S–H–J–K–F–P–E–R–D–O–C–Q–B–N
   b. H–S–M–L–G–A–N–B–Q–C–O–P–F–E–D–R–K–J
10. 1–3–4–5–6–7–8–5–3–2–1
11. D
12. 2–1–6–5–4–3–7
13. C
14. 2–1–6–5–4–7–3–2
15. E
16. 7–6–5–4–3–2–1
17. A
18. **a.** 2–1–3–4–5–2–3
    b. Cannot be done (odd vertices)
    c. 1–2–3–4–5
    d. 1–3–4–5–2–1
19. **a.** Join 4 to 7 and 3 to 8.
    b. Euler circuit = 3–2–1–3–4–1–5–4–7–6–10–7–8–10–9–8–3; Hamiltonian cycle = 3–2–1–5–4–7–6–10–9–8–3
20. **a.** Y–R–N–Y–S–L–N–L
    b. An Euler trail
21. **a.** L–S–Y–R–N–Y–L–N–L
    b. An Euler circuit
22. **a.** R–Y–S–L–N
    b. A Hamiltonian path
23. **a.** C–D–A–E–F–B
    b. C–A–E–F–B–D–C
    c. Hamiltonian path, Hamiltonian cycle

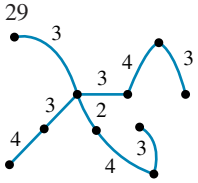
Exercise 9.5 Trees and their applications

1. A–C–B; A–D–C–B; A–E–D–C–B
2. C
3. **b**, **d**
4. A–C–B; A–D–B; A–C–D–B; A–D–C–B; A–E–D–B; A–E–D–C–B
5. 17 km
6. 16 minutes
7. **a.** 28 km **b.** 17 km
8. 28 km
9. 

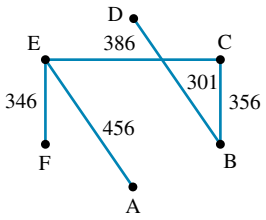
10. B
11. B
12. **a.** 32


b. 86


(1 of 2 possible answers)
13. 30 km
14. 41 km
15. E
16. **a.** A–B–F–H; A–B–F–G–H; A–B–F–G–E–H
    b. 22
17. C
18. D
19. 44 km
20. **a.** 5 **b.** 18 **c.** 7 **d.** 19
21. **a.** 12 + 13 + 18 + 24 = 67 **b.** 6 + 8 + 9 + 16 + 15 = 54
    c. 5 + 10 + 14 + 26 = 55 **d.** 4 + 4 + 6 + 7 = 21
22. **a.** 49


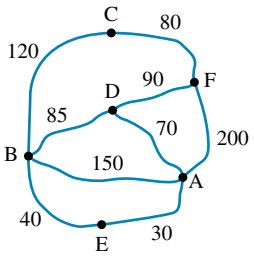
b. 29


23. a.



b. 1845 passengers

24. a.



Entrance/exit

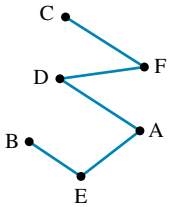
i. 9

ii. 6

iii. A–4, B–4, C–2, D–3, E–2, F–3

b. 430 m

c.



d.

	A	B	C	D	E	F
A	0	1	0	1	1	1
B	1	0	1	1	1	0
C	1	1	0	0	0	1
D	1	0	1	0	0	0
E	1	1	0	0	0	0
F	1	0	1	1	0	0

e. E–A–D–F–C–B–E

9.6 Review: exam practice

Multiple choice

1. E	**2.** C	**3.** D	**4.** D
5. C	**6.** C	**7.** C	**8.** B
9. C	**10.** A	**11.** C	

Short answer

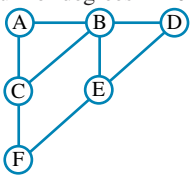
1. a. $V = \{A, B, C, D, E, F, G, H\}$

$E = \{(A, B), (A, C), (A, G), (A, H), (B, C), (B, D), (B, E), (D, E), (E, F), (F, G), (G, H)\}$

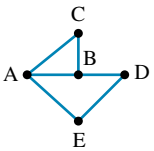
b.

	A	B	C	D	E	F	G	H
A	0	1	1	0	0	0	1	1
B	1	0	1	1	1	0	0	0
C	1	1	0	0	0	0	0	0
D	0	1	0	0	1	0	0	0
E	0	1	0	1	0	1	0	0
F	0	0	0	0	1	0	1	0
G	1	0	0	0	0	1	0	1
H	1	0	0	0	0	0	1	0

2. Diagram may be somewhat different from figure.
Sum of degrees = 16



3. a.



b. $V = 5, E = 6, F = 3$, thus $5 = 6 - 3 + 2$

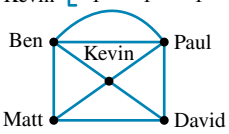
4. A–B–C–D–B–F–D–E–F–A

5. A–B–C–D–E–F–A or A–F–E–D–C–B–A

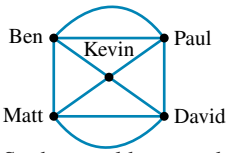
6. a.

	Ben	Paul	Matt	David	Kevin
Ben	0	2	1	0	1
Paul	2	0	0	1	1
Matt	1	0	0	1	1
David	0	1	1	0	1
Kevin	1	1	1	1	0

b.



7. a.

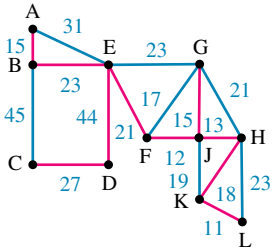


b. So they could start and finish at the same house.

c. M–D–P–B–M–K–P–B–K–D–M

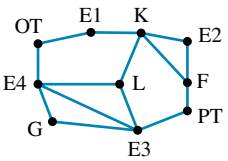
8. 1–2–3–4–5–6–7–8–9

9. Distance = 199 km



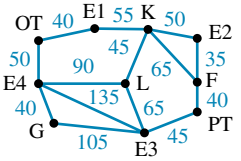
Extended response

1. a.

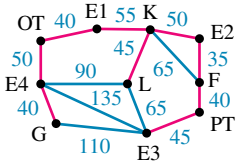


b. Connect Flower bed to Lake An Euler circuit:
E2–K–L–F–K–E1–OT–E4–L–E3–E4–G–E3–PT–F–E2
Others possible

c.



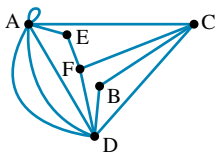
d. Minimum spanning tree; distance = 400 m



e. E2–K–L–E4, distance = 185 m

2. a. 1 **b.** 3

c. 7 **d.** 6

e. 

f. Answers will vary.

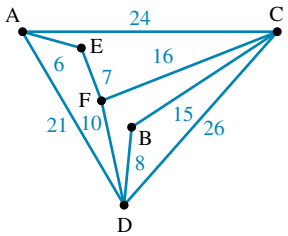
3. a. With the loops removed there are no paths leaving any camp site and coming directly back to the camp site.

b. There are no walks between these camp sites.

c.

	A	B	C	D	E	F
A	0	–	24	21	6	–
B	–	0	15	8	–	–
C	24	15	0	26	–	16
D	21	8	26	0	–	10
E	6	–	–	–	0	7
F	–	–	16	10	7	0

d. No, the matrix is symmetrical about the zeros.

e. 

4. a. A–E–F–D–A–C–D–B–C–F. (Others are possible but they must start at either A or F.)

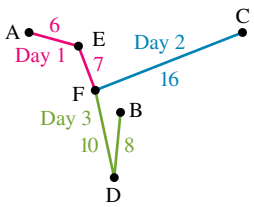
b. Euler path

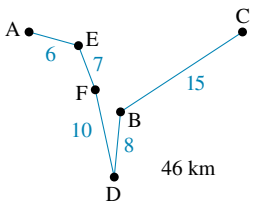
c. No, there are two vertices of odd degree.

d. i. A path from A to F

ii. Yes

e. The walks would most likely have intersections.

5. a. 

b. 

c. The requirement that they must return to camp and cover no more than 20 km per day.

6. a. Yes. No edges cross.

b. $V = 10$, $E = 15$, $F = 7$, so $10 = 15 - 7 + 2$

c.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

d. Euler path. Several options are possible, but the path must start at vertex 2 or 5 and finish at the other (as there are two vertices of odd degree).
2–1–3–4–7–10–9–7–6–3–2–5–8–9–6–5

e. Euler circuit. Not possible as all vertices must be of an even degree. Another street needs to be added between intersections 2 and 5.

f. One possible path is 1–2–3–4–7–6–5–8–9–10.
A Hamiltonian path

g. Shortest path is 1–3–6–9–10 and is 450 m.

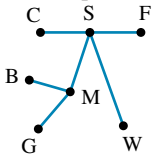
7. a. $V = 7$, $E = 11$, $F = 6$, so $7 = 11 - 6 + 2$

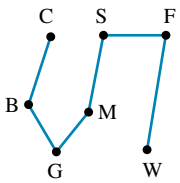
b.

	M	G	B	C	S	F	W
M	0	1	1	0	1	0	0
G	1	0	1	0	0	0	1
B	1	1	0	1	1	0	0
C	0	0	1	0	1	0	0
S	1	0	1	1	0	1	1
F	0	0	0	0	1	0	1
W	0	1	0	0	1	1	0

c. A Hamiltonian cycle M–G–W–F–S–C–B–M and M–B–C–S–F–W–G–M

d. Several possibilities



e. 

Total distance = 510 km
Of little use because they need to start and finish in Melbourne and flight paths would backtrack.

f. 240 km

TOPIC 10
Directed graphs and networks

10.1 Overview

10.1.1 Introduction

Directed graphs show all the existing connections between vertices (or nodes) and also the direction in which movement can take place from one vertex to another. The world wide web is a directed network, where websites are the vertices and the communication between websites are the edges. Not all websites communicate with each other and some websites can send information to another website, but the receiving website cannot send information back. The first paper written on this branch of mathematics was published by Leonard Euler in 1736, having been asked to solve the famous problem of the seven bridges of Konigsberg. Konigsberg (Kaliningrad), a city in Russia, had seven bridges that crossed the Pregel river and linked the mainland to two islands. Euler was asked to provide a walk around the city that crossed each of the bridges only once. He solved the problem using directed graphs.

Graph theory is used to solve problems in computer science, business and organisational structures, transportation, including trains and the trucking industry, logistics, chemistry, physics, electric circuits, biology, neuro pathways and housing development infrastructure, such as, water and electricity supplies. In the modern world, a company such as Netflix uses algorithms based on the mathematics of networks to collect data on their customer's activity and hence ensure their recommendations are highly personalised. Google maps and other GPS systems use directed graphs to calculate the shortest route for a particular journey.

LEARNING SEQUENCE

10.1 Overview
10.2 Critical path analysis
10.3 Critical path analysis with backward scanning and crashing
10.4 Network flow
10.5 Assignment problems and bipartite graphs
10.6 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 Critical path analysis

10.2.1 Activity charts and networks

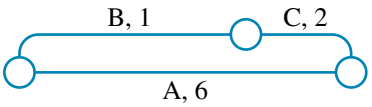
In any process, ranging from our daily schedule to major construction operations, tasks need to be completed within a certain period of time.

Consider Frieda's morning schedule, where she needs to eat her cooked breakfast, download her email and read her email. The first two tasks take 6 minutes and 1 minute respectively, while the last takes 2 minutes. Frieda needs to complete all these tasks in 7 minutes. How might she accomplish this?

Clearly, she needs to be able to do some tasks simultaneously. Although this seems like a simple problem, let us look at what might happen each minute.

Time	Activity	Activity
1st minute		Download email
2nd minute	Eat breakfast	
3rd minute	Eat breakfast	
4th minute	Eat breakfast	
5th minute	Eat breakfast	Read email
6th minute	Eat breakfast	Read email
7th minute	Eat breakfast	

More complex activities require more planning and analysis. A network diagram can be used to represent the 'flow' of activities.



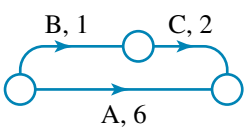
In the figure at right, the **edges** of our network represent the three activities of downloading (B), reading (C) and eating (A). The left **vertex** (or **node**) represents the start of all activity, the right vertex the end of all activity and the middle vertex indicates that activity B must occur before activity C can begin. In other words, activity B is the **immediate predecessor** of activity C.

Another way of representing this information is in an activity chart.

Activity letter	Activity	Predecessor	Time (min)
A	Eat breakfast	—	6
B	Download email	—	1
C	Read email	B	2

This chart also shows that activity B (downloading) is the immediate predecessor of activity C (reading), and that activities B and A have no predecessors.

An alternative network diagram is also shown.



The activities can be undertaken only in a certain sequence, so arrowheads are placed on the edges. Because of the implied direction, these networks are called **directed graphs** or *directed networks*. (The edges in a directed graph represent a *one-way path* between the vertices, as compared with undirected graphs where the edges represent a two-way path between the vertices.)

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

We can use the network diagram to help Frieda reduce the total time spent on the tasks. If the tasks were spread out in a straight line, so that no two tasks were completed at the same time, then they would take her 9 minutes. The diagram shows that some of Frieda's tasks can be carried out simultaneously. Let us investigate the time savings that can be made.

To determine the time saving, first determine the **earliest start time** for each activity.

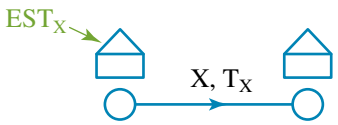
10.2.2 Forward scanning

By **forward scanning** through a network we can calculate the earliest start times for each activity and the earliest completion time for the whole project.

> **The *earliest start time* (EST) is the earliest that any activity can be started after all prior activities have been completed.**

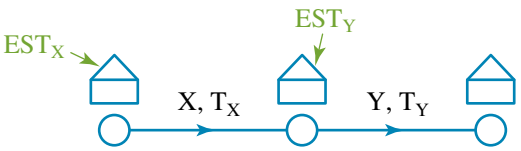
The EST is determined by looking at all the previous activities, starting with the immediate predecessors and working back to the start of the project. An activity can start no earlier than the *completion* of such predecessors. Obviously, the EST for the first activity is 0.

The EST can be recorded on a network diagram by using triangles and boxes, as shown in the diagram at right. The activity is represented by the edge between the vertices. The duration (T_X) of the activity is represented as the number above the edge. The earliest start time of activity X(EST_X) is recorded in the triangle preceding the edge.



When a network includes two or more activities, the same labelling process is used.

The purpose of the boxes beneath the triangles will be explained in a later section.



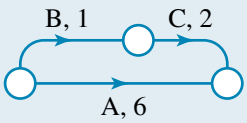
WORKED EXAMPLE 1

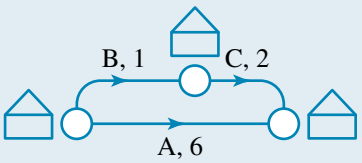
Use forward scanning to determine the earliest completion time for Frieda's initial three tasks.

THINK

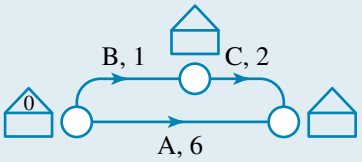
1. Begin with the network diagram.
2. Add boxes and triangles near each of the vertices.
3. The earliest start time (EST) for each vertex is entered in the appropriate triangle. Vertices with no immediate predecessors are given the value of zero.

WRITE/DRAW

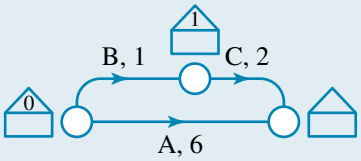




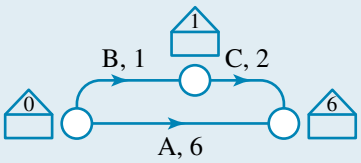
As activities B and A have no immediate predecessor then their earliest start time is zero.



4. Move to another vertex and enter the earliest start time (EST) in its triangle. In the case of activity C, it must wait one minute while its immediate predecessor, B, is completed.
5. The last vertex's earliest start time is entered. When more than one edge joins at a vertex then the earliest start time is the largest value of the paths to this vertex. This is because all tasks along these paths must be completed before the job is finished.
   There are two paths converging at the final vertex. The top path takes 3 minutes to complete and the bottom, 6 minutes. The larger value is entered in the triangle.
6. The earliest completion time is the value in the triangle next to the end vertex.



Path B – C = 1 + 2
= 3 minutes
Path A = 6 minutes



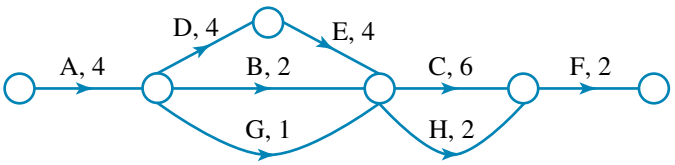
All tasks can be completed in 6 minutes.

It is important for anybody planning many tasks to know which tasks can be delayed and which tasks must be completed immediately. In Worked example 1, the eating must be commenced immediately if the 6-minute time is to be attained, whereas downloading the email could be delayed three minutes and there would still be enough time for it to be read while eating.

Let us now extend Frieda's activity chart to a more complex set of activities for her morning routine.

Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Download email	A	1
H	Read email	B, E, G	2
Total time			25

The network diagram for these activities is shown below.



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Units 3 & 4 > AOS M2 > Topic 2 > Concept 1

Graphs of directed networks Summary screen and practice questions

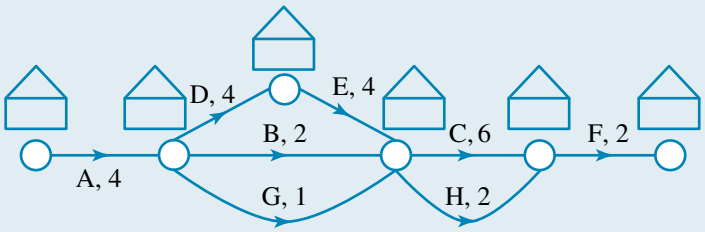
WORKED EXAMPLE 2

Using all the activities listed in Frieda's morning routine, find the earliest completion time and hence identify those tasks that may be delayed without extending the completion times.

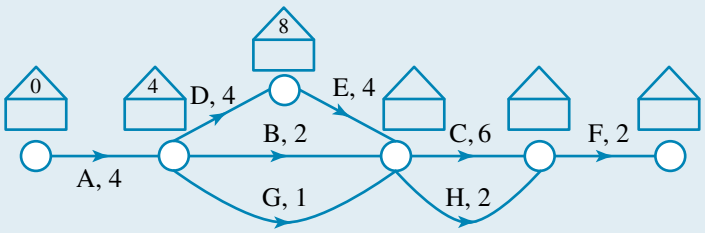
THINK

WRITE/DRAW

1. Add the boxes and triangles to the directed network diagram.

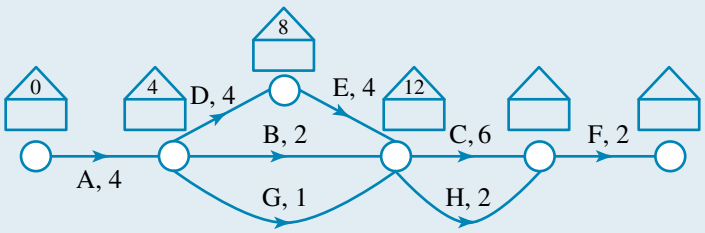


2. Begin forward scanning. The earliest start time (EST) for the first three vertices in the path can be entered immediately.



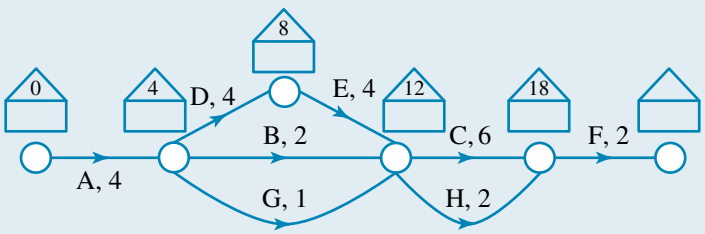
3. Calculate the time values for the paths to the fourth vertex. Enter the largest value (or longest time) into the appropriate triangle.

A–D–E = 4 + 4 + 4
= 12 minutes
A–B = 4 + 2
= 6 minutes
A–G = 4 + 1
= 5 minutes



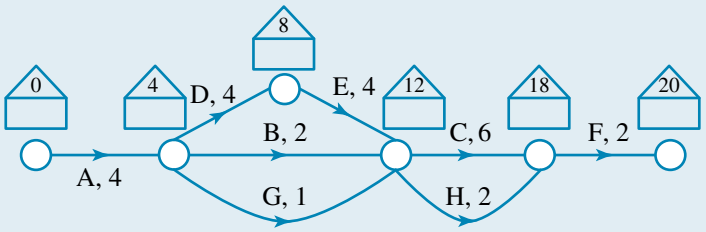
4. Repeat step 3 for the next vertex. Note that calculations begin by using the time from the previous vertex (12 minutes).

A–E–C = 12 + 6
= 18 minutes
A–E–H = 12 + 2
= 14 minutes



5. There is only one path to the last activity (F). Add its time requirement to that of the previous vertex (18 minutes).

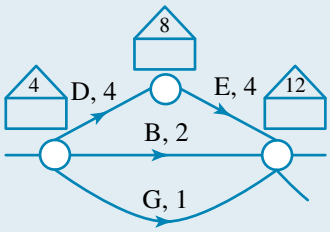
A–C–F = 18 + 2
= 20 minutes
Earliest completion time is 20 minutes.



6. The time in the last triangle indicates the earliest completion time.

Earliest completion time = 20 minutes

7. Identify sections of the network where there was a choice of paths. There are two such sections of the network. Examine the first one (the 4th vertex).

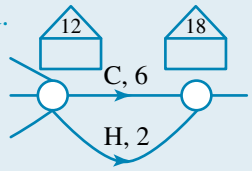


8. List and total the time for each path through this section of the network. Activities on the path with the largest value cannot be delayed.

D–E = 4 + 4 = 8 minutes
B = 2 minutes
G = 1 minute
Activities B and G can be delayed.

9. Repeat step 8 for the next section identified in step 7.

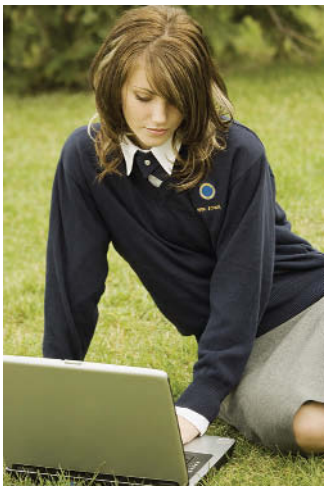
C = 6 minutes
H = 2 minutes
Activity H can be delayed.



10.2.3 Critical paths

The path through the network which follows those activities that cannot be delayed without causing the entire project to be delayed is called the **critical path**.

Therefore the critical path for the activities listed in Frieda's morning routine would be A–D–E–C–F. It is easily seen that this path takes the longest time (20 minutes).

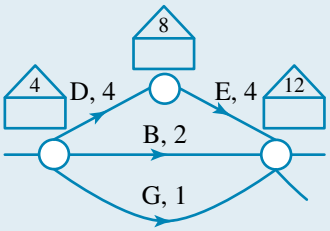


10.2.4 Float time and latest start time

Float time is the difference in time between those paths that cannot be delayed and those that can. When planning projects, paths with float time are often delayed if there is a cost saving, otherwise, they are done as soon as possible if this is more appropriate. The **latest start time** for such activities is defined as the latest time they may begin without delaying the project.

WORKED EXAMPLE 3

Work out the float time for activities B and G in Worked example 2, and hence identify the latest starting time for these activities.



THINK	WRITE
1. List the alternative paths for the section containing activities B and G and the times for these alternatives.	D–E = 4 + 4 = 8 minutes B = 2 minutes G = 1 minute
2. Subtract the smaller times separately from the maximum time.	Float time for activity B = 8 − 2 = 6 minutes Float time for activity G = 8 − 1 = 7 minutes
3. Look up the earliest completion time for the activity on the critical path and subtract the activity times.	D–E is on the critical path. Earliest completion time = 12 minutes Latest start time for activity B = 12 − 2 = 10 minutes Latest start time for activity G = 12 − 1 = 11 minutes

The float times indicate the amount of time for which these activities can be delayed without delaying the completion of all tasks. Furthermore, activity B could begin up to 6 minutes (4+6) after the start of the critical activity (D), while G could begin up to 7 minutes (4 + 7) after the same critical activity (D). There will be a more formal treatment of float time in the next section.

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Critical path network construction Summary screen and practice questions

10.2.5 Drawing network diagrams

Let us now look at how networks are prepared from activity charts. As an example, we shall see how Frieda's morning schedule network was prepared.

WORKED EXAMPLE 4

From the activity chart below, prepare a network diagram of Frieda's morning schedule.

Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Download email	A	1
H	Read email	B, E, G	2
Total time			25

THINK

1. Begin the diagram by drawing the starting vertex.

2. a. Examine the table looking for activities that have no predecessors. There must be at least one of these. Why?

   b. This activity becomes the first edge and is labelled with its activity letter and arrowhead.

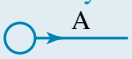
3. a. List all activities for which A is the immediate predecessor.

   b. Add a vertex to the end of the edge for activity A.

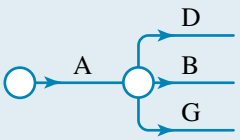
   c. Create one edge from this vertex for each of the listed activities.
   Label these edges.
   Note: The end vertex for each of these activities is not drawn until either you are certain that it is not the immediate predecessor of any later activities, or all activities have been completed.

WRITE/DRAW

Activity A has no predecessors.



Activity B has A as an immediate predecessor.
Activity D has A as an immediate predecessor.
Activity G has A as an immediate predecessor.



4. Repeat step 3 for activity D. Since it is the *only* immediate predecessor of activity E, this can be added to the diagram. Otherwise, activity E could not be added yet.

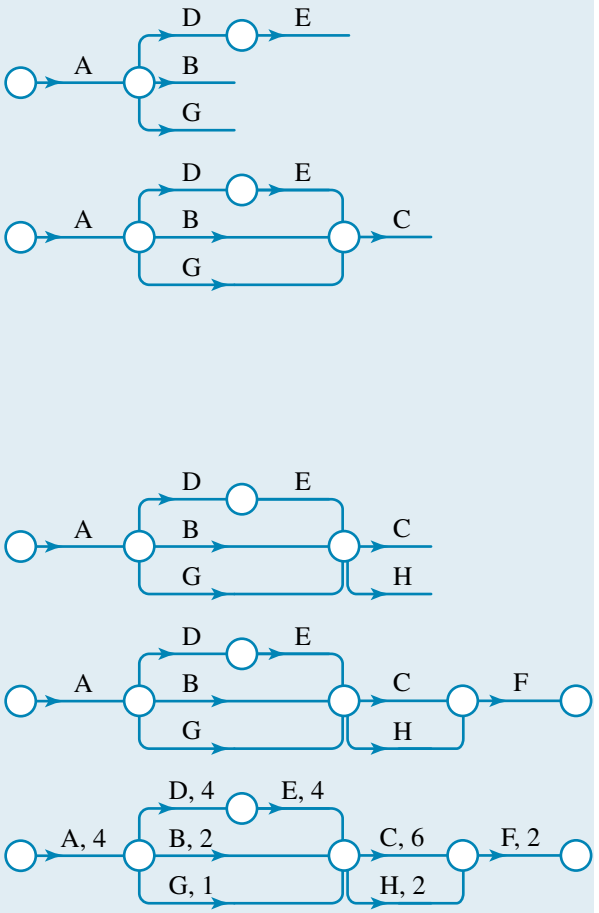
5. a. Repeat step 3 for activities B and G. They have no activities for which they are the only predecessors. Since activity C is preceded by all of B, G and E, join all the edges at a single vertex.

   b. Add activity C after this joining vertex. Note that activity H is also preceded by all of B, G and E but *not* by activity C.

6. Determine whether activity C and H are independent of each other. Since they are independent, activity H starts from the same vertex as activity C.

7. The last activity is F, which has C and H as its immediate predecessors. Therefore join C and H with a vertex, then add an edge for F. Since F is the final activity, also add the end vertex.

8. Add the time required for each activity next to its letter.

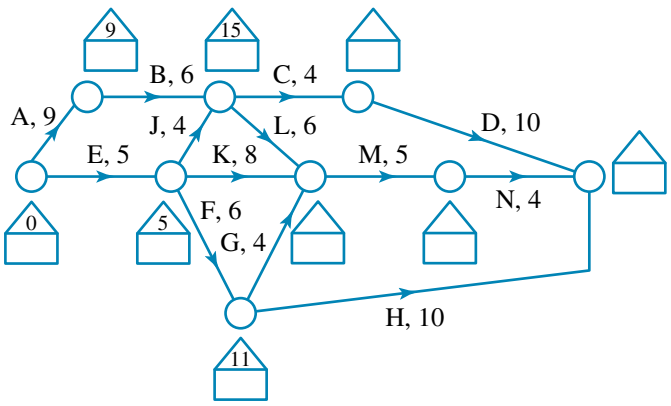


on Resources

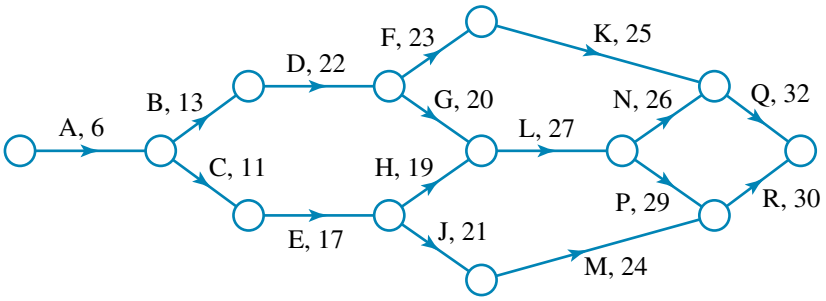
Interactivity: Critical path analysis (int-6290)

Exercise 10.2 Critical path analysis

1. **WE1** Complete a forward scan for the critical path network shown. Determine the earliest completion time.



2. The project plan for a new computer software program is shown in the figure below. Time is measured in days. Determine the earliest completion time.



3. **WE2** From question **1**, what is the maximum time that path J can be delayed without increasing the earliest completion time?
4. From question **2**, what is the maximum time that path K can be delayed without increasing the earliest completion time?
5. **WE3** For the network in question **1**, complete a backward scan and determine:
   a. the critical path
   b. the float time for non-critical activities.
6. For the network in question **2**, determine the float time for non-critical activities.
7. **WE4** Prepare a network diagram from the activity chart.

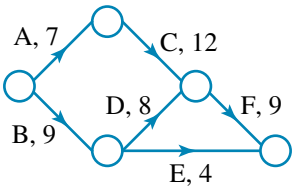
Activity	Immediate predecessor
D	—
E	D
F	D
G	E, F

8. Prepare a network diagram from the activity chart.

Activity	Immediate predecessor
N	—
O	N
P	O, T
Q	P
R	—
S	N

Activity	Immediate predecessor
T	S, Y
U	O, T
V	O, T
W	V
X	Y
Y	R
Z	X

9. MC Consider the network diagram shown. Times shown are in minutes.



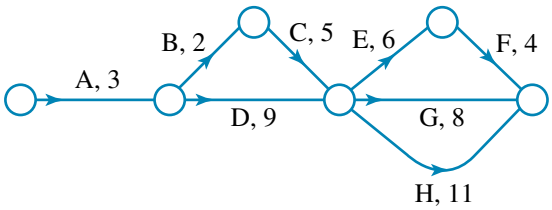
a. Which of the following statements is true?

A. Activity A is an immediate predecessor of F.
B. Activity D is an immediate predecessor of F.
C. Activity F must be done before activity D.
D. Activity F must be done before activity E.
E. Activity D is an immediate predecessor of E.

b. The minimum time taken to complete all activities is:

A. 19 minutes
B. 21 minutes
C. 23 minutes
D. 28 minutes
E. 49 minutes

10. Refer to the diagram shown.

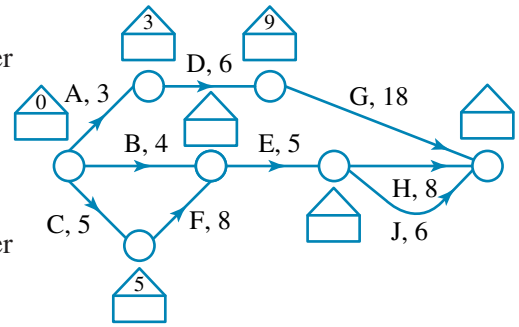


a. Use forward scanning to determine the earliest completion time.

b. Identify tasks that may be delayed without increasing the earliest completion time.

11. Determine the critical path for the network in question **9**.

12. MC Refer to the network diagram shown.



a. The number required in the triangle above the vertex after activities B and F is:

A. 0
B. 4
C. 5
D. 8
E. 13

b. The number required in the triangle above the vertex after activity E is:

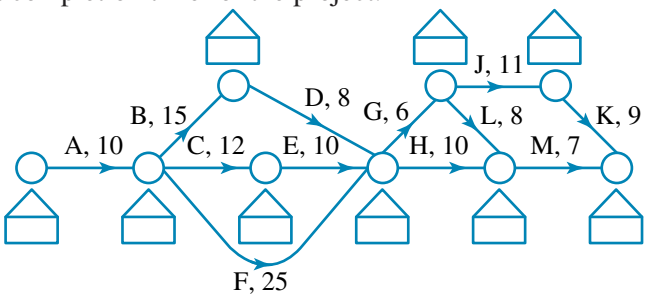
A. 5
B. 9
C. 10
D. 18
E. none of these

c. The earliest completion time for all tasks is:

A. 27
B. 24
C. 21
D. 18
E. 15

13. **a.** Find the earliest start time for each vertex in the network shown.

b. Hence, find the earliest completion time for the project.



14. From the network diagram in question **9**, produce an activity chart.

15. From the network diagram in question **12**, produce an activity chart.

16. From the network in question **13**, produce an activity chart.

17. For the network in question 12:
    a. find the critical path
    b. determine which activities have float time and hence calculate their float times
    c. determine the latest start time for all non-critical activities.
18. For the network in question 13:
    a. find the critical path
    b. determine which activities have float time.
19. Prepare a network diagram from each of the activity charts.

a.

Activity	Immediate predecessor
A	–
B	–
C	A

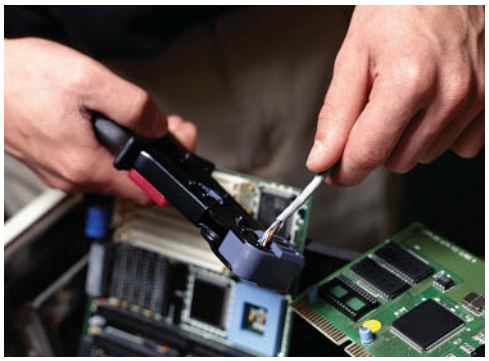
b.

Activity	Immediate predecessor
A	—
B	A
C	A
D	C
E	B
F	B
G	F
H	D, E, G
I	J, H
J	D, E, G

20. When a personal computer is being assembled the following processes must be performed.

Activity letter	Activity	Predecessor	Time (min)
A	Install memory board	—	2
B	Install hard drive	A	20
C	Test hard drive	B, E	4
D	Install I/O ports	A	5
E	Install DVD drive	D	3
F	Test DVD drive	E	5
G	Install operating system	C, F	10
H	Test assembled computer	G	12
Total time			61

    a. Construct a network diagram.
    b. Determine the minimum time in which all tasks could be completed.



10.3 Critical path analysis with backward scanning and crashing

With more complex projects requiring the coordination of many activities, it is necessary to record more information on the network diagrams and to display the information more formally using charts.

In the previous section the float times and the critical path were worked out using somewhat informal methods. In this section a more formal method will be shown to enable float times to be calculated and the critical path to be determined. This method involves **backward scanning**.

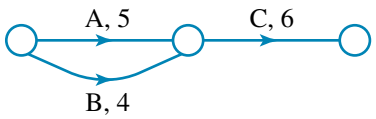
10.3.1 Requirements for critical path analysis

Along with the informal rules and techniques already developed in the previous section, we need to define two more rules that must be followed in order to successfully complete a critical path analysis of a network.

Rule 1. Two vertices can be connected directly by a maximum of one edge.

Consider the following activity table, and associated network diagram.

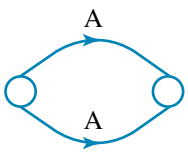
Activity letter	Immediate predecessor	Time (min)
A	—	5
B	—	4
C	A, B	6

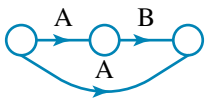


For activity C to have both A and B as predecessors, activities A and B must be drawn as parallel edges. Clearly this does not meet the requirement of Rule 1, which allows for only one edge (activity) connecting two vertices.

Violation of this rule does not affect forward scanning and the calculation of minimum completion time but will cause problems when identifying the critical path using the method of backward scanning described later in this section. A method for dealing with parallel edges will be suggested below.

Rule 2. An activity must be represented by exactly one edge. Consider the two network diagrams shown here.





The left-hand drawing indicates two separate flows along the same edge. If A were a water pipe, how could you keep the two flows separate? The right-hand example suggests that A can happen at the same time as B while still being its immediate predecessor.

10.3.2 Backward scanning

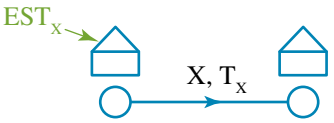
To complete critical path analysis, a procedure called **backward scanning** must be performed.

Backward scanning starts at the end vertex and moves backward through the network subtracting the time of each edge from the earliest start time of each succeeding vertex.

When two or more paths are followed back to the same vertex the smallest such difference is recorded. The results of each backward scanning step yield the **latest start time** for each activity. Latest start time is the latest time an activity can start without delaying the project.

Earliest finish time (EFT) for an activity is equal to the earliest start time (EST) of the activity plus its duration (T).

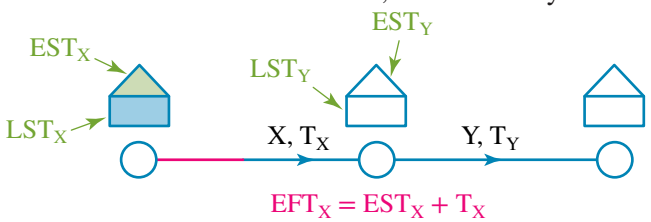
The following diagram illustrates the activity X.



The earliest finish time for activity X (EFT_X) will be the earliest start time for activity X (EST_X) plus the duration of activity X (T_X).

$$EFT_X = EST_X + T_X$$

The following diagram illustrates two activities X and Y, where activity Y directly follows activity X.



As previously established, EST_X is represented by the triangle preceding activity X (in pink on the previous diagram). The latest start time for activity X (LST_X) is represented by the blue box preceding the activity. The same applies for activity Y and all following activities.

Note that EFT cannot be read from the triangles or boxes; it must be calculated.

Float time, also called 'slack', is the maximum time that an activity can be delayed without delaying a subsequent activity on the critical path and thus affecting the earliest completion time.

From the above, it can be seen that there is a relationship between float time and the other quantities, namely:

Float time for activity X = Latest start time for activity Y (LST_Y) – Earliest Start Time for activity X (EST_X) – duration of activity X (T_X)

That is: Float $time_X = LST_Y - EST_X - T_X$

where activity Y directly follows activity X.

The technique of backward scanning is best explained with an example.

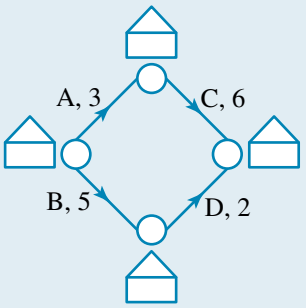
studyon

Units 3 & 4 > AOS M2 > Topic 2 > Concept 8

Critical path location Summary screen and practice question

WORKED EXAMPLE 5

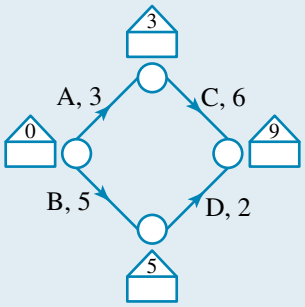
The network diagram shown has been constructed for a project manager. Use forward and backward scanning to clearly display the critical path and to list any float times.



THINK

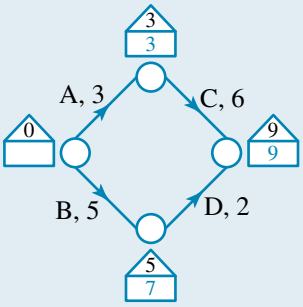
WRITE/DRAW

1. Forward scan through the network and record the earliest start time (EST) for each activity in the appropriate triangle.



2. a. Begin backward scanning. Start at the end vertex and trace backwards along all paths from this vertex.

Along path C: 9 – 6 = 3
Along path D: 9 – 2 = 7

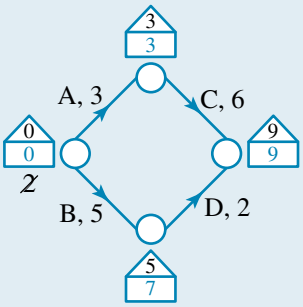


b. Subtract the times of the activities along each path from the earliest finish time (EST = 9) and record the value in the box at the previous vertex. These values are the latest start times (LST) for the activities leaving this vertex.

Latest start time for activity C (LST_C) = 3
Latest start time for activity D (LST_D) = 7

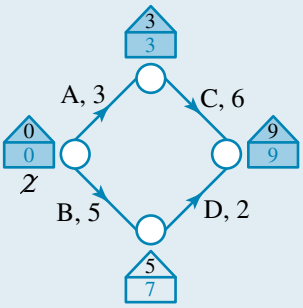
3. Repeat the process backwards through the diagram. Where two (or more) paths come together (activities A and B), record the *smaller* value in the box.

Along path A: 3 – 3 = 0
Along path B: 7 – 5 = 2
Smaller value = 0



4. The critical path can now be clearly identified. It is the path that has the same numbers in both the triangles and boxes at any vertex. Remember to include *all* such vertices in the critical path.

Critical path is A–C (as shown in blue).



5. a. Float times are calculated now. Construct a table with the headings shown.

Activity X	Activity time (T_X)	Earliest start time of this activity (EST_X)	Latest start time of the following activity (LST_Y)	Float time
A	3	0	3	0
B	5	0	7	2
C	6	3	9	0
D	2	5	9	2

b. Record the times from the triangles in the earliest start times (EST) column, the times in the boxes in the latest start time (LST_Y) column as well as the activity times (T_X). Calculate float times using the equation: $Float_X = LST_Y - EST_X - T_X$
In this example the float times are also the differences between the corresponding times in the boxes and triangles. This is not the rule in the general case.

For activity D: $Float = 9 - 5 - 2 = 2$
For activity C: $Float = 9 - 3 - 6 = 0$
For activity B: $Float = 7 - 0 - 5 = 2$
For activity A: $Float = 3 - 0 - 3 = 0$

Worked example 5 is fairly simple as the critical path could easily be determined by direct inspection. There is only one path that is not on the critical path, therefore the calculation of float time is also simple. In the real world, the problems are more complicated and so require the use of the formal method. Float times are important for the efficient management of any project. They enable the manager to determine what delays can be tolerated in the project. For example, the manager in charge of a building site is able to tell sub-contractors that they have a time window in which they must work. The sub-contractors can then arrange their schedules to incorporate this time window.

WORKED EXAMPLE 6

The chart shown has been given to an operations manager. The activities have been simplified to letter names. The manager is required to find all critical activities and the earliest completion time for the project by:

a. creating a network diagram
b. completing a forward scan and determining the earliest completion time
c. completing a backward scan and identifying the critical path
d. calculating the float times for each activity.

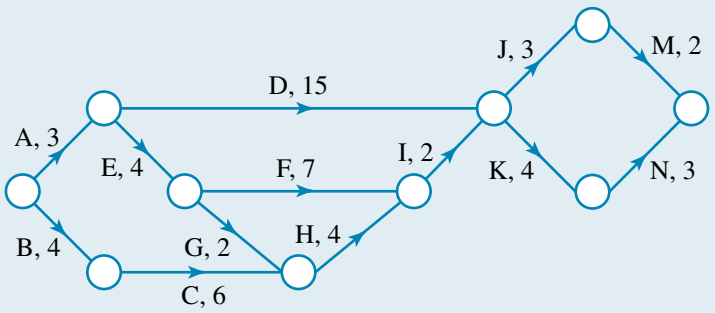
Activity letter	Immediate predecessor	Time (days)
A	—	3
B	—	4
C	B	6
D	A	15
E	A	4
F	E	7
G	E	2
H	C, G	4
I	F, H	2
J	D, I	3
K	D, I	4
M	J	2
N	K	3

THINK

a. Construct the network diagram from the table.

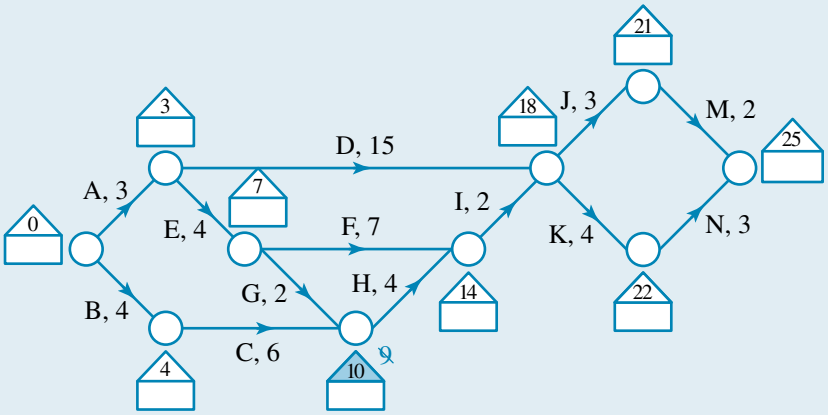
WRITE/DRAW

a.



b. 1. Draw boxes and triangles at each vertex. Forward scan through the network. Start at zero for the first vertex and then add the times taken for a path (T_X) and write it in the triangle at the next vertex (EST_Y). Adding the times of the paths (T_X) to the times in the triangle at the previous vertex (EST_X) gives the next value to be entered (EST_Y).
When two paths converge at a vertex, the larger time value is entered as all immediate predecessors need to be completed before the next activity can begin.

b. A = 3 days
A – E = 3 + 4 = 7 days
B = 4 days
The blue triangle (EST_H) may be reached by following two paths:
A–E–G = 3 + 4 + 2
= 9 days
B–C = 4 + 6
= 10 days, thus the larger of the times, 10, is entered in the triangle.

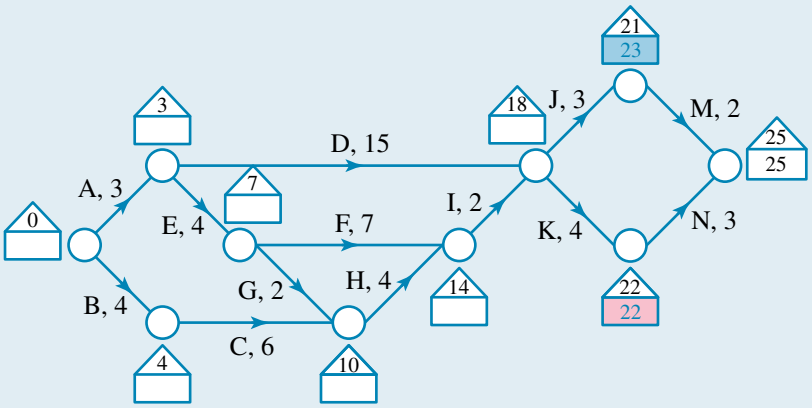


2. The earliest completion time can be read from the last triangle.

Earliest completion time = 25 days

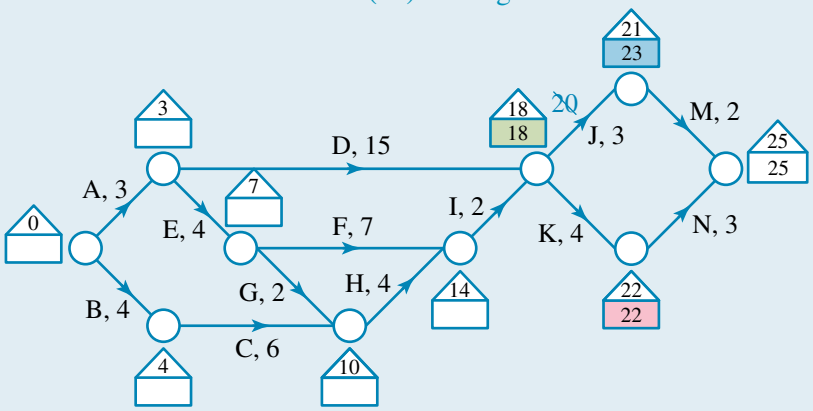
c. 1. Starting at the end vertex, begin the backward scan. Enter the earliest completion time in the last box. Subtract the times of any paths ending at this vertex (M and N) from the value in the last box and enter the result in the appropriate boxes to calculate the latest start time (LST) of that activity.

c. Blue box value (LST_M) = 25 – 2
= 23 days
Pink box value (LST_N) = 25 – 3
= 22 days

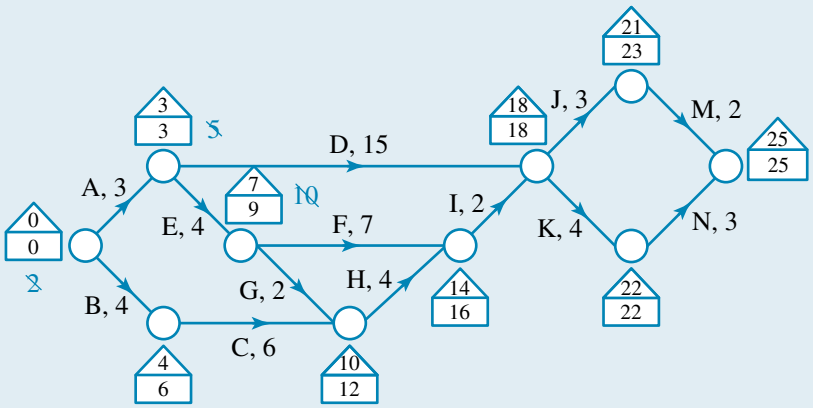


2. Repeat the process backwards through the network diagram. Where the paths converge the smaller value is entered in the box.

Backtracking to the green box (LST_J) via activity
J = 23 – 3
= 20
Backtracking to the green box (LST_K) via activity
K = 23 – 4
= 18
Enter the smaller of the values (18) in the green box.

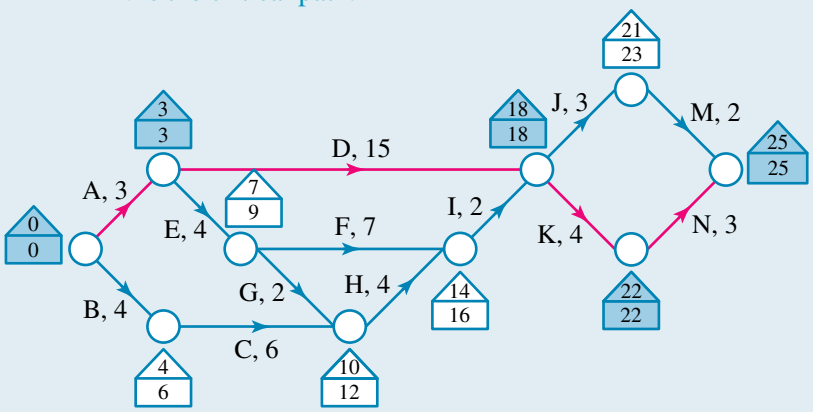


3. Repeat the process back through all paths.



4. vertices which have the earliest start time (triangles) equal to the latest start times (boxes) are identified as being on the critical path.

A–D–K–N is the critical path.



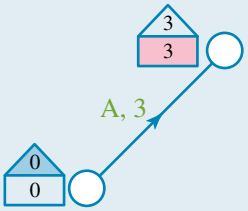
d. 1. a. The float times for each of the activities are calculated using the formula.

b. The section showing activity A from part **c**, step 4 clearly shows the latest start time of the following activity (LST_Y) (pink), the earliest start time (EST_X) (blue) and the activity time (T_X) (green).

2. a. The best way to keep organised and to calculate float times is to set up a table.

b. Add columns for earliest start times (EST), latest start times (LST) and float times to the original table. Repeat step 1 for all activities.

d. Float time of X = $LST_Y - EST_X - T_X$
Float (A) = 3 − 0 − 3 = 0
A result of zero indicates that activity A is on the critical path with no float time available.



Activity letter	Immediate predecessor	T_X	EST_X	LST_Y	Float time$_X$ = $LST_Y - EST_X - T_X$
A	—	3	0	3	3 − 0 − 3 = 0
B	—	4	0	6	6 − 0 − 4 = 2
C	B	6	4	12	12 − 4 − 6 = 2
D	A	15	3	18	18 − 3 − 15 = 0
E	A	4	3	9	9 − 3 − 4 = 2
F	E	7	7	16	16 − 7 − 7 = 2
G	E	2	7	12	12 − 7 − 2 = 3
H	C, G	4	10	16	16 − 10 − 4 = 2
I	F, H	2	14	18	18 − 14 − 2 = 2
J	D, I	3	18	23	23 − 18 − 3 = 2
K	D, I	4	18	22	22 − 18 − 4 = 0
M	J	2	21	25	25 − 21 − 2 = 2
N	K	3	22	25	25 − 22 − 3 = 0

Note that all activities that were on the critical path have float times of zero. It is important to note that if even a single activity is 'floated' by having its start delayed, then the entire network diagram should be re-drawn and float times recalculated.

If the manager employed extra workers for a critical activity, its duration time could be reduced, hence reducing the completion time for the project. The reduction in the duration time of an activity is called **crashing**. Crashing may result in a different critical path. This will be explored further in a later section.

10.3.3 Dummy activities

A dummy activity is an edge that must be added to avoid a network with two or more activities having the same name or occurring in parallel.

Earlier in this section we set up two rules:

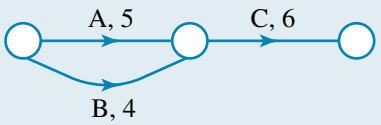
Rule 1. Two vertices can be connected directly by a maximum of one edge.

Rule 2. An activity must be represented by exactly one edge.

A table and a drawing were presented in which there were parallel edges (breaking Rule 1). A method, explained by example, will be given to overcome this problem.

WORKED EXAMPLE 7

Introduce a dummy edge to the network diagram shown.

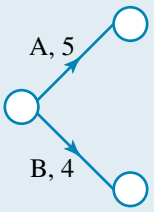


Activity letter	Immediate predecessor	Time (min)
A	—	5
B	—	4
C	A, B	6

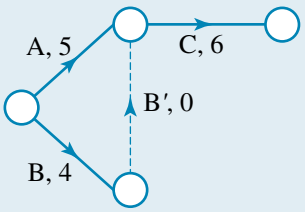
THINK

1. Construct edges A and B.
   C must follow both A and B. This is clearly a problem causing parallel edges (violating Rule 1 for networks).

WRITE/DRAW



2. Construct C after activity A. Introduce a dummy activity (B′) label and allocate a time of zero.
   Note: Use a dotted line to show dummy activity.
   Therefore, not not only is A the immediate predecessor of C, but B (via B′ with a time = 0) is also effectively the immediate predecessor of C.



The introduction of the dummy activity with a time value of zero enables scanning to take place along both edges, A and B. Additionally, the critical path can be shown more clearly.

WORKED EXAMPLE 8

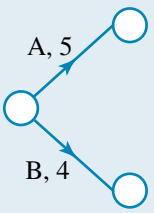
From the table shown construct a network diagram.

Activity letter	Immediate predecessor	Time (days)
A	—	5
B	—	4
C	A, B	4
D	B	6

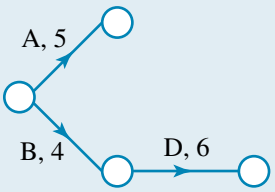
THINK

1. Construct edges A and B.

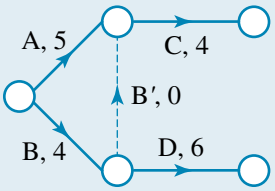
WRITE/DRAW



2. Construct edge D following B as it has only one immediate predecessor.



3. C must follow both A and B (clearly a problem). Construct C after activity A. Introduce a dummy activity (B′) from B to C and allocate it a time of zero.



Worked example 8 provides a method of not only avoiding parallel edges but also avoiding A being shown as the immediate predecessor of D, which (from the table) it clearly is not.

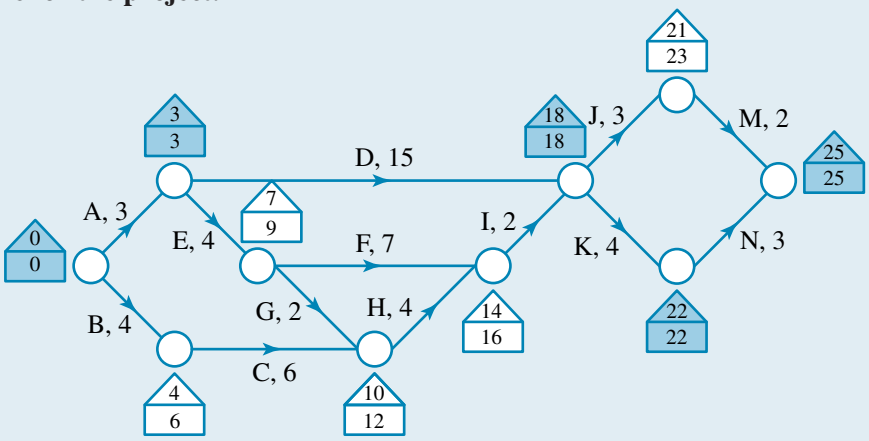
Once any required dummy activities have been defined, it is possible, using forward and backward scanning, to determine the earliest completion time for the project and float times for each non-critical activity, as per the methods of Worked example 6.

10.3.4 Crashing

As discussed earlier, crashing is a method of speeding up the completion time of a project by shortening the critical path. Follow the same method as in previous sections to calculate the new critical path and minimum completion time.

WORKED EXAMPLE 9

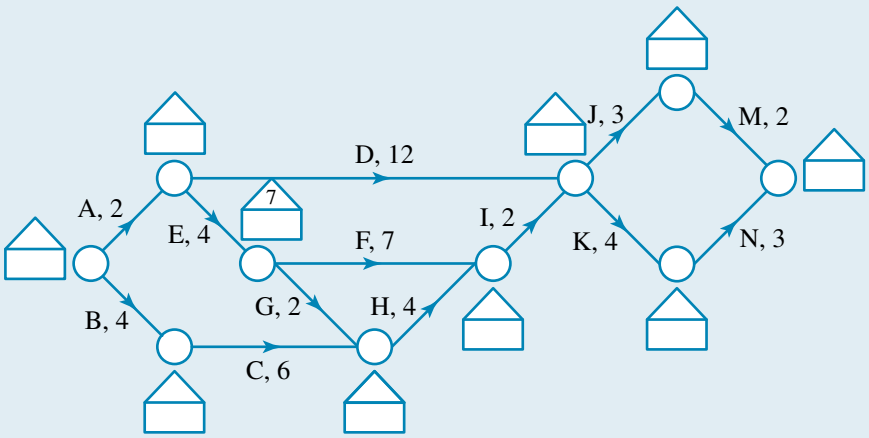
Take the critical path found in the network in Worked example 6 reproduced here. To shorten the overall completion time of the project, activity A is to be shortened to 2 days and activity D is to be shortened to 12 days. Determine the new critical path and the new minimum completion time for the project.



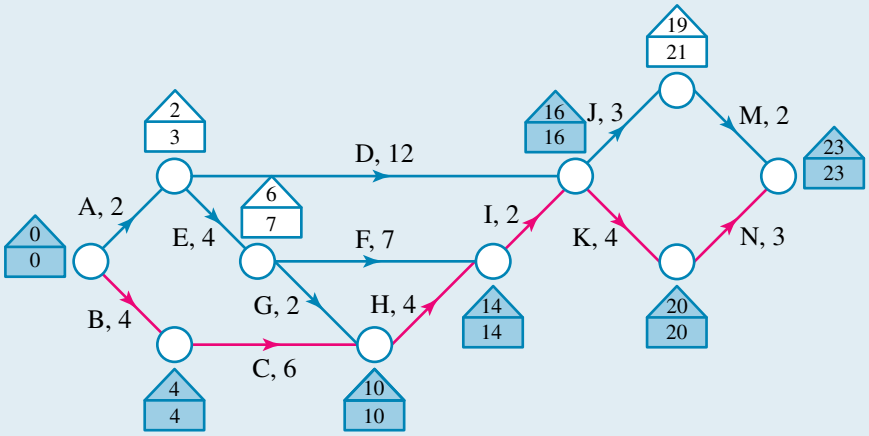
THINK

1. Redraw the network with the new completion times.

WRITE/DRAW



2. Recalculate the earliest start times (EST) and latest start times (LST) by completing forward and backward scanning.



3. Write the answer.

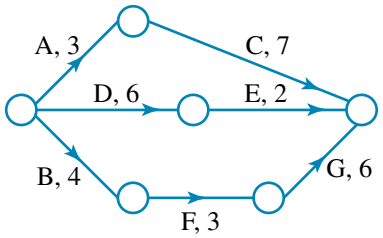
The new critical path is B–C–H–I–K–N. The minimum finishing time is 23 days.

on Resources

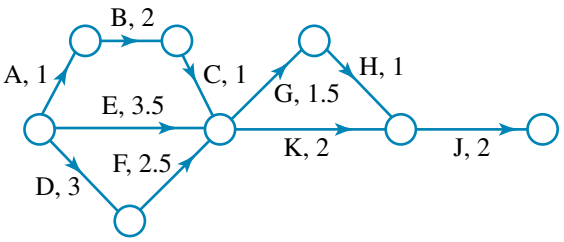
Interactivity: Crashing (int-6286)

Exercise 10.3 Critical path analysis with backward scanning and crashing

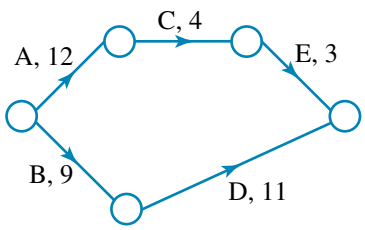
1. **WE5** For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times for non-critical activities. Times are in hours.



2. Perform a backward scan on the network shown. Determine:
   a. the critical path
   b. float times for non-critical activities.



3. **WE6** The manufacturing of bicycles can be considered as a 7-step process:
   A — Collect all the parts — 12 minutes
   B — Paint frame — 35 minutes (requires A to be completed first)
   C — Assemble brakes — 16 minutes (requires A to be completed first)
   D — Assemble gears — 20 minutes (requires B to be completed first)
   E — Install brakes — 12 minutes (requires C to be completed first)
   F — Install seat — 5 minutes (requires C to be completed first)
   G — Final assembly — 18 minutes (requires D and E to be completed first)
   a. Construct an activity chart.
   b. Construct a network diagram.
   c. Determine the earliest completion time using forward and backward scanning.
   d. Determine the critical path.
4. **MC** In the bicycle manufacturing system described in question 3, activities with float time are:
   A. A, B, C, D, E, F, G
   B. A, B, D, G
   C. C, E, F
   D. C only
   E. none
5. For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times. Times are in minutes.



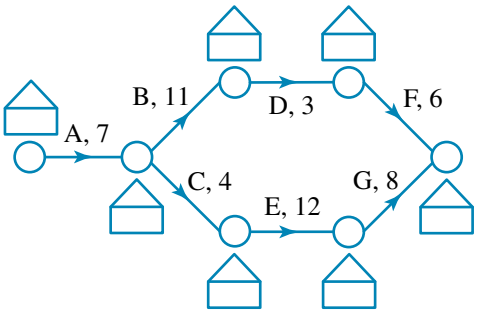
6. Complete the figure by forward and backward scanning and hence:

a. determine the earliest completion time

b. indicate the critical path.

Times are in days.

c. Imagine now that activity E can be completed in 9 days. How does this affect the answers to parts **a** and **b**?



7. MC The float time for activity D in question **6** is:

A. 1 day **B.** 2 days **C.** 3 days **D.** 4 days **E.** 7 days

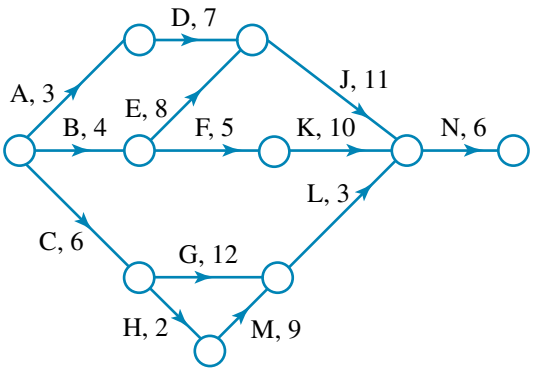
8. From the network diagram shown:

a. produce an activity table

b. complete a forward scan and hence determine the earliest completion time

c. complete a backward scan and hence determine the critical path

d. use the activity table from **a** to calculate float times for all non-critical activities.



9. From the following activity table:

a. construct a network diagram

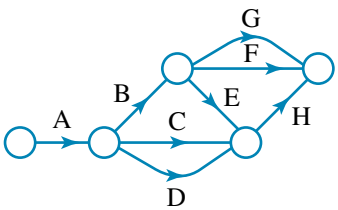
b. determine the earliest completion time

c. by forward and backward scanning, determine the critical path

d. determine the float time for all non-critical activities.

Activity	Immediate predecessor	Activity Time (h)
A	—	5
B	—	3
C	A	4
D	A	7
E	B	4
F	C	9
G	D, E	5
H	G	3
J	G	6
K	F, H	4

10. WE7 Re-draw the network diagram shown, inserting any necessary dummy activities so that Rule 1 for critical path problems is not violated.



11. Consider the following activity table.

Activity	Immediate predecessor	Activity Time (min)
A	—	7
B	—	6
C	A, B	8
D	C	12
E	C	7
F	D, E	9

By creating one (or more) dummy activities, construct a proper critical path network diagram.

12. WE8 From the following activity table construct a network diagram and indicate the location and direction of the dummy activity.

Activity	Immediate predecessor	Activity Time (h)
A	—	3
B	—	5
C	A	7
D	B	7
E	B, C	1
F	D, E	2

13. Convert the following activity table into a network diagram.

Activity	Immediate predecessor	Activity Time (h)
A	—	1000
B	—	600
C	—	800
D	A, B	1100
E	B	400
F	C	100
G	C	600
H	D, E	1600

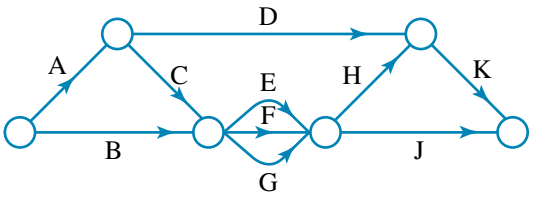
14. WE9 From your network diagram in question **11**, determine:

a. the earliest completion time

b. the critical path

c. float times for non-critical activities.

15. Re-draw the network diagram shown, inserting any necessary dummy activities so that rule 1 for critical path problems is not violated.



16. From the following activity table:

a. construct a network diagram, adding any dummy activities that may be required

b. determine the earliest completion time

c. determine the critical path by forward and backward scanning

d. determine the float time for all non-critical activities.

Activity	Immediate predecessor	Activity Time (h)
A	—	11
B	—	9
C	A	2
D	A	5
E	B	12
F	C	3
G	D	3
H	E	4
J	E, F, G	7

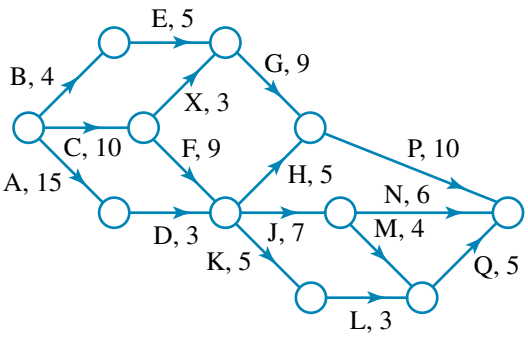
17. From the network diagram at right (activity times are recorded in days):

a. forward scan to determine the earliest completion time

b. backward scan to determine the critical path

c. determine the float time for activity X

d. imagine that crashing results in J being completed in 5 days. How does this affect the earliest completion time and the critical path?

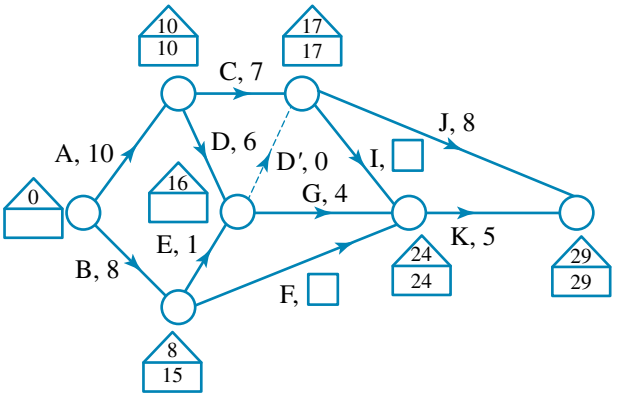


18. Given the network diagram at right, determine the:

a. four missing values in the boxes

b. critical path

c. float time for activity F.

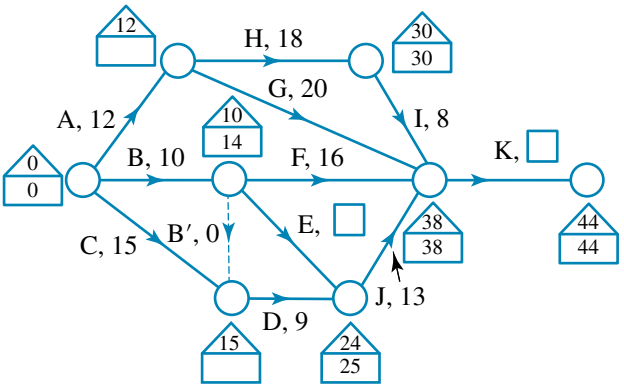


19. Given the network diagram at right, determine the:

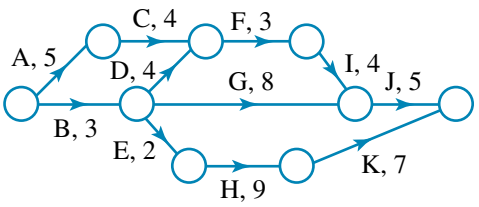
a. four missing values in the boxes

b. critical path

c. float time for activity F.



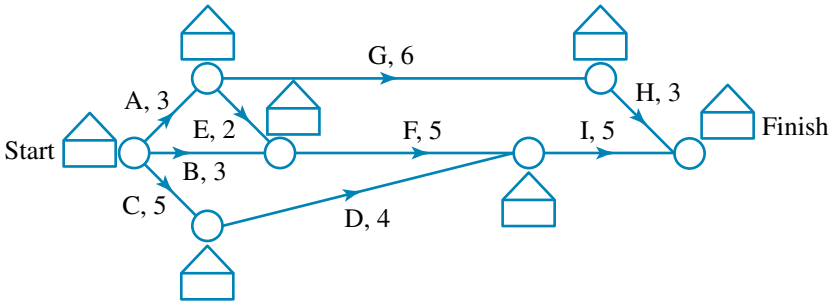
20. The network diagram shows the activities, along with the times (in hours), needed to complete a particular project. Also drawn below is the corresponding activity chart showing earliest start times rather than activity times. There is an activity X, which is yet to be drawn on the network diagram.



Activity	Immediate predecessor(s)	EST
A	—	0
B	—	0
C		5
D	B	3
E	B	3
F		9
G	B	
H	E	5
I	F	12
J	G, I, X	17
K	H	
X	C, D	8

a. Use the information in the network diagram to complete the table above by filling in the shaded boxes.
b. Draw and label activity X on the network diagram, including its direction and time.
c. What is the critical path?
d. Determine the latest start time for activity H.

21. A school is building a new library. The separate stages required for construction and the number of weeks taken to complete them are shown on the directed network.



a. What is the earliest start time for stage F?
b. What is the minimum time it will take to build the library?
c. What is the critical path?
d. What is the slack time for stage H?
e. If the overall time of the project was to be reduced, which stages could be shortened?
f. If stage A was reduced to 2 weeks and stage F was also reduced to 2 weeks, what would be the new critical path and the new minimum time for completion of the library?

10.4 Network flow

There are many examples of networks throughout the modern world. The road and rail systems that link all parts of our country, such as the one in the image, and the airline flight paths that not only link us to other places within the country but also to places overseas, are examples of networks that move people and products. Other networks such as the information superhighway (the internet), telephone lines and the postal system allow the transfer of information.

All these networks have common attributes. The attribute dealt with in this section is **network flow**. There are networks that allow for the flow to be in both directions along a path and those that allow for flow in one direction only. The *direction* of flow needs to be displayed clearly on any diagram: simple arrows on the path suffice. The *quantity* of flow is just as important.

10.4.1 Flow capacities and maximum flow

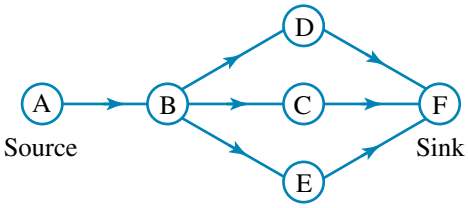
The network's starting vertex(s) is called the **source**. This is where all flows commence. The flow goes through the network to the end vertex(s) which is called the **sink**.

The **flow capacity** (capacity) of an edge is the amount of flow that an edge can allow through if it is not connected to any other edges.

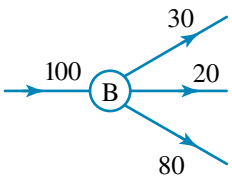
The **inflow** of a vertex is the total of the flows of all edges leading into the vertex.

The **outflow** of a vertex is the *minimum* value obtained when one compares the inflow to the sum of the capacities of all the edges leaving the vertex.

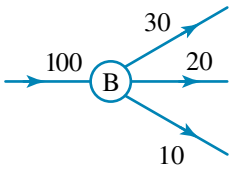
Consider the following figures.



All flow commences at A. It is therefore the source. All flow converges on F indicating it is the sink.



B has an inflow of 100 but now the capacity of the edges leaving B is 130 (80 + 20 + 30). The outflow from B is now 100.



B has a maximum inflow of 100. The flow capacity of the edges leaving B is 30 + 20 + 10 = 60. The outflow is the smaller of these two inflow/outflow values, which is 60.

WORKED EXAMPLE 10

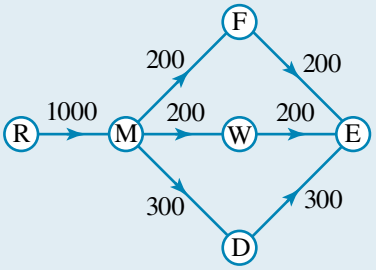
Convert the information presented in the following table to a network diagram, clearly indicating the direction and quantity of the flow.

From	To	Quantity (litres per minute)	Demand (E)*
Rockybank Reservoir (R)	Marginal Dam (M)	1000	—
Marginal Dam (M)	Freerange (F)	200	200
Marginal Dam (M)	Waterlogged (W)	200	200
Marginal Dam (M)	Dervishville (D)	300	300

THINK

Construct and label the required number of vertices. *Note:* The vertices are labelled with the names of the source of the flow and the corresponding quantities are recorded on the edges. Link the vertices with edges and record flow direction and quantity on these.

WRITE/DRAW



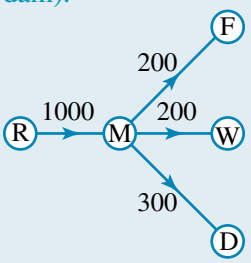
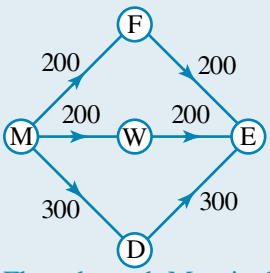
Note: In Worked example 10 the location Demand (E) does not exist. It is preferable for a network diagram to have both a single source and a single sink, so the Demand (E) was included to simplify the diagram. The reason for this will become clear in the following worked examples.

Worked example 10 is a simple case of a network in which the direction and quantity of flow are evident. Such a network diagram allows for analysis of the flow in the network; it allows us to see if various edges in the network are capable of handling the required flow.

The *flow capacity of the network* is the total flow possible through the entire network.

WORKED EXAMPLE 11

Use the information from Worked example 10 to determine, by inspection:
a. the flow capacity of the network
b. whether the flow through the network is sufficient to meet the demand of all towns.

THINK	WRITE/DRAW
a. 1. Examine the flow into and out of the Marginal Dam vertex. Record the smaller of the two at the vertex. This is the maximum flow through this point in the network.	a. Even though it is possible for the reservoir to send 1000 L/min (in theory), the maximum flow that the dam can pass on is 700 L/min (the minimum of the inflow and the sum of the capacities of the edges leaving the dam). 
2. In this case the maximum flow through Marginal Dam is also the maximum flow of the entire network.	Maximum flow is 700 L/min.
b. 1. Determine that the maximum flow through Marginal Dam meets the total flow demanded by the towns.	b.  Flow through Marginal = 700 L/min Flow demanded = 200 + 300 + 200 = 700 L/min
2. If the requirements of step 1 are able to be met then determine that the flow into each town is equal to the flow demanded by them.	By inspection of the table in Worked example 10, all town inflows equal town demands (capacity of edges leaving the town vertices).

Excess flow capacity

Consider what would happen to the system if Rockybank Reservoir continually discharged 1000 L/min into Marginal Dam while its output remained at 700 L/min.

Such flow networks enable future planning. Future demand may change, the population may grow or a new industry that requires more water may come to one of the towns. Worked example 12 will examine such a case.

Excess flow capacity is the surplus of the capacity of an edge less the flow into the edge.

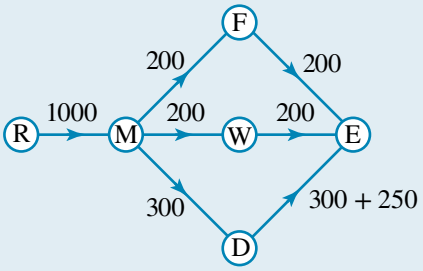
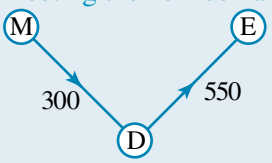
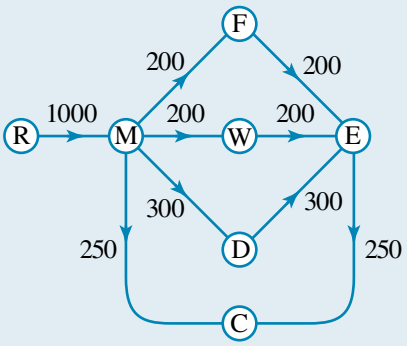
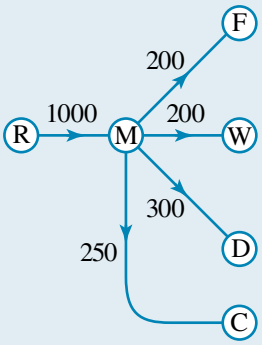
WORKED EXAMPLE 12

A new dairy factory (Creamydale (C)) is to be set up on the outskirts of Dervishville. The factory will require 250 L/ min of water.

a. Determine whether the original flow to Dervishville is sufficient.

b. If the answer to part a is no, is there sufficient flow capacity into Marginal Dam to allow for a new pipeline to be constructed directly to the factory from Marginal Dam to meet their demand?

c. Determine the maximum flow through the network if the new pipeline is constructed.

THINK	WRITE/DRAW
a. 1. Add the demand of the new factory to Dervishville's original flow requirements. If this value exceeds the flow into Dervishville then the new demand cannot be met.	**a.** 
2. The new requirements exceed the flow.	The present network is not capable of meeting the new demands. 
b. 1. Reconstruct the network including a new edge for the factory after Marginal Dam.	**b.** 
2. Repeat step 1 from Worked example 11 to find the outflow of vertex M.	Marginal Dam inflow = 1000 Marginal Dam outflow = 200 + 200 + 300 + 250 = 950 

3. Determine if the flow is sufficient.

There is excess flow capacity of 300 into Marginal Dam which is greater than the 250 demanded by the new factory. The existing flow capacity to Marginal Dam is sufficient.

c. This answer can be gained from part **b** step 2 above.

c. The maximum flow through the new network is 950.

The maximum flow through most simple networks can be determined using these methods, but more complex networks require different methods.

studyon

Units 3 & 4 > AOS M2 > Topic 2 > Concept 3

Network flow Summary screen and practice questions

10.4.2 Minimum cut–maximum flow method

To determine the maximum flow, the network first needs to be divided or **cut** into two parts.

A *cut* in a network diagram is a line drawn through a number of edges which stops *all* flow from the source to the sink.

The value of the cut is the total flow of the edges that are cut.

The *minimum cut* is the cut with the minimum value.

The *maximum flow* through a network is equal to the value of the minimum cut.

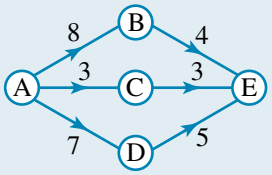
WORKED EXAMPLE 13

For the network diagram shown:

a. make three cuts

b. calculate the value of each cut

c. determine the value of all possible cuts to give the value of the minimum cut and hence the maximum flow through the network.

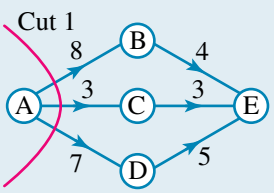


THINK

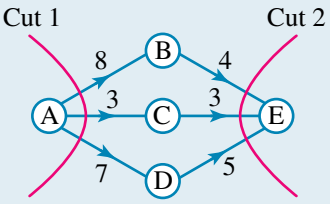
a. 1. Isolate the source, A, by cutting any edges leaving it.
Cut 1 = {AB, AC, AD}

WRITE/DRAW

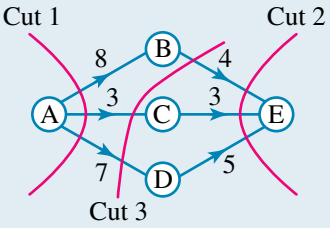
a.



2. Isolate the sink, E, by cutting any edges leading into it.
Cut 2 = {BE, CE, DE}



3. Place a cut through each of the three paths going from A to E.
Cut 3 = {AD, AC, BE}

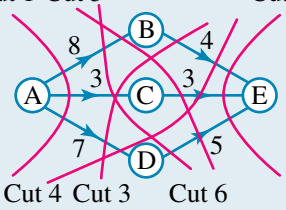


b. Add the values of the edges crossed by each of the cuts in **a**.

b. Value of cut 1 = 8 + 3 + 7
= 18
Value of cut 2 = 4 + 3 + 5
= 12
Value of cut 3 = 7 + 3 + 4
= 14

c. 1. Identify the cuts in the diagram.
Cut 1 = {AB, AC, AD}
Cut 2 = {BE, CE, DE}
Cut 3 = {AD, AC, BE}
Cut 4 = {AD, CE, BE}
Cut 5 = {AB, CE, DE}
Cut 6 = {AB, AC, DE}

c. Cut 1 Cut 5 Cut 2



2. Calculate the value of all cuts.

Value of cut 1 = 8 + 3 + 7
= 18
Value of cut 2 = 4 + 3 + 5
= 12
Value of cut 3 = 7 + 3 + 4
= 14
Value of cut 4 = 7 + 3 + 4
= 14
Value of cut 5 = 8 + 3 + 5
= 16
Value of cut 6 = 8 + 3 + 5
= 16

3. Select the minimum value as the maximum flow.

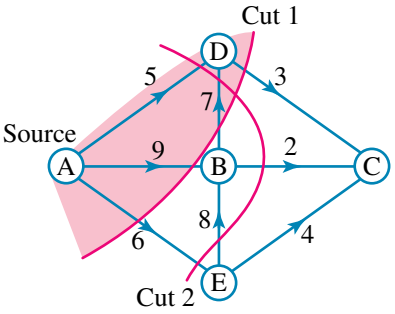
The value of the minimum cut is 12, therefore the maximum flow through the network has a value of 12.

As can be seen in Worked example 13, ensuring that all cuts have been made is a complicated procedure. The diagram becomes cluttered.

There are two cuts missing from the diagram. Can you find them?

Performing cuts at the source and sink first enables an upper limit for the value of the minimum cut to be set. In part **b**, it was clear that the value of the minimum cut in the diagram must be less than or equal to 12. In this case it was 12.

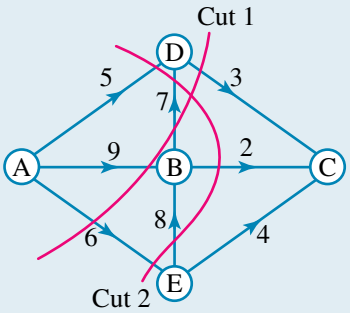
In some networks it is possible to produce a cut in which an edge actually heads back *inside* the cut rather than being directed out of the cut, as is required and as all edges in Worked example 13 do. If an edge does this, then its flow value is ignored in the calculation of the cut value. The inside of a cut is the side on which the source vertex lies. Sometimes the inside of a cut is shaded.



The shaded section represents the inside of cut 1.

WORKED EXAMPLE 14

Determine the values of the cuts made on the network diagram.



THINK	WRITE
1. Determine the overall direction of the flow.	The flow is from A to C, so the inside of the cut is on the left-hand side where the source vertex, A, is.
2. Determine the edges that are crossed by the cuts and identify any which head back into the cut.	Cut 1 = {AE, AB, BD, DC} BD heads back inside the cut, so ignore its flow value. Cut 2 = {AD, BD, BC, BE, AE} BE heads back inside the cut, so ignore its flow value. Note that for this cut, BD heads outside the cut.
3. Calculate the cut values.	Cut 1 = 6 + 9 + 3 = 18 Cut 2 = 5 + 7 + 2 + 6 = 20

Resources

Interactivity: Network flow (int-6287)
Interactivity: Network flow cuts (int-6288)

Exercise 10.4 Network flow

Convert the flow tables for questions **1** and **2** into network diagrams, clearly indicating the direction and quantity of the flow.

1. WE10

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	V	100
U	V	50

2.

From	To	Flow capacity
E	F	8
E	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	K	8
H	K	8

3. WE11 From the network in question **1**, determine

a. the flow capacity

b. whether the flow rate through the network is sufficient to meet the demand.

4. From the network in question **2**, determine

a. the flow capacity

b. whether the flow rate through the network is sufficient to meet the demand.

For questions **5** and **6**

a. introduce new edges, as shown, to the network diagrams produced in questions **1** and **2**

b. calculate the new network flow capacities.

5. WE12

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	V	100
U	V	50
S	T	100

6.

From	To	Flow capacity
E	F	8
E	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	K	8
H	K	8
E	K	10

7. Convert the following flow tables into network diagrams, clearly indicating the direction and quantity of the flow.

a.

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300

b.

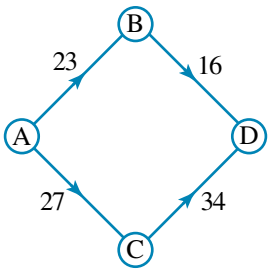
From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	P	12
R	P	12

8. For vertex B in the network, state:

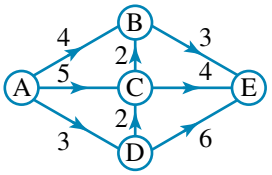
a. the inflow at B

b. the edge capacities flowing out of B

c. the outflow from B.



9. Repeat question **8** for the network shown below.



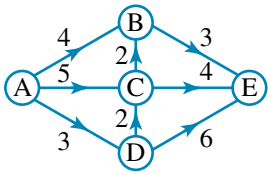
10. For each of the networks in question **7**, determine:

i. the flow capacity

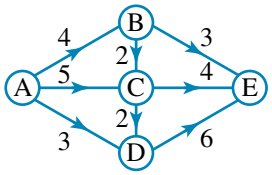
ii. whether the flow through the network is sufficient to meet the demand.

11. Convert the following flow diagrams to tables as in question **7**.

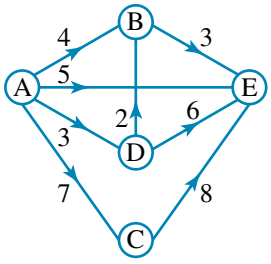
a.



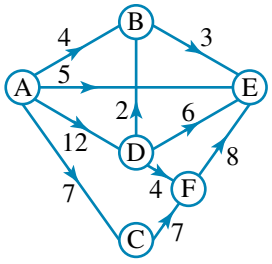
b.



c.



d.



12. Calculate the capacity of each of the networks in question **11**.

13. **i.** Introduce new edges, as shown, to each of the network diagrams produced in question **7**.

ii. Calculate the new network flow capacities.

a.

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300
B	E	100

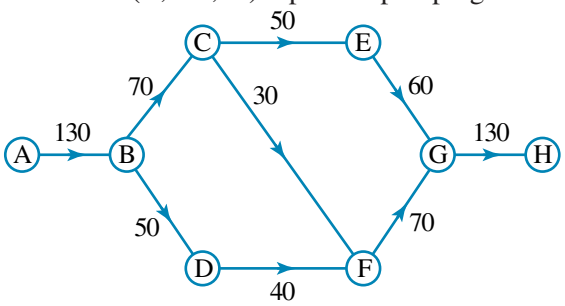
b.

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	P	12
R	P	12
N	P	5

14. **MC** In question **13b** the outflow from N is:

A. 5 **B.** 20 **C.** 15 **D.** 25 **E.** 18

15. **WE13** Consider the network flow diagram shown, representing the flow of water between a reservoir (A) and a town (H). The other vertices (B, … , G) represent pumping stations.



Using the minimum cut method, determine the maximum flow through the network.

16. **a.** Which of the four cuts shown in the network at right are invalid?

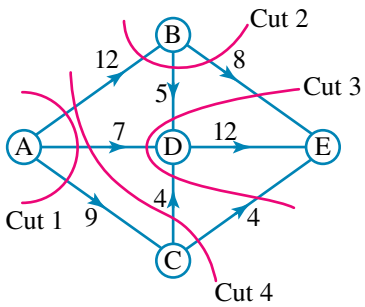
b. Determine the value of each cut.

c. What is the minimum cut and hence the maximum flow?

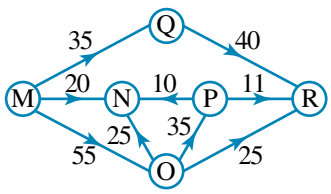
For the networks in questions **17** and **18**:

d. draw in the line of minimum cut

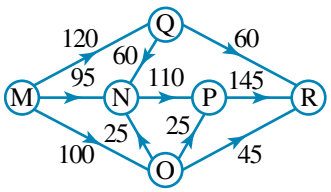
e. state the maximum flow.



17. **WE14**



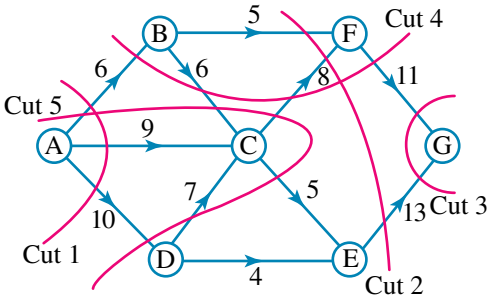
18.



19. **a.** Which, if any, of the five cuts in the network diagram are invalid?

b. Determine the value of each cut.

c. What is the minimum cut and hence the maximum flow?

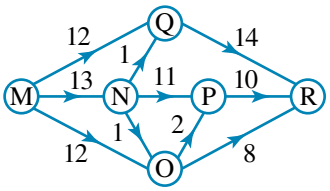


20. For each of the following networks:

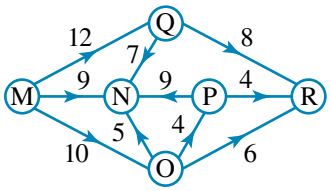
i. draw in the line of minimum cut

ii. state the maximum flow.

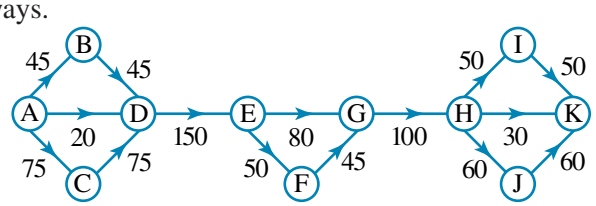
a.



b.



21. A network diagram of streets and freeways flowing into a city at peak hour is shown. Copy the diagram and clearly label the freeways.



22. Refer to the diagram in question **21** and explain what would happen to the traffic at:

a. vertex E

b. vertex H.

c. Add a new edge to the diagram in question **21** so that bottlenecks would not occur at peak hour.

10.5 Assignment problems and bipartite graphs

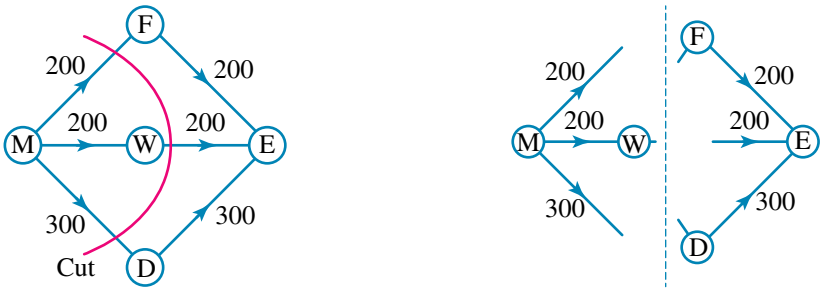
With network flow problems, the task was to find the maximum flow through a network. Another possible problem could be to determine the *exact* flows through each path in a network, so that as much of the capacity of each edge as possible is used. This is a class of situation called assignment (or allocation) problems. For example, we could be generating electricity at three power plants for distribution to five towns, so that each town gets the required amount of electricity, regardless of the plant it came from. To start this technique we need to define the concept of **bipartite graphs**.

10.5.1 Bipartite graphs

In the previous section on network flow, the minimum cut method was introduced to help solve the flow problem. Consider the cut through the network as dividing the network into two parts: 'flow in and flow out', or 'supply and demand'. Imagine separating the graph along the cut into its two parts. This graph is known as a bipartite graph.

A bipartite graph is one where the vertices can be separated into two types of vertex — supply and demand.

Consider a typical network flow problem with a cut defined in the figure on the left. By separating the graph, as in the figure on the right, we have divided the graph into two parts. The supply vertices are M and W, while the demand vertices are F, D and E.



10.5.2 Representing information as bipartite graphs

In many practical situations not only flows, but goods, money and even people can be represented by bipartite graphs.

WORKED EXAMPLE 15

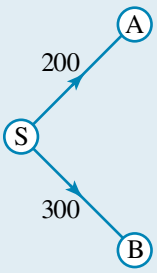
The Shiny Shoe Company make shoes at two factories, one at Alphaville producing 200 pairs per day, the other at Beta City producing 300 pairs per day. It ships them to three distributors: Fartown, who want 50 pairs per day, Giver River, who want 250 pairs per day and Hamlet, who want 200 pairs per day. Represent this information as a bipartite graph.

THINK

1. Determine all the supply vertices, starting with a single 'supply' vertex.
2. Represent the supply side of the graph.

WRITE/DRAW

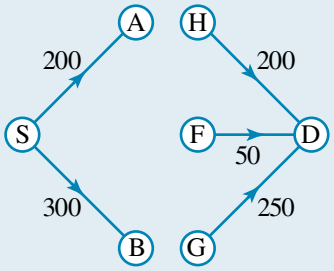
Supply vertices are S, A and B.



3 Determine all the demand vertices, ending with a single 'demand' vertex.

4 Add the demand side of the graph.

Demand vertices are F, G, H and D.



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Bipartite graphs Summary screen and practice questions

The next section concerns the problem of allocating the 'flow' of shoes between the suppliers and the distributors.

Sometimes the supply and demand vertices have quite distinct types. For example, the supply vertices might be students, while the demand vertices could be the subjects that they study, or even the sports they play. The key to bipartite graphs is that there is a separation between the two sides and that there is some sort of flow from supply to demand. The separation does not always have to be literal, as in the next example.

WORKED EXAMPLE 16

The following table lists four students and the four subjects offered in the Science program in Year 12.

a. Represent this information as a bipartite graph.

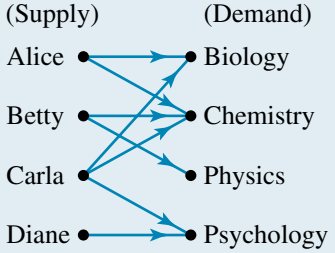
b. Determine whether the following statements are true or false.

i. Alice studies more Science subjects than Carla does.

ii. Between Carla and Betty, all Science subjects are studied.

iii. Between Alice and Betty, more subjects are studied than by Carla.

Student	Subjects taken
Alice	Biology, Chemistry
Betty	Chemistry, Physics
Carla	Biology, Chemistry, Psychology
Diane	Psychology

THINK	WRITE/DRAW
a. 1. Determine the supply vertices and represent the supply side of the graph. In this case they are (arbitrarily) assigned to the students.	**a.** Supply vertices are Alice, Betty, Carla and Diane. (Supply) Alice • Betty • Carla • Diane •
2. Determine the demand vertices. Represent the demand side and show the connections as arrows joining supply vertices to demand vertices. Note there is no quantity of flow between supply and demand.	Demand vertices are Biology, Chemistry, Physics and Psychology. 
b. i. 1. Determine the number of subjects studied by Alice and Carla.	**b. i.** Alice studies two subjects (Biology and Chemistry). Carla studies three subjects (Biology, Chemistry, Psychology).
2. Determine the truth of the statement.	Clearly, the statement is false.
ii. 1. Determine the number of subjects studied by Betty and Carla.	**ii.** Betty studies Chemistry and Physics. Carla studies Biology, Chemistry and Psychology.
2. Determine the truth of the statement.	Clearly, the statement is true.
iii. 1. Determine the subjects studied between Alice and Betty, and Carla.	**iii.** Alice and Betty between them study three subjects (Biology, Chemistry and Physics). Carla studies three subjects (Biology, Chemistry, Psychology).
2. Determine the truth of the statement.	Clearly the statement is false (as both groups study three subjects).

Note that in part **b iii**, the fact that Alice and Betty both studied Chemistry does not mean that two (distinct) subjects were studied.

10.5.3 The allocation problem

Consider a situation where there are a number of jobs to be done and the same number of people to do them. However, each person can do each job in a different amount of time (or at a different cost). How can the jobs be allocated: one per person, so that time (or cost) is minimised? This is known as **optimal allocation**. Sometimes the optimal allocation may be obvious, but in most cases it is not, and may even require some trial and error work.

In the general case the jobs and people can be put in an allocation matrix as in the following worked example.

	Job X	Job Y	Job Z
Alan	10	4	9
Bob	8	11	10
Carl	9	8	7

So, Alan would take 10 hours to do job X, 4 hours to do job Y and 9 hours to do job Z. Similarly, Bob would take 8 hours to do job X, … and so on.

The first step is to perform row reduction by subtracting the smallest value in each row from all the numbers in that row. This may produce the optimal allocation in one step.

This is the simplest possible case: the optimal allocation is almost obvious.

Remember to calculate the total time (or cost) and ensure it is indeed the minimum.

WORKED EXAMPLE 17

A building site has three more jobs to be done by the three remaining workers, Alan, Bob and Carl. The times taken by each to do the three jobs are given in the following table. Determine the optimal allocation and hence state the minimum time.

	Job X	Job Y	Job Z
Alan	10	4	9
Bob	8	11	10
Carl	9	8	7

THINK

1. Generate or write the matrix of people against job times.

WRITE/DRAW

	X	*Y*	*Z*
A	10	4	9
B	8	11	10
C	9	8	7

2. Step 1a. Perform row reduction.
   Find the smallest value in each row, and subtract it from all numbers in that row.

The smallest number in row *A* is 4.
The smallest number in row *B* is 8.
The smallest number in row *C* is 7.

3. Subtract 4 from each number in row *A*.
   Subtract 8 from each number in row *B*.
   Subtract 7 from each number in row *C*.

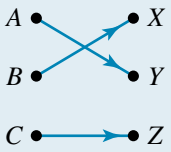
	X	*Y*	*Z*
A	6	0	5
B	0	3	2
C	2	1	0

4. Step 1b. Attempt an allocation.
   Cover all the zeroes with the smallest number of straight (horizontal/vertical, but not diagonal) lines. If the number of lines equals the number of jobs to be allocated, go to step 2.
   If not, further steps to be taken will be shown in another worked example.

	X	Y	Z
A	~~6~~	~~0~~	~~5~~
B	~~0~~	~~3~~	~~2~~
C	~~2~~	~~1~~	~~0~~

In this case there are three lines and three jobs.

5. Step 2a. Produce a bipartite graph.
   Show *all* possible allocations, where there are zeroes connecting people to jobs.



6. Step 2b. List the possible allocations and determine the smallest total.

There is only 1 possible allocation:
$A \rightarrow Y, B \rightarrow X, C \rightarrow Z$
Total time $= 4 + 8 + 7$
$= 19$ hours
Hence, the minimum time is 19 hours.

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Optimal allocation Summary screen and practice questions

10.5.4 The Hungarian algorithm

Unfortunately, there are two cases where the basic algorithm will not produce the optimal allocation:

1. when there are not enough zeros to draw the right number of straight lines in step 1b
2. when there are too many zeros so that one person may be allocated more than one job in step 2a.

In either case, we need to use the **Hungarian algorithm** which involves the following steps, replacing step 2 of the previous example.

Step 2. Perform column reduction (when there are not enough zeros). This is similar to row reduction and is performed when step 1 has not produced an optimal allocation. This may produce an optimal allocation using the method of steps 1b and 1c. If not, proceed to step 3.

Step 3. Modify the original matrix according to the Hungarian algorithm. This involves an addition and subtraction operation involving various quantities in the matrix from step 2 and will be detailed in the worked example below. This should produce an optimal allocation.

Step 4. Produce a bipartite graph and list all possible allocations.

WORKED EXAMPLE 18

Amy, Beth, Cate and Dana offer quotes on how much each of them will charge to complete four different jobs, P, Q, R and S. The table summarises these charges (in dollars). Use the Hungarian algorithm to minimise the total cost to complete the four jobs, one job per person.

	P	Q	R	S
Amy	17	24	42	21
Beth	25	18	19	20
Cate	29	14	31	22
Dana	11	20	17	14

THINK

WRITE/DRAW

1. Step 1a. Perform row reduction.
   Row A: Subtract 17
   Row B: Subtract 18
   Row C: Subtract 14
   Row D: Subtract 11

	P	Q	R	S
A	0	7	25	4
B	7	0	1	2
C	15	0	17	8
D	0	9	6	3

2. Step 1b. Attempt an allocation.
   Draw the minimum number of straight lines to cover all the zeros.

	P	Q	R	S
A	0	7	25	4
B	7	0	1	2
C	15	0	17	8
D	0	9	6	3

Only 2 lines — cannot continue allocation.

3. Step 2a. Perform column reduction.
   Column P: Subtract 0 (do nothing)
   Column Q: Subtract 0 (do nothing)
   Column R: Subtract 1
   Column S: Subtract 2

	P	Q	R	S
A	0	7	24	2
B	7	0	0	0
C	15	0	16	6
D	0	9	5	1

4. Step 2b. Attempt an allocation.
   Draw the minimum number of straight lines to cross out all the zeros.

	P	Q	R	S
A	0	7	24	2
B	7	0	0	0
C	15	0	16	6
D	0	9	5	1

Only 3 lines are required — cannot continue allocation.

5. Step 3. Perform the Hungarian algorithm.
   Step 3a. Find the smallest uncovered number from step 2b.
   Step 3b. Add this number to all covered numbers in the matrix from step 2b. At the intersections of straight lines, add this number twice (circled at right).

The smallest uncovered number is 1.

	P	Q	R	S
A	1	8	24	2
B	9	2	1	1
C	16	1	16	6
D	1	10	5	1

6. Step 3c. Subtract the overall smallest number from all the numbers in the matrix.

The smallest overall number is 1.

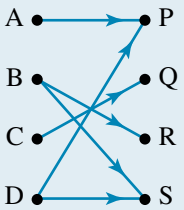
	P	Q	R	S
A	0	7	23	1
B	8	1	0	0
C	15	0	15	5
D	0	9	4	0

Step 3d. Attempt an allocation. Repeat step 3 until a possible allocation is found.

In this case there are 4 lines and 4 jobs, so go to step 4.

	P	Q	R	S
A	0	7	23	1
B	8	1	0	0
C	15	0	15	5
D	0	9	4	0

Step 4a. Produce a bipartite graph. Show all possible allocations, where there are zeros connecting people to jobs.



7. Step 4b. List possible allocations and determine the smallest total.

- By inspection, Cate must be allocated Q (she's the only one at Q).
- Beth must be allocated R (she's the only one at R).
- Therefore Dana must be allocated S (she's the only one left at S).
- Therefore Amy must be allocated P.

Amy → P, Beth → R, Cate → Q, Dana → S

Minimum cost $= 17 + 19 + 14 + 14$
$= \$64$

(numbers from the original matrix in the question)

studyon

Units 3 & 4 > AOS M2 > Topic 2 > Concept 6

Hungarian algorithm Summary screen and practice questions

Sometimes the optimal allocation is carried out to maximise a quantity such as score. In this process, all elements in the matrix are subtracted from the largest one first, and in doing so, modifying the situation to a minimisation problem. From then on, the procedure is exactly the same as that set out in Worked example 18.

on Resources

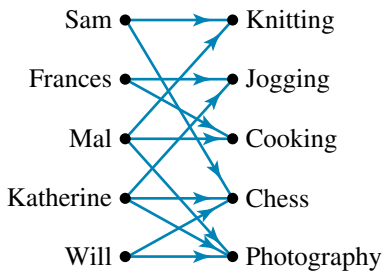
Interactivity: The allocation problem and the Hungarian algorithm (int-6289)

Exercise 10.5 Assignment problems and bipartite graphs

1. **WE15** An electricity company produces 4000 kWh, 5000 kWh and 6000 kWh at its three hydroelectric plants. This electricity is supplied to four towns as follows: 20% to Town A, 25% to Town B, 15% to Town C and the rest to Town D. Represent this information with a bipartite graph.
2. An oil company supplies petrol to three towns, A, B and C. Within town A there are two sub-depots where petrol is stored temporarily before being delivered to service stations. The other two towns have only a single depot. Town A gets 30 000 and 40 000 litres per week for each of its sub-depots, while town B gets 10 000 litres and town C gets 25 000 litres for its depots. However, the demand for petrol is as follows; town A: 60 000, town B: 15 000 and town C: 30 000 litres.
   a. Represent this information in a bipartite graph.
   b. Suggest a delivery system so that all towns get the required amount of petrol each week.
3. **WE16** Five diners (Albert, Brian, Chris, David and Earl) go to the pub for dinner and place the orders as shown in the table. Represent this information as a bipartite graph.

Diner	Dishes
Albert	Soup, fish
Brian	Fish
Chris	Soup, beef, desert
David	Beef, dessert
Earl	Fish, dessert

4. **MC** From the information in question 3, which of the following statements is true?
   A. Albert and Brian between them have more kinds of dishes than Chris and David.
   B. Chris and David between them have tried all the dishes.
   C. David and Earl between them have more kinds of dishes than Brian and Chris.
   D. Brian and Chris between them have more kinds of dishes than David and Earl.
   E. All the above statements are false.
5. A publisher produces 1000 copies per month of the latest bestseller by Wolf Thomas at two factories. The first factory produces 400 copies. The books are then distributed to two states (Queensland and Victoria), with Queensland requiring 350 copies and Victoria getting the rest. Represent this information as a bipartite graph.
6. **MC** Five students have various hobbies as indicated by the bipartite graph.
   From this graph it can be said that:
   A. Sam and Mal have all the hobbies between them
   B. Mal and Frances, in total, have more hobbies than Sam and Will
   C. Mal and Sam each have the same number of hobbies
   D. Katherine and Sam, in total, have fewer hobbies than Frances and Mal
   E. Will had fewer hobbies than all other students



For questions 7 and 8, perform row reduction on the matrices, which represent times (in hours), and attempt an optimal allocation for the minimum time. State the minimum time.

7. **WE17** $\begin{bmatrix} 6 & 3 & 7 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 3 & 7 & 3 \\ 9 & 4 & 6 & 5 \\ 5 & 6 & 7 & 8 \\ 4 & 8 & 3 & 5 \end{bmatrix}$

For questions **9** and **10**, perform an optimal allocation on the matrices by:

1. row reduction
2. column reduction
3. the Hungarian algorithm.

State the minimum total allocation and at which stage each matrix was solved.

9. **WE18** $\begin{bmatrix} 6 & 9 & 9 & 4 \\ 10 & 9 & 9 & 7 \\ 4 & 9 & 6 & 3 \\ 5 & 8 & 8 & 6 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 23 & 19 & 4 \\ 11 & 29 & 6 & 14 \\ 21 & 17 & 14 & 13 \\ 20 & 27 & 22 & 8 \end{bmatrix}$

11. Perform row reduction on the matrix, which represents times (in hours), and attempt an optimal allocation for the minimum time. State the minimum time.

$$\begin{bmatrix} 16 & 14 & 20 & 13 \\ 15 & 16 & 17 & 16 \\ 19 & 13 & 13 & 18 \\ 22 & 26 & 20 & 24 \end{bmatrix}$$

12. Draw the bipartite graph from question **11**.

13. A mother wishes to buy presents (a game, a doll and a toy truck) for her three children from three different stores. The game's cost is ($30, $45, $40) from the three stores, the doll's cost is ($50, $50, $60) while the truck's cost is ($35, $30, $30). Show that the optimal allocation yields a total of $110.

14. **MC** Given the matrix shown, the total value of the optimal allocation is:

A. 9 **B.** 11 **C.** 15 **D.** 16 **E.** 19

$$\begin{bmatrix} 7 & 3 & 7 \\ 3 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}$$

15. A government department wishes to purchase five different cars for its fleet. For reasons of fairness it must not purchase more than one car from any dealer. It receives quotes from the dealers according to the following table (cost in thousands of dollars).

a. Perform row and column reduction.

b. Draw the bipartite graph after reduction.

c. **i.** Find the optimal allocation.

ii. State the minimum cost.

	D1	D2	D3	D4	D5
Car 1	20	15	17	16	18
Car 2	17	15	19	17	16
Car 3	18	19	16	19	16
Car 4	19	19	17	21	17
Car 5	24	19	17	17	17

16. Perform an optimal allocation on the following matrices by:

1. row reduction
2. column reduction
3. the Hungarian algorithm.

State the minimum total allocation and at which stage each matrix was solved.

a. $\begin{bmatrix} 10 & 15 & 12 & 17 \\ 17 & 21 & 19 & 14 \\ 16 & 22 & 17 & 19 \\ 23 & 26 & 29 & 27 \end{bmatrix}$

b. $\begin{bmatrix} 12 & 10 & 11 & 13 & 11 \\ 11 & 11 & 13 & 12 & 12 \\ 12 & 16 & 13 & 16 & 12 \\ 9 & 10 & 9 & 11 & 9 \\ 14 & 11 & 11 & 11 & 11 \end{bmatrix}$

17. A company wishes to hire four computer programmers (A, B, C and D) to develop four software packages. Because of time constraints each programmer can accept at most one job. The quotes (in weeks) given by the four programmers are shown in the following table.

	Job 1	Job 2	Job 3	Job 4
A	30	40	50	60
B	70	30	40	70
C	60	50	60	30
D	20	80	50	70

a. Perform row (and if necessary, column) reduction.

b. Perform the Hungarian algorithm (if necessary).

c. Display possible allocations using a bipartite graph.

d. Determine:

i. the optimal allocation

ii. the total time required to complete the four tasks.

18. Four workers (Tina, Ursula, Vicky, Wendy) need to be optimally allocated four tasks (one per worker). The times (in minutes) that each worker takes are shown in the table.

	Job 1	Job 2	Job 3	Job 4
T	100	50	35	55
U	60	45	70	55
V	40	70	50	30
W	70	50	70	70

Perform an optimal allocation stating:

a. the tasks allocated to each worker

b. the total time required to complete all tasks.

19. David Lloyd George High School will be competing in a Mathematics competition. There are four categories: Algebra, Calculus, Functions and Geometry and four possible competitors: Ken, Louise, Mark and Nancy — one per category. To determine the best competitor for each category, a test is given and the results (as percentages) for each student recorded.

	Algebra	Calculus	Functions	Geometry
Ken	60	78	67	37
Louise	45	80	70	90
Mark	60	35	70	86
Nancy	42	66	54	72

a. Determine the row-reduced matrix.
b. Determine the column-reduced matrix.
c. Determine the optimal allocation using a bipartite graph.
d. Determine the average score for the team.

20. In a Meals-on-Wheels program in a remote region of Victoria, there are four elderly people (P1, P2, P3, P4) who require meals to be delivered. There are four volunteers (V1, V2, V3, V4) who live at varying distances (specified in kilometres) away.

	P1	P2	P3	P4
V1	13	17	14	23
V2	8	12	17	9
V3	9	17	14	11
V4	21	16	13	14

a. Determine the optimum allocation.
b. State the total distance travelled given that each volunteer delivers exactly one meal.

10.6 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

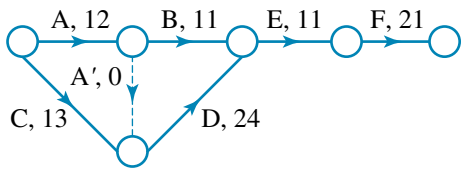
Multiple choice

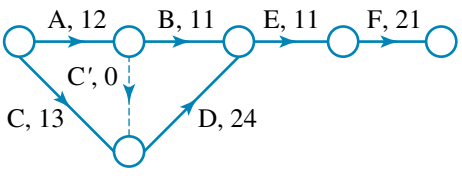
Questions **1** to **4** refer to the following table:

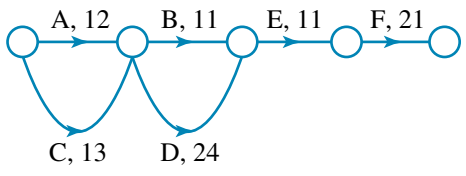
Activity	Time	Immediate predecessor
A	12	—
B	11	A
C	13	—
D	24	A, C
E	11	D, B
F	21	E

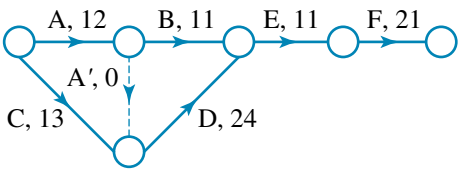
1. MC Using the table, the activities that come before activity E are:
A. D and B **B.** A, D and B **C.** A, B, C and D
D. A, C and D **E.** F

2. MC The correct diagram for the table is:

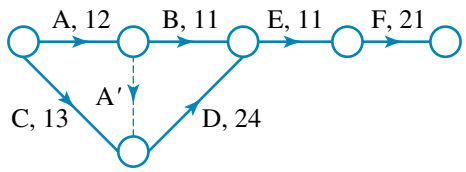
A.


B.


C.


D.


E.



3. **MC** The earliest start time for activity E is:

A. 23 **B.** 24 **C.** 37 **D.** 36 **E.** none of these

4. **MC** The earliest completion time for the network is:

A. 56 **B.** 55 **C.** 68 **D.** 69 **E.** 70

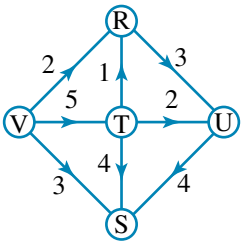
5. **MC** The times missing from the table for activities B and C respectively are:

Activity	Activity time	Earliest start time	Earliest finish time	Float time
A	3	0	3	0
B		0	7	2
C	6	3		0
D	2	5	9	2

A. 5 and 9 **B.** 5 and 8 **C.** 0 and 9 **D.** 9 and 5 **E.** 9 and 0

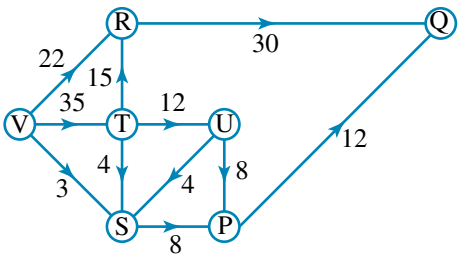
6. **MC** The source and sink, respectively, for the network flow diagram are:

A. V and U
B. U and V
C. S and V
D. V and S
E. R and S



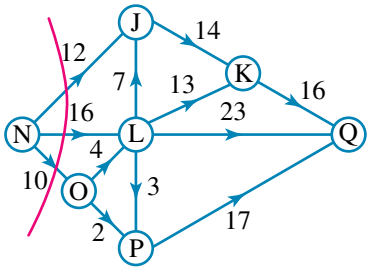
7. **MC** The outflow from vertex T is:

A. 35
B. 27
C. 31
D. 24
E. more than 1 possible outflow

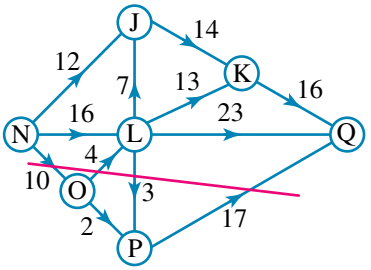


8. **MC** The diagram that shows the minimum cut is:

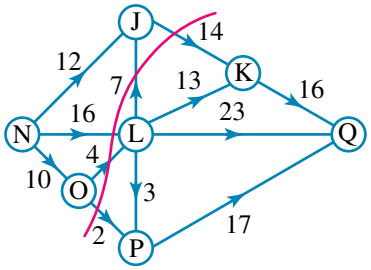
A.



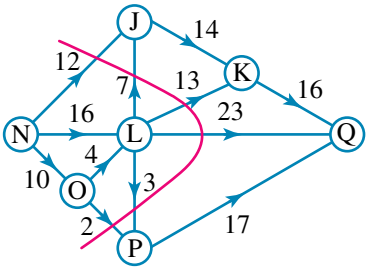
B.



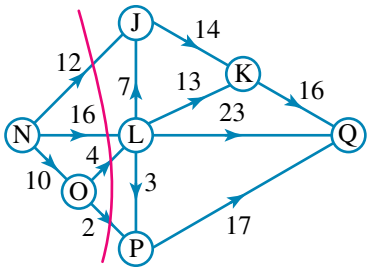
C.



D.



E.

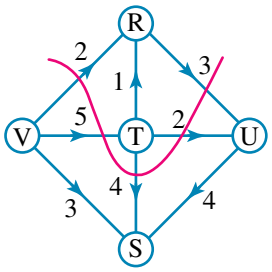


9. **MC** Which of the following do *not* show valid cuts?

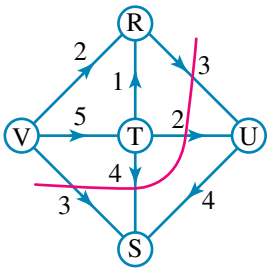
A. **a** and **c** **B.** **d** and **e** **C.** **a** and **d** **D.** **c** only

E. They are all cuts.

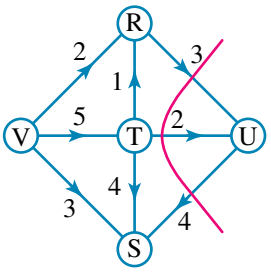
a.



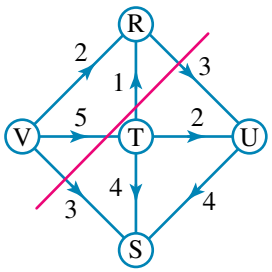
b.



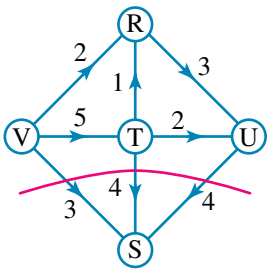
c.



d.



e.



Questions **10** and **11** refer to the following matrix.

$$\begin{bmatrix} 15 & 11 & 16 & 6 \\ 13 & 5 & 2 & 12 \\ 4 & 2 & 11 & 14 \\ 12 & 7 & 12 & 10 \end{bmatrix}$$

10. MC Which of the following is the row-reduced matrix for the matrix shown above?

A. $\begin{bmatrix} 9 & 5 & 10 & 0 \\ 11 & 3 & 2 & 10 \\ 2 & 0 & 9 & 12 \\ 5 & 0 & 5 & 5 \end{bmatrix}$ **B.** $\begin{bmatrix} 11 & 6 & 14 & 0 \\ 9 & 3 & 0 & 6 \\ 0 & 0 & 9 & 8 \\ 8 & 5 & 10 & 4 \end{bmatrix}$ **C.** $\begin{bmatrix} 9 & 5 & 10 & 0 \\ 11 & 3 & 0 & 10 \\ 2 & 0 & 9 & 12 \\ 5 & 0 & 5 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 9 & 5 & 10 & 6 \\ 11 & 3 & 2 & 10 \\ 2 & 2 & 9 & 12 \\ 7 & 5 & 5 & 5 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

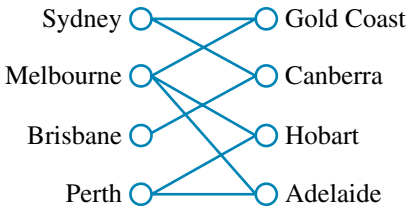
11. MC After column reduction is performed to the matrix obtained in question **10**, the resultant matrix is:

A. $\begin{bmatrix} 7 & 5 & 10 & 0 \\ 9 & 3 & 0 & 10 \\ 0 & 0 & 9 & 12 \\ 3 & 0 & 5 & 3 \end{bmatrix}$ **B.** $\begin{bmatrix} 7 & 3 & 8 & 1 \\ 9 & 1 & 0 & 5 \\ 0 & 0 & 7 & 7 \\ 5 & 3 & 3 & 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 7 & 5 & 8 & 0 \\ 9 & 3 & 0 & 10 \\ 0 & 0 & 7 & 12 \\ 3 & 0 & 3 & 5 \end{bmatrix}$

D. $\begin{bmatrix} 11 & 6 & 14 & 0 \\ 9 & 3 & 0 & 6 \\ 0 & 0 & 9 & 8 \\ 8 & 5 & 10 & 4 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

12. MC For the graph of flights between cities, which statement is not true?

A. There are more flights leaving Melbourne and Brisbane than there are leaving Sydney and Perth.
B. Brisbane has the least flights leaving it.
C. All destinations have the same number of flights arriving.
D. Melbourne has twice as many flights leaving it as does Brisbane.
E. Sydney has twice as many flights leaving it as does Brisbane.



Short answer

The following information relates to questions **1** to **6**. The following precedence table has been provided for the movement of stock within a store during a refit. The junior manager has been given the task of planning the operation so that the total time taken by the project is at a minimum and that staff are used most efficiently. All time is in hours.

Task	Time	Immediate predecessor
A	3	—
B	5	—
C	4	B
D	8	A, B
E	1	D, G
F	1	E
G	2	C
H	3	D
J	4	H
K	3	F, J

1. Produce a network diagram from the information in the table.
2. Determine, by forward scanning, the earliest completion time for the refit.
3. Perform a backward scan and clearly show the critical path.
4. Copy the table, adding columns for earliest start, earliest finish and float times. Complete the table.
5. If the overall time of the project was to be reduced, which stages could be shortened? Explain your answer.
6. If task D was reduced to 5 hours, what would be new critical path and the new minimum time for completion of the project?
7. From the following flow table produce a network flow diagram.

From	To	Flow quantity
A	B	13
A	C	6
B	C	10
B	D	4
C	D	3
C	E	14
D	F	10
E	F	15

8. a. Show the minimum cut on your diagram from question 7.
   b. Hence, state the maximum flow.

The table is for use in questions 9 and 10.

From	To	Flow quantity
A	B	13
A	C	6
A	G	16
B	C	10
B	D	4
B	G	2
C	D	3
C	E	14
D	F	10
E	F	15
G	D	3
G	H	10
H	F	13

9. Draw the network flow diagram for the table.
10. a. Find the maximum flow.
    b. Show the minimum cut.

The following data refer to questions 11 to 14. A gymnastics team of four — Annie, Beth, Consuela and Dianelle — has been selected. The team has been rated on four pieces of apparatus as follows:

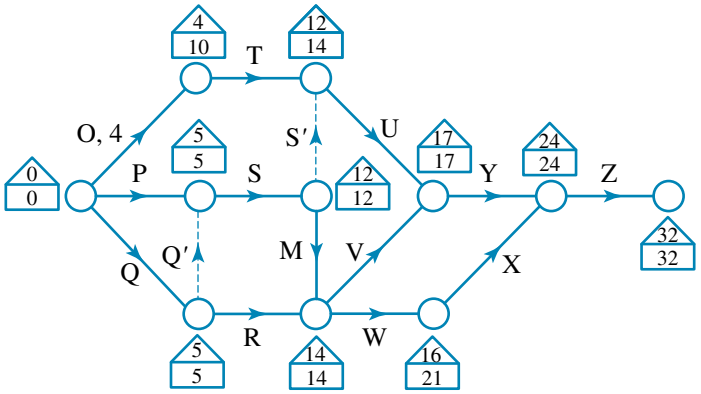
	Vault	Uneven	Beam	Floor
Annie	5	7	5	9
Beth	6	10	10	7
Consuela	7	5	3	8
Dianelle	7	8	8	9

11. Convert the table into a matrix for a minimisation problem.
12. Perform a row reduction on the matrix.
13. Perform a column reduction on the matrix produced for question 12.
14. a. Apply the Hungarian algorithm.
    b. Hence, state the optimum team make-up.

Extended response

1. The following network diagram was produced by a manager of a construction project. Unfortunately, he is not very competent and has forgotten to write the times for the activities on the diagram.

   He has also lost most of the information from the original table that was used to produce the diagram. All the information that's left is in the table on the next page. He has decided that if he is to lose his job then you, as his junior, will go too, unless you can fill in the information missing from the diagram and the table.

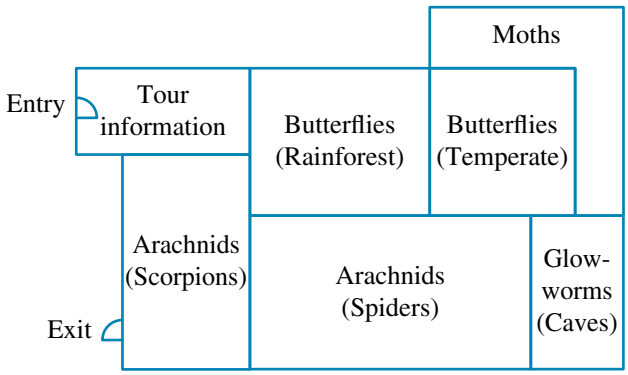


Activity	Immediate predecessor	Activity time	Earliest start time	Latest finish time	Float time
O	—				
P	—				1
Q	—				
R	Q				6
S	P, Q				
T	O				6
U	S, T				2
M	S				
V	R, M				
W	R, M				5
X	W				5
Y	U, V				
Z	X, Y				

   a. Copy the diagram and indicate the (one) critical path.
   Copy the table.
   b. First complete the missing float time column in the table.
   c. Complete the earliest start time column in the table.
   d. Complete the latest finish time column in the table.
   e. Now, complete the time column for each activity and put these times into your copy of the diagram.

2. A Lepidoptera and Arachnid building is to be set up at the zoo. The floor plan is shown in the diagram.

   The building is to be designed so that the people can flow through in one direction only. Each doorway will open only one way and is designed to ensure that there is no mixing of the exhibits.



a. Draw the doors leading from one section to the next, clearly indicating in which direction they open. (The entry and exit doors have been completed for you.)

An analysis of the times spent in similar exhibits in zoos in other cities has provided the following table.

Section from	Section to	Arrival rate
Entry	Tour information	12
Tour information	Rainforest butterflies	13
Rainforest butterflies	Temperate butterflies	12
Temperate butterflies	Moths	2
Temperate butterflies	Glow-worms	4
Temperate butterflies	Arachnids	2
Moths	Glow-worms	4
Glow-worms	Spiders	6
Spiders	Scorpions	5
Scorpions	Exit	12

b. If the doors at the entry can be represented by an edge (A) with a capacity of 12, convert the information given in the chart and plan to a network flow diagram using the letters A to J.

c. **i.** Draw the minimum cut.

ii. State at what rate people should be admitted so that they can flow smoothly through the building.

3. Imagine it is the year 2031, and the Australian Democrats have won the Federal election with 71 members in the House of Representatives and 29 Senators. The party has decided to do things differently when it comes to selecting its top four officials, namely Prime Minister, Treasurer, Foreign Minister and Speaker of the House. It will allow each of the 100 members of the caucus to 'elect' them according to the following rules:

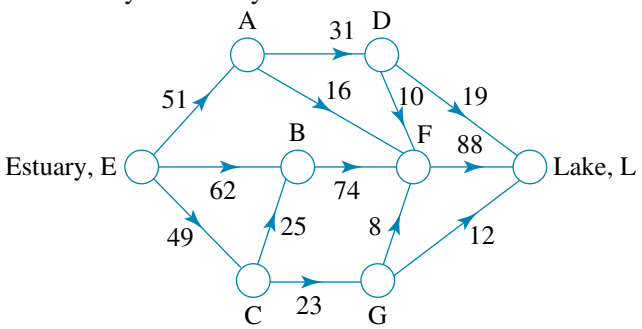
- Anyone can nominate for *all* positions.
- Members cast one vote for *each* position, but can pick a person for two or more positions.
- The four winners will be selected so that the *total* number of votes received for the four positions is a maximum.

There were five candidates in all — Alice Anderson, Boris Bologna, Cristina Colokis, Daniel Davis and Emily Eastwood — and the results of the poll were:

	Prime Minister	Treasurer	Foreign Minister	Speaker
A. Anderson	48	35	31	22
B. Bologna	1	10	5	4
C. Colokis	8	35	31	22
D. Davis	2	9	15	39
E. Eastwood	41	11	17	8

a. Since there are five people and only four positions, what should be done to the matrix so that an allocation algorithm can be applied?
b. Since the problem requires a maximum total, what should be done next so that the allocation algorithm can be applied?
c. Perform row reduction on your modified matrix.
d. Perform column reduction (if required) on your modified matrix.
e. Perform the Hungarian algorithm (if required) on your modified matrix.
f. Determine the optimal allocation.
g. Discuss the implications of the result in terms of candidates who got the most votes for a position (for example, D. Davis for Speaker) actually getting the position.
h. If your answer to part **g** was negative, discuss why this might have happened.

4. The network diagram represents the flow of salmon up the main tributaries of a river system during their spawning run, which leads from the estuary, E, to the lake, L. The figures represent the number of fish that pass along each section of the system every hour.



a. Which are the source and sink vertices?
b. Determine the inflow and outflow for all other vertices.
c. Does a pattern exist between the inflows and outflows? If so, give a possible reason for the pattern.
d. Determine the flow capacity of the river network.
e. Find the maximum flow for the network.

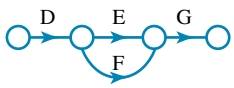
studyon

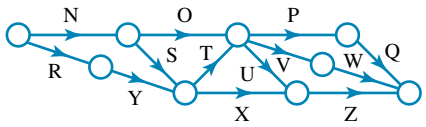
Units 3 & 4 Sit exam

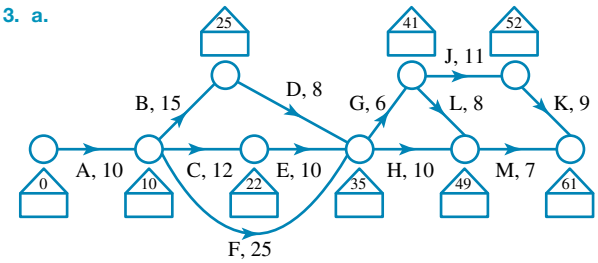
Answers

Topic 10 Directed graphs and networks

Exercise 10.2 Critical path analysis

1. 30 minutes
2. 147 days
3. 10
4. 58 days
5. a. Critical path = A–B–L–M–N
   b. Float (D) = 30 – 19 – 10 = 1
   Float (C) = 20 – 15 – 4 = 1
   Float (H) = 30 – 11 – 10 = 9
   Float (F) = 17 – 5 – 6 = 6
   Float (G) = 21 – 11 – 4 = 6
   Float (E) = 11 – 0 – 5 = 6
   Float (K) = 21 – 5 – 8 = 8
   Float (J) = 15 – 5 – 4 = 6
6. Float (C) = 25 – 6 – 11 = 8 days
   Float (E) = 42 – 17 – 17 = 8 days
   Float (F) = 90 – 41 – 23 = 26 days
   Float (H) = 61 – 34 – 19 = 8 days
   Float (J) = 93 – 34 – 21 = 38 days
   Float (K) = 115 – 64 – 25 = 26 days
   Float (M) = 117 – 55 – 24 = 38 days
   Float (N) = 115 – 88 – 26 = 1 day
   Float (Q) = 147 – 114 – 32 = 1 day
7. 

8. 

9. a. B b. D
10. a. 23 minutes
    b. B, C, E, F, G
11. A–C–F
12. a. E b. D c. A
13. a. 

    b. 61 minutes

14.

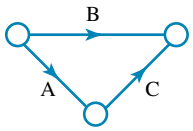
Activity	Immediate predecessor	Activity time
A	—	7
B	—	9
C	A	12
D	B	8
E	B	4
F	C , D	9

15.

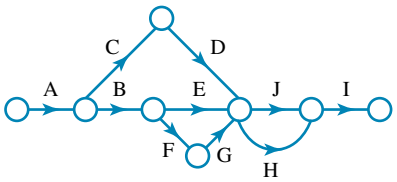
Activity	Immediate predecessor	Activity time
A	—	3
B	—	4
C	—	5
D	A	6
E	B, F	5
F	C	8
G	D	18
H	E	8
J	E	6

16.

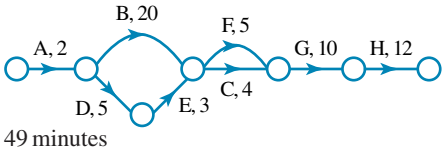
Activity	Immediate predecessor	Activity time
A	—	10
B	A	15
C	A	12
D	B	8
E	C	10
F	A	25
G	D, E, F	6
H	D, E, F	10
J	G	11
K	J	9
L	G	8
M	H, L	7

17. a. A–D–G
    b. Float (H) = 1, Float (J) = 3,
    Float (E) = 1,
    Float (B) = 10, Float (C) = 1,
    Float (F) = 1
    c. Activity B can be delayed 10 minutes,
    activity C can be delayed 1 minute,
    activity E can be delayed 1 minute,
    activity F can be delayed 1 minute,
    activity H can be delayed 1 minute,
    activity J can be delayed 3 minutes.
18. a. A–F–G–J–K b. M, L, H, C, E, B, D
19. a. 

b.



20. a.



b. 49 minutes

Exercise 10.3 Critical path analysis with backward scanning and crashing

1. Critical path = B–F–G; Float (C) = 3 h,
Float (E) = 5 h, Float (A) = 3 h,
Float (D) = 5 h

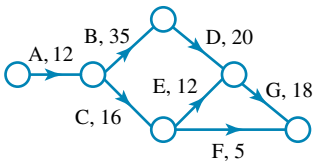
2. a. The critical path is D–F–G–H–J.

b. Float(K) = 8 − 5.5 − 2 = 0.5 hours
Float(C) = 5.5 − 3 − 1 = 1.5 hours
Float(B) = 4.5 − 1 − 2 = 1.5 hours
Float(A) = 2.5 − 0 − 1 = 1.5 hours
Float(E) = 5.5 − 0 − 3.5 = 2 hours

3. a.

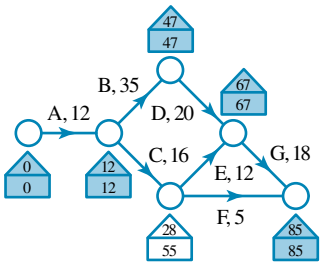
Activity	Activity	Immediate predecessor	Activity time
A	Collect parts	—	12
B	Paint frame	A	35
C	Assemble brakes	A	16
D	Assemble gears	B	20
E	Install brakes	C	12
F	Install seat	C	5
G	Final assembly	D, E	18

b.



c. 85 minutes

d.



Critical path = A–B–D–G

4. C

5. Critical path = B–D; Float (E) = 1 min,
Float (C) = 1 min, Float (A) = 1 min

6. a. 31 days

b. Critical path = A–C–E–G

c. The completion time is reduced to 28 days.
The critical path is not affected.

7. D

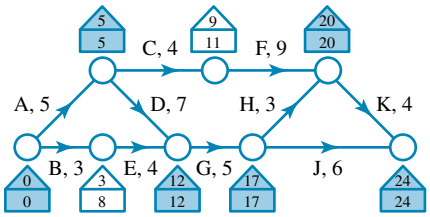
8. a, d

Activity	Immediate predecessor	Activity time	Float time
A	—	3	2
B	—	4	0
C	—	6	2
D	A	7	2
E	B	8	0
F	B	5	4
G	C	12	2
H	C	2	3
J	D, E	11	0
K	F	10	4
L	G, M	3	2
M	H	9	3
N	J, K, L	6	0

b. 29

c. B–E–J–N

9. a.

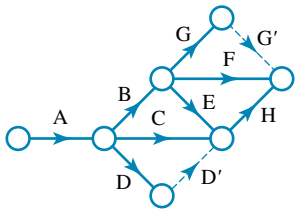


b. 24 hours

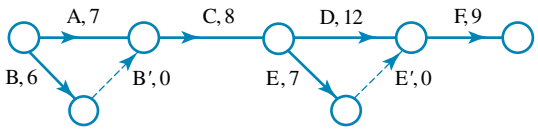
c. A–D–G–H–K

d. Float (B) = 5, Float (E) = 5,
Float (C) = 2,
Float (F) = 2,
Float (J) = 1

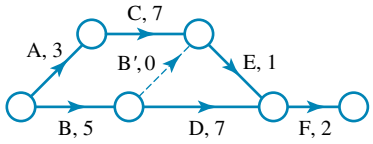
10.



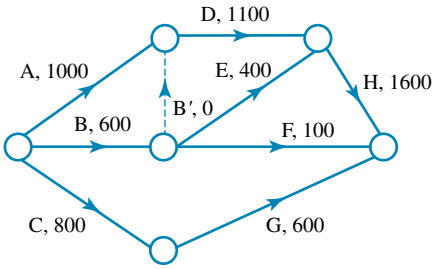
11.



12.



13.

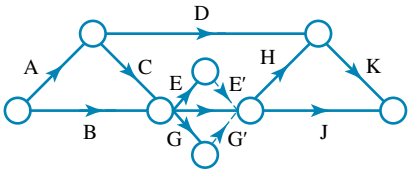


14. a. 36 minutes

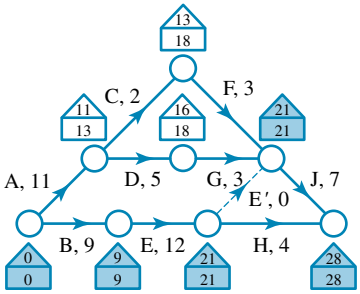
b. A–C–D–F

c. Float (E) = 27 − 15 − 7 = 5 minutes
Float (B) = 7 − 0 − 6 = 1 minutes

15.



16. a.



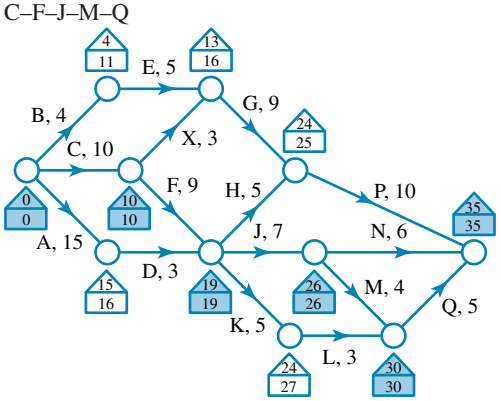
b. 28 h

c. B–E–E′–J

d. Float (A) = 2, Float (C) = 5,
Float (F) = 5, Float (D) = 2,
Float (G) = 2, Float (H) = 3

17. a. 35 days

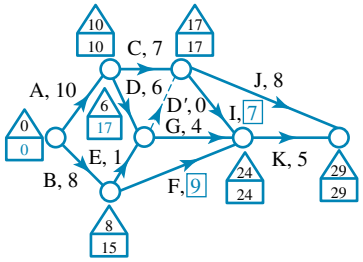
b. C–F–J–M–Q



c. 3 days

d. When J is reduced to 5 days, the earliest completion time is reduced to 34 days. The critical path becomes C–F–H–P.

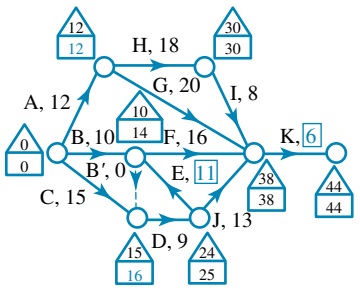
18. a.



b. A–C–I–K

c. Float (F) = 7

19. a.

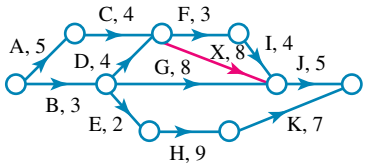


b. A–H–I–K

c. Float (F) = 12

20. a. C: A; F: C, D; EST G = 3; EST K = 14

b.



c. A–C–X–J

d. Latest start time for H = 6

21. a. 5 weeks

b. 15 weeks

c. A–E–F–I

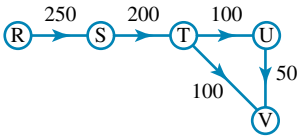
d. 3 weeks

e. A, E, F, I

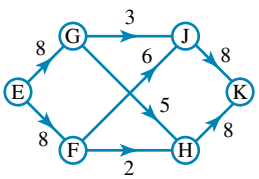
f. C–D–I, 14 weeks

Exercise 10.4 Network flow

1.



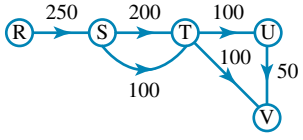
2.



3. a. 150 **b.** Yes

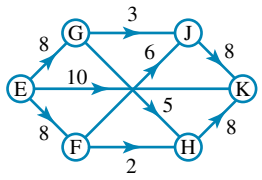
4. a. 15 **b.** No

5. a.



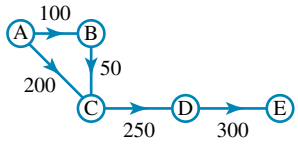
b. 150

6. a.

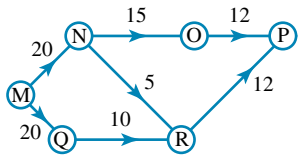


b. 25

7. a.



b.



8. a. 23 **b.** 16 **c.** 16

9. a. 6 **b.** 3 **c.** 3

10. a. i. 250 **ii.** No

b. i. 24 **ii.** Yes

11. a.

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
C	B	2
C	E	4
D	C	2
D	E	6

b.

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
B	C	2
C	E	4
D	C	2
D	E	6

c.

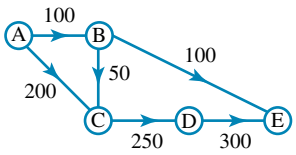
From	To	Flow capacity
A	B	4
A	C	7
A	D	3
A	E	5
B	E	3
C	E	8
D	B	2
D	E	6

d.

From	To	Flow capacity
A	B	4
A	C	7
A	D	12
A	E	5
C	F	7
D	B	2
D	E	6
D	F	4
F	E	8

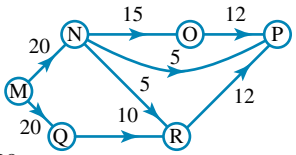
12. a. 10 **b.** 10 **c.** 18 **d.** 22

13. a. i.



ii. 300

b. i.



ii. 29

14. B

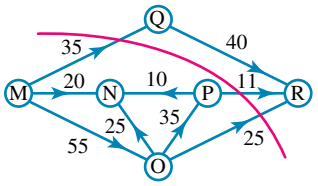
15. Minimum cut = maximum flow = 110

16. a. Cut 2

b. Cut 1 = 28, cut 3 = 28, cut 4 = 27

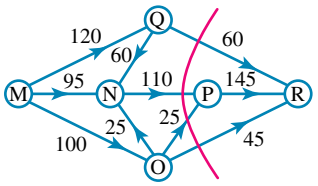
c. Maximum flow = 24

17. a.



b. Maximum flow = 71

18. a.



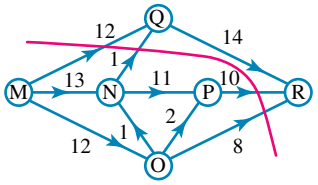
b. Maximum flow = 240

19. a. Cut 4

b. Cut 1 = 25, cut 2 = 26, cut 3 = 24, cut 5 = 29

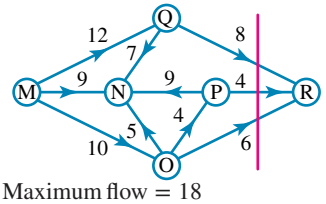
c. Maximum flow = 20

20. a. i.



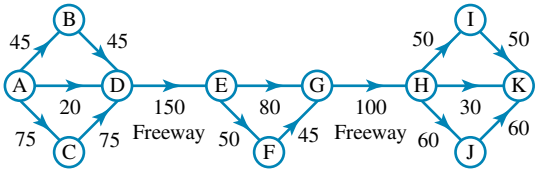
ii. Maximum flow = 31

b. i.



ii. Maximum flow = 18

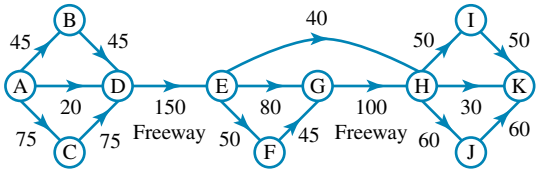
21.



22. a. There would be a traffic jam.

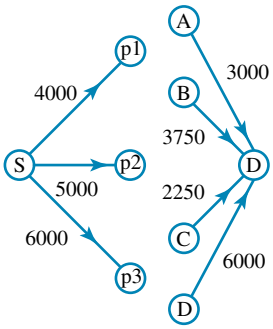
b. The traffic should flow smoothly as the inflow is less than the capacity of flows leading from H.

c.

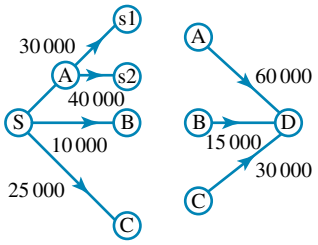


Exercise 10.5 Assignment problems and bipartite graphs

1.

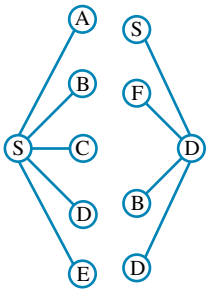


2. a.



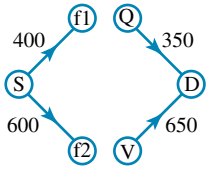
b. Send 30 000 from S1 to A, 30 000 from S2 to A, 10 000 from S2 to B, 5000 from B to B, 5000 from B to C and 25 000 from C to C. (This may not be the cheapest method.)

3.



4. D

5.



6. B

7. 7

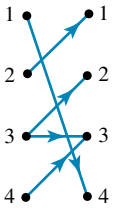
8. 15

9. 24 — Hungarian algorithm

10. 36 — column reduction

11. 61

12.

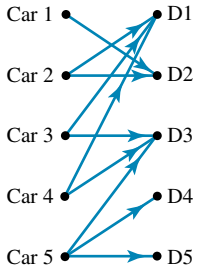


13. Answers will vary.

14. B

15. a.
$$\begin{bmatrix} 3 & 0 & 2 & 1 & 3 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 5 & 2 & 0 & 0 & 0 \end{bmatrix}$$

b.



c. i. C1 → D2, C2 → D1, C5 → D4, C3 → D3, C4 → D5 or C1 → D2, C2 → D1, C5 → D4, C3 → D5, C4 → D3

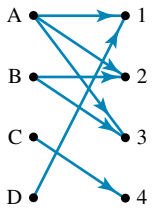
ii. Total = $82 000

16. a. 67 — column reduction **b.** 53 — row reduction

17. a.
$$\begin{bmatrix} 0 & 10 & 10 & 30 \\ 40 & 0 & 0 & 40 \\ 30 & 20 & 20 & 0 \\ 0 & 60 & 20 & 50 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 0 & 0 & 20 \\ 50 & 0 & 0 & 40 \\ 40 & 20 & 20 & 0 \\ 0 & 50 & 10 & 40 \end{bmatrix}$$ (one possible result)

c.



d. i. A → 2, B → 3, C → 4, D → 1 or A → 3, B → 2, C → 4, D → 1

ii. Total = 130 minutes

18. a. T → 3, U → 1, V → 4, W → 2

b. 175 minutes

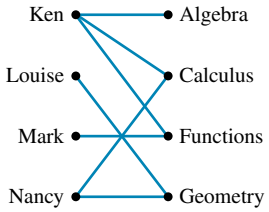
19. a.

	A	*C*	*F*	*G*
K	18	0	11	41
L	45	10	20	0
M	26	51	16	0
N	30	6	18	0

b.

	A	*C*	*F*	*G*
K	0	0	0	41
L	27	10	9	0
M	8	51	5	0
N	12	6	7	0

c.



Ken → Algebra, Louise → Geometry, Mark → Functions, Nancy → Calculus

d. 71.5%

20. a. V1 → P2, V2 → P4, V3 → P1, V4 → P3 or V1 → P3, V2 → P4, V3 → P1, V4 → P2

b. Total = 48 km

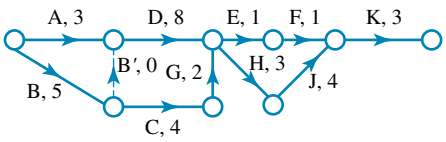
10.6 Review: exam practice

Multiple choice

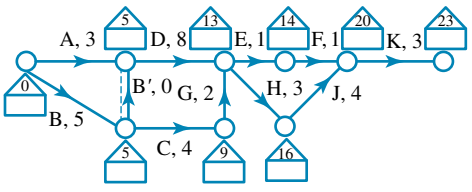
1. C	**2.** A	**3.** C	**4.** D
5. A	**6.** D	**7.** E	**8.** E
9. A	**10.** C	**11.** A	**12.** D

Short answer

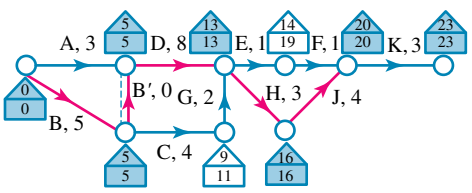
1.



2.



3.



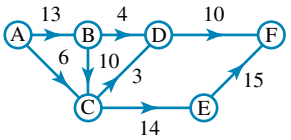
4.

Activity X	TX	ESTX	LSTY	Float time	Immediate predecessor
A	3	0	5	2	—
B	5	0	5	0	—
C	4	5	11	2	B
D	8	5	13	0	A, B
E	1	13	19	5	D, G
F	1	14	20	5	E
G	2	9	13	2	C
H	3	13	16	0	D, G
J	4	16	20	0	H
K	3	20	23	0	F, J

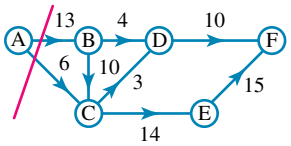
5. B, D, H, J, K

6. New critical path: B–C–G–H–J–K; new minimum completion time = 21h

7.

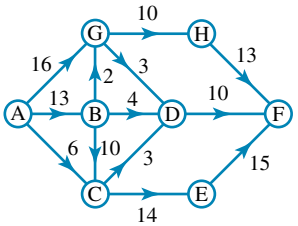


8. a.



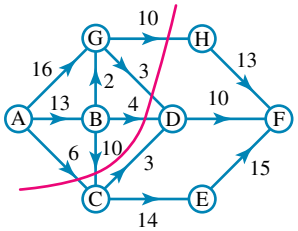
b. The maximum flow is 19.

9.



10. a. The maximum flow is 33.

b.



11.

	Vault	Uneven	Beam	Floor
Annie	5	3	5	1
Beth	4	0	0	3
Consuela	3	5	7	2
Dianelle	3	2	2	1

12.

	Vault	Uneven	Beam	Floor
Annie	4	2	4	0
Beth	4	0	0	3
Consuela	1	3	5	0
Dianelle	2	1	1	0

13.

	Vault	Uneven	Beam	Floor
Annie	3	2	4	0
Beth	3	0	0	3
Consuela	0	3	5	0
Dianelle	1	1	1	0

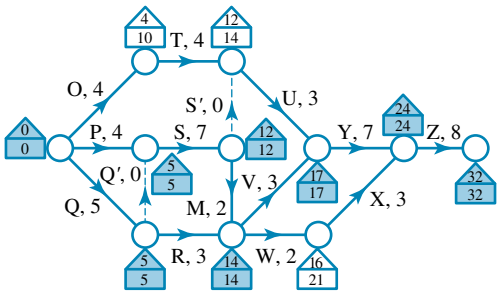
14. a.

	Vault	Uneven	Beam	Floor
Annie	2	1	3	0
Beth	3	0	0	4
Consuela	0	3	5	1
Dianelle	0	0	0	0

b. A → F, B → U, C → V, D → B or A → F, B → B, C → V, D → U

Extended response

1.

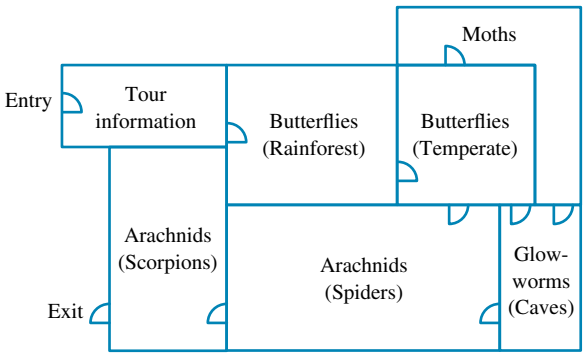


a. Critical path = Q–Q′–S–M–V–Y–Z

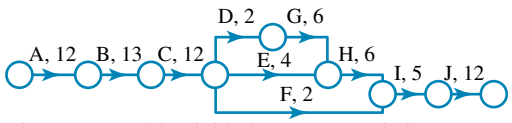
b, c, d, e.

Activity X	Immediate predecessor	TX	ESTX	LSTY	Float time
O	—	4	0	10	6
P	—	4	0	5	1
Q	—	5	0	5	0
R	Q	3	5	14	6
S	P, Q	7	5	12	0
T	O	4	4	14	6
U	S, T	3	12	17	2
M	S	2	12	14	0
V	R, M	3	14	17	0
W	R, M	2	14	21	5
X	W	3	16	24	5
Y	U, V	7	17	24	0
Z	X, Y	8	24	32	0

2. a.

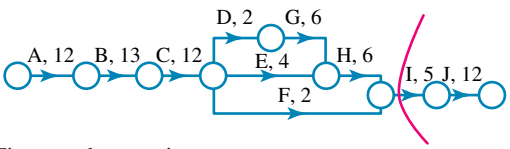


b.



c. The room would quickly become crowded.

d. i.



ii. Five people per minute

3. a. Add a 'dummy' position, giving each candidate 0 votes for it.

b. Subtract each element from the largest in the matrix (48). The resultant matrix is:

	PM	Treas	FS	Spkr	Dummy
A	0	13	17	26	48
B	47	38	43	44	48
C	40	13	17	26	48
D	46	39	33	9	48
E	7	37	31	40	48

c.

	PM	Treas	FS	Spkr	Dummy
A	0	13	17	26	48
B	9	0	5	6	10
C	27	0	4	13	35
D	37	30	24	0	39
E	0	30	24	33	41

d.

	PM	Treas	FS	Spkr	Dummy
A	0	13	13	26	38
B	9	0	1	6	0
C	27	0	0	13	25
D	37	30	20	0	29
E	0	30	20	33	31

e.

	PM	Treas	FS	Spkr	Dummy
A	0	0	0	13	25
B	22	0	1	6	0
C	40	0	0	13	25
D	50	30	20	0	29
E	0	17	7	20	18

f. *B* → Dummy (it's the only 0 in the Dummy column)
D → Speaker (it's the only 0 in the Speaker column)
E → PM (it's the only 0 in the row)
A → Treasurer, *C* → Foreign Secretary or
C → Treasurer, *A* → Foreign Secretary

g. PM not given to the top vote getter.
Because of the tie between *A* and *C*, either the Treasurer or the Foreign Secretary not given to the top vote getter. *D* definitely deserved Speaker, *B* definitely deserved Dummy.

h. Although apparently democratic, it 'rewards' candidates who are relatively unpopular over the range of positions (particularly candidate *E*) who only get a lot of votes for 1 position, and 'punishes' those (particularly candidate *A*) who do get lots of votes across the positions. One could argue, in this case, that *E* got PM as she could do the 'least damage' in that job!

4. a. Source = estuary, *E*; sink = lake, *L*

b.

Vertex	Inflow	Outflow
A	51	47
B	87	74
C	49	48
D	31	29
F	108	88
G	23	20

c. The outflow for every vertex is less than the inflow. This may be due to the death of some fish on their way upstream or it may be due to the fish slowing down, the further upstream they go.

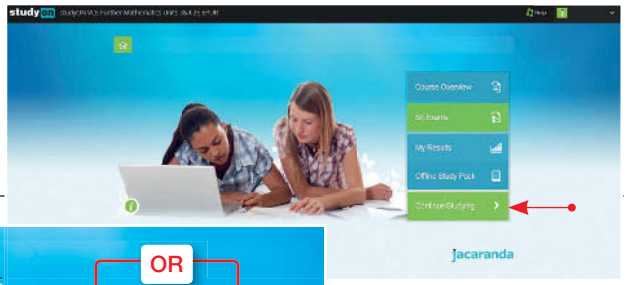
d. 162 fish/hour

e. 119 fish/hour

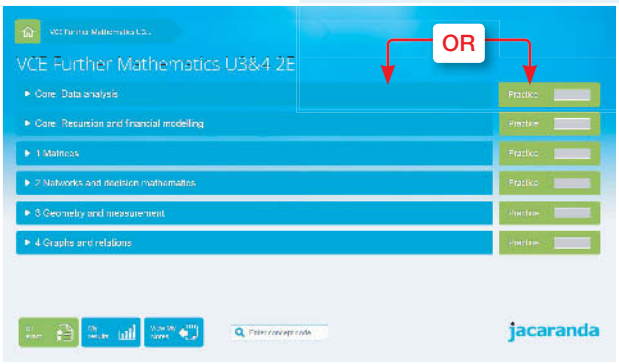
REVISION: AREA OF STUDY 2 Networks and decision mathematics

TOPICS 9 and 10

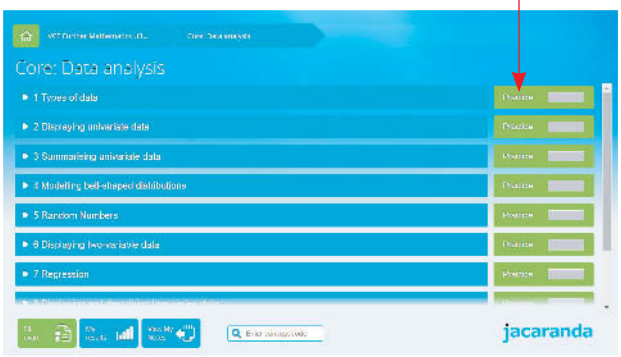
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



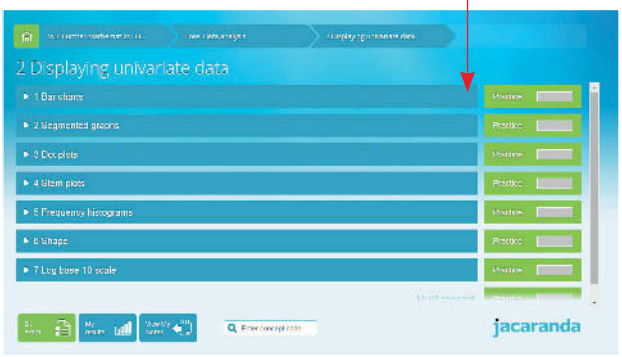
- Select your **course** *Further Mathematics VCE Units 3 & 4* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



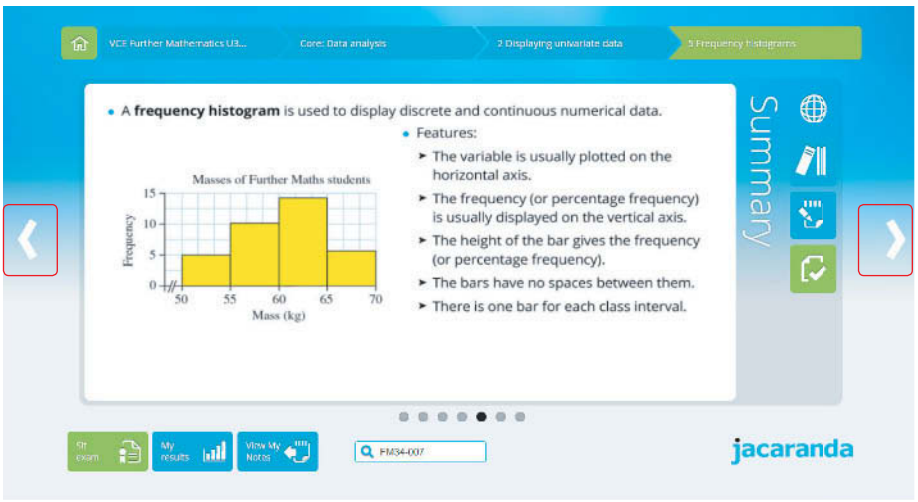
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 11
Geometry and measurement

11.1 Overview

11.1.1 Introduction

Geometry is the study of spatial relationships and is one of the two branches of pre-modern mathematics. The other is arithmetic. The first formal written mathematics was the book called 'The elements' by the Greek mathematician, Euclid. His work on two and three dimensional geometry is still the basis for the geometry studied in schools today.

Euclid's work was published in 400 BCE but there is evidence that an understanding of Geometry was known much earlier.

Ancient Egyptians needed an understanding of geometry to build the pyramids at Giza. The required measurements of the pyramids needed to be calculated prior to the commencement of construction to ensure consistent and accurate angles and side lengths.

The Bent pyramid at Dashur in Egypt was partially constructed but the dimensions of the base of the pyramid and angle of the sides meant that the weight of materials created problems and construction was halted for a time. When building resumed, the angle of the sloping sides was decreased and hence the pyramid has a bend. This could not have been achieved without some knowledge and application of geometry.

Practical applications of two and three dimensional geometry are still widely used today in such areas as construction, art and design and in computer graphics, including special effects in the making of movies.

LEARNING SEQUENCE

11.1 Overview
11.2 Review of properties of angles, triangles and polygons
11.3 Review of area and perimeter of composite shapes
11.4 Total surface area
11.5 Volume of prisms, pyramids and spheres
11.6 Similar figures
11.7 Similar triangles
11.8 Triangulation — similarity
11.9 Area and volume scale factors
11.10 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.1.2 Kick off with CAS

Calculating areas with CAS

We can use CAS to define formulas which allow us to quickly and efficiently calculate the areas of different shapes.

1. Use CAS to define and save the formula for calculating the area of a trapezium ($A = \frac{1}{2}(a + b) \times h$).
2. Use your formula to calculate the area of trapeziums with the following lengths and heights.
   a. $a = 7$ cm, $b = 15$ cm, $h = 11$ cm
   b. $a = 7.8$ cm, $b = 9.4$ cm, $h = 4$ cm
3. Use your formula to calculate the height of trapeziums with the following areas and lengths.
   a. $A = 63$ cm^2, $a = 6$ cm, $b = 12$ cm
   b. $A = 116.25$ mm^2, $a = 13$ mm, $b = 18$ mm
4. Use CAS to define and save the formula for calculating the area of a circle ($A = \pi r^2$).
5. Use your formula to calculate the area of circles with the following radii, correct to 2 decimal places.
   a. Radius = 4 cm b. Radius = 7.5 cm
6. Use your formula to calculate the radii of circles with the following areas, correct to 1 decimal place.
   a. Area = 181.5 cm^2 b. Area = 47.8 m^2

on Resources

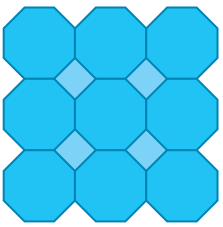
Please refer to the Resources section in the Prelims of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

11.2 Review of properties of angles, triangles and polygons

11.2.1 Geometry

Geometry is an important area of study. Many professions and tasks require and use geometrical concepts and techniques. Besides architects, surveyors and navigators, all of us use it in our daily lives — for example, to describe shapes of objects, directions on a car trip and space or position of a house. Much of this area of study is assumed knowledge gained from previous years of study.

In this module, we will often encounter problems in which some of the information we need is not clearly given. To solve the problems, some missing information will need to be deduced using the many common rules, definitions and laws of geometry. Some of the more important rules are presented in this topic.

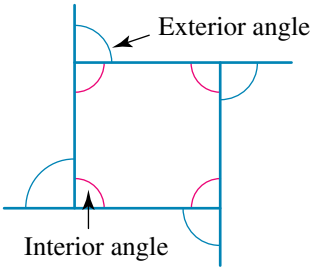


11.2.2 Interior angles of polygons

For a regular polygon (all sides and angles are equal) of n sides, the interior angle is given by $180° - \frac{360°}{n}$.

For example, for a square the interior angle is:

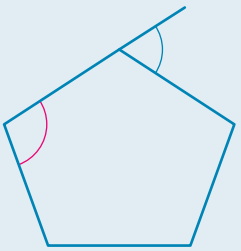
$$180° - \frac{360°}{4} = 180° - 90°$$
$$= 90°.$$



The exterior angle of a regular polygon is given by $\frac{360°}{n}$.

WORKED EXAMPLE 1

Calculate the interior and exterior angle of the regular polygon shown.



THINK

1. This shape is a regular pentagon, a 5-sided figure.
   Substitute $n = 5$ into the interior angle formula.
2. Substitute $n = 5$ into the exterior angle formula.
3. Write your answer.

WRITE

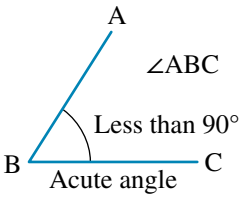
$$\text{Interior angle} = 180° - \frac{360°}{5}$$
$$= 180° - 72°$$
$$= 108°$$
$$\text{Exterior angle} = \frac{360°}{5}$$
$$= 72°$$

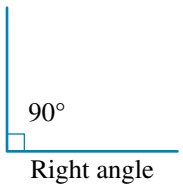
A regular pentagon has an interior angle of 108° and an exterior angle of 72°.

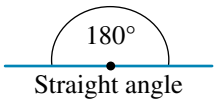
11.2.3 Geometry rules, definitions and notation rules

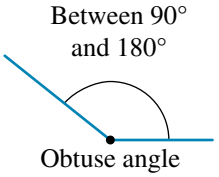
The following geometry rules and notation will be most valuable in establishing unknown values in the topics covered and revised in this module.

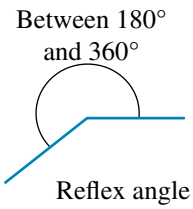
11.2.4 Definitions of common terms



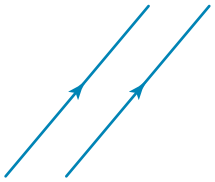


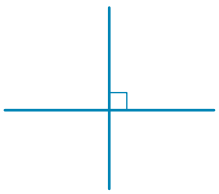


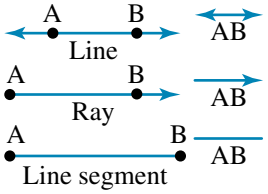




Reflex angle

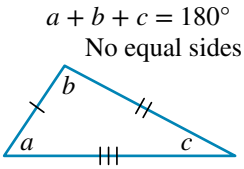

Parallel lines


Perpendicular lines

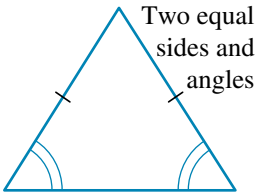


Line segment

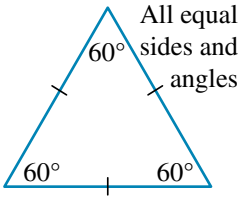
11.2.5 Some common notations and rules



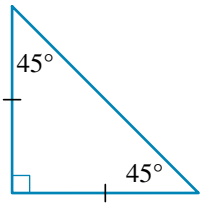
Scalene triangle



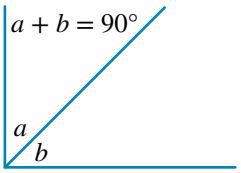
Isosceles triangle



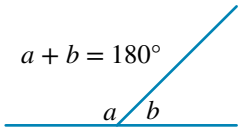
Equilateral triangle



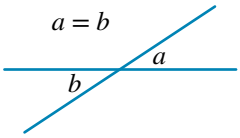
Right-angled
isosceles triangle



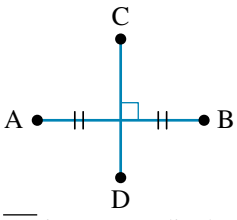
Complementary angles



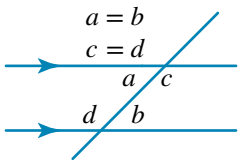
Supplementary angles



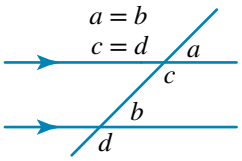
Vertically opposite
angles



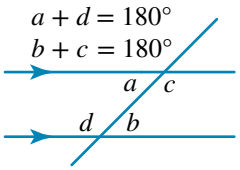
$\overline{CD}$ is a perpendicular
bisector of $\overline{AB}$.



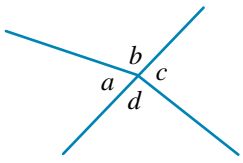
Alternate angles



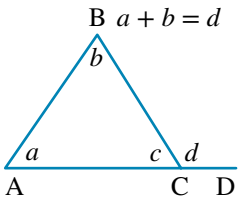
Corresponding angles



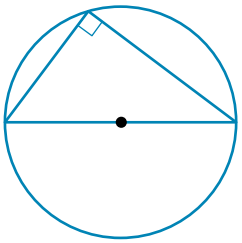
Co-interior angles



$a + b + c + d = 360°$

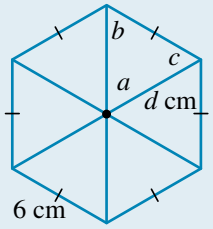


∠BCD is an exterior angle.


Right angle at the
circumference in
a semicircle

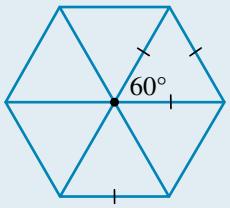
WORKED EXAMPLE 2

Calculate the values of the pronumerals in the polygon shown.



THINK

1. This shape is a regular hexagon. The angles at the centre are all equal.
2. The other two angles in the triangle are equal.



3. The 6 triangles are equilateral triangles; therefore, all sides are equal.

WRITE

$a = \frac{360°}{6}$

$= 60°$

$a + b + c = 180°$

$b = c$

So:

$60 + 2b = 180°$

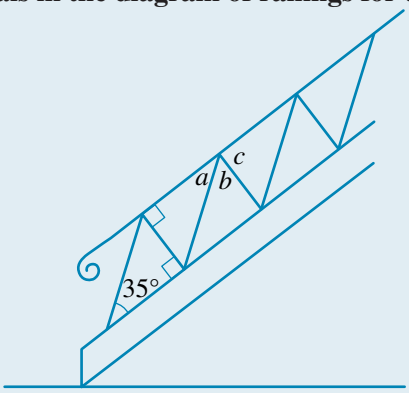
$b = 60°$

$c = 60°$

$d\,\text{cm} = 6\,\text{cm}$

WORKED EXAMPLE 3

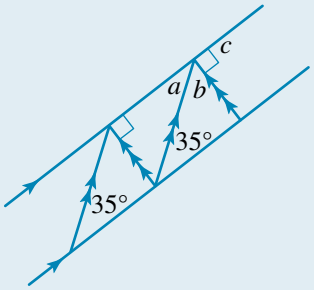
Calculate the missing pronumerals in the diagram of railings for the set of stairs shown.



THINK

1. Recognise that the top and bottom of the stair rails are parallel lines.

WRITE/DRAW

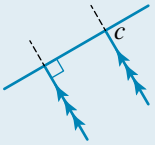


2. To find the unknown angle a, use the alternate angle law and the given angle.

Given angle $= 35°$

$a = 35°$

3. Using the corresponding angle law and the given right angle, recognise that the unknown angle c is a right angle.



$c = 90°$

4. Use the straight angle rule to find the unknown angle b.

$a + b + c = 180°$

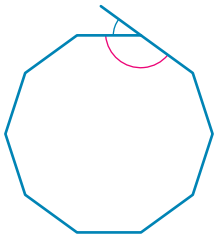
$35° + b + 90° = 180°$

$b = 180° - 125°$

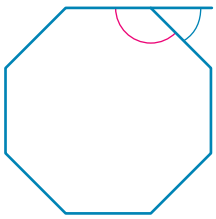
$= 55°$

Exercise 11.2 Review of properties of angles, triangles and polygons

1. **WE1** Calculate the interior and exterior angles for a regular nonagon.
2. Calculate the interior and exterior angles for the following regular polygon.



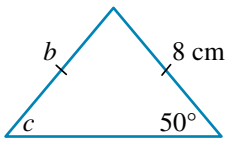
3. Calculate the interior and exterior angles for each of the following regular polygons.
   - **a.** Equilateral triangle
   - **b.** Regular quadrilateral
   - **c.** Hexagon
   - **d.**



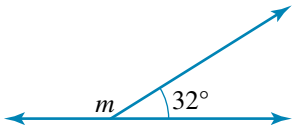
   - **e.** Heptagon

For questions **4–7** calculate the value of the pronumeral in the following figures.

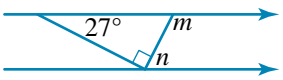
4. **WE2**



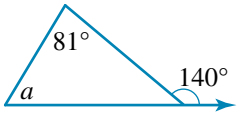
5.



6. **WE3**

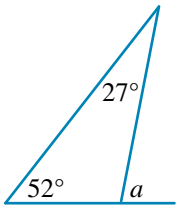


7.

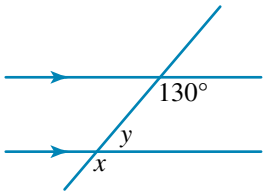


8. Calculate the value of the pronumerals in the following figures.

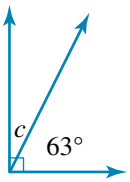
a.



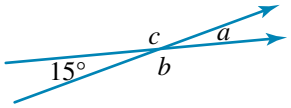
b.



c.

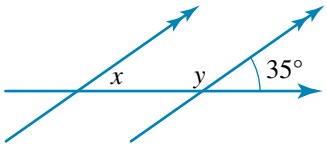


d.

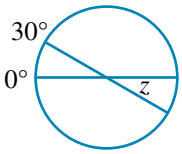


9. Calculate the value of the pronumerals in the following figures.

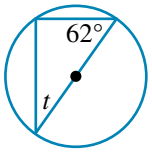
a.



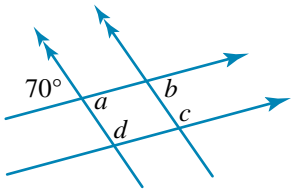
b.



c.



d.



10. Name the regular polygon that has the given angle(s).

a. Interior angle of 108°, exterior angle of 72°

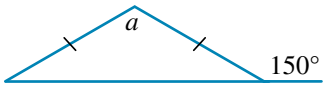
b. Interior angle of 150°, exterior angle of 30°

c. Interior angle of 135°, exterior angle of 45°

d. Interior angle of 120°

e. Exterior angle of 120°

11. MC In the figure, the value of a is:



A. 30° **B.** 75° **C.** 90°

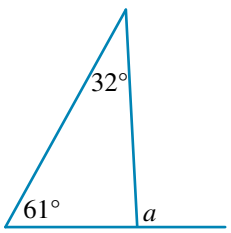
D. 120° **E.** 150°

12. MC An isosceles triangle has a known angle of 50°. The largest possible angle for this triangle is:

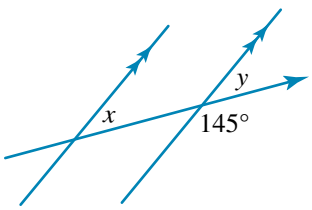
A. 80° **B.** 130° **C.** 90° **D.** 65° **E.** 50°

13. Find the values of the pronumerals in each of the following figures.

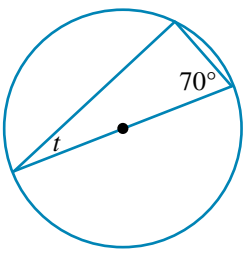
a.



b.

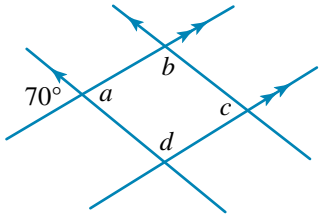


c.

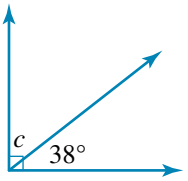


14. Find the values of the pronumerals in each of the following figures.

a.

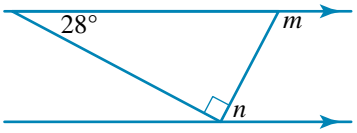


b.

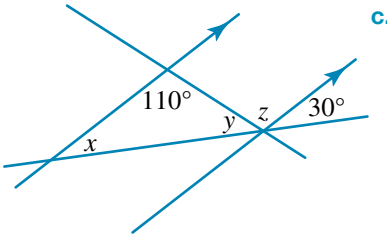


15. Find the values of the pronumerals in each of the following figures.

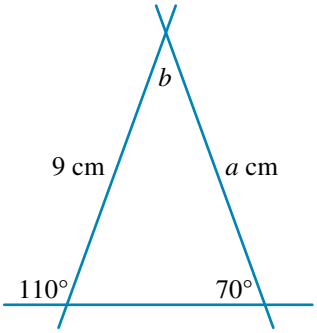
a.



b.

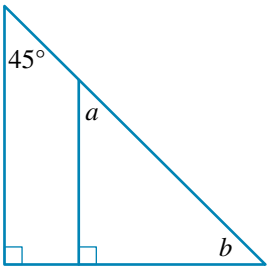


c.

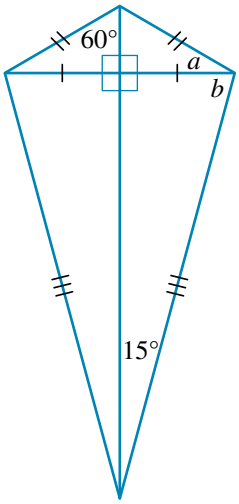


16. Find the values of the pronumerals in each of the following figures.

a.

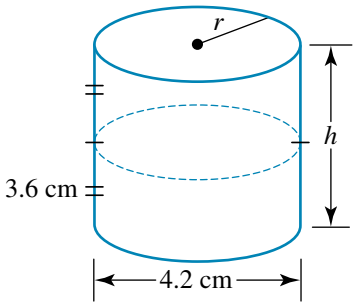


b.

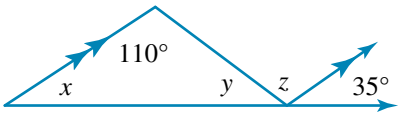


17. Calculate the unknown pronumerals.

a.

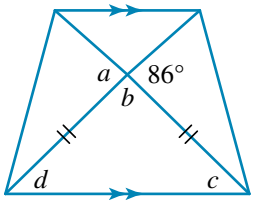


b.

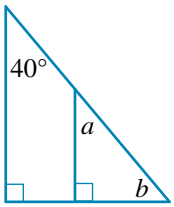


18. Find the values of the pronumerals.

a.

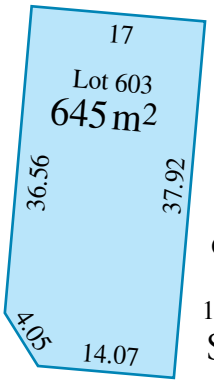


b.

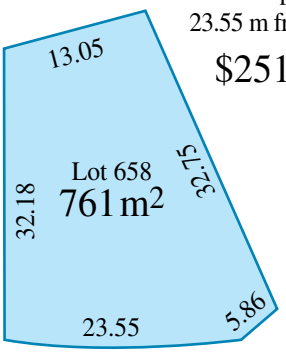


11.3 Review of area and perimeter of composite shapes

Much of our world is described by **area** (the amount of space enclosed by a closed figure) and **perimeter** (the distance around a closed figure).



Corner block with wide 17 m frontage $147 000



Corner block with expansive 23.55 m frontage $251 000

Some examples are the area of a house block, the fencing of a block of land, the size of a bedroom and the amount of paint required to cover an object. In this section we will review the more common shapes.

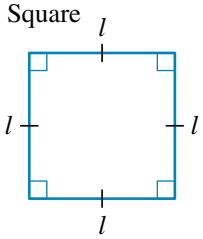
11.3.1 Perimeter

Perimeter is the distance around a closed figure.

Some common rules are:

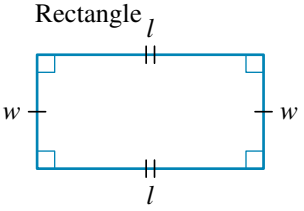
1. For squares, the perimeter $= 4l$

Square



2. For rectangles, the perimeter $= 2(l + w)$

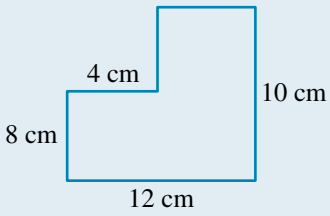
Rectangle



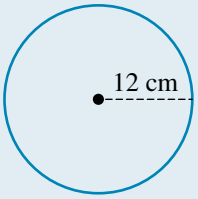
3. Circumference is the distance around a circle; circumference $= 2\pi r$

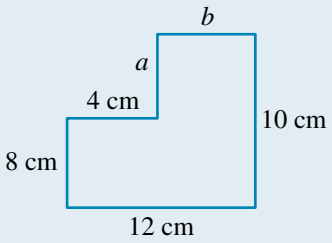
WORKED EXAMPLE 4

a. Find the perimeter of the closed figure shown.



b. Calculate the circumference of the following shape correct to 2 decimal places.



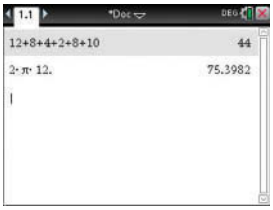
THINK	WRITE
a. 1. To find the perimeter we need the lengths of each side around the closed figure. So we need to find a and b.	
2. Find the length of a.	$a = 10 - 8$ $= 2\text{ cm}$
3. Find the length of b.	$b = 12 - 4$ $= 8\text{ cm}$
4. Calculate the perimeter by adding all the lengths around the outside of the figure.	Perimeter $= 12 + 8 + 4 + 2 + 8 + 10$ $= 44\text{ cm}$
5. Write your answer.	The perimeter of the closed figure is 44 cm.
b. 1. Write the formula for the circumference of a circle.	$C = 2\pi r$
2. Substitute the radius into the equation, using the π key on your calculator.	$C = 2 \times \pi \times 12$ $= 75.398\,223\,69$ $\approx 75.40\text{ cm}$
3. Write the answer correct to 2 decimal places.	The circumference is 75.40 cm.

TI | THINK

a, b 1. On a Calculator page, complete the entry lines as:
- **a.** $12+8+4+2+8+10$, press ENTER.
- **b.** $2 \times \pi \times 12$, press ENTER.

3. The answer appears on the screen.

WRITE



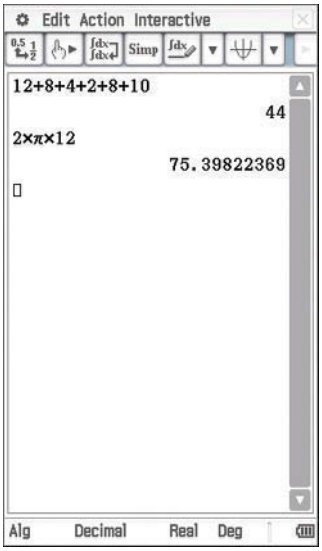
a. The perimeter of the closed figure is 44 cm.
b. The circumference of the circle is 75.40, correct to 2 decimal places.

CASIO | THINK

1. On a MAIN screen, complete the entry lines as:
- **a.** $12+8+4+2+8+10$, press EXE.
- **b.** $2 \times \pi \times 12$, press EXE.

3. The answer appears on the screen.

WRITE

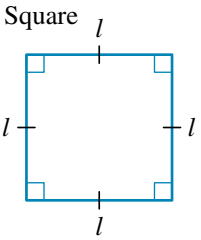


a. The perimeter of the closed figure is 44 cm.
b. The circumference of the circle is 75.40, correct to 2 decimal places.

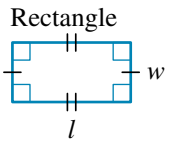
11.3.2 Area of common shapes

The areas of shapes commonly encountered are:

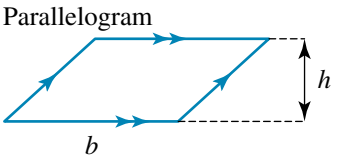
1. Area of a square:
$$A = \text{length}^2 = l^2$$



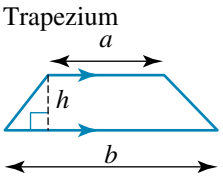
2. Area of a rectangle:
$$A = \text{length} \times \text{width} = l \times w$$



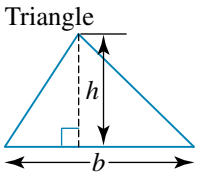
3. Area of a parallelogram:
$$A = \text{base} \times \text{height} = b \times h$$



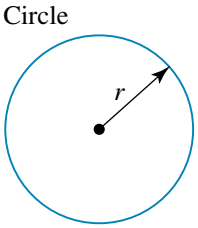
4. Area of a trapezium:
$$A = \frac{1}{2}(a+b) \times h$$



5. Area of a triangle: $A = \frac{1}{2} \times b \times h$



6. Area of a circle: $A = \pi r^2$

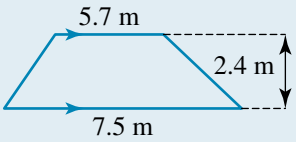


Area is measured in mm^2, cm^2, m^2, km^2 and hectares.

$$1 \text{ hectare} = 100 \text{ m} \times 100 \text{ m}$$
$$= 10\,000 \text{ m}^2$$

WORKED EXAMPLE 5

Calculate the area of the garden bed given in the diagram (correct to the nearest square metre).



THINK

a. 1. The shape of the garden is a trapezium. Use the formula for area of a trapezium. Remember that the lengths of the two parallel sides are a and b, and h is the perpendicular distance between the two parallel sides.

2. Substitute and evaluate.

3. Write your answer.

WRITE

Area of a trapezium $= \frac{1}{2}(a + b) \times h$

$a = 7.5 \quad b = 5.7 \quad h = 2.4$

$$A = \frac{1}{2}(7.5 + 5.7) \times 2.4$$
$$= \frac{1}{2} \times 13.2 \times 2.4$$
$$= 15.84 \text{ m}^2$$

The area of the garden bed is approximately 16 square metres.

11.3.3 Composite areas

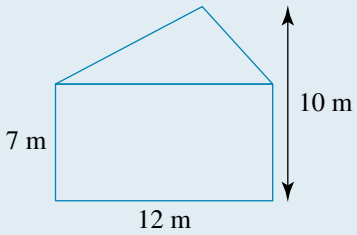
Often a closed figure can be identified as comprising two or more different common figures. Such figures are called **composite figures**. The area of a composite figure is the sum of the areas of the individual common figures.

Area of a composite figure = sum of the areas of the individual common figures

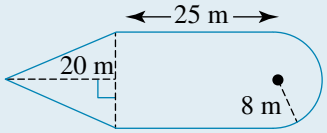
$$A_{\text{comopsite}} = A_1 + A_2 + A_3 + A_4 + \cdots$$

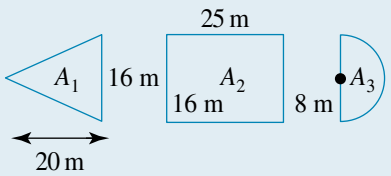
WORKED EXAMPLE 6

a. Calculate the area of the composite shape shown.



b. Calculate the area of the hotel foyer from the plan given (correct to the nearest square metre).



THINK	WRITE
a. 1. Identify the two shapes.	The shapes are a rectangle and a triangle.
2. Calculate the area of the rectangle.	$A_{\text{rectangle}} = l \times w$ $= 12 \times 7$ $= 84$
3. Calculate the area of the triangle.	$A_{\text{triangle}} = \frac{1}{2} b \times h$ $= \frac{1}{2} \times 12 \times (10 - 7)$ $= 6 \times 3$ $= 18$
4. Sum the two areas together.	$A_{\text{total}} = A_{\text{rectangle}} + A_{\text{triangle}}$ $= 84 + 18$ $= 102$
5. Write your answer.	The total area of the composite shape is $102\,\text{m}^2$.
b. 1. The shape is composite and needs to be separated into two or more common shapes: in this case, a rectangle, a triangle and half of a circle.	 Area of foyer $= A_1 + A_2 + A_3$

2. Find the area of each shape. (The width of the rectangle and the base of the triangle is twice the radius of the circle, that is, 16 metres.)

A_1 = area of triangle

$= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 16 \times 20$

$= 160\ \text{m}^2$

A_2 = area of rectangle

$= l \times w$

$= 25 \times 16$

$= 400\ \text{m}^2$

A_3 = area of half of a circle

$= \frac{1}{2} \times \pi \times r^2$

$= \frac{1}{2} \times \pi \times 8^2$

$= 100.53\ \text{m}^2$

3. Add together all three areas for the composite shape.

$\text{Area of foyer} = A_1 + A_2 + A_3$

$= 160 + 400 + 100.53$

$= 660.53\ \text{m}^2$

4. Write your answer.

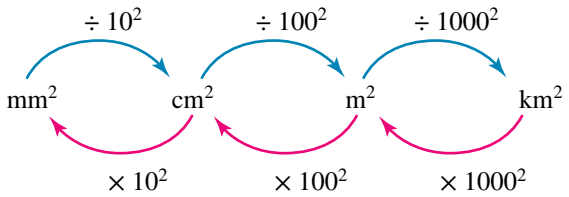
The area of the hotel foyer is approximately 661 m^2.

11.3.4 Conversion of units of area

Often the units of area need to be converted, for example, from **cm^2** to **m^2** and vice versa.

1. **To convert to smaller units, for example, m^2 to cm^2, multiply ($\times$).**
2. **To convert to larger units, for example, mm^2 to cm^2, divide ($\div$).**

Some examples are:



a. $1\ \text{cm}^2 = 10\ \text{mm} \times 10\ \text{mm}$

$= 100\ \text{mm}^2$

b. $1\ \text{m}^2 = 100\ \text{cm} \times 100\ \text{cm}$

$= 10\,000\ \text{cm}^2$

c. $1\ \text{km}^2 = 1000\ \text{m} \times 1000\ \text{m}$

$= 1\,000\,000\ \text{m}^2$

d. 1 hectare = $10\,000\ \text{m}^2$

WORKED EXAMPLE 7

Convert 1.12 m^2 to square centimetres (cm^2).

THINK

1. To convert from m^2 to cm^2, multiply by 100^2 or 10 000.
2. Write your answer.

WRITE

$1.12\ m^2 = 1.12 \times 10\,000\ cm^2$

$= 11\,200\ cm^2$

$1.12\ m^2$ is equal to 11 200 square centimetres (cm^2).

WORKED EXAMPLE 8

Convert 156 000 m^2 to:
a. km^2 **b. hectares.**

THINK

a. 1. To convert from m^2 to km^2 divide by 1000^2 or 1 000 000.

2. Write the answer in correct units.

b. 1. 10 000 m^2 equals 1 hectare. To convert from m^2 to hectares, divide by 10 000.

2. Write the answer.

WRITE

a. $156\,000\ m^2 = 156\,000 \div 1\,000\,000\ km^2$

$= 0.156\ km^2$

$156\,000\ m^2 = 0.156$ square kilometres (km^2)

b. $156\,000\ m^2 = \frac{156\,000}{10\,000}$ hectares

$= 15.6$ hectares

$156\,000\ m^2 = 15.6$ hectares

Resources

Interactivity: Area and perimeter (int-6274)
Interactivity: Conversion of units of areas (int-6269)

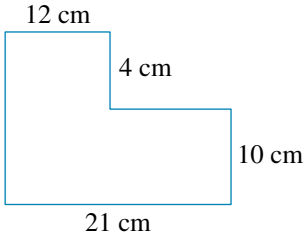
studyon

Units 3 & 4 > AOS M3 > Topic 1 > Concept 6

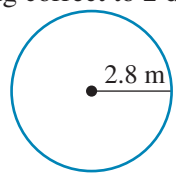
Linear dimensions in 2D and 3D Summary screen and practice questions

Exercise 11.3 Review of area and perimeter of composite shapes

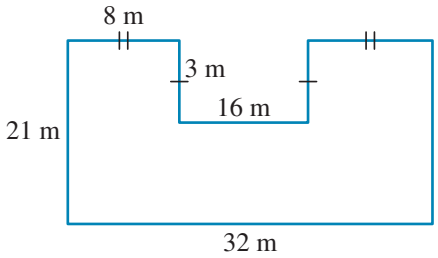
1. a. WE4 Calculate the perimeter of the closed figure shown.



b. Calculate the circumference of the following correct to 2 decimal places.

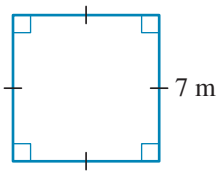


2. Calculate the perimeter of the closed figure shown.

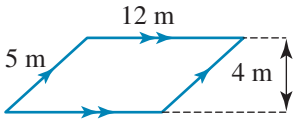


3. Calculate the perimeters of the following figures (correct to the nearest whole units).

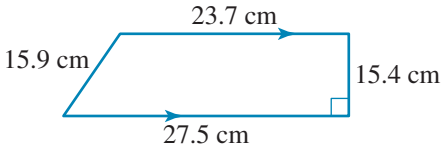
a.



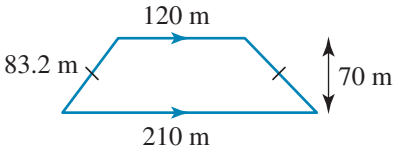
b.



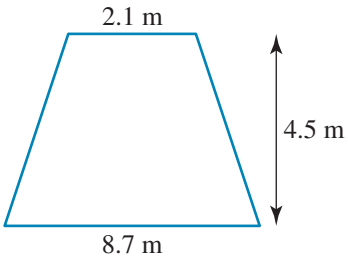
c.



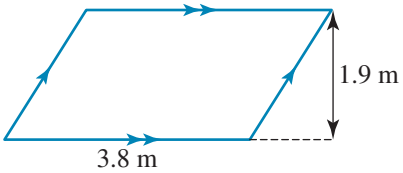
d.



4. WE5 Calculate the area of the backyard given in the diagram (correct to the nearest square metre).

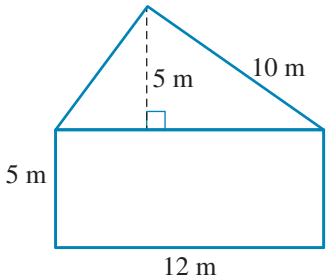


5. Calculate the area of the shape shown (correct to the nearest square metre).

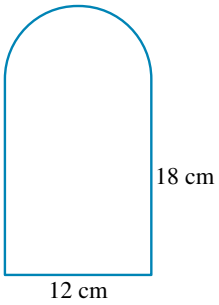


6. Calculate the areas of the closed figures in question **3**.

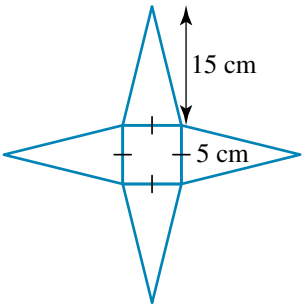
7. WE6 **a.** Calculate the area of the composite shape shown (correct to the nearest square metre).



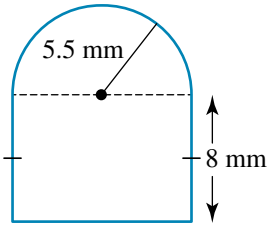
b. Calculate the perimeter of the following shape (correct to 1 decimal place).



8. a. Calculate the area of the composite shape shown.

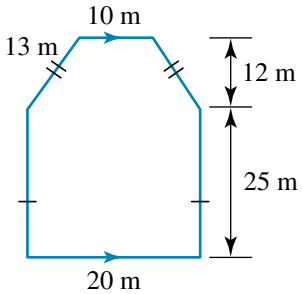


b. Find the area of the following figure (correct to the nearest whole unit).

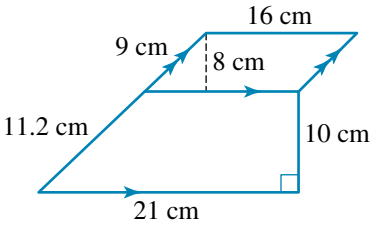


9. Calculate the areas of the following figures (correct to 1 decimal place).

a.



b.



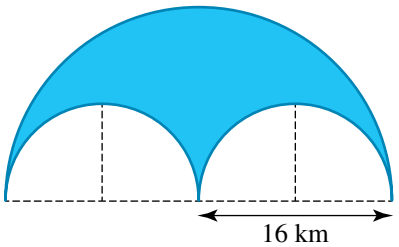
10. Calculate the perimeters of the closed figures in question **9**.

11. **WE7** Convert 2.59 m^2 to square centimetres (cm^2).

12. Convert 34.56 cm^2 to square millimetres (mm^2).

13. **WE8** Convert 2 500 000 m^2 to square kilometres^2 (km^2).

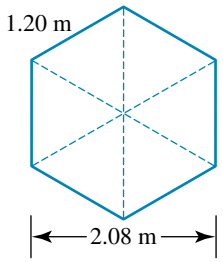
14. Calculate the area of the shaded shape shown at right, correct to 2 decimal places.



15. Convert the following areas to the units given in brackets.

a. 20 000 mm^2 (cm^2)
b. 320 000 cm^2(m^2)
c. 0.035 m^2 (cm^2)
d. 0.035 m^2 (mm^2)
e. 2 500 000 m^2 (km^2)
f. 357 000 m^2 (hectares)
g. 2 750 000 000 mm^2 (m^2)
h. 0.000 06 km^2 (m^2)

16. Find the area of the regular hexagon shown in the diagram (correct to 2 decimal places, in m^2).

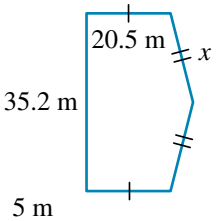


17. MC The perimeter of the enclosed figure shown is 156.6 metres.

The unknown length, x, is closest to:

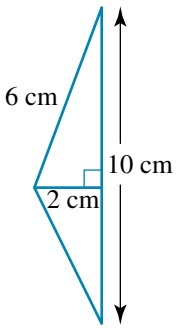
A. 20.5 m **B.** 35.2 m **C.** 40.2 m

D. 80.4 m **E.** 90.6 m

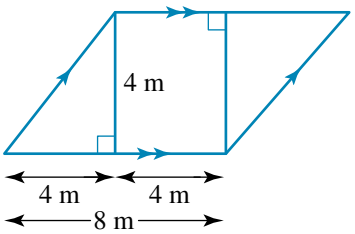


18. Find the area of each of the following figures.

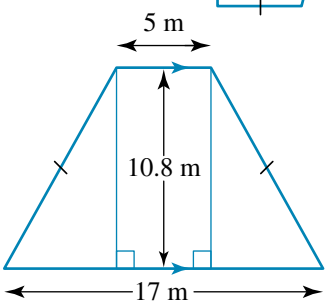
a.



b.



c.



19. Calculate the perimeter of each of the shapes in question **18**. Give your answers correct to the nearest whole number.

20. Convert the following areas to the units given in the brackets.

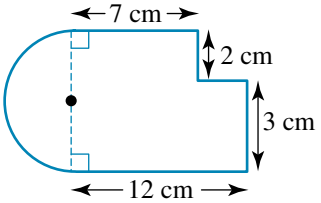
a. 45 000 cm^2 (m^2)

b. 0.71 m^2 (mm^2)

c. 216 000 m^2 (hectares)

d. 0.0737 km^2 (m^2)

21. MC The perimeter of the figure shown, in centimetres, is:



A. 34

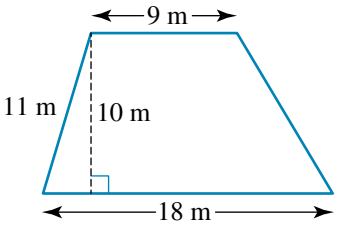
B. $24 + 5\pi$

C. $24 + 2.5\pi$

D. $29 + 5\pi$

E. $29 + 2.5\pi$

22. Find the perimeter, in millimetres, of the shape at right. Give your answer correct to the nearest whole number.



23. On the set of a western movie, a horse is tied to a railing outside a saloon bar. The railing is 2 metres long; the reins are also 2 metres long and are tied at one of the ends of the railing.

a. Draw a diagram of this situation.

b. To how much area does the horse now have access (correct to 1 decimal place)? The reins are now tied to the centre of the railing.

c. Draw a diagram of this situation.

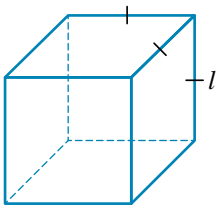
d. To how much area does the horse have access (correct to 1 decimal place)?

11.4 Total surface area

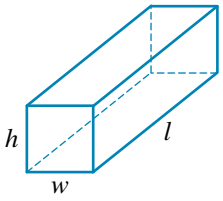
The **total surface area** (TSA) of a solid object is the sum of the areas of the surfaces.

In some cases we can use established formulas of very common everyday objects. In other situations we will need to derive a formula by using the net of an object.

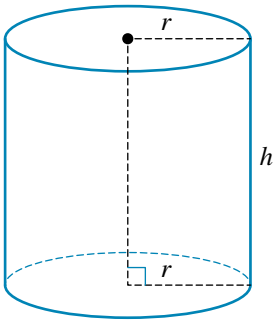
11.4.1 Total surface area formulas of common objects



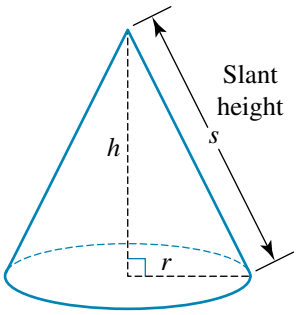
Cube:
TSA = $6l^2$



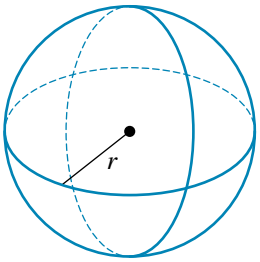
Cuboid:
TSA = $2(lw + lh + wh)$



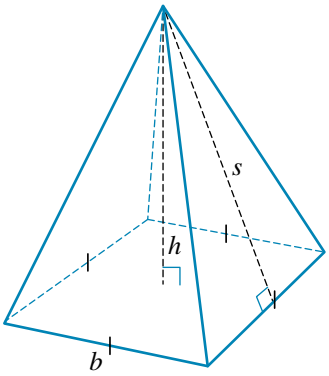
Cylinder:
TSA = $2\pi r\,(r + h)$



Right cone:
TSA = $\pi r\,(r + s)$,
where s is the slant height



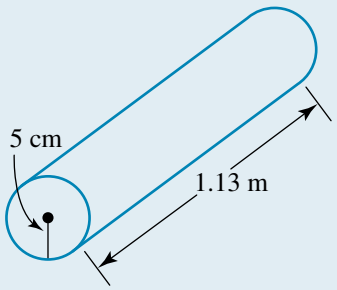
Sphere:
TSA = $4\pi r^2$



Right square pyramid:
TSA = $4b + 2bs$,
where s is the slant height

WORKED EXAMPLE 9

Calculate the total surface area of a poster tube with a length of 1.13 m and a radius of 5 cm. Give your answer correct to the nearest 100 cm^2.



THINK

1. A poster tube is a cylinder. Express all dimensions in centimetres.
   Remember 1 metre equals 100 centimetres.

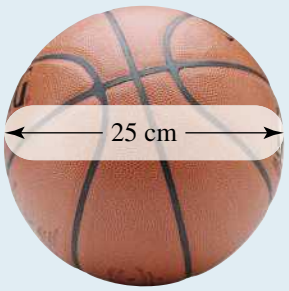
WRITE

TSA of a cylinder = $2\pi r(r + h)$
Radius, $r = 5$ cm
Height, $h = 1.13$ m
$= 113$ cm

2. Substitute and evaluate. Remember BODMAS.	$\text{TSA} = 2 \times \pi \times 5(5 + 113)$ $= 2 \times \pi \times 5 \times 118$ ≈ 3707.08
3. Write your answer.	The total surface area of a poster tube is approximately 3700 cm^2.

WORKED EXAMPLE 10

Calculate the total surface area of a size 7 basketball with a diameter of 25 cm. Give your answer correct to the nearest 10 cm^2.



THINK	WRITE
1. Use the formula for the total surface area of a sphere. Use the diameter to find the radius of the basketball and substitute into the formula.	TSA of sphere $= 4\pi r^2$ Diameter $= 25$ cm Radius $= 12.5$ cm $\text{TSA} = 4 \times \pi \times 12.5^2$ ≈ 1963.495
2. Write your answer.	Total surface area of the ball is approximately 1960 cm^2.

WORKED EXAMPLE 11

A die used in a board game has a total surface area of 1350 mm^2. Calculate the linear dimensions of the die (correct to the nearest millimetre).

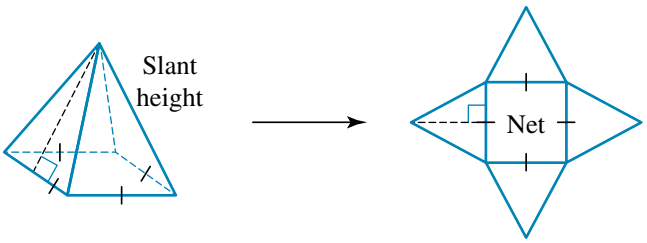
THINK	WRITE
1. A die is a cube. We can substitute into the total surface area of a cube to determine the dimension of the cube. Divide both sides by 6.	$\text{TSA} = 6 \times l^2$ $= 1350 \text{ mm}^2$ $1350 = 6 \times l^2$ $l^2 = \frac{1350}{6}$ $= 225$
2. Take the square root of both sides to find l.	$l = \sqrt{225}$ $= 15$ mm
3. Write your answer.	The dimensions of the die are 15 mm × 15 mm × 15 mm.

11.4.2 Total surface area using a net

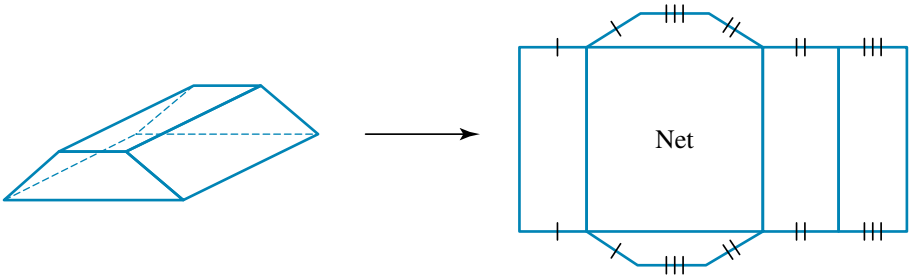
If the object is not a common object or a variation of one, such as an open cylinder, then it is easier to generate the formula from first principles by constructing a net of the object.

A net of an object is a plane figure that represents the surface of a 3-dimensional object.

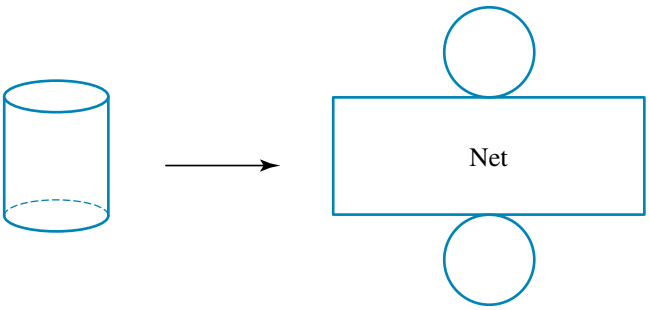
Right squre pyramid



Trapezoidal prism

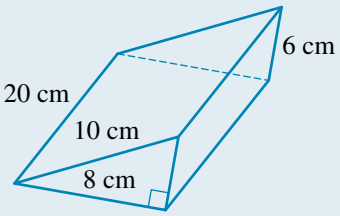


Right cylinder



WORKED EXAMPLE 12

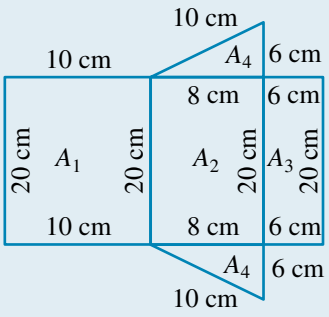
Calculate the total surface area of the triangular prism shown in the diagram.



THINK

1. Form a net of the triangular prism, transferring all the dimensions to each of the sides of the surfaces.

WRITE/DRAW



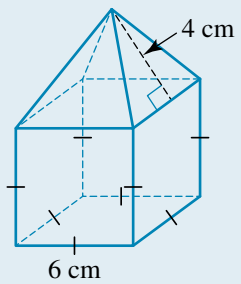
2. Identify the different-sized common figures and set up a sum of the surface areas. The two triangles are the same.	$\text{TSA} = A_1 + A_2 + A_3 + 2 \times A_4$ $A_1 = l_1 \times w_1$ $= 20 \times 10$ $= 200\,\text{cm}^2$ $A_2 = l_2 \times w_2$ $= 20 \times 8$ $= 160\,\text{cm}^2$ $A_3 = l_3 \times w_3$ $= 20 \times 6$ $= 120\,\text{cm}^2$ $A_4 = \frac{1}{2} \times b \times h$ $= \frac{1}{2} \times 8 \times 6$ $= 24\,\text{cm}^2$
3. Sum the areas.	$\text{TSA} = A_1 + A_2 + A_3 + 2 \times A_4$ $= 200 + 160 + 120 + 2 \times 24$ $= 528\ \text{cm}^2$
4. Write your answer.	The total surface area of the triangular prism is $528\ \text{cm}^2$.

Total surface area of composite solids

When calculating the total surface area of composite solids, be careful to include only the surfaces that form the outer part of the solid.

WORKED EXAMPLE 13

Calculate the total surface area of the following solid.



THINK	WRITE
1. Identify the components of the composite solid.	The composite solid consists of a cube of length 6 cm and a right square pyramid with base length 6 cm and slant height 4 cm.
2. Use the formulas to calculate the surface areas of the component parts. Remember to exclude internal surfaces.	TSA of a cube (minus top) $= 5l^2$ $= 5 \times 6^2$ $= 5 \times 36$ $= 180\,\text{cm}^2$ TSA of a pyramid (minus bottom) $= 2bs$ $= 2 \times 6 \times 4$ $= 48\,\text{cm}^2$

3. Add the areas together to obtain the total surface area. $180 + 48 = 228\,\text{cm}^2$
4. Write your answer. The total surface area of the solid is $228\,\text{cm}^2$.

on Resources

Interactivity: Surface area (int-6477)

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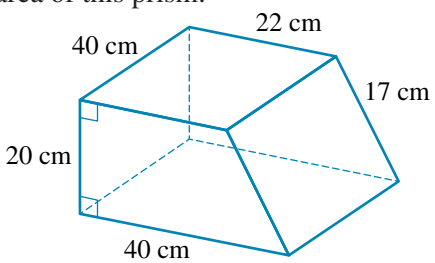
Units 3 & 4 > AOS M3 > Topic 1 > Concept 7

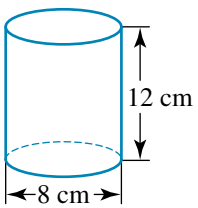
Surface area and volume Summary screen and practice questions

Exercise 11.4 Total surface area

1. **WE9** Calculate the total surface area of a closed cylinder with a radius of 1.2 cm and a height of 6 cm. Give your answer correct to 1 decimal place.
2. Calculate the total surface area of a closed cone with a radius of 7 cm and a slant height of 11 cm. Give your answer correct to 1 decimal place.
3. **WE10** Find the total surface area of a sphere with a radius of 0.8 m. Give your answer correct to 1 decimal place.
4. Find the total surface area of a cube with side length 110 cm.
5. **WE11** Find the unknown length of a cube, given that the total surface area is $255\,\text{m}^2$. Give your answer correct to 1 decimal place.
6. Find the unknown radius of a sphere, given that the total surface area is $785.12\,\text{cm}^2$. Give your answer correct to 1 decimal place.
7. **WE12** Calculate the total surface area of this prism.



8. Calculate the surface area of an open cylindrical can that is 12 cm high and 8 cm in diameter (correct to 1 decimal place) by completing the following steps.
   a. Form a net of the open cylinder, transferring all the dimensions to each of the surfaces.
   b. Identify the different-sized common figures and set up a sum of the surface areas. The length of the rectangle is the circumference of the circle.
   c. Add the areas.
   d. Write your answer.

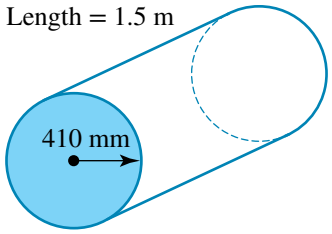


9. Calculate the total surface area for each of the solids **a** to **d** from the following information. Give answers correct to 1 decimal place where appropriate.

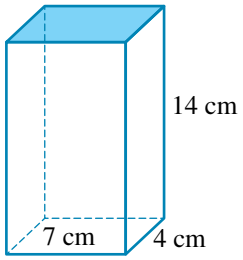
- **a.** A cube with side lengths of 135 mm
- **b.** A cuboid with dimensions of 12 m × 5 m × 8 m ($l \times w \times h$)
- **c.** A sphere with a radius of 7.1 cm
- **d.** An opened cylinder with a diameter of 100 mm and height of 30 mm

10. Calculate the total surface area of the objects given in the diagrams. Give answers correct to 1 decimal place.

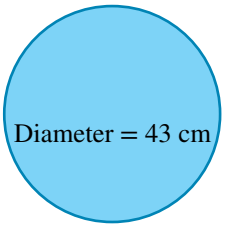
a. Length = 1.5 m



b.



c.

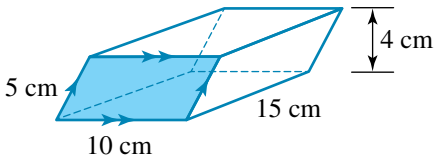


11. Calculate the unknown dimensions, given the total surface area of the objects. Give answers correct to 1 decimal place.

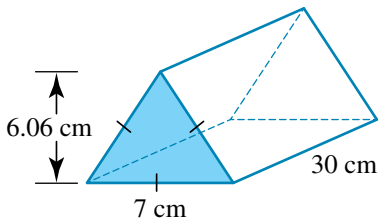
- **a.** Length of a cube with a total surface area of 24 m^2
- **b.** The radius of a sphere with a total surface area of 633.5 cm^2
- **c.** Length of a cuboid with width of 12 mm, height of 6 cm and a total surface area of 468 cm^2
- **d.** Diameter of a playing ball with a total surface area of 157 630 cm^2

12. Calculate the total surface areas for the objects given in the diagrams. Give answers correct to 1 decimal place.

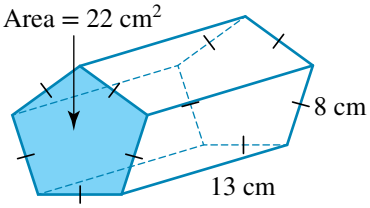
a.



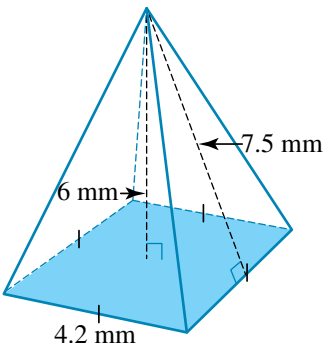
b.



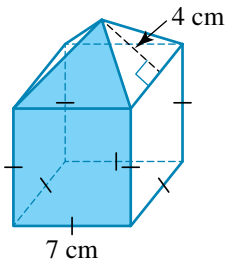
c. Area = 22 cm^2



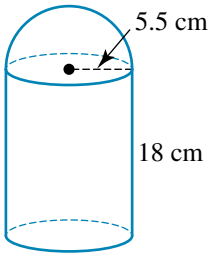
d.



13. **WE13** Calculate the total surface area of the following solid.

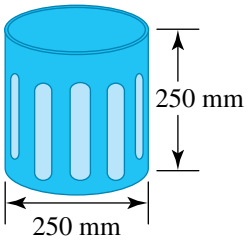


14. Calculate the total surface area of the following solid, correct to 2 decimal places.

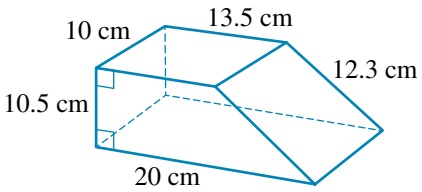


15. Calculate the total surface area of each of the objects in the diagrams. Give answers correct to 1 decimal place.

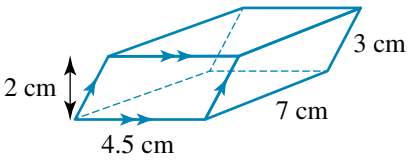
a.



b.

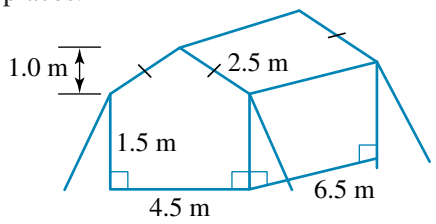


c.



16. A concrete swimming pool is a cuboid with the following dimensions: length of 6 metres, width of 4 metres and depth of 1.3 metres. What surface area of tiles is needed to line the inside of the pool? (Give your answer in m^2 and cm^2.)

17. What is the total area of canvas needed for the tent (including the base) shown in the diagram? Give your answer correct to 2 decimal places.



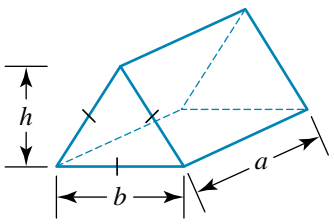
18. MC The total surface area of a 48 mm diameter ball used in a game of pool is closest to:

A. 1810 mm^2
B. 2300 mm^2
C. 7240 mm^2
D. 28 950 mm^2
E. 115 800 mm^2

19. MC The total surface area of a square-based pyramid with a base length of 3.55 cm and a slant height of 6.71 cm is closest to:

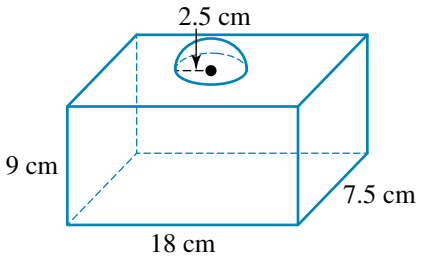
A. 107.88 cm^2 **B.** 60.24 cm^2 **C.** 60.25 cm^2 **D.** 92.66 cm^2 **E.** 92.67 cm^2

20. MC The formula for the total surface area for the object shown is:



A. $\frac{1}{2}abh$
B. $2 \times \frac{1}{2}bh + ab + 2 \times ah$
C. $3\left(\frac{1}{2}bh + ab\right)$
D. $\frac{1}{2}bh + 3ab$
E. $bh + 3ab$

21. Calculate the total surface area of the following solid, correct to 2 decimal places.



22. A baker is investigating the best shape for a loaf of bread. The shape with the smallest surface area stays freshest. The baker has come up with two shapes: a rectangular prism with a 12 cm square base and a cylinder with a round end that has a 14 cm diameter.
    a. Which shape stays fresher if they have the same overall length of 32 cm?
    b. What is the difference between the total surface areas of the two loaves of bread?

11.5 Volume of prisms, pyramids and spheres

The most common volumes considered in the real world are the volumes of prisms, pyramids, spheres and objects that are a combination of these. For example, people who rely on tank water need to know the capacity (volume) of water that the tank is holding.

Volume is the amount of space occupied by a 3-dimensional object.

The units of volume are mm^3 (cubic millimetres), cm^3 (cubic centimetres or cc) and m^3 (cubic metres).

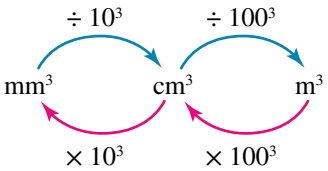
$$1000\ mm^3 = 1\ cm^3$$
$$1\,000\,000\ cm^3 = 1\ m^3$$

Another measure of volume is the litre, which is used primarily for quantities of liquids but also for capacity, such as the capacity of a refrigerator or the size of motor car engines.

$$1\ \text{litre} = 1000\ cm^3$$
$$1000\ \text{litres} = 1\ m^3$$

11.5.1 Conversion of units of volume

Often the units of volume need to be converted, for example, from cm^3 to m^3 and vice versa.



WORKED EXAMPLE 14

Convert 1.12 cm^3 to mm^3.

THINK

1. To convert from cm^3 to mm^3 multiply by 10^3 or 1000. (That is, 1 cm^3 equals 1000 mm^3.)
2. Write the answer in the correct units.

WRITE

$1.12\ cm^3 = 1.12 \times 1000\ mm^3$
$= 1120\ mm^3$
$1.12\ cm^3$ is equal to $1120\ mm^3$.

WORKED EXAMPLE 15

Convert 156 000 cm^3 to:
a. m^3 **b. litres.**

THINK

a. 1. To convert from cm^3 to m^3 divide by 100^3 or 1 000 000. (That is, 1 000 000 cm^3 equals 1 m^3.)

2. Write the answer in correct units.

b. 1. 1000 cm^3 is equivalent to 1 litre; therefore, to convert from cm^3 to litres, divide by 1000.

2. Write the answer.

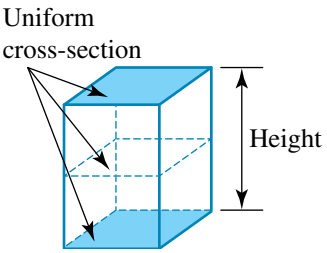
WRITE

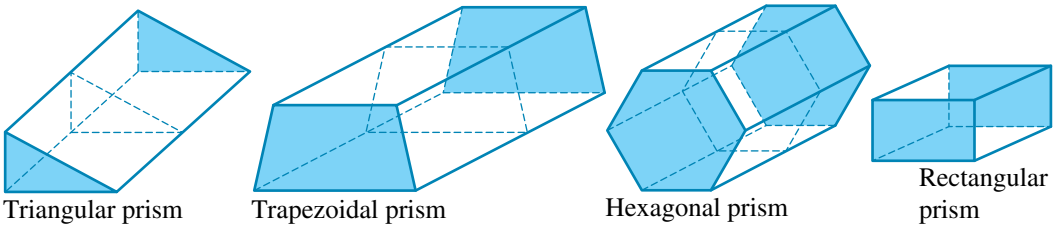
a. $156\,000\ cm^3 = 156\,000 \div 1\,000\,000\ m^3$
$= 0.156\ m^3$
$156\,000\ cm^3 = 0.156$ cubic metres (m^3)

b. $156\,000\ cm^3 = \frac{156\,000}{1000}$ litres
$= 156$ litres
$156\,000\ cm^3 = 156$ litres

11.5.2 Volume of prisms

A prism is a **polyhedron** with a uniform cross-section.
It is named in accordance with its uniform cross-sectional area.





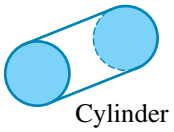
To find the volume of a prism we need to determine the area of the uniform cross-section (or base) and multiply by the height. This is the same for all prisms.
Volume of a prism, V_{prism}, can be generalised by the formula:

$$V_{prism} = \text{area of uniform cross-section} \times \text{height}$$
$$V = AH$$

For example:

$$V_{rect.\ prism} = A_{rect.} \times H$$
$$V_{triangular\ prism} = A_{triangle} \times H$$

Note: Although cylinders are not prisms, they have a uniform cross-section (which is a circle). Therefore, the same formula can be applied to find volume of a cylinder.

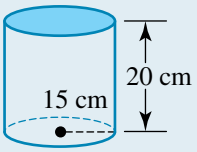

Cylinder

$$V_{\text{cylinder}} = A_{\text{circle}} \times H$$
$$= \pi r^2 H$$

In fact, the formula $V = A \times H$ can be applied to any solid with a uniform cross-section of area A.

WORKED EXAMPLE 16

Calculate the volume of the object shown. Give your answer correct to the nearest cm^3.



THINK

1. The object has a circle as a uniform cross-section. It is a cylinder.
   The area of the base is the area of a circle = πr^2.
   Volume is cross-sectional area times height.
2. Write your answer.

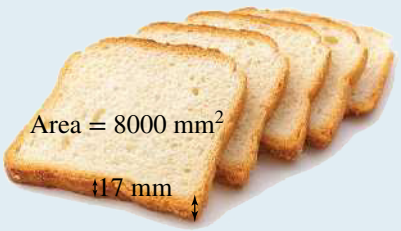
WRITE

$V_{\text{cylinder}} = A \times H$, where $A_{\text{circle}} = \pi r^2$
$= \pi \times r^2 \times H$
$= \pi \times 15^2 \times 20$
$= 4500\pi$
$\approx 14\,137.1669\ \text{cm}^3$

The volume of the cylinder is approximately $14\,137\ \text{cm}^3$.

WORKED EXAMPLE 17

Calculate (correct to the nearest mm^3) the volume of the slice of bread with a uniform cross-sectional area of 8000 mm^2 and a thickness of 17 mm.



THINK

1. The slice of bread has a uniform cross-section.
   The cross-section is not a common figure but its area has been given.
2. Write your answer.

WRITE

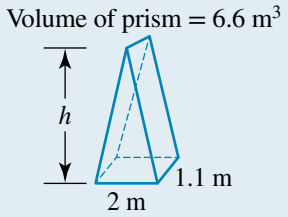
$V = A \times H$
where $A = 8000\ \text{mm}^2$
$V = 8000\ \text{mm}^2 \times 17\ \text{mm}$
$= 136\,000\ \text{mm}^3$

The volume of the slice of bread is $136\,000\ \text{mm}^3$.

Given the volume of an object, we can use the volume formula to find an unknown dimension of the object by transposing the formula.

WORKED EXAMPLE 18

Calculate the height of the triangle (correct to 1 decimal place) from the information provided in the diagram.

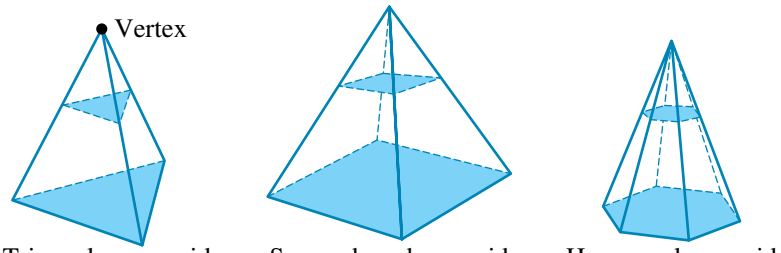


THINK	WRITE
1. The volume of the object is given, along with the width of the triangular cross-section and the height of the prism.	$V = 6.6\,\text{m}^3$, $H = 1.1\,\text{m}$, $b = 2\,\text{m}$ $V = A \times H$, where $A = \frac{1}{2} \times b \times h$ $V = \frac{1}{2} \times b \times h \times H$
2. Substitute the values into the formula and solve for h.	$6.6 = \frac{1}{2} \times 2 \times h \times 1.1$ $= 1.1h$ $h = \frac{6.6}{1.1}$ $= 6$
3. Write your answer.	The height of the triangle in the given prism is 6 meters.

11.5.3 Volume of pyramids

A pyramid is a polyhedron, where the base is any polygon and all other faces are triangles meeting at the vertex.

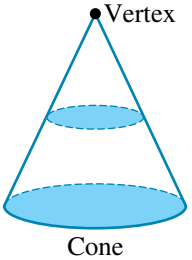
The name of the pyramid is related to the shape of the polygon at the base.



The shape of the cross-section of the pyramid remains unchanged, but its size reduces as it approaches the vertex.

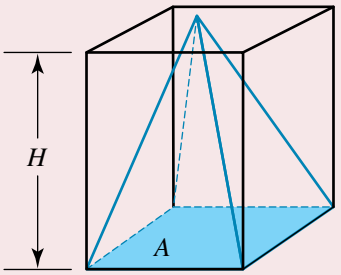
Similarly, for cones, the shape of the cross-section is always the same (a circle), but its size reduces as we move from base towards the vertex.

The volume of a pyramid is always one-third of the volume of a prism with the same base and same height, H. This holds for all pyramids and cones.



Volume of a pyramid, V_{pyramid}, and the value of a cone, V_{cone} can be found by using the formulae:

$$V_{\text{pyramid}} = \frac{1}{3} \times \text{area of the base} \times \text{height}$$
$$= \frac{1}{3}AH$$
$$V_{\text{cone}} = \frac{1}{3}\pi r^2 H$$

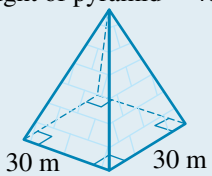


The height of a pyramid, H, is sometimes called the altitude.

WORKED EXAMPLE 19

Calculate the volume of the pyramid shown (correct to the nearest m^3).

Height of pyramid = 40 m



THINK

1. The pyramid has a square base. Calculate the area of the base, and then substitute the area into the volume formula.
2. Write your answer.

WRITE

$$V_{\text{pyramid}} = \frac{1}{3} \times A \times H, \quad \text{where } A_{\text{square}} = l^2$$
$$= \frac{1}{3} \times l^2 \times H$$
$$= \frac{1}{3} \times 30^2 \times 40$$
$$= 12\,000 \text{ m}^3$$

The volume of the pyramid is $12\,000 \text{ m}^3$.

11.5.4 Volume of spheres and composite objects

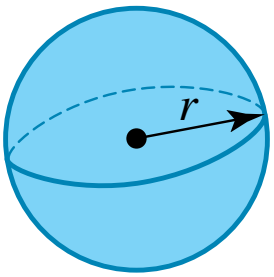
Volume of a sphere

Spheres are unique (but common) objects that deserve special attention.

The formula for the volume of a sphere is:

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere.



Volume of composite objects

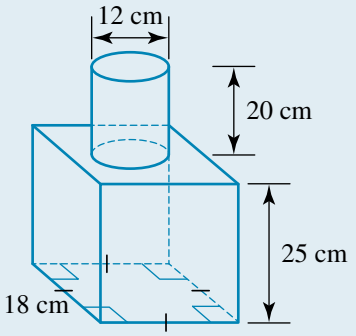
Often the object can be identified as comprising two or more different common prisms, pyramids or spheres. Such figures are called composite objects. The volume of a composite object is found by adding the volumes of the individual common figures or deducting volumes. For example, each of the structures on the right can be roughly modelled as the sum of a cylinder and a cone.

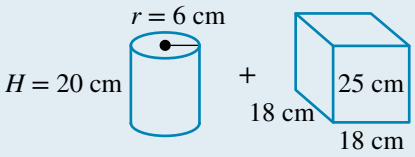
Volume of a composite object = the sum of the volumes of the individual components.

$$V_{composite} = V_1 + V_2 + V_3 + \ldots \left(\text{or } V_{composite} = V_1 - V_2\right)$$

WORKED EXAMPLE 20

Calculate the capacity of the container shown. (Give your answer correct to the nearest litre.)



THINK	WRITE/DRAW
1 The object consists of a cylinder and a square-based prism.	
2 The volume of the composite object is the sum of the volumes of the cylinder and the prism.	$V_{composite}$ = volume of cylinder + volume of square − based prism $= A_{circle} \times H_{cylinder} + A_{square} \times H_{prism}$ $= (\pi r^2 \times H_c) + (l^2 \times H_p)$ $= (\pi \times 6^2 \times 20) + (18^2 \times 25)$ $= 2261.946\,711 + 8100$ $= 10\,361.946\,711 \text{ cm}^3$
3 Convert to litres using the conversion of 1000 cm^3 equals 1 litre.	$10\,362 \text{ cm}^3 = 10.362$ litres
4 Write your answer.	The capacity of the container is 10 litres, correct to the nearest litre.

on Resources

Interactivity: Volume (int-6476)
Interactivity: Conversion of units of volume (int-6272)

Exercise 11.5 Volume of prisms, pyramids and spheres

1. **WE14** Convert 0.35 cm^3 to mm^3.
2. Convert 4800 cm^3 to m^3.
3. **WE15** Convert 1.6 m^3 to litres.
4. Convert 250 000 mm^3 to cm^3.

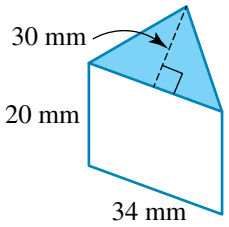
5. Convert the volumes to the units specified.

a. $530\,cm^3$ to m^3 **b.** $56\,000\,cm^3$ to litres **c.** 15 litres to cm^3

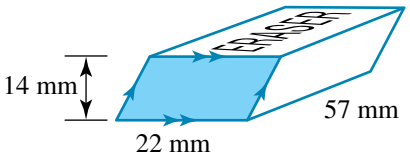
d. $72.1\,m^3$ to litres **e.** $0.0023\,cm^3$ to mm^3 **f.** $0.000\,57\,m^3$ to cm^3

g. $140\,000\,mm^3$ to litres **h.** $16\,000\,mm^3$ to cm^3

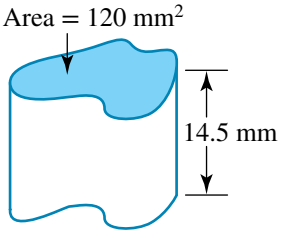
6. **WE16** Calculate the volume of the following solid correct to the nearest whole unit.



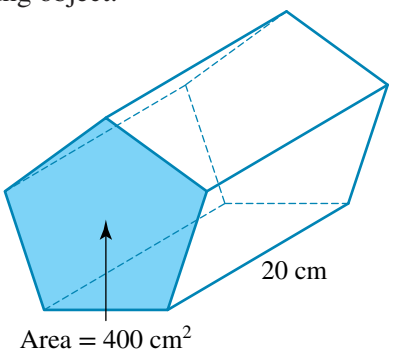
7. Calculate the volume of the following solid correct to the nearest whole unit.



8. **WE17** Calculate the volume of the following object (correct to 2 decimal places).

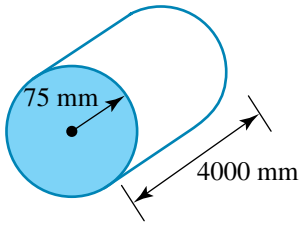


9. Calculate the volume of the following object.

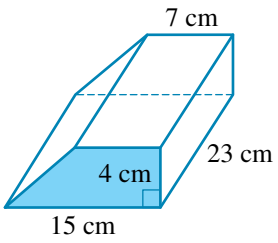


10. Calculate the volume of the following solids correct to the nearest whole unit.

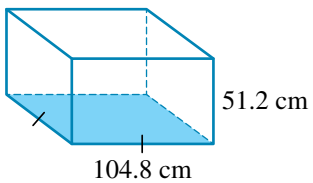
a.



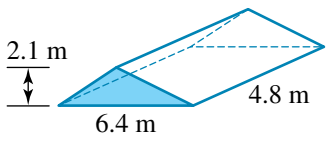
b.



c.

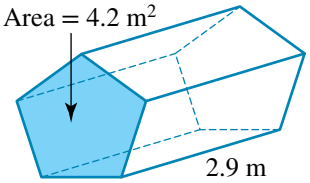


d.

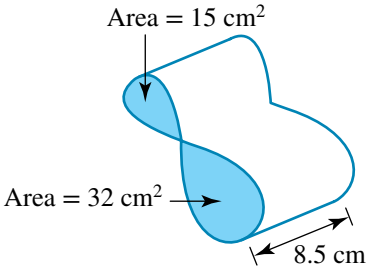


11. Calculate the volume of the following objects (correct to 2 decimal places).

a. Area = 4.2 m^2

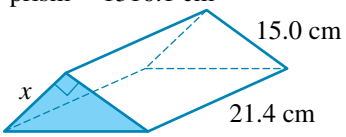


b. Area = 15 cm^2



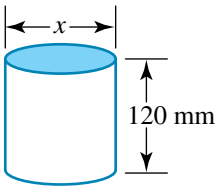
12. **WE18** Calculate the measurement of the unknown dimension (correct to 1 decimal place).

Volume of triangular prism = 1316.1 cm^3



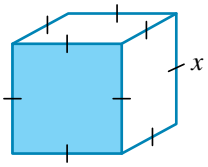
13. Calculate the measurement of the unknown dimension (correct to 1 decimal place).

Volume of cylinder = 150 796.4 mm^3

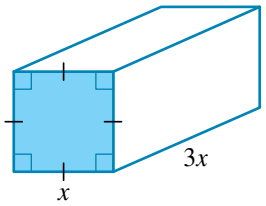


14. Calculate the measurement of the unknown dimension (correct to 1 decimal place).

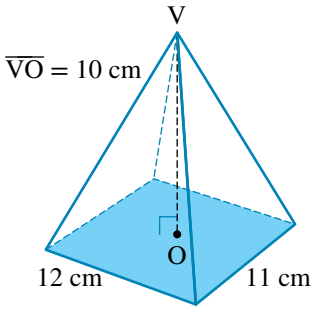
a. Volume of cube = 1.728 m^3



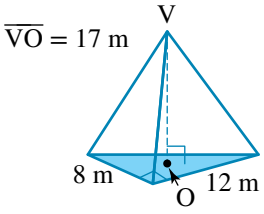
b. Volume of prism = $10\frac{1}{8}$ litres



15. **WE19** Calculate the volume of the object (correct to the nearest whole unit).

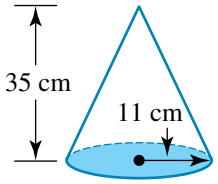


16. Calculate the volume of the object (correct to the nearest whole unit).

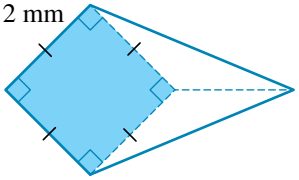


17. Calculate the volume of these objects (correct to the nearest whole unit).

a.

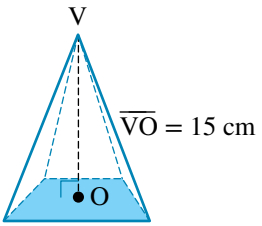


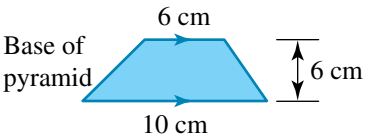
b.



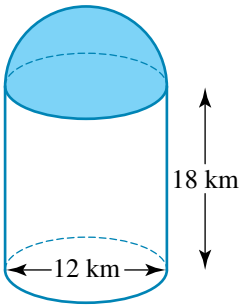
Altitude of square-based pyramid = 18 mm

c.

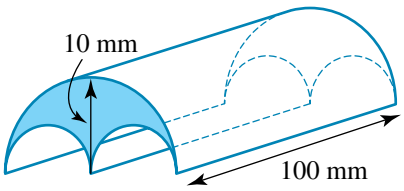




18. **WE20** Calculate the volume of the figure below. Give your answer in m^3 correct to 3 significant figures.

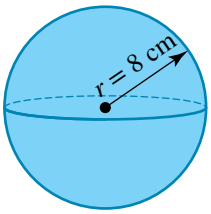


19. Calculate the volume of the figure. Give your answer in m^3 correct to 4 significant figures.

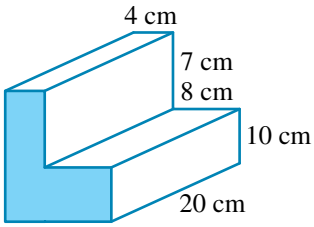


20. Calculate the volume of these objects (correct to the nearest whole unit).

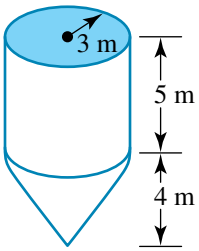
a.



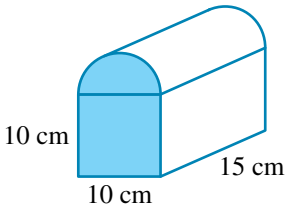
b.



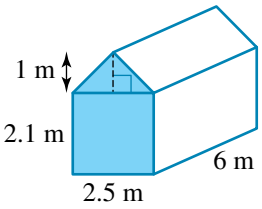
c.



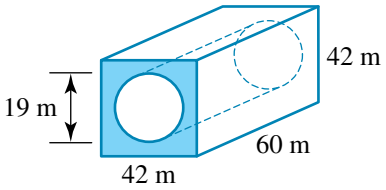
d.



e.



f.



21. **a.** Calculate the volume of a cube with sides 4.5 cm long.

b. Calculate the volume of a room, 3.5 m by 3 m by 2.1 m high.

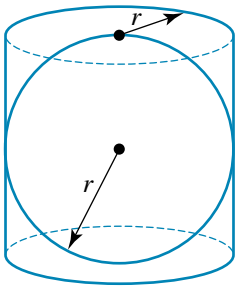
c. Calculate the radius of a baseball that has a volume of 125 cm^3.

d. Calculate the height of a cylinder that is 20 cm in diameter with a volume of 2.5 litres (correct to the nearest unit).

e. Calculate the height of a triangular prism with a base area of 128 mm^2 and volume of 1024 mm^3.

f. Calculate the depth of water in a swimming pool that has a capacity of 56 000 litres. The pool has rectangular dimensions of 8 metres by 5.25 metres.

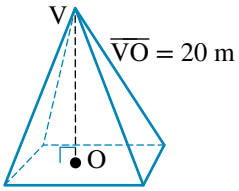
22. MC The ratio of the volume of a sphere to that of a cylinder of similar dimensions, as shown in the diagram, is best expressed as:



A. $\frac{4}{3}$ **B.** $\frac{2}{3}$ **C.** $\frac{\frac{4}{3}r}{h}$

D. $\frac{3}{4}$ **E.** $\frac{3}{2}$

23. MC If the volume of the square-based pyramid shown is 6000 m^3, then the perimeter of the base is closest to:



A. 900 m **B.** 20 m **C.** 30 m

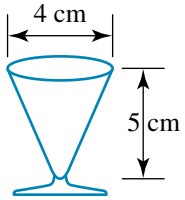
D. 80 m **E.** 120 m

24. MC A tin of fruit is 13 cm high and 10 cm in diameter. Its volume, correct to 1 decimal place, is:

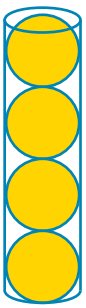
A. 1021.0 cm^3 **B.** 510.5 cm^3 **C.** 1021.4 cm^3

D. 1020.1 cm^3 **E.** 4084.1 cm^3

25. The medicine cup shown has the shape of a cone with a diameter of 4 cm and a height of 5 cm (not including the cup's base). Calculate the volume of the cone correct to the nearest millilitre, where 1 cm^3 = 1 mL.



26. Tennis balls have a diameter of 6.5 cm and are packaged in a cylinder that can hold four tennis balls. Assuming the balls just fit inside the cylinder, calculate:



a. the height of the cylindrical can

b. the volume of the can (correct to 1 decimal place)

c. the volume of the four tennis balls (correct to 1 decimal place)

d. the volume of the can occupied by air

e. the fraction of the can's volume occupied by the balls.

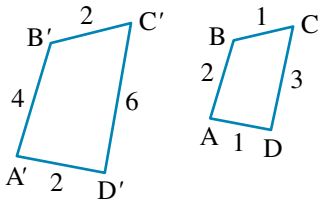
11.6 Similar figures

Objects that have the same shape but different size are said to be **similar**.

For two figures to be similar, they must have the following properties:

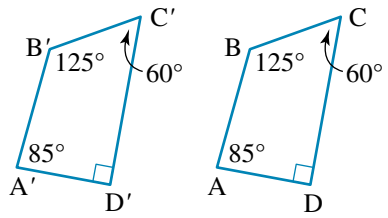
1. The ratios of the corresponding sides must be equal.

$$\frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{C'D'}}{\overline{CD}} = \frac{\overline{A'D'}}{\overline{AD}} = \text{common ratio}$$



2. The corresponding angles must be equal.

$$\angle A = \angle A' \quad \angle B = \angle B' \quad \angle C = \angle C' \quad \angle D = \angle D'$$



11.6.1 Scale factor, *k*

A measure of the relative size of the two similar figures is the **scale factor**. The scale factor is the common ratio of the corresponding sides and quantifies the amount of enlargement or reduction one figure undergoes to transform into the other figure. The starting shape is commonly referred to as the *original* and the transformed shape as the *image*.

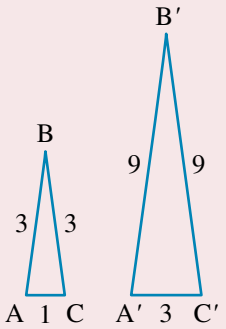
Scale factor, k, is the amount of enlargement or reduction and is expressed as integers, fraction or scale ratios.

For example, $k = 2$, $k = \frac{1}{12}$ or 1 : 10 000.

Scale factor, k = $\dfrac{\text{length of image}}{\text{length of original}}$

$$= \frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{C'A'}}{\overline{CA}}$$

where for enlargements k is greater than 1 and for reductions k is between 0 and 1.

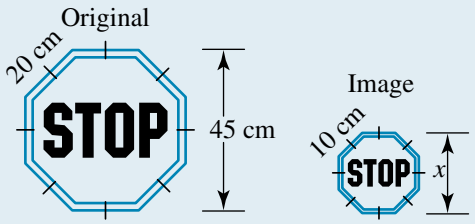


For $k = 1$, the figures are exactly the same shape and size and are referred to as **congruent**.

Enlargements and reductions are important in many aspects of photography, map making and modelling. Often, photographs are increased in size (enlarged) to examine fine detail without distortion, while house plans are an example of a reduction to a scale; for example, 1 : 25.

WORKED EXAMPLE 21

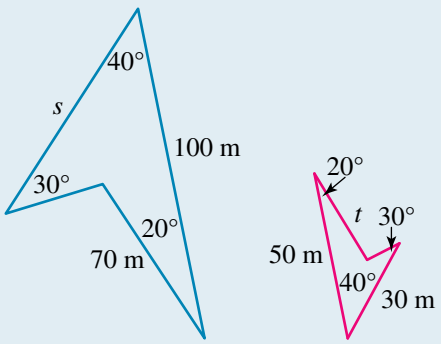
For the similar shapes shown:
a. find the scale factor for the reduction of the shape
b. find the unknown length in the smaller shape.

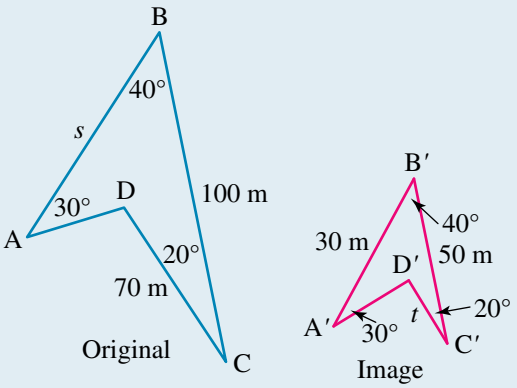


THINK	WRITE
a. 1. As it is a reduction, the larger shape is the original and the smaller shape is the image.	**a.** Scale factor, $k = \dfrac{\text{length of image}}{\text{length of original}}$
2. The two shapes have been stated as being similar, so set up the scale factor, k.	$= \dfrac{\overline{A'B'}}{\overline{AB}}$
	$= \dfrac{10\text{ cm}}{20\text{ cm}}$
	$= \dfrac{1}{2}$
b. 1. Use the scale factor to determine the unknown length as all corresponding lengths are in the same ratio.	**b.** Scale factor, $k = \dfrac{1}{2}$
	$k = \dfrac{\text{length of image}}{\text{length of original}}$
	$\dfrac{1}{2} = \dfrac{x}{45\text{ cm}}$
	$x = \dfrac{1}{2} \times 45\text{ cm}$
	$= 22.5\text{ cm}$
2. Write your answers.	The scale factor of reduction is $\dfrac{1}{2}$ and the unknown length on the smaller shape is 22.5 cm.

WORKED EXAMPLE 22

The figures given are similar.
Find the lengths of the two unknown sides s and t.



THINK	WRITE/DRAW
1. First, redraw the figures so that they are oriented the same way. (This will help to identify corresponding sides and angles.) Label the vertices of the 1st figure A, B, C and D and corresponding vertices of the 2nd figure A′, B′, C′ and D′ respectively.	
Since it is known that the figures are similar, ratios of the corresponding side lengths must be equal. Both length, $\overline{BC}$ and $\overline{B'C'}$ are known. These can be used to find the value of s, as the ratio of $\overline{BA}$ to $\overline{B'A'}$ must be the same.	$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{B'A'}}{\overline{BA}}$ $\frac{50}{100} = \frac{30}{s}$ $s = \frac{100 \times 30}{50}$ $s = 60\text{ m}$
3. Now find the value of t. (The ratio of $\overline{DC}$ to $\overline{D'C'}$ must be the same as the ratio of $\overline{BC}$ to $\overline{B'C'}$.)	$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{D'C'}}{\overline{DC}}$ $\frac{50}{100} = \frac{t}{70}$ $t = \frac{50 \times 70}{100}$ $t = 35\text{ m}$

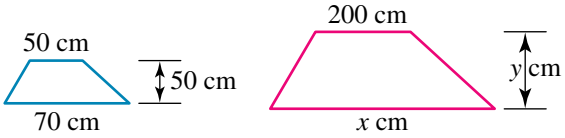
studyon

Units 3 & 4 | AOS M3 | Topic 1 | Concept 5

Similarity Summary screen and practice questions

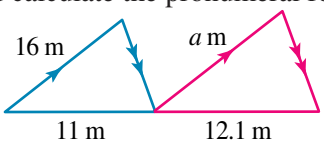
Exercise 11.6 Similar figures

1. **WE21** For the pair of similar shapes, calculate:
   a. the scale factor
   b. the value of x and y.

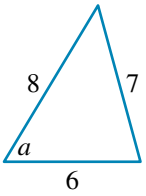


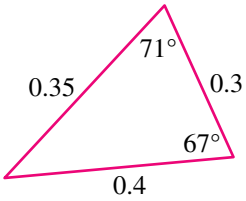
2. A tree is 7 mm long in a photograph, while it stands 3 m tall in real life. What is the scale factor?

3. **WE22** Determine the scale factor and calculate the pronumeral for the following figure.



4. Determine the scale factor and calculate the pronumeral for the following figure.

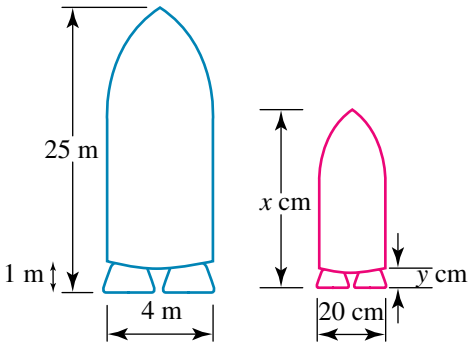




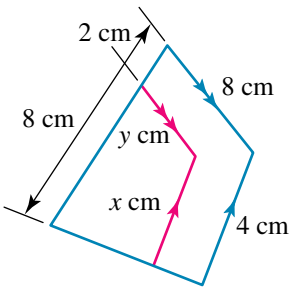
5. For each of these pairs of similar shapes, calculate:

i. the scale factor **ii.** the value of x and y.

a.

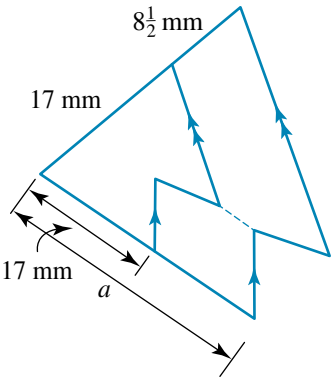


b.

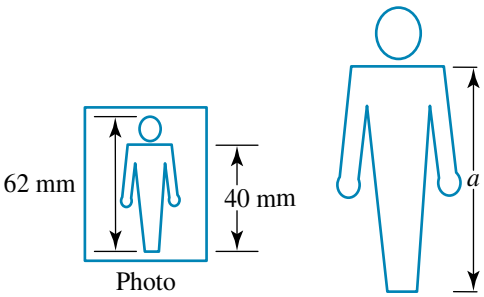


6. For each of the following pairs of similar figures, calculate the value of a.

a.



b.



Height of person = 186 cm

7. A photo has the dimensions 10 cm by 12 cm. The photo is enlarged by a factor of 2.5. Calculate the new dimensions of the photo.

8. A set of model cars is made using the scale ratio 1 : 12. Calculate:

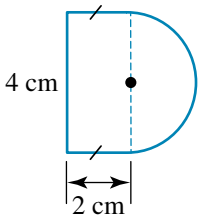
a. the length of a real car if the model is 20 cm long (in metres correct to 1 decimal place)

b. the height of a real car if the model is 3 cm high (correct to the nearest centimetre)

c. the length of a model if the real car is 3 metres long.

9. The dimensions of a student's room are 4300 mm by 3560 mm. A scale diagram of the room is to be drawn on an A4 sheet, using the scale ratio 1 : 20. Calculate the dimensions of the scale drawing of the room and state whether the drawing should be landscape or portrait on the A4 sheet.

10. **MC** The scale used to draw the diagram is 1 : 25. The perimeter of the real object is closest to:



A. 464 cm **B.** 514 cm **C.** 357 cm
D. 14.28 cm **E.** 150 cm

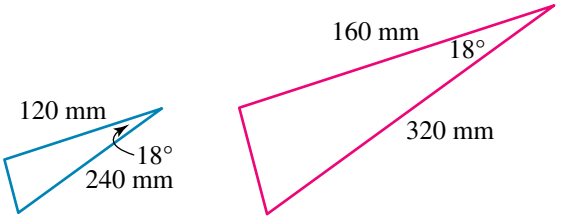
11. **MC** A 1 : 27 scale model of a truck is made from clay. What is the length of the tray on the original truck, if length of the tray on the model is 27 cm?

A. 1 cm **B.** 100 cm **C.** 270 cm **D.** 540 cm **E.** 729 cm

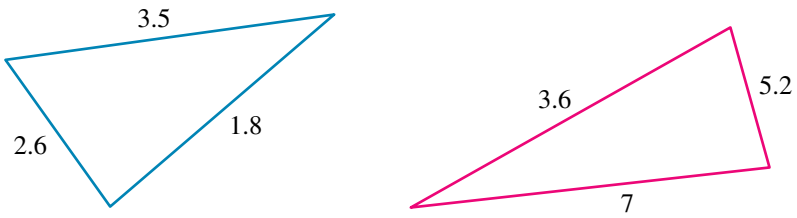
12. **MC** A scale factor of 0.2 is:

A. a reduction with a scale of 1 cm = 2 cm
B. an enlargement with a scale of 1 cm = 0.2 cm
C. an enlargement with a scale of 1 cm = 5 cm
D. a reduction with a scale of 1 cm = 5 cm
E. a reduction with a scale of 1 cm = 20 cm

13. Determine the scale factor of the figure.



14. Determine the scale factor of the pair of triangles.



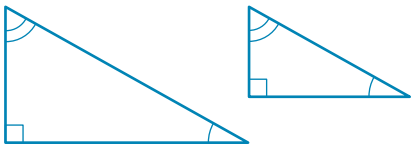
15. Jane wants a larger clock in her house. The current clock has a radius of 20 cm, but Jane wants one 2.5 times its size. Calculate the circumference of the new clock.

16. The distance from Melbourne to Ballarat is 113 km. If a map used a scale ratio of 1:2 500 000, what is the distance on the map that represents this distance correct to 1 decimal place?

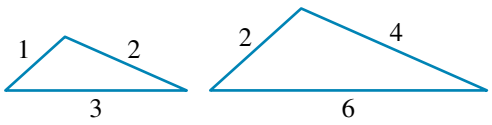
11.7 Similar triangles

Similar triangles can be used in a variety of situations. For example, with the aid of similar triangles, we could find the heights of trees and buildings or the width of rivers and mountains. Two triangles are similar if one of the following is true:

1. All three corresponding angles are equal (AAA).

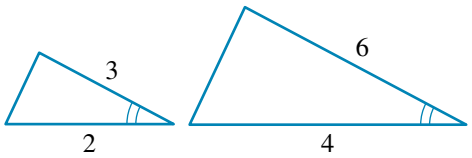


2. All three corresponding pairs of sides are in the same ratio (linear scale factor) (SSS).



$$\text{Scale factor} = \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = 0.5 = k$$

3. Two corresponding pairs of sides are in the same ratio and the included angles are equal (SAS).



$$\text{Scale factor} = \frac{6}{3} = \frac{4}{2} = 2 = k$$

As in the previous section, we use the known values of a pair of corresponding sides to determine the scale factor (sf) for the similar triangles.

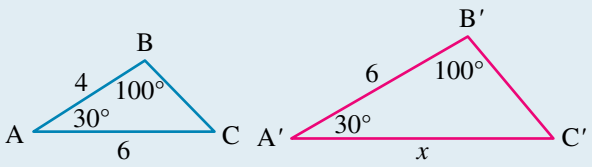
$$\textbf{Scale factor, } k = \frac{\textbf{length of side of image}}{\textbf{length of corresponding side of original}}$$

WORKED EXAMPLE 23

For the triangles shown:

a. **State the rule that proves that the triangles are similar and determine the scale factor.**

b. **Find the value of the pronumeral, x.**



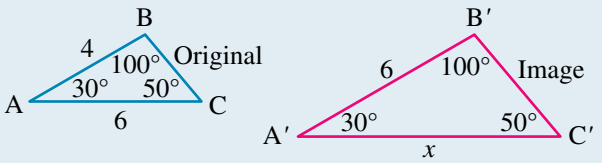
THINK

a. 1. Two corresponding angles are equal. The third angle is not given but can be easily found using the rule that all angles in a triangle add to 180°. State the rule that proves similarity.

WRITE/DRAW

a. $\angle A = \angle A' = 30°$
$\angle B = \angle B' = 100°$
$\angle C = \angle C' = 180° - (30° + 100°)$
$= 50°$
$\therefore \Delta ABC$ is similar to $\Delta A'B'C'$ because all three corresponding angles are equal (AAA).

2. Always select the triangle with the unknown length, x, as the image. Evaluate the scale factor by selecting a pair of corresponding sides with both lengths known.



$$\text{Scale factor, } k = \frac{\text{length of side of image}}{\text{length of corresponding side of original}}$$
$$= \frac{\overline{A'B'}}{\overline{AB}}$$
$$= \frac{6}{4}$$
$$= 1.5$$

b. 1. Use the scale factor to find the unknown length, x.

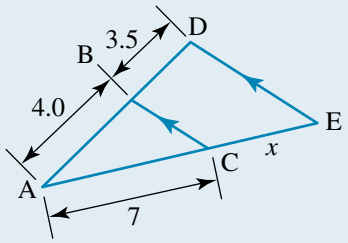
b. Scale factor, $k = 1.5$
$$1.5 = \frac{\overline{A'C'}}{\overline{AC}}$$
$$1.5 = \frac{x}{6}$$
$$x = 1.5 \times 6$$
$$= 9$$

2. Write the answer.

The value of the pronumeral, x, is 9.

WORKED EXAMPLE 24

For the given triangles, find the value of the pronumeral, x.

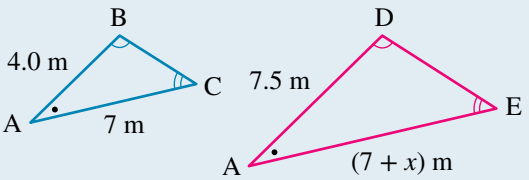


All measurements in metres

THINK

1. For clear analysis, separate the two triangles.
Note: The lengths of the sides $\overline{AE}$ and $\overline{AD}$ are the sum of the given values.
Establish that the two triangles are similar using an appropriate rule.

WRITE/DRAW



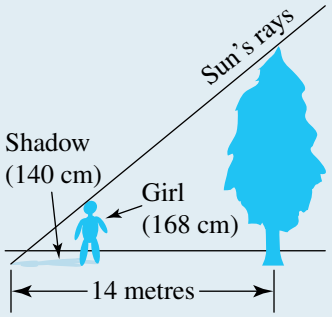
$\overline{AD} = 4.0 + 3.5$
$= 7.5$ m
$\overline{AE} = (7 + x)$ m
$\angle A = \angle A$ (common)
$\angle B = \angle D$ (corresponding angles are equal)
$\angle C = \angle E$ (corresponding angles are equal)
$\therefore \Delta ABC$ is similar to ΔADE (AAA).

2. Since the triangles are similar, the ratios of the corresponding sides are equal.	$\frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$ $\frac{7+x}{7} = \frac{7.5}{4}$
3. Cross-multiply and solve for x.	$4(7+x) = 7 \times 7.5$ $28 + 4x = 52.5$ $4x = 24.5$ $x = 6.125$
4. Write the answer including units.	The value of x is 6.125 metres.

There are many practical applications of similar triangles in the real world. It is particularly useful for determining the lengths of inaccessible features, such as the height of tall trees or the width of rivers. This problem is overcome by setting up a triangle similar to the feature to be examined, as shown in the next example.

WORKED EXAMPLE 25

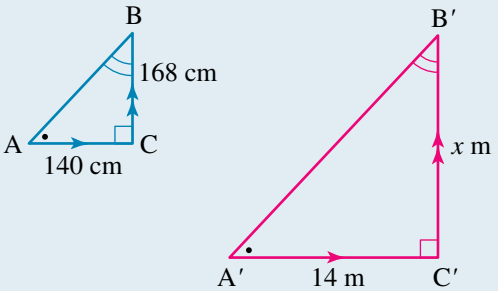
Find the height of the tree shown in the diagram. Give the answer correct to 1 decimal place.



THINK

1. For clear analysis, separate the two triangles. Establish that the two triangles are similar. (Assuming both the tree and the girl are perpendicular to the ground, they form parallel lines.)

WRITE/DRAW



$\angle A = \angle A'$ (common angles)
$\angle B = \angle B'$ (corresponding)
$\angle C = \angle C'$ (corresponding)
is similar to $\Delta A'B'C'$ (AAA).
$\therefore \Delta ABC$

2. Since the triangles are similar, their corresponding sides must be in the same ratio.
Note: Convert all measurements to metres, as it is the most appropriate unit for the height of a tree. So 140 cm = 1.4 m and 168 cm = 1.68 m.

$\frac{14}{1.4} = \frac{x}{1.68}$

3. Solve for x.

$$10 = \frac{x}{1.68}$$
$$x = 10 \times 1.68$$
$$= 16.8 \text{ m}$$

4. Write the answer including units.

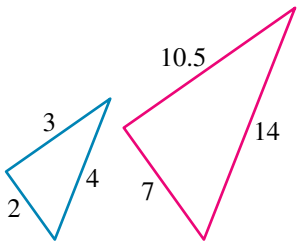
The height of the tree is 16.8 metres.

on Resources

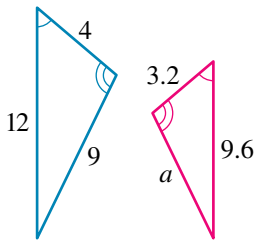
Interactivity: Similar triangles (int-6273)

Exercise 11.7 Similar triangles

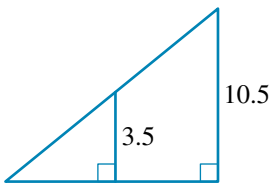
1. a. WE23 State the rule (SSSor AAA or SAS) that proves that the triangles shown below are similar and determine the scale factor (expressed as an enlargement $k > 1$).



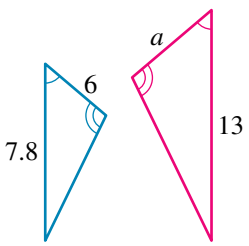
b. Find the value of the pronumeral a.



2. a. State the rule (SSSor AAA or SAS) that proves that the triangles below are similar and determine the scale factor (expressed as an enlargement $k > 1$).

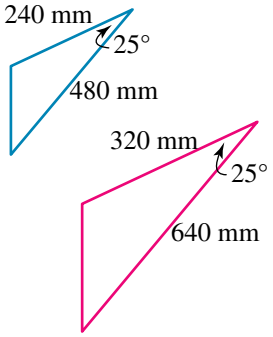


b. Find the value of the pronumeral a.

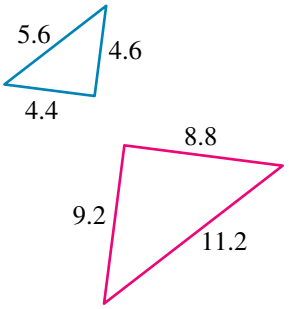


3. State the rule (SSSor AAA or SAS) that proves that the triangles in each pair are similar and determine the scale factor (expressed as an enlargement $k > 1$).

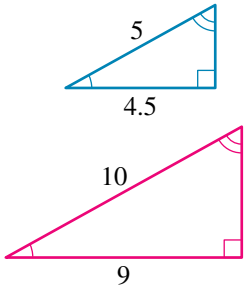
a.



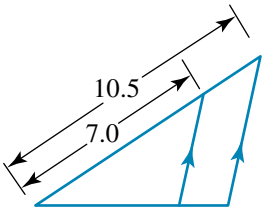
b.



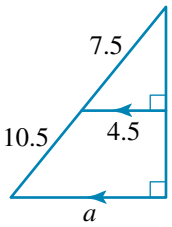
c.



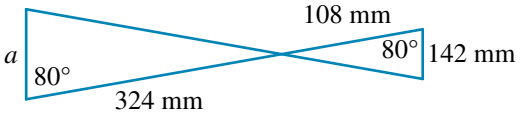
d.



4. **WE24** Find the value of the pronumeral a.

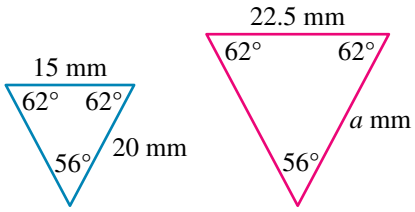


5. Find the value of the pronumeral a.

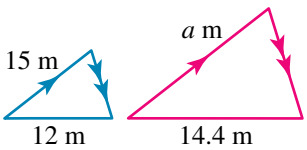


6. For the given pairs of similar triangles, find the value of the pronumeral a .

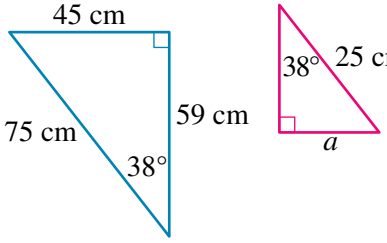
a.



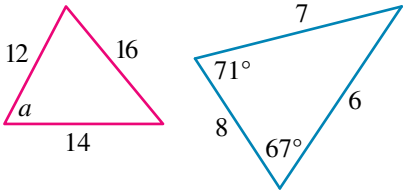
b.



c.

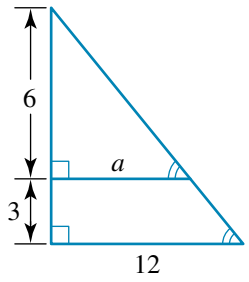


d.

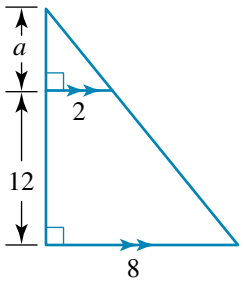


7. For the given pairs of triangles, find the value of the pronumeral a.

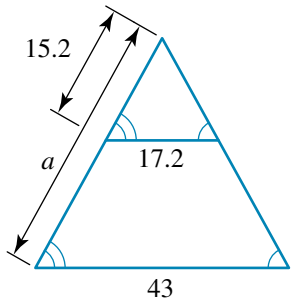
a.



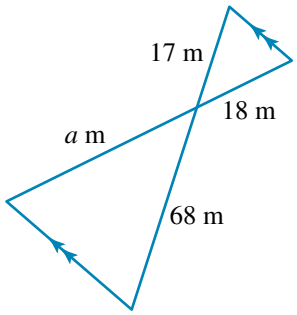
b.



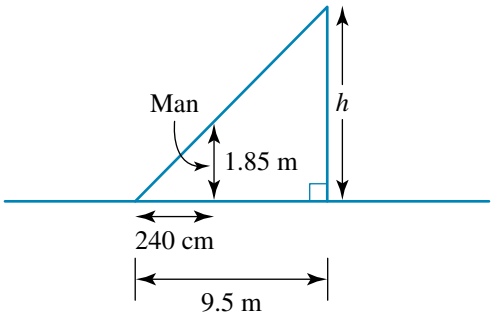
c.



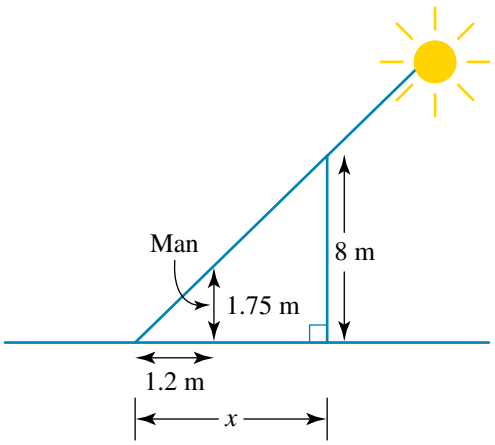
d.



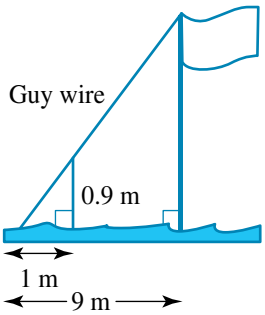
8. **WE25** Find the height h, to the nearest centimetre, of the billboard shown in the diagram.



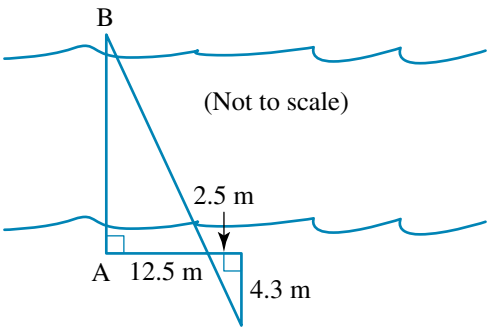
9. Find the length of the shadow cast by the telecommunications pole, correct to 1 decimal place.



10. Find the height (correct to the nearest centimetre) of the flagpole shown in the diagram.

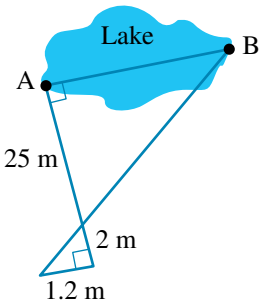


11. Find the length (correct to 1 decimal place) of the bridge, $\overline{AB}$, needed to span the river, using similar triangles as shown.



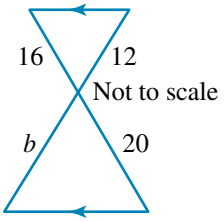
12. The shadow of a tree is 4 metres and at the same time the shadow of a 1-metre stick is 25 cm. Assuming both the tree and stick are perpendicular to the horizontal ground, what is the height of the tree?

13. Find the width of the lake (correct to the nearest metre) using the surveyor's notes.

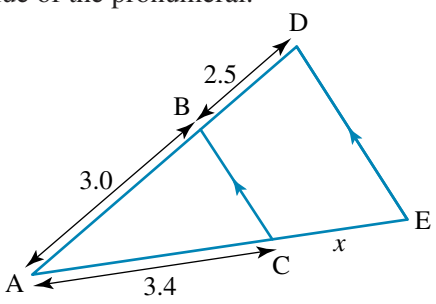


14. MC In the given diagram, the length of side b is closest to:

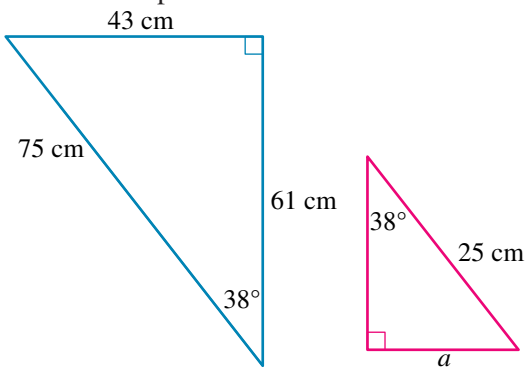
A. 24
B. 22
C. 16
D. 15
E. 9.6



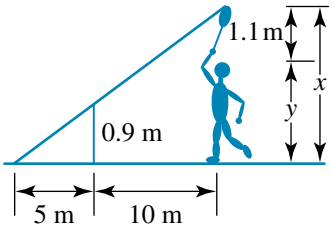
15. For the following triangles, state the rule (SSS, AAA or SAS) that proves that the pair of triangles is similar, and then calculate the value of the pronumeral.



16. For the following triangles, state the rule (SSS, AAA or SAS) that proves that the pair of triangles is similar, and then calculate the value of the pronumeral.



Questions 17 and 18 refer to the following information.
A young tennis player's serve is shown in the diagram.
Assume the ball travels in a straight line.



17. MC The height of the ball x, just as it is hit, is closest to:
   - **A.** 3.6 m
   - **B.** 2.7 m
   - **C.** 2.5 m
   - **D.** 1.8 m
   - **E.** 1.6 m

18. MC The height of the player, y, as shown is closest to:
   - **A.** 190 cm
   - **B.** 180 cm
   - **C.** 170 cm
   - **D.** 160 cm
   - **E.** 150 cm

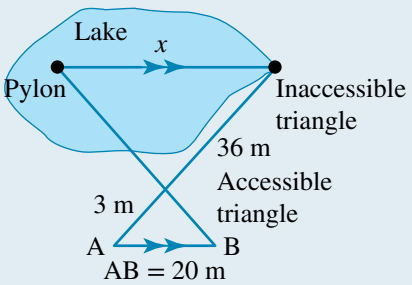
11.8 Triangulation — similarity

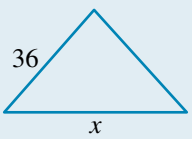
We can solve triangulation problems by using similar triangles. There are situations where a triangle can be constructed in an area that is accessible so as to determine the dimensions of a similar triangle in an inaccessible region.

1. We need two corresponding lengths to establish the scale factor between the two similar triangles. A second accessible side will be used to scale up or down to the corresponding inaccessible side.
2. For similar triangles use the following rules as proof:
   - **a.** AAA — all corresponding angles are the same
   - **b.** SSS — all corresponding sides are in the same ratio
   - **c.** SAS — two corresponding sides are in the same ratio with the same included angle.

WORKED EXAMPLE 26

Find the unknown length, x, from the pylon to the edge of the lake.



THINK	WRITE
1. Identify that the two triangles are similar (proof: AAA rule).	
2. Draw the triangles separately, highlighting the corresponding sides.	

3. In similar triangles, all corresponding sides are in the same ratio.

$$\frac{3}{36} = \frac{20}{x}$$

4. Transpose the equation to get the unknown by itself and evaluate.

$$x = \frac{20 \times 36}{3}$$

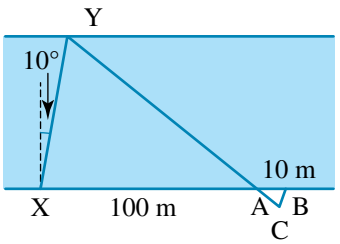
$$= 240 \text{ metres}$$

5. Write the length and include units with the answer.

The distance from the edge of the lake to the pylon is 240 metres.

Exercise 11.8 Triangulation — similarity

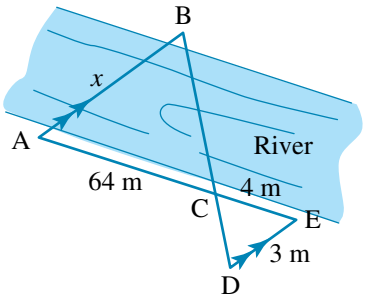
1. **WE26** A bridge is to be built across a river at an angle of 10 degrees to the river between points X and Y, as shown in the figure at right.



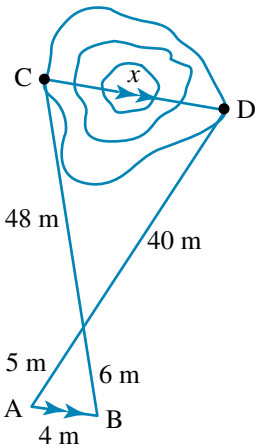
A surveyor walks along the bank 100 m from X to A, then a further 10 m to B. He then walks to point C, so that a line of sight can be drawn through A and Y from C.
If the distance BC = 3.1 m, what will the length of the bridge be?

2. A young boy wishes to determine the height of a tall building. He stands 100 m away from the building and puts a 1-metre stick vertically in the ground. He then waits until the sun is in a position where the shadow of the stick and the shadow of the building end at the same point. This point is 1.7 m from the base of the stick. Determine the height of the building.

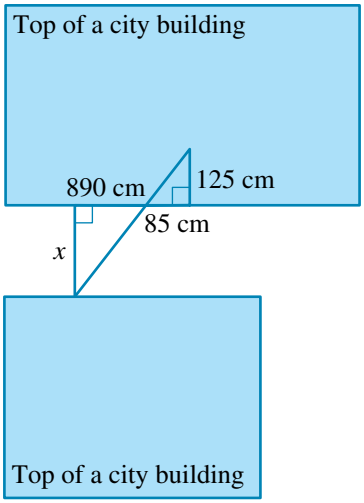
3. In the figure shown, find the length of the proposed bridge, AB.



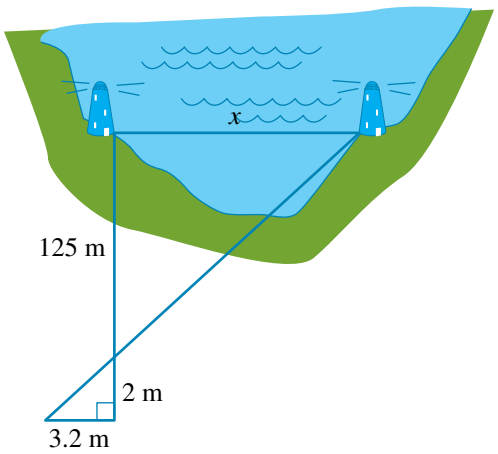
4. In the figure shown, find the length of the base of a hillside from C to D.



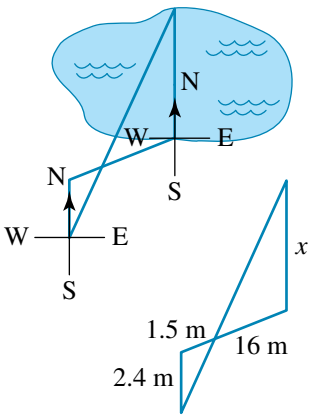
5. In the figure shown, find the perpendicular gap between the two city buildings.



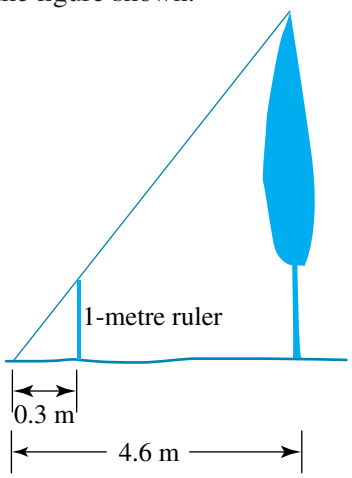
6. Find the distance between the two lighthouses in the figure shown.



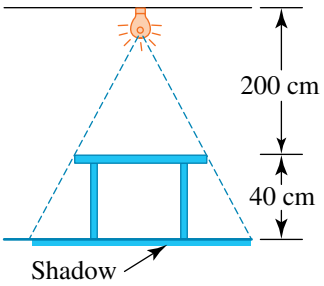
7. Find the distance across the lake in the figure shown.



8. Find the height of the cyprus tree in the figure shown.



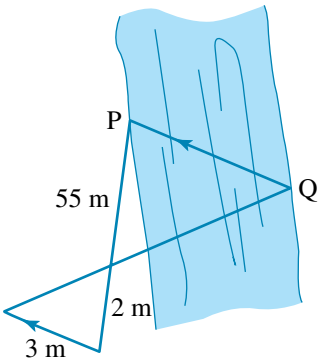
9. Find the width (correct to the nearest centimetre) of the shadow under the round table which has a diameter of 115 centimetres.



10. MC The distance across a river is to be determined using the measurements outlined.

The width from P to Q is closest to:

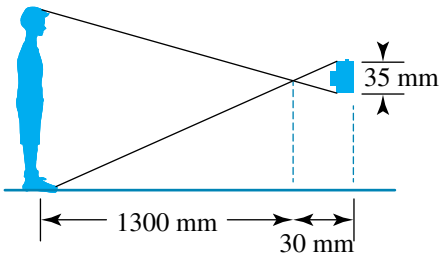
A. 37 m
B. 60 m
C. 83 m
D. 113 m
E. 330 m



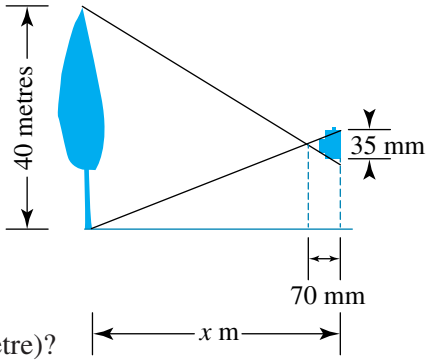
11. MC The shadow formed on the ground by a person who is 2 m in height was 5 m. At the same time a nearby tower formed a shadow 44 m long. The height of the tower, correct to the nearest metre, is:

A. 18 m **B.** 20 m **C.** 51 m **D.** 110 m **E.** 123 m

12. In the figure shown, find the height (correct to the nearest centimetre) of the person being photographed.



13. In the figure shown, find the minimum distance from the tree to the camera, x metres, so that the tree is completely in the photo.



14. A girl is looking through her window.
    - **a.** She is standing 2 metres from the window which is 2.4 metres wide. What is the width of her view:
        - **i.** 300 metres from the window (correct to the nearest metre)?
        - **ii.** 1.5 kilometres from the window (correct to the nearest 100 metres)?
        - **iii.** 6 kilometres from the window (correct to the nearest kilometre)?
    - **b.** She is now standing 1 metre from the window. What is the width of her view:
        - **i.** 300 metres from the window (correct to the nearest metre)?
        - **ii.** 1.5 kilometres from the window (correct to the nearest 100 metres)?
        - **iii.** 6 kilometres from the window (correct to the nearest kilometre)?

11.9 Area and volume scale factors

An unknown area or volume of a figure can be found without the need to use known formulas such as in Exercises 11.3 and 11.7. We have seen that two figures that are similar have all corresponding lengths in the same ratio or (linear) scale factor, k. The same can be shown for the area and volume of two similar figures.

11.9.1 Area of similar figures

If the lengths of similar figures are in the ratio $a : b$ or k, then the areas of the similar shapes are in the ratio $a^2 : b^2$ or k^2. Following are investigations to support this relationship.

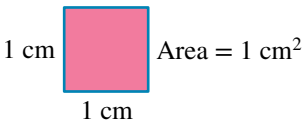
Different length ratios (or scale factors) of a square

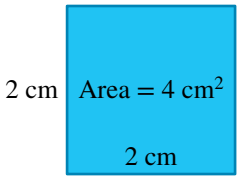
$$\frac{\text{Length of blue square}}{\text{Length of pink square}} = \frac{2\text{ cm}}{1\text{ cm}}$$
$$= 2$$
$$= k$$

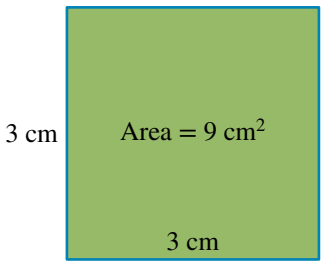
$$\frac{\text{Area of blue square}}{\text{Area of pink square}} = \frac{4\text{ cm}^2}{1\text{ cm}^2}$$
$$= 4$$
$$= 2^2$$
$$= k^2$$

$$\frac{\text{Length of green square}}{\text{Length of pink square}} = \frac{3\text{ cm}}{1\text{ cm}}$$
$$= 3$$
$$= k$$

$$\frac{\text{Area of green square}}{\text{Area of pink square}} = \frac{9\text{ cm}^2}{1\text{ cm}^2}$$
$$= 9$$
$$= 3^2$$
$$= k^2$$







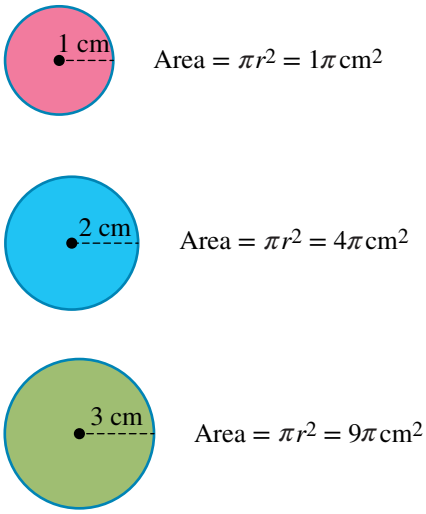
Different length ratios (or scale factors) of a circle

$$\frac{\text{Radius length of blue circle}}{\text{Radius length of pink circle}} = \frac{2\,\text{cm}}{1\,\text{cm}} = 2 = k$$

$$\frac{\text{Area of blue circle}}{\text{Area of pink circle}} = \frac{4\pi\,\text{cm}^2}{1\pi\,\text{cm}^2} = 4 = 2^2 = k^2$$

$$\frac{\text{Radius length of green circle}}{\text{Radius length of pink circle}} = \frac{3\,\text{cm}}{1\,\text{cm}} = 3 = k$$

$$\frac{\text{Area of green circle}}{\text{Area of pink circle}} = \frac{9\pi\,\text{cm}^2}{1\pi\,\text{cm}^2} = 9 = 3^2 = k^2$$



As we can see, as long as two figures are similar, then the area ratio or scale factor is the square of the linear scale factor, k. The same applies for the total surface area.

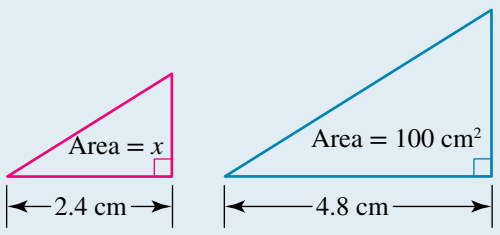
Area scale ratio or factor (asf) = $\frac{\text{area of image}}{\text{area of original}}$
= square of linear scale factor (lsf)
= (lsf)^2
= k^2

The steps required to solve for length, area or volume (investigated later) using similarity are:

1. Clearly identify the known corresponding measurements (length, area or volume) of the similar shape.
2. Establish a scale factor (linear, area or volume) using known measurements.
3. Convert to an appropriate scale factor to determine the unknown measurement.
4. Use the scale factor and ratio to evaluate the unknown.

WORKED EXAMPLE 27

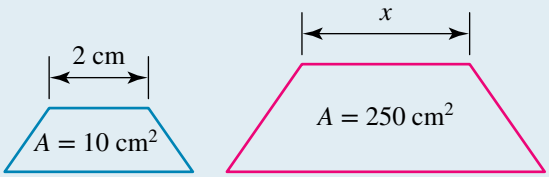
For the two similar triangles shown, find the area, x cm^2, of the small triangle.



THINK	WRITE
1. Determine a scale factor, in this instance the linear scale factor, from the two corresponding lengths given. It is preferred that the unknown triangle is the image.	$\text{Linear scale factor} = \frac{\text{length of small triangle (image)}}{\text{length of large triangle (original)}}$ $k = \frac{2.4\text{ cm}}{4.8\text{ cm}}$ $= \frac{1}{2}$
2. Determine the area scale factor.	$\text{Area scale factor} = k^2$ $= \left(\frac{1}{2}\right)^2$ $= \frac{1}{4}$
3. Use the area scale factor to find the unknown area.	$\text{Area scale factor} = \frac{\text{area of small triangle (image)}}{\text{area of large triangle (original)}}$ $\frac{1}{4} = \frac{x\text{ cm}^2}{100\text{ cm}^2}$
4. Transpose the equation to get the unknown by itself.	$x = \frac{1}{4} \times 100$ $= 25$
5. Write your answer.	The area of the small triangle is 25 cm^2.

WORKED EXAMPLE 28

For the two similar shapes shown, find the unknown length, *x* cm.



THINK	WRITE
1. Determine a scale factor, in this instance the area scale factor, as both areas are known. It is preferred that the triangle with the unknown dimension is stated as the image.	$\text{Area scale factor} = \frac{\text{area of image (large trapezium)}}{\text{area of original (small trapezium)}}$ $k^2 = \frac{250\text{ cm}^2}{10\text{ cm}^2}$ $= 25$
2. Determine the linear scale factor.	$\text{Linear scale factor} = \sqrt{k^2}$ $k = \sqrt{25}$ $= 5$
3. Use the linear scale factor to find the unknown length.	$\text{Linear scale factor} = \frac{\text{length of image (large trapezium)}}{\text{length of original (small trapezium)}}$ $5 = \frac{x\text{ cm}}{2\text{ cm}}$

4. Transpose the equation to get the unknown by itself. $x = 5 \times 2$
$= 10$
5. Write your answer. The length, x, is 10 cm.

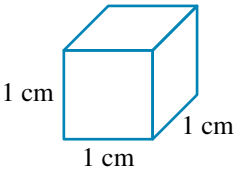
11.9.2 Volume of similar figures

If the lengths of similar figures are in the ratio $a : b$ or k, then the volume of the similar shapes are in the ratio $a^3 : b^3$ or k^3. The following is an investigation of two different objects, cubes and rectangular prisms.

A cube

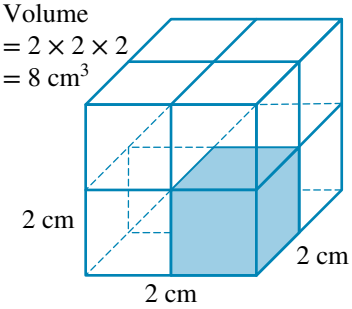
$$\frac{\text{Length of large cube}}{\text{Length of small cube}} = \frac{2\text{ cm}}{1\text{ cm}}$$
$$= 2$$
$$= k$$

Volume $= 1 \times 1 \times 1$
$= 1\text{ cm}^3$



$$\frac{\text{Volume of large cube}}{\text{Volume of small cube}} = \frac{8\text{ cm}^3}{1\text{ cm}^3}$$
$$= 8$$
$$= 2^3$$
$$= k$$

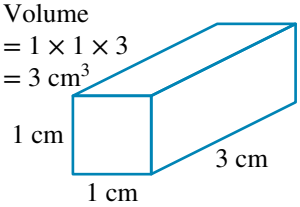
Volume
$= 2 \times 2 \times 2$
$= 8\text{ cm}^3$



A rectangular prism

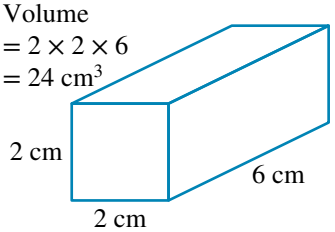
$$\frac{\text{Length of small prism}}{\text{Length of large prism}} = \frac{3\text{ cm}}{6\text{ cm}}$$
$$= \frac{1}{2}$$
$$= k$$

Volume
$= 1 \times 1 \times 3$
$= 3\text{ cm}^3$



$$\frac{\text{Volume of small prism}}{\text{Volume of large prism}} = \frac{3\text{ cm}^3}{24\text{ cm}^3}$$
$$= \frac{1}{8}$$
$$= \left(\frac{1}{2}\right)^3$$
$$= k^3$$

Volume
$= 2 \times 2 \times 6$
$= 24\text{ cm}^3$

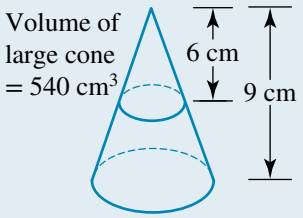


From above, as long as two figures are similar, then the volume ratio or scale factor is the cube of the linear scale factor, k.

$$\text{Volume scale factor (vsf)} = \frac{\text{volume of image}}{\text{volume of original}}$$
$$= \text{cube of linear scale factor (lsf)}$$
$$= (\text{lsf})^3$$
$$= k^3$$

WORKED EXAMPLE 29

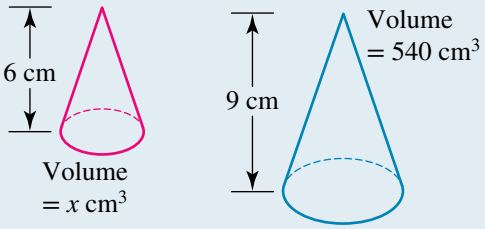
For the two similar figures shown, find the volume of the smaller cone.



THINK

1. Draw the two figures separately.

WRITE/DRAW



2. Determine a scale factor, in this instance the linear scale factor, from the two corresponding lengths given. It is preferred that the triangle with the unknown volume is the image.

$$\text{Linear scale factor} = \frac{\text{length of small triangle (image)}}{\text{length of large triangle (original)}}$$
$$k = \frac{6\text{ cm}}{9\text{ cm}}$$
$$= \frac{2}{3}$$

3. Determine the volume scale factor.

$$\text{Volume scale factor} = k^3$$
$$= \left(\frac{2}{3}\right)^3$$
$$= \frac{8}{27}$$

4. Use the volume scale factor to find the unknown length.

$$\text{Volume scale factor} = \frac{\text{volume of small cone (image)}}{\text{volume of large cone (original)}}$$
$$\frac{8}{27} = \frac{x\text{ cm}^3}{540\text{ cm}^3}$$

5. Transpose the equation to get the unknown by itself.

$$x = \frac{8}{27} \times 540$$
$$= 160$$

6. Write your answer.

The volume of the smaller cone is 160 cm^3.

We can use the relationship between linear, area and volume scale factors to find any unknown in any pair of similar figures as long as a scale factor can be established.

Given	Then	
Linear scale factor (lsf) $= k$ Example $(k = 2) = 2$	Area scale factor $= k^2$ $= 2^2$ $= 4$	Volume scale factor $= k^3$ $= 2^3$ $= 8$
Area scale factor (asf) $= k^2$ Example $= 4$	Linear scale factor $= \sqrt{k^2}$ $= \sqrt{4}$ $= 2$	Volume scale factor $= k^3$ $= 2^3$ $= 8$
Volume scale factor (vsf) $= k^3$ Example $= 8$	Linear scale factor $= \sqrt[3]{k^3}$ $= \sqrt[3]{8}$ $= 2$	Area scale factor $= k^2$ $= 2^2$ $= 4$

WORKED EXAMPLE 30

For two similar triangular prisms with volumes of 64 m^3 and 8 m^3, find the total surface area of the larger triangular prism, if the smaller prism has a total surface area of 2.5 m^2.

THINK

1. Determine a scale factor, in this instance the volume scale factor, from the two known volumes. It is preferred that the larger unknown triangular prism is stated as the image.
2. Determine the area scale factor. For ease of calculation, change volume scale factor to linear scale factor first and then to area scale factor.
3. Use the area scale factor to find the total surface area.
4. Transpose the equation to get the unknown by itself.
5. Write your answer.

WRITE

$$\text{Volume scale factor} = \frac{\text{volume of larger prism (image)}}{\text{volume of smaller prism (original)}}$$

$$k^3 = \frac{64\text{ m}^3}{8\text{ m}^3}$$

$$= 8$$

$$\text{Linear scale factor} = k$$

$$= \sqrt[3]{k^3}$$

$$= \sqrt[3]{8}$$

$$= 2$$

$$\text{Area scale factor} = k^2$$

$$= 2^2$$

$$= 4$$

$$\text{Area scale factor} = \frac{\text{area of larger prism (image)}}{\text{area of smaller prism (original)}}$$

$$4 = \frac{x\text{ m}^2}{2.5\text{ m}^2}$$

$$x = 4 \times 2.5$$

$$= 10$$

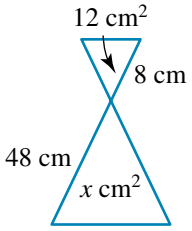
The total surface area of the larger triangular prism is 10 m^2.

Resources

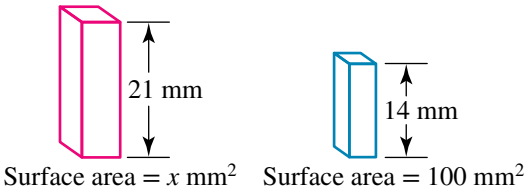
Interactivity: Area scale factor (int-6478)
Interactivity: Volume scale factor (int-6479)

Exercise 11.9 Area and volume scale factors

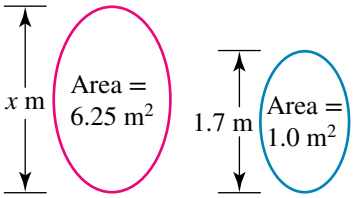
1. **WE27** Find the unknown area for the pair of similar figures.



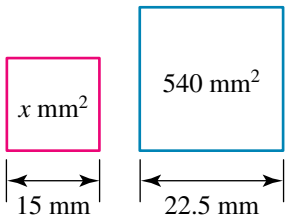
2. Find the unknown area for the pair of similar figures.



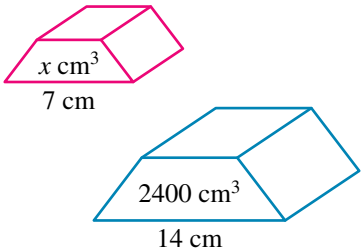
3. **WE28** Find the unknown length for the pair of similar figures.



4. Two similar trapezium-shaped strips of land have an area of 0.5 hectares and 2 hectares. The larger block has a distance of 50 metres between the parallel sides. Find the same length in the smaller block.
5. Find the unknown area for the pair of similar figures.

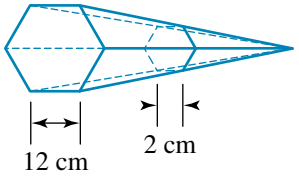


6. **WE29** Find the unknown volume in the pair of similar objects.



7. Find the unknown volume in the pair of similar objects.

Volume of small pyramid = 40 cm^3

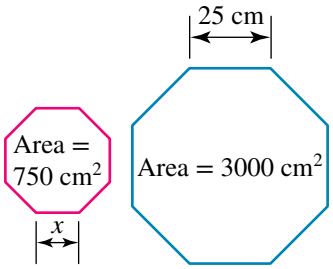


8. Complete the following table of values.

Linear scale factors, k	Area scale factors, k^2	Volume scale factors, k^3
2		8
	16	
3		
		125
	100	
	64	
		0.027
	36	
0.1		
100		
	0.16	
	400	

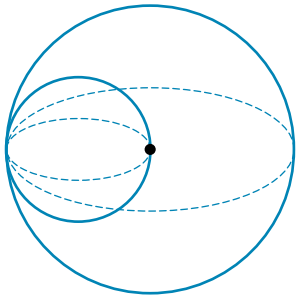
9. a. Find the unknown length for the pair of similar figures.

b. Two photographs have areas of 48 cm^2 and 80 cm^2. The smaller photo has a width of 6 cm. Find the width of the larger photo.



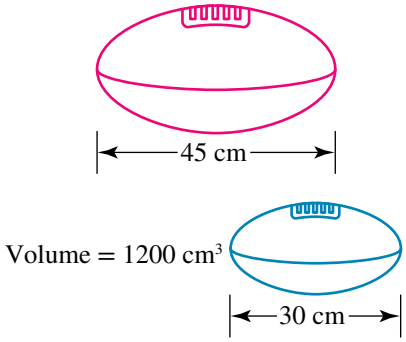
10. Find the unknown volume for the following pairs of similar objects.

a.



Volume of large sphere = 8 litres

b.



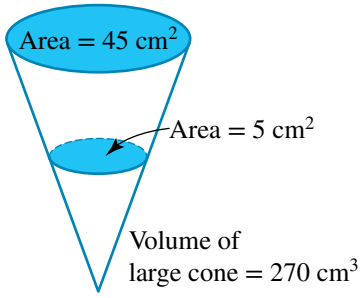
11. **WE30** For a baseball with diameter of 10 cm and a basketball with a diameter of 25 cm, calculate the total surface area of the baseball if the basketball has a total surface area of 1963.5 cm^2.

12. For a 14-inch car tyre and 20-inch truck tyre that are similar, calculate the volume (correct to the nearest litre) of the truck tyre if the car tyre has a volume of 70 litres.

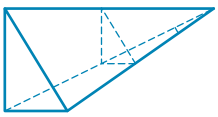
13. **a.** For two similar triangular pyramids with volumes of 27 m^3 and 3 m^3, calculate the total surface area of the larger triangular prism if the smaller prism has a total surface area of 1.5 m^2.

b. For two similar kitchen mixing bowls with total surface areas of 1500 cm^2 and 3375 cm^2, calculate the capacity of the larger bowl if the smaller bowl has a capacity of 1.25 litres (correct to the nearest quarter of a litre).

14. **a.** Calculate the volume of the small cone.



b. Calculate the volume of the larger triangular pyramid.

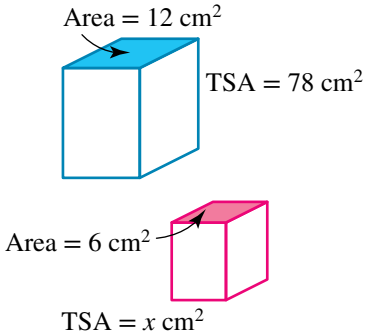


TSA of small pyramid = 200 cm^2

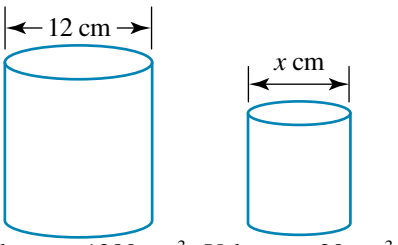
Volume of small pyramid = 1000 cm^3

TSA of large pyramid = 288 cm^2

c. Calculate the total surface area of the small prism.



d. Calculate the diameter of the small cylinder.



Volume = 1280 cm^3 Volume = 20 cm^3

15. What is the area ratio of:

a. **i.** two similar squares with side lengths of 3 cm and 12 cm

ii. two similar circles with diameters of 9 m and 12 m

iii. two similar regular pentagons with sides of 16 cm and 20 cm

iv. two similar right-angled triangles with bases of 7.2 mm and 4.8 mm?

b. Calculate the volume ratios from the similar shapes given in part **a**.

16. MC The ratio of the volume of the blue portion to the volume of the pink portion is:

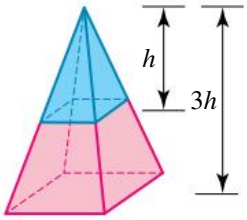
A. 1 : 3

B. 1 : 8

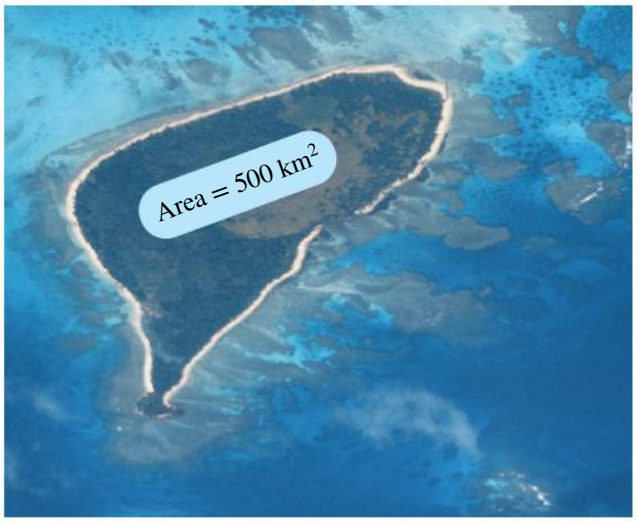
C. 1 : 9

D. 1 : 26

E. 1 : 27



17. An island in the Pacific Ocean has an area of 500 km^2. What is the area of its representation on a map drawn to scale of 1 cm ⇔ 5 km?



18. Two statues of a famous person used 500 cm^3 and 1.5 litres of clay. The smaller statue stood 15 cm tall. What is the height of the other statue (correct to the nearest centimetre)?

19. A 1 : 12 scale model of a car is created from plaster and painted.

a. If the actual car has a volume of 3.5 m^3, calculate the amount of plaster needed for the model to the nearest litre.

b. The model needed 25 millilitres of paint. How much paint would be needed for the actual car (correct in litres to 1 decimal place)?

20. **i.** The ratio of the volume of two cubes is 27 : 8. What is the ratio of:

a. the lengths of their edges

b. the total surface area?

ii. A cone is filled to half its height with ice-cream. What is the ratio of ice-cream to empty space?

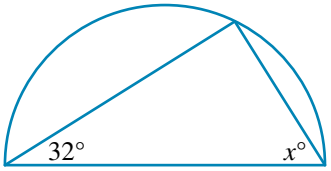
11.10 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC For the triangle shown in a semicircle, x is:

A. 32° **B.** 58°
C. 68° **D.** 90°
E. none of the above

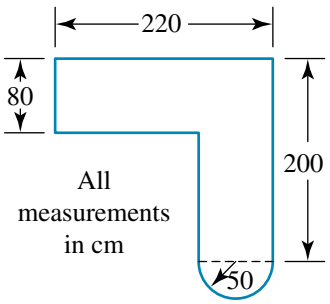


2. MC A triangle ΔABC has the following values: $\overline{AB} = 10$ cm, $\overline{AC} = 12$ cm where $\overline{AB}$ and $\overline{AC}$ are perpendicular. The area of the triangle is:

A. 120 cm^2 **B.** 30 cm^2 **C.** 240 cm^2 **D.** 121 cm^2 **E.** 60 cm^2

3. MC The area of the kitchen bench shown in the plan is closest to:

A. $1250\pi + 19\,600$ cm^2
B. $1250\pi + 37\,600$ cm^2
C. $1250\pi + 29\,600$ cm^2
D. $2500\pi + 29\,600$ cm^2
E. 30 100 cm^2

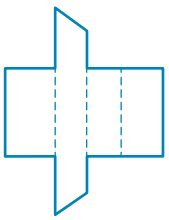


4. MC The total surface area of a closed cylinder with a radius of 40 cm and a height of 20 cm is given by:

A. $2 \times \pi \times 20 \times (40)$ **B.** $2 \times \pi \times 40 \times (40)$ **C.** $2 \times \pi \times 40 \times (100)$
D. $2 \times \pi \times 40 \times (60)$ **E.** $2 \times \pi \times 20 \times (60)$

5. MC The net of an object is shown in the diagram. An appropriate name for the object is:

A. rectangular prism
B. rectangular pyramid
C. triangular prism
D. triangular pyramid
E. trapezoidal prism

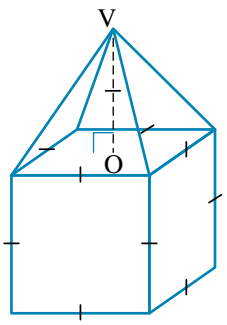


6. MC The volume of a sphere with a diameter of 15 cm is closest to:

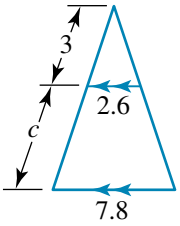
A. 560π cm^3 **B.** 900 cm^3 **C.** 3600 cm^3 **D.** 4500 cm^3 **E.** 3600 cm^3

7. MC The volume of the composite object, given that $\overline{VO} = 10$ cm, is closest to:

A. 1000 cm^3
B. 1300 cm^3
C. 1500 cm^3
D. 2000 cm^3
E. 10 000 cm^3

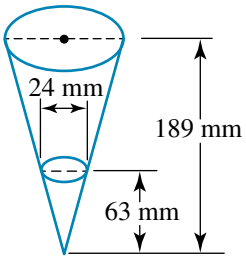


8. MC In the triangle shown, the value of c is:



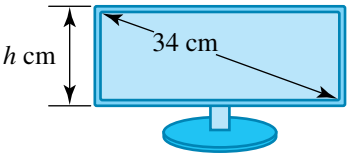
A. 3
B. 6
C. 9
D. 12
E. 4

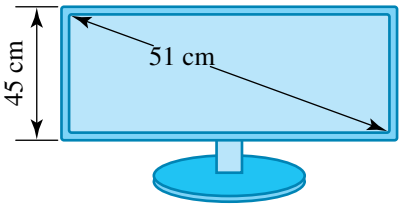
9. MC The circumference of the circle at the base of the larger cone is closest to:



A. 113 mm
B. 151 mm
C. 226 mm
D. 302 mm
E. 459 mm

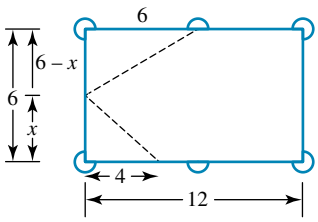
10. MC The diagonal distance on the plasma screen is used to specify the different sizes available. If the height on a 51 cm plasma is 45 cm, then a similar 34 cm television has a height, h, which is closest to:





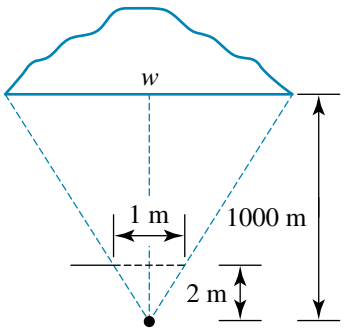
A. 67 cm
B. 45 cm
C. 34 cm
D. 30 cm
E. 26 cm

11. MC The diagram shows the path of a pool ball into the middle pocket of a 12 by 6 rectangular billiard table. To achieve this, the expression for the value of x is:



A. $\frac{6}{4} = \frac{6-x}{x}$
B. $\frac{4}{6} = \frac{6-x}{x}$
C. $\frac{6}{4} = \frac{x-6}{x}$
D. $\frac{12}{6} = \frac{6-x}{x}$
E. $\frac{6}{4} = \frac{2+x}{x}$

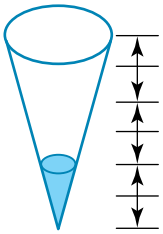
12. MC Jennifer is standing 2 metres directly in front of her bedroom window, which is 1 metre wide. The width (w) of her view of a mountain range 1 kilometre from her window is (correct to the nearest metre):



A. 1002 metres
B. 1000 metres
C. 499 metres
D. 501 metres
E. 500 metres

13. MC The large cone is filled to one-third of its height with water as shown. The ratio of the volume of water to air is:

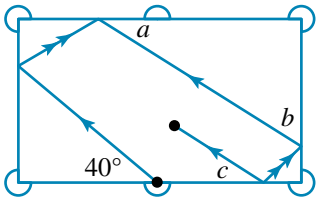
A. 1 : 27
B. 1 : 26
C. 27 : 1
D. 1 : 9
E. 1 : 3



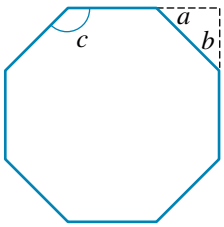
Short answer

1. For each of the figures, calculate the values of the pronumerals.

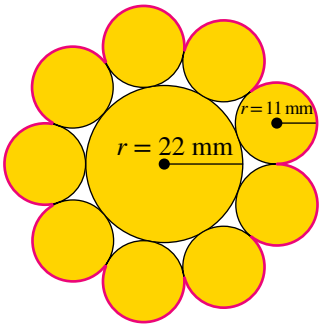
a.



b.

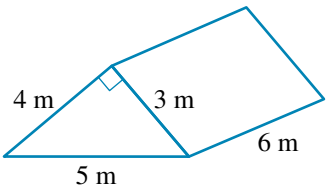


2. Calculate the outer perimeter (shown in red) and shaded area of the flower.

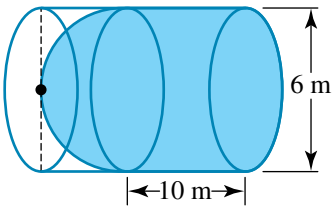


3. For the triangular prism:

a. Sketch an appropriate net for the given solid prism.
b. Transfer the units appropriately to the net from part **a**.
c. Calculate the total surface area of the object.



4.



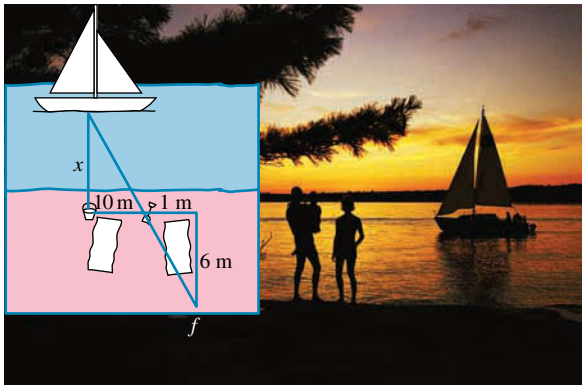
a. What is the volume contained by the solid and framed sections (correct to 1 decimal place)?
b. What is the volume of the solid part only?
c. What is the total surface area of the solid part only (correct to 1 decimal place)?

5. The dimensions of a rectangular prism tub are 30 cm by 20 cm by 15 cm. The tub is filled completely with water and then transferred into a cylinder tank that is 10 cm in radius and 40 cm tall. How high is the water level in the cylinder?

6. Two ladders are placed against the wall at the same angle. They are 2 metres and 3 metres long. If the taller ladder reaches 2.1 metres up the wall, how far up will the second ladder reach (correct to 1 decimal place)?

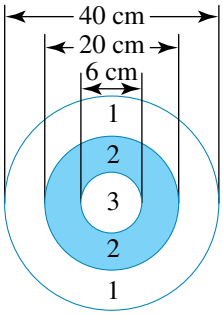
7. A yacht is an unknown distance from the shore. A family sitting on the beach at F make the measurements as shown in the diagram. How far is it to the yacht (correct to the nearest metre)?

8. A plan is drawn to scale of 1 : 50 000. Find:



   a. the length in centimetres on the plan that represents 1 km
   b. the area in hectares of a region represented by 4 cm^2 on the plan
   c. the area on the plan of a region of 25 hectares.

9. A 3-ring dartboard has dimensions as shown. (Give all answers to 1 decimal place.)



   a. What is the total area of the dartboard?
   b. What is the area of the bullseye (inner circle)?
   c. What is the area of the 2-point middle ring?
   d. Express each area of the three rings as a percentage of the total area (where appropriate correct to 2 decimal places).

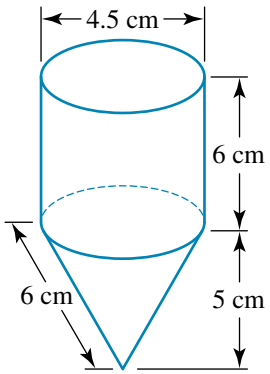
Extended response

Use the following information to answer questions **1** and **2**.

A rectangular block of modelling clay has dimensions of 30 cm by 20 cm by 10 cm.

1. a. What is the volume of the block of clay?
   b. Express, in litres, your answer from question **1 a**.
   c. What is the total surface area of the clay?
2. The entire block of clay is remoulded to the shape of a cylinder with a height of 30 cm.
   a. Find the diameter of the cylindrical block of clay (correct to 2 decimal places).
   b. Find the new total surface area of the clay when moulded as a cylinder (correct to the nearest cm^2).
   c. What fraction of the volume needs to be removed to turn the cylindrical block into a cone with the same diameter and height?

3. Clay is moulded to the shape shown to represent a 1:100 scale model of a grain silo.



   a. Find the volume of clay needed to make a scale model grain silo (correct to 1 decimal place).
   b. Find the actual volume of the grain silo. Express your answer correct to the nearest cubic metre.
   c. What is the ratio of the volume of the model to the volume of the actual grain silo?
   d. If the scale model has a total surface area of 143.14 cm^2, find the total surface area of the actual silo.

4. Another silo, half the size of the silo in question **3**, is to be built. What fraction will this smaller silo be in volume compared to the larger silo?

5. The rectangular rear window of a car has an area of 1.28 m^2.
   a. Find the height of the rear window if its length is 160 centimetres (correct to the nearest centimetre).
   b. A wiper blade is 50 centimetres long and just reaches the top of the window as it makes a semicircular sweep. The base of the wiper is situated at the bottom centre of the rear window.
      i. Draw a diagram of the situation.
      ii. Find the area of the window swept by the wiper (correct to the nearest cm^2).
      iii. Find the percentage of the window's area not swept by the wiper.
   c. The manufacturer decides to increase the wiper length by 10 centimetres
      i. Find the new area of the window that is swept by the wiper (correct to the nearest cm^2).
      ii. Find the percentage of the window's area that is not swept by the wiper.

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Answers

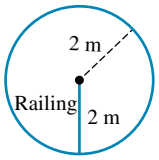
Topic 11 Geometry and measurement

Exercise 11.2 Review of properties of angles, triangles and polygons

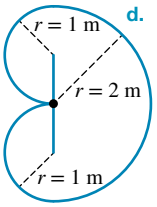
1. 140°, 40°
2. 144°, 36°
3. **a.** 60°, 120° **b.** 90°, 90° **c.** 120°, 60° **d.** 135°, 45° **e.** $128\frac{4}{7}°, 51\frac{3}{7}°$
4. $b = 8$ cm, $c = 50°$
5. 148°
6. $m = 117°$, $n = 63°$
7. 59°
8. **a.** 79° **b.** $x = 130°$, $y = 50°$ **c.** 27° **d.** $a = 15°$, $b = 165°$, $c = 165°$
9. **a.** $x = 35°$, $y = 145°$ **b.** 30° **c.** 28° **d.** $a = 70°$, $b = 110°$, $c = 70°$, $d = 110°$
10. **a.** Pentagon (5-sided) **b.** Dodecagon (12-sided) **c.** Octagon **d.** Hexagon **e.** Equilateral triangle
11. D
12. A
13. **a.** $a = 93°$ **b.** $x = 35°$, $y = 35°$ **c.** $t = 20°$
14. **a.** $a = 70°$, $b = 110°$, $c = 70°$, $d = 110°$ **b.** $c = 52°$
15. **a.** $m = 118°$, $n = 62°$ **b.** $x = 30°$, $y = 40°$, $z = 110°$ **c.** $a = 9$ cm (isosceles triangle), $b = 40°$
16. **a.** $a = 45°$, $b = 45°$ **b.** $a = 30°$, $b = 75°$
17. **a.** $h = 7.2$ cm, $r = 2.1$ cm **b.** $x = 35°$, $y = 35°$, $z = 110°$
18. **a.** $a = 86°$, $b = 94°$, $c = d = 43°$ **b.** $a = 40°$, $b = 50°$

Exercise 11.3 Review of area and perimeter of composite shapes

1. **a.** 70 cm **b.** 17.59 m
2. 112 m
3. **a.** 28 m **b.** 34 m **c.** 82.5 cm **d.** 496.4 m
4. 24 m^2
5. 7 m^2
6. **a.** 49 m^2 **b.** 48 m^2 **c.** 394 cm^2 **d.** 11 550 m^2
7. **a.** $A_{total} = 90\ m^2$ **b.** 66.85 cm
8. **a.** $A_{total} = 175\ cm^2$ **b.** 136 mm^2
9. **a.** 680 m^2 **b.** 313 cm^2
10. **a.** 106 m **b.** 76.2 cm
11. 25 900 cm^2
12. 3456 mm^2
13. 2.5 km^2
14. 201.06 km^2
15. **a.** 200 cm^2 **b.** 32 m^2 **c.** 350 cm^2 **d.** 35 000 mm^2 **e.** 2.5 km^2 **f.** 35.7 hectares **g.** 2750 m^2 **h.** 60 m^2
16. 3.74 m^2
17. C
18. **a.** 10 cm^2 **b.** 32 m^2 **c.** 118.8 m^2
19. **a.** 21 cm **b.** 27 m **c.** 47 m
20. **a.** 4.5 m^2 **b.** 710 000 mm^2 **c.** 21.6 hectares **d.** 73 700 m^2
21. E
22. 48 932 mm
23. **a.**



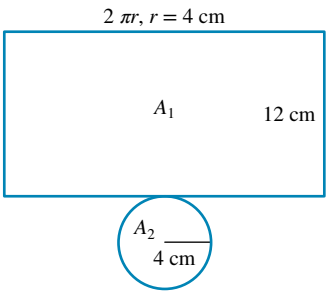
b. 12.6 m^2 **c.**



d. 9.4 m^2

Exercise 11.4 Total surface area

1. 54.3 cm^2
2. 395.8 cm^2
3. 8.0 m^2
4. 72 600 cm^2
5. 6.5 m
6. 7.9 cm
7. 5200 cm^2
8. **a.**



b. TSA $= A_1 + A_2$

$A_1 = 2\pi r \times h$

$= 2 \times \pi \times 4 \times 12$

$= 301.59\ cm^2$

$A_2 = \pi \times r^2$

$= \pi \times 4^2$

$= 50.27\ cm^2$

c. TSA $= A_1 + A_2$
$= 301.59 + 50.27$
$= 351.86\text{ cm}^2$

d. The total surface area of the open cylindrical can is 351.9 cm^2, correct to 1 decimal place.

9. a. $109\,350\text{ mm}^2$ **b.** 392 m^2
c. 633.5 cm^2 **d.** $17\,278.8\text{ mm}^2$

10. a. 4.9 m^2
b. 364 cm^2
c. 5808.8 mm^2 or 58.0 cm^2

11. a. 2 m **b.** 7.1 cm
c. 31.5 cm **d.** 224.0 cm

12. a. 530 cm^2 **b.** 672.4 cm^2
c. 564 cm^2 **d.** 80.6 mm^2

13. 301 cm^2

14. 907.13 cm^2

15. a. $245\,436.9\text{ mm}^2$ **b.** 914.8 cm^2 **c.** 123 cm^2

16. 50 m^2, $500\,000\text{ cm}^2$

17. 99.25 m^2

18. C

19. B

20. E

21. 748.63 cm

22. a. Cylinder loaf **b.** 108.7 cm^2

Exercise 11.5 Volume of prisms, pyramids and spheres

1. 350 mm^3

2. 0.0048 m^3

3. 1600 litres

4. 250 cm^3

5. a. $0.00\,053\text{ m}^3$ **b.** 56 litres **c.** $15\,000\text{ cm}^3$
d. 72 100 litres **e.** 2.3 mm^3 **f.** 570 cm^3
g. 0.14 litres **h.** 16 cm^3

6. $10\,200\text{ mm}^3$

7. $17\,556\text{ mm}^3$

8. 1740 mm^3

9. 8000 cm^3

10. a. $70\,685\,835\text{ mm}^3$ **b.** 1012 cm^3
c. $562\,332\text{ cm}^3$ **d.** 32 m^3

11. a. 12.18 m^3 **b.** 399.5 cm^3

12. 8.2 cm

13. 40 mm

14. a. 1.2 m **b.** 15 cm

15. 440 cm^3

16. 272 m^3

17. a. 4435 cm^3 **b.** 864 mm^3 **c.** 240 cm^3

18. $2.49 \times 10^{12}\text{ m}^3$

19. $7.854 \times 10^{-6}\text{ m}^3$

20. a. 2145 cm^3 **b.** 2960 cm^3 **c.** 179 m^3
d. 2089 cm^3 **e.** 39 m^3 **f.** $88\,828\text{ m}^3$

21. a. 91.125 cm^3 **b.** 22.05 m^3 **c.** 3.1 cm
d. 8 cm **e.** 8 mm **f.** $1\frac{1}{3}$ m

22. B

23. E

24. A

25. 21 mL

26. a. 26 cm **b.** 862.8 cm^3 **c.** 575.2 cm^3
d. 287.6 cm^3 **e.** $\frac{2}{3}$

Exercise 11.6 Similar figures

1. a. $k = 4$ **b.** $x = 280$, $y = 200$

2. 0.002 333

3. $k = 1.1$, $a = 17.6$

4. $k = 0.05$, $a = 67°$

5. a. i. 0.05 **ii.** $x = 125$, $y = 5$
b. i. 0.8 **ii.** $x = 3.2\text{ cm}$, $y = 6.4\text{ cm}$

6. a. 25.5 mm **b.** 1.2 m

7. 25 cm by 30 cm

8. a. 2.4 m **b.** 36 cm **c.** 25 cm

9. 215 mm by 178 mm, landscape

10. C

11. E

12. D

13. $\frac{4}{3}$

14. 2

15. 314.16 cm

16. 4.52 cm

Exercise 11.7 Similar triangles

1. a. SSS, $k = 3.5$ **b.** 7.2

2. a. AAA, $k = 3$ **b.** 10

3. a. SAS, $k = \frac{4}{3}$ or $1.\dot{3}$ **b.** SSS, $k = 2$
c. AAA or SAS, $k = 2$ **d.** AAA, $k = 1.5$

4. 10.8

5. 426

6. a. 30 **b.** 18 **c.** 15 **d.** 42

7. a. 8 **b.** 4 **c.** 38 **d.** 72

8. 732 cm

9. 5.5 m

10. 810 cm

11. 21.5 metres

12. 16 m

13. 15 m

14. D

15. AAA; $k = \frac{11}{6}$; $x = 2.83$

16. AAA; $k = \frac{1}{3}$; $a = 14.3\text{ cm}$

17. B

18. D

Exercise 11.8 Triangulation — similarity

1. 31 m
2. 59.82 m
3. 48 m
4. 32 m
5. 13.09 m
6. 200 m
7. 25.6 m
8. 15.33 m
9. 138 cm^2
10. C
11. A
12. 152 cm
13. 80.07 m
14. **a.** **i.** 362 m **ii.** 1800 m **iii.** 7 km
    b. **i.** 722 m **ii.** 3600 m **iii.** 14 km

Exercise 11.9 Area and volume scale factors

1. 432 cm^2
2. 225 mm^2
3. 4.25 m
4. 25 m
5. 240 mm^2
6. 300 cm^3
7. 8640 cm^3
8.

Linear scale factors k	Area scale factors k^2	Volume scale factors k^3
2	4	8
4	16	64
3	9	27
5	25	125
10	100	1000
8	64	512
0.3	0.09	0.027
6	36	216
0.1	0.01	0.001
100	10 000	1 000 000
0.4	0.16	0.064
20	400	8000

9. **a.** 12.5 cm **b.** 7.75 cm
10. **a.** 1 litre **b.** 4050 cm^3
11. 314.2 cm^2
12. 204 litres
13. **a.** 6.5 m^2 **b.** 4.25 litres
14. **a.** 10 cm^3 **b.** 1728 cm^3 **c.** 39 cm^2 **d.** 3 cm
15. **a.** **i.** 16 **ii.** $\frac{16}{9}$ **iii.** $\frac{25}{16}$ **iv.** $\frac{4}{9}$
    b. **i.** 64 **ii.** $\frac{64}{27}$ **iii.** $\frac{125}{64}$ **iv.** $\frac{8}{27}$
16. D
17. 20 cm^2
18. 22 cm
19. **a.** 2 litres **b.** 3.6 litres
20. **i.** **a.** 3:2 **b.** 9:4 **ii.** 1:7

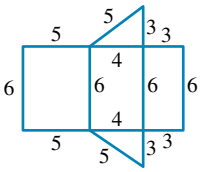
11.10 Review: exam practice

Multiple choice

1. B
2. E
3. C
4. D
5. E
6. A
7. B
8. B
9. C
10. D
11. A
12. E
13. B

Short answer

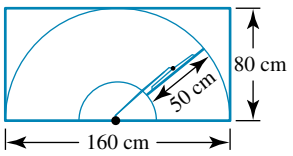
1. **a.** $a = 40°$, $b = 50°$, $c = 40°$
   b. $a = 45°$, $b = 45°$, $c = 135°$
2. Area = 4942 mm^2, perimeter = 311 mm
3. **a, b**



   b. 84 m^2
4. **a.** 367.6 m^3 **b.** 339.3 m^3 **c.** 273.3 m^2
5. 28.6 cm
6. 1.4 m
7. 60 m
8. **a.** 2 cm **b.** 100 hectares **c.** 1 cm^2
9. **a.** 1256.6 cm^2 **b.** 28.3 cm^2 **c.** 285.9 cm^2
   d. 75%, 22.75%, 2.25%

Extended response

1. **a.** 6000 cm^3 **b.** 6 litres **c.** 2200 cm^2
2. **a.** 15.96 cm **b.** 1904 cm^2 **c.** $\frac{2}{3}$
3. **a.** 121.9 cm^3 **b.** 122 m^3
   c. 1 : 1000000 **d.** 143.14 m^2
4. $\frac{1}{8}$
5. **a.** 80 cm
   b. **i.**



   ii. 8639 cm^2
   iii. 32.5%
   c. **i.** 9425 cm^2 **ii.** 26.4%

TOPIC 12
Geometry and trigonometry

12.1 Overview

12.1.1 Introduction

Trigonometry is a branch of mathematics that is used to solve problems involving the relationships between the sides and angles of triangles. It builds on previous knowledge of geometrical concepts including the properties of parallel lines and triangles. Like many of the advances in mathematics, it was the study of the stars that led the ancient Greeks to further discoveries in trigonometry. Hipparchus, who lived in the second century BC, spent time studying the movements of the sun, moon and planets and is thought to be the first person to develop trigonometry to the extent that he calculated the length of the lunar month to within one second. Other achievements of his included an accurate method of predicting solar eclipses and the production of one of the first star catalogs. Today trigonometry is used in many areas, for example, surveying, construction, engineering, architecture, search and rescue, orienteering, GPS systems, design and computer graphics.

LEARNING SEQUENCE

12.1 Overview
12.2 Two- and three-dimensional Pythagoras' theorem
12.3 Trigonometric ratios
12.4 The sine rule
12.5 Ambiguous case of the sine rule
12.6 The cosine rule
12.7 Area of triangles
12.8 Angles of elevation and depression
12.9 Bearings and specification of location
12.10 Triangulation – cosine and sine rules
12.11 Review: exam practice

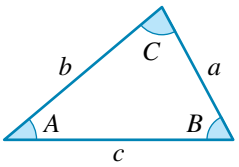
Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

12.1.2 Kick off with CAS

Exploring the sine rule with CAS

The sine rule can be used to find unknown side lengths and angles in triangles.
For any triangle ABC, the sine rule states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Before completing any questions involving angles on your CAS, ensure that your angle setting is in DEGREEs mode.

1. Use CAS to determine the value of the following:
   - a. i. $\sin(40°)$ ii. $\sin(140°)$
   - b. i. $\sin(25°)$ ii. $\sin(155°)$
2. a. What do you notice about your answers to questions 1a and 1b? Can you spot a relationship between the given angles?
   - b. Use CAS to explore if this works for other pairs of angles with the same relationship.
3. Use the sine rule to determine the size of angle B in the domain $0° < 90°$ given:

$$a = 16.2,\ b = 25.1,\ A = 33°$$

4. Which angle between 90° and 180° gives the same sine value as your answer to question 3?
5. Is your answer to question 4 a potential second solution to your triangle (from question 3)? Try to draw a triangle with these given values.

on Resources

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

12.2 Two- and three-dimensional Pythagoras' theorem

12.2.1 Pythagoras' theorem

Pythagoras' theorem is attributed to the Greek mathematician and philosopher, Pythagoras, around 500 BC. (However, the principle was known much earlier, and it seems that even the pyramid builders of ancient Egypt used the theorem in constructing the pyramids.)

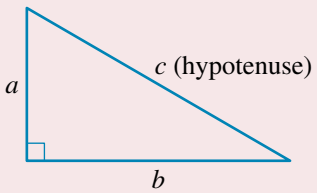
The theorem describes the relationship between the lengths of the sides of all **right-angled triangles**.

Pythagoras' theorem states that the square of the hypotenuse is equal to the sum of the squares of the other two sides, or

$$c^2 = a^2 + b^2$$

and, therefore, to find c,

$$c = \sqrt{a^2 + b^2}$$



where c is the longest side or hypotenuse and a and b are the two shorter sides.

Note: Because the equation $c^2 = a^2 + b^2$ has become a standard way of expressing Pythagoras' theorem, we often adjust the labelling convention to use c for the hypotenuse no matter how the opposite (right) angle and vertex is labelled. However, this will always be clear from the diagram.

The longest side is always opposite the largest angle (90° for right-angled triangles) and similarly, the shortest side is opposite the smallest angle.

To find one of the shorter sides (for example, side a), the formula is rearranged to:

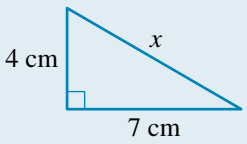
$$a^2 = c^2 - b^2$$

and so to find a,

$$a = \sqrt{c^2 - b^2}.$$

WORKED EXAMPLE 1

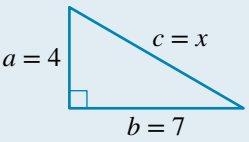
Find the length of the unknown side (correct to 1 decimal place) in the right-angled triangle shown.



THINK

1. Note that the triangle is right-angled and we need to find the unknown *length*, given the other two lengths.
2. Label the sides of the triangle, using the convention that c is the hypotenuse.
3. Substitute the values into the appropriate formula.
4. Write the answer using the correct units and to the appropriate degree of accuracy.

WRITE/DRAW



$c^2 = a^2 + b^2$
$x^2 = 4^2 + 7^2$
$= 16 + 49$
$= 65$
$x = \sqrt{65}$
$= 8.0623$

Alternatively,
$c = \sqrt{a^2 + b^2}$
$x = \sqrt{4^2 + 7^2}$
$= \sqrt{16 + 49}$
$= \sqrt{65}$

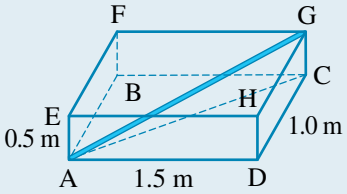
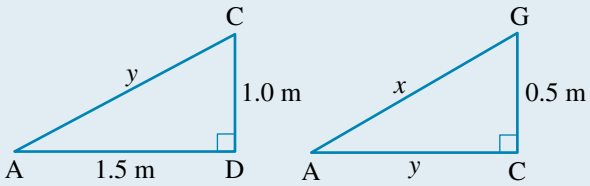
The unknown side's length is 8.1 cm, correct to 1 decimal place.

12.2.2 Three-dimensional Pythagoras' theorem

Many practical situations involve three-dimensional objects with perpendicular planes and therefore the application of Pythagoras' theorem. To solve three-dimensional problems, a carefully drawn and labelled diagram will help. It is also of benefit to identify right angles to see where Pythagoras' theorem can be applied. This enables you to progress from the known information to the unknown value(s).

WORKED EXAMPLE 2

Correct to the nearest centimetre, what is the longest possible thin rod that could fit in the boot of a car? The boot can be modelled as a simple rectangular prism with the dimensions of 1.5 metres wide, 1 metre deep and 0.5 metres high.

THINK	WRITE/DRAW
1. Draw a diagram of the rectangular prism.	
2. Identify the orientation of the longest object — from one corner to the furthest diagonally opposite corner. In this case, it is AG.	
3. Identify the two right-angled triangles necessary to solve for the two unknown lengths.	
4. Draw the triangles separately, identifying the lengths appropriately.	
5. Calculate the length of diagonal AC.	$c^2 = a^2 + b^2$ $y^2 = 1.5^2 + 1.0^2$ $= 2.25 + 1$ $= 3.25$ $y = \sqrt{3.25}$ $= 1.803$ (correct to 3 decimal places) The length of AC is 1.8 metres (correct to 1 decimal place).
6. Calculate the length of diagonal AG, using the calculated length for AC. *Note:* To avoid *rounding error*, use the most accurate form, which is the surd $\sqrt{3.25}$.	$c = \sqrt{a^2 + b^2}$ (alternative form) $x = \sqrt{0.5^2 + (\sqrt{3.25})^2}$ $= \sqrt{0.25 + 3.25}$ $= \sqrt{3.5}$ $= 1.8708$ (m)
7. Write the answer using the correct units and level of accuracy.	The longest rod that could fit in the car boot is 187 centimetres, correct to the nearest centimetre.

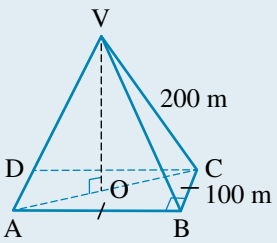
WORKED EXAMPLE 3

To find the height of a 100-metre square-based pyramid, with a slant height of 200 metres as shown, calculate the:

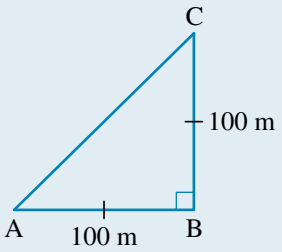
a. length of AC (in surd form)

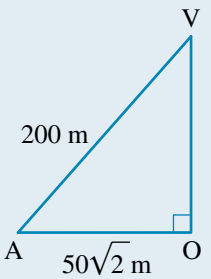
b. length of AO (in surd form)

c. height of the pyramid VO (correct to the nearest metre).



THINK	WRITE
a. Calculate the length of diagonal AC in the right-angled triangle, ABC. Write surds in their simplest form.	**a.** $c = \sqrt{a^2 + b^2}$ (alternative form) $AC = \sqrt{100^2 + 100^2}$ $= \sqrt{20\,000}$ $= \sqrt{10\,000} \times \sqrt{2}$ $= 100 \times \sqrt{2}$ $= 100\sqrt{2}$ The length of AC is $100\sqrt{2}$ metres.
b. AO is half the length of AC.	**b.** Length of AO is $\frac{100\sqrt{2}}{2}$ or $50\sqrt{2}$ metres.
c. 1. Calculate the height of the pyramid, VO, in the right-angled triangle, VOA.	**c.** $a = \sqrt{a^2 + b^2}$ (alternative form) $VO = \sqrt{200^2 - (50\sqrt{2})^2}$ $= \sqrt{40\,000 - 5000}$ $= \sqrt{35\,000}$ $= 187.0829$
2. Write the answer using the correct units and level of accuracy.	The height of the pyramid, VO, is 187 metres, correct to the nearest metre.





on Resources

Interactivity: Pythagoras' theorem (int-6473)

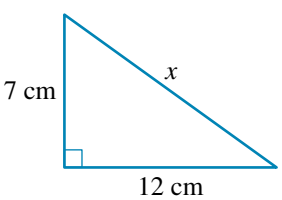
studyon

Units 3 & 4 > AOS M3 > Topic 1 > Concept 1

Right-angled triangles Summary screen and practice questions

Exercise 12.2 Two- and three-dimensional Pythagoras' theorem

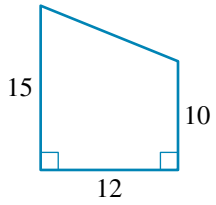
1. **WE1** Find the length of the unknown side (correct to 1 decimal place) in the right-angled triangle shown.



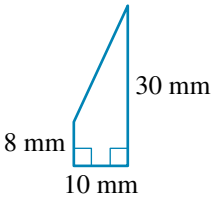
2. **WE2** Find the maximum horizontal distance (correct to the nearest metre) a ship could drift from its original anchored point, if the anchor line is 200 metres long and it is 50 metres to the bottom of the sea from the end of the anchor line on top of the ship's deck.
3. An extension ladder is used to paint windows on a multi-level house. The ladder extends fully to 6 metres and to be safe the base of the ladder is kept 1.5 metres from the house. At full extension how far vertically can the ladder reach to paint the windows (correct to 1 decimal place)?

4. Calculate the lengths of the sloping sides in the following objects. Give your answers correct to 2 decimal places where appropriate. (Remember to construct a suitable right-angled triangle.)

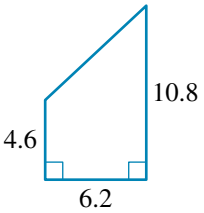
a.



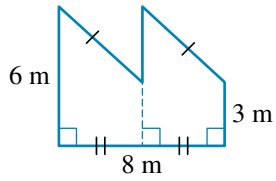
b.



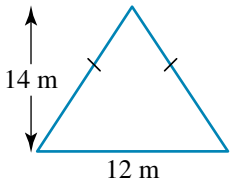
c.



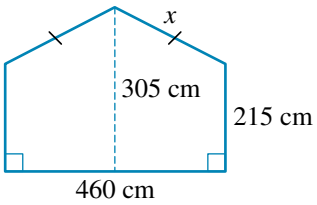
d.



e.

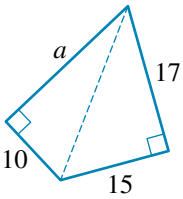


f.

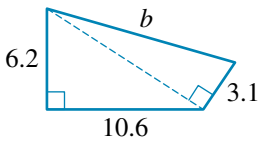


5. Calculate the values of the pronumerals correct to 1 decimal place.

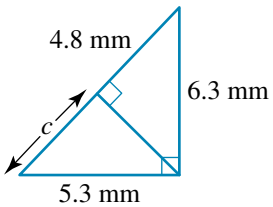
a.



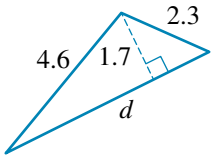
b.



c.

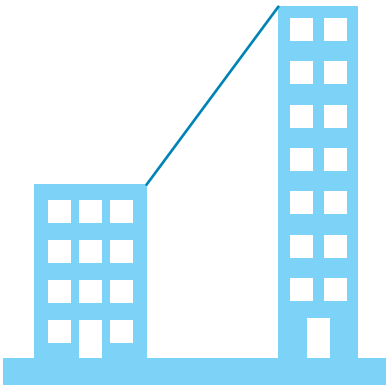


d.

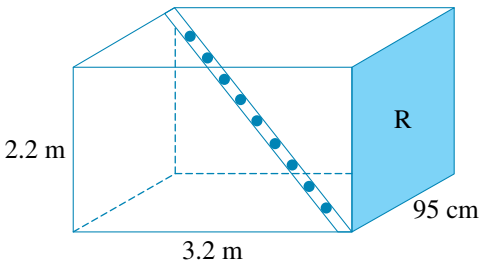


6. **MC** To the nearest metre, the length of cable that would connect the roofs of two buildings that are 40 metres and 80 metres high respectively and are 30 metres apart is:

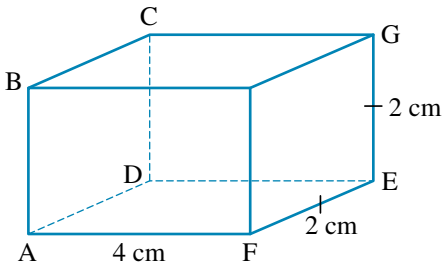
A. 40 metres
B. 45 metres
C. 50 metres
D. 55 metres
E. none of these



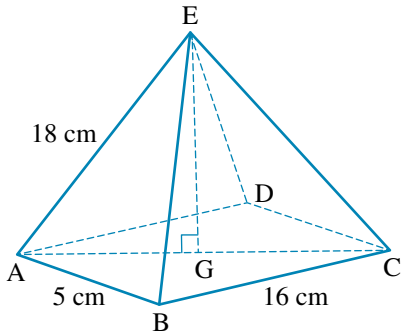
7. **WE2** A wooden plank of the greatest possible length is placed inside a garden shed. Use the diagram to calculate the length of the plank of wood, correct to 1 decimal place.



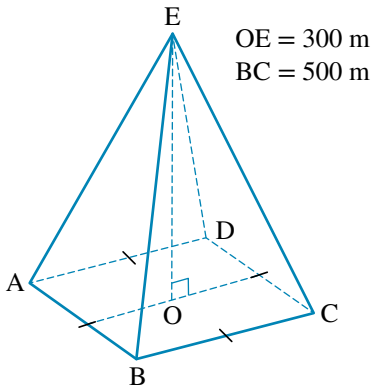
8. Correct to 2 decimal places, calculate the length of:
   a. AE
   b. AG.



9. **WE3** Find:
   a. the length AC (in surd form)
   b. the length AG (in surd form)
   c. the height of the pyramid EG (correct to the nearest cm).

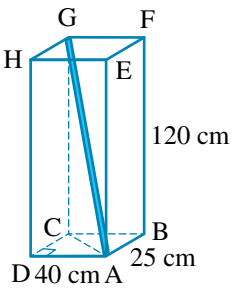


10. Use the diagram of the pyramid to answer this question.
    a. What is the length of the diagonal AC correct to 2 decimal places?
    b. What is the length of EB correct to 2 decimal places?

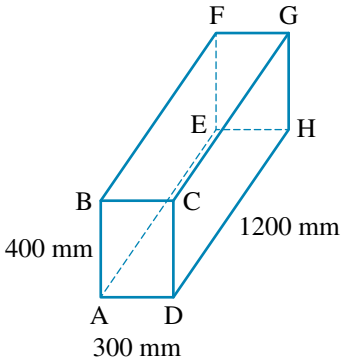


11. Correct to the nearest centimetre, what is the longest thin rod that could fit inside a cube with side length 2 m?
12. Correct to the nearest centimetre, what is the longest drumstick that could fit in a rectangular toy box whose dimensions are 80 cm long by 80 cm wide by 60 cm high?
13. For each of the prisms shown, calculate:
    i. the length of AC correct to the nearest whole number
    ii. the length of AG correct to the nearest whole number.

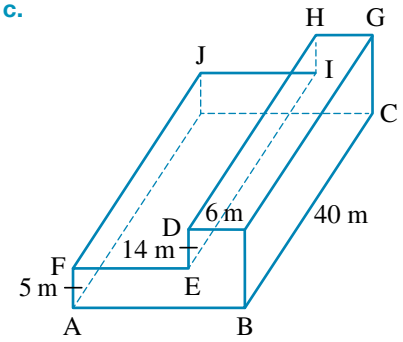
a.



b.



c.

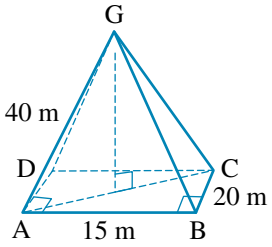


14. For each of the pyramids shown, calculate:

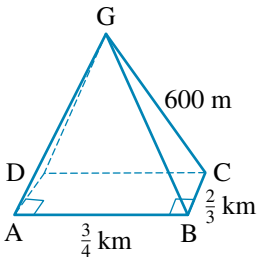
i. the length of AC

ii. the perpendicular height.

a.

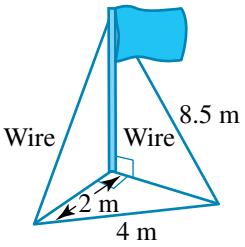


b.



15. A 3.5-metre long ramp rises to a height of 1.2 metres. How long (correct to 1 decimal place) is the base of the ramp?

16. **MC** Two guide wires are used to support a flagpole as shown. The height of the flagpole would be closest to:



A. 3 m

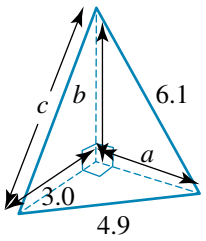
B. 8 m

C. 12 m

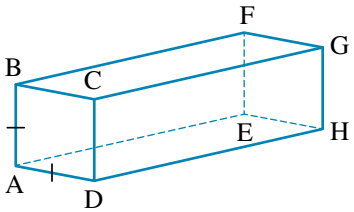
D. 21 m

E. 62 m

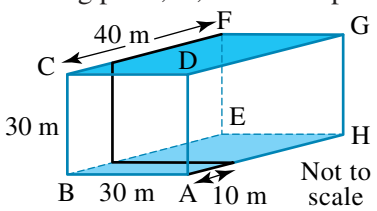
17. Find the values of the pronumerals (correct to 1 decimal place) in the pyramid at right.



18. Find the lengths of AB and DH (correct to 2 decimal places), where AC = 7.00 m and CH = 15.00 m.

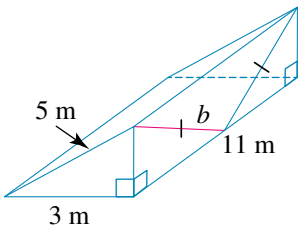


19. A man moves through a two-level maze by following the solid black line, as shown in the diagram. What is the *direct* distance from his starting point, A, to his end point, F (to the nearest metre)?

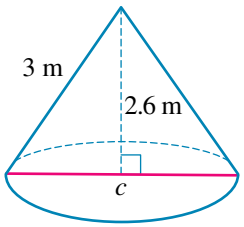


20. In each of the following typical building structures find the length of the unknown cross-brace shown in pink. Give your answers correct to 1 decimal place.

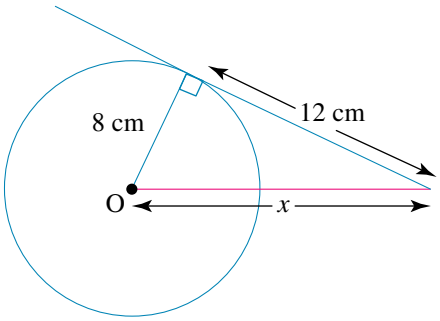
a.



b.



c.



21. For the coffee table design at right, find the length of the legs (correct to the nearest millimetre) if the coffee table is to be:

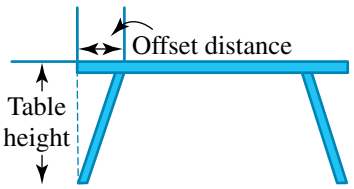
a. 500 mm off the ground

b. 700 mm off the ground

and the legs are offset from the vertical by a distance of:

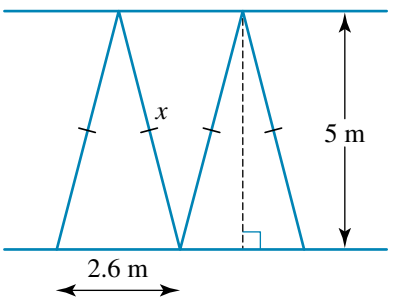
i. 100 mm

ii. 150 mm.

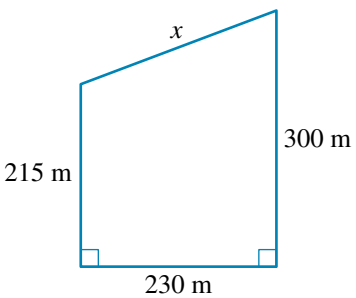


22. Calculate the length of the pronumeral in each of the following. (Where applicable, construct a suitable right-angled triangle.) Give your answer correct to 2 decimal places.

a.



b.



12.3 Trigonometric ratios

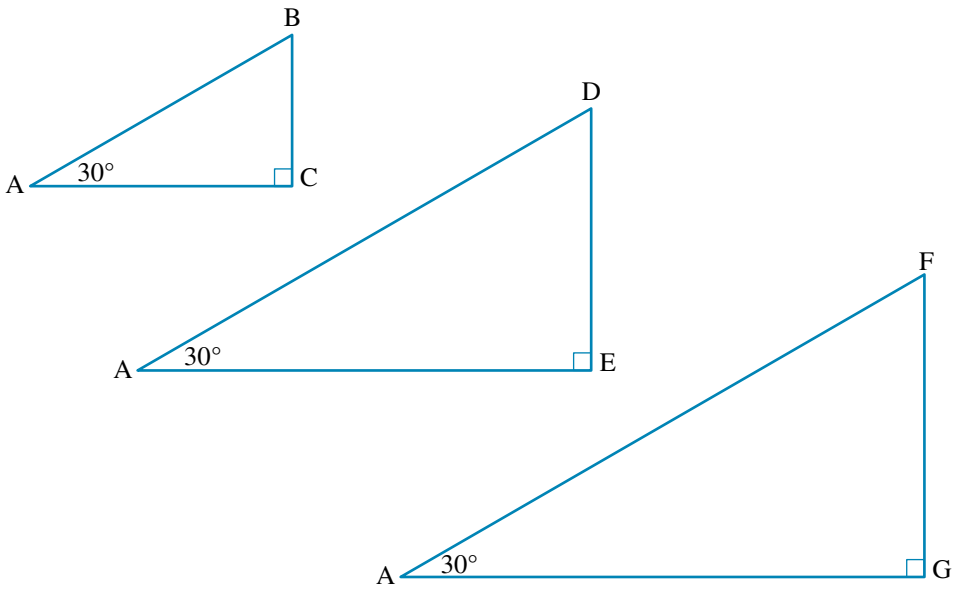
Trigonometric ratios include the sine ratio, the cosine ratio and the tangent ratio; three ratios of the lengths of sides of a right-angled triangle dependent on a given acute angle.

12.3.1 Labelling convention

For the trigonometric ratios the following labelling convention should be applied:

1. The hypotenuse is opposite the right angle (90°).
2. The **opposite side** is directly opposite the given angle, θ.
3. The **adjacent side** is next to the given angle, θ.

Consider the three triangles shown here. We know from the previous topic on similarity that ΔABC, ΔADE and ΔAFG are similar because the corresponding angles are the same. Therefore, the corresponding sides are in the same ratio (scale factor).



12.3.2 Ratio of lengths of sides

Copy and complete the table by identifying and measuring the lengths of the three sides for each of the three triangles shown. Evaluate the ratios of the sides.

	Length of side			Ratio of lengths of sides		
Triangle	**Opposite**	**Adjacent**	**Hypotenuse**	$\frac{\textbf{Opposite}}{\textbf{Hypotenuse}}$	$\frac{\textbf{Adjacent}}{\textbf{Hypotenuse}}$	$\frac{\textbf{Opposite}}{\textbf{Adjacent}}$
ABC						
ADE						
AFG						

Notice that for each of the ratios, for example $\frac{\text{opposite}}{\text{hypotenuse}}$, the value is the same for all three triangles. This is the same for all right-angled triangles with the same acute angle.

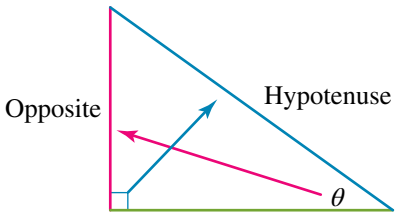
Trigonometric ratios are used in right-angled triangles:
1. **to find an unknown length, given an angle and a side**
2. **to find an unknown angle, given two lengths.**

12.3.3 Sine ratio

The sine ratio is defined as follows:

The sine of an angle $= \frac{\text{length of opposite side}}{\text{length of hypotenuse side}}$.

In short,

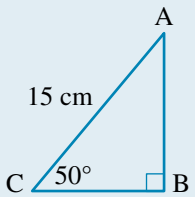


$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin(\theta) = \frac{\text{O}}{\text{H}} \quad \text{[SOH]}$$

WORKED EXAMPLE 4

Find the length (correct to 1 decimal place) of the line joining the vertices A and B in the triangle.



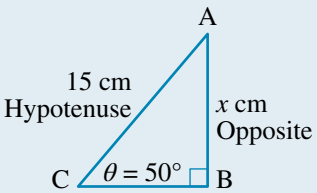
THINK

1. Identify the shape as a right-angled triangle with a given length and angle. Label the sides as per the convention for trigonometric ratios.

2. Identify the appropriate trigonometric ratio, namely the sine ratio, from the given information.

3. Substitute into the formula.

WRITE



Angle $= 50°$
Opposite side $= x$ cm
Hypotenuse $= 15$ cm [SOH]

$$\sin(\theta) = \frac{\text{length of opposite side}}{\text{length of hypotenuse side}}$$

$$\sin(\theta) = \frac{\text{O}}{\text{H}}$$

$$\sin(50°) = \frac{x}{15}$$

4. Isolate x and evaluate.

$$15 \times \sin(50°) = \frac{x}{15} \times 15$$
$$x = 15 \times \sin(50°)$$
$$= 15 \times 0.766$$
$$= 11.491$$

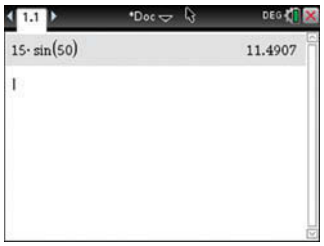
5. Write the answer using the correct units and level of accuracy.

The length of the line joining vertices A and B is 11.5 centimetres, correct to 1 decimal place.

TI | THINK

1. On a Calculator page, complete the entry line as:
   $15 \times \sin(50.0)$
   Press ENTER.

WRITE



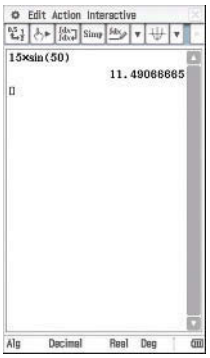
2. The answer appears on the screen.

The length of the line joining vertices A and B is 11.5 cm, correct to 1 decimal place.

CASIO | THINK

1. On the MAIN screen, complete the entry line as:
   $15 \times \sin(50)$
   Press EXE.

WRITE



2. The answer appears on the screen.

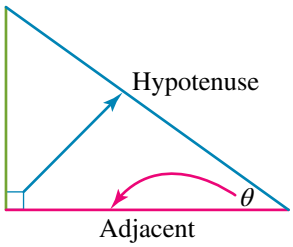
The length of the line joining vertices A and B is 11.5 cm, correct to 1 decimal place.

12.3.4 Cosine ratio

The cosine ratio is defined as follows:

$$\text{The cosine of an angle} = \frac{\text{length of adjacent side}}{\text{length of hypotenuse side}}.$$

In short,

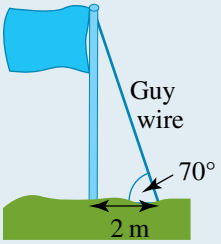
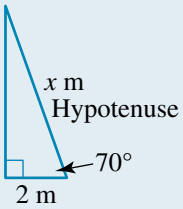


$$\cos(\theta) = \frac{\textbf{adjacent}}{\textbf{hypotenuse}}$$
$$\cos(\theta) = \frac{\mathbf{A}}{\mathbf{H}} \quad \textbf{[CAH]}$$

In Worked example 4 the sine ratio was used to find the unknown length. The cosine ratio can be used in the same way, if it is required.

WORKED EXAMPLE 5

Find the length of the guy wire (correct to the nearest centimetre) supporting a flagpole, if the angle of the guy wire to the ground is 70° and it is anchored 2 metres from the base of the flagpole.

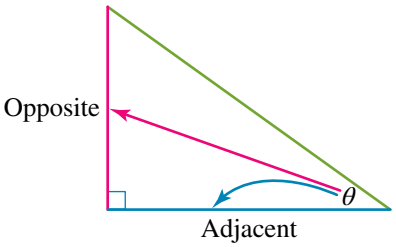
THINK	WRITE/DRAW
1. Draw a diagram to represent the situation and identify an appropriate triangle.	
2. Label the diagram with the given angle and the given side to find an unknown side in a right-angled triangle.	
3. Choose the appropriate trigonometric ratio, namely the cosine ratio.	Angle = 70° Adjacent side = 2 m Hypotenuse = x m [CAH]
4. Substitute into the formula.	$\cos(\theta) = \frac{A}{H}$ $\cos(70°) = \frac{2}{x}$
5. Isolate x and evaluate.	$\frac{1}{\cos(70°)} = \frac{x}{2}$ $x = \frac{2}{\cos(70°)}$ $= 5.8476$
6. Write the answer using the correct units and level of accuracy.	The length of the guy wire is 5.85 metres or 585 centimetres, correct to the nearest centimetre.

12.3.5 Tangent ratio

The tangent ratio is defined as follows:

The tangent of an angle $= \frac{\text{length of opposite side}}{\text{length of adjacent side}}$.

In short,

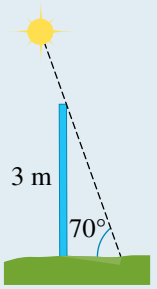
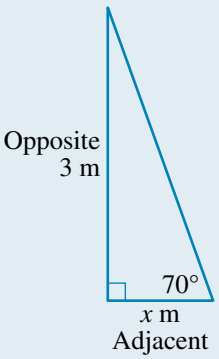


$$\tan(\theta) = \frac{\textbf{opposite}}{\textbf{adjacent}}$$

$$\tan(\theta) = \frac{\textbf{O}}{\textbf{A}} \quad \textbf{[TOA]}$$

WORKED EXAMPLE 6

Find the length of the shadow (correct to 1 decimal place) cast by a 3-metre tall pole when the angle of the sun to the horizontal is 70°.

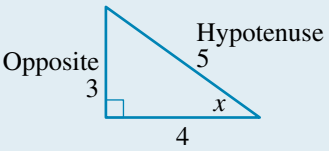
THINK	WRITE/DRAW
1. Draw a diagram to represent the situation and identify an appropriate triangle.	
2. Label the diagram with the given angle and the given side in order to find an unknown side in a right-angled triangle.	
3. Identify the appropriate trigonometric ratio, namely the tangent ratio.	Angle = 70° Opposite side = 3 m Adjacent side = x m [TOA]
4. Substitute into the formula.	$\tan(\theta) = \frac{\text{O}}{\text{A}}$ $\tan(70°) = \frac{3}{x}$
5. Isolate x and evaluate.	$\frac{1}{\tan(70°)} = \frac{x}{3}$ $x = \frac{3}{\tan(70°)}$ $= 1.0919$
6. Write the answer using the correct units and level of accuracy.	The length of the shadow is approximately 1.1 metres, correct to 1 decimal place.

12.3.6 Finding an unknown angle

If the lengths of the sides of a triangle are known, unknown angles within the triangle can be found.

WORKED EXAMPLE 7

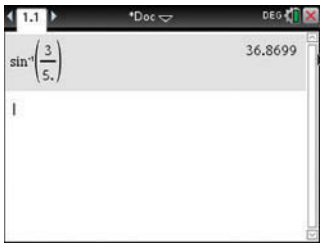
Find the smallest angle (correct to the nearest degree) in a 3, 4, 5 Pythagorean triangle.

THINK	WRITE/DRAW
1. The smallest angle is opposite the smallest side. Label the sides as given by convention for trigonometric ratios.	
2. All side lengths are known, therefore, any one of the 3 ratios can be used. Choose one ratio, for example, sine ratio.	Angle $= x$ Opposite side $= 3$ Hypotenuse $= 5$ [SOH]
3. Substitute into the formula.	$\sin(\theta) = \frac{\text{O}}{\text{H}}$ $\sin(x) = \frac{3}{5}$
4. Convert the ratio to a decimal.	$= 0.6$
5. Evaluate x.	$x = \sin^{-1}(0.6)$ $= 36.87°$
6. Write the answer using the correct units and level of accuracy.	The smallest angle is 37°, correct to the nearest degree.

TI | THINK

1. On a Calculator page, complete the entry line as: $\sin^{-1}\left(\frac{3.0}{5.0}\right)$
   Press ENTER.
2. The answer appears on the screen.

WRITE

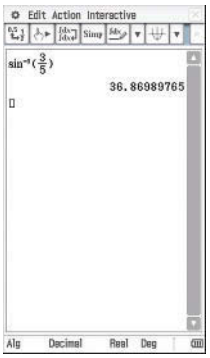


The smallest angle is 37°, correct to the nearest degree.

CASIO | THINK

1. On the MAIN screen, complete the entry line as: $\sin^{-1}\left(\frac{3}{5}\right)$
   Press EXE.
2. The answer appears on the screen.

WRITE



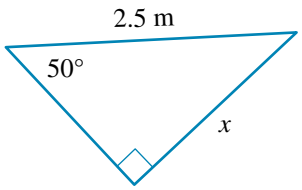
The smallest angle is 37°, correct to the nearest degree.

Resources

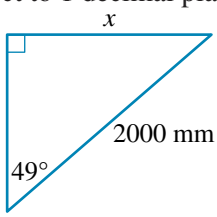
Interactivity: Trigonometric ratios (int-2577)

Exercise 12.3 Trigonometric ratios

1. **WE4** Find the length of the unknown side (correct to 1 decimal place) in the following triangle.

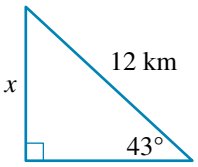


2. Find the length of the unknown side (correct to 1 decimal place) in the following triangle.

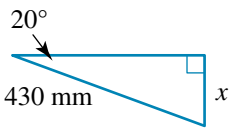


3. Find the length of the unknown side (correct to 1 decimal place) in each of the following triangles.

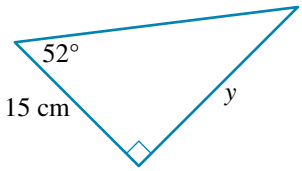
a.



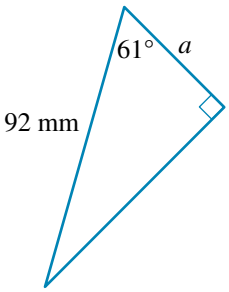
b.



c.



d.

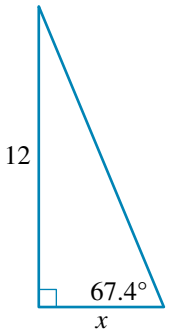


4. **WE5** A boat is moored in calm waters with its depth sounder registering 14.5 m. If the anchor line makes an angle of 72° with the vertical, what is the length of line (correct to the nearest metre) that is out of the boat?

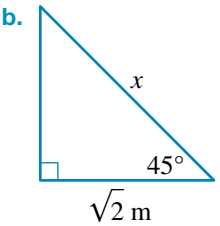
5. Find the length of the ramp (correct to the nearest centimetre), if the angle the ramp makes to the ground is 32° and the ramp covers 3.6 m horizontally.

6. Find the value of the missing side (correct to 1 decimal place) of the following triangles.

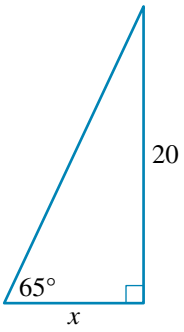
a.



b.



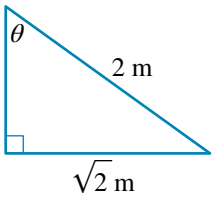
c.



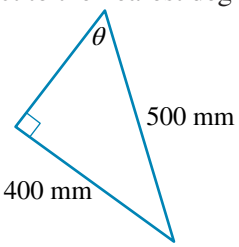
7. **WE6** Find the length of a shadow (correct to 1 decimal place) cast by a 4.5 m tall flag pole when the angle of the sun to the horizontal is 63°.

8. A person is hoping to swim directly across a straight river from point A to point B, a distance of 215 m. The river carries the swimmer downstream so that she actually reaches the other side at point C. If the line of her swim, AC, makes an angle of 67° with the river bank, find how far (correct to the nearest metre) downstream from point B she finished.

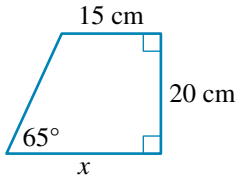
9. **WE7** Find the size of the unknown angle (correct to the nearest degree) in the triangle on the right.

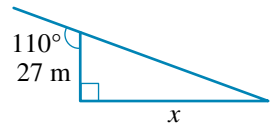


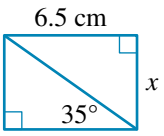
10. Find the size of the unknown angle (correct to the nearest degree) in the triangle below.



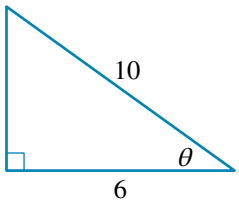
11. Find the value of the unknown sides (correct to 1 decimal place) of the following shapes.

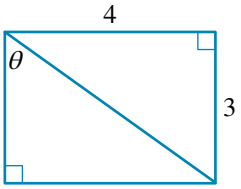
a.


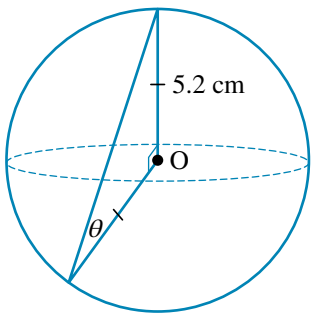
b.


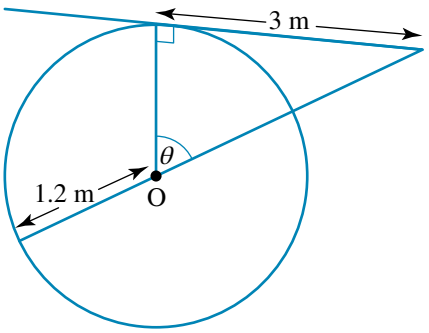
c.


12. Find the size of the unknown angle (correct to the nearest degree) in each of the triangles.

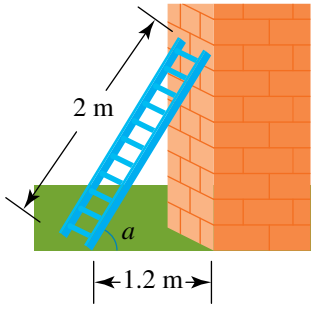
a.


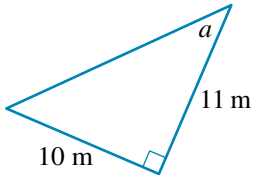
b.


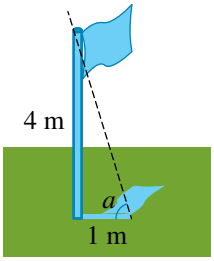
c.


d.


13. Find the values of the unknown angle, a (correct to the nearest degree).

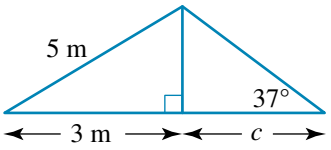
a.


b.


c.


14. Find the sizes of the two acute angles in a 6, 8, 10 Pythagorean triangle.

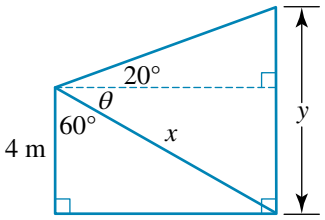
15. MC The correct expression for the value of c in the figure at right is:



A. $\frac{\tan(37°)}{4}$ **B.** $\frac{\cos(37°)}{4}$ **C.** $\frac{5}{\tan(37°)}$

D. $\frac{4}{\tan(37°)}$ **E.** $\frac{4}{\sin(37°)}$

16. In the diagram on the right find θ (correct to the nearest degree), x and y (both correct to 1 decimal place).



17. A 1.9 m javelin is thrown so that 15 cm of its pointy end sticks into the ground. The sun is directly overhead, casting a shadow of 90 cm in length. Determine the angle (correct to the nearest degree) that the javelin makes with the ground.

18. A hot air balloon is hovering in strong winds, 10 m vertically above the ground. It is being held in place by a taut 12 m length of rope from the balloon to the ground. Find the angle (correct to the nearest degree) that the rope makes with the ground.

19. A ramp joins two points, A and B. The horizontal distance between A and B is 1.2 m, and A is 25 cm vertically above the level of B.

a. Find the length of the ramp (in metres correct to 2 decimal places).

b. Find the angle that the ramp makes with the horizontal.

20. A cable car follows a direct line from a mountain peak (altitude 1250 m) to a ridge (altitude 840 m). If the horizontal distance between the peak and the ridge is 430 m, find the angle of descent (correct to the nearest degree) from one to the other.

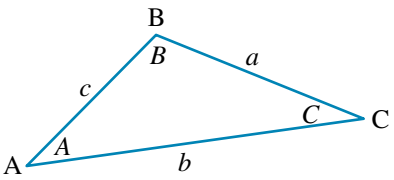
12.4 The sine rule

12.4.1 Introduction — sine and cosine rules

Often the triangle that is apparent or identified in a given problem is *non-right-angled*. Thus, Pythagoras' theorem or the trigonometric ratios are not as easily applied. The two rules that can be used to solve such problems are:

1. the **sine rule**
2. the **cosine rule**.

For the sine and cosine rules the following labelling convention should be used.



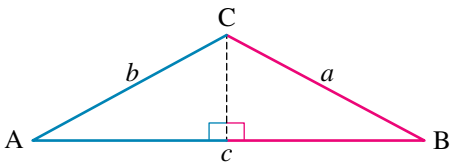
Angle A is opposite side a (at vertex A).

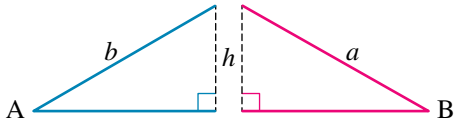
Angle B is opposite side b (at vertex B).

Angle C is opposite side c (at vertex C).

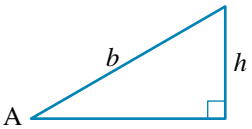
Note: To avoid cluttered diagrams, only the vertices (A, B and C) are usually shown and are used to represent the angles A, B and C.

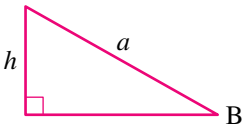
All triangles can be divided into two right-angled triangles.





Earlier, we saw that the new side, h, can be evaluated in two ways.





$$\sin(A) = \frac{h}{b}$$
$$h = b \times \sin(A)$$
$$\sin(B) = \frac{h}{a}$$
$$h = a \times \sin(B)$$

If we equate the two expressions for h:

$$b \times \sin(A) = a \times \sin(B)$$

and rearranging the equation, we obtain:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

Using a similar approach the Sine Rule is:

$$\frac{\boldsymbol{a}}{\boldsymbol{\sin(A)}} = \frac{\boldsymbol{b}}{\boldsymbol{\sin(B)}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin(C)}}$$

Similarly, if the triangle is labelled using other letters, for example STU, then:

$$\frac{s}{\sin(S)} = \frac{t}{\sin(T)} = \frac{u}{\sin(U)}$$

This can be generalised as follows: in any triangle, the ratio of side length to the sine of the opposite angle is constant.

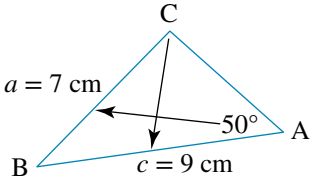
The sine rule is used if you are given:

1. **two angles and one opposite side**

   or

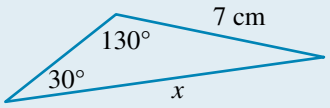
2. **an angle and its opposite side length (a complete ratio) and one other side.**

For example, in triangle ABC at right, $a = 7$ cm, $A = 50°$ and $c = 9$ cm. Angle C could then be found using the sine rule.



WORKED EXAMPLE 8

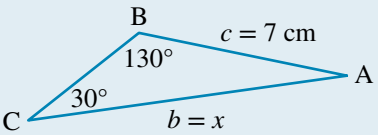
Find the unknown length, x cm, in the triangle (correct to 1 decimal place).



THINK

1. Label the triangle appropriately for the sine rule.
2. We have the angle opposite to the unknown side and a known $\frac{\text{side}}{\text{angle}}$ ratio, therefore, the sine rule can be used.
3. Substitute known values into the two ratios.
4. Isolate x and evaluate.
5. Write the answer.

WRITE/DRAW



$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$b = x$ $\quad B = 130°$

$c = 7\text{ cm}$ $\quad C = 30°$

$$\frac{x}{\sin(130°)} = \frac{7}{\sin(30°)}$$

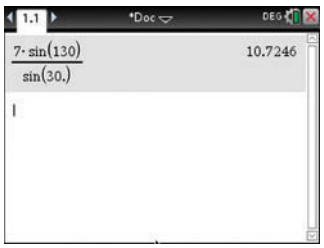
$$x = \frac{7 \times \sin(130°)}{\sin(30°)}$$
$$= 10.7246$$
$$\approx 10.7$$

The unknown length is 10.7 cm, to 1 decimal place.

TI | THINK

1. On a Calculator page, complete the entry line as:
   $$\frac{7 \times \sin(130.0)}{\sin(30.0)}$$
   Press ENTER.
2. The answer appears on the screen.

WRITE

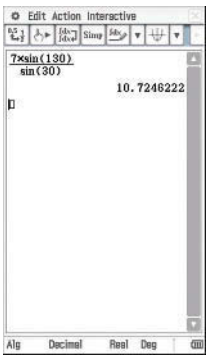


The unknown length is 10.7 cm, correct to 1 decimal place.

CASIO | THINK

1. On the MAN screen, complete the entry line as:
   $$\frac{7 \times \sin(130)}{\sin(30)}$$
   Press EXE.
2. The answer appears on the screen.

WRITE

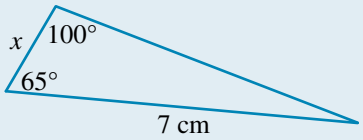


The unknown length is 10.7 cm, correct to 1 decimal place.

Sometimes it is necessary to find the third angle in a triangle in order to apply the sine rule.

WORKED EXAMPLE 9

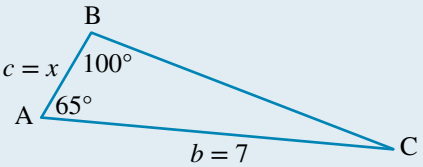
Find the unknown length, x cm (correct to 2 decimal places).



THINK

1. Label the triangle appropriately for the sine rule.
2. Calculate the third angle because it is opposite the unknown side.
3. Write the sine rule and identify the values of the pronumerals.
4. Substitute the known values into the rule.
5. Isolate x and evaluate.
6. Write the answer.

WRITE/DRAW



$C = 180° - (65° + 100°)$
$= 15°$

$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

$c = x \quad C = 15°$

$b = 7 \quad B = 100°$

$\frac{x}{\sin(15°)} = \frac{7}{\sin(100°)}$

$x = \frac{7 \times \sin(15°)}{\sin(100°)}$

$= 1.8397$

The unknown length is 1.84 cm, correct to 2 decimal places.

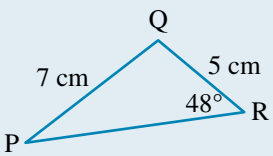
WORKED EXAMPLE 10

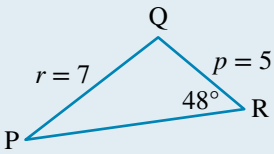
For a triangle PQR, find the unknown angle (correct to the nearest degree), P, given $p = 5$ cm, $r = 7$ cm and $R = 48°$.

THINK

1. Draw the triangle and assume it is non-right-angled.
2. Label the triangle appropriately for the sine rule (it is just as easy to use the given labels).

WRITE/DRAW





3. Confirm that it is the sine rule that can be used as you have the angle opposite to the unknown angle and a known $\frac{\text{side}}{\text{angle}}$ ratio.

$$\frac{p}{\sin(P)} = \frac{r}{\sin(R)}$$

4. Substitute known values into the two ratios.

$$p = 5 \quad P = ?$$
$$r = 7 \quad R = 48°$$
$$\frac{5}{\sin(P)} = \frac{7}{\sin(48°)}$$

5. Isolate $\sin(P)$.

$$\frac{\sin(P)}{5} = \frac{\sin(48°)}{7}$$
$$\sin(P) = \frac{5 \times \sin(48°)}{7}$$

6. Evaluate the angle (inverse sine) and include units with the answer.

$$P = \sin^{-1}\left(\frac{5 \times \sin(48°)}{7}\right)$$
$$= 32.06°$$
$$\approx 32°$$

The unknown angle is 32°, correct to the nearest degree.

Sometimes the angle required for the sine rule is not given. In such cases simply subtract the two known angles from 180°, as was done in step 2 of Worked example 9.

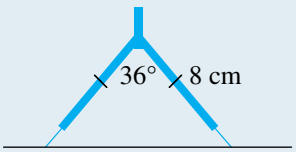
WORKED EXAMPLE 11

A pair of compasses (often called a compass) used for drawing circles has two equal legs joined at the top. The legs are 8 centimetres long. If it is opened to an included angle of 36 degrees between the two legs, find the radius of the circle that would be drawn (correct to 1 decimal place).

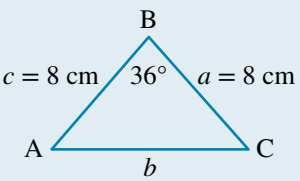
THINK

1. Draw the situation and identify that the triangle is non-right-angled.

WRITE/DRAW



2. Draw the triangle separately from the situation and label it appropriately.
The sine rule cannot be used straight away as we do not have both a known angle and known length opposite to the known angle. Therefore, we need to find either $\angle A$ or $\angle C$ first.
This is an isosceles triangle since $a = c$; therefore $\angle A = \angle C$. Using the fact that the angle sum of a triangle is 180°, find $\angle A$ and $\angle C$.



$$180° = \angle A + \angle B + \angle C$$
$$= x + 36° + x$$
$$2x = 180° - 36°$$
$$= 144°$$
$$x = 72° \text{ and, therefore, } \angle A = \angle C = 72°$$

3. Write the formula for the sine rule and identify the values of the pronumerals.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
$$b = y \qquad B = 36°$$
$$c = 8 \qquad C = 72°$$

4. Substitute the known values into the formula.

$$\frac{y}{\sin(36°)} = \frac{8}{\sin(72°)}$$

5. Transpose the equation to get the unknown by itself.

$$y = \frac{8 \times \sin(36°)}{\sin(72°)}$$

6. Evaluate y, correct to 1 decimal place, and include the units.

$$y \approx 4.9$$

The radius of the circle is 4.9 cm, correct to 1 decimal place.

on Resources

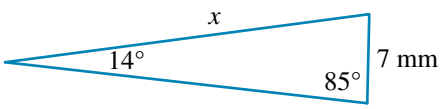
Interactivity: The sine rule (int-6275)

studyon

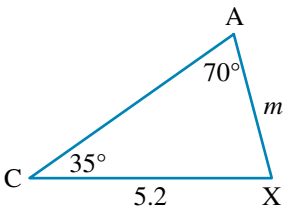
Units 3 & 4 > AOS M3 > Topic 1 > Concept 2 > **The sine rule** Summary screen and practice questions

Exercise 12.4 The sine rule

1. WE8 Find the unknown length, x.

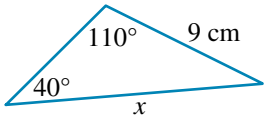


2. MC The length of side m is nearest to:
   - **A.** 3.2
   - **B.** 3.1
   - **C.** 3.6
   - **D.** 5.8
   - **E.** 3.0

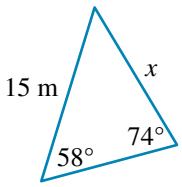


3. Find the unknown length, x, in each of the following.

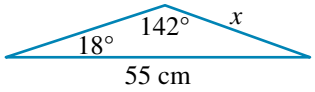
a.



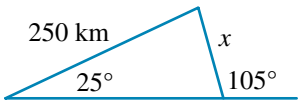
b.



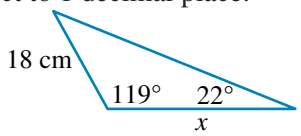
c.



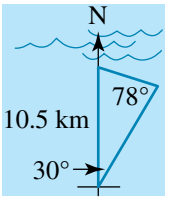
d.



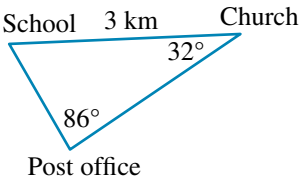
4. **WE9** Find the unknown length, x, correct to 1 decimal place.



5. A sailing expedition followed a triangular course as shown. Find the total distance covered in the round trip.

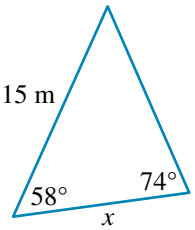


6. The relative positions of the school, church and post office in a small town are shown at the vertices of the triangle.
Find the straight-line distance between the school and the post office (correct to 1 decimal place).

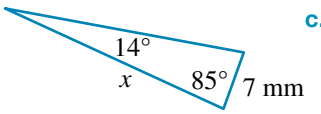


7. Find the unknown length, x (correct to 1 decimal place) in each case.

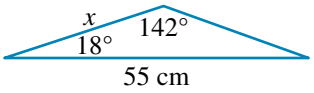
a.



b.



c.



8. **WE10** In ΔPQR, find the unknown angle, R, given $p = 48$, $q = 21$ and $\angle P = 110°$, correct to the nearest degree.

9. Construct a suitable triangle from the following instructions and find all unknown sides and angles. One of the sides is 23 cm; the smallest side is 15 cm; the smallest angle is 28°.

10. For the following questions give answers, correct to the nearest degree.

a. In ΔABC, find the unknown angle, B, given $b = 6$, $c = 6$ and $\angle C = 52°$.

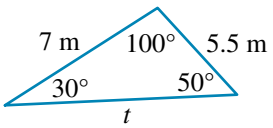
b. In ΔLMN, find the unknown angle, M, given $m = 14.1$, $n = 27.2$ and $\angle N = 128°$.

c. In ΔSTU, find the unknown angle, S, given $s = 12.7$, $t = 16.3$ and $\angle T = 45°$.

d. In ΔPQR, find the unknown angle, P, given $p = 2$, $r = 3.5$ and $\angle R = 128°$.

e. In ΔABC, find the unknown angle, A, given $b = 10$, $c = 8$ and $\angle B = 80°$.

11. **MC** The correct expression for the value of t in the given triangle is:



A. $\dfrac{7 \times \sin(100°)}{\sin(30°)}$

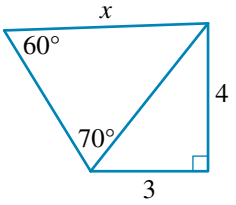
B. $\dfrac{5.5 \times \sin(100°)}{\sin(30°)}$

C. $\dfrac{5.5 \times \sin(30°)}{\sin(100°)}$

D. $\dfrac{5.5 \times \sin(100°)}{\sin(50°)}$

E. $\dfrac{7 \times \sin(50°)}{\sin(100°)}$

12. **MC** The value of x (correct to 1 decimal place) in the given figure is:



A. 4.3

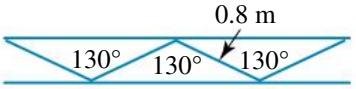
B. 4.6

C. 5.4

D. 3.3

E. 3.6

13. **WE11** Steel trusses are used to support the roof of a commercial building. The struts in the truss shown are each made from 0.8 m steel lengths and are welded at the contact points with the upper and lower sections of the truss.

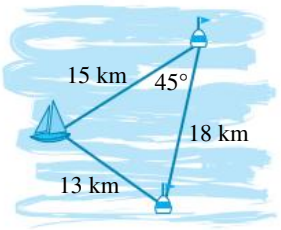


a. On the lower section of the truss, what is the distance (correct to the nearest centimetre) between each pair of consecutive welds?

b. What is the height (correct to the nearest centimetre) of the truss?

14. A logo is in the shape of an isosceles triangle with the equal sides being 6.5 cm long and the equal angles 68°. Use the sine rule to find the length (correct to 1 decimal place) of the unknown side.

15. **MC** A yacht sails the three-leg course shown. The largest angle between any two legs within the course, to the nearest degree, is:

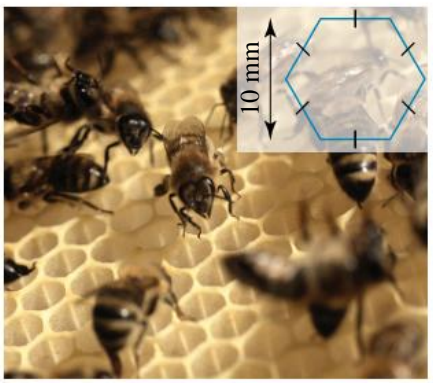


A. 34°
B. 55°
C. 45°
D. 78°
E. 90°

16. **MC** The correct expression for angle S in the given triangle is:

A. $\sin^{-1}\left(\frac{40 \times \sin(41°)}{30}\right)$
B. $\cos^{-1}\left(\frac{40 \times \cos(41°)}{30}\right)$
C. $\sin^{-1}\left(\frac{30 \times \sin(41°)}{40}\right)$
D. $\sin^{-1}\left(\frac{41 \times \sin(41°)}{30}\right)$
E. $\sin^{-1}\left(\frac{30}{41 \times \sin(41°)}\right)$

17. Find the perimeter of the beehive cell shown.



18. A rope is pegged at one end into the ground, pulled tightly up over a branch and pegged into the ground at the other end. It is known that one peg-to-branch length of rope is 8 m and it makes an angle of 39° with the ground. The other end of the rope makes an angle of 48° with the ground. Find (correct to 1 decimal place):

a. the length of the rope

b. the distance between the two pegs.

19. A playground swing, which is 2.3 m long, makes an angle of 74°, at its swing point, in one complete swing. Determine the horizontal distance (in metres correct to 1 decimal place) between the extreme positions of the swing seat.

20. A scenic flight leaves Geelong and flies west of north for the 80 km direct journey to Ballarat. At Ballarat the plane turns 92° to the right to fly east of north to Kyneton. From here the plane again turns to the right and flies the 103 km straight back to Geelong.

a. Determine the angle (in degrees correct to 1 decimal place) through which the plane turned at Kyneton.

b. Find the distance (correct to the nearest km) of the direct flight from Ballarat to Kyneton.

12.5 Ambiguous case of the sine rule

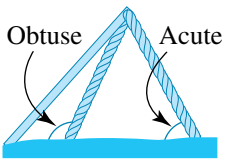
Using CAS, investigate the values for each of these pairs of sine ratios:

- sin(30°) and sin(150°)
- sin(110°) and sin(70°).

You should obtain the same number for each value in a pair.

Similarly, sin(60°) and sin(120°) give an identical value of 0.8660.

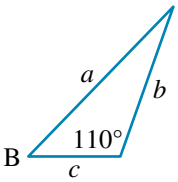
Now try to find the inverse sine of these values; for example, $\sin^{-1}(0.8660)$ is 60°. The obtuse (greater than 90°) angle is not given by the calculator. When using the inverse sine function on your calculator, the calculator will give only the acute angle.

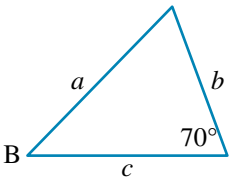


A rope attached to a pole can be anchored in two possible positions.

The situation is illustrated practically in the diagram above, where the sine of the acute angle equals the sine of the obtuse angle.

Therefore always check your diagram to see if the unknown angle is the acute or obtuse angle or perhaps either. This situation is illustrated in the two diagrams on the next page. The triangles have two corresponding sides equal, a and b, as well as angle B. The sine of 110° also equals the sine of 70°; however, the side c is quite different. It is worth noting that this ambiguity occurs when the smaller known side is opposite the known acute angle. That is, an ambiguous case occurs if $\angle B < 90°$ and $a \sin B \times \angle b < \angle a$:





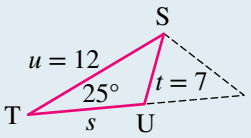
WORKED EXAMPLE 12

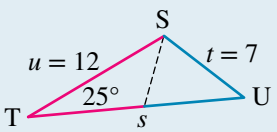
Correct to the nearest degree, find the angle, U, in a triangle, given $t = 7, u = 12$ and angle T is 25°.

THINK

1. Draw a suitable sketch of the triangle given. As the length of s is not given, side t can be drawn two different ways. Therefore angle U could be either an acute or an obtuse angle. Label the triangles appropriately for the sine rule. (It is just as easy to use the given labels.)
2. Identify that it is the sine rule that can be used as you have the side opposite to the unknown angle and a known $\frac{\text{side}}{\text{angle}}$ ratio.
3. Substitute the known values into the two ratios.

WRITE/DRAW





$$\frac{t}{\sin(T)} = \frac{u}{\sin(U)}$$

$t = 7 \qquad T = 25°$

$u = 12 \qquad U = ?$

$$\frac{7}{\sin(25°)} = \frac{12}{\sin(U)}$$

4. Transpose the equation to get the unknown by itself.

$$\frac{\sin(U)}{12} = \frac{\sin(25°)}{7}$$
$$\sin(U) = \frac{12 \times \sin(25°)}{7}$$

5. Evaluate the angle (inverse sine). Note that the value is an *acute* angle but it may also be an *obtuse* angle.

$$\sin(U) = 0.724\,488$$
$$U = 46.43°$$

6. Calculate the obtuse angle.

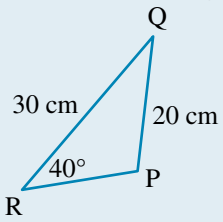
$$\text{or } U = 180° - 46.43°$$
$$= 133.57°$$

7. Write the answer, giving both the acute and obtuse angles, as not enough information was given (the information was ambiguous) to precisely position side *t*.

The angle U is either 46° or 134°, correct to the nearest degree.

WORKED EXAMPLE 13

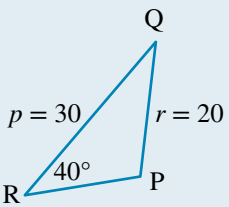
In the obtuse-angled triangle PQR, find the unknown angle (correct to the nearest degree), *P*.



THINK

1. Label the triangle appropriately for the sine rule. (It is just as easy to use the given labels.)

WRITE/DRAW



2. Identify that the sine rule is used as you have the side opposite to the unknown angle and a known $\frac{\text{side}}{\text{angle}}$ ratio.

$$\frac{p}{\sin(P)} = \frac{r}{\sin(R)}$$
$$p = 30 \quad P = ?$$
$$r = 20 \quad R = 40°$$

3. Substitute the known values into the two ratios.

$$\frac{30}{\sin(P)} = \frac{20}{\sin(40°)}$$

4. Transpose the equation to get the unknown by itself.

$$\frac{\sin(P)}{30} = \frac{\sin(40°)}{20}$$
$$\sin(P) = \frac{30 \times \sin(40°)}{20}$$

5. Evaluate the angle (inverse sine). Note that the value is an acute angle while in the diagram given it is an obtuse angle.

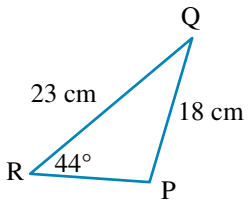
$\sin(P) = 0.964\,18$
$P = 74.62°$

6. Calculate the obtuse angle.

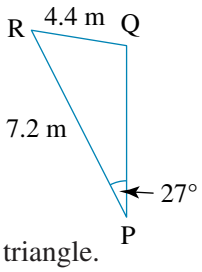
$P = 180° - 74.62°$
$= 105.38°$
The angle P is 105°, correct to the nearest degree.

Exercise 12.5 Ambiguous case of the sine rule

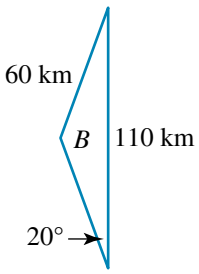
1. **WE12** To the nearest degree, find the angle, U, in a triangle, given $t = 7, u = 8$ and angle T is 27°.
2. To the nearest degree, find the angle, U, in a triangle, given $t = 12, u = 16$ and angle T is 33°.
3. **WE13** In the obtuse-angled triangle PQR shown, find the unknown angle (correct to the nearest degree), P.



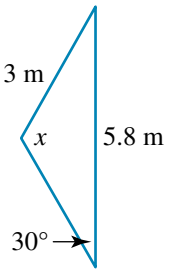
4. In the obtuse-angled triangle PQR shown, find the unknown angle (correct to the nearest degree), Q.



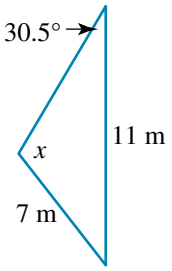
5. In ΔABC, find the unknown angle, B, given $b = 10.8, c = 6$ and $\angle C = 26°$.
6. In ΔSTU, find the unknown angle, S, given $t = 12.7, s = 16.3$ and $\angle T = 45°$.
7. In ΔPQR, find the unknown angle, P, given $p = 3.5, r = 2$ and $\angle R = 12°$.
8. In ΔLMN, find the unknown angle, M, given $n = 0.22$ km, $m = 0.5$ km and $\angle N = 18°$.
9. Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.



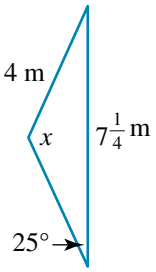
10. Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.



11. Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.

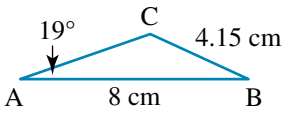


12. Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.



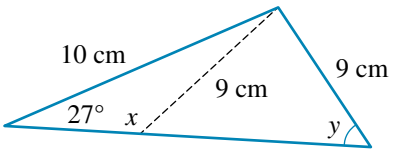
13. MC In the triangle given, angle C is (correct to the nearest degree):

A. 38° **B.** 39° **C.** 78°

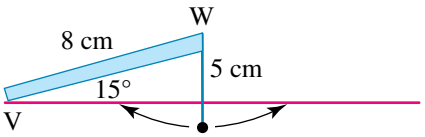


D. 141° **E.** 142°

14. Find the two unknown angles shown in the diagram (correct to 1 decimal place).



15. Look at the swinging pendulum shown.

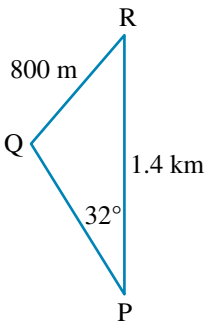


a. Draw the two possible positions of the bob at the level of the horizontal line.

b. Find the value of the angle, W, at these two extreme positions.

c. Find the smallest and largest distances between vertex V and the bob.

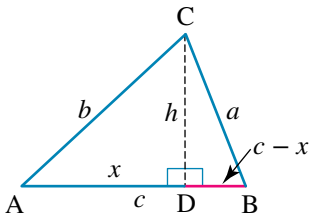
16. If a boat travelled the path shown:



a. what is the obtuse angle between the PQ and QR legs of the trip?

b. what is the distance travelled from P to Q?

12.6 The cosine rule



The **cosine rule** is derived from a non-right-angled triangle divided into two right-angled triangles in a similar way to the derivation of the sine rule. The difference is that, in this case, Pythagoras' theorem and the cosine ratio are used to develop it.

The triangle ABC in the figure has been divided into two right-angled triangles with base sides equal to x and $(c - x)$.

In ΔACD, $h^2 = b^2 - x^2$

and in ΔBCD, $h^2 = a^2 - (c - x)^2$ (Pythagoras' theorem)

Equating expressions for h^2,

$$b^2 - x^2 = a^2 - (c - x)^2$$

$$a^2 = b^2 - x^2 + (c - x)^2$$

$$= b^2 - x^2 + c^2 - 2cx + x^2$$

$$a^2 = b^2 + c^2 - 2cx \qquad [1]$$

Now, from ΔACD, $\cos(A) = \frac{x}{b}$

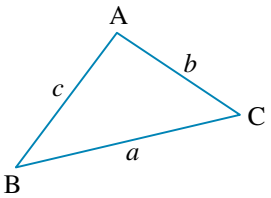
$$x = b\cos(A)$$

Substitute this value of x into [1] above.

$$a^2 = b^2 + c^2 - 2c\,[b\cos(A)]$$

So, the cosine rule can be written as:

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \times \cos(A)}$$



In a similar way to that above, it can be shown that:

$$b^2 = a^2 + c^2 - 2ac \times \cos(B)$$

$$c^2 = a^2 + b^2 - 2ab \times \cos(C)$$

Also, if the triangle is labelled using other letters, for example STU, then:

$$s^2 = t^2 + u^2 - 2tu \times \cos(S)$$

The formula may be transposed in order to find an unknown angle.

$$\boldsymbol{\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}}$$

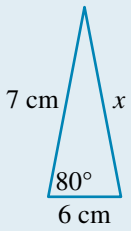
or alternatively, $\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$ and $\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$.

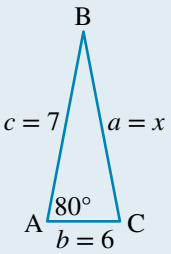
The cosine rule is used to find:

1. **an unknown length when you have the lengths of two sides and the angle in between**
2. **an unknown angle when you have the lengths of all three sides.**

WORKED EXAMPLE 14

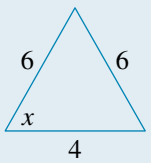
Find the unknown length (correct to 2 decimal places), x, in the triangle.

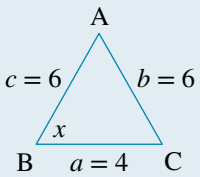


THINK	WRITE/DRAW
1. Identify the triangle as non-right-angled. 2. Label the triangle appropriately for the sine rule or cosine rule.	
3. Identify that it is the cosine rule that is required as you have the two sides and the angle in between.	$b = 6 \quad A = 80°$ $c = 7 \quad a = x$
4. Substitute the known values into the cosine rule formula and evaluate the right-hand side.	$a^2 = b^2 + c^2 - 2bc \times \cos(A)$ $x^2 = 6^2 + 7^2 - 2 \times 6 \times 7 \times \cos(80°)$ $= 36 + 49 - 84 \times \cos(80°)$ $= 70.4136$
5. Remember to get the square root value, x.	$x = \sqrt{70.4136}$ $= 8.391$
6. Write the answer, rounding off to the required number of decimal places and including the units.	$x = 8.39$ The unknown length is 8.39 cm, correct to 2 decimal places.

WORKED EXAMPLE 15

Find the size of angle x in the triangle, correct to the nearest degree.



THINK	WRITE/DRAW
1. Identify the triangle as non-right-angled. 2. Label the triangle appropriately for the sine rule or cosine rule.	
3. As all three sides are given, the cosine rule should be used. Write the rule and identify the values of the pronumerals.	$\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$ $a = 4,\ b = 6,\ c = 6,\ B = x$
4. Substitute the known values into the formula and simplify.	$\cos(x) = \dfrac{4^2 + 6^2 - 6^2}{2 \times 4 \times 6}$ $\cos(x) = \dfrac{16}{48}$

5. Evaluate $x[x = \cos^{-1}(0.3333)]$.

$\cos(x) = 0.3333$
$x = 70.53°$
$x \approx 71°$

6. Round to the nearest degree and state your answer.

The angle x is 71°, correct to the nearest degree.

Resources

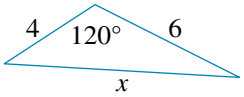
Interactivity: The cosine rule (int-6276)

studyon

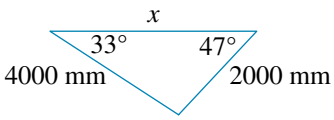
Units 3 & 4 > AOS M3 > Topic 1 > Concept 3 **The cosine rule** Summary screen and practice questions

Exercise 12.6 The cosine rule

1. **WE 14** Find the unknown length, correct to 2 decimal places.

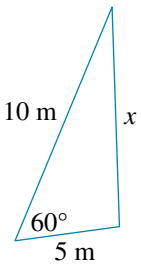


2. Find the unknown length, correct to 2 decimal places.

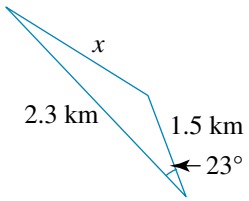


3. Find the unknown length in each of the following (correct to 2 decimal places).

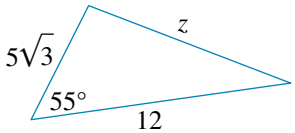
a.



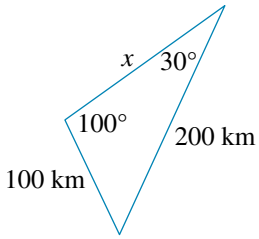
b.



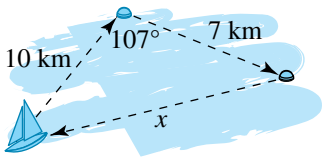
c.



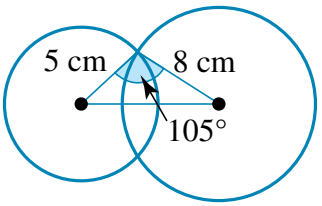
d.



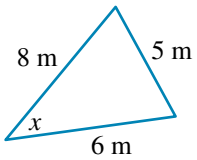
4. During a sailing race, the boats followed a triangular course as shown. Find the length, x, of the third leg (correct to 1 decimal place).



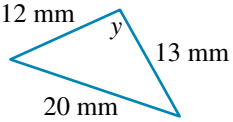
5. Two circles, with radii 5 cm and 8 cm, overlap as shown. If the angle between the two radii that meet at the point of intersection of the circumferences is 105°, find the distance between the centres of the circles (correct to 1 decimal place).

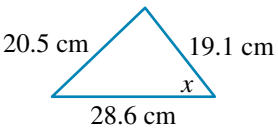


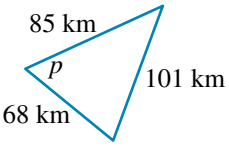
6. **WE15** Find the size of the unknown angle (correct to the nearest degree).



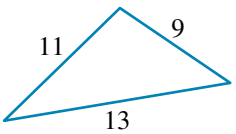
7. Construct a suitable triangle from the following instructions and find all unknown sides and angles. Two sides are 23 cm and 15 cm and the angle in between is 28°.
8. Find the size of the unknown angle in each of the following (correct to the nearest degree).

a. 

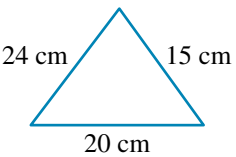
b. 

c. 

9. Consider the sailing expedition course in question 4. Find the two unknown angles (correct to the nearest degree) in the triangular course.
10. For the triangle shown, find all three unknown angles (correct to the nearest degree).

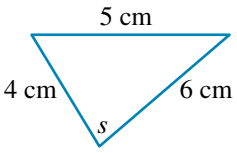


11. For the following questions, give answers correct to 1 decimal place.
    a. For ΔABC, find the unknown side, b, given $a = 10\,\text{km}$, $c = 8\,\text{km}$ and $\angle B = 30°$.
    b. For ΔABC, find the unknown angle, B, given $a = b = 10$ and $c = 6$.
    c. For ΔABC, find the unknown side, c, given $a = 7\,\text{m}$, $b = 3\,\text{m}$ and $\angle C = 80°$.
    d. For ΔSTU, find the unknown angle, S, given $t = 12.7$, $s = 16.3$ and $u = 24.5$.
    e. For ΔPQR, find the unknown angle, P, given $p = 2$, $q = 3.5$ and $r = 2.5$.
    f. For ΔABC, find the unknown side, a, given $b = 260$, $c = 120$ and $\angle A = 115°$.
12. **MC** In the triangle given, the largest angle is:



A. 39° **B.** 45° **C.** 56°
D. 85° **E.** 141°

13. MC The correct expression for angle s is:

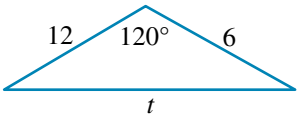


A. $\cos^{-1}\left(\frac{6^2+4^2-5^2}{2\times6\times4}\right)$ **B.** $\cos^{-1}\left(\frac{4^2+5^2-6^2}{2\times4\times5}\right)$

C. $\cos^{-1}\left(\frac{4^2-6^2+5^2}{2\times4\times6}\right)$ **D.** $\cos^{-1}\left(\frac{4^2+6^2-5^2}{2\times4\times5}\right)$

E. $\cos^{-1}\left(\frac{5^2+6^2-4^2}{2\times5\times6}\right)$

14. MC The correct expression for the value of t is:

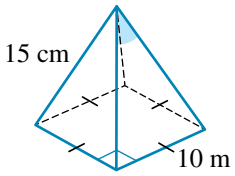


A. $\sqrt{180+144\cos(120°)}$ **B.** $\sqrt{180-120}$

C. $\sqrt{180-144\times0.5}$ **D.** $\sqrt{180-72}$

E. $\sqrt{180+72}$

15. MC The 4 surface angles at the vertex of a regular square-based pyramid are all the same. The magnitude of these angles for the pyramid shown at right (correct to the nearest degree) is:



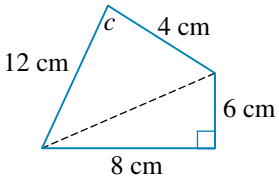
A. 1° **B.** 34°

C. 38° **D.** 39°

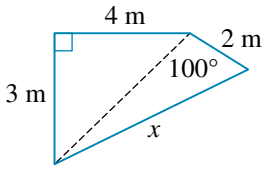
E. 71°

16. Find the unknown values.

a.



b.



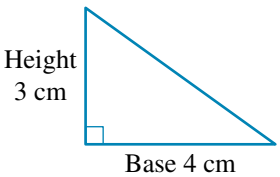
12.7 Area of triangles

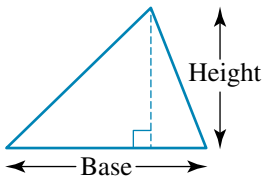
Three possible methods can be used to find the area of a triangle:

Method 1. When the two known lengths are perpendicular to each other we would use:

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{base} \times \text{height}$$

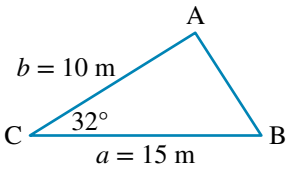
$$A = \frac{1}{2}bh$$

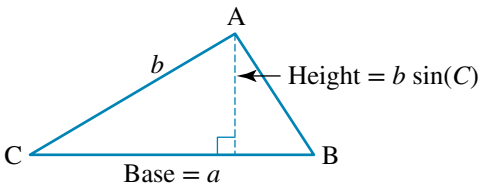




Method 2. When we are given *two lengths and the angle in between* we would use:

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times a \times b \times \sin(C)$$
$$A = \frac{1}{2}ab\sin(C)$$





$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times a \times b \sin(C)$

Method 3. When all three sides are known we would use:

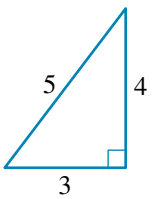
$$\text{Area}_{\text{triangle}} = \sqrt{s(s-a)(s-b)(s-c)} \text{ where the semi-perimeter, } s = \frac{(a+b+c)}{2}.$$

This formula is known as Heron's formula. It was developed by Heron (or Hero) of Alexandria, a Greek mathematician and engineer who lived around 62 AD (CE).

Let us find the area of the triangle at right to demonstrate that all three formulas provide the same result.

For the 3, 4, 5 triangle, the most appropriate method is method 1 because it is a right-angled triangle.

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$A = \frac{1}{2} \times 3 \times 4$$
$$= 6$$



The other two methods may also be used.

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times a \times b \times \sin(C)$$
$$A = \frac{1}{2} \times 3 \times 4 \times \sin(90°)$$
$$= 6 \times 1$$
$$= 6$$

$$\text{Area}_{\text{triangle}} = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{(a+b+c)}{2}$$

$$A = \sqrt{6(6-3)(6-4)(6-5)} \qquad = \frac{(3+4+5)}{2}$$

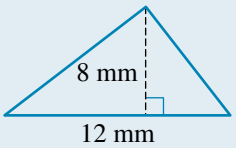
$$= \sqrt{6 \times 3 \times 2 \times 1} \qquad = \frac{12}{2}$$

$$= \sqrt{36} \qquad = 6$$

$$= 6$$

WORKED EXAMPLE 16

Find the area of the triangle shown.



THINK

1. The two given lengths are perpendicular. Write the most appropriate formula for finding the area.
2. Substitute the known values into the formula.
3. Write the answer using correct units.

WRITE

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{base} \times \text{height}$$

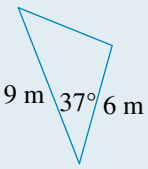
$$= \frac{1}{2} \times 12 \times 8$$

$$= 48$$

The area of the triangle is $48\ \text{mm}^2$.

WORKED EXAMPLE 17

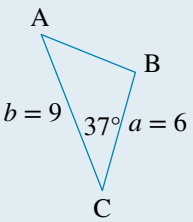
Find the area of the triangle (correct to 2 decimal places).



THINK

1. Identify the shape as a triangle with two known sides and the angle in between.
2. Identify and write down the values of the two sides, a and b, and the angle in between them, C.
3. Identify the appropriate formula and substitute the known values into it.
4. Write the answer using correct units.

WRITE/DRAW



$a = 6$
$b = 9$
$C = 37°$

$$\text{Area}_{\text{triangle}} = \frac{1}{2}ab\sin(C)$$

$$= \frac{1}{2} \times 6 \times 9 \times \sin(37°)$$

$$= 16.249$$

The area of the triangle is $16.25\ \text{m}^2$, correct to 2 decimal places.

WORKED EXAMPLE 18

Find the area of a triangle PQR (correct to 1 decimal place), given $p = 6$, $q = 9$ and $r = 4$, with measurements in centimetres.

THINK

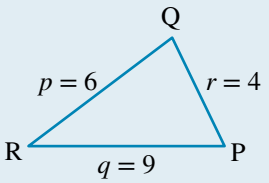
1. All three sides of the triangle have been given; therefore, Heron's formula can be used to find the area.

2. Write the values of the three sides, a, b and c, and calculate the semi-perimeter value, s.

3. Substitute the known values into Heron's formula and evaluate.

4. Write the answer, using the correct units.

WRITE/DRAW



$a = p = 6,\ b = q = 9,\ c = r = 4$

$$s = \frac{(a + b + c)}{2}$$
$$= \frac{(6 + 9 + 4)}{2}$$
$$= 9.5$$

$$\text{Area}_{\text{triangle}} = \sqrt{s(s - a)(s - b)(s - c)}$$
$$= \sqrt{9.5(9.5 - 6)(9.5 - 9)(9.5 - 4)}$$
$$= \sqrt{9.5 \times 3.5 \times 0.5 \times 5.5}$$
$$= \sqrt{91.4375}$$
$$\text{Area} = 9.5623$$

The area of triangle PQR is $9.6\,\text{cm}^2$, correct to 1 decimal place.

on Resources

- **Interactivity:** Area of triangles (int-6483)
- **Interactivity:** Using Heron's formula to find the area of a triangle (int-6475)

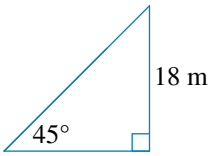
studyon

Units 3 & 4 > AOS M3 > Topic 1 > Concept 4

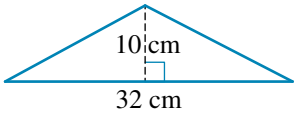
Area of a triangle Summary screen and practice questions

Exercise 12.7 Area of triangles

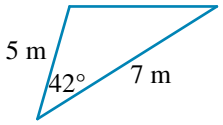
1. **WE16** Find the area of the triangle shown.



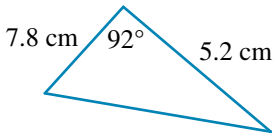
2. Find the area of the triangle shown.



3. **WE17** Find the area of the triangle shown (correct to 2 decimal places).



4. Find the area of the triangle shown (correct to 2 decimal places).

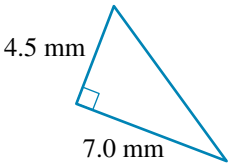


5. **WE18** Find the area of ΔABC (correct to 1 decimal place) given $a = b = 10\,\text{cm}$ and $c = 6\,\text{cm}$.

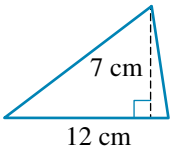
6. Find the area of an equilateral triangle with side lengths of 10 cm.

7. Find the areas of the following triangles (correct to 1 decimal place).

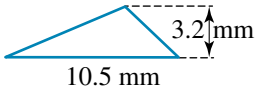
a.



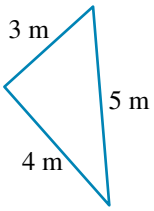
b.



c.

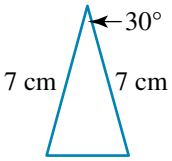


d.

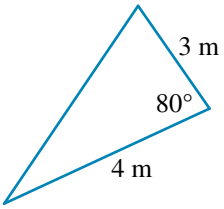


8. Find the areas of the following triangles (correct to 1 decimal place).

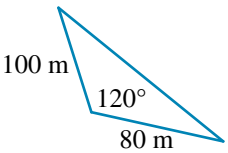
a.



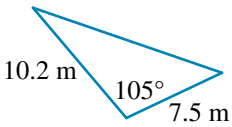
b.



c.

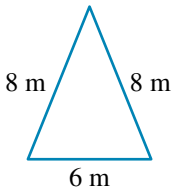


d.

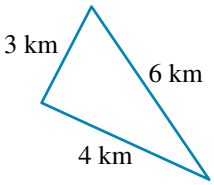


9. Find the areas of the following triangles (correct to 1 decimal place).

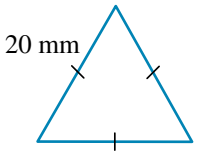
a.



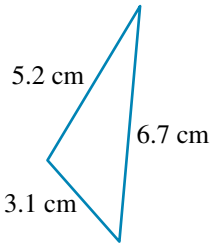
b.



c.

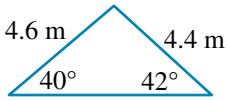


d.

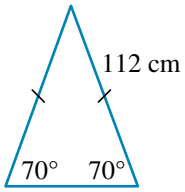


10. Find the areas of the following triangles (correct to 1 decimal place).

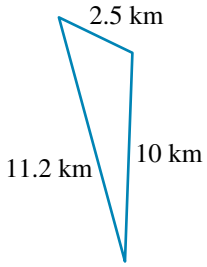
a.



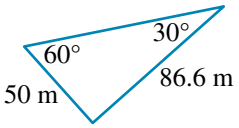
b.



c.



d.



11. Find the area of each of the following triangles. (Give all answers correct to 1 decimal place.)

a. ΔABC, given $a = 10\,\text{km}$, $c = 8\,\text{km}$ and $\angle B = 30°$

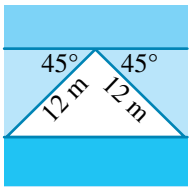
b. ΔABC, given $a = 7\,\text{m}$, $b = 3\,\text{m}$, $c = 8.42\,\text{m}$ and $\angle C = 108°$

c. ΔSTU, given $t = 12.7\,\text{m}$, $s = 16.3\,\text{m}$ and $u = 24.5\,\text{m}$

d. ΔPQR, given $p = 2\,\text{units}$, $q = 3.5\,\text{units}$ and $r = 2.5\,\text{units}$

e. ΔABC, given $b = 260\,\text{cm}$, $c = 120\,\text{cm}$ and $\angle A = 90°$

12. A triangular arch has supporting legs of equal length of 12 metres as shown in the diagram. What is its area?



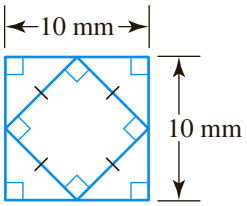
13. From the diagram given at right,

a. find the area of:

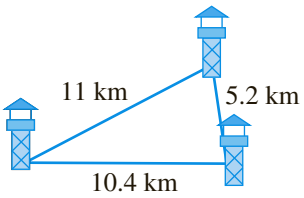
i. one of the triangles

ii. all of the triangles.

b. Use another technique to verify your answer in **a i**.



14. Find the area of the state forest as defined by the three fire-spotting towers on the corners of its boundary.



15. MC If the perimeter of an equilateral triangle is 210 metres, its area is closest to:

A. 2100 m^2 **B.** 2450 m^2 **C.** 4800 m^2 **D.** 5500 m^2 **E.** 1700 m^2

16. MC The correct expression for the area of the shape shown is:

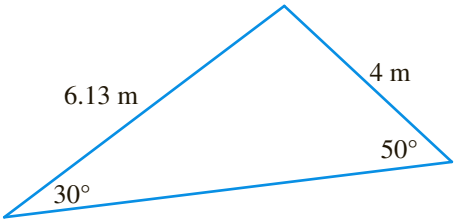
A. $\frac{1}{2} \times 6.13 \times 4 \times \sin(80°)$

B. $\frac{1}{2} \times 6.13 \times 4 \times \cos(100°)$

C. $\frac{1}{2} \times 6.13 \times 4 \times \sin(100°)$

D. $\frac{1}{2} \times 6.13 \times 4$

E. none of the above



17. The correct expression for the area of the octagon at right is:

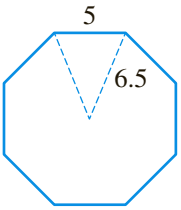
a. $195 \times \sin(45°)$

b. $169 \times \sin(45°)$

c. $195 \times \sin(60°)$

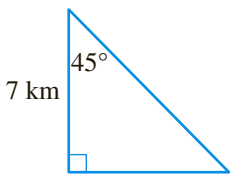
d. $338 \times \sin(60°)$

e. $5 \times 6.5 \times \sin(67.5°)$

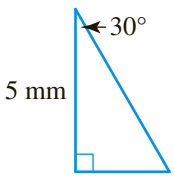


18. Find the area of the following triangles.

a.



b.

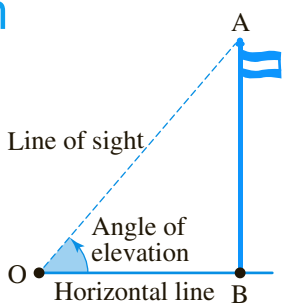


12.8 Angles of elevation and depression

One method for locating an object in the real world is by its position above or below a horizontal plane or reference line.

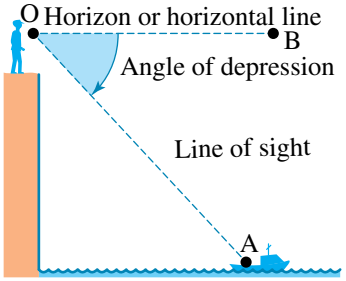
The **angle of elevation** is the angle above the horizon or horizontal line.

Looking up at the top of the flagpole from position O, the angle of elevation, ∠AOB, is the angle between the horizontal line OB and the line of sight OA.



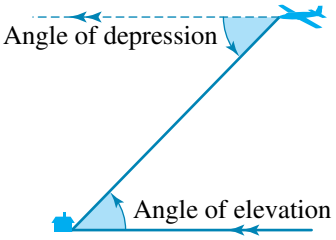
The **angle of depression** is the angle *below* the horizon or horizontal line.

Looking down at the boat from position O, the angle of depression, ∠AOB, is the angle between the horizontal line, OB, and the line of sight, OA.



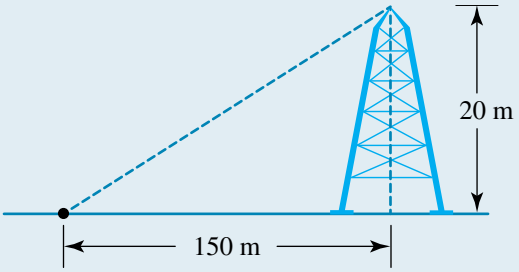
Angles of elevation and depression are in a vertical plane.

We can see from the diagram the angle of depression given from one location can give us the angle of elevation from the other position using the alternate angle law.



WORKED EXAMPLE 19

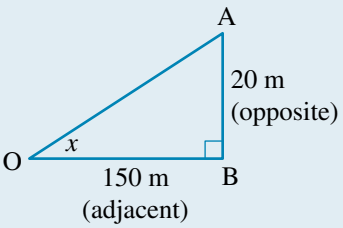
Find the angle of elevation (correct to 2 decimal places) of the tower measured from the road as given in the diagram.



THINK

1. The angle of elevation is ∠AOB. Use ΔAOB and trigonometry to solve the problem.

WRITE/DRAW



2. The problem requires the tangent ratio. Substitute the values and simplify.

$$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$
$$= \frac{\text{opposite}}{\text{adjacenet}}$$
$$\tan(x) = \frac{20}{150} = 0.133\,33$$

3. Evaluate x and round correct to 2 decimal places.

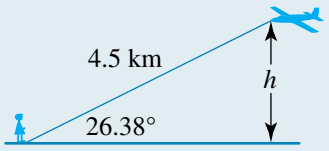
$$x = \tan^{-1}(0.133\,33)$$
$$= 7.5946°$$
$$\approx 7.59°$$

4. Write the answer.

From the road the angle of elevation to the tower is 7.59°, correct to 2 decimal places.

WORKED EXAMPLE 20

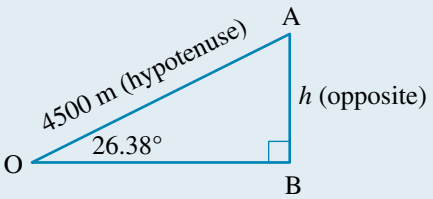
Find the altitude of a plane (correct to the nearest metre) if the plane is sighted 4.5 km directly away from an observer who measures its angle of elevation as 26.38°.



THINK

1. Draw a suitable diagram. Change distance to metres.

WRITE/DRAW



2. Use the sine ratio and simplify.

$$\sin(\theta) = \frac{\text{length of opposite side}}{\text{length of hypotenuse side}}$$
$$= \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(26.38°) = \frac{h}{4500}$$
$$h = 4500 \times \sin(26.38°)$$

3. Evaluate.

$$= 1999.45$$

4. Write the answer in correct units.

The plane is flying at an altitude of 2000 m, correct to the nearest metre.

WORKED EXAMPLE 21

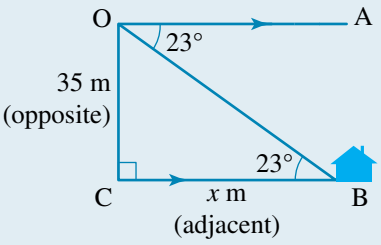
The angle of depression from the top of a 35-metre cliff to a house at the bottom is 23°. How far from the base of the cliff is the house (correct to the nearest metre)?

THINK

1. Draw a suitable diagram.

WRITE/DRAW

2. Angle of depression is ∠AOB. Use the alternate angle law to give the angle of elevation ∠CBO.



3. Use the tangent ratio. Substitute into the formula and evaluate.

$$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$
$$= \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(23°) = \frac{35}{x}$$
$$\frac{1}{\tan(23°)} = \frac{x}{35}$$
$$x = \frac{35}{\tan(23°)}$$
$$= 82.4548 \ldots$$

4. Write the answer in correct units.

The distance from the house to the base of the cliff is 82 metres, correct to the nearest metre.

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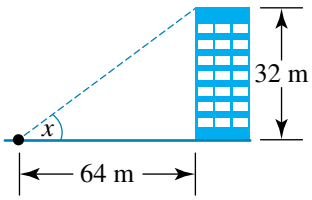
Units 3 & 4 > AOS M3 > Topic 1 > Concept 8

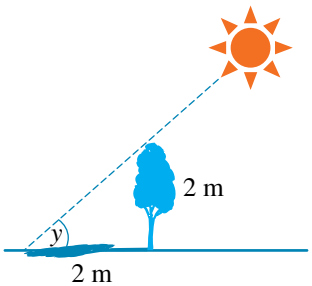
Angles in the horizontal and vertical planes Summary screen and practice questions

Exercise 12.8 Angles of elevation and depression

1. WE19 Find the value of x, correct to 2 decimal places.

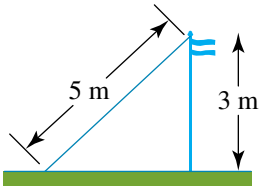


2. Find the value of y, correct to 2 decimal places.

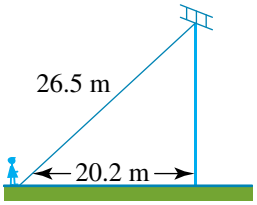


3. Find the angle of elevation (correct to 2 decimal places) in the following situations.

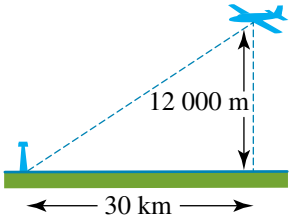
a.



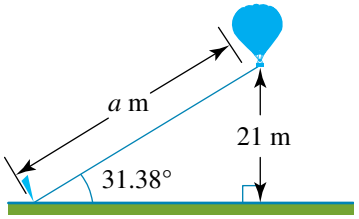
b.



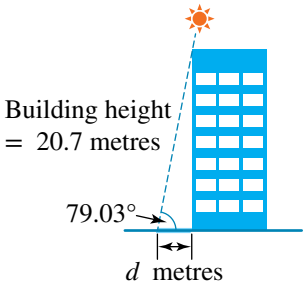
c.



4. **WE20** Find the value of a, correct to 2 decimal places.

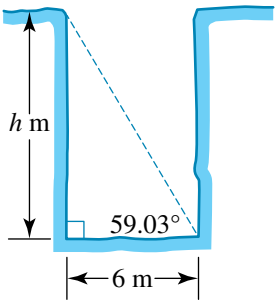


5. Find the value of d, correct to 2 decimal places.

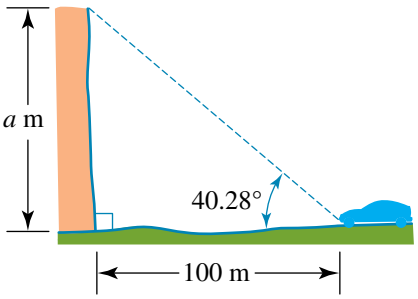


6. Find the values of the pronumerals (correct to the nearest metre).

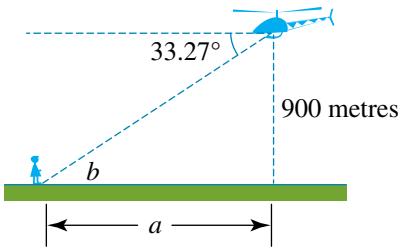
a.



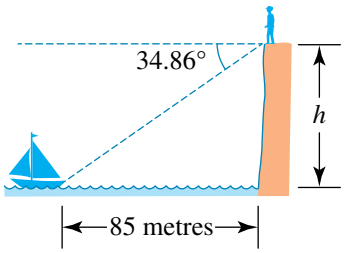
b.



7. **WE21** Find the values of a and b, correct to 2 decimal places.



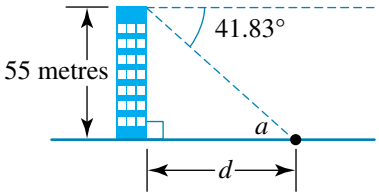
8. Find the value of h, correct to 2 decimal places.



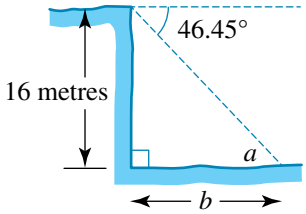
9. The angle of elevation of the sun at a particular time of the day was 49°. What was the length of a shadow cast by a 30-metre tall tower at that time?

10. Find the values of these pronumerals (correct to 2 decimal places).

a.

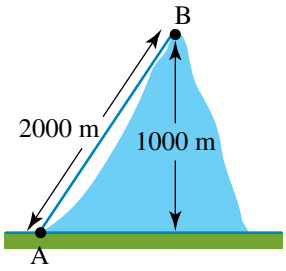


b.

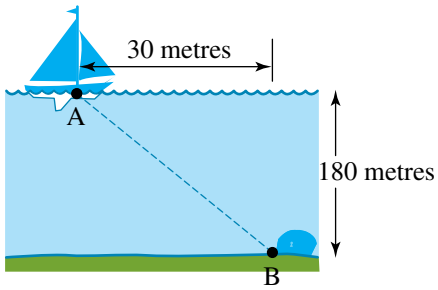


11. Find the angle of elevation or depression from an observer positioned at point A to the object at point B in each situation shown below, correct to two decimal places. State clearly whether it is an angle of depression or elevation.

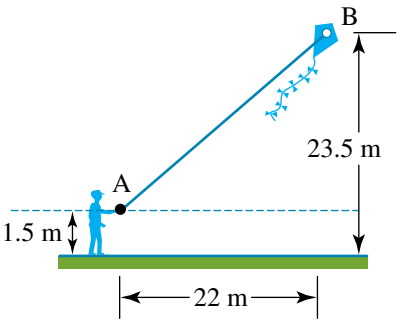
a.



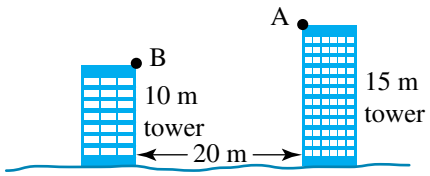
b.



c.



d.



12. A hole has a diameter of 4 metres and is 3.5 metres deep. What is the angle of depression from the top of the hole to the bottom opposite side of the hole?

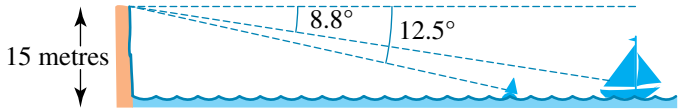
13. MC The angle of elevation of the top of a tree from a point 15.2 m from the base of the tree is 52.2°. The height of the tree is closest to:

A. 12 m **B.** 15 m **C.** 19 m
D. 20 m **E.** 25 m

14. MC A supporting wire for a 16-m high radio tower is 23.3 m long and is attached at ground level and to the top of the tower. The angle of depression of the wire from the top of the tower is:

A. 34.48° **B.** 43.37° **C.** 46.63° **D.** 55.52° **E.** 45.16°

15. The angle of depression to a buoy from the top of a 15-metre cliff is 12.5°. A boat is observed to be directly behind but with an angle of depression of 8.8°.



Find (correct to the nearest metre):

a. the distance to the buoy from the base of the cliff

b. the distance between the boat and the buoy.

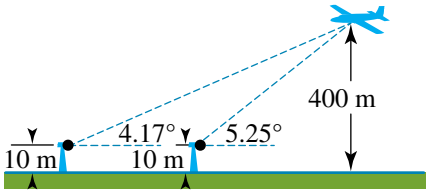
16. Two buildings are 50 metres apart. Building A is 110 metres high. Building B is 40 metres high.

a. Find the angle of elevation from the bottom of building A to the top of building B.

b. Find the angle of depression from the top of building A to the bottom of building B.

c. Find the angle of depression from the top of building B to the bottom of building A.

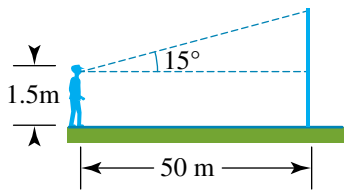
17. Watchers in two 10-metre observation towers each spot an aircraft at an altitude of 400 metres. The angles of elevation from the two towers are shown in the diagram. (Assume all three objects are in a direct line.)



a. What is the horizontal distance between the nearest tower and the aircraft (correct to the nearest 10 metres)?

b. How far apart are the two towers from each other (correct to the nearest 100 metres)?

18. A boy standing 1.5 metres tall measures the angle of elevation to the top of the goalpost using an inclinometer.



a. If the angle was 15° when measured 50 metres from the base of the goalpost, how tall is the goalpost?

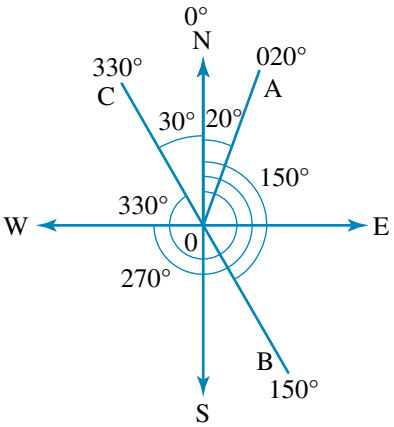
b. If the angle of elevation to the top of the goalpost is now 55.5°, how far is the boy from the base of the goalpost?

c. The angle of elevation is measured at ground level and is found to be 45°. Find the distance from the base of the goalpost to where the measurement was made.

d. The result in part **c** is the same as the height of the goalpost. Explain why.

12.9 Bearings and specification of location

Bearings are used to locate the position of objects or the direction of a journey on a two-dimensional horizontal plane. Bearings or directions are straight lines from one point to another.



12.9.1 Three-figure bearings

Three-figure bearings are always taken in a clockwise direction from North and are specified using three digits as shown at right.

A is on a bearing of 020° from O.

B is on a bearing of 150° from O.

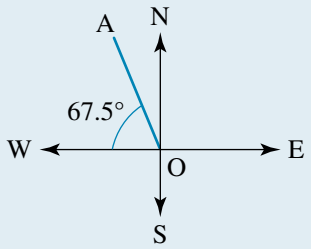
C is on a bearing of 330° from O.

To specify a three-figure bearing, first consider the following:

1. **the angle is measured from north**
2. **the angle is measured in a clockwise direction to the bearing line**
3. **the angle of rotation may take any value from 0° to 360°**
4. **for bearings less than 100°, use three digits with the first digit being a zero to indicate it is a bearing, for example, 045°, 078°.**

WORKED EXAMPLE 22

Specify the bearing of A.



THINK

Find the angle from north to the bearing line in a clockwise direction. The bearing of west is 270°.

WRITE

Angle required = 270° + 67.5°
= 337.5
The bearing of A is 337.5°.

WORKED EXAMPLE 23

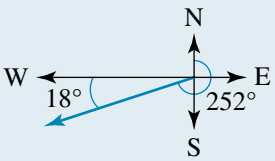
Draw a suitable diagram to represent 252°.

THINK

A bearing of 252° is more than 180° and less than 270°, so the direction lies between south and west. Find the difference between the bearing and west (or south). Draw the 4 main standard bearings and add the bearing line. Add the angle from west (or south).

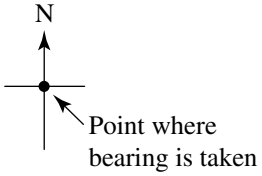
WRITE/DRAW

Difference from west = 270° − 252°
= 18°



12.9.2 Navigation and specification of locations

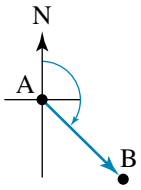
In most cases when you are asked to solve problems, a carefully drawn sketch of the situation will be given. When a problem is described in words only, very careful sketches of the situation are required. Furthermore, these sketches of the situation need to be converted to triangles with angles and lengths of sides included. This is so that Pythagoras' theorem, trigonometric ratios, areas of triangles, similarity and sine or cosine rules may be used.

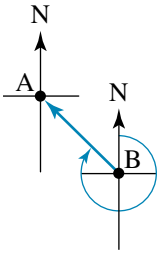


Hints:

1. Carefully follow given instructions.
2. It can be helpful to draw the **compass rose** at the starting point of the direction requested.

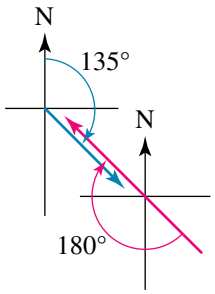
3. Key words are *from* and *to*. For example:
   The bearing *from* A *to* B (see diagram left) is very different from the bearing *from* B *to* A (see diagram right).

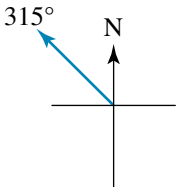




4. When you are asked to determine the direction to return directly back to an initial starting point, it is a 180° rotation or difference. For example, to return directly back after heading north, we need to change the direction to head south.
   Other examples are:
   Returning directly back after heading 135°

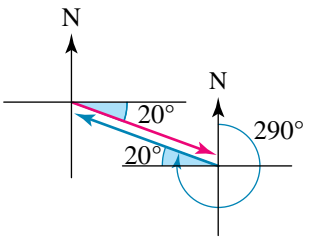
$$\text{New bearing} = 135° + 180°$$
$$= 315°$$

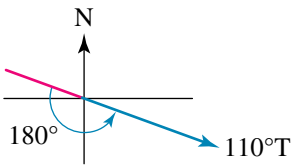




Returning directly back after heading 290°

$$\text{New bearing} = 290° - 180°$$
$$= 110°$$



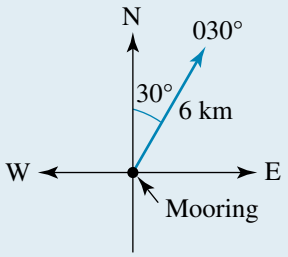


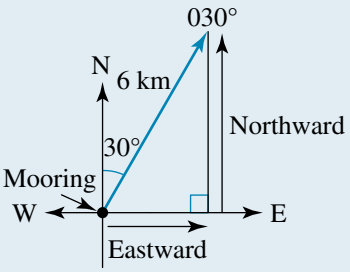
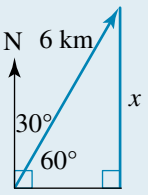
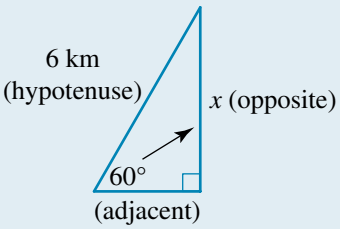
WORKED EXAMPLE 24

A ship leaves port, heading 030° for 6 kilometres as shown.

a. How far north or south is the ship from its starting point (correct to 1 decimal place)?

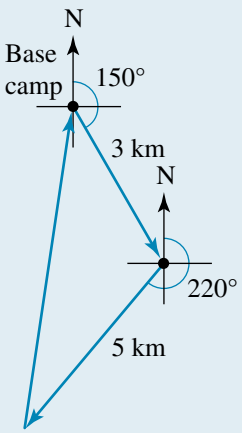
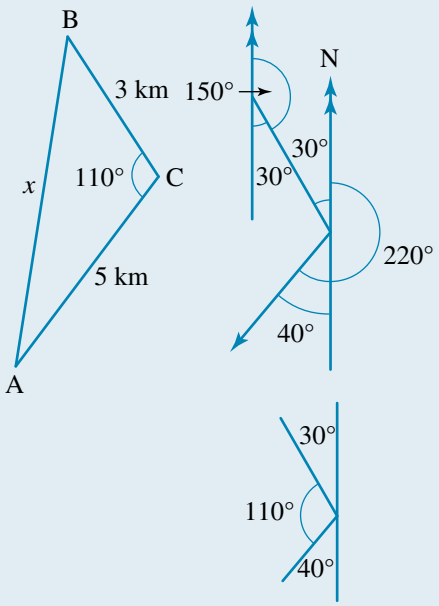
b. How far east or west is the ship from its starting point (correct to 1 decimal place)?



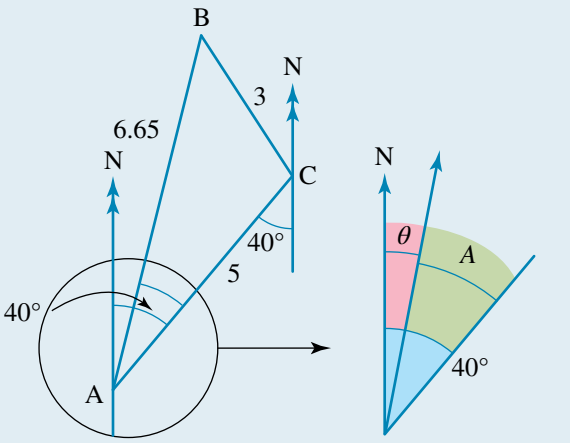
THINK	WRITE/DRAW
a. 1. Draw a diagram of the journey and indicate or superimpose a suitable triangle.	**a.** 
2. Identify the side of the triangle to be found. Redraw a simple triangle with the most important information provided. Use the bearing given to establish the angle in the triangle, that is, use the complementary angle law.	 $90° - 30° = 60°$
3. As the triangle is right-angled, the sine ratio can be used to find the distance north.	
4. Substitute and evaluate.	$\sin(\theta) = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse side}}$ $= \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\sin(60°) = \dfrac{y}{6}$ $x = 6 \times \sin(60°)$ $= 6 \times 0.8660$ $= 5.196$
5. State the answer to the required number of decimal places.	The ship is 5.2 km north of its starting point, correct to 1 decimal place.
b. 1. Use the same approach as in part **a**. This time the trigonometric ratio is cosine to find the distance east, using the same angle.	**b.** $\cos(\theta) = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse side}}$ $= \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\cos(60°) = \dfrac{y}{6}$ $y = 6 \times \cos(60°)$ $= 6 \times 0.5$ $= 3.0$
2. Answer in correct units and to the required level of accuracy.	The ship is 3.0 km east of its starting point, correct to 1 decimal place.

WORKED EXAMPLE 25

Soldiers on a reconnaissance set off on a return journey from their base camp. The journey consists of three legs. The first leg is on a bearing of 150° for 3 km; the second is on a bearing of 220° for 5 km.
Find the direction (correct to the nearest degree) and distance (correct to 2 decimal places) of the third leg by which the group returns to its base camp.

THINK	WRITE/DRAW
1. Draw a diagram of the journey and indicate or superimpose a suitable triangle.	
2. Identify the side of the triangle to be found. Redraw a simple triangle with the most important information provided. ($\angle BCA = 180° - 30° - 40°$. $\therefore \angle BCA = 110°$)	
3. The cosine rule can be used to find the length of the side AB, as we are given two sides and the angle in between.	$a = 3$ km, $b = 5$ km, $C = 110°$, $c = x$ km
4. Substitute the known values into the cosine rule and evaluate.	$c^2 = a^2 + b^2 - 2ab \times \cos(C)$ $x^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos(100°)$ $= 44.260\,604$ $x = \sqrt{44.260\,604}$ $= 6.65$

5. For direction, we need to find the angle between the direction of the second and third legs first, that is, ∠BAC. Once the size of ∠BAC (or simply, ∠A) is established, it can be subtracted from 40° to find angle θ. This will give the bearing for the third leg of the journey. Since all 3 side lengths in ΔABC are now known, use the cosine rule to find ∠A.



$a = 3, \quad b = 5, \quad c = 6.65 \text{ or } \sqrt{44.260\,604}$

6. Substitute the known values into the rearranged cosine rule.
Note: Use the most accurate form of the length of side c.

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2 \times b \times c}$$
$$= \frac{5^2 + 44.260\,604 - 3^2}{2 \times 5 \times \sqrt{44.260\,604}}$$
$$= 0.9058$$
$$A = 25.07°$$
$$= 25° \text{(correct to the nearest degree)}$$

7. Calculate the angle of the turn from the north bearing.

$$\theta = 40° - 25°$$
$$= 15°$$

Bearing is 015°.

8. Write the answer in correct units and to the required level of accuracy.

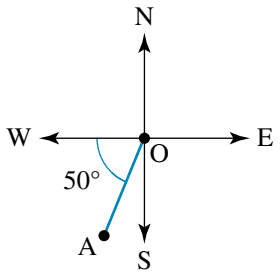
The distance covered in the final leg is 6.65 km, correct to 2 decimal places, on a bearing of 015°, correct to the nearest degree.

Resources

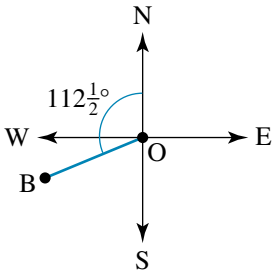
Interactivity: Bearings (int-6481)

Exercise 12.9 Bearings and specification of location

1. WE22 Specify the bearing of A from O.



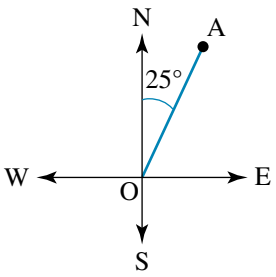
2. Specify the bearing of B from O.



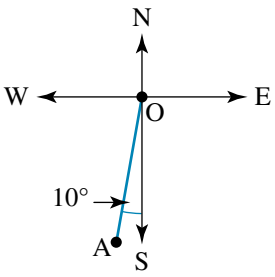
3. **WE23** Draw a suitable diagram to represent 080° direction.

4. Specify the following bearings of A from O.

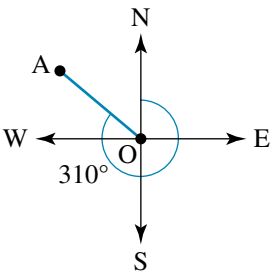
a.



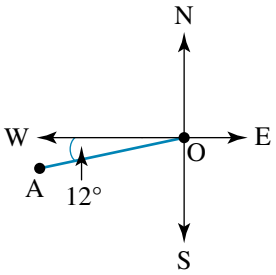
b.



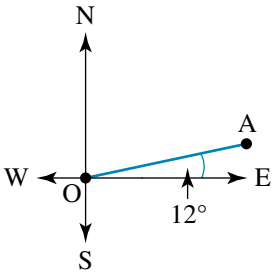
c.



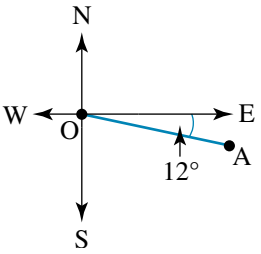
d.



e.



f.

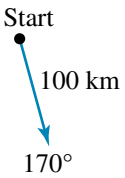


5. **MC** The direction of the first leg of a hiking trip was 220°. For the second leg, the hikers turn 40° right. The new direction for the second leg of the hike is:

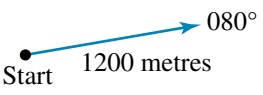
A. 270° **B.** 180° **C.** 260° **D.** 040° **E.** 280°

6. A hiker heads out on the direction 018°. After walking 5 km they adjust their bearing 12° to their right and on the last leg of their hike they adjust their bearing 5° to their left. Calculate the bearing of their last leg.

7. **WE24** For the diagram at right, find how far north or south and east or west the end point is from the starting point (correct to 1 decimal place).

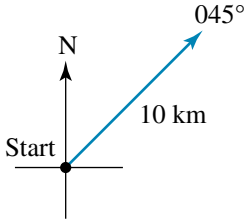


8. For the diagram at right, find how far north or south and east or west the end point is from the starting point (correct to 1 decimal place).

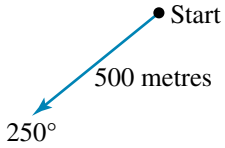


9. For the following, find how far north or south and east or west the end point is from the starting point (correct to 1 decimal place).

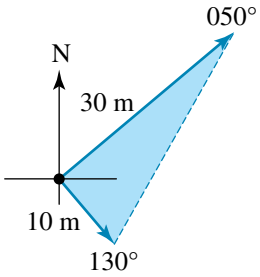
a.



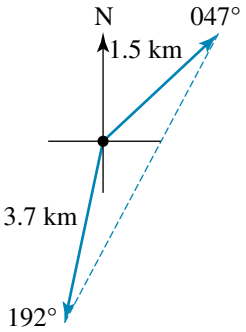
b.



10. Find the length of the unknown side for the triangle (correct to the nearest unit).

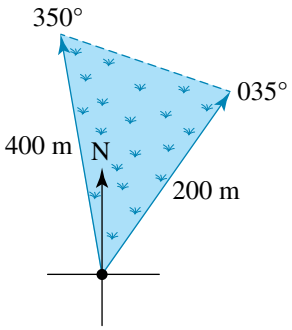


11. Find the length of the unknown side for the triangle (correct to the nearest unit).

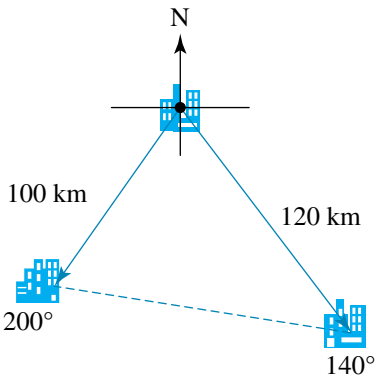


12. Find the length of the unknown side for each of the triangles (correct to the nearest unit).

a.



b.



13. **WE25** A scout troop travels on a hike of 2 km with a bearing of 048°. They then turn and travel on a bearing of 125° for 5 km, then camp for lunch. How far is the camp from the starting point? Give the answer correct to 2 decimal places.

14. Find the bearing required in question **13** to get the scout troop back to their starting point. Give your answer correct to the nearest degree.

15. In a pigeon race, the birds start from the same place. In one race, pigeon A flew 35 km on a bearing of 295° to get home, while pigeon B flew 26 km on a bearing of 174°.

a. MC The distance between the two pigeons' homes is closest to:

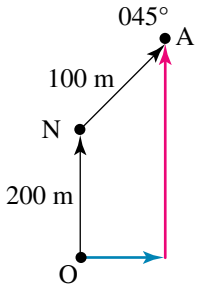
A. 13 km **B.** 18 km **C.** 44 km **D.** 50 km **E.** 53 km

b. MC The bearing of pigeon A's home from pigeon B's home is closest to:

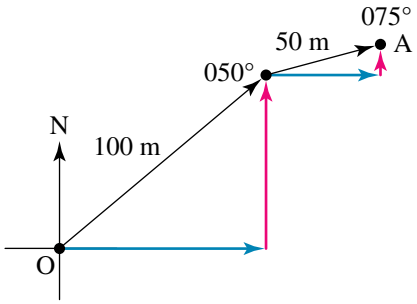
A. 332° **B.** 326° **C.** 320° **D.** 208° **E.** 220°

16. For each of the following, find how far north/south and east/west position A is from position O.

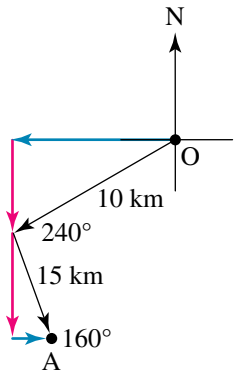
a.



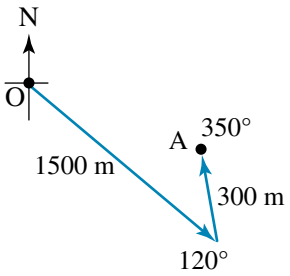
b.



c.



d.



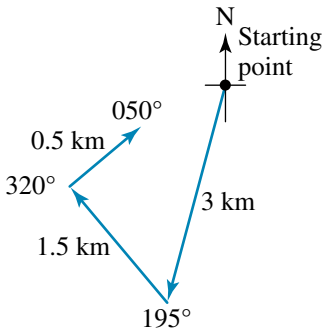
17. For the hiking trip shown in the diagram, find (correct to the nearest metre):

a. how far south the hiker is from the starting point

b. how far west the hiker is from the starting point

c. the distance from the starting point

d. the direction of the final leg to return to the starting point.



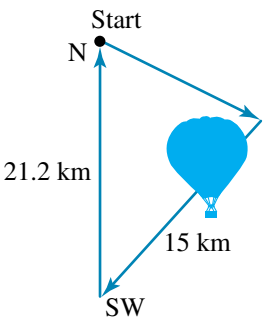
18. Captain Cook sailed from Cook Island on a bearing of 010° for 100 km. He then changed direction and sailed for a further 50 km on a bearing of SE to reach a deserted island.

a. How far from Cook Island is Captain Cook's ship (correct to the nearest kilometre)?

b. Which direction would have been the most direct route from Cook Island to the deserted island (correct to 2 decimal places)?

c. How much shorter would the trip have been using the direct route?

19. A journey by a hot-air balloon is as shown. The balloonist did not initially record the first leg of the journey. Find the direction and distance for the first leg of the balloonist's journey.



20. A golfer is teeing off on the 1st hole. The distance and direction to the green is 450 metres on a bearing of 190°. If the tee shot of the player was 210 metres on a bearing of 220°, how far away from the green is the ball and in what direction should she aim to land the ball on the green with her second shot? (Give the distance correct to the nearest metre and the direction correct to the nearest degree.)

21. A boat begins a journey on a bearing of 063.28° and travels for 20 km.

a. How far east of its starting point is it?
It then changes to a bearing of 172.43° and travels for a further 35 km.

b. Through what angle did the boat turn?

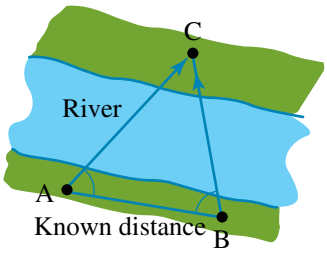
c. How far is it now from its starting point?

d. What is the bearing of its end point from the starting point?

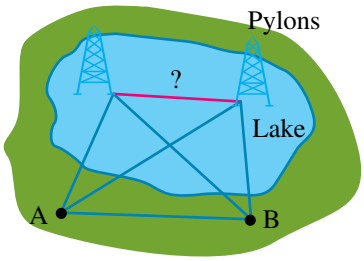
12.10 Triangulation — cosine and sine rules

In many situations, certain geographical or topographical features are not accessible to a survey. To find important locations or features, triangulation is used. This technique requires the coordination of bearings from two known locations, for example, fire spotting towers, to a third inaccessible location, the fire (see Worked example 26).

1. Triangulation should be used when:
   a. the distance between two locations is given and
   b. the direction from each of these two locations to the third inaccessible location is known.

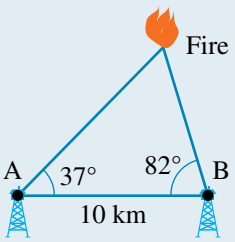


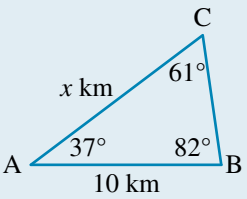
2. For triangulation:
   a. the sine rule is used to find distances from the known locations to the inaccessible one
   b. the cosine rule may be used occasionally for locating a fourth inaccessible location.



WORKED EXAMPLE 26

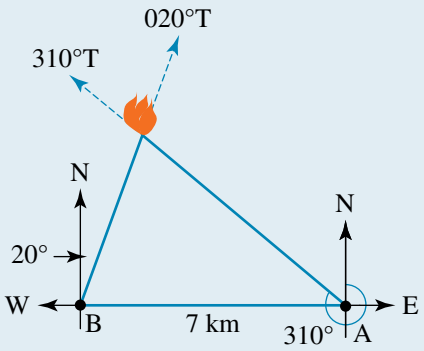
How far (correct to 1 decimal place) is the fire from Tower A?



THINK	WRITE/DRAW
1. Draw a triangle and identify it as a non-right-angled triangle with a given length and two known angles. Determine the value of the third angle and label appropriately for the sine rule.	
	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
	where $c = 10\,\text{km}$ $C = 180° - (37° + 82°)$ $= 61°$ $b = x$ $B = 82°$
2. Substitute into the formula and evaluate.	$\frac{x}{\sin(82°)} = \frac{10}{\sin(61)°}$ $x = \frac{10 \times \sin(82)°}{\sin(61)°}$ $= 11.322$
3. Write the answer using correct units and to the required level of accuracy.	The fire is 11.3 km from Tower A, correct to 1 decimal place.

WORKED EXAMPLE 27

Two fire-spotting towers are 7 kilometres apart on an east–west line. From Tower A a fire is seen on a bearing of 310°. From Tower B the same fire is spotted on a bearing of 020°. Which tower is closest to the fire and how far is that tower from the fire (correct to 1 decimal place)?

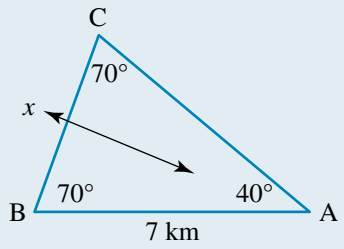
THINK	WRITE/DRAW
1. Draw a suitable sketch of the situation described. It is necessary to determine whether Tower A is east or west of Tower B.	

2. Identify the known values of the triangle and label appropriately for the sine rule.
   The shortest side of a triangle is opposite the smallest angle. Therefore, Tower B is closest to the fire. Use the sine rule to find the distance of Tower B from the fire.

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

where

$a = x$
$c = 7\text{ km}$
$A = 40°$
$C = 180° - (70° + 40°)$
$= 70°$



3. Substitute into the formula and evaluate.
   Note: ΔABC is an isosceles triangle, so Tower A is 7 km from the fire.

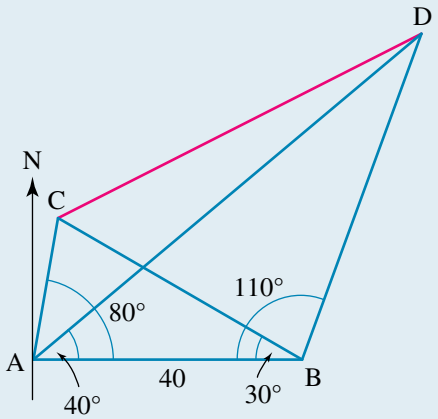
$$\frac{x}{\sin(40)°} = \frac{7}{\sin(70)°}$$
$$x = \frac{7 \times \sin(40)°}{\sin(70)°}$$
$$= 4.788\,282\text{ km}$$

4. Write the answer in the correct units.

Tower B is closest to the fire at a distance of 4.8 km, correct to 1 decimal place.

WORKED EXAMPLE 28

From the diagram find:
a. the length of CD (correct to 1 decimal place)
b. the bearing from C to D (correct to 1 decimal place).

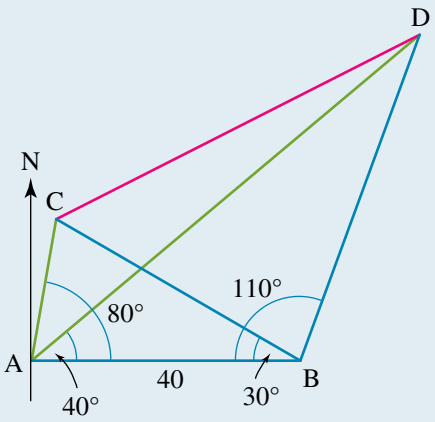


THINK

a. 1. To evaluate the length of CD, we need to first determine the lengths of AC and AD. (Alternatively, we can find the lengths of BC and BD.)

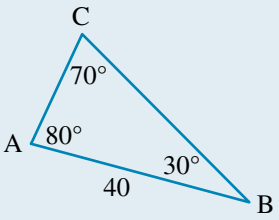
WRITE/DRAW

a.



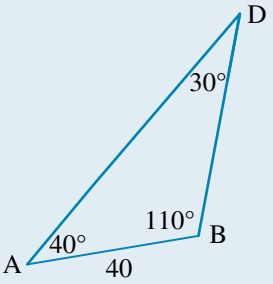
2. Label ΔABC for the sine rule and evaluate the length of AC.
($C = 180° - 30° - 80°$
$= 70°$)

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
$b = \overline{AC}, B = 30°$
$c = 40, \ C = 70°$
$$\frac{\overline{AC}}{\sin(30)°} = \frac{40}{\sin(70)°}$$
$$\overline{AC} = \frac{40 \times \sin(30)°}{\sin(70)°}$$
$$= 21.283\,555$$

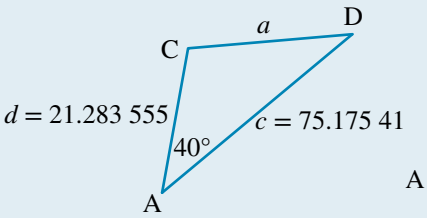


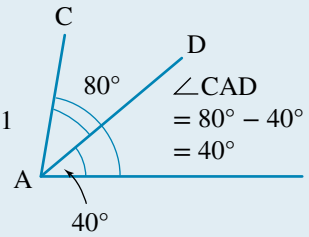
3. Label ΔABD for the sine rule and evaluate the length of AD.
($D = 180° - 40° - 110°$
$= 30°$)

$$\frac{b}{\sin(B)} = \frac{d}{\sin(D)}$$
$b = \overline{AD}, B = 110°$
$d = 40, \ D = 30°$
$$\frac{\overline{AD}}{\sin(110)°} = \frac{40}{\sin(30)°}$$
$$\overline{AD} = \frac{40 \times \sin(110)°}{\sin(30)°}$$
$$= 75.175\,41$$



4. Draw ΔACD, which is needed to find the length of CD. Use the two given angles to find ∠CAD. Now label it appropriately for the cosine rule.





$a = \overline{CD}$
$d = 21.283\,55$
$A = 40°$
$c = 75.175\,41$

5. Substitute into the formula and evaluate.

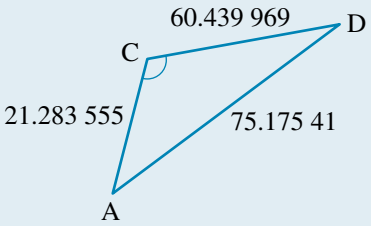
$$a^2 = c^2 + d^2 - 2cd \times \cos(A)$$
$$\overline{CD}^2 = 75.175\,41^2 + 21.283\,555^2 - 2 \times 75.175\,41 \times 21.283\,555 \times \cos(40°)$$
$$\overline{CD} = \sqrt{3652.9898}$$
$$= 60.439\,969$$

6. Write the answer in the correct units.

The length of $\overline{CD}$ is 60.4 units, correct to 1 decimal place.

b. 1. Redraw ΔACD and label it with the known information.

b.



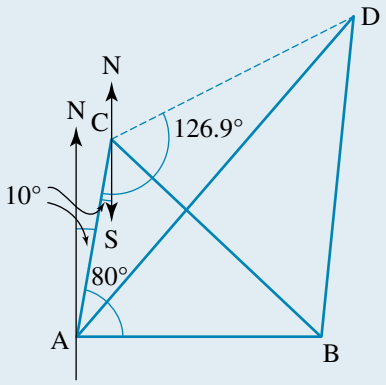
2. Bearing required is taken *from* point C, so find ∠ACD by using the cosine rule first.

$a = 60.439\,969$
$c = 75.175\,41$
$d = 21.283\,555$

3. Substitute into the rearranged cosine rule and evaluate angle C.

$$\cos(C) = \frac{a^2 + d^2 - c^2}{2 \times a \times d}$$
$$= \frac{(60.439\,969)^2 + (21.283\,555)^2 - (75.175\,41)^2}{2 \times 60.439\,969 \times 21.283\,555}$$
$$= -0.6007$$
$$C = 126.9°$$

4. Redraw the initial diagram (from the question) with known angles at point C in order to find the actual bearing angle.



5. Determine the angle from south to the line CD.

$\angle SCD = 126.9° - 10°$
$= 116.9°$

6. Determine the bearing angle.

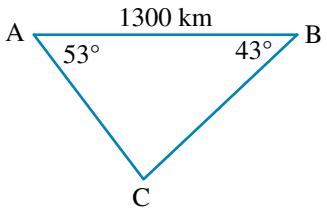
$\angle NCD = 180° - 116.9°$
$= 63.1°$

7. Write the bearing of D from C.

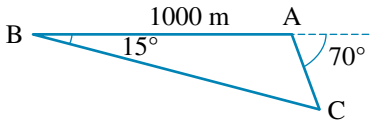
The bearing of D from C is 063.1°, correct to 1 decimal place.

Exercise 12.10 Triangulation — cosine and sine rules

1. **WE26** Find the distance from A to C (correct to 1 decimal place).

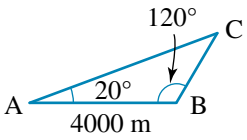


2. Find the distance from A to C (correct to 1 decimal place).

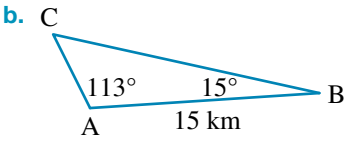


3. Find the distance from A to C in each case (correct to 1 decimal place).

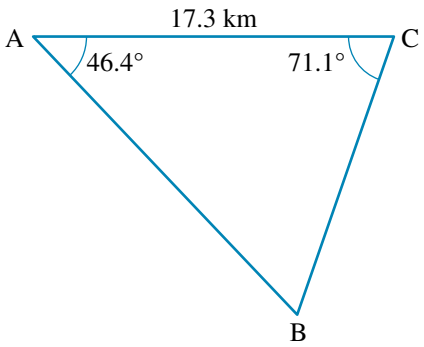
a.



b.



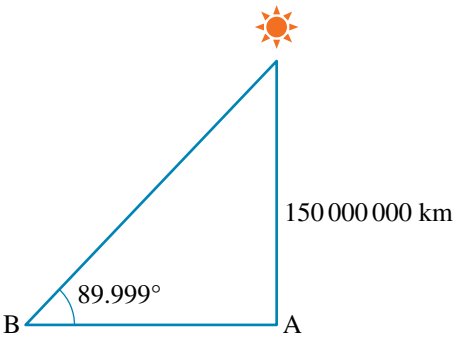
4. **WE27** A boat (B) is spotted by 2 lighthouses (A and C), which are 17.3 km apart. The angles measured by each lighthouse are shown in the figure at right.



Determine the distance from each lighthouse to the boat.

5. Two ancient astronomers lived in different towns (A and B) and wished to know the distance between the towns. At the same time they measured the angle of the sun, as shown in the diagram at right. Assuming the distance of the astronomer from town A to the sun was 150 000 000 km, find the distance between the towns (i.e. the distance AB).

(Note that due to the nature of the angles, this is not an accurate way to measure this distance.)



6. **a.** Two fire-spotting towers are 17 kilometres apart on an east–west line. From Tower A, a fire is seen on a bearing of 130°. From Tower B, the same fire is spotted on a bearing of 200°. Which tower is closest to the fire and how far is that tower from the fire?

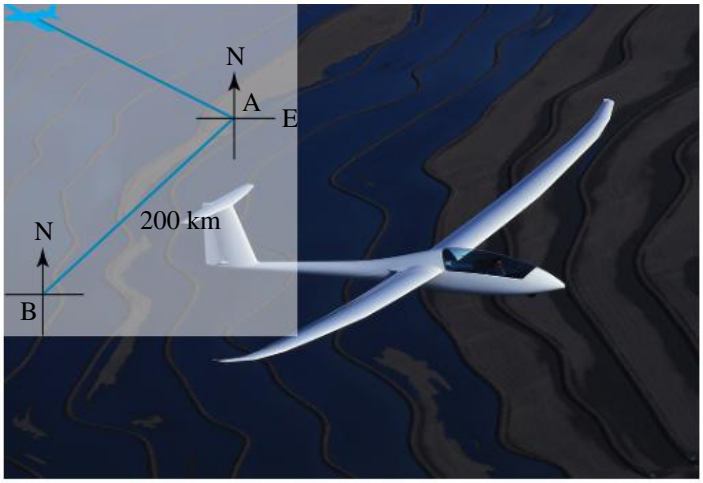
b. Two fire-spotting towers are 25 kilometres apart on a north–south line. From Tower A, a fire is reported on a bearing of 082°. Spotters in Tower B see the same fire on a bearing of 165°. Which tower is closest to the fire and how far is that tower from the fire?

7. Two lighthouses are 20 km apart on a north–south line. The northern lighthouse spots a ship on a bearing of 100°. The southern lighthouse spots the same ship on a bearing of 040°.

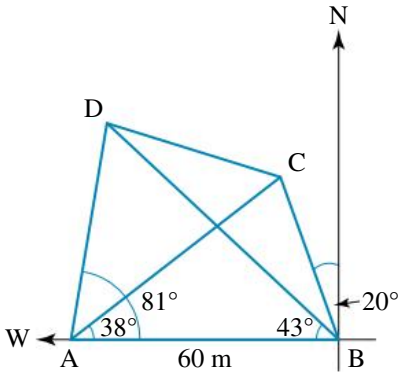
a. Find the distance from the northern lighthouse to the ship.

b. Find the distance from the southern lighthouse to the ship.

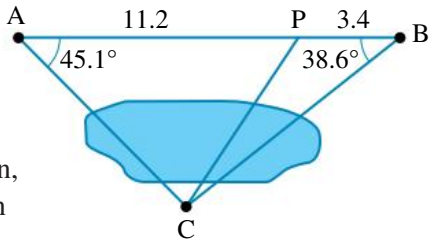
8. Two air traffic control towers detect a glider that has strayed into a major air corridor. Tower A has the glider on a bearing of 315°. Tower B has the glider on a bearing of north. The two towers are 200 kilometres apart on a NE line as shown. To which tower is the glider closer? What is the distance?



9. **WE28** Given the information in the diagram, find the length of CD.

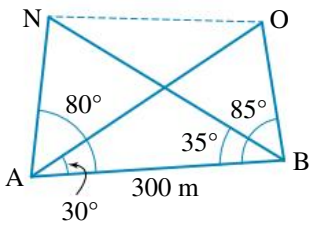


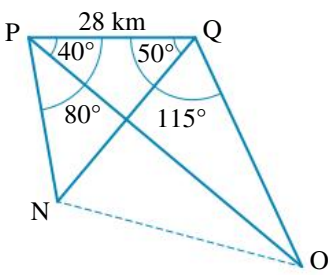
10. Three towns, A, B and C, are situated as shown in the diagram at right.

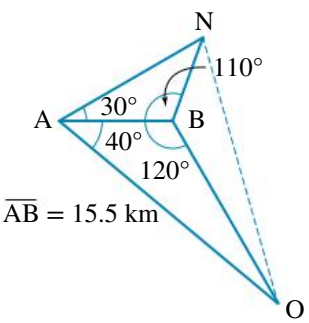


A mountain range separates town C from the other two, but a road is to be built, starting at P and tunnelling through the mountain in a straight line to C. From geographic information, the angles (45.1°, 38.6°) and distances (11.2 km, 3.4 km) between points are determined as shown in the figure.
Find the angle the road PC makes with the line AB (∠APC).

11. Find the value of line segment NO in each case (correct to 1 decimal place).

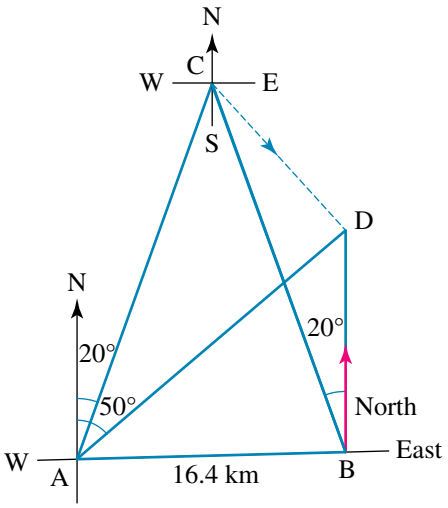
a.


b.


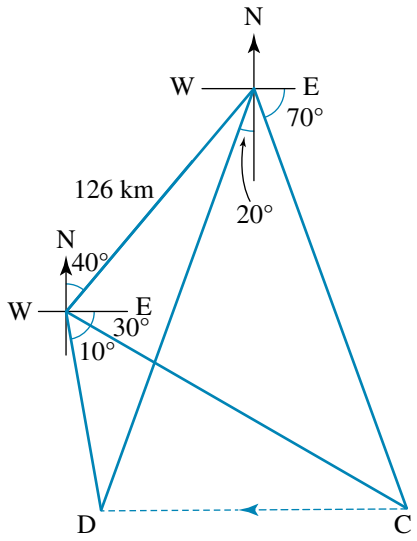
c.


12. Find the distance and bearing from C to D (both correct to 1 decimal place).

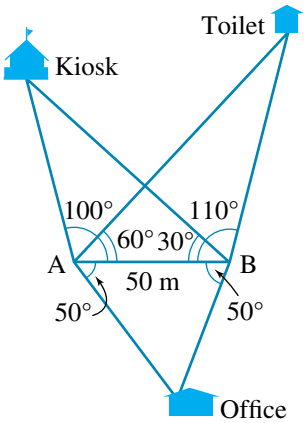
a.



b.

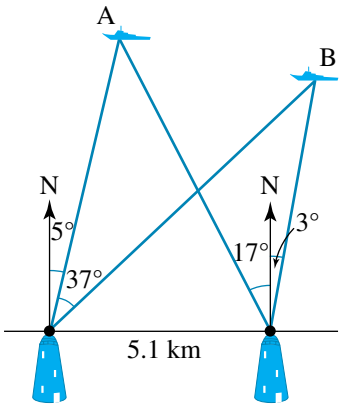


13. A student surveys her school grounds and makes the necessary measurements to 3 key locations as shown in the diagram.

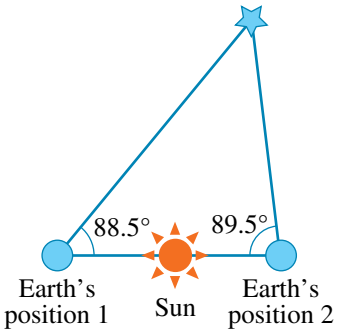


- **a.** Find the distance to the kiosk from:
  - **i.** location A
  - **ii.** location B.
- **b.** Find the distance to the toilets from:
  - **i.** location A
  - **ii.** location B.
- **c.** Find the distance from the toilet to the kiosk.
- **d.** Find the distance from the office to location A.

14. From the diagram below, find the distance between the two ships and the bearing from ship A to ship B.



15. An astronomer uses direction measurements to a distant star taken 6 months apart, as seen in the diagram at next page (which is not drawn to scale). The known diameter of the Earth's orbit around the Sun is 300 million kilometres. Find the closest distance from Earth to the star (correct to the nearest million kilometres).



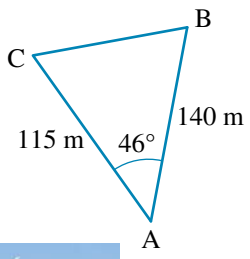
16. MC Two girls walk 100 metres from a landmark. One girl heads off on a bearing of 136°, while the other is on a bearing of 032°. After their walk, the distance between the two girls, correct to the nearest metre, is closest to:

A. 123 m **B.** 158 m **C.** 126 m **D.** 185 m **E.** 200 m

17. MC Two ships leave the same port and sail the same distance, one ship on a bearing of NW and the other on a bearing of SSE. If they are 200 kilometres apart, what was the distance sailed by each ship?

A. 100 km **B.** 101 km **C.** 102 km **D.** 202 km **E.** 204 km

18. In the swimming component of a triathlon, competitors have to swim around the buoys A, B and C, as marked in the figure, to end at point A.



a. What is the shortest leg of the swimming component?

b. What is the angle at buoy B?

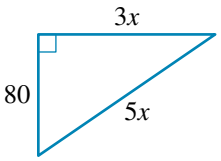
c. What is the total distance travelled?

12.11 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC For the triangle shown, the value of x is:

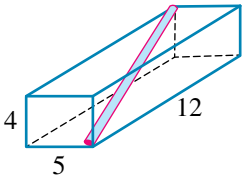


A. 4 **B.** 10 **C.** 20 **D.** 6.7 **E.** 30

2. MC A 15 cm long straw is the longest that can fit into a cylindrical can with a radius of 6 cm. The height of the can, in centimetres, is closest to:

A. 8 **B.** 9 **C.** 15 **D.** 16 **E.** 17

3. MC A rectangular box has a rod positioned as shown in the diagram. The expression that would enable the angle the rod makes with the base of the box to be found, is:

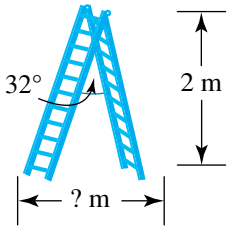


A. $\tan(\theta) = \frac{5}{12}$ **B.** $\sin(\theta) = \frac{4}{13}$

C. $\tan(\theta) = \frac{4}{12}$ **D.** $\tan(\theta) = \frac{4}{13}$

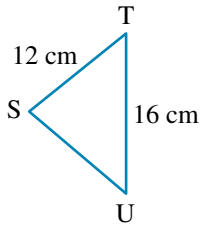
E. $\cos(\theta) = \frac{5}{12}$

4. MC A stepladder is erected as shown. How far apart (correct to 2 decimal places) at the base are the two legs?



A. 1.15 m
B. 1.41 m
C. 1.50 m
D. 2.00 m
E. 6.97 m

5. MC Given ST = 12 cm, TU = 16 cm and $\sin(U) = \frac{3}{4}$, then $\sin(S)$ equals:



A. $\frac{12}{16}$

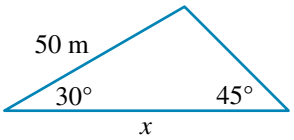
B. $\frac{16}{12}$

C. $\frac{16}{9}$

D. $\frac{4}{3}$

E. 1

6. MC Find the value of the pronumeral, correct to the nearest metre.

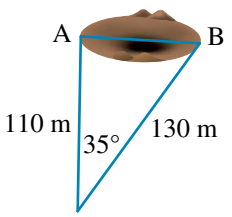


A. 35 **B.** 45
C. 50 **D.** 62
E. 68

7. MC In a triangle ABC where $b = 10$, $c = 20$ and $B = 26°$, C (correct to the nearest degree) could be:

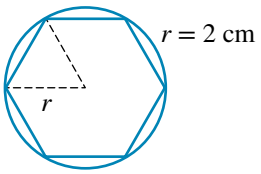
A. 61° **B.** 62° **C.** 63° **D.** 63° or 117° **E.** 61° or 119°

8. MC To find the distance across a large excavation, measurements were found as shown in the diagram. The distance, AB, across the excavation is closest to:



A. 75 metres
B. 74 metres
C. 100 metres
D. 120 metres
E. none of these

9. **MC** A regular hexagon is inscribed in a circle of radius 2 cm. The perimeter of the hexagon, in centimetres, is:

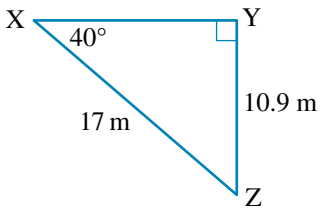


A. 4π
B. 12
C. 16
D. 17
E. 18

10. **MC** A right-angled isosceles triangle has a longest side of 141 metres. The other two equal sides have a value closest to:

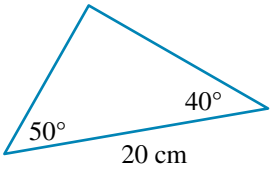
A. 200 m
B. 100 m
C. 50 m
D. 120 m
E. none of these

11. **MC** The area of triangle XYZ (correct to the nearest m^2) is:



A. 55
B. 170
C. 45
D. 85
E. 71

12. **MC** The area of the triangle at right is closest to:



A. 96 cm^2
B. 97 cm^2
C. 98 cm^2
D. 99 cm^2
E. 100 cm^2

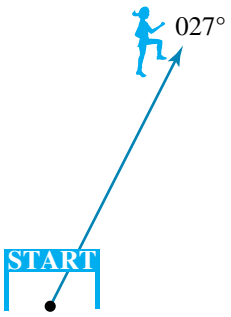
13. **MC** A fly is hovering above a frog sitting on the ground, at an angle of elevation of 42.33° and 12 cm directly above the ground. The minimum length (to 1 decimal place) that the frog's tongue needs to be to touch the fly is:

A. 8.1 cm **B.** 12.0 cm **C.** 13.2 cm **D.** 16.2 cm **E.** 17.8 cm

14. **MC** A helicopter has its 8-metre rescue ladder hanging vertically from the doorway of the craft. The ladder's free end is just touching the surface of the floodwater below. From the top of the ladder the angle of depression of a stranded person in the water, clinging to a pole, is 54°. The distance that the helicopter must fly horizontally toward the person to rescue him is:

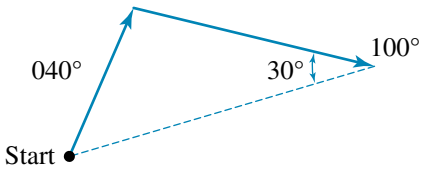
A. 4.7 m **B.** 5.8 m **C.** 16.5 m **D.** 9.9 m **E.** 11.0 m

15. **MC** The bearing required to return to the starting line in the situation is:



A. 027°
B. 153°
C. 207°
D. 117°
E. 063°

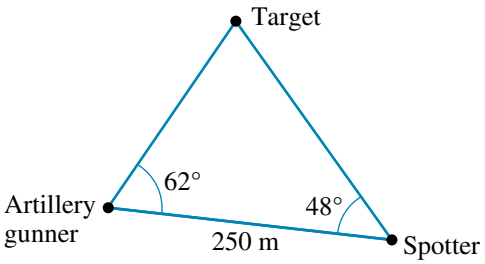
16. **MC** For the final leg of the journey shown, the bearing to return to the start is:



A. 070° **B.** 130°
C. 250° **D.** 110°
E. 230°

17. MC The horizontal distance (to the nearest metre) that the artillery gunner shown needs to fire to hit the target is closest to:

A. 198 m
B. 210 m
C. 235 m
D. 250 m
E. 266 m



18. MC Two boats, P and Q, are to rendezvous at port R. At the moment, boat P is 15 km due west of Q. For boat P, port R has a bearing of 049° and for boat Q, the bearing of the port is 334°. The distance (to the nearest kilometre) that Q has to travel is:

A. 10 km **B.** 12 km **C.** 13 km **D.** 15 km **E.** 18 km

Short answer

1. A 2.5 m long ladder is placed up against a wall and reaches to a height of 2.4 m. Find the distance that the legs of the ladder are from the base of the wall.

2. A 190 mm square ceramic floor tile is to be cut diagonally. What is the exact length of the cut to be made?

3. A boat sails directly northwards for 11 km before turning towards the east and sailing 60 km. At this point the boat is 61 km from the starting point. On the second leg of the trip, did the boat sail directly eastwards?

4. A cube with 8 cm sides is internally braced. What is the length of the longest brace that could be placed inside the cube? Express your answer in surd form.

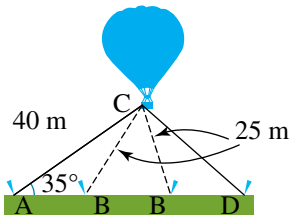
5. A staircase is to rise by 2250 mm from the ground floor to the first level of a house. The maximum angle of elevation allowed for the stairs is 50°.

a. What is the length of the base of the staircase (correct to the nearest mm)?
b. What is the length of the staircase (correct to the nearest mm)?

6. A car badge to be fitted on a bonnet is of an isosceles triangle design.

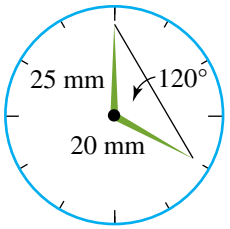
a. If the height of the badge is not to be more than 30 mm, what is the maximum length of the base of the badge (correct to the nearest mm), if the equal angles are 25°?
b. If the longest side is to be set at 100 mm, what is the length of the other two equal sides, if the two equal angles are still 25°?

7. A hot-air balloon is anchored to the ground at points A and D as shown in the following diagram. A 25-metre length of excess rope is dropped to the ground from the balloon. It is then tied to the ground, at a point B, as a further safety measure.



a. What are the smallest and largest angles that can be made at point C by the two lengths of rope, AC and BC (correct to the nearest degree)?
b. Using these values, determine the furthest and closest positions (correct to 1 decimal place) possible for point B from point A.

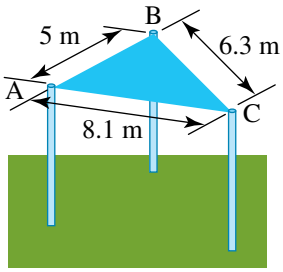
8. The hour hand of a clock is 20 mm long and the minute hand is 25 mm. The clock face below shows the time at 4 o'clock. What is the distance between the tips of both hands (correct to the nearest mm)?



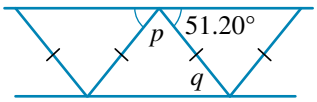
9. An undercover patio is covered with a sail as shown in the diagram.

a. At which support pole is the smallest angle?

b. What are the angles made by the sail at each of the support poles (correct to the nearest degree)?



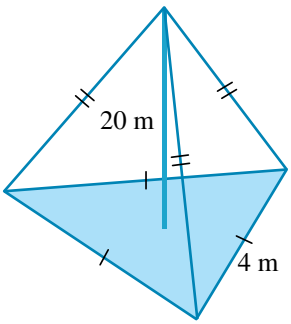
10. A steel truss, which is to be used to reinforce the roof in a building, is designed as shown in the diagram. Find the values of p and q.



11. Three guy wires are used to support a 20-metre tall pole, as shown. Find:

a. the length of a guy wire (correct to 1 decimal place)

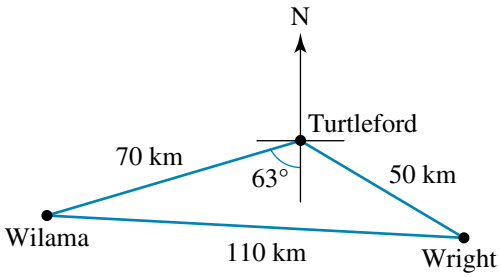
b. the angle of elevation of the guy wires.



12. A country valley is bounded by three major roads as shown.

a. What is the direction from Turtleford to Wright if Turtleford to Wilama is 243°T ?

b. What is the bearing of Wilama from Wright?



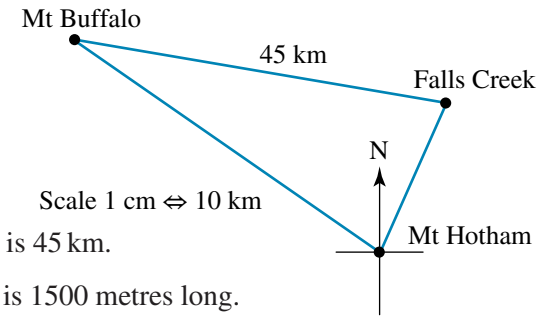
13. Three communication towers are located as shown.

a. Find the distance from the tower at Mt Hotham to:

i. Falls Creek

ii. Mt Buffalo.

b. Find the direction from Mt Hotham to Falls Creek, given the direction from Mt Hotham to Mt Buffalo is 330° and the distance from Falls Creek to Mt Buffalo is 45 km.



14. A ski run rises 120 metres from the baseline and the run is 1500 metres long.

a. What is the angle of elevation of the slope (correct to 1 decimal place)?

b. What is the gradient of the slope (expressed as a scale ratio, for example 1:20)?

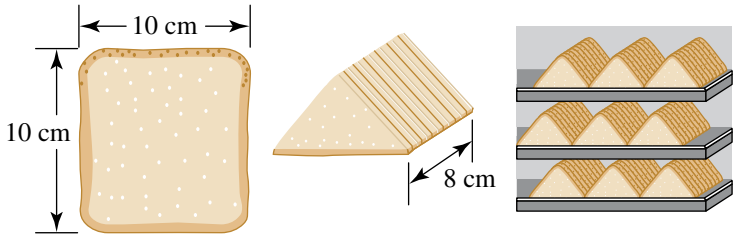
15. A bushwalker travels from home on a bearing of 020° for 8 km, then on a bearing of 130° for 11 km. Find the distance the bushwalker is from home.

16. From the result of question **15**, determine the bearing required for the bushwalker to return home.

17. Two firespotting towers are on an east–west line 15 km apart. Tower A spots a fire on a bearing of 053°, while Tower B spots it on a bearing of 008°. Determine the distance of each tower from the fire.

Extended response

1. A sandwich bar uses bread that is roughly 10 cm square. The bread slices are cut into four equal triangles and packaged in a cardboard box with the triangles arranged as shown.



a. **i.** What is the total length of the two cuts required to make four triangular pieces?

ii. What is the area of the triangular face of the packaged sandwich (correct to 1 decimal place)? The completed sandwiches are placed on shelves as shown above.

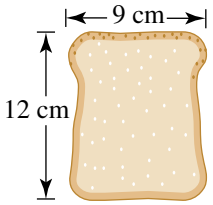
iii. What is the smallest possible gap required between shelves for the sandwiches to fit? To maintain sandwich freshness, the owner is advised to prepare sandwiches so that the surface area is minimised.

b. **i.** Show that the surface area of packaged, triangular sandwiches is close to 243 cm^2.

ii. Would cutting the sandwiches into four small equal square pieces reduce the surface area? If so, by how much (correct to the nearest cm^2)? Draw a suitable diagram(s).

iii. Find the volume of the sandwich package.

c. An alternative is to use bread which has a rectangular shape as shown, and to prepare it as triangular pieces.

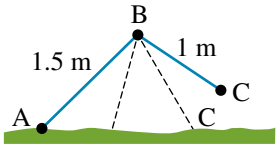


i. What is one disadvantage of using rectangular slices of bread for making four triangular sandwich pieces?

ii. The four angles at the centre of the bread just after making the two cuts are no longer right angles. Find the value of the largest angle.

iii. If the four triangular pieces are also to be packaged, what is the smallest possible area of the triangular face of the cardboard box?

2. Two thin rods are hinged together and the end of one rod is hinged to the ground, while the end of the other rod is free, as shown in the diagram.



a. Lucie conducts an investigation of the triangle formed. She starts by investigating the formation of a right-angled triangle.

i. At what distance from A (correct to 2 decimal places) must Lucie place end C so that a right-angled triangle is formed at C? Remember that the two rods can move, although they are fixed at A.

ii. What is the angle made by the 1.5 m rod with the ground (correct to the nearest degree)?

iii. Using your answer from part **ii**, what is the value of the other acute angle?

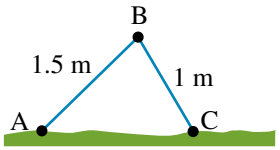
b. Lucie now brings end C to a position 1 m from end A.

i. State the type of triangle formed.

ii. What is the size of the largest angle formed in this triangle (correct to the nearest degree)?

iii. If the largest angle is now to be 110°, what is the new distance from A to C (correct to 3 decimal places)?

c. An alternative is to move end C away from A, as shown right. How far is end C from A, if ∠ABC is to be 110° (in metres correct to 1 decimal place)?



d. Lucie now investigates the area of the triangle made in each situation.

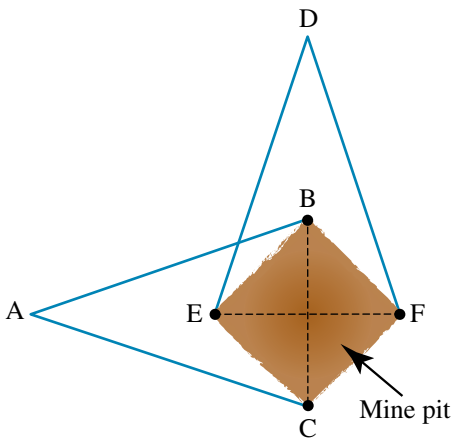
i. What is the area of the triangle in part **a i** (in m^2 correct to 2 decimal places)?

ii. What is the area of the triangle in part **b iii** (in m^2 correct to 2 decimal places)?

iii. What is the area of the triangle in part **c** (in m^2 correct to 2 decimal places)?

e. A third rod, 3 metres long, is connected at point B to the right-angled triangle formed in part **a**. Its free end rests on the ground. What is the horizontal distance between B and the end of this third rod (correct to the nearest cm)?

3. The diagram represents a plan view (looking down onto) of an open-cut mine, which is roughly in the shape of a parallelogram.

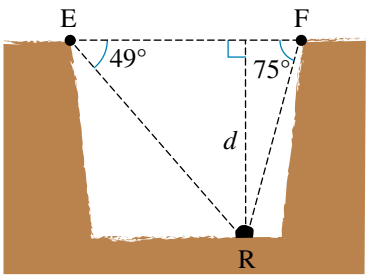


A mine surveyor has been asked to determine the dimensions of this pit.

a. From A, she measures the distances, AB and AC, to either side of the pit and also the angle in between. She finds that: AB = 86 m, AC = 97 m and ∠BAC = 46°. Find the pit's width, BC (correct to the nearest metre).

b. From D, she completes a similar exercise to find the length of the pit, EF. This time she finds: DE = 102 m, DF = 111 m and ∠EDF = 53°.

The following diagram is a cross-sectional view of the pit along its length.



The surveyor needs to find the depth, *d*. To do this she has located a large boulder, R, at the bottom of the pit and found that the angle between the horizontal, EF, and ER is 49°, and the angle between FE and FR is 75°. Find (correct to the nearest metre):

c. the distance ER

d. the depth, *d*.

e. The surveyor's last task is to find the area of the opening of the pit. For this calculation she simply measures the sides of the pit opening. She finds that:

$$\text{EB} = \text{CF} = 71\text{ m and BF} = \text{EC} = 45\text{ m}.$$

Find the area, to the nearest square metre.

Note that in a parallelogram the diagonals bisect each other.

4. The laws of geometry are evident in the life of bees.

a. In a cross-sectional view of a beehive with its regular hexagonal cells:

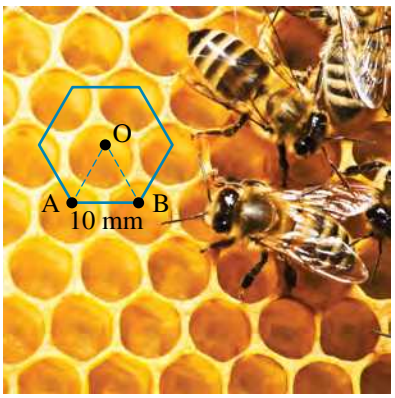
i. find the height of ΔAOB (correct to 3 decimal places)

ii. find the length of line AO

iii. find the area of ΔAOB and thus the area (correct to 1 decimal place) of one hexagonal cell.

In a beehive slab there are 500 cells.

Each cell is 12 mm deep.



b. i. Find the volume of honey (correct to 1 decimal place) that would be stored in a single cell.

ii. Find the volume of honey collected from one slab of beehive. Express your answer to the nearest cubic centimetre.

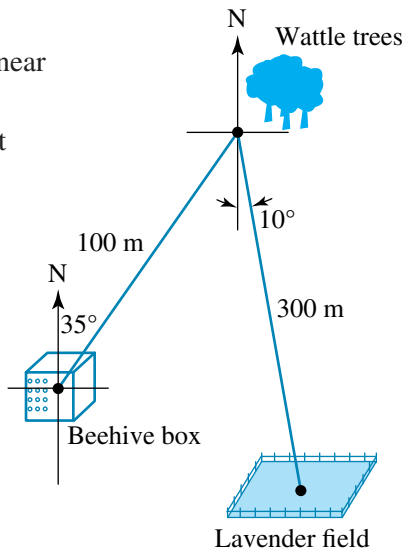
The beekeeper decides to trial a synthetic beehive where the linear dimensions of the hexagonal cell are doubled.

c. i. What is the scale factor for the new artificial hive compartment compared to the original?

ii. What is the area (correct to 1 decimal place) of the new artificial hexagon?

iii. What is the volume of honey that would be stored in a single artificial hexagonal cell? (Answer in cubic centimetres correct to 1 decimal place.)

The beekeeper wishes the bees to collect the nectar from two prime locations — a field of lavender and a group of wattle trees. A map of their respective locations is shown.

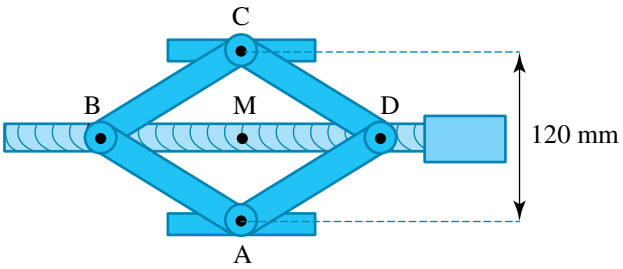


d. **i.** Find the distance (correct to the nearest metre) the bees have to travel to the lavender field from the hive box.

ii. If the direction from the hive to the wattle trees is 035°, what is the direction (correct to the nearest degree) of the lavender field from the hive box?

iii. How far north (correct to the nearest metre) must the beekeeper move the hive box so that it is directly west from the wattle trees?

5. Every car should carry a jack. One type of jack used to raise a car is a scissor-jack. A simple diagram of a scissor-jack is given.



AB = BC = CD = AD = 200 mm

The threaded rod is rotated to increase or decrease the length of the line segment BD.

a. **i.** In the triangle BCD, M is the midpoint of BD. What is the length of CM, given that AC = 120 mm?

ii. If ∠BCD = 160°, what is the length of BD (correct to the nearest millimetre)?

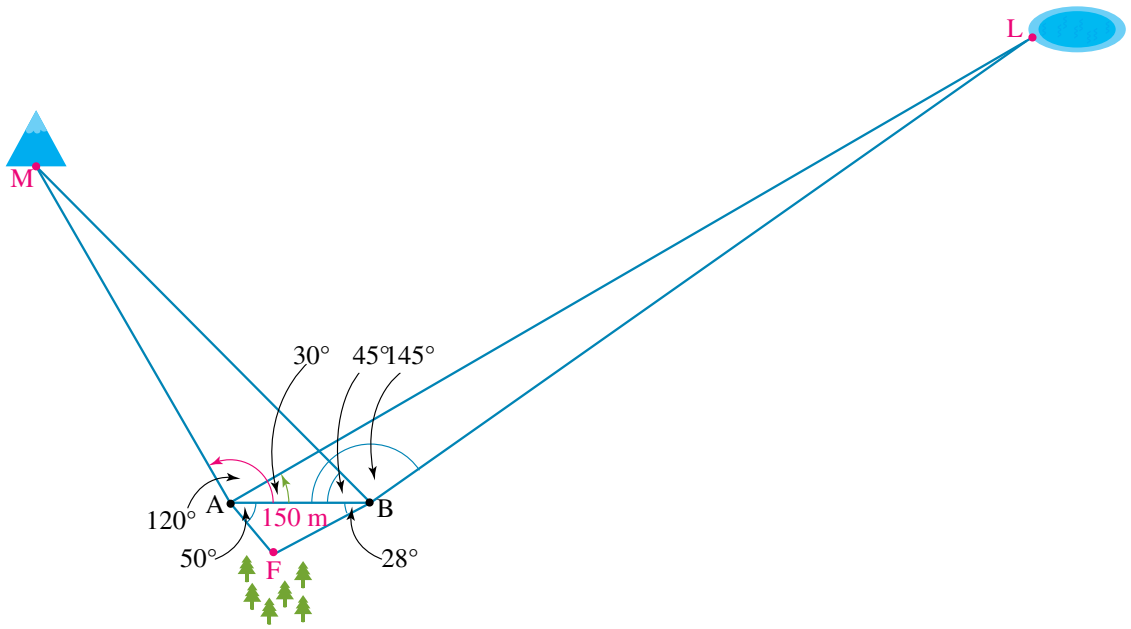
iii. What is the size of ∠MBC, given that ∠BCD = 160°?

The jack is raised by reducing the length of the line segment BD.

b. **i.** If the height of the jack, AC, is raised to 250 mm, what is the length of BD (correct to the nearest millimetre)?

ii. If ∠MBC is 70°, what is the length of BD and what is the height of the jack?

6. A tourist is trying to estimate distances to the three nearby locations surrounding his camping spot: the lake, the forest and the foot of the mountain. Using two points A and B that are 150 m apart, he measures several key angles as shown on the diagram.



a. Find (correct to the nearest meter) the distance to the forest from:
   i. Point A **ii.** Point B.

b. Find (correct to the nearest meter) the distance to the lake from:
   i. Point A **ii.** Point B.

c. Find (correct to the nearest meter) the distance to the foot of the mountain from:
   i. Point A **ii.** Point B.

d. Use your answers to parts **b** and **c** to find the distance (correct to the nearest metre) between the foot of the mountain and the lake.

e. If the line AB runs East–West, find:
   i. the bearing of the mountain from point A
   ii. the bearing of the lake from point B.

studyon

Units 3 & 4 Sit exam

Answers

Topic 12 Geometry and trigonometry

Exercise 12.2 Two- and three-dimensional Pythagoras' theorem

1. 13.9 cm
2. 194 m
3. 5.8 m
4. **a.** 13 **b.** 24.17 mm **c.** 8.77
   d. 5 m **e.** 15.23 m **f.** 246.98 cm
5. **a.** 20.3 **b.** 12.7 **c.** 3.4 mm **d.** 5.8
6. C
7. 4.0 m
8. **a.** 4.47 cm **b.** 4.90 cm
9. **a.** $\sqrt{281}$ cm **b.** $\frac{\sqrt{281}}{2}$ cm **c.** 16 cm
10. **a.** 707.11 m **b.** 463.68 m
11. 346 cm
12. 128 cm
13. **a.** **i.** $\sqrt{2225}$ cm or 47.2 cm
    ii. $\sqrt{16\,625}$ cm or 128.9 cm
    b. **i.** 500 mm **ii.** 1300 mm
    c. **i.** 44.72 m **ii.** 45.83 m
14. **a.** **i.** 25 m **ii.** 38 m
    b. **i.** 1.00 km **ii.** 332 m
15. 3.3 m
16. B
17. $a = 3.9$, $b = 4.7$, $c = 5.6$
18. AB = 4.95 m, DH = 14.16 m
19. 58 m
20. **a.** 6.8 m **b.** 3.0 m **c.** 14.4 cm
21. **a.** **i.** 510 mm **ii.** 522 mm
    b. **i.** 707 mm **ii.** 716 mm
22. **a.** $x = 5.17$ m **b.** $x = 245.20$ m

Exercise 12.3 Trigonometric ratios

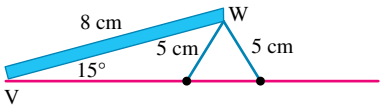
1. 1.9 m
2. 1509.4 mm
3. **a.** 8.2 km **b.** 147.1 mm
   c. 19.2 cm **d.** 44.6 mm
4. 47 m
5. 425 cm
6. **a.** 5.0 **b.** 2 m **c.** 9.3
7. 2.3 m
8. 91 m
9. 45°
10. 53°
11. **a.** 24.3 cm **b.** 74.2 m **c.** 4.6 cm
12. **a.** 53° **b.** 53°
    c. 45° **d.** 68°
13. **a.** 53° **b.** 42° **c.** 76°
14. 37°, 53°
15. D
16. $\theta = 30°$, $x = 8$ m, $y = 6.5$ m
17. 59°
18. 56°
19. **a.** 1.23 m **b.** 11.8°
20. 44°

Exercise 12.4 The sine rule

1. 28.8 mm
2. A
3. **a.** 13.2 cm **b.** 13.2 m
   c. 27.6 cm **d.** 109.4 km
4. 30.2 cm
5. 26.1 km
6. 1.6 km
7. **a.** 11.6 m **b.** 28.6 mm **c.** 30.6 cm
8. 46°
9. 46°, 106°, 30.7 cm
10. **a.** 52° **b.** 24° **c.** 33°
    d. 27° **e.** 48°
11. B
12. C
13. **a.** 145 cm **b.** 34 cm
14. 4.9 cm
15. D
16. A
17. 34.6 mm
18. **a.** 14.8 m **b.** 10.8 m
19. 2.8 m
20. **a.** 129.1° **b.** 68 km

Exercise 12.5 Ambiguous case of the sine rule

1. 31° or 149°
2. 47° or 133°
3. 117°
4. 132°
5. 52.1° or 127.9°
6. 65.2° or 114.8°
7. 21.3° or 158.7°
8. 44.6° or 135.4°
9. 141°
10. 105°
11. 127°
12. 130°
13. D
14. $y = 30.3°$ and $x = 149.7°$

15. a. 

b. 9.5° or 140.5°
c. 3.2 cm or 12.3 cm

16. a. 112° b. 0.89 km or 890 m

Exercise 12.6 The cosine rule

1. 8.72
2. 4772.67 mm
3. a. 8.66 m b. 1.09 km
   c. 9.99 units d. 155.85 km
4. 13.8 km
5. 10.5 cm
6. 39°
7. 12.03 cm, 116.2°, 35.8°
8. a. 106° b. 46° c. 82°
9. 44°, 29°
10. 43°, 80°, 57°
11. a. 5.0 km b. 72.5° c. 7.1 m
    d. 37.2° e. 34.0° f. 329.2
12. D
13. A
14. E
15. D
16. a. 51.3° b. 5.7 m

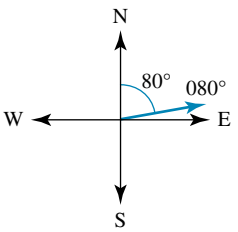
Exercise 12.7 Area of triangles

1. 162 m^2
2. 160 cm^2
3. 11.71 m^2
4. 20.27 cm^2
5. 28.6 cm^2
6. 43.3 cm^2
7. a. 15.8 mm^2 b. 42.0 cm^2
   c. 16.8 mm^2 d. 6.0 m^2
8. a. 12.3 cm^2 b. 5.9 m^2
   c. 3464.1 m^2 d. 36.9 m^2
9. a. 22.2 m^2 b. 5.3 km^2
   c. 173.2 mm^2 d. 7.8 cm^2
10. a. 10.0 m^2 b. 4031.6 cm^2
    c. 11.5 km^2 d. 2165 m^2
11. a. 20.0 km^2 b. 10.0 m^2
    c. 94.0 m^2 d. 2.4 $units^2$
    e. 15 600 cm^2
12. 72 m^2
13. a. i. 12.5 mm^2 ii. 50 mm^2
14. 26.8 km^2
15. A
16. C
17. B
18. a. 24.5 km^2 b. 7.22 mm^2

Exercise 12.8 Angles of elevation and depression

1. 26.57°
2. 45°
3. a. 36.87° b. 40.34° c. 21.80°
4. 40.33 m
5. 4.01 m
6. a. 10 m b. 85 m
7. $b = 33.27°$, $a = 1371.68$ m
8. 59.21 m
9. 26 metres
10. a. $a = 41.83°$, $d = 61.45$ m
    b. $a = 46.45°$, $b = 15.21$ m
11. a. Elevation 30°
    b. Depression 80.54°
    c. Elevation 45°
    d. Depression 14.04°
12. 41.19°
13. D
14. B
15. a. 68 m b. 29 m
16. a. 38.66° b. 65.56° c. 38.66°
17. a. 4240 m b. 1100 m
18. a. 14.9 m
    b. 9.2 m
    c. 14.9 m
    d. A right-angled isosceles triangle

Exercise 12.9 Bearings and specification of location

1. 220°
2. 247.5°
3. 

4. a. 025° b. 190° c. 310° d. 258° e. 078° f. 102°
5. C
6. 025°T
7. 98.5 km south, 17.4 km east
8. 208.4 km north, 1181.8 m east
9. a. 7.1 km north, 7.1 km east
   b. 171.0 m south, 469.8 m west
10. 30 m
11. 5 m
12. a. 295 m b. 111 km
13. 5.79 km

14. 286°

15. a. E b. C

16. a. 271 m north, 71 m east
    b. 77.2 m north, 124.9 m east
    c. 19.1 km south, 3.5 m west
    d. 454.6 m south, 1246.9 m east

17. a. 1427 m b. 1358 m
    c. 1970 m d. ~044°

18. a. 82 km b. 039.87°
    c. 68 km

19. 15 km, 225°

20. 288 m, 169°

21. a. 17.86 km b. 109.15°
    c. 34.14 km d. 119.02°

Exercise 12.10 Triangulation — cosine and sine rules

1. 891.5 km

2. 316.0 m

3. a. 5389.2 m b. 4.9 km

4. CB = 14.12 km, AB = 18.45 km

5. 2618 km

6. a. Tower B, 11.6 km b. Tower A, 6.5 km

7. a. 14.8 km b. 22.7 km

8. Tower A (200 km)

9. 40.87 m

10. 53.9°

11. a. 253.6 m b. 42.6 km
    c. 38.0 km

12. a. 12.0 km, 136.1°
    b. 801.8 km, 222.6°

13. a. i. 33 m ii. 64 m
    b. i. 271 m ii. 249 m
    c. 247 m
    d. 39 m

14. 8.1 km, 148.53°

15. 8593 million km

16. B

17. C

18. a. BC b. 54° c. 357.26 m

12.11 Review: exam practice

Multiple choice

1. C	2. B	3. D	4. A
5. E	6. E	7. E	8. A
9. B	10. B	11. E	12. C
13. E	14. B	15. C	16. C
17. A	18. A		

Short answer

1. 0.7 m

2. $190\sqrt{2}$ mm

3. Yes, because 11, 60, 61 is a Pythagorean triad, so the triangle is right-angled.

4. $\sqrt{192}$ cm

5. a. 1888 mm b. 2937 mm

6. a. 129 mm b. 55 mm

7. a. 32°, 78° b. 42.6 m, 23.1 m

8. 39 mm

9. a. C b. 38°, 51°, 91°

10. $p = 77.60°$, $q = 51.20°$

11. a. 20.1 m b. 83.41°

12. a. 127.71° b. 279.57°

13. a. i. 20 km ii. 45 km
    b. 312.84°

14. a. 4.6° b. 2 : 25

15. 11.2 km

16. 267.84°

17. Tower A = 21.01 km
    Tower B = 12.77 km

Extended response

1. a. i. 28.3 cm ii. 25.0 cm^2 iii. 5 cm
   b. i. $25 + 25 + 80 + 40\sqrt{2} + 40\sqrt{2} = 243\ cm^2$
   ii. Yes: 210 cm^2 or a reduction of 33 cm^2
   iii. 200 cm^2
   c. i. Two differently shaped triangles
   ii. 106.26°
   iii. 36 cm^2

2. a. i. 1.12 m ii. 42° iii. 48°
   b. i. Obtuse isosceles triangle
   ii. 97°
   iii. 0.827 m
   c. 2.1 m
   d. i. 0.56 m^2 ii. 0.39 m^2 iii. 0.70 m^2
   e. 283 cm

3. a. BC = 72 m b. EF = 95 m c. ER = 111 m
   d. $d = 84$ m e. 3042 m^2

4. a. i. 8.660 mm ii. 10 mm
   iii. 43.3 mm^2, 259.8 mm^2
   b. i. 3117.7 mm^3 ii. 1559 cm^3
   c. i. 1 : 2 ii. 1039.2 mm^2 iii. 24.9 cm^3
   d. i. 240 m ii. 153° iii. 82 m

5. a. i. 60 mm ii. 394 mm iii. 10°
   b. i. 312 mm ii. 137 mm, 376 mm

6. a. i. 72 m ii. 117 m
   b. i. 987 m ii. 861 m
   c. i. 410 m ii. 502 m
   d. 1069 m
   e. i. 330° ii. 55°

TOPIC 13
Spherical geometry

13.1 Overview

13.1.1 Introduction

It has been known for over 2000 years that the earth is round. The great Greek philosopher, Aristotle, observing the movements of the stars, noted that small changes in position resulted in a different view of the stars. If the earth was flat, everyone would see the same stars at any given time, but those in the northern hemisphere see different stars to those in the southern hemisphere. Sailors from the ancient world also noted the curved nature of the earth's surface. As they sailed towards a land mass with a mountain, more of the mountain was revealed as they approached the land.

The first full view of the earth was the famous 'Blue marble' shot taken by the crew of Apollo 17 on December 7, 1972 as they left the Earth's orbit for the Moon.

Although it looks like a perfect sphere from space, the earth is not a perfect sphere; it is squashed at the poles and bulges slightly at the equator. This shape is called an oblate spheroid. The flattening at the poles and bulge at the centre is thought by scientists to be caused by the force of the rotation of the earth. GPS and other satellites have allowed scientists to measure the size and shape of the earth to within a centimetre.

When calculating distances between two points on the surface of the earth the assumption that the earth is spherical in shape and at the same elevation must be made. This will not be an exact distance, but a close approximation.

LEARNING SEQUENCE

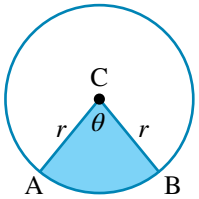
13.1 Overview
13.2 Circle mensuration
13.3 Right-angled triangles within a sphere
13.4 The earth as a sphere (Great circles and small circles)
13.5 Time zones
13.6 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

13.1.2 Kick off with CAS

Calculating circular areas with CAS

The area of a sector of a circle can be calculated using the formula, Area of sector $= \frac{\theta}{360^\circ} \times \pi \times r^2$, where θ is the angle subtended at the centre of the circle by the arc AB as shown in the diagram.



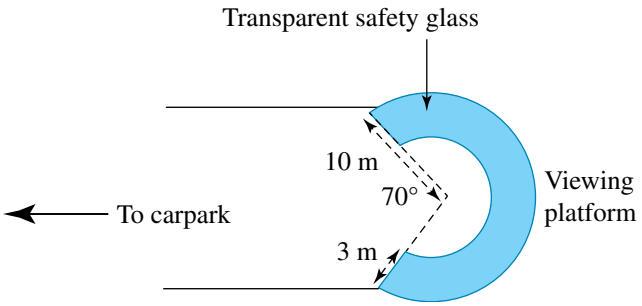
After considering how the formula for the sector of a circle was determined, can you determine the formula for the length of the arc of a circle?

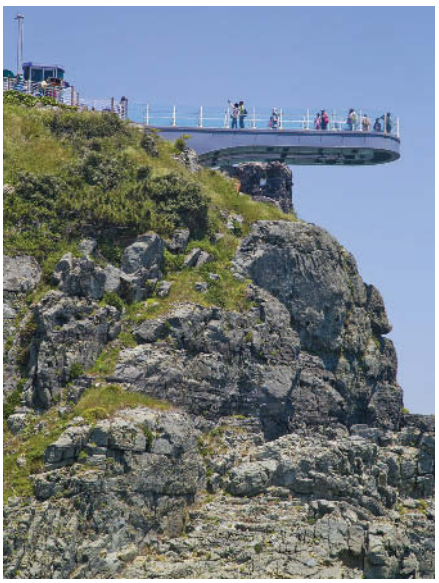
We can use CAS to define formulas which allow us to quickly and efficiently calculate the areas of different shapes.

1. Use CAS to define and save the formula for calculating the area of a sector
   $A_{sector} = \frac{\theta}{360^\circ} \pi r^2$.
2. Use CAS to define and save the formula for calculating the length of an arc of a circle.
   Arc length $= \frac{\theta}{360^\circ} \times 2\pi r$.
3. A company intends to build a transparent viewing platform on a scenic mountain top, similar to the one depicted in the photograph below.

   After some initial research they have decided that a viewing platform that has curved sides rather than straight sides will look more interesting and provide room for a greater number of people to stand directly in front of the railing.

   In the diagram below, the shaded area represents the area of transparent glass needed to complete the viewing platform.





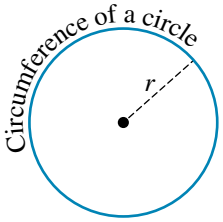
   Calculate the surface area of safety glass needed to complete the structure.
4. Calculate the length of the curved outside railing.
5. Circular structures are very costly. If the company had built a rectangular platform of transparent safety glass 3 metres wide on 3 sides of a central opaque square 4 metres by 4 metres, would there have been less viewing room against the railing for the visitors?

on Resources

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

13.2 Circle mensuration

In section 11.2 the formula for calculating the circumference of a circle was given.



Circumference (C) is the perimeter of a circle.

$$C = 2 \times \pi \times \text{radius}$$
$$= 2\pi r$$

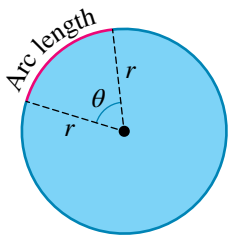
studyon

Units 3 & 4 > AOS M3 > Topic 2 > Concept 1

Circle mensuration Summary screen and practice questions

13.2.1 Arc length

- An **arc** is part of the circumference of a circle.
- The **arc length** of a circle is calculated by finding the circumference of a circle and multiplying by the fraction of 360° that the angle forms at the centre of the circle.



$$\text{Arc length} = \frac{\theta}{360^\circ} \times 2\pi r$$

where r = radius of the circle and θ = the angle the ends of the arc makes with the centre of the circle.

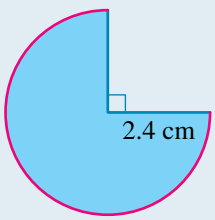
on Resources

Interactivity: Arc length (int-6270)

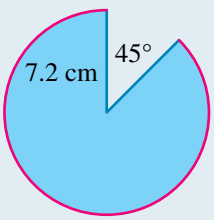
WORKED EXAMPLE 1

Calculate the arc lengths for each of the following shapes, correct to 2 decimal places.

a.



b.



THINK

a. 1. Determine the angle the ends of the arcs make with the centre.

WRITE

a. $\theta = 360° - 90°$
$= 270°$

2. Substitute the values into the arc length equation.

$$\text{Arc length} = \frac{\theta}{360^\circ} \times 2\pi r$$
$$= \frac{270}{360} \times 2 \times \pi \times 2.4$$
$$= 11.3097$$
$$\approx 11.31 \text{ cm (2 decimal places)}$$

3. Write the answer.

$$\text{Arc length} = 11.31 \text{ cm}$$

b. 1. Determine the angle the ends of the arcs make with the centre.

b. $\theta = 360^\circ - 45^\circ$
$= 315^\circ$

2. Substitute the values into the arc length equation.

$$\text{Arc length} = \frac{\theta}{360^\circ} \times 2\pi r$$
$$= \frac{315^\circ}{360^\circ} \times 2 \times \pi \times 7.2$$
$$= 39.5841$$
$$\approx 39.58 \text{ cm (2 decimal places)}$$

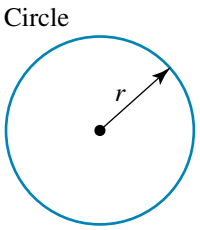
3. Write the answer.

$$\text{Arc length} = 39.58 \text{ cm}$$

13.2.2 Area of circles, sectors and segments

In section 11.2, the formula for calculating the area of a circle was given.

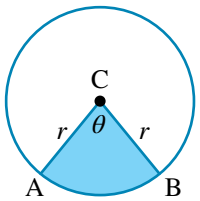
Area of a circle



$$A = \pi \times \text{radius}^2$$
$$= \pi r^2$$

Sectors

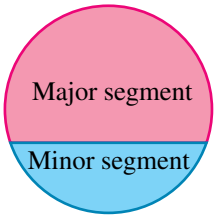
A **sector** is a part of the area of a circle, as shown by the shaded area in the diagram. The angle θ, at the centre of the circle, subtends the arc AB. The area of the sector CAB is calculated using:



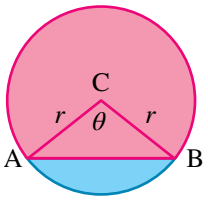
$$A_{\text{sector}} = \frac{\theta}{360^\circ} \pi r^2$$

Segments

A **segment** is formed by joining two points on the circumference with a straight line, known as a chord. This forms two areas, with the smaller area being the minor segment and the larger area the major segment. This is shown by the blue (minor segment) and pink (major segment) sections in the diagram.



To work out the area of the minor segment, first draw a triangle to the centre of the circle. This creates a sector. The following equation can then be used to calculate the area of the minor segment:

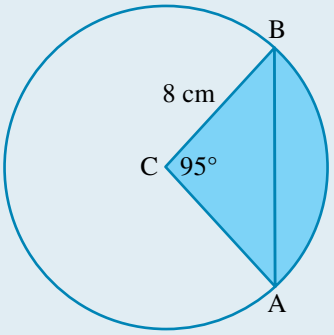


$$\begin{aligned} A_{\text{Minor segment}} &= A_{\text{sector}} - A_{\text{triangle}} \\ &= \frac{\theta}{360^\circ}\pi r^2 - \frac{1}{2}r^2 \sin\theta \\ &= r^2\left(\frac{\theta}{360^\circ}\pi - \frac{1}{2}\sin\theta\right) \end{aligned}$$

WORKED EXAMPLE 2

From the circle shown, which has a radius of 8 cm, calculate:
a. the area of the blue sector
b. the area of the minor segment of the circle.

Give your answers correct to 2 decimal places.

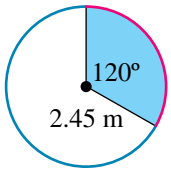


THINK	WRITE
a. 1. Use the formula for the area of a sector in degrees.	**a.** $A_{\text{Sector}} = \frac{\theta}{360^\circ} \times \pi r^2$
	$r = 8$
	$\theta = 95^\circ$
2. Substitute the quantities into the formula and simplify.	$A_{\text{Sector}} = \frac{\theta}{360^\circ} \times \pi r^2$
	$= \frac{95^\circ}{360^\circ} \times \pi(8)^2$
	$= 53.0580$
3. Write the answer correct to 2 decimal places.	$A_{\text{Sector}} = 53.06\text{ cm}^2$
b. 1. Use the formula for the area of a segment in degrees.	**b.** $A_{\text{Minor segment}} = r^2\left(\frac{\theta}{360^\circ}\pi - \frac{1}{2}\sin(\theta)\right)$
	$r = 8$
	$\theta = 95^\circ$
2. Substitute the quantities into the formula and simplify.	$A_{\text{Minor segment}} = 8^2\left(\frac{95^\circ}{360^\circ}\pi - \frac{1}{2}\sin 95\right)$
	$= 21.1798...$
3. Write the answer correct to 2 decimal places.	$A_{\text{Minor segment}} = 21.18\text{ cm}^2$

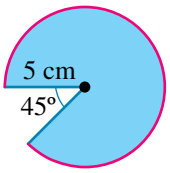
Exercise 13.2 Circle mensuration

1. **WE1** Calculate the arc length of each of the following shapes, correct to 2 decimal places.

a.

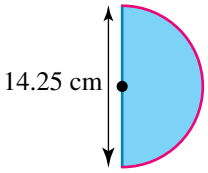


b.

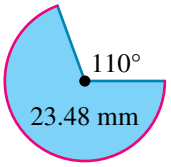


2. Calculate the arc length of each of the following shapes, correct to 2 decimal places.

a.

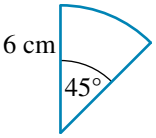


b.

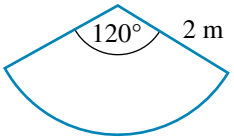


3. Calculate the areas of the following shapes.

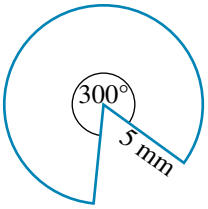
a.



b.



c.

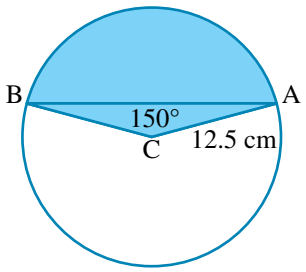


4. **WE2** From the circle shown, which has a radius of 12.5 cm, calculate:

a. the area of the minor sector

b. the area of the minor segment of the circle.

Give your answers correct to 2 decimal places.

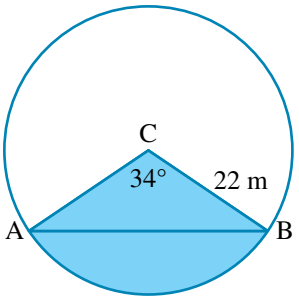


5. From the circle shown which has a radius of 22 m, calculate:

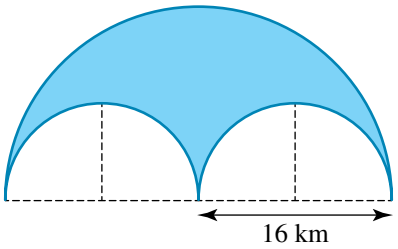
a. the area of the minor sector

b. the area of the minor segment of the circle.

Give your answers correct to 2 decimal places.



6. Calculate the area of the shaded shape shown, correct to 2 decimal places.

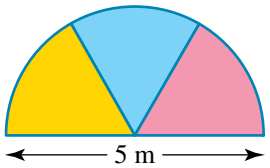


7. MC A minor sector is formed by an angle of 123° in a circle of diameter 9 cm. The area of the minor sector is closest to:

A. 86.94 cm^2 **B.** 63.62 cm^2 **C.** 41.88 cm^2 **D.** 15.17 cm^2 **E.** 21.74 cm^2

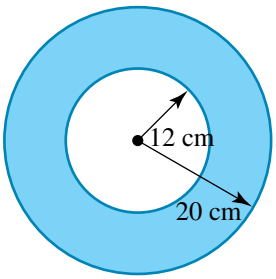
8. Calculate the difference in area between a sector of a circle formed by an angle of 50° and a sector of a circle formed by an angle of 70° in a circle of radius 3 cm. Give your answer correct to 2 decimal places.

9. A church window is made in the shape of a semicircle with three equal stained glass sections. Calculate the area of one of the stained glass sections. Give your answer correct to 2 decimal places.

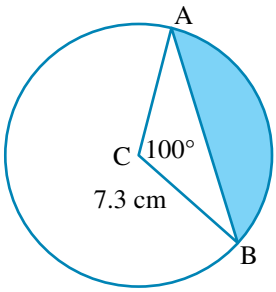


10. Calculate the areas of the shaded region in the following shapes, correct to 2 decimal places:

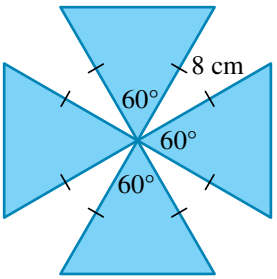
a.



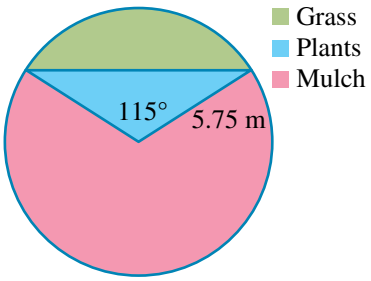
b.



c.



11. A circular garden bed is to be landscaped with grass, plants and mulch. Given the circular garden is designed with the shown dimensions:



a. calculate the area of the grassed area, correct to 2 decimal places

b. calculate the area of the mulched area, correct to 2 decimal places

c. calculate the area of the planted area, correct to 2 decimal places.

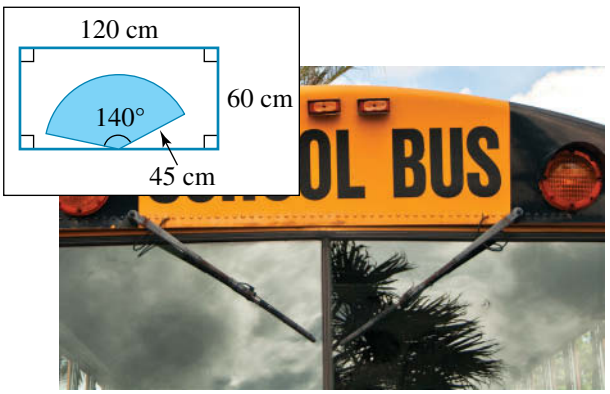
d. If the cost of grass, plants and mulch are $40, $55 and $15 per metre respectively, calculate the cost of landscaping the circular garden correct to the nearest cent.

12. Early-model vehicles had a single windscreen-wiper blade to remove water from the wind-screen. (The bus at right has two single blades of this type.)

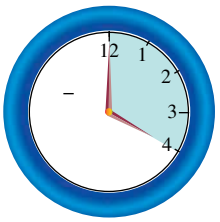
Using the dimensions given in the diagram:

a. What area (to the nearest whole number) did the blade cover?

b. What percentage (to 1 decimal place) of the windscreen was cleared?



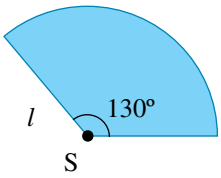
13. A minute hand on a clock face moving from the number 12 to the number 4 position sweeps out a sector of area 104.72 cm^2. What is the length of the minute hand to the nearest cm?



14. MC A circle of diameter 10 cm is divided into 9 equal sectors. The area of one sector is closest to

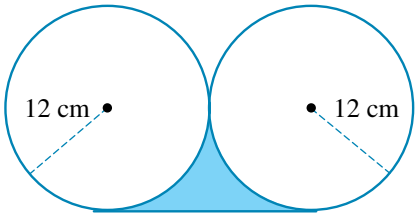
A. 8 cm^2 **B.** 9 cm^2 **C.** 21 cm^2 **D.** 34 cm^2 **E.** 35 cm^2

15. A search light on the front of a surf life saving club covers an area in the shape of a sector as shown in the diagram at right.



The search light is positioned at S and can turn through an angle of 130°. The shaded area in the diagram is the area covered by the searchlight. If the shaded area is 280 m^2, what is the maximum distance, l, that the searchlight can reach? (Give your answer correct to 2 decimal places.)

16. The diagram below shows two identical circles of radius 12 cm just touching each other. Calculate the area of the shaded region. Give your answer correct to two decimal places.



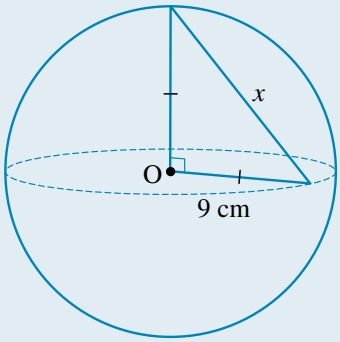
13.3 Right-angled triangles within a sphere

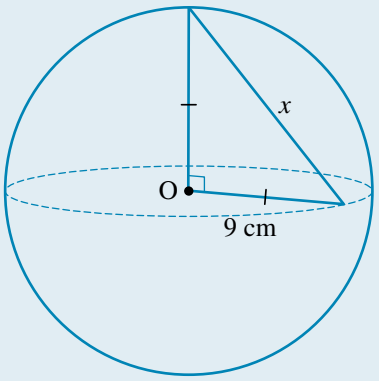
Trigonometry and Pythagoras' theorem can be used to solve problems involving the solution of right-angled triangles within a sphere.

$$a^2 + b^2 = c^2$$

WORKED EXAMPLE 3

Calculate the length of x in the following sphere. Give your answer correct to two decimal places.



THINK	WRITE
1. A right angle has been formed at the centre of the circle. Pythagoras' theorem can be used.	**a.** 
2. Write Pythagoras' theorem and substitute the values on the diagram for a, b and c.	$c^2 = a^2 + b^2$ $a = 9$ $b = 9$ $c = x$
3. Calculate the value of x.	$x^2 = 9^2 + 9^2$ $x^2 = 81 + 81$ $= 162$ $x = \sqrt{162}$ $x = 12.73\,\text{cm}$

WORKED EXAMPLE 4

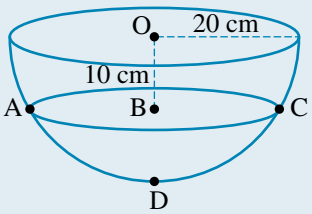
A hemispherical block of wood of radius 20 cm is sliced so that the distance OB is 10 cm

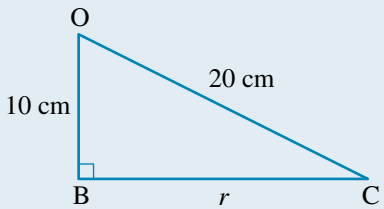
a. Find the radius AB of the circular cut, correct to 2 decimal places.

b. Find the circumference of the new circle that passes through A and C.

c. Find the area of this circular cut.

d. Find the distance from A to C passing through D. Give your answer correct to 2 decimal places.



THINK	WRITE
a. A right-angled triangle OBC is formed which has a hypotenuse of 20 cm (since the radius is 20 cm) and one side of 10 cm, so Pythagoras' theorem may be used.	 $r^2 + 10^2 = 20^2$ $r^2 = 400 - 100$ $r = \sqrt{300}$ $\approx 17.32\,\text{cm}$

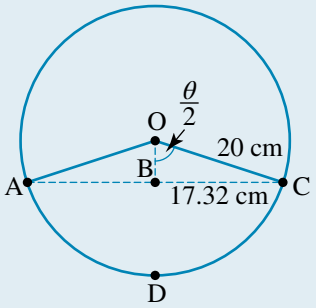
b. Use the rule for circumference of a circle, $C = 2\pi r$.

$$C = 2 \times \pi \times 17.32$$
$$= 108.83 \text{ cm}$$

c. 1. Use the rule for area of a circle, $A = \pi r^2$.

$$A = \pi \times 17.32^2$$
$$= 942.48 \text{ cm}^2$$

2. Using the great circle through ADC, the angle subtending the minor arc AC is required.



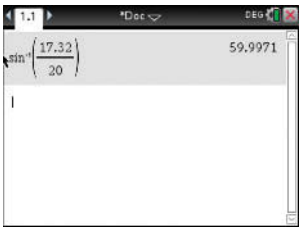
$$\sin\left(\frac{\theta}{2}\right) = \frac{17.32}{20}$$
$$\frac{\theta}{2} = 60°$$
$$\theta = 120°$$

$$\text{length of minor arc AC} = \frac{\theta}{180°} \times \pi \times r$$
$$l = \frac{120°}{180°} \times \pi \times 20$$
$$\approx 41.89 \text{ cm}$$

TI | THINK

c. 1. On a Calculator page, press trig and then complete the entry line as:
$\sin^{-1}\left(\frac{17.32}{20}\right)$
Press ENTER.

WRITE



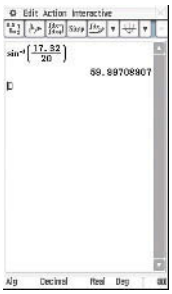
2. The answer appears on the screen.

$$\frac{\theta}{2} = 59.9971$$
$$\frac{\theta}{2} = 60°$$
$$\theta = 120°$$

CASIO | THINK

c. 1. On the MAIN screen, tap Trig and then complete the entry line as:
$\sin^{-1}\left(\frac{17.32}{20}\right)$
Press EXE.

WRITE

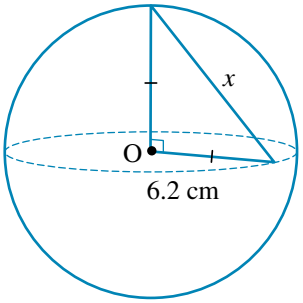


2. The answer appears on the screen.

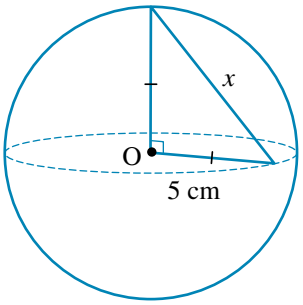
$$\frac{\theta}{2} = 59.9971$$
$$\frac{\theta}{2} = 60°$$
$$\theta = 120°$$

Exercise 13.3 Right-angled triangles within a sphere

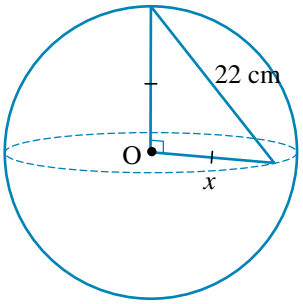
1. **WE3** Calculate the length of x in the following sphere, correct to 2 decimal places.



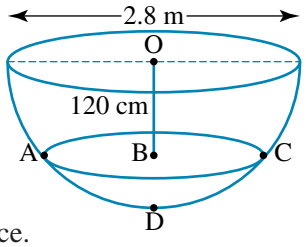
2. Calculate the length of the pronumeral in the following sphere, correct to 2 decimal places.



3. Calculate the length of x in the following sphere, correct to 2 decimal places.



4. **WE4** A hemispherical block of wood of diameter 2.8 m is sliced so that the distance OB is 120 cm.



- **a.** Find the radius AB of the circular cut correct to 1 decimal place.
- **b.** Find the circumference of the new circle that passes through A and C, correct to 2 decimal places.
- **c.** Find the area of this circular cut, correct to 2 decimal places.
- **d.** Find the distance from A to C passing through D, correct to 1 decimal place.

5. Calculate the radius of the small circle formed when a circular cut is made 1.2 m from the centre of a sphere of diameter 4.6 m. Give your answer correct to two decimal places.

6. Calculate the circumference of the small circle formed 12 cm from the centre of a sphere of radius 18 cm. Give your answer correct to 3 significant figures.

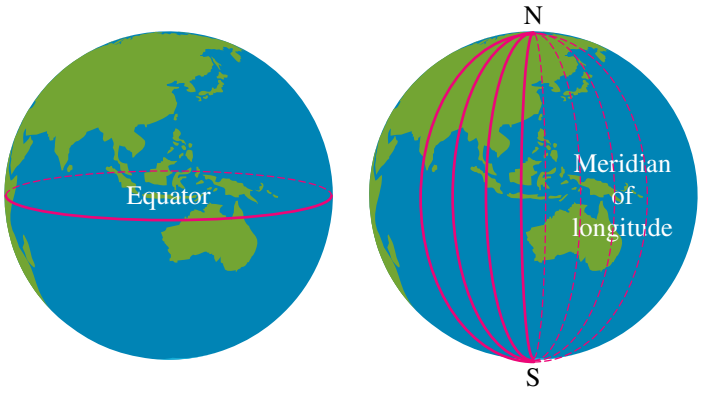
7. A ping-pong ball of radius 2 cm has 2 holes made in it and a string is threaded through the holes. If the length of string needed to go from one hole to the other is 3 cm, find the distance from the centre of the ping-pong ball to the middle of the string. Give your answer correct to 2 decimal places.
8. A hemispherical fish bowl of radius 15 cm contains water to a depth of 12 cm. Find the surface area of the water, correct to 1 decimal place.

13.4 The earth as a sphere

A **great circle** is a circle that divides a sphere into two equal hemispheres, where the centre of the great circle is the centre of the sphere.

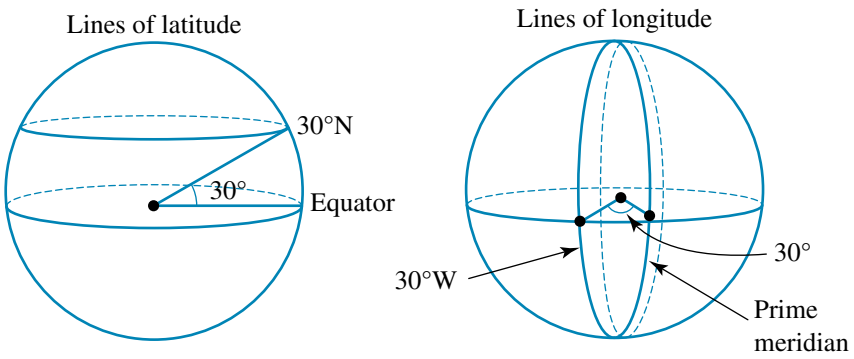
As with a circle, the circumference of a great circle can be calculated by using $C = 2\pi r$.

The great circles that we will focus on are those that are related to Earth. These are the lines of **longitude** (also known as **meridians**), which pass through the north and south poles. The equator is also a great circle.



13.4.1 Latitude and longitude

Parallels of latitude run from east to west and measure the angle north or south of the equator. The meridians, or lines of longitude, measure the angle east or west of the prime meridian, which passes through the Greenwich Observatory (0° longitude) in London, England.

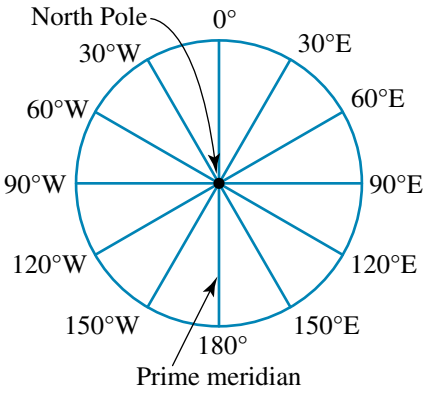


Measuring distances with great circles between any two points on the Earth with the same longitude, or between any two points on the Equator, is along a great circle.

To conduct calculations in relation to great circles on Earth, consider the centre of the circle as the centre of the Earth, with the radius being 6400 km.

The shortest distance between two points on the Earth that have the same longitude, or which are both on the Equator, is the great circle arc length between those points.

This is a birds eye view from directly above the North Pole



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Units 3 & 4 | AOS M3 | Topic 2 | Concept 2

Earth modelled by a sphere Summary screen and practice questions

on Resources

Interactivity: Great circles (int-6271)

WORKED EXAMPLE 5

Calculate the distance travelled in the following situations correct to the nearest kilometre:
a. an aeroplane that travels along the equator from a point of longitude 48°W to 22°E
b. a boat that travels from a point M: 36°S, 138°W to point N: 40°S, 138°W.

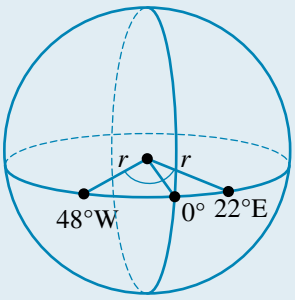
THINK

a. 1. Both points lie on the equator, which is a great circle. The angle between the two points is made of 2 parts, from the starting point to 0°, then from 0° to the finishing point.

2. The distance between the two points is the arc length. Substitute θ and r into the arc length formula and simplify.

WRITE

a.



$\theta = 48° + 22°$
$= 70°$
$r = 6400\,\text{km}$

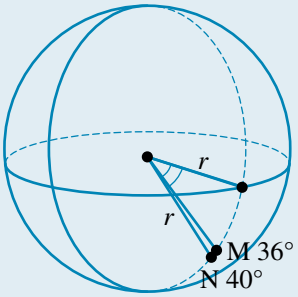
$$\text{arc length} = \frac{\theta}{360°} \times 2\pi r$$
$$= \frac{70°}{360°} \times 2\pi \times 6400$$
$$= 7819.075...$$

3. Answer the question correct to the nearest kilometre.

The aeroplane travelled 7819 km.

b. 1. Observing that M and N lie on the same longitude 138°W, path MN is along a great circle. Since both latitudes are S, the angle θ is formed by their difference.

b.



$\theta = 40° - 36°$
$= 4°$

2. The distance between the two points is the arc length. Substitute θ and r into the arc length formula and simplify, using the radius of the Earth as 6400 km.

$$\text{arc length} = \frac{\theta}{360°} \times 2\pi r$$
$$= \frac{4°}{360°} \times 2\pi \times 6400$$
$$= 446.804...$$

3. Answer the question correct to the nearest kilometre.

The boat travelled 447 km.

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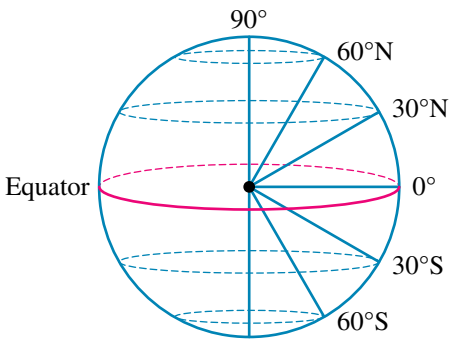
Distance between points on a sphere Summary screen and practice questions

13.4.2 The shortest distance

Measuring distances with small circles

With the exception of the equator (which is a great circle), lines of latitude form small circles which have a radius smaller than 6400 km.

To calculate the distance along a parallel of latitude, first the radius of the small circle must be found using trigonometric ratios, and then the formula for arc length may be used.



WORKED EXAMPLE 6

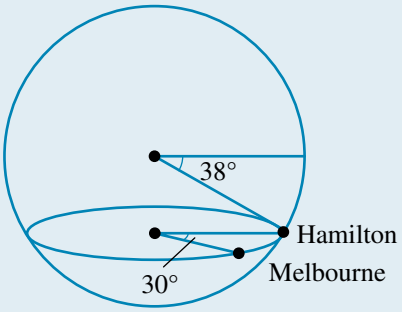
Calculate the distance along the parallel of latitude between Melbourne, Australia 38°S, 145°E and Hamilton, New Zealand 38°S, 175°E. Give your answer correct to the nearest kilometre.

THINK

a. Both points lie on the 38th parallel south. The angle between the two points is the difference between the longitudes.

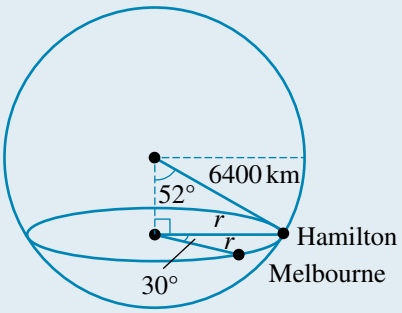
WRITE

a.



$175° - 145° = 30°$

b. The radius of the Earth is 6400 km. The radius of the small circle is calculated using the right-angled triangle and the complementary angle.



$$\sin\theta = \frac{\text{O}}{\text{H}}$$

$$\sin 52° = \frac{r}{6400}$$

$$r = 6400 \sin 52°$$

$$= 5043.2688... \text{ km}$$

c. Using the radius of the small circle and the angle of 30°, calculate the distance using arc length.

$$l = \frac{\theta}{180°} \times 2\pi r$$

$$= \frac{30°}{180°} \times 2\pi \times 5043.2688...$$

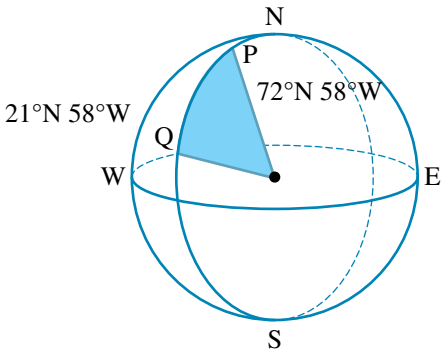
$$= 5281.29887$$

$$\approx 5281 \text{ km}$$

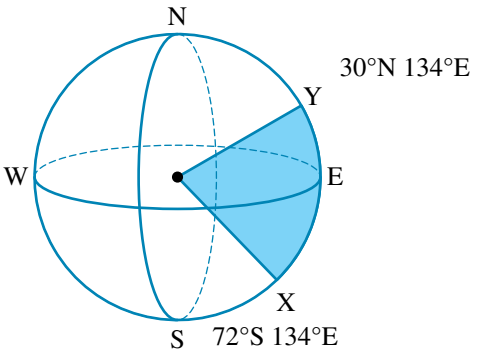
Exercise 13.4 The earth as a sphere

1. **WE5** Calculate the distance travelled in the following situations correct to the nearest kilometre:
   a. an aeroplane that travels along the equator from a point of longitude 23°W to 35°E
   b. a boat that travels from a point M: 12°S, 100°W to point N: 58°S, 100°W.
2. Calculate the distance travelled in the following situations correct to the nearest kilometre:
   a. a plane that travels along the equator from a point of longitude 67°W to 15°E
   b. a yacht that travels from a point P: 36°S, 98°E to point Q: 81°S, 98°E.
3. Calculate the distance travelled by an aeroplane travelling along the equator from a point of longitude 36°W to 123°E. Give your answer correct to the nearest km.
4. Calculate the distance travelled by a cruise ship that travels from point P: 18°N, 25°W to point Q: 30°S, 25°W. Give your answer correct to the nearest km.
5. Calculate the great circle distances shown in these figures. Give your answers correct to the nearest kilometre.

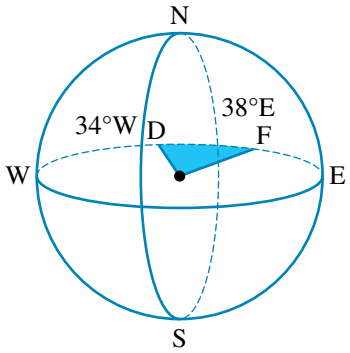
a.



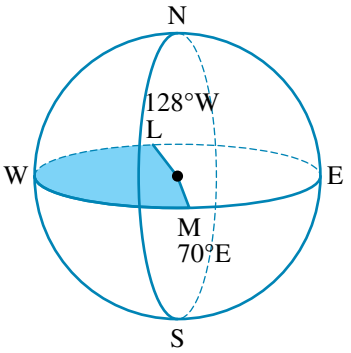
b.



c.



d.



6. Calculate the shortest distance, correct to the nearest kilometre, between two points with the same longitude that form the following angles at the centre of the Earth.
   a. 12° b. 3° c. 0.5° d. 0.08°
7. Calculate the distance between two points on the equator that form the following angles at the centre of the Earth. Give your answers correct to the nearest kilometre.
   a. 15.75° b. 10.31°
8. Use meridians to calculate the shortest distance, correct to the nearest kilometre, between the following cities and the North Pole.
   a. London (51.51°N, 0.13°W)
   b. Rome (41.90°N, 12.50°E)
   c. Dallas (32.78°N, 96.80°W)

9. Melbourne (37.83°S, 145°E) and Port Douglas (16.48°S, 145°E) are on the same meridian. What is the shortest distance (correct to the nearest kilometre) between the two towns?
10. The cities of Washington in the USA and Lima in Peru are 5625 km apart. They both lie on the same meridian. What is the angle formed at the centre of the Earth by the two cities? Give your answer correct to 2 decimal places.
11. The cities of Port Hedland in Australia and Nanjing in China are 3795 km apart. They both lie on the same line of longitude. What is the angle formed at the centre of the Earth by the two cities? Give your answer correct to 2 decimal places.
12. Use meridians to calculate the shortest distance, correct to the nearest kilometre, between the following cities and the equator.
    a. Bangalore (12.97°N, 77.57°E)
    b. São Paulo (23.55°S, 46.63°W)
    c. Tokyo (35.68°N, 139.68°E)
13. Find the distance, correct to the nearest kilometre, between Moscow 56°N, 38°E and Copenhagen 56°N, 13°E along the parallel of latitude.
14. Find the distance, correct to the nearest kilometre, between Prague 50°N, 14°E and Krakow 50°N, 20°E along the parallel of latitude.

Use the information in the table to answer questions **15** to **18**.

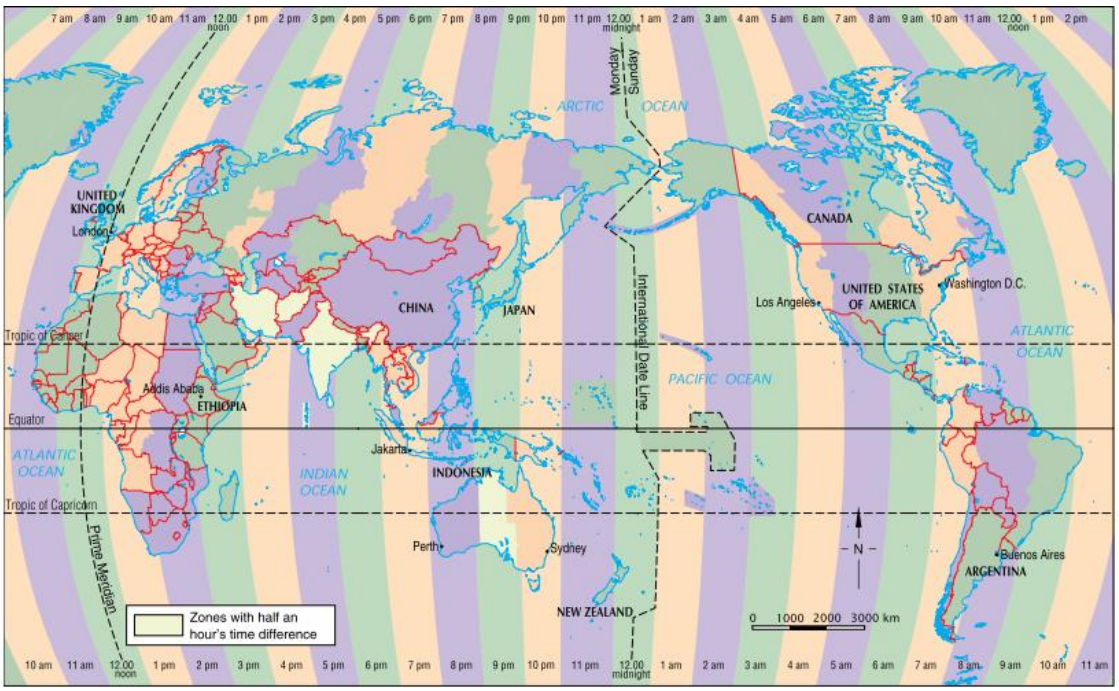
City	Latitude	Longitude
Sydney	34°S	151°E
New York	41°N	74°W
Cape Town	34°S	19°E
Naples	41°N	14°E
Budapest	47°N	19°E
Prague	50°N	14°E

15. a. Which cities are on the same meridian of longitude?
    b. If two points represented these cities, what would be the angle subtended between them at the centre of the Earth?
16. a. Which cities are on the same parallel of latitude?
    b. If two points represented these cities, what would be the angle subtended between them at the centre of the Earth?
17. a. Calculate the radius, correct to 2 decimal places, of the circle which forms the 34°S parallel of latitude.
    b. Calculate the distance, correct to the nearest kilometre, between Sydney and Cape Town along this parallel of latitude.
18. a. Calculate the radius of the circle, correct to 2 decimal places, which forms the 41°N parallel of latitude.
    b. Calculate the distance, correct to the nearest kilometre, between New York and Naples along this parallel of latitude.
19. WE6 a. An aircraft flies from Tampa Florida 28°N, 82°W to Corpus Christi Texas 28°N, 97°W. How far, correct to the nearest kilometre, has it flown along the parallel of latitude?
    b. From Texas it then flies to Oklahoma City 35°N, 97°W. Calculate the total distance, correct to the nearest kilometre, travelled by the aircraft.
20. An aircraft begins at Melbourne Australia and travels non-stop around the 38°S parallel of latitude until it returns to Melbourne. How far has it travelled correct to the nearest kilometre?

21. Melbourne and Hokkaido Island in Japan lie on the same line of longitude and are 9094 km apart. If Melbourne has a latitude of 37.83°S, find the latitude of Hokkaido Island, which is north of Melbourne. Give your answer correct to 2 decimal places.
22. The cities of Perth and Beijing lie on the same line of longitude and are 7985 km apart. If Perth has a latitude of 31.95°S, find the latitude of Beijing, which is north of Perth. Give your answer correct to 2 decimal places.

13.5 Time zones

The world is divided into 24 different **time zones**.



If you move from one time zone to another in an easterly direction, you need to move your clock forwards. If you move from one time zone to another in a westerly direction, you need to move your clock backwards.

The **International Date Line** is the line on Earth where a calendar day officially starts and ends. If you cross the International Date Line in an easterly direction, your calendar goes back one day. If you cross the International Date Line in a westerly direction, your calendar goes forward one day.

The time in England used to be called Greenwich Mean Time (GMT), but is now known as Coordinated Universal Time (UTC). It is referred to as Zulu time in the military.

UTC is the reference point for every time zone in the world. Time zones are either ahead of or behind UTC.

For example, the time difference between Melbourne and Los Angeles is 18 hours, not taking into account Pacific Daylight Time (PDT). This is because Melbourne is 10 hours ahead of UTC (UTC+10) and Los Angeles is 8 hours behind UTC (UTC–8)

WORKED EXAMPLE 7

What time is it in Melbourne (UTC+10) if the time in London is (UTC) 9.00 am?

THINK	WRITE
1 Melbourne is 10 hours ahead of London. Add 10 hours to the London time.	The time in London is 9.00 am. The time in Melbourne is 9.00 am + 10 hours.
2 Calculate the time in Melbourne.	The time in Melbourne is 7.00 pm.

13.5.1 Longitude and time difference

We spoke about longitude with great circles in section 13.4. Longitude is also linked to time differences around the world. As with the time zones map of the world shown at the start of this section, each 1 hours time difference is linked to a change of 15° in longitude. This enables us to calculate time differences between countries and cities if we know the difference in their longitude.

WORKED EXAMPLE 8

Calculate the time difference between Perth with a longitude of 116°E and Cairns with a longitude of 146°E.

THINK	WRITE
1. Find the difference in their longitudes.	Difference $= 146° - 116°$ $= 30°$
2. Each 15° represents a time difference of 1 hour.	$\frac{30°}{15°} = 2$ Therefore passes through two 1 hour time differences.
3. Write the time difference.	The difference between Perth time and Cairns time is 2 hours.

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Time zones Summary screen and practice questions

Exercise 13.5 Time zones

1. WE7 What time is it in Melbourne (UTC + 10) if the time in London (UTC) is 11.45 am?
2. What time is it in Los Angeles (UTC − 8) if the time in Sydney (UTC + 10) is 3.35 pm?
3. a. How does the local time change as you move around the world in a westerly direction?
   b. How does the local time change as you move around the world in an easterly direction?
4. What is the time in Melbourne (UTC + 10) if the time in London (UTC) is:
   a. 11.33 am b. noon c. 2.55 pm d. 7.14 am?
5. What is the time in Los Angeles (UTC − 8) if the time in Melbourne (UTC + 10) is:
   a. 2.30 pm b. noon c. 6.05 am d. midnight?
6. Your friend in New York wishes to call you. If you wish to receive the telephone call at 6.00 pm in Melbourne (UTC + 10), at what time in New York (UTC − 5) should your friend make the call?
7. To have the time zones work in Santa's favour:
   a. to which country should he deliver his presents first
   b. what direction should he travel from his first country?
8. A plane flew from Melbourne to London. Some of the details of the trip are listed.
   - Departure date: Friday 5 August
   - Departure time: 8.00 am
   - Travel time: 25 hours

   a. What was the time and date in Melbourne (UTC + 10) when the plane arrived in London (UTC)?
   b. What was the local time (UTC) when the plane arrived?

9. WE8 Calculate the time difference between Perth with a longitude of 116°E and Hamilton (NZ) with a longitude of 176°E.
10. Calculate the time difference between Memphis (USA) with a longitude of 90°W and Philadelphia (USA) with a longitude of 75°W.
11. MC The time difference between a city with a longitude of 42° compared to a city with a longitude of 87° is:
    A. 1.5 hours B. 2 hours C. 2.5 hours
    D. 3 hours E. 4 hours

The following information relates to questions 12 and 13.

The city of Memphis, USA is positioned at longitude 90°W and latitude 35°N, compared to the city of Los Angeles, USA at longitude 120°W and latitude 35°N.

12. Determine the time difference between the two cities.
13. If the plane departed Memphis at 9.30 am local time and the journey took 3 hours and 32 minutes, at what time did it arrive at Los Angeles in Los Angeles time?

14. The city of Melbourne has a longitude of 145°E and the city of Hamburg (Germany) has a longitude of 10°E.

If it takes 22 hours and 45 minutes to fly from Melbourne to Hamburg (including stopovers), at what time would a tourist land in Hamburg if their flight left Melbourne at 5.45 pm?

The following information relates to questions **15** and **16**.

A cargo ship travels from Perth (115°E longitude and latitude 33°S) to Cape Town, South Africa (20°E longitude and latitude 33°S). The ship set sail at 4.00 am on 3 April and travelled at an average speed of 15 m/s

15. If the journey from Perth to Cape Town was 8700 km, calculate the duration of the trip correct to the nearest minute.

16. **a.** Find the time in Perth when the ship arrived at Cape Town.

b. Find the time difference between Perth and Cape Town (to the nearest hour).

c. Find the local time of the ship's arrival.

13.6 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC An arc of a circle is formed by an angle of 80° in a circle of radius 10 cm. The length of the arc is closest to:

A. 14 cm **B.** 13.5 cm **C.** 13 cm **D.** 12.5 cm **E.** 12 cm

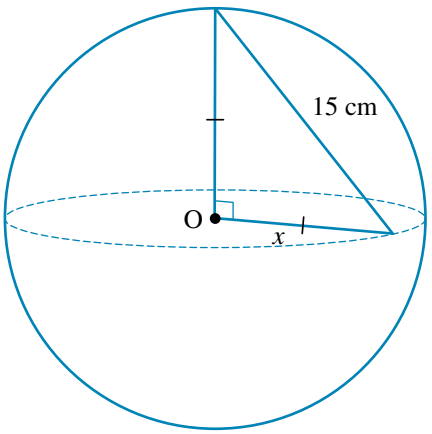
2. MC The area of a minor sector of a circle formed by an angle of 67°in a circle of radius 22 cm is closest to:

A. 160 cm^2 **B.** 212 cm^2 **C.** 488 cm^2 **D.** 283 cm^2 **E.** 282 cm^2

3. MC The area of a minor sector of a circle formed by an angle of 135° is 180 cm^2. The radius of the circle is closest to:

A. 12 cm **B.** 12.4 cm **C.** 152.7 cm **D.** 152.8 cm **E.** 160.2 cm

4. MC In the sphere below, the length of x is:



A. 15 cm **B.** 10 cm **C.** 225 cm **D.** 112.5 cm **E.** 10.6 cm

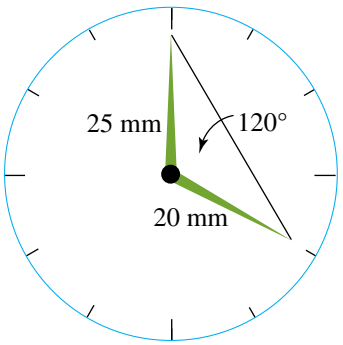
5. MC Perth is at approximately (31°S, 115°E) while Hong Kong is approximately at (22°N, 115°E). The distance between Perth and Hong Kong is approximately:

A. 19 km **B.** 98 km **C.** 1000 km **D.** 3547 km **E.** 5920 km

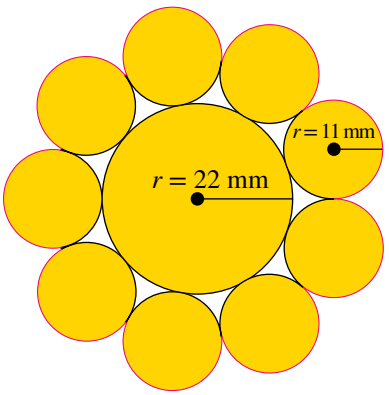
6. MC Rachel is a flight navigator. She is responsible for calculating the distance between Stockholm (60°N, 18°E) and Budapest (47°N, 18°E). Rachel's answer would be closest to:
   A. 1375 km **B.** 1452 km **C.** 2833 km **D.** 11 890 km **E.** 11 952 km
7. MC The shortest distance between two places A and B on the earth's surface with coordinates A (40°S, 130°E) and B (40°S, 159°E) is closest to:
   A. 1300 km **B.** 1867 km **C.** 2470 km **D.** 11 254 km **E.** 12 112 km
8. MC A plane leaves London (50°N, 0°) at 9:00 am Sunday, London time and flies to Sydney (33°S, 150°E). The flight takes 20 hours. Calculate the time in Sydney when the plane arrives.
   A. 3.00 pm Monday **B.** 7.00 pm Monday **C.** 3.00 am Monday
   D. 9.00 am Monday **E.** 9.00 pm Monday
9. MC What is the radius of the small circled formed when a circular cut is made 33 cm from the centre of a sphere of diameter 1 m?
   A. 0.11 m **B.** 0.38 m **C.** 0.45 m **D.** 0.94 m **E.** 3.14 m
10. MC If the time difference between Melbourne and Perth is 4 hours, then which of the following is true?
    A. The difference in latitude is about 60°. **B.** The difference in latitude is about 15°.
    C. The difference in latitude is about 45°. **D.** The difference in longitude is about 60°.
    E. The difference in longitude is about 45°.

Short answer

1. Calculate the arc length (correct to 2 decimal places) of a sector with radius 12.5 cm and an internal angle of 135°.
2. Calculate the area of the sector created in question 1, correct to 2 decimal places.
3. The end-points of the radii that form a sector in question 1 are joined with the straight line, thus dividing the sector into a triangle and the minor segment. Calculate the area of the minor segment, correct to two decimal places.
4. The hour hand of a clock is 20 mm long and the minute hand is 25 mm. The clock face at right shows the time at 4 o'clock. What is the distance between the tips of both hands (correct to the nearest mm)?



5. Calculate the difference between the circumferences of a small circle formed 10 cm and a small circle formed 15 cm from the centre of a sphere of radius 20 cm. Give your answer correct to 2 decimal places.
6. Calculate the outer perimeter (shown in red) and shaded area of the flower.



7. Calculate the shortest distance from Newcastle, Australia (latitude 33°S, longitude 153°E) to Mendoza, Argentina (latitude 33°S, longitude 72°W), correct to the nearest kilometre.
8. Estimate the time difference between Newcastle and Mendoza from question **7** to the nearest hour.

Extended response

1. **a.** A maker of a new perfume is considering a spherical shape for the perfume bottle.

If the initial prototype of the bottle has a radius of 3.5 cm, what is the total volume of the bottle, including the lid? Remember the equation to calculate volume of a sphere is $V = \frac{4}{3}\pi r^3$. Give your answer correct to the nearest cm^3.

b. Marketing information suggests that new perfume sales increase if there is also a handbag size perfume bottle available as a promotion.

The perfume is expensive to produce so the company will only consider giving away $\frac{1}{8}$ of the volume of the large bottle. What is the radius of the small bottle of perfume?

2. **a.** The lid of the perfume bottle in question **1a** has a circular base. The centre of the base is 2 cm from the centre of the bottle. What is the radius of the base of the lid? Give your answer correct to two decimal places.

b. A gold stripe of 4 cm is to be painted on the rim of the lid. What percentage of the rim of the lid will have the gold stripe?

3. The perfume company wants to set up a meeting between their sales team in Sydney (33°S, 150°E) and Los Angeles (34°N, 120°W). What is the time in Los Angeles when it is 8.00 am on Wednesday in Sydney?

studyon

Units 3 & 4 Sit exam

Answers

Topic 13 Spherical geometry

Exercise 13.2 Circle mensuration

1. a. 5.13 m b. 27.49 cm
2. a. 22.38 cm b. 102.45 mm
3. a. 14.14 cm^2 b. 4.19 m^2 c. 65.45 m^2
4. a. 204.53 cm^2 b. 165.47 cm^2
5. a. 143.61 m^2 b. 8.28 m^2
6. 201.06 km^2
7. E
8. 1.57 cm^2
9. 3.27 m^2
10. a. 804.25 cm^2 b. 20.26 cm^2 c. 110.85 cm^2
11. a. 18.20 m^2 b. 70.69 m^2
    c. 14.98 m^2 d. $2612.25
12. a. 2474 cm^2 b. 34.4%
13. 10 cm
14. B
15. 15.71 m
16. 61.81 cm^2

Exercise 13.3 Right-angled triangles within a sphere

1. 8.77 cm
2. 7.07 cm
3. 15.56 cm
4. a. 72.1 cm b. 453.09 cm
    c. 16336.28 cm^2 d. 151.5 cm
5. 1.96 cm
6. 84.3 cm
7. 1.32 cm
8. 678.6 cm^2

Exercise 13.4 The earth as a sphere

1. a. 6479 km b. 5138 km
2. a. 9159 km b. 5027 km
3. 17 760 km
4. 5362 km
5. a. 5697 km b. 11 394 km
    c. 8042 km d. 22 117 km
6. a. 1340 km b. 335 km
    c. 56 km d. 9 km
7. a. 1759 km b. 1152 km
8. a. 4299 km b. 5373 km c. 6392 km
9. 2385 km
10. 50.36°
11. 33.97°
12. a. 1449 km b. 2631 km c. 3985 km
13. 1521 km
14. 431 km
15. a. Cape Town and Budapest
    Naples and Prague
    b. Cape Town and Budapest – 81°
    Naples and Prague – 9°
16. a. Sydney and Cape Town
    New York and Naples
    b. Sydney and Cape Town – 132°
    New York and Naples – 88°
17. a. 5305.84 km b. 12 224 km
18. a. 4830.14 km b. 7419 km
19. a. 1479 km b. 2261 km
20. 31 688 km
21. 43.58°N
22. 39.54°N

Exercise 13.5 Time zones

1. Melbourne time is 9.45 pm
2. Los Angeles time is 9.35 pm the day before.
3. a. Backwards b. Forwards
4. a. 9.33 pm b. 10.00 pm
    c. 12.55 pm d. 5.14 pm
5. a. 8.30 pm the day before
    b. 6.00 pm the day before
    c. 12.05 pm the day before
    d. 6.00 am
6. 3.00 am
7. a. New Zealand b. Westerly direction
8. a. 9.00 am Saturday 6 August (Melbourne time)
    b. 11.00 pm Friday 5 August
9. 4 hours difference
10. 1 hours difference
11. D
12. 2 hours time difference
13. 11.02 am
14. 7.30 am the next day
15. 6 days, 17 hours and 7 minutes
16. a. 9.07 pm on 9 April
    b. 6 hours
    c. 3.07 pm on 9 April

13.6 Review exam practice

Multiple choice

1. A	**2.** D	**3.** B	**4.** E	**5.** E
6. B	**7.** C	**8.** A	**9.** B	**10.** D

Short answer

1. 29.45 cm

2. Area of sector = 184.08 cm^2

3. Area of the segment = 128.84 cm^2

4. 39 mm

5. 25.71 cm

6. Area 4942 mm^2, perimeter 311 mm

7. 15 080 km

8. 9 hours' time difference

Extended response

1. a. 180 cm^3 **b.** 1.75 cm

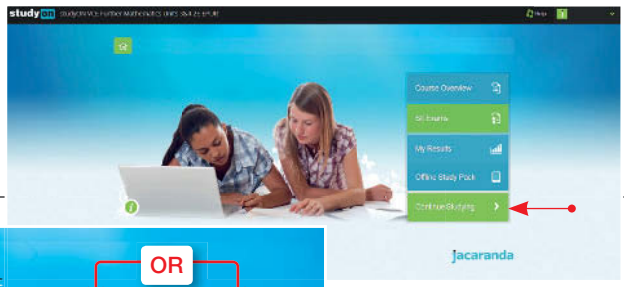
2. a. 2.87 cm **b.** 22.16%

3. 2.00 pm Tuesday

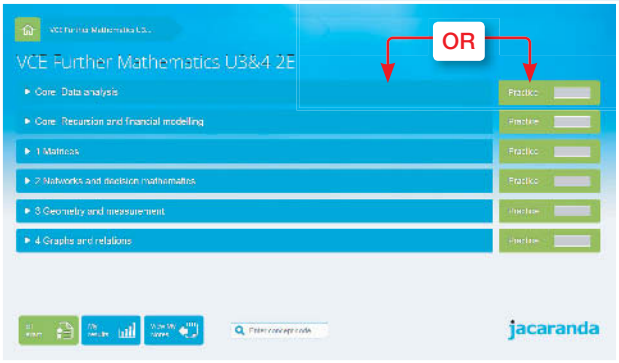
REVISION: AREA OF STUDY 2 Geometry and measurment

TOPICS 11 to 13

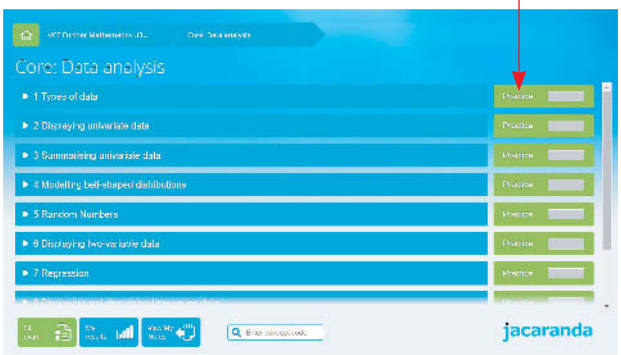
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



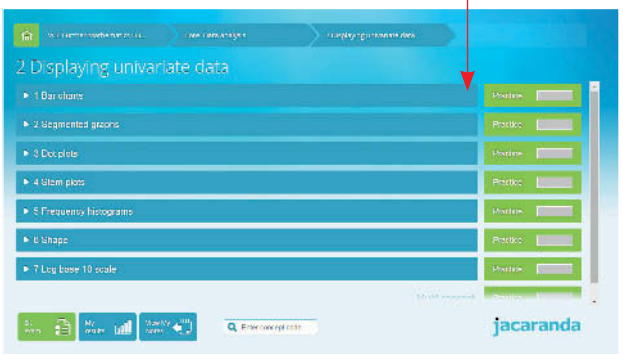
- Select your **course** *Further Mathematics VCE Units 3 & 4* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



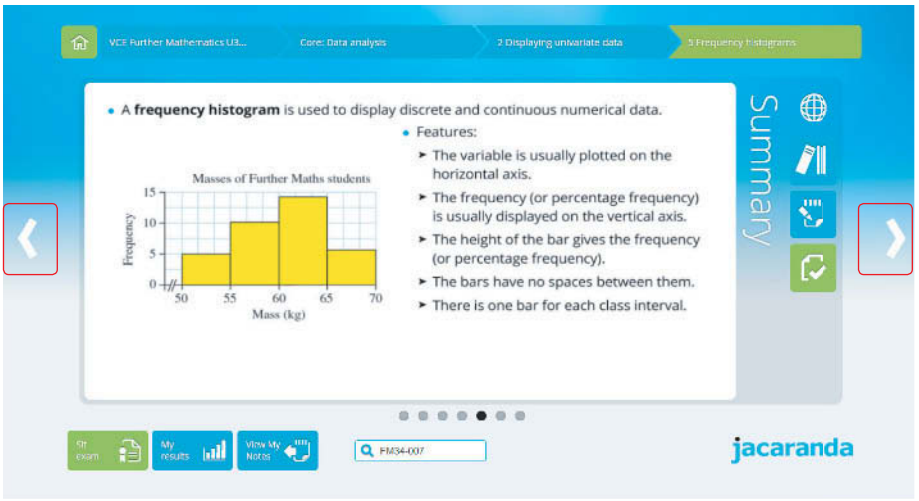
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 14
Construction and interpretation of graphs

14.1 Overview

14.1.1 Introduction

Graphs are a means of providing visual representations of within data. Large amounts of data are very unwieldly but once organised into suitable graphs can be analysed and interpreted. Every organisation, large and small, use graphs to assist decision making. Corporations and businesses rely on graphs to provide important information to all their stakeholders in weekly, monthly and yearly reports. Scientists, economists, sales and marketing personnel, government departments, educators, traffic engineers and the media, to name just a few, all rely on graphs to make future predictions and hence convince stake holders that change may be necessary or even essential.

Recent data has shown that from Christmas 2016 to Christmas 2017, online spending on significant retail days, such as, Click Frenzy, Black Friday, Cyber Monday and the Amazon AU launch has demonstrated a significant increase. This increase is clearly shown in the graph of the data. The visual impact of these spikes is hard to ignore and can help retailers both online and offline to consider what affect this may have on their business in subsequent years and make decisions accordingly.

It is important that everyone has the skills and knowledge to interpret the graphs that are providing information about all aspects of life, including health, study scores, mobile phone accounts, electricity and gas usage and billing, housing costs, consumer trends and environmental issues. This knowledge at both a personal and global level ensures informed consumers and citizens.

LEARNING SEQUENCE

14.1 Overview
14.2 Constructing and interpreting straight-line graphs
14.3 Line segments and step functions
14.4 Simultaneous equations and break-even point
14.5 Interpreting non-linear graphs
14.6 Constructing non-linear relations and graphs
14.7 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

14.1.2 Kick off with CAS

Solving simultaneous equations with CAS

Simultaneous equations are a set of two (or more) equations that contain the same variables. By solving simultaneous equations we are finding the solution(s) which satisfy all of the equations within the set.

To solve simultaneous equations graphically we can sketch graphs of the equations on the same set of axes. The intersection of the two graphs is the solution to the set of simultaneous equations.

1. Using CAS, sketch the graph of $y = 4x - 3$.
2. Using CAS, sketch the graph of $y = 6 - 3x$ on the same set of axes as the graph in question 1.
3. Use CAS to find the point of intersection of both of the graphs you have sketched.
4. Confirm your solution to question 3 by using CAS to solve the pair of simultaneous equations ($y = 4x - 3$ and $y = 6 - 3x$) algebraically on a calculator page.
5. Use CAS to sketch the graphs of $y = 2x + 5$ and $y = 2x - 1$ on the same set of axes.
6. Is there a point of intersection between the two graphs you plotted in question 5? What is the relationship between the two equations?

on Resources

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

14.2 Constructing and interpreting straight-line graphs

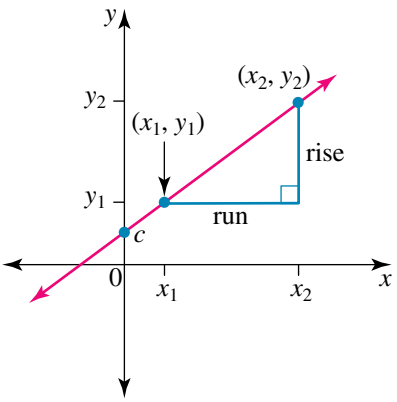
Equations and graphs are used to study the relationship between two variables such as distance and speed, tax payable and income, or radioactivity and time.

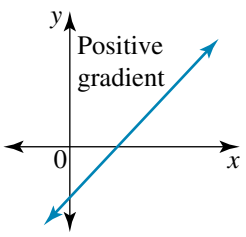
If a relationship exists between the variables, one can be said to be a **function** of the other. A function can be described by a table, a rule or a graph. A function whose graph is a straight line is called a **linear function**.

The general equation of a straight line is $y = mx + c$, where m is the gradient and c is the y-intercept.

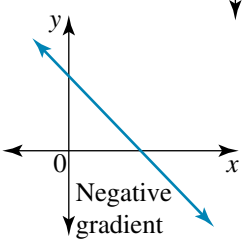
The ***y*-intercept** (c) is the value of y where the graph cuts the y-axis.
The **gradient** (or slope) of a straight line is denoted by m where

$$\text{gradient } (m) = \frac{\text{rise}}{\text{run}} \text{ or } m = \frac{y_2 - y_1}{x_2 - x_1}.$$





A graph that rises as x increases will have a positive gradient.



A graph that falls as x increases will have a negative gradient.

WORKED EXAMPLE 1

Consider the equation $y = 3x + 4$.
a. What is the value of y when $x = 2$?
b. What is the value of x when $y = 13$?
c. Does the point (2, 4) lie on the graph of $y = 3x + 4$?
d. Does the point (5, 19) lie on the graph of $y = 3x + 4$?
e. State the value of the gradient and the y-intercept.

THINK	WRITE
a. Substitute $x = 2$ into the equation.	a. When $x = 2$, $y = 3 \times 2 + 4$ $= 10$
b. Substitute $y = 13$ into the equation and solve for x.	b. When $y = 13$, $13 = 3x + 4$ $3x = 9$ $x = 3$
c. Substitute 2 for x and 4 for y. If this makes the equation true then the point lies on the line; otherwise it does not lie on the line.	c. Does $4 = 3 \times 2 + 4$? 4 does not equal 10. The equation is not true, therefore the point (2, 4) does not lie on the line.
d. Substitute 5 for x and 19 for y. If this makes the equation true then the point lies on the line; otherwise it does not lie on the line.	d. Does $19 = 3 \times 5 + 4$? The equation is true, therefore the point (5, 19) does lie on the line.
e. Compare the given equation with that of the general equation of a straight line, $y = mx + c$. The gradient is the coefficient of the x-term, m, and the y-intercept is the constant term, c.	e. The gradient is 3 and the y-intercept is 4.

WORKED EXAMPLE 2

Two points A (1, 7) and B (3, 13) lie on the same line. Use the points A and B to calculate the gradient of the line.

THINK	WRITE
1. Write the gradient formula and match points A and B with (x_1, y_1) and (x_2, y_2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1) = (1, 7)$ and $(x_2, y_2) = (3, 13)$
2. Substitute the values into the formula and evaluate.	$m = \frac{13 - 7}{3 - 1}$ $= \frac{6}{2}$ $= 3$ The gradient is 3.

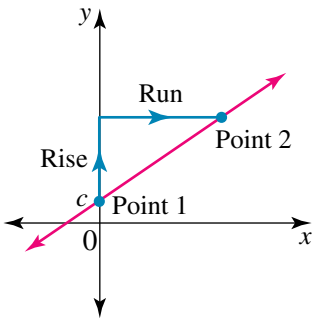
14.2.1 Sketching straight-line graphs

In previous years of study you have learned that only one straight line can be drawn through two distinct points. Therefore, the graph of a straight line can be obtained by plotting and joining together any two points on the line. Thus, to sketch a straight-line graph by hand, we first need to find the coordinates of any two points on the line. This can be done in a number of different ways. The three most common methods are outlined as follows.

Gradient–intercept method

This method is used if the equation is in $y = mx + c$ form.

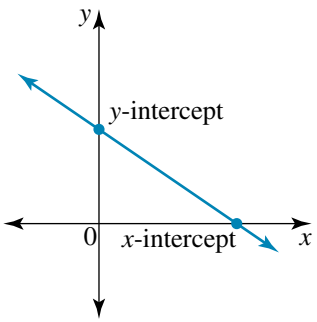
1. The first point plotted is the y-intercept, given by the value of c. Plot it on a set of axes.
2. The gradient is given by $m = \frac{\text{rise}}{\text{run}}$. Write the gradient as a fraction and identify the values of the rise and the run.
3. To obtain the second point, start from the y-intercept and move up (or down) and across, as suggested by the gradient.
4. Join the two points together with a straight line and label the graph.



x- and y-intercept method

This method is used if the equation is in $ax + by = c$ form, or if you are required to show both the x- and y-intercepts.

1. If a point is on the y-axis, its x-coordinate is 0. To find the y-intercept, substitute 0 for x and solve the resultant equation.
2. If a point is on the x-axis, its y-coordinate is 0. To find the x-intercept, substitute 0 for y and solve the resultant equation.
3. Plot the x-intercept and the y-intercept on a set of axes.
4. Join the two points together with a straight line and label the graph.

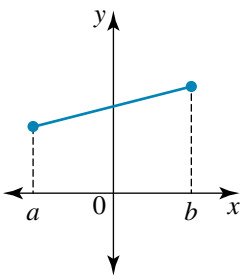


Sketching a line over a required interval

If a graph needs to be sketched between two given x-values, its end points must be shown. Since only two points are needed to sketch a line, we can obtain the coordinates of the end points and join them together. To construct a graph of a straight line between a and b, follow these steps:

1. Rearrange the equation to make y the subject.
2. Substitute each of the two given x-values (that is, a and b) into the equation and find corresponding values of y.
3. Plot the end points on a set of axes.
4. Join the two points together with a straight line and label the graph.

Worked example 3 illustrates the use of these three methods.



WORKED EXAMPLE 3

Sketch the graph of each of the following equations.

a. $y = 3x + 4$ **b.** $y = -\frac{2}{3}x$ **c.** $2x + 3y = 6$

d. $x - y = 3$ **between** $x = -2$ **and** $x = 6$.

THINK

a. 1. The equation is in $y = mx + c$ form, so use the gradient-intercept method. State the value of the y-intercept.

2. Write the gradient as a fraction and identify the values of the rise and the run.

3. Draw a set of axes and plot the y-intercept. To obtain the second point, start from the y-intercept and move up and across as suggested by the positive gradient; that is, 3 units up and 1 unit right. The second point is $(1, 7)$.

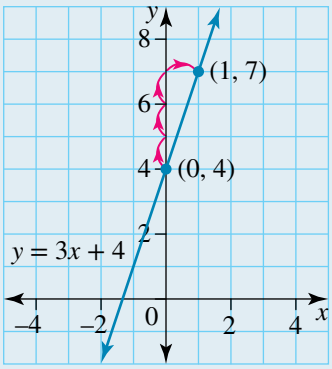
4. Join the two points with a straight line and label the graph.

WRITE/DRAW

a. $y = 3x + 4$

$c = 4$, so the y-intercept is $(0, 4)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{3}{1}$$



b. 1. The equation is in $y = mx + c$ form, so use the gradient-intercept method. State the value of the y-intercept. Since the value of c is 0, the line passes through the origin.

2. Write the gradient as a fraction and identify the values of the rise and the run. Note that the negative sign always belongs to the rise.

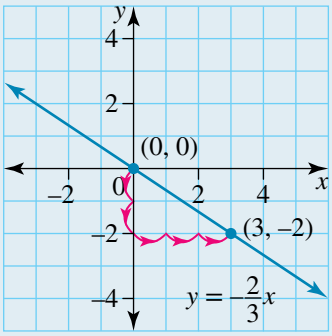
3. Draw a set of axes and plot they-intercept. To obtain the second point, start from the origin and move down and across as suggested by the negative gradient; that is, 2 units down and 3 units right. The second point is $(3, -2)$.

4. Join the two points with a straight line and label the graph.

b. $y = -\frac{2}{3}x$

$c = 0$, so the y-intercept is $(0, 0)$.

$$m = \frac{\text{rise}}{\text{run}} = -\frac{2}{3}$$



c. 1. The equation is in $ax + by = c$ form, so find the x- and y-intercepts.

2. Find the y-intercept by substituting $x = 0$ into the equation and solving.

c. $2x + 3y = 6$

$2(0) + 3y = 6$

$3y = 6$

$y = 2$

3. State the coordinates of the y-intercept.

The coordinates of the y-intercept are $(0, 2)$.

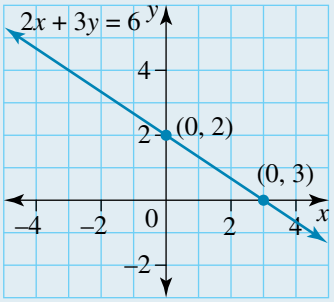
4. Find the x-intercept by substituting $y = 0$ into the equation and solving.

$$2x + 3y = 6$$
$$2x + 3(0) = 6$$
$$2x = 6$$
$$x = 3$$

5. State the coordinates of the x-intercept.

The coordinates of the x-intercept are $(3, 0)$.

6. Plot the intercepts on a set of axes and join them together with a straight line. Label the graph.



d. 1. The graph needs to be sketched between two x-values. Rearrange the equation to make y the subject.

d. $x - y = 3$
$$-y = -x + 3$$
$$y = x - 3$$

2. Substitute each of the two given x-values (that is, -2 and 6) into the equation and find corresponding values of y.

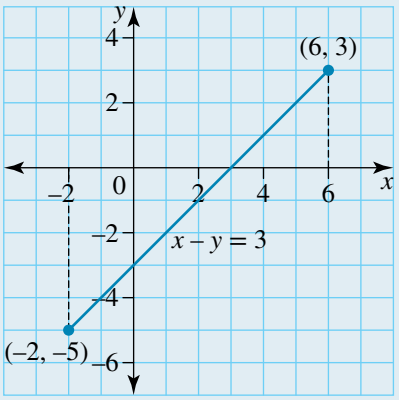
When $x = -2$:
$$y = x - 3$$
$$= -2 - 3$$
$$= -5$$

When $x = 6$:
$$y = x - 3$$
$$= 6 - 3$$
$$= 3$$

3. State the coordinates of the end points.

The end points are $(-2, -5)$ and $(6, 3)$.

4. Plot the end points on a set of axes and join them with a straight line. Label the graph. *Note*: Do not extend the line beyond the end points.



14.2.2 Applications of straight-line graphs

Many real-life situations involve variables whose relationship is linear and hence can be described by a linear rule and represented graphically by a straight line.

When modelling a linear relationship, remember that the y-intercept represents the value of the function when $x = 0$. In most situations, it represents the initial (or original) value of something, a fixed cost or flag fall.

The gradient represents the rate of change of the y-value with respect to x. It shows the change (increase or decrease) in y, as x increases by 1 unit. For example, let the equation $V = 1000 - 200t$ describe the volume

(in litres) of water in a tub t minutes after a plug is removed. The y-intercept shows that the initial amount of water in the tub is 1000 litres. The gradient means that the volume of water decreases (since the gradient is negative) at a rate of 200 litres every minute.

WORKED EXAMPLE 4

Mikaela works as a car salesperson. She is paid a retainer of $150 a week and receives 2% commission on her sales.

a. How much does Mikaela earn in a week in which she sold cars worth $40 000 in total?

b. Construct a table to describe the relationship between earnings and sales using sales of $0, $10 000, $20 000 and $80 000.

c. Write an equation that relates earnings (E) to sales (S).

d. State the value of the gradient and the y-intercept and interpret their meaning.

e. Draw a graph of the relationship between earnings and sales. That is, draw a graph of E versus S.

THINK

a. Calculate earnings by finding the sum of the retainer and the commission.

$$2\% = \frac{2}{100}$$
$$= 0.02$$

b. For each of the four values of sales, repeat the working of **a** opposite.

c. Use the pattern seen in **a** and **b** to relate E and S. Writing the relationship in words may assist in finding the equation.

d. 1. Compare the equation relating earnings to sales with that of the general equation of a straight line, $y = mx + c$. The gradient is the coefficient of the x-term, m, and the y-intercept is the constant term, c.

2. Interpret the meaning of the gradient and the y-intercept.

WRITE/DRAW

a. Earnings = retainer + commission
= retainer + 2% of sales
$= 150 + 2\% \times 40\,000$
$= 150 + 0.02 \times 40\,000$
$= 150 + 800$
$= 950$
Mikaela earns $950.

b.

Sales	Earnings
0	150
10 000	350
20 000	550
80 000	1750

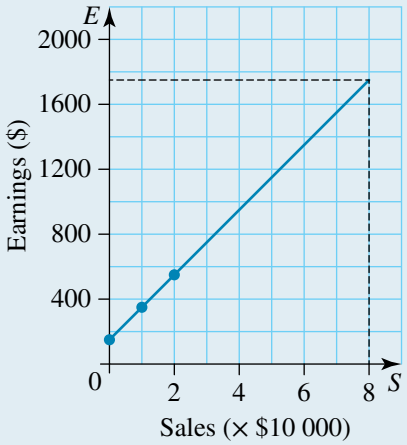
c. Earnings = retainer + 2% of sales
$E = 150 + 0.02 \times S$
or $E = 0.02S + 150$

d. The gradient is 0.02 and the y-intercept is 150.

The gradient represents the commission rate. It shows that for every $100 worth of sales, Mikaela earns $2 in commission. The y-intercept represents the value of the retainer; that is, the amount Mikaela is paid if no sales are made.

e. Earnings depends on sales. Thus sales is the *explanatory* variable and goes on the horizontal axis. Do not crowd the axes by writing the sales values in thousands — use a legend and write only the number of multiples of ten thousand.

e.



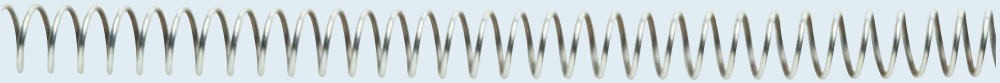
14.2.3 Extrapolation and interpolation

Two terms used widely in interpreting data are **extrapolation** and **interpolation**. Extrapolation means to examine the relationship between the variables by extending it beyond the data. Interpolation means to infer the relationship between distinct data points. Worked example 5 should make these meanings clear.

WORKED EXAMPLE 5

The tension (measured in Newtons) in a spring is linearly related to the extension of the spring (measured in centimetres). Some values relating tension (T) and extension (x) are given:

x	2	5	7
T	50	125	175



a. Plot these data.
b. By extrapolating from the data, predict the tension when the extension is 10 cm.
c. By interpolating, predict the tension when the extension is 4 cm.

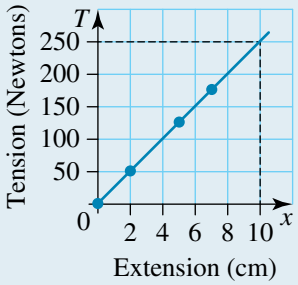
THINK

a. Plot the data on a set of axes that shows x-values up to 10. Draw a line through these points, extending it beyond the given values.

b. Extrapolation involves inference beyond the data range. Use the graph to find T when x is 10.

WRITE/DRAW

a.



b. When $x = 10$, $T = 250$
When the spring is stretched 10 cm, the tension is 250 newtons.

c. Interpolation involves inference between the data points. Use the graph to find T when x is 4.

c. When $x = 4$, $T = 100$
When the spring is stretched 4 cm, the tension is 100 newtons.

Resources

- **Interactivity:** Equations of straight lines (int-6485)
- **Interactivity:** Linear graphs (int-6484)
- **Interactivity:** Constructing and interpreting straight-line graphs (int-6280)

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Units 3 & 4 > AOS M4 > Topic 1 > Concepts 1 & 8

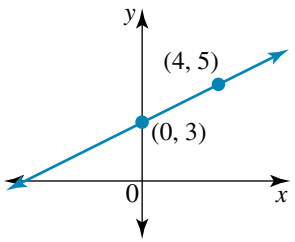
Straight-line graphs Summary screen and practice questions

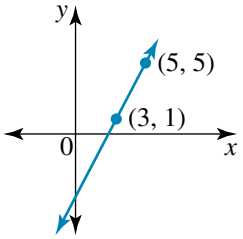
Interpolation and extrapolation Summary screen and practice questions

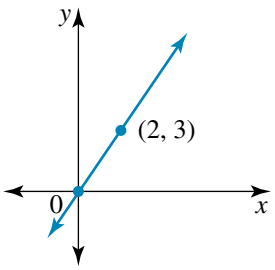
Exercise 14.2 Constructing and interpreting straight-line graphs

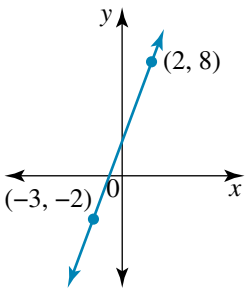
1. WE1 Consider the equation of the line $y = 2x + 5$.
   - **a.** What is the value of y when $x = 4$?
   - **b.** What is the value of x when $y = 17$?
   - **c.** Does the point $(3, 12)$ lie on the line?
   - **d.** State the value of the gradient and the y-intercept.
2. Consider the equation $5x - 2y = 10$.
   - **a.** Find the value of y when $x = 2$.
   - **b.** Find the value of x when $y = 5$.
   - **c.** Does the point $(-8, -25)$ lie on the line?
3. Consider the equation of the line $y = -2x + 4$.
   - **a.** What is the value of y when $x = 2$?
   - **b.** What is the value of x when $y = -4$?
   - **c.** Does the point $(0, 4)$ lie on the line?
   - **d.** State the value of the gradient and the y-intercept.
4. Consider the equation of the line $2y + 3x = 12$.
   - **a.** What is the value of y when $x = 2$?
   - **b.** What is the value of x when $y = 6$?
   - **c.** Does the point $(3, 2)$ lie on the line?
   - **d.** State the value of the gradient and the y-intercept.
5. WE2 Two points P$(2, 5)$ and Q$(4, 13)$ lie on the same line.
   Calculate the gradient of the line.
6. The points $(5, -3)$ and $(-2, 4)$ lie on a line. Calculate the gradient of this line.
7. Consider the equation of the line $2y + x = 8$.
   - **a.** What is the value of y when $x = 2$?
   - **b.** What is the value of x when $y = 1$?
   - **c.** Does the point $(3, 2)$ lie on the line?
   - **d.** Using the two points whose coordinates were found in **a** and **b**, calculate the gradient.

8. Calculate the gradient of the line shown in each of the following.

a.


b.


c.


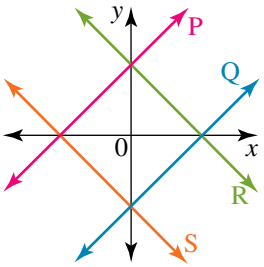
d.


9. **WE3** Sketch the graphs of the following functions.
   a. $y = 3x - 4$
   b. $y - 3x = 12$
   c. $y = \frac{3}{4}x$
   d. $4x - 3y = 12$ between $x = -2$ and $x = 4$

10. Sketch the graph of the following functions.
   a. $5y + x = 10$
   b. $y = -x + 3$
   c. $y = 3x$
   d. $y = 3x - 2$, between $x = -2$ and $x = 2$

11. Sketch the graphs of the following functions.
   a. $3x + y = 12$
   b. $y = -2x + 8$
   c. $y = -2x$
   d. $y = 2x + 1$ between $x = -2$ and $x = 2$

12. **MC** The graphs of the following lines are shown at right.



   i. $y + x = 4$
   ii. $y = x + 4$
   iii. $y + x + 4 = 0$
   iv. $y = x - 4$

   The best match between graphs P to S and equations i to iv is:

   A. P → iv Q → i R → iii S → ii
   B. P → i Q → iii R → ii S → iv
   C. P → iv Q → i R → ii S → iii
   D. P → iv Q → iii R → i S → ii
   E. P → ii Q → iv R → i S → iii

13. **WE4** Greta sells boats and she is paid $275 per week plus 3% of her weekly sales.
   a. How much does Greta receive if the value of her weekly sales is $12 000?
   b. Construct a table to show Greta's weekly pay for the following values of weekly sales: $5000, $10 000, $15 000 and $20 000.
   c. Write an equation that relates her weekly pay (P) and the value of her weekly sales (S).
   d. Draw a graph of the relationship between weekly pay and weekly sales. That is, draw a graph of P versus S. Place S on the horizontal axis.
   e. State the value of the gradient and the y-intercept, and interpret their meaning.

14. Sam, who works as a car salesperson, is paid a retainer of $200 a week and receives 3% commission on her sales.

a. Copy and complete this table.

Sales	$0	$20 000	$40 000	$60 000
Earnings				

b. Write an equation that relates earnings (E) to sales (S).

c. Draw the graph of earnings (E) versus sales (S) using the information in **a**.

15. **WE5** For a farm water tank, the volume of water in the tank (V) is related linearly to the depth of water in the tank (D) for values of D greater than 450 mm.

Depth (mm)	**Volume (L)**
500	2500
700	4500
1000	7500

a. Plot these data using suitable axes.

b. By extrapolating from the data, predict the depth needed to give a volume of 13 000 litres.

c. By interpolating from the data, predict the volume when the depth of the water is 600 mm.

16. The distance covered (in metres) of an object over a period of time (in seconds) is shown in the table.

Time (s)	2	4	5	7
Distance (m)	8	16	20	28

a. Plot these data.

b. By extrapolating from the data, predict the distance covered when 9 seconds have elapsed.

c. By interpolating, predict the distance covered when the time elapsed is 6 seconds.

17. An object that falls freely due to gravity increases its speed (S) by 9.8 m/s each second. Assume that at the start the speed of the object was 0.

a. How fast is the object travelling after 3 seconds?

b. How long has the object been falling when its speed is 78.4 metres per second?

c. If t stands for the number of seconds the object has been falling, write a formula that relates S to t.

d. Draw a graph of S versus t.

18. Visa car rentals charge $75 per day plus $15 per hundred kilometres.

a. How much would it cost to rent a car for one day if the car travelled 345 km?

b. The bill for one day's rental came to $142.50. How many kilometres did the car travel?

c. Sketch a graph of the cost of renting the car for one day (C) versus the number of kilometres travelled (d).

19. A psychology experiment is testing the relationship between the number of errors (n) made by a child on a task and the time taken to complete the task in seconds (t). The results of five trials of the experiment are given in the table below.

If the number of errors made is linearly related to the time taken, which experiment does not fit the pattern?

Experiment	Time taken (seconds)	Number of errors
A	30	10
B	45	6
C	36	9
D	48	7
E	42	8

20. Janine sells cosmetics at a department store. She knows she is paid a retainer plus commission on sales but she is not sure of the exact rates. For three weeks she records her wages and the value of product she sold during that week.

Week	Sales	Wages
1	$1200	$494
2	$750	$440
3	$880	$455.60

a. In week 4 her sales totalled $1000. Predict her wage for that week.
b. Write a formula to calculate Janine's weekly wage in terms of her weekly sales.

21. In a one-day international cricket match, each team bats for 50 overs. After 20 overs Australia's score had reached 108 runs. If the target for Australia is 250 runs at the end of 50 overs:
a. how many runs per over does the team need to score for the remaining 30 overs?
b. Write an equation to relate the score (S) to the number of overs completed (n).
c. The graph of S versus n could be a straight line. What would be the gradient of that straight line?

22. Taxi hire charges are shown in the diagram.
a. Calculate the cost of travelling 15 km in a taxi.
b. If the taxi fare was $20.80, how far did the taxi travel?
c. Write an equation relating the fare (F) to the distance (d).
d. Draw a graph relating F to d.
e. What is the slope of this graph?

14.3 Line segments and step functions

In this section we consider graphs which are made from multiple straight lines. We also consider graphs which are comprised of discrete sets of points.

WORKED EXAMPLE 6

When a real estate agent sells a property, he earns commission at the following rate:
1.5% on the first 20 000
0.9% on the remainder.

a. Calculate the commission earned on sales of:
i. $10 000 ii. $20 000 iii. $30 000 iv. $40 000.

b. Draw a line segment graph of commission (C) versus the value of the sales (S) up to a sales value of $40 000.

c. Give a reason for the difference of the gradients of the two segments.

THINK

a. i. For any amount up to and including $20 000, the commission is 1.5% of the amount.

ii. The commission is 1.5% of the amount.

iii. $30 000 is more than $20 000, so calculate 1.5% on $20 000 plus 0.9% on the remainder over $20 000; that is, $10 000.

iv. $40 000 is more than $20 000, so calculate 1.5% on $20 000 plus 0.9% on the remainder over $20 000; that is, $20 000.

b. 1. Since the commission rate changes when sales exceed $20 000, the graph will consist of two segments. One segment for all sales up to and including $20 000 and the other for sales of more than $20 000 and up to and including $40 000.

2. The starting point of the first segment is (0, 0) (since there is no commission when there are no sales). The end point of this segment is (20 000, 300) (as was found in part **a (ii)**). Plot these points and join them with a straight line.

WRITE/DRAW

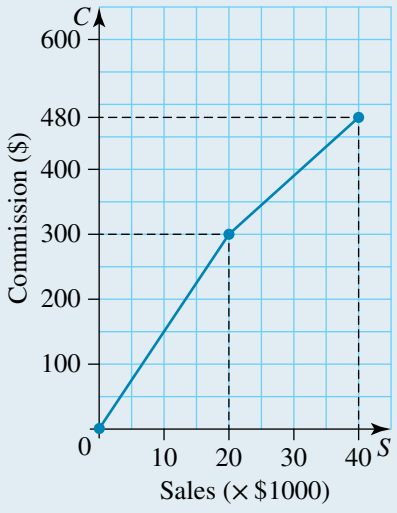
a. i. Commission on $10 000
$= 1.5\%$ of 10 000
$= 0.015 \times 10\,000$
$= \$150$

ii. Commission on $20 000
$= 1.5\%$ of 20 000
$= 0.015 \times 20\,000$
$= \$300$

iii. Commission on $30 000
$= 1.5\%$ of $20 000 + 0.9\%$ of $10 000
$= 0.015 \times 20\,000 + 0.009 \times 10\,000$
$= 300 + 90$
$= \$390$

iv. Commission on $40 000
$= 1.5\%$ of 20 000 $+ 0.9\%$ of 20 000
$= 0.015 \times 20\,000 + 0.009 \times 20\,000$
$= 300 + 180$
$= \$480$

b.



3. The second segment starts where the first segment ends. The end point of the second segment is (40 000, 480) (as was found in part **a** (**iv**)). Plot this end point and join it to the first segment.

c. In this case, the gradient is the rate of commission. When the sales exceed $20 000, the commission rate decreases — this results in a line segment that is less steep.

c. The gradient of each line segment represents the commission rates. Since the rates on values lower than $20 000 differ from the rates on values greater than $20 000, the gradients also differ.

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Units 3 & 4 > AOS M4 > Topic 1 > Concept 2 **Segment graphs** Summary screen and practice questions

WORKED EXAMPLE 7

An electrician charges the rates shown.

a. Calculate the charges for service calls of the following durations: 10 minutes, 25 minutes, 40 minutes, 55 minutes, 70 minutes and 85 minutes.

b. Draw a step graph of charges (C) versus the time of the service call in minutes (t) for calls of up to 120 minutes.

THINK

a. 'Part thereof' means that whether the repairs take 3 minutes, 10 minutes or 30 minutes, you are charged for 30 minutes. For a service call up to 30 minutes, the electrician charges $35 call-out fee plus $30. For a service call over 30 minutes long and up to 60 minutes, the charge is $35 call-out fee plus 2 lots of $30. Finally, for a service call over 60 minutes and up to 90 minutes, the charge is $3 call-out fee plus 3 lots of $30.

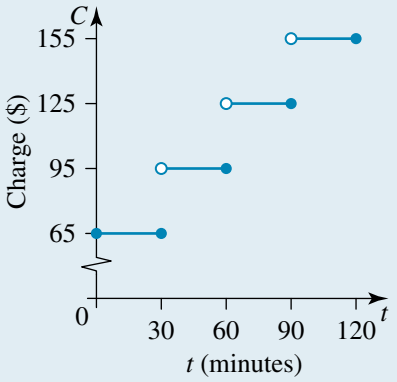
WRITE/DRAW

a.

Time	Charge
10 min	$35 + 30 = \$65$
25 min	$35 + 30 = \$65$
40 min	$35 + 2 \times 30 = \$95$
55 min	$35 + 2 \times 30 = \$95$
70 min	$35 + 3 \times 30 = \$125$
85 min	$35 + 3 \times 30 = \$125$

b. Draw the graph. Place t along the horizontal axis as it is the explanatory variable. The graph consists of four horizontal sections (steps). This reflects the fact that the fixed fee is charged for any amount of time up to and including 30 minutes; then a different fixed fee for any amount of time above 30 minutes and up to and including 60 minutes, and so on. Note that where the end point of the step is included in the step, it is shown as a full circle (closed circle), whereas if the end point is not included, it is shown as an empty circle (open circle).

b.



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Units 3 & 4 > AOS M4 > Topic 1 > Concept 3 **Step graphs** Summary screen and practice questions

In the previous example, the explanatory variable, time (t), was a **continuous variable**; that is, t can take any values — 2, 15.5, 87 — and so on. In the next example we consider a situation where the explanatory variable can take whole number values only. A variable that is not continuous is called a **discrete variable**. If the data are discrete, the points on the graph are not joined together.

WORKED EXAMPLE 8

A bakery sells bread rolls for 50 cents each or $2.50 for 6.
a. Calculate the cost of 4, 5, 6, 7, 11, 12, 13, 17, 18 and 19 bread rolls.
b. Sketch this information on a graph for up to 24 rolls.

THINK

a. Calculate the cost for each number of bread rolls. If the number of rolls is above 5, it can be formed by combining pack(s) of 6 rolls and single rolls. For example, 7 rolls can be bought as a pack of 6 plus 1 single roll; 17 rolls as 2 packs of 6 plus 5 single rolls etc.

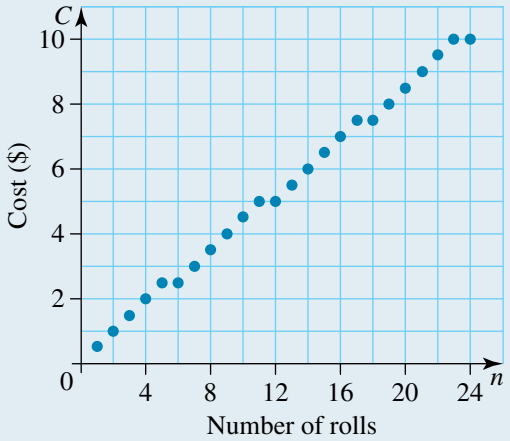
WRITE/ DRAW

a.

Number of rolls	Cost
4	$4 \times 50c = \$2.00$
5	$5 \times 50c = \$2.50$
6	$\$2.50$
7	$\$2.50 + 1 \times 50c = \$3.00$
11	$\$2.50 + 5 \times 50c = \$5.00$
12	$2 \times \$2.50 = \$5.00$
13	$2 \times \$2.50 + 1 \times 50c = \$5.50$
17	$2 \times \$2.50 + 5 \times 50c = \$7.50$
18	$3 \times \$2.50 = \$7.50$
19	$3 \times \$2.50 + 1 \times 50c = \$8.00$

b. Sketch the information using the number of bread rolls as the explanatory variable. Note that the points on the graph are not joined together as the data are discrete.

b.



Resources

Interactivity: Piecewise linear graphs (int-6486)
Interactivity: Step functions (int-6281)

Exercise 14.3 Line segments and step functions

1. **WE6** Suppose a real estate agent is paid commission at the following rates: 1.5% on the first $40 000 and 1% on the remainder.
   a. Calculate the commission due on sales of $20 000, $30 000, $40 000, $50 000 and $60 000.
   b. Draw a graph of commission (C) versus sales (S).
   c. Give a reason for the difference of the slopes of the two segments.
2. Beena, who runs a real estate agency, sells a property and earns commission at the following rate:

   2% on the first $25 000
   0.8% on the remainder.

   Calculate the commission earned by Beena on sales of:
   a. $10 000 **b.** $25 000 **c.** $35 000 **d.** $45 000.
   e. Construct a graph of commission (C) versus the value of sales (S).

   This information relates to questions **3** and **4**. The amount of electricity used around the home is measured in kilowatt hours (kWh). A light bulb left on for 10 hours will consume about 1 kWh of power. The local power supplier charges at the following rates.

Power	Cost per kWh ($)
First 400 kWh	0.20
Next 1000 kWh	0.15
Remaining kWh	0.10

3. Copy and complete the following table by calculating the cost due for each of the consumptions.

Consumption (kWh)	Power bill ($)
200	
400	
600	
1000	
1500	

4. Using the data in the table, draw a graph of power bill versus consumption.
5. WE7 A plumber charges these rates:

$55 call out fee
$40 per half hour or part thereof.

   a. Using this information, construct the graph of charges (C) versus time of the service call, in minutes (t), for calls of up to 120 minutes.
   b. Calculate the charges for these service calls:
   12 minutes, 23 minutes, 44 minutes, 56 minutes, 73 minutes, 87 minutes.
6. A telephone company charges users at a rate of 25 cents for each completed 30 seconds. This implies a call of less than 30 seconds is free.
   a. Copy and complete this table for the calls shown.
   b. Construct the graph of cost versus length of call for calls of up to 120 seconds.

Length of call (seconds)	Cost (cents)
15	
30	
45	
60	
75	
90	
105	
120	

7. An electrician charges at the following rates:

$45 call out fee plus
$35 per half hour or part thereof.

   a. Calculate the charges for the following service calls:
   20 minutes, 30 minutes, 45 minutes, 60 minutes, 80 minutes and 90 minutes.
   b. Draw a graph of charges (C) versus time of the service call in minutes (t) for calls of up to 90 minutes.

8. A mobile phone company charges users a rate of 15 cents for each completed 20 seconds of the call. This means a call of less than 20 seconds is free.
   a. Draw a table with the following headings and complete it for calls of 10, 20, 30, 40, 50, 60, 70, 80, and 90 seconds.

Length of call (seconds)	Cost (cents)

   b. Draw the graph of cost versus length of call for calls of up to 90 seconds.

9. WE8 A country bakery sells buns for 40 cents each or 6 for $2.00.
   a. Using this information, construct a graph for buying up to 20 buns using the number of buns as the independent variable.
   b. Calculate the cost of 2, 4, 6, 8 and 10 buns.

10. The Explorindo Travel Company specialises in surfing tours of remote islands in Indonesia. They will take individuals but prefer to deal with groups of people. They have the following charges for a holiday package:

1 person	$900
2 people	$1650
Each extra person	$600

   a. Draw a table with the following headings and complete it for costs for 1, 2, 3, 6, 8 and 10 people.

Number of people	Total cost ($)

   b. Draw the graph of total cost versus number of people. (Include only the number of people discussed in part a.)

11. MC Which of the following is clearly a discrete variable?
   A. the time taken for a phone call
   B. the number of CDs in a collection
   C. the commission earned on sales
   D. the power consumed by a hot-water jug
   E. a person's weight

12. MC On a particular visit, a tradesperson charges the householder $100. It was noted he arrived at 11.30 am. Which of the following departure times could be correct?

   A. 11.38 am B. 11.48 am C. 12.05 pm D. 12.20 pm E. 12.35 pm

13. This is the 2011–12 tax table.

Taxable income	Tax on this income
0–$6000	Nil
$6001–$37 000	15c for each $1 over $6000
$37 001–$80 000	$4650 plus 30c for each $1 over $37 000
$80 001–$180 000	$17 550 plus 37c for each $1 over $80 000
$180 001 and over	$54 550 plus 45c for each $1 over $180 000

Use this tax table to calculate the amount of tax paid in 2012 by people with taxable incomes of:

a. $4000 **b.** $7000 **c.** $20 000 **d.** $35 000 **e.** $40 000
f. $60 000 **g.** $100 000 **h.** $200 000.

14. a. Using the values calculated in question **13** and any other values, draw the graph of income tax versus taxable income.

b. Use the graph to estimate the income tax payable on a taxable income of:

i. $24 000 **ii.** $95 000.

Confirm your answers with calculations.

15. This is the 2017–2018 tax table.

Taxable income	Tax on this income
0–$18 200	Nil
$18 201–$37 000	19c for each $1 over $18 200
$37 001–$87 000	$3572 plus 32.5c for each $1 over $37 000
$87 001–$180 000	$19 822 plus 37c for each $1 over $87 000
$180 001 and over	$54 232 plus 45c for each $1 over $180 000

Use this tax table to calculate the amount of tax paid in 2018 by people with taxable incomes of:

a. $4000 **b.** $7000 **c.** $20 000 **d.** $35 000 **e.** $40 000
f. $60 000 **g.** $100 000 **h.** $200 000.

16. a. Use the values calculated in question **15** to draw a graph of income tax versus taxable income.

b. Use the graph to estimate the income tax payable on a taxable income of:

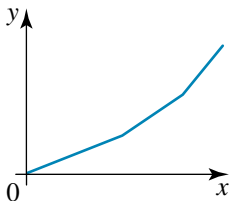
i. $24 000
ii. $95 000.

Confirm your answers with calculations.

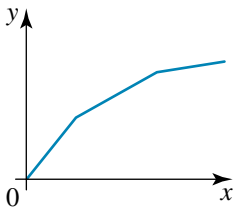
c. Compare your answers to questions **14b** and **16b**.

17. What situation could be described by the following graphs? In your response, clearly identify the variables used on both axes and explain how the graph represents that situation. For example, the first graph could measure the profit versus the number of units of production.

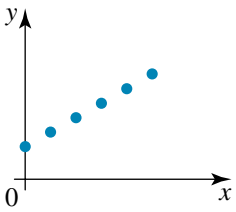
a.



b.



c.

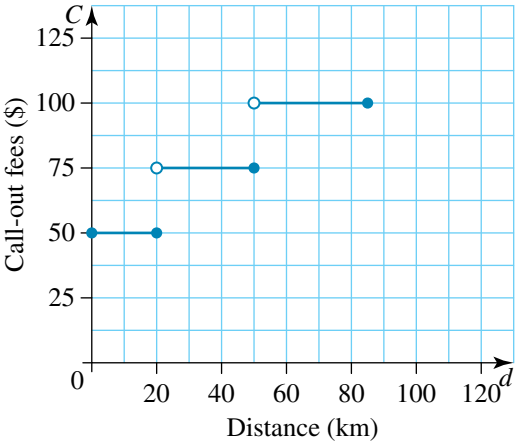


18. 'Fix-it-fast' is a photocopier repair service for schools in North-central Victoria. The call-out charge depends on the distance the repair person has to travel. The call-out fees for distances up to 85 km are shown on the following graph.

a. **i.** What is the call-out fee for a distance of 40 km?

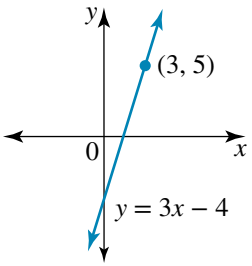
ii. What is the maximum distance travelled for a call-out fee of $50?

b. Copy the shown graph and add this information to it. A call-out fee of $125 is charged for schools at a distance of more than 85 km but less than 120 km.



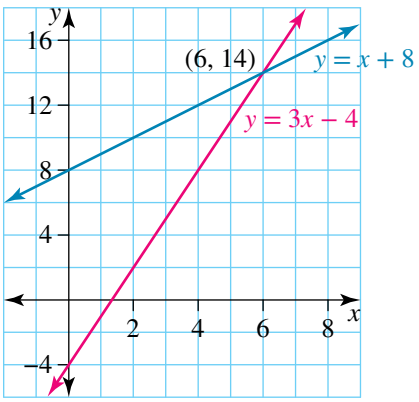
14.4 Simultaneous equations and break-even point

In this topic, we have seen that when an equation such as $y = 3x - 4$ is graphed it forms a straight line. Each point on the line will have an x- and a y-coordinate and these values of x and y satisfy the equation. The reverse is also true. Each pair, x and y, which satisfy the equation will be the coordinates of a point on the line.



14.4.1 Solving linear simultaneous equations

If we draw the graph of $y = 3x - 4$ and $y = x + 8$ on the same set of axes the result is shown at right.



The point where the lines intersect (6, 14) is called the **simultaneous solution** because $x = 6$ and $y = 14$ satisfies both equations.

In this example we found the simultaneous solution graphically; that is, from the graph. It is also possible to find the simultaneous solution algebraically, either by the substitution method or the elimination method.

WORKED EXAMPLE 9

Find the simultaneous solutions to these equations using algebraic methods.

a. $y = 3x - 4$ and $y = x + 8$ **b.** $2x + 3y = 13$ and $x - 4y = 1$

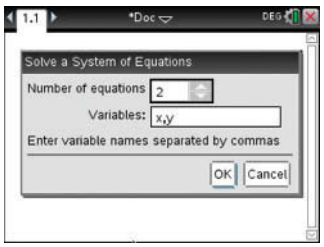
THINK	WRITE
a. 1. Since both equations start with $y =$, solve using the substitution method. Use equation [1] to substitute for y in equation [2].	**a.** $y = 3x - 4$ [1] $y = x + 8$ [2] Substitute [1] into [2].
2. Solve equation [3].	$3x - 4 = x + 8$ [3] $3x - x = 8 + 4$ $2x = 12$ $x = 6$
3. Find y by substituting $x = 6$ into one of the original equations.	Substitute $x = 6$ into [2]: $y = 6 + 8$ $= 14$
4. State the solution.	The solution is (6, 14).
b. 1. If one of the equations was rearranged to a $y = mx + c$ format, the substitution method could be used. However, it will be more efficient to use the elimination method in this case.	**b.** $2x + 3y = 13$ [1] $x - 4y = 1$ [2]
2. Multiply equation [2] by 2 so that the x-coefficients of both equations are the same.	[2] $\times 2$: $2x - 8y = 2$ [3]
3. Eliminate x by subtracting [3] from [1].	[1] $-$ [3]: $11y = 11$
4. Solve to find y.	$y = 1$
5. Find x by substituting $y = 1$ into one of the equations (into equation [2] is easiest).	**c.** Substitute $y = 1$ into [2]: $x - 4 \times 1 = 1$ $x - 4 = 1$ $x = 5$
6. State the solution.	The solution is (5, 1).

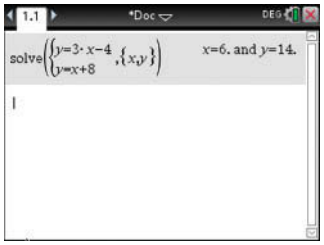
TI | THINK

a. 1. On a Calculator page, press MENU and then
3: Algebra
7: Solve System of Equations
1: Solve System of Equations
Complete the fields as shown.
Press ENTER.
Complete the simultaneous equations template as:
$y = 3x - 4$
$y = x + 8$
Press ENTER.

2. The answer appears on the screen.

WRITE





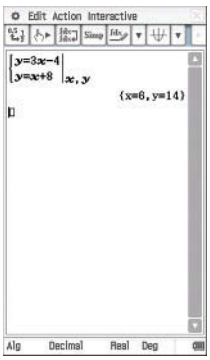
The solution is (6, 14).

CASIO | THINK

a. 1. On the MAIN screen, open the keyboard and locate the simultaneous template in Math1.
Insert the first equation in the top box. Insert the second equation in the bottom box.
Insert x, y in the box on the right.
Press EXE.

2. The answer appears on the screen.

WRITE



The solution is (6, 14).

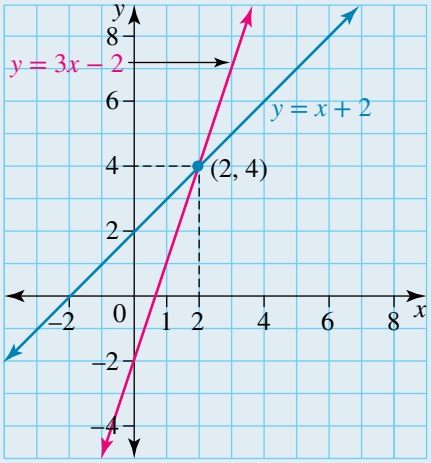
WORKED EXAMPLE 10

Find the simultaneous solution of $y = 3x - 2$ and $y = x + 2$ using a graphical method.

THINK

1. Draw both graphs on the one set of axes.

WRITE/DRAW



2. The simultaneous solution is the point where the graphs intersect.

The point of intersection is (2, 4) so the simultaneous solution is (2, 4) or $x = 2$, $y = 4$.

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Units 3 & 4 > AOS M4 > Topic 1 > Concept 5

Algebraic solutions to simultaneous equations Summary screen and practice questions

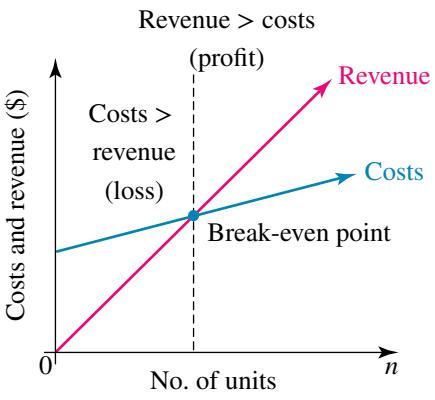
14.4.2 Break-even analysis

The aim of most businesses is to make a profit. The profit depends on the costs associated with the business (labour, raw materials and plant) and its revenue (the money it earns through sales). It represents the difference between the revenue and the costs.

Profit = revenue − costs

It is evident that a profit will occur if the revenue exceeds the costs. However, if the costs exceed the revenue, a loss will result. Finally, if the costs equal the revenue, there will be neither a profit nor a loss. This is referred to as a **break-even point**.

The diagram shows the graph of a cost function and a revenue function, drawn on the same set of axes. The point of intersection of the two lines represents the point at which costs and revenue are equal; that is, the break-even point. To the left of the break-even point, the cost line is above the revenue line. This means that the costs are higher than the revenue and will result in a loss. To the right of the break-even point, the cost line is below the revenue line. This means that costs are lower than the revenue and will result in a profit.



WORKED EXAMPLE 11

The cost associated with publishing a particular maths book (C) is given by

$$C = 12n + 24\,000$$

where n represents the number of books. The revenue (R) made from selling n books is given by $R = 28n$. Both C and R are in dollars.

a. Copy and complete the following table.

Number of books (n)	Costs ($)	Revenue ($)
500		
1000		
1500		
2000		

b. Sketch the graph of the costs (C) versus number (n), and the graph of revenue (R) versus number (n) on the same set of axes.

c. How many books need to be published and sold so that the revenue equals the costs?

d. State the coordinates of the break-even point and interpret its meaning.

THINK

a. 1. Write the equations for the costs (C) and the revenue (R).

2. Calculate the value of C and R for each value of n given in the table.

3. Use the results from the calculations in step 2 to complete the table.

WRITE/DRAW

a. $C = 12n + 24\,000$ $\qquad R = 28n$

When $n = 500$,
$C = 12 \times 500 + 24\,000$ $\qquad R = 28 \times 500$
$= \$30\,000$ $\qquad = \$14\,000$

When $n = 1000$,
$C = 12 \times 1000 + 24\,000$ $\qquad R = 28 \times 1000$
$= \$36\,000$ $\qquad = \$28\,000$

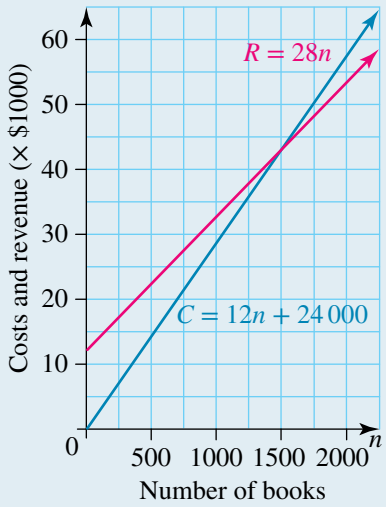
When $n = 1500$,
$C = 12 \times 1500 + 24\,000$ $\qquad R = 28 \times 1500$
$= \$42\,000$ $\qquad = \$42\,000$

When $n = 2000$,
$C = 12 \times 2000 + 24\,000$ $\qquad R = 28 \times 2000$
$= \$48\,000$ $\qquad = \$56\,000$

Number of books (n)	Costs ($)	Revenue ($)
500	30 000	14 000
1000	36 000	28 000
1500	42 000	42 000
2000	48 000	56 000

b. The number of books (n) is the explanatory variable, so place it on the horizontal axis. The graphs of the cost and revenue functions are straight lines, so they can be constructed by plotting any two points from the table then joining them with a straight line.

b.



c. From the table (or the graph), we can see that when $n = 1500$, the costs and the revenue are both the same.

c. 1500 books need to be published and sold so that the revenue equals the costs.

d. The break-even point is the point where the costs equal the revenue. On the graph, it is the point of intersection of the two lines.

d. The coordinates of the break-even point are (1500, 42 000). The cost associated with publishing and the revenue made from selling 1500 books is $42 000. If less than 1500 books are sold, the costs are higher than the revenue (loss). If more than 1500 books are sold, the revenue is higher than the costs (profit).

Note that the break-even point in the previous worked example could have been found using CAS or algebraically.

on Resources

- **Interactivity:** Solving simultaneous equations using substitution (int-6453)
- **Interactivity:** Break-even points (int-6454)

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Units 3 & 4 > AOS M4 > Topic 1 > Concepts 4 & 6

Graphical solutions to simultaneous equations Summary screen and practice questions

Break-even analysis Summary screen and practice questions

Exercise 14.4 Simultaneous equations and break-even point

1. **WE9** Solve the following simultaneous equations.

$$y = 10x - 7$$
$$y = 2x + 1$$

2. Solve the following simultaneous equations.

$$6x - 11y = 2$$
$$5x - 9y = 1$$

3. Find the simultaneous solution, algebraically, to:
   a. $y = 4x - 4$, $y = 2x$
   b. $y = 3x - 4$, $y = 2x - 2$
   c. $2y + x = 5$, $3y - 2x = 4$
4. **WE10** Find the simultaneous solution of $y = 2x - 3$ and $y = -x$ using a graphical method.
5. Find the simultaneous solution of $y = -2x - 3$ and $y = x + 3$ using a graphical method.
6. Find the simultaneous solution of each pair of linear equations in question **3** using a graphical method.
7. Find the simultaneous solution of $y = 3x - 10$ and $x = 2y + 5$ using the substitution method.
8. Find the simultaneous solution of $3x - y = -8$ and $2x + 2y = 0$ using the elimination method.
9. Find the simultaneous solution of $y = x$ and $y = -x$ using a graphical method.
10. Find the simultaneous solution of $-3x + 2y = 1$ and $-2x + 3y = 9$ using the elimination method.
11. **WE11** The cost of manufacturing toys (C) is related to the number of toys produced (n), by the formula $C = 600 + 3n$. The revenue (R) made from selling n toys is $R = 7n$. Both C and R are in dollars.
    a. Copy and complete the following table.

Number of toys	Cost ($)	Revenue ($)
50		
100		
150		
200		

    b. Sketch the graph of cost (C) versus number (n) and the graph of revenue (R) versus number (n) on the same set of axes.
    c. How many toys need to be produced before revenue equals cost?
    d. State the coordinates of the break-even point and interpret its meaning.
12. The cost of manufacturing basketballs (C) is related to the number of basketballs produced (n) by the formula $C = 2800 + 4n$. The revenue (R) made from selling n basketballs is $R = 14n$.
    a. Copy and complete the table below.

Number of basketballs	Cost ($)	Revenue ($)
150		
200		
250		
300		
350		

b. Using the information supplied, construct the graph of cost (C) versus number (n), and the graph of revenue (R) versus number (n), on the same set of axes.

c. Using the graph, write the number of basketballs produced before revenue equals cost to 'break even'.

13. Consider the phone connection plans shown at right.

Plan A: $15 monthly fee plus 60 cents a minute for calls.
Plan B: $25 monthly fee plus 30 cents a minute for calls.

a. Copy and complete the following table.

Call time (minutes)	Cost ($) — Plan A	Cost ($) — Plan B
10		
20		
30		
40		

b. Sketch the graph of cost versus time for each of the two plans on the same set of axes.

c. How many minutes of calls would you need to make for both plans to cost the same?

14. The cost of manufacturing electronic components (C) is related to the number of components produced (n), by the formula $C = 6000 + 2.5n$. The revenue (R) made from selling n components is $R = 4.5n - 8000$. Both C and R are in dollars.

a. Copy and complete the following table.

Number of components	Cost ($)	Revenue ($)
5000		
10 000		
15 000		
20 000		

b. Sketch the graph of cost (C) versus number (n) and the graph of revenue (R) versus number (n) on the same set of axes.

c. How many components need to be produced before revenue equals cost?

15. A new employee, whose job it is to sell a software package, is offered two different salary plans by Minitech:
Plan A: $400 per week plus $25 for each package sold
Plan B: $150 per week plus $45 for each package sold.

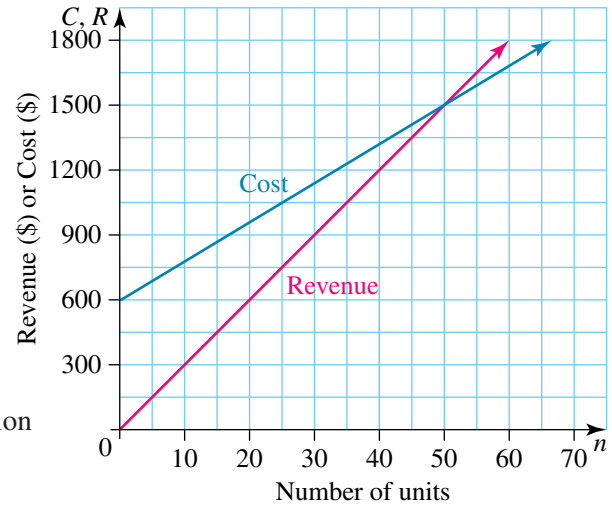
a. Copy and complete the following table.

Number of packages sold	Salary ($) Plan A	Salary ($) Plan B
5		
10		
15		
20		

b. Sketch the graph of salary versus number of packages sold for both Plan A and Plan B on the same set of axes.

c. How many packages need to be sold before Plan B is the better choice?

16. MC The statement that best matches the graph is:



- **A.** revenue exceeds costs when more than 50 units are sold
- **B.** revenue exceeds costs when less than 50 units are sold
- **C.** revenue exceeds costs when more than 1500 units are sold
- **D.** revenue exceeds costs when less than 1500 units are sold
- **E.** revenue exceeds costs when exactly 50 units are sold.

17. A factory producing mattresses finds that the equation linking cost in dollars (C) and the number of mattresses produced (m) is $C = 800m + 100\,000$.

- **a.** What is the cost of producing 10 mattresses?
- **b.** How many mattresses could be produced for $124 000?

18. From question **17**, if all mattresses are sold, the revenue in dollars (R) from the sale of m mattresses is $R = 1800m$.

- **a.** Draw graphs of C and R on the same set of axes for $0 \leq m \leq 200$.
- **b.** Determine the number of mattresses which would need to be sold for the factory to break even.
- **c.** From the two equations given, write a profit equation (P).
- **d.** Calculate the profit if 175 mattresses were produced and sold.

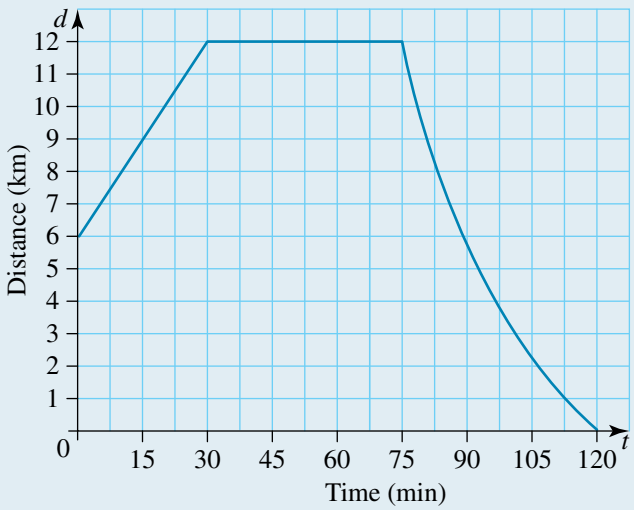
14.5 Interpreting non-linear graphs

Why are graphs used so widely in papers, magazines, journals, in education, in government, and in sales and marketing? Graphs are used because they have the capacity to convey a significant amount of information effectively.

To get the most from a graph, a user needs to learn to read and interpret information presented graphically. In this section, we will look at **non-linear graphs**. A non-linear graph is a graph which is not a straight line.

WORKED EXAMPLE 12

The graph shows the distance a cyclist is from her home over a period of 120 minutes.



- **a.** **How far from her home was the cyclist at the start of the time period?**
- **b.** **At what speed did the cyclist travel for the first 30 minutes?**
- **c.** **Describe the motion of the cyclist after 45 minutes.**
- **d.** **What was the cyclist's furthest distance from home?**
- **e.** **When did the cyclist turn for home?**
- **f.** **How long did the cyclist take to get home on the way back?**

THINK

a. The start of the time period means $t = 0$ min. The graph shows a distance of 6 km when $t = 0$.

b. Use the relationship speed $= \frac{\text{distance}}{\text{time}}$.
The cyclist travels from 6 km to 12 km in 30 minutes or 0.5 hour.

c. Between $t = 30$ min and $t = 75$ min the distance from home remains at 12 km.

d. Observe the furthest distance from home from the graph.

e. From the graph it can be seen that the distance the cyclist is from home decreases when $t = 75$ min.

f. The cyclist began travelling home when $t = 75$ min. The cyclist reached home when $t = 120$ min.

WRITE

a. The cyclist was 6 km from home at the start of the time period.

b. Speed $= \frac{\text{distance}}{\text{time}}$
$= \frac{6\text{ km}}{0.5\text{ h}}$
$= 12$ km/h

c. At 45 minutes into the time period, the cyclist is not moving away from or closer to her home.

d. The furthest distance from home is 12 km.

e. The cyclist turns for home after 75 min.

f. Time taken to travel home $= (120 - 75)$ min
$= 45$ min

WORKED EXAMPLE 13

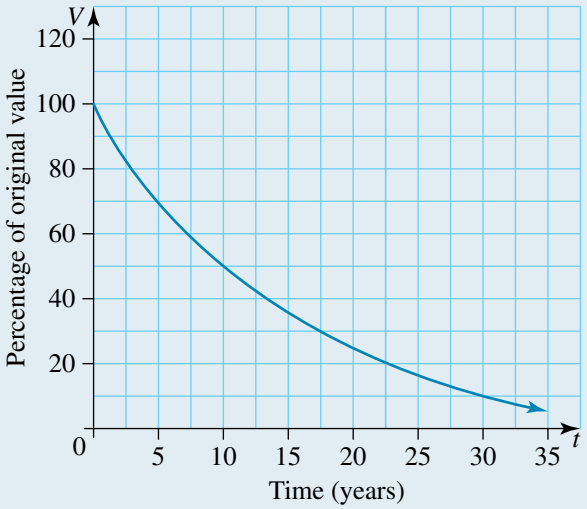
The value of a car originally worth $30 000 decreases over time. The graph describes the value of the car as a percentage of its starting price.

a. What is the value of the car, in percentage terms, after 5 years?

b. The 'half-life' is the time taken for the value to decrease by half. How long does it take for the car to lose half its original value?

c. How long does it take for the car to fall in value from 50% of its original value to 25% of its original value?

d. Estimate, in dollar terms, the value of the car after 30 years.



THINK

a. Read directly from the graph.

b. 'Half' means 50%. Reading from the graph, it takes 10 years for the value of the car to fall to 50%.

c. From the graph, the car is worth 50% of the original price at $t = 10$ and it is worth 25% at $t = 20$. Note that the half-life is also the time taken to fall from 50% to 25%.

WRITE

a. The value of the car after 5 years is approximately 70% of its original value.

b. The time taken to fall in value by 50% (half-life) is 10 years.

c. Time taken to fall from 50% to 25% of original value $= 20 - 10$ years
$= 10$ years

d. As the half-life is 10 years, use this to calculate the value after 30 years. That is, after 3 half-lives. Calculate 12.5% of $30 000 to obtain the value of the car in dollar terms.

d. Half-life = 10 years
Therefore, the value after:
10 years is 50%
20 years is 25%
30 years is 12.5%.
The value after 30 years
= 12.5% of $30 000
= 0.125 × $30 000
= $3750

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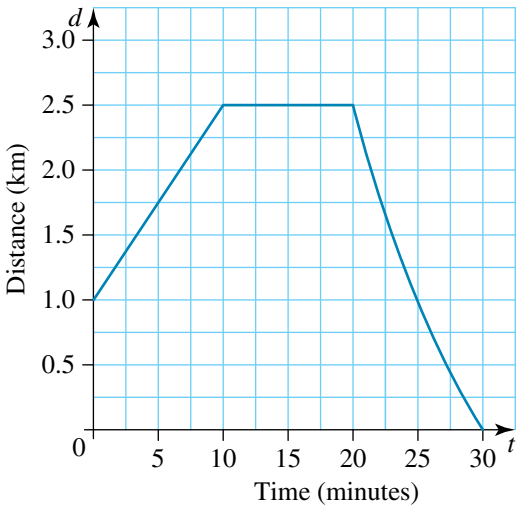
Units 3 & 4 > AOS M4 > Topic 1 > Concept 1 **Non-linear graphs** Summary screen and practice questions

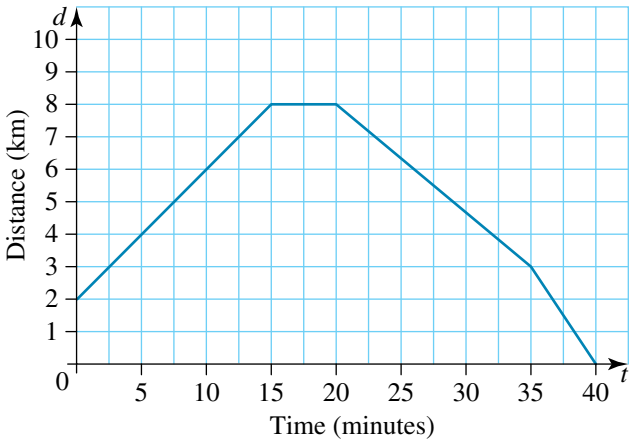
Exercise 14.5 Interpreting non-linear graphs

1. **WE12** The graph shows the distance a skateboarder is from the skate park over a period of 30 minutes.



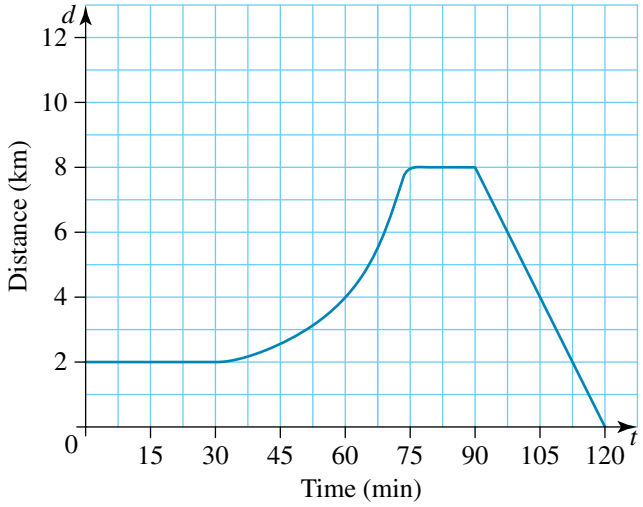
- **a.** How far from the skate park was the boarder at the start of the time period?
- **b.** At what speed did the boarder travel for the first 10 minutes?
- **c.** Describe the motion of the boarder after 15 minutes.
- **d.** What was the boarder's furthest distance from the skate park?
- **e.** When did the boarder start travelling towards the skate park?
- **f.** How long did it take the boarder to get back to the skate park on the way back?

2. The graph at right shows the distance a runner is from the finish line over a period of 40 minutes.

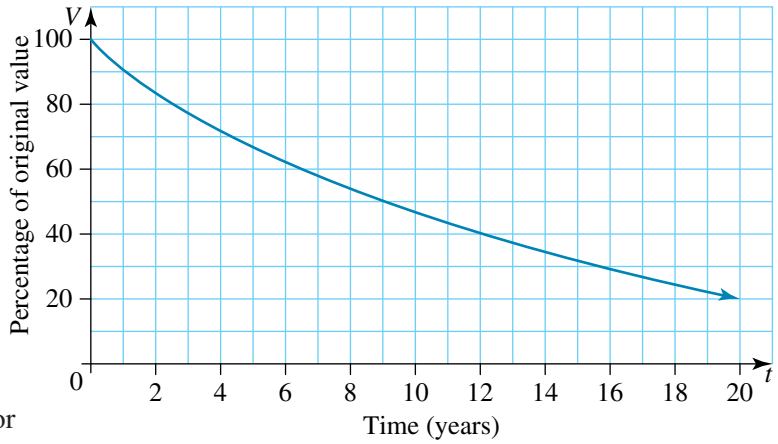


- **a.** How far from the finish line was the runner at the start of the time period?
- **b.** At what speed did the runner travel for the first 15 minutes?
- **c.** Describe the motion of the runner after 25 minutes.
- **d.** What was the runner's furthest distance from the finish line?
- **e.** When did the runner start travelling towards the finish line?
- **f.** How long did it take the runner to get back to the finish line on the way back?

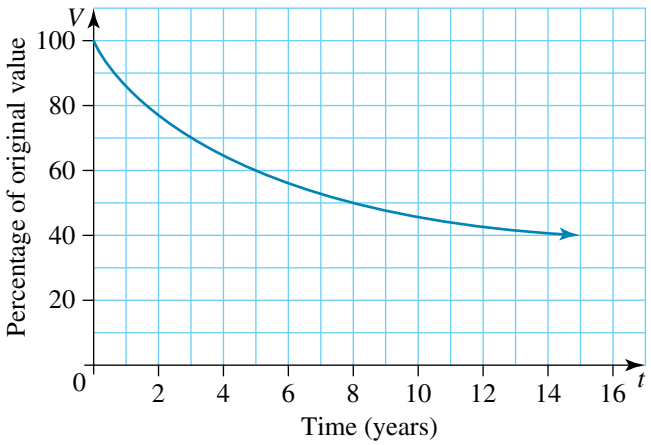
3. The graph shown gives the distance of a cyclist from her home over a period of 120 minutes.
   a. How far from her home was the cyclist at the start of the time period?
   b. At what speed did the cyclist travel for the first 30 minutes?
   c. Describe the motion of the cyclist after 7 minutes.
   d. What was the cyclist's furthest distance from home?
   e. When did the cyclist begin to travel home?
   f. How long did the cyclist take to get home?
   g. What was the average speed of the cyclist on the journey home?



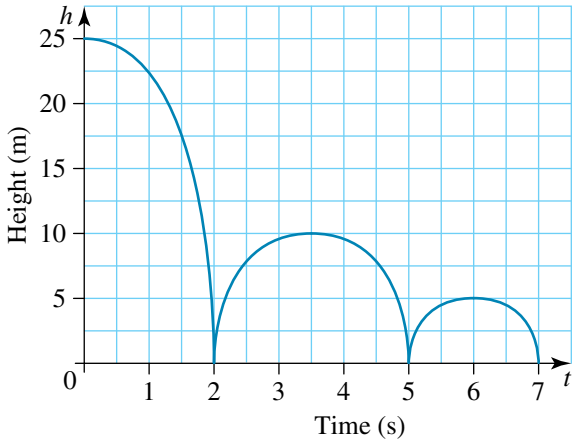
4. **WE13** The value of a ute originally worth $42 000 decreases over time. The graph describes the value of the car as a percentage of its starting price.
   a. What is the value of the ute, in percentage terms, after 5 years?
   b. The 'half-life' is the time taken for the value to decrease by half. How long does it take for the car to lose half its original value?
   c. How long does it take for the car to fall in value from 50% of its original value to 25% of its original value?
   d. Estimate, in dollar terms, the value of the car after 20 years.



5. The value of a boat originally worth $37 500 decreases over time. The graph drawn describes the value of the boat as a percentage of its starting price.
   a. What is the value of the boat, in percentage terms, after 3 years?
   b. The 'half-life' is the time taken for the value to decrease by half. How long does it take for the boat to lose half its original value?
   c. How long does it take for the boat to fall in value from 50% of its original value to 40% of its original value?
   d. Estimate, in dollar terms, the value of the boat after 15 years.

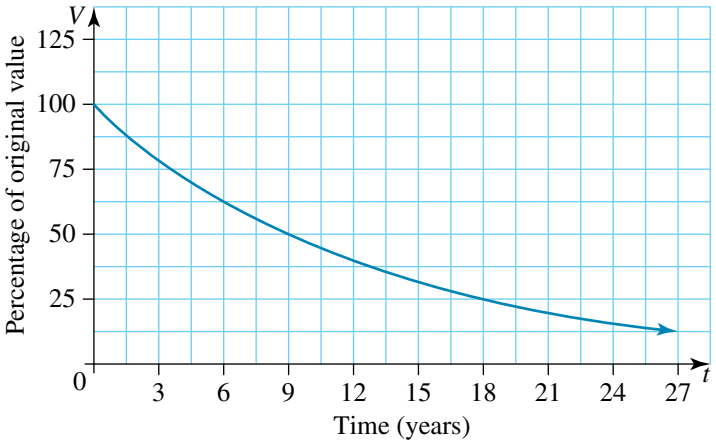


6. In an experiment, a ball is dropped from a height of 20 m. The height of the ball above ground level, as it bounces, is given in the graph.



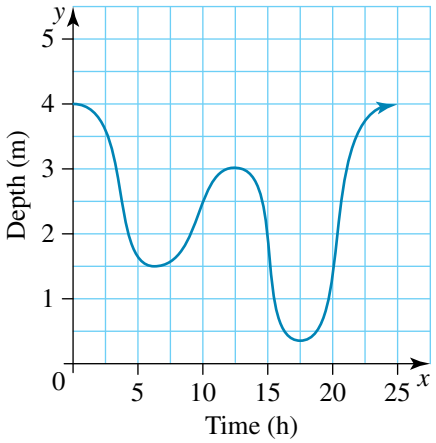
- **a.** How long did it take the ball to reach the ground for the first time?
- **b.** How long was the ball in the air between the second and third bounce?
- **c.** What total distance is travelled by the ball in 7 seconds?
- **d.** Calculate the average speed of the ball for the first 2 seconds.

7. The value of a car, originally worth $32 000, decreases over time. The graph describes the value of the car as a percentage of its original price.



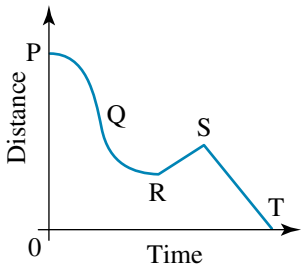
- **a.** What is the value of the car, as a percentage of its original price, after 6 years?
- **b.** The 'half-life' is the time taken for the value to decrease by half. How long does it take for the car to lose half its original value?
- **c.** How long does it take the car to fall in value from 50% of its original value to 25% of its original value?
- **d.** Estimate, in dollars, the value of the car after 27 years.

8. The depth of water across a sand bar varies according to the tide. This is shown in the graph.



- **a.** Approximately how long after high tide is the next low tide?
- **b.** How deep is the water at the first low tide?
- **c.** What is the lowest value in the depth of the water across the bar?
- **d.** What is the difference in the heights of the first and second high tide?
- **e.** If the first high tide is at 9.40 am Tuesday, at what time approximately will the first high tide occur on Wednesday?
- **f.** The water must be at least 1.5 m deep for Judy's boat to travel across the bar. Between what times, after the first high tide, can the boat *not* travel across the bar?

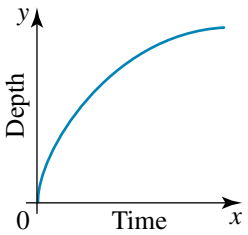
9. MC The graph shown describes the distance from home of an object as it travels.



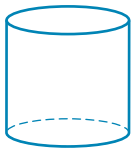
The point at which the object is moving with the greatest speed is:

A. P **B.** Q **C.** R **D.** S **E.** T

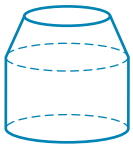
10. MC Water is poured into a container at a constant rate. The depth of water in the vessel is described in the graph. From the options following, the vessel that best matches the rate at which the vessel fills with water is:



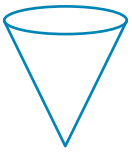
A.



B.

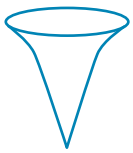


C.



D.

E.



11. Carbon-14 is a radioactive element which breaks down over time. It has a half-life of about 6000 years; that is, every 6000 years, half of the material present breaks down.

a. Copy and complete the table for the amount of carbon-14 (C-14) present.

Time (years)	0	6000	12 000	18 000
Amount of carbon-14 units	800			

b. Plot the graph using these data.

c. Use the graph to estimate the time taken for the amount of C-14 to fall to 600 units.

d. Use extrapolation to estimate the amount of C-14 present after 24 000 years.

12. The following table displays the profit/loss a company makes from selling x units of toothpaste.

Units of toothpaste sold (× 1000)	0	1	2	3	4	5
Profit (× $1000)	−5	−6	−5	−2	1	9

a. Plot the graph using these data.

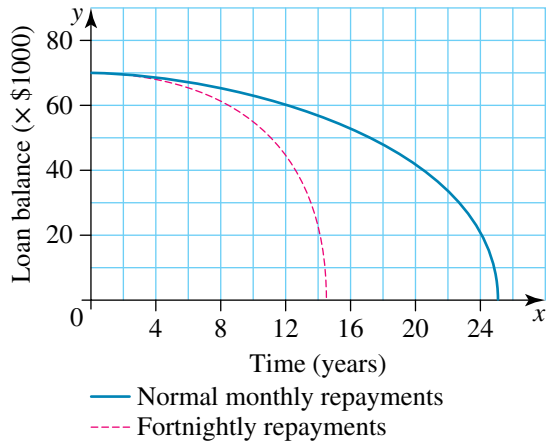
b. Use your graph to estimate how many units of toothpaste need to be sold in order to break even.

13. Yasmin borrows $70 000. She can pay a certain amount each month or half that value each fortnight. The graph shows the progress of the loan under both systems of payment.

a. How much time does it take to pay off the loan using monthly repayments?

b. How long does it take to pay off the loan using fortnightly repayments?

c. Using monthly repayments, after how many years has the balance fallen to 50% of its original value?

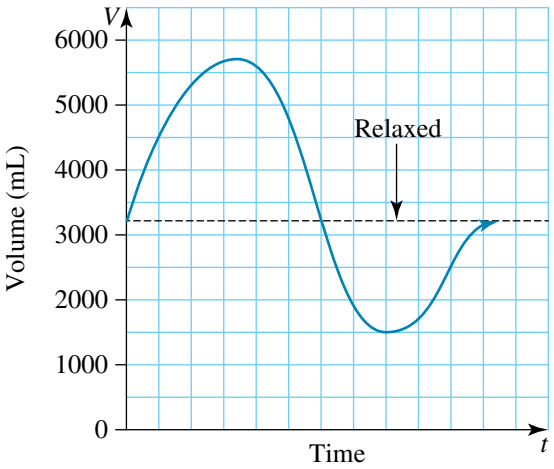


14. The graph shows the volume of air in a person's lungs during a cycle of breathing.

a. What is the maximum volume of the lungs?

b. When relaxed, how much air is contained in the lungs?

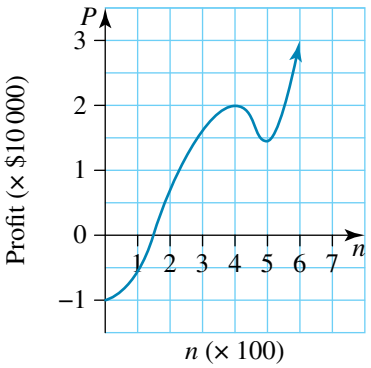
c. How much air is exhaled during the breathing cycle?



15. The graph at right shows the profits made by a manufacturing firm versus the number of units produced.

a. What profit is made when 400 units are produced?

b. What is the result if 0 units are produced?



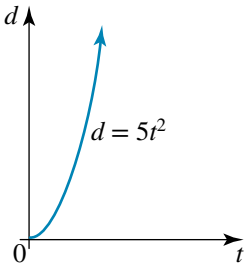
16. **a.** Using the graph from question **15**, how many units need to be produced to break even?

b. Can you suggest an explanation for the dip in profits in the region $n = 500$?

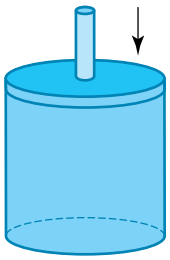
14.6 Constructing non-linear relations and graphs

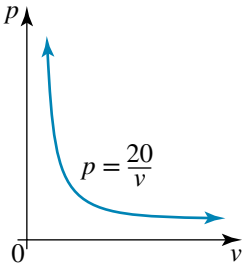
There are many important non-linear relationships where the graphs are not a straight line. Some examples follow.

A rivet is accidentally dropped from a tall pylon under construction. The relationship between the distance the rivet has fallen (d), and the time taken (t) is $d = 5t^2$.

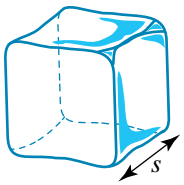


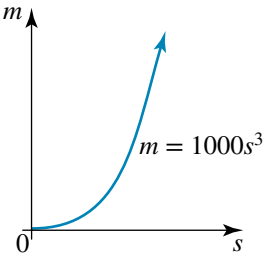
The gas in a piston is compressed. The relationship between the pressure (p) and the volume (v), is $p = 20v^{-1}$ or $p = \frac{20}{v}$.





The mass of a cubic block of ice (m) is related to the length of the side of the block (s) by the formula $m = 1000s^3$.

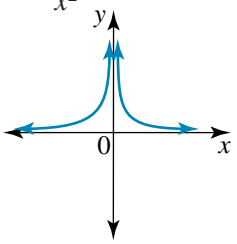




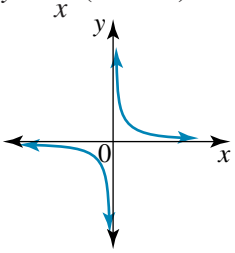
14.6.1 Graphical representation of relations of the form $y = kx^n$

Given a non-linear equation of the form $y = kx^n$, the easiest way to graph it is to use CAS. Alternatively, from the equation a table of values can be produced and this can be used to sketch a graph of the relationship. The general shape of the graphs of $y = kx^n$ for $n = -2, -1, 1, 2, 3$ are given here.

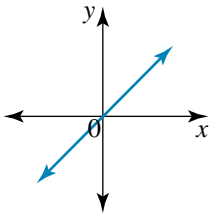
$y = \frac{k}{x^2}$ $(n = -2)$



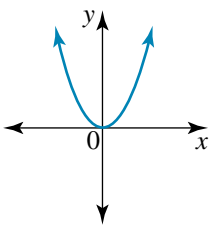
$y = \frac{k}{x}$ $(n = -1)$



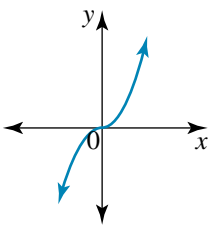
$y = kx$ $(n = 1)$



$y = kx^2$ $(n = 2)$



$y = kx^3$ $(n = 3)$



The value of k in these relationships is known as the **constant of proportionality**.

WORKED EXAMPLE 14

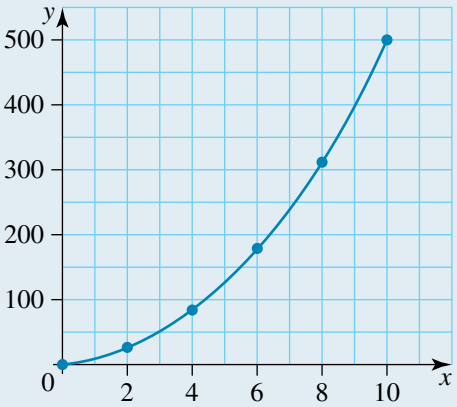
A research scientist has discovered that her data are related according to the equation $y = 5x^2$. Construct a table of values to draw the graph of the equation $y = 5x^2$ for x between 0 and 10.

THINK

1. Substitute values of x into $y = 5x^2$ to obtain corresponding y-values. For example, if $x = 4$ then
   $y = 5 \times 4^2$
   $= 5 \times 16$
   $= 80$.
2. Plot the points and draw a smooth curve through these points.

WRITE/DRAW

x	0	2	4	6	8	10
y	0	20	80	180	320	500



14.6.2 Finding a non-linear relationship using linear graphs

If you know the equation, it is reasonably straightforward to produce a table of values. The reverse process (that is, discovering a formula for a non-linear relationship from a table of values) is more difficult.

If the relationship between x and y is of the form $y = kx^n$, then plotting y against x^n will produce a straight line from the origin. The gradient of this line will equal the value of k, the constant of proportionality. The following algorithm can be used for finding the rule of the relationship, if the value of n is known.

1. Plot y against x^n. The result must be a straight line, coming from the origin.
2. Select any two points on the line to calculate the value of the gradient.
3. Since the gradient represents the value of k, substitute it into $y = kx^n$ to give the rule for the relationship.

Note that if the value of n is not known, plot y against x first. The shape of the graph will indicate the possible value(s) of n. Test this value by plotting y against x^n. If a straight line is produced, proceed as above; otherwise try a different value of n.

WORKED EXAMPLE 15

By graphing y against x^2 find the equation for the relationship between x and y if the equation is of the form $y = kx^2$.

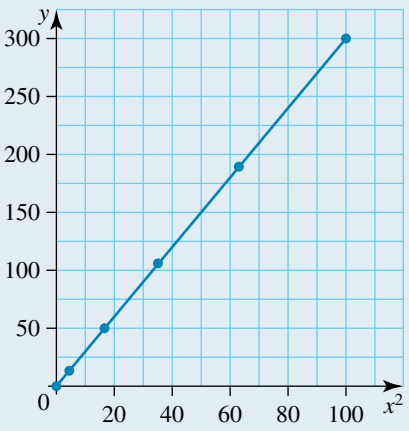
x	0	2	4	6	8	10
y	0	12	48	108	192	300

THINK

1. Since it is known that the relationship is of the form $y = kx^2$, draw up a table of values for x^2 and y.
2. Plot the values of x^2 on the horizontal axis and y on the vertical axis and join with a smooth line.
   (A straight line from the origin confirms that the relationship is of the form $y = kx^2$.)
3. Choose two points on the line, say (0, 0) and (100, 300), and calculate the gradient.
4. The gradient represents the value of k in $y = kx^2$. Replace k with 3 to state the equation for the relationship.

WRITE/DRAW

x^2	0	4	16	36	64	100
y	0	12	48	108	192	300



$$\text{Gradient} = \frac{300 - 0}{100 - 0}$$
$$= \frac{300}{100}$$
$$= 3$$

$y = kx^2$

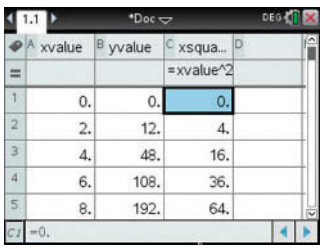
$k = 3$

Therefore, $y = 3x^2$.

TI | THINK

1. On a Lists & Spreadsheets page, enter the data into the Columns A and B and label the columns as *xvalue* and *yvalue*. Label Column C as *xsquared* and in the formula cell, complete the entry line as:
   $= xvalue^2$
   Press ENTER.

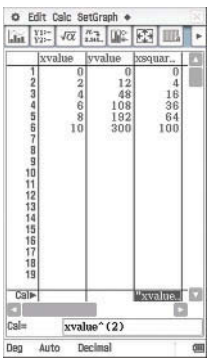
WRITE



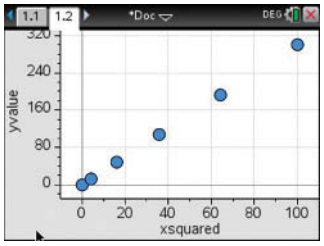
CASIO | THINK

1. On the Statistics screen, enter the data into List1 and List2 and rename the lists as *xvalue* and *yvalue*. Label List3 as *xsquared* and in the formula cell, complete the entry line as:
   $xvalue^2$
   Press EXE.

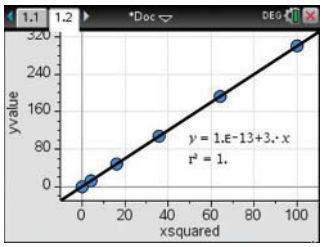
WRITE



2. Press Ctrl + I and select:
5: Data & Statistics
Move the cursor to the horizontal axis and select *xsquared*.
Move the cursor to the vertical axis and select *yvalue*.
The points produce a straight line, so this confirms that the relationship is of the form $y = kx^2$.



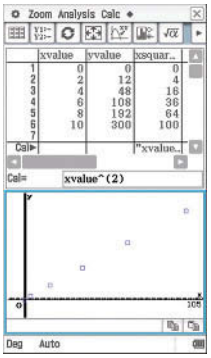
Press MENU
4: Analyze
6: Regression
2: Show Linear $(a + bx)$
The gradient b represents the value of k in $y = kx^2$.



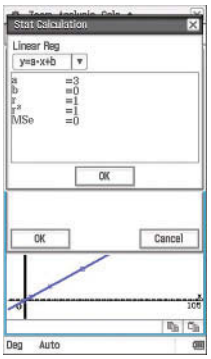
3. The answer appears on the screen.

$k = 3$
Therefore, $y = 3x^2$.

2. Tap:
SetGraph
Settings
Complete the Set StatGraphs fields as:
Draw: On
Type: Scatter
XList: main\xsquared
YList: main\yvalue
Freq: 1
Mark: square
Tap Set.
Tap the graph icon.



Tap:
Calc
Regression
Linear Reg
Complete the Set Calculation fields as:
XList: main\squared
YList: main\yvalue
Freq: 1
Copy Formulae: y1
Copy Residual: Off
Tap OK.
The gradient a represents the value of k in $y = kx^2$.



3. The answer appears on the screen.

$k = 3$
Therefore, $y = 3x^2$.

studyon

Units 3 & 4 > AOS M4 > Topic 1 > Concepts 9 & 10

***y* = *kx*^{n}** Summary screen and practice questions

Constant of proportionality Summary screen and practice questions

Exercise 14.6 Constructing non-linear relations and graphs

1. **WE14** A statistician found that his data are related according to the equation $y = 3x^2$. Construct a table of values and draw the graph of the equation $y = 3x^2$ for values of x between 0 and 10.
2. A researcher found that her data are related according to the equation $y = \frac{2}{x}$. Construct a table of values and draw the graph of the equation $y = \frac{2}{x}$ for values of x between 2 and 8.
3. Construct a table of values to draw the graphs of the following equations. (Use values of x from -5 to 5.)
   a. $y = x^2$
   b. $y = 2x^2$
   c. $y = 2x^3$
   d. $y = 0.5x^3$
   e. $y = \frac{0.5}{x}$
   f. $y = \frac{1}{x^2}$

   Use CAS to check your answers.

4. Construct a table of values to draw the graphs of the following equations. (Use values of x from -5 to 5.)

a. $y = 0.5x^2$ **b.** $y = x^3$ **c.** $y = \frac{1}{x}$ **d.** $y = \frac{2}{x}$

e. $y = \frac{2}{x^2}$ **f.** $y = \frac{0.5}{x^2}$

Use CAS to check your answers.

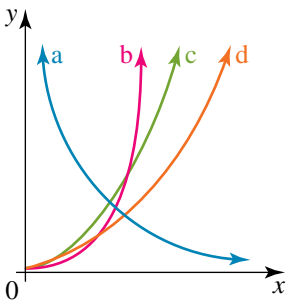
5. Write the letters **a** to **d** in your book. Against each letter that identifies a graph, write the equation which best matches that graph.

$y = x^2$

$y = 2x^2$

$y = x^3$

$y = \frac{3}{x}$



6. **WE15** By graphing y against x^2 find the equation for the relationship between x and y if the equation is of the form $y = kx^2$.

x	0	2	4	6	8	10
y	0	8	32	72	128	200

7. By graphing y against x^2, find the equation for the relationship between x and y if the equation is of the form $y = kx^2$.

x	0	2	4	6	8	10
y	0	10	40	90	160	250

8. By graphing y against x^3, find the equation of the relationship between x and y, if the equation is of the form $y = kx^3$.

x	0	2	4	6	8
y	0	12	96	324	768

9. The Safety Council conducted research on the braking distance of vehicles and its relationship to the speed of the vehicle. The following data were obtained.

Speed (s) (km/h)	30	45	60	80	100
Braking distance (d) (metres)	7.5	16.9	30	53.3	83.3

a. Plot d versus s^2.

b. What is the equation relating d and s?

10.

x	0.1	0.4	1	2	5
y	200	50	20	10	4

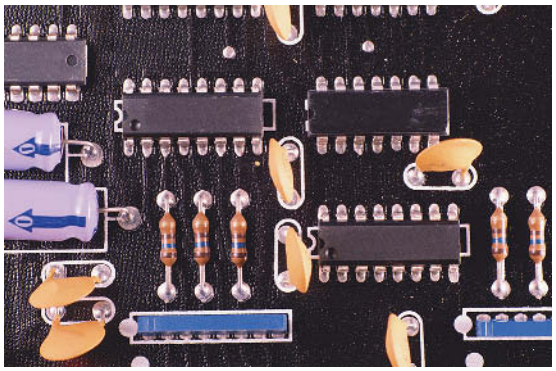
a. Plot these values on a set of axes.

b. Plot y versus $\frac{1}{x}$ and draw a straight line through the points.

c. What is the slope of the line?

d. Deduce the relationship between y and x.

11. In a physics experiment a student measured the current (I) flowing through a resistor for different values of the resistance (R) and obtained the following data.



Resistance (R) (ohms)	100	200	1000	1500
Current (I) (milliamps)	300	150	30	20

a. Plot values of I versus $\frac{1}{R}$.

b. Deduce a relationship between I and R.

12. Examine the following table of values.

x	0	2	4	6	8
y	0	1	8	27	64

a. Plot these values on a set of axes.

b. Plot y versus x^3 and draw a straight line through the points.

c. What is the slope of the line?

d. Deduce the relationship between y and x.

13. It is suspected that for male adults, mass (m) is related to height (h) by a formula like $m = kh^3$. Use the data in the table to find a relationship between m and h.

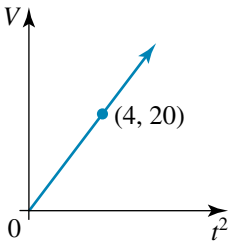
Mass (kg)	45	55	71	84
Height (cm)	150	160	175	185

14. The intensity of light drops off as you move away from the source. The relationship is of the form $I = \frac{k}{r^2}$. If the intensity (I) is 50 when $r = 20$, find:

a. the value of k

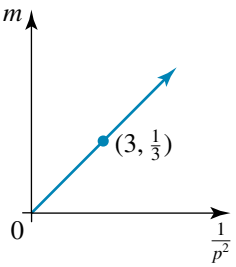
b. the distance at which the intensity of light falls below 35.

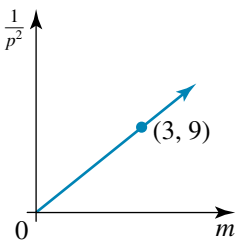
15. MC Which of the following equations describes the relationship shown on the graph at right?

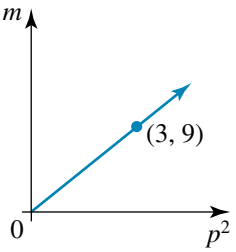


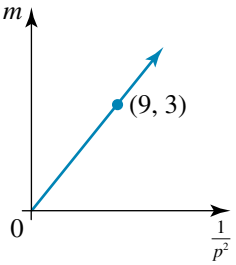
A. $V = 10t^2$
B. $V = 10t$
C. $V = 5t^2$
D. $V = 5t$
E. $V^2 = 5t$

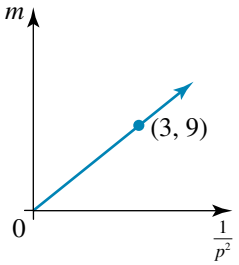
16. MC Which of the graphs shows the relationship $m = \frac{3}{p^2}$?

A.


B.


C.


D.


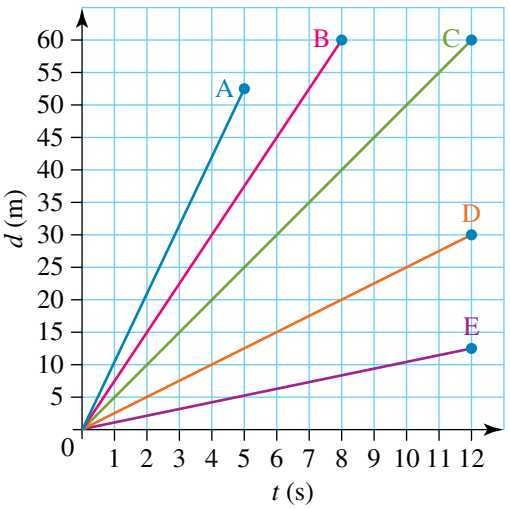
E.


14.7 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

Questions **1** to **3** relate to the following information. Karl and Lisa are travelling with a group of friends along the footpath and roadside at a constant speed. Lisa starts with Karl but, riding a bike at a constant speed, she moves faster than Karl. After 4 seconds, Lisa has moved 30 meters more than Karl.



1. MC The line that represents Karl is:
A. Line A **B.** Line B
C. Line C **D.** Line D
E. Line E

2. MC The line that represents the fastest person is:
A. Line A **B.** Line B
C. Line C **D.** Line D
E. Line E

3. MC The speed represented by line C is:
A. 60 m/s **B.** 12 m/s **C.** 5 m/s
D. 1 m/s **E.** 0.2 m/s

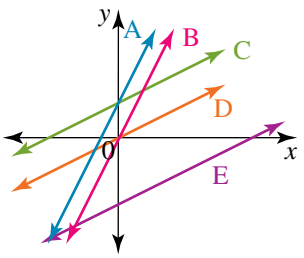
4. MC Match each of the following equations to the graph that fits best.

a. $y = 2x$

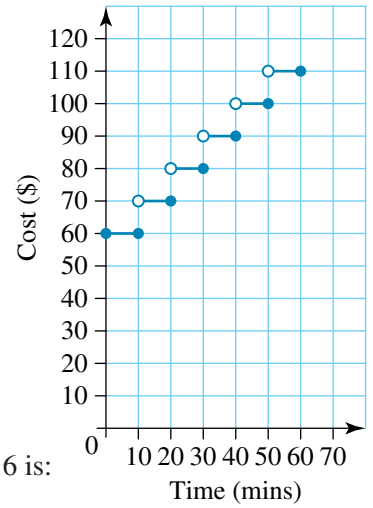
b. $y = 2x + 2$

c. $2y = x$

d. $2y = x + 4$



5. MC A plumber charges a call-out fee plus an amount for each 10 minutes or part thereof needed to complete a job. The graph below shows the cost charged for jobs up to 60 minutes in length.



a. The cost charged by the plumber for a job taking 20 minutes is:

A. \$60 **B.** \$70 **C.** \$80 **D.** \$90 **E.** \$100

b. The cost charged by the plumber for a job taking 35 minutes is:

A. \$80 **B.** \$85 **C.** \$90 **D.** \$95 **E.** \$100

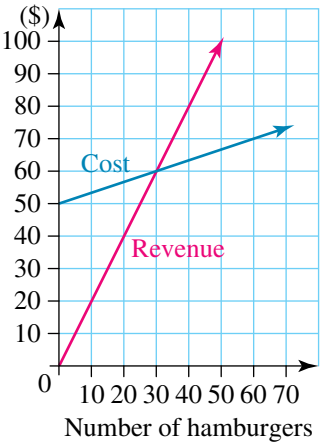
c. The call-out fee is:

A. \$40 **B.** \$50 **C.** \$60 **D.** \$70 **E.** \$80

6. MC The solution to the simultaneous equations $y = 5x - 3$ and $y = 2x + 6$ is:

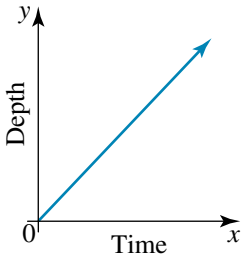
A. (1, 2) **B.** (2, 10) **C.** (2, 7) **D.** (3, 12) **E.** (3, 4)

7. MC The graph shows the relationships for revenue and cost against the number of hamburgers sold at a school fete.
The number of hamburgers that need to be sold to break even is:

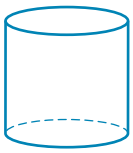


A. 0 **B.** 20 **C.** 30 **D.** 40 **E.** 50

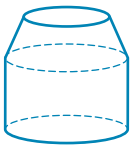
8. MC Water is poured into a container at a constant rate. The graph shows the depth of water in the container versus time. The container below that best matches this graph is:



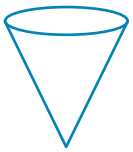
A.



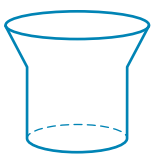
B.



C.

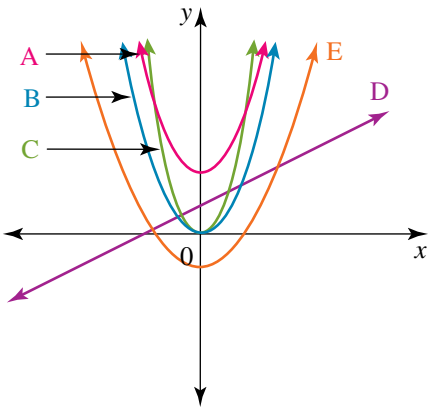


D.



E.

Questions **9** to **11** relate to the graph that shows the following (in random order):
$y = x^2$, $y = 2x^2$, $y = x^2 + 2$, $y = x^2 - 1$ and $y = x + 1$.



9. MC Which graph best matches the equation $y = x^2$?

10. MC Which graph best matches the equation $y = x^2 + 2$?

11. MC Which graph best matches the equation $y = 2x^2$?

12. MC A publisher produces a restaurant guide each year. To produce x copies, the cost is C dollars, where $C = 15\,000 + 15x$. If all of the x copies produced are sold, then the revenue gained is R dollars, where $R = 25x$. Which of the following statements is not true?

A. The cost and revenue equations are linear.

B. The selling price for each copy of the guide is $25.

C. It will cost $30 000 to produce 1000 copies of the guide.

D. The revenue from selling 1000 copies is $15 000.

E. The revenue is more than the cost if 1600 copies of the guide are sold.

Short answer

1. Bruno is an insurance salesman. He is paid $150 per week plus $120 for each insurance policy he sells.

a. How much does Bruno receive if he sells 3 policies in a week?

b. One week Bruno is paid $630. How many policies did he sell in that week?

c. Write a rule that relates Bruno's weekly pay (P) and the number of policies he sells (n).

d. Draw a graph of P versus n.

2. For the equation of the line $y = 3x - 3$:

a. what is the value of y when $x = 4$?

b. what is the value of x when $y = 18$?

c. does the point $(0, -3)$ lie on the line?

d. Using the two points whose coordinates were found in **a** and **b**, calculate the gradient.

3. Sketch the graphs of the following functions between $x = -5$ and $x = 5$.

a. $y = 4x - 3$ **b.** $y + 2x = 6$ **c.** $y = 2x + 2$ **d.** $2y - x = 12$

4. A car salesperson is paid commission at the rate of 1.75% on the first $40 000 worth of cars that he or she sells in a month, and 1.2% on the remainder.

a. Copy and complete the following table by calculating the commission due each month.

Sales	Commission
$20 000	
$30 000	
$40 000	
$50 000	
$60 000	

b. Using the data in the table, draw a graph of commission versus sales.

5. For each of the following pairs of simultaneous equations, find the solution using algebraic methods.

a. $y = 4x - 2$
$y = 3x$

b. $2y + x = 4$
$3y - 2x = 4$

6. The cost of manufacturing surfboards depends on labour, plant and raw materials. The monthly cost (C) is related to the number produced (n) by the equation $C = 4000 + 280n$.
The revenue (R) made from selling n surfboards is $R = 450n$.
Both C and R are in dollars.

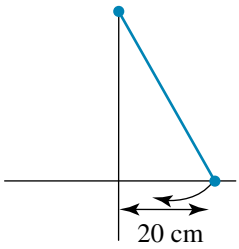
a. Copy and complete the following table.

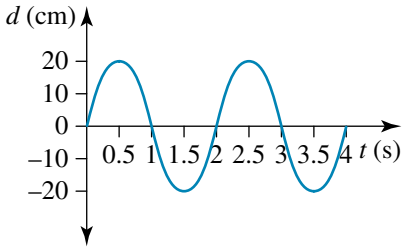
Number of surfboards made	Cost ($)	Revenue ($)
10		
20		
30		
40		

b. Sketch the graph of cost (C) versus number (n) and the graph of revenue (R) versus number (n) on the same set of axes.

c. How many surfboards need to be produced before the break-even point is reached?

7. A pendulum swings back and forth as shown in the diagram. The distance the bob of the pendulum is from its rest position is given as a function of time in the graph.





a. How long does it take the pendulum to return to its starting position?

b. Give the position of the pendulum after 1.5 seconds.

c. Predict the position of the pendulum after 22.5 seconds.

8. It is known that y is related to x by the equation $y = kx^2$.

a. Graph y versus x^2.

b. Deduce the exact relationship of y versus x.

x	2	4	5
y	12	48	75

Extended response

1. One day, Rachel and Nathan decided to travel to Mt Buffalo. For the first three hours they were driving at an average speed of 50 km/h, while for the last two hours they managed to increase their average speed to 60 km/h.
   a. Copy and complete the table below to show Rachel and Nathan's distance from home at various times of their journey.

Number of hours travelled (t)	Distance (km) from home (D)
1	
2	
3	
4	
5	

   b. Use the data from the table to represent the trip graphically.
   c. Name the type of graph formed in part **b**.
   d. Find the gradient and the y-intercept of the first section of the graph and hence, state the equation that relates time and distance travelled over the first 3 hours of the trip.
   e. Interpret the value of the gradient and the y-intercept in part **d**.
   f. Explain why the second section of the graph is steeper than the first.
   g. State the distance of Mt Buffalo from Rachel and Nathan's home.

2. Alex owns a small cake shop. To attract more customers, he decides to start making and selling coffee in addition to his cakes. The total cost of an espresso machine, cups and spoons adds up to $7500. In addition to this, Alex estimates that it will cost him an average of 20 cents to make each cup of coffee. He intends to sell the coffee at $2.70 per cup.
   a. Find a function that gives the total cost $C\,(\text{in } \$)$ of making n cups of coffee.
   b. Find a function that gives the revenue $R\,(\text{in } \$)$ from selling n cups of coffee.
   c. Draw the graphs of the cost and the revenue functions on the same set of axes.
   d. Find the coordinates of the point of intersection of the two graphs using algebraic methods.
   e. Interpret the meaning of the point of intersection.
   f. What is the smallest number of cups of coffee that Alex needs to make and sell to start making a profit?
   g. Write the function for the profit $P\,(\text{in } \$)$ in terms of n.
   h. How many cups of coffee does Alex have to make and sell in order to make a profit of $2000?

3. In business, the rule of 72 is used to predict the growth of an investment. The rule states that if the growth rate of an investment is $r\%$ p.a., then the time taken for the investment to double is $72 \div r$ years.
It is estimated that a share portfolio, initially worth \$75 000, will grow at an annual rate of 9%.

a. Use the rule of 72 to predict the value of the investment for the following years.

Time (years)	0	8	16
Value	\$75 000		

b. Plot the data and draw a smooth curve through the points.

c. Use this graph to estimate the value of the investment after 10 years.

d. What do you predict the value of the investment to be in 40 years' time?

4. Deduce the relationship between y and x.

x	0	2	4	6	8
y	0	12	96	324	768

(*Hint*: It is of the form $y = kx^n$.)

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Units 3 & 4 Sit exam

Answers

Topic 14 Construction and interpretation of graphs

Exercise 14.2 Constructing and interpreting straight-line graphs

1. **a.** 13 **b.** 6 **c.** No **d.** $m = 2, c = 5$

2. **a.** $y = 0$ **b.** $x = 4$ **c.** Yes

3. **a.** 0 **b.** 4 **c.** Yes **d.** $m = -2, c = 4$

4. **a.** 3 **b.** 0 **c.** No **d.** $m = \frac{-3}{2}, c = 6$

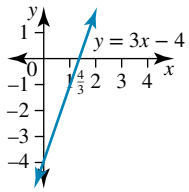
5. 4

6. -1

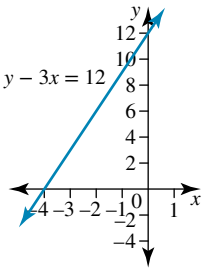
7. **a.** 3 **b.** 6 **c.** No **d.** -0.5

8. **a.** 0.5 **b.** 2 **c.** 1.5 **d.** 2

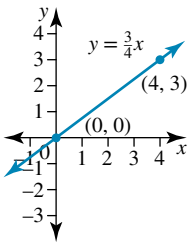
9. **a.**



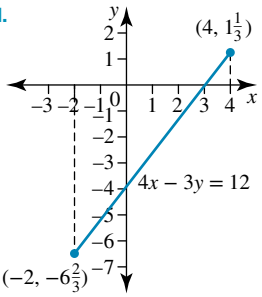
b.



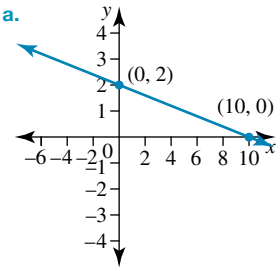
c.



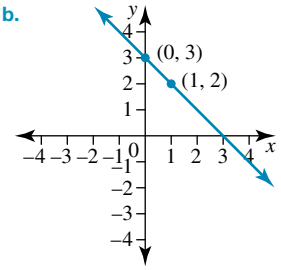
d.



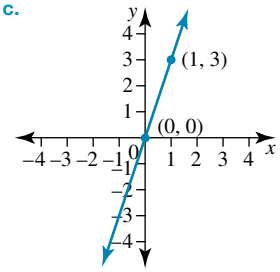
10. **a.**



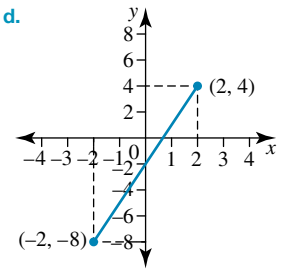
b.



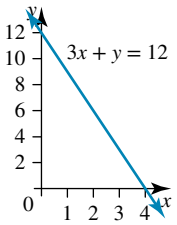
c.



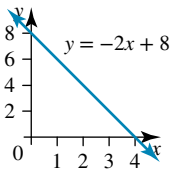
d.



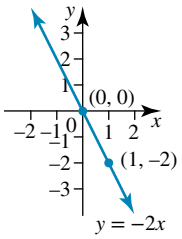
11. **a.**



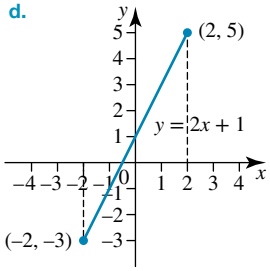
b.



c.



d.



12. E

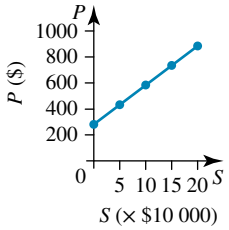
13. **a.** \$635

b.

Weekly sales	Weekly pay
5000	\$425
10 000	\$575
15 000	\$725
20 000	\$875

c. $P = 275 + 0.03S$

d.



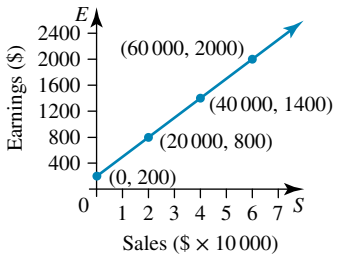
e. $m = 0.03$; y-intercept $= 275$; retainer amount

14. **a.**

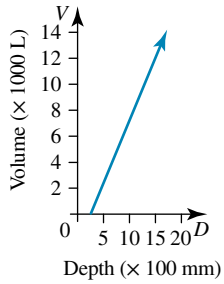
Sales	0	\$20 000	\$40 000	\$60 000
Earnings	\$200	\$800	\$1400	\$2000

b. $E = 200 + 0.03S$

c.



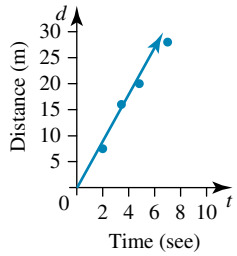
15. a.



b. 1550 mm

c. 3500 L

16. a.



b. $d = 36$ m

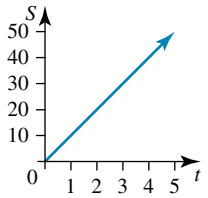
c. $d = 24$ m

17. a. 29.4 m/s

b. 8 s

c. $S = 9.8t$

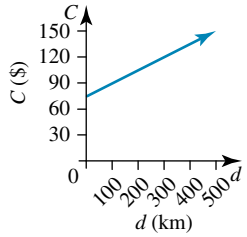
d.



18. a. $126.75

b. 450 km

c.



19. B

20. a. $470

b. $W = 350 + \frac{3S}{25}$

21. a. 4.73

b. $S = 108+(n-20)\times 4.73$

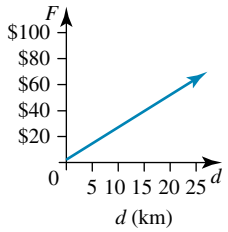
c. 4.73

22. a. $40.30

b. 7.2 km

c. $F = 2.8 + 2.5d$

d.



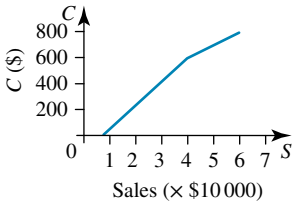
e. 2.5

Exercise 14.3 Line segments and step functions

1. a.

Sales ($)	Commission ($)
20 000	300
30 000	450
40 000	600
50 000	700
60 000	800

b.



c. Different commission rates mean different gradients.

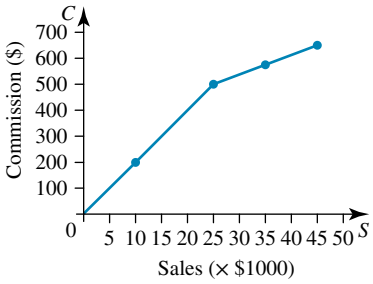
2. a. $200

b. $500

c. $580

d. $660

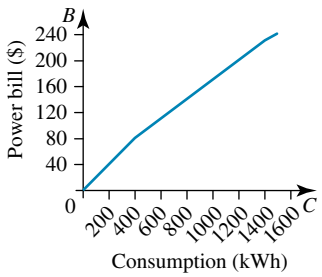
e.



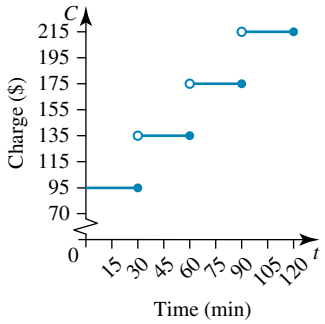
3.

Consumption (kWh)	Power bill
200	$40
400	$80
600	$110
1000	$170
1500	$240

4.



5. a.



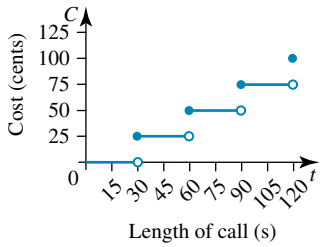
b.

Time	Charge
12 min	$55 + $40 = $95
23 min	$55 + $40 = $95
44 min	$55 + 2 × $40 = $135
56 min	$55 + 2 × $40 = $135
73 min	$55 + 3 × $40 = $175
87 min	$55 + 3 × $40 = $175

6. a.

Length of call (seconds)	Cost (cents)
15	0
30	25
45	25
60	50
75	50
90	75
105	75
120	100

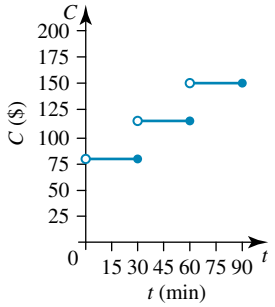
b.



7. a.

Time of service (minutes)	Cost
20	$80
30	$80
45	$115
60	$115
80	$150
90	$150

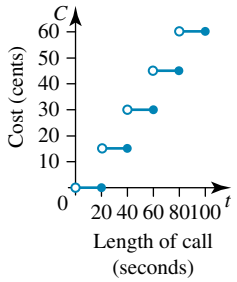
b.



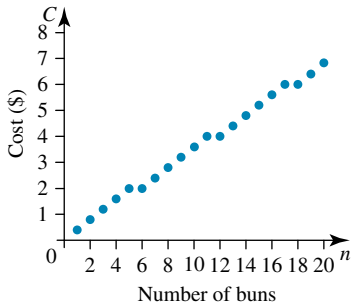
8. a.

Length of call (seconds)	Cost (cents)
10	0
20	15
30	15
40	30
50	30
60	45
70	45
80	60
90	60

b.



9. a.



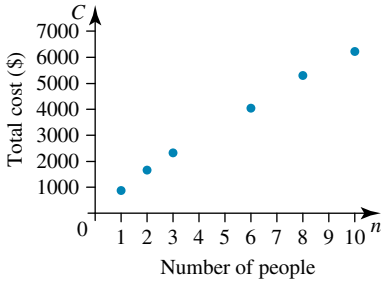
b.

Number of buns	Cost
2	2 × 40c = $0.80
4	4 × 40c = $1.60
6	$2.00
8	$2.00 + 2 × 40c = $2.80
10	$2.00 + 4 × 40c = $3.60

10. a.

Number of people	Total cost
1	$900
2	$1650
3	$2250
6	$4050
8	$5250
10	$6450

b.

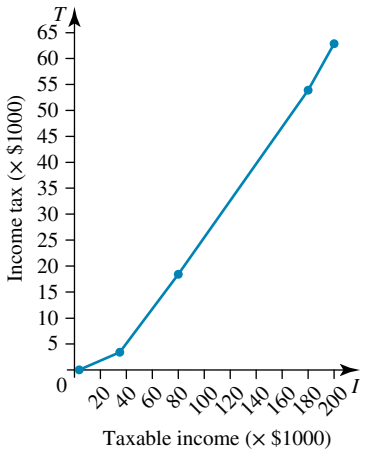


11. B

12. B

13. a. $0 **b.** $150 **c.** $2100
d. $4350 **e.** $5550 **f.** $11 550
g. $24 950 **h.** $63 550

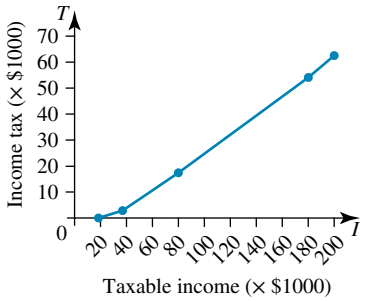
14. a.



b. i. $2700 **ii.** $23 100

15. a. $0 **b.** $0 **c.** $342
d. $3192 **e.** $4547 **f.** $11 047
g. $24 632 **h.** $63 232

16. a.

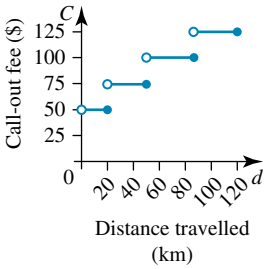


b. i. Approx $1100
ii. Approx $23 000
c. Answers will vary.

17. Answers will vary.

18. a. i. $75
ii. 20 km

b.



Exercise 14.4 Simultaneous equations and break-even point

1. (1, 3)

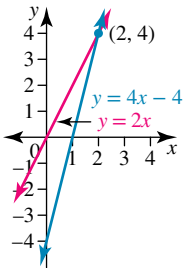
2. (−7, −4)

3. a. (2, 4) **b.** (2, 2) **c.** (1, 2)

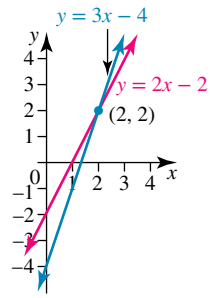
4. (1, −1)

5. (−2, 1)

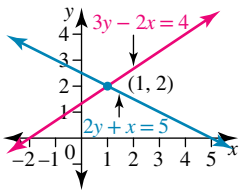
6. a.



b.



c.



7. (3, −1)

8. (−2, 2)

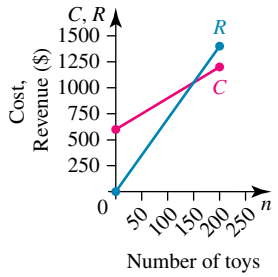
9. (0, 0)

10. (3, 5)

11. a.

Number of toys	Cost ($)	Revenue ($)
50	750	350
100	900	700
150	1050	1050
200	1200	1400

b.



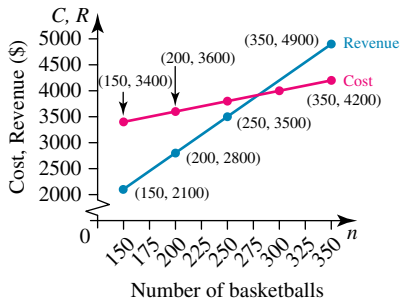
c. 150

d. (150, 1050). If 150 toys are produced, the cost and revenue is equal to $1050. No profit nor loss is made.

12. a.

Number of basketballs	Cost ($)	Revenue ($)
150	$3400	$2100
200	$3600	$2800
250	$3800	$3500
300	$4000	$4200
350	$4200	$4900

b.

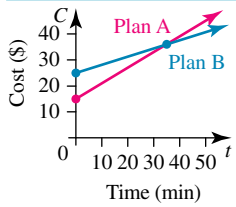


c. Number of basketballs (n) = 280 required to break even

13. a.

Call time (minutes)	Cost Plan A	Cost Plan B
10	$21	$28
20	$27	$31
30	$33	$34
40	$39	$37

b.

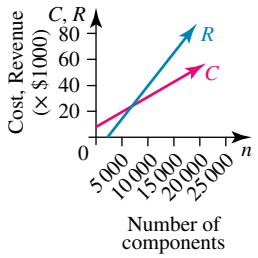


c. Approx. $33\frac{1}{3}$ minutes

14. a.

Number of components	Cost ($)	Revenue ($)
5000	18 500	14 500
10 000	31 000	37 000
15 000	43 500	59 500
20 000	56 000	82 000

b.

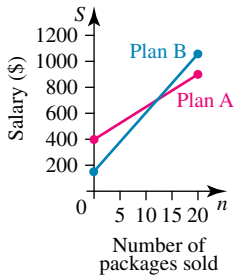


c. 7000

15. a.

Number of packages sold	Salary Plan A	Salary Plan B
5	$525	$375
10	$650	$600
15	$775	$825
20	$900	$1050

b.

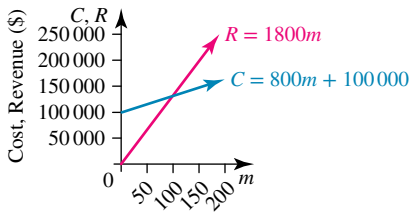


c. 13

16. A

17. a. $108 000 **b.** 30

18. a.



b. 100

c. $P = 1000m - 100\,000$

d. $75 000

Exercise 14.5 Interpreting non-linear graphs

1. a. 1.0 km

b. 0.15 km/min

c. The boarder is not moving.

d. 2.5 km

e. 20 mins

f. 10 mins

2. a. 2 km

b. 0.4 km/min

c. The runner is moving towards the finish line.

d. 8 km

e. 20 mins

f. 20 mins

3. a. 2 km

b. 0 km/h

c. Stationary (not moving)

d. 8 km

e. 90 min

f. 30 min
g. 16 km/h

4. a. ≈ 70%
b. 9 years
c. 9 years
d. $8400

5. a. ≈ 70%
b. 8 years
c. 7 years
d. $15000

6. a. 2 s **b.** 2 s **c.** 50 m **d.** 10 m/s

7. a. ≈ 62%
b. 9 years
c. 9 years
d. $4000

8. a. About 6 hours
b. 1.5 m
c. 0.5 m
d. 1 m
e. 10.40 am
f. Between about 15 hours and 21 hours after the first high tide

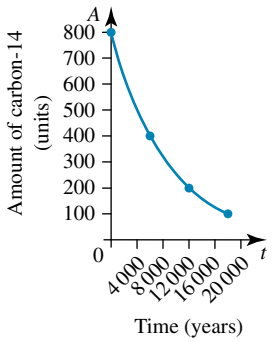
9. B

10. C

11. a.

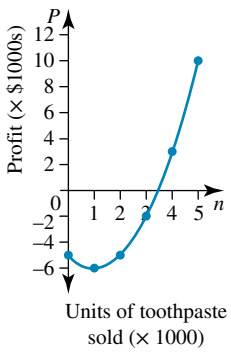
Time (number of years)	Amount of carbon-14 (units)
0	800
6000	400
12 000	200
18 000	100

b.



c. 2500 years
d. 50 units

12. a.



b. 3450

13. a. 25 years
b. 14.5 years
c. Approximately 21 years

14. a. 5750 ml **b.** 3100 ml **c.** 4250 ml

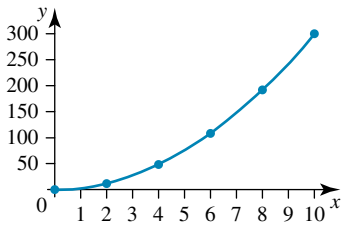
15. a. $20 000
b. Loss of $10 000

16. a. 150
b. May need to employ extra staff or install new plant.

Exercise 14.6 Constructing non-linear relations and graphs

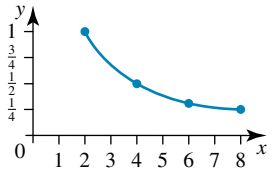
1. $y = 3x^2$

x	0	2	4	6	8	10
y	0	12	48	108	192	300

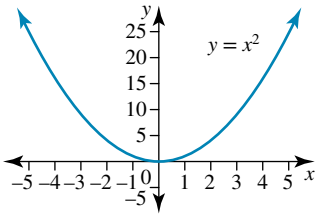


2.

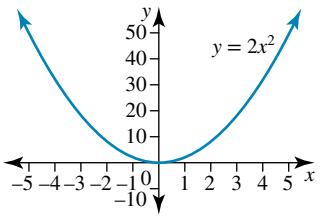
x	2	4	6	8
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



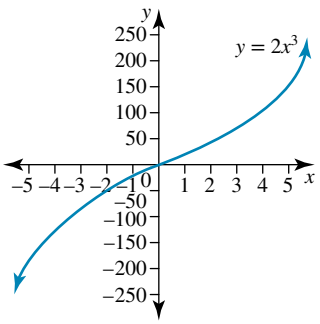
3. a.



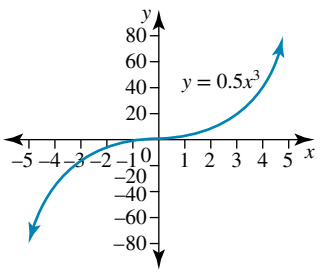
b.



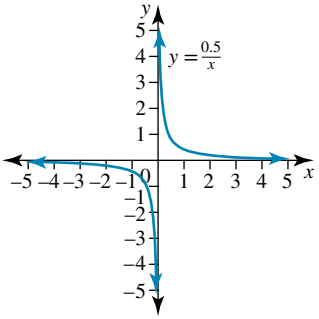
c.



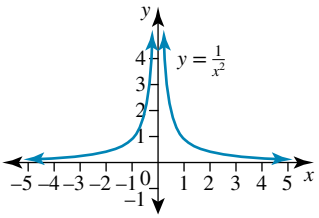
d.



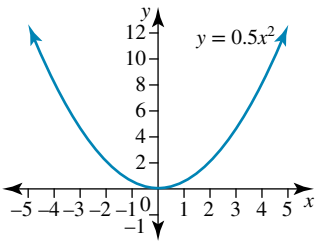
e.



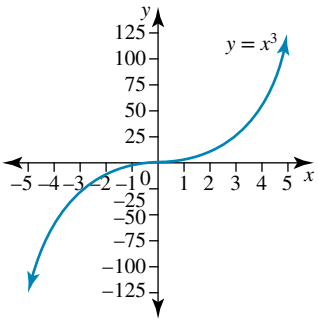
f.



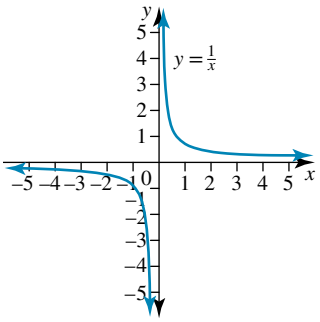
4. a.



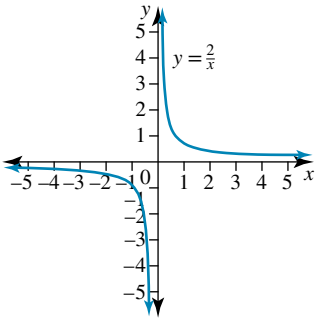
b.



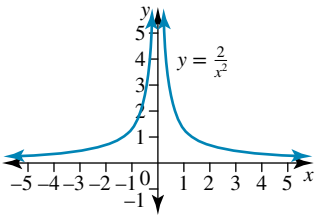
c.



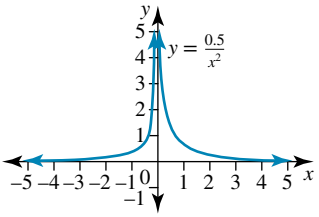
d.



e.



f.



5. a. $y = \frac{3}{x}$ **b.** $y = x^3$

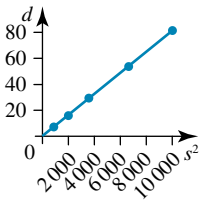
c. $y = 2x^2$ **d.** $y = x^2$

6. $y = 2x^2$

7. $y = 2.5x^2$

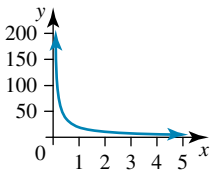
8. $y = 1.5x^3$

9. a.

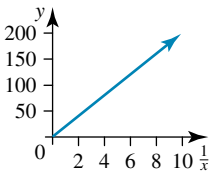


b. $d = 0.0083s^2$

10. a.



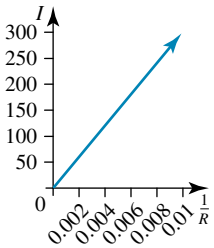
b.



c. 20

d. $y = \frac{20}{x}$

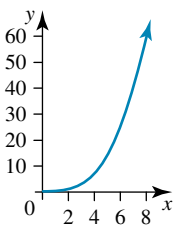
11. a.



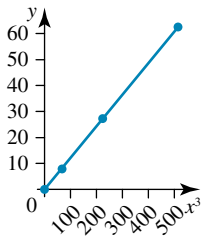
b. $I = \frac{30\,000}{R}$

R	100	200	1000	1500
$\frac{1}{R}$	0.01	0.005	0.001	0.000 67
I	300	150	30	20

12. a.



b.



x	0	2	4	6	8
x^3	0	8	64	216	512
y	0	1	8	27	64

c. 0.125

d. $y = 0.125x^3$

13. $m = \dfrac{h^3}{75\,000}$

14. a. $k = 20\,000$ **b.** 23.9

15. C

16. E

14.7 Review: exam practice

Multiple choice

1. D

2. A

3. C

4. a. B **b.** A **c.** D **d.** C

5. a. B **b.** C **c.** B

6. D

7. C

8. A

9. B

10. A

11. C

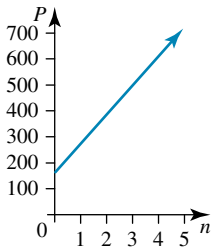
12. D

Short answer

1. a. \$510 **b.** 4

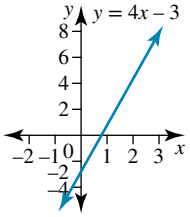
c. $P = 150 + 120n$

d.

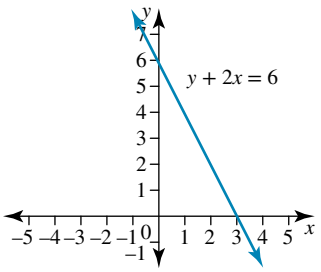


2. a. 9 **b.** 7 **c.** Yes **d.** 3

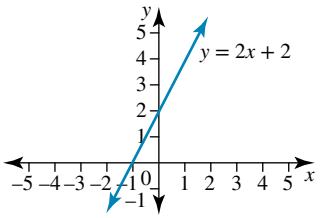
3. a.



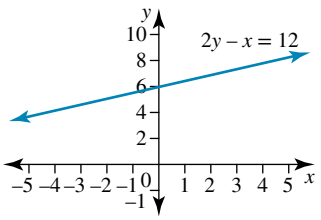
b.



c.



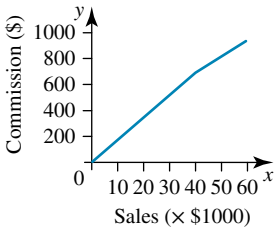
d.



4. a.

Sales	Commission
\$20 000	\$350
\$30 000	\$525
\$40 000	\$700
\$50 000	\$820
\$60 000	\$940

b.

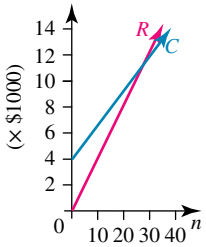


5. a. $x = 2, y = 6$ **b.** $x = \frac{4}{7}, y = \frac{12}{7}$

6. a.

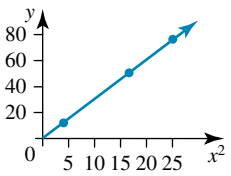
Number of surfboards made	Cost (\$)	Revenue (\$)
10	6800	4500
20	9600	9000
30	12 400	13 500
40	15 200	18 000

b.



c. About 23 (That is, when 23 boards are produced, small loss occurs when 24 boards we produced; and sold, small profit occurs.)

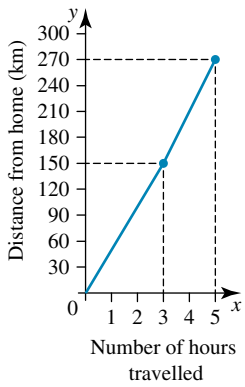
7. a. 1 second
b. 20 cm to the left
c. 20 cm to the right

8. a.


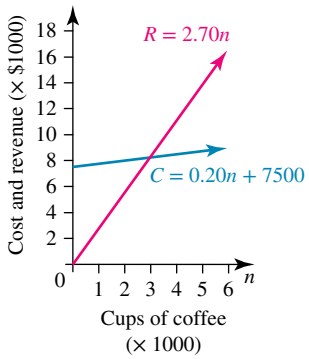
b. $y = 3x^2$

Extended response

1. a. 50 km, 100 km, 150 km, 210 km, 270 km

b.


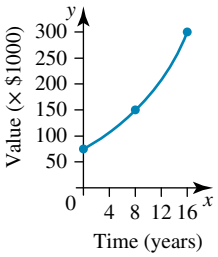
c. Line segment graph
d. $m = 50$; y-intercept $= 0$; $D = 50t$
e. The gradient represents the average speed (50 km/h) during the first section of the trip. The y-intercept represents the initial distance from home; since it is 0, this means that they have started their trip from home.
f. The steepness is determined by the gradient; in this case the gradient represents the average speed. The second section is steeper because the average speed increased from 50 km/h to 60 km/h.
g. 270 km

2. a. $c = 0.2n + 7500$
b. $R = 2.7n$
c.


d. (3000, 8100)
e. The point of intersection is the break-even point. It shows that by making and selling 3000 cups of coffee, neither a profit nor a loss will be made.
f. 3001
g. $P = 2.5n - 7500$
h. 3800

3. a.

Time (years)	0	8	16
Value	$75 000	$150 000	$300 000

b.


c. $170 000
d. $2.4 million

4. $y = 1.5x^3$

TOPIC 15
Linear programming

15.1 Overview

15.1.1 Introduction

Linear programming was formally developed by George Dantzig in 1947 to assist the US Air force to solve planning problems. As the industrial age gathered momentum in the post-war years, linear programming has been used extensively by manufacturing, business and the social sciences to assist with scheduling, dealing with constraints and capacities and to maximise profit or output and minimise cost and input.

The introduction of powerful computers has enabled linear programming problems to solve very difficult problems in a very short time. These problems may involve thousands of constraints and variables. For example, an airline needs to provide schedules for flights and personnel. The airline would require a maximum profit with minimum cost. The constraints could include, operational budgets, available runways and airport resources, available staff, available aircraft, flight path restrictions, legal number and length of work shifts and required maintenance on aircraft. Once all constraints have been identified, equations can be generated and a computer used to solve this very complex problem in a matter of seconds.

Linear programming is also used to design diet programs and diet foods where the objectives might be to maximise nutrients and taste and minimise cost and kilojoules. Constraints could include amounts of sugar, fat and specific dietary requirements.

Linear programming is a powerful mathematical modelling techniques that can solve very complex real life problems.

LEARNING SEQUENCE

15.1 Overview
15.2 Linear inequalities
15.3 Simultaneous linear inequalities
15.4 Linear programming
15.5 Applications
15.6 Review: exam practice

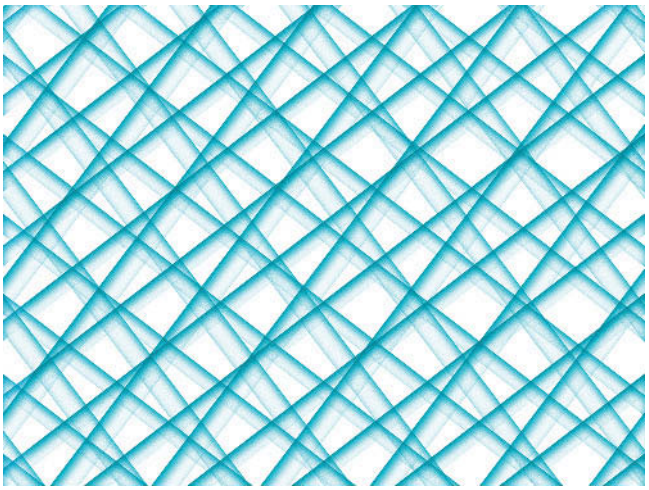
Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

15.1.2 Kick off with CAS

Shading regions with CAS

When graphing a linear equation, the Cartesian plane is split into two regions (one on either side of the line). Linear inequalities replace the equals sign in an equation with an inequality sign ($\leq$, $\geq$, $<$ or $>$). This means that one of the regions the Cartesian plane is split into satisfies the inequality, whereas the other region doesn't satisfy the inequality.

1. Using CAS, sketch the graph of $y < 12 - 3x$.
2. Using any appropriate method, determine which region of the graph (shaded or unshaded) satisfies the inequality, i.e. on which side of the line do the points make the inequality true?
3. Using CAS, sketch the graphs of $y < 12 - 3x$ and $y < \dfrac{12 - x}{3}$ on the same set of axes.
4. Determine the point of intersection of the two lines sketched in question **3**.
5. Determine which region of the graphs plotted in question **3** is the required region; that is, which region satisfies both of the inequalities.
6. Use CAS to plot the following system of linear inequalities:
   $y < 10 - x$
   $y < 4x$
   $y \geq 0$
7. Use CAS to find all the corner points of the required region for the system of linear inequalities plotted in question **3**.



on Resources

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

15.2 Linear inequalities

To begin our work in linear programming, we first have to develop some skills with **linear inequalities**. Recall that linear *equations* come in two forms:

$$y = mx + c$$

and $$ax + by < c$$

For example, $y = 3x - 4$ is a linear equation, as is $3x + 4y = -9$.
Linear *inequalities* also have two forms:

$$y < mx + c$$

and $$ax + by < c$$

For example, $y < 3x - 4$ is a linear inequality, as is $5x + 6y \geq -9$. It is the second form of equation that we will be chiefly working with in this topic. The only difference between equations and inequalities is that the equals sign (=) is replaced with an 'inequality' sign. The possible inequality signs are:

$\geq$ greater than or equal to
$>$ greater than
$<$ less than
$\leq$ less than or equal to.

15.2.1 Graphing a linear inequality

The steps for graphing a linear inequality are the same as those for linear equations, with one extra step. The graph of an inequality is a **region** or *half-plane* while the graph of a linear equation is a straight line.

Step 1. Temporarily treat the inequality as an equation to find the boundary of the region.

Step 2. Find two points using the equation to assist in graphing the boundary. Pick any value of x and find the corresponding value of y. Pick another value of x and find the corresponding value of y. This value of x should be *reasonably far away* from the first one chosen.

Step 3. Plot the points from step 2 and join them with a straight line. Extend the straight line to the extent of your graph paper. A dashed line should be used for a $<$ or $>$ inequality and a solid line for a $\leq$ or $\geq$ inequality.

Step 4. Shade in a region, either above the graph or below the graph, depending upon the inequality sign given. The convention is to shade the region *not* required. Any point inside the shaded region is *not a solution* to the inequation. Thus, the unshaded (or clear) area is called the *solution region* or the *feasible region. Note:* Some CAS calculators *shade* the solution region.

WORKED EXAMPLE 1

Graph the linear inequality $3x + 4y \geq -9$.

THINK	WRITE/DRAW
1. Find the boundary of the region.	
a. Write the inequality as an equation.	The boundary is $3x + 4y = -9$.

b. Pick any value for x and find the corresponding y-value.

c. Substitute the x-value into the equation and solve for y.

Let $x = 0$

$3(0) + 4y = -9$

$4y = -9$

$y = -\frac{9}{4}$

$= -2.25$

First point is $(0, -2.25)$.

d. Pick another value for x and find the corresponding y-value.

e. Substitute the x-value into the equation and solve for y.

Let $x = 9$

$3(9) + 4y = -9$

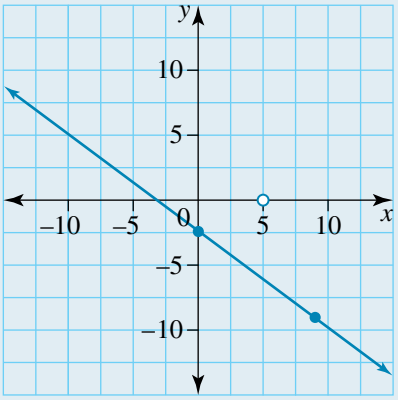
$27 + 4y = -9$

$4y = -36$

$y = -9$

Second point is $(9, -9)$.

f. Plot the two points on a graph and join them with a straight line. A solid line is used since the inequality contains a $\geq$ sign. Observe how the line is extended the full extent of the graph paper. This line divides the coordinate plane into two 'half-planes'.



2. Find the required region.

a. Select a test point above the line to work out the region or half-plane required.

b. Substitute the x- and y-values into the inequality. If the inequality is satisfied (or true) then the half-plane containing the test point is required. If not, the other half-plane is required.

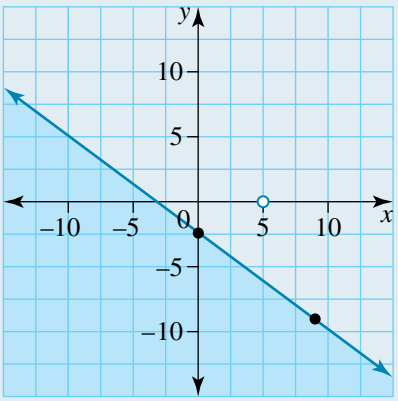
Test point: $(5, 0)$

$3x + 4y \geq -9$?

$3(5) + 4(0) \geq -9$?

$15 \geq -9$ is true, so the region required is above the line.

3. Shade in the region not required (that is, the half-plane below the line) and add the legend.



☐ Region required

Note that in Worked example 1, because the inequality was 'greater than or equal to', the solution region also included the straight line itself (shown with a solid line).

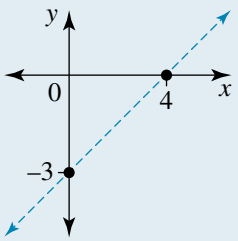
If the inequality had been just 'greater than' then the region would not have included the straight line points (a dashed line would have been used).

However, most problems in this topic will be of the '... equal to' type.

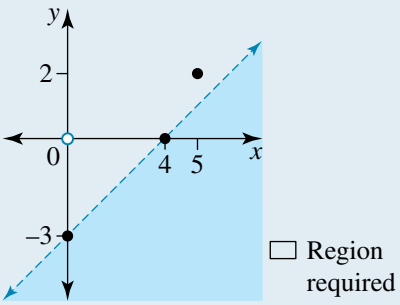
A faster alternative to steps 1(b) to 1(e) in Worked example 1 is to find the x- and y-intercepts to the boundary line. First substitute $x = 0$ and find the corresponding value of y and then substitute $y = 0$ and find the corresponding value of x. Plot these two points to get the straight line. This is called the x- and *y-intercept method*.

WORKED EXAMPLE 2

Graph the inequality $3x - 4y < 12$ using the x- and y-intercept method and show that (5, 2) is a solution.

THINK	WRITE/DRAW
1. Find the boundary.	
a. Write the equation of the boundary.	The boundary of the region is $3x - 4y = 12$.
b. Find the y-intercept by substituting $x = 0$ into the equation and finding the corresponding value of y.	When $x = 0$, $3(0) - 4y = 12$ $-4y = 12$ $y = -3$ Point is $(0, -3)$.
c. Find the x-intercept by substituting $y = 0$ into the equation and finding the corresponding value of x.	When $y = 0$ $3x - 4(0) = 12$ $3x = 12$ $x = 4$ Point is $(4, 0)$
d. Plot the two points on a graph and join them with a straight, dashed line.	
2. Find the required region.	
a. Select a test point, say (0, 0), to work out the half-plane required.	Test point: (0, 0)
b. Substitute the x- and y-values into the inequality. If the inequality is satisfied (or true) then the half-plane containing the test point is required. If not, the other half-plane is required.	$3x - 4y < 12$ $3(0) + 4(0) < 12?$ $0 < 12$ is true, so the region required is above the line.

3. Shade in the region not required, that is, the half-plane below the line.



4. Plot the given point (5, 2) to see whether it is within the required region.
Alternatively: Substitute the co-ordinates of the given point (5, 2) into the inequality to see if it will make a true statement.

The point (5, 2) lies inside the unshaded region. Therefore, it is a solution to the inequality.
Test point (5, 2):
$3(5) - 4(2) < 12?$
$15 - 8 < 12?$
$7 < 12$ is true, therefore (5, 2) is a solution to the inequality.

on Resources

Interactivity: Linear inequalities in two variables (int-6488)

studyon

Units 3 & 4 > AOS M4 > Topic 2 > Concepts 1 & 2

Variables, constraints and inequalities Summary screen and practice questions
Graphing inequalities Concept summary and practice questions

Exercise 15.2 Linear inequalities

1. **WE1** Graph the solution to the linear inequality $4x + 7y \leq 28$.
2. Graph the solution to the linear inequality $50x - 65y \geq 650$.
3. Graph the solution to the linear inequality $4x - 5y \geq 10$.
4. Graph the solutions to the following linear inequalities.
   a. $6x + 2y > 12$
   b. $4x + y \leq 5$
   c. $3x + 2y \geq 0$
   d. $x - y > 20$
5. **WE2** Graph the inequality $7x - 12y \leq 84$ and show that $(4, -3)$ is a solution.
6. Graph the inequality $-5x - 3y \geq 20$ and show that $(-10, -5)$ is a solution.
7. Graph the inequality $4x + 5y \leq 20$ and show that $(1, 1)$ is a solution.

8. Graph the inequality $6x - 4y + 20 > 0$ and show that $(10, 5)$ is a solution.
9. Graph the inequality $2x + 3y \geq 6$ and show that $(7, 0)$ is a solution.
10. Graph the inequality $1.5x - 3.5y < 2.7$.
11. MC Which one or more of the points below is/are solutions to the inequality in question 10?
    A. $(5, -5)$ **B.** $(5, 5)$
    C. $(10, 2)$ **D.** $(-1, -2)$
    E. $(10, 6)$
12. Graph the inequality $20x + 30y \geq 300$.
13. MC The point which is *not* a solution to the inequality in question 12 is:
    A. $(10, 5)$ **B.** $(5, 8)$ **C.** $(5, 5)$ **D.** $(1, 10)$
    E. $(2, 10)$
14. Graph the solution to the linear inequality $4x - 5y \geq 40$.
15. Graph the linear inequality $2x + 7y \leq 0$ and show that $(-5, 1)$ is inside the solution region.
16. Graph the solution to the linear inequality $-1.8x + 0.4y \leq 14.4$.

15.3 Simultaneous linear inequalities

As with linear equations, it is possible to solve groups of linear inequalities *simultaneously*; that is, to find the solution that satisfies more than one inequality *at the same time*. You may recall that there are several methods for solving simultaneous linear equations; some of these techniques will be useful for inequations. Note that simultaneous linear inequalities are also called *systems of linear inequalities*.

15.3.1 Graphical solution of simultaneous linear inequalities

To solve for two linear inequalities, follow these steps.

Step 1. Graph the solution for the first linear inequality
Step 2. Graph the solution for the second linear inequality.
Step 3. The solution to both inequalities is the intersection of the solution regions from Steps 1 and 2.

WORKED EXAMPLE 3

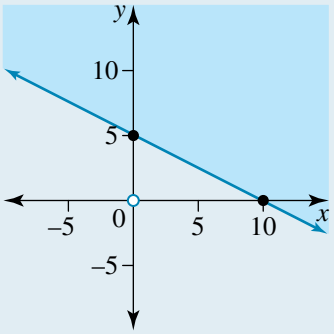
Graph the solution to the simultaneous linear inequalities:

$$x + 2y \leq 10 \quad [1]$$
$$2x + y \leq 10 \quad [2]$$

THINK	WRITE/DRAW
1. Write the boundary equation for region [1].	The boundary for region [1] is $x + 2y = 10$.
2. Find the x- and y-intercepts of the boundary line.	When $x = 0$, $0 + 2y = 10$ $y = 5$ $(0, 5)$ When $y = 0$, $x + 2(0) = 10$ $x = 10$ $(10, 0)$

3. Plot the straight line and determine the region of the solution by using a test point.

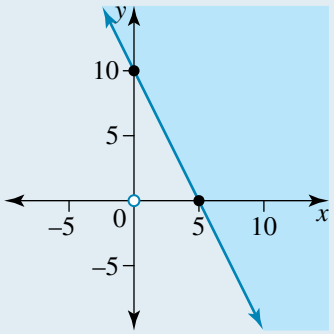
Test point: (0, 0)
$x + 2y \leq 10$
$0 + 2(0) \leq 10$ is true,
so (0, 0) is inside the solution region.



Region required

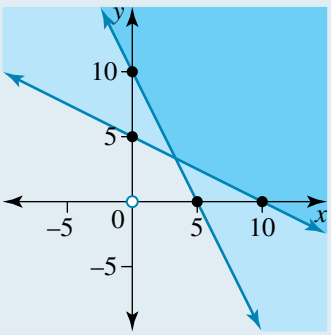
4. Repeat steps 1–3 for region [2].

The boundary for region [2] is $2x + y = 10$.
When $x = 0$,
$2(0) + y = 10$
$y = 10$ (0, 10)
When $y = 0$,
$2x + 0 = 10$
$2x = 10$
$x = 5$ (5, 0)
Test point: (0, 0)
$2x + y \leq 10$
$2(0) + 0 \leq 10$ is true, so
(0, 0) is inside the region.



Region required

5. Determine the overlapping areas of the two regions. This can be done graphically, by showing the two regions on the one set of axes. The clear area is the overlap of the two solution regions and represents the solution to the simultaneous linear inequalities.



Region required

Note that in Worked example 3, there were three *non-solution regions*:
1. the region above and to the left of the first line,
2. the region above and to the right of the second line, and
3. the region where the two exclusion zones overlap.

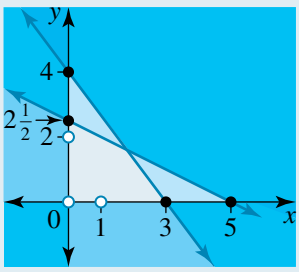
Notes

1. The convention of shading the region that is *not* required makes it easier to identify overlapping exclusion zones; however, CAS may shade the *required* region. Use test points to confirm which style is being used.
2. In this topic, where more exclusion zones overlap, the regions will be shown in increasingly darker shades of blue.
3. In subsequent sections, we examine systems of three or more linear inequalities, but the methods are identical.
4. Examination questions usually ask for the *required* region to be shaded. Always read the question carefully and include a legend with your graphical solutions.

WORKED EXAMPLE 4

Graph the solution to the following system of linear inequalities.

$$x + 2y \leq 5 \quad [1]$$
$$4x + 3y \leq 12 \quad [2]$$
$$x \geq 0 \quad [3]$$
$$y \geq 0 \quad [4]$$

THINK	WRITE/DRAW
1. Write the boundary equations for each region and find the intercepts or relevant information to sketch each line.	The boundary for region [1] is $x + 2y = 5$. When $x = 0$, $y = 2.5$ $(0, 2.5)$ When $y = 0$, $x = 5$ $(5, 0)$ The boundary for region [2] is $4x + 3y = 12$. When $x = 0$, $y = 4$ $(0, 4)$ When $y = 0$, $x = 3$ $(3, 0)$ The boundary for region [3] is $x = 0$. This is a vertical line through $x = 0$. The boundary for region [4] is $y = 0$. This is a horizontal line through $y = 0$.
2. Use test points to identify the required region for each inequality. Note that $(0, 0)$ cannot be used for inequalities [3] and [4] as this point lies on the boundary for each.	For $x + 2y \leq 5$: test point $(0, 0)$ $(0) + 2(0) \leq 5$ is true, so $(0, 0)$ is inside the region. For $4x + 3y \leq 12$: test point $(0, 0)$ $4(0) + 3(0) \leq 12$ is true, so $(0, 0)$ is inside the region. For $x \geq 0$: test point $(1, 0)$ $1 \geq 0$, so $(1, 0)$ is inside the region. For $y \geq 0$: test point $(0, 2)$ $2 \geq 0$, so $(0, 2)$ is inside the region.
3. Show each region on the one set of axes. Remember to shade the region not required in each case. The clear area is the solution for this system of inequalities.	 ☐ Region required

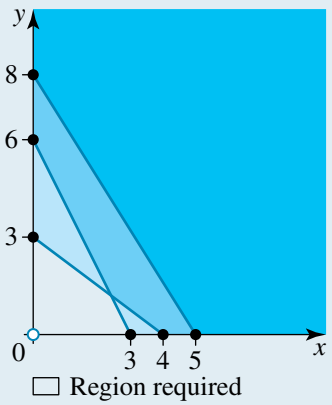
Observe that the solution region in Worked example 4 is a polygon (in this case a quadrilateral). This is a feature of all of the subsequent problems we will study. Generally speaking, if there are n linear inequalities, we can get a polygon of up to n sides. The solution region is also called the **feasible region**.

Sometimes the overlap between solution regions is not so obvious.

WORKED EXAMPLE 5

Graph the solution to the following system of linear inequalities.

$$8x + 5y \leq 40 \quad [1]$$
$$3x + 4y \leq 12 \quad [2]$$
$$2x + y \leq 6 \quad [3]$$
$$x \geq 0 \quad [4]$$
$$y \geq 0 \quad [5]$$

THINK	WRITE/DRAW
1. Find the intercepts of the boundary lines and use a test point to determine which side of the boundary to shade. Note that inequations [4] and [5] indicate that we are interested only in positive values of x and y, as well as when x or y equal zero; that is, non-negative values of x and y.	Region [1]: $8x + 5y = 40$ has intercepts of (0, 8) and (5, 0) with a test point of (0, 0) inside the feasible region. Region [2]: $3x + 4y = 12$ has intercepts (0, 3) and (4, 0) with a test point of (0, 0) inside the feasible region. Region [3]: $2x + y = 6$ has intercepts (0, 6) and (3, 0) with a test point of (0, 0) inside the feasible region.
2. Graph the solution to each of the inequalities on the same set of axes. Remember to shade the regions not required. The clear area is the solution region for this system of inequalities. *Note:* When we are interested in only positive values of x and y, it is usual not to extend lines beyond the axes, or to shade in the negative quadrants.	

What Worked example 5 indicates is that *not all* the inequalities are part of the solution region. In Worked example 5, inequality [1] has no role to play. However, this is usually not obvious until all the inequalities have been graphed.

Note: In the exercises that follow, you will notice that the majority of the systems contain inequalities $x \geq 0$ and $y \geq 0$. These two inequalities describe the first quadrant of the Cartesian plane, including the origin and the positive sections of the x- and y-axis.

on Resources

Interactivity: Graphing a linear inequality (simultaneous inequalities) (int-6283)

Exercise 15.3 Simultaneous linear inequalities

1. **WE3** Graph the solution to the following system of inequalities.
$$2x - 3y \leq 12$$
$$5x + 4y \leq 40$$

2. Graph the solution to the following system of inequalities.
$$-4x - y \geq 12$$
$$x + 3y \leq 21$$

3. **WE4** Graph the solution to the following system of inequalities.
$$2x + 3y \leq 24$$
$$3x + 2y \leq 18$$
$$x \geq 0$$
$$y \geq 0$$

4. Graph the solution to the following system of inequalities.
$$5x + 12y \leq 120$$
$$6x + 5y \leq 60$$
$$x \geq 0$$
$$y \geq 0$$

5. **WE5** Graph the solution to the following system of inequalities.
$$2x + y \leq 2$$
$$x + 3y \leq 3$$
$$3x + 2y \leq 6$$
$$x \geq 0$$
$$y \geq 0$$

6. Graph the solution to the following system of inequalities.
$$3x + 5y \leq 75$$
$$x + y \leq 20$$
$$10y + 4x \leq 100$$
$$x \geq 0$$
$$y \geq 0$$

For questions **7** – **15**, graph the solution to the given system of linear inequalities.

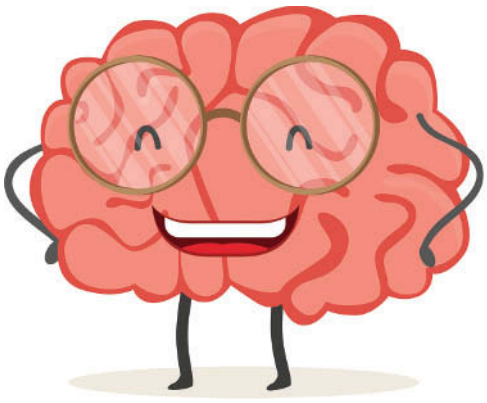
7. $5x + 3y \leq 15$
$3x + 5y \leq 15$

8. $5x + 3y \leq 15$
$-x + y \leq 2$

9. $2x + 3y \leq 6$
$2x - 5y \geq 10$

10. $6x + 8y \geq 0$
$3x + 4y \leq 12$

11. $8x + 3y \leq 24$
$5x + 7y \leq 35$
$x \geq 0$
$y \geq 0$



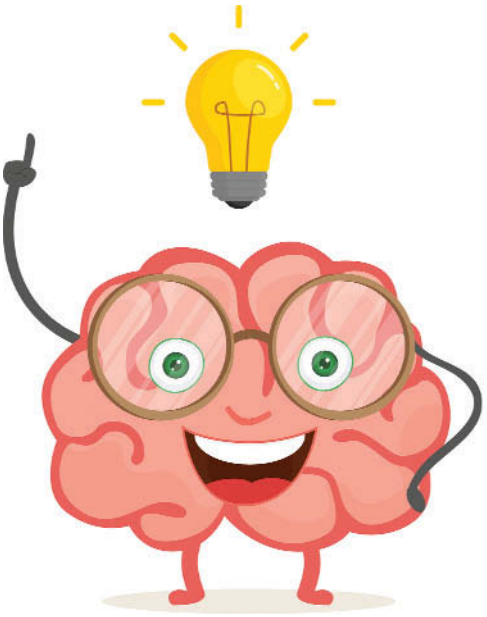
12. $8x + 2y \leq 16$
$4x + 5y \leq 20$
$x \geq 0$
$y \geq 0$

13. $2x + 5y \leq 10$
$7x + 3y \leq 21$
$x \geq 0$
$y \geq 0$

14. $3x + 2y \geq 6$
$x + y \leq 6$
$-2x + 6y \leq 6$
$x \geq 0$
$y \geq 0$

15. $10x + 3y \leq 30$
$7x + 5y \leq 35$
$3x + 7y \leq 21$
$x \geq 0$
$y \geq 0$

16. $10x + 3y \geq 30$
$7x + 5y \leq 35$
$3x + 7y \geq 21$
$x \geq 0$
$y \geq 0$

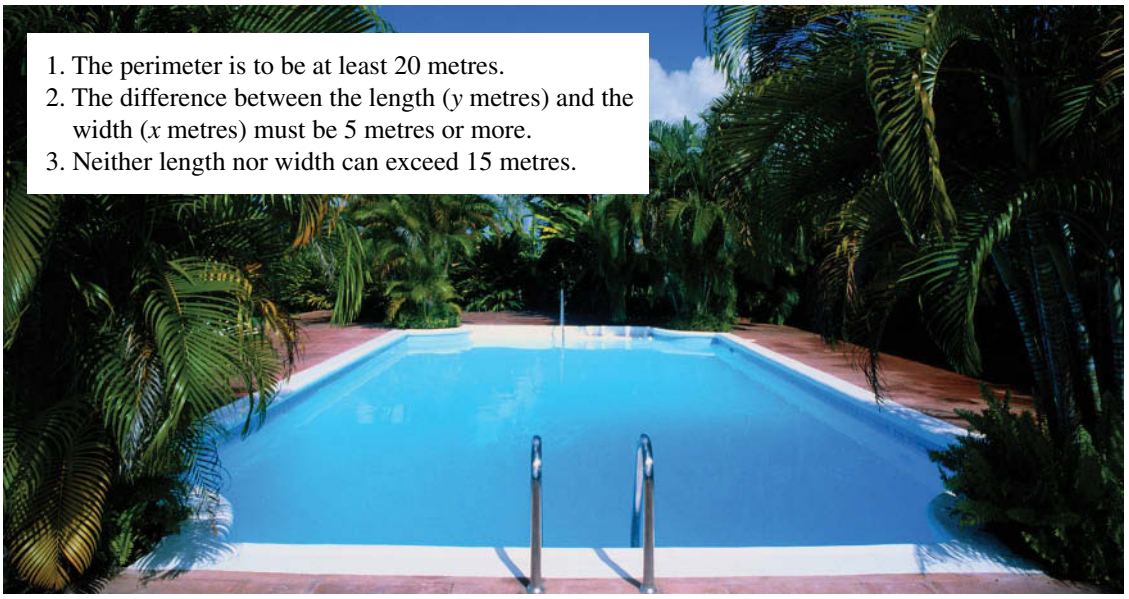


17. MC $3x + y \geq 3$, $5x + 4y \leq 20$, $14x + 4y \leq 28$, $x \geq 0$, $y \geq 0$

The point that is part of the solution to the system of linear inequalities above is:

A. (1, 5) **B.** (2, 1) **C.** (3, 0) **D.** (1, 3) **E.** (0, 6)

18. A rectangular swimming pool is constructed with the following restrictions in mind:



a. Write the restrictions in terms of algebraic inequalities.
b. Graph the solution to the system of linear inequalities.
c. Show that a pool of length of 10 m and a width of 10 m is not possible.

15.4 Linear programming

Now that you have the skills necessary for graphing inequations, you are in a position to tackle the main purpose of this topic: solving **linear programming** problems.

What is a linear programming problem? It consists of 3 vital components:

1. a set of *variables* (in this topic we shall always have exactly two) called **decision variables**
2. a set of restrictions or **constraints** on the values of these variables
3. a *function* (called the **objective function**) of these two variables that we wish to make either as large as possible (maximise) or as small as possible (minimise).

Generally, by convention, x and y are used to represent the two variables.

15.4.1 The decision variables

In the general linear programming problem there are many variables. In the Further Mathematics course we are restricted to only two, so that Cartesian graphical methods can be employed. In real-life engineering situations (linear programming is part of a study known as industrial engineering), linear programming problems with up to 100 variables have been solved. In some cases the variables must be integers (whole numbers), while in other problems they can take on any real value; sometimes it may not be obvious which case applies.

15.4.2 The constraints

The constraints are expressed as a set of inequalities, similar to those we encountered in the previous section. Let us consider a scenario which we will be working on in the next few examples. In this case there will be three constraints; but the procedures outlined in Worked examples 6, 7 and 8 would be exactly the same as for only two constraints.

WORKED EXAMPLE 6

A company produces clothes washers and dryers.
Each product requires assembly in three different factories before it can be sold.
Each factory requires different amounts of time to work on each washer or dryer, according to the following table.

Use of time	Factory A	Factory B	Factory C
Time spent on washer	2 hours	4 hours	2 hours
Time spent on dryer	1 hour	2 hours	2 hours
Maximum time the factory is available each day	24 hours	16 hours	12 hours

From this table we conclude that it takes $2 + 4 + 2 = 8$ hours to produce a single washer and $1 + 2 + 2 = 5$ hours to produce a single dryer. Convert these times into a series of constraints. The aim is to investigate the number of washers and dryers produced each day.

THINK	WRITE
1. Identify the decision variables.	Let x = number of washers made daily
a. What is the number of washers produced each day?	
b. What is the number of dryers produced each day? (You could just as easily choose x for dryers and y for washers.)	Let y = number of dryers made daily
2. Consider the daily capacity of factory A.	For factory A: Time spent on x washers = $2x$ hours Time spent on y dryers = $1y$ hours Maximum time spent on items = 24 hours
3. Turn this into an inequality. (Because 24 hours is a maximum, we could use less time, therefore the 'less than or equal sign' is used.)	$2x + 1y \leq 24$
4. Repeat steps 2 and 3 for factory B.	For factory B: Time spent on x washers = $4x$ hours Time spent on y dryers = $2y$ hours Maximum time spent on items = 16 hours $4x + 2y \leq 16$
5. Repeat steps 2 and 3 for factory C.	For factory C: Time spent on x washers = $2x$ hours Time spent on y dryers = $2y$ hours Maximum time spent on items = 12 hours $2x + 2y \leq 12$
6. Consider other (physical) constraints. Obviously, both x and y cannot be negative, since they represent a *number* of washers and dryers.	$x \geq 0$ $y \geq 0$
7. List all the constraints together.	$2x + y \leq 24$ $4x + 2y \leq 16$ $2x + 2y \leq 12$ $x \geq 0$ $y \geq 0$

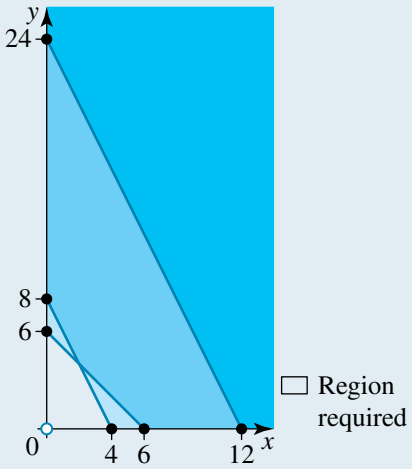
It may or may not be reasonable to insist that x and y be integers. Is it possible to make 2.4 washers in a day? The answer to this is not necessarily obvious from the question, so for now, assume that x and y can take any positive value.

15.4.3 Graphing the constraints

The next stage in the process is to graph the constraints as we did in the previous section on simultaneous linear inequalities, and determine the overlapping solution region.

WORKED EXAMPLE 7

Graph the constraints from Worked example 6.

THINK	WRITE/DRAW
1. List the constraints.	$2x + y \leq 24$ $4x + 2y \leq 16$ $2x + 2y \leq 12$ $x \geq 0$ $y \geq 0$
2. a. Since $x \geq 0$ and $y \geq 0$, we need to graph only in the first quadrant. b. Find intersection points with axes. c. Demonstrate that (0, 0) is in the solution region of all three inequalities.	$2x + y = 24$ has intercepts (0, 24) and (12, 0). Since $2(0) + 0 \leq 24$, then (0, 0) is inside the solution region. $4x + 2y = 16$ has intercepts (0, 8) and (4, 0). Since $4(0) + 2(0) \leq 16$, then (0, 0) is inside the solution region. $2x + 2y = 12$ has intercepts (0, 6) and (6, 0). Since $2(0) + 2(0) \leq 12$, then (0, 0) is inside the solution region.
3. Graph each equation and shade the non-solution region for each inequality. (It is the 2nd and 3rd inequalities that ultimately determine the overlapped solution region, shown as a clear area.)	

Note the characteristic polygonal shape of the solution region. This will play a crucial role in the next stage.

15.4.4 The objective function

As mentioned at the start of this section, the third component of a linear programming problem is the *objective function*. This is a function of the two decision variables usually relating to a cost, distance or some other relationship between them. The object in linear programming is to maximise (or in some cases minimise) the value of this function within the solution region. In theory, we would have to check *every point* in our solution region to see which one gave us the maximum (or minimum) value required. Even if we restricted ourselves

to integers only, this could take a long time. Fortunately there is a theorem (which will not be proved) that states:

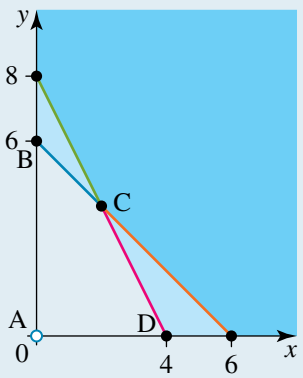
> **The maximum (or minimum) of the objective function occurs at one of the vertices (corner points) of the solution region polygon.**

Sliding line method: By equating your objective function to different values, it will create a number of parallel lines on your graph. If this line touches or passes through the feasible region then it is a possible solution. As shown, the maximum (or minimum) of the objective function occurs at one of the corners. So this line can slide up and down (keeping the same gradient) to find the required solution.

Corner point methods: Since the maximum (or minimum) of the objective function occurs at one of the corners, the corner point method is an efficient process to use. To do this, evaluate the objective function at each of the corner points to find the maximum (or minimum).

WORKED EXAMPLE 8

When selling its clothes washers and dryers, the company refered to in Worked examples 6 and 7 can make a profit of $150 on each washer and $100 on each dryer.
How many of each should be made to maximise the company's daily profit?

THINK	WRITE/DRAW
1. List the decision variables.	Let x be the number of washers, and y the number of dryers.
2. Identify the objective function. $150 is made on each washer, $100 on each dryer.	Profit $= 150 \times$ number of washes $+ 100 \times$ number of dryers $= 150x + 100y$
3. Label the vertices (corner points) of the solution region polygon.	
4. Find the coordinates of the vertices.	
a. The coordinates of the points A, B and D can be read directly from the graph.	Point A: (0, 0) Point B: (0, 6) Point D: (4, 0)
	Equation 1: $4x + 2y = 16$ [1]
	Equation 2: $2x + 2y = 12$ [2]

b. Point C is the point of intersection of the two lines, whose equations are $4x + 2y = 16$ and $2x + 2y = 12$.

Therefore, to find the coordinates of point C, solve these equations simultaneously.

c. Eliminate one of the variables (y) by subtraction and solve for the other variable (x).

$2x = 4$ [1] –[2]

$x = 2$

d. Substitute back into one of the original equations [1] or [2], to find the other variable (y).

Substitute into [1]: $4(2) + 2y = 16$

$8 + 2y = 16$

$2y = 8$

$y = 4$

Point C: (2, 4)

5. Since the maximum profit will occur at one of the vertices, evaluate the objective function for each vertex in the solution region.

Profit = $150x + 100y$

Point A(0, 0): profit = $150(0) + 100(0)$
$= 0$

Point B(0, 6): profit = $150(0) + 100(6)$
$= 600$

Point C(2, 4): profit = $150(2) + 100(4)$
$= 700$

Point D(4, 0): profit = $150(4) + 100(0)$
$= 600$

6. Find the maximum profit.

Maximum profit ($700) occurs when $x = 2$, $y = 4$. Thus, the company should make 2 washers and 4 dryers per day.

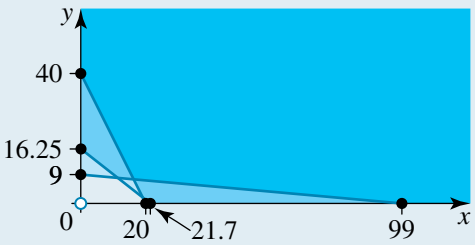
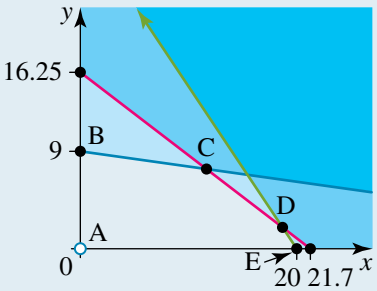
Let us now combine all the stages into a single example.

WORKED EXAMPLE 9

A company is deciding how much to invest in two new calculator models, graphics (x) and CAS (y). They will base their decision on the capacity of their workforce to design, assemble and ship the devices. They wish to ship as many calculators as possible. The workforce restrictions are outlined in the following table. (All restrictions are per thousand calculators.)

	Design phase	Assembly phase	Shipping
Graphics	1 worker	3 workers	2 workers
CAS	11 workers	4 workers	1 worker
Total capacity	99 workers	65 workers	40 workers

Use linear programming methods to determine the total number of calculators (in thousands) that should be built.

THINK	WRITE/DRAW
1. Identify the decision variables.	Let x = number of graphics calculators (in thousands) Let y = number of CAS calculators (in thousands)
2. State the constraints. Clearly both x and y must not be negative.	Design: $1x + 11y \leq 99$ [1] Assembly: $3x + 4y \leq 65$ [2] Shipping: $2x + 1y \leq 40$ [3] Non-negative values: $x \geq 0$ $y \geq 0$
3. Identify the objective function. The question stated that the company wishes to make *as many calculators* as possible.	Objective function: Number = $x + y$
4. Graph the constraint inequations and identify the feasible region. Because of non-negative restriction on x and y, draw the graph in the 1st quadrant only.	 ☐ Region required
The second graph shows an enlarged version of the feasible region.	 ☐ Region required
5. Find the coordinates of the vertices of the feasible region. (Coordinates of points A, B and E can be read directly from the graph.)	Point A: (0, 0); point B: (0, 9); point E: (20, 0) Point C: intersection of $x + 11 = 99$ [4] and $3x + 4y = 65$ [5]
a. Solve simultaneously $x + 11y = 99$ and $3x + 4y = 65$ to find the coordinates of point C.	
b. Multiply [4] by 3 to have the same x-coefficient as [5].	3× [4]: $3x + 33y = 297$
c. Subtract the 2 equations.	3× [4] − [5]: $29y = 232$
d. Solve for y.	$y = 8$
e. Substitute the value of y into equation [4].	Substitute into [4]: $x + 11(8) = 99$
f. Solve for x.	$x + 88 = 99$ $x = 11$
	Point C: (11, 8)

g. Similarly, solve simultaneously $3x + 4y = 65$ and $2x + y = 40$ to find coordinates of point D.

h. Multiply [7] by 4 to have the same y-coefficient as [6].

i. Subtract the two equations.

j. Solve for x.

k. Substitute the value of x into equation [6].

l. Solve for y.

Point D: intersection of $3x + 4y = 65$ [6]
and $2x + y = 40$ [7]

$4\times$ [7]: $8x + 4y = 160$

$4 \times [7] - [6]$: $5x = 95$

$x = 19$

Substitute into [6]: $3(19) + 4y = 65$

$57 + 4y = 65$

$4y = 8$

$y = 2$

Point D: (19, 2)

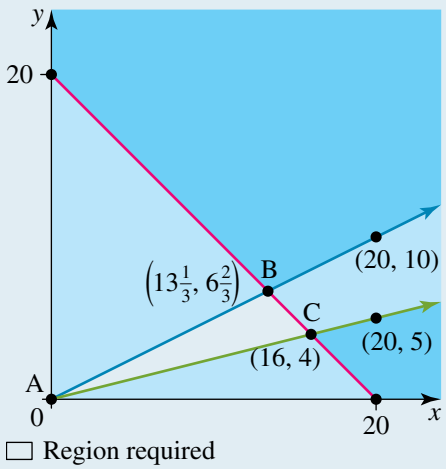
6 Substitute coordinates of each corner point into the objective function and choose the maximum. (Remember that x and y are in thousands.)

Objective function:
Number $= x + y$
Point A: (0, 0), number $= 0 + 0$
$= 0$
Point B: (0, 9), number $= 0 + 9$
$= 9$
Point C: (11, 8), number $= 11 + 8$
$= 9$
Point D: (19, 2), number $= 19 + 2$
$= 21^*$ *Maximum
Point E: (20, 0), number $= 20 + 0$
$= 20$
So, the company should make 19 000 graphics and 2000 CAS calculators.

Sometimes the constraints are related to each other, rather than a constant as in the previous examples.

WORKED EXAMPLE 10

Jenny spends some of her spare time outside school hours playing netball and working a part-time job. Each week she has a maximum of 20 hours available for the two activities.
The number of hours that she spends playing sport is less than or equal to four times the hours she spends working.
The hours spent at sport are greater than or equal to twice the hours spent at work.
Find the time spent on each activity during a week in order to maximise Jenny's earnings if she earns $2.50 per hour playing netball and $6 per hour working. Also find her maximum earnings.

THINK	WRITE/DRAW
1. Identify the decision variables.	x = number of hours spent at sport y = number of hours spent at work
2. State the constraints. Maximum of 20 hours available. (Clearly both x and y are non-negative.)	$x + y \leq 20$ [1] $x \leq 4y$ [2] $x \geq 2y$ [3] $x \geq 0$ $y \geq 0$
3. Identify the objective function. (The objective is to maximise earning.)	Objective function: Earnings $= 2.5x + 6y$
4. Graph the constraints.	$x + y = 20$ has intercepts $(0, 20)$ and $(20, 0)$, since $0 + 0 \leq 20$ is true, $(0, 0)$ is inside the solution region.
The boundary lines for constraints [2] and [3] pass through the origin. Therefore, a second point is required to sketch each line. Also a test point other than $(0, 0)$ must be selected for these constraints.	$x = 4y$ has an intercept of $(0, 0)$. When $x = 20$, $y = 5$ $(20, 5)$ Consider $(20, 0)$ as a test point. Since $20 \leq 4(0)$ is untrue, then $(20, 0)$ is outside the solution region. $x = 2y$ has an intercept of $(0, 0)$. When $x = 20$, $y = 10$ $(20, 10)$ Consider $(20, 0)$ as a test point. Since $20 \geq 2(0)$ is true, then $(20, 0)$ is inside the solution region.
Identify the feasible region overall. We are dealing with the 1st quadrant only.	
5. Identify the coordinates of the vertices of the feasible region.	Point A: $(0, 0)$
a. To find coordinates of point B, solve $x + y = 20$ and $x = 2y$ simultaneously.	Point B: intersection of $x + y = 20$ [4] and $x = 2y$ [5] [5] becomes $x - 2y = 0$ [6] [4] − [6]: $3y = 20$ $y = 6\frac{2}{3}$

Substitute into [5]: $x = 2 \times 6\frac{2}{3}$

$x = 13\frac{1}{3}$

Point B: $\left(13\frac{1}{3}, 6\frac{2}{3}\right)$

b. To find coordinates of point C, solve $x + y = 20$ and $x = 4y$ simultaneously.

Point C: intersection of $x + y = 20$ [7]

and $x = 4y$ [8]

[8] becomes $x - 4y = 0$ [9]

[7] − [9]: $5y = 20$

$y = 4$

Substitute into [8]: $x = 4(4)$

$x = 16$

Point C: $(16, 4)$

6. Substitute coordinates of each point into the objective function to find the maximum earnings.

Objective function:

Earnings $= 2.5x + 6y$

Point A: $(0, 0)$

Earning $= 2.5(0) + 6(0)$

$= 0$

Point B: $\left(13\frac{1}{3}, 6\frac{2}{3}\right)$

Earnings $= 2.5\left(13\frac{1}{3}\right) + 6\left(6\frac{2}{3}\right)$

$= 73.33$

Point C: $(16, 4)$

Earnings $= 2.5(16) + 6(4) = 64$

So, Jenny should play netball for $13\frac{1}{3}$ hours and work for $6\frac{2}{3}$ hours for maximum earnings of $73.33.

on Resources

Interactivity: Linear programming: corner point method (int-6282)

studyon

Units 3 & 4 > AOSM4 > Topic 2 > Concept 3

Objective function Summary screen and practice questions

Exercise 15.4 Linear programming

1. **WE6** The Danish-Australian Club is organising a camping trip to the Snowy Mountains. A large campsite has 2-person cabins available at $40 per night and 3-person cabins at $45 per night. The club has a budget of $3000 to accommodate up to 180 people.
   a. Define the decision variables.
   b. Determine the constraint based upon the total club budget.
   c. Determine the constraint based upon the number of members of the club.

2. An investment banker has a maximum of $1 000 000 to invest on behalf of a group of doctors. She can either purchase bonds or invest in real estate mortgages. However, the doctors, being conservative, insist that the amount invested in bonds be *at least* twice as much as that invested in mortgages.

   Based on past experience, the fund manager knows that bonds will return 6% profit and real estate will return 11%.
   a. State the decision variables.
   b. State the constraints.
3. **WE7** The objective function from question **1** is based upon the total number of cabins used. Graph the constraints from question **1**.
4. State the objective function from question **2** and graph the constraints.
5. **WE8** Use linear programming techniques (sliding line or corner point) to find the values of the decision variables that minimise the total number of cabins rented from question **1**.
6. Use linear programming techniques (sliding line or corner point), to determine the optimal 'mix' of investments from question **2**; that is, determine maximum return.
7. **WE9** A company is deciding how much to invest in two new TV models, flat screen (x) and curved screen (y). They will base their decision on the capacity of their workforce to design, assemble and ship the devices. They wish to ship as many TVs as possible. The workforce restrictions are shown in the following table (all restrictions are per thousand TVs).

	Design	Assembly	Shipping
Flat	2	5	3
Curved	15	7	3
Total capacity	120	75	50

   Use linear programming (shifting line or corner point) methods to determine the number of TVs (in thousands) that should be built.
8. Two teachers work together to produce a Mathematics text consisting of *text material* and *tests*. For each chapter of text material and for each test, the teachers devote a certain amount of time (in hours) to the task. Furthermore, each teacher can devote time only up to a maximum number of hours. The data is summarised in the table below.

	Teacher A	Teacher B
Text chapter	20	10
Test	8	2
Maximum time available	320	100

   Let x be the number of text chapters and y be the number of tests.

If the publisher pays $1500 for each chapter and $250 for each test, how many of each should the teachers produce in order to maximise their income?

9. WE10 Kyle spends his leisure time playing basketball and football. Each week he has a maximum of 30 hours available for his two sports.

The number of hours he spends playing football is less than or equal to three times the hours he spends on basketball.

The hours spent at football are greater than or equal to two times the hours spent at basketball. Find the time spent on each activity during a week in order to maximise Kyle's earning if he earns $6.50 per hour playing football and $10 per hour playing basketball. Also find his maximum earnings.

10. A furniture manufacturer makes 2 kinds of sofas: leather and cloth. Each leather sofa requires 9 workers and each cloth sofa requires 8 workers. A maximum of 207 workers are available to make sofas each day. Furthermore, shipping and packaging cloth sofas requires 5 workers, while leather sofas require only 2 workers. There are 75 workers available for this task each day.

Let x = number of leather sofas.

Let y = number of cloth sofas.

If the profit for a leather sofa is $400 and the profit for a cloth sofa is $370, how many of each should be produced each day?

11. A factory produces two different models of transistor radio. Each model requires two workers to assemble it. The time taken by each worker varies according to the following table.

	Worker 1	**Worker 2**
Model *A*	5 minutes	5 minutes
Model *B*	18 minutes	4 minutes
Maximum time available for each worker	360 minutes	150 minutes

The company makes $2.50 on each of Model A sold and $4.00 on each of Model B sold.

a. Define the decision variables.

b. Write the constraints as linear inequalities.

c. Graph the constraints and indicate the solution region.

d. State the objective function.

e. Use linear programming methods to find how many of each radio should be made to maximise takings.

12. At a car spray-painting workshop, each car receives three coats of paint, all of which have to be completed within one day. There are two types of car: sedan and utility. The times taken for each are listed in the following table.

	Stage of painting		
	1st stage	**2nd stage**	**3rd stage**
Sedan	5 minutes	9 minutes	6 minutes
Utility	7 minutes	8 minutes	5 minutes
Total time available for each stage	140 minutes	183 minutes	120 minutes

a. Define the decision variables.

b. Write the constraints as linear inequalities.

13. From the data in question 12,
    a. graph the constraints and indicate the feasible region
    b. state the objective function
    c. use linear programming methods to work out how many of each vehicle type should be sprayed to maximise the total number of vehicles sprayed.
14. A large electronics firm must decide on the number of different models of computer monitor and models of hard disk drive that it will manufacture. For each model of each type there are the following budgetary constraints for different departments. (All budgetary numbers are in thousands of dollars.)

	Department			
	Marketing	**Development**	**Design**	**Production**
Computer monitor	7	11	22	11
Disk drive	8	7	12	3
Total budget available	320	319	594	264

    a. Define the decision variables.
    b. Write the constraints as linear inequalities.
15. From the data in question 14,
    a. graph the constraints and indicate the solution region
    b. use linear programming methods to find how many models of monitor and disk drive should be made to maximise profit, given that each monitor model provides a profit of $150 000 and each disk drive model provides a profit of $120 000.
16. Consider the washer and dryer problem in Worked examples 6–8. How would the solution change if the profit per washer was modified to $120 and the profit per dryer was modified to $130?
17. The Swiss Army Diet requires that a person eat only two foods: apples and cheese. A dieter is told that they may eat as much food as they wish, subject to the dietary constraints listed in the table.

	Amount of vitamin X	**Amount of vitamin Y**
An apple	3 units	3 units
A piece of cheese	10 units	4 units
Number of units	Maximum of 98	Minimum of 50

    Each apple also supplies 100 calories, while each piece of cheese also supplies 85 calories. Use linear programming techniques to determine the number of apples and pieces of cheese needed for the *minimum* calorie intake.
18. A paint shop mixes two kinds of paint — indoor paint and outdoor paint — subject to the following constraints:
    1. At least 30 litres of paint in total must be mixed.
    2. Indoor paint requires 3 units of dye, while outdoor paint requires 8 units of dye and there is a *maximum* of 100 units of dye available.

    Indoor paint yields a profit of $2 50 per litre, while outdoor paint yields $3.25 per litre. Use linear programming techniques to determine the volume of indoor and outdoor paint to be mixed to maximise the shop's profit.

19. A flower grower sells roses and tulips. She has enough land to grow no more than 3000 of these plants altogether, subject to the following constraints:

1. The number of roses that she grows is less than or equal to 3 times the number of tulips grown.
2. The number of roses is greater than or equal to twice the number of tulips grown.

If she sells roses for $15 each and tulips for $6 each, find the number of each that would maximise her income from the sale of these plants.

20. An investment fund manager is given a maximum of $100 000 to invest. She invests in shares and gold according to the following constraints:

1. The value of the shares must be less than or equal to 2 times the value of gold.
2. The value of the shares must be greater than or equal to $\frac{1}{2}$ the value of gold.

Historically, shares have returned 10% (they increase in value by 10%) per year, while gold has returned 16% per year. Use linear programming techniques to determine the manager's investment strategy to maximise profit (or return).

21. MC The required region shown in the following graph is best described by which set of inequalities?

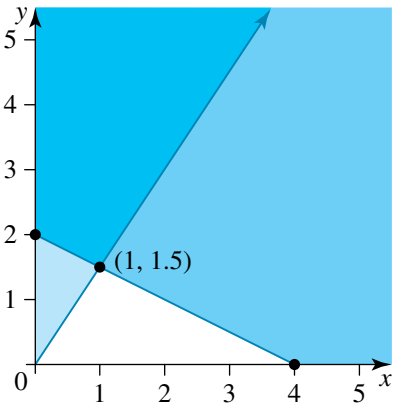
A. $2x + 4y \geq 8$
$2y - 3x \geq 0$
$x \geq 0$
$y \geq 0$

B. $4x + 2y \leq 0$
$3y - 2x \leq 0$
$x \geq 0$
$y \geq 0$

C. $2x + 4y \leq 8$
$2y - 3x \leq 0$
$x \geq 0$
$y \geq 0$

D. $4x + 2y \geq 0$
$3y - 2x \geq 0$
$x \geq 0$
$y \geq 0$

E. $2x + 4y < 8$
$2y - 3x < 0$
$x > 0$
$y > 0$

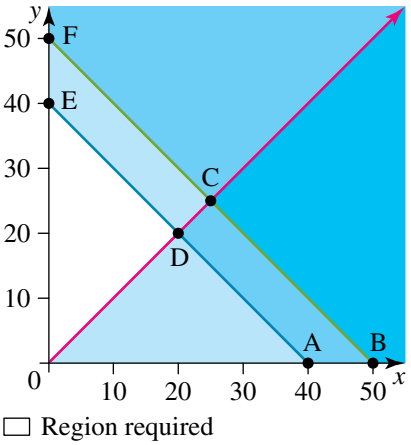


22. A small, used-car yard stocks sedans and wagons. Due to limited space in the yard, they cannot have more than 50 cars at any one time. Since they know that sedans are more popular than wagons, they decide never to have more wagons than sedans in stock. On average, they can purchase a wagon for $12 000 and a sedan for $12 500. To minimise insurance costs, they cannot have more than $490 000 worth of stock at any given time.

Let x = the number of wagons and y = the number of sedans in the car yard.

Use a linear programming method (sliding line or corner point) to answer the following.

a. Write down the 5 constraints involved.
b. The constraints are sketched on the graph as shown. Identify the inequalities with the boundary lines OC, BF and AE.
c. Write down the coordinates of each of the points C and D.
d. If sedans make an average profit of $3500 and wagons make an average profit of $4000, optimise the number of each to stock to maximise profit.



15.5 Applications

The applications in this section are particularly suited for the case where there are only two decision variables (x and y). In practice, when more decision variables are allowed, these problems can be much more sophisticated, but this is beyond the scope of Further Mathematics.

15.5.1 Blending problems

Blending consists of combining raw material from several different sources into a single composite. The raw materials contain one or more chemicals (or other components) in varying amounts, which are then restricted in some way in the blended composite.

WORKED EXAMPLE 11

A miller can buy wheat from three suppliers: Airy Farm, Berry Farm and Cherry Farm. In each case the wheat is contaminated with two things — bran and husks. When combined, the wheat must contain no more than 5% bran and no more than 4% husks. The miller wishes to make 50 tonnes of wheat in total, by purchasing serve from each farm. Each farm's wheat contains the following same amounts of bran and husks.

	Bran	Husks	Cost per tonne
Airy Farm	3%	5%	$70
Berry Farm	5%	2%	$60
Cherry Farm	7%	6%	$40
Maximum amount of contaminant allowed	5%	4%	

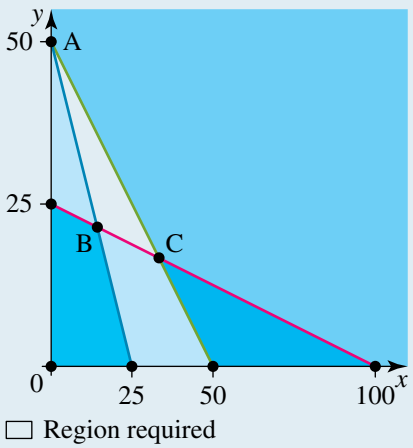
Airy Farm charges $70 per tonne, Berry Farm charges $60 per tonne and Cherry Farm charges $40 per tonne.
Use linear programming techniques (sliding line or corner point) to find the amount of wheat to purchase from each supplier in order to keep costs to a minimum. That is:

a. identify the decision variables
b. define the constraint inequality caused by the restriction on bran
c. define the constraint inequality caused by the restriction on husks
d. define the remaining constraints
e. graph the solution region
f. define the objective function
g. find the amount of wheat purchased from each supplier.

THINK	WRITE/DRAW
a. Identify the decision variables. Note that there are exactly 50 tonnes of wheat needed so if x and y come from Airy and Berry farms, then the rest must come from Cherry Farm.	**a.** Let x = tonnes of wheat purchased from Airy Farm. Let y = tonnes of wheat purchased from Berry Farm. Let $(50 - x - y)$ = tonnes of wheat purchased from Cherry Farm.
b. a. Define the constraint due to bran.	**b.** 3% of x + 5% of y + 7% of $(50 - x - y) \leq$ 5% of 50 tonnes
b. Multiply the inequality by 100.	$0.03x + 0.05y + 0.07(50 - x - y) \leq 0.05(50)$
c. Expand and collect like terms.	$3x + 5y + 7(50 - x - y) \leq 5(50)$ $3x + 5y + 350 - 7x - 7y \leq 250$
d. Multiply the inequality by -1, so the sign changes.	$-4x - 2y + 350 \leq 250$ $-4x - 2y \leq -100$ $4x + 2y \geq 100$
c. Define the constraint due to husks.	**c.** 5% of x + 2% of y + 6% of $(50 - x - y) \leq$ 4% of 50 tonnes $0.05x + 0.02y + 0.06(50 - x - y) \leq 0.04(50)$ $5x + 2y + 6(50 - x - y) \leq 4(50)$ $5x + 2y + 300 - 6x - 6y \leq 200$ $-x - 4y + 300 \leq 200$ $-x - 4y \leq -100$ $x + 4y \geq 100$
d. Complete the list of constraints. Note also that we need to look in the 1st quadrant only, since all supplies of wheat must be non-negative.	**d.** $x \geq 0$ $y \geq 0$
Supply of wheat from Cherry Farm must be non-negative too.	$50 - x - y \geq 0$ or $-x - y \geq -50$ or $x + y \leq 50$

e. 1. Graph the solution region.

e.



2. Find the coordinates of the vertices of the solution region.

a. Point B is the intersection of $4x + 2y = 100$ and $x + 4y = 100$.

b. Multiply [2] by 4 to eliminate x from the simultaneous equations.

c. Solve for y.

d. Substitute the value of y into [1] and solve for x.

e. Point C is the intersection of $x + 4y = 100$ and $x + y = 50$.

f. Subtract equations to eliminate x and solve for y.

g. Substitute the value of y into [3] and solve for x.

Point A is $(0, 50)$.

$$4x + 2y = 100 \quad [1]$$
$$x + 4y = 100 \quad [2]$$

$4 \times [2]$: $4x + 16y = 400$

$4 \times [2] - [1]$: $14y = 300$

$$y = 21.43$$

Substitute into [1]: $4x + 2(21.43) = 100$

$$4x + 42.86 = 100$$
$$x = 14.29$$

Thus, point B is $(14.29, 21.43)$.

$$x + 4y = 100 \quad [2]$$
$$x + y = 50 \quad [3]$$

[2] – [3]: $3y = 50$

$$y = 16.67$$

Substitute into [3]: $x + 16.67 = 50$

$$x = 33.33$$

Thus, point C is $(33.33, 16.67)$.

f. Define the objective function. The desired result is the values of x and y which produce the minimum cost.

f. $\text{Cost} = \$70x + \$60y + \$40(50 - x - y)$
$= 70x + 60y + 2000 - 40x - 40y$
$\text{Cost} = 30x + 20y + 2000$

g. 1. a. Substitute the coordinates of each vertex into the objective function.

b. Determine the minimum cost.

g. Point A: $x = 0$, $y = 50$
$\text{Cost} = 30(0) + 20(50) + 2000$
$= \$3000$

Point B: $x = 14.29$, $y = 21.43$
$\text{Cost} = 30(14.29) + 20(21.43) + \$2000$
$= \$2857.30$

Point C: $x = 33.33$, $y = 16.67$
$\text{Cost} = 30(33.33) + 20(16.67) + 2000$
$= \$3333.30$

2. Determine the amounts from each supplier.	Since point B gives minimum costs, purchase: 14.29 tonnes from Airy (x) 21.43 tonnes from Berry (y) 14.28 tonnes from Cherry ($50 - x - y$).

In order for there to be exactly two decision variables, there must be exactly three sources of material. In Worked example 11 these were the three farms. If only two sources were available to provide the blended composite, then the solution would be a line segment, not a feasible region. There is *no limit* on the number of constraints (bran and husks); there can be as many of these as necessary.

15.5.2 Transportation problems

Consider a company which produces a single item in a number of factories for a number of showrooms. Transportation problems are concerned with how many *from* each factory should be sent *to* each showroom, so that each showroom's demand for the product is fully met. Furthermore, there are different shipping costs between the factories and the showrooms. In Further Mathematics we are restricted to two factories and two distributors.

WORKED EXAMPLE 12

Birmingham Bicycle Builders makes bicycles at two factories and ships them to two different distributors.

	Shipping	
	Factory A	**Factory B**
Distributor X	$6	$7
Distributor Y	$3	$5

Factory A produces 100 bicycles; Factory B produces 150. Distributor X wants 70 bicycles; Distributor Y wants 90. Furthermore, the shipping costs between factories (in dollars per bicycle) and distributors are shown in the table above.

Use linear programming techniques to determine the number of bicycles shipped between factories and distributors in order to minimise total shipping cost. That is:

a. define the decision variables and summarise in a table

b. define the constraints based on the fact that all shipments must be non-negative

c. graph the solution region

d. define the objective function

e. determine the best way to ship to minimise cost.

THINK

a. 1. Define the decision variables. (The controlling influence is what the distributors want.)

- **a.** Distributor X needs 70 bicycles. Find how many it gets from each factory.
- **b.** Proceed similarly for distributor Y.

2. Summarise the decision variables, supply and demand data in a table.

WRITE/DRAW

a. Let x = number of bicycles shipped to distributor X from factory A.
Therefore $70 - x$ is the number shipped to X from factory B.
Let y = number of bicycles shipped to distributor Y from factory A.
Therefore $90 - y$ is the number shipped to Y from factory B.

	Factory A	Factory B	Total demand from each distributor
Distributor X	x	$70 - x$	70
Distributor Y	y	$90 - y$	90
Total supply from each factory	100	150	

b. 1. Define constraints based on the fact that all shipments must be non-negative amounts.

b. $x \geq 0$
$y \geq 0$
$70 - x \geq 0$ or $x \leq 70$
$90 - y \geq 0$ or $y \leq 90$

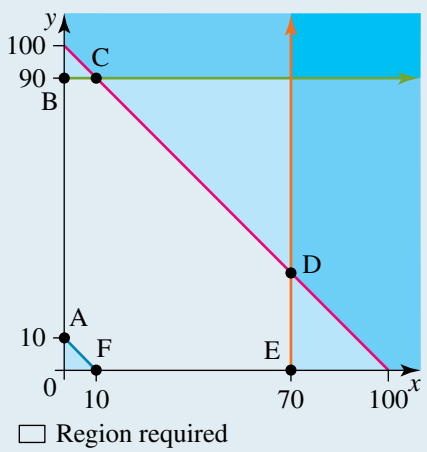
2. Define the remaining constraints on each factory's supply.

Factory A: $x + y \leq 100$
Factory B: $70 - x + 90 - y \leq 150$
$160 - x - y \leq 150$
$-x - y \leq -10$
$x + y \geq 10$

c. 1. a. Graph the solution region. Note that there are six constraints, but that the first two force us to look at the 1st quadrant only.

- **b.** Label the vertices of the feasible region.

c.



2. Determine the coordinates of the vertices. The intersection of a diagonal line with a vertical or horizontal line is done easily without using simultaneous linear equations as follows.

Point A: (0, 10)
Point B: (0, 90)
Point C: (10, 90)
Point D: (70, 30)
Point E: (70, 0)
Point F: (10, 0)

For point C:
substitute $y = 90$ into
$x + y = 100$.
For point D:
substitute $x = 70$ into
$x + y = 100$.

d. Determine the objective function from the shipping cost table.

d. Factory A to distributor X:
x bicycles at \$6 per bicycle
Factory A to distributor Y:
y bicycles at \$3 per bicycle
Factory B to distributor X:
$(70 - x)$ at \$7 per bicycle
Factory B to distributor Y:
$(90 - y)$ at \$5 per bicycle
$\text{Total} = 6x + 3y + 7(70 - x) + 5(90 - y)$
$= 6x + 3y + 490 - 7x + 450 - 5y$
$= -x - 2y + 940$

e. 1. Determine the cost for each vertex and find the minimum cost.

e. Point A: cost $= -(0) - 2(10) + 940 = 920$
Point B: cost $= -(0) - 2(90) + 940 = 760$
Point C: cost $= -(10) - 2(90) + 940 = 750*$
$*$Minimum
Point D: cost $= -(70) - 2(30) + 940 = 810$
Point E: cost $= -(70) - 2(0) + 940 = 870$
Point F: cost $= -(10) - 2(0) + 940 = 930$

2. Determine the value of each variable, and thus, the amount to ship from each factory to each distributor.

Minimum costs occur when:
$x = 10, y = 90$.
Ship 10 from factory A to distributor X.
Ship 90 from factory A to distributor Y.
Ship $(70 - 10) = 60$ from factory B to distributor X.
Ship $(90 - 90) = 0$ from factory B to distributor Y.

Note that the demand from each distributor is *always* met. In Worked example 12, distributor X received $10 + 60 = 70$ bicycles, while distributor Y received $90 + 0 = 90$. However, there is a surplus in factory B of 90 bicycles. This surplus could be shipped to a third distributor. Under what conditions will there be no surplus?

15.5.3 Manufacturing problems

Manufacturing problems involve manufacturing more than one item (in Further Mathematics, exactly two items) and having to share resources such as money, staff, parts or time. Worked examples 6 to 9 in the previous section demonstrated such problems, as did questions **11** to **14** of Exercise 15.4.

15.5.4 Limitations of linear programming techniques

Many of the linear programming problems we have encountered required that the decision variables be positive integers; for example, they referred to quantities like the number of bicycles shipped from a factory or the number of calculators manufactured. However, what happens if the solution region for such a problem has vertices that are not integers? Consider the following example.

A manufacturer makes two products — blankets and cookware sets — that need to be shipped in special containers. Each blanket weighs 3.7 kg and has a volume of 6300 cm^3. Each cookware set weighs 25 kg and has a volume of 5000 cm^3. The shipping containers can hold 250 kg and have a maximum volume of 200 000 cm^3. Use linear programming techniques to determine the largest number of items that can be shipped in one container for optimal efficiency.

The decision variables are: let x = number of blankets shipped
let y = number of cookware sets shipped.

The constraints are: $x \geq 0$
$y \geq 0$
$3.7x + 25y \leq 250$
$6300x + 5000y \leq 200\,000$ or
$6.3x + 5y \leq 200$ (dividing by 1000)

The solution region is shown in the diagram.

The vertices are found as follows.

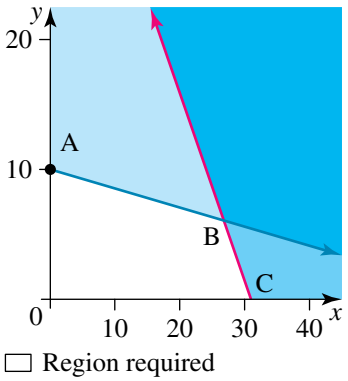
The intersection of the two lines shown (point B) is found by solving the simultaneous linear equations:

$3.7x + 25y = 250$

$6.3x + 5y = 200$

The solution is $x = 26.98,\ y = 6.01$.

The other vertices are (0, 10) and (31.7, 0).



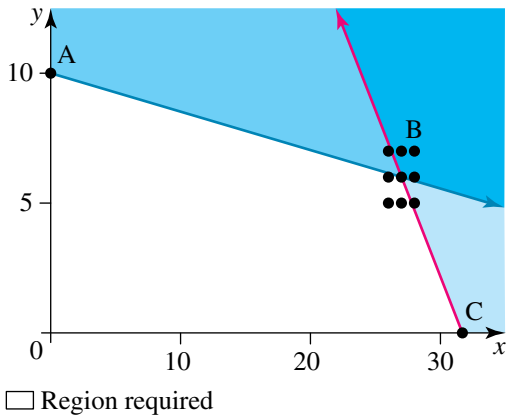
The objective function, which we wish to maximise, is:

Objective function: number = $x + y$.

The values of the objective function are:

Point A: (0, 10); number = 10
Point B: (26.9, 6.01); number = 32.91* *Maximum
Point C: (31.7, 0); number = 31.7

Note that the 'optimal' solution (26.9, 6.01) has non-integer values for x and y. How is it possible to ship 26.9 blankets and 6.01 cooking sets? This is a class of problem known as *integer programming*, which is beyond the scope of Further Mathematics; however, a possible solution is given as follows.



Consider the integer points closest to the optimal solution: (27, 6) and its nearest 8 neighbours, (26, 7), (27, 7), (28, 7), … (28, 5).

Check *each one* in turn to see if they satisfy the constraints and choose the one that has the largest objective function value.

The completion of this problem is left as question **16** in Exercise 15.5.

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Units 3 & 4 › AOS M4 › Topic 2 › Concept 4

Solving linear programming Summary screen and practice questions

Exercise 15.5 Applications

1. **WE11** A gardener can buy grass blends from three suppliers: Greener Grass, Green Lawns and Green Turf. In each case the grass contains a number of grasses; two of these are tall fescue and ryegrass. When combined, the grass can contain no more than 25% tall fescue and no more than 15% ryegrass. The gardener needs to use 2 tonnes of grass in total, by purchasing from each supplier. Each supplier contains the following amounts of tall fescue and ryegrass.

	Tall fescue	Ryegrass	Cost per tonne
Greener Grass	20%	18%	$100
Green Lawns	25%	12%	$90
Green Turf	30%	16%	$75
Maximum allowed	25%	15%	

Use linear programming techniques (sliding line or corner point) to find the amount of grass to purchase from each supplier in order to keep costs to a minimum. That is:

   a. define the decision variables
   b. define the constraints
   c. graph the solution region
   d. define the objective function
   e. determine the amount of grass purchased from each nursery.

2. A smoothie store buys orange juice from suppliers A, B and C. In each case the juice contains pulp and seeds. When combined, the juice must contain no more than 7% pulp and no more than 4% seeds. The owner wishes to purchase a total of 150 litres of juice from three suppliers. Each supplier's juice contains the following amounts of pulp and seeds.

	Pulp	Seeds	Cost per litre (c)
Supplier A	10%	4%	45
Supplier B	8%	6%	60
Supplier C	5%	3%	75
Maximum allowed	7%	4%	

Use linear programming techniques (sliding line or corner point) to find the amount of juice to purchase from each supplier in order to keep costs to a minimum. That is:

a. define the decision variables
b. define the constraints
c. graph the solution region
d. define the objective function
e. determine the amount of juice purchased from each supplier.

3. **WE12** Fast Grow nurseries grow plants at two nurseries and ship them to two different distributors.

Nursery A produces 750 plants; Nursery B produces 900 plants.

Distributor X wants 600 plants while Distributor Y wants 700 plants.

The shipping costs between nurseries (in dollars per plant) and distributors are shown in the following table.

	Nursery A	Nursery B
Distributor X	\$0.70	\$0.80
Distributor Y	\$0.60	\$0.65

Use linear programming techniques (sliding line or corner point) to determine the number of plants shipped between nurseries and distributors in order to minimise total shipping cost. That is:

a. define the decision variables and summarise in a table
b. define the constraints based on the fact that all shipments must be non-negative
c. graph the solution region
d. define the objective function
e. determine the best way to ship to minimise cost.

4. Digi Electronics produces digital radios at two factories and ships them to two different distributors. Each factory produces 27 radios per day.

Distributor X requires 23 radios while Distributor Y requires 19 radios.

The shipping costs (in dollars per radio) between the factories and the distributors are shown in the following table:

	Factory A	Factory B
Distributor X	\$7	\$12
Distributor Y	\$5	\$10

Use linear programming techniques (sliding line or corner point) to determine the number of radios shipped between factories and distributors in order to minimise total shipping cost. That is:

a. define the decision variables
b. define the constraints based on the fact that all shipments must be non-negative
c. graph the solution region
d. define the objective function
e. determine the best way to ship to minimise cost.

5. A local restaurant owner buys his orange juice from three suppliers: A, B and C. In each case the juice contains unwanted pulp and seeds. When combined, the juice must contain no more than 5% pulp and no more than 3% seeds. The owner wishes to purchase a total of 100 litres of juice from among the three suppliers. Each supplier's juice contains the following amounts of pulp and seeds:

	Pulp	Seeds	Cost per litre (C)
Supplier A	7%	5%	35
Supplier B	6%	3%	45
Supplier C	3%	2%	55
Maximum amount of contaminant allowed	5%	3%	

Supplier A charges $0.35 per litre, supplier B charges $0.45 per litre and supplier C charges $0.55 per litre. Use linear programming methods to find the amount of juice to buy from each supplier in order to minimise the cost. That is:

a. identify the decision variables
b. define the constraint inequation caused by the restriction on pulp
c. define the constraint inequation caused by the restriction on seeds
d. define the remaining constraints.

6. Using the information provided and the answers calculated in question **5**,

a. graph the solution region
b. define the objective function
c. find the amount of juice purchased from each supplier.

7. A canner of fruit salad can buy from three different suppliers, A, B and C, subject to the following restrictions on the minimum amount of peaches and cherries in the salad.

	Ingredient		Cost per kg ($)
	Peaches	Cherries	
Supplier A	23%	5%	1.10
Supplier B	20%	3%	0.90
Supplier C	30%	7%	1.40
Minimum amount of ingredient allowed	22%	4%	

Use linear programming techniques (sliding line or corner point) to find the amount of salad to buy from each supplier to make up a total of 100 kg with the minimum cost.

8. Kumquat Computers produces computers at its two factories and ships them to two distributors. Each factory produces 35 computers per day.
Distributor X requires 30 computers; distributor Y requires 25.
The shipping costs (in dollars per computer) between the factories and the distributors are shown in the table:

	Factory 1	Factory 2
Distributor X	$25	$55
Distributor Y	$20	$35

Use linear programming methods (sliding line or corner point) to find the number of computers shipped between factories and distributors in order to minimise the cost. That is:

a. define the decision variables
b. define the constraints based on the fact that all shipments must be non-negative
c. graph the solution region
d. define the objective function
e. determine the best way to ship to minimise cost.

9. A manufacturer of refrigerators has two factories: the first produces 60 refrigerators per week, the second produces 25 per week. Their two distributors require 30 and 45 refrigerators respectively. The shipping costs (in dollars per refrigerator) are given in the following table.

	Factory 1	Factory 2
Distributor X	$18	$24
Distributor Y	$20	$27

Use linear programming techniques (sliding line or corner point) to determine the best way to ship to minimise cost.

10. A shoemaker has two factories which produce 50 and 80 dozen pairs each. These are shipped to two factory outlets, Outlet X and Outlet Y, which require 50 and 30 dozen pairs each respectively. The shipping costs (in dollars per dozen pairs) are given in the following table.

	Factory 1	Factory 2
Outlet X	$14	$11
Outlet Y	$16	$8

Use linear programming techniques (sliding line or corner point) to determine the best way to ship to minimise cost.

The following information relates to question **11**, **12** and **13**.
A cereal retailer obtains a particular type of cereal from three different suppliers, P, Q and R, subject to the following restrictions on the amounts of sultanas and dried apricots.

	Sultanas	Apricots	Cost per kg ($)
Supplier P	15%	12%	1.20
Supplier Q	19%	10%	0.90
Supplier R	12%	15%	1.10
Minimum amount of ingredient needed	14%	11%	

The total amount of cereal required is 100 kg.

11. MC If x = the number of kilograms of cereal from Supplier P and y = the number of kilograms of cereal from Supplier Q, then the constraint inequality caused by the restriction on sultanas is:

A. $15x + 12y \leq 1.20$ **B.** $3x + 7y \geq 200$

C. $3x + 7y \leq 200$ **D.** $3x + 5y \geq 400$

E. $3x + 5y \leq 400$

12. MC The inequality which is not a constraint for this situation is:

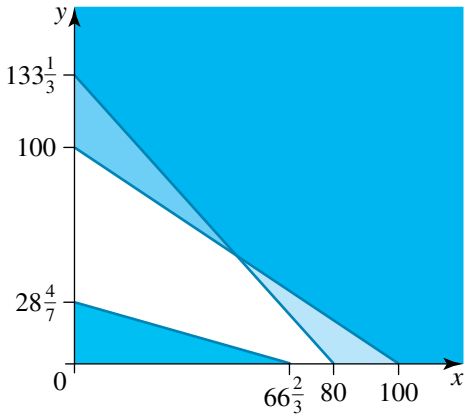
A. $3x + 5y \leq 400$ **B.** $x \geq 0$

C. $y \geq 0$ **D.** $x + y \geq 100$

E. $x + y \leq 100$

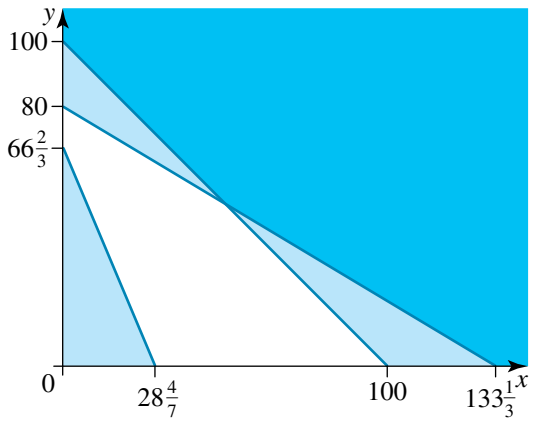
13. MC The feasible region for this situation is:

A.



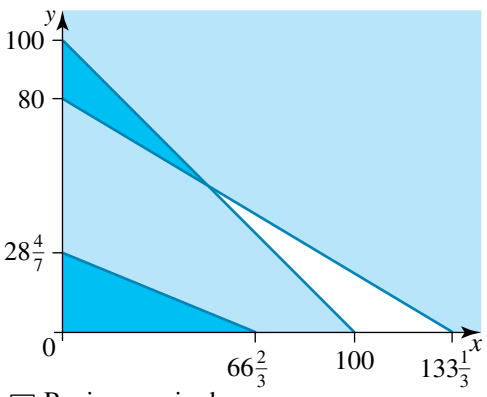
☐ Region required

B.



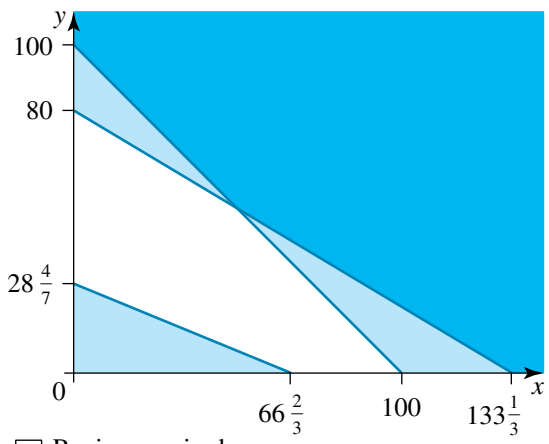
☐ Region required

C.



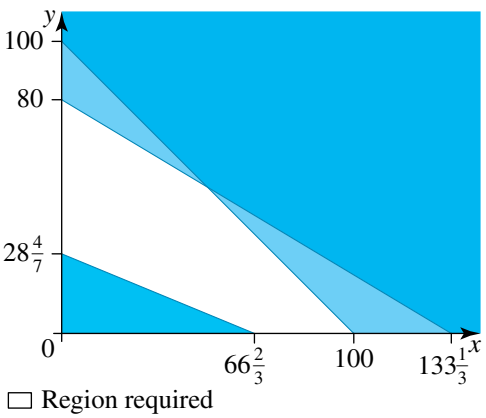
☐ Region required

D.



☐ Region required

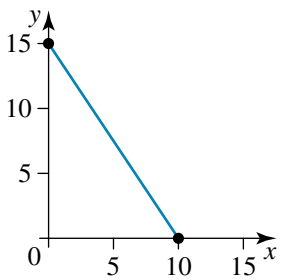
E.



14. MC The line shown below has the equation of the form $2y + 3x = 6c$.

The value of c must therefore be:

A. 2
B. 5
C. 6
D. -6
E. -5

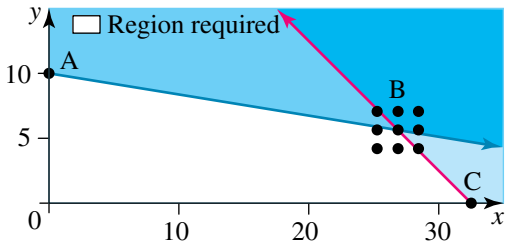


15. A refinery gets its petrol from three oil wells. Each well provides oil with a certain amount of lead and iso-octane according to the following table. The blended product must contain a *maximum* of 3.5% lead and a *minimum* of 65% iso-octane.

	Ingredient		
	Lead	**Iso-octane**	**Cost per litre($)**
Oil well A	4%	70%	0.24
Oil well B	2%	60%	0.22
Oil well C	6%	80%	0.26
Amount of ingredient allowed	3.5% maximum	65% minimum	

Use linear programming techniques (sliding line or corner point) to determine how much to buy, at minimum cost, from each well per 100 litres of petrol refined.

16. Refer back to the text example on pages 844–5, which illustrated a limitation of the linear programming techniques used in this topic. The optimal solution involved non-integers but the situation required integer solutions. The 9 integer points surrounding the optimal solution are shown in the diagram $\{(26, 7), (27, 7), (28, 7), ..., (28, 5)\}$.



The constraints are: $x \geq 0$

$y \geq 0$

$3.7x + 25y \leq 250$

$6.3x + 5y \leq 200$

The objective function is: number = $x + y$.

Check each integer point to see if the values satisfy the constraints and determine which one maximises the objective function. That is, find the maximum numbers of blankets and cookware sets that can be shipped.

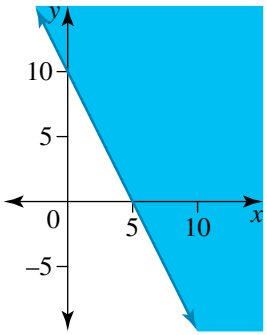
15.6 Review: exam practice

A summary of this topic is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC The region shown represents the inequality:

A. $x + 2y \geq 10$
B. $x + 2y \leq 10$
C. $2x + y \geq 10$
D. $2x + y \leq 10$
E. $2x - y \geq 10$



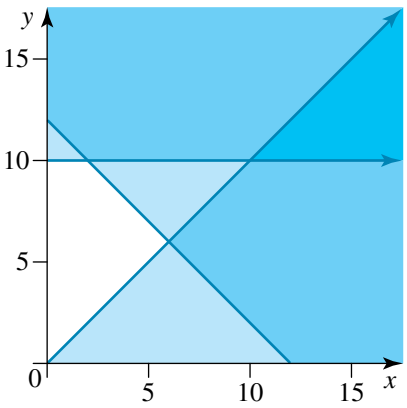
Region required

2. MC Consider the inequality $4x + 3y \leq 12$. Which of the following statements is true? Of the three points (0, 0), (3, 0) and (0, 4):

A. (3, 0) and (0, 4) are the only ones in the solution region.
B. (0, 0) and (3, 0) are the only ones in the solution region.
C. (0, 4) and (0, 0) are the only ones in the solution region.
D. (0, 0) is the only one in the solution region.
E. All 3 are in the solution region.

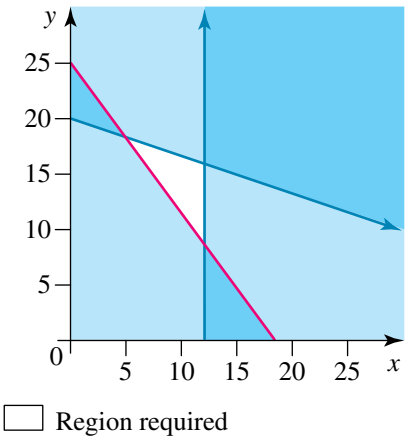
3. MC The clear area in the figure below represents the intersection of the solution regions of three inequalities in the 1st quadrant. One of the inequalities is $x - y \leq 0$. The other two inequalities are:

A. $y \leq 10$ and $x + y \leq 0$
B. $y \leq 10$ and $x + y \leq 12$
C. $y \leq 10$ and $x + y \geq 12$
D. $y \geq 10$ and $x + y \leq 12$
E. $y \geq 10$ and $x + y \geq 12$



Region required

4. MC Consider the graph shown.



a. The pink line indicates the boundary of the region with inequality:

A. $25x + 18y \leq 450$
B. $25x + 18y \geq 450$
C. $9x + 5y \leq 125$
D. $9x + 5y \geq 125$
E. none of these

b. The point (0, 20) lies on the boundary of the region with inequality:

A. $2x + 5y \leq 100$
B. $2x + 4y \geq 100$
C. $5x + 2y \geq 100$
D. $5x + 2y \leq 100$
E. none of these

c. If the objective function is Profit $= x + 2y$, then the values of x and y which maximise profit are:

A. (5, 18)
B. (12, 8.2)
C. (12, 15.2)
D. (12, 8.3)
E. (0, 20)

5. MC Refer to the figure in question **3**.

a. Two of the vertices of the solution region are (0, 0) and (0, 10). The other two vertices are:

A. (10, 2) and (5, 5)
B. (2, 10) and (5, 6)
C. (2, 10) and (6, 5)
D. (2, 10) and *d* (6, 6)
E. (2, 12) and (6, 6)

b. If these inequalities were from a linear programming problem where the objective function is Profit $= -2.3x + 5.1y$, then the values of x and y which maximise profit are:

A. (0, 0)
B. (0, 10)
C. (10, 2)
D. (6, 6)
E. (2, 10)

6. MC At a fruit farm the cost of strawberries (x) is \$3.00 per punnet and the cost of raspberries (y) is \$2.50 per punnet. A supermarket wishes to purchase no more than 150 punnets and wishes to pay no more than \$400. A pair of inequalities which satisfies these constraints is:

A. $2x + 2y \leq 400$
$5x + 6y \leq 400$

B. $x + y \leq 150$
$x + y \leq 400$

C. $6x + 5y \leq 800$
$x + y \leq 150$

D. $5x + 6y \leq 800$
$x + y \leq 150$

E. $6x + 5y \leq 800$
$2x + 2y \leq 150$

7. MC In a blending problem with only two decision variables, the maximum number of constraints on these variables is:

A. 1 constraint inequality
B. 2 constraint inequalities
C. 3 constraint inequalities
D. 4 constraint inequalities
E. unlimited

8. MC A horticulturist wants to produce an organic fertiliser which is a blend of fertilisers from Farmer Higgs, Farmer Hoggett and Farmer Hubbard. The fertilisers are contaminated with grass seeds and the basis of the fertiliser is sheep manure. Each fertiliser contains the following amounts of seeds and manure. The costs are also shown.

	Seeds	Manure	Cost per tonne
Farmer Higgs	5%	60%	$45
Farmer Hoggett	2%	30%	$40
Farmer Hubbard	7%	60%	$50
Amount of constituent	5%maximum	50%minimum	

The horticulturist wishes to produce 50 tonnes of the blended fertiliser at minimum cost.
Let x = the number of tonnes of fertiliser from Higgs and y = the number of tonnes of fertiliser from Hoggett.

a. The constraint inequality caused by the restriction on seeds is:

A. $2x + 5y \leq 100$ **B.** $x + y \leq 50$
C. $5x + 2y \geq 100$ **D.** $x \geq 0$
E. $2x + 5y \geq 100$

b. The constraint inequality caused by the restriction on manure is:

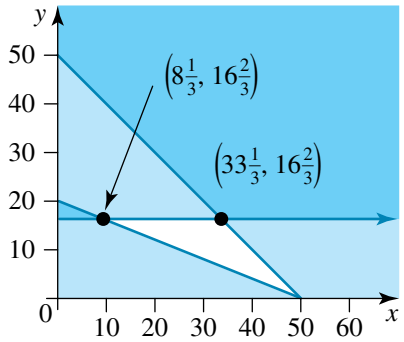
A. $2x + 5y \geq 100$ **B.** $3y \leq 50$
C. $y \geq 0$ **D.** $3y \geq 50$
E. $12x + 3y \leq 50$

c. The objective function would be:

A. Cost $= 50 - 5x - 10y$ **B.** Cost $= 2500 - 95x - 90y$
C. Cost $= 2500 + 44x + 39y$ **D.** Cost $= 2500 - 5x - 10y$
E. Cost $= 45x + 40y - 50$

d. At right is the feasible region for this situation.
The number of tonnes of fertiliser obtained from Farmer Hubbard is closest to:

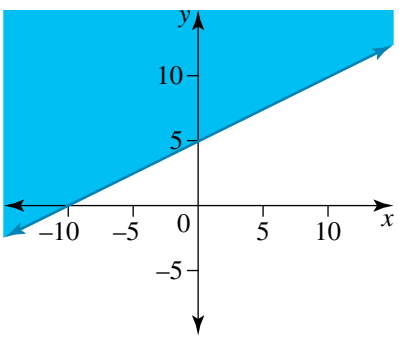
A. 0
B. 8
C. 16
D. 33
E. 50



Region required

Short answer

1. State the inequality of the region shown in the diagram.



Region required

2. a. Sketch the region represented by the inequality $2x + 3y > 24$.

b. Sketch the region represented by the simultaneous linear inequalities $-x + 2y \leq 10$ and $2x + 3y \leq 24$.

c. Find the intersection point of the two lines that form boundaries for the two regions in question **2b**.

3. A manufacturer of school desks requires three workers to build each desk. There are two models of desk; plain and computer. The time spent on each desk by each worker is outlined in the following table.

	Worker A	Worker B	Worker C
Plain desk	15 minutes	20 minutes	30 minutes
Computer desk	50 minutes	40 minutes	30 minutes
Total time available for each worker	350 minutes	360 minutes	450 minutes

   a. Define the decision variables.
   b. State the constraints on these variables.
4. Sketch the feasible region for the data in question 3
5. For the desk manufacturer in question 3, a plain desk results in a profit of $50, while a computer desk earns $90 profit.
   a. State the objective function.
   b. Find the coordinates of the vertices of the feasible region.
   c. Find the number of each type of desk to manufacture to yield maximum profit.
6. A drug company can purchase its ingredients from three suppliers. The ingredients must contain a minimum amount of three chemicals: Alchemia, Biologica and Chemoblast. Each supplier's mix, the minimum requirements and the costs are displayed in the following table.

	Alchemia	Biologica	Chemoblast	Cost per kg
Xerxes Chemicals	2%	4%	1%	$30
Yayhoo Drugs	3%	8%	17%	$22
Zorro Manufacturing	8%	9%	2%	$26
Minimum amount of chemical required	4%	6%	2.65%	

   Altogether, the company wants to purchase 100 kg of ingredients. By letting x represent the amount of ingredients purchased from Xerxes and y be the amount of ingredients purchased from Yayhoo, state the *constraints* on the variables.
7. The Prime Car Company requires batteries for its cars which are manufactured in two factories. Supplier A can ship 31 batteries per day, while supplier B can ship 29.
   Factory 1 requires 23 batteries per day; factory 2 requires 19.

	Supplier A	Supplier B
Factory 1	$3	$13
Factory 2	$11	$7

   The shipping costs are summarised in the table.
   a. Define the decision variables for this problem.
   b. List the constraints on these variables given that all shipments must be non-negative amounts.
   c. List the constraints due to the fact that each supplier has a maximum amount of batteries.

Extended response

1. The Clive Norman electrical store wishes to stock electric stoves and microwaves. They have a maximum of $12 000 to spend. Each microwave costs $250, while each stove costs $300, and they can be sold to consumers for $310 and $370 respectively. However, the supplier states that it takes 6 hours to make a microwave, but 4 hours to make a stove, and that they can give only 240 hours to meeting Clive Norman's demand.
   a. State the decision variables.
   b. State the constraint imposed by the supplier's time limit.

c. State the constraint imposed by Clive Norman's investment limit.
d. Graph the solution region.
e. State the objective function.
f. Determine the best number of each model to order to maximise profits.

2. Complete the blending problem that was posed in question 6 in the 'Short answer' section for the production of 100 kilograms of drug at minimum cost. The relevant data are replicated in the table.

	Alchemia	Biologica	Chemoblast	Cost per kg
Xerxes Chemicals	2%	4%	1%	$30
Yayhoo Drugs	3%	8%	17%	$22
Zorro Manufacturing	8%	9%	2%	$26
Minimum amount of chemical required	4%	6%	2.65%	

3. A manufacturer ships car engines to two distributors from its two factories.
Factory 1 can produce 300 engines per month; factory 2 produces 600.
Distributor 1 requires 200 engines per month, while distributor 2 requires 450.
The shipping costs between factory and distributor are shown in the following table.

	Factory 1	Factory 2
Distributor 1	$15	$20
Distributor 2	$20	$15

a. Define the decision variables.
b. State the constraints.
c. Graph the solution region.
d. Define the objective function.
e. Find the optimal values of the decision variables.
f. Unfortunately, Factory 2's output drops from 600 to 350. This means that there is no surplus production. Discuss what happens to the solution region and the resulting optimal values of the decision variables.

4. The Fairday Fitness Equipment Company manufactures treadmills and exercise bikes. The company wishes to make and sell as many of each as possible but is restricted by the time (in hours) available each month for assembling, testing and transporting these items, as set out in the table.

	Assembly	Testing	Transport
Treadmills	1	2	3
Exercise bikes	3	1	4
Maximum time available (hours)	240	200	400

a. Use linear programming techniques to determine the maximum number of treadmills and exercise bikes that should be made and sold.
b. If the items are transported by air instead of by road, the time required is drastically reduced. It will then require only 1 hour to transport each of the items, and the maximum time available for transporting items per month will be reduced to only 120 hours. What effect will these changes have on the number of items transported?

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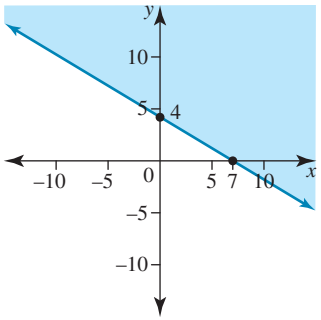
Answers

Topic 15 Linear programming

Unless stated otherwise, for all graphical answers to topic, ☐ indicates the region required.

Exercise 15.2 Linear inequalities

1.



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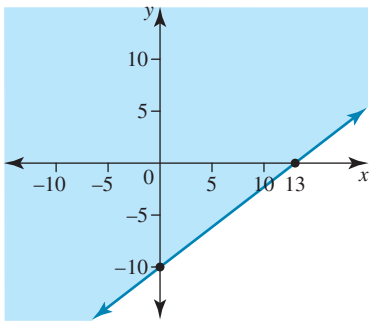

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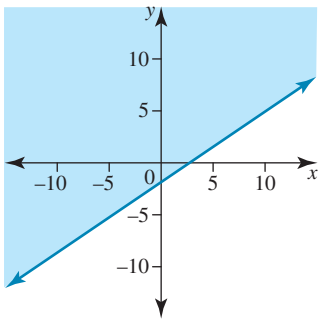
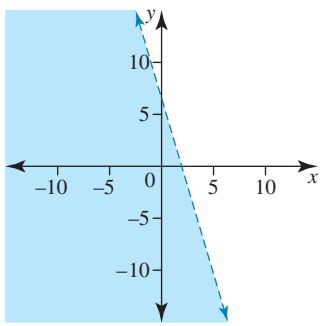


4. a.



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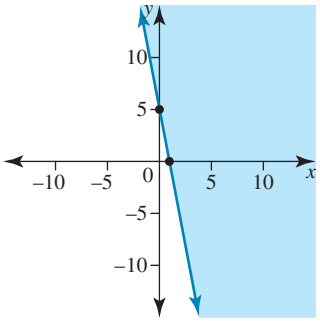

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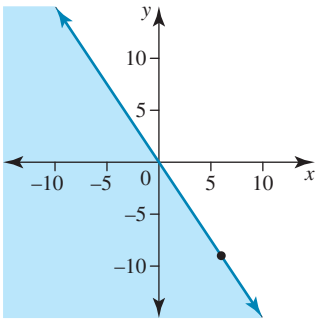

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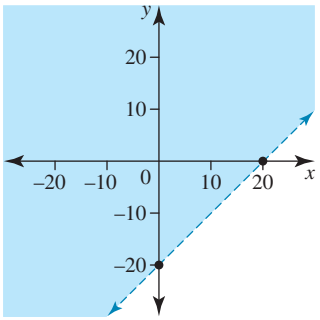

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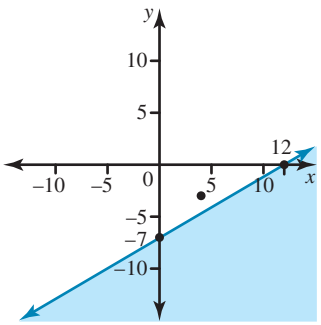

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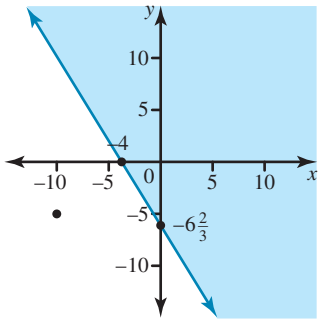

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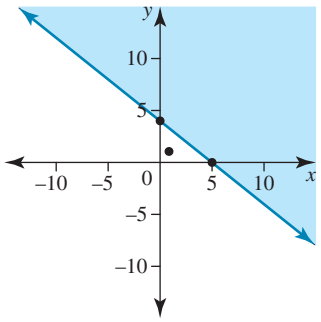

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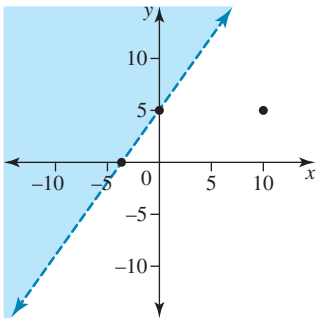

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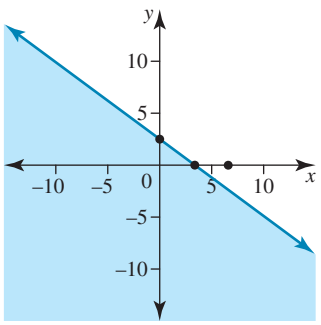

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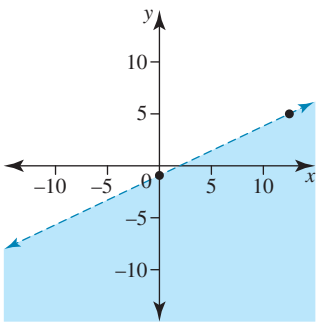
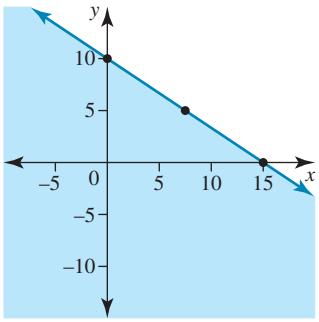


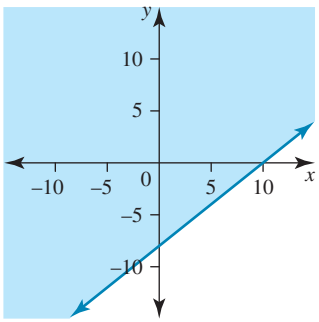
11. B, E

12.



13. C

14.



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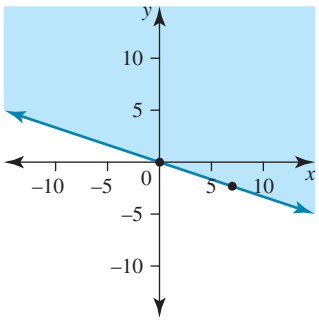

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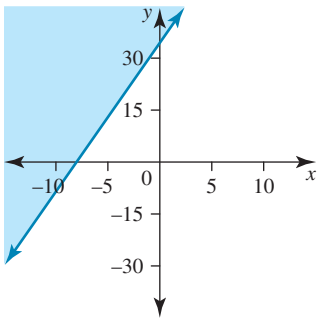

Exercise 15.3 Simultaneous linear inequalities

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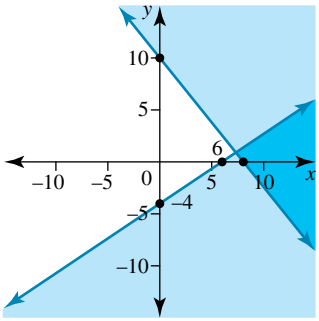

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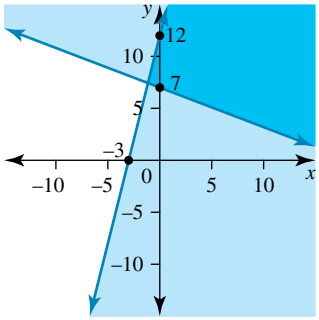

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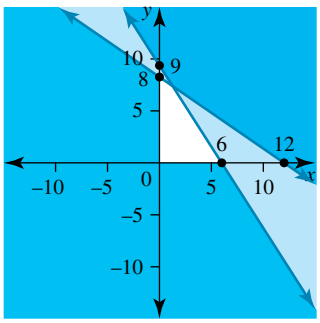

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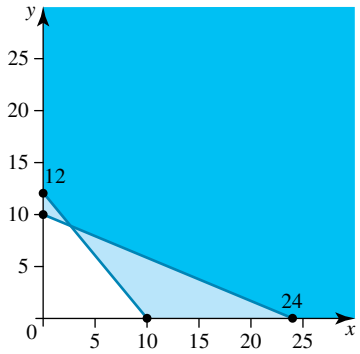

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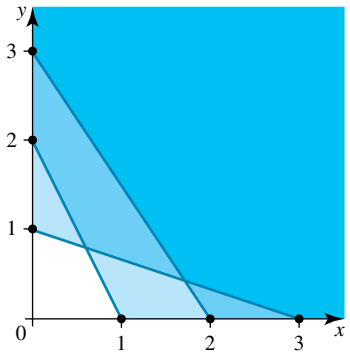

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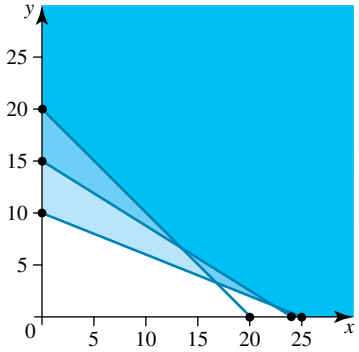

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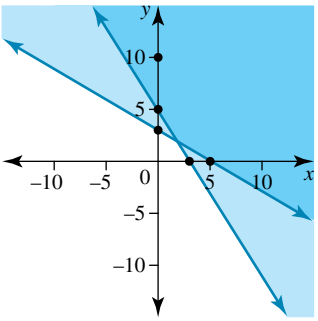

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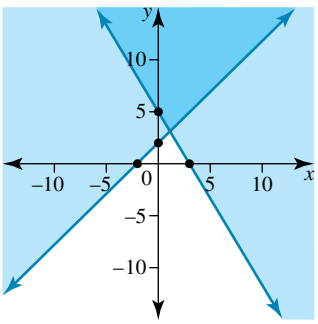

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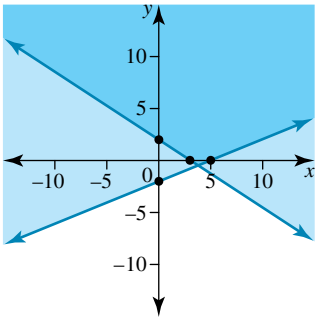

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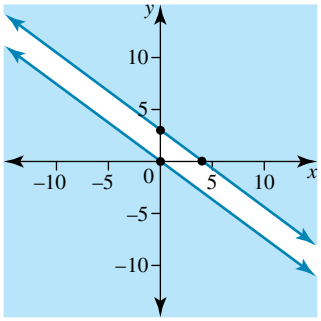

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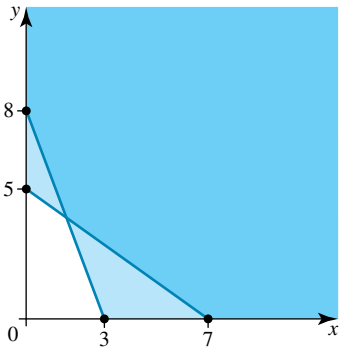

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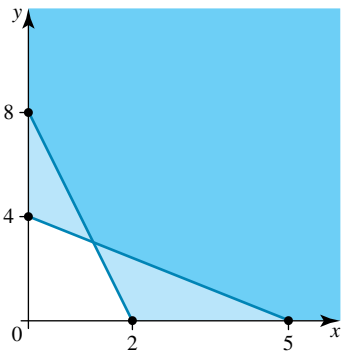

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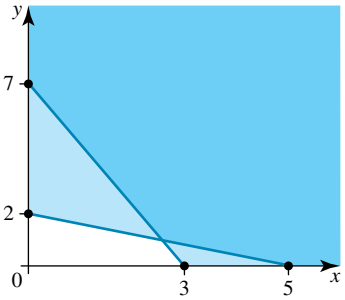

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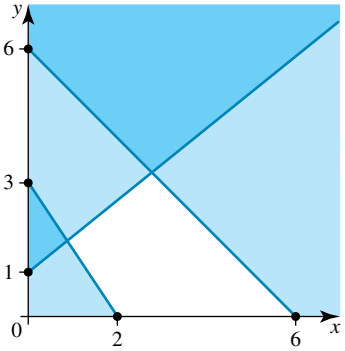

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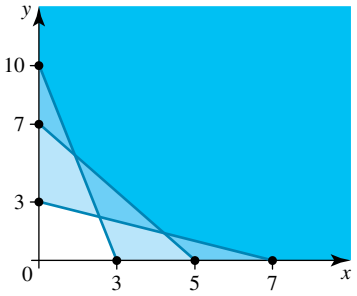

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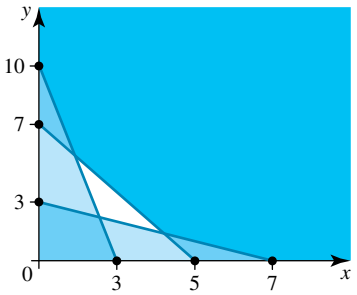
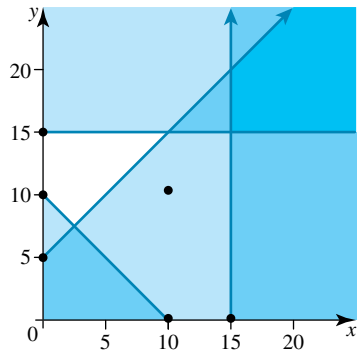


17. D

18. a. $2x + 2y \geq 20, -x + y \geq 5, x \leq 15, y \leq 15, x \geq 0, y \geq 0$

b.



c. (10, 10) does not lie in the solution region.

Exercise 15.4 Linear programming

1. a. x = the number of 2-person cabins rented,
y = the number of 3-person cabins rented

b. $40x + 45y \leq 3000$

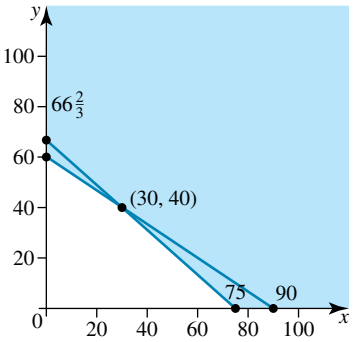
c. $2x + 3y \leq 180$

2. x = amount invested in bonds,
y = amount invested in real estate mortgages

a. $x + y \leq 1\,000\,000$

b. $x \geq 2y$ or $x - 2y \geq 0$

3.



4. Total return = $0.06x + 0.11y$

5. The club should rent 60 3-person cabins (and not rent any 2-person cabins).

6. The optimal 'mix' is to invest one-third in real estate (at 11%) and the rest in bonds (at 6%) in order to maximise return.

7. x = number of flat screen TVs,
y = number of curved screen TVs
Objective function: number = $x + y$
The company should make 15 000 flat screen TVs and 0 curved screen TVs.

8. Teacher A: $20x + 8y \leq 320$
Teacher B: $10x + 2y \leq 100$
10 text chapters and 0 tests

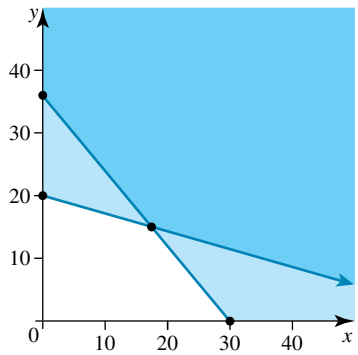
9. x = number of hours on football,
y = number of hours on basketball
Objective function: earnings = $6.5x + 10y$
Kyle should play football for 20 hours and basketball for 10 hours to maximise his earnings at $230.

10. Staff to build the sofas: $9x + 8y \leq 207$
Shipping and packaging constraint: $2x + 5y \leq 75$
The optimal solution is when 15 leather and 9 cloth sofas are produced.

11. a. x = number of model A made
y = number of model B made

b. $x \geq 0, y \geq 0, 5x + 18y \leq 360, 5x + 4y \leq 150$

c.



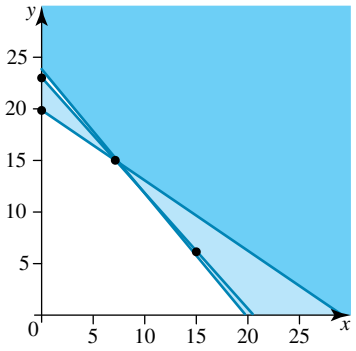
d. Profit = $2.5x + 4y$

e. $x = 18, y = 15$, profit = $105

12. a. x = number of sedans sprayed
y = number of utilities sprayed

b. $x \geq 0, y \geq 0, 5x + 7y \leq 140, 9x + 8y \leq 183, 6x + 5y \leq 120$

13. a.



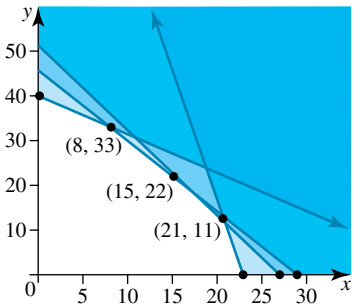
b. Number = $x + y$

c. $x = 7, y = 15$, vehicles = 22

14. a. x = models of monitor
y = models of disk drive

b. $x \geq 0, y \geq 0, 7x + 8y \leq 320, 11x + 7y \leq 319, 22x + 12y \leq 594, 11x + 3y \leq 264$

15. a.

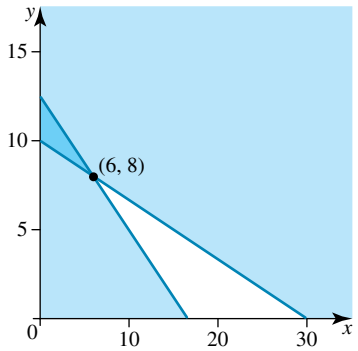


b. Profit = $150\,000x + 120\,000y$

$x = 8, y = 33$, profit = $5 160 000

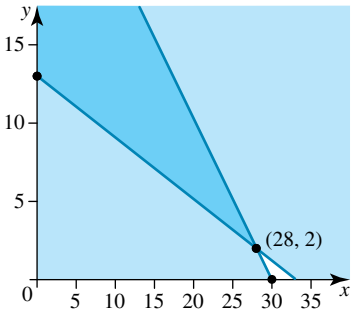
16. The solution would be $x = 0, y = 6$, with a profit of $780.

17. x = number of apples
y = number of pieces of cheese
Constraints: $x \geq 0, y \geq 0, 3x + 10y \leq 98, 3x + 4y \geq 50$
Objective function:
Calorie intake = $100x + 85y$
Solution: $x = 6, y = 8$, calories = 1280

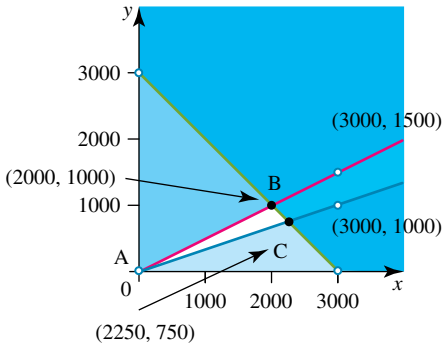


18. x = indoor paint
y = outdoor paint
Constraints: $x \geq 0, y \geq 0, 3x + 8y \leq 100, x + y \geq 30$
Objective function:
Profit = $2.50x + 3.25y$

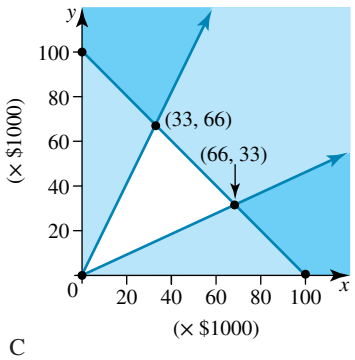
Solution: $x = 33.33$, $y = 0$, profit = \$83.33



19. x = number of roses
y = number of tulips
Constraints: $x \geq 0$, $y \geq 0$, $x + y \leq 3000$, $x \leq 3y$, $x \geq 2y$
Objective function: income = $15x + 6y$
Solution: $x = 2250$, $y = 750$,
Income = \$38 250



20. x = value of shares
y = value of gold
Constraints: $x \geq 0$, $y \geq 0$, $x + y \leq 100\,000$, $x - 2y \leq 0$, $x - 0.5y \geq 0$
Objective function:
Profit = $0.1x + 0.16y$
Solution: $x = 33\,333.33$,
$y = 66\,666.67$, profit = \$14 000



21. C

22. a. $x + y \leq 50$
$12\,000x + 12\,500y \leq 490\,000$
$y \geq x$
$x \geq 0$
$y \geq 0$

b. OC is $y \geq x$
BF is $x + y \leq 50$
AE is $12\,000x + 12\,500y \leq 490\,000$

c. C (25, 25)
D (20, 20)

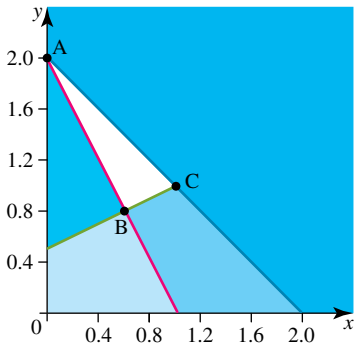
d. 20 sedans and 20 wagons

Exercise 15.5 Applications

1. a. Let x = tonnes of grass from Greener Grass
Let y = tonnes of grass from Green Lawns
Let $(2 - x - y)$ = tonnes of grass from Green Turf

b. $10x + 5y \geq 10$, $4y - 2x \geq 2$, $x \geq 0$, $y \geq 0$, $2 - x - y \geq 0$ or $x + y \leq 2$

c.



Point A: (0, 2),
Point B: $\left(\frac{3}{5}, \frac{4}{5}\right)$,
Point C: (1, 1)

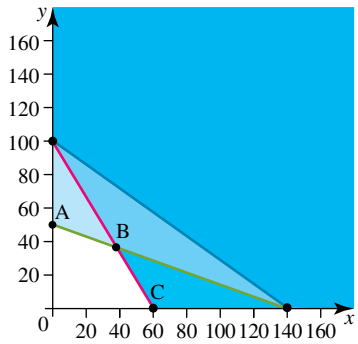
d. Cost = $25x + 15y + 150$

e. $\frac{3}{5}$ of a tonne from Greener Grass, $\frac{4}{5}$ of a tonne from Green Lawns, $\frac{3}{5}$ of a tonne from Green Turf

2. a. x = litres of orange juice from Supplier A
y = litres of orange juice from Supplier B
$(150 - x - y)$ = litres of orange juice from Supplier C

b. $5x + 3y \leq 300$, $x + 3y \leq 150$, $x \geq 0$, $y \geq 0$, $150 - x - y \geq 0$ or $x + y \leq 150$

c.



Point A: (0, 50)
Point B: (37.5, 37.5)
Point C: (60, 0)

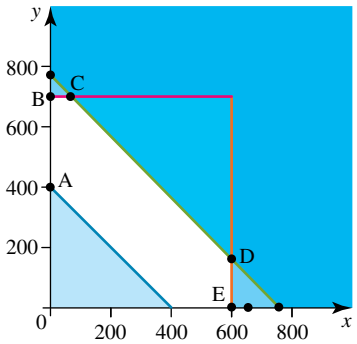
d. Cost = $11\,250 - 30x - 15y$

e. 60 litres from Supplier A
0 litres from Supplier B
90 litres from Supplier C

3. a.

	Nursery A	Nursery B	Total
Distributor X	x	$(600 - x)$	600
Distributor Y	y	$(700 - y)$	700
Total	750	900	

b. $x \geq 0$
$y \geq 0$
$600 - x \geq 0$ or $x \leq 600$
$700 - y \geq 0$ or $y \leq 700$
Nursery A: $x + y \leq 750$
Nursery B: $x + y \geq 400$



c. Point A : (0, 400)
Point B : (0, 700)
Point C : (50, 700)
Point D : (600, 150)
Point E : (600, 0)
Point F : (400, 0)

d. Total cost $= 935 - 0.1x - 0.05y$

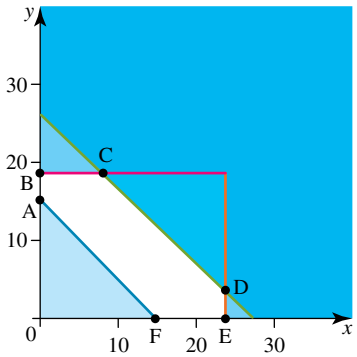
e. Minimum cost is at Point D.
Ship 600 from nursery A to distributor X.
Ship 150 from nursery A to distributor Y.
Ship 0 from nursery B to distributor X.
Ship 550 from nursery B to distributor Y.

4. a.

	Factory A	Factory B	Total
Distributor X	x	$(23 - x)$	23
Distributor Y	y	$(19 - y)$	19
Total	27	27	

b. $x \geq 0$ $y \geq 0$
$23 - x \geq 0$ or $x \leq 23$
$19 - y \geq 0$ or $y \leq 19$
Factory A: $x + y \leq 27$
Factory B: $x + y \geq 15$

c.



Point A: (0, 15)
Point B: (0, 19)
Point C: (8, 19)
Point D: (23, 4)
Point E: (23, 0)
Point F: (15, 0)

d. Total cost $= 466 - 5x - 5y$

e. Minimum cost is at point C or D: (8, 19) or (23, 4).
Ship 8 or 23 from factory A to distributor X.
Ship 19 or 4 from factory A to distributor Y.
Ship 15 or 0 from factory B to distributor X.
Ship 0 or 15 from factory B to distributor Y.

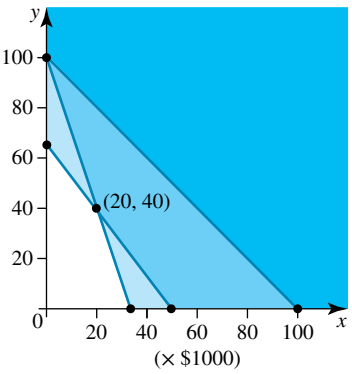
5. a. x = litres purchased from supplier A
y = litres purchased from supplier B
$100 - x - y$ = litres purchased from supplier C

b. $4x + 3y \leq 200$

c. $3x + y \leq 100$

d. $x \geq 0$, $y \geq 0$, $x + y \leq 100$

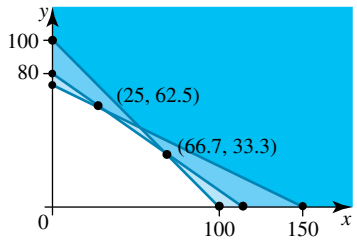
6. a.



b. Cost $= -20x - 10y + 5500$ (in cents)

c. $x = 20$, $y = 40$. Cost = $47.00. Purchase 20 litres from A, 40 litres from each of B and C.

7. x = kg purchased from supplier A
y = kg purchased from supplier B
$100 - x - y$ = kg purchased from supplier C
Constraints: $7x + 10y \leq 800$, $2x + 4y \leq 300$, $x \geq 0$, $y \geq 0$, $x + y \leq 100$

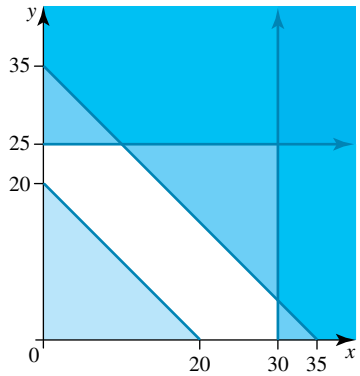


Objective function:
Cost $= -0.3x - 0.5y + 140$
$x = 25$, $y = 62.5$, Cost = $101.25
Purchase 25 kg from A, 62.5 kg from B and 12.5 kg from C.

8. a. x = computers from factory 1 to distributor X
y = computers from factory 1 to distributor Y
$(30 - x)$ = computers from factory 2 to distributor X
$(25 - y)$ = computers from factory 2 to distributor Y

b. $x \geq 0$, $y \geq 0$, $x \leq 30$, $y \leq 25$, $x + y \leq 35$, $x + y \geq 20$

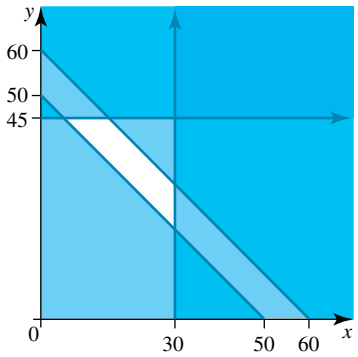
c.



d. Cost $= -30x - 15y + 2525$

e. $x = 30$, $y = 5$, cost $= \$1550$
Ship 30 from factory 1 to distributor X, 5 from factory 1 to distributor Y, 20 from factory 2 to distributor Y

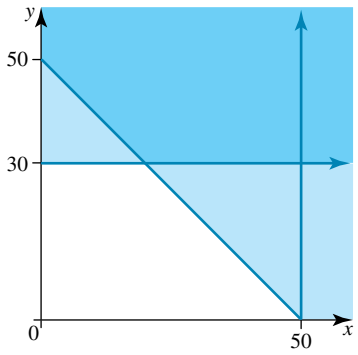
9. Variables:
x = refrigerators from factory 1 to distributor X
y = refrigerators from factory 1 to distributor Y
$(30 - x)$ = refrigerators from factory 2 to distributor X
$(45 - y)$ = refrigerators from factory 2 to distributor Y



$x \geq 0$, $y \geq 0$, $x \leq 30$, $y \leq 45$,
$x + y \leq 60$, $x + y \geq 50$
Cost $= -6x - 7y + 1935$
$x = 15$, $y = 45$, cost $= \$1530$

Ship 15 from factory 1 to distributor X, 45 from factory 1 to distributor Y, 15 from factory 2 to distributor X.

10. Variables:
x = dozen pairs of shoes from factory 1 to distributor X
y = dozen pairs of shoes from factory 1 to distributor Y
$(50 - x)$ = dozen pairs of shoes from factory 2 to distributor X



$(30 - y)$ = dozen pairs of shoes from factory 2 to distributor Y
$x \geq 0$, $y \geq 0$, $x \leq 50$, $y \leq 30$, $x + y \leq 50$, $x + y \geq 0$
Cost $= 3x + 8y + 790$
$x = 0$, $y = 0$. $x = 0$, $y = 0$
Cost $= \$790$
Ship 50 from factory 2 to distributor X, 30 from factory 2 to distributor Y

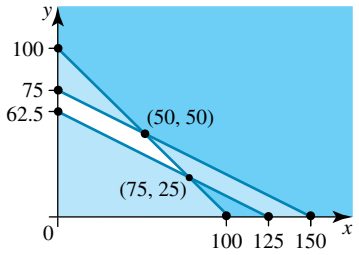
11. B

12. D

13. D

14. B

15. x = litres purchased from supplier A
y = litres purchased from supplier B
$100 - x - y$ = litres purchased from supplier C
Constraints: $2x + 4y \geq 250$,
$x + 2y \leq 150$, $x \geq 0$,
$y \geq 0$, $x + y \leq 100$



Objective function:
Cost $= -0.02x - 0.04y + 26$
$x = 0$, $y = 75$, cost $= \$23$
$x = 50$, $y = 50$, cost $= \$23$
Purchase 75 L from B and 25 L from C (none from A) or purchase 50 L from A and 50 L from B (none from C).

16. Feasible solutions are (26, 5), (27, 5), (26, 6). Choose either of the last 2 points ($x + y = 32$).
The maximum number of blankets and cookware sets that can be shipped is 32.

15.6 Review: exam practice

Multiple choice

1. D

2. E

3. B

4. **a.** B **b.** A **c.** C

5. **a.** D **b.** B

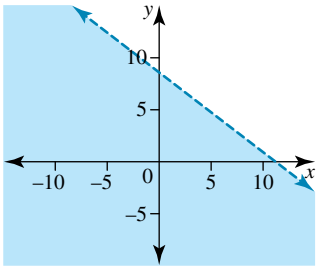
6. C

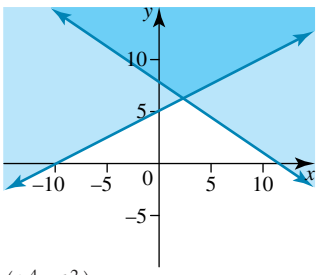
7. E

8. **a.** E **b.** B **c.** D **d.** A

Short answer

1. $-x + 2y \leq 10$

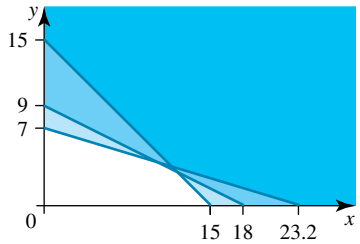
2. a.


b.


c. $\left(2\frac{4}{7}, 6\frac{2}{7}\right)$

3. a. x = plain, y = computer

b. $x \geq 0,\ y \geq 0,\ 15x + 50y \leq 350,$
$20x + 40y \leq 360,\ 30x + 30y \leq 450$

4.


5. a. Profit $= 50x + 90y$

b. (0, 0), (0, 7), (10, 4), (12, 3), (15, 0)

c. $x = 12, y = 3$, profit = \$870

6. $6x + 5y \leq 400;\ 5x + y \leq 300;\ -x + 15y \geq 65$

7. a. Let x be the amount from supplier A to factory 1,
$(23 - x)$ be the amount from supplier B
Let y be the amount from supplier A to factory 2,
$(19 - y)$ be the amount from supplier B

b. $x \geq 0,\ y \geq 0,\ x \leq 23,\ y \leq 19$

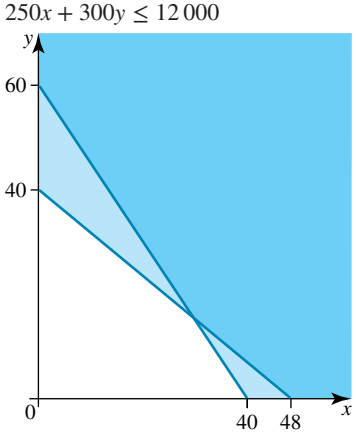
c. $x + y \leq 31, x + y \geq 13$

Extended response

1. a. x = number of microwaves, y = number of stoves

b. $6x + 4y \leq 240$

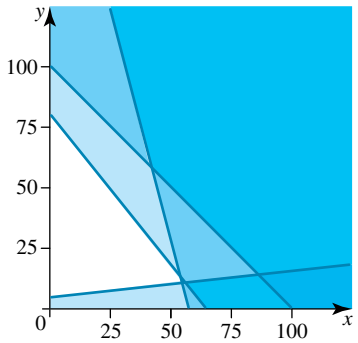
c. $250x + 300y \leq 12\,000$

d.


e. Profit $= 60x + 70y$

f. $x = 30, y = 15$, profit = \$2850

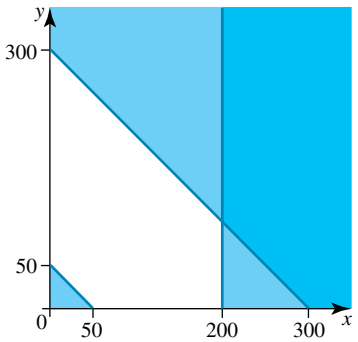
2. Let x = amount (kg) from Xerxes, y = amount (kg) from Yayhoo and $100 - x - y$ = amount (kg) from Zorro
Constraints: $6x + 5y \leq 400,$
$5x + y \leq 300,\ -x + 15y \geq 65,$
$x \geq 0,\ y \geq 0,\ x + y \leq 100$
Objective function:
Cost $= 2600 + 4x - 4y$



Solution: $x = 0,\ y = 80,$ cost = \$2280
None supplied by Xerxes, 80 kg supplied by Yayhoo and 20 kg by Zorro

3. a. Let x = shipment from factory 1 to distributor 1, $(200 - x)$ from factory 2 to distributor 1
Let y = shipment from factory 1 to distributor 2, $(450 - y)$ from factory 2 to distributor 2

b. $x \geq 0,\ y \geq 0,\ x \leq 200,\ y \leq 450,\ x + y \geq 50,\ x + y \leq 300$

c.


d. Cost $= -5x + 5y + 10\,750$

e. $x = 200,\ y = 0$, cost = \$9750

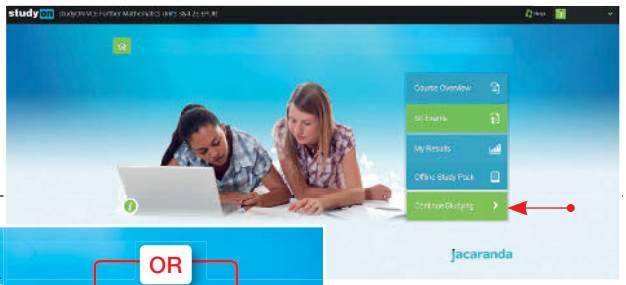
f. The solution region 'collapses' to just the line $x + y = 300$ between $x = 0$ and $x = 200$.
The objective function remains unchanged. In theory, any value of x along this line is possible, but by examining the objective function, the larger the x-value, the smaller the cost, so $x = 200,\ y = 100$ is the optimal solution, with a cost of \$10 250.

4. a. 80 treadmills, 40 exercise bikes

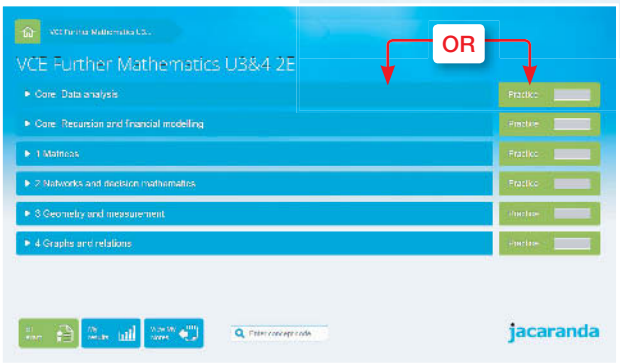
b. Number of treadmills and bikes transported could be any combination of 60 of each to 80 and 40 respectively, as long as the sum equals 120 items.

TOPICS 14 and 15

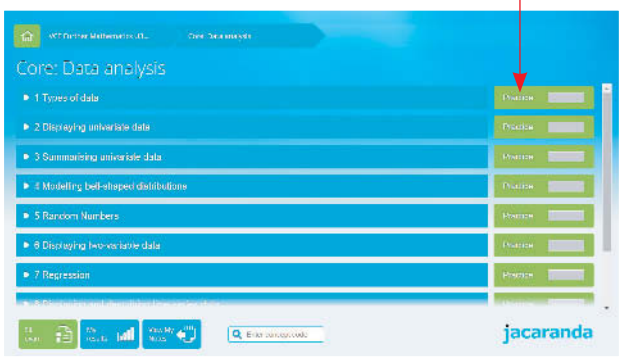
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



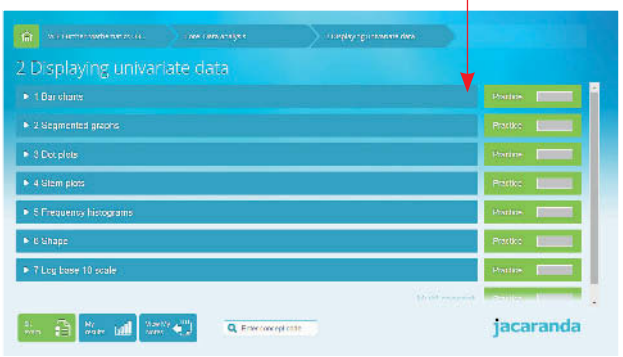
- Select your **course** *Further Mathematics VCE Units 3 & 4* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



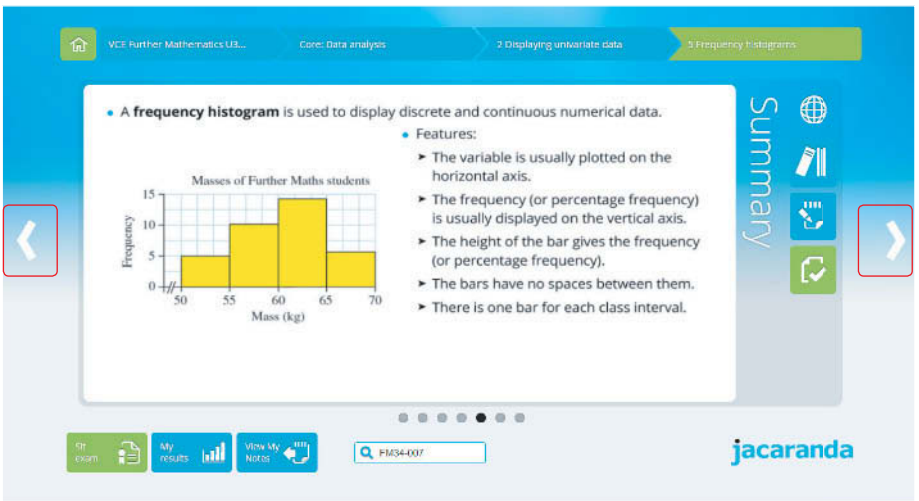
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



GLOSSARY

adjacency matrix A matrix that displays all of the possible one-stage pathways for a network. The sum of each row is equal to the outdegree of each vertex, and the sum of each column is equal to the indegree of each vertex.

adjacent side The side next to the given angle (θ) in a right-angled triangle that is not the hypotenuse.

angle of depression The angle between a horizontal line from the observer to an object that is below the horizontal line.

angle of elevation The angle between a horizontal line from the observer to an object that is above the horizontal line.

annuity An investment that has regular and constant payments over a period of time.

annuity investment An investment that has regular deposits made over a period of time, resulting in the growth of the principal amount.

arc Part of the circumference of a circle.

arc length Calculated by finding the circumference of a circle and multiplying by the fraction of the angle that it forms at the centre of the circle: Arc length $= \frac{\theta}{360} \times 2\pi r$.

area The two-dimensional space enclosed by a closed figure.

association table A table that summarises how time periods are converted to numerical values, so that a regression line can be calculated for time series.

associative A property whereby changing the placement of brackets in an expression does not alter the result; for example, $(A + B) + C = A + (B + C)$ shows that addition is associative.

back-to-back stem plot A display used to compare two different sets of data. Back-to-back stem plots share the same stem, with one data set appearing on the left of the stem and the other data set appearing on the right of the stem.

backward scanning A process of determining the critical path in a network. Backward scanning starts at the end node and moves backward through a network subtracting the time of each edge from the earliest start time of each succeeding node.

bar charts A display of the categories of data on one axis (usually the horizontal axis) and the frequency of the data on the other axis (usually the vertical axis).

bearings The direction of a fixed point, or the path of an object, from the point of observation.

bias Bias occurs when some individuals are more likely to be selected for study than others, which may result in a sample that is not representative of the whole population.

binary matrix A matrix that consists only of elements that are either 0 or 1.

bipartite graphs A graph where the nodes can be separated into two types of node — supply and demand.

bivariate data Data that contain two variables, where one variable affects the other variable.

break-even point The point at which revenue equals the cost of production.

bridge An edge in a connected graph whose deletion will no longer cause the graph to be connected.

categorical data Data that can be organised into groups or categories and is often an 'object', 'thing' or 'idea'. Examples include brand names, colours, general sizes and opinions. Categorical data can be nominal or ordinal.

causation The measure of how much the change in one variable is caused by another variable.

circuit A trail beginning and ending at the same vertex.

coefficient matrix A matrix containing the coefficients of variables in a set of simultaneous equations.

coefficient of determination This is the square of the product-moment correlation coefficient (r^2), and is a measure of how much a change in the response variable is influenced by a change in the explanatory variable.

coincidence Two (or more) lines or shapes that lie exactly on top of each other.

column matrix A matrix that has only one column.

column permutation A matrix that has the same elements in each column as another matrix, but with the columns in a different order.

communication matrices A matrix created from a communication network, in which a 1 represents a vertex that is able to communicate with another vertex.

communication network A network that has the ability of the arrows in it to travel both ways, for example, a set of people where they can have a one-way or two-way communication link.

commutative A property whereby changing the order of terms in an expression does not alter the result; for example, $A + B = B + A$ shows that addition is commutative.

complete graph A graph where there is an edge connecting each vertex to all other vertices in the graph, without parallel edges or loops.

composite figures A closed figure that comprises of two or more different common figures.

conditional distributions A sub-population (sample) of a distribution, which is found in the middle of the table.

confounding variables Confounding variables occur when it is impossible to draw conclusions by observing the relationship between explantory and responses variables.

congruent Objects that are exactly the same size and shape.

constant of proportionality Denoted as k, a value that is equal to the ratio of y to x for any data pair. It is the rate at which y varies with x, or the gradient.

constraints The set of linear inequalities that define a problem.

continuous Numerical data that can take any value that lies within an interval.

continuous variable A variable that can take any value, for example a time variable.

corner point method The method that uses the principle that the maximum or minimum value of the objective function must lie at one of the corners (vertices) of the feasible region.

correlation A measure of the strength of the linear relationship between two variables.

cosine rule A rule that can enable unknown side lengths and angles of triangles to be determined if we know either two side lengths and an included angle, or all three side lengths: $a^2 = b^2 + c^2 - 2bc \times \cos(A)$ or $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$.

crashing The reduction in the duration time of an activity, which causes the critical path to be shortened.

critical path The path through the network which follows those activities that cannot be delayed without causing the entire project to be delayed.

cycle A path beginning and ending at the same vertex.

cyclic time series Data that show fluctuations upwards and downwards, but not according to season.

decision variables A set of variables (usually two) that are used in a linear programming problem.

degenerate graph A graph with no edges, also known as a null graph.

degree (of a vertex) The number of edges that are directly connected to a vertex.

dependent equations Equations in which there is no unique solution, because the graphs of the equations coincide; in other words, they are the same line.

deseasonalising A technique involving calculating seasonal indices, which modifies the original time series in order to remove the effect of seasonal variation and exposing any other fluctuations which may be 'hidden' by seasonal variation.

determinant In the 2×2 matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ the determinant (det A or $|A|$) is the difference between the product of the diagonal elements: $ad - bc$. If $A = 0$, the inverse matrix will not exist.

diagonal matrix A square matrix that only has non-zero elements on the main diagonal.

Dijkstra's algorithm A method for determining the shortest path between a given vertex and each of the other vertices, often done in a tabular format.
directed graph A graph where it is only possible to move along the edges of the graph in one direction.
discrete Numerical data that is counted in exact values, with the values often being whole numbers.
discrete variable A variable that is not continuous and usually can take only whole number values. If the data are discrete, the points on the graph are not joined together.
dominance matrix A matrix that is the result of adding the matrices that represent the one-stage pathways (the adjacency matrix, A) and two-stage pathways (A^2). The dominance matrix consists of all the possible one- and two-stage pathways in the network.
dominant vertex The vertex that holds the most influence over all the other vertices. It belongs to the row that has the highest sum in the dominance matrix.
dot plots A display of discrete numerical data or categorical data. A scaled horizontal axis is used and each data value is indicated by a dot above this scale, resulting in a set of vertical 'lines' of evenly-spaced dots.
dummy activity An edge that must be added to avoid a network with two or more activities having the same name or occurring in parallel.
earliest finish time (EFT) Equal to the earliest start time (EST) of the activity plus its duration (T).
earliest start time (EST) The earliest time that any activity can be started after all prior activities have been completed.
edge A connection between vertices.
effective annual interest rate A rate used to compare the annual interest between loans with different compounding periods, such as daily, weekly or monthly: $r = \left(1 + \frac{i}{n}\right)^n - 1$.
effective life The length of time that an asset is useful to a business.
elements The entries in a matrix.
Euler circuit A circuit in which every edge is used once.
Euler trail A trail in which every edge is used once.
Euler's formula The mathematical relationship between the vertices, edges and faces of planar graphs: $V = E - F + 2$.
excess flow capacity The surplus of the capacity of an edge less the flow into the edge.
explanatory variable The variable represented on the x-axis and used in a bivariate analysis to explain or predict a difference in the response variable.
extrapolation Making a prediction from a line of best fit that appears outside the parameters of the original data set.
faces The distinct areas created by the non-intersecting edges and vertices of planar graphs.
feasible region Every required point that is a possible solution for the problem.
first-order recurrence relation A relation that relates a term in a sequence to the previous term in the same sequence.
flat rate method A method of deprecation in which an item's value decreases by a fixed amount each unit time interval, generally each year. The depreciating value may be a dollar amount or a percentage of the original cost price.
float time The difference in time between those paths that cannot be delayed and those that can.
flow capacity The flow capacity of an edge is the amount of flow that an edge can allow through if it is not connected to any other edges.
forecasting The process of predicting occurrences in the future, based on the behaviour of existing data.
forward scanning The process of looking through a network to calculate the earliest start time for each activity and the earliest completion time for the whole project.
frequency table A display that tabulates data according to the frequencies of predetermined groupings.
function Relations in which ordered pairs of unique numbers can be identified by a given rule or equation.

gradient Also known as the slope, it determines the change in the y-value for each change in x-value. This measures the steepness of a line as the ratio $m = \frac{\text{rise}}{\text{run}}$. If (x_1, y_1) and (x_2, y_2) are two points on the line, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

great circle A circle that divides two spheres into two equal hemispheres, where the centre of the great circle is the centre of the sphere.

Hamiltonian cycle A cycle that reaches all vertices of a network.

Hamiltonian path A path that reaches all vertices of a network.

histograms A display of continuous numerical data similar to a bar chart where the width of each column represents a range of data values, and the height represents their frequencies.

Hungarian algorithm A method for finding the optimal allocation, where the row reduction and column reduction will not produce the correct answer.

hypotenuse The longest side of a right-angled triangle. The hypotenuse will be opposite the right angle.

identity matrix A square matrix in which all of the elements on the diagonal line from the top left to bottom right are 1 and all of the other elements are 0.

immediate predecessor Something that must occur before another thing can happen is an immediate predecessor.

inconsistent equations Equations in which there is no solution, because the graphs of the equations never meet; in other words, they are parallel.

indegree The number of edges moving into a vertex.

inflow The inflow of a node is the total of the flows of all edges leading into the node.

initial state The original, first state from which the next state's probability is conditional on.

initial state matrix A column vector with an entry for each state of the system. Also known as a distribution vector, and denoted by S_0.

initial value The first term in a sequence, from which the rest of the terms in a recurrence relation are generated.

intercept The point at which a graph crosses one of the axes on a Cartesian plane.

interest only loans A loan where the borrower makes only the minimum repayment equal to the interest charged on the loan.

International Date Line The line on Earth where a calendar day officially starts and ends.

interpolation Making a prediction from a line of best fit that appears within the parameters of the original data set.

interquartile range The difference between the upper and lower quartiles of a data set.

isolated vertex A vertex that is not connected to any other vertex, and hence with degree 0.

isomorphic networks Networks that have the same number of vertices and edges, with corresponding vertices having identical degrees and connections.

latest start time The difference in time between those paths that cannot be delayed and those that can.

linear function A relation between up to two variables of degree 1 that produces a straight line.

linear inequality A linear function with an inequality sign.

linear programming A method used to achieve the best outcome in a given situation in a mathematical model whose requirements are represented by linear inequalities.

linear relationship A relation between up to two variables of degree 1 that produces a straight line.

logarithmic (log) form The log of a number is the power of 10 that creates this number, i.e. $\log(10^n) = n$. This is used to make a logarithmic scale, a scale that uses logarithms instead of normal numbers to mark intervals. In base 10 each unit increase or decrease represents a tenfold increase or decrease in the quantity being measured.

longitude Lines of longitude, also known as the meridians, measure the angle east or west of the prime meridian, which passes through Greenwich Observatory.

loop An edge in a graph that joins a vertex to itself.
lower triangular matrix A type of square matrix in which all the elements above the main diagonal are zero.
marginal distributions The sums (totals) of the rows or the columns in a two-way table that are found in the margins of the table.
Markov system A system that illustrates the changing nature of an event between different states, according to fixed probabilities.
matrix A rectangular array of rows and columns that is used to store and display information.
maximum spanning tree The spanning tree that covers the greatest distance for a network.
mean Commonly referred to as the average, it is calculated by dividing the sum of the data values by the number of data values. Represented by either $\bar{x}$ (sample) or μ (population).
median The middle value of an ordered data set if there are an odd number of values, or halfway between the two middle values if there are an even number of values.
median smoothing A technique for smoothing random or cyclical variation, where a group of points is replaced by the median of that group. This process is often done graphically. Median smoothing is preferred where there are small data sets or sets with many outliers.
meridian The meridians, also known as the lines of longitude, measure the angle east or west of the prime meridian, which passes through Greenwich Observatory.
minimum spanning tree The spanning tree that covers the least distance for a network.
mode The category or data value/s that have the highest frequency. It is the most frequently occurring value in a data set.
moving-mean smoothing A technique for smoothing random or cyclical variation, which uses the fact that the means of data can be used to represent the original data. Moving-mean smoothing is preferred for data sets with few random fluctuations.
multiplicative inverse A square matrix that when multiplied with another, results in the identity matrix. In other words, if $AA^{-1} = A^{-1}A = I$, then A^{-1} is called the multiplicative inverse of A.
negatively skewed Distributions with higher frequencies on the right-hand side of the display.
network flow An attribute that describes whether a network allows for the flow to be in both directions along a path, or in one direction only.
networks Systems of interconnected objects.
node A point on a network that may be connected to other nodes via edges.
nominal Categorical data that has no natural order or ranking.
non-linear graph A graph that is not a straight line.
non-linear relationship When the relation between variables forms a shape or curve rather than a straight line.
normal distributions A particular kind of distribution in which the data is symmetrical and has a bell shape centred on the mean.
objective function A function of the variables in a linear programming problem.
one-stage pathways A pathway between two vertices that includes one edge only.
opposite side The side directly opposite the given angle, θ.
optimal allocation Allocation that gives the most efficient result (for example, minimising time or cost).
order (of a matrix) The number of rows, m, and columns, n, of a matrix expressed as $m \times n$.
ordinal Categorical data that can be arranged into categories that have a natural order or ranking.
outdegree The number of edges moving away from a vertex.
outflow The outflow of a node is the minimum value obtained when one compares the inflow to the sum of the capacities of all the edges leaving the node.
parallel boxplots A display where two or more boxplots share the same scale to enable comparisons between data sets.

parallel dot plots A display where two or more dot plots share the same scale to enable comparisons between data sets.

parallel edges Where two (or more) edges connect the same pair of vertices. All the edges count towards the degree of the vertex.

parallels of latitude Imaginary lines around the Earth that are parallel to the equator.

path A walk in which no vertices are repeated, except possibly the start and finish.

Pearson's product–moment correlation coefficient A measure of the strength of a linear trend that is designated with the letter r and associated with a numerical value between -1 and $+1$. Values close to positive or negative '1' indicate strong linear trends, while values close to zero indicate weak or no linear trends.

perimeter The distance around a closed figure.

permutation matrix An $n \times n$ matrix that is a row or column permutation of the identity matrix. Permutation matrices re-order the rows or columns of another matrix via matrix multiplication.

perpetuity An annuity where a permanently invested sum of money provides regular payments which continue indefinitely.

planar graph A graph that can be drawn so that it has no intersecting edges.

polygon A two-dimensional shape consisting of at least three straight sides.

polyhedron A three-dimensional solid with flat faces.

population All the units or individuals of interest; the total set of observations that could be made.

positively skewed Distributions with higher frequencies on the left-hand side of the display.

Prim's algorithm A set of logical steps that can be used to identify the minimum spanning tree for a weighted connected graph.

quartiles These divide a set of data into quarters. The lower quartile (Q_1) is the median of the lower half of an ordered data set. The upper quartile (Q_3) is the median of the upper half of an ordered data set. The middle quartile (Q_2) is the median of the whole data set.

range A measure of the spread of a numerical data set determined by calculating the difference between the smallest and largest values.

reachability The concept of how it is possible to go from one vertex in a directed network to another.

rectangular matrix A matrix in which the elements are arranged in a rectangle (as opposed to a square).

reducing balance depreciation A method of depreciation, sometimes called 'diminishing value' depreciation, where the value of an asset is reduced by a fixed percentage of its previous value. This is an application of compound interest and the depreciated future value can be calculated using:

$$V_n = V_0\left(1 - \frac{r}{100}\right)^n.$$

region Also known as a half plane, a region is a section of the Cartesian plane covering all relevant points for an inequality.

re-seasonalise The process of removing the effect of deseasonalising in order to produce more accurate predictions.

residual plots Where the residuals are plotted against the original x-values. This indicates whether or not a linear relationship exists between the two variables under investigation.

response variable The variable represented on the x-axis and used in a bivariate analysis to explain or predict a difference in the response variable.

right-angled triangle A triangle that has a right angle.

row matrix A matrix that has only one row.

row permutation A matrix that has the same elements in each row as another matrix, but with the rows in a different order.

sample A limited number of individuals selected from the population with the purpose of representing characteristics of the whole population.

scalar A scalar quantity can be any real number (as opposed to a matrix or matrix elements), such as negative or positive numbers, fractions or decimals.

scalar multiplication Each element of the matrix is multiplied by the same number, called a 'scalar'. A scalar quantity can be any real number, such as negative or positive numbers, fractions or decimals.

scale factor A measure of the relative size of two similar figures and represented by k. It is the amount of enlargement or reduction and is expressed as integers, fractions or scale ratios.

scatterplot A visual display of bivariate data.

seasonal indices A measure of by what factor a particular season is above or below the average of all seasons for the cycle.

seasonal time series When data fluctuate during the year, as seasons change. The term *seasonal* is not specific to seasons of the year; it could also relate to other constant periods of highs and lows. The key feature is that the seasons occur at the same time each cycle.

sector A part of the area of a circle: $A = \frac{\theta}{360}\pi r^2$.

segment Formed by joining two points on the circumference with a straight line known as a chord. Each of the resulting areas is a segment.

segmented bar chart A single bar that is divided up into segments. Each segment represents a particular group of the data. For univariate data, the segmented bar chart represents all the data being studied, and if comparing two categorical variables, multiple columns are used.

sequence A related set of objects or events that follow each other in a particular order.

shortest path The path from one vertex to another that covers the shortest distance possible.

significant figures The number of digits that would occur in a if the number was expressed in scientific notation as $a \times 10^b$ or as $a \times 10^{-b}$.

similar figures Objects that have exactly the same shape but different size.

simple random sample A sample that is a random selection from the population and each individual has an equal chance of being selected for the sample. The selections must also be independent.

simultaneous equations A system of equations where the solutions for the values of the unknowns must satisfy each equation.

simultaneous solution The point where the graphs of a set of simultaneous equations intersect. The values of this point satisfy all the equations.

sine rule A rule that can enable unknown side lengths and angles of triangles to be determined if we know either two side lengths and one opposite side, or an angle and its opposite side length (a complete ratio) and one other side: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$. This can be used to find the unknown side length or angle in non-right-angled triangles.

singular matrix A matrix whose determinant is equal to 0, and hence does not have an inverse.

sink A network's end node(s).

sliding line method A method to find the optimal solution(s) to a linear programming problem by changing the y-intercept of the objective function. Sliding the line up to meet the last point it touches in the feasible region will identify the maximum value of the function. Sliding the line down to meet the last point it touches in the feasible region will identify the minimum value of the function.

slope Indicates the change in the response variable as the explanatory variable increases by 1 unit.

source A network's starting node(s).

spanning trees A tree that is a subgraph that includes all of the vertices of the original graph.

square matrix A matrix that has the same number of rows and columns.

standard deviation A measure of the spread of continuous numerical data around the mean.

state matrices Matrices that represent a particular state in a Markov system after a particular number of transitions from the initial state.

steady state matrix The fixed matrix, T^{∞}, which is found by taking higher and higher powers of the transition matrix, T, until the elements show no noticeable difference from the previous matrix.

stem plot An arrangement used for numerical data where it is grouped according to its numerical place value (the 'stem') and then displayed horizontally as a single digit (the 'leaf').

subgraphs All or part of a graph.

symmetric When a data set displays a single peak and the data trail off both sides of this peak in roughly the same fashion.

symmetrical matrix A matrix in which all its elements are symmetric with respect to the main diagonal, i.e. where a matrix is equivalent to its transpose. A symmetrical matrix must be square.

time series Data sets where the horizontal variable is time and, generally, where time goes up in even increments such as hours, days, weeks or years. Time series are used to see how a quantity varies with time.

time zones Regions of the world for which a common standard time is used.

total surface area (TSA) The combined total of the areas of each individual surface that forms a solid object.

trail A walk in which no edges are repeated.

transition matrix A matrix that summarises the probabilities of a Markov system.

trees A simple connected graph with no circuits.

triangular matrix A type of square matrix in which either all the elements above or all the elements below the main diagonal are zero.

trigonometric ratios A ratio of the lengths of two sides of a right-angled triangle.

two-stage pathways A pathway between two vertices that includes exactly two edges.

two-way (or contingency) tables A table in which the data are organised about two different categorical variables. The rows group one category and the columns group another category.

unit cost method A method of depreciating an asset according to its use – the more it is used, the faster it will depreciate: $V_n = V_0 - nd$.

univariate data Data that contain one variable.

upper triangular matrix A type of square matrix in which all the elements below the main diagonal are zero.

variance A measure of the spread of a data set from the mean: $s^2 = \dfrac{\sum f(x - \bar{x})^2}{\left(\sum f\right) - 1}$ (sample variance).

walk Any route taken through a network, including those that repeat edges and vertices.

weighted graph A graph with values attached to the edges.

***y*-intercept** The point where the graph of the equation crosses the y-axis. This occurs when $x = 0$.

zero matrix A square matrix that consists entirely of '0' elements.

***z*-score** Also called the standardised score, it indicates the position of a certain score in relation to the mean. A z-score of 0 indicates the score is equal to the mean, a negative z-score indicates a score below the mean and a positive z-score indicates a score above the mean.

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