

NELSON QMATHS

GENERAL
MATHEMATICS

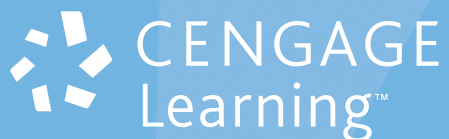
YEAR

12

Ross Brodie
Regina Edwards
Stephen Swift

4TH EDITION





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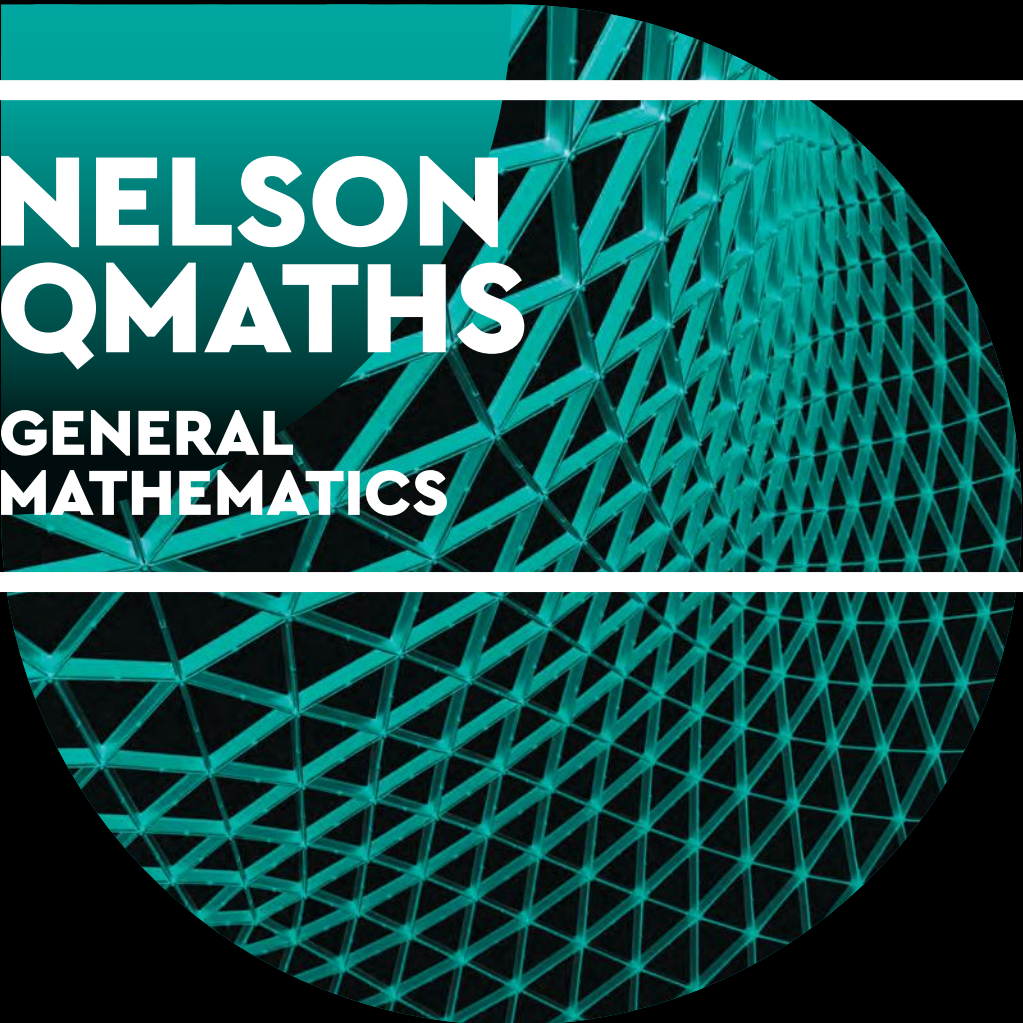
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Nelson QMaths 12 General Mathematics

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Level 7, 80 Dorcas Street

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Cengage Learning New Zealand

Unit 4B Rosedale Office Park

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CONTENTS

PREFACE	vi
SYLLABUS REFERENCE GRID	vii
ABOUT THIS BOOK	viii
LIST OF SYMBOLS AND ABBREVIATIONS	xiii
LIST OF MATHEMATICAL VERBS	xiv

UNIT 3

1

ASSOCIATIONS BETWEEN VARIABLES

2

1.01 Bivariate data	4
1.02 Percentage tables	10
1.03 Interpreting associations	16
1.04 Scatterplots	21
1.05 Describing associations	25
1.06 The correlation coefficient	30
Chapter summary	38
Chapter review	41

2

ARITHMETIC AND GEOMETRIC SEQUENCES

46

2.01 Arithmetic sequences	48
2.02 General term of an arithmetic sequence	54
2.03 Modelling with arithmetic sequences	59
2.04 Geometric sequences	64
2.05 General term of a geometric sequence	69
2.06 Modelling with geometric sequences	76
Chapter summary	83
Chapter review	85

3

TIME SERIES

88

3.01 Time series plots	90
3.02 Time series patterns	94
3.03 Outliers	98
3.04 Smoothing time series data	103
Chapter summary	111
Chapter review	112

PRACTICE EXAMINATION 1	115

MEASURING AND EXPLAINING ASSOCIATIONS 122

4.01	Response and explanatory variables	124
4.02	Relationships between variables	127
4.03	Least-squares linear models	130
4.04	Residual plots	136
4.05	The coefficient of determination	143
4.06	Interpolation and extrapolation	146
4.07	Association and causation	150
4.08	Applications of associations	154
	Chapter summary	157
	Chapter review	159

ANALYSING TIME SERIES DATA 162

5.01	Seasonal indices	164
5.02	Deseasonalisation	168
5.03	Using seasonal indices	174
5.04	Modelling long-term trends	178
5.05	Applications of time series data	183
	Chapter summary	189
	Chapter review	191

EARTH GEOMETRY AND TIME ZONES 194

6.01	Latitude and longitude	196
6.02	Maps and position coordinates	201
6.03	Distances on the Earth	206
6.04	Longitude and time zones	213
6.05	Applications of time zones	216
	Chapter summary	221
	Chapter review	222

PRACTICE EXAMINATION 2	225

UNIT 4

INVESTMENTS AND LOANS 232

7.01	Compound interest	234
7.02	Rests and interest	241
7.03	Effective interest rates	250
7.04	Future value and present value	254
7.05	Reducing balance loans	260
7.06	Loan calculations	270
7.07	Modifying loans	275
	Chapter summary	286
	Chapter review	288

NETWORKS, PATHS AND TREES 292

8.01	Graphs and networks	294
8.02	Applications of networks	305
8.03	Adjacency matrices	316
8.04	Planar graphs and Euler's formula	323
8.05	Eulerian graphs	330
8.06	The travelling salesperson problem ...	338
8.07	Spanning trees	344
8.08	Minimum spanning trees	349
8.09	Flow networks	361
8.10	The maximum flow–minimum cut theorem	367
	Chapter summary	374
	Chapter review	376

PRACTICE EXAMINATION 3	385

ANNUITIES AND PERPETUITIES 392

9.01	Annuities	394
9.02	Ordinary annuities	402
9.03	Superannuation	410
9.04	Perpetuities	418
9.05	Applications of annuities	422
	Chapter summary	429
	Chapter review	431

NETWORKS, PROJECTS AND ASSIGNMENTS 436

10.01	Project networks	438
10.02	Earliest and latest starting times	448
10.03	Critical paths	457
10.04	Float times	463
10.05	Applications of project networks	470
10.06	Bipartite graphs	477
10.07	Optimum assignment problems	486
	Chapter summary	498
	Chapter review	501

PRACTICE EXAMINATION 4	511

ANSWERS	519

GLOSSARY/INDEX	579

PREFACE

Nelson QMaths, Queensland's longest-running senior mathematics series, has been rewritten for the new syllabuses and assessment procedures for implementation from 2019. Based on the Australian Curriculum, four new senior mathematics courses have been introduced into Queensland schools.

- Essential Mathematics
- General Mathematics
- Mathematical Methods
- Specialist Mathematics

With the introduction of external QCE examinations, *Nelson QMaths* will have a renewed focus on assessment, and will include features such as topic tests, practice examinations, *ExamView* question banks and chapter quizzes, video tutorials and worked solutions to all exercise questions. In this book, teachers will find familiar features such as clear worked examples, graded exercises, strong syllabus coverage, Investigations, Technology, chapter summaries, chapter reviews and a glossary/index. We wish all teachers and students using this book every success in embracing the new mathematics courses.

ABOUT THE AUTHORS

Ross Brodie has taught mathematics and been Head of Mathematics at a variety of regional and urban schools in Queensland. He is an experienced teacher and author, having co-written *Nelson QMaths 7-10*, *New QMaths 11-12* and *Nelson Senior Maths 11-12* (Australian Curriculum).

Regina Edwards teaches mathematics and science at Cleveland District State High School where she coordinates International Baccalaureate High Level Mathematics and Physics. She was a district panellist for Mathematics C and Physics, and has worked in the Information Technology industry as an analyst.

Stephen Swift has taught Mathematics, Science and Computing in urban and regional schools, most recently as Head of Mathematics

at Wellington Point State High School. He is the lead author of *Nelson QMaths 7-10*, *New QMaths 11-12* and *Nelson Senior Maths 11-12* (Australian Curriculum).

CONTRIBUTING AUTHORS

Sharon Kirkby (Moreton Bay College) wrote many of the *NelsonNet* worksheets.

John Drake, **Katie Jackson** and **Joanne Wagner** created the video tutorials.

Rashmi Bhagwati and **Trisha Goss** wrote the topic tests.

Steven Purse and **Greg Neal** wrote the worked solutions to all exercise sets.

Roger Walter wrote the *ExamView* questions.

SYLLABUS REFERENCE GRID

Topics and subtopics	Nelson QMaths 12 General Mathematics chapter
UNIT 3: BIVARIATE DATA, SEQUENCES AND CHANGE, EARTH GEOMETRY	
Bivariate data analysis	
Identifying and describing associations between two categorical variables	1 Associations between variables
Identifying and describing associations between two numerical variables	1 Associations between variables
Fitting a linear model to numerical data	4 Measuring and explaining associations
Association and causation	4 Measuring and explaining associations
Time series analysis	
Describing and interpreting patterns in time series data	3 Time series
Analysing time series data	3 Time series 5 Analysing time series data
Growth and decay in sequences	
The arithmetic sequence	2 Arithmetic and geometric sequences
The geometric sequence	2 Arithmetic and geometric sequences
Earth geometry and time zones	
Locations on the Earth	6 Earth geometry and time zones
Time zones	6 Earth geometry and time zones
UNIT 4: INVESTING AND NETWORKING	
Loans, investments and annuities	
Compound interest loans and investments	7 Investments and loans
Reducing balance loans	7 Investments and loans
Annuities and perpetuities	9 Annuities and perpetuities
Graphs and networks	
Graphs, associated terminology and the adjacency matrix	8 Networks, paths and trees
Planar graphs, paths and cycles	8 Networks, paths and trees
Networks and decision mathematics	
Trees and minimum connector problems	10 Networks, projects and assignments
Project planning and scheduling using critical path analysis	10 Networks, projects and assignments
Flow networks	10 Networks, projects and assignments
Assigning order and the Hungarian algorithm	10 Networks, projects and assignments

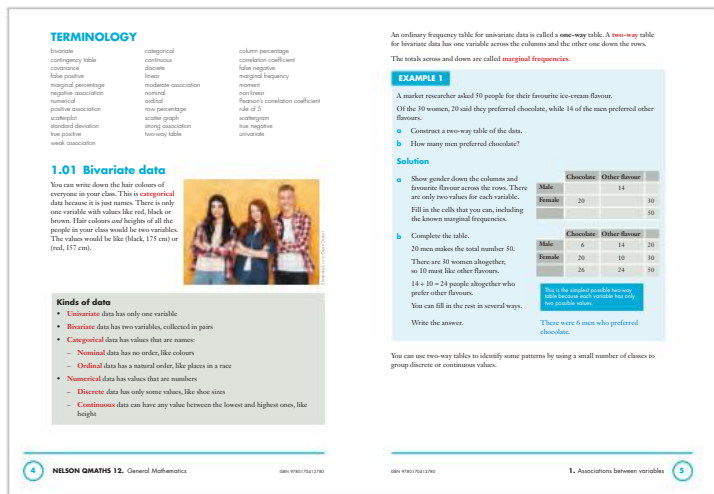
ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of syllabus subject matter (© Queensland Curriculum and Assessment Authority 2019)



- Terminology** previews the key words and phrases from within the chapter.



IN EACH CHAPTER

- Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.
- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the glossary at the back of the book.
- Graded exercises include **Problem solving** questions, are linked to worked examples and include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Technology** promotes ICT in the classroom, using spreadsheets and the Internet.

Contingency table percentages

You can work out percentages in a **contingency table**:

- by row (or **row percentages**) based on the totals on the right of the rows
- by column (or **column percentages**) based on totals at the bottoms of the columns
- by table total (or **table total percentages**) based on the grand total at the bottom right
- Percentages by rows or columns are called **marginal percentages**

EXAMPLE 5

The table shows the preferences of people for ice-cream flavours.

	Chocolate	Other flavour	
Male	6	14	20
Female	20	10	30
	26	24	50



Find:

- the percentage of females liking chocolate.
- the percentage of people who are female chocolate lovers.
- the percentage of chocolate lovers who are female.

Solution

You must identify the total correctly to get the right answers.

- This is 'out of' females. $\% \text{ by row} = \frac{20}{30} \times 100\% = 67\%$
Write the answer. About 67% of females like chocolate.
- This is 'out of' those surveyed. $\% \text{ by table total} = \frac{20}{50} \times 100\% = 40\%$
Write the answer. 40% are female chocolate lovers.
- This is 'out of' chocolate lovers. $\% \text{ by column} = \frac{20}{26} \times 100\% = 77\%$
Write the answer. About 77% of those who prefer chocolate are female.

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1. Associations between variables

11

INVESTIGATION

CLASS ASSOCIATIONS

- Collect the following data from your class:
Shoe size, height, forearm length (elbow to wrist), age in months, number of siblings, hair length, number living in home
- There are many possible scatterplots that you can draw from the data.
- Work in groups of 2 or 3 to draw the scatterplots; your teacher will assign the scatterplots for each group.
- Comment on any patterns in the scatterplots.

Exercise 1.04 Scatterplots

- This table shows the values of some bivariate data.

X	10.3	14.5	16.4	13.7	18.1	14.3	17.6	19	24.7	26.3	27.6	23.6
Y	51	53.1	41.5	42.4	41.3	26.9	30.6	22.1	24.5	13.4	10.2	16.9

- Make a scatterplot of the data with X horizontal.
- Comment on the pattern.

- This table shows the values of some bivariate data.

A	27	35	34	32	47	44	56	45	49	58
B	56	33	13	34	64	45	27	15	24	44

- Make a scatterplot of the data with A horizontal.
- Comment on the pattern.

- This table shows the values of some bivariate data.

P	5.6	6.9	6.8	7.4	7.7	9.6	10.6	9.5	9.3	11.6	12	12.2
Q	11.5	11.7	12.7	12.5	12.7	14.2	13	13.9	13.8	14.8	15.4	14.7

- Make a scatterplot of the data with P horizontal.
- Comment on the pattern.

- This table shows the values of some bivariate data.

V	0.7	1.2	1.4	2	2.1	2.4	2.8	2.9	2.9	3.9	4
W	12.7	13.3	17.6	14.1	15.1	13.1	23.4	18.5	17.6	26.7	25.9

- Make a scatterplot of the data with V horizontal.
- Comment on the pattern.

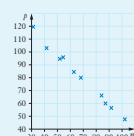
24

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Solution

- Plot the points on suitable axes.



- Describe the result.

The points form a very clear straight line sloping downwards, so there is a strong negative linear association between m and p .

TECHNOLOGY

Spreadsheet: Scatterplots

You can use a spreadsheet to draw the scatterplot in Example 13.

- Type the data from Example 13 into columns A and B in the spreadsheet.

A	B
11.5	120
11.7	100
12.7	80
12.5	70

- Move the mouse to cell A2.

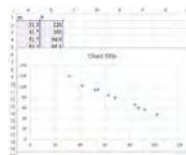
- Hold down the shift button and move the mouse down to B11 to highlight the data.

- Release the shift button and click the mouse on the 'Insert' tab.

- Click on the scatterplot icon.

- Choose the icon with no lines.

The graph will appear.



You can click on the graph and choose how to format the graph.
Your teacher will tell you what format is required.

28

NELSON QMATHS 12. General Mathematics

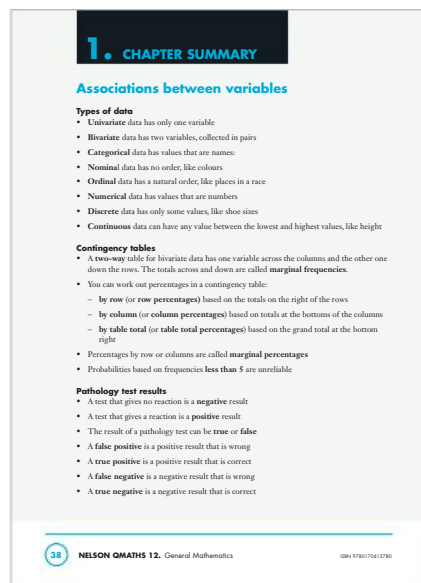
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AT THE END OF EACH CHAPTER

- **Chapter summary** summarises the key concepts, skills and formulas of the chapter.
- **Chapter review** contains revision exercises and are linked to worked examples.
- **Practice examinations** after every 2–3 chapters revise the skills and knowledge of those chapters.

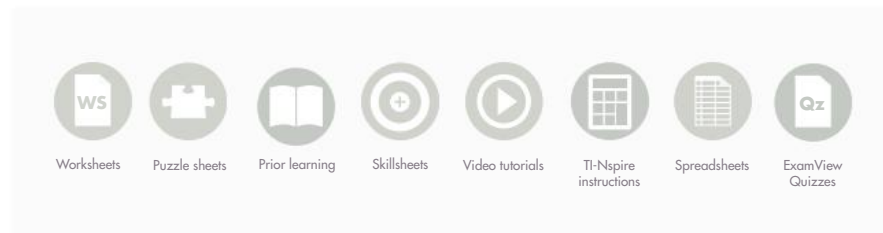
AT THE END OF THE BOOK

- **Glossary and index** includes a comprehensive dictionary of course terminology.
- **Answers** (with worked solutions on the teacher website).



NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, www.nelsonnet.com.au. These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Prior learning** exercises that revise prerequisite skills for the chapter
- **Skillsheets** of examples and exercises of prerequisite skills and knowledge
- **Video tutorials:** worked examples explained by ‘flipped classroom’ teachers
- **Spreadsheets:** *Excel* files
- **ExamView quizzes:** interactive and self-marking



NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at www.nelsonnet.com.au, contains:

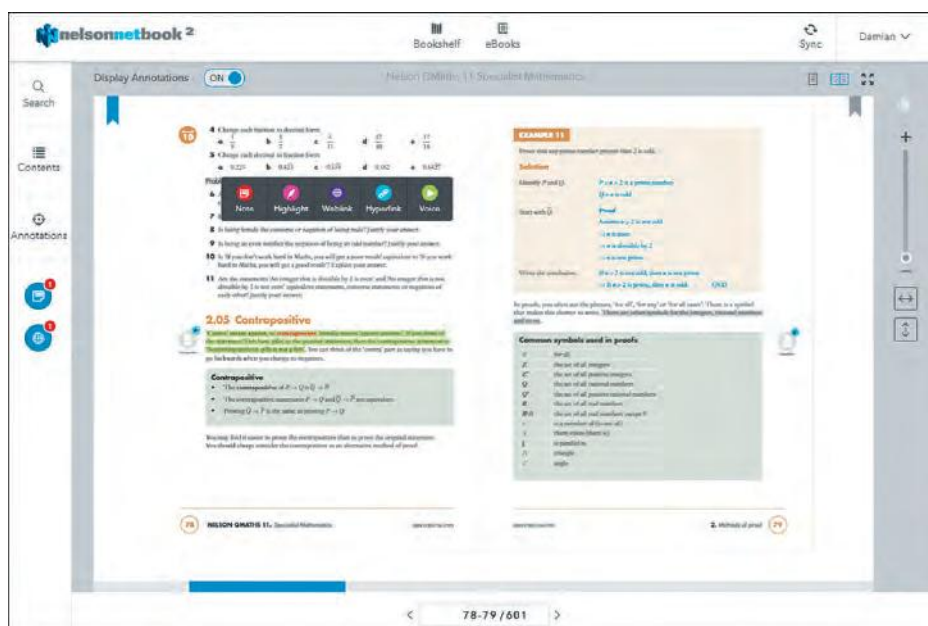
- **Teaching plan**, in Microsoft Word and PDF formats
- **Topic tests**, in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- **ExamView** exam-writing software and questionbanks
- **Resource Finder**: search engine for *NelsonNet* resources

Note: Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom. Contact your Cengage Education Consultant for information about access codes and conditions.

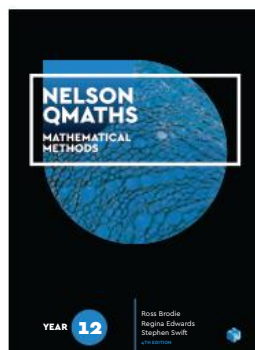
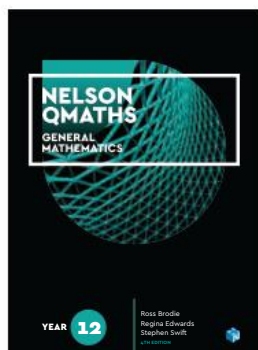
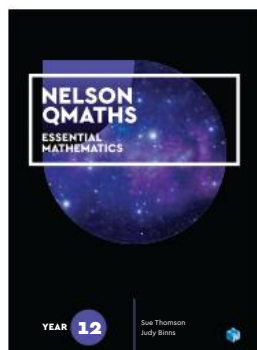
NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on *NelsonNet*.

- To each page of NelsonNetBook you can add notes, voice and sound bites, highlighting, weblinks and bookmarks
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students



NELSON QMATHS 11–12 SERIES



LIST OF SYMBOLS AND ABBREVIATIONS

$=$	is equal to	$\sqrt{\quad}$	square root, radical sign	(x, y)	Cartesian coordinates, ordered pair
$\neq$	is not equal to	SA	surface area	%	percentage
$\approx$	is approximately equal to	CSA	curved surface area	cos	trigonometric cosine ratio
$<$	is less than	AUD	Australian dollar	sin	trigonometric sine ratio
$>$	is greater than	GST	goods and services tax	tan	trigonometric tangent ratio
$\leq$	is less than or equal to	a_{24}	element in row 2, column 4 of a matrix	$\bar{x}$	mean
$\geq$	is greater than or equal to	$S37^\circ W$	a compass bearing	Σ	the sum of
$\pm$	plus or minus	$217^\circ T$	a true bearing	Q_1	first quartile or lower quartile
π	$\pi \approx 3.14$	$\overline{A}, A'$	not A , the complement of A	Q_2	median (second quartile)
$\equiv$	is congruent to	S	sample space	Q_3	third quartile or upper quartile
$\parallel$	is similar to	$n(A)$	the number of elements in A	IQR	interquartile range
$\dots$	and so on	$P(A)$	the probability of A	s	standard deviation
$^\circ$	degree	LHS	left-hand side	σ	standard deviation of a population
$\angle$	angle	RHS	right-hand side		
$\triangle$	triangle	m	gradient of a straight line		
$\parallel$	is parallel to	c	y -intercept of a straight line		
$\perp$	is perpendicular to				
$\therefore$	therefore				

LIST OF MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study and state in detail the relationship of parts of a situation

apply: use knowledge or a procedure in a given situation

calculate: find a numerical value

comment: express an opinion or judgement about a statement or calculation

communicate: transmit information to others

compare: state similarities and differences and their significance

consider: take into account

construct: draw an accurate diagram or logically arrange items or ideas

convert: change from one form to another

define: give the meaning of or identify in exact terms

demonstrate: show to be correct

describe: state the features of a situation, object, pattern, event, etc.

determine: find the answer or make a decision using evidence

discuss: give reasons based on evidence for and against a position or proposal

establish: introduce and develop a result

evaluate: find the value of or state the application, strengths and limitations of a solution

examine: state the details and assumptions of a situation

expand: remove brackets or change a product to an extended sum of terms

explain: state the meaning in logical detail

explore: examine or state the details and assumptions of a situation

factorise: convert to factors or change a sum of terms to a product of factors

formulate: give mathematical expression to an idea or situation

hence find/prove: find an answer or prove a result using previous answers or information supplied

identify: state the type, name or distinguishing feature of an item or situation

interpret: state a conclusion or trend from given information

investigate: establish facts, trends or conclusions from collected information

isolate a variable: express a formula or equation with one variable only, usually on the left-hand side (LHS)

justify: give reasons or evidence for an answer or conclusion

modify: change to accommodate different information

obtain: find an answer or conclusion

prove: use logical steps to establish the truth of

recall: remember (and state)

recognise: use knowledge to identify features of a situation

show that: in questions where the answer is given, to use mathematical reasoning to prove that the answer is true

simplify: reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

sketch: draw a diagram that shows the general shape and includes relevant features

solve: find the answer or explanation for a problem, particularly the values of variables

substitute: replace a variable by a number to calculate an answer

translate (to mathematical form): express a situation as mathematical relationships

verify: check a solution or result, usually referring to the given situation

write/state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

1.

ASSOCIATIONS BETWEEN VARIABLES

You may have heard that people with green eyes often have red hair or that people with blue eyes are more likely to have fair hair. Is this really true? Or is it just a myth? It is obvious that tall people weigh more than short people because adults are generally taller and heavier than children. What about people of comparable ages?

In these cases, you are looking for an **association** between the variables. In this chapter you will see some ways to test for these associations.

- 1.01 Bivariate data
- 1.02 Percentage tables
- 1.03 Interpreting associations
- 1.04 Scatterplots
- 1.05 Describing associations
- 1.06 The correlation coefficient
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Identifying and describing associations between two categorical variables

- define bivariate data
- construct two-way frequency tables and determine the associated row and column sums and percentages
- use an appropriately percentaged two-way frequency table to identify patterns that suggest the presence of an association
- describe an association in terms of differences observed in percentages across categories in a systematic and concise manner, and interpret this in the context of the data

Identifying and describing associations between two numerical variables

- construct a scatterplot to identify patterns in the data suggesting the presence of an association
- describe an association between two numerical variables in terms of direction (positive/negative), form (linear) and strength (strong/moderate/weak)
- calculate and interpret the correlation coefficient (r) to quantify the strength of a linear association using Pearson's correlation coefficient



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

bivariate
contingency table
covariance
false positive
marginal percentage
negative association
numerical
positive association
scatterplot
standard deviation
true positive
weak association

categorical
continuous
discrete
linear
moderate association
nominal
ordinal
row percentage
scatter graph
strong association
two-way table

column percentage
correlation coefficient
false negative
marginal frequency
moment
non-linear
Pearson's correlation coefficient
rule of 5
scattergram
true negative
univariate



Two-way
tables



Two-way
tables

1.01 Bivariate data

You can write down the hair colours of everyone in your class. This is **categorical** data because it is just names. There is only one variable with values like red, black or brown. Hair colours *and* heights of all the people in your class would be two variables. The values would be like (black, 175 cm) or (red, 157 cm).



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Kinds of data

- **Univariate** data has only one variable
- **Bivariate** data has two variables, collected in pairs
- **Categorical** data has values that are names:
 - **Nominal** data has no order, like colours
 - **Ordinal** data has a natural order, like places in a race
- **Numerical** data has values that are numbers
 - **Discrete** data has only some values, like shoe sizes
 - **Continuous** data can have any value between the lowest and highest ones, like height

An ordinary frequency table for univariate data is called a **one-way table**. A **two-way table** for bivariate data has one variable across the columns and the other one down the rows.

The totals across and down are called **marginal frequencies**.

EXAMPLE 1

A market researcher asked 50 people for their favourite ice-cream flavour.

Of the 30 women, 20 said they preferred chocolate, while 14 of the men preferred other flavours.

- a Construct a two-way table of the data.
- b How many men preferred chocolate?

Solution

- a Show gender down the columns and favourite flavour across the rows. There are only two values for each variable.
Fill in the cells that you can, including the known marginal frequencies.

	Chocolate	Other flavour	
Male		14	
Female	20		30
			50

- b Complete the table.
20 men makes the total number 50.

There are 30 women altogether, so 10 must like other flavours.

$14 + 10 = 24$ people altogether who prefer other flavours.

You can find the remaining data in several ways.

Write the answer.

	Chocolate	Other flavour	
Male	6	14	20
Female	20	10	30
	26	24	50

This is the simplest possible two-way table because each variable has only two possible values.

There were 6 men who preferred chocolate.

You can use two-way tables to identify some patterns by using a small number of classes to group discrete or continuous values.

EXAMPLE 2

The heights of a sample of Year 12 students and the numbers of computers in their homes are shown below.

Adam 185 cm, 3	Asha 168 cm, 4	Bonnie 157 cm, 2	Charmaine 172 cm, 5
Chris 194 cm, 1	Colin 165 cm, 3	David 178 cm, 2	Elena 176 cm, 1
Harry 169 cm, 0	Helga 158 cm, 3	Janita 162 cm, 4	Lee 183 cm, 2
Mary 164 cm, 1	Mustafa 167 cm, 2	Mirza 179 cm, 2	Pam 160 cm, 2
Peter 177 cm, 1	Phoong 166 cm, 3	Roberto 185 cm, 4	Robyn 156 cm, 3
Sam 164 cm, 4	Stefan 172 cm, 3	Thomas 167 cm, 2	Xing 155 cm, 1

- Construct a two-way table with height in 3 classes.
- Identify any patterns.

Solution

- Choose the height classes.

Use short (< 166 cm), average ($166\text{--}175$ cm) and tall (> 175 cm) for heights.

Use tally marks to fill in the table to avoid mistakes.

Do Adam, Asha, Bonnie, Charmaine, and so on.

		Height (cm)		
		Short	Average	Tall
No. of computers	0			
	1			
	2			
	3			
	4			
	5			

Complete the table and then replace the tally marks by the frequencies.

		Height (cm)			
		Short	Average	Tall	
No. of computers	0	0	1	0	1
	1	2	0	3	5
	2	2	2	3	7
	3	3	2	1	6
	4	2	1	1	4
	5	0	1	0	1
		9	7	8	24

- Note any patterns.

It looks as if there might be more tall people with only a few computers, but this is probably a coincidence.

INVESTIGATION

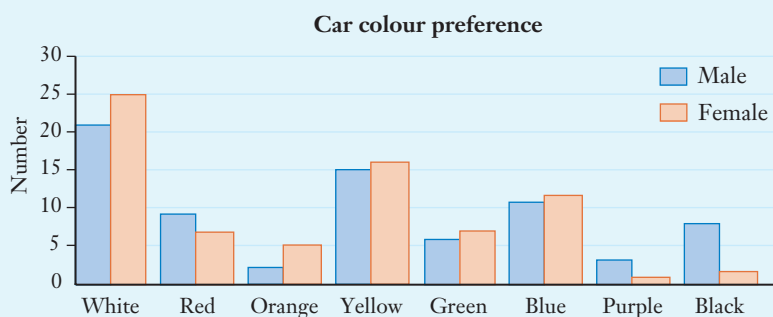
SUBJECT CHOICES AND CAREERS

- Try to categorise the subject choices of students in your class as ‘mainly science’, ‘mainly arts’, ‘mainly commerce’ or ‘other’.
- Choose some career orientations for next year, such as ‘TAFE’, ‘apprentice’, ‘university’, ‘employment’.
- Gather information from your class and draw up a two-way table. You may want to extend your data gathering to another class if it can be arranged by your teacher.
- Can you make any conclusions from your table?
- Discuss the difficulties of classifying the information in a meaningful way.

You can use a side-by-side column graph to show the data in a two-way table, or vice-versa.

EXAMPLE 3

The column graph below shows car colour preferences. Construct a two-way table for the data.



Solution

Put gender across and the colours down.

Read the frequencies for each cell from the column graph.

	Male	Female	Total
White	21	25	46
Red	9	7	16
Orange	2	5	7
Yellow	15	16	31
Green	6	7	13
Blue	11	12	23
Purple	3	1	4
Black	8	2	10
Total	75	75	150

Exercise 1.01 Bivariate data

Example

1

- 1 At Kirra beach, 50 people were asked whether they were staying locally or visiting and whether they intended to surf, swim or lie on the beach. Of the 31 people staying locally, 12 were surfers and 4 were beach-liers. Altogether, there were 18 surfers and 10 beach-liers.

- a Construct a two-way table.
- b How many swimmers were visiting?

Example

2

- 2 The data below shows the results of a survey on the amount of support for logging of old native hardwood timber in state forests.

NSW:	20 for, 56 against	SA:	12 for, 48 against
Vic:	18 for, 63 against	WA:	24 for, 39 against
Qld:	25 for, 43 against	Tas:	15 for, 32 against
NT:	8 for, 23 against	ACT:	2 for, 25 against

Construct a two-way table.

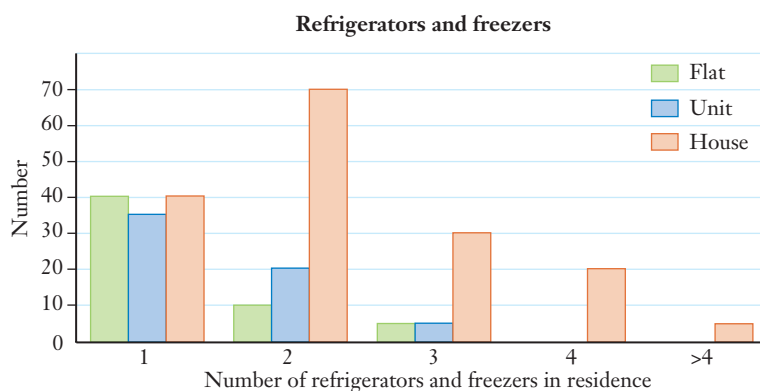


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Example

3

- 3 Construct a two-way table from this side-by-side column graph showing the results of a survey about types of residences and the number of refrigerators and freezers in them.



4 A two-way table was constructed as shown below, but only partially completed.

- a How many non-members were there?
- b How many women were there?

	Male	Female	
Book club member	20		52
Non-member	8	15	

5 Construct a side-by-side column graph from the following two-way table showing the numbers of people driving different types of vehicles and their opinions about whether there should be more police cars patrolling the roads or not.

	More patrols needed	No more needed
Car	30	15
Motorcycle	8	6
Truck	20	15
Delivery van	6	18

6 The results of a survey of 40 men and 40 women on how they use sunscreen are given in the two-way table below. Construct a side-by-side column graph from the results.

	At work	At home	On the beach
Men	7	12	25
Women	4	8	30

Problem solving

7 The data below shows the results of a survey of the preferred music of teachers and students in a school. The first letter shows whether the preferred music is Classical (C), Jazz (J), Easy-listening (E), Country and Western (CW) or Rock (R). The age of the person is shown next. Choose suitable age groupings and show the information as a two-way table. Are there any patterns?

C, 17	E, 13	E, 48	R, 51	C, 26	E, 18	CW, 30
R, 14	R, 50	J, 54	R, 15	R, 16	E, 18	CW, 15
CW, 53	R, 13	E, 43	E, 37	J, 18	CW, 16	R, 18
J, 33	R, 15	E, 16	E, 30	CW, 27	R, 15	R, 33
C, 16	R, 14	E, 17	E, 18	J, 38	CW, 30	J, 33
C, 55	CW, 51	J, 18	R, 53	C, 51	E, 18	CW, 15
J, 25	R, 34	R, 17	J, 14	C, 16	C, 29	E, 38
R, 17	R, 46	R, 39	CW, 32	J, 17	CW, 16	J, 15
J, 51	C, 16	E, 15	R, 13	CW, 16	E, 17	C, 15
J, 30	R, 18	C, 35	R, 52	C, 38		

- 8** The data below shows the results of a survey of blue-collar (B), white-collar (W) and management (M) workers, and the number of years of education completed, including school, apprenticeships, university and TAFE. Compile a two-way table. Are there any patterns?

B, 15	W, 13	B, 13	W, 14	B, 13	W, 15	W, 16	M, 17	W, 16	W, 13
B, 15	M, 19	W, 12	W, 16	M, 17	W, 15	W, 13	W, 14	B, 11	M, 14
W, 15	M, 17	M, 17	W, 15	W, 12	W, 12	B, 15	B, 14	W, 13	W, 15
B, 15	W, 14	B, 11	W, 13	M, 17	W, 13	W, 13	W, 12	W, 16	W, 16
W, 13	M, 17	B, 12	B, 12	W, 14	M, 15	B, 15	W, 14	W, 15	B, 13
W, 16	M, 19	W, 15	B, 13	B, 12	M, 15	B, 11	M, 15	B, 12	B, 15
W, 15	B, 13	W, 14	W, 16	M, 14	W, 15	W, 12	B, 15	M, 18	W, 13
W, 12	W, 15	B, 14	W, 16	M, 18	W, 12	W, 13	W, 15	B, 15	M, 19
W, 13	B, 12	W, 16	M, 19	B, 12	W, 12	M, 18	B, 13	M, 16	B, 15
W, 14	B, 13	M, 18	W, 12	W, 12	B, 13	W, 16	B, 11	B, 13	W, 14



Percentage
tables

1.02 Percentage tables

You can work out probabilities using two-way tables. It is important to understand what the probabilities or percentages represent.

EXAMPLE 4

What is the total for each percentage described?

- a** The percentage of dog-owners who also like cats.
- b** The percentage of pet-owners who have a dog.
- c** The probability that a cat-owner also has a dog.

Solution

a 'Percentage of' shows the total.	The total is the number of dog-owners.
b 'Percentage of' shows the total.	The total is the number of pet-owners.
c 'Probability that' shows the total.	The total is the number of cat-owners.

A two-way table has two sets of marginal frequency totals, and a grand total at the bottom right. There are 3 ways to work out percentages (relative frequencies or probabilities).

Contingency table percentages

You can work out percentages in a **contingency table**:

- by row (or **row percentages**) based on the totals on the right of the rows
- by column (or **column percentages**) based on totals at the bottom of the columns
- by table total (or **table percentages**) based on the grand total at the bottom right
- Percentages by rows or columns are called **marginal percentages**

EXAMPLE 5

The table shows the preferences of people for ice-cream flavours.

	Chocolate	Other flavour	
Male	6	14	20
Female	20	10	30
	26	24	50



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Compassionate Eye Foundation

Find:

- the percentage of females liking chocolate.
- the percentage of people who are female chocolate lovers.
- the percentage of chocolate lovers who are female.

Solution

You must identify the total correctly to get the right answers.

- a** This is 'out of' females.

Write the answer.

$$\% \text{ by row} = \frac{20}{30} \times 100\% \approx 67\%$$

About 67% of females like chocolate.

- b** This is 'out of' those surveyed.

Write the answer.

$$\% \text{ by table total} = \frac{20}{50} \times 100\% = 40\%$$

40% are female chocolate lovers.

- c** This is 'out of' chocolate lovers.

Write the answer.

$$\% \text{ by column} = \frac{20}{26} \times 100\% \approx 77\%$$

About 77% of those who prefer chocolate are female.

A survey of 123 Year 12 students at schools in Roma found only 2 good surfers. Does this mean you can be sure that only 1.6% of Roma students are good surfers? What if some students who had just moved from the Gold Coast were away that day? A small error can make a big difference to percentages.

Rule of 5

- Probabilities based on frequencies **less than 5** are unreliable
- You should combine cells in two-way tables to make sure all frequencies are at least 5
- Expected frequencies less than 5 should be treated as unreliable

You can use the **rule of 5** to work out how to group data to get meaningful results.

EXAMPLE 6

People across Queensland were asked if they would agree to daylight saving in summer. The results from various towns were as follows. 1Y = 1 said yes, 2N = 2 said no.

Bamaga	1Y	2N	Mt Isa	2Y	3N
Brisbane	19Y	9N	Nambour	6Y	3N
Cairns	9Y	16N	Rockhampton	4Y	7N
Gold Coast	11Y	6N	Roma	4Y	7N
Ipswich	9N	5N	Townsville	7Y	13N

Show the results as a two-way table. Calculate percentages to show any pattern of results.

Solution

Consider possible groupings.

Bamaga, Mt Isa, Nambour, Roma and Rockhampton have less than 5 for at least one of Y or N. These results need to be combined.

4 of these are outside the south-east corner of Queensland.

Choose groupings.

Combine Bamaga and Cairns as Far north coast; Mt Isa and Roma as West; Rockhampton and Townsville as Mid-north coast; and Brisbane, Ipswich and Nambour as Greater Brisbane.

Construct the two-way table.

	Yes	No	
Far north	10	18	28
West	6	10	16
Mid-north coast	11	20	31
Greater Brisbane	34	17	51
Gold Coast	11	6	17
	72	71	143

Comment on any patterns.

There was no overall preference one way or the other in this survey, but the first 3 areas have a different pattern from Brisbane and the Gold Coast.

Work out percentages by rows.

	Yes	No	
Far north	35.7	64.3	100
West	37.5	62.5	100
Mid-north coast	35.5	64.5	100
Greater Brisbane	66.7	33.3	100
Gold Coast	64.7	35.3	100
	50.3	49.7	100

Comment on the final contingency table.

People outside the south-east corner of Queensland opposed daylight saving about 2 : 1, while those in the south-east corner supported it about 2 : 1.

You could combine the data into 2 categories: South-East Queensland and Other.

Exercise 1.02 Percentage tables

- 1 What is the total for each percentage described?
 - a The percentage of single parents who are men.
 - b The percentage of men who are single parents.
 - c The percentage of accidental injuries occurring at sports games.
 - d The percentage of sports games with accidental injuries.
 - e The percentage of low-income families buying houses.
- 2 What total should be used for each percentage?
 - a The percentage interviewed who favour daylight saving and are women.
 - b The percentage of women who favour daylight saving.
 - c The percentage of those in favour of daylight saving who are women.
 - d The percentage of teenagers who own pets.
 - e The percentage of pet-owners who are teenagers.

Example
4

- 3 This table shows the number of physical assaults on Queensland people aged 15 and over, according to different categories.

Queensland physical assaults (thousands)				
Location	By men		By women	
	On men	On women	On men	On women
At home	258.9	44.5	34.7	119.0
At work	21.8	141.8	23.8	7.3
At licensed premises	24.1	42.6	7.7	4.3
Outside	11.3	167.9	18.9	18.4
Other	30.0	47.1	9.4	10.6

Source: Australian Bureau of Statistics

Use the table to find the percentage of assaults:

- a** at home by women **b** by women at home
c on men by women **d** by women on men
e at licensed premises by men **f** by men at work
g on men at work.

- 4 The table shows the results of a survey of employees' preferences for the length of their lunch-break (with appropriate changes to finishing time). Use the table to find the percentage of:

	Lunch-break		
	$\frac{1}{2}$ hour	$\frac{3}{4}$ hour	1 hour
Workshop	14	20	12
On-site	20	14	6
Office	6	8	13

- a** workshop workers who prefer 1 hour
b office workers who prefer $\frac{1}{2}$ hour
c those who prefer $\frac{1}{2}$ hour who are office workers
d all workers who prefer $\frac{1}{2}$ hour and are office workers
e on-site workers who prefer 1 hour.

- 5 The table shows the results of quality testing samples of toys produced by different production lines in a factory. Use the table to find the percentage of:

	Quality control	
	Pass	Fail
Line A	35	6
Line B	42	8
Line C	59	8

- a** failures that are from line A
b Line A that fail quality control
c total production from line A
d good production that is from line A
e Line C that fail quality control.

- 6 Voters were asked for their feelings about a republic, as well as their voting intentions for the next election. Of the Labor voters, 28 were in favour of a republic, 17 were against and 4 didn't care. Of the Liberal/National voters, 25 were in favour, 27 were against and 3 didn't care. Of those who favoured minor parties, 8 were in favour, 6 were against and 7 didn't care.
- a** Construct a two-way table of this information.
b Calculate percentages and state any noticeable patterns in the results.

Problem solving

- 7** The data below shows the results of a survey of people in different states regarding their support for oil exploration on the Great Barrier Reef.

NSW:	Y Y Y N N N N N N N	SA:	Y Y N N N N N N N
Vic:	Y Y N N N N N N N N	WA:	Y Y Y Y N N N N N N
Qld:	Y Y Y Y N N N N N N N	Tas:	Y N N N N
NT:	Y Y Y N N N N	ACT:	Y N N N N

Show the information as a two-way table, and use percentages to find any noticeable patterns.

- 8** People were asked for their age and the number of takeaway meals eaten in the last week. The age is given first in the results below.

35, 2	55, 1	58, 2	51, 0	39, 1
52, 1	26, 0	52, 2	23, 2	59, 0
24, 3	24, 4	48, 0	41, 4	17, 1
19, 1	32, 3	26, 4	26, 4	59, 3
39, 2	21, 1	65, 2	59, 0	46, 2
41, 1	22, 3	46, 3	56, 3	25, 2
24, 2	26, 4	38, 3	59, 2	17, 7
46, 3	32, 5	31, 4	43, 1	17, 3

Use suitable age groupings to construct a percentaged two-way table to find any noticeable patterns.



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- 9** Some people in Cairns gave information about their ages and the numbers of brothers and sisters they had. The following information was collected, where the age is given first and then the number of siblings.

Alanna 28, 2	Allan 55, 1	Andrew 18, 1	Anita 29, 0	Ann-Marie 41, 2
Astrid 43, 1	Bjorn 16, 0	Bruce 24, 2	Carlo 35, 3	Chris 53, 2
Denise 28, 3	Duke 37, 3	Elaine 35, 0	Eleni 34, 1	Jenny 19, 0
Jodie 28, 6	Joy 33, 2	Kandai 34, 2	Kayo 16, 1	Ken 19, 3
Louise 22, 2	Lynne 17, 4	Michael 17, 1	Miguel 22, 2	Paul 34, 2
Peter 35, 0	Rebecca 46, 5	Sandra 58, 4	Shaia 23, 6	Shayne 16, 0
Tom 16, 4	Trudy 44, 0	Wendy 32, 5		

Use a percentaged two-way table to find any noticeable patterns.

1.03 Interpreting associations

Exact probabilities are more useful than general comparisons. For example, $P(\text{Chemotherapy cure}) = 0.85$ is better data than the statement ‘chemotherapy has a higher chance of curing cancer’. You can use marginal percentages to compare probabilities.



Contingency
tables

EXAMPLE 7

A study of the deaths of 10 000 men found that 2500 of them were smokers. The number of men who died of lung cancer was 715, and 630 of these were smokers.

- Construct a contingency table for smoking and lung cancer deaths.
- Compare the risk of dying from lung cancer for smokers and non-smokers.

Solution

- Put the known data into a contingency table.

Cause of death	Smoker	Non-smoker	
Lung cancer	630		715
Other reason			
	2500		10 000

Use the totals to complete the table.

Cause of death	Smoker	Non-smoker	
Lung cancer	630	85	715
Other reason	1870	7415	9285
	2500	7500	10 000

- To compare results for smokers and non-smokers, calculate the column percentages.

Cause of death	Smoker	Non-smoker	
Lung cancer	25.2	1.1	7.2
Other reason	74.8	98.9	92.9
	100	100	100

Compare the risks.

Smokers have about a 25.2% chance of dying from lung cancer, compared with 1.1% for non-smokers.

Quantify the comparison.

$$25.2\% \div 1.1\% \approx 22$$

Write the answer.

Smokers are about 22 times as likely to die from lung cancer as non-smokers.

Research has now shown smoking causes changes in the lungs that often leads to cancer. This research proves the link between smoking and lung cancer.

There are many medical/pathology tests that your doctor can use. The number of red cells and different kinds of white cells in the blood are often counted. Other tests look for *markers* in your blood of different conditions or diseases. These chemicals are waste products from metabolic processes, bacteria or parasites. A pathology test uses a reagent that reacts with the marker chemical in a characteristic way. This could be a coloured precipitate, like those used in school chemistry experiments.

Pathology test results

- A test that gives no reaction has a **negative** result
- A test that gives a reaction has a **positive** result
- The result of a pathology test can be **true** or **false**
- A **false positive** is a positive result that is wrong
- A **true positive** is a positive result that is correct
- A **false negative** is a negative result that is wrong
- A **true negative** is a negative result that is correct

A false negative is the most serious error in a pathology test. Tests are often set up to minimise false negatives, so may give a lot of false positives.

EXAMPLE 8

About 2% of people suffer from diabetes at some time, and the test is about 99% accurate. The accuracy is not affected by the presence or absence of the disease.

- Set up a two-way table for test results and disease presence.
- Find the probability of a false positive.
- Find the probability of a false negative.
- Compare the probabilities of true and false positives.



Getty Images/vchal

Solution

- Set up the table with 2% with diabetes and 98% without diabetes.
Calculate 99% of 2% for true positives and 99% of 98% for the true negatives.

	Positive	Negative	
Diabetes	1.98%		2%
No diabetes		97.02%	98%
			100%

Complete the other cells by subtraction.

You might want to do the wrong results in red.

	Positive	Negative	
Diabetes	1.98%	0.02%	2
No diabetes	0.98%	97.02%	98
	2.96%	97.04%	100

- b** Recalculate the percentages by column to get the desired probability.

	Positive	Negative	
Diabetes	66.9	0.021	2
No diabetes	33.1	99.979	98
	100	100	100

Write the answer.

The probability that a positive result is wrong is about 33%.

- c** Read the answer from the table.

The probability that a negative result is wrong is about 0.02%.

- d** Write the answer.

About 66% of positive results are true and about 33% are false, so true positives are about twice as likely as false positives.

For rare conditions, false positives often outnumber true positives. Doctors consider all the symptoms to make a diagnosis. They do not rely on pathology tests. They use *all* of the evidence, including symptoms, medical history and clinical experience. Pathology tests and biopsies are used to confirm a diagnosis.

Exercise 1.03 Interpreting associations

Example
7

- 1** A study of causes of death in Australia found that, from 54 000 deaths, 9400 resulted from heart attack. Of those who died from heart attack, 3020 were obese. A total of 5840 of the 54 000 studied were obese.
 - a** Construct a contingency table for heart attack and obesity.
 - b** Calculate the probabilities of dying of heart attack for obese and non-obese people.
 - c** Compare the risks of dying of heart attack for obese and other people.

Example
8

- 2** An over-the-counter pregnancy test kit has an accuracy of 90%. About 30% of women who buy and use the kit are actually pregnant. Use an appropriately percentaged contingency table to find the probability of:

- a** a false positive
- b** a false negative

Compare the probabilities of true and false positives.

- 3** The contingency table below shows the return-to-work plans of women from white-collar jobs on maternity leave.

	Full-time	Part-time	Not returning	
Professional	3%			
Administration	5%	7%		25%
Clerical		18%		63%
	25%		44%	100%

- Copy and complete the table.
 - What is the probability that a woman is a professional planning to return to part-time work?
 - Find the probability that a professional woman plans to return to part-time work.
 - Find the probability that a woman planning to return to part-time work is a professional.
 - From 130 women on maternity leave, how many would you expect to be clerical workers planning to return to full-time work?
 - From 90 women on maternity leave, how many would you expect to be professionals not planning to return to work?
- 4** The table below shows the percentages of people interviewed about TV-watching preferences, by age.

	Age				
	≤ 30	31–40	41–50	> 50	
Current affairs	12	10	11	14	47
Sport	10	7	8	6	31
Drama	1	3	5	13	22
	23	20	24	33	100

- How many people from a similar group of 150 would prefer to watch drama?
- How many people from a similar group of 150 would be aged over 50?
- How many people from a similar group of 150 would be aged under 30 and prefer to watch drama?
- What is the expected number who would prefer to watch sport from 50 people aged 41–50?
- What is the expected number from 90 current affairs enthusiasts who would be aged 31–40?

Problem solving

- 5** An expensive pathology test is only about 80% accurate, but doctors normally order the test only if other symptoms of the disease are present. It is thought that, of patients for whom the test is ordered, about 30% actually have the disease.

Compare the probabilities of true and false positives.

- 6** This table shows support for paid maternity leave. A student concluded that ‘30% of women would vote for paid maternity leave’. Explain why the conclusion is incorrect and find the correct result.

	Paid maternity leave		
	Yes	No	
Men	15%	35%	50%
Women	30%	20%	50%
	45%	55%	100%

- 7** A market survey of preferred drinks among children grouped them as young (< 5), medium age (5–10) or older (> 10) with the following results:

Young preferring soft drinks:	5%
Young preferring milk drinks:	14%
Young preferring fruit juices:	6%
Medium age preferring soft drinks:	13%
Medium age preferring milk drinks:	16%
Medium age preferring fruit juices:	26%
Older preferring soft drinks:	10%
Older preferring milk drinks:	4%
Older preferring fruit juices:	6%

A student concluded that:

- a** 26% of medium age children preferred fruit juices
- b** older children were twice as likely as young children to prefer soft drinks
- c** 9% of those who preferred milk drinks were older children.

For each conclusion, state if the student was correct, and give the correct result if they were wrong.



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1.04 Scatterplots

For bivariate data where both variables are numerical, it makes sense to draw a graph to show the data.



Scatterplots
and
associations



A page of
scatterplots

Scatterplots

With a **scatterplot** (also called a **scatter graph** or **scattergram**):

- use ordered pairs of numerical bivariate data
- make one variable horizontal and the other vertical
- choose scales so the axes needed are about the same length

EXAMPLE 9

The table below shows the values of some bivariate data with variables P and Q .

P	1	3	6	7	2	3	4	5	6	7	8	9
Q	1	6	7	11	2	5	7	8	11	14	16	16

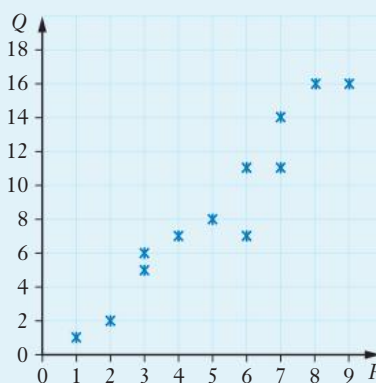
- Construct a scatterplot of the data with P on the horizontal axis.
- Comment on the pattern.

Solution

- Choose appropriate scales.

Go up in 1s for P and up in 2s for Q so the scales are about the same size.

Plot the points on a graph with P horizontal.



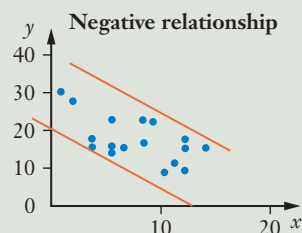
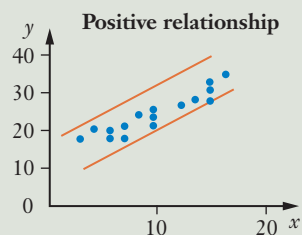
- Make a comment.

The values of P and Q vary in roughly the same way. Ordered pairs with low values of P also have low values of Q . The pattern continues for medium and high values.

The data in Example 9 vary in a similar way. This means that variables are roughly proportional to each other. You say that there is a **positive association** between P and Q .

Variable associations

- A pattern in a scatterplot suggests an **association** between bivariate numerical variables
- In a **positive association**, the low values of the variables tend to be in the same pairs, as do the high values; the graph slopes upward from left to right
- In a **negative association**, the low values of one variable tend to be in pairs with high values of the other; the graph slopes downward from left to right



EXAMPLE 10

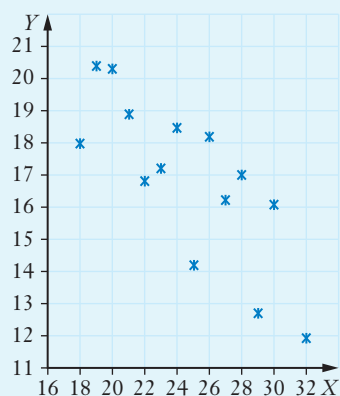
The table below shows the values of some bivariate data with variables X and Y .

X	18	19	20	21	22	23	24	25	26	27	28	29	30	32
Y	18	20.4	20.3	18.9	16.8	17.2	18.5	14.2	18.2	16.2	17	12.7	16.1	11.9

- Construct a scatterplot of the data with X on the horizontal axis.
- Comment on the pattern.

Solution

- Choose appropriate scales.
Go up in 2s for X and up in 1s for Y so the scales are about the same size.
Plot the points with X horizontal.



- Make a comment.

The points slope roughly downward from left to right, so there is a negative association between X and Y .

EXAMPLE 11

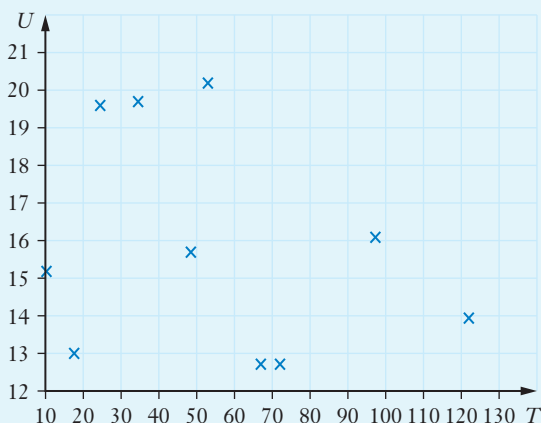
The table below shows the values of some bivariate data with variables T and U .

T	10.4	17.6	24.5	34.5	48.7	53.1	67.1	72	97.3	122
U	15.2	13	19.6	19.7	15.7	20.2	12.7	12.7	16.1	13.9

- Make a scatterplot of the data with T on the horizontal axis.
- Comment on the pattern.

Solution

- Choose appropriate scales.
Go up in 10s for T and in 1s for U so the scales are about the same size.
Plot the points with T horizontal.



- Make a comment.

There is no pattern, so there is no apparent association between T and U .

INVESTIGATION

CLASS ASSOCIATIONS

Collect the following data from your class:

Shoe size, height, forearm length (elbow to wrist), age in months, number of siblings, hair length, number living in home

There are many possible scatterplots that you can draw from the data.

Work in groups of 2 or 3 to draw the scatterplots; your teacher will assign the scatterplots for each group.

Comment on any patterns in the scatterplots.

Exercise 1.04 Scatterplots

Examples

9, 10

Example

11

- 1 This table shows the values of some bivariate data.

X	10.3	14.5	16.4	13.7	18.1	14.3	17.6	19	24.7	26.3	27.6	23.6
Y	51	53.1	41.5	42.4	41.3	26.9	30.6	22.1	24.5	13.4	10.2	16.9

- a** Make a scatterplot of the data with X on the horizontal axis.
b Comment on the pattern.

- 2 This table shows the values of some bivariate data.

A	27	35	34	32	47	44	56	45	49	58
B	56	33	13	34	64	45	27	15	24	44

- a** Make a scatterplot of the data with A on the horizontal axis.
b Comment on the pattern.

- 3 This table shows the values of some bivariate data.

P	5.6	6.9	6.8	7.4	7.7	9.6	10.6	9.5	9.3	11.6	12	12.2
Q	11.5	11.7	12.7	12.5	12.7	14.2	13	13.9	13.8	14.8	15.4	14.7

- a** Make a scatterplot of the data with P on the horizontal axis.
b Comment on the pattern.

- 4 This table shows the values of some bivariate data.

V	0.7	1.2	1.4	2	2.1	2.4	2.8	2.9	2.9	3.9	4
W	12.7	13.3	17.6	14.1	15.1	13.1	23.4	18.5	17.6	26.7	25.9

- a** Make a scatterplot of the data with V on the horizontal axis.
b Comment on the pattern.

- 5 This table shows the values of some bivariate data.

X	19	15	18	24	28	33	26	30	34	39	44	47	45	43
Y	133	130	148	123	120	97	106	87	112	84	95	67	71	44

- a** Make a scatterplot of the data with X on the horizontal axis.
b Comment on the pattern.

Problem solving

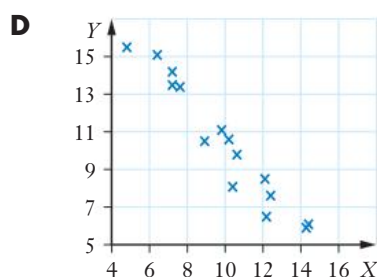
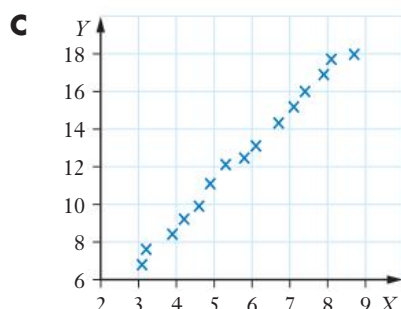
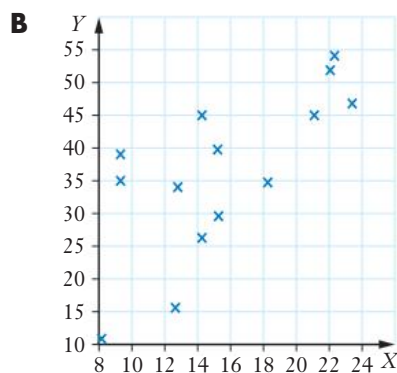
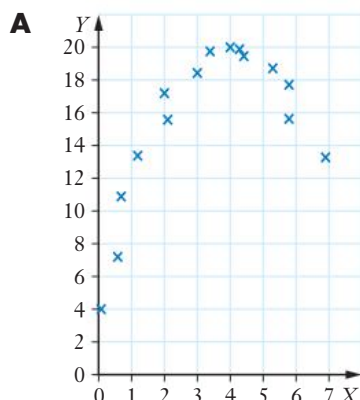
- 6 Some people were asked how many takeaway meals they had the previous week. The data below shows their ages and the number of meals that they reported.

35, 2	55, 1	58, 2	51, 0	39, 1	52, 1	26, 0	52, 2	23, 2	59, 0
24, 3	24, 4	48, 0	41, 4	17, 1	19, 1	32, 3	26, 4	26, 4	59, 3
39, 2	21, 1	65, 2	59, 0	46, 2	41, 1	22, 3	46, 3	56, 3	25, 2
24, 2	26, 4	38, 3	59, 2	17, 7	46, 3	32, 5	31, 4	43, 1	17, 3

Use a scatterplot to examine the data.

1.05 Describing associations

Consider the following scatterplots.



A has a clear association, but it is not a straight line. **C** and **D** have clear associations that are straight lines, but the line is much clearer in **C**. Graph **B** could be a straight line going up from left to right, but it is not very clear.

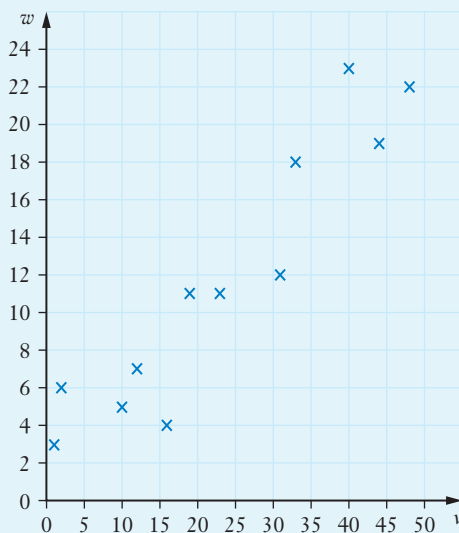
Describing associations

- A **non-linear** association is not a straight line
- A **linear** association is a straight line
- A **positive** association goes upward from left to right
- A **negative** association goes downward from left to right
- A **strong association** has points that make an almost perfect line
- A **moderate association** has an upward or downward trend that can be shown by a straight line through the points. Almost all the points are close to such a line
- A **weak association** has a pattern, but no clear line

In the above, **A** is a non-linear association, **B** is a weak positive linear association, **C** is a strong positive linear association and **D** is a moderate negative linear association.

EXAMPLE 12

The scatterplot shows the values of some bivariate data. Describe any apparent association between v and w .

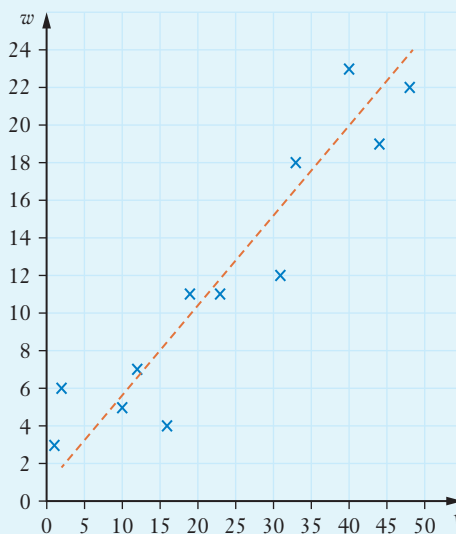


Solution

Describe the points.

Try drawing a line.

The points have a clear trend upwards from left to right.



Describe the result.

Write the answers.

All the points are fairly close to a straight line sloping upwards.

There is a moderate positive linear association between v and w .

EXAMPLE 13

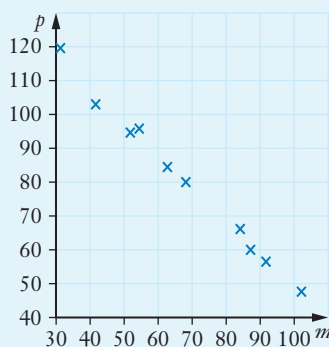
The table below shows the values of some bivariate data.

- a Draw a scatterplot of the data.
- b Describe any apparent association.

m	31.3	41.7	51.7	54.4	62.8	68.2	84.2	87.2	91.9	102.1
p	120	103	94.9	95.8	84.5	80.1	66.2	60.1	56.7	47.8

Solution

- a Plot the points on suitable axes.



- b Describe the result.

The points form a very clear straight line sloping downwards, so there is a strong negative linear association between m and p .



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TECHNOLOGY

Spreadsheet: Scatterplots

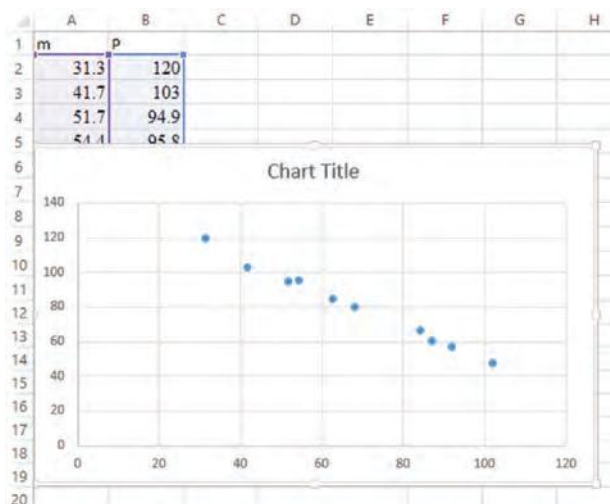
You can use a spreadsheet to draw the scatterplot in Example 13.

- 1 Type the data from Example 13 into columns A and B in the spreadsheet.

	A	B	C	D	E
1	m	p			
2	31.3	120			
3	41.7	103			
4	51.7	94.9			
5	54.4	95.8			

- 2 Move the mouse to cell A2.
- 3 Hold down the shift button and move the mouse down to B11 to highlight the data.
- 4 Release the shift button and click the mouse on the 'Insert' tab.
- 5 Click on the scatterplot icon.
- 6 Choose the icon with no lines.

The graph will appear.



You can click on the graph and choose how to format the graph.

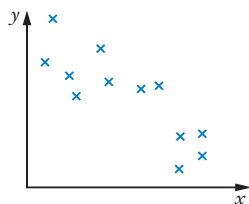
Your teacher will tell you what format is required.

Exercise 1.05 Describing associations

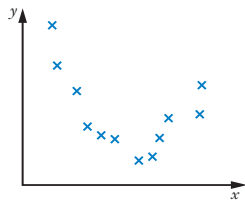
1 Describe any apparent associations in each scatterplot.

Example
12

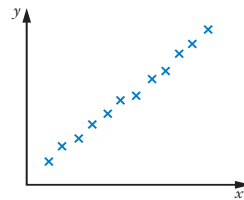
a



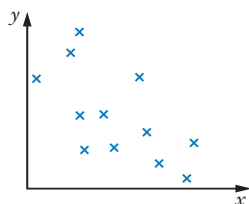
b



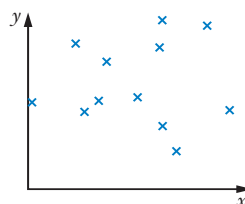
c



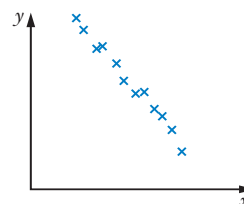
d



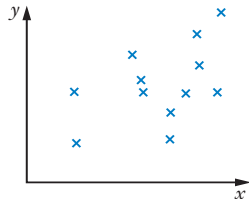
e



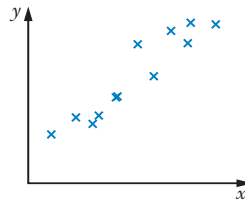
f



g



h



2 Use a scatterplot of the values below to describe any apparent association.

m	4.7	6.4	7.2	7.4	8.3	8.3	10.1	10.5	10.7	11.7	12	12.8
T	8.8	7.1	6.7	6.9	7.4	6.3	5.9	6.2	6.8	5.4	4.8	4.8

Example
13

3 Use a scatterplot of the values below to describe any apparent association.

p	2	4	5	6	8	10	11	13	14	16	18	19	20	22	24
q	3	7	10	15	18	23	28	32	39	42	48	50	56	62	66

4 Use a scatterplot of the values below to describe any apparent association.

x	3.5	0.8	4.2	10.5	7.7	13.3	14.9	15.9	15.7	21.4
y	1.2	11	10.8	5.1	15.3	5	12.5	16.8	19.9	8.5

- 5 Use a scatterplot of the values below to describe any apparent association.

P	0.6	1.6	3.1	4.1	5.3	6.2	6.7	7.7	8.7	10.3	10.8	11.8	13.1	14.2
V	39.1	36.3	33.5	33.2	31.1	28.2	24.9	23.5	23	19.9	18.5	17.1	13.7	11.3

Problem solving

- 6 The ages of the parents of 23 Year 12 students are shown below. Use a scatterplot to describe any association.

Mother	50	53	40	39	44	36	53	56	58	48	50	40
Father	52	55	42	42	47	39	56	59	58	49	49	41

Mother	45	47	44	49	45	41	52	54	37	43	45	
Father	50	46	50	53	49	44	58	62	39	43	45	

- 7 The highest frequency, f , of sound that people aged 20 to 80 years old could hear is shown in the table. Use a scatterplot to describe any association.

Age	20	25	31	37	43	48	51	57	62	70	76	79
f (kHz)	24	25	21	22	21	18	18	16	19	16	16	14

- 8 An insurance company compared the quotes of two panel beaters for different car repairs. Use the table below to draw a scatterplot and describe any association.

Panel beater 1	865	1107	1350	1379	1093	1197	1589	1822
Panel beater 2	783	996	928	1310	1614	1313	1474	1488

Panel beater 1	1929	1911	2193	2141	2379	2170	2421	
Panel beater 2	1966	1641	2323	2295	2118	2599	2710	

1.06 The correlation coefficient

You can use a scatterplot to describe a linear association as weak, moderate or strong. Scatterplots are tedious to draw. Other people may not agree with your choice of weak, moderate or strong.

A *measurement* is a better way of describing the strength of an association.

Recall that the **standard deviation** measures the scatter or spread of data about the mean. A scatter graph *shows* how close bivariate data is to a straight line.

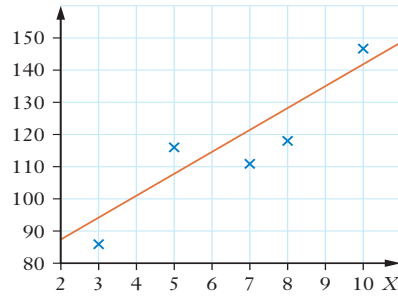
This table shows the ages and heights of a group of girls.

Name	Skye	Zoe	Mila	Hope	Arwen
Age (years)	3	5	7	8	10
Height (cm)	86	116	111	118	147

The graph is shown on the right.

How close are the points to the straight line?

The average age is 6.6 years old and the average height is 115.6 cm. Skye is below the average for both by 3.6 years and 29.6 cm. You can write this as $(-3.6, -29.6)$. You can write $(-1.6, 0.4)$, $(0.4, -4.6)$, $(1.4, 2.4)$ and $(3.4, 31.4)$ for Zoe, Mila, Hope and Arwen respectively.



The line in the graph goes upward.

Skye, Hope and Arwen have age and height on the same side of the mean.

The variables of almost all pairs will be on the same side of the average for a strong positive association. The products of the differences with the average will almost all be positive.

The variables of almost all pairs will be on opposite sides of the average for a strong negative association. The products of the differences with the average will almost all be negative.

For a weak association, there will be much more variation in the signs.

This means that the average of the differences will be positive for a positive association and negative for a negative association. The average will be bigger for a strong association and smaller for a weak association.

For the age and height above, the average of the products is

$$\begin{aligned}
 \frac{\sum (x - \bar{x})(y - \bar{y})}{n} &= \frac{(-3.6) \times (-29.6) + (-1.6) \times 0.4 + 0.4 \times (-4.6) + 1.4 \times 2.4 + 3.4 \times 31.4}{5} \\
 &= \frac{105.56 - 0.64 - 1.86 + 3.36 + 106.76}{5} \\
 &= 42.84
 \end{aligned}$$

Remember that the Σ sign means ‘the sum of’.

This average is called the **covariance** of the age and height. It measures how close the points are to a straight line. You should write it as 42.84 year-cm because you have multiplied years and cm.

You can make it a coefficient by dividing by the standard deviations of the age and height. This eliminates the years and cm as well.

The standard deviations of the age and height are about 2.42 and 19.4 respectively.

$$\frac{42.84}{2.42 \times 19.4} \approx 0.91. \text{ This is the } \textbf{correlation coefficient} \text{ for the age and height of the group.}$$

Pearson's correlation coefficient

- The **moment** of xy about (a, b) is $(x - a) \times (y - b)$
- The **covariance** $COV(x, y)$ of bivariate data $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ is the average of the moments of $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ about the averages $\bar{x}$ and $\bar{y}$ of the x and y values
- The **Pearson's correlation coefficient** r is the covariance divided by the standard deviations of the x and y values;

$$r = \frac{COV(x, y)}{SD(x) \times SD(y)} = \frac{1}{n} \sum \left(\frac{x_i - \bar{x}}{\sigma_x} \right) \left(\frac{y_i - \bar{y}}{\sigma_y} \right) = \frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right)$$

- Note that if you use the sample standard deviations, which are divided by $n - 1$ instead of n , then you have to use the sample covariance, where the total of the product moments is divided by $n - 1$ instead of n , as shown above
- The correlation coefficient is always a number from -1 to 1
- A positive correlation coefficient means that there is a positive association
- A negative correlation coefficient means that there is a negative association
- A correlation coefficient of -1 or 1 means that there is a perfect straight line

You can use a scientific calculator, spreadsheet or other technology to find the correlation coefficient.

EXAMPLE 14

The covariance of x and y is -14.35 and their standard deviations are 7.204 and 3.097 respectively.

- a** What is their correlation coefficient?
- b** What can you say about the variables?

Solution

- a** Write the formula.

$$r = \frac{COV(x, y)}{SD(x) \times SD(y)}$$

Substitute values and calculate.

$$\begin{aligned} &= \frac{-14.35}{7.204 \times 3.097} \\ &= -0.6432\dots \end{aligned}$$

Write the answer.

The correlation coefficient is about -0.64 .

- b** Write what the correlation coefficient means.

There is a negative association between the variables x and y .

It doesn't make sense to give a correlation coefficient to more than 2 decimal places unless it would round off to 1 or -1 .

TECHNOLOGY

Correlation coefficients on a scientific calculator

You can find the correlation coefficient of this data on your calculator.

p	4	5	6	8	10	11	13	14	16	18
q	7	10	15	18	23	28	32	39	42	48

TI Scientific calculator

- Press (DATA) and enter the values of p and q in the L1 and L2 columns respectively.
- Then press 2nd data [stat] and choose 2: 2-Var Stats.
- Make sure it has XDATA: L1 and YDATA: L2 and then use the cursor to go down to CALC and press enter.
- Move down to $r=$ to read the result 0.995...
- Press [2nd] [mode] [quit] to exit the statistics calculations.

Casio Scientific calculator

- Press [mode], 2: [stat] and 2: A+BX.
- Enter the values of p and q in the X and Y columns respectively.
- Press AC then press [SHIFT] 1 [stat], 5: Reg, 3: r and press [=] to read the result of 0.995...
- Press [mode], 1: COMP to exit the statistics mode.

Sharp Scientific calculator

- Press [mode], 1 and choose 1 (LINE): Linear regression calculation.
- Enter the values of p and q in pairs by pressing
4, [sto→] (x,y), 7, M+ (DATA), 5, [sto→] (x,y), 10, M+ (DATA), etc.
until you have entered all the values of p and q .
- Then press [ALPHA], [D] to read the result for r of 0.995...

Make sure you can use your calculator effectively to find the correlation coefficient.



Correlation

The correlation coefficient

As a general rule, you can use the following to interpret correlation coefficients:

- $r = 1$: perfect positive association
- r from 0.7 up: strong positive association
- r from 0.4 up: moderate positive association
- r from 0.2 up: weak positive association
- r between -0.2 and 0.2 : little or no association
- r from -0.2 down: weak negative association
- r from -0.4 down: moderate negative association
- r from -0.7 down: strong negative association
- $r = -1$: perfect negative association

Remember that correlation has *nothing to do with cause and effect*. All it shows is how close the variables are to a straight line.

For example, you may have heard of the old legend about storks bringing babies. In many countries of Europe, there is a strong correlation between birth rates and stork populations (0.85 in Copenhagen after the Second World War).



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This does not mean that more storks in an area will *cause* more babies to be born. It could be that more housing construction in an area attracts young couples and also provides more nesting sites for storks.

EXAMPLE 15

Find the correlation coefficient for the following data and interpret the result.

x	5	10	15	20	25	30	35	40
y	65	65.7	89.9	97.6	45.4	37.7	68.5	54

Solution

Use your scientific calculator to find the correlation coefficient.

Use the appropriate range.

You should get a value of $-0.386\dots$ if you have entered the data correctly.

There is a weak negative correlation between x and y .

TECHNOLOGY

Spreadsheet: Correlation calculations

You can use a spreadsheet to find the covariance, standard deviation and correlation coefficient of the data below.

a	20	25	31	37	43	48	51	57	62	70	76	79
f	24	25	21	22	21	18	18	16	19	16	16	14

1 Put titles in cells A1 to E1.

2 Enter the data in A2 to B13.

3 Move back to cell C2.

4 Type the following

`=COVARIANCE.S(A2:A13,B2:B13)`

and press enter.

5 Move to cell D2.

6 Type the following

`=STDEV.S(A2:A13)`

and press enter.

7 Move to cell E2.

8 Type the following

`=STDEV.S(B2:B13)`

and press enter.

9 Move to cell F2.

10 Type the following

`=C2/(D2*E2)`

and press enter.

a	f	COV(a, f)	SD(a)	SD(f)	r
20	24				
25	25				
31	21				

a	f	COV(a, f)	SD(a)	SD(f)	r
20	24	-63.0757576	19.5841231	3.459725	-0.93092944
25	25				
31	21				
37	22				
43	21				
48	18				
51	18				
57	16				
62	19				
70	16				
76	16				
79	14				

The correlation coefficient is about -0.93.

You can keep this spreadsheet and change the data in columns A and B to work out the answers for other problems. Don't forget to change the formulas for different values of n .

You may notice there is a COVARIANCE.P function and a STDEV.P function. These are for populations instead of samples. It doesn't make any difference to the correlation coefficient which functions you use, as long as you use the same kind for all the calculations.

Exercise 1.06 The correlation coefficient

Examples
14, 15

- 1 The covariance of a and b is 6.6 and their standard deviations are 2.1 and 3.4.
 - a What is the correlation coefficient?
 - b What can you say about any apparent association?
- 2 The covariance of m and p is -8.9 and their standard deviations are 5.1 and 6.3.
 - a What is the correlation coefficient?
 - b Describe any apparent association.
- 3 The covariance of a and b is 28.6 and their standard deviations are 7.9 and 5.9.
 - a What is the correlation coefficient?
 - b What can you say about any apparent association?
- 4 The covariance of y and z is -12.5 and their standard deviations are 2.4 and 6.1.
 - a What is the correlation coefficient?
 - b Describe any apparent association.
- 5 The covariance of p and q is 15.2 and their standard deviations are 9.7 and 10.2.
 - a What is the correlation coefficient?
 - b What can you say about any apparent association?
- 6 Use a spreadsheet to find the covariance, standard deviations and correlation coefficient of the variables in the table below and describe the association.

g	8	10	12	14	16	18	20	22	24	26	28
f	55	67.1	64	77	64.6	57.2	83.7	77.6	70.6	78.2	80.1

- 7 Use a spreadsheet to find the covariance, standard deviations and correlation coefficient of the variables in the table below and describe the association.

t	8.4	6.6	9.3	14.4	19.2	24.3	26.6	32.3	34.4	42.7
v	102	59.4	57.1	86.5	67.1	72.1	31.7	37.3	57.2	67.1

- 8 Use your calculator to find the correlation coefficient of the variables in the table below and describe the association.

m	20	35	42	54	73	90
E	64	28	52	28	19	6

- 9 Use your calculator to find the correlation coefficient of the variables in the table below and describe the association.

p	9	10	11	10	12	12	16	14
q	20	25	18	19	30	18	31	15

- 10** Use your calculator to find the correlation coefficient of the variables in the table below and describe the association.

<i>r</i>	68.2	74.2	76.9	98	87.4	112.9	113.8	119.3
<i>v</i>	79.9	151	62.9	162	97.9	57.2	85.3	55.4

- 11** Use your calculator to find the correlation coefficient of the variables in the table below and describe the association.

<i>a</i>	9.9	10.8	11.3	11.7	12.3	12.8	13.4	13.2	14.4	14.4
<i>b</i>	127	129	130	128	130.2	130.1	131	130.7	131.9	130.3

- 12** Use your calculator to find the correlation coefficient of the variables in the table below and describe the association.

<i>x</i>	17.7	44.2	94.7	91.2	85.1	120.4	163.7	124	149	173.7
<i>y</i>	17.5	15	27.4	18.9	26.9	27.2	21.6	24.3	21	37.5

Problem solving

- 13** The mouse population in a field of wheat was estimated by trapping them at night and counting them. The population was monitored every 3 days for a month and the following results were obtained.

Time (days)	0	3	6	9	12	15	18	21	24	27	30
Pop ('00)	0.6	0.9	1.1	1	1.2	1.4	1.6	2	2.2	2.2	2.5

Find if there is any association and comment on your results.

- 14** The scores of 15 Year 12 students doing both Maths Methods and English at the end of Term 1 exams were as follows.

Maths	E-	D-	D-	D	D	D+	C-	C-	C-	C+	B-	B	B+	B+	A+
English	D+	D-	D	D+	C-	B	E	D	B-	C+	A+	C	C	A+	A+

Find if there is any association and comment on your results.

1. CHAPTER SUMMARY

Associations between variables

Types of data

- **Univariate** data has only one variable
- **Bivariate** data has two variables, collected in pairs
- **Categorical** data has values that are names
 - **Nominal** data has no order, like colours
 - **Ordinal** data has a natural order, like places in a race
- **Numerical** data has values that are numbers
 - **Discrete** data has only some values, like shoe sizes
 - **Continuous** data can have any value between the lowest and highest values, like height

Contingency tables

- A **two-way** table for bivariate data has one variable across the columns and the other one down the rows. The totals across and down are called **marginal frequencies**.
- You can work out percentages in a contingency table:
 - **by row** (or **row percentages**) based on the totals on the right of the rows
 - **by column** (or **column percentages**) based on totals at the bottom of the columns
 - **by table total** (or **table total percentages**) based on the grand total at the bottom right
- Percentages by rows or columns are called **marginal percentages**
- Probabilities based on frequencies **less than 5** are unreliable

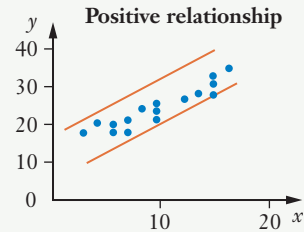
Pathology test results

- A test that gives no reaction has a **negative** result
- A test that gives a reaction has a **positive** result
- The result of a pathology test can be **true** or **false**
- A **false positive** is a positive result that is wrong
- A **true positive** is a positive result that is correct
- A **false negative** is a negative result that is wrong
- A **true negative** is a negative result that is correct

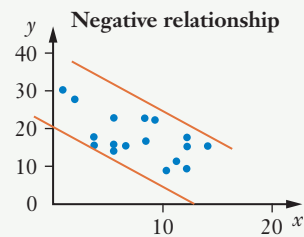
Scatterplots

Also called a **scatter graph** or **scattergram**

- Use ordered pairs of numerical bivariate data for a **scatterplot**
- Make one variable horizontal and the other vertical
- Choose scales so the axes needed are about the same length
- A pattern in a scatterplot suggests an **association** between bivariate numerical variables
- In a **positive** association, the low values of the variables tend to be in the same pairs, as do the high values; the graph slopes upward from left to right



- In a **negative** association, the low values of one variable tend to be in pairs with high values of the other; the graph slopes downward from left to right



- A **non-linear** association is not a straight line
- A **linear** association is a straight line
- A **positive** association goes upward from left to right
- A **negative** association goes downward from left to right
- A **strong** association has points that make an almost perfect line
- A **moderate** association has an upward or downward trend that can be shown by a straight line through the points. Almost all the points are close to such a line
- A **weak** association has a pattern, but no clear line

Pearson's correlation coefficient

- The **moment** of xy about (a, b) is $(x - a) \times (y - b)$
- The **covariance** $COV(x, y)$ of bivariate data $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ is the average of the moments of $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ about the averages $\bar{x}$ and $\bar{y}$ of the x and y values
- The **Pearson's correlation coefficient** r is the covariance divided by the standard deviations of the x and y values;

$$r = \frac{COV(x, y)}{SD(x) \times SD(y)} = \frac{1}{n} \sum \left(\frac{x_i - \bar{x}}{\sigma_x} \right) \left(\frac{y_i - \bar{y}}{\sigma_y} \right) = \frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right)$$

- The covariance and standard deviation calculations must be of the same kind: either all sample or all population calculations. The answers are the same because the ns or $(n - 1)s$ cancel out.
- The correlation coefficient is always a number from -1 to 1 .
 - $r = 1$: perfect positive association, a perfect straight line
 - r from 0.7 up: strong positive association
 - r from 0.4 up: moderate positive association
 - r from 0.2 up: weak positive association
 - r between -0.2 and 0.2 : little or no association
 - r from -0.2 down: weak negative association
 - r from -0.4 down: moderate negative association
 - r from -0.7 down: strong negative association
 - $r = -1$: perfect negative association, a perfect straight line
- The correlation coefficient has *nothing to do with cause and effect*. All it shows is how close the variables are to a straight line

1. CHAPTER REVIEW

Associations between variables

- 1** From 540 males studying first-year Economics at the University of Queensland, 210 lived on the Southside. There were 640 females in the course and 650 people living on the Northside.

Example
1

- a** Construct a two-way table.
- b** How many males were living on the Northside?
- c** How many students were living on the Southside?

- 2** Some Year 12 students were asked what Mathematics course they do and what they want to do next year. The results are shown below. Art College and the Conservatorium are included in Uni. E = Essential Maths, G = General Maths, M = Maths Methods.

Example
2

M, Work	E, TAFE	G, Uni	G, Uni	E, Work
G, Uni	M, Uni	E, TAFE	M, Uni	G, TAFE
G, Work	M, Uni	E, Work	G, Apprenticeship	M, Work
G, Work	M, Uni	E, Uni	G, Uni	M, Apprenticeship
G, Work	E, Apprenticeship	M, Uni	E, Apprenticeship	G, Uni
M, TAFE	G, TAFE	M, TAFE	M, Uni	G, TAFE
G, Apprenticeship		E, TAFE	E, Work	

- a** Construct a two-way table.
- b** How many students intend to get a job next year?

Questions 3 and 4 use the following information.

The two-way table below shows the numbers of students from a Year 12 General Maths class with 24 students in the class who study Biology, Ancient History or Geography. These subjects are on different lines, so cannot be studied at the same time.

	Biology	Ancient History	Geography	Total
Male	4		4	10
Female		3		12
Total	7			

- 3** The number of females studying Geography is:

A 2 **B** 3 **C** 4 **D** 5 **E** 6

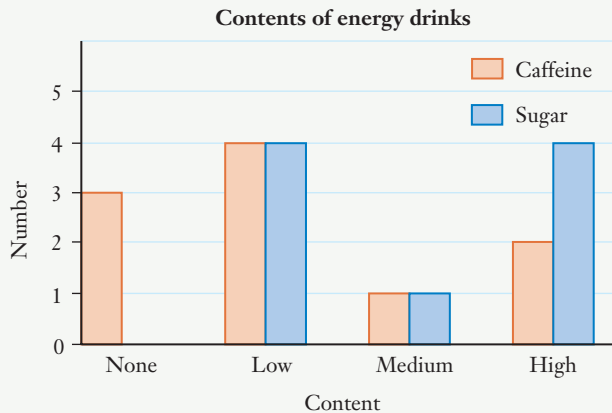
Examples
1, 2

- 4** The number of students who do the only other subject on the same line is:

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Example
3

- 5** Construct a two-way table from this graph. It shows the caffeine and sugar content of some popular ‘energy’ drinks.



Example
4

- 6** State the total for the probability or percentage that is to be calculated in each statement.
- The percentage of the crowd at a Sydney FC–Brisbane Roar game who are Brisbane fans.
 - The probability that a randomly chosen teenager from Kalkadoon SHS has a pet dog.
 - The probability that a Brisbane supporter on a train is female.

Example
5

- 7** The table below shows the hair and eye colours of a group of Year 12 students.

		Hair colour			
		Black	Brown	Fair	Red
Eye colour	Brown	3	13	4	1
	Hazel	2	9	3	1
	Green	1	2	1	0
	Blue	4	16	5	1
	Grey	1	2	1	0

- Find the percentage of brown-haired people with blue eyes.
- Find the probability that someone has red hair and blue eyes.

- 8** Some Year 12 students were asked what time they got up and whether they preferred morning or night showers. The results are summarised below.

Example
6

Time	Morning shower	Night shower
Before 5:45	2	0
Before 6:00	3	0
Before 6:15	4	2
Before 6:30	3	4
Before 6:45	5	2
Before 7:00	4	3
Before 7:15	4	4
Before 7:30	3	3
Before 7:45	3	4
Before 8:00	2	5
Before 8:15	0	6
Before 8:30	0	3

Redraw the table and calculate percentages to find a pattern.

- 9** In a nursing home, 70 out of 240 women aged over 65 suffered from osteoporosis. From answers to a questionnaire about their lifestyles between the ages of 20 and 60, it was found that 60 of the women had had very active lifestyles, including 10 of those with osteoporosis. The lifestyles of the remaining women were classified as sedentary.
- Construct a contingency table for this information.
 - How many of those with sedentary lifestyles developed osteoporosis?
 - Find the probabilities of developing osteoporosis for those with very active and sedentary lifestyles.
 - Compare the probabilities of developing osteoporosis for women with very active lifestyles to those with sedentary lifestyles.
- 10** A test for a rare blood disorder has an accuracy of 98% and 0.4% of the population have the disease. Use a contingency table to find the probability that:
- a positive result is false
 - a negative result is false.

Example
7

Example
8

Examples
9,10

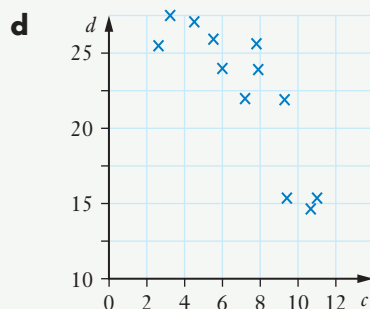
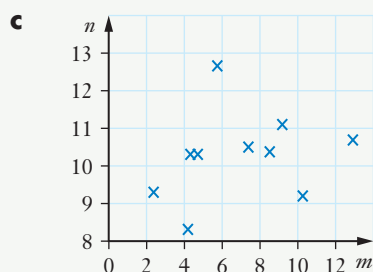
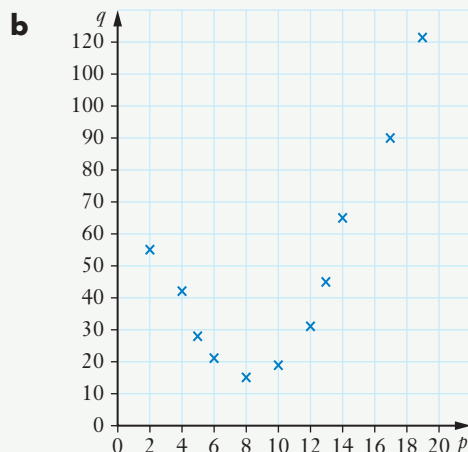
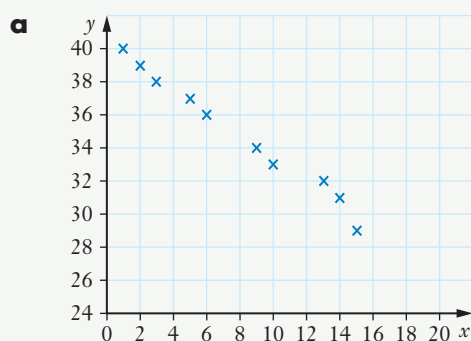
Example
11

- 11** This table shows the values of some bivariate data.

<i>A</i>	17	19	21	32	35	42	49	53	59	65
<i>B</i>	157	168	150	127	204	168	182	213	257	229

- Construct a scatterplot of the data with *A* horizontal.
- Comment on the pattern.

12 Describe any apparent associations in each scatterplot.



13 The covariance of two variables is 27.8 and the standard deviations of the variables are 6.3 and 7.1. The correlation coefficient is approximately:

- A** 31.3 **B** 0.22 **C** 0.26 **D** 0.62 **E** 1.61

14 Find the correlation coefficient for each set of statistics described and state the meaning of any association.

- a** Covariance of x and y is 25.8, standard deviations of x and y are 5.7 and 9.3.
b Covariance of a and b is -16.2 , standard deviations of a and b are 4.7 and 11.5.
c Covariance of p and q is 17.6, standard deviations of p and q are 3.5 and 5.8.

15 Find the correlation coefficient for each table of bivariate data and comment on any association.

a

p	30	36	43	49	55	61	68	74
q	38	40	36	30	36	31	28	23

b

v	1	3	8	9	10	10	16	18	19	19
w	33	36	51	36	40	35	50	44	36	39

Problem solving

- 16** The results of a survey on the amount of homework that high school students say they do each night are shown below in the order: Year level, time in minutes.

9, 90	9, 70	11, 30	8, 30	7, 30	9, 70	11, 150
12, 80	9, 30	9, 70	8, 10	12, 60	8, 30	8, 50
9, 80	9, 80	9, 20	11, 80	11, 40	11, 40	8, 20
12, 180	7, 15	12, 20	10, 90	10, 90	9, 30	12, 90
12, 40	10, 80	7, 15	12, 60	12, 130	7, 20	8, 50
12, 70	7, 15	12, 100	7, 10	12, 120	10, 70	8, 40
7, 10	9, 90	12, 120	7, 10	10, 30	7, 10	7, 10
10, 30	11, 140	7, 15	12, 180	7, 10	7, 15	11, 40
10, 20	8, 30	12, 70	10, 30	12, 110	11, 120	9, 20
8, 10	12, 140	11, 140	7, 10	8, 10	8, 50	11, 150
8, 30	11, 110	9, 60	11, 110	12, 160	10, 60	12, 150
7, 10	8, 10	9, 80	11, 120	8, 20	10, 40	9, 30
9, 80	9, 40	11, 70	11, 50	11, 150	9, 70	10, 30
7, 15	7, 20	8, 50	10, 10	8, 20	12, 30	

Construct a two-way table using suitable time categories to identify any patterns.

- 17** A large Brisbane high school has 275 students in Year 12. There are 46 short female students and 26 tall female students. There are 130 male students, of whom 65 are of average height and 42 are tall. Construct a contingency table and find the probabilities that a female student is tall and the probability that a tall student is female.
- 18** A blood test for dengue fever is about 87% accurate. The disease occurs in only about 0.01% of the population in Australia. A doctor may order the test because of other symptoms, and about 0.5% of such patients do have the disease. Compare the probability of true positives and false positives for tests ordered because of a patient's symptoms.
- 19** The loss of yield from mouse plagues in Australian wheat crops were estimated as follows. The mouse plagues occurred at different stages of the crop cycle.

Loss	3	29	34	52	78	114	112	134	162	168
Population (mice/ha)	6	4	18	28	14	42	36	55	67	73

What, if any, is the association between mouse population and loss of a wheat crop?



Practice quiz

2.

ARITHMETIC AND GEOMETRIC SEQUENCES

When you drop a ball from a height, it bounces up to only part of that height. The rebound height reduces at each bounce. A particular tennis ball dropped onto concrete rebounds to 60% of its previous height. How many bounces will it take to reduce to 20 cm if it starts at a height of 2 m? This is a **sequence** of heights. In this chapter, you will study two kinds of sequences to answer questions like this.

- 2.01 Arithmetic sequences
- 2.02 General term of an arithmetic sequence
- 2.03 Modelling with arithmetic sequences
- 2.04 Geometric sequences
- 2.05 General term of a geometric sequence
- 2.06 Modelling with geometric sequences
- Chapter summary
- Chapter review



SYLLABUS SUBJECT MATTER

The arithmetic sequence

- use recursion to generate an arithmetic sequence
- display the terms of an arithmetic sequence in both tabular and graphical form and demonstrate that arithmetic sequences can be used to model linear growth and decay in discrete situations
- use the rule for the n th term using $t_n = t_1 + (n - 1)d$, where t_n represents the n th term of the sequence, t_1 = first term, n = term number and d = common difference of a particular arithmetic sequence from the pattern of the terms in an arithmetic sequence, and use this rule to make predictions
- use arithmetic sequences to model and analyse practical situations involving linear growth or decay, such as analysing a simple interest loan or investment, calculating a taxi fare based on the flag fall and the charge per kilometre, or calculating the value of an office photocopier at the end of each year using the straight-line method or the unit cost method of depreciation

The geometric sequence

- use recursion to generate a geometric sequence
- display the terms of a geometric sequence in both tabular and graphical form and demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations
- use the rule for the n th term using $t_n = t_1 r^{(n-1)}$, where t_n represents the n th term of the sequence, t_1 = first term, n = term number and r = common ratio of a particular geometric sequence from the pattern of the terms in the sequence, and use this rule to make predictions
- use geometric sequences to model and analyse (numerically or graphically only) practical problems involving geometric growth and decay (logarithmic solutions not required), such as analysing a compound interest loan or investment, the growth of a bacterial population that doubles in size each hour or the decreasing height of the bounce of a ball at each bounce; or calculating the value of office furniture at the end of each year using the declining (reducing) balance method to depreciate



Prior learning

TERMINOLOGY

accelerated depreciation method
common ratio
depreciation
exponential growth
geometric progression, GP
linear growth
prime cost method
salvage value
straightline method
unit production cost method

arithmetic sequence,
arithmetic progression, AP
compound interest
double declining balance method
exponential relationship
geometric sequence
 n th term
recursive
scrap value
term

book value
common difference
declining balance method
exponential decay
general term
linear decay
prime cost
reducing balance method
sequence
unit cost method



Arithmetic
progressions

2.01 Arithmetic sequences

A **mathematical sequence** is a list of numbers in order. There is a first number, a second number, a third number, and so on. Some sequences have simple rules you can use to work out the list. You can probably work out the rule for the sequence 5, 6, 8, 11, 15, 20, 26, ...

Sequences

- A **sequence** is a list of numbers in a particular order.
- Each member of the sequence is called a **term**, usually written as $t_1, t_2, t_3, t_4, \dots$. The subscript tells you where it is in the sequence.
- Sometimes the **first term** is written as a , so $a = t_1$.
- The **n th term** is written as t_n .

INVESTIGATION

SEQUENCE RULES

Work out a rule for each of the sequences below. There could be more than one answer for some of them. Your teacher might want you to work in small groups.

5, 6, 8, 11, 15, 20, 26, ...

37, 72, 26, 61, 15, 50, 4, 39, 92, ...

5, 10, 20, 40, 80, ...

29, 27, 25, 23, 21, 19, ...

1, 3, 6, 10, 15, 21, 28, ...

8, 4, 2, $1, \frac{1}{2}, \frac{1}{4}, \dots$

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

7, 11, 15, 19, 23, 27, 31, ...

48, 85, 59, 96, 70, 8, 81, 19, 92, ...

1, -5, -11, -17, -23, -29, ...

Make up some sequences of your own for other students to try. Try theirs!

You get the terms of 7, 10, 13, 16, ... by adding 3 onto a term to get the next one. You can write this as $t_n = t_{n-1} + 3$, or 'the n th term is the $(n - 1)$ th term plus 3'. This is a **recursive** definition of the sequence.

Arithmetic sequences

- A **recursive** definition of a sequence shows how each term is calculated from the one before it. t_1 is usually given as well.
- The difference between successive terms of an **arithmetic sequence** (or **arithmetic progression, AP**), is a constant.
- The difference is called the **common difference**; it is given by

$$d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \dots \text{ or } d = t_{n+1} - t_n.$$
- The terms are given by the recursive rule $t_{n+1} = t_n + d$ or $t_n = t_{n-1} + d$.
- You can find any term from the first term and common difference.

EXAMPLE 1

Find the eighth term of each of the following sequences. State if they are arithmetic sequences and give the common difference if they are.

a 5, 9, 13, 17, 21, ...

b 5, 10, 20, 40, 80, ...

c 10, 8, 6, 4, 2, ...

d 1, 4, 9, 16, 25, ...

e 17, 11, 5, -1, -7, ...

Solution

a Look for a pattern.

Each term is 4 more than the previous term.

Write the rule in symbols.

$t_n = t_{n-1} + 4$, so there is a common difference.

Find the eighth term.

$$t_5 = 21$$

$$t_6 = 21 + 4 = 25$$

$$t_7 = 25 + 4 = 29$$

$$t_8 = 29 + 4 = 33$$

Write the answer.

The eighth term is 33. It is an arithmetic sequence with a common difference of 4.

b	Look for a pattern.	Each term is twice the previous term.
	Write the rule in symbols.	$t_n = 2 \times t_{n-1}$, so there isn't a common difference.
	Find the eighth term.	$t_5 = 80$ $t_6 = 2 \times 80 = 160$ $t_7 = 2 \times 160 = 320$ $t_8 = 2 \times 320 = 640$
	Write the answer.	The eighth term is 640. It is not an arithmetic sequence.
c	Look for a pattern.	Each term is 2 less than the previous term.
	Write the rule in symbols.	$t_n = t_{n-1} + (-2)$, so there is a common difference.
	Find the eighth term.	$t_5 = 2$ $t_6 = 2 + (-2) = 0$ $t_7 = 0 + (-2) = -2$ $t_8 = -2 + (-2) = -4$
	Write the answer.	The eighth term is -4 . It is an arithmetic sequence with a common difference of -2 .
d	Look for a pattern.	Each term is the square of the term number.
	Write the rule in symbols.	$t_n = n^2$, so there isn't a common difference.
	Find the eighth term.	$t_8 = 8^2 = 64$
	Write the answer.	The eighth term is 64. It is not an arithmetic sequence.
e	Look for a pattern.	Each term is 6 less than the previous term.
	Write the rule in symbols.	$t_n = t_{n-1} + (-6)$, so there is a common difference.
	Find the eighth term.	$t_5 = -7$ $t_6 = -7 + (-6) = -13$ $t_7 = -13 + (-6) = -19$ $t_8 = -19 + (-6) = -25$
	Write the answer.	The eighth term is -25 . It is an arithmetic sequence with a common difference of -6 .

You can show an arithmetic sequence in a table or as a graph.

EXAMPLE 2

An arithmetic sequence has a first term of 3 and a common difference of 6.
Show the first 8 terms:

- a in a table
- b as a graph.

Solution

- a Make a table and put in the term numbers and first term.

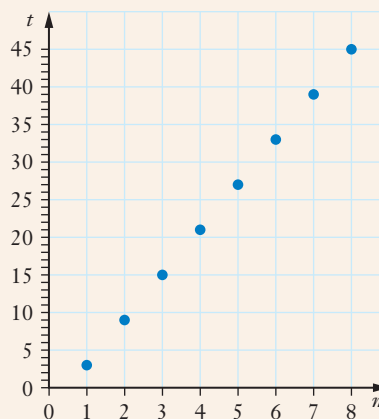
n	1	2	3	4	5	6	7	8
t_n	3							

Add 6 to each term to get the next one.

n	1	2	3	4	5	6	7	8
t_n	3	9	15	21	27	33	39	45

- b Plot the graph with n horizontal and t vertical.

Notice that the points of the sequence make a perfect straight line.



Grass trees are an iconic part of the Australian bush, but they grow very slowly. It is common for them to grow about 9 mm each year. A 2 m high grass tree is probably over 200 years old. Growth of this nature is an example of **linear growth**.



Shutterstock.com/Philip Schubert

Linear growth or decay

Growth or reduction by the same amount in each period is called **linear growth** or **linear decay**.

EXAMPLE 3

A grass tree on the east coast of Queensland grows by 9 mm each year. When first measured, it has a height of 0.6 m. How high will it be in another 5 years?

Solution

Write the change from 1 year to the next.

$$h_n - h_{n-1} = 9 \text{ mm}$$

What is the sequence of heights?

The heights have a common difference, so they form an arithmetic sequence.

Write the rule for the height.

$$h_n = h_{n-1} + 9 \text{ mm}$$

Find the 6th term (5 after the first).

$$t_1 = 0.6 \text{ m}$$

$$t_2 = 0.6 \text{ m} + 0.009 \text{ m} = 0.609 \text{ m}$$

$$t_3 = 0.609 \text{ m} + 0.009 \text{ m} = 0.618 \text{ m}$$

$$t_4 = 0.618 \text{ m} + 0.009 \text{ m} = 0.627 \text{ m}$$

$$t_5 = 0.627 \text{ m} + 0.009 \text{ m} = 0.636 \text{ m}$$

$$t_6 = 0.636 \text{ m} + 0.009 \text{ m} = 0.645 \text{ m}$$

Write the answer.

The grass tree will be 0.645 m high in another 5 years.

Exercise 2.01 Arithmetic sequences

Example
1

- 1** Find the eighth term of each sequence, state if it is an arithmetic sequence and give the common difference if it is.

a 1.2, 2.4, 3.6, 4.8, 6.0, ...

b 3, 6, 12, 24, 48, ...

c 20, 18, 16, 14, 12, ...

d 3, 4, 6, 9, 13, ...

e $5, 5\frac{1}{4}, 5\frac{1}{2}, 5\frac{3}{4}, 6, \dots$

f 9, 8.6, 8.2, 7.8, 7.4, ...

g 4, 6, 7, $7\frac{1}{2}$, $7\frac{3}{4}$, ...

h 16, 22, 28, 34, 40, ...

i 7, 3, -1, -5, -9, ...

j 40, 60, 90, 135, 202.5, ...

- 2** Find the eighth term of each sequence, state if it is an arithmetic sequence and give the common difference if it is.

a 2, 5, 4, 7, 6, ...

b 2, 2, 4, 6, 10, ...

c 11, 14, 17, 20, 23, ...

d 98, 95, 92, 89, 86, ...

e 9, 8.4, 7.8, 7.2, 6.6, ...

f 27, 9, 3, $1, \frac{1}{3}, \dots$

g $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, \dots$

h 0.1, 0.2, 0.3, 0.5, 0.8, ...

i 2, 2.2, 2.22, 2.222, 2.2222, ...

j $2\frac{5}{6}, 2\frac{3}{4}, 2\frac{2}{3}, 2\frac{7}{12}, 2\frac{1}{2}, \dots$

- 3** An arithmetic sequence has a first term of 35 and a common difference of -3. Show the first 8 terms:

a in a table

b as a graph.

- 4** An arithmetic sequence has a first term of 1.6 and a common difference of 0.2. Show the first 8 terms:

a in a table

b as a graph.

- 5** An arithmetic sequence has a first term of 2 and a common difference of -0.5. Show the first 8 terms:

a in a table

b as a graph.

- 6** An arithmetic sequence has a first term of 7 and a common difference of 1.2. Show the first 8 terms:

a in a table

b as a graph.

- 7** A granite boulder in Girraween National park has a diameter of 4.3 m. It loses 4 cm a year due to exfoliation ('onion-skin' weathering).

a Show in a table its diameter in metres for the first 8 years.

b How long would it take to reduce to a diameter of 3.9 m?

Example
2

Example
3

Problem solving

- 8** A dam west of Winton reduces in level by 1.6 cm a week due to evaporation. Use by cattle reduces it by 3.2 cm a week as well. The depth of the dam after a recent rain was 2.4 m. How long will it take to reduce to half this level if cattle continue to use it and there is no rain?
- 9** Samantha saves \$6 a week. She started her savings with \$20 from her Nan. How much will she have after 3 months? State any assumptions you make.
- 10** A piece of coral broken off a reef is 1 cm in diameter. The piece gets smaller as it is rolled around by tides and waves. Each year, the diameter reduces by about 0.7 mm. How long will it take to reduce to sand grains (diameter between 0.07 mm and 2 mm)?



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Arithmetic
sequences

2.02 General term of an arithmetic sequence

The sequence 7, 13, 19, 25, ... is an arithmetic sequence with a common difference of 6.

Notice that: $t_1 = 7 = 7 + 0 \times 6$

$$t_2 = 7 + 6 = 7 + 1 \times 6$$

$$t_3 = 7 + 6 + 6 = 7 + 2 \times 6$$

$$t_4 = 7 + 6 + 6 + 6 = 7 + 3 \times 6$$

and so on.

Each term is $7 + (\text{one less than the term number}) \text{ times } 6$.

This is a general rule for the n th term of an arithmetic sequence.

The **general term** (**n th term**) of an arithmetic sequence with first term t_1 and common difference d is given by

$$t_n = t_1 + (n - 1)d$$

EXAMPLE 4



Arithmetic
sequences

Find the indicated term of each AP.

a The sequence with first term 6 and common difference 4 (t_{15})

b 28, 23, 18, 13, ... (t_{12})

Solution

a Write the values of the variables.

$$t_1 = 6, d = 4, n = 15$$

Write the formula.

$$t_n = t_1 + (n - 1)d$$

Substitute values and calculate the answer.

$$\begin{aligned} t_{15} &= 6 + (15 - 1) \times 4 \\ &= 6 + 14 \times 4 \\ &= 6 + 56 \\ &= 62 \end{aligned}$$

Write the answer.

The 15th term is 62.

b Write the values of the variables.

$$t_1 = 28, d = -5, n = 12$$

Write the formula.

$$t_n = t_1 + (n - 1)d$$

Substitute values and find the result.

$$\begin{aligned} t_{12} &= 28 + 11 \times (-5) \\ &= 28 + (-55) \\ &= -27 \end{aligned}$$

Write the answer.

The 12th term is -27.

You can use the formula for the general term to find the number of terms needed to pass a particular number.

EXAMPLE 5

How many terms of the sequence 14, 17, 20, 23, ... does it take to exceed 300?

Solution

Write the known variables, assuming that a term actually equals 300.

$$t_1 = 14, d = 3, t_n = 300$$

Write the formula.

$$t_n = t_1 + (n - 1)d$$

Substitute values and solve the equation.

$$300 = 14 + (n - 1) \times 3$$

$$3(n - 1) = 300 - 14$$

$$3(n - 1) = 286$$

$$n - 1 = \frac{286}{3} = 95\frac{1}{3}$$

$$n = 96\frac{1}{3}$$

You need to choose whether the 96th or 97th term is correct.

$$t_{96} = 14 + 95 \times 3$$

$$= 299$$

$$t_{97} = 14 + 96 \times 3$$

$$= 302$$

Make sure you have the correct answer by checking either side of the value you get.

Write the answer.

The 97th term is the first over 300.



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You may need to solve *simple* simultaneous equations for some problems.

EXAMPLE 6

The third term of an AP is 8 and the thirteenth term is 5.

What are the first term and common difference?

Solution

Write the known variables.

$$t_3 = 8, t_{13} = 5$$

Write the formula.

$$t_n = t_1 + (n - 1)d$$

Substitute values.

$$8 = t_1 + 2d \quad [1]$$

$$5 = t_1 + 12d \quad [2]$$

Rearrange equation [1].

$$t_1 = 8 - 2d$$

Substitute in equation [2].

$$5 = (8 - 2d) + 12d$$

Simplify and solve.

$$8 + 10d = 5$$

$$10d = 5 - 8$$

$$10d = -3$$

$$d = -0.3$$

Substitute the value of d to find t_1 .

$$t_1 = 8 - 2d$$

$$= 8 - 2 \times (-0.3)$$

$$= 8.6$$

Write the answer.

The first term is 8.6 and the common difference is -0.3 .

TECHNOLOGY

Checking your answers

Use your scientific calculator to check your answers.

In Example 3, you should check that your answers give the right terms using the formula $t_n = t_1 + (n - 1)d$.

For t_3 , enter $8.6 + (3 - 1) \times (-0.3) \rightarrow$

For t_{13} , enter $8.6 + (13 - 1) \times (-0.3) \rightarrow$

Exercise 2.02 General term of an arithmetic sequence

Example

4

1 Find the indicated term of each arithmetic progression described.

- a First term 11 and common difference -4 (t_{10})
- b First term 9 and common difference 0.6 (t_{12})
- c First term -5 and common difference 1.6 (t_{28})
- d First term 7 and common difference -5 (t_{15})
- e First term 1.9 and common difference 0.06 (t_{40})

2 Find the indicated term of each AP.

- a $4, 10, 16, 22, \dots$ (t_{11})
- b $24, 22.6, 21.2, 19.8, \dots$ (t_{25})
- c $-5, -12, -19, -26, \dots$ (t_{30})
- d $3.4, 5.3, 7.2, 9.1, \dots$ (t_{16})
- e $\frac{2}{15}, \frac{1}{3}, \frac{8}{15}, \frac{11}{15}, \dots$ (t_{22})

Example

5

3 Which is the first term of the arithmetic sequence $3, 10, 17, 24, \dots$ that is more than 400?

4 Which is the first term of the arithmetic sequence $103, 99, 95, 91, \dots$ that is less than 30?

5 How many terms of the AP $4.5, 4.38, 4.26, 4.14, \dots$ are needed for it to be less than 2?

6 How many terms of the AP $-16, -15.2, -14.4, -13.6, \dots$ does it take to exceed 14?

Example

6

7 The ninth term of an AP is 12 and the thirty-fourth term is 57. What are the first term and common difference?

8 The twelfth term of an AP is 24 and the fourth term is 40. Find the first term and common difference.

9 Find the first term and common difference of an arithmetic sequence with tenth term 34.5 and seventeenth term 26.8.

Problem solving

10 The third term of an AP is 11 and the fifteenth term is 14.6. What is the 35th term?

11 The eleventh term of an AP is 22 and the nineteenth term is 19.6. What is the first term to be less than -3 ?

12 The 12th term of an arithmetic sequence is 11 and the 52nd term is 27. What is the sum of the 19th and 20th terms?

2.03 Modelling with arithmetic sequences

Common applications of arithmetic sequences involve the same increase or decrease at each step of a process. The usual formula for the n th term of an AP is $t_n = t_1 + (n - 1)d$.

In many applications it is better to use the **alternative formula**:

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\&= t_1 - d + nd \\&= t_0 + nd, \text{ where } t_0 = t_1 - d \text{ is the value at } n = 0.\end{aligned}$$

EXAMPLE 7

Emily has an investment account of \$500 that pays 4.5% p.a. flat-rate interest.

- a Find the interest for 1 year.
- b Find an expression for the value of the investment after n years.
- c How much will she have in the account after 10 years?

Solution

- a Calculate the interest.

$$\begin{aligned}\text{Interest for 1 year} &= 4.5\% \text{ of } \$500 \\&= 0.045 \times \$500 \\&= \$22.50\end{aligned}$$

Write the answer.

The interest for one year is \$22.50.

- b State the way the value changes.

Flat-rate interest is based on the original amount, so the increase is the same each year. The value is an AP.

Write the variables.

$$\begin{aligned}t_0 &= \$500 \\d &= \$22.50\end{aligned}$$

Write the alternative formula.

$$t_n = t_0 + nd$$

Substitute values and simplify.

$$\begin{aligned}\text{Value after } n \text{ years} &= \$500 + n \times \$22.50 \\&= \$(500 + 22.5n)\end{aligned}$$

- c Substitute $n = 10$ and find the value.

Write the answer.

$$\begin{aligned}\text{Value after 10 years} &= \$ (500 + 22.5 \times 10) \\ &= \$725\end{aligned}$$

She will have \$725 in the account after 10 years, assuming she does not deposit or withdraw anything, and there are no fees.

Depreciation

An owner-driver needs to replace his vehicle when it becomes too old and costly to maintain. The value of the vehicle **depreciates** over time.

A new prime mover could cost \$240 000. It probably has a useful life of 8 years or 1.2 million km before it has to be replaced. Its value when it is replaced might be \$60 000.

You can work out its loss of value each year or for each (say) 100 000 km.

$$\text{Loss of value over life} = \$240\,000 - \$60\,000 = \$180\,000$$

$$\text{Loss of value each year} = \frac{\$180\,000}{8} = \$22\,500/\text{year}$$

$$\text{Loss of value each km} = \frac{\$180\,000}{1\,200\,000} = \$0.15/\text{km}$$

$$\text{Loss of value each 100 000 km} = \$0.15 \times 100\,000 = \$15\,000/100\,000\text{ km}$$

Suppose the truck travelled 310 000 km in the first 2 years. Its value after 2 years would be:

$$\$240\,000 - \$22\,500 \times 2 = \$195\,000 \text{ or}$$

$$\$240\,000 - \$15\,000 \times 3.1 = \$193\,500, \text{ depending on the method used.}$$

Depreciation

- **Depreciation** is the loss of value of an asset (item) over its working life
- The cost of the new item is called its **prime cost**
- The **salvage value** (**scrap value**) is its value at the end of its working life.
- The **straight-line method** (**prime cost method**) uses the formula

$$\text{Depreciation} = \frac{\text{Prime cost} - \text{Salvage value}}{\text{Life in years}} / \text{year}$$

- The **unit cost method** (**unit production cost method**) uses the formula

$$\text{Depreciation} = \frac{\text{Prime cost} - \text{Salvage value}}{\text{Units of production}} / \text{unit}$$

- The **book value** of an item is its value after depreciation

EXAMPLE 8

An office printer costing \$3600 has a useful life of 4 years or 120 000 pages. It has no scrap value.

- a Find the annual depreciation.
- b Find the depreciation per 10 000 pages.
- c Write an expression for its book value after n years of use.
- d Write an expression for its value after printing n pages.

Solution

- a Write the formula.

Substitute values and calculate the answer.

Write the answer.

$$\begin{aligned}\text{Depreciation} &= \frac{\text{Prime cost} - \text{Salvage value}}{\text{Life in years}} \\ &= \frac{\$3600 - 0}{4} \\ &= \$900/\text{year}\end{aligned}$$

The annual depreciation is \$900.

- b Find the number of units of production.

Write the formula.

Substitute values and calculate the answer.

Write the answer.

$$\text{Units of production} = \frac{120\,000}{10\,000} = 12$$

$$\begin{aligned}\text{Depreciation} &= \frac{\text{Prime cost} - \text{Salvage value}}{\text{Units of production}} \\ &= \frac{\$3600 - 0}{12} \\ &= \$400/10\,000 \text{ units}\end{aligned}$$

The depreciation is \$400/10 000 pages.

- c State the method.

Write the alternative formula.

Substitute values and simplify.

Write the answer.

The value changes by the same amount each year, so it is an AP with $t_1 = \$3600$ and $d = -\$900$.

$$t_n = t_0 + nd$$

$$\begin{aligned}\text{Value after } n \text{ years} &= \$3600 + n \times (-\$900) \\ &= \$(3600 - 900n)\end{aligned}$$

The book value after n years is $\$(3600 - 900n)$.

d State the method.

The value changes by the same amount each unit, so it is an AP with $t_1 = \$3600$ and

$$\begin{aligned}d &= -\$400/\text{unit} \\&= -\frac{400}{10\,000} \\&= -\$0.04/\text{page}\end{aligned}$$

Write the formula.

$$t_n = t_0 + nd$$

Substitute values and simplify.

$$\begin{aligned}\text{Value after } n \text{ pages} &= \$3600 + n \times (-\$0.04) \\&= \$(3600 - 0.04n)\end{aligned}$$

Write the answer.

The value after n pages is $\$(3600 - 0.04n)$.

The book values of these methods of depreciation form arithmetic sequences with A_0 = prime cost and d = depreciation.

Exercise 2.03 Modelling with arithmetic sequences

Example
7

- 1** Joshua has \$2300 in a special savings account that pays 5.3% interest.
 - a** Find the interest for 1 year.
 - b** Find an expression for the value of the investment after n years.
 - c** How much will he have in the account after 7 years?
- 2** Alice is buying a small used car for \$8590. She gets finance from the dealer at 13.5% flat-rate interest over 3 years with 5% loan insurance. She pays equal monthly instalments.
 - a** How much interest does she pay?
 - b** How much is the loan insurance?
 - c** How much does she pay altogether?
 - d** What is her monthly instalment?
 - e** Write an expression for the amount she owes after n months.
 - f** How much does she still owe after 2 years?
- 3** The flagfall (starting fee) at night for a taxi is \$6.30 and the cost rate is \$2.17/km.
 - a** Write an expression for the fare for a trip of n km at night.
 - b** What is the cost of a 26 km trip starting at 1 a.m.?



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- 4** A collating and stapling photocopier cost \$6890 when new. It has a useful life of 6 years or 150 000 pages. It has a scrap value of \$800.

- a** Find the annual depreciation.
- b** Find the depreciation per 10 000 pages.
- c** Write an expression for its value after n years of use.
- d** Write an expression for its value after printing n pages.
- e** Find its value after 3 years using the annual depreciation.
- f** Find its value after printing 89 000 copies.



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Problem solving

- 5** Taj has been practising for the 100 m race at the athletics carnival that will be held in three weeks time. Three weeks ago he ran 24 seconds, a fortnight ago it was 20 seconds, last week it was 18 seconds and this week it was 17 seconds.
 - a** If he keeps improving at this rate, what time could he expect to run in the carnival?
 - b** Assuming nothing changes, what is the best time he could hope for in the future?
- 6** Mila saves \$20 a week. She currently has \$240 saved. How long will it take her to reach her target of \$800?
- 7** A water tank can hold 8000 litres, but after a long dry spell there are only 1300 litres in it. It started raining steadily at 10 a.m. and by 11:30 a.m. the tank was up to 2500 litres. At what time will the tank be full? State any assumptions you make.
- 8** Jack has put \$3200 into a savings account paying 4.8% p.a. flat-rate interest. How much will he have after n years? State any assumptions you make.
- 9** Chloe has borrowed \$6590 to buy a car. The flat-rate interest is 14.6% p.a. and loan insurance is 4.5% p.a. She has signed a contract to pay it off over 4 years with monthly instalments. Write an expression for the amount she owes after n months and find what she owes after 3 years.
- 10** Lachlan has bought a new van to do deliveries. The van cost \$42 800 and has to be replaced every 5 years under his delivery contract. Its salvage value is estimated to be \$18 000 after 5 years. Write an expression for its value after n years and find its value in 3 years.



2.04 Geometric sequences

Each term of the sequence 5, 15, 45, 135, ... is 3 times the previous term. You can write the recursive rule $t_n = 3t_{n-1}$ or $t_{n+1} = 3t_n$, where $t_1 = 5$.

The sequence -28, 14, -7, $3\frac{1}{2}$, ... has the same kind of rule: $t_n = -\frac{1}{2}t_{n-1}$, where $t_1 = -28$.

Geometric sequences

- A **geometric sequence** (or **geometric progression, GP**) is a sequence with terms given by the recursive rule $t_n = rt_{n-1}$ for a constant r
- r is called the **common ratio** and is given by $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \dots$ or $r = \frac{t_{n+1}}{t_n}$

You can check if a sequence is geometric by checking for a common ratio.

EXAMPLE 9

Determine whether or not each of the following is a GP and, if it is, state the common ratio.

- a 75, 15, 3, $\frac{3}{5}$, ...
- b 8, -12, 18, -27, ...
- c 4, -8, -16, 32, ...
- d 200, 210, 220.5, 231.525, ...

Solution

- a Check the ratios of successive terms.

$$\frac{t_2}{t_1} = \frac{15}{75} = \frac{1}{5}$$

$$\frac{t_3}{t_2} = \frac{3}{15} = \frac{1}{5} \checkmark$$

$$\frac{t_4}{t_3} = \frac{0.6}{3} = \frac{1}{5} \checkmark$$

State the conclusion.

It has a common ratio, $r = \frac{1}{5}$, so it is a GP.

- b** Check the ratios of successive terms.

$$\frac{t_2}{t_1} = \frac{-12}{8} = -1.5$$

$$\frac{t_3}{t_2} = \frac{18}{-12} = -1.5 \checkmark$$

$$\frac{t_4}{t_3} = \frac{-27}{18} = -1.5 \checkmark$$

State the conclusion.

It has a common ratio, $r = -1.5$, so it is a GP.

- c** Check the ratios of successive terms.

If you find a change, you don't need to keep checking.

$$\frac{t_2}{t_1} = \frac{-8}{4} = -2$$

$$\frac{t_3}{t_2} = \frac{-16}{-8} = 2 \times$$

State the conclusion.

There is no common ratio, so it is not a GP.

- d** Check the ratios of successive terms.

$$\frac{t_2}{t_1} = \frac{210}{200} = 1.05$$

$$\frac{t_3}{t_2} = \frac{220.5}{210} = 1.05 \checkmark$$

$$\frac{t_4}{t_3} = \frac{231.525}{220.5} = 1.05 \checkmark$$

State the conclusion.

It has a common ratio, $r = 1.05$, so it is a GP.

You can show the terms of a GP in a table or as a graph. You can use repetition on your calculator to find successive terms. When you press **EXE**, **enter** or **=** on your calculator, it repeats the last operation.

EXAMPLE 10

Show the first 6 terms of each geometric sequence described in a table and as a graph.

- a** First term 5, common ratio 0.2.
- b** First term 4, common ratio 3.

Solution

- a** Enter the first term on your calculator.

$$5 \text{ [enter]} \rightarrow 5$$

Now do the multiplication.

$$\times 0.2 \text{ [enter]} \rightarrow 1$$

Press **[EXE]**, **[enter]** or **[=]** again.

$$\text{[enter]} \rightarrow 0.2$$

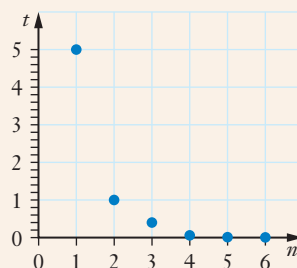
Keep going.

$$\text{[enter]} \rightarrow 0.04$$

Make a table from your values.

n	1	2	3	4	5	6
t_n	5	1	0.2	0.04	0.008	0.0016

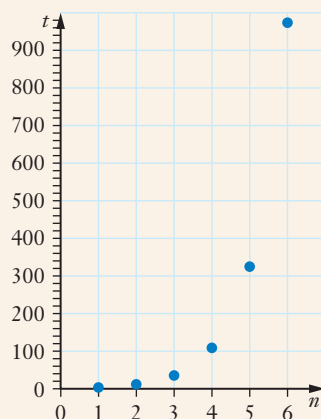
Plot the graph.



- b** Use your calculator to find the terms.

n	1	2	3	4	5	6
t_n	4	12	36	108	324	972

Plot the graph.



The graph of the terms of a GP with a positive common ratio shows points on a curved line.

You can use $t_n = rt_{n-1}$ or $t_{n+1} = rt_n$ to move forward in a GP. You can rearrange them to get $t_{n-1} = \frac{t_n}{r}$ or $t_n = \frac{t_{n+1}}{r}$. You can use these arrangements to go backwards.

EXAMPLE 11

Write down the first 6 terms of each geometric sequence.

- a** Sixth term 300, common ratio 5.
- b** Third term 4.5, common ratio 0.3.

Solution

- a** Use the constant divider on your calculator to go backwards.

$$300 \text{ [enter]} \rightarrow 300$$

$$\div 5 \text{ [enter]} \rightarrow 60$$

$$\text{[enter]} \rightarrow 12$$

etc...

Write the sequence. Remember you went backwards.

0.096, 0.48, 2.4, 12, 60, 300, ...

- b** Go backwards for the first 2 terms.

$$4.5 \text{ [enter]} \rightarrow 4.5$$

$$\div 0.3 \text{ [enter]} \rightarrow 15$$

$$\text{[enter]} \rightarrow 50$$

Now go forwards.

$$\times 0.3 \text{ [enter]} \rightarrow 15$$

$$\text{[enter]} \rightarrow 4.5$$

$$\text{[enter]} \rightarrow 1.35$$

$$\text{[enter]} \rightarrow 0.405$$

etc...

Write the sequence.

50, 15, 4.5, 1.35, 0.405, 0.1215, ...

Exercise 2.04 Geometric sequences

Example

9

1 Determine whether each sequence is a GP and, if it is, state the common ratio.

a 2, 4, 8, 16, ...

b 1, 4, 9, 25, ...

c 8, -4, 2, -1, ...

d $6, 2, \frac{1}{2}, \frac{1}{6}, \dots$

e $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{54}, \dots$

f $12, 8, 5\frac{1}{3}, 3\frac{5}{9}, \dots$

2 Determine whether each sequence is a GP and, if it is, state the common ratio.

a $2\frac{1}{2}, 4\frac{1}{4}, 8\frac{1}{8}, 16\frac{1}{16}, \dots$

b -5, 25, -125, 625, ...

c 80, 40, 20, -10, ...

d $6, -9, 13\frac{1}{2}, -20\frac{1}{4}, \dots$

e 5, 15, 45, 135, ...

f 32, 24, 18, 13.5, ...

Example

10

3 Show the first 6 terms of the geometric sequence with a first term of 7 and a common ratio of 2 in a table and as a graph.

4 Show the first 6 terms of the geometric sequence with a first term of 200 and a common ratio of 0.8 in a table and as a graph.

5 Show the first 6 terms of the geometric sequence with a first term of 6 and a common ratio of -2 in a table and as a graph.

6 Show the first 6 terms of the geometric sequence with a first term of 10 and a common ratio of $-\frac{1}{2}$ in a table and as a graph. What can you say about the graph of a GP with a negative common ratio?

Example

11

7 Write down the first 6 terms of the geometric sequence with a fifth term of 18 and a common ratio of $\frac{1}{3}$.

8 Write down the first 6 terms of the geometric sequence with a third term of 5 and a common ratio of 4.

9 Write down the first 6 terms of the geometric sequence with a sixth term of 12 and a common ratio of 0.2.

Problem solving

10 Show that the sequence $5a^4, 20a^3, 80a^2, 320a$ is geometric and find the common ratio.

11 \$5400 is invested at compound interest of 6.5% p.a. over 4 years. Show that the values of the investment at the end of each year form a GP.

12 The value of a painting appreciates (grows) by 6% each year. Show that the values of the painting at the end of each year form a GP.

2.05 General term of a geometric sequence

The sequence 4, 10, 25, 62.5, ... is a geometric sequence with a common ratio of 2.5.

Notice that: $t_1 = 4 = 4 \times 1 = 4 \times 2.5^0$ (Remember that $a^0 = 1$)

$$t_2 = 10 = 4 \times 2.5 = 4 \times 2.5^1$$

$$t_3 = 25 = 4 \times 2.5 \times 2.5 = 4 \times 2.5^2$$

$$t_4 = 62.5 = 4 \times 2.5 \times 2.5 \times 2.5 = 4 \times 2.5^3$$

and so on.

Each term is 4×2.5 to the power of one less than the term number.

This is a general rule for the n th term of a geometric progression.

The **general term** (**n th term**) of a geometric sequence with first term t_1 and common ratio r is given by $t_n = t_1 r^{(n-1)}$.

You can see that this is true for the geometric sequence: 4, 10, 25, 62.5, ...

$$t_1 = t_1 \times r^0 \quad (\text{Remember that } r^0 = 1)$$

$$t_2 = t_1 \times r = t_1 \times r^1$$

$$t_3 = t_1 \times r \times r = t_1 \times r^2$$

$$t_4 = t_1 \times r \times r \times r = t_1 \times r^3$$

and so on.

From the pattern, you can see that:

$$t_5 = t_1 \times r^4$$

$$t_6 = t_1 \times r^5$$

$$t_7 = t_1 \times r^6$$

...

$$\text{so } t_n = t_1 \times r^{(n-1)}$$



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Geometric
sequences



Geometric
progressions 1

**EXAMPLE 12**

Find the indicated term in each geometric sequence.

- a** $t_1 = 10, r = 1.2$, fifth term
b $t_1 = 36, r = \frac{1}{3}$, t_7
c $t_1 = 3.25$, third term = 0.52, sixth term
d $r = 5$, third term = 36, fifth term

Solution

- a** Write the formula.

$$t_n = t_1 r^{(n-1)}$$

Substitute values and use your calculator.

$$\begin{aligned} t_5 &= 10 \times 1.2^{(5-1)} \\ &= 10 \times 1.2^4 \\ &= 20.736 \end{aligned}$$

Write the answer.

The fifth term is 20.736.

- b** Write the first term and common ratio.

$$t_1 = 36, r = \frac{1}{3}$$

Write the formula.

$$t_n = t_1 r^{n-1}$$

Substitute values and use your calculator. You can use the decimal-fraction key to get a fractional answer.

$$\begin{aligned} t_7 &= 36 \times \left(\frac{1}{3}\right)^6 \\ &= \frac{36}{3^6} \\ &= 0.0493\dots \text{ or } \frac{4}{81} \end{aligned}$$

It could be shown as $f \blacktriangleleft d$, $a^{b/c}$
 or $S \div D$.

Write the answer.

The seventh term is 0.0493... or $\frac{4}{81}$.

It may be easier to write some answers as fractions instead of recurring decimals.

c Write the known variables.

$$t_1 = 3.25, t_3 = 0.52$$

Write the formula.

$$t_n = t_1 r^{n-1}$$

Substitute values and solve for r .

$$0.52 = 3.25 \times r^2$$

$$r^2 = \frac{0.52}{3.25} = 0.16$$

Remember that even powers of positive and negative numbers are both positive.

$$\begin{aligned} r &= \pm\sqrt{0.16} \\ &= 0.4 \text{ or } -0.4 \end{aligned}$$

Write the first term and common ratio.

$$t_1 = 3.25, r = 0.4 \text{ or } -0.4$$

Write the formula.

$$t_n = t_1 r^{n-1}$$

Substitute and find the possible values.

$$\begin{aligned} t_6 &= 3.25 \times 0.4^5 \text{ or } 3.25 \times (-0.4)^5 \\ &= 0.033\ 28 \text{ or } -0.033\ 28 \end{aligned}$$

Write the answer.

The sixth term is either 0.033 28 or -0.033 28

d Write the known variables.

$$r = 5, t_3 = 36$$

Write the formula.

$$t_n = t_1 r^{(n-1)}$$

Substitute values and solve for a .

$$36 = t_1 \times 5^2$$

$$25t_1 = 36$$

$$t_1 = \frac{36}{25} = 1.44$$

Write the first term and common ratio.

$$t_1 = 1.44, r = 5$$

Write the formula.

$$t_n = t_1 r^{(n-1)}$$

Substitute and find the value.

$$\begin{aligned} t_5 &= 1.44 \times 5^4 \\ &= 900 \end{aligned}$$

OR Calculate t_5 using t_3 and r .

$$\begin{aligned} t_5 &= t_3 \times r \times r \\ &= 36 \times 5 \times 5 \\ &= 900 \end{aligned}$$

Write the answer.

The fifth term is 900.

You can find all the terms of a geometric progression from *any* 2 terms.

EXAMPLE 13

The 2nd and 5th terms of a GP are 48 and $\frac{3}{4}$ respectively. Find the first term and common difference.

Solution

Write the formula.

$$t_n = t_1 r^{(n-1)}$$

Substitute values for the known terms.

$$48 = t_1 r^1 \quad [1]$$

$$\frac{3}{4} = t_1 r^4 \quad [2]$$

Solve to find a and r .

Method 1

Isolate t_1 from the lower term, [1].

$$t_1 = \frac{48}{r}$$

Substitute in the other term, [2].

$$\frac{3}{4} = \frac{48}{r} \times r^4$$

Simplify.

$$\frac{3}{4} = 48r^3$$

Divide by 48 and find the cube root.

$$r^3 = \frac{1}{64} \text{ so } r = \frac{1}{4}$$

Substitute r in the lower term [1] to find a .

$$48 = t_1 \times \left(\frac{1}{4}\right)^1 = \frac{1}{16}$$

$$t_1 = 48 \times 4 = 192$$

Check the values of a and r in the original terms.

$$t_2 = t_1 r^1 = 192 \times \left(\frac{1}{4}\right)^1 = \frac{192}{4} = 48 \quad \checkmark$$

$$t_5 = t_1 r^4 = 192 \times \left(\frac{1}{4}\right)^4 = \frac{192}{256} = \frac{3}{4} \quad \checkmark$$

Write the answer.

The first term is 192 and the common ratio is $\frac{1}{4}$.

Method 2

Divide the lower term, [1], into the higher, [2].

$$\frac{\frac{3}{4}}{48} = \frac{t_1 r^4}{t_1 r^1}$$

Simplify.

$$\frac{1}{64} = r^3$$

Find the cube root.

$$r = \frac{1}{4}$$

Then continue as in Method 1.

When a question has several steps, use your calculator to check your answer.

If you make an error in an exam:

- Check quickly (no more than a minute)
- If you can't see the error straight away, come back later
- You will often see it straight away when you come back.

EXAMPLE 14

Find the first term to exceed 800 in the geometric sequence with a first term of 500 and a common ratio of 1.053.

Solution

Write the first term and common ratio.

$$t_1 = 500, r = 1.053$$

Write the formula.

$$t_n = t_1 r^{(n-1)}$$

Substitute values.

$$t_n = 500 \times 1.053^{n-1}$$

Use your calculator to try some values of n .

$$t_5 = 500 \times 1.053^4 = 614.728\dots$$

$$t_{10} = 500 \times 1.053^9 = 795.839\dots$$

$$t_{11} = 500 \times 1.053^{10} = 838.018\dots$$

Write the answer.

The first term of the sequence to exceed 800 is $t_{11} \approx 838.018\dots$



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TECHNOLOGY

Geometric sequence

A spreadsheet is very useful for sequences. The calculations for questions like Example 14 are particularly tedious.

Open a spreadsheet and complete the following.

Type 'First term =' into cell A1.

Type 'Common ratio =' into cell A2.

Type 500 into cell B1.

Type 1.053 into cell B2.

Widen column A so both '=' signs are visible.

Type 'Term number' into cell A3.

Type 'Term value' into cell B3.

Type 1 into cell A4.

Type '= B1' into cell B4.

Type '= A4 + 1' into cell A5.

Type '= B4*\$B\$2' into cell B5. The dollar signs make the cell reference absolute.

It won't change when you copy and paste it.

Now highlight cells A5 and B5 and copy them.

Highlight cells A6 down to, say, A20 and paste.

Save your spreadsheet as, say, GP Terms to use in the future.

Try changing the values in cells B1 and B2.

	A	B
1	First term =	500
2	Common ratio =	1.053
3	Term number	Term value
4	1	500
5	2	526.5
6	3	554.4045
7	4	583.787939
8	5	614.728699
9	6	647.30932
10	7	681.616714
11	8	717.7424
12	9	755.782747
13	10	795.839233
14	11	838.018712
15	12	882.433704
16	13	929.20269

Exercise 2.05 General term of a geometric sequence

Example
12

1 Find the indicated term in each geometric sequence.

a $t_1 = 6, r = 3$, sixth term

b $t_1 = 5, r = -2$, eighth term

c $t_1 = 1.4, r = 1.5$, fifth term

d $t_1 = -3, r = -4$, seventh term

e $t_1 = -20, r = 0.3$, sixth term

f $t_1 = 10, r = 1.2$, fifth term

2 Find the indicated term in each geometric sequence.

a $t_1 = 28, r = -5.2, t_4$

b $t_1 = 0.6, r = 7, t_7$

c $t_1 = -6, r = \frac{1}{3}, t_5$

d $t_1 = 52, r = -\frac{1}{4}, t_6$

e $t_1 = -7, r = \frac{1}{2}, t_{10}$

f $t_1 = 500, r = 1.05, t_6$

3 Find the indicated term in each geometric sequence.

a $t_1 = 12$, fifth term = 192, tenth term

b $t_1 = 1.08$, fourth term = -0.04 , sixth term

c $t_1 = -36$, sixth term = 3515.625, fourth term

d $t_1 = 162, t_3 = 8$, fifth term

e $t_1 = 80, t_4 = -1\frac{1}{4}$, sixth term

f $t_1 = \frac{8}{625}, t_6 = 1\frac{1}{4}, t_4$

4 Find the indicated term in each geometric sequence.

a $r = 2, t_3 = 48, t_7$

b $r = 1.5$, fourth term = 81, second term

c $r = -\frac{1}{4}$, third term = $1\frac{1}{2}$, fifth term

d $r = 5$, fourth term = 10, t_7

e $r = \frac{1}{3}, t_5 = \frac{4}{9}$, second term

f $r = 1.2$, third term = 540, sixth term

5 The second and 5th terms of a GP are -80 and 0.64 respectively. Find the first term and common difference.

Example
13

6 A GP has $t_3 = 20$ and $t_6 = 160$. Find the first term and common difference.

7 The 3rd and 5th terms of a GP are $2\frac{1}{4}$ and $\frac{9}{16}$ respectively. Find the first term and common difference.

8 The 2nd and 7th terms of a GP are 0.4 and 1250 respectively. Find the first term and common difference.

9 Find the first term to exceed 300 in the geometric sequence with a first term of 200 and a common ratio of 1.04.

Example
14

10 Find the first term to exceed 1200 in the geometric sequence with a first term of 6 and a common ratio of 3.

- 11** Find the first term to be below 1 in the geometric sequence with a first term of 80 and a common ratio of $\frac{1}{2}$.
- 12** Find the first term to be below -100 in the geometric sequence with a first term of 28 and a common ratio of $-\frac{5}{4}$.

Problem solving

- 13** A geometric sequence has a first term of 8 and a fourth term of 27. What is the first term to exceed 200?
- 14** Moss on some wet paving increases in area by 20% each week during winter. At the beginning of June, it covers an area of 15 cm^2 . How long does it take to reach a size of 35 cm^2 ?
- 15** The population of mice in a wheat field increases geometrically as the crop ripens. When first checked, there are estimated to be 30 mice per hectare, but 2 weeks later there are thought to be 39 mice/ha. A significant drop in yield occurs if the population exceeds 100 mice/ha. How much longer will it take for this to happen?



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Modelling
with
sequences

2.06 Modelling with geometric sequences

The general term of a geometric sequence is $t_n = t_1 r^{(n-1)}$. The sequence begins at t_1 .

For some models, it is better to regard the sequence beginning at t_0 when $n = 0$.

In this case, the sequence is $t_n = t_0 r^n$, where t_0 is the value at $n = 0$.



Modelling
arithmetic and
geometric
sequences

Exponential growth and decay

- $A_n = A_0 r^n$, where $r > 0$, is called an **exponential relationship**
- For $r < 1$, it is called **exponential decay**
- For $r > 1$, it is called **exponential growth**
- A_0 is the value at $n = 0$

The relationship is called **exponential** because the variable n is the exponent (index).

INVESTIGATION

EXPONENTIAL GROWTH OF BACTERIA

A bacterial colony on a round agar plate has an initial area of 0.5 mm^2 . It doubles in size each hour.

- Make a table for the area each hour for up to 10 hours.
- Draw a graph of the area each hour.
- Describe the shape of the graph for the exponential growth of the colony.
- How long will it take for the colony to cover the plate, which has a diameter of 10 cm?
- What will happen to the colony when it exhausts the nutrients on the plate?
- What can you conclude about the exponential growth of natural systems?

A compound interest loan or investment is calculated on the *current balance*. This means that an investment of \$200 at 7% annual interest will get \$14 interest at the end of the first year. In the second year, the interest will be calculated on the total of \$214, so the interest will be $0.07 \times \$214 = \$14.98$, making a total of \$228.98. Each year, the amount will be 107% of the previous year, so you can calculate it as $1.07 \times \$200 = \$214$ and $1.07 \times \$214 = \$228.98$.

The amounts form a geometric sequence with a common ratio of 1.07.

Notice that $1.07 = 1 + \text{interest rate as a decimal}$.

Compound interest

- The amounts produced by **compound interest** at the rate i form a geometric sequence with common ratio $(1 + i)$
- For a rate at $r\%$, $i = \frac{r}{100}$
- It is useful to write the sequence from term 0, with A_0 = initial amount
- The amount after n periods (years), $A_n = A_0(1 + i)^n$
- Amounts invested at compound interest grow exponentially
- Interest compounds each year unless otherwise stated

EXAMPLE 15

\$2500 is invested at an interest rate of 6.3% p.a.

- a** What is it worth after 7 years?
- b** What is it worth after 10 years?
- c** What is it worth after 15 years?
- d** How long does it take to double in value?

Solution

- a** Write the first term and common ratio. $A_0 = \$2500, r = 1.063$

Write the general term.
$$A_n = A_0 r^n$$
$$= 2500 \times 1.063^n$$

Find the value after 7 years.
$$A_7 = 2500 \times 1.063^7$$
$$\approx \$3834.18$$

Write the answer. After 7 years, the investment is worth \$3834.18.

- b** Find the value after 10 years.
$$A_{10} = 2500 \times 1.063^{10}$$
$$\approx \$4605.46$$

Write the answer. After 10 years, the investment is worth \$4605.46.

- c** Find the value after 15 years.
$$A_{15} = 2500 \times 1.063^{15}$$
$$\approx \$6250.85$$

Write the answer. After 15 years, the investment is worth \$6250.85.

- d** It must be between 10 and 15 years. Try different values of n to get \$5000.
$$A_{13} = 2500 \times 1.063^{13} \approx \$5531.88$$
$$A_{12} = 2500 \times 1.063^{12} \approx \$5204.02$$
$$A_{11} = 2500 \times 1.063^{11} \approx \$4895.60$$

Write the answer. It takes 12 years for the investment to reach at least \$5000, double the original amount.

INVESTIGATION

DOUBLING OF AN INVESTMENT

Is there a rule to find the time it takes for an investment to double in size at compound interest?

- Find the number of years it takes for an investment of \$1000 to reach \$2000 at interest rates of 2%, 5%, 7%, 10%, 15%, 20% (you could use a spreadsheet).
- Make a table of your results.
- What can you conclude about the time it takes for an investment to double?

Any quantity that goes up or down by the same percentage for each term grows or decays exponentially. The terms form a geometric sequence.

EXAMPLE 16

A tennis ball dropped onto concrete rebounds to 60% of its previous height. It starts from a height of 2 m.

- a What height does it reach after 3 bounces?
- b What height does it reach after 10 bounces?
- c How many bounces will it take to reduce to a height of 20 cm?

Solution

- a State the nature of the heights.

The rebound heights are each 0.6 times the previous heights, so they form a GP.

Write the first term and common ratio.

$$t_1 = 200, r = 0.6$$

Find the rebound height after 3 bounces.

$$\begin{aligned}t_n &= t_1 r^n \\t_3 &= 200 \times 0.6^3 \\&= 43.2\end{aligned}$$

Write the answer.

Its rebound height after 3 bounces is 43.2 cm.

- b** Find the height after 10 bounces.

$$t_{10} = 200 \times 0.6^{10} \\ \approx 1.21$$

Write the answer.

Its rebound height after 10 bounces is 1.2 cm.

- c** It must be between 3 and 10. Try some values.

$$t_4 = 200 \times 0.6^4 = 25.92 \\ t_5 = 200 \times 0.6^5 = 15.552$$

Write the answer.

It takes 5 bounces before the rebound height reduces to (less than) 20 cm.

You've already seen the straight-line method of depreciation. In that method, the book value decreases by the same *amount* each year. This gives an arithmetic sequence of values.

What happens if you decrease the book value by the same *percentage* each year?

Consider a \$4800 set of reception chairs for a waiting room losing 12% in value each year.

After 1 year they will be worth 88% of \$4800 = $4800 \times 0.88 = \$4224$

After 2 years they have a book value of 88% of \$4224 = $4224 \times 0.88 = \$3717.12$

The value at the start of a year is multiplied by 0.88 to get the value at the end of the year.

This makes a geometric sequence with a common ratio of 0.88.

Depreciation using the declining balance method

- The **declining (reducing, diminishing) balance method** of depreciation reduces the book value by the same percentage each year
- The book values are a geometric sequence with A_0 = initial value and

$$r = 1 - \frac{\text{depreciation}\%}{100}$$

The depreciation percentage for the reducing balance method is often set as *double* the straight-line percentage depreciation. This is sometimes called the **double declining balance method** or **accelerated depreciation method**.

EXAMPLE 17

The office furniture cost \$12 000 and has an effective life of 8 years. There is little or no scrap value. Use the double declining balance method to find its depreciated value after

- a** 2 years **b** 5 years **c** 8 years

Solution

- a** Find the straight-line depreciation.

$$\begin{aligned}\text{Straight-line depreciation} &= \frac{12\,000 - 0}{8} \\ &= \$1500\end{aligned}$$

Find the percentage.

$$\begin{aligned}\text{Percentage} &= \frac{1500}{12\,000} \times 100\% \\ &= 12.5\%\end{aligned}$$

Find the double declining balance percentage.

$$\begin{aligned}\text{Double declining balance} \\ \text{percentage} &= 2 \times 12.5\% \\ &= 25\%\end{aligned}$$

Find the common ratio.

$$r = 1 - 0.25 = 0.75$$

Find the value after 2 years.

$$\begin{aligned}A_2 &= A_0 r^2 \\ &= \$12\,000 \times 0.75^2 \\ &= \$6750\end{aligned}$$

Write the answer.

The depreciated value after 2 years is \$6750.

- b** Find the value after 5 years.

$$\begin{aligned}A_5 &= \$12\,000 \times 0.75^5 \\ &= \$2847.656\dots\end{aligned}$$

Write the answer.

The depreciated value after 5 years is about \$2848.

- c** Find the value after 8 years.

$$\begin{aligned}A_8 &= \$12\,000 \times 0.75^8 \\ &= \$1201.354\dots\end{aligned}$$

Write the answer.

The depreciated value after 8 years is about \$1201.

The Australian Taxation Office allows businesses to choose either the straight-line method or the double declining balance method of depreciation. They allow some assets to be written off immediately.

Exercise 2.06 Modelling with geometric sequences

Example

15

- 1 Anis invested \$5000 at 8.2% p.a. compound interest. What is the investment worth after:
a 3 years? **b** 5 years? **c** 12 years?
- 2 Coralie invested \$8200 at 3.6% p.a. compound interest. What is the investment worth after:
a 2 years? **b** 6 years? **c** 20 years?
- 3 Penny lent Sam \$4000 at 7.2% p.a. compound interest. Sam did not make any payments for 4 years. How much does he owe Penny by this time?
- 4 Tino borrowed \$7500 at 6.8% p.a. compound interest, but did not pay anything back for 6 years. How much does he owe then?

Example

16

- 5 A 'super' ball dropped from a height of 3 m rebounds to 2.5 m. Assuming that it rebounds the same percentage each time, find the height it rebounds to after:
a 3 bounces **b** 5 bounces **c** 10 bounces
- 6 A microphone is placed too close to the speaker, so it begins to emit a squeal. The feedback loop increases the volume by 3% each millisecond. Find the percentage increase in volume after
a 4 milliseconds **b** 8 milliseconds **c** 20 milliseconds

Example

17

- 7 Carmichael bought a new truck for \$124 000 and used an annual depreciation of 12% to estimate its value for tax purposes. What is its book value after:
a 3 years? **b** 5 years? **c** 10 years?
- 8 Silvia paid \$8500 for an office computer system and used depreciation of 30% each year to calculate its book value. What is the book value after:
a 2 years? **b** 3 years? **c** 5 years?

Problem solving

- 9 How many years would it take for an investment of \$7000 to triple in value at 8% p.a. compound interest?
- 10 Sarah put a weight on the end of a piece of elastic. When she hung it from a door frame, the weight was at a height of 0.85 m. She pulled it down to a height of 0.5 m. She let it go and it went up and then fell down to a height of 0.65 m. How many cycles would it take to settle to within 1 cm of the resting position? State any assumptions you make.
- 11 How long would it take a milling machine costing \$28 590 to depreciate to half its initial value if its scrap value is about \$5000 after 10 years and the double declining balance method of depreciation is used?
- 12 A tennis ball bounces to 40% of its previous height at each bounce. It starts from a height of 1.8 m. How many bounces does it take to stop? State any assumptions you make.

2. CHAPTER SUMMARY

Arithmetic and geometric sequences

- A **sequence** is a list of numbers in a particular order.
- Each member of the sequence is called a **term**, usually written as $t_1, t_2, t_3, t_4, \dots$
- Sometimes the **first term** is written as a , so $a = t_1$.
- The **n th term** is written as t_n .
- A **recursive** definition of a sequence shows how each term is calculated from the one before it. The difference between successive terms of an **arithmetic sequence** (or **arithmetic progression, AP**), is a constant.
- The difference is called the **common difference** and is given by
$$d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \dots \text{ or } d = t_{n+1} - t_n \text{ so } t_{n+1} = t_n + d \text{ or } t_n = t_{n-1} + d.$$
- Growth or reduction by the same amount in each period is called **linear growth** or **linear decay**.
- The **general term (n th term)** of an arithmetic sequence with first term t_1 and common difference d is given by $t_n = t_1 + (n - 1)d$
- The **alternative formula** is $t_n = t_0 + nd$, where t_0 is the value at $n = 0$
- The graph of an AP is a straight line
- **Depreciation** is the loss of value of an asset (item) over its working life
- The cost of a new item is called its **prime cost**
- The **salvage value (scrap value)** is its value at the end of its working life.
- The **straight-line method (prime cost method)** uses the formula

$$\text{Depreciation} = \frac{\text{Prime cost} - \text{Salvage value}}{\text{Life in years}} / \text{year}$$

- The **unit cost method (unit production cost method)** uses the formula

$$\text{Depreciation} = \frac{\text{Prime cost} - \text{Salvage value}}{\text{Units of production}} / \text{unit}$$

- The **book value** of an item is its value after depreciation
- A **geometric sequence** (or **geometric progression, GP**) is a sequence with terms given by $t_n = rt_{n-1}$ for a constant r
- r is called the **common ratio** and is given by $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \dots$ or $r = \frac{t_{n+1}}{t_n}$
- The **general term (n th term)** of a geometric sequence with first term t_1 and common ratio r is given by $t_n = t_1 r^{(n-1)}$

- The graph of a GP with a positive common ratio is a curved line
- $A_n = A_0 r^n$, where $r > 0$ is called an **exponential relationship**
 - For $r < 1$, it is called **exponential decay**
 - For $r > 1$, it is called **exponential growth**
- The amounts produced by **compound interest** at the rate i form a geometric sequence with common ratio $(1 + i)$. For a rate at $r\%$, $i = \frac{r}{100}$
- The amount after n periods (years) is given by $A_n = A_0(1 + i)^n$, where A_0 = initial amount
- Amounts invested at compound interest grow exponentially
- Any quantity that goes up or down by the same percentage for each term grows or decays exponentially. The terms form a geometric sequence
- The **declining balance (reducing) balance method** of depreciation reduces the book value by the same percentage each year
- The book values are a geometric sequence with A_0 = initial value and

$$r = 1 - \frac{\text{depreciation}\%}{100}$$
- The depreciation percentage for the reducing balance method is often set as *double* the straight-line percentage depreciation. This is sometimes called the **double declining balance method** or **accelerated depreciation method**

2. CHAPTER REVIEW

Arithmetic and geometric sequences

- 1** Find the eighth term of each sequence, state if it is an arithmetic sequence and give the common difference if it is.
- a** 2, 5, 9, 15, ...
 - b** 1, -7, -15, -23, ...
 - c** 0.2, 0.6, 1, 1.4, ...
- 2** What is the next term of the sequence 5, 11, 18, 26, 35?
- A** 41 **B** 43 **C** 45 **D** 47 **E** 49
- 3** An arithmetic sequence has first term 5 and common difference 3. Show the first 8 terms:
- a** in a table
 - b** as a graph.
- 4** A radiata pine grows an average of about 90 cm each year. When first measured, a radiata pine had a height of 5 m.
- a** How old was it when it was first measured?
 - b** How tall is it likely to be in another 8 years?
 - c** How long will it take until it is harvested at a height of about 35 m?
- 5** The 7th term of the arithmetic sequence with a first term of 5 and a common difference of 4 is
- A** 9 **B** 27 **C** 34 **D** 29 **E** 33
- 6** Find the 15th term of an arithmetic sequence with a first term of 1.7 and a common difference of 0.6.
- 7** Which is the first term of the sequence 21, 23.4, 25.8, 28.2, ... to exceed 400?
- 8** The 5th term of an AP is 11 and the 9th term is 17. What are the first term and common difference?
- 9** Kylie invested \$600 at 5.8% p.a. flat-rate interest.
- a** Find the interest for one year.
 - b** Find an expression for the value of the investment after n years.
 - c** How long will it take for the investment to reach \$900?

Example
1

Example
1

Example
2

Example
3

Example
4

Example
4

Example
5

Example
6

Example
7

Example
8

10 Andrew, a business representative, bought a new car for \$23 999. He plans to sell the car in 5 years, when he estimates it will be worth about \$7000. Use the straight-line method of depreciation to find its value after:

- a** 2 years **b** 3 years **c** 4 years.

Example
8

11 Ziad is a leather worker with an industrial sewing machine he bought for \$7300. He thinks the scrap value after 2000 hours work will be \$2100. Use the unit cost method to find its value after 1300 hours of work.

Example
9

12 The common ratio of the geometric sequence 5, -15, 45, -135, ... is

- A** -3 **B** 3 **C** $\frac{1}{3}$ **D** $-\frac{1}{3}$ **E** None of these

Example
9

13 Find whether or not each of the following is a geometric sequence. State the common ratio for those that are GPs.

- a** 3.35, 6.7, 13.4, 26.8, ...
b 1, 2, 6, 24, ...
c 5, 10, 15, 20, ...

Example
10

14 Show the first 6 terms of a GP with first term 4 and common ratio 1.6 as

- a** a table **b** a graph.

Example
11

15 A geometric progression has a 5th term of 81.92 and a 7th term of 52.4288. Find the first 6 terms.

Example
12

16 Find the indicated term in each geometric sequence.

- a** $t_1 = 12, r = 0.6$, 5th term
b $r = 3, t_4 = 16.2$, t_7

Example
13

17 The 3rd and 5th terms of a GP are 2.4 and 9.6. Find the:

- a** common ratio
b first term
c 8th term.

Example
14

18 The first term of a geometric sequence is 9 and the common ratio is 1.2. What is the first term to exceed 100?

Example
15

19 Krystina invested \$800 at 7.2% compound interest. What will her investment be worth in

- a** 3 years? **b** 5 years? **c** 10 years?

- 20** A bridge bounces by 1 cm as some soldiers begin to march across it. The bounce increases by 20% each minute as the soldiers continue to march across. After 8 minutes, the soldiers are given the order to break step.
- a** How much will the bridge be bouncing by then?
 - b** What would happen if the order was not given for 12 minutes?

- 21** A new bain-marie costing \$4900 is installed in a takeaway shop. It depreciates at 12% each year. What is its book value after:
- a** 3 years?
 - b** 5 years?
 - c** 8 years?

Problem solving

- 22** A stone on a riverbed is about 20 cm in diameter. As it rolls along the bed, it reduces in size by about 0.3 cm each year. How long will it take to reduce to the size of a pebble (20 mm or less)? State any assumptions you make.
- 23** The 16th term of an AP is 8 and the 25th term is 11.6. What is the sum of the 40th and 41st terms?
- 24** Sandy is a florist whose delivery van cost \$42 000 when new, 3 years ago. Its current book value is about \$28 500, using the straight-line method of depreciation. She intends to replace the van when it is 8 years old. What is its scrap value?
- 25** The value of an antique dresser appreciates by 6% each year. It is currently worth \$1400. How long will it take to appreciate to \$2000?
- 26** An algal bloom on a lake covered 50 m^2 when it was first noticed. 3 weeks later, it covered an area of 400 m^2 . What area would it cover in another 6 weeks? State any assumptions you make.
- 27** How long will an investment of \$6000 at 8% p.a. compound interest take to reach a value of \$15 000?
- 28** The double declining value method is used to depreciate the value of a combine harvester bought for \$170 000. After 2 years, its book value is \$108 800. Its useful life is about 10 years. What is its scrap value?

3.

TIME SERIES

The number of people on a beach changes with the time of year, the weather and so on. An ironman event or surfing contest will bring many people. A graph of the number of people at a beach will have highs and lows. Some of these will be similar from year to year. Other peaks will occur during school holidays. These are examples of trends. This chapter is about time series and trends. These are important in making predictions.

- 3.01 Time series plots
- 3.02 Time series patterns
- 3.03 Outliers
- 3.04 Smoothing time series data
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Describing and interpreting patterns in time series data

- construct time series plots
- describe time series plots by identifying features such as trend (long-term direction), seasonality (systematic, calendar-related movements) and irregular fluctuations (unsystematic, short-term fluctuations), and recognise when there are outliers, e.g. one-off unanticipated events

Analysing time series data

- smooth time series data by using a simple moving average, including the use of spreadsheets to implement this process



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

centred moving average
outlier
time series

irregular time series
seasonal time series
time series plot

moving average
smoothing
trend

3.01 Time series plots

Information that is recorded every day, every month or at any regular period is called a **time series**. Weather information is a time series because it is recorded regularly. Most of the information collected by the Australian Bureau of Statistics (ABS) is time series information. This is also true of the Bureau of Meteorology (BOM) and many other government departments. Businesses record daily, weekly and monthly sales and expenses so they can plan for the future.

INVESTIGATION

TIME SERIES AT SCHOOL

Your teacher may be able to get a daily total of school attendance and absence.

- What other time series are there at your school?
- Are there any trends in absenteeism over the week?
- Are there more absences on particular days?
- Are there more absences at particular times of the year?
- Can you think of any reasons for patterns of absences?

Time series

- A **time series** is a set of data collected at regular periods of time
- A **time series plot** is a graph of a time series with time on the horizontal axis. You normally join the points to make a line graph

You can use a graph of time series to look for a pattern.

EXAMPLE 1

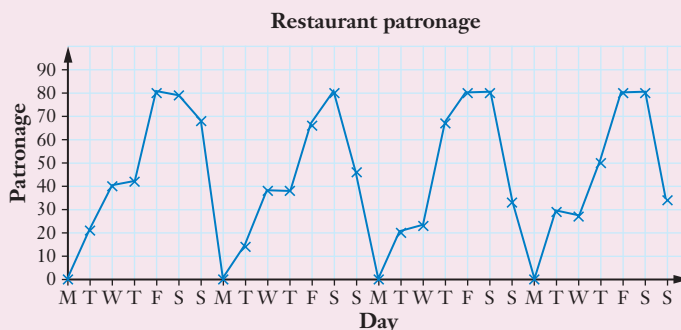
This table shows the number of customers at a restaurant every day over four weeks.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Week 1	0	21	40	42	80	79	68
Week 2	0	14	38	38	66	80	46
Week 3	0	20	23	67	80	80	33
Week 4	0	29	27	50	80	80	34

- Plot the data.
- Can you make any conclusions?

Solution

- Plot with time on the horizontal axis.



- Write any obvious conclusions.

The restaurant never has more than 80 patrons and never has any patrons on Monday. It could have a maximum capacity of 80 and be shut on Mondays. It does best on Fridays and Saturdays.

When drawing a time series plot, make the time axis appropriate for the data.

EXAMPLE 2

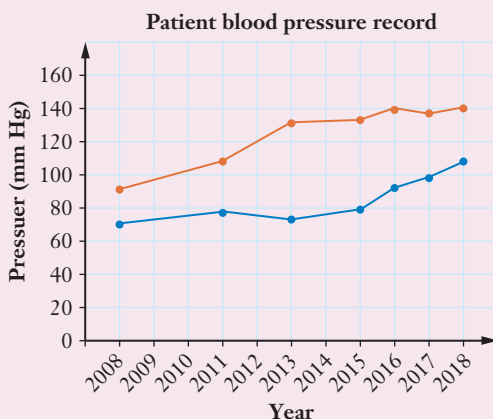
The blood pressure of a middle-aged man over a number of years is shown below. The first number is the systolic pressure and the second is the diastolic pressure. Systolic pressure is the maximum pressure generated by the heart and diastolic pressure indicates the pressure in the arteries when the heart rests between beats.

Year	2008	2011	2013	2015	2016	2017	2018
BP	91/70	108/77	131/73	133/79	139/92	137/98	140/108

- Plot the data.
- What conclusions can you make?

Solution

- a** Plot the pressures against time on the horizontal axis.



- b** State any conclusions.

There has been a gradual increase in blood pressure.

The normal range of blood pressure is from about 90/60 to 120/80.

Examples
1, 2

Exercise 3.01 Time series plots

- 1** The average maximum and minimum temperatures in °C for each month in Mackay are shown.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Max	30.4	30.2	29.3	27.7	25.3	23.2	22.7	23.8	26.0	28.3	29.5	30.8
Min	23.1	23.2	22.0	19.4	15.7	12.8	11.3	12.0	14.7	18.2	20.5	22.2

Source: Bureau of Meteorology, <http://www.bom.gov.au/climate/data/index.shtml>.

- a** Plot the data. **b** Can you make any conclusions?
- 2** The annual number of births in Queensland over the period 1993–2016 is shown.

Year	1993	1994	1995	1996	1997	1998	1999	2000
Births	46 778	46 578	46 484	47 769	46 965	47 046	46 503	47 278
Year	2001	2002	2003	2004	2005	2006	2007	2008
Births	47 678	47 771	48 342	49 940	51 707	52 695	61 306	63 168
Year	2009	2010	2011	2012	2013	2014	2015	2016
Births	66 149	64 523	63 253	63 837	63 354	63 066	61 745	61 841

Source: Australian Bureau of Statistics

- a** Plot the data.
b Can you make any conclusions?

- 3** The average incomes of wage and salary earners of two places in Queensland over 6 years is shown.

Year	2005	2006	2007	2008	2009	2010
Longreach	33 559	35 191	37 373	40 255	40 831	43 245
Mackay	43 659	46 522	49 005	53 109	55 320	60 163

- a** Plot the data.
b Can you make any conclusions?



Alamy Stock Photo/JELLE vanderwolf

- 4** Delia checked her pulse every minute during and after exercising for 8 minutes. The results are shown below in beats per minute.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Pulse	60	70	78	90	110	120	134	135	136	118	80	70	65	63

- a** Plot the data.
b Can you make any conclusions?

- 5** A courier driver has kept records of weekly income over 6 months (26 weeks).

Week	1	2	3	4	5	6	7	8	9
Income \$	690	644	552	529	782	1012	1104	989	736
Week	10	11	12	13	14	15	16	17	18
Income \$	667	46	345	460	782	805	805	782	644
Week	19	20	21	22	23	24	25	26	
Income \$	621	874	920	966	828	713	621	690	

- a** Plot the data.
b Can you make any conclusions?



Time series
plots

3.02 Time series patterns

Time series often have a pattern. There are 3 basic types of time series patterns.

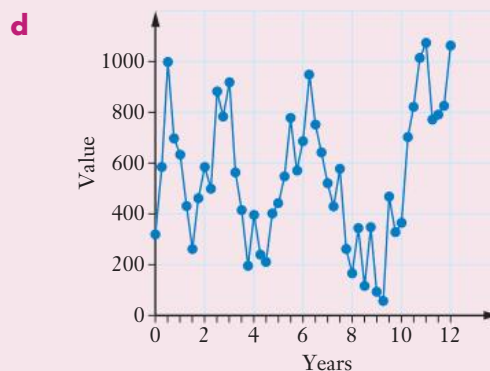
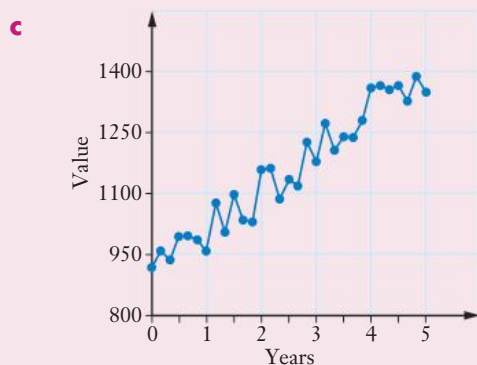
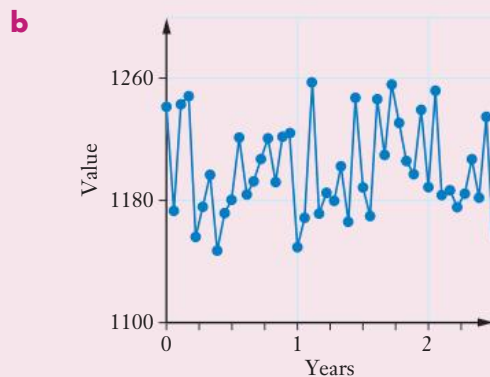
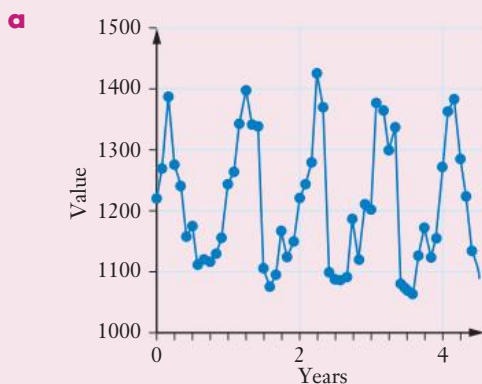
Time series patterns

- A **trend** has a gradual increase or decrease. The trend could be a straight line (linear) or otherwise
- A **seasonal** variation rises and falls in the same way each year. The changes through the year are often related to the seasons
- Variation that does not follow a pattern is described as **irregular**.

Real time series usually combine several patterns, such as seasonal and trend variations. Even a strong trend, seasonal or cyclic variation will have irregularities.

EXAMPLE 3

Describe each time series.



Solution

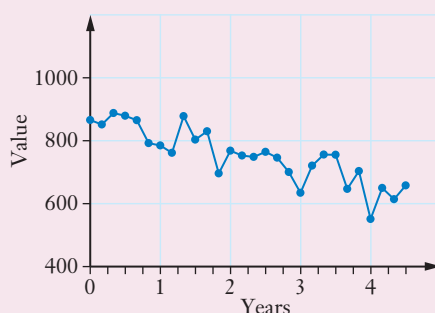
a Each year has the same pattern of variation.	There is seasonal variation.
b There is no pattern in the variation.	There is irregular variation.
c The values gradually increase.	There is an upward trend.
d There are peaks and troughs over the years.	There is a cyclic pattern.

You need to ignore minor variations and look for the overall pattern.

It is more difficult to work out combined variations.

EXAMPLE 4

Describe this time series.



Solution

Look at the general direction of the graph.	There is a downward trend.
Look at the pattern in each year.	There is a seasonal variation.
Write the conclusion.	The time series has a seasonal variation with an overall downward trend.

Exercise 3.02 Time series patterns

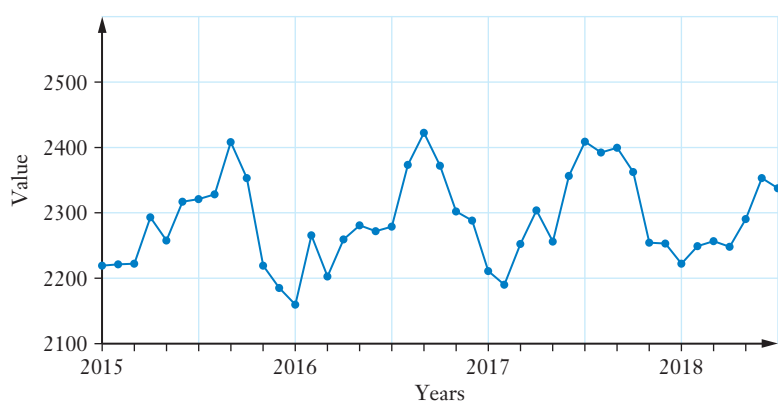
Examples
3, 4

Describe each time series shown.

1



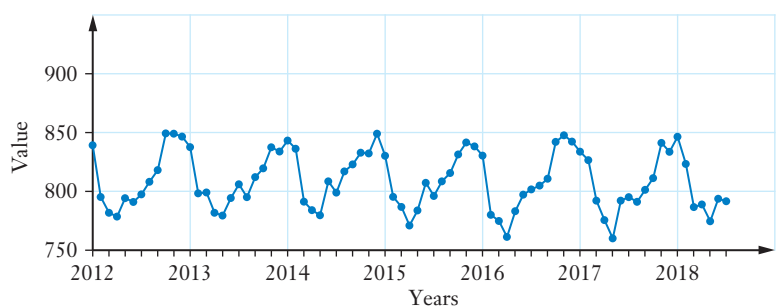
2



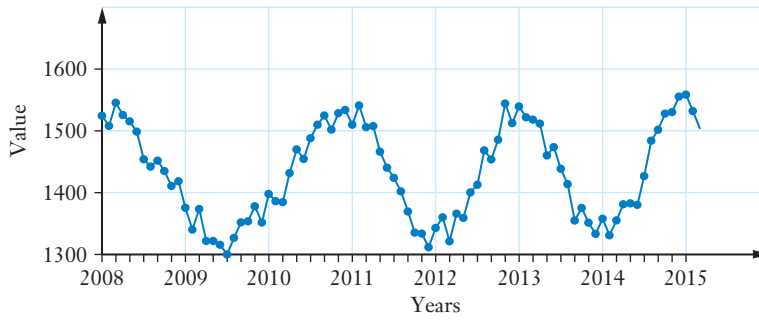
3



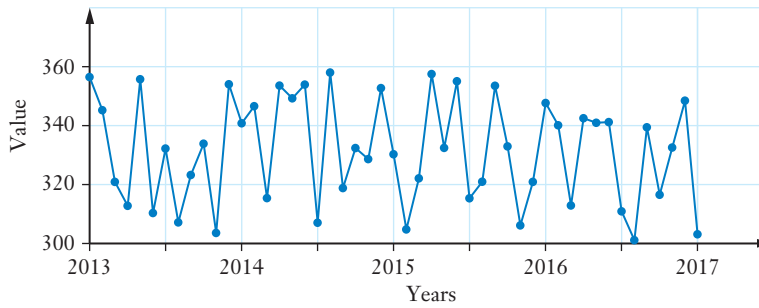
4



5



6

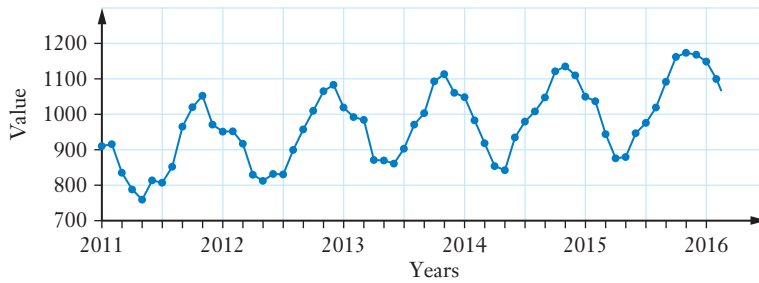


7



Problem solving

8



9





3.03 Outliers

In any time series, there are likely to be values that stand out from the rest of the series. These observations can make it difficult to predict future values.

Outliers

An **outlier** is an observation that is dramatically different from the rest of the time series.

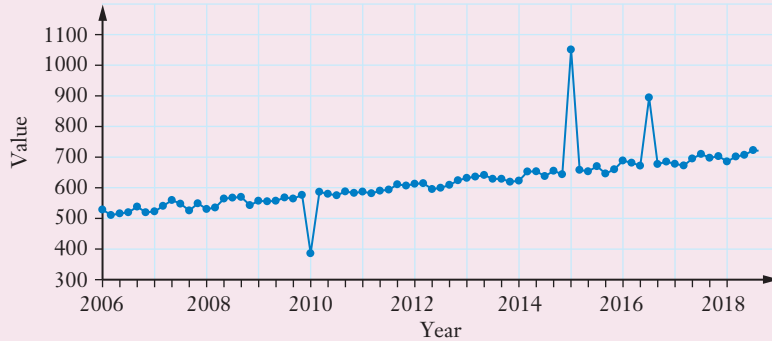


Shutterstock.com/JeremyRichards

The easiest way to detect any outliers is by looking at the time series plot. Look for one-off changes from the normal variation.

EXAMPLE 5

Identify any outliers in this time series.



Solution

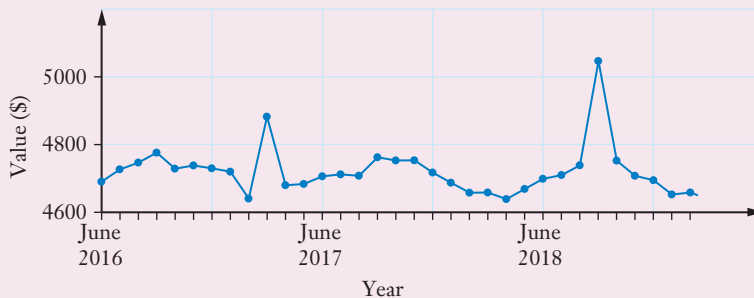
There are 3 values that dramatically differ from the trend.

Assuming labels are at the beginning of the years, there are outliers in January 2010, January 2015 and June 2016.

If the labels are at midyear, the outliers would be in June 2010, June 2015 and January 2017.

EXAMPLE 6

Identify any outliers in this time series.



Solution

There is a seasonal variation, but there are two extraordinary data points.

There are outliers in March 2017 and October 2018.

INVESTIGATION

STOCK EXCHANGE CHARTS

Is there any way of predicting outliers or substantial changes on the stock exchange?

If you could make such predictions, you could make money by anticipating the changes.

If a particular company was due to get a large rise in its share price, you could buy shares before the rise. Selling them after the rise would make you a profit.

If you could predict a fall, you could buy shares and pay for them after the fall, but sell them before the fall. This is called short trading.

There are many graphs of share prices on the stock exchange.

- Get some graphs of share prices on the stock exchange.
- Cover the later part of the graph and try to predict any change.
- Do this for as many graphs as you can.
- What conclusion can you make about predictions from charts?



Shutterstock.com/Poznyakov

Many mathematicians think that share and currency prices have so many influences that they constitute a *random walk*. This means that within any long-term trends, the changes in prices are random and cannot be predicted.

INVESTIGATION

REAL TIME SERIES OUTLIERS

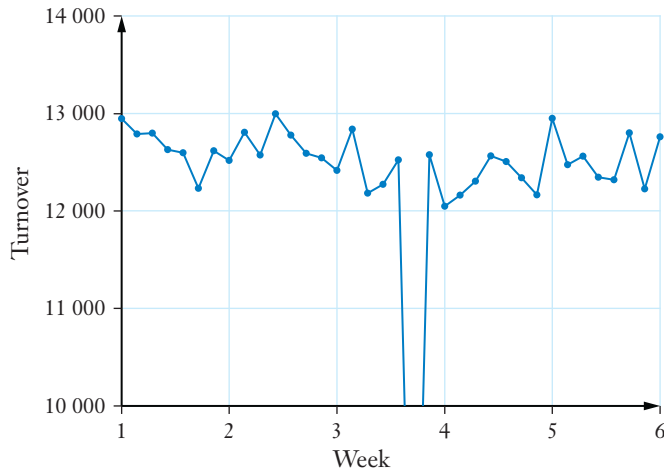
The ABS has many time series available on its website. The time series generally have both raw data and data that has been adapted in some way. Obtain some raw data time series from the website and look for outliers in the series. Employment statistics use smaller samples than many of the series, so are more likely to have outliers.

The Australian stock exchanges have data available on very short periods, such as a minute. These may also have outliers evident.

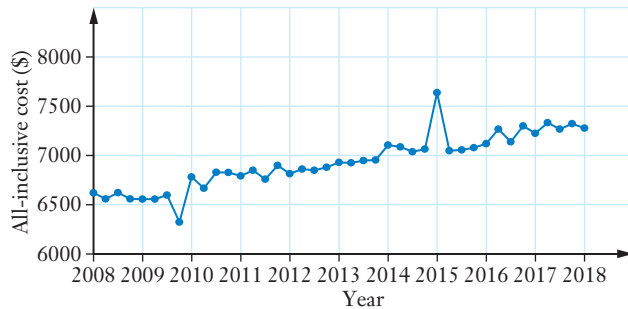
Your teacher might want you to write a report.

Exercise 3.03 Outliers

- 1 Identify any outliers in this time series that shows daily takings of a petrol station over a 5-week period. Each week starts at Monday.

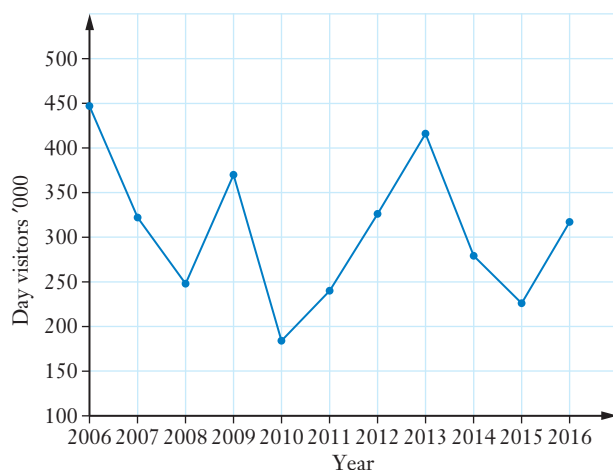


- 2 Identify any outliers in this time series of the average price of all-inclusive 7-day packaged holidays in Asia. The data was collected quarterly, starting in March 2008.



Shutterstock.com/Sean Pavone

- 3** Identify any outliers in this time series that shows the numbers of day visitors to the Whitsundays region, starting with the 2006–07 financial year.



Source: Queensland Government Statisticians Office

- 4** Identify any outliers in this time series of cattle prices in cents/kg live weight at fortnightly intervals, starting from 30 June, 2016.



- 5** Identify any outliers in this time series of the cost of supplemental bulk cattle feed, adjusted for inflation. The chart starts in January 2014.



3.04 Smoothing time series data

Unemployment and other reports that you see on TV often quote an actual number and a trend number. This means that some variation has been taken out of the data.



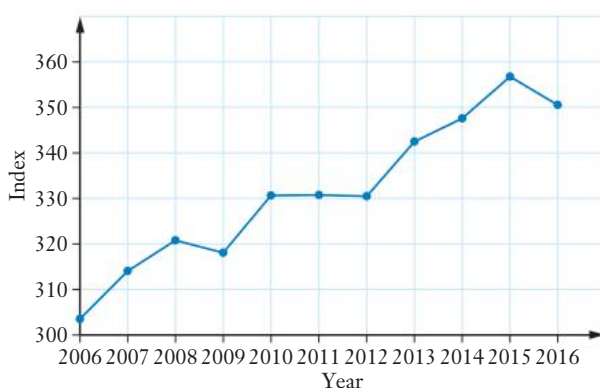
Data smoothing

Smoothing means removing variation from data to make important trends more obvious.

There are many methods of smoothing data. The **moving average** is the simplest method and is effective for data that does not have seasonal variation.

Consider the data shown in the table and graph below.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Index	304	314	321	318	331	331	343	348	357	351	357



It is obvious that the index value is gradually increasing. What happens to the data if you average each of the last 3 values? You can only start at 2008.

For 2008, average of the last 3 values = $\frac{304 + 314 + 321}{3} = 313$

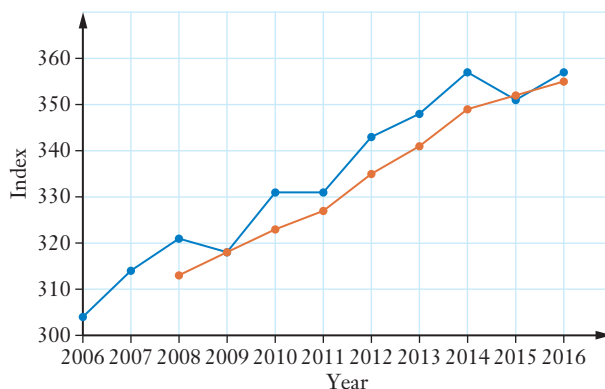
For 2009, average of the last 3 values = $\frac{314 + 321 + 318}{3} = 317.666\dots$

Doing this for all the data, you get the following table.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Index	304	314	321	318	331	331	343	348	357	351	357
Average			313	318	323	327	335	341	349	352	355

Redrawing with the averages in red, you get the graph on the right.

The graph of the red points is much smoother than the original data. Notice that the red line is also below the original blue line. This is because the trend is increasing. The average of the last 3 points biases the result towards the earlier values.



Moving averages

- A **moving average** smooths data using average values
- A **three-point moving average** uses the last 3 values
- A **centred moving average** puts the average in the middle of the time used



Moving averages

EXAMPLE 7

- a** Calculate 5-year centred moving averages for the data below.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Index	304	314	321	318	331	331	343	348	357	351	357

- b** Draw a graph showing the original and smoothed data.
c Comment on the graphs.

Solution

- a** Work out the average for the first 5 years.

$$\text{Average} = \frac{304 + 314 + 321 + 318 + 331}{5} = 317.6$$

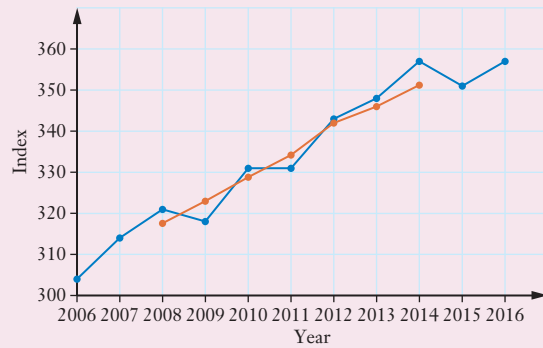
Work out the average for the next 5-year period.

$$\text{Average} = \frac{314 + 321 + 318 + 331 + 331}{5} = 323$$

Complete the other averages and place them in the table.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Index	304	314	321	318	331	331	343	348	357	351	357
Average			318	323	329	334	342	346	351		

- b** Draw the graph, showing both the original and smoothed data.



- c** Compare the lines.

The smoothed data shows the trend much more clearly than the original information.

The smoothed data shown in red above is smoother and closer to the original line than the 3-year moving average shown for the same data shown at the top of the previous page.



Gelby Images/ginevre

TECHNOLOGY

Moving average spreadsheets

A spreadsheet makes the calculation of moving averages much easier.

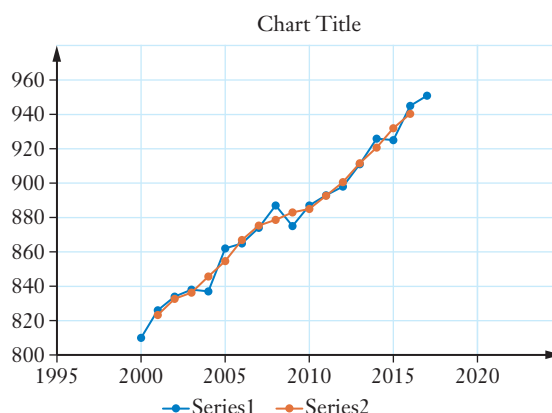
Follow the instructions below to find centred 3-year moving averages for the data in the table below. It shows house prices for a particular Brisbane suburb.

Average auction prices of properties \$ '000									
Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Price	810	826	834	838	837	862	865	874	887
Year	2009	2010	2011	2012	2013	2014	2015	2016	2017
Price	875	887	893	898	911	926	925	945	951

- Open a new spreadsheet and put the headings 'Year', 'Price' and 'Average' in cells A1, B1 and C1
- Copy the data into the first two columns
- Type the formula for the average into cell C3: $= (B2+B3+B4)/3$
- Press return to activate the formula and get 823.333...
- Highlight cell C3 (click your mouse button on it)
- Copy
- Highlight cells C4–C18 (Start at cell C4 and hold the mouse button down while you move down to cell C18)
- Paste to get all the values calculated

	A	B	C
1	Year	Price	Average
2	2000	810	
3	2001	826	$= (B2+B3+B4)/3$
4	2002	834	

You can also draw the graph in your spreadsheet.



You can click on parts of the graph to format it.

EXAMPLE 8

Time	Nov-2012	May-2013	Nov-2013	May-2014	Nov-2014	May-2015
Male	1329.00	1371.40	1338.20	1374.20	1331.50	1312.80
Female	816.70	804.40	824.50	845.00	864.80	905.20
Time	Nov-2015	May-2016	Nov-2016	May-2017	Nov-2017	
Male	1334.60	1355.80	1353.80	1386.80	1405.80	
Female	889.70	901.70	886.20	898.10	891.70	

Source: Australian Bureau of Statistics

- Use a spreadsheet to calculate 3-month centred moving averages for the average weekly earnings of males and females in Queensland from May 2013 to November 2017.
- Draw a graph of the data, showing both the original and smoothed data.
- Comment on the results.

Solution

- Enter the data in columns A2–A12, B2–B12 and C2–C12.

Time	Male	Female
Nov-2012	1329.00	816.70
May-2013	1371.40	804.40
Nov-2013	1338.20	824.50

Enter the formula $=(B2+B3+B4)/3$ in D3 and copy it down to D12. Then copy it to cells E3 to E12.

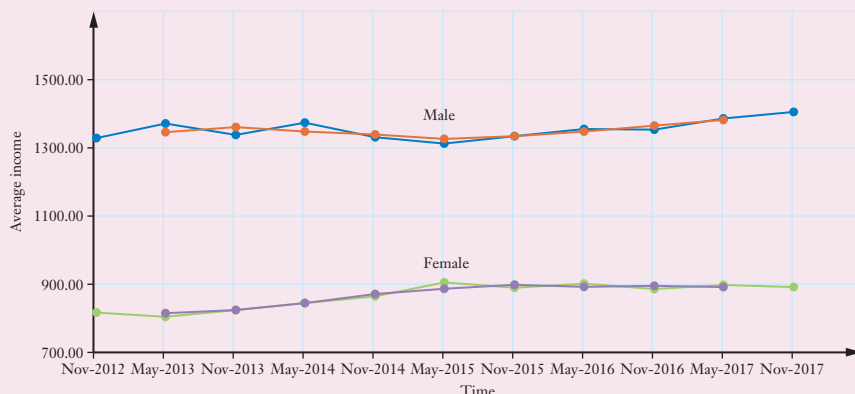
Time	Male	Female		
Nov-2012	1329.00	816.70		
May-2013	1371.40	804.40	1346.2	815.2
Nov-2013	1338.20	824.50	1361.267	824.6333

Round the data to the nearest 10 cents.

Smoothed data

Time	Nov-2012	May-2013	Nov-2013	May-2014	Nov-2014
Male		1346.20	1361.30	1348.00	1339.50
Female		815.20	824.60	844.80	871.70
Time	May-2015	Nov-2015	May-2016	Nov-2016	May-2017
Male	1334.40	1348.10	1365.50	1382.10	1396.30
Female	898.90	892.50	895.30	892.00	894.90

- b** Draw the graph, with both male and female, original and smoothed data.



- c** Make comments.

The average female income is much less than the average male income over this period.

Smoothing does not change the difference between male and female incomes.

Exercise 3.04 Smoothing time series data

Example
7

- 1** The table below shows the annual turnover of a Queensland retailer over 12 years.

Year	2007	2008	2009	2010	2011	2012
Turnover \$'000	1206	1229	1234	1259	1243	1275
Year	2013	2014	2015	2016	2017	2018
Turnover \$'000	1282	1283	1292	1293	1319	1329

- a** Calculate 3-year centred moving averages.
b Calculate 5-year centred moving averages.
c Draw a graph showing the original and both smoothed sets of data.
d Comment on the graphs.
- 2** The table below shows the average annual profit of an Australian confectionery maker over a 15-year period.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Profit \$'000	3406	3514	3459	3495	3571	3534	3537	3551
Year	2009	2010	2011	2012	2013	2014	2015	
Profit \$'000	3635	3673	3700	3702	3697	3681	3704	

- a** Calculate 3-year centred moving averages.
b Calculate 7-year centred moving averages.
c Draw a graph showing the original and both smoothed sets of data.
d Comment on the graphs.

- 3** The table below shows the number of passengers using buses in regional Queensland over the period 2008–2015.

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2008–09	3 020 749	2 858 003	2 766 600	2 893 915
2009–10	3 181 291	2 912 592	2855 117	3 058 620
2010–11	3 305 231	2 946 470	2885 767	2 985 207
2011–12	3 240 303	2 917 110	2841 066	2 685 566
2012–13	3 084 170	2 778 107	2798 180	2 905 462
2013–14	3 236 489	2 811 101	2797 966	2 905 647
2014–15	3 017 381	2 764 974	2779 259	2 872 799

- Use a spreadsheet to calculate 7-quarter-centred moving averages.
- Draw a graph showing the original and smoothed sets of data.
- Comment on the graphs.



Source: Fairfax Syndication/Paul Harris

- 4** The table below shows the total monthly distance, in thousands of kilometres, driven by drivers for a Brisbane-based courier over a 2-year period.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Distance	456	457	455	477	469	492	490	511	509	528	520	530
Month	J	F	M	A	M	J	J	A	S	O	N	D
Distance	533	559	552	572	568	577	595	589	613	613	625	623

- Use a spreadsheet to calculate 3-month centred moving averages.
- Draw a graph showing the original and smoothed data.
- Comment on the graphs.

5 This table shows the number of students in Queensland schools from 1993 to 2017.

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001
Number	543 731	548 531	556 112	567 134	575 106	584 625	594 584	602 912	610 810
Year	2002	2003	2004	2005	2006	2007	2008	2009	2010
Number	619 587	629 771	638 955	647 254	657 797	697 903	706 462	717 988	724 956
Year	2011	2012	2013	2014	2015	2016	2017		
Number	733 652	747 682	761 411	773 309	784 224	794 815	806 555		

- Use a spreadsheet to calculate 5-year centred moving averages, rounded to the nearest thousand.
 - Draw a graph showing the original and smoothed data.
 - Comment on the graphs.
- 6** The table below shows the production of potatoes in Queensland for 1980–2011 in thousands of tonnes.

Year	80	81	82	83	84	85	86	87	88	89	90
Total	127	113	129	120	124	117	113	133	120	115	122
Year	91	92	93	94	95	96	97	98	99	00	01
Total	119	113	125	118	109	104	115	122	108	119	116
Year	02	03	04	05	06	07	08	09	10	11	
Total	118	119	119	120	94	88	99	98	95	88	

- Use a spreadsheet to calculate 5-year centred moving averages.
- Draw a graph showing the original and smoothed data.
- Comment on the graphs.

Problem solving

- 7** This table shows the annual production, in thousands of tonnes, of a Queensland coal mine. It has been in operation for a long time and needs to produce at least 200 000 tonnes per year to remain viable. Use 5-year moving averages to predict when it will close. Are there any difficulties with your prediction?

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Production	1589	1421	1598	1110	1346	1007	847	852	667	631	670

- 8** The table below shows the production of sugar, in kilotonnes, in Australia from 2001 to 2017. Use a 3-year moving average to predict the amount of sugar that will be produced in 2025. Are there any problems with your prediction?

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009
Sugar	4162	4987	5398	5045	5234	5063	5026	4763	4634
Year	2010	2011	2012	2013	2014	2015	2016	2017	
Sugar	4472	3610	3683	4248	4364	4572	4920	4804	

3. CHAPTER SUMMARY

Time series

- A **time series** is a set of data collected at regular periods of time
- A **time series plot** is a graph of a time series with time on the horizontal axis. You normally join the points to make a line graph
- A **trend** has a gradual increase or decrease. The trend could be a straight line (linear) or otherwise
- A **seasonal** variation rises and falls in the same way each year. The changes through the year are often related to the seasons
- An **irregular** time series does not follow a pattern
- An **outlier** is an observation that is dramatically different from the rest of the time series
- **Smoothing** means removing variation from data to make important trends more obvious
- A **moving average** smooths data using average values
- A **three-point** moving average uses the last 3 values
- A **centred** moving average has the average in the middle of the time used

3. CHAPTER REVIEW

Time series

Example
1

- 1** The number of bookings at Anya's hair salon each day are shown below for a month.

Day	W	Th	F	S	S	M	T	W	Th	F	S	S	M	T	W
Bookings	16	15	12	10	0	10	11	15	17	20	15	3	9	14	14
Day	Th	F	S	S	M	T	W	Th	F	S	S	M	T	W	Th
Bookings	18	18	10	0	11	9	17	18	17	8	0	10	11	12	16

- a** Make a time-series plot of the data.
b Can you make any conclusions?

Example
2

- 2** The average maximum and minimum daily temperatures each month for Thargominda are below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Max °C	36.9	35.6	33.2	28.4	23.3	19.7	19.2	21.6	26.0	30.1	33.4	36.2
Min °C	23.7	23.1	20.2	15.3	10.7	7.1	5.6	7.1	11.0	15.4	18.9	23.7

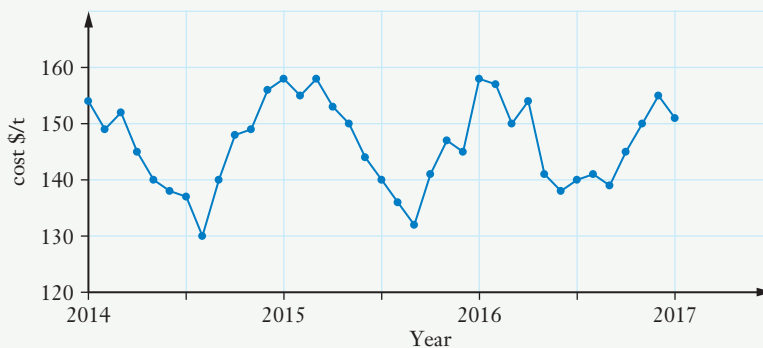
- a** Plot the temperatures for Thargominda on the same axes.
b Can you make any conclusions?

- 3** Which of the following is not a time-series description?

A seasonal **B** jagged **C** irregular **D** trend **E** cyclic

Examples
3, 4

- 4** Describe this time series starting in January 2014.



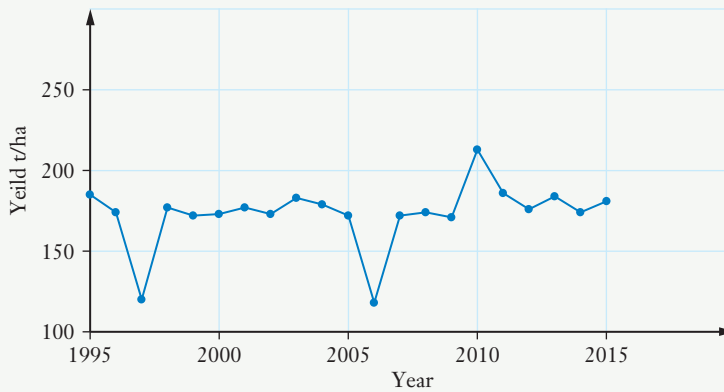
5 Describe this time series.

Examples
3, 4



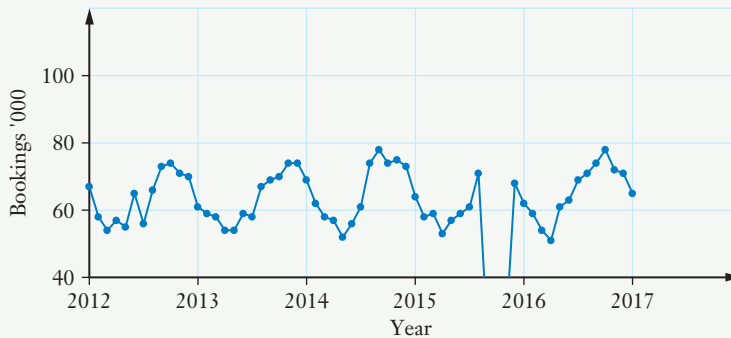
6 Identify any outliers in this time series.

Examples
5, 6



7 Identify any outliers in the following time series starting in January 2012.

Examples
5, 6



Example
7

- 8 a** Use 3-year centred moving averages to smooth the data in the table below that shows cases of a particular disease requiring hospitalisation.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018
Cases	624	612	625	621	642	634	651	653	647

- b** Draw a plot of both the original and the smoothed data.
c Comment on the graphs.

Example
8

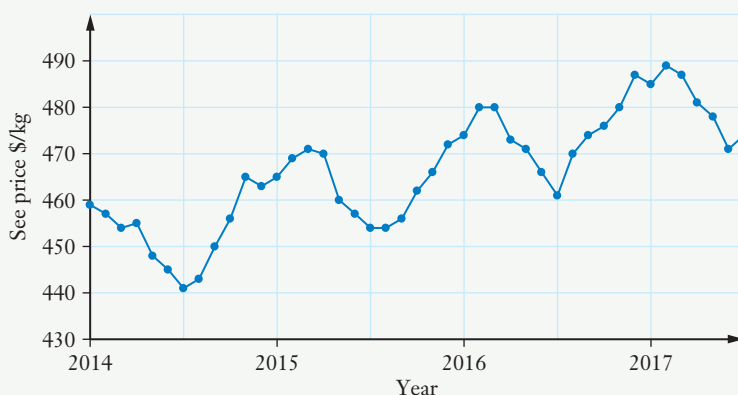
- 9 a** Use a spreadsheet to calculate 3-month moving averages for the reported cases of a particular disease shown below.

2014	J	F	M	A	M	J	J	A	S	O	N	D
Cases	779	935	491	1362	1931	2533	2785	2941	2386	1237	974	1036
2015	J	F	M	A	M	J	J	A	S	O	N	D
Cases	714	176	402	1412	1786	2004	2677	2035	1839	1598	1286	1022
2016	J	F	M	A	M	J	J	A	S	O	N	D
Cases	451	7	117	986	1009	2547	2402	2465	1521	1170	754	978
2017	J											
Cases	634											

- b** Draw a plot of the original and smoothed data on the same axes.
c Comment on the graphs.

Problem solving

- 10** Describe this time series.



- 11** This table shows the average annual profit in \$/ha of a crop from the year 2000.

Year	00	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15
Profit	272	265	254	246	264	253	236	256	236	229	230	233	222	244	218	207

Use smoothed data to predict the value in 2025 and evaluate your assessment.

Qz

Practice quiz

Practice examination 1 ●○○○

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved scientific calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (2 marks)

The brands of scientific calculators used by students in Year 12 General Maths classes from 2 different schools were as follows.

Sandy Hills SHS, with 47 General students, had 23 TIs, 20 Casios and the rest were Sharps.

Mt Flute SHS, with 74 General students, had 47 TIs, 16 Sharps and the rest were Casios.

Construct a two-way table of the data.

Question 2 (1 mark)

What are the 2 types of categorical data?

- A** numeric and continuous
- B** discrete and continuous
- C** nominal and ordinal
- D** nominal and discrete
- E** ordinal and numeric

Question 3 (3 marks)

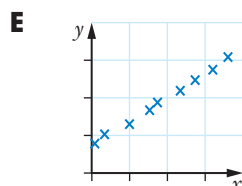
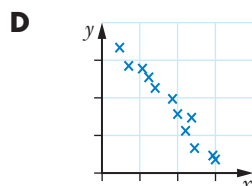
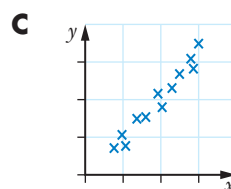
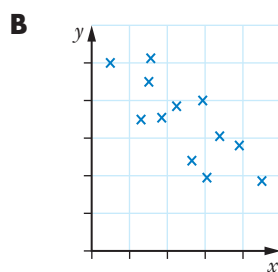
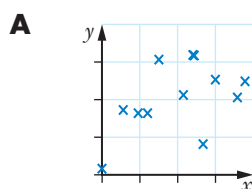
- a** Copy and complete the marginal frequencies of the following table of toast preferences.

	Dark	Medium	Light
Male	9	6	4
Female	2	5	4

- b** Rewrite the table with percentages to look for differences between males and females.
- c** Is there any pattern to the differences?

Question 4 (1 mark)

Which of the following scatterplots shows a moderate negative association?

**Question 5** (3 marks)

- a** Construct a scatterplot of this table of values.

x	6.4	7.3	7.7	9.6	9.2	11	10.5	11.8	13	12.3	13.6	14.5
y	16.9	17.3	19.2	18.8	16.7	17.5	18.4	18.9	21.2	21.5	26	24.3

- b** Identify any association.

Question 6 (2 marks)

The standard deviations of 2 variables are 5.3 and 7.2 and the covariance is -32.5 .

- a What is the correlation coefficient?
- b What can you say about the variables?

Question 7 (4 marks)

State whether each sequence is arithmetic, geometric or neither. Where appropriate, state the common difference or common ratio.

- a 13, 5, -3 , -11 , ...
- b 2, 5, 9, 14, ...
- c 1, 3, 6, 18, ...
- d 4, 6, 9, 13.5, ...

Question 8 (1 mark)

What is the 16th term of an arithmetic sequence with a first term of 8 and a common difference of 3?

Question 9 (2 marks)

How many terms of the sequence 3, 10, 17, 24, ... does it take to exceed 400?

Question 10 (1 mark)

What is the 7th term of a geometric sequence with a first term of 80 and a common ratio of $\frac{1}{2}$?

Question 11 (2 marks)

The 5th and 6th terms of a geometric sequence are 8 and 36 respectively. What is the second term?

Question 12 (2 marks)

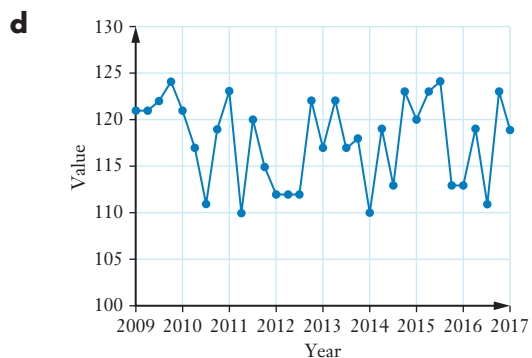
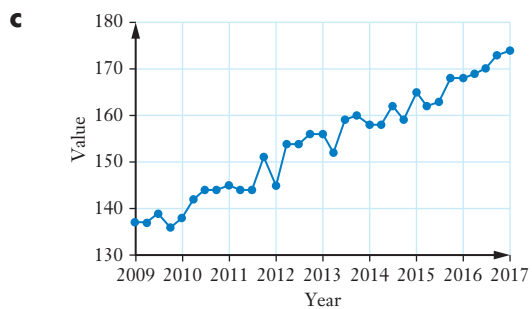
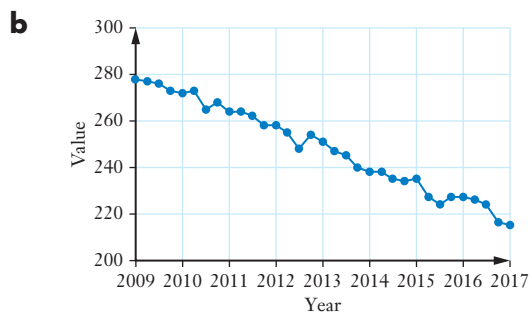
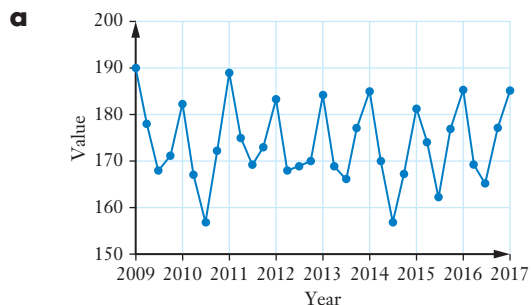
The DVD sales in Australia from 2007 to 2013 are shown below.

Year	2007	2008	2009	2010	2011	2012	2013
Sales (millions)	95	81	84	71	59	54	54

- a Make a plot of the data.
- b What conclusions can you make?

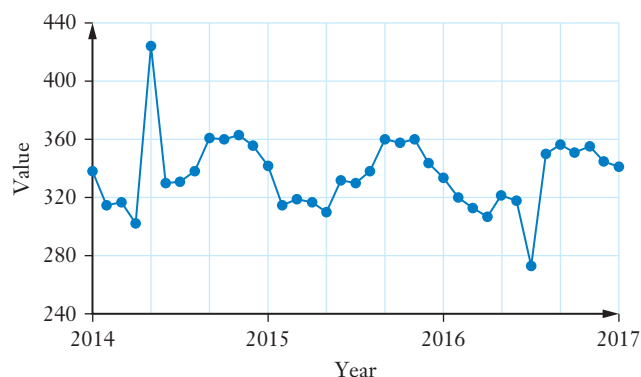
Question 13 (2 marks)

Describe each time series.



Question 14 (1 mark)

This time series starts in January 2014.



Which statement is true about the outliers in this time series?

- A** There are no outliers.
- B** There is one outlier, in April 2014.
- C** There are two outliers, in May 2014 and March 2016.
- D** There is one outlier, in July 2016.
- E** There are two outliers, in May 2014 and July 2016.

Question 15 (3 marks)

a Calculate 3-year centred averages of the following data.

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Value	226	225	241	237	241	241	246	262	259
Year	2009	2010	2011	2012	2013	2014	2015	2016	
Value	268	275	269	278	282	279	282	285	

- b** Plot the original and averaged data on the same axes.
- c** Comment on the graphs.

Question 16 (4 marks)

Some variegated bromeliads were put in full sun all day, some in full shade all day and some in full sun for about half the day. Originally, they all had leaves that were about half-green and half-red. After 2 months, the leaf colours were assessed. The results were as follows:

Full sun code = S

Half-sun code = H

Full shade code = U

Mainly red = r

Half-red = h

Mostly green = g

Fr	Fr	Fr	Fg	Fr	Fh	Fh	Fr	Fh	Fh	Fh	Fr	Fr
Fg	Hg	Hh	Hh	Hg	Hr	Hh	Hh	Hh	Hg	Hh	Hh	Hh
Hg	Hh	Hg	Hg	Hg	Hg	Hh	Hh	Hg	Hg	Hr	Hh	Hg
Hh	Hg	Hg	Hh	Hh	Uh	Ug	Ug	Ug	Ug	Uh	Ug	Ug
Uh	Ug	Ug	Uh	Ug	Uh	Uh	Uh	Ug	Uh			

Use probabilities from a contingency table to compare the probabilities of the bromeliads being mostly red in different sun conditions.

Question 17 (3 marks)

The area covered by an oil slick grows by 15% each day. On the first day, it covered an area of 450 m^2 . How long does it take to reach a size of 900 m^2 ?

Question 18 (3 marks)

Plot and describe the time series shown below.

2012	Q1	Q2	Q3	Q4	2013	Q1	Q2	Q3	Q4	2014	Q1	Q2	Q3	Q4
Value	210	173	190	230	Value	210	188	192	218	Value	185	175	183	206

2015	Q1	Q2	Q3	Q4	2016	Q1	Q2	Q3	Q4	2017	Q1	Q2	Q3	Q4
Value	171	155	158	190	Value	164	153	153	187	Value	161	156	143	190

Question 19 (4 marks)

The table below shows the annual occupancy rate and the numbers of vehicles sold in Queensland. The occupancy rate is for accommodation places with over 15 rooms. The figures for each are from the same region during the period 2001–2016.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Occupancy '000	2721	2703	2847	3056	3197	3251	3361	3380
Vehicle sales '000	35	40	46	53	60	57	62	62
Year	2009	2010	2011	2012	2013	2014	2015	2016
Occupancy '000	3243	3274	3367	3457	3383	3252	3533	3653
Vehicle sales '000	53	58	54	63	64	61	64	65

Find any association between accommodation occupancy and vehicles sales and assess its strength.

Question 20 (3 marks)

A transport company can choose the straight-line method or double declining-balance method of depreciation. A prime mover cost \$140 000 new and has a scrap value of \$45 000 after 6 years. For each depreciation method, compare the book values 3 years after purchase.

Question 21 (3 marks)

The data below shows expenditure in Queensland on petroleum exploration.

2015	Mar	Jun	Sep	Dec
Expenditure (\$millions)	154.4	153.5	90.4	62.0
2016	Mar	Jun	Sep	Dec
Expenditure (\$millions)	28.7	23.5	23.2	35.7
2017	Mar	Jun	Sep	Dec
Expenditure (\$millions)	38.3	57.7	45.3	41.9

Use smoothed data to predict the expenditure in the March quarter of 2020 and evaluate your prediction.

END OF EXAMINATION

4.

MEASURING AND EXPLAINING ASSOCIATIONS

From their own playing days, rugby coaches know what works best on the field. They also need to know what works best off the field. Should players spend more time on general fitness or practising skills? How much time should players spend on the scrum machine? Good coaches know the answers to these questions. They know how to motivate their teams.

In medical science, which medicine doses work best for different patients? What mixture of endurance and strength training is best for your health?

In this chapter you will learn about statistical methods that can be used to help answer questions like these.

- 4.01 Response and explanatory variables
- 4.02 Relationships between variables
- 4.03 Least-squares linear models
- 4.04 Residual plots
- 4.05 The coefficient of determination
- 4.06 Interpolation and extrapolation
- 4.07 Association and causation
- 4.08 Applications of associations
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Fitting a linear model to numerical data

- identify the response variable and the explanatory variable
- use a scatterplot to identify the nature of the relationship between variables
- model a linear relationship by fitting a least-squares line to the data
- use a residual plot to assess the appropriateness of fitting a linear model to the data
- interpret the intercept and slope of the fitted line
- use, not calculate, the coefficient of determination (R^2) to assess the strength of a linear association in terms of the explained variation
- use the equation of a fitted line to make predictions
- distinguish between interpolation and extrapolation when using the fitted line to make predictions, recognising the potential dangers of extrapolation

Association and causation

- recognise that an observed association between two variables does not necessarily mean that there is a causal relationship between them
- identify and communicate possible non-causal explanations for an association, including coincidence and confounding due to a common response to another variable
- solve practical problems by identifying, analysing and describing associations between two categorical variables or between two numerical variables



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

coefficient of determination
contingency table
extrapolation
least-squares model
moderate association
 R^2
response variable
straight-line relationship
y-intercept

coincidence
dependent variable
independent variable
linear association
negative association
residual
scatterplot
strong association

confounding variable
explanatory variable
interpolation
mechanism
positive association
residual plot
slope
weak association

4.01 Response and explanatory variables

Is studying before tests likely to help you do better? Or does doing better in tests help you to study before them? When you examine change in variables, you need to decide which comes first. Which variable could cause the change? Which variable is changed as a result?

Response and explanatory variables

- A **response variable** (**dependent variable**) is changed by another variable
- An **explanatory variable** (**independent variable**) is not changed by another variable. The response variable could depend on the explanatory variable(s)

You can often work out which variable is which by putting the variables in the sentence: 'Changes of the *explanatory variable* cause changes in the *response variable*'.

EXAMPLE 1

For each pair of variables, decide which is more likely to be the response variable and which is the explanatory variable.

- a** Temperature during the day and number of people at the beach.
- b** The weight of tomatoes picked and the time pickers spend working.

Solution

- a** Try putting the variables in each position of the test sentence.

Pick the answer that makes more sense.

Changes of the temperature cause changes in the number of people at the beach.

Changes in the number of people at the beach cause changes of the temperature.

The response variable is the number of people at the beach. The explanatory variable is the temperature.

- b** Try putting the variables in each position of the test sentence.

Pick the answer that makes more sense.

The weight of tomatoes picked causes changes in the time pickers spend working.

The time employees spend working cause changes in the weight of tomatoes picked.

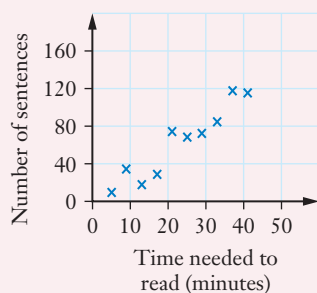
The response variable is the weight of tomatoes picked. The explanatory variable is the time pickers spend working.

When you draw a **scatterplot**, put the explanatory variable on the horizontal axis. The response variable goes on the vertical axis.

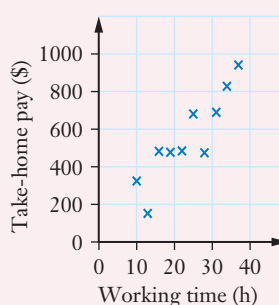
EXAMPLE 2

State whether or not each scatterplot has been correctly drawn.

a



b



Solution

- a** Which is the response variable?

Write the answer.

The time taken to read something depends on the number of sentences, so the time is the response variable.

The scatterplot is the wrong way around.

- b** Which is the response variable?

Write the answer.

Take-home pay depends on the time spent working, so take-home pay is the response variable.

The scatterplot is correctly drawn.

Exercise 4.01 Response and explanatory variables

Example

1

- 1 In each pair of variables, decide which is more likely to be the response variable and which is the explanatory variable. Give the explanatory variable first.
- a Evaporation rate, temperature
 - b Time, mass of ice melted
 - c Cost, mass of flour
 - d Number of hens, number of eggs laid
 - e Erosion of beach, number of storms



Getty Images/Dorling Kindersley

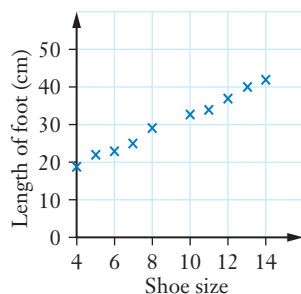
- 2 In each pair of variables, identify the response and explanatory variables.
- a Rainfall, river flow
 - b Seedlings eaten, number of birds
 - c Time spent training, 100 m time
 - d Sawdust, logs milled
 - e Sawdust, time to sweep

Example

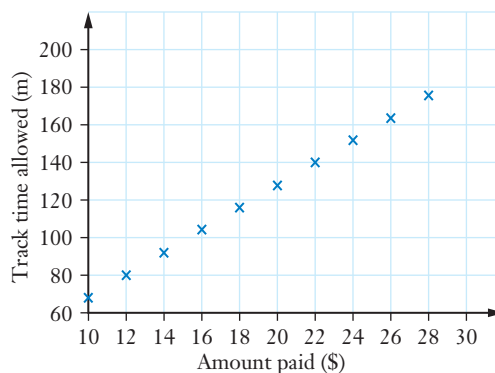
2

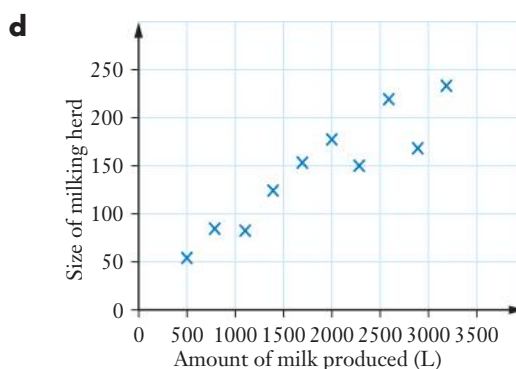
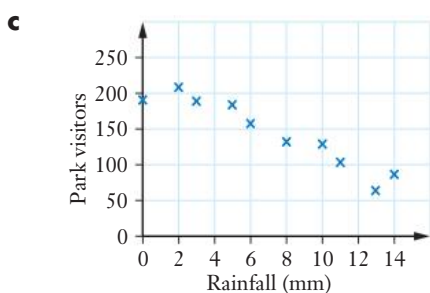
- 3 State whether or not each scatterplot has been correctly drawn.

a



b





Problem solving

- 4** Choose at least 5 pairs of variables from the following list. Each pair has to be an explanatory and response variable in that order.

Age of student	Cost of vegetable	Speed of yacht
Weight of vegetable	Height of student	Type of vegetable
Floor area tiled	Hair colour of student	Number of tiles available
Length of yacht	Size of army	Size of tiles
Cost of army	Sail area of yacht	Crew size of army tank

- 5** State which variables in question 4 could not be explanatory or response variables.

4.02 Relationships between variables

You looked at associations between variables in Chapter 1. An association between variables can suggest there is a **relationship**. However, it does not mean that there is necessarily a cause-and-effect relationship. For example, there is a strong positive relationship between ice-cream sales and drownings at Australian beaches. This does not mean that one causes the other.

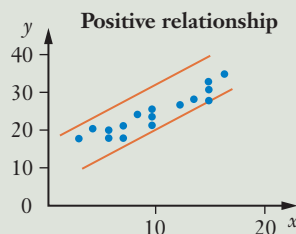
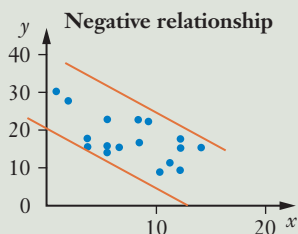
As shown in Chapter 1, you can use scatterplots to identify associations. If you can identify one variable as explanatory and the other as response, then there is a possible relationship.



Relationships
between
variables

Identifying possible relationships

- Draw a scatterplot of the bivariate numerical variables
- A pattern suggests an association
- A graph that slopes down from left to right suggests a **negative** association
- A graph that slopes upwards from left to right suggests a **positive** association



- A **linear** association is a straight line
- A **strong** association has points that make an almost perfect line
- A **moderate** association has an upward or downward trend that can be shown by a straight line through the points. Almost all the points are close to such a line
- A **weak** association has a pattern, but no clear line
- A moderate or strong linear association can be modelled by a **straight-line relationship** between the variables

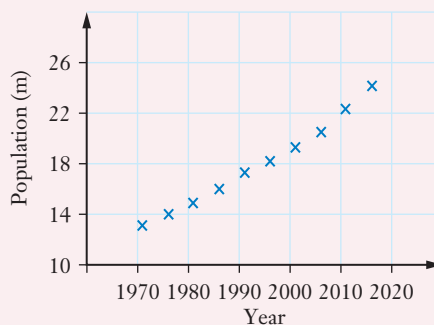
EXAMPLE 3

Draw a scatterplot of the Australian population (in millions) between 1971 and 2016 to determine if there is a relationship.

Year	1971	1976	1981	1986	1991	1996	2001	2006	2011	2016
Population (mil)	13.1	14	14.9	16	17.3	18.2	19.3	20.5	22.3	24.2

Solution

Make the year the horizontal variable.



Write the answer.

There appears to be a strong positive relationship between the year and the population.

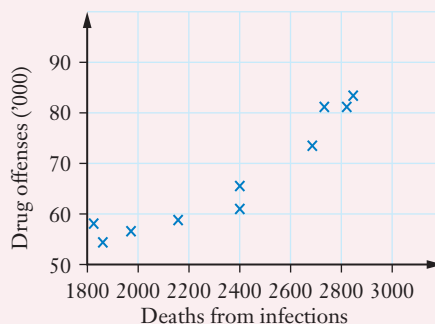
EXAMPLE 4

Draw a scatterplot of deaths from infections and drug offences (in thousands) for 2007–2016 to determine if there is a possible relationship.

Deaths	1857	1969	1826	2153	2400	2394	2682	2732	2843	2818
Drugs '000	54	56	58	58.7	61	65	73	81	83	81

Solution

Make the deaths from infections the horizontal variable.



Write the answer.

There is a moderate to strong association. Deaths from infections and drug offences are not possible response and explanatory variables for each other. There is no relationship.

Exercise 4.02 Relationships between variables

- 1 Draw a scatterplot of average monthly temperature (in °C) and humidity at 9 a.m. for Normanton to determine if there is a relationship.

Temperature	29.5	29.45	29.3	28.5	25.7	23.3	22.9	24.25	27.45	30.05	31.2	30.95
Humidity %	75	76	66	58	50	49	46	44	44	43	49	61

Examples
3, 4

- 2 Draw a scatterplot of days dieting and weight loss to determine if there is a relationship.

Days	6	8	8	11	15	17	19	18	21	23
Loss (kg)	0	0.9	1.4	1.5	2.2	1.7	1.1	2.7	1.5	2.3

- 3 Draw a scatterplot of age and weight to determine if there is a relationship.

Age	23	37	21	34	46	53	64	70	87	78
Weight (kg)	77.4	85.9	81.9	85	75.7	74.1	78.5	74.3	72.6	74.2

- 4 Draw a scatterplot of cost (in thousands of dollars) and house size to determine if there is a relationship.

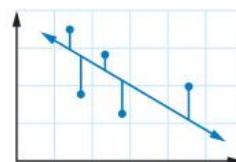
Size (m ²)	153	195	220	264	274	341	384	427	462	484
Cost (\$'000)	139	80	446	397	327	416	694	598	766	599

- 5 Draw a scatterplot of weekly income and weight to determine if there is a relationship.

Income \$/week	300	600	700	750	800	900	1050	1020	1300	1500
Weight kg	92	78	87	65	86	60	108	83	95	75

4.03 Least-squares linear models

If you know there is a linear relationship of two variables, there is more you can do. You can work out the equation of the relationship. Mathematicians say the **least-squares model** is the best fit of a linear relationship. You can work out the slope and **y-intercept** of the line as shown below.



The least-squares model makes the squares of the **vertical distances** of the points from the line the smallest they can possibly be.

Least-squares linear model calculations

- Find the total of all the x scores, shown as $\sum x$, and the average $\bar{x} = \frac{\sum x}{n}$
- Find the total of all the y scores, shown as $\sum y$, and the average $\bar{y} = \frac{\sum y}{n}$
- Find the total of all the products of x and y , shown as $\sum xy$
- Find the total of all the squares of the x scores, shown as $\sum x^2$
- The equation of the line is $y = bx + a$, where b = slope and a = y-intercept
- The **slope** is given by

$$b = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = r \frac{s_y}{s_x} = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

- The **y-intercept** is given by $a = \frac{\sum y - b \sum x}{n} = \bar{y} - b\bar{x}$

Note: There are many ways to calculate the least-squares model, but they all give the same answer. The line is also expressed as $y = ax + b$, with the symbols for the slope and y-intercept swapped. Calculators use this form.

EXAMPLE 5

- a** Find the least-squares linear relationship between x and y .
- b** Draw the scatterplot and least-squares line.

x	10	14	24	27	35
y	16	26	32	42	48

Solution

- a** Find the total of the x scores.

$$\begin{aligned}\sum x &= 10 + 14 + 24 + 27 + 35 \\ &= 110\end{aligned}$$

Find the total of the y scores.

$$\begin{aligned}\sum y &= 16 + 26 + 32 + 42 + 48 \\ &= 164\end{aligned}$$

Find the total of the products.

$$\begin{aligned}\sum xy &= 10 \times 16 + 14 \times 26 + 24 \times 32 + 27 \times 42 + 35 \times 48 \\ &= 160 + 364 + 768 + 1134 + 1680 \\ &= 4106\end{aligned}$$

Find the total of the squares of the x scores.

$$\begin{aligned}\sum x^2 &= 10^2 + 14^2 + 24^2 + 27^2 + 35^2 \\ &= 100 + 196 + 576 + 729 + 1225 \\ &= 2826\end{aligned}$$

Find the gradient and y -intercept.

$$\begin{aligned}\text{Slope, } b &= \frac{n \sum xy - \sum x \sum y}{n \sum (x^2) - (\sum x)^2} \\ &= \frac{5 \times 4106 - 110 \times 164}{5 \times 2826 - 110^2} \\ &= \frac{2490}{2030} \\ &= 1.2266... \\ \text{y-intercept, } a &= \frac{\sum y - b \sum x}{n} \\ &= \frac{164 - 1.2266... \times 110}{5} \\ &= 5.8147...\end{aligned}$$

Write the answer.

$$\begin{aligned}y &= bx + a \\ y &\approx 1.23x + 5.81\end{aligned}$$

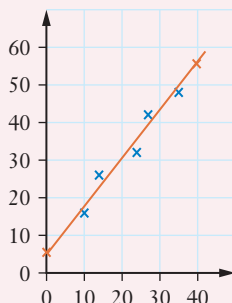
- b** Work out a couple of points on the line.

When $x = 0, y \approx 5.81$

When $x = 40, y \approx 55.01$

Draw the scatterplot. Plot the points and draw in the least-squares line.

The least-squares line makes a very good fit.



The calculations for the least-squares model are very tedious. You can use your scientific calculator to save time. Even if you are required to do calculations by hand, you should use a calculator to check.

EXAMPLE 6

- a** Use a scientific calculator to find the least-squares linear relationship between the variables shown below.

x	2	4	6	7	8	10	10	12	13	15
y	12	12	13	16	21	21	30	33	34	34

- b** Draw the scatterplot and the least-squares line.

Solution

a TI Scientific calculator

Press data and enter the values in the L1 and L2 columns respectively.

Then press **2nd** **DATA** (STAT) and choose 2: 2-Var Stats.

Make XDATA: L1 and YDATA: L2, if they are not already, and use the cursor to go down to CALC and press **ENTER**.

Move the cursor down to get the values of the slope (2.14) and y -intercept (3.95).

Press **2nd** **MODE** (quit) to exit the statistics calculations.

Casio Scientific calculator

Press **MODE**, 2: STAT and 2: A+BX

Enter the values in the X and Y columns respectively.

Press **AC** then press **SHIFT** 1 (STAT), 5: Reg, 3: r and press **=** to get the values of the slope (2.14) and y -intercept (3.95).

Press **MODE**, 1: COMP to exit the statistics mode.

Sharp Scientific calculator

Press **MODE**, 1 and choose 1 (LINE): Linear regression calculation.

Enter the values in pairs by pressing 4, **STO** (x,y), 7, **M+** (DATA), 5, **STO** (x,y), 10, **M+** (DATA), etc., until you have entered all the values.

Then press **ALPHA**, **RCL** to get the values of the slope (2.14) and y -intercept (3.95).

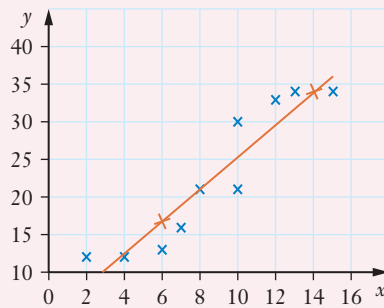
Press **MODE** 0 to exit statistics calculations.

All calculators

Write the answer.

$$y \approx 2.14x + 3.95$$

- b** Draw the scatterplot and use the equation to calculate some points. Use these points to draw the least-squares line.



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TECHNOLOGY

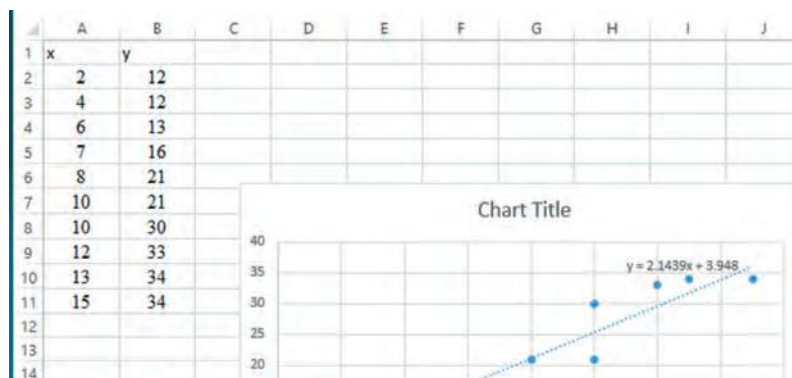
Least-squares models on a spreadsheet

You can use a spreadsheet to find and plot the least-squares model of a set of data.

For the data in Example 6:

x	2	4	6	7	8	10	10	12	13	15
y	12	12	13	16	21	21	30	33	34	34

- Open a spreadsheet and enter 'x' and 'y' in cells A1 and B1.
- Enter the x scores in cells A2 to A11.
- Enter the corresponding y scores in columns B2 to B11.
- Highlight A2–B11 and click on the INSERT tab.
- Click on the Scatter graph symbol and choose the 'points only' version.
- Highlight the points on the graph by clicking on one of them.
- Right-click and choose 'Add Trendline'.
- Choose the default, Linear, and tick the 'Display Equation on chart' box.
- Close the 'Format Trendline' dialogue box.
- You can add titles and tidy the scales.



Exercise 4.03 Least-squares linear models

For each table of values:

- a Find the least-squares linear relationship *without* using the statistics mode of your calculator.
- b Draw the scatterplot and the least-squares line.

1

x	7	10	12	13	15
y	10.1	10.3	11.6	12.6	13.9

Example
5

2

x	2	4	5	7	8
y	32	33	28	23	21

3

x	2	3	6	7	9
y	41	46	47	51	57

For each table of values:

- a Find the least-squares linear relationship using the statistics mode of your calculator.
- b Draw the scatterplot and the least-squares line.

4

x	0	2	5	7	10	12	15
y	32	32	35	58	81	65	83

Example
6

5

x	0	2	4	6	8	9	11	13	15
y	74	68	60	66	64	60	47	47	47

6

x	5	7	8	10	11	13	14	16	17	19
y	9.7	6	10.5	12.3	12.7	12.6	15	20.7	19.2	25

7

x	2	3	5	6	8	9	10	12	13	15
y	4.4	3.2	3.9	4.4	4.3	6.3	5.2	7.2	7	6.7

8

x	12	15	18	22	25	28	31	34	37	41	44	47
y	31	27.1	28.2	33.7	32.2	23.4	20.4	18.2	16.2	17.9	17	18.7

Problem solving

- 9 The opening times and day's takings of a shop are shown in the table. Find the least-squares linear relationship and give some possible explanations for your findings.

Hours opened, h	3	4	5	6	7	8	9	10	11	12
Takings \$'000, t	1.5	1.8	1.9	2.4	2.2	2.4	3	3	3.4	3.3

- 10 The moisture content of timber from the time it is first cut to when it is partly seasoned is shown in the table below. Find the least-squares linear relationship and estimate how long it takes to fully season the timber. Fully seasoned timber has a moisture content of about 12%.

Moisture, m %	0	1	2	3	4	5	6	7	8	9
Time, t months	129	139	117	114	123	101	85	63	95	47



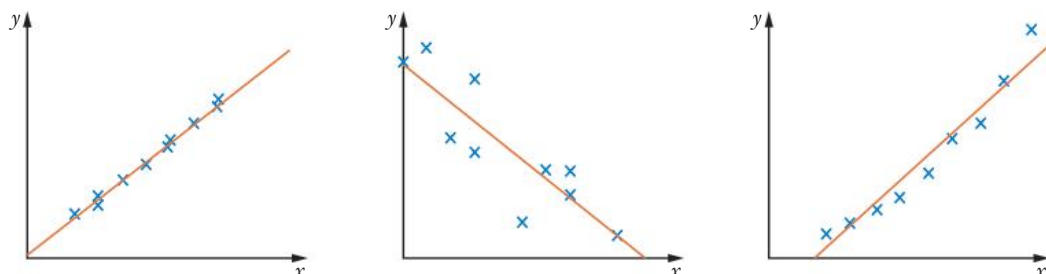
Shutterstock.com/Tony Boggett



Residual plots

4.04 Residual plots

You can see that some least-squares models are better than others.



The first line above is almost perfect. The middle line has 3 points on one side and 4 on the other. The third line has a strange pattern, with the points in the middle below the line and the points on the ends above the line. It looks a bit more like a curve than a straight line.

Consider the data below.

x	3	4	5	6	7	8	9	10	11
y	20.5	16.8	11.2	13.4	12.3	4.6	3.7	0.9	4.3

The least squares line has the equation $y \approx 25.646 - 2.272x$.

The scatterplot and line is shown on the right.

For $x = 3$, the line is at $y \approx 25.646 - 2.272 \times 3 = 18.83$.

For $x = 4$, it is at $y \approx 25.646 - 2.272 \times 4 = 16.558$.

For $x = 5$, it is at $y \approx 25.646 - 2.272 \times 5 = 14.286$, and so on.

You can make a table of the values *predicted* by the least squares line.

x	3	4	5	6	7	8	9	10	11
y	18.83	16.558	14.286	12.014	9.742	7.47	5.198	2.926	0.654

You can work out the distances of the points from the line by subtracting the predictions.

For $x = 3$, the difference is $20.5 - 18.83 = 1.67$.

For $x = 4$, it is $16.8 - 16.558 = 0.242$.

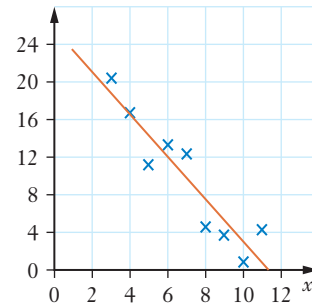
For $x = 5$, it is $11.2 - 14.286 = -3.086$, and so on.

You can make a table of the differences between the points and the line.

x	3	4	5	6	7	8	9	10	11
Difference	1.67	0.242	-3.086	1.386	2.558	-2.87	-1.498	-2.026	3.646

The differences add up to 0.022. If you use greater accuracy, you get very close to 0.

Notice that differences of the points above the line are positive and those below are negative.



Residuals

- A **residual** is the difference between a data point and the value predicted by the line of best fit (least-squares line)
- A plot of the residuals against the explanatory variable is called a **residual plot**

EXAMPLE 7

The data below has a least-squares model of $y \approx 1.843x + 3.718$.

x	2	3	3	4	5	5	5	6	7	7	8	9
y	8.2	8.4	10.2	9.2	12.6	12.3	14.9	14.5	16.3	16.3	19.9	19.8

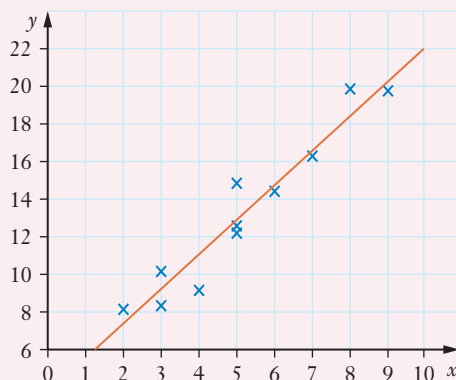
- Calculate the residuals, correct to 2 decimal places.
- Draw the scatterplot and least-squares line.
- Draw the residual plot.
- Comment on the results.

Solution

- For (2, 8.2), find
 $8.2 - (1.843 \times 2 + 3.718)$.
 Repeat for all the other values.

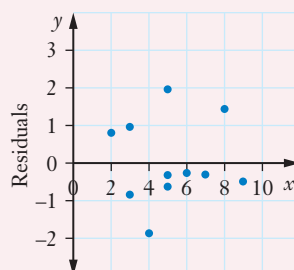
x	2	3	3	4	5	5
Residual	0.8	-0.85	0.95	-1.89	-0.33	-0.63
x	5	6	7	7	8	9
Residual	1.97	-0.28	-0.32	-0.32	1.44	-0.51

- Draw the scatterplot and line.



- Plot the residual values against the x values.

The residuals are all small compared to the y -values.



- Comment on their positions.

The residuals are evenly scattered on both sides of the horizontal axis with no clear pattern.

INVESTIGATION

RESIDUAL PLOTS

Set A

x	0.5	0.8	1	1.1	1.5	1.4	1.8	2	2.1	2.5	2.6	2.9
y	1	0.7	1.1	1.1	1.3	1.1	1.7	2.2	2.7	2.6	3.6	3.9

Set B

x	2	2	3	4	5	5	5	6	6	7	8	8
y	8.6	9.2	9.8	11.5	14.3	11.3	18	21.4	18.5	9.6	12.7	14.4

Set C

x	1	3	4	5	6	8	9	11	12	14	15	16
y	16.1	14.4	14.1	14.2	12.9	19.2	9.2	7.4	7.5	3.8	2.2	1.4

- Find least-squares models for each table of values.
- Draw the scatterplot and the least-squares line for each set.
- Find the residuals for each set.
- Draw the residual plot for each set.

What do you notice about the residual plots?

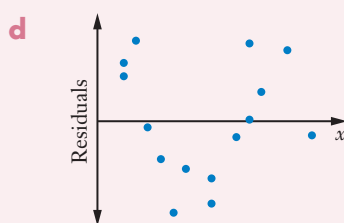
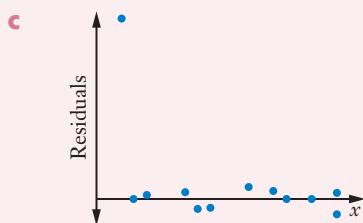
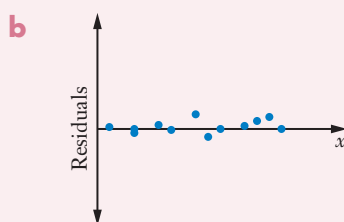
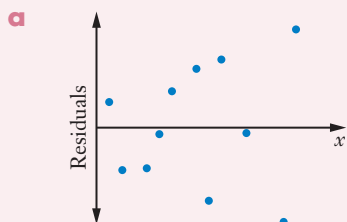
How do the residual plots relate to the scatterplots?

Residual plots and straight-line models

- A residual plot with points scattered close to both sides of the axis shows that the model is appropriate for a straight line
- A residual plot with points in a curved line or with a pattern that could be a curve shows that a straight-line model is not appropriate
- A residual plot with points diverging from the axis in one direction shows that a straight-line model is not appropriate
- A residual plot with one or several outliers indicates that there are outliers in the original data. If they are valid points, the model may not be appropriate

EXAMPLE 8

For each residual plot, state whether a straight-line model is appropriate.



Solution

a Consider the residual points.

Write the answer.

The residuals spread out to the right.

The straight-line model is not appropriate.

b Consider the residual points.

Write the answer.

The points are scattered close to both sides of the axis.

The straight-line model is appropriate.

c Consider the residual points.

Write the answer.

There is an outlier.

If it is a valid point, the model is not appropriate.

d Consider the residual points.

Write the answer.

The points have a pattern that could be a curve.

The straight-line model is not appropriate.

Exercise 4.04 Residual plots

Example
7

1 The data below has a least-squares model of $y \approx 1.5212x + 12.842$.

x	10	11	12	13	14	15	16	17	18	19
y	29	28	33	32	33	35	38	38	42	41

a Calculate the residuals, correct to 2 decimal places.

b Draw the residual plot.

c Comment on the results.

- 2** The data below has a least-squares model of $y \approx 30.473 - 0.5489x$.

x	4	5	6	8	10	11	13	15	16	19
y	28	28	28	26	24	25	23	22	21	21

- Calculate the residuals, correct to 2 decimal places.
- Draw the residual plot.
- Comment on the results.

- 3** The data below has a least-squares model of $y \approx 0.5361x - 8.1186$.

x	24	26	27	29	30	31	34	35	36	38
y	5	8	7	9	1	9	10	10	13	13

- Calculate the residuals, correct to 2 decimal places.
- Draw the residual plot.
- Comment on the results.

- 4** The data below has a least-squares model of $y \approx 0.3445x + 14.049$.

x	7	8	10	11	14	14	15	17	19	20
y	18	15	14	30	5	20	36	-1	3	47

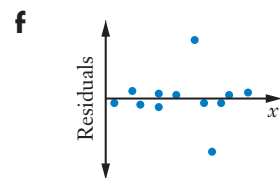
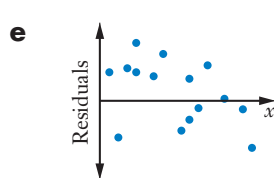
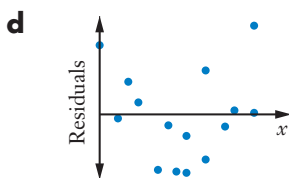
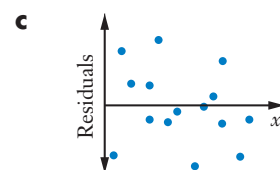
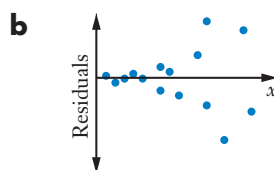
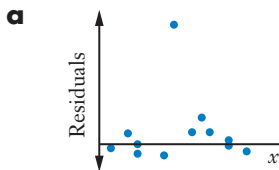
- Calculate the residuals, correct to 2 decimal places.
- Draw the residual plot.
- Comment on the results.

- 5** The data below has a least-squares model of $y \approx 25.619x - 143.02$.

x	4	5	8	9	13	14	15	18	20	22
y	12	33	93	43	147	117	180	303	394	527

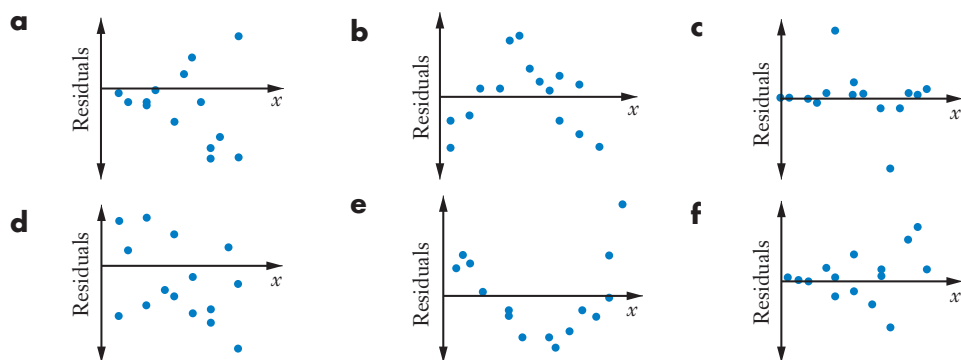
- Calculate the residuals, correct to 2 decimal places.
- Draw the residual plot.
- Comment on the results.

- 6** For each residual plot, state whether or not a straight-line model is appropriate. Assume that all points are valid.



Example
8

- 7** For each residual plot, state whether or not a straight-line model is appropriate. Assume that all points are valid.



Problem solving

For each table of values:

- a** Find the least-squares straight-line model for the data.
b Draw the residual plot and use it to state if the model is appropriate.

8

x	1	2	5	7	10	11	12	14	17	19	21	23
y	25.1	20.5	17.7	17.4	15.8	10.9	21.3	8.8	8.3	6.4	0.9	2

9

x	1	3	5	7	9	11	13	16	16	19	21	22
y	10.8	10.1	13	14.4	21.1	25.8	24.8	26.1	29.9	38	40.7	39.8

10

x	1	1.7	2	2.7	3.2	4.1	4.3	5.1	5.5	6.4	6.7	7.4
y	39.9	36.6	40	40	32.7	35.4	27.6	25.7	23.2	14.2	7.1	0.6



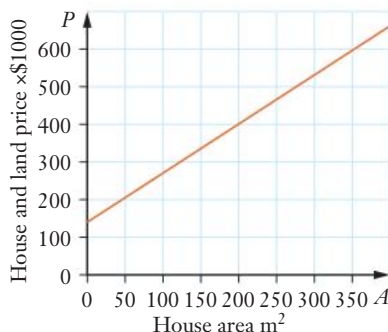
NRL Photos

4.05 The coefficient of determination

Consider the relationship $P = 1.3A + 140$, where A is in square metres and P is in thousands of dollars. This was the relationship for the prices of new house-and-land packages in the outer suburbs of Brisbane in 2018. A is the floor area of the house.

If you draw the relationship, it looks like the graph on the right.

The values of 1.3 and 140 in the relationship are the slope and P -intercept (the value when $A = 0$) of the line respectively.



Coefficient of determination

The P -intercept of 140 means that it would cost about \$140 000 to buy a house-and-land package with a floor area of 0 m^2 . In other words, \$140 000 is the cost of the land and service connections. The slope of 1.3 means that as the house size increases, it costs about \$1300 for every extra square metre.

Straight-line relationship coefficients

For the straight line $y = bx + a$,

- b is the **slope (gradient)** of the line. It shows the rise or fall in the value of y for every horizontal step of 1 unit
- a is the y -intercept. It shows the value of y when the value of x is 0
- For any real relationship, the interpretation depends on what the variables represent

EXAMPLE 9

The relationship between the time and cost of hiring a small car in a regional Queensland town is given by $C = 78d + 205$, where d is the number of days and \$ C is the cost of hire.

- What are the slope and intercept of the relationship?
- What do the values mean?

Solution

- Write the general formula for the equation with this relationship.

$$y = bx + a$$

$$C = 78d + 205$$

Identify the values of a and b .

$$b = 78, a = 205$$

Write the answer.

The slope is 78 and the intercept is 205.

- Write the meaning of the values.

There is a fixed charge of \$205 to hire a car, then \$78 needs to be paid for every day that the car is hired for.

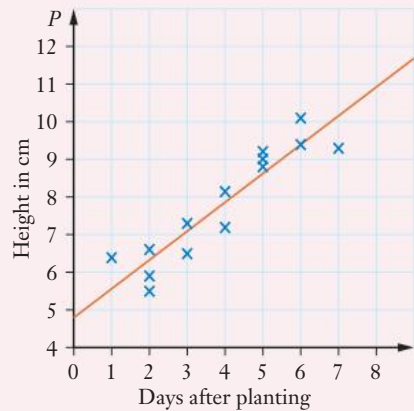


EXAMPLE 10

The least-squares line for the growth of seedlings over a week is shown on the right.

Seedlings that died have been ignored.

- Find the slope and intercept of the line.
- Write the equation of the line.
- What do the slope and intercept mean?



Solution

- Choose 2 points on the line.

(2, 6.4) and (7, 10.2) are on the line.

Work out the slope and y -intercept.

$$\text{Slope} = \frac{10.2 - 6.4}{7 - 2} = \frac{3.8}{5} = 0.76$$
$$y\text{-intercept} \approx 4.8$$

- Use $y = bx + a$.

The equation is $h = 0.76d + 4.8$

- Write the meaning of the slope.

The seedlings grew about 0.76 cm per day.

Write the meaning of the y -intercept.

The seedlings were about 4.8 cm high when first planted.

In Chapter 1, you used the correlation coefficient r to measure the strength of an association. You can use the square of the correlation coefficient, r^2 or R^2 , to measure how well a relationship explains the variables.

The coefficient of determination

- The value of R^2 (r^2) is called the **coefficient of determination**
- R^2 measures how much the response variable is determined by the explanatory variable. It is the proportion of the response variable that changes in the same way as the explanatory variable
- $R^2 = 0$ means that the points are randomly scattered and the response variable has no relationship with the explanatory variable
- $R^2 = 1$ means that the points are exactly on the line, so the explanatory variable completely determines the value of the response variable

EXAMPLE 11

The coefficient of determination between the English and Mathematics grades of Year 12 students is about 0.45. Explain what this means.

Solution

Give the numerical meaning.

The English and Mathematics grades have 45% in common. Looking at it the other way around, 55% of the English grade is *not* determined by the Mathematics grade.

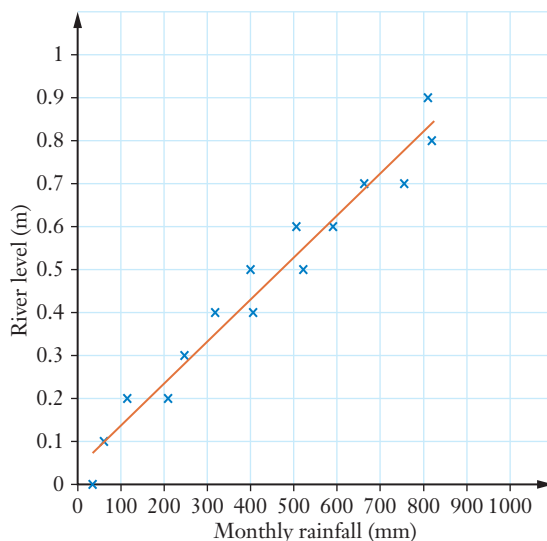
Interpret the mathematical meaning.

Most of the English and Mathematics grades are independent, but there is something else that helps increase both of them.

There is a common factor in all test achievement. This is sometimes called the *g*-factor for *general* ability.

Exercise 4.05 The coefficient of determination

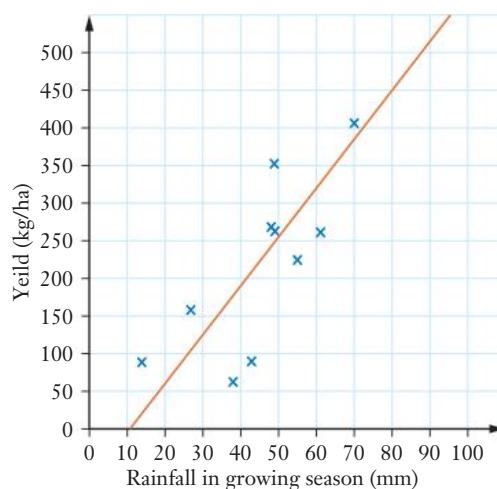
- 1 The relationship between the weight of pumpkins and their market price over six months is given $P = 0.7w + 0.85$, where w is in kilograms and P is in dollars.
 - a What are the slope and intercept of the relationship?
 - b What do the values mean?
- 2 The age and weight of men between the ages of 20 and 60 has the relationship $w = 0.24a + 64$, where a is in years and w is in kilograms.
 - a What are the slope and intercept of the relationship?
 - b What do the values mean?
- 3 The scatterplot and line shows the amount of rain in a month and the level of the river in a Queensland town.
 - a Find the gradient and y -intercept of the line.
 - b Write the equation of the line.
 - c What do the slope and y -intercept mean?



Example
9

Example
10

- 4 The rainfall and yield of peanut crops in Queensland is shown in the graph.
- Find the gradient and y -intercept of the line.
 - Write the equation of the line.
 - What do the slope and y -intercept mean?



Example
11

- 5 Explain the mathematical meaning of each coefficient of determination.
- 0
 - 0.2
 - 0.5
 - 0.8
 - 1

Problem solving

- 6 The least-squares line of best fit was worked out for the populations and areas of a selection of countries. The equation of the line was $P = 0.000\ 009A + 49.7$, where A is the area in square kilometres and P is the population in millions. The value of R^2 is 0.26. What does this mean?



Interpolation
and
extrapolation

4.06 Interpolation and extrapolation

In the last section, the y -intercept of a fitted line was sometimes below the values of the data. When you find the y -intercept in those cases, you are outside the range of the data. This means that the y -intercept is actually a prediction. You can use a fitted line to find values that 'average' the data or go outside the range of the data.

Interpolation and extrapolation

- Interpolation** means using a fitted line to predict a value within the range of the data
- Extrapolation** means using a fitted line to predict a value outside the range of the data
- Extrapolation can be invalid for values too different from the data

EXAMPLE 12

The age and weight of Australian men between the ages of 20 and 60 has the relationship $w = 0.24a + 64$, where a is in years and w is in kilograms. Use the equation to predict the average weight of a:

- a** 35-year-old male
- b** 50-year-old male
- c** 10-year-old male
- d** 65-year-old male
- e** 40-year-old Tongan male

Solution

- a** Write the relationship.

Substitute $a = 35$ and calculate the result.

Write the answer.

$$\begin{aligned}w &= 0.24a + 64 \\&= 0.24 \times 35 + 64 \\&= 72.4\end{aligned}$$

The average weight of a 35-year-old male is about 72 kg.

- b** Write the relationship.

Substitute $a = 50$ and calculate the result.

Write the answer.

$$\begin{aligned}w &= 0.24a + 64 \\&= 0.24 \times 50 + 64 \\&= 76\end{aligned}$$

The average weight of a 50-year-old male is about 76 kg.

- c** Write the relationship.

Substitute $a = 10$ and calculate the result.

Write the answer.

$$\begin{aligned}w &= 0.24a + 64 \\&= 0.24 \times 10 + 64 \\&= 66.4\end{aligned}$$

The average weight of a 10-year-old male is about 66 kg. See note next page.

- d** Write the relationship.

Substitute $a = 65$ and calculate the result.

Write the answer.

$$\begin{aligned}w &= 0.24a + 64 \\&= 0.24 \times 65 + 64 \\&= 79.6\end{aligned}$$

The average weight of a 65-year-old male is about 80 kg.

- e** Write the relationship.

Substitute $a = 40$ and calculate the result.

Write the answer.

$$\begin{aligned}w &= 0.24a + 64 \\&= 0.24 \times 40 + 64 \\&= 73.6\end{aligned}$$

The average weight of a 40-year-old Tongan male is about 74 kg. See note next page.

In Example 12, the interpolated weights for the 35-year-old and 50-year-old are probably correct. The extrapolated weight for the 65-year-old is also probably right. However, the value for the 10-year-old is almost certainly wrong. It is outside the data range and you know that the growth rate of a human from birth to about 18 years old is very different from their growth after 20 years of age. In addition, the weight for the Tongan might be incorrect because the data is for Australian males. Typical Tongans might have a different build to typical Australians.

EXAMPLE 13

The relationship between the weight of pumpkins and the market price is $P = 0.7w + 0.85$, where w is in kilograms and P is in dollars. Use the relationship to predict the prices of the following pumpkins and state any problems with your predictions.

- a A 8.3 kg Windsor Black
- b A 9.5 kg Queensland Blue
- c A 682 kg VanKooter



Getty Images/BruceBlock

Solution

- a Write the relationship.

Substitute $w = 8.3$ and calculate the result.

Write and assess the answer.

$$\begin{aligned} P &= 0.7w + 0.85 \\ &= 0.7 \times 8.3 + 0.85 \\ &= 6.66 \end{aligned}$$

This is within the normal range of pumpkins, so the price is about \$6.70.

- b Write the relationship.

Substitute $w = 9.5$ and calculate the result.

Write and assess the answer.

$$\begin{aligned} P &= 0.7w + 0.85 \\ &= 0.7 \times 9.5 + 0.85 \\ &= 7.5 \end{aligned}$$

This is within the normal range of pumpkins, so the price is about \$7.50.

- c Write the relationship.

Substitute $w = 682$ and calculate the result.

Write and assess the answer.

$$\begin{aligned} P &= 0.7w + 0.85 \\ &= 0.7 \times 682 + 0.85 \\ &= 478.25 \end{aligned}$$

This must be one of the giant pumpkins grown for competitions. It wouldn't be sold, so there is no price.

An incorrect extrapolation can be obvious, even without knowing the data range.

Exercise 4.06 Interpolation and extrapolation

- 1 There is a census in Australia every 5 years, in those years ending in a 1 or 6. The population in the census years from 1971 to 2016 is given by $P = 0.239y - 458$, where y is the year and P is the population in millions. Use the relationship to predict the population in:
 - a 2000 b 2013 c 2020 d 2030 e 2050
- 2 The moisture content of timber at a mill was checked. It was tested each month while drying 2 to 9 months after milling. The relationship was $w = 144 - 9.3t$, where t = time in months, w = moisture as a percentage of dry weight and $R^2 = 0.72$. The moisture content of timber should be about 12% of dry weight for it to be used.
 - a What was the moisture content after $3\frac{1}{2}$ months?
 - b What was the moisture content after 1 month?
 - c What was the moisture content when it was milled?
 - d Predict the moisture content after 12 months.
 - e Predict the moisture content after 2 years. Is there any problem with this prediction?
- 3 There is a relationship between the temperature and humidity in a North Queensland town. It is given by $h = 4.1T - 50$, where T is the temperature in $^{\circ}\text{C}$ and h is the percentage humidity. The data used was the average temperature and humidity each month of the year. $R^2 = 0.82$. Predict the humidity on days with the following temperatures and state any problems.
 - a 30°C b 25°C c 40°C d 0°C
- 4 The cost of building an average house on a flat block in south-east Queensland in 2017 was given by $C = 1700a - 100\,000$, where a is the area in square metres and C is the cost in dollars. $R^2 = 0.77$. The data used was for houses from 150 m^2 to 500 m^2 . Predict the costs for houses with the following areas and state any problems.
 - a 200 m^2 b 400 m^2 c 100 m^2 d 900 m^2

Problem solving

- 5 The data below shows the highest frequency of sounds that some people between the ages of 25 and 70 could hear.

Age	25	31	37	43	48	51	57	62	70
$f(\text{kHz})$	25	21	22	21	18	18	16	19	16

Use the data to predict the highest frequency that people of ages 15, 45 and 90 could hear and state any problems with your predictions.

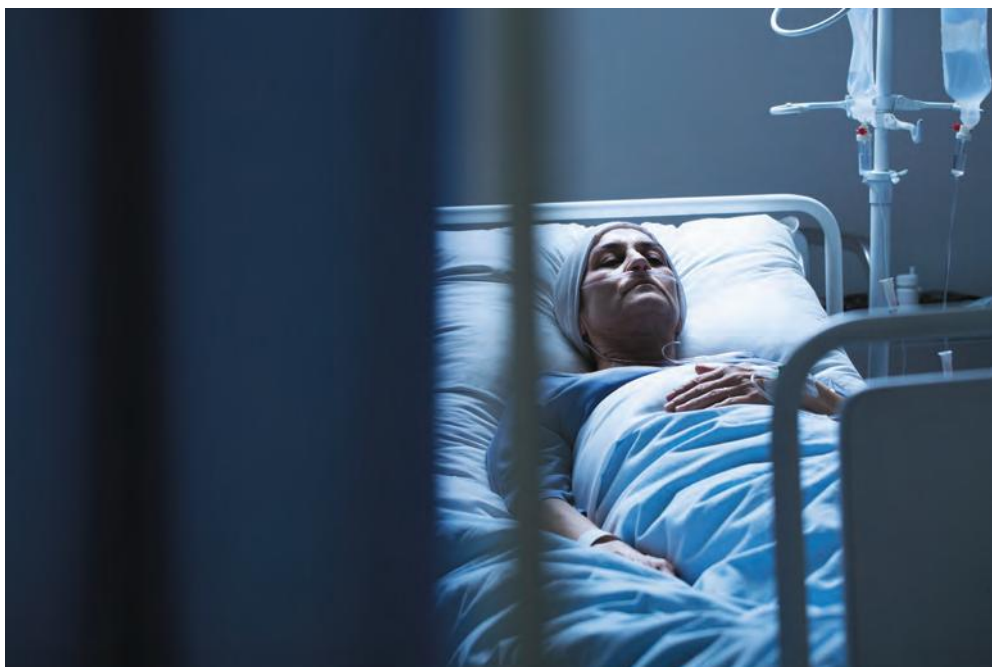
4.07 Association and causation

People use statistics to find associations between different variables. This is in the hope of finding what causes a particular response variable. For example, circulatory diseases kill or disable many Australians each year. Doctors and medical researchers spend a lot of time and money on research in the hope finding the causes.

Over 60 years ago, doctors found a strong association between smoking and the chance of getting lung cancer. It wasn't until much later that the changes in the lung cells produced by smoking showed that smoking actually causes lung cancer.

There is a very strong relationship between the number of deaths in Australia and the total amount of cash and savings held by the Australian population. The equation is $S = 26.3D - 3136$, with $R^2 = 0.975$. D is the annual number of deaths in thousands and S is the amount held in billions. The data used was from the years 2000–2016. It is obvious that deaths do not cause people to save more. However, the very high coefficient of determination suggests there must be some link. The simple explanation is that both are linked to the population. As the population rises, so do the number of deaths and the total of household cash and savings.

Unfortunately, a strong association or relationship between two variables *does not mean* that one causes the other.



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Association and causation

- An association between variables does not mean that one causes the other
- A **mechanism** is needed to show causation. There have to be specific steps between a change in one variable and the change in the other
- A third variable for which they are both responses is called a **confounding variable**. You can say they are **confounded** by the common explanatory variable
- Associations can also be a **coincidence**, particularly over a short time period

The relationship between household savings and deaths is confounded by the population.

EXAMPLE 14

Over the period from 1950 to 1980 there was a very strong relationship between the consumption of carrots in Australia and the rate of lead poisoning.

However, from 2000 to now, the association has markedly declined in strength.

- a** State if you think this is likely to be a causal relationship.
- b** Give detailed reasons for your answers, including reasons for the decline.

[Hint: Before 1980, lead was used as a petrol additive.]



Shutterstock.com/Tish1

Solution

- a** Perhaps both are linked to something that increased in the period.
- b** Car ownership increased over the period. In the 1950s, most households had at most one car.

It seems unlikely that there is a causal relationship in either direction.

It is likely that the population and the increase in the rate of car ownership while they used lead is a confounding variable.

EXAMPLE 15

Since about 1990, the populations of frogs and honey bees in Europe have both declined. There is an association between the populations. Identify some non-causal reasons that could account for this association.

Solution

The size of cities has increased all over the world.

Both populations could have been affected by habitat destruction. This would be a confounding variable.

The number of people has increased.

They could both have declined as a result of the increase in human activity. The human population would be a confounding variable.

They may have both been affected by infectious diseases.

Both having diseases in the population would be a coincidence.

If there is no sensible causal link and no confounding variable, you should suspect a coincidence.



Getty Images/Natali_Mis

Exercise 4.07 Association and causation

- 1 In Tasmania, since the 1990s, there has been a decrease in the population of Tasmanian devils and an increase in the planting of pyrethrum daisies. These have a strong negative association.
 - a State if you think there is a causal association.
 - b Suggest some possible reasons for the association.

Examples
14, 15



Shutterstock.com/Susan Flashman

Problem solving

- 2 There is a strong association between the number of vehicles and the number of doctor consultations in Australia. Identify some possible reasons.
- 3 There is a strong association between the number of industrial accidents and the amount of coal exported from Australia. Identify some possible reasons.
- 4 Over the years since 1980, there has been an association between the rates of serious cases of AIDS and influenza. Identify some possible reasons.
- 5 There is a negative association between the increase in cattle numbers in Australia and the loss of rainforest. Identify some possible reasons.
- 6 There is a negative association between the rate of stomach ulcer cases and the rate of sports injuries in Australia since about 1980. Identify some possible reasons.

4.08 Applications of associations

In Chapter 1, you used **contingency tables** (percentaged two-way tables) to identify associations between categorical variables. In this chapter, you have used scatterplots to identify associations between numerical variables. The aim of both displays is to find how changes in one variable affect another. Both techniques are used for research in science, economics, medicine and other areas. It is important to remember that a statistical finding does not *prove* that a change in one variable causes a change in another.

EXAMPLE 16

The table records the occurrence of skin cancer among a group of older people with different skin types.

	Light skin	Dark skin
Cancer	31	34
No skin cancer	65	280

Analyse the table to find any significant differences between skin types.

Solution

Find the row and column totals.

	Light	Dark	Totals
Cancer	31	34	65
No cancer	65	280	345
Totals	96	314	410

You are interested in the cancers for different skin types, so find the column percentages in the contingency table.

	Light	Dark	Totals
Cancer	32.3%	10.8%	
No cancer	67.7%	89.2%	
Totals	100%	100%	

Compare the percentages for different skin types.

32.3% of people with light skin had a skin cancer, but only 10.8% of those with dark skin had one.

Work out a probability comparison.

$$32.3\% \div 10.8\% \approx 3$$

Write the answer in a simple form.

People with light skin were 3 times as likely to get skin cancer as those with dark skin.

Summary statements like the answer to Example 16 help people understand statistical evidence. It might also help to make a statement such as: 'Light-skinned people have a 32% chance of developing a skin cancer by age ...'

Numerical analysis of data also lets you make predictions.

EXAMPLE 17

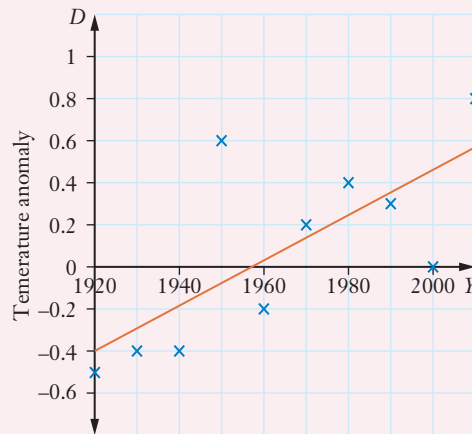
The difference, D , in $^{\circ}\text{C}$ between the average surface temperature and the long-term average surface temperature in a northern coastal region of Queensland every 10 years is shown below.

Y	1920	1930	1940	1950	1960	1970	1980	1990	2000	2010
D	-0.5	-0.4	-0.4	0.6	-0.2	0.2	0.4	0.3	0	0.8

Use the table to find a relationship for the data and predict the average surface temperature change by 2100.

Solution

Draw a scatterplot and find the least-squares line of best fit.



Find the least-squares line of best fit.

$$D = 0.0108Y - 21.118, \text{ with } R^2 = 0.52$$

Use the equation to predict the temperature variation in 2100.

$$= 0.0108 \times 2100 - 21.118$$

$$\approx 1.6$$

Write the answer.

On current trends, the average surface temperature will be 1.6° higher in 2100.

Exercise 4.08 Applications of associations

Problem solving

Example
16

- 1** A survey of whether or not people in Queensland have asthma is shown.

Asthma	Male	Female
Yes	23	25
No	214	181

Analyse the table to find any significant differences between males and females.

- 2** The table below shows the results of an Australian survey in 2007 of whether or not people were overweight or obese.

	Not overweight	Overweight	Overweight but not obese
Males	94	89	53
Females	108	75	42

Analyse the table to find any significant differences between males and females.

Example
17

- 3** The numbers of people in Australia with dementia over 2011–2017 is shown.

Year	2011	2012	2013	2014	2015	2016	2017
People (thousands)	102	104	111	113	115	117	124

Use the data to predict the numbers of people with dementia in 2050.

- 4** Does the data in question 2 suggest that the rate of dementia is increasing?
- 5** The number of mobile phone users in Australia in millions between 2011 and 2017 is shown in the table.

Year	2011	2012	2013	2014	2015	2016	2017
Phone users	16.6	17.5	17.5	18.2	18.6	19	19.4

Use the data to predict the number of users in 2030 and discuss any problems with the prediction.

- 6** The Australian population in millions over the period from 2011 to 2017 is shown.

Year	2011	2012	2013	2014	2015	2016	2017
Population	22.3	22.7	23.1	23.5	23.9	24.2	24.6

Use the data with that in question 5 to investigate the rate of mobile phone ownership in Australia. Do your results suggest that mobile phone sales in Australia are mostly about replacement? Are there any problems with your results?

4. CHAPTER SUMMARY

Measuring and explaining associations

Variables and associations

- A **response variable** (**dependent variable**) is changed by another variable
- An **explanatory variable** (**independent variable**) is not changed by another variable. The response variable could depend on the explanatory variable(s)
- A scatterplot that slopes down from left to right suggests a **negative** association
- A scatterplot that slopes upwards from left to right suggests a **positive** association
- A **linear** association is a straight line
- A **strong** association has points that make an almost perfect line
- A **moderate** association has an upward or downward trend that can be shown by a straight line through the points. Almost all the points are close to such a line
- A **weak** association has a pattern, but no clear line
- A moderate or strong linear association can be modelled by a straight-line **relationship** between the variables

Straight-line models

- **To find a least-squares model**
 - find the total of all the x scores, shown as $\sum x$
 - find the total of all the y scores, shown as $\sum y$
 - find the total of all the products of x and y , shown as $\sum xy$
 - find the total of all the squares of the x scores, shown as $\sum x^2$
 - the equation of the line is $y = bx + a$, where b = slope and a = y -intercept
 - the slope is given by $b = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = r \frac{s_y}{s_x} = \frac{n \sum xy - \sum x \sum y}{n \sum (x^2) - (\sum x)^2}$
 - the y -intercept is given by $a = \frac{\sum y - b \sum x}{n} = \bar{y} - b\bar{x}$
- You can use a scientific calculator to find a least-squares model

Residuals

- A **residual** is the difference between a data point and the value predicted by the line of best fit (least-squares line)

- A plot of the residuals against the explanatory variable is called a **residual plot**
- A residual plot with points scattered close to both sides of the axis shows that the model is appropriate for a straight line
- A residual plot with points in a curved line or points in a line on one side of the axis shows that a straight-line model is not appropriate
- A residual plot with points diverging from the axis in one direction shows that a straight-line model is not appropriate
- A residual plot with one or several outliers indicates that there are outliers in the original data. If they are valid points, the model may not be appropriate

Meaning and causation

- For the straight line $y = bx + a$
 - b is the slope (gradient) of the line. It shows the rise or fall in the value of y for every horizontal step of 1 unit
 - a is the y -intercept. It shows the value of y when the value of x is 0
 - For any real relationship, the interpretation depends on what the variables represent
- The value of R^2 (or r^2) is called the **coefficient of determination**
- R^2 measures how much of the response variable is determined by the explanatory variable. It is the proportion of the response variable that is due to the explanatory variable
- $R^2 = 0$ means that the points are randomly scattered and the response variable has no relationship with the explanatory variable
- $R^2 = 1$ means that the points are exactly on the line, so the explanatory variable completely determines the value of the response variable

Prediction and validity

- **Interpolation** means using a fitted line to predict a value within the range of the data
- **Extrapolation** means using a fitted line to predict a value outside the range of the data
- Extrapolation can be invalid for values too different from the data
- An association between variables does not mean that one causes the other
- A **mechanism** is needed to show causation. There have to be specific steps between a change in one variable and the change in the other
- A third variable for which they are both responses is called a **confounding** variable. You can say they are **confounded** by the common explanatory variable
- Associations can also be a **coincidence**, particularly over a short time period
- **Contingency tables** (percentaged two-way tables) are used to identify associations between categorical variables

4. CHAPTER REVIEW

Measuring and explaining associations

1 For each pair of variables, identify the response variable and the explanatory variable.

- a Dryness of clothes, time hanging in the sun
- b Diameter of pipe, maximum flow of water
- c Top speed of car, size of motor

Example
1

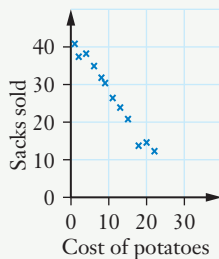
2 Which one of these variables cannot be a response variable?

- A The population of Australia
- B The most popular type of coffee in Australia
- C The number of 19–24 year-olds in Australia who are unemployed
- D The size of the Australian economy
- E The total amount of money held by banks in Australia

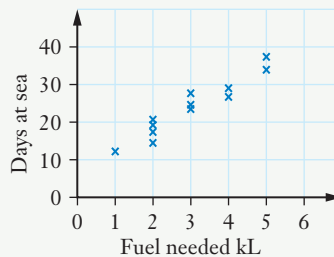
Example
1

3 State whether each scatterplot has been correctly drawn.

a



b



Example
2

4 Draw a scatterplot of Queensland car sales and housing costs to determine if there is a relationship.

Examples
3, 4

Year	2004	2006	2008	2010	2012	2014	2016
Car sales '000	940	971	1007	1013	1060	1122	1175
Housing \$/w	215	238	261	273	287	292	290

5 The values of 2 variables are shown in the table below.

Example
5

x	1	3	5	6	8
y	31	56	76	103	121

- a Find the least-squares linear relationship without using the statistics mode of your calculator.
- b Draw the scatterplot and the least-squares line.

Example
6

- 6** The values of two variables are shown in the table below.

x	3	4	6	7	8	9	11	12	13	15
y	53	50	48	50	49	47	41	42	37	35

- Use your calculator to find the least-squares linear relationship.
- Draw the scatterplot and the least-squares line.

Example
7

- 7** What are the residuals from a scatterplot and line of best fit?
- The values that are not on the straight line
 - The values that are on the straight line
 - The distances between successive points that are not on the straight line
 - The horizontal distances between the points and the line
 - The vertical distances between the points and the line

Example
7

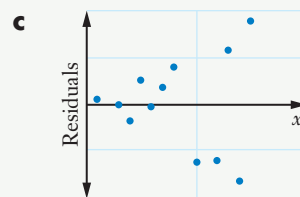
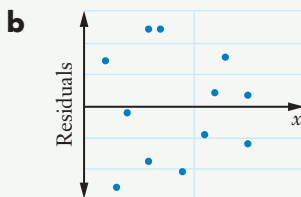
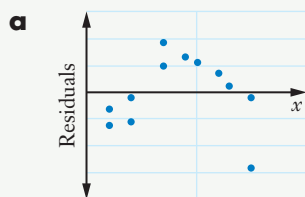
- 8** The data below has a least-squares model of $y \approx 1.843x + 3.718$.

x	1	2	4	5	7	8	10	11	12	13
y	5	11	17	11	20	20	20	22	31	29

- Calculate the residuals, correct to 2 decimal places.
- Draw the residual plot.
- Comment on the results.

Example
8

- 9** For each residual plot, state whether or not the straight-line model is appropriate.



Example
9

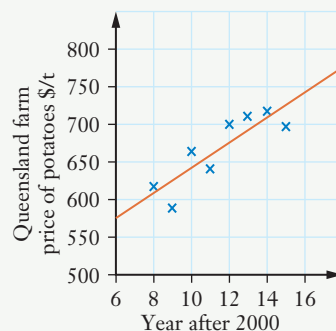
- 10** The relationship between the time a repairer spends fixing a washing machine and the cost of the call is given by $C = 1.4T + 125$, where T is the time in minutes and C is the cost of the call in dollars. Parts are additional.

- What are the slope and intercept of the relationship?
- What do the values mean?

Example
10

- 11** The least-squares line of farm prices of potatoes in \$/tonne is shown. The equation of the line is $y = 17x + 425$

- What is the slope and y -intercept of the line?
- What do the slope and y -intercept mean?



5.

ANALYSING TIME SERIES DATA

The weather changes from summer to autumn to winter to spring and back to summer every year. You know that an unusually warm day in January is much hotter than one in June. How can you tell which is the more unusual? The weather bureau has records of the average temperature for each month. You could just look at the average monthly maximums.

Some vegetables won't grow in Queensland in summer but you can still buy them fresh. They are brought in from southern states or overseas so they cost more. This is why the prices of vegetables change with the seasons.

In Chapter 3, you learned to recognise different patterns in time series data. In this chapter, you will learn how to predict quantities that are seasonal or have a trend.

- 5.01 Seasonal indices
- 5.02 Deseasonalisation
- 5.03 Using seasonal indices
- 5.04 Modelling long-term trends
- 5.05 Applications of time series data
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Analysing time series data

- calculate seasonal indices by using the average percentage method
- deseasonalise a time series by using a seasonal index, including the use of spreadsheets to implement this process
- fit a least-squares line to model long-term trends in time series data, using appropriate technology
- solve practical problems that involve the analysis of time series data



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

December quarter
June quarter
March quarter
September quarter

deseasonalisation
least-squares method
quarter
slope

intercept
linear trend
seasonal index
trend

5.01 Seasonal indices

A seasonal time series varies according to the time of year. You can work out the variation by looking at previous years. You put the data into a usable form by making a **seasonal index**.

Seasonal index construction from one year

Follow the steps below to make a seasonal index from monthly data for one year.

- Find the average for the year
- Divide the amount for each month by the average for the year

You can express the index as a decimal or as a percentage.

EXAMPLE 1

The monthly values of a variable for one year are shown in the table.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	635	618	563	477	396	351	359	332	404	479	552	601

Find the monthly index.

Solution

Find the average for the year.

$$\begin{aligned}\text{Total} &= 635 + 618 + 563 + 477 + \dots \\ &= 5767\end{aligned}$$

$$\text{Average} = \frac{5767}{12} \approx 480.6$$

Divide the value for each month by the average.

$$\text{Jan} = \frac{635}{480.6} \approx 1.32$$

$$\text{Feb} = \frac{618}{480.6} \approx 1.29$$

$$\text{Mar} = \frac{563}{480.6} \approx 1.17$$

$$\text{Apr} = \frac{477}{480.6} \approx 0.99$$

$$\text{May} = \frac{396}{480.6} \approx 0.83$$

$$\text{Jun} = \frac{351}{480.6} \approx 0.73$$

$$\text{Jul} = \frac{359}{480.6} \approx 0.75$$

$$\text{Aug} = \frac{332}{480.6} \approx 0.69$$

$$\text{Sep} = \frac{404}{480.6} \approx 0.84$$

$$\text{Oct} = \frac{479}{480.6} \approx 1.00$$

$$\text{Nov} = \frac{552}{480.6} \approx 1.15$$

$$\text{Dec} = \frac{601}{480.6} \approx 1.25$$

Write the answer.

J	F	M	A	M	J	J	A	S	O	N	D
1.32	1.29	1.17	0.99	0.83	0.73	0.75	0.69	0.84	1.00	1.15	1.25

Example 1 looks like there might be a ‘blip’ in July. The value is higher than both June and August, which is against the trend. You can even out variations by working out the index over several years.

Seasonal index

Follow the steps below to make a seasonal index from data for more than one year.

- Find the average for each year
- Divide the amount for each quarter or month by the average for its year
- Find the average for each quarter or month

Example 2 shows the construction of a quarterly index from data over 3 years. The **quarters** of a year are as below:

Quarter 1 (**Q1**): January to March, the **March quarter**

Quarter 2 (**Q2**): April to June, the **June quarter**

Quarter 3 (**Q3**): July to September, the **September quarter**

Quarter 4 (**Q4**): October to December, the **December quarter**

EXAMPLE 2

This table shows the quarterly values of a variable over 3 years.

2015	1st	2nd	3rd	4th
Value	763	754	724	670

2016	1st	2nd	3rd	4th
Value	783	759	721	684

2017	1st	2nd	3rd	4th
Value	781	761	738	693

Find the quarterly index.

Solution

Find the averages for each year.

$$2015 \text{ average} = \frac{763 + 754 + 724 + 670}{4} = 727.75$$

$$2016 \text{ average} = \frac{783 + 759 + 721 + 684}{4} = 736.75$$

$$2017 \text{ average} = \frac{781 + 761 + 738 + 693}{4} = 743.25$$

Find 2015s quarterly index values.

$$\frac{763}{727.75} \approx 1.048, \quad \frac{754}{727.75} \approx 1.036,$$

$$\frac{724}{727.75} \approx 0.995, \quad \frac{670}{727.75} \approx 0.921$$

Find 2016s quarterly index values.

$$\frac{783}{736.75} \approx 1.063, \quad \frac{759}{736.75} \approx 1.030,$$

$$\frac{721}{736.75} \approx 0.979, \quad \frac{684}{736.75} \approx 0.928$$

Find 2017s quarterly index values.

$$\frac{781}{743.25} \approx 1.051, \quad \frac{761}{743.25} \approx 1.024,$$

$$\frac{738}{743.25} \approx 0.993, \quad \frac{693}{743.25} \approx 0.932$$

Find the averages of each quarterly value.

$$1\text{st quarter average} = \frac{1.048 + 1.063 + 1.051}{3} = 1.054$$

$$2\text{nd quarter average} = \frac{1.036 + 1.030 + 1.024}{3} = 1.03$$

$$3\text{rd quarter average} = \frac{0.995 + 0.979 + 0.993}{3} = 0.989$$

$$4\text{th quarter average} = \frac{0.921 + 0.928 + 0.932}{3} = 0.927$$

Write the answer.

Quarter	1st	2nd	3rd	4th
Index	1.054	1.03	0.989	0.927

INVESTIGATION

SEASONAL VARIATION

For the data in Example 2:

- work out the averages of each quarter for the 3 years
- find the average of the quarterly averages
- find the index for the combined data.
- What do you find?
- Which approach is easier?

Exercise 5.01 Seasonal indices

- 1 The quarterly values of a variable for 2010 are shown.

Quarter	1st	2nd	3rd	4th
Value	480	586	608	537

Find the quarterly index.

- 2 The quarterly values of a variable for 2015 are shown.

Quarter	1st	2nd	3rd	4th
Value	246	235	223	242

Find the quarterly index.

- 3 The monthly values of a variable for one year are shown.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	234	214	220	207	222	227	216	239	248	253	263	255

Find the monthly index.

- 4 The monthly values of a variable for one year are shown.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	125	128	145	148	165	175	163	170	154	151	130	138

Find the monthly index.

Example
1

- 5 The quarterly values of a variable over 2 years are shown.

2016	1st	2nd	3rd	4th
Value	451	419	405	454

2017	1st	2nd	3rd	4th
Value	462	424	405	449

Find the quarterly index.

- 6 The quarterly values of a variable over 3 years are shown.

2015	1st	2nd	3rd	4th
Value	176	183	174	178

2016	1st	2nd	3rd	4th
Value	193	169	167	179

2017	1st	2nd	3rd	4th
Value	191	167	165	177

Find the quarterly index.

- 7 The quarterly values of a variable over 4 years are shown.

2012	1st	2nd	3rd	4th
Value	91	91	92	101

2013	1st	2nd	3rd	4th
Value	79	90	95	84

2014	1st	2nd	3rd	4th
Value	80	107	101	96

2015	1st	2nd	3rd	4th
Value	87	97	92	93

Find the quarterly index.

- 8 Use the monthly values of the variable shown below for 2 years to find the monthly index.

2017	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	352	381	344	353	326	332	308	324	329	359	359	365

2018	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	378	384	382	340	335	338	343	303	330	322	339	342

5.02 Deseasonalisation

You can use a seasonal index to remove the seasonal variation from a time series. When you do this, you can see the other variations more clearly.

Deseasonalisation

Deseasonalisation is the smoothing of seasonal variation from a time series, completed by dividing each data point by the seasonal index value for that time.



Deseasonalisation



Seasonality

For quarterly data, you need a quarterly index.

EXAMPLE 3

- a Use the index to deseasonalise the time series below.

Quarter	Q1	Q2	Q3	Q4
Index	0.665	1.164	1.292	0.879

2015	Q1	Q2	Q3	Q4
Value	134	237	258	171

2016	Q1	Q2	Q3	Q4
Value	133	235	251	182

- b Make a scatterplot of the original data and the smoothed data on the same axes.
c Comment on the results.

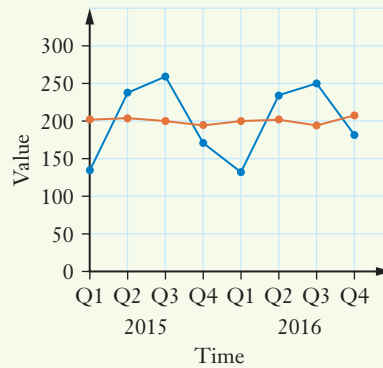
Solution

- a Divide the value for each quarter by the index to get the indexed value.
- $$\frac{134}{0.665} \approx 202, \quad \frac{237}{1.164} \approx 204, \quad \frac{258}{1.292} \approx 200, \quad \frac{171}{0.879} \approx 195$$
- $$\frac{133}{0.665} \approx 200, \quad \frac{235}{1.164} \approx 202, \quad \frac{251}{1.292} \approx 194, \quad \frac{182}{0.879} \approx 207$$

Write the answer.

Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4
202	204	200	195	200	202	194	207

- b Draw the plot.



- c Make your comment.

The deseasonalised series is much smoother than the original.

Deseasonalisation of monthly data uses the same method, but there is more data.

EXAMPLE 4

The monthly index for a time series based on the years 2014–2016 is shown below.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Index	1.115	1.099	1.051	0.998	0.945	0.89	0.892	0.898	0.947	1.009	1.053	1.103

- a** Use the index to deseasonalise the time series below.

2017	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	149	140	133	133	118	119	118	119	126	127	138	140

- b** Make a scatterplot of the original data and the smoothed data on the same axes.
c Comment on the results.

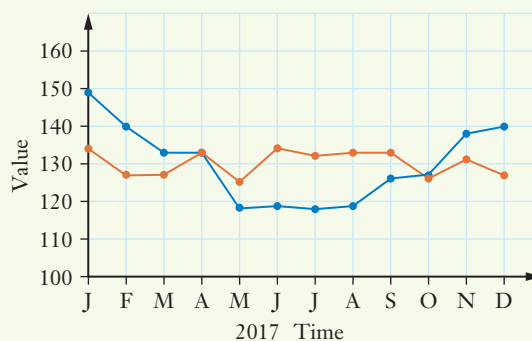
Solution

- a** Use the index to smooth the values
- $$\frac{149}{1.115} \approx 134, \frac{140}{1.099} \approx 127, \frac{133}{1.051} \approx 127, \frac{133}{0.998} \approx 133 \dots$$

Write the answer.

2017	Jan	Feb	Mar	Apr	May	Jun
DS	134	127	127	133	125	134
2017	Jul	Aug	Sep	Oct	Nov	Dec
DS	132	133	133	126	131	127

- b** Draw the plot.



- c** Make a comment.

The deseasonalised values are smoother than the original values.

TECHNOLOGY

Deseasonalisation using a spreadsheet

The index and data for two years are shown below.

Quarter	Q1	Q2	Q3	Q4
Index	0.998	1.035	0.996	0.971

Data

2014	Q1	Q2	Q3	Q4
Value	790	810	785	766

2015	Q1	Q2	Q3	Q4
Value	771	805	783	763

You can use a spreadsheet to deseasonalise the data as follows.

- Type 'Index' into cell A1.
- Type Q1, Q2, Q3 and Q4 into cells A2, B2, C2 and D2.
- Copy the index into cells A3, B3, C3 and D3.
- Type 'Year' into cell A4.
- Type 2014 into cell B4.
- Type Q1, Q2, Q3, and Q4 into cells A5, B5, C5 and D5.
- Type the data for 2014 into cells A6 to D6.
- Type 'Deseasonalised' into cell A7.
- Type '=A6/A3' into cell A8, '=B6/B3' into cell B8, '=C6/C3' into cell C8, '=D6/D3' into cell D8.
- Print your spreadsheet.
- Save your spreadsheet as 'Quarterly deseasoner' in your folder.
- Now type 2015 instead of 2014 and enter the data for 2015 into cells A6 to D6.
- Print your spreadsheet.

	A	B	C	D
1	Index			
2	Q1	Q2	Q3	Q4
3	0.998	1.035	0.996	0.971
4	Year	2015		
5	Q1	Q2	Q3	Q4
6	771	805	783	763
7	Deseasonalised			
8	772.5451	777.7778	786.1446	785.7878

You can use your spreadsheet for quarterly data in the future. You can construct a similar spreadsheet for monthly data.

Exercise 5.02 Deseasonalisation

Example

3

- 1 The quarterly index for a particular variable is shown below.

Quarter	Q1	Q2	Q3	Q4
Index	0.952	1.028	1.043	0.977

- a Use the index to deseasonalise the time series of the same variable below.

2016	Q1	Q2	Q3	Q4
Value	242	265	261	253

2017	Q1	Q2	Q3	Q4
Value	236	262	267	249

- b Make a scatterplot of the original data and the smoothed data on the same axes.
c Comment on the results.

- 2 The quarterly index for a particular variable is shown below.

Quarter	Q1	Q2	Q3	Q4
Index	1.094	0.957	0.888	1.062

- a Use the index to deseasonalise the time series of the same variable below.

2017	Q1	Q2	Q3	Q4
Value	163	142	133	164

2018	Q1	Q2	Q3	Q4
Value	169	142	136	157

- b Make a scatterplot of the original data and the smoothed data on the same axes.
c Comment on the results.

- 3 The quarterly index for a particular variable is shown below.

Quarter	Q1	Q2	Q3	Q4
Index	0.955	1.047	1.045	0.953

- a Use the index to deseasonalise the time series of the same variable below.

2014	Q1	Q2	Q3	Q4
Value	617	677	681	619

2015	Q1	Q2	Q3	Q4
Value	613	679	682	617

- b Make a scatterplot of the original data and the smoothed data on the same axes.
c Comment on the results.

4 The monthly index for a particular variable is shown below.

Month	J	F	M	A	M	J
Index	1.12	1.093	1.061	0.99	0.943	0.905

Month	J	A	S	O	N	D
Index	0.879	0.903	0.934	1.005	1.064	1.101

a Use the index to deseasonalise the time series of the same variable below.

2014	J	F	M	A	M	J	J	A	S	O	N	D
Value	232	233	225	205	199	191	188	192	198	211	220	236

2015	J	F	M	A	M	J	J	A	S	O	N	D
Value	236	235	227	208	201	188	188	188	197	215	227	231

b Make a scatterplot of the original data and the smoothed data on the same axes.

c Comment on the results.

5 The monthly index for a particular variable is shown below.

Month	J	F	M	A	M	J
Index	1.002	0.948	0.904	0.89	0.906	0.952

Month	J	A	S	O	N	D
Index	0.996	1.053	1.097	1.105	1.092	1.055

a Use the index to deseasonalise the time series of the same variable below.

2016	J	F	M	A	M	J	J	A	S	O	N	D
Value	369	341	333	329	326	343	364	382	397	403	402	386

2017	J	F	M	A	M	J	J	A	S	O	N	D
Value	362	342	326	330	329	349	367	383	395	403	400	385

b Make a scatterplot of the original data and the smoothed data on the same axes.

c Comment on the results.

Problem solving

6 Quarterly data for 2 years are shown below. Calculate and apply the seasonal index to all of the data and compare plots of the original and deseasonalised data.

2016	Q1	Q2	Q3	Q4
Value	304	330	305	292

2017	Q1	Q2	Q3	Q4
Value	307	329	303	298

- 7** Quarterly data for 4 years is shown below. Calculate and apply the seasonal index to all of the data and compare plots of the original and deseasonalised data.

2013	Q1	Q2	Q3	Q4
Value	115	105	113	137
2015	Q1	Q2	Q3	Q4
Value	120	106	120	138

2014	Q1	Q2	Q3	Q4
Value	120	100	114	131
2016	Q1	Q2	Q3	Q4
Value	120	103	116	135

- 8** Monthly data for 3 years are shown below. Calculate and apply the seasonal index to all of the data and compare plots of the original and deseasonalised data.

2015	J	F	M	A	M	J	J	A	S	O	N	D
Value	810	858	890	908	889	855	809	758	725	706	729	753
2016	J	F	M	A	M	J	J	A	S	O	N	D
Value	817	863	892	905	903	867	806	761	723	714	723	766
2017	J	F	M	A	M	J	J	A	S	O	N	D
Value	804	864	895	901	893	867	808	761	716	719	732	761

5.03 Using seasonal indices

A purely seasonal pattern is not common. In most cases, there will be trends.

For long-term trends, you can use annual averages. For data over periods of about 5–10 years, it is often useful to deseasonalise before analysing a trend. You should use all the data available to find the seasonal index.



Seasonal indices

EXAMPLE 5

The quarterly sales of sports utility vehicles in Queensland are shown below.

2015	Q1	Q2	Q3	Q4
Sales	19 044	21 281	20 257	20 520
2016	Q1	Q2	Q3	Q4
Sales	20 882	22 565	21 308	20 030
2017	Q1	Q2	Q3	Q4
Sales	20 869	25 051	21 680	21 246

- Find the seasonal index.
- Deseasonalise the data.
- Plot the deseasonalised data.
- If there is any trend, identify what it is.
- Suggest possible reasons for the trend.

Solution

- a** Find the quarterly averages.

Find the average.

Divide each quarter by the average to find the index.

Add to check the index.

Q1 average = 20 265, Q2 average $\approx$ 22 966

Q3 average $\approx$ 21 082, Q4 average $\approx$ 20 599

Average for year = 21 228

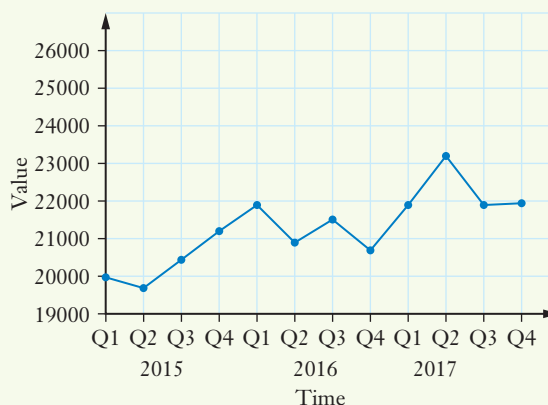
Quarter	Q1	Q2	Q3	Q4
Index	0.955	1.082	0.993	0.970

$0.955 + 1.082 + 0.993 + 0.970 = 4$, so the index is probably correct.

- b** Divide each year by the index to deseasonalise.

2015	Q1	Q2	Q3	Q4
Sales	19 941	19 668	20 400	21 155
2016	Q1	Q2	Q3	Q4
Sales	21 866	20 855	21 458	20 649
2017	Q1	Q2	Q3	Q4
Sales	21 852	23 152	21 833	21 903

- c** Plot the deseasonalised values.



- d** The plot gradually goes upwards. **There is a positive trend.**

- e** State some possible reasons.

You should think of as many possible reasons as you can. The real reasons for a trend may be a combination of factors.

Some possible reasons for the upward trend are:

- the population is increasing
- the economy is improving, so people have more money to spend
- people are buying more cars per family
- sports utility vehicles are becoming more popular.

INVESTIGATION

SIMPLIFYING SEASONAL CALCULATIONS

- Use the data from Example 5 for this investigation.
- Find the grand total of all the data values.
- Find the totals for each quarter.
- Divide the quarterly totals by the grand total.
- Multiply by 4 (because there are 4 quarters in a year).
- What do you find?
- Can you make a list of steps to calculate a seasonal index more easily?



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Exercise 5.03 Using seasonal indices

Example
5

- 1** The quarterly sales of non-passenger vehicles in Queensland are shown below.

2015	Q1	Q2	Q3	Q4
Sales	12 839	15 561	12 756	14 595

2016	Q1	Q2	Q3	Q4
Sales	14 367	16 826	13 793	13 973

2017	Q1	Q2	Q3	Q4
Sales	13 910	18 416	15 494	17 246

- Find the seasonal index.
- Deseasonalise the data.
- Plot the deseasonalised data.
- If there is any trend, identify what it is.
- Suggest possible reasons for the trend.

Problem solving

- 2** Deseasonalise the manufacturing industry profits below to identify any trend.

2015	Q1	Q2	Q3	Q4
Before-tax profit (\$million)	3164	4000	5261	5203
2016	Q1	Q2	Q3	Q4
Before-tax profit (\$million)	2845	4176	5096	5972
2017	Q1	Q2	Q3	Q4
Before-tax profit (\$million)	4533	5177	5685	6690

- 3** Deseasonalise the wages and salaries (millions of dollars) for the manufacturing industry below to identify any trend.

2015	Q1	Q2	Q3	Q4
Wages	12 888	13 384	13 374	13 554
2016	Q1	Q2	Q3	Q4
Wages	12 816	13 176	12 959	13 082
2017	Q1	Q2	Q3	Q4
Wages	12 223	12 960	13 064	13 287



Getty Images/Drazen

- 4** Deseasonalise the following numbers of loans to owner-occupiers to build houses to identify any trend.

	J	F	M	A	M	J	J	A	S	O	N	D
2015	925	1062	1220	1138	1238	1206	1224	1030	1237	1160	1186	1205
2016	913	1025	1138	1173	1211	1259	1082	1243	1351	1243	1347	1181
2017	972	1069	1346	1161	1491	1567	1440	1544	1280	1244	1399	1167

- 5** Deseasonalise the number of unemployed Australian youths below to identify any trend.

Unemployed 15–19-year-olds (thousands)

	J	F	M	A	M	J	J	A	S	O	N	D
2015	334	315	313	283	270	259	277	263	258	249	236	270
2016	301	276	274	257	254	260	269	260	256	247	260	293
2017	294	301	306	269	255	261	262	261	259	240	243	279

5.04 Modelling long-term trends

In Chapter 3, you used a moving average to smooth a time series. In Chapter 4, you calculated the least-squares model for a linear association. The equation of a linear trend can be calculated in the same way.

Linear trend least-squares calculation

- Find the totals and averages of the x and y values: $\sum x$, $\bar{x}$, $\sum y$ and $\bar{y}$
- Find the totals of x^2 and xy : $\sum x^2$ and $\sum xy$
- Find the slope of the line as $b = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = r \frac{s_y}{s_x} = \frac{n \sum xy - \sum x \sum y}{n \sum (x^2) - (\sum x)^2}$
- Find the intercept as $a = \bar{y} - b\bar{x}$
- Write the equation of the line: $y = bx + a$

EXAMPLE 6

The data below shows the values of a variable over 7 years.

Year	2011	2012	2013	2014	2015	2016	2017
Value	264	276	279	286	311	313	322

- Find the least-squares model.
- Predict the value in 2022.
- Is the prediction likely to be accurate?

Solution

- Rewrite the data with 2010 as year zero.

Year	1	2	3	4	5	6	7
Value	264	276	279	286	311	313	322

Find the totals and averages.

$$\text{Sum of years} = 1 + 2 + 3 + \dots = 28$$

$$\text{Average} = \frac{28}{7} = 4$$

$$\text{Sum of values} = 264 + 276 + 279 + \dots = 2051$$

$$\text{Average} = \frac{2051}{7} = 293$$

Find the sum of the squares of the years.

$$\begin{aligned}\sum x^2 &= 1^2 + 2^2 + 3^2 + \dots \\ &= 140\end{aligned}$$

Find the sum of the products.

$$\begin{aligned}\sum xy &= 1 \times 264 + 2 \times 276 + 3 \times 279 + \dots \\ &= 8484\end{aligned}$$

Use the slope formula.

$$\begin{aligned}b &= \frac{n \sum xy - \sum x \sum y}{n \sum (x^2) - (\sum x)^2} \\ &= \frac{7 \times 8484 - 28 \times 2051}{7 \times 140 - 28^2} \\ &= \frac{1960}{196} \\ &= 10\end{aligned}$$

The slope is the same, whatever the starting point. Use the exact values.

$$\begin{aligned}a &= \bar{y} - b\bar{x} \\ &= 293 - 10 \times (2010 + 4), \text{ correcting the average year} \\ &= -19847\end{aligned}$$

Using the year correction makes the calculations much easier.

Write the equation.

$$y \approx 10x - 19847$$

Write the answer.

$$V \approx 10Y - 19847$$

b Use the year equation to predict the value in 2025.

$$\begin{aligned}V &\approx 10Y - 19847 \\ &= 10 \times 2025 - 19847 \\ &= 403\end{aligned}$$

Write the answer.

The value in 2025 will be about 403.

c Comment on the accuracy of the answer.

The data is only over 7 years and 2025 is 8 years further on, so it is well outside the range of the data. It may not be accurate.

You can use your scientific calculator to find the least-squares line.

EXAMPLE 7

The data below shows the value of a variable over 16 years.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Value	106	126	126	135	138	141	154	170
Year	2009	2010	2011	2012	2013	2014	2015	2016
Value	178	204	215	228	233	226	245	251

- a Use your calculator to find the least-squares line.
- b Predict the value in 2025.
- c Is the prediction likely to be accurate?

Solution

a TI Scientific calculator

Press data and delete any old data.

Enter the years and values in the L1 and L2 columns respectively.

Then press **2nd** **DATA** (STAT) and choose 2: 2-Var Stats.

Make XDATA: L1 and YDATA: L2 if they are not already, and use the cursor to go down to CALC and press enter.

Move the cursor down to read the values of the gradient 10.091 and y -intercept $-20\ 888$.

Press **2nd** **MODE** (quit) to exit the statistics calculations.

Casio Scientific calculator

Press **MODE**, 2: STAT and 2: A+BX.

Delete any old data.

Enter the years and values in the X and Y columns respectively,

Press **AC**, then press **SHIFT** 1 (STAT), 5: Reg, 3: r and press **=** to read the values of the gradient 10.091 and y -intercept $-20\ 888$.

Press **MODE**, 1: COMP to exit the statistics mode.

Sharp Scientific calculator

Press **MODE**, 1 and choose 1 (LINE): Linear regression calculation.

Delete any old data.

Enter the years and values in pairs by pressing 4, **STO** (x,y), 7, **M+** (DATA), 5, **STO** (x,y), 10, **M+** (DATA), etc., until you have entered all the values.

Then press **ALPHA**, **RCL** to get the values of the gradient 10.091 and y-intercept -20 888.

Press **MODE** 0 to exit statistics calculations.

All calculators

Write the equation of the line.

$$V \approx 10.091Y - 20\,088$$

b Substitute 2025 into the equation.

$$V \approx 10.091 \times 2025 - 20\,088$$

Calculate the value.

$$\approx 346$$

Write the answer.

The value in 2025 will be about 346.

c Comment on the accuracy.

The data covers 16 years. 2025 is 9 years after the last year, so it is a reasonable extrapolation. The value is likely to be accurate unless conditions change.

The equation of the least-squares model can be written as $y = bx + a$ or $y = ax + b$. The only difference is that the symbols for slope and intercept are exchanged.



Getty Images Plus / Stock / Denis Tangney Jr

Exercise 5.04 Modelling long-term trends

Example

6

- 1 The data below shows the values of a variable over 5 years.

Year	2013	2014	2015	2016	2017
Value	102	117	116	118	122

- a Find the least-squares model without using the statistics mode of your calculator.
- b Predict the value in 2020.
- c Is the prediction likely to be accurate?

- 2 The data below shows the values of a variable over 8 years.

Year	2009	2010	2011	2012	2013	2014	2015	2016
Value	184	194	203	212	217	223	225	234

- a Find the least-squares model without using the statistics mode of your calculator.
- b Predict the value in 2030.
- c Is the prediction likely to be accurate?

- 3 The data below shows the values of a variable over 6 years.

Year	2012	2013	2014	2015	2016	2017
Value	612	625	667	679	710	739

- a Find the least-squares model without using the statistics mode of your calculator.
- b Predict the value in 2022.
- c Is the prediction likely to be accurate?

Example

7

- 4 The data below shows the values of a variable over 10 years.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Value	208	208	229	231	239	241	259	259	269	284

- a Use your calculator to find the least-squares model.
- b Predict the value in 2030.
- c Is the prediction likely to be accurate?

- 5 The data below shows the values of a variable over 20 years.

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
Value	108	113	116	118	123	129	133	136	140	141
Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Value	146	151	153	157	162	167	171	176	181	183

- a Use your calculator to find the least-squares model.
- b Predict the value in 2030.
- c Is the prediction likely to be accurate?

- 6 The data below shows the values of a variable over 15 years.

Year	2003	2004	2005	2006	2007	2008	2009	2010
Value	224	231	234	241	249	251	259	266
Year	2011	2012	2013	2014	2015	2016	2017	
Value	269	276	283	290	293	303	304	

- a Use your calculator to find the least-squares model.
- b Predict the value in 2025.
- c Is the prediction likely to be accurate?

5.05 Applications of time series data

In many cases, real data has both a seasonal variation and a long-term trend. For long-term predictions, it is usually sufficient to predict the value for the year. In this case, only the trend needs to be used. However, for some predictions you should use both.

Time series with a trend and seasonal variation

- For long-term predictions, the trend to predict the base value for the year is usually sufficient
- For short-term predictions, use both the seasonal and trend variations
- For cases where the seasonal variation is greater than the trend variation over the year, use both the seasonal and trend variations
- The trend is applied before seasonal variation to make predictions

The Australian Bureau of Statistics and other organisations that collect long-term data use more complicated methods of analysing data than are shown in this book, however, the principles are the same.

The Consumer Price Index (CPI) is published by the ABS each year. It is used as a measure of inflation. It is also used to make decisions about increases in pensions and government allowances.

The CPI was reset to 100 in the 2011–12 financial year. The CPI shows the changes in the general purchasing power of your money.

EXAMPLE 8

A constructed variable had an average value of 123 in the year 1999–2000. The trend of the time series is given by $V = 0.624Y - 1125$. The index of the quarterly seasonal change is shown below.

Quarter	Q1 (Jan–Mar)	Q2 (Apr–Jun)	Q3 (Jul–Sep)	Q4 (Oct–Dec)
Index	1.05	0.96	0.95	1.04

- a Use the trend to find the value at the beginning of 2020.
- b Use the seasonal index to find the value at the end of June in 2020.
- c Find the value at the beginning of October in 2022.

Solution

- a Write the trend formula. $V = 0.624Y - 1125$

Substitute 2020 and calculate the trend value at the beginning of 2020.
$$= 0.624 \times 2020 - 1125$$
$$= 135.48$$

Write the answer. The trend value is about 135.
- b Apply the index for Q2.
$$\text{Q2, 2020 value} = 135.48 \times 0.96$$
$$= 130.0608$$

Write the answer. The value would be about 130.
- c Apply the trend. $V = 0.624Y - 1125$
$$= 0.624 \times 2022 - 1125$$
$$= 136.728$$

Apply the index for Q3.
$$\text{Q3, 2022 value} = 136.728 \times 0.95$$
$$= 129.8916$$

Write the answer. The value at the beginning of October in 2022 would be about 130.

To find both the seasonal index and the trend for a time series, you should reverse the order of their application. That is, find the seasonal index first, and then apply the trend to the deseasonalised values.

In Example 9, a simplified method of finding the seasonal index is used.

EXAMPLE 9

The values of a variable over three years are shown below.

- a Find the seasonal index.
- b Find the trend.
- c Use the information to predict the value at the beginning of 2020.

2014	Q1	Q2	Q3	Q4
Value	320	310	291	309

2015	Q1	Q2	Q3	Q4
Value	327	317	298	320

2016	Q2	Q2	Q2	Q2
Value	342	316	305	323

Solution

- a Find the totals for each quarter.

$$\text{Q1 total} = 989$$

$$\text{Q2 total} = 943$$

$$\text{Q3 total} = 894$$

$$\text{Q4 total} = 952$$

Find the grand total.

$$\text{Grand total} = 3778$$

Divide each quarter's total by the grand total and multiply by 4 (quarters) to get the index numbers.

$$\text{Q1 index value} = 4 \times \frac{989}{3778} \approx 1.047$$

$$\text{Q2 index value} = 4 \times \frac{943}{3778} \approx 0.998$$

$$\text{Q3 index value} = 4 \times \frac{894}{3778} \approx 0.947$$

$$\text{Q4 index value} = 4 \times \frac{952}{3778} \approx 1.008$$

Write the answer.

Quarter	Q1	Q2	Q3	Q4
Index	1.047	0.998	0.947	1.008

- b** Deseasonalise by dividing each value by the index.

	Q1	Q2	Q3	Q4
2014	306	311	307	307
2015	312	318	315	317
2016	327	317	322	320

Find the averages of each year.

$$2014 \text{ average} = 307.75$$

$$2015 \text{ average} = 315.5$$

$$2016 \text{ average} = 321.5$$

Use your calculator to find the values of a and b .

$$b = 6.875, a = -13\,538$$

Write the answer.

$$\text{The trend line is } V = 6.875Y - 13\,538.$$

- c** The beginning of 2020 is the end of 2019.

$$\begin{aligned} V &= 6.875 \times 2019 - 13\,538 \\ &= 342.625 \end{aligned}$$

Apply the Q4 index.

$$\begin{aligned} \text{Q4 value} &= 1.008 \times 342.625 \\ &= 345.366 \end{aligned}$$

Write the answer.

The value at the beginning of 2020 will be about 345.

Many of the variables constructed by the Australian Bureau of Statistics are not linear. This course deals only with linear long-term trends and seasonal variations.



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Exercise 5.05 Applications of time series data

- 1** A variable had a value of 236 at the beginning of 2015 and a long-term trend equation of $V = 1.4Y - 2485$. The quarterly seasonal indices are shown below.

Example
8

Quarter	Q1	Q2	Q3	Q4
Index	0.91	1.06	1.09	0.94

- a** Use the trend to find the value at the end of 2020.
b Use the seasonal index to find the value at the end of March in 2021.
c Use the trend to find the value at the beginning of 2025.
d Use the seasonal index to find the value at the beginning of October in 2025.
- 2** A variable has a long-term trend equation of $V = 0.85Y - 1589$. The quarterly seasonal index is as shown below.

Quarter	Q1	Q2	Q3	Q4
Index	0.96	1.02	1.04	0.98

- a** Find the value at the beginning of July in 2025.
b What was the value at the end of September in 2005?
- 3** A variable has a long-term trend equation of $V = 0.76Y - 1100$. The monthly seasonal index is as shown below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	0.997	1.039	1.055	1.03	1.014	0.997	0.981	0.947	0.972	0.981	0.989	0.997

- a** Find the value at the beginning of August in 2023.
b What was the value at the end of February in 2004?
- 4** The values of a variable over 4 years are shown below.

2014	Q1	Q2	Q3	Q4
Value	272	281	277	270

2015	Q1	Q2	Q3	Q4
Value	283	291	285	279

2016	Q1	Q2	Q3	Q4
Value	288	294	291	285

2017	Q1	Q2	Q3	Q4
Value	290	298	293	285

- a** Find the seasonal index.
b Find the trend.
c What value would you predict at the end of June in 2023?
d What would the value have been at the beginning of October in 2008?

Example
9

5 The values of a variable over 4 years are shown below.

2012	Q1	Q2	Q3	Q4
Value	386	369	382	400

2013	Q1	Q2	Q3	Q4
Value	384	382	392	416

2014	Q2	Q2	Q2	Q2
Value	399	388	398	422

2015	Q1	Q2	Q3	Q4
Value	413	397	410	435

- a** Find the seasonal index.
- b** Find the trend.
- c** Predict the value at the beginning of April in 2022.
- d** What would the value have been at the end of September in 2006?

Problem solving

- 6** The slope of the long-term trend of a variable is 1.24 and the value at the beginning of 2017 was 246. Predict the value at the beginning of 2025.



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5. CHAPTER SUMMARY

Analysing time series data

- The **March quarter** (first quarter, Q1) is from January to March
- The **June quarter** (second quarter, Q2) is from April to June
- The **September quarter** (third quarter, Q3) is from July to September
- The **December quarter** (fourth quarter, Q4) is from October to December
- To find the **seasonal index** of a time series:
 - find the average for each year
 - divide the amount for each quarter or month by the average for its year
 - find the average for each quarter or month
- **Deseasonalisation** is the smoothing of seasonal variation from a time series
 - Divide each value by the seasonal index for that time to deseasonalise a time series
 - You should deseasonalise data to find long-term trends
- A seasonal index can also be found by
 - finding the totals for each quarter or month
 - finding the grand total
 - dividing the total for each quarter or month by the grand total
 - multiplying the result by 4 (for quarters) or 12 (for months)
- To find a **linear trend** using the **least-squares method**
 - find the totals and averages of the x and y values: $\sum x$, $\bar{x}$, $\sum y$ and $\bar{y}$
 - find the totals of x^2 and xy : $\sum x^2$ and $\sum xy$
 - find the slope of the line as $b = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = r \frac{s_y}{s_x} = \frac{n \sum xy - \sum x \sum y}{n \sum (x^2) - (\sum x)^2}$
 - find the intercept as $a = \bar{y} - b\bar{x}$
 - write the equation of the line: $y = bx + a$
- You can simplify the calculation of the slope by changing the zero of the years, but you still use the actual year to find the intercept
- The equation of the least-squares line is sometimes written as $y = ax + b$, with the symbols for slope and intercept swapped
- Real-world data often involves both seasonal and trend variation

- For long-term predictions, using the trend to predict the base value for the year is usually sufficient
- For short-term predictions, use both the seasonal and trend variations
- For cases where the seasonal variation is greater than the trend variation over the year, use both the seasonal and trend variations
- The trend is applied before seasonal variation to make predictions
- The seasonal index is found before the trend when you analyse a time series

5. CHAPTER REVIEW

Analysing time series data

- 1 The values of a variable over a year are as shown.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	344	347	320	306	277	257	257	258	284	307	333	346

Construct the monthly index.

Example
1

- 2 The values of a variable over 4 years are as shown.

2010	Q1	Q2	Q3	Q4	2011	Q1	Q2	Q3	Q4
Value	160	163	190	173	Value	151	171	186	166

2012	Q1	Q2	Q3	Q4	2013	Q1	Q2	Q3	Q4
Value	153	167	183	166	Value	157	165	181	163

Construct the quarterly index.

Example
2

- 3 A seasonal index:
- A shows the way inflation changes
 - B shows how a variable changes across a year
 - C is an average of the values of a variable over a year
 - D shows a trend
 - E is calculated by the Australian Bureau of Statistics

Examples
1, 2

- 4 A seasonal index:
- A adds up to 4 if it is quarterly and 12 if it is monthly
 - B can add up to any number
 - C adds up to the number of years
 - D adds up to 1
 - E has pairs of values that each add up to 1

Examples
1, 2

- 5 A quarterly index is shown below.

Quarter	Q1	Q2	Q3	Q4
Index	0.994	0.936	0.99	1.08

- a Use the index to deseasonalise the time series below.

2016	Q1	Q2	Q3	Q4	2017	Q1	Q2	Q3	Q4
Value	266	263	277	303	Value	275	258	275	301

- b Make a scatterplot of the original data and the smoothed data on the same axes.

Example
3

Example
4

- 6 The monthly index for a time series is shown below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	1.105	1.096	1.074	1.023	0.975	0.945	0.937	0.951	0.977	0.935	0.976	1.006

Use the index to deseasonalise the data below.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	181	178	173	169	158	154	155	157	159	167	174	184

Example
6

- 7 The table shows the values of a variable over 5 years.

Year	2013	2014	2015	2016	2017
Value	122	127	138	142	149

- Find the least-squares model without using the statistics mode of your calculator.
- Predict the value in 2025.
- Is the prediction likely to be accurate?

Example
7

- 8 The table shows the values of a variable over 14 years.

Year	2002	2003	2004	2005	2006	2007	2008
Value	342	316	310	302	315	320	342
Year	2009	2010	2011	2012	2013	2014	2015
Value	358	338	365	352	384	368	374

- Find the least-squares model.
- Predict the value in 2025.
- Is the prediction likely to be accurate?

Examples
8, 9

- 9 The values of a variable over 4 years are shown below.

2013	Q1	Q2	Q3	Q4
Value	346	334	332	335

2014	Q1	Q2	Q3	Q4
Value	353	340	341	345

2015	Q1	Q2	Q3	Q4
Value	357	350	347	354

2016	Q1	Q2	Q3	Q4
Value	364	353	350	351

- Find the seasonal index.
- Find the trend.
- Use the information to predict the value at the beginning of October in 2020.

Problem solving

- 10** Quarterly data for 4 years is shown below. Calculate and apply the seasonal index to all of the data and compare plots of the original and deseasonalised data.

2014	Q1	Q2	Q3	Q4
Value	449	466	474	466

2015	Q1	Q2	Q3	Q4
Value	449	458	480	460

2016	Q1	Q2	Q3	Q4
Value	450	464	480	462

2017	Q1	Q2	Q3	Q4
Value	451	469	474	464

- 11** The index of earnings calculated by the Australian Bureau of Statistics over 4 years is shown.

2014	Q1	Q2	Q3	Q4
Value	117.7	118.2	119.3	119.9

2015	Q1	Q2	Q3	Q4
Value	120.3	120.8	121.8	122.3

2016	Q1	Q2	Q3	Q4
Value	122.7	123.1	124.1	124.5

2017	Q1	Q2	Q3	Q4
Value	124.9	125.4	126.4	126.9

Find the seasonally adjusted data and comment on the results.

- 12** The average weekly earnings in Australia over 7 years were as follows.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Earnings (\$/week)	996.1	1033.7	1081.3	1114.2	1128.7	1145.7	1163.5	1191.5

Use the data to find a trend and predict the average weekly earnings in 2020 and 2030. Are there any problems with your predictions?



Practice quiz

EARTH GEOMETRY AND TIME ZONES

6.

EARTH GEOMETRY AND TIME ZONES

Until recently, it wasn't possible to know the exact positions of ships and aircraft at any instant. The worldwide **Global Navigation Satellite System (GNSS)** allows you to determine positions extremely accurately. The United States' Global Positioning System (GPS) and the Russian equivalent GLONASS are examples of GNSS. Various forms of GNSS have been used for military applications since the 1960s, but civil use only began in the 1990s. GNSS now has a vast array of applications, including locating vessels lost at sea, assisting emergency vehicles in searches and helping freight and transport companies keep track of their fleets. Car navigation systems use GNSS to show locations on a street map and assist with navigation.

- 6.01 Latitude and longitude
- 6.02 Maps and position coordinates
- 6.03 Distances on the Earth
- 6.04 Longitude and time zones
- 6.05 Applications of time zones
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Locations on the Earth

- define the meaning of great circles
- define the meaning of angles of latitude and longitude in relation to the equator and the prime meridian
- locate positions on Earth's surface given latitude and longitude, e.g. using a globe, an atlas, GPS and other digital technologies
- state latitude and longitude for positions on Earth's surface and world maps (in degrees only)
- use a local area map to state the position of a given place in degrees and minutes
- calculate angular distance (in degrees and minutes) and distance (in kilometres) between two places on Earth on the same meridian using $D = 111.2 \times \text{angular distance}$
- calculate angular distance (in degrees and minutes) and distance (in kilometres) between two places on Earth on the same parallel of latitude using $D = 111.2 \cos(\theta) \times \text{angular distance}$
- calculate distances between two places on Earth, using appropriate technology

Time zones

- define Greenwich Mean Time (GMT), International Date Line and Coordinated Universal Time (UTC)
- understand the link between longitude and time
- determine the number of degrees of longitude for a time difference of one hour
- solve problems involving time zones in Australia and in neighbouring nations, making any necessary allowances for daylight saving, including seasonal time systems used by Aboriginal peoples and Torres Strait Islander peoples
- solve problems involving GMT, International Date Line and UTC
- calculate time differences between two places on Earth
- solve problems associated with time zones, such as online purchasing, making phone calls overseas and broadcasting international events
- solve problems relating to travelling east and west incorporating time zone changes, such as preparing an itinerary for an overseas holiday with corresponding times



Prior learning

TERMINOLOGY

Australian Central Standard Time (ACST)
 Australian Western Standard Time (AWST)
 Coordinated Universal Time (UTC)
 great circles
 latitude
 parallels of latitude
 standard time zones

Australian Eastern Standard Time (AEST)
 chord
 daylight saving
 Greenwich Mean Time (GMT)
 longitude
 Prime Meridian
 time zone

circle of illumination
 equator
 International Date Line
 meridian
 small circles



Latitude and longitude



Australian coordinates



Positions on the globe

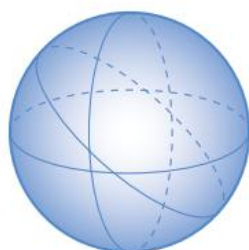


Map of the world

6.01 Latitude and longitude

Great circles are drawn on the surface with their centres at the centre of the Earth.

Small circles do not have their centres at the centre of the Earth.



Great circles



Small circles

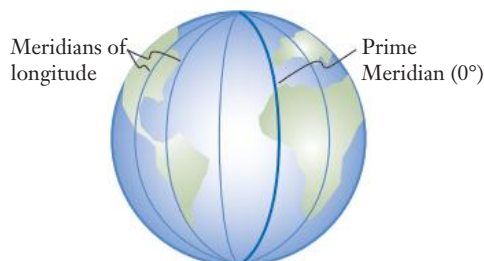
A position on the Earth is described by its **latitude** and **longitude**. Latitude and longitude are both stated as angles because they are measured by angles at the centre of the Earth between circles drawn on the Earth's surface. A **meridian** is half a great circle drawn between the North and South Poles. The **Prime Meridian** of longitude is the meridian passing through Greenwich, England.

The **equator** is the great circle halfway from the North Pole to the South Pole.

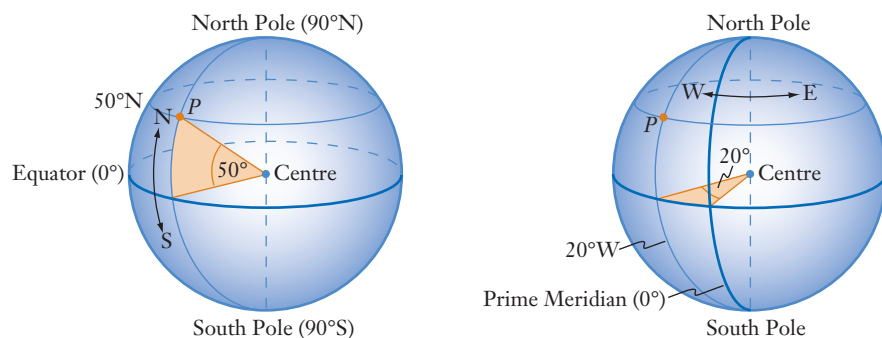
Parallels of latitude are small circles parallel to the equator with their centres at the axis through the North and South Poles.

Your **latitude** is the angle between the equator and the parallel of latitude through your position. You measure it north or south from the equator.

Your **longitude** is the angle between the Prime Meridian and the meridian passing through your position. You measure it to the east or west so the angle is less than 180° .

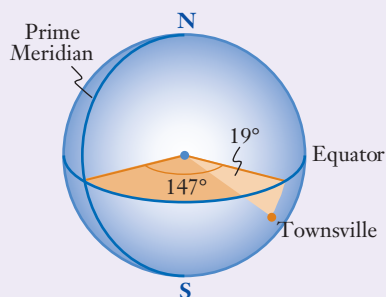


The position of point P below is $50^{\circ}\text{N } 20^{\circ}\text{W}$. The latitude is written first and then the longitude, without a comma between them.



EXAMPLE 1

Use the following diagram to find the latitude and longitude of Townsville. The Prime Meridian and equator are both shown on the diagram.



Solution

Townsville is shown as about 147° to the east of the Prime Meridian and about 19° south of the equator. State the answer.

Townsville is located at approximately $19^{\circ}\text{S } 147^{\circ}\text{E}$.

To find the latitude and longitude of a place, you look up the position in an **atlas**. You can get more precise information from government mapping authorities or the Internet.

EXAMPLE 2

Use an atlas to find the latitude and longitude of Alice Springs.

Solution

Large atlases will have the most precise position. State the answer.

Alice Springs is located at approximately $24^{\circ}\text{S } 134^{\circ}\text{E}$.

INVESTIGATION

USING GOOGLE EARTH TO FIND COORDINATES OF A PLACE

You can use Google Earth to find locations on the Earth.

- 1 Open Google Earth on your computer.
 - Type 'London' into the 'Search' field at the top left and press Enter. Wait for the location to be shown.
 - Now type 'Brisbane' into the 'Search' field and press Enter. Wait for the location to be shown.
 - Now type 'Townsville' into the 'Search' field and press Enter. Wait for the location to be shown.



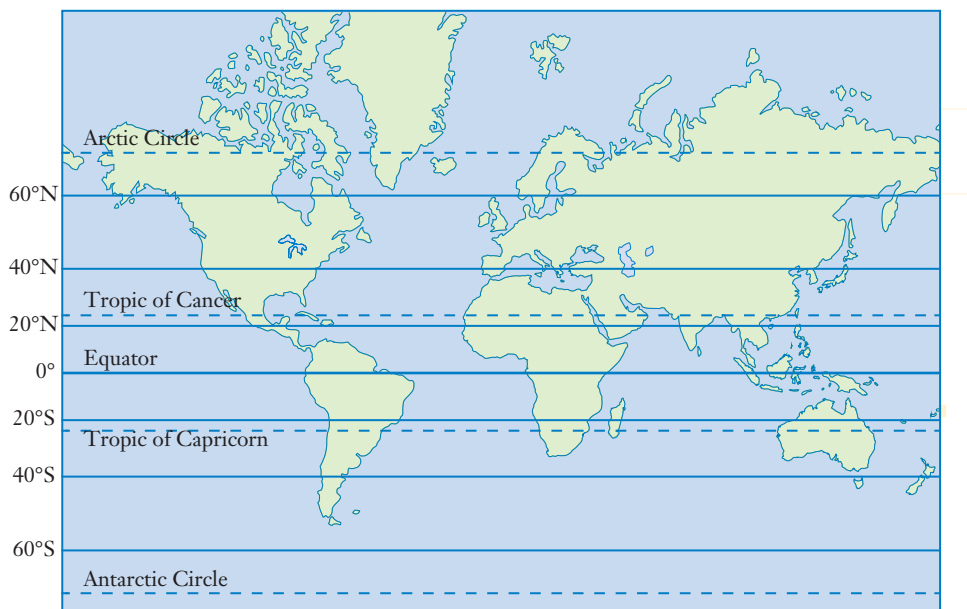
Source: Alamy Stock Photo/Ian Dagnall

- 2 The latitude and longitude of the location are shown in the bottom right-hand corner. This is the location of the pointer. Move the pointer around and notice how the position shown changes. The position shown is accurate to within a metre.
- 3 Try zooming in and out to change the detail that is shown. See whether you can find your own house and note its precise location.

Exercise 6.01 Latitude and longitude

- 1
 - a Are all small circles the same length?
 - b Which parallel of latitude has the greatest length?
 - c What is at latitude 90°S?
 - d Are all meridians of longitude the same length?
- 2 The 80° meridian of longitude passes through the Indian Ocean. Would it be east or west?
- 3 Use the map below to match each of the countries with its corresponding latitude from the following list. (Consult an atlas if you are unsure of the positions of countries.)

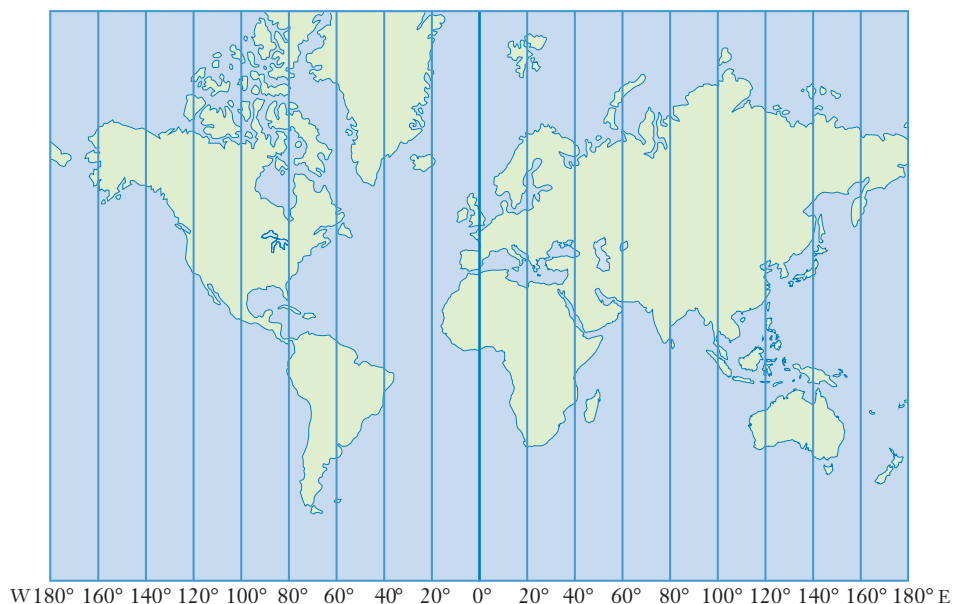
40°S 5°S 50°N 20°N 65°N 0° 20°S 45°N 50°N 25°S



a Zimbabwe	b Sweden
c Myanmar	d Singapore
e Mongolia	f Kenya
g Paraguay	h New Zealand
i Canada	j Argentina

- 4** Use the map below to match each of the countries named with its corresponding longitude from the following list. (Consult an atlas if you are unsure of the positions of countries.)

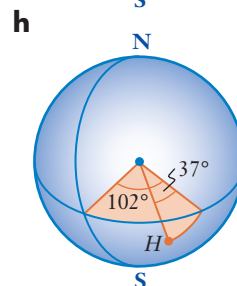
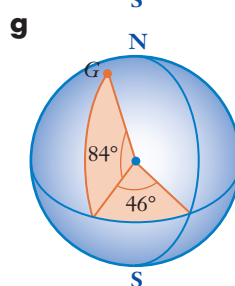
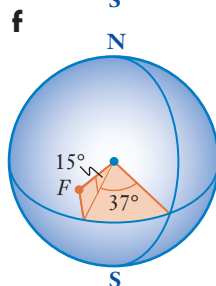
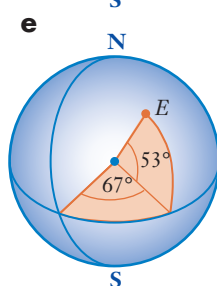
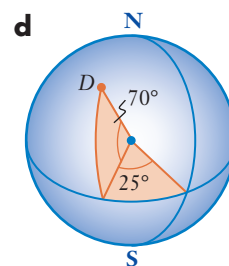
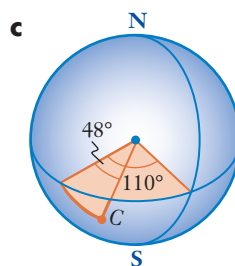
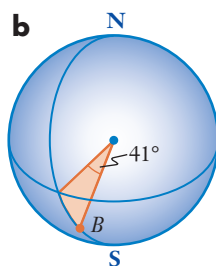
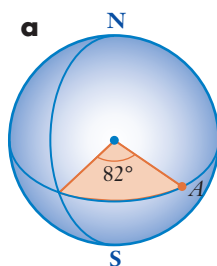
40°E 100°E 60°E 70°W 20°W 140°E 90°W 15°E 170°E 80°E



- a** Sri Lanka **b** Iceland **c** Japan **d** Syria **e** Honduras
f Thailand **g** South Africa **h** Turkmenistan **i** Peru **j** Austria

Example
1

- 5** State the latitude and longitude of each point shown on the diagrams below. The Prime Meridian and equator are shown on each diagram.

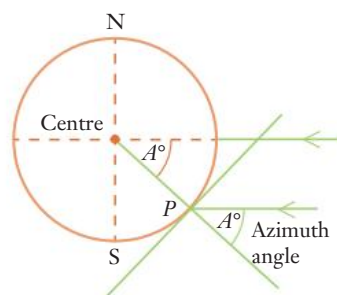


- 6 Use an atlas, Google Earth or another website to state the position of:
- | | | | |
|----------------------|---------------------|-------------------|--------------------------------|
| a Cairo | b Helsinki | c Brisbane | d Gibraltar |
| e Dublin | f Hong Kong | g Darwin | h San Juan, Puerto Rico |
| i Mexico City | j Montevideo | | |
- 7 Use an atlas or Google Earth to name the place situated at:
- | | | | |
|---------------------|--------------------|---------------------|--------------------|
| a 24°S 134°E | b 51°N 4°E | c 23°S 43°W | d 17°N 96°E |
| e 11°N 107°E | f 33°S 71°W | g 31°N 121°E | h 23°N 82°W |
| i 50°N 14°E | j 56°N 13°E | | |

Problem solving

- 8 The angle between the vertical and the direction of the Sun is called the **azimuth**.

At point P on the diagram, the azimuth is A° . At midday on 21 March and 21 September the Sun is directly overhead at the equator. At this time in Brisbane, the azimuth is 27.5° .



- a** What is the latitude of Brisbane?
b What is the azimuth for New York (41°N 74°W) at the same time?

- 9 Longreach Airport is exactly on the Tropic of Capricorn (latitude 23.5°S). On 22 December, the Sun is directly overhead at noon, so its azimuth is 0° . Brisbane is at 27.5°S 153°E and Cairns is further north at 17°S 146°E . To the nearest 0.5° , what is the azimuth of the Sun on 22 December:

- a** in Brisbane? **b** in Cairns?

- 10 During the Australian winter, the Sun moves north of the equator, so that on 21 June it is over the Tropic of Cancer (latitude $23^\circ 5'\text{N}$). To the nearest 0.5° , what is the azimuth of the Sun on 21 June:

- a** in Brisbane? **b** in Cairns?

6.02 Maps and position coordinates

You have seen that latitude and longitude coordinates are expressed in degrees. If more accuracy is needed, you can use decimal values. The original way of subdividing degrees was to divide one degree into 60 minutes and each minute into 60 seconds. Note that these minutes and seconds are not related to time. They are just a naming convention.



Degrees, minutes and seconds

- 1 degree = 60 minutes
- 1 minute = 60 seconds

Minutes of degrees have the symbol ' and seconds of minutes have the symbol ". For example, the position of the Port of Brisbane, expressed in degrees, minutes and seconds is $27^{\circ} 22' 30''\text{S}$ $153^{\circ} 9' 37''\text{E}$. You would say this as 27 degrees, 22 minutes, 30 seconds south, 153 degrees, 9 minutes, 37 seconds east. On maps of large areas, seconds are not used because minutes provide enough accuracy. Seconds are used on maps of very small areas.

EXAMPLE 3

Use the maritime chart on the following page to determine the position of:

- a** Dunwich **b** Welsby Light

Solution

- a** Refer to the chart. The values of latitude are displayed along the bottom and the longitude values are displayed on the right-hand side. Dunwich is in the lower middle section of the map. Look for the closest horizontal line and trace this line out to the right to find the value of longitude.

Dunwich is located at approximately $27^{\circ} 30'\text{S}$.

Do the same for latitude, this time tracing the closest vertical line out to the bottom. For more accuracy, you can use the increment lines to determine the seconds. Since there are 5 increments, each increment represents 12 seconds, since $5 \times 12 = 60$. In this case, the first increment to the right of the $153^{\circ} 24'\text{E}$ longitude is used.

Dunwich is located at approximately $153^{\circ} 24' 12''\text{E}$.

Combine and state the answer in degrees, minutes and seconds.

The location of Dunwich is approximately $27^{\circ} 30' 0''\text{S}$ $153^{\circ} 24' 12''\text{E}$.

- b** Identify the value of latitude.

Welsby Light is located at approximately $27^{\circ} 25'\text{S}$.

Do the same for longitude.

Welsby Light is located at approximately $153^{\circ} 25'\text{E}$.

Combine and state the answer.

The location of Welsby Light is approximately $27^{\circ} 25'\text{S}$ $153^{\circ} 25'\text{E}$.



Source: © Copyright The State of Queensland (Department of Transport and Main Roads) 2014, <http://creativecommons.org/licenses/by/3.0/au>

Example 3 showed how the location of a place can be pinpointed by referring to a map. Sometimes it is necessary to reverse the procedure and use a map to determine what landmark is located at some given coordinates.

EXAMPLE 4

Use the previous maritime chart to determine what landmark is located at:

a $27^{\circ} 31'S 153^{\circ} 23'E$

b $27^{\circ} 30' 0''S 153^{\circ} 27' 12''E$

Solution

- a** Refer to the chart. First identify the latitude value $27^{\circ} 31'S$ on the right and trace this line across the map.

This is the bottom solid horizontal line.

Next, identify the longitude value $153^{\circ} 23'E$ at the bottom and trace this line upwards until it intersects the line of latitude.

This is the second line from the left solid vertical line.

State the answer.

The landmark at $27^{\circ} 31'S 153^{\circ} 23'E$ is Goat Island.

- b** Trace the lines of latitude and longitude to find the intersection point.

This is the second line from the bottom horizontal line and the first increment to the left of the far right solid vertical line.

State the answer.

The landmark at $27^{\circ} 30' 0''S 153^{\circ} 27' 12''E$ is a radar tower.

INVESTIGATION

USING GOOGLE EARTH TO FIND A PLACE WITH GIVEN COORDINATES

You can use Google Earth to find the position coordinates of a place. You can also use Google Earth to find the place that has some given coordinates. If you enter the coordinates into the search bar, Google Earth will take you to the corresponding position. Enter the following coordinates and determine the associated landmarks.

- $25^{\circ} 20' 40''S 131^{\circ} 2' 5''E$
- $51^{\circ} 30' 2''N 0^{\circ} 7' 28''W$
- $37^{\circ} 49' 8''N 122^{\circ} 28' 42''W$
- $48^{\circ} 51' 28''N 2^{\circ} 17' 41''E$

Refer to the following chart for questions 1 and 2.

SMB-2



- ### Example 3

- 2 Determine what is located at each position.
 - a $27^{\circ} 30'S 153^{\circ} 19'E$
 - b $27^{\circ} 31'S 153^{\circ} 16'E$
- 3 Use Google Earth to determine the coordinates of each place.
 - a Sydney Harbour Bridge
 - b Colosseum in Rome
 - c The White House in Washington DC
- 4 Use Google Earth to determine what is at each position.
 - a $33^{\circ} 57' 46''S 18^{\circ} 24' 44''E$
 - b $51^{\circ} 26' 1''N 0^{\circ} 12' 51''W$
 - c $43^{\circ} 4' 35''N 79^{\circ} 4' 18''W$

Problem solving

- 5 Use the maritime chart on page 205 to find the difference in latitude and longitude between Hope Banks and Amity Banks.
- 6 Use the maritime chart on page 205 to identify the latitudes and longitudes which contain the marina at Cleveland.



Distances
on the Earth

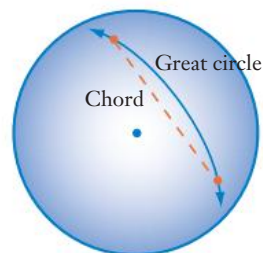
6.03 Distances on the Earth

The Earth rotates once a day, from west to east. You are carried with the Earth, taking 24 hours for one complete circle. The circle is a parallel of latitude, so unless you are at the equator, it is a small circle. The gravity of the Earth makes it close to a sphere, but the spinning makes it fattest at the equator and thinnest at the poles. The radius at the equator is 6378.14 km, but the radius at the poles is only 6356.75 km.

The average radius is good enough for nearly all calculations of distances on the Earth.

The average radius of the Earth is about 6371 km.

The *shortest* distance between 2 points is a straight line. However, you cannot tunnel through the Earth to make a straight-line path between positions on the Earth's surface. The shortest distance on the *surface* of the Earth will be directly above the line going through the Earth. It will be a curve following a great circle route, because the straight line is a **chord** of the great circle directly above it.



Great circles

- The shortest distance between any 2 points on the Earth is part of a great circle.
- On a great circle, $1^\circ \approx 111.2 \text{ km}$

INVESTIGATION

GREAT CIRCLE ROUTES

Visit the websites of major international airlines to see whether their air routes are displayed. The Great Circle Mapper website allows you to find great circle air routes by identifying the departure and destination locations. The Qantas website can be used to find international routes from Sydney.



Map courtesy of Qantas

Try to find the routes from Hawaii to Sydney, San Francisco and Tokyo. What shape are the routes as they appear on the map?

Find the route from London to San Francisco. Why does it seem to go off the top of the map?

Now work with a partner, using a small globe.

Use a piece of string to find the shortest surface routes between different places on the Earth. What do you find about the routes?

Compare with your findings from examining maps of air routes. What do you conclude?

Points that lie on the same meridian are on a great circle. You can calculate the distance between them using the difference in their latitudes. Great circles have a radius of about 6371 km, so their circumference is

$$2 \times \pi \times 6371 \approx 40\,030 \text{ km}$$

Travelling through an angle of 1 degree on a great circle will be a distance of $\frac{1}{360}$ of the circumference. Dividing 40 030 km by 360, you find that this distance is 111.194... km.

EXAMPLE 5

- a** Find the distance between Auckland Island (51°S 166°E) and Noumea (22°S 166°E).
- b** Find the time it would take to travel the distance in a boat averaging 15 km/h.

Solution

- a** Since the 2 positions have the same longitude, they are on the same meridian, which is a great circle. The angle travelled is given by the difference in latitudes.

$$\begin{aligned}\text{Latitude difference} &= (51 - 22)^\circ \\ &= 29^\circ\end{aligned}$$

Calculate the distance to the nearest kilometre.

$$\begin{aligned}\text{Distance} &\approx 111.2 \times 29^\circ \\ &\approx 3225 \text{ km}\end{aligned}$$

State the result.

The distance between Auckland Island and Noumea is about 3225 km.

- b** Write down the rule for speed.

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

Substitute known values.

$$15 \text{ km/h} \approx \frac{3225}{\text{time}}$$

Rearrange and evaluate.

$$\begin{aligned}\text{Time} &\approx \frac{3225}{15 \text{ km/h}} \\ &= 215 \text{ h}\end{aligned}$$

State the result.

The trip between Auckland Island and Noumea would take about 215 hours (almost 9 days).

Once you know the radius of a parallel of latitude, you can work out the distance around the Earth at that latitude and the speed of rotation of a point on the Earth at that latitude.

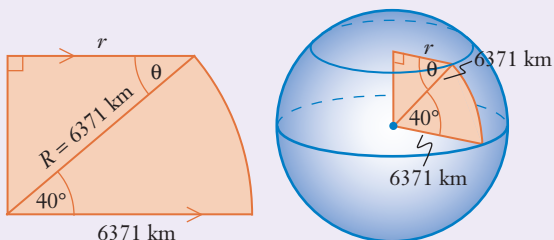
EXAMPLE 6

- a What is the radius of the parallel of latitude at 40°N ?
- b Calculate the circumference of the parallel of latitude at 40°N .
- c Find the speed of rotation of points at 40°N .

Solution

- a Start by drawing a diagram showing the equator, the latitude angle and the radius of the Earth.

Next draw a sketch using the triangle to work out the radius of the circle, r .



Calculate θ using alternate angles.

$$\theta = 40^\circ$$

r is adjacent to θ and R is the hypotenuse.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Use $\cos(\theta)$ to calculate r .

Rewrite using information from the sketch.

$$\cos(40^\circ) = \frac{r}{R}$$

Rearrange and substitute for R .

$$r = 6371 \times \cos(40^\circ)$$

Evaluate and round.

$$\approx 4880 \text{ km}$$

State the result.

The radius of the parallel of latitude at 40°N is about 4880 km.

- b The radius of the parallel of latitude at 40°N is about 4880 km. Write down the rule for circumference C .

$$C = 2\pi r$$

Substitute for r .

$$C = 2 \times \pi \times 4880 \text{ km}$$

Evaluate and round.

$$\approx 30\,662 \text{ km}$$

State the result.

The parallel of latitude at 40°N has a circumference of 30 662 km.

- c Any point on the Earth travels around it in 24 hours. Write down the rule for speed.

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

Substitute known values.

$$\approx \frac{30\,662 \text{ km}}{24 \text{ h}}$$

Evaluate and round.

$$\approx 1278 \text{ km/h}$$

State the result.

Points at 40°N rotate at about 1278 km/h.

The circumference of a parallel of latitude is $2\pi \times (\text{radius of the parallel of latitude})$.

The radius of the parallel of latitude at latitude θ is $R \times \cos(\theta)$, where R is the average radius of the Earth. Compared to a great circle, distances along a parallel of latitude are reduced by the factor $\cos(\theta)$.

On a parallel of latitude, $1^\circ \approx 111.2 \cos(\theta)$ km, where θ is the angle of latitude.

EXAMPLE 7

Find the distance along the parallel of latitude between Bowen (20°S 148°E) and Port Hedland (20°S 119°E).

Solution

Write the formula.

$$\text{Distance for } 1^\circ \approx 111.2 \cos(\theta)$$

Substitute the latitude.

$$= 111.2 \cos(20^\circ)$$

Calculate the difference in longitude.

$$\text{Longitude difference} = (148 - 119)^\circ = 29^\circ$$

Calculate the distance for a difference of 29° of longitude at a latitude of 20° .

$$\begin{aligned}\text{Distance} &\approx 29 \times 111.2 \times \cos(20^\circ) \\ &\approx 29 \times 111.2 \times 0.9397\end{aligned}$$

Evaluate and round.

$$\approx 3030 \text{ km}$$

State the result.

It is about 3030 km along the parallel of latitude from Bowen to Port Hedland.



Alamy Stock Photo/Genevieve Vallee

INVESTIGATION

GREAT CIRCLE DISTANCES

You have seen how distances are calculated along meridians and along parallels of latitude. The distance along a parallel of latitude is not actually the shortest distance between points. The shortest distance is *always* along a great circle route and parallels of latitude are not great circles (except for the equator).

The formula below can be used to calculate the great circle distance d between any two points A ($\text{Lat}_1, \text{Lon}_1$) and B ($\text{Lat}_2, \text{Lon}_2$).

$$d \approx 111.2 \times \cos^{-1} [\cos (\text{Lon}_d) \times \cos (\text{Lat}_1) \times \cos (\text{Lat}_2) + \sin (\text{Lat}_1) \times \sin (\text{Lat}_2)]$$

Lon_d is the difference between longitudes of points A and B .

Lon_d must be less than 180° .

For example, the shortest distance between Flinders Island ($40^\circ\text{S } 148^\circ\text{E}$) and Perth ($32^\circ\text{S } 116^\circ\text{E}$) is given by:

$$\begin{aligned} d &\approx 111.2 \times \cos^{-1} [\cos (148^\circ - 116^\circ) \times \cos (40^\circ) \times \cos (32^\circ) + \sin (40^\circ) \times \sin (32^\circ)] \\ &\approx 111.2 \times \cos^{-1} [\cos (32^\circ) \times \cos (40^\circ) \times \cos (32^\circ) + \sin (40^\circ) \times \sin (32^\circ)] \\ &\approx 111.2 \times \cos^{-1} [0.550\,928 + 0.340\,626] \\ &\approx 111.2 \times 26.930\,868 \\ &\approx 2995 \text{ km} \end{aligned}$$

So the shortest distance from Flinders Island to Perth is about 2995 km.

1 Work in pairs to calculate the shortest distances between different positions on the Earth. Compare the distance along the parallel of latitude with the shortest distance between:

- Bowen ($20^\circ\text{S } 148^\circ\text{E}$) and Port Hedland ($20^\circ\text{S } 119^\circ\text{E}$)
- Shanghai ($31^\circ\text{N } 121^\circ\text{E}$) and Jerusalem ($32^\circ\text{N } 35^\circ\text{E}$)
- Canberra ($35.5^\circ\text{S } 149^\circ\text{E}$) and Montevideo ($35^\circ\text{S } 56^\circ\text{W}$)
- Los Angeles ($34^\circ\text{N } 118^\circ\text{W}$) and Tokyo ($36^\circ\text{N } 140^\circ\text{E}$)

2 Discuss the differences in distance with your partner.

Exercise 6.03 Distances on the Earth

Express answers to the nearest whole number.

Example
5

- 1 Find the distance between Melbourne ($38^{\circ}\text{S } 145^{\circ}\text{E}$) and Wewak, Papua New Guinea ($4^{\circ}\text{S } 145^{\circ}\text{E}$).
- 2 What is the distance from Boston ($42^{\circ}\text{N } 71^{\circ}\text{W}$) to Santiago, Chile ($33^{\circ}\text{S } 71^{\circ}\text{W}$)?
- 3 Find the distance between Stockholm ($59^{\circ}\text{N } 18^{\circ}\text{E}$) and Cape Town ($34^{\circ}\text{S } 18^{\circ}\text{E}$). How long would it take to fly between these cities at 600 km/h ?

Example
6

- 4 Find the radius, circumference and speed of rotation of a point with latitude:
a 30°N **b** 75°N **c** 85°S

Example
7

- 5 What is the distance from Brisbane ($27^{\circ}\text{S } 153^{\circ}\text{W}$) to Corrientes, Argentina ($27^{\circ}\text{S } 59^{\circ}\text{W}$) along the parallel of latitude?
- 6 What is the distance between the Indian cities Aurangabad ($20^{\circ}\text{N } 75^{\circ}\text{E}$) and Puri ($20^{\circ}\text{N } 86^{\circ}\text{E}$) along the parallel of latitude?
- 7 What is the distance between Montevideo ($35^{\circ}\text{S } 56^{\circ}\text{W}$) and Canberra ($35^{\circ}\text{S } 149^{\circ}\text{E}$) along the parallel of latitude?

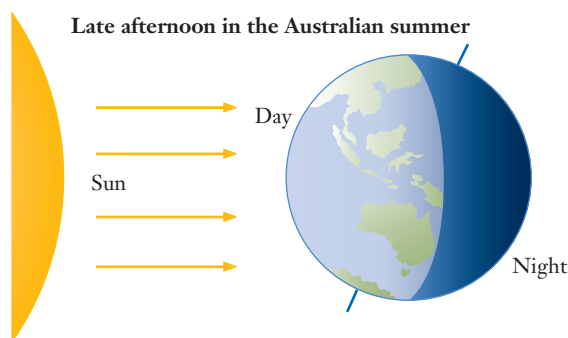
Problem solving

- 8 How long would it take a light aircraft to fly at 210 km/h from Barcelona ($41^{\circ}\text{N } 2^{\circ}\text{E}$) to Tirana, Albania ($41^{\circ}\text{N } 20^{\circ}\text{E}$) along the parallel of latitude?
- 9 A plane flying at 400 km/h travelled directly north for 6 hours and 40 minutes before making an emergency landing. If the plane took off from Hobart ($43^{\circ}\text{S } 147^{\circ}\text{E}$), where did it land?
- 10 Find the time taken for a migrating bird to fly from Townsville ($19^{\circ}\text{S } 147^{\circ}\text{E}$) to Chimoio, Mozambique ($19^{\circ}\text{S } 33^{\circ}\text{E}$) if it flies along the parallel of latitude at 19 km/h .
- 11 Calculate the radius and circumference of the parallel of latitude and the speed of rotation for:
 - a** Brisbane ($27^{\circ}\text{S } 153^{\circ}\text{E}$)
 - b** Townsville ($19^{\circ}\text{S } 147^{\circ}\text{E}$)
 - c** Cape Canaveral ($28^{\circ}\text{N } 81^{\circ}\text{W}$)
 - d** Bamaga on Cape York ($11^{\circ}\text{S } 142^{\circ}\text{E}$)

6.04 Longitude and time zones

The side of the Earth facing the Sun is illuminated, so it is daylight on that side. As the Earth rotates, different parts face the Sun. The **circle of illumination** sweeps around the Earth once a day, bringing dawn at one edge and sunset at the other.

You can work out the time difference between 2 places using the difference in longitudes. Since $360^\circ \div 24 = 15^\circ$, each hour corresponds to a difference in longitude of 15° . Additionally, since $24 \times 60 \div 360 = 4$, each degree of change of longitude gives a time difference of 4 minutes.



- 1 hour $\equiv 15^\circ$ of longitude
- 1° of longitude $\equiv 4$ minutes

EXAMPLE 8

Find the time difference between longitudes separated by:

a 60°

b 78°

Solution

- a** There are 15° of longitude per hour and 1° of longitude per 4 minutes. Divide 60° by 15° to determine the hours.

$$\frac{60^\circ}{15^\circ} = 4$$

State the result.

There are 4 hours between longitudes separated by 60° .

- b** Divide 78° by 15° to determine the hours.

$$\frac{78^\circ}{15^\circ} = 5 \text{ with remainder } 3^\circ$$

Multiply the remainder 3° by 4 to determine the minutes.

$$3^\circ \times 4 = 12$$

Combine and state the result.

There are 5 hours and 12 minutes between longitudes separated by 78° .

Before railways were built, every town set its own time. Dawn and dusk were used to set the town's time. For example, when it is dawn in Brisbane, it is still dark in Toowoomba. It is dawn in Toowoomba about 4 minutes later. It is another 20 minutes before it is dawn in Townsville. 'Railway time' was first used around 1840 in England. It became 'London time' and then '**Greenwich Mean Time (GMT)**'. **Coordinated Universal Time (UTC)** is adjusted to take account of changes in the Earth's rotation. There is now no real difference between GMT and UTC. The world is now divided into **standard time zones** with the clocks set 1 hour apart in neighbouring time zones. These **time zones** are placed approximately in 15° increments of longitude.

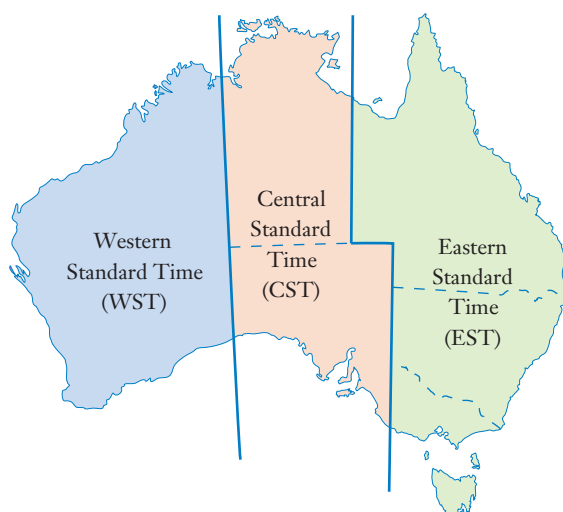
For example, between Johannesburg ($26^\circ\text{S } 28^\circ\text{E}$) and London ($51^\circ\text{N } 0^\circ\text{E}$) there is $28^\circ - 0^\circ = 28^\circ$ of longitude which equates to approximately 2 time zones ($2 \times 15^\circ = 30^\circ$). These 2 cities are therefore in time zones that are 2 hours apart.

Australia is divided into 3 time zones. In addition, some states change the clocks in summer to include **daylight saving**, but this is not a standard time zone. The eastern and western Australian standard time zones are part of an international system of time zones.

The **Australian Eastern Standard Time (AEST)** zone covers the whole of Queensland, New South Wales, the ACT, Victoria and Tasmania. It is 10 hours ahead of UTC (or GMT at 0° longitude).

The **Australian Central Standard Time (ACST)** zone covers South Australia and the Northern Territory, but it is not an international standard time zone. It is only half an hour behind Eastern Standard Time.

The **Australian Western Standard Time (AWST)** zone is 2 hours behind Eastern Standard Time and covers Western Australia. It is 8 hours ahead of UTC.



Visit the Qlock world time website. Enter some cities from Australia and other countries into the box below the map to find out their current times. There are many websites that give the time and date of places relative to UTC, such as Time and Date, and World Time Zone.

Clocks in a time zone are set so that solar noon (when the sun is at its highest altitude) is at 12 o'clock in the middle of the zone. The date is changed at the **International Date Line**, which passes through the Pacific Ocean. All time zones are given with reference to Coordinated Universal Time (UTC). Areas with eastern longitudes are 'ahead' of Greenwich in time, and areas with western longitudes are 'behind' Greenwich. For example, Brisbane's time zone is UTC + 10 while Los Angeles is UTC – 8.

INVESTIGATION

TIME ZONES

The theoretical time difference between 2 places on the Earth is often different from the actual time difference. The difference in longitude between Mackay (21°S 149°E) and Perth (32°S 116°E) is $149^\circ - 116^\circ = 33^\circ$. Thus, the theoretical time difference between Perth and Mackay is $33 \times 4 \text{ minutes} = 132 \text{ minutes}$, or 2 hours and 12 minutes. The time difference according to the websites given above is 2 hours.

- 1 Work in groups of 3 or 4 to find the theoretical and actual time differences between your city and New York, London, Paris, Moscow, Tokyo, Jakarta, San Francisco and Johannesburg.
- 2 Use a website of your choice to investigate the time differences between places of interest to you.

Exercise 6.04 Longitude and time zones

- 1 What is the time difference between longitudes separated by:
a 90°? **b** 74°? **c** 32°? **d** 153°? **e** 6°?
- 2 Calculate the *theoretical* time difference between each pair of cities.
a Brisbane and Perth **b** Sydney and Copenhagen
c Melbourne and Johannesburg **d** Townsville and New York
e Adelaide and Berlin
- 3 Now use a website of your choice to determine the *actual* time difference between the pairs of cities from question 2. Compare the differences in times and determine which pairs have the largest and smallest differences.

Example
8



Applications
of time zones



Table of
time zones



Map of the
world

6.05 Applications of time zones

When travelling across time zones, you need to take into account the change between local times at your origin and destination and adjust the time accordingly.

EXAMPLE 9

It takes about 5 hours to fly from Brisbane to Perth.

- a If you take off at 8 a.m. in Brisbane, at what time will you arrive in Perth?
- b If you take off at 8 a.m. in Perth, at what time will you arrive in Brisbane?

Solution

- a Calculate the arrival time in Perth in terms of Brisbane time.

Perth is UTC + 8 and Brisbane is UTC + 10. Therefore, Perth is 2 hours behind Brisbane. Adjust for the difference.

State the result.

$$\begin{aligned}\text{Arrival time} &= 8 \text{ a.m.} + 5 \text{ h} \\ &= 1 \text{ p.m.}\end{aligned}$$

$$\begin{aligned}\text{Perth time} &= 1 \text{ p.m.} - 2 \text{ h} \\ &= 11 \text{ a.m.}\end{aligned}$$

The flight will arrive in Perth at 11 a.m. local time, or 1 p.m. Brisbane time.

- b Calculate the arrival time in Brisbane in terms of Perth time.

Brisbane is 2 hours ahead of Perth. Adjust for the difference.

State the result.

$$\begin{aligned}\text{Arrival time} &= 8 \text{ a.m.} + 5 \text{ h} \\ &= 1 \text{ p.m.}\end{aligned}$$

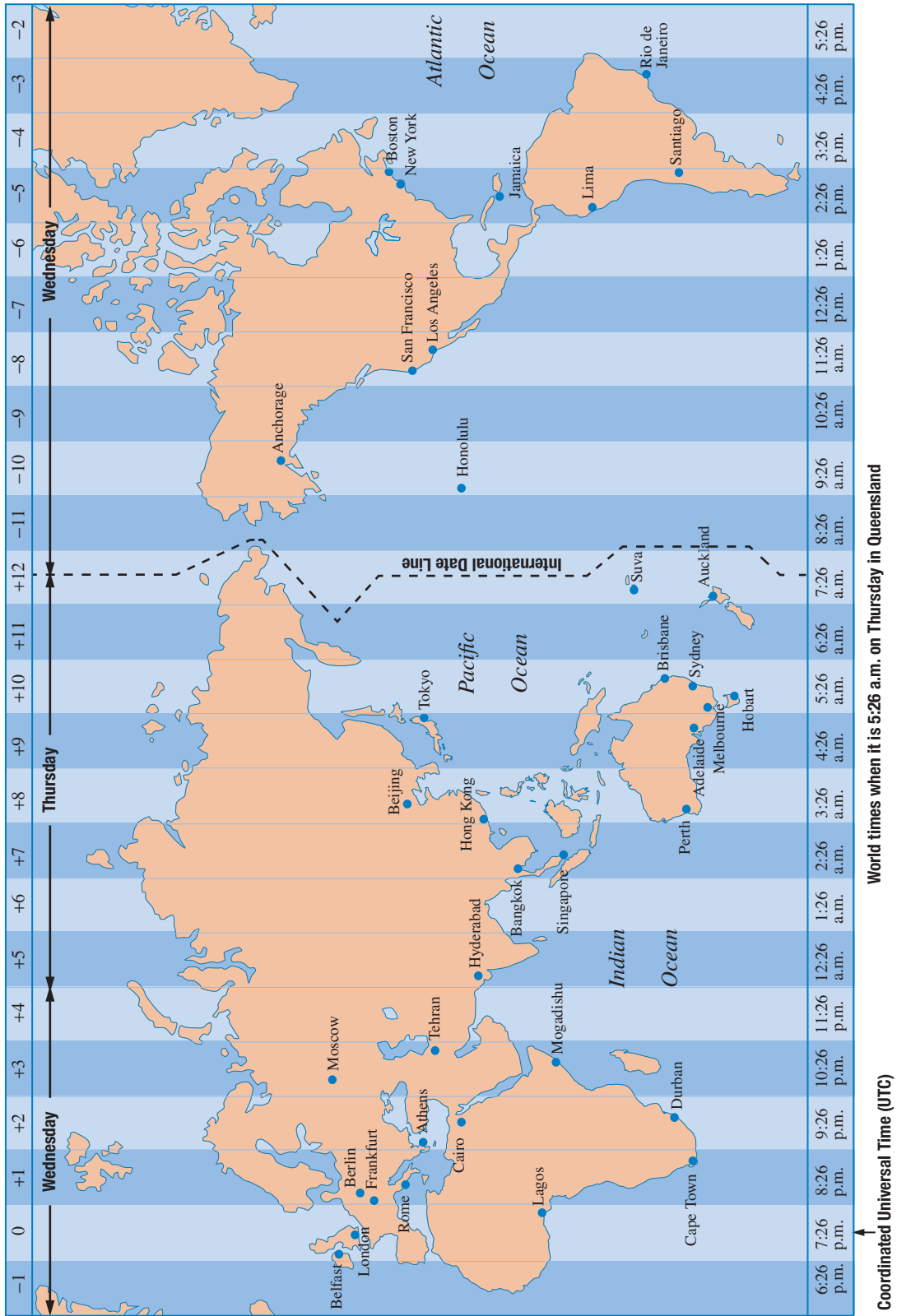
$$\begin{aligned}\text{Brisbane time} &= 1 \text{ p.m.} + 2 \text{ h} \\ &= 3 \text{ p.m.}\end{aligned}$$

The flight will arrive in Brisbane at 3 p.m. local time, or 1 p.m. Perth time.

You can get confused when you cross the International Date Line. On a plane, you can land in Los Angeles 'before' taking off in Brisbane, provided you refer to local time. If you visit the Qantas website or those of other international airlines, you will be able to compare departure and arrival dates and times for overseas flights.



Getty Images/Stock/undefined undefined





International
time zones



International
time zones

EXAMPLE 10

A flight from Brisbane to Los Angeles takes about 13 hours. The flight leaves Brisbane at 1:30 p.m. on Sunday. When will it arrive in Los Angeles:

- a in Brisbane time?
- b in Los Angeles time?

Solution

- a Calculate the arrival time in Los Angeles in terms of Brisbane time.

State the result.

$$\begin{aligned}\text{Arrival time} &= 1:30 \text{ p.m. Sunday} + 13 \text{ h} \\ &= 2:30 \text{ a.m. Monday}\end{aligned}$$

The flight will arrive in Los Angeles at 2:30 a.m. the next day (Monday), Brisbane time.

- b Find Brisbane and Los Angeles times in terms of UTC. (Refer to the map on page 217.)

Calculate the time difference between Los Angeles and Brisbane.

Brisbane is UTC + 10 and Los Angeles is UTC – 8.

$$\begin{aligned}\text{Time difference} &= (-8) - (+10) \\ &= -18 \text{ hours}\end{aligned}$$

Therefore, Los Angeles is 18 hours *behind* Brisbane.

Convert Brisbane time of arrival to Los Angeles local time.

$$\begin{aligned}\text{Arrival time} &= 2:30 \text{ a.m. (Monday)} - 18 \text{ h} \\ &= 8:30 \text{ a.m. (Sunday)} \\ &\quad \text{Los Angeles time.}\end{aligned}$$

State the result.

The flight will arrive in Los Angeles at 8:30 a.m. (Sunday) local time.

The above example demonstrates a paradox where you arrive 5 hours ‘before’ you depart. You effectively ‘gained’ 18 hours by travelling from an eastern to a western time zone in 13 hours. Of course, the opposite would be true if you were flying from Los Angeles to Brisbane since you would be ‘losing’ 18 hours by travelling from a western to an eastern time zone. In this case, the arrival time in Brisbane (local time) would be 13 hours (flight time) + 18 hours (time zone difference) = 31 hours after departure from Los Angeles (local time).

The aircraft will fly the shortest route between Brisbane and Los Angeles across the Pacific Ocean. From Brisbane, the aircraft begins in an eastern time zone and flies east. However, it crosses the International Date Line (IDL) and enters the western time zones. The opposite is true for the return flight where the aircraft begins in a western time zone and flies west, crossing the IDL and entering the eastern time zones.

Exercise 6.05 Applications of time zones

Use the time zone map shown on page 217.

1 Calculate the time in Brisbane when it is:

a 9:00 p.m. in Mogadishu, Somalia	b 3:20 p.m. in Suva, Fiji
c 2:45 p.m. in Belfast, Northern Ireland	d 10:15 a.m. in Rome, Italy
e 10:15 a.m. in Hong Kong	

Example
9



Getty Images / iStock / wvalakite

2 An email is sent at 11:00 a.m. on Wednesday from Sydney. Assuming that there are no transmission delays, work out the time and day when it will arrive in:

a Suva, Fiji	b Bangkok, Thailand	c Santiago, Chile
d Athens, Greece	e Cape Town, South Africa	f Lagos, Nigeria
g Lima, Peru	h Beijing, China	

Example
10

Problem solving

3 A plane leaves Brisbane at 7:00 a.m. on Saturday for Narita airport, Tokyo. The direct flight takes 9 hours and 30 minutes. When it arrives, what will be the time in:

a Brisbane?	b Tokyo?

4 A plane leaves Sydney at 1:20 p.m. on Tuesday for Heathrow airport, London. The flight takes 25 hours and 45 minutes, including a stopover at Changi airport, Singapore. When it arrives, what will be the day and time in:

a Sydney?	b London?

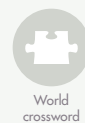
Answer the following questions in local time (the time at the destination).

- 5** It takes $13\frac{1}{2}$ hours (including a stopover) to fly from Sydney to Taipei, Taiwan. If a plane leaves Sydney at 2:30 p.m. on Monday, when will it arrive at Taoyuan airport in Taipei?
- 6** With stopovers, it takes 26 hours and 45 minutes to fly from Brisbane to Vancouver, Canada. If a plane leaves Brisbane at 6:30 a.m. Wednesday, when will it get to Vancouver?
- 7** It takes 26 hours and 40 minutes to fly from Charles De Gaulle airport in Paris to Sydney, including a stopover in Hong Kong. If a flight leaves Paris at 7:30 p.m. on Sunday, when will it arrive in Sydney?
- 8** A direct flight from Los Angeles to Brisbane takes 12 hours and 25 minutes. At what time will a flight leaving Los Angeles at 10 a.m. arrive in Brisbane?
- 9** It takes $24\frac{3}{4}$ hours, including stopovers, to fly from Frankfurt in Germany to Brisbane. When would a flight leaving Frankfurt at 7 a.m. on Wednesday arrive in Brisbane?
- 10** A flight leaves Brisbane at 4:30 p.m. on Tuesday and flies to Rome. When does it arrive, if the flight takes 23 hours and 45 minutes (including stopovers)?



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6. CHAPTER SUMMARY



Earth geometry and time zones

- The **latitude** of a point is the angle between the equator and the parallel of latitude passing through the point. It is measured north or south from the equator
- The **longitude** of a point on the Earth is the angle between the Prime Meridian and a meridian passing through the point. It is measured east or west of the Prime Meridian
- A position is stated with latitude first and then the longitude in the form 50°N 20°W
- Degrees can be subdivided into minutes and seconds
 - 1 degree = 60 minutes
 - 1 minute = 60 seconds
- Degrees, minutes and seconds are expressed in the form 27° 22' 30"
- The shortest distance between any two points on the Earth is part of a great circle
- On a great circle, $1^\circ \approx 111.2 \text{ km}$
- On a parallel of latitude, $1^\circ \approx 111.2 \cos(\theta) \text{ km}$, where θ is the angle of latitude
- $d \approx 111.2 \times \cos^{-1} [\cos(\text{Lon}_d) \times \cos(\text{Lat}_1) \times \cos(\text{Lat}_2) + \sin(\text{Lat}_1) \times \sin(\text{Lat}_2)]$ is used to calculate the great circle distance d between any two points A ($\text{Lat}_1, \text{Lon}_1$) and B ($\text{Lat}_2, \text{Lon}_2$), where Lon_d is the difference between longitudes of points A and B .
- When considering time:
 - 1 hour $\equiv 15^\circ$ of longitude
 - 1° of longitude $\equiv 4$ minutes
- The world is divided into **standard time zones** with the clocks set 1 hour apart in neighbouring time zones
- Time zones are determined with reference to Coordinated Universal Time (UTC) with UTC + zones to the east and UTC – zones to the west
- The International Date Line (IDL) is the line at which the date is changed. Crossing the IDL must be taken into account when comparing time zones

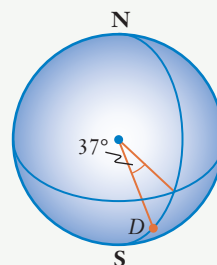
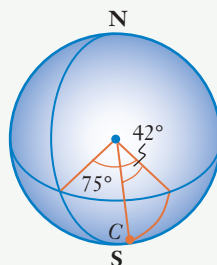
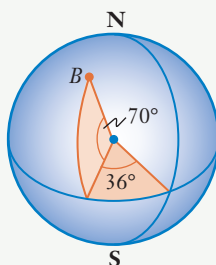
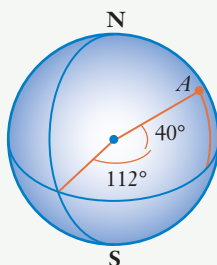
6. CHAPTER REVIEW

Earth geometry and time zones

Example

1

- 1 State the latitude and longitude of points A , B , C and D . The Prime Meridian and equator are shown on each diagram.



Example

2

- 2 Use an atlas to find the position of each city.

a Sydney **b** Rome **c** Kuala Lumpur **d** Kyoto

Example

5

- 3 Find the distance between Budapest ($48^\circ\text{N } 19^\circ\text{E}$) and Cape Town ($34^\circ\text{S } 19^\circ\text{E}$) along the meridian.

Example

6

- 4 What is the radius of the parallel of latitude at:

a 35°N **b** 55°S

Example

6

- 5 Calculate the circumference and speed of rotation of the parallel of latitude at:

a 35°N **b** 55°S

Example

7

- 6 Find the distance between Perth ($32^\circ\text{S } 116^\circ\text{E}$) and Newcastle ($32^\circ\text{S } 152^\circ\text{E}$) along the parallel of latitude.

Refer to the maritime chart next page for questions 7 and 8.

Example

3

- 7 Determine the position of each place in degrees and minutes.

a Point Vernon **b** North Shore Point

Example

4

- 8 Determine what is located at each position.

a $25^\circ 16'\text{S } 152^\circ 40'\text{E}$ **b** $25^\circ 14'30''\text{S } 152^\circ 32'30''\text{E}$



9 What is the time difference between longitudes separated by:

a 75° **b** 36° **c** 8°

10 Calculate the time in Cairns when it is:

a 5:45 a.m. in Perth **b** 1:25 p.m. in London
c 3:30 p.m. in Los Angeles

11 An email is sent at 9:00 a.m. on Wednesday from Melbourne to multiple locations. What will the local time be when it is received at each city?

a Frankfurt, Germany	b Cairo, Egypt
c Rio de Janeiro, Brazil	d Moscow, Russia

- 12** A plane takes 9 hours and 15 minutes to fly from Perth, WA, to Durban, South Africa. If it leaves Perth at 8:50 a.m., at what time does it arrive in Durban?
- 13** A plane leaves Brisbane at 2:40 p.m. on Monday bound for Boston on the eastern coast of the USA. The trip takes 26 hours, including stopovers. What is the local time and day when it arrives?
- 14** A fairy penguin swims along a line of latitude from Albany, WA (35°S 118°E) to Glenelg, SA (35°S 138°E), except that it has to detour around Cape Spencer. The detour adds 35 minutes to its trip, but otherwise it swims at an average of 10 km/h. If it leaves Albany at 9 a.m. on Wednesday, what is the time and day in Albany when it arrives at Glenelg?
- 15** A flight to Belfast, Northern Ireland, leaves Brisbane at 7:50 a.m. on Wednesday. The plane lands 29 hours later in Belfast. What is the time and day in Belfast on arrival?



Practice examination 2 ●●○○

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved scientific calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

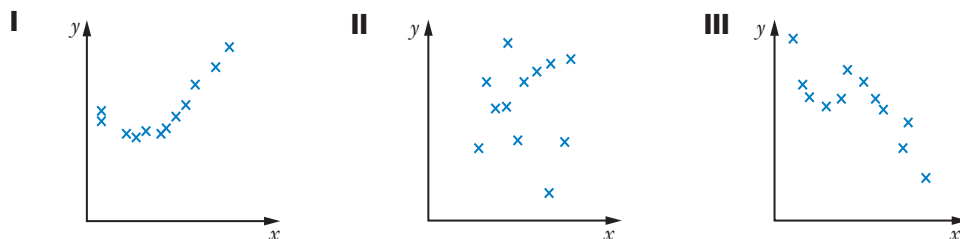
Question 1 (1 mark)

Which of the following could be explanatory and response variables, in that order?

- A** Temperature, state of matter (solid, liquid, gas or plasma)
- B** Type of flame, temperature
- C** Number of seeds planted, mass of crop
- D** Temperature of solar hot water, temperature of day
- E** Season, cost of vegetables

Question 2 (1 mark)

The plots of some variables are shown below.



Which of the following is the most correct?

- A** I, II and III show associations.
- B** Only I shows an association.
- C** Only III shows an association.
- D** I and III show associations.
- E** II and III show associations.

Question 3 (2 marks)

Draw a scatterplot of these adult heights and weights to identify any relationship.

Height (cm)	143	160	140	173	175	169	179	165	177	176	187	190
Weight (kg)	54	58	67	89	45	71	81	89	93	83	68	104

Question 4 (4 marks)

Find the least-squares relationship between the variables in the table below without using the statistics mode of your calculator.

x	3	4	7	8	10
y	4	10	7	17	21

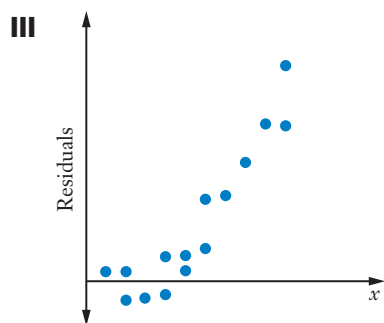
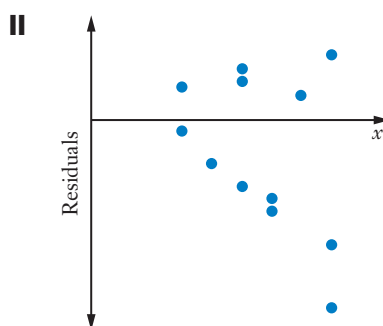
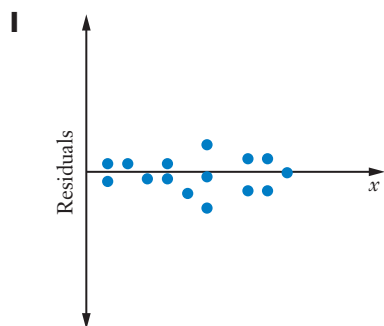
Question 5 (2 marks)

The least-squares line of the table data is $y = 1.61x - 0.9$. Find the residuals.

x	3	5	6	7	9	10
y	4	5	10	13	11	16

Question 6 (1 mark)

Which of the following plots of residuals shows a strong linear relationship?



- A** I and III
- B** III only
- C** I only
- D** II and III
- E** I and II

Question 7 (3 marks)

A study of seagrass growth and pollution levels had a least-squares model of $s = 60 - 5t$, where s was the growth in cm^2/year and t was the water pollution in parts per million (ppm).

- What do the y -intercept and slope mean?
- Predict the growth when the pollution level is 4 ppm.

Question 8 (2 marks)

The quarterly values of a variable are as follows.

2014	Q1	Q2	Q3	Q4
Value	202	184	205	207

2015	Q1	Q2	Q3	Q4
Value	193	197	204	204

2016	Q1	Q2	Q3	Q4
Value	205	185	199	203

2017	Q1	Q2	Q3	Q4
Value	202	191	196	209

Find the quarterly index.

Question 9 (2 marks)

The monthly index of a time series is as follows.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	1.061	1.054	1.04	1.022	0.999	0.986	0.985	0.993	0.997	0.938	0.961	0.965

Use the index to deseasonalise the data below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Value	533	521	529	510	490	492	489	492	508	502	528	526

Question 10 (4 marks)

The data below shows the values of a variable over the years 2010–2017.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Value	300	305	319	318	332	323	344	340

- Find the least-squares model of the data.
- Use your model to predict the value in 2025.
- Explain whether or not your prediction is likely to be accurate.

Question 11 (3 marks)

What is the time difference between each pair of longitudes?

- a 30°E and 45°E
- b 22°W and 64°E

Question 12 (2 marks)

Find the distance between Bucharest, Romania (44°N 26°E) and Gaborone, Botswana (24°S 26°E) along the meridian.

Question 13 (3 marks)

The time in Brisbane is 18 hours ahead of Los Angeles. A flight departs Brisbane at 1 p.m. on Sunday and arrives in Los Angeles 14 hours later. Determine the local arrival time and day in

- a Los Angeles
- b Brisbane

Question 14 (2 marks)

The annual Australian production of carrots and the number of hairdressers in Brisbane in 2000–2017 have a positive association, with an R^2 value of 0.65. Explain the association.

Question 15 (3 marks)

The daily watering time and yield were studied for the dry part of the year in 2018 near Bundaberg. 20 farms were used to form a least-squares equation of $Y = 0.6T + 6.7$, where T is in hours and Y is in tonne/ha, with $R^2 = 0.75$. The 20 farms watered between 2 and 6 hours each day in 2017.

Predict the yield for a crop near Bundaberg watered for 12 hours a day in 2020 and explain any problems with your prediction.

Question 16 (2 marks)

A variable has a seasonal index as follows.

Month	J	F	M	A	M	J	J	A	S	O	N
Index	0.998	0.998	1.037	1.033	1.022	1.034	1.033	1.026	1.01	0.938	0.943

The long-term trend is $V = 3.5Y - 6896$.

Find the value at the end of March in 2022.

Question 17 (3 marks)

Oslo, Norway, is located at $60^\circ\text{N } 11^\circ\text{E}$. If Jon lives in Oslo, calculate to the nearest whole number:

- the radius of the parallel of latitude on which he stands
- the length of the parallel of latitude on which he stands
- the speed at which he is rotating.

Question 18 (5 marks)

The prices paid for some rural Queensland properties (in thousands of dollars) are shown.

Size (acres)	30	10	25	104	5	34	10	40	260
Price (\$'000)	460	358	470	420	411	360	330	450	700

Two other properties in the same general area have 48 acres and 617 acres.

Use the data to predict the prices these will sell for and evaluate your predictions.

Question 19 (2 marks)

The values of an index are shown.

Year	2012	2013	2014	2015	2016	2017	2018
Index	124	127	131	133	134	136	137

What is most likely to be the base year of the index?

Question 20 (2 marks)

Kiev, Ukraine is located at 51°N 31°E . Estimate how many hours ahead of UTC Kiev's time zone should be set.

Question 21 (2 marks)

To recover from jet lag, airline pilots are advised to have one day's rest per time zone travelled west and 1.5 days rest per time zone travelled east. Consider airline pilots who fly between Johannesburg (26°S 28°E) and Perth (32°S 116°E). How many days recovery would they require:

- a** when arriving in Johannesburg?
- b** when arriving in Perth?

END OF EXAMINATION

7.

INVESTMENTS AND LOANS

It is likely that at some point in your life you will need to borrow money for some significant reason like buying a car or a property. It is important that you understand the different types of loans and investments that are available. A reducing balance loan involves the repayment of a small portion of the amount borrowed and interest by regular payments so there is nothing owing at the end of the term of the loan. It is also important that you understand that how and when you make repayments affects the amount of interest that is payable. This knowledge can help you to make informed decisions and save you money.

- 7.01 Compound interest
- 7.02 Rests and interest
- 7.03 Effective interest rates
- 7.04 Future value and present value
- 7.05 Reducing balance loans
- 7.06 Loan calculations
- 7.07 Modifying loans
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Compound interest loans and investments

- use a recurrence relation $A_{n+1} = rA_n$ to model a compound interest loan or investment, and investigate (numerically and graphically) the effect of the interest rate and the number of compounding periods on the future value of the loan or investment, e.g. payday loan
- calculate the effective annual rate of interest and use the results to compare investment returns and cost of loans when interest is paid or charged daily, monthly, quarterly or six-monthly
- solve problems involving compound interest loans or investments, e.g. determining the future value of a loan, the number of compounding periods for an investment to exceed a given value, the interest rate needed for an investment to exceed a given value

Reducing balance loans (compound interest loans with periodic repayments)

- use a recurrence relation, $A_{n+1} = rA_n - R$ (where R = monthly repayment) to model a reducing balance loan and investigate (numerically or graphically) the effect of the interest rate and repayment amount on the time taken to repay the loan
- solve problems involving reducing balance loans, e.g. determining the monthly repayments required to pay off a housing loan



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

compound interest

fixed-term loan

instalment

nominal interest rate

present value of money

reducing balance loan

rest

compounding period

flexible-term loan

interest

payment

principal

reducing interest loan

simple interest

effective interest rate

future value of money

loan schedule

periodic interest rate

recurrence relation

repayment



Compound
interest
puzzle

7.01 Compound interest

The price of borrowing or lending money is called **interest**. You have previously calculated **simple interest** using the formula:

$$I = Pin$$

where I is the interest, P is the **principal** (amount borrowed), i is the annual interest rate as a decimal and n is the number of years.

Suppose you borrow \$200 from a relative, and agree to pay 5% simple interest for one year.

$$\begin{aligned} I &= Pin \\ &= \$200 \times 0.05 \times 1 \\ &= \$10 \end{aligned}$$



Alamy Stock Photo/Takatashi Kurikawa

This means you need to repay your relative \$200 plus \$10 interest or \$210 at the end of the year.

Most interest calculations use **compound interest**. It is called compound interest because the interest earned for the first year is added to the principal, and in the second year interest is earned on the total of principal and interest. This process is repeated for the life of the loan or investment. This means that the interest compounds.

EXAMPLE 1

\$5000 is invested at 4% p.a. compounded yearly. Calculate:

- a the total value of the investment at the end of 3 years
- b the compound interest earned after 3 years

Solution

- a** Write the values of the variables for year 1. $P = 5000, i = 0.04, n = 1$
- Calculate the interest for year 1. $I = Pin$
- Substitute for known values. $= 5000 \times 0.04 \times 1$
- Evaluate. $= 200$
- Calculate the amount (A) of the investment at the end of year 1. $A_1 = 5000 + 200$
- $= 5200$
- Write the values of the variables for year 2. $P = 5200, i = 0.04, n = 1$
- Calculate the interest for year 2. $I = Pin$
- Substitute for known values. $= 5200 \times 0.04 \times 1$
- Evaluate. $= 208$
- Calculate the value of the investment at the end of year 2. $A_2 = 5200 + 208$
- $= 5408$
- Write the values of the variables for year 3. $P = 5408, i = 0.04, n = 1$
- Calculate the interest for year 3. $I = Pin$
- Substitute for known values. $= 5408 \times 0.04 \times 1$
- Evaluate. $= 216.32$
- Calculate the value of the investment at the end of year 3. $A_3 = 5408 + 216.32$
- $= 5624.32$
- State the result. The total value of the investment at the end of 3 years is \$5624.32.
- b** The interest earned is the difference between the value of the investment at the end of 3 years and the initial value. $\text{Interest} = 5624.32 - 5000$
- $= 624.32$
- State the result. The compound interest earned after 3 years is \$624.32.

The compound interest does not need to be compounded annually. Any period can be used. The period of time after which interest is added back to the principal is known as the **compounding period** (or **rest**).

The **periodic interest rate**, $i = \frac{\text{annual rate}}{\text{number of periods per year}}$.

The value of a compound interest investment at the end of each time period forms the terms of a geometric sequence. You can write each term as A_n , where A is the value of the investment and n is the number of compounding periods. Consider the value of the investment in Example 1 as shown below.

From the table, you can see that:

$$A_0 = A_0$$

$$A_1 = A_0 \times 1.04$$

$$A_2 = A_1 \times 1.04$$

$$A_3 = A_2 \times 1.04$$

n	A_n
0	5000
1	$5000 + 5000 \times 0.04 = 5000 \times 1.04 = 5200$
2	$5200 + 5200 \times 0.04 = 5200 \times 1.04 = 5408$
3	$5408 + 5408 \times 0.04 = 5408 \times 1.04 = 5624.32$

This is a geometric sequence where the first term (a) is A_0 , and the common ratio (r) is 1.04.

This sequence can be described by the recurrence relation $A_{n+1} = rA_n$, where $A_0 = 5000$.

The value of a compound interest investment at the end of the n th compounding period is given by the **recurrence relation**:

$$A_{n+1} = rA_n$$

where: A_0 = the principal

A_n = the value of the investment at the end of n compounding periods

$$r = 1 + i$$

i = periodic interest rate as a decimal

EXAMPLE 2

\$30 000 is invested at 5% p.a. compounded yearly. Write the recurrence relation that shows the value of the investment at the end of each year.

Solution

State the value of A_0 .

$$A_0 = 30\,000$$

Calculate r .

$$i = 0.05$$

Write the recurrence relation.

$$A_{n+1} = (1 + i)A_n, \text{ where } A_0 = \text{the principal}$$

Substitute for known values.

$$A_{n+1} = 1.05A_n, \text{ where } A_0 = 30\,000$$

You can use this definition to simplify compound interest calculations.

EXAMPLE 3

\$45 000 is invested at 6.3% p.a. compounding yearly.

- a What is the investment worth at the end of 8 years?
- b How much interest was earned in the 5th year of the investment?

Solution

- a Calculate r .

$$i = 0.063$$

Write the recurrence relation.

$$A_{n+1} = (1 + i)A_n, \text{ where } A_0 = \text{the principal}$$

Substitute for known values.

$$A_{n+1} = 1.063A_n, \text{ where } A_0 = 45\,000$$

Use your scientific calculator to find A_1 .

$$\begin{aligned} A_1 &= 1.063 \times A_0 \\ &= 1.063 \times 45\,000 \\ &= 47\,835 \end{aligned}$$

Now repeat the process up to A_8 . (Use the unrounded values for all calculations.)

$$\begin{aligned} A_2 &= 1.063 \times A_1 = 1.063 \times 47\,835 \\ &= 50\,848.60\dots \\ A_3 &= 1.063 \times A_2 = 1.063 \times 50\,848.60\dots \\ &= 54\,052.06\dots \\ A_4 &= 1.063 \times A_3 = 1.063 \times 54\,052.06\dots \\ &= 57\,457.34\dots \\ A_5 &= 1.063 \times A_4 = 1.063 \times 57\,457.34\dots \\ &= 61\,077.16\dots \\ A_6 &= 1.063 \times A_5 = 1.063 \times 61\,077.16\dots \\ &= 64\,925.02\dots \\ A_7 &= 1.063 \times A_6 = 1.063 \times 64\,925.02\dots \\ &= 69\,015.29\dots \\ A_8 &= 1.063 \times A_7 = 1.063 \times 69\,015.29\dots \\ &= 73\,363.26\dots \end{aligned}$$

State the result.

The value of the investment at the end of 8 years is approximately \$73 363.26.

- b** The interest earned in the 5th year is the difference between the values of the investment at the end of years 5 and 4.

State the result.

Interest earned in 5th year

$$\begin{aligned} &= A_5 - A_4 \\ &= 61\,077.16... - 57\,457.34... \\ &= 3619.8128... \end{aligned}$$

Approximately \$3619.81 interest was earned in the 5th year.

TECHNOLOGY

Constant multiplier function

The constant multiplier function on a calculator can be used as a shortcut for calculations like those shown in Example 3. Since the previous amount is always multiplied by the same number, you put in that number as the constant multiplier and press the equals key the required number of times.

On many calculators, the constant multiplier is obtained by keying in the desired multiplier and then pressing the multiply key twice. You need to work out how to do this for your calculator.

The following shows how this would be done for the investment in Example 3.

Enter the constant multiplier. 1.063 $\times \times \rightarrow$

Now key in the starting amount. 45000

Press $=$ to find the amount after 1 year. $= \rightarrow$

Press $=$ again for the amount after 2 years. $= \rightarrow$

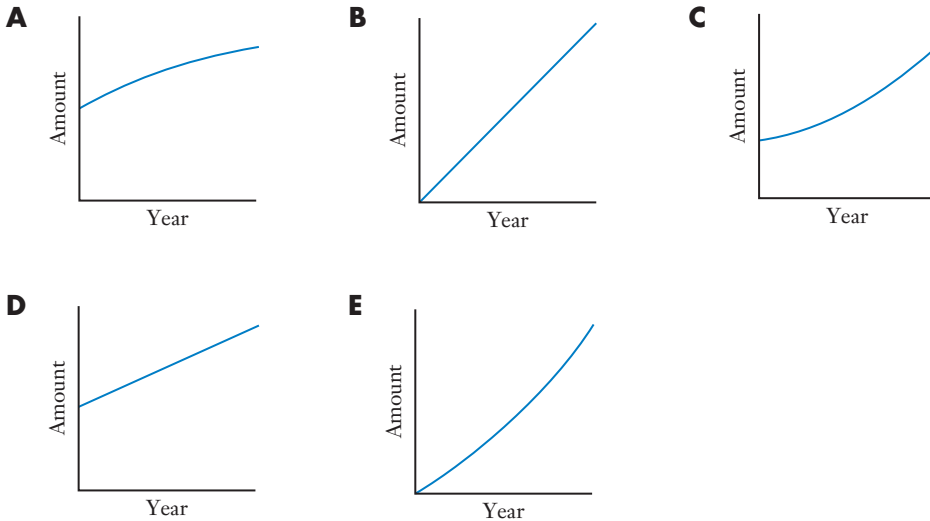
And again for the amount after 3 years. $= \rightarrow$

And again for the amount after 4 years. $= \rightarrow$

And so on.

Exercise 7.01 Compound interest

- 1 An investment of \$20 000 is made at a fixed rate of interest compounding monthly over a number of years. Which graph best represents the amount of the investment at the end of each year?



- 2 The recurrence relation $A_{n+1} = 1.08A_n$, where $A_0 = 12\ 000$, represents the yearly value of an investment of

- A** \$12 000 invested at 1.08% p.a. compounded yearly
B \$8000 invested at 12% p.a. compounded yearly
C \$12 000 invested at 10.8% p.a. compounded yearly
D \$12 000 invested at 8% p.a. compounded yearly
E \$10 800 invested at 8% p.a. compounded yearly

- 3 The amount of money that should be invested at 8% p.a. interest, compounding annually, if you require \$10 000 in three years time, is closest to:

- A** \$5000 **B** \$8000 **C** \$12 000
D \$2000 **E** \$15 000

- 4 Liesel invested \$19 000 at 6% p.a. compounded annually for 4 years. Calculate:

- a** the interest earned in the first year
b the total value of the investment at the end of the first year
c the interest earned in the second year
d the total value of the investment at the end of the second year
e the total value of the investment at the end of 4 years
f the total interest earned on the investment

Example
1

- 5** Ethan placed \$15 000 in a term deposit returning 3.5% p.a. compounded yearly for 5 years. Calculate:
- a** the interest earned in the first year
 - b** the total value of the term deposit at the end of the first year
 - c** the interest earned in the third year
 - d** the total value of the term deposit at the end of 5 years
 - e** the total interest earned on the term deposit

Example
2

- 6** \$10 000 is invested at 6.8% p.a. compounded yearly. Write the recurrence relation that shows the value of the investment at the end of each year.
- 7** Write the recurrence relation that shows the value of the investment at the end of each year for each of the following investments.
- a** \$46 000 is invested at 8.2% compounded yearly
 - b** \$32 800 is invested at 5.3% compounded yearly
 - c** \$112 000 is invested at 7.25% compounded yearly
 - d** \$18 500 is invested at 4.6% compounded yearly
 - e** \$26 950 is invested at 3.75% compounded yearly

Example
3

- 8** \$24 000 is invested at 5.8% p.a. compounding yearly.
- a** What is the investment worth at the end of 5 years?
 - b** How much interest was earned in the 3rd year of the investment?
- 9** A company invests \$250 000 at 6.2% p.a. compounding yearly.
- a** What is the investment worth at the end of 7 years?
 - b** How much interest was earned in the 4th year of the investment?

Problem solving

- 10** Sophie invests \$12 000 at 8% p.a. compounding six-monthly.
- a** What is the investment worth at the end of 2 years?
 - b** How much interest was earned at the end of 2 years?
- 11** Tyler invests \$26 000 at 6.4% p.a. compounding quarterly. How much interest was earned in the second year of the investment?

7.02 Rests and interest

You saw in the previous section that the value of a compound interest investment at the end of the n th compounding period (rest) is given by the recurrence relation:

$$A_{n+1} = (1+i)A_n$$

where $A_0 = P$, the principal

A_n = the value of the investment at the end of n compounding periods

i = periodic interest rate as a decimal.

The value of an investment at the end of each compounding period can be calculated as follows.

	A_n
A_0 : Start	P
A_1 : Period 1	$P(1+i)$
A_2 : Period 2	$P(1+i) \times (1+i) = P(1+i)^2$
A_3 : Period 4	$P(1+i)^2 \times (1+i) = P(1+i)^3$
A_4 : Period 4	$P(1+i)^3 \times (1+i) = P(1+i)^4$
.	
.	
.	
A_n : Period n	$P(1+i)^{n-1} \times (1+i) = P(1+i)^n$

This gives the general formula for compound interest.

Compound interest formula

$$A = P(1+i)^n$$

where: A = amount of the investment after n compounding periods (A_n)

P = principal or initial value (A_0)

i = periodic interest rate as a decimal

n = number of compounding periods

Interest rates are usually stated as per annum (p.a.) amounts. It is important to remember that if the compounding period is not yearly, the values of n and i need to be adjusted accordingly.



Using spreadsheets
to calculate interest

EXAMPLE 4

The parents of a newborn child deposit \$10 000 in an interest-bearing account that pays 8% p.a. interest, compounded 6-monthly. How much will be in the account on the child's 17th birthday?

Solution

Write the compound interest formula.

$$A = P(1 + i)^n$$

Calculate i .

$$i = \frac{\text{annual rate}}{\text{number of periods per year}}$$

$$\begin{aligned} i &= \frac{0.08}{2} \\ &= 0.04 \end{aligned}$$

Calculate n .

$n = \text{number of years} \times \text{periods per year}$

$$\begin{aligned} n &= 17 \times 2 \\ &= 34 \end{aligned}$$

Identify the variables.

$$P = 10\,000, i = 0.04, n = 34$$

Substitute into the compound interest formula.

$$\begin{aligned} A &= P(1 + i)^n \\ &= 10\,000(1 + 0.04)^{34} \end{aligned}$$

Evaluate.

$$= 37\,943.163\dots$$

State the result.

The account will be worth approximately \$37 943.16.

TECHNOLOGY

Power calculations

The calculator key y^x is used for power calculations. The following shows how to use this key to complete the calculation $10\,000(1 + 0.04)^{34}$ from Example 4.

1	+	0.04	=	→	1.04
y^x		34	=	→	3.79431634059
$\times$		10000	=	→	37943.1634059

You can also use the bracket keys to work out the answer in one long calculation, but this often leads to error.

Changing the number of compounding periods (rests) per annum affects the total interest earned on the principal amount invested.

EXAMPLE 5

\$15 000 is deposited in an interest-bearing account with an annual interest rate of 6% p.a. Calculate the value of the investment after 5 years if interest is compounded:

- a** yearly **b** monthly

Solution

- a** Identify the variables. $P = 15\,000, i = 0.06, n = 5$
- Write the compound interest formula. $A = P(1 + i)^n$
- Substitute for known values. $= 15\,000(1 + 0.06)^5$
- Evaluate. $= 20\,073.383 \dots$
- Round off and state the result. The value of the investment is about \$20 073.38.
-
- b** Calculate i . $i = \frac{0.06}{12} = 0.005$
- Calculate n . $n = 5 \times 12 = 60$
- Identify the variables. $P = 15\,000, i = 0.005, n = 60$
- Substitute into the compound interest formula. $A = P(1 + i)^n$
- $= 15\,000(1 + 0.005)^{60}$
- Evaluate. $= 20\,232.752 \dots$
- Round off and state the result. The value of the investment is about \$20 232.75.

From the previous example you can see that different amounts of interest are earned when the same principal is invested for different compounding periods. The interest earned using yearly and monthly rests when \$15 000 is invested for 5 years, is shown below.

Yearly rests: Interest = \$20 073.38 – \$15 000 = \$5073.38

Monthly rests: Interest = \$20 232.75 – \$15 000 = \$5232.75

Shorter compounding periods mean that interest is added back to the principal more often and so the interest earned is higher.

You can also use the compound interest formula in reverse to find an unknown principal or interest rate.

EXAMPLE 6

- a Areti has the opportunity to invest some money at 9% p.a. compound interest with monthly rests. How much should she invest if she wants to have \$10 000 available in 5 years?
- b Two years ago, Tanh placed \$8300 in an investment paying compound interest with quarterly rests. If his money has grown to \$9500, what interest rate p.a. did he earn?

Solution

- a Write the formula for compound interest. $A = P(1 + i)^n$
 List known values. $A = \$10\,000$
 Calculate other values. $i = \frac{0.09}{12} = 0.0075$
 $n = 12 \times 5 = 60$
 Substitute for A , i and n . $10\,000 = P \times (1 + 0.0075)^{60}$
 Simplify. $10\,000 = P \times 1.5656\dots$
 Rearrange. $P = 10\,000 \div 1.5656\dots$
 Evaluate and round off. $\approx \$6387.00$
 Write the result. Areti should invest \$6387.00.

b Write the formula for compound interest.	$A = P(1 + i)^n$
List known values.	$A = \$9500, P = \$8300$
Calculate other values.	$n = 4 \times 2 = 8$
Substitute into the formula.	$9500 = 8300 \times (1 + i)^8$
Rearrange and simplify.	$(1 + i)^8 = \frac{9500}{8300} = 1.1445\dots$
Use a calculator to find the 8th root.	$(1 + i) = \sqrt[8]{1.1445\dots} \approx 1.0170\dots$
Isolate i .	$i = 1.0170\dots - 1$
Evaluate.	$= 0.0170\dots$
i is the interest rate (as a decimal) for the compounding period.	$i = 0.0170\dots$ per quarter
Calculate the annual rate and round off.	$R = 4 \times 0.0170\dots \approx 0.0681$ per year
State the result.	Tanh earned about 6.81% p.a. interest.



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TECHNOLOGY

Compound interest calculations

You can use a spreadsheet to help carry out compound interest calculations. Open a blank spreadsheet.

- 1 Enter the headings 'Annual interest rate (R)' into cell B2, 'Principal (P)' into B3, 'Rests per year (k)' into B4, 'End of year' into A6, 'Principal' into B6, 'Yearly interest' into C6 and 'Total interest' into D6.
 - 2 Next, enter the years for which interest will be calculated by placing '1' into cell A7, '=A7+1' and copying cell A8 down to A16.
 - 3 The principal at the end of each year will be different because it will have interest earned for the year added in. Begin by entering the formula '=C\$3*(1+C\$2/C\$4)^(A7*C\$4)' into cell B7 and copying cell B7 down to B16.
 - 4 Enter the yearly interest calculation by placing '=B7-C3' into cell C7, '=B8-B7' into C8 and copying cell C8 down to C16.
 - 5 Finally, enter the calculation for total interest by placing '=B7-C\$3' into cell D7 and copying cell D7 down to D16. Next, format cell C3 and the region B7:D16 as currency with 2 decimal places and the '\$' symbol, using the Format command in the toolbar.
- Use the spreadsheet to carry out some compound interest calculations by entering an annual rate as a decimal into cell C2, a principal into C3 and the number of rests per year into C4. Try different values for interest and principal and note the effect.
 - A copy of the spreadsheet is on NelsonNet. Compare yours with the one provided.
 - You can change the years over which the calculations are made by entering the year of your choice into cell A7. Try beginning at a year other than year 1.



COMPOUND INTEREST				
1				
2		Interest rate (i)	0.08	
3		Principal (P)	\$1,000.00	
4		Rests per year (k)	365	
5				
6	End of year	Principal	Yearly Interest	Total interest
7	1	\$1,083.28	\$83.28	\$83.28
8	2	\$1,173.49	\$90.21	\$173.49
9	3	\$1,271.22	\$97.73	\$271.22
10	4	\$1,377.08	\$105.86	\$377.08
11	5	\$1,491.76	\$114.68	\$491.76
12	6	\$1,615.99	\$124.23	\$615.99

A more sophisticated version of this spreadsheet is also provided on NelsonNet. Use it to check your answers to the following exercise.

INVESTIGATION

DOUBLING YOUR MONEY

How long does it take to double, triple, or quadruple your money at compound interest?

The formula $n = \frac{70}{R}$ can be used to work out approximately how many years

it takes to double your money at an annual interest rate of $R\%$ compounded annually.

Work in groups of 3 or 4 to decide on the accuracy of the formula.

- Each person should work out the times it takes to double \$10 000 to \$20 000 with different interest rates. One person does the rates from 4% to 7%, another 8% to 11%, and so on.
- Work out the actual times using trial and error or the compound interest spreadsheet on NelsonNet and compare them with the results obtained by using the above formula.
- Try changing the number of rests in a year to see how that affects the accuracy.
- Discuss your findings as a group and then with the whole class. How accurate is the formula?

Extension

The formula is derived from something called the Maclaurin series. It is possible to work out corresponding formulas for the times to triple, quadruple, etc. the principal.

In general, the time taken to increase the money by a factor f is given by

$$n = \frac{100 \times \ln(f)}{R}$$

where $\ln(f)$ is the natural logarithm of the factor f . The 'ln' function can be found on your calculator.

Work in your groups to test the accuracy of the formula for a factor other than 2.



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Exercise 7.02 Rests and interest

- 1 The compound interest earned if \$15 000 is invested at 7% p.a. compounded quarterly for 6 years is:

A \$22 746 **B** \$22 510 **C** \$22 801 **D** \$21 442 **E** \$22 666

- 2 Which of the following will give the amount an investment of \$30 000 at 9% p.a. compounded quarterly will grow to in 5 years?

A $30\,000\left(1 + \frac{0.9}{100}\right)^{20}$ **B** $30\,000\left(1 + \frac{9}{100}\right)^5$ **C** $30\,000\left(1 + \frac{9}{400}\right)^5$

D $30\,000\left(1 + \frac{9}{400}\right)^{20}$ **E** $30\,000\left(1 + \frac{0.9}{400}\right)^{20}$

- 3 Grace has \$10 000 to invest for 5 years. Which of the following will give her the greatest return?

A 8.4% p.a. compounded weekly **B** 8.4% p.a. simple interest
C 8.4% p.a. compounded annually **D** 8.4% p.a. compounded quarterly
E 8.4% p.a. compounded daily

- 4 Find the final amount and interest gained for each investment, where interest is compounded annually.

a \$2000 at 6% p.a. for 5 years **b** \$8000 at 4.5% p.a. for 4 years
c \$7000 at 5.7% p.a. for 8 years **d** \$500 at 8.1% p.a. for 9 years
e \$2140 at 10.25% p.a. for 7 years **f** \$60 at 9% p.a. for 50 years

Example

4

- 5 Find the final amount and interest gained for each investment at compound interest.

a \$1500 at 7% p.a. for 2 years with monthly rests
b \$2840 at 3.9% p.a. for 4 years with monthly rests
c \$9000 at 11% p.a. for 8 years with quarterly rests
d \$800 at 12.75% p.a. for 4 years with weekly rests
e \$5000 at 7.8% p.a. for 2 years with daily rests

Example

5

- 6 Find the final amount and interest gained for each investment at compound interest and determine which is the best investment.

a \$7500 at 8.8% p.a. for $2\frac{1}{2}$ years with monthly rests
b \$7500 at 8.8% p.a. for $2\frac{1}{2}$ years with quarterly rests
c \$7500 at 15% p.a. for $1\frac{1}{2}$ years with weekly rests
d \$7500 at 15% p.a. for $1\frac{1}{2}$ years with monthly rests

Example

6

- 7 If it takes 4 years for \$2000 to accumulate to \$4000 compounded annually, find the compound interest rate per annum.

- 8** After 8 years, \$1000 will accumulate to \$3000 compounded annually. What is the interest rate per annum?
- 9** At what compound interest rate p.a. should \$7500 be invested to achieve a total value of \$10 500 after 5 years compounded quarterly?
- 10** Money is invested at 6.5% p.a. compounded monthly. How much should be invested to have \$10 500 available in 4 years?
- 11** How much should be invested at 3.9% p.a. compounded weekly to have \$20 000 available in 7 years?

Problem solving

- 12** How long will it take \$9000 to accumulate to at least \$15 000 at 16% p.a. compounded annually?
- 13** About how long would it take for \$100 to accumulate to \$1000 at 15% p.a. compound interest?
- 14** Ashok with \$20 000 to invest for a period of 5 years is offered 2 options by a financial institution.

Option A: 7.8% p.a. with annual rests

Option B: 7.6% p.a. with weekly rests

Which of the investments would provide a better return for Ashok? Give reasons for your answer.

- 15** Tania's credit union pays 5% p.a. interest (compounded daily). How long would it take \$10 000 to increase to \$10 500 in this account?
- 16** How long will it take to double your money at 8% p.a. compound interest compounded annually? (You can work this out using a principal of \$10 000.)
- 17** Zoe wants to invest some money for a period of 7 years. She can invest her funds at 5.9% p.a. with annual rests or at 5.2% p.a. with daily rests. Which of these investments would provide a better return? Give reasons for your answer.
- 18** Nikita had \$5000 in an investment and after 4 years with monthly rests it had grown to \$6300. What interest rate was Nikita getting?
- 19** How much should Renee invest in an account paying 6% p.a. interest calculated half-yearly to have \$12 500 in 7 years time?
- 20** How long would it take Jarrod to triple his investment at 7.5% p.a. interest, calculated monthly?



Effective
interest rates

7.03 Effective interest rates

The interest rate that is stated for a loan or investment is also known as the **nominal interest rate**. The amount of interest paid in a compound interest calculation depends primarily on the nominal interest rate and the term. However, it is also affected by the frequency of the compounding period.



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INVESTIGATION

COMPARING INTEREST PAYMENTS

Work in groups of four for this investigation.

- 1 Each person in the group is to work out the amount of interest he/she would get by investing \$100 000 at 12% compound interest over 2 years, using a different compounding period: annual, monthly, weekly or daily. In each case, add the interest into the account when it is paid.
- 2 Compare the total interest that each person gets. Draw a graph showing the number of compounding periods in 1 year on the horizontal axis and the amount of interest on the vertical axis.
- 3 Discuss the effect of a more frequent compound interest payment at the same nominal rate.
- 4 Which would be better: 15% compounded daily or 16% compounded annually?
- 5 What do you think would happen to the amount of interest if it were paid each hour or each second? Investigate this by extending the graph.

In order to compare compound interest rates, you must take into account the number of compounding periods in a year. The **effective interest rate** takes the number of rests into account. It is the rate that, if compounded annually, would provide the same interest payment as a rate that is compounded more frequently.

As a decimal, it is the interest that would be paid on \$1 invested for 1 year. As a percentage, it is the interest that would be paid on \$100 invested for 1 year. You can use the effective rate to compare rates of interest for which the compounding periods are different.

EXAMPLE 7

Calculate the effective rate of interest for an investment earning 8% p.a. compounded monthly.

Solution

To work out the effective rate as a percentage, calculate the interest on \$100 for 1 year.

Write the formula for compound interest. $A = P(1 + i)^n$

List known values. $P = \$100$

Calculate i . Keep the value in your calculator. $i = \frac{0.08}{12} = 0.00666\dots$

Calculate n . $n = 12 \times 1 = 12$

Substitute into the formula. $A = 100 \times 1.00666\dots^{12}$

Evaluate and round off. $\approx \$108.30$

Calculate interest. $I \approx \$108.30 - \$100 = \$8.30$

State the result. The interest over the year is about \$8.30, so the effective rate of interest is about 8.3% p.a.

It is possible to develop a formula to work out the effective interest rate.

Effective interest rate

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

where $i_{\text{effective}}$ = the effective interest rate as a decimal

i = annual interest rate as a decimal

n = number of compounding periods in a year

EXAMPLE 8

What is the effective rate of interest for an investment earning 7% p.a. compounded monthly?

Solution

Write the formula for effective interest.

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

Write the number of rests.

$$n = 12$$

Calculate $\frac{i}{n}$. Keep the value in your calculator.

$$\frac{i}{n} = \frac{0.07}{12} = 0.005833\dots$$

Substitute into the formula.

$$i_{\text{effective}} = (1 + 0.005833\dots)^{12} - 1$$

Evaluate and round off.

$$\approx 0.0723$$

State the result.

The effective interest rate is about 7.23%.

EXAMPLE 9

Ella is saving for a new car and wants to invest the money she has saved in an interest-bearing deposit. She is offered the following rates from different banks:

Offer 1: 5.4% p.a. compounded weekly

Offer 2: 5.3% p.a. compounded daily

Offer 3: 5.6% p.a. compounded monthly

Which offer will give Ella the best return?

Solution

Calculate the effective interest rate for each offer.

Offer 1

Write the formula for effective interest.

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

Write the number of rests.

$$n = 52$$

Calculate $\frac{i}{n}$. Keep the value in your calculator.

$$\frac{i}{n} = \frac{0.054}{52} = 0.001\,038\dots$$

Substitute into the formula.

$$i_{\text{effective}} = (1 + 0.001\,038\dots)^{52} - 1$$

Evaluate and round off.

$$\approx 0.0555$$

State the result.

The effective interest rate is about 5.55%.

Offer 2

Write the number of rests.

$$n = 365$$

Calculate $\frac{i}{n}$.

$$\frac{i}{n} = \frac{0.053}{365} = 0.000\,145\dots$$

Substitute into the formula.

$$i_{\text{effective}} = (1 + 0.000\,145\dots)^{365} - 1$$

Evaluate and round off.

$$\approx 0.0544$$

State the result.

The effective interest rate is about 5.44%.

Offer 3

Write the number of rests.

$$n = 12$$

Calculate $\frac{i}{n}$.

$$\frac{i}{n} = \frac{0.056}{12} = 0.004\,666\dots$$

Substitute into the formula.

$$i_{\text{effective}} = (1 + 0.004\,666\dots)^{12} - 1$$

Evaluate and round off.

$$\approx 0.0575$$

State the result.

The effective interest rate is about 5.75%.

Write the answer.

Ella should choose offer 3 because it pays a higher effective interest rate.

Exercise 7.03 Effective interest rates

- 1 Which of the following has the highest effective interest rate?
 - A 7.4% p.a. compounding monthly
 - B 7.3% p.a. compounding quarterly
 - C 7.2% p.a. compounding weekly
 - D 7.5% p.a. compounding annually
 - E 7.1% p.a. compounding daily
- 2 Calculate the effective interest rate for each rate given without using the effective interest rate formula.
 - a 9% p.a. compounding monthly
 - b 11% p.a. compounding weekly
 - c 12% p.a. compounding half-yearly
 - d 6% p.a. compounding daily

Example
7

Example
8

- 3** Calculate the effective interest rate for each rate given using the effective interest rate formula.
- a** 7.5% p.a. compounding monthly **b** 13.4% p.a. compounding daily
c 8.4% p.a. compounding quarterly **d** 8% p.a. compounding fortnightly
- 4** Mia invests \$10 000 at 8.2% p.a. compounded weekly. What is the effective rate of interest?
- 5** Find the effective interest rate for each of the following.
- a** 8.05% p.a. compounding weekly **b** 10.13% p.a. compounding monthly
c 14.24% p.a. compounding fortnightly **d** 17.11% p.a. compounding daily

Example
9

- 6** Liam has a choice of 3 investments at his bank. He can invest at 7.6% p.a. compounded quarterly, 7.5% p.a. compounded monthly or 7.4% p.a. compounded daily. Calculate the effective rate for each option and decide which is best.
- 7** Evie inherits \$30 000 from her aunt. She is offered 3 investment options:
- a term deposit at 7.5% p.a. compounded daily
 - insurance bonds at 7.7% p.a. compounded monthly
 - government bonds at 7.9% p.a. compounded annually.

Which investment should Evie choose if she wants to earn the most interest?

- 8** Determine which investment would provide the better return: 6.08% p.a. compounding quarterly or 5.94% p.a. compounding daily?

Problem solving

- 9** Without performing any calculations, determine which investment would have a higher effective rate of interest: 5.7% p.a. compounded fortnightly or 5.7% p.a. compounded every 6 months. Give reasons for your answer.



Future and
present
values

7.04 Future value and present value

The values of things change over time. Anyone who has bought a new car and tried to sell it later knows that its value decreases quite quickly. Other commodities, such as real estate and money invested in term deposits, generally increase in value over time.

Even money changes in value. This may at first seem strange if you think of, say, a \$100 note. Many people would consider that a \$100 note now has the same value as a \$100 note 1 year from now. Unfortunately, even though the note is the same, you can do more with the money now than you can in the future, because inflation decreases its buying power.

When you invest money at compound interest, the value of your investment increases with time. The compound interest formula can be rewritten as a **future value** formula.

Future value of money

$$FV = PV(1 + i)^n$$

where: FV = **future value of money**

PV = present value of money

i = periodic interest rate as a decimal

n = number of compounding periods

If no compounding period (rest) is stated, assume it is 1 year.

EXAMPLE 10

Divya is going to invest \$8000 at 7% p.a. compound interest with monthly rests. What will be the value of her investment in 11 years time?

Solution

Write the formula for FV .

$$FV = PV(1 + i)^n$$

Calculate i . Keep the value in your calculator.

$$i = \frac{0.07}{12} = 0.005\ 833\ldots$$

Calculate the number of rests.

$$n = 12 \times 11 = 132$$

Identify any known values.

$$PV = \$8000$$

Substitute for known values.

$$FV = 8000 \times (1 + 0.005\ 833\ldots)^{132}$$

Keep the value in your calculator.

$$= 8000 \times 2.1549\ldots$$

Evaluate and round off.

$$\approx 17\ 239.52$$

Write the answer.

The value of Divya's investment in 11 years time will be about \$17 239.52.

You can also work out the **present value** of some future amount. You could use the previous method in reverse or use a new formula with the present value as its subject.

Present value of money

$$PV = FV(1 + i)^{-n}$$

where: PV = **present value of money**

FV = future value of money

i = periodic interest rate as a decimal

n = number of compounding periods



Present value

EXAMPLE 11

How much must Toni invest at 5% p.a. compound interest to obtain \$7800 in 4 years time?

Solution

Write the present value formula.

$$PV = FV(1 + i)^{-n}$$

The compounding period is not stated, so assume it is annual.

$$i = 0.05$$

Calculate the number of rests.

$$n = 4 \times 1 = 4$$

Identify any known values.

$$FV = \$7800$$

Substitute for known values.

$$PV = 7800 \times (1.05)^{-4}$$

Evaluate and round off.

$$= 6417.0793\dots$$

$$\approx 6417.08$$

Write the answer.

Toni must invest about \$6417.08 to obtain the future value of \$7800 in 4 years time.

You need to be careful when using negative powers on the calculator. The steps used on a scientific calculator are as follows:

$$1.05 \quad y^x \quad (-) \quad 4 \quad = \rightarrow 0.822702474$$

$$0.822702474 \quad \times \quad 7800 \quad = \rightarrow 6417.079304$$

Check your calculator manual if this sequence does not work.

You take out insurance to protect yourself against the financial impact of some unfortunate future event. There are many different types of insurance, including medical, house, contents, car, loss of income, travel and life.

Some forms of life insurance require you to make contributions during your working life. After an agreed number of years, the insurance policy provides a single lump sum payout. Lump sum payouts usually seem to be very large amounts. You need to work out the present value (buying power) of the payout to see whether the policy is a good investment.



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EXAMPLE 12

Matilda is 25 and has just started paying \$5000 a year into a life insurance fund. She is promised a lump sum of \$450 000 when she turns 65 and the policy matures. If the average interest rate over this period is 5.5% p.a., what is the present value of the payout?

Solution

Write the present value formula.

$$PV = FV(1 + i)^{-n}$$

Identify the known values.

$$FV = 450\,000$$

Calculate i .

$$i = 0.055$$

Calculate the number of rests.

$$n = 40 \times 1 = 40$$

Substitute.

$$PV = 450\,000 \times (1.055)^{-40}$$

Keep the value in your calculator.

$$= 450\,000 \times 0.11746 \dots$$

Evaluate.

$$= 52\,858.414 \dots$$

Round off and write the answer.

The present value of the payout is about \$52 858, which doesn't seem nearly as much as \$450 000. It is the present buying power of the lump sum.

Exercise 7.04 Future value and present value

- 1** \$5000 is invested at 6% p.a. compounding quarterly. The value of the investment in 5 years time is given by:
- A** $5000(1 + 0.06)^5$ **B** $5000(1 + 0.06)^{20}$
C $5000(1 + 0.015)^{20}$ **D** $5000(1 + 0.015)^4$
E $5000(1 + 0.04)^6$
- 2** An investment that was made 4 years ago is now worth \$8000. The money was invested at 4.8% p.a. compounding monthly. The original value of the investment is given by:
- A** $5000(1 + 0.048)^{-48}$ **B** $5000(1 + 0.004)^{-12}$
C $5000(1 + 0.04)^{-48}$ **D** $8000(1 + 0.004)^{-48}$
E $5000(1 + 0.048)^{-4}$
- 3** Calculate the future value of each investment.
- a** \$5000 invested at 7% p.a. compounding annually for 10 years
b \$9000 invested at 5% p.a. compounding monthly for 7 years
c \$12 700 invested at 6.8% p.a. compounding weekly for 12 years
d \$23 000 invested at 8.2% p.a. compounding quarterly for 6.5 years
e \$17 800 invested at 4.9% p.a. compounding daily for 5 years and 9 months
- 4** Hannah signs an employment contract that guarantees an annual increase of 6% p.a. subject to satisfactory performance of duties. If Hannah now earns \$570 a week, how much will she earn in 5 years time if she performs at a satisfactory level?
- 5** Connor has \$3000 invested at 10% p.a. interest calculated quarterly. What will be the value of his investment in 12 years?
- 6** An amount of \$10 000 is invested at 7.25% p.a. calculated weekly. Assuming that there are 52 weeks in a year, what will the investment grow to after $4\frac{1}{2}$ years?
- 7** Calculate the present value of each promised investment return.
- a** \$10 000 paid after 6 years invested at 5% p.a. compounding annually
b \$8000 paid after 10 years invested at 3.5% p.a. compounding fortnightly
c \$25 000 paid after 11 years invested at 6.7% p.a. compounding weekly
d \$19 500 paid after 8 years 6 months invested at 5.9% p.a. compounding monthly
e \$32 900 paid after 11.7 years invested at 8.2% p.a. compounding daily

Example
10

Example
11

- 8** What is the present value of \$7600 in 8 years time if the interest rate is 8.4% p.a.?
- 9** Aneka needs to have \$3700 available in 5 years time. She can get 7.8% p.a. interest. How much must she invest now?
- 10** Sienna wants to obtain \$12 500 in 6 years time. If the compound interest rate available is 8.3% p.a., how much must she invest now to reach her goal?
- 11** An inheritance of \$30 000 is held in an interest-bearing trust account for Harrison, who is currently $7\frac{1}{2}$ years old. The trust is guaranteed to earn 4.55% p.a. compound interest calculated quarterly. How much will Harrison collect from the trust when he is entitled to claim it on his 21st birthday?
- 12** Scarlett has an insurance policy that promises a payout of \$300 000 in 35 years time. She is required to pay \$2500 a year into the policy. If the average interest rate over this period is 6.8% p.a., what is the present value of the payout?
- 13** Hunter pays \$75 a month into an insurance policy. He will retire after 40 years, at which time the policy will pay a total of \$250 000. If the average interest rate for that period is 7.2% p.a., what is the present value of the payout?
- 14** Emily expects to get a \$15 000 payout from some government bonds in 5 years time. If the interest rate is 11.5% p.a., what is the present value of this payout?
- 15** Cooper is planning a trip overseas and knows that he will need \$15 000 to pay for the trip.
- a** If Cooper can earn 8.05% p.a. calculated monthly, how much does he need to invest now if he wants to be able to pay for his holiday in $1\frac{1}{2}$ years time?
 - b** How much does he need to invest if he can earn only 4.25% p.a. interest calculated monthly?

Example
12

Problem solving

- 16** Lily is very ambitious and thinks that she should double her income in the next 5 years. What percentage increase must she get each year to achieve her objective?
- 17** Marine biologists discover a Lyngbya outbreak in Moreton Bay. The Lyngbya bloom is initially measured to cover 220 m^2 , but a day later the bloom is found to cover 260 m^2 .
- a** What is the percentage growth rate (per day) of the bloom correct to 2 decimal places?
 - b** If it continues unchecked at this rate, what area (to the nearest square metre) will the bloom cover after 1 week?
 - c** How many days will it take to completely cover 1 km^2 ? ($1 \text{ km}^2 = 1\,000\,000 \text{ m}^2$)

7.05 Reducing balance loans

Almost everyone will borrow money at some stage in their life. Whenever you borrow money, the lender will charge you interest for the use of this money. When you borrow for large purchases such as a car or a house, the amount of interest you will pay becomes significant.

A loan is repaid by **instalments** or **repayments** spread evenly over the term of the loan.

For loans using compound interest, the interest is worked out using the amount still owing at the beginning of each compounding period or rest. These are often called **reducing interest** or **reducing balance loans**.

You have seen that the value of a compound interest investment at the end of the n th compounding period is given by the recurrence relation:

$$A_{n+1} = rA_n$$

where: A_n = the value of the investment at the end of n compounding periods

$$r = 1 + i \text{ and } i \text{ is the periodic interest rate as a decimal.}$$

This formula is adapted to make a formula to calculate the value of a reducing balance loan.

Value of a reducing balance loan

$$A_{n+1} = rA_n - R$$

where: A_0 = the initial value of the loan

A_n = the value of the loan at the end of n compounding periods

$$r = 1 + i$$

i = periodic interest rate as a decimal

R = the repayment per compounding period

As the loan is paid off, the amount of interest payable in each period reduces. You usually make fixed repayments, so the amount paid off the principal increases as the amount of interest in the payment decreases. The progress of payments is shown in a **loan schedule** with columns for the time, principal, interest, payment and balance. The interest is calculated before the payment is subtracted from the principal.

EXAMPLE 13

Charlotte has a loan of \$15 000 at 9% reducing interest. Interest compounds monthly and she makes repayments of \$153 a month.

- a Draw up a schedule showing the progress of the loan over the first 6 months.
- b Summarise the position of Charlotte's loan after 6 months, including the total interest paid, the total repayments made, the reduction of the principal and the balance of the loan.

Solution

- a Calculate i .

$$i = \frac{0.09}{12} = 0.0075$$

State any known values.

$$A_0 = 15\,000, R = 153$$

State the formula.

$$A_{n+1} = rA_n - R$$

Calculate the balance after the first month.

$$A_1 = rA_0 - R$$

Substitute known values.

$$= (1.0075) \times 15\,000 - 153$$

Evaluate.

$$= 14\,959.50$$

Now repeat this process and draw up a table showing the principal, interest, payment and balance of the loan for each month.

Month ($n + 1$)	Principal (A_n)	Principal + interest (rA_n)	Interest	Repayment	Balance ($A_{n+1} = rA_n - R$)
1	15 000	$15\,000 \times 1.0075$ $= 15\,112.50$	112.50	153	$15\,112.50 - 153$ $= 14\,959.50$
2	14 959.50	$14\,959.50 \times 1.0075$ $= 15\,071.70$	112.20	153	$15\,071.70 - 153$ $= 14\,918.70$
3	14 918.70	$14\,918.70 \times 1.0075$ $= 15\,030.59$	111.89	153	$15\,030.59 - 153$ $= 14\,877.59$
4	14 877.59	$14\,877.59 \times 1.0075$ $= 14\,989.17$	111.58	153	$14\,989.17 - 153$ $= 14\,836.17$
5	14 836.17	$14\,836.17 \times 1.0075$ $= 14\,947.44$	111.27	153	$14\,947.44 - 153$ $= 14\,794.44$
6	14 794.44	$14\,794.44 \times 1.0075$ $= 14\,905.40$	110.96	153	$14\,905.40 - 153$ $= 14\,752.40$

b Calculate total interest.

$$\begin{aligned}\text{Interest} &= 112.50 + 112.20 + 111.89 \\ &\quad + 111.58 + 111.27 + 110.96 \\ &= 670.40\end{aligned}$$

Calculate the total repayments.

$$\begin{aligned}\text{Repayments} &= 153 \times 6 \\ &= 918\end{aligned}$$

Read the loan balance from the table.

$$\text{Loan balance} = 14\,752.40$$

Calculate the reduction in the principal.

$$\begin{aligned}\text{Reduction in principal} &= 15\,000 - 14\,752.40 \\ &= 247.60\end{aligned}$$

State the result.

After 6 months, Charlotte has reduced the amount owed to \$14 752.40. She has made payments totalling \$918, paid \$670.40 in interest and reduced the principal by \$247.60.

Example 12 shows how the interest reduces each month. This particular loan would take 15 years to repay, but it is very tedious to work through a repayment schedule by hand to find the time taken to repay the loan. Banks and other institutions use computers to calculate the progress of a loan.

Changes in interest rate and extra repayments make a big difference to the amount of interest and time taken to pay off a large loan. Loans used to buy property are called mortgages and are normally paid off using reducing interest. If you do not repay a mortgage, the lender can sell the property to recover the loan.



Alamy Stock Photo/Ian Allenden

INVESTIGATION

LOAN SCHEDULES USING SPREADSHEETS

A computer spreadsheet can be used to show a loan schedule. Work in groups of 2 or 3 to set up and use a loan schedule as below (also on NelsonNet). The procedure is set out in steps for a loan of \$15 000 at 9% compounding monthly with monthly repayments of \$153.



Step 1: Set the spreadsheet labels as shown below. Bold and centre the labels in rows 1 and 7 and format C2:C5 as numbers with 2 decimal places.

	A	B	C	D	E
1	Nelson Qmaths 12 General		LOAN SCHEDULE		
2	Annual interest rate (%)		9.00		
3	Payments per year		12.00		
4	Original principal		15000.00		
5	Payment amount		153.00		
6	Rest interest rate (decimal)				
7	Time	Principal	Interest	Payment	Balance

Step 2: Format cell C6 as a number with 4 decimal places and the cells below row 7 as numbers with 2 decimal places. Type in the formulas shown to calculate the interest and subtract the payment as time progresses. You may be able to speed up the process from row 9 by using a COPY command. Extend the sheet so that it shows payments for 1 year.

	A	B	C	D	E
1	Nelson Qmaths 12 General		LOAN SCHEDULE		
2	Annual interest rate (%)		9		
3	Payments per year		12		
4	Original principal		15000		
5	Payment amount		153		
6	Rest interest rate (decimal)		=C2/C3/100		
7	Time	Principal	Interest	Payment	Balance
8	1	=C4	=B8*\$C\$6	=\$C\$5	=B8+C8-D8
9	=A8+1	=E8	=B9*\$C\$6	=\$C\$5	=B9+C9-D9
10	=A9+1	=E9	=B10*\$C\$6	=\$C\$5	=B10+C10-D10
11	=A10+1	=E10	=B11*\$C\$6	=\$C\$5	=B11+C11-D11

Step 3: Calculate.

	A	B	C	D	E
1	Nelson Qmaths 12 General		LOAN SCHEDULE		
2	Annual interest rate (%)		9.00		
3	Payments per year		12.00		
4	Original principal		15000.00		
5	Payment amount		153.00		
6	Rest interest rate (decimal)		0.0075		
7	Time	Principal	Interest	Payment	Balance
8	1	\$15,000.00	\$112.50	\$153.00	\$14,959.50
9	2	\$14,959.50	\$112.20	\$153.00	\$14,918.70
10	3	\$14,918.70	\$111.89	\$153.00	\$14,877.59
11	4	\$14,877.59	\$111.58	\$153.00	\$14,836.17
12	5	\$14,836.17	\$111.27	\$153.00	\$14,794.44
13	6	\$14,794.44	\$110.96	\$153.00	\$14,752.40
14	7	\$14,752.40	\$110.64	\$153.00	\$14,710.04

Step 4:

- Try changing the interest rate. What happens when the rate is more than 12% p.a.?
- Restore the original values and change the number of payments each year. Comment on the results.
- Restore the original values and try different regular payment amounts. What is the smallest regular payment that must be made so that the balance of the loan reduces?
- What happens if you change the original principal?
- Restore the original values and adjust the spreadsheet to make payments every fortnight at half the amount (26 payments a year). Check the total after the same amount of time.

Discuss your results and contrast the effects of reducing interest with flat rate interest.

As an extension to this activity, try to find the effect of an extra payment (of a large amount) on the length of the loan and on the total amount of interest paid.

You may want to know the amount still owing on a loan after you have made some payments.

EXAMPLE 14

On 16 January, a couple borrowed \$50 000 from a bank to complete some renovations on their home. They borrowed the money at 10.2% p.a. Interest is calculated monthly and they must pay \$931.34 each month for 6 years to repay the loan.

- a How much was owing on 16 May in the same year?
- b How much interest had been paid to the bank during this period?

Solution

- a Calculate i .

$$i = \frac{0.102}{12} = 0.0085$$

Each month, interest is charged and a repayment is deducted.

Calculate the amount owing on 16 February.

$$\begin{aligned} A &= (1 + i) \times 50\,000 - 931.34 \\ &= 1.0085 \times 50\,000 - 931.34 \\ &= 49\,493.66 \end{aligned}$$

Calculate the amount owing on 16 March. $A = 1.0085 \times 49\,493.66 - 931.34$
 $= 48\,983.02$

Calculate the amount owing on 16 April. $A = 1.0085 \times 48\,983.02 - 931.34$
 $= 48\,468.04$

Calculate the amount owing on 16 May. $A = 1.0085 \times 48\,468.04 - 931.34$
 $= 47\,948.68$

State the result. There was still \$47 948.68 owing on 16 May.

- b** The interest paid is the difference between the total of the repayments made and the reduction in the amount owing on the loan.

Calculate the total repaid. Total repaid $= 4 \times 931.34$
 $= 3725.36$

Calculate the reduction in the amount owing. Reduction in loan balance
 $= 50\,000 - 47\,948.68$
 $= 2051.32$

Calculate the interest paid. Interest paid $= 3725.36 - 2051.32$
 $= 1674.04$

State the result. A total of \$1674.04 in interest had been paid.

From Example 14, you can see that the balance of the loan at the end of a month is calculated using the balance of the previous month. You can use the recurrence relation with $A_0 =$ amount borrowed. Think of this as the amount owing at the end of 0 years.

EXAMPLE 15

Aisha borrows \$35 000 to set up a takeaway coffee shop she is leasing. The loan is at 9.6% p.a. with interest added monthly. Her monthly repayments are \$600.



Getty Images/mangpor_2004

- a Write a recurrence relation that Aisha could use to calculate the amount owing after n months.
- b How much does Aisha still owe after 1 year?
- c If Aisha could have secured a loan at 7.5% p.a., how much would she still owe after 1 year with the same repayments?

Solution

- a Write the recurrence relation.

$$A_{n+1} = rA_n - R$$

Calculate i .

$$i = \frac{0.096}{12} = 0.008$$

State any known values.

$$R = 600, r = 1.008$$

Substitute.

$$A_{n+1} = 1.008A_n - 600$$

- b You will need to construct a table or spreadsheet to calculate the remaining solutions.

Open a spreadsheet and enter the headings 'n' in Cell A1, 'Start value' in B1 and 'End value' in C1.

n = the number of months for the loan

Start value = the value of the loan at the start of the month

End value = the value of the loan at the end of the month.

Enter the first 12 months as follows:

Enter 1 in cell A2 and 2 in cell A3. Block cells A2 and A3 and then copy this down to A13.

Start value:

The initial value of the loan (A_0) is 35 000, so enter this amount in cell B2. The value of the loan at the start of month 2 is the same as the value at the end of month 1, so enter '=C2' in cell B3. Now copy cell B3 down to B13.

End value:

The value of the loan at the end of month 1 is $A_1 = rA_0 - R$, so enter '=1.008*B2 - 600' in cell C2. Now copy cell C2 down to C13.

Read the value from the spreadsheet and state the result.

- c Calculate i .

State any known values.

Substitute.

Use a spreadsheet as previously described but change the entry in cell C2 to '=1.00625*B2 - 600' and then copy it down to cell C13.

State the result.

	A	B	C
1	n	Start value	End value
2	1	35000	34680
3	2	34680	34357.44
4	3	34357.44	34032.29952
5	4	34032.29952	33704.55792
6	5	33704.55792	33374.19438
7	6	33374.19438	33041.18793
8	7	33041.18793	32705.51744
9	8	32705.51744	32367.16158
10	9	32367.16158	32026.09887
11	10	32026.09887	31682.30766
12	11	31682.30766	31335.76612
13	12	31335.76612	30986.45225

Aisha still owes \$30 986.45 after 1 year.

$$i = \frac{0.075}{12} = 0.00625$$

$$R = 600, r = 1.00625$$

$$A_{n+1} = 1.00625A_n - 600$$

	A	B	C
1	n	Start value	End value
2	1	35000	34618.75
3	2	34618.75	34235.11719
4	3	34235.11719	33849.08667
5	4	33849.08667	33460.64346
6	5	33460.64346	33069.77248
7	6	33069.77248	32676.45856
8	7	32676.45856	32280.68643
9	8	32280.68643	31882.44072
10	9	31882.44072	31481.70597
11	10	31481.70597	31078.46663
12	11	31078.46663	30672.70705
13	12	30672.70705	30264.41147

Aisha still owes \$30 264.41 after 1 year.

From the solution in Example 15, you can see that by obtaining an interest rate that is 2.1% p.a. lower, she would owe \$722.04 less after 1 year.

Exercise 7.05 Reducing balance loans

- 1** An amount of \$18 000 is borrowed at a rate of 8.4% p.a. interest calculated monthly. The loan is repaid using monthly payments of \$350. If A_n is the amount owing after n payments, which of the following is the recurrence relation for this loan?
- A** $A_{n+1} = 0.084A_n - 350$, where $A_0 = 18\,000$
B $A_{n+1} = 0.06A_n - 18\,000$, where $A_0 = 350$
C $A_{n+1} = 0.7A_n - 350$, where $A_0 = 18\,000$
D $A_{n+1} = 0.84A_n - 350$, where $A_0 = 18\,000$
E $A_{n+1} = 0.007A_n - 350$, where $A_0 = 18\,000$
- 2** Draw up a loan schedule showing the progress of each loan for the time specified.
- a** \$25 000 at 7% compounded monthly with monthly payments of \$306 for 4 months
b \$64 000 at 9% compounded monthly with monthly payments of \$720 for 6 months
c \$100 000 at 7.8% compounded monthly with monthly payments of \$1000 for 6 months
d \$100 000 at 7.8% compounded fortnightly with fortnightly payments of \$500 for 6 months (13 payments)
e \$85 000 at 12% compounded fortnightly with fortnightly payments of \$800 for 6 months
f \$70 000 at 15% compounded monthly with monthly payments of \$980 for 6 months
g \$70 000 at 15% compounded monthly with monthly payments of \$1500 for 6 months
h \$70 000 at 15% compounded monthly with monthly payments of \$900 for 6 months
- 3** Liam borrowed \$30 000 on 3 June to buy a new car. The loan has an interest rate of 12.7% p.a. with interest calculated monthly and monthly repayments of \$620.
- a** How much does Liam owe on 3 November in the same year?
b How much interest has Liam paid during this period?
- 4** Nada borrowed \$24 000 to buy new computers for her business. The terms of the loan are that interest is charged at 9.8% p.a. compounded monthly and regular monthly repayments of \$600 will be made by Nada. Nada decides to review her loan after 5 months.
- a** How much does she still owe?
b By how much has Nada reduced the loan?
c How much interest has Nada paid?
- 5** A loan of \$6000 is repaid in quarterly instalments of \$433.20 at a rate of 7% p.a. with interest debited quarterly. Calculate:
- a** the amount still owing after the 4th repayment
b the total interest charged during the first 5 quarters.
- 6** Tam borrows \$1000 from a short-term lender at a rate of 17% p.a. with interest calculated weekly. She repays \$55.12 per week. Calculate:
- a** how much Tam owes after the 6th repayment
b the total interest charged during the first 6 weeks.

Example
13

Example
14

- 7** Lucas wants to travel overseas for a holiday and so he takes out a personal loan for \$9000 from his credit union. The loan is at 12.6% p.a. with interest added monthly and his repayments are \$180 per month.
- Write a recurrence relation to calculate the amount owing after n months for the loan.
 - How much does Lucas still owe after 3 years?
- 8** Betty and Bill are 'grey nomads' who borrow \$26 000 to buy a caravan to travel around Australia. The terms of the loan are that interest is charged at 13.7% p.a. with interest added monthly and Betty and Bill will make monthly repayments of \$450.



Alamy Stock Photo/travellinglight

- Write a recurrence relation to calculate the amount owing by Betty and Bill after n months.
 - How much do Betty and Bill owe after 4.5 years?
- 9** Tahlia borrows \$5000 at 6.7% p.a. with interest added fortnightly. She repays the loan with fortnightly repayments of \$70.84.
- Write a recurrence relation to calculate the amount owed by Tahlia after n fortnights.
 - How much does she owe after 20 weeks?

Problem solving

- 10** \$17 000 is borrowed at 8.9% p.a. with interest added fortnightly. Repayments are also made fortnightly.
- Write a recurrence relation to calculate the amount owing on the loan after n fortnights with repayments of \$194.56.
 - How long will it take to pay off the loan with this level of repayment?
- 11** \$25 000 is borrowed at 9.7% p.a. with interest added monthly. Repayments are also made monthly.
- Write a recurrence relation to calculate the amount owing on the loan with repayments of \$527.50.
 - How long will it take to pay off the loan with this level of repayment?
 - How much longer will it take to repay the loan if the repayments are reduced to \$450 per month and what is the value of the final repayment?



Loan
calculations

7.06 Loan calculations

You can use loan schedules and recurrence relations to make a range of calculations for reducing balance loans. Recurrence relations require a large number of calculations to work out solutions.

There are formulas you can use for these calculations.

Reducing balance loan formulas

Repayment value (Q)

$$Q = \frac{Ar^n(r-1)}{r^n-1}$$

Loan balance (B)

$$B_n = Ar^n - \frac{Q(r^n-1)}{r-1}$$

where: Q = the repayment amount per period

A = the amount borrowed (principal) of a loan

B_n = the balance of the loan after n repayments

$$r = 1 + i$$

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

EXAMPLE 16

Ava and Aiden decide to borrow \$80 000 from their credit union at 11.7% p.a. compounded monthly to complete some home renovations. Calculate:

- a** the monthly repayment if the loan is paid off after 12 years
- b** the total amount repaid
- c** the total interest paid.

Solution

- a** State the reducing balance loan formula.

$$Q = \frac{Ar^n(r-1)}{r^n-1}$$

Calculate i .

$$i = \frac{0.117}{12} = 0.00975$$

State known values.

$$A = 80\,000, r = 1 + i = 1.00975$$

$$n = 12 \times 12 = 144$$

Substitute.

$$Q = \frac{80000 \times 1.00975^{144} (1.00975 - 1)}{1.00975^{144} - 1}$$

Evaluate.

$$= 1036.2535 \dots$$

State the result.

The monthly repayment is about \$1036.25.

- b** They make 144 payments of \$1036.25.

$$\begin{aligned}\text{Total repaid} &= 144 \times 1036.25 \\ &= 149\,220\end{aligned}$$

State the result.

Ava and Aiden repay \$149 220.

- c** The interest is the difference between the amount repaid and the principal.

$$\begin{aligned}\text{Interest} &= \text{Amount repaid} - \text{principal} \\ &= 149\,220 - 80\,000 \\ &= 69\,220\end{aligned}$$

State the result.

A total of \$69 220 is paid in interest.

EXAMPLE 17

\$55 000 is borrowed at 7.8% p.a. interest compounding fortnightly. If fortnightly repayments of \$430 are made, what is the balance of the loan after 2 years?

Solution

State the reducing balance loan formula.

$$B_n = Ar^n - \frac{Q(r^n - 1)}{r - 1}$$

Calculate i .

$$i = \frac{0.078}{26} = 0.003$$

State known values.

$$A = 55\,000$$

$$r = 1 + i = 1.003$$

$$n = 2 \times 26 = 52$$

Substitute.

$$B_{52} = 55\,000 \times 1.003^{52} - \frac{430(1.003^{52} - 1)}{1.003 - 1}$$

Evaluate.

$$= 64\,270.43 \dots - 24\,159.30 \dots$$

State the result.

The balance is about \$40 111.13.

You can't use the reducing balance loan formulas to calculate the number of payments.

EXAMPLE 18

Arianna took out a loan for \$20 000 at a rate of 8.2% p.a. compounded monthly. How long will she take to pay off the loan if she makes monthly payments of \$500?

Solution

Write the recurrence relation.

$$A_n = rA_{n-1} - R$$

Calculate i .

$$i = \frac{0.082}{12}$$

State any known values.

$$R = 500, r = 1 + \frac{0.082}{12}$$

Substitute.

$$A_n = \left(1 + \frac{0.082}{12}\right) A_{n-1} - 500$$

Open a spreadsheet and enter the headings 'n' in Cell A1, 'Start value' in B1 and 'End value' in C1 as described in Example 15.

Enter the 12 months

Enter 1 in cell A2 and 2 in cell A3. Block cells A2 and A3 and then copy this down to A50.

Start value

The initial value of the loan (A_0) is 20 000, so enter this amount in cell B2. The value of the loan at the start of month 2 is the same as the value at the end of month 1, so enter '=C2' in cell B3. Now copy cell B3 down to B50.

End value

The value of the loan at the end of month 1 is $rA_{n-1} - R = rA_0 - R$, so enter '= (1 + 0.082/12)*B2 - 500' in cell C2.

Now copy cell C3 down to C50.

	A	B	C	D
1	n	Start value	End value	
2	1	20000	19636.66667	
3	2	19636.66667	19270.85056	
4	3	19270.85056	18902.5347	
5	4	18902.5347	18531.70202	
6	5	18531.70202	18158.33532	
7	6	18158.33532	17782.41728	
8	7	17782.41728	17403.93046	
9	8	17403.93046	17022.85732	
10	9	17022.85732	16639.18018	
11	10	16639.18018	16252.88124	
12	11	16252.88124	15863.9426	
13	12	15863.9426	15472.34621	
14	13	15472.34621	15078.0739	

Look down column C of the spreadsheet.

When the loan balance changes from positive to negative, the loan has been paid out.

45	45	1423.294622	933.0204687
46	46	933.0204687	439.3961086
47	47	439.3961086	-57.6013514
48	48	-57.6013514	-557.994961
49	49	-557.994961	-1061.80793

State the number of payments.

47 payments are needed to pay out the loan.

Calculate the time required.

The loan will be paid out in 3 years and 11 months.

Exercise 7.06 Loan calculations

- 1** Find the size of the repayment for each loan.
 - a** \$10 000 borrowed at 9% p.a. compounding monthly for 5 years with monthly repayments
 - b** \$2000 borrowed at 11.5% p.a. compounding weekly for 1 year with weekly repayments
 - c** \$250 000 borrowed at 5.8% p.a. compounding quarterly for 25 years with quarterly repayments
 - d** \$85 000 borrowed at 6.7% p.a. compounding fortnightly for 8 years with fortnightly repayments
 - e** \$380 000 borrowed at 7.2% p.a. compounding monthly for 30 years with monthly repayments
- 2** Find the balance of the loan at the time indicated in each situation.
 - a** \$20 000 borrowed at 8% p.a. compounding monthly with monthly repayments of \$500, after 2 years
 - b** \$150 000 borrowed at 6.8% p.a. compounding fortnightly with fortnightly repayments of \$900, after 3.5 years
 - c** \$75 000 borrowed at 12.6% p.a. compounding quarterly with quarterly repayments of \$2800, after 2.5 years
 - d** \$3500 borrowed at 7.9% p.a. compounding weekly with weekly repayments of \$120, after 11 weeks
 - e** \$220 000 borrowed at 5.5% p.a. compounding monthly with monthly repayments of \$2400, after 5 years and 10 months
- 3** How long will it take to repay each loan?
 - a** \$15 000 borrowed at 9.2% p.a. compounding monthly with monthly repayments of \$400
 - b** \$900 borrowed at 14.5% p.a. compounding weekly with weekly repayments of \$60
 - c** \$310 000 borrowed at 6.2% p.a. compounding monthly with monthly repayments of \$2400
 - d** \$62 000 borrowed at 10.8% p.a. compounding fortnightly with fortnightly repayments of \$750
 - e** \$48 000 borrowed at 8.7% p.a. compounding quarterly with quarterly repayments of \$1500
- 4** Ryan takes out a loan for \$20 000 so he can buy a car. The finance company offers Ryan a loan at 12.9% p.a. compounded monthly. Calculate:
 - a** the monthly repayment if the loan is paid off after 5 years
 - b** the total amount repaid
 - c** the total interest paid

Example
16

Example
17

Example
18

- 5** Olivia needed to buy furniture for her flat so she borrowed \$5000 at 8.6% p.a. interest compounding fortnightly. If she paid fortnightly repayments of \$75, what is the balance of the loan after 2 years?
- 6** Jackson borrowed \$32 000 at a rate of 10.7% p.a. compounded monthly to buy new equipment for his café. How long will he take to pay off the loan if he makes monthly payments of \$640?
- 7** A loan of \$165 000 is taken out over 20 years at a rate of 12.7% p.a. with interest calculated monthly. The loan is repaid with monthly instalments of \$1897.95. Find the amount still owing after:
 - a** 5 years
 - b** 10 years
 - c** 15 years

Problem solving

- 8** Addison has a reducing balance loan of \$14 000 with interest charged at 9.8% p.a., calculated monthly.
 - a** Find the monthly repayment and the total interest charged if the loan is fully repaid in:
 - i** 2 years
 - ii** 3 years
 - iii** 4 years
 - iv** 4.5 years.
 - b** What happens to the repayment value and the total interest repaid as the time taken to repay the loan increases?
- 9** Layla borrowed \$24 000. She agrees to repay the reducing balance loan over 6 years with monthly instalments at 9.1% p.a. compounding monthly. Find:
 - a** the instalment value
 - b** the principal repaid and the interest paid during:
 - i** the 15th repayment
 - ii** the 40th repayment
 - c** Compare the amount paid off the principal of the loan and the amount of interest paid as the number of payments increases.
- 10** Jacob borrowed \$12 000 with interest on his reducing balance loan and has been charged at 7.5% p.a. calculated monthly. He has been paying \$290.15 each month and currently still owes \$7431.75. How long ago did Jacob borrow the money?

7.07 Modifying loans

Most reducing balance loans you have dealt with so far have been **fixed-term loans**. Many people prefer to take out **flexible-term loans**. This means that the interest rate, instalment value or the frequency of payment may vary. Changing the terms of a reducing balance loan will affect the time taken to pay off the loan and the total amount of interest paid. The most common change to the terms of a loan is the interest rate.

EXAMPLE 19

Zoe borrows \$80 000 at 7.5% p.a. compounding monthly over 9 years with monthly repayments of \$1020. After 4 years, the interest rate is reduced to 6% p.a.

- a What is the new repayment amount required for Zoe to pay off the loan in the same period of time?
- b How much does Zoe save due to the reduction in the interest rate?

Solution

- a State the reducing balance loan formula.

$$B_n = Ar^n - \frac{Q(r^n - 1)}{r - 1}$$

Calculate i .

$$i = \frac{0.075}{12} = 0.00625$$

State the known values.

$$A = 80\,000$$

$$r = 1 + i = 1.00625$$

$$n = 4 \times 12 = 48$$

Substitute.

$$B_{48} = 80\,000 \times 1.00625^{48}$$

$$- \frac{1020(1.00625^{48} - 1)}{1.00625 - 1}$$

Evaluate

$$= 107\,887.93... - 56\,891.38...$$

State the result.

The balance is about \$50 996.55.

Now calculate the payment required to repay the remaining debt in the time remaining, that is, $9 - 4 = 5$ years.

- State the reducing balance loan formula.

$$Q = \frac{Ar^n(r - 1)}{r^n - 1}$$

Calculate i .

$$i = \frac{0.06}{12} = 0.005$$

State the known values.

$$A = 50\,996.55$$

$$r = 1 + i = 1.005$$

$$n = 5 \times 12 = 60$$

Substitute.

$$Q = \frac{50\,996.55 \times 1.005^{60} (1.005 - 1)}{1.005^{60} - 1}$$

Evaluate.

$$= 985.9061\dots$$

State the result.

Repayments of \$985.91 will be required to repay the remainder of the loan.

- b** Calculate the difference between the old and the new monthly payment.

$$\begin{aligned}\text{Saving} &= 1020 - 985.91 \\ &= 34.09\end{aligned}$$

Calculate the total saving made.

$$\begin{aligned}\text{Total saving} &= 34.09 \times 60 \\ &= 2045.40\end{aligned}$$

State the result.

Zoe saves a total of \$2045.40 due to the decrease in the interest rate.

Some people increase or decrease the size of their loan payments as their circumstances change.



Getty Images/FotoCamera

EXAMPLE 20

Elijah takes out a \$150 000 loan at 9.25% p.a. compounding monthly with monthly payments of \$1284.58.

a How long will it take Elijah to repay the loan?

After 6 years, Elijah finds that he can afford to repay \$1600 a month.

b How does this affect the period required to repay the loan?

c How does this affect the total interest paid?

Solution

a Write the recurrence relation.

$$A_{n+1} = rA_n - R$$

Calculate i .

$$i = \frac{0.0925}{12}$$

State any known values.

$$R = 500, r = 1 + \frac{0.0925}{12}$$

Substitute.

$$A_{n+1} = \left(1 + \frac{0.0925}{12}\right) A_n - 1284.58$$

Open a spreadsheet and enter the headings 'n' in Cell A1, 'Start value' in B1 and 'End value' in C1 as described in Example 15.

Fill out the values for columns A and B as previously described. You won't know how far to copy the values yet.

	A	B	C	D
2	1	150000	149871.67	
3	2	149871.67		
4	3			
5	4			
6	5			

The value of the loan at the end of month 1 is $A_1 = rA_0 - R$, so enter ' $=(1 + 0.0925/12)*B2 - 1284.58$ ' in cell C2.

Copy the values in each column down until the value in column C changes from positive to negative.

298	297	5032.604121	3786.817112
299	298	3786.817112	2531.42716
300	299	2531.42716	1266.360244
301	300	1266.360244	-8.45822867
302	301	-8.45822867	-1293.10343
303	302	-1293.10343	-2587.6511
304	303	-2587.6511	-3892.17758
305	304	-3892.17758	-5206.75078

State the result.

The loan will be repaid in 300 months (25 years).

- b** Calculate the amount of the loan still owing after 6 years.

State the reducing balance loan formula.

$$B_n = Ar^n - \frac{Q(r^n - 1)}{r - 1}$$

Calculate i . Store this value on your calculator.

$$i = \frac{0.0925}{12} = 0.007708\dots$$

State the known values.

$$A = 150\,000$$

$$r = 1 + i = 1.007708\dots$$

$$n = 6 \times 12 = 72$$

Substitute.

$$B_{72} = 150\,000 \times 1.007708\dots^{72} - \frac{1284.58(1.007708\dots^{72} - 1)}{1.007708\dots - 1}$$

Evaluate.

$$= 260\,735.67\dots - 123\,026.01\dots$$

State the result.

\$137 709.66 is still owing on the loan after 6 years.

Now find how long it takes to repay the remaining loan balance with the increased repayment.

Open a spreadsheet and enter the headings 'n' in Cell A1, 'Start value' in B1 and 'End value' in C1 as described previously.

Fill out the values for columns A and B as previously described.

The initial value of the loan is now 137 709.66, so enter this value in B1.

The repayment has increased to \$1600, so change the value entered in C2 to '=(1 + 0.0925/12)*B2 - 1600'.

	A	B	C
1	n	Start value	End value
2	1	137709.66	137171.172
3	2	137171.172	136628.5331
4	3	136628.5331	136081.7114
5	4	136081.7114	135530.6745
6	5	135530.6745	134975.3902

Copy the values in each column down until the value in column C changes from positive to negative.

140	139	5996.041365	4444.216201
141	140	4444.276267	2878.53423
142	141	2878.53423	1300.722932
143	142	1300.722932	-289.250662
144	143	-289.250662	-1891.4803
145	144	-1891.4803	-3506.06046
146	145	-3506.06046	-5133.08635

State the result.

It will take 142 months (11 years and 10 months) to repay the remainder of the loan.

Find the total time needed to repay the loan with the increased payment.

$$\begin{aligned}\text{Time to repay loan} &= 72 + 142 \\ &= 214 \text{ months}\end{aligned}$$

Find the difference between the original time and the new time.

$$\begin{aligned}\text{Time difference} &= 300 - 214 \\ &= 86 \text{ months}\end{aligned}$$

Write the conclusion.

By increasing the repayment to \$1600 after 6 years, the length of the loan is reduced by 86 months (7 years 2 months).

- c** Find the total amount repaid with the original payment.

$$\begin{aligned}\text{Total repaid with original payment} \\ &= \$1284.58 \times 300 \\ &= \$385\,374\end{aligned}$$

Find the total amount repaid with the increased payment after 6 years.

$$\begin{aligned}\text{Total repaid with increased payment} \\ &= \$1284.58 \times 72 + \$1600 \times 142 \\ &= \$319\,689.76\end{aligned}$$

The difference between the total amounts repaid is the amount of interest saved.

$$\begin{aligned}\text{Interest saved} \\ &= \$385\,374 - \$319\,689.76 \\ &= \$65\,684.24\end{aligned}$$

State the result.

Elijah saves \$65 684.24 by increasing his payment to \$1600 after 6 years.

Making a lump sum payment can reduce the total interest paid significantly.

EXAMPLE 21

Sophie borrows \$12 000 at 11.2% p.a. compounded monthly to buy a car. She agrees to pay \$229.64 a month for 6 years to repay the loan. After 2 years, Sophie receives a bonus of \$3400 and she decides to use this to reduce her loan balance. She then continues with her regular payments. How much interest does Sophie save by doing this?

Solution

Find the amount owing on the loan after 2 years.

State the reducing balance loan formula. $B_n = Ar^n - \frac{Q(r^n - 1)}{r - 1}$

Calculate i . Store this value on your calculator. $i = \frac{0.112}{12} = 0.00933\dots$

State known values. $A = 12\,000$
 $r = 1 + i = 1.00933\dots$
 $n = 2 \times 12 = 24$

Substitute. $B_{24} = 12\,000 \times 1.00933\dots^{24}$
$$- \frac{229.64(1.00933\dots^{24} - 1)}{1.00933\dots - 1}$$

Evaluate. $= 14\,997.263\dots - 6145.461\dots$
 $= 8851.802\dots$

State the result. \$8851.80 is still owing after 2 years.

Calculate the loan balance after Sophie reduces it. $\text{Reduced loan balance} = \$8851.80 - \$3400$
 $= \$5451.80$

Calculate the time required to repay the reduced loan balance.

Write the recurrence relation. $A_{n+1} = rA_n - R$

State any known values. $R = 229.64, r = 1 + \frac{0.112}{12}$

Substitute. $A_{n+1} = \left(1 + \frac{0.112}{12}\right) A_n - 229.64$

Open a spreadsheet and enter the headings 'n' in Cell A1, 'Start value' in B1 and 'End value' in C1 as described previously.

Fill out the values for columns A and B as previously described. You won't know how far to copy the values yet.

The value of the loan at the end of month 1 is $rA_{n-1} - R = rA_0 - R$, so enter ' $=(1 + 0.112/12)*B2 - 229.64$ ' in cell C2.

	A	B	C	D
1	n	Start value	End value	
2	1	5451.8	5273.043467	
3	2	5273.043467		
4	3			
5	4			
6	5			
7	6			

Copy the values in each column down until the value in column C changes from positive to negative.

23	24	668.0455514	444.6406432
26	25	444.6406432	219.1506225
27	26	219.1506225	-8.44397167
28	27	-8.44397167	-238.162782
29	28	-238.162782	-470.025635
30	29	-470.025635	-704.052544

State the result.

The loan will take another 27 months to repay.

Find the total amount repaid with the original payment.

Total repaid with original payment

$$= \$229.64 \times 72$$

$$= \$16\,534.08$$

Find the length of the loan when the balance is reduced by \$3400 after 2 years.

$$\text{Length of loan} = 24 + 27$$

$$= 51 \text{ months}$$

Find the total amount repaid when the balance is reduced by \$3400 after 2 years.

Total repaid with reduced balance

$$= \$229.64 \times 51 + \$3400$$

$$= \$15\,111.64$$

The difference between the total amounts repaid is the amount of interest saved.

$$\text{Interest saved} = \$16\,534.08 - \$15\,111.64$$

$$= \$1422.44$$

State the result.

Sophie saves \$1422.44 by reducing the loan balance by \$3400 after 2 years.

INVESTIGATION



Housing loan

HOUSING LOANS USING A SPREADSHEET

Open the spreadsheet called 'Housing loan' on NelsonNet. The top part of the spreadsheet shows that a housing loan of \$150 000 at 9% reducing interest will take 20 years to repay at \$1350 per month.



Getty Images/FatCamera

Use the spreadsheet to work out the loan balance (amount to be repaid) after 18 payments.

The bottom part of the spreadsheet shows what happens when an extra payment is then made. How does making a payment of \$2000 after the 18th payment change the loan balance and the time needed to repay the loan?

Experiment with the spreadsheet by changing:

- the interest rate
- the monthly payment
- the value of the extra payment.

(Note: If the spreadsheet records an error, it means that either the interest rate or the monthly payment you have entered is not acceptable because it would take too long to repay the loan.)

Write a short report outlining the effect of interest rate changes on mortgage repayments.

Exercise 7.07 Modifying loans

- 1 A loan is repaid in 60 monthly payments. Which of the following payments will reduce the principal by the greatest amount?
A 1st **B** 10th **C** 30th **D** 31st **E** 60th
- 2 An amount is borrowed with interest charged at 7.5% p.a. compounded monthly. Which of the following loan terms would result in the greatest amount of interest being paid?
A 10 years **B** 20 years **C** 5 years **D** 15 years **E** 1 year
- 3 An amount of \$10 000 is borrowed at an interest rate of 6.7% p.a. compounding monthly. The loan is repaid with monthly payments of \$196.60 over 5 years. Which of the following statements is incorrect?
A The number of repayments is 60.
B Increasing the monthly repayment to \$220 will decrease the length of the loan.
C If the interest rate increases to 7% p.a. and the repayment stays the same, the length of the loan will decrease.
D If the length of the loan is increased to 7 years, the monthly payment required will decrease.
E If the length of the loan is increased to 7 years, the total amount of interest paid will increase.
- 4 Oliver borrows \$25 000 at 6.4% p.a. compounding monthly over 7 years with monthly repayments.
 - a How much is Oliver's monthly repayment?
 - b How much does he repay in total?
 - c How much interest does Oliver pay?
 - d After 3 years, the interest rate is increased to 7.9% p.a. and remains the same for the rest of the loan. What is the new repayment amount required for Oliver to pay off the loan in the same period of time?
 - e How much more does it cost Oliver to pay out the loan due to the increase in the interest rate?

Example
19



Shutterstock.com/Nestor Rzhnick

- 5** Poppy borrows \$108 000 at 8.8% p.a. compounding quarterly over 12 years with quarterly repayments.
- a** What is Poppy's quarterly repayment?
 - b** After 5 years, the interest rate decreases to 6.5% p.a. and remains the same for the rest of the loan. What is the new repayment amount required for Poppy to pay off the loan in the same period of time?
 - c** How much does Poppy save due to the decrease in the interest rate?

Example
20

- 6** Xavier borrows \$94 000 at 8.8% p.a. compounding fortnightly with fortnightly repayments of \$582.22.
- a** How long will it take Xavier to repay the loan?
 - b** After 4 years, Xavier finds that he can afford to repay \$710 a fortnight. How many more payments of \$710/fortnight will Xavier need to repay the loan?
 - c** How much interest did Xavier save by increasing his repayment?
- 7** Ellie borrows \$58 000. She agrees to quarterly repayments of \$1749.55 for 12 years at 6.5% p.a. compounded quarterly. However, after 2 years Ellie finds that she can only afford to make quarterly repayments of \$1400.
- a** What amount is still owing after 2 years?
 - b** How much more time will Ellie need to pay out the loan?
 - c** How much extra interest will Ellie need to pay?

Example
21

- 8** Luke takes out a loan of \$180 000 at 7.6% p.a. compounded monthly to buy a home unit. He pays \$1531.67 a month for 18 years to repay the loan. After 5 years, Luke inherits \$8000 from his grandmother's estate.
- a** How much is still owing on the loan at this time?
 - b** If Luke uses his inheritance to reduce the loan balance and continues to make his current repayments, how much interest will Luke save?
- 9** Piper borrows \$120 000 at 9.2% p.a. compounded monthly. She pays \$1231.44 a month for 15 years to repay the loan. After 7 years, Piper needs access to \$11 800 to pay for some urgent repairs to her home.
- a** How much is still owing on the loan at this time?
 - b** If Piper increases her loan balance by \$11 800 and continues to make her current repayments, how much additional interest will Piper pay on the loan?

Problem solving

- 10 a** Find the monthly repayment on a loan of \$150 000 at 9.25% p.a. reducing monthly over 20 years.
- b** Calculate the value of the fortnightly payment that would result in the same annual amount being repaid as the monthly repayments for this loan.
- c** How much earlier is the loan paid off by making fortnightly repayments with interest calculated fortnightly?
- 11** Nate needs a new ute for his landscaping business so he borrows \$45 000 at 12.7% p.a. with interest calculated monthly. He agrees to make repayments of \$896.23 to service the loan. After 3 years, the interest rate drops to 10.9% p.a. and Nate takes the opportunity to pay \$4000 off the balance of the loan. He continues with the same monthly repayment.
- a** By how much did these actions reduce the time taken to repay the loan compared with the original arrangement?
- b** How much interest did Nate save due to these actions?
- 12** Some lenders recommend the use of fortnightly payments for mortgages. They claim that making half the monthly payment each fortnight considerably reduces the interest and time taken to pay off the mortgage. Why would making fortnightly payments in this way reduce the time taken to pay off the loan?

7. CHAPTER SUMMARY

Investments and loans

- **Interest** is the price of borrowing or lending money
- **Simple interest** is calculated using the formula $I = Pin$, where I is the interest, P is the **principal** (amount borrowed), i is the annual interest rate as a decimal and n is the number of years
- **Compound interest** results when interest earned for a **compounding period** (or **rest**) is added to the principal; the interest for the next compounding period is earned on the total of principal and interest
- The **periodic interest rate**, $i = \frac{\text{annual rate}}{\text{number of periods per year}}$
- The value of a compound interest investment at the end of the n th compounding period is given by the recurrence relation:

$$A_{n+1} = rA_n$$

where: A_0 = the principal

A_n = the value of the investment at the end of n compounding periods

$$r = 1 + i$$

i = periodic interest rate as a decimal

- Compound interest formula

$$A = P(1 + i)^n$$

where: A = amount of the investment after n compounding periods (A_n)

P = principal or initial value (A_0)

i = periodic interest rate as a decimal

n = number of compounding periods

- The interest rate that is stated for a loan or investment is also known as the **nominal interest rate**
- The **effective interest rate**, e , takes the number of rests into account and is calculated using the formula

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

where $i_{\text{effective}}$ = the effective interest rate

i = annual interest rate as a decimal

n = number of compounding periods in a year

- **Present and future value of money**

$$PV = FV(1 + i)^{-n}$$

$$FV = PV(1 + i)^n$$

where: FV = future value of money

PV = present value of money

i = periodic interest rate as a decimal

n = number of compounding periods

- A loan is repaid by **instalments** or **repayments** spread evenly over the term of the loan
- A **reducing interest** or **reducing balance loan** is one in which the interest is calculated using the amount still owing at the beginning of each compounding period
- **Value of a reducing balance loan**

$$A_{n+1} = rA_n - R$$

where: A_0 = the initial value of the loan

A_n = the value of the loan at the end of n compounding periods

$$r = 1 + i$$

i = periodic interest rate as a decimal

R = the repayment per compounding period

- The progress of payments is shown in a **loan schedule**
- Reducing balance loan formulas

$$\text{Repayment value, } Q = \frac{Ar^n(r-1)}{r^n-1}$$

$$\text{Loan balance, } B_n = Ar^n - \frac{Q(r^n-1)}{r-1}$$

where: Q = the repayment amount per period

A = the amount borrowed (principal) of a loan

B_n = the balance of the loan after n repayments

$$r = 1 + i$$

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

- For **fixed-term loans**, the interest rate, instalment value or the frequency of payments are constant, while for **flexible-term loans**, they may vary

7. CHAPTER REVIEW

Investments and loans

Example

1

- 1 \$8000 is invested at 6% p.a. compounded yearly. Without using the compound interest formula, calculate:
- a the total value of the investment at the end of 3 years
 - b the total interest earned after 3 years

Example

2

- 2 Which investment has its yearly value described by the recurrence relation $A_{n+1} = 1.12A_n$, where $A_0 = 20\,000$?
- A \$20 000 invested at 0.12% p.a. compounded yearly
 - B \$20 000 invested at 1.12% p.a. compounded yearly
 - C \$11 200 invested at 0.12% p.a. compounded yearly
 - D \$20 000 invested at 12% p.a. compounded yearly
 - E \$11 200 invested at 12% p.a. compounded yearly

Example

3

- 3 \$26 000 is invested at 5.4% p.a. compounding yearly. Use a recurrence relation to calculate:
- a how much the investment is worth at the end of 7 years
 - b the interest that was earned in the 4th year of the investment

Example

4

- 4 \$15 000 is invested in a term deposit that pays 5% p.a. interest compounded monthly. Using the compound interest formula, the value of this investment after 9 years is closest to:
- A \$15 571 B \$23 500 C \$18 500 D \$20 700 E \$33 500

Example

5

- 5 \$22 000 is deposited in an interest-bearing account with an annual interest rate of 8%. Use the compound interest formula to calculate the difference in the value of the investment after 6 years, if interest is compounded yearly and monthly.

Example

6

- 6 Use the compound interest formula to calculate:
- a the amount that needs to be invested at 7% p.a. compounded quarterly to have \$25 000 available in 6 years
 - b the compound interest rate that was applied if an amount of \$9500 grew to \$12 500 over 3 years, if the interest was compounded monthly

Example

7

- 7 Without using the effective interest rate formula, calculate the effective rate of interest for an investment earning 6% p.a. compounded quarterly.

Example

8

- 8 Using the effective interest rate formula, the effective rate of interest for an investment earning 8% p.a. compounded monthly is closest to:
- A 8.3% B 8.5% C 8.2% D 8.4% E 8.1%

- 9** An investor is looking for the best return from 3 different offers. Which of the following interest rates would be best for the investor?
- A** 7.51% p.a. compounded weekly
 - B** 7.45% p.a. compounded daily
 - C** 7.59% p.a. compounded monthly
- 10** Use the future value formula to find the amount to which \$10 500 will grow after 7 years at 8% p.a. compound interest with quarterly rests.
- 11** Use the present value formula to find how much must be invested at 7.5% p.a. compound interest with fortnightly rests to obtain \$15 000 in 5 years time.
- 12** Isla pays \$3500 a year into a life insurance fund. She is promised a lump sum of \$320 000 after 38 years. If the average interest rate over this period is 6.2% p.a., calculate the present value of the payout.
- 13** Flynn borrowed \$19 000 at 9% p.a. interest compounding monthly with monthly repayments of \$251.
- a** Draw up a schedule showing the progress of the loan over the first 6 months.
 - b** State the total interest paid, the total repayments made, the reduction of the principal and the balance of the loan after 6 months.
- 14** On 7 April, Lucas and Lily borrow \$220 000 from a bank to buy a home unit. They borrow the money at 8.3% p.a. Interest is calculated monthly and they must pay \$1881.45 each month for 20 years to repay the loan. Use a recurrence relation to calculate:
- a** the amount still owing on the loan on 7 September in the same year
 - b** the total interest that has been paid to the bank during this period.
- 15** Mason borrowed \$42 000 to buy new equipment for his landscaping business. The loan is at 11.3% p.a. with interest added monthly. His monthly repayments are \$919.
- a** Write a recurrence relation that Mason could use to calculate the amount owing after n months.
 - b** How much does Mason still owe after 3 years?
 - c** If Mason could have secured a loan at 9.3% p.a., how much would he still owe after 3 years with the same repayments?
- 16** Ivy borrows \$55 000 at 10.9% p.a. compounded monthly from her bank to carry out some major home repairs. Calculate:
- a** the monthly repayment if the loan is paid off after 7 years
 - b** the total amount repaid
 - c** the total interest paid

Example
9

Example
10

Example
11

Example
12

Example
13

Example
14

Example
15

Example
16

Example
17

17 A fruit shop owner borrows \$28 000 at 12.2% p.a. interest compounding quarterly to upgrade a refrigeration unit. If fortnightly repayments of \$1892 are made, what is the balance of the loan after 3 years?

Example
18

18 \$42 000 is borrowed at a rate of 11.4% p.a. compounded fortnightly. How long will it take to pay off the loan if fortnightly payments of \$320.87 are made?

Example
19

19 Hudson borrowed \$65 000 at 8.2% p.a. compounding monthly over 8 years with monthly repayments of \$925.51. After 3 years, the interest rate is reduced to 7% p.a.

- a** If Hudson wants to pay off the loan in the same period of time, what monthly repayment should he make?
- b** How much does Hudson save on his loan?

Example
20

20 Zara borrows \$125 000 at 10.3% p.a. compounding monthly with monthly payments of \$1586.28.

a How long will it take Zara to repay the loan?

After 4 years, Zara increases her to repayments \$2000 a month.

b How does this affect the period required to repay the loan?

c How does this affect the total interest paid?

Example
21

21 Lucas borrows \$18 000 at 10.6% p.a. compounded monthly to buy a jet ski. He agrees to pay \$338.94 a month for 6 years to repay the loan. After 3 years, Lucas reduces the loan balance by \$3950 by making a lump sum payment. If he continues to make his regular payments, how much interest does Lucas save?



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Problem solving

22 A loan of \$25 000 is borrowed at 11.5% p.a. interest compounded annually. The loan is repaid with 3 annual instalments leaving a balance of \$6714.46 paid at the end of the 4 years. Determine the value of the annual payments.

23 Two banks offer the following investment packages:

- Western Mutual: 8.2% p.a. compounded annually, fixed for 6 years.
- Eastpac: 6.3% p.a. compounded annually, fixed for 10 years.

If Carrie invests \$50 000 with Western Mutual, how much would she have to invest with Eastpac to produce the same amount as Western Mutual at the end of the investment period?

24 Ella borrowed \$38 000 at 11.3% p.a. compounded monthly. She elects to repay the loan over 8 years with monthly instalments of \$603.09. After 3 years, Ella decides that she would like to repay the loan in full in the next $3\frac{1}{2}$ years. Find the monthly repayment needed to achieve her goal.



Practice quiz

8

NETWORKS, PATHS AND TREES

The branch of mathematics called **graph theory** uses diagrams with points and lines to plan and analyse networks and connections. You can use graph theory to plan road networks, railway systems, computer cabling layouts, pipeline systems, scheduling of large building projects, and designing computer chips.

Using graph theory, you can find the optimal route for a particular situation. This could be the least expensive, involve the least amount of time or cover the least distance to connect networks.

- 8.01 Graphs and networks
- 8.02 Applications of networks
- 8.03 Adjacency matrices
- 8.04 Planar graphs and Euler's formula
- 8.05 Eulerian graphs
- 8.06 The travelling salesperson problem
- 8.07 Spanning trees
- 8.08 Minimum spanning trees
- 8.09 Flow networks
- 8.10 The maximum flow–minimum cut theorem
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Graphs, associated terminology and the adjacency matrix

- understand the meanings of the terms graph, edge, vertex, loop, degree of a vertex, subgraph, simple graph, complete graph, bipartite graph, directed graph (digraph), arc, weighted graph and network
- identify practical situations that can be represented by a network and construct such networks, e.g. trails connecting camp sites in a national park, a social network, a transport network with one-way streets, a food web, the results of a round-robin sporting competition
- construct an adjacency matrix from a given graph or digraph

Planar graphs, paths and cycles

- understand the meaning of the terms planar graph and face
- apply Euler's formula, $v + f - e = 2$, to solve problems relating to planar graphs
- understand the meaning of the terms walk, trail, path, closed walk, closed trail, cycle, connected graph and bridge
- investigate and solve practical problems to determine the shortest path between two vertices in a weighted graph (by trial-and-error methods only)
- understand the meaning of the terms Eulerian graph, Eulerian trail, semi-Eulerian graph, semi-Eulerian trail and the conditions for their existence, and use these concepts to investigate and solve practical problems, e.g. the Königsberg bridge problem, planning a garbage bin collection route
- understand the meaning of the terms Hamiltonian graph and semi-Hamiltonian graph and use these concepts to investigate and solve practical problems (by trial-and-error methods only), e.g. planning a sightseeing tourist route around a city, the travelling-salesman problem

Trees and minimum connector problems

- understand the meaning of the terms tree and spanning tree
- identify practical examples
- identify a minimum spanning tree in a weighted connected graph, e.g. using Prim's algorithm
- use minimal spanning trees to solve minimal connector problems, e.g. minimising the length of cable needed to provide power from a single power station to substations in several towns

Flow networks

- solve small-scale network flow problems including the use of the 'maximum-flow minimum-cut' theorem, e.g. determining the maximum volume of oil that can flow through a network of pipes from an oil storage tank to a terminal



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

adjacency matrix
bipartite graph
chain
connected
degree (of a vertex)
disconnected graph
edge
graph
inflow capacity
loop
node
planar graph
route
sink
subgraph
undirected
walk

adjacent vertices
branch
circuit
cut
digraph
Euler's formula
flow capacity
Hamiltonian cycle
isolated vertex
network
outflow capacity
Prim's algorithm
semi-Eulerian graph
spanning tree
trail
valid cut
weighted

arc
capacity
complete graph
cycle
directed
Eulerian graph
flow network
Hamiltonian path
isomorphic graphs
network flow
path
residual flow capacity
simple graph
source
tree
vertex



Types of
graphs

8.01 Graphs and networks

In this chapter, a **graph** is defined as a set of points called **vertices** (or **nodes**) connected by lines called **edges** (or **arcs**), with each edge joining a pair of vertices. Although edges are often drawn as straight lines, curves can also be used.

The vertices are A, B, C, D, E and F .

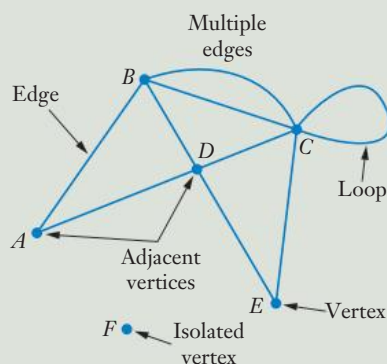
The edges are $AB, AD, BD, BC, DE, DC, CE$ and CC .

Two vertices connected by an edge are called **adjacent vertices**.

A **loop** is an edge that connects a vertex to itself.

The graph has **multiple edges** if two or more edges connect the same pair of vertices.

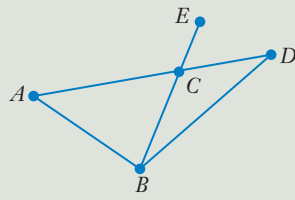
An **isolated vertex** is one that is not connected to any other vertex.



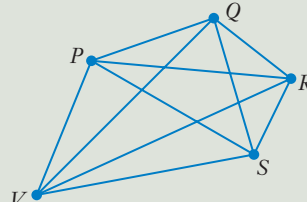
You usually label vertices so they can be identified. You can classify graphs using the way that the vertices are connected. The edges of a graph can intersect without there being a vertex.

A **simple graph** is one where the vertices are connected by no more than one edge.

A **complete graph** has an edge connecting each vertex to all other vertices in the graph.

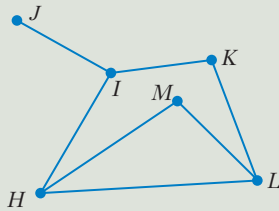


Simple graph

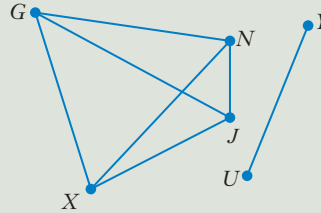


Complete graph

If it is possible to reach every vertex of a graph by moving along the edges, it is called a **connected graph**; otherwise, it is a **disconnected graph**.

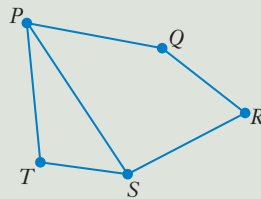


Connected graph

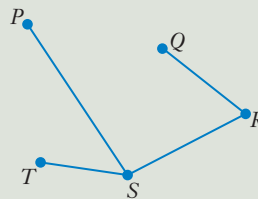


Disconnected graph

A **subgraph** is a graph that is all or part of another graph without any new edges or vertices.



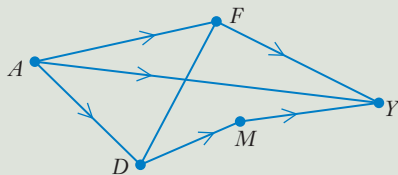
Graph



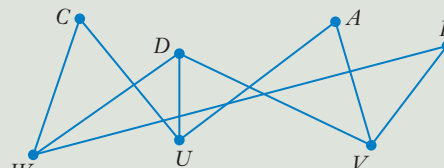
Subgraph

In a **directed graph** each edge has a direction indicated by an arrow.

A **bipartite graph** is a graph whose set of vertices can be divided into two distinct subsets so that every edge of the graph connects a vertex in one subset and a vertex in the other.



Directed graph

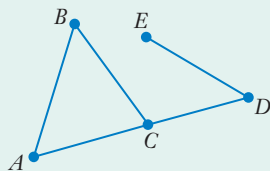


Bipartite graph

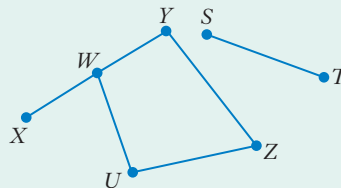
EXAMPLE 1

Use the graphs below to answer the questions that follow.

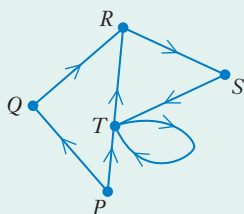
a



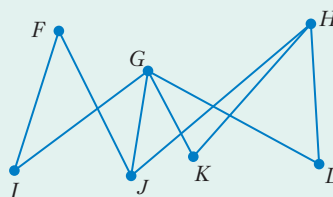
b



c



d



- i** Identify the number of vertices and edges in each graph.
- ii** Identify any connected graph(s).
- iii** Identify any simple graph(s).
- iv** Identify any directed graph(s).
- v** Identify any bipartite graph(s).

Solution

- i** Graph **a** has 5 points and 5 lines connecting these points.

Graph **a** has 5 vertices and 5 edges.

Graph **b** has 5 points connected by 5 lines and 2 points connected by 1 line.

Graph **b** has 7 vertices and 6 edges.

Graph **c** has 5 points connected by 6 lines and one loop.

Graph **c** has 5 vertices and 7 edges.

Graph **d** has 7 points and 9 lines connecting these points.

Graph **d** has 7 vertices and 9 edges.

- ii** Try tracing the outline of each graph to include every vertex without lifting your pencil.

Graphs **a**, **c** and **d** are connected graphs.

- iii** Find any graph in which each pair of vertices is only connected by one edge.

Graphs **a** and **b** are simple graphs.

- iv Find any graphs in which each edge has a direction indicated by an arrow.
- v Find any graph where the vertices can be divided into two distinct groups so that every edge of the graph connects a vertex in one group and a vertex in the other group.

Graph **c** is a directed graph.

Graph **d** is a bipartite graph.

A **network** is a type of graph consisting of a collection of vertices connected by edges. Networks are classified according to their characteristics.

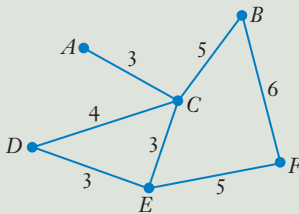
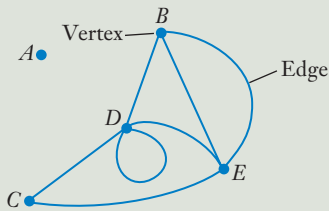
A **finite network** has a limited number of vertices and edges.

In a **directed network**, each edge has a *direction*.

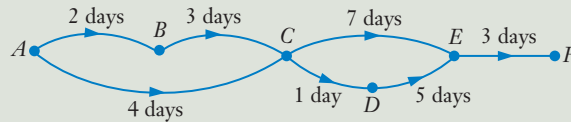
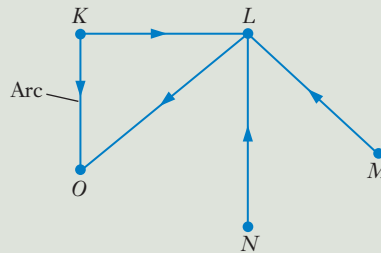
An **undirected network** has edges with no directions.

In some networks, a number is given for the 'length' of each edge. The length may be a distance, time, cost or some other quantity. These are called **weighted networks**.

Undirected networks



Directed networks



Some mathematicians reserve the term 'network' for graphs that have a length for each edge. In this book, 'network' is used for all graphs consisting of vertices (nodes) and edges (arcs).

Specific terms are used to describe the ways in which the vertices of a network connect. These terms are derived from common English expressions but have exact mathematical meanings.

A **branch** is a vertex that has more than one exit. It has more than 2 edges joined to it.

A **path** or **chain** joins a starting vertex to a finishing vertex via a continuous series of edges. For directed networks, a path must have the edges in the same direction. You can travel along a path.

A **circuit** or **cycle** is a path that returns to its starting point. For directed networks, a circuit must have the edges in the same direction. You can travel right around a circuit.

A **simple** chain, path, cycle or circuit does not cross over itself, so does not repeat vertices.

In a **connected network**, every vertex has at least one chain connecting it to every other vertex. A **disconnected network** is not connected.

A **cycle network** or **circuit network** consists of a single simple cycle or circuit.

A **tree** is a subgraph of a network that has no cycles, loops or multiple edges.

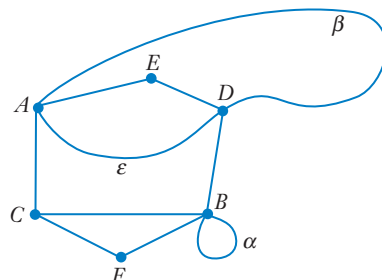
A **spanning tree** is a tree that contains every vertex of a network.

In many network problems, loops or multiple edges are not permitted. The loops are omitted and only the shortest (or longest) edge is used.

In the network on the right, there is a branch at C and the loop α starts and finishes at B .

A and D have multiple edges β and ϵ joining them.

A is adjacent to D , C and E .



In directed networks, you can distinguish a path from a chain, or a circuit from a cycle, by the direction of the edges.

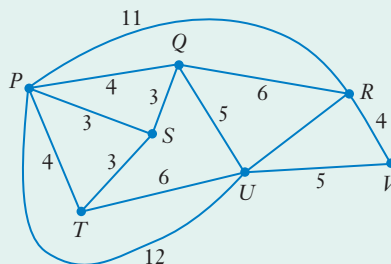
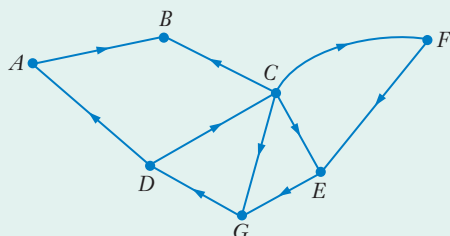
In undirected networks, a chain is also a path, because the edges can be traversed in either direction. Similarly, a cycle must also be a circuit. The terms 'chain' and 'cycle' are redundant in undirected networks, so are not generally used.

EXAMPLE 2

Use the diagrams below to classify each of the following.

a $ABCEFC$, $DCFE$, $DCEF$, $ABCD$, $DCFEGD$

b $SQUTPUR$, $QSPRVURQ$, $PQRVUTP$



Solution

a $ABCDEF$ is a network in which each edge has a direction.

$ABCDEF$ is a directed network.

$ABCEFC$ connects a sequence of vertices, but the edges are not all in the same direction. It crosses over itself, so is not simple.

$ABCEFC$ is a chain.

$DCFE$ has a connected sequence of edges in the same direction and does not cross over itself.

$DCFE$ is a simple path.

$DCEF$ has a connected sequence of edges, but they are not all in the same direction. It does not cross over itself.

$DCEF$ is a simple chain.

$ABCD$ returns to its starting point without crossing itself, but the edges are not all in the same direction.

$ABCD$ is a simple cycle.

$DCFEGD$ returns to its starting point without crossing over itself and the edges are all in the same direction.

$DCFEGD$ is a simple circuit.

b $PQRSTUV$ is a network in which no edges have direction.

$PQRSTUV$ is an undirected network.

$SQUTPUR$ is connected in order, but crosses over itself, so it is not simple.

$SQUTPUR$ is a path.

$QSPRVURQ$ returns to its starting point, but crosses over itself, so it is not simple.

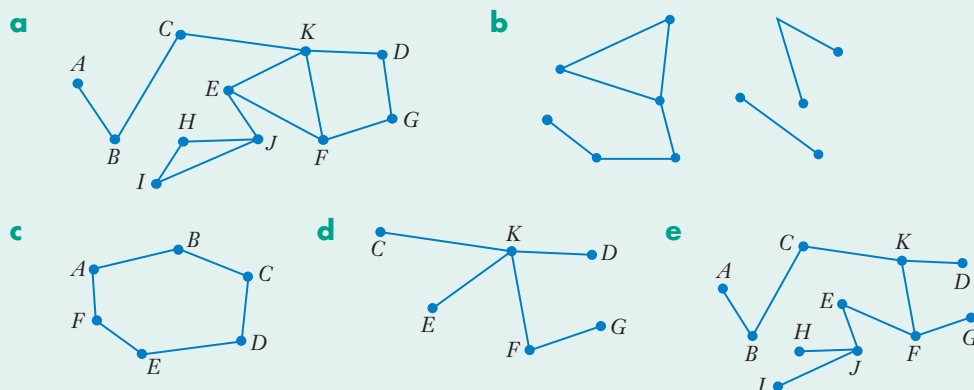
$QSPRVURQ$ is a circuit.

$PQRVUTP$ returns to its starting point and does not cross over itself.

$PQRVUTP$ is a simple circuit.

EXAMPLE 3

Classify each network.



Solution

All 5 networks are undirected.

a All vertices of the network are joined.

It is a connected network.

b Not all the vertices in the network are connected and it is in 3 separate parts.

It is a disconnected network with 3 parts.

c This is a path that returns to its starting point and does not cross over itself. The network is a single simple circuit.

It is a circuit network.

d The network has no circuits and is part of network **a**.

It is a tree that is a subnetwork of part **a**.

e The network has no circuits and includes all vertices of network **a**.

It is a spanning tree of part **a**.

INVESTIGATION

VERTICES AND EDGES

Work in groups of 2 or 3 people for this investigation.

- 1 Each person in the group should make up 3 or 4 connected networks on a sheet of paper.
- 2 Swap papers so that each person has someone else's networks. Now remove some of the edges so that each network is trimmed down to a spanning tree.
- 3 Redraw the trees on new pieces of paper with the names of the originator and trimmer on the top.
- 4 As a group, look at the trees and try to decide whether any of the trees are the same.
- 5 Now count the vertices and edges on each tree. Write the number of vertices and the number of edges next to each tree.
- 6 Discuss your findings about the numbers of vertices and edges on trees. Can you give a general rule for the numbers of vertices and edges?

As a class group, discuss when trees should be considered the same. As a class group, discuss when you should consider networks to be the same.



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Exercise 8.01 Graphs and networks

1 Which of these is not a bipartite graph?

A



B



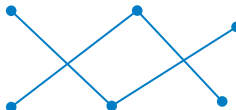
C



D



E



2 A complete graph has 5 vertices. How many edges does it have?

A 7

B 8

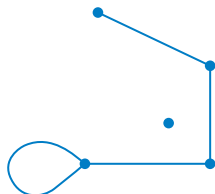
C 9

D 10

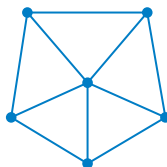
E 11

3 Count the number of vertices and edges in each graph.

a



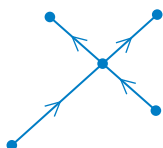
b



c



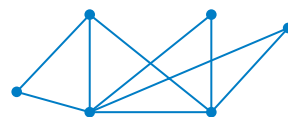
d



e



f



4 List the graphs shown in question 3 that are:

a connected

b complete

c directed

d simple

e bipartite

f disconnected

5 Answer the following questions about this graph.

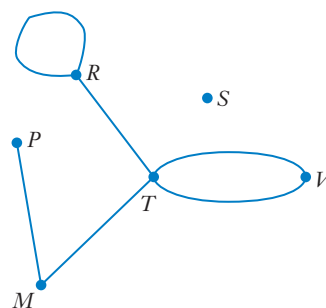
a Name the vertex adjacent to P .

b The loop connects which vertex to itself?

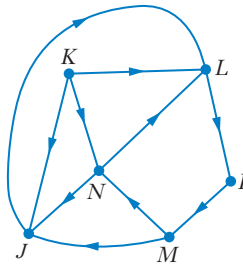
c Name the isolated vertex.

d Name the vertices connected by multiple edges.

e Name the vertex adjacent to M .

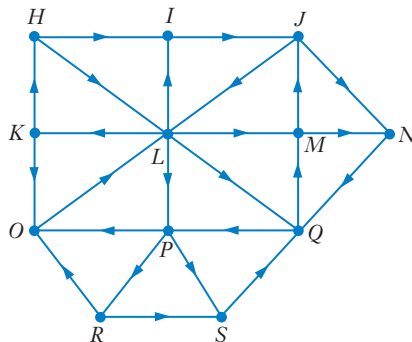


a	<i>ABCD</i>	b	<i>ABCDBEA</i>	c	<i>ABEA</i>
d	<i>ABDCBE</i>	e	<i>KLNK</i>	f	<i>KLIM</i>
g	<i>JLIMJ</i>	h	<i>KNLKJ</i>	i	<i>JLIMN</i>
j	<i>KLNJMNK</i>				

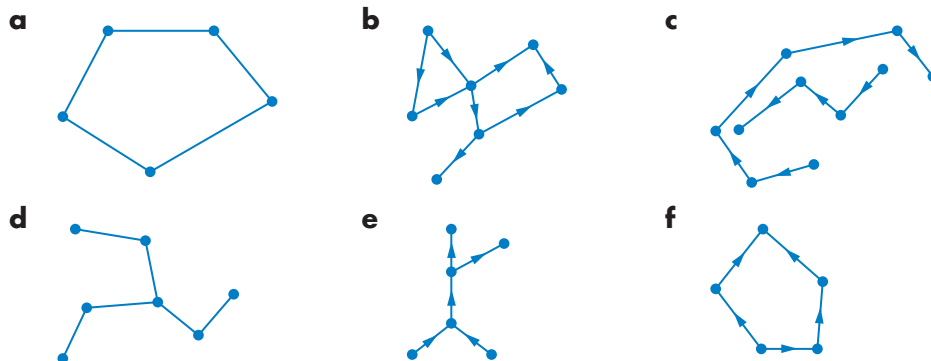


-

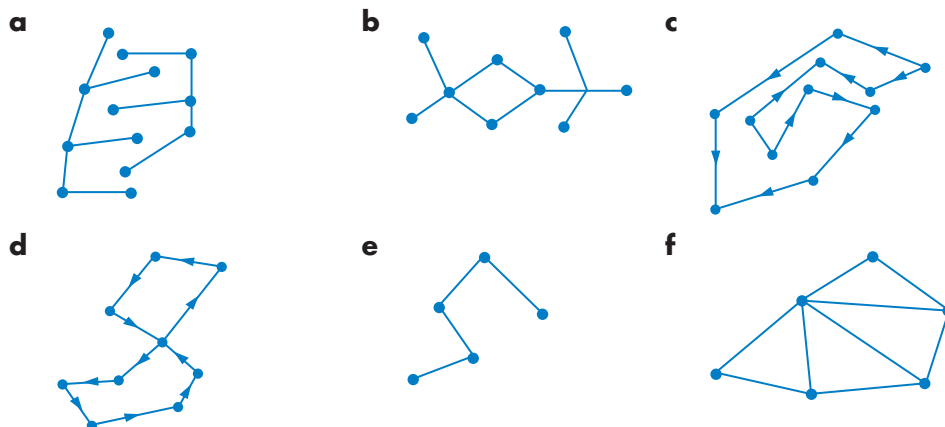
a	<i>ABCEBDG</i>	b	<i>ABEDGFA</i>	c	<i>BEDBAGCB</i>
d	<i>FDE</i>	e	<i>LPSQNJL</i>	f	<i>NMJIHKLJN</i>
g	<i>KHLIJLQPOLK</i>	h	<i>MNQPRS</i>	i	<i>HLIJNQMJLP</i>
j	<i>ORPOL</i>				



- 9 Classify each network as directed or undirected; connected or disconnected; and cycle, circuit, tree or mixed type.



- 10 Classify each network as directed or undirected; connected or disconnected; and cycle, circuit, tree or mixed type.



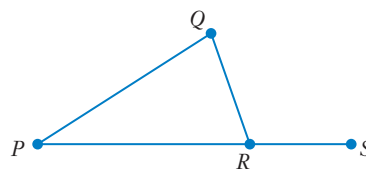
Problem solving

- 11 The graph shown here can be described using the vertices (V) and edges (E) as:

$$V = \{P, Q, R, S\} \text{ and } E = \{PQ, QR, PR, RS\}.$$

Draw a graph for each of the following:

- a $V = \{A, B, C\}$ and $E = \{AB, BC\}$
- b $V = \{W, X, Y, Z\}$ and $E = \{WX, XY, YW\}$
- c $V = \{P, Q, R, S\}$ and $E = \{PS, QS, SR\}$
- d $V = \{M, N, O, P, Q, R\}$ and $E = \{MN, NO, OP, PQ, QM, MR, NR, OR, PR, QR\}$.
- e $V = \{V, T, U, W, X\}$ and $E = \{TV, TV, WT, WU, VW, XV, XW, VU\}$
- f $V = \{H, K, M, L, N\}$ and $E = \{HK, KM, HL, LM, LN, NM\}$
- g $V = \{F, K, J, L, M, P\}$ and $E = \{FJ, FL, JK, LK, PM\}$
- h $V = \{A, H, Q, M, V\}$ and $E = \{AH, QH, MQ, MM\}$



- 12** A medical practice consists of 4 doctors: Aria, Benjamin, Hannah and Oscar. The practice does not operate on a Sunday. The availability of the doctors to work on particular days is:

Aria is available on Tuesday, Thursday and Saturday.

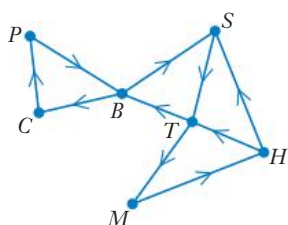
Benjamin is available every day.

Hannah is available on Wednesday, Thursday and Friday.

Oscar is available on Monday and Saturday.

Draw a graph to model this situation.

- 13** One-way streets connect various points of interest in a city as shown below.



Key:

S = School

B = Bus stop

C = Cinema

T = Train station

P = Petrol station

H = Hospital

M = Museum

- Is it possible to travel from the cinema to every other location without passing a location more than once?
- Describe 2 routes from the museum to the bus stop.
- Is it possible to travel from the school to every other location without passing a location more than once?
- How many different routes are there from the bus stop to the hospital?

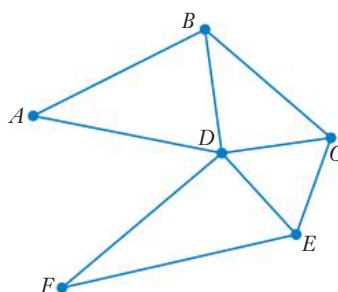
8.02 Applications of networks

If you want to analyse and compare graphs, you can check the number of edges that are directly connected to a particular vertex. This is called the **degree** of the vertex and is written $\deg(V)$, where V is the label of the vertex.

In this diagram, for example, $\deg(A) = 2$ and $\deg(D) = 5$.

$$\begin{aligned}
 \text{Sum of degrees} &= \deg(A) + \deg(B) + \deg(C) + \deg(D) \\
 &\quad + \deg(E) + \deg(F) \\
 &= 2 + 3 + 3 + 5 + 3 + 2 \\
 &= 18
 \end{aligned}$$

You will see that there are 9 edges – half the sum of the degrees of the vertices.



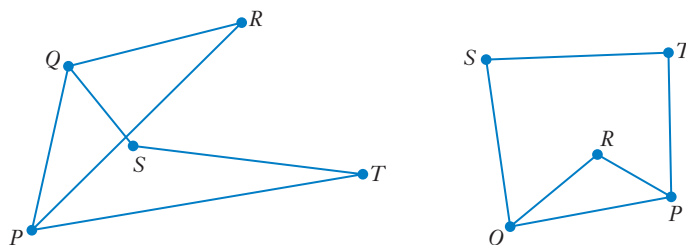
Degree sum of a network

For any network:

Sum of the degrees of the vertices = $2 \times$ total number of edges

If a vertex has a loop, you need to count the start and the end.

The 2 graphs below don't look much like each other. There is a way to determine if the graphs show the same information.



Each graph has 5 vertices.

The degree of corresponding vertices in each graph is also the same.

The connections associated with each vertex in the first graph are shown below.

Vertex	Connections		
<i>P</i>	<i>Q</i>	<i>R</i>	<i>T</i>
<i>Q</i>	<i>P</i>	<i>S</i>	<i>R</i>
<i>R</i>	<i>Q</i>	<i>P</i>	
<i>S</i>	<i>Q</i>	<i>T</i>	
<i>T</i>	<i>S</i>	<i>P</i>	

If you check, you will see that the connections of corresponding vertices are the same for the second graph.

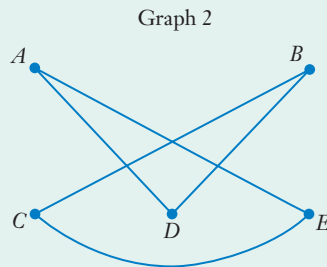
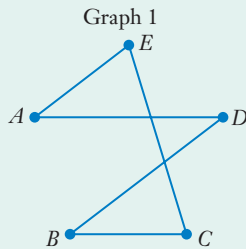
These graphs represent the same information and are called **isomorphic**.

Isomorphic graphs

Isomorphic graphs have the same number of vertices and edges, and corresponding vertices have identical degrees and connections.

EXAMPLE 4

Determine if these 2 graphs are isomorphic.



Solution

State the number of vertices and edges for each graph.

Graph 1: 5 vertices, 5 edges

Graph 2: 5 vertices, 5 edges

State the degree of the vertices.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
Graph 1	2	2	2	2	2
Graph 2	2	2	2	2	2

Identify the vertex connections for each graph.

Vertex	Connections	
<i>A</i>	<i>D</i>	<i>E</i>
<i>B</i>	<i>D</i>	<i>C</i>
<i>C</i>	<i>B</i>	<i>E</i>
<i>D</i>	<i>A</i>	<i>B</i>
<i>E</i>	<i>A</i>	<i>C</i>

The connections are the same for each graph.

State the result.

The 2 graphs have the same number of vertices and edges, and the same degree and connections for corresponding vertices. This means that the graphs are isomorphic.

Graphs are used to represent many different types of connections. Consider a group of 5 people in a room who are meeting for the first time. Each person shakes hands with every other person once. How many handshakes are there in total? This is known as the **handshake problem** and it has a number of applications in mathematics.



Shutterstock.com/Olena Yakobchuk

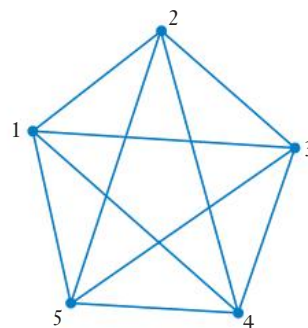
This situation can be represented by a graph where the vertices represent people and the edges represent handshakes, as shown here.

What type of graph is this?

If you count the number of edges (handshakes), you get 10.

Since each person shakes hands with every other person, why isn't the answer $5 \times 4 = 20$?

The reason the answer isn't 20 is that each pair of people only shake hands *once*. So the number of handshakes is $\frac{5 \times 4}{2} = 10$.



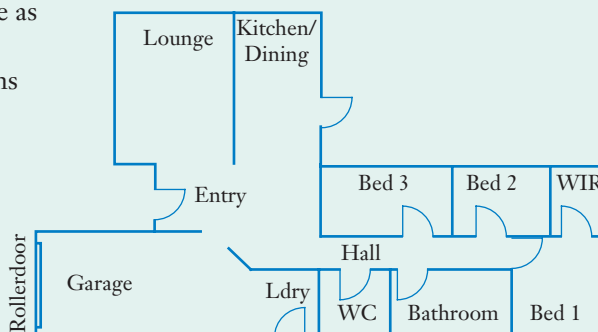
In general, if there are n people and each person shakes hands with every other person once, the number of handshakes is given by:

$$\text{Number of handshakes} = \frac{n(n-1)}{2}$$

While roads and phone lines form obvious networks, you can treat many other structures as networks. Any structure that has connections of some kind between points can be modelled as a network.

EXAMPLE 5

Model the house plan shown here as a network, showing the doors (doorways) as edges and the rooms as vertices.



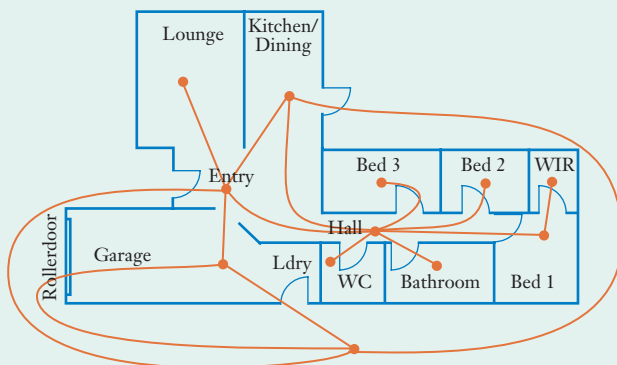
Solution

Use a pencil or overlay.

Put a dot over each room (including the Entry and Hall).

Put one dot on the outside.

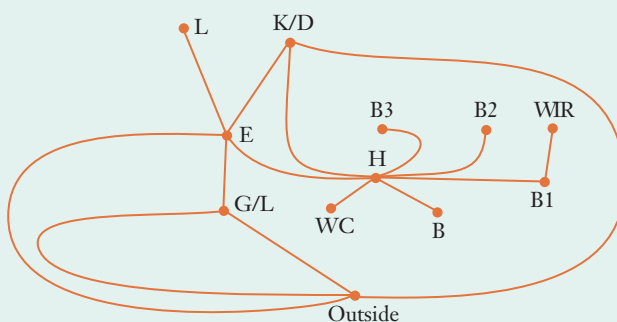
Connect the dots through the doorways.



Redraw the network without the plan.

Show the connections clearly.

Label each vertex. There are 2 connections from the Garage/Laundry to the Outside.



EXAMPLE 6

The map of a large island is shown on the right. Use a network to find the smallest number of colours that can be used to colour the map so that every border has a different colour on each side.



Solution

Replace each area by a small circle, and show each border as a connection.

The outline and division lines can then be discarded.



Start by colouring (red) a circle on the network. Then colour as many unconnected circles as possible with the same colour.



Colour another vertex with a different colour (green) and any unconnected circles with this colour.



Colour the last vertex with a different colour (blue).



Write the answer.

3 colours are needed.

Exercise 8.02 Applications of networks

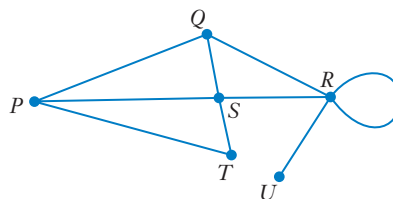
Questions 1 and 2 refer to this graph.

1 The degree of vertex S in the figure is:

- A** 1 **B** 2 **C** 3
D 4 **E** 5

2 The sum of the degrees of the vertices is:

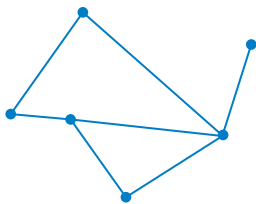
- A** 14 **B** 18 **C** 16



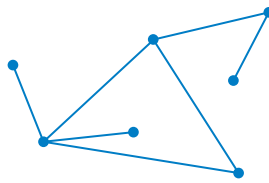
- D** 17 **E** 15

3 For each graph, verify that the sum of the degrees of the vertices is twice the number of edges.

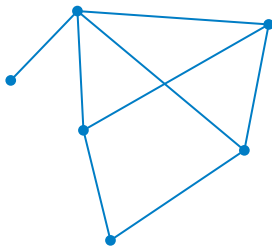
a



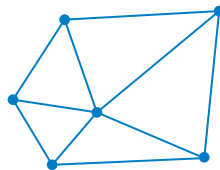
b



c

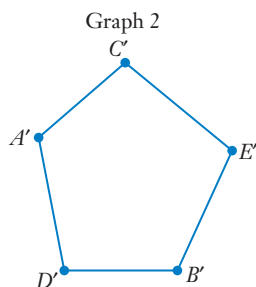
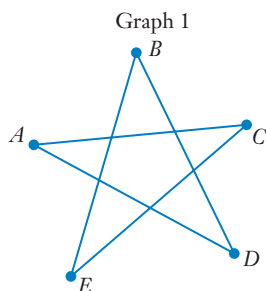


d

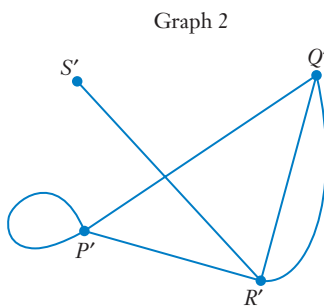
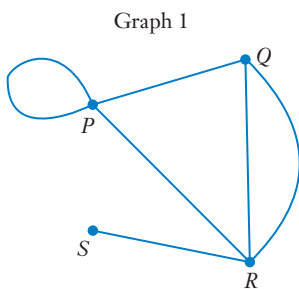


4 Show that the graphs in each pair are isomorphic.

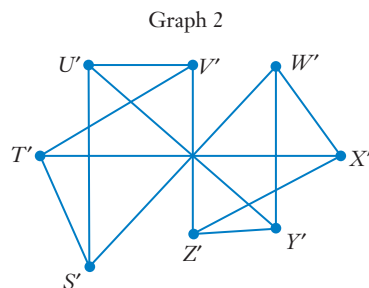
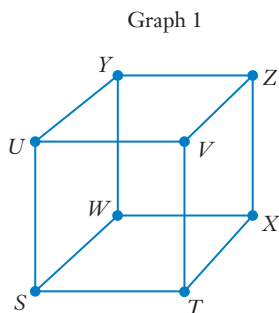
a



b



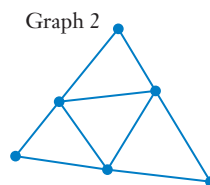
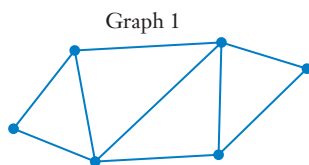
c



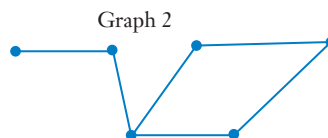
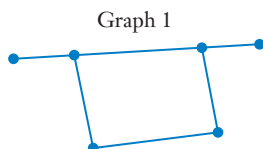
Example
4

5 Show that the graphs in each pair are not isomorphic.

a



b



6 A netball carnival has 6 teams competing over 2 days. The roster for the tournament is shown below.

	Tigers	Bullets	Raiders	Chargers	Lightning	Victory
Tigers			✓		✓	✓
Bullets			✓	✓		✓
Raiders	✓	✓		✓		
Chargers		✓	✓		✓	
Lightning	✓			✓		✓
Victory	✓	✓			✓	

Draw a graph to represent this information.

7 A squash club is holding a round robin tournament with their top 6 players. In a round robin tournament each player competes against every other player once only.

a Number the players 1 through 6 and draw a graph to represent this information.

b What type of graph is this?

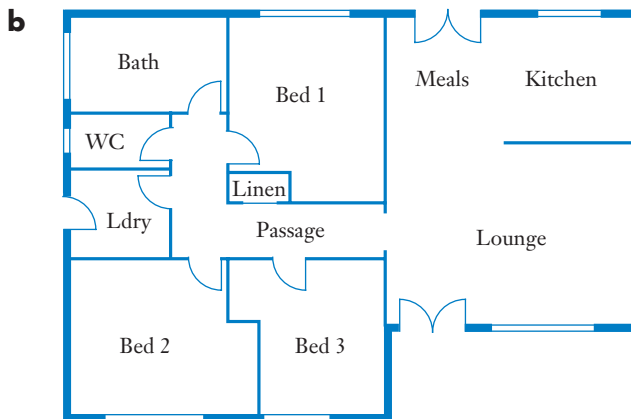
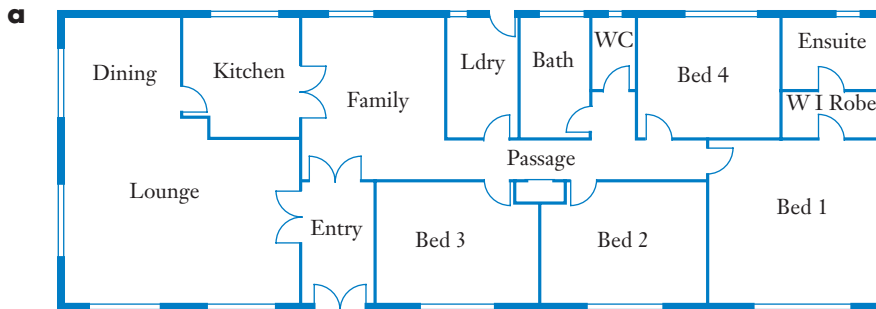
c How many games will be played in total?

d Consider a round robin tournament with 5 teams. Without drawing a graph, how many games will be played?

8 There are 15 people at a meeting and each person shakes hands with every other person. How many handshakes are there altogether?

- 9 Draw each building plan as a network, showing the outside as one vertex and doorways as connections between rooms.

Example
5



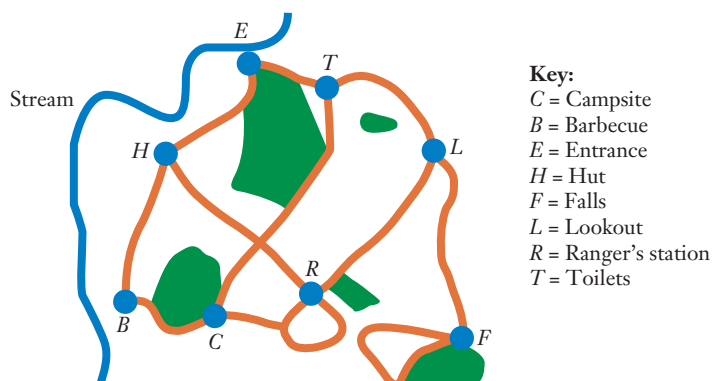
- 10 Use networks to find the number of colours needed to colour each map so that every border has a different colour on each side.

Example
6



Problem solving

11 Draw a graph to represent the following tourist map.



12 A camp site used by scouts has various tracks for walking. The tracks connect various places at the camp site as follows:

Main hut to beach

Flying fox to main hut

Barbecue site to flying fox

Main gate to main hut

Scoutmaster's hut to beach

Beach to flying fox

Main hut to barbecue site

Barbecue site to main gate

Main hut to Scoutmaster's hut

Draw a network showing the paths.



Alamy Stock Photo/Petr Bonek

13 Detective-Sergeant Kelly was investigating a bank robbery. He knew that Peter Pinder was involved and decided to do a map of his acquaintances in order to narrow the search for accomplices. Peter was a known associate of Luke Fellowes and David Alsop. David was known to have been involved in a previous job with John French, Andrew Smith and Kerry Stubbs. Kerry had previously shared a cell with Peter Pinder and Felix Goldschmeid at different times, and Felix was arrested the previous year in company with Luke Fellowes and Jason Chan. Jason's wife was Larry Smart's sister. Larry was a friend of Kerry Stubbs and John Kinder.

- a** Draw a network showing people as vertices and the acquaintanceships as edges.
- b** If three people were involved in the robbery, who should DS Kelly check first?
- c** If DS Kelly found that one of the people involved in the robbery was previously unknown to Peter, who should he check?

14 John married Mary in 1890 and they had four children: David, Andrew, Elizabeth and Julia. In 1901, Mary died and John later remarried. He and his second wife, Celia, had two children: Charles and Margaret.

David married Henrietta and had three children: Stephen, Michael and Victoria.

Andrew never married.

Elizabeth and Julia both married sons of the same couple: Giuseppe and Maria.

Elizabeth married Giorgio and had two children: Antonio and Yvette.

Julia married Enzo and had one child: Julianna.

Draw a directed network of the family tree, showing each member of the families as a vertex and using arcs to indicate parenthood.

15 A set menu at a dinner has the choices shown on the right.

- a** Assuming that everyone chooses one item from each course, draw a directed network showing the possible sequences of courses.
- b** Redraw the network assuming that some people do not have the entree.
- c** How many different meals are possible if you have one dish from every course?

MENU	
APPETISER OR SOUP	
	Grilled Bocconcini Cheese
	Grilled Tomato Bruschetta
	Scallops with Lemon Butter
	Laksa
	Pumpkin Soup
ENTREE	
	Vegetable Lasagne
	Cajun Chicken Salad
	Smoked Salmon
MAIN	
	Pork Fillet with Crisp Pancetta
	Roast Beef with Sweet Potato
	Pumpkin Risotto
	Peppered Garfish with Lemon Rice Pilaf
DESSERT	
	White Chocolate Tiramisu
	Sticky Date Pudding
	Mascarpone Tart
	Citrus Tart



Adjacency
matrices

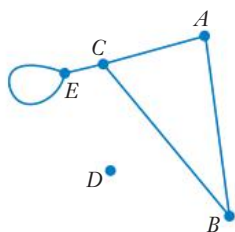


Adjacency
matrices

8.03 Adjacency matrices

You can represent the vertices and edges of a graph or network using a table or a matrix. Use a '1' to show that there is one edge connecting two adjacent vertices, a '0' to show that there is no edge joining two vertices, a '2' to show that there are two edges connecting two adjacent vertices and so on. A loop only counts as 1 edge. Because the matrix describes adjacent vertices, it is called an **adjacency matrix**.

The graph below has been used to draw a table showing vertices and edges, giving the adjacency matrix shown.



	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	0	0
C	1	1	0	0	1
D	0	0	0	0	0
E	0	0	1	0	1

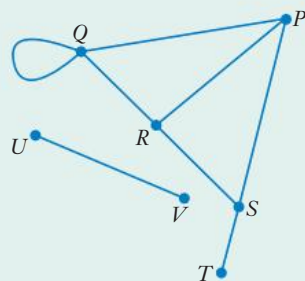
$$\begin{matrix} & A & B & C & D & E \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

By looking at the matrix, you can see that:

- the '1' in the $E-E$ cell shows there is a loop at E
- the '0' in the cells of the 4th row and column shows D is an isolated vertex.

EXAMPLE 7

Represent this graph as an adjacency matrix.



Solution

Set up a blank matrix with the vertex names across the top and on the left-hand side.

$$\begin{matrix} & P & Q & R & S & T & U & V \\ \begin{matrix} P \\ Q \\ R \\ S \\ T \\ U \\ V \end{matrix} & \begin{bmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{matrix}$$

Begin with vertex P . P is connected to Q , R and S only. So place a '1' in the cells that correspond to Q , R and S in the row and column for P .

	P	Q	R	S	T	U	V
P		1	1	1			
Q	1						
R	1						
S	1						
T							
U							
V							

Now look at Q . Q is connected to P , R and itself (with a loop). So place a '1' in the cells that correspond to P , R and Q in the row and column for Q . The Q – P cell has already been filled, so it doesn't need to be filled again.

	P	Q	R	S	T	U	V
P		1	1	1			
Q	1	1	1				
R	1	1					
S	1						
T							
U							
V							

Continue this process for the remaining vertices.

	P	Q	R	S	T	U	V
P		1	1	1			
Q	1	1	1				
R	1	1		1			
S	1		1		1		
T				1			
U						1	
V						1	

None of the remaining cells have any connections. Complete the matrix by placing a '0' in each of the remaining cells.

	P	Q	R	S	T	U	V
P	0	1	1	1	0	0	0
Q	1	1	1	0	0	0	0
R	1	1	0	1	0	0	0
S	1	0	1	0	1	0	0
T	0	0	0	1	0	0	0
U	0	0	0	0	0	0	1
V	0	0	0	0	0	1	0

You can construct a graph from its adjacency matrix.

EXAMPLE 8

Construct a graph that has the adjacency matrix shown here.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>A</i>	0	3	1	0	0
<i>B</i>	3	0	1	0	0
<i>C</i>	1	1	0	1	0
<i>D</i>	0	0	1	0	0
<i>E</i>	0	0	0	0	0

Solution

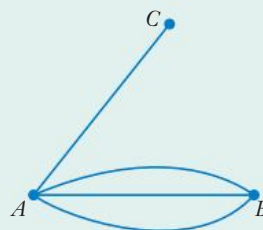
Start with vertex *A*. It has 3 edges connecting it to *B*.

Draw 3 edges joining *A* and *B*.



A also has one edge connecting it to *C*.

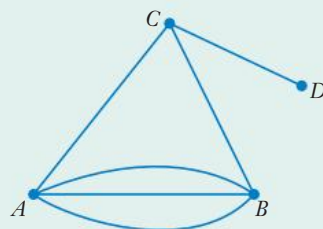
Draw an edge joining *A* and *C*.



C has one edge connecting it to *A* and one connecting it to *D*.

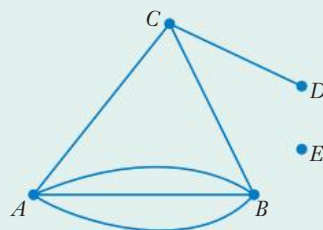
You only need to draw the edge joining *C* and *D* because the edge joining *C* to *A* has already been drawn in.

C also has one edge connecting it to *B*. Draw the edge connecting *C* to *B*.



The only remaining vertex, *E*, has no connections with any other vertex, so it is isolated.

Draw in the isolated vertex anywhere that is convenient.



There may be other geometric configurations you can draw from the adjacency matrix shown in the previous example, but each one will be isomorphic (or equivalent) to the one shown.

Exercise 8.03 Adjacency matrices

Use the adjacency matrices below to answer questions 1–3 that follow.

Matrix 1

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

Matrix 2

$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Matrix 3

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix 4

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 3 & 0 \end{bmatrix}$$

Matrix 5

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- 1 The matrix that corresponds to a graph that has a loop is:

A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4 **E** Matrix 5

- 2 The matrix that corresponds to a complete graph is:

A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4 **E** Matrix 5

- 3 The matrix that corresponds to a disconnected graph is:

A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4 **E** Matrix 5

- 4 Without constructing a graph, the number of edges in the graph represented by this adjacency matrix is:

A 2 **B** 3 **C** 4
D 5 **E** 6

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- 5 Without constructing a graph, the number of edges in the graph represented by this adjacency matrix is:

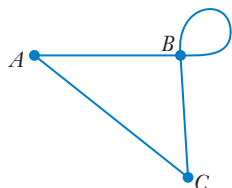
A 12 **B** 10 **C** 17
D 9 **E** 15

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

6 For each graph:

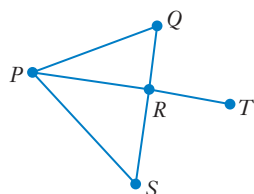
- i** confirm that the associated adjacency matrix is correct
- ii** calculate the sum of the degrees of the vertices
- iii** compare each answer to part **ii** with the sum of the cells of the associated adjacency matrix.

a



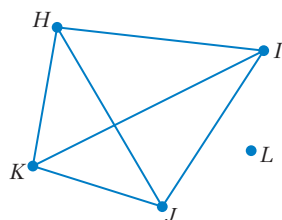
$$\begin{array}{c} A \quad B \quad C \\ A \left[\begin{array}{ccc} 0 & 1 & 1 \\ B & 1 & 1 & 1 \\ C & 1 & 1 & 0 \end{array} \right] \end{array}$$

b



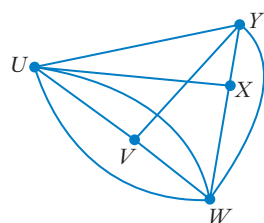
$$\begin{array}{c} P \quad Q \quad R \quad S \quad T \\ P \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ Q & 1 & 0 & 1 & 0 & 0 \\ R & 1 & 1 & 0 & 1 & 1 \\ S & 1 & 0 & 1 & 0 & 0 \\ T & 0 & 0 & 1 & 0 & 0 \end{array} \right] \end{array}$$

c



$$\begin{array}{c} H \quad I \quad J \quad K \quad L \\ H \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ I & 1 & 0 & 1 & 1 & 0 \\ J & 1 & 1 & 0 & 1 & 0 \\ K & 1 & 1 & 1 & 0 & 0 \\ L & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

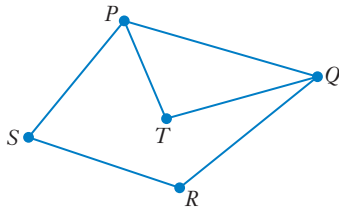
d



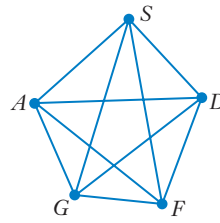
$$\begin{array}{c} U \quad V \quad W \quad X \quad Y \\ U \left[\begin{array}{ccccc} 0 & 1 & 2 & 1 & 1 \\ V & 1 & 0 & 1 & 0 & 1 \\ W & 2 & 1 & 0 & 1 & 1 \\ X & 1 & 0 & 1 & 0 & 1 \\ Y & 1 & 1 & 1 & 1 & 0 \end{array} \right] \end{array}$$

7 Represent each graph as an adjacency matrix.

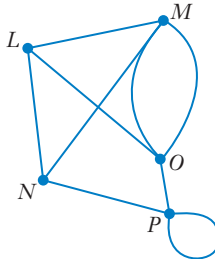
a



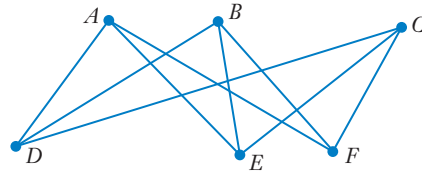
b



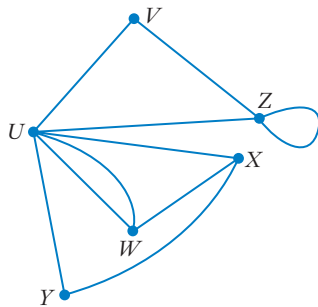
c



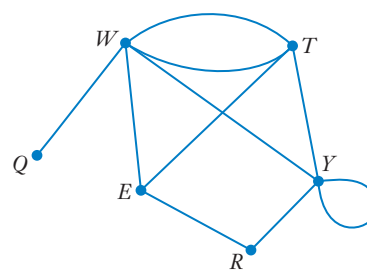
d



e



f



8 Construct a graph that corresponds to each adjacency matrix.

a

$$\begin{matrix} & A & B & C \\ A & \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\ B & \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \\ C & \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

b

$$\begin{matrix} & P & Q & R & S \\ P & \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \\ Q & \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \\ R & \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \\ S & \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

c

$$\begin{matrix} & W & X & Y & Z \\ W & \begin{bmatrix} 0 & 2 & 1 & 0 \end{bmatrix} \\ X & \begin{bmatrix} 2 & 0 & 0 & 1 \end{bmatrix} \\ Y & \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \\ Z & \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

d

$$\begin{matrix} & H & I & J & K & L & M \\ H & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \\ I & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ J & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \\ K & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ L & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\ M & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

e

$$\begin{matrix} & R & S & T & U & V \\ R & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix} \\ S & \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \end{bmatrix} \\ T & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \end{bmatrix} \\ U & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix} \\ V & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

f

$$\begin{matrix} & A & B & C & D & E & F \\ A & \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \end{bmatrix} \\ B & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ C & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ D & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \\ E & \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ F & \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 0 \end{bmatrix} \end{matrix}$$

Problem solving

9 Copy and complete each adjacency matrix.

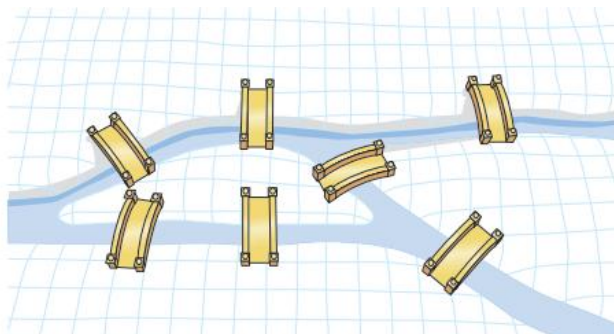
a
$$\begin{bmatrix} 0 & 1 & & 1 \\ & 0 & & \\ 1 & 1 & 0 & 1 \\ & 1 & & 0 \end{bmatrix}$$

b
$$\begin{bmatrix} 0 & & 0 & & \\ 1 & 1 & 1 & 0 & \\ & & 0 & & \\ 0 & & 1 & 0 & \end{bmatrix}$$

c
$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

d
$$\begin{bmatrix} 0 & & & 0 & & \\ 1 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & & & \\ & & 1 & 0 & & \\ 1 & 0 & 0 & 1 & 0 & \end{bmatrix}$$

- 10 There is a famous maths problem that originated in the the medieval town of Königsberg. The Pregel river flowed through the town around a central island before dividing into two. The 4 parts of the town were linked by 7 bridges as shown on the diagram.

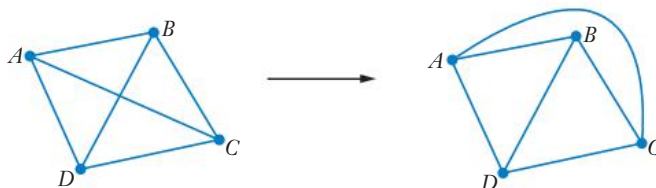


The problem was to work out if it was possible to find a route through Königsberg, beginning and ending at the same point, that crossed each bridge exactly once.

- Draw a graph of this situation using the vertices of the graph to represent the parts of the city and the edges of the graph to represent the bridges.
- Construct an adjacency matrix for the graph.
- Work out a method of using the graph to find a solution to the problem.
- What do you conclude about the solution of this problem?

8.04 Planar graphs and Euler's formula

Many graphs and networks have edges that cross. Some graphs with edges that cross can be redrawn so that they have no intersecting edges, as shown below.



If you imagine that the edges are like elastic bands, you can visualise how edge AC is able to be stretched so that it still joins A and C but does not cross the edge BD . Both graphs shown here represent the same information regarding vertices and connections.

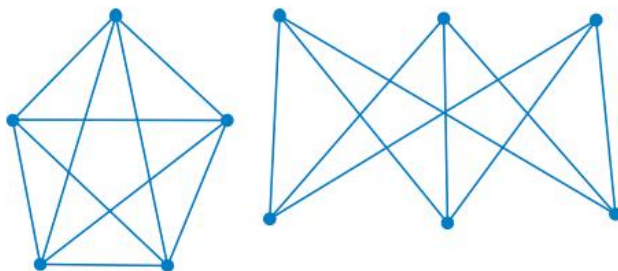
A graph that can be drawn without any crossing edges is called a **planar graph**.

There is no set method for working out if a graph with crossing edges can be drawn as a planar graph. You need to use trial and error. In some cases, you need to move the vertices to different positions so that the connecting edges don't cross. In these cases, you need to ensure the connections remain the same.

INVESTIGATION

PLANAR GRAPHS

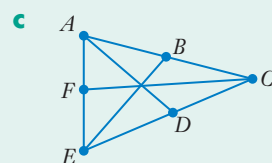
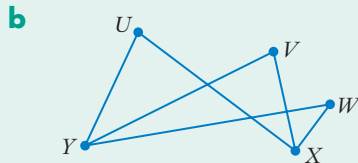
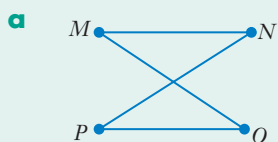
Examine each graph shown below.



- 1 Work in small groups to determine if it is possible to redraw either of these as a planar graph.
- 2 Compare your findings with other groups.

EXAMPLE 9

Redraw each graph as a planar graph if possible.

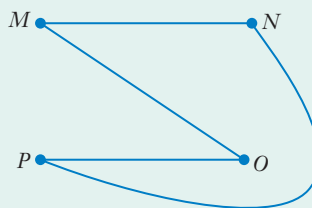


Solution

- a** MO and NP cross.

The current form of the graph is non-planar.

The edge NP can be drawn so that it does not cross MO .



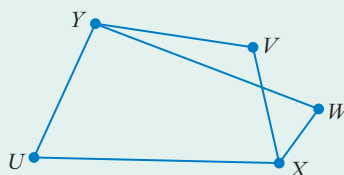
State the result.

The graph is planar because it can be drawn so that no edges cross.

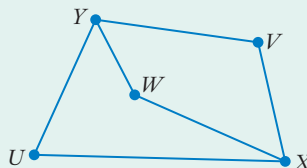
- b** UX and VY cross and UX crosses VY and YW .

The current form of the graph is non-planar.

Two crossings could be eliminated by swapping vertices U and Y . Redraw the graph and check that all the edges are connected to the correct vertices, i.e. $U-Y$, $U-X$, $Y-V$, $Y-W$, $X-W$ and $V-X$.



Place vertex W inside the quadrilateral and retain its connections to Y and X to eliminate the remaining crossings.



State the result.

The graph is planar because it can be drawn so that no edges cross.

c AD , FC and EB cross.

The current form of the graph is non-planar.

Using trial and error, some crossed edges can be eliminated but it is not possible to remove all crossed edges.

The graph is non-planar.

In the previous example, you should note that the planar graphs shown are only one of several possible answers in each part.

Planar graphs can be used to represent a three-dimensional solid with flat faces as a two-dimensional figure.

A cube can be used to demonstrate how this works.

Consider the cube shown on the right. There are 8 vertices – A, B, C, D, E, F, G and H and 12 edges – $AB, BC, CD, \dots$

Each vertex is connected to three other vertices.

You can move vertices A, B, C and D and retain their relevant connections as shown to form a two-dimensional projection of the cube.

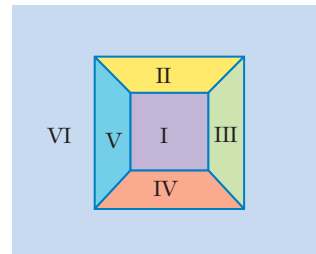
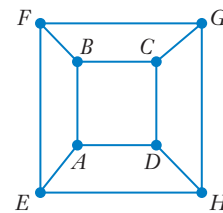
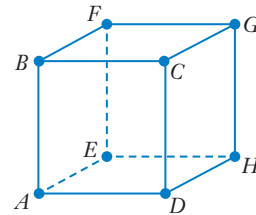
All of the connections between the vertices of the cube have been retained in the new graph. You can see that it is a planar graph because no edges cross.

You can divide the graph into 6 regions – I, II, III, IV, V and VI as shown on the right.

Each region represents a face of the cube. One of the faces is infinite because it isn't bounded.

There is a mathematical relationship between the vertices, edges and faces of planar graphs.

This relationship was established by Leonhard Euler (pronounced 'oiler'), an 18th century Swiss mathematician, and is known as **Euler's formula**.



Euler's formula

For all planar graphs:

$$\text{Vertices} + \text{faces} - \text{edges} = 2$$

$$v + f - e = 2$$

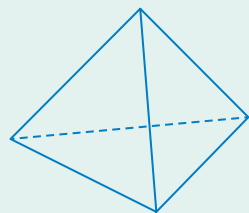


Alamy Stock Photo/North Wind Picture Archives

You can rearrange Euler's formula to make v , f or e the subject if required.

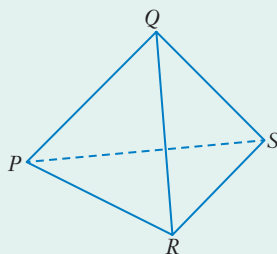
EXAMPLE 10

- a** Represent the tetrahedron on the right as a planar graph.
- b** Verify Euler's formula using the graph.

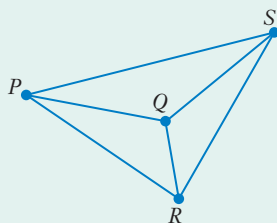


Solution

- a** Name the vertices.



Move Q so that it lies within face PRS and retain all connections.



- b** Count the vertices, edges and faces.

$$v = 4$$

$$e = 6$$

$$f = 4$$

Substitute into Euler's formula.

$$v + f - e = 4 + 4 - 6$$

$$= 2$$

State the result.

Euler's formula is satisfied for the tetrahedron.

You can use Euler's formula to determine the number of vertices, edges or faces for any planar graph.

EXAMPLE 11

Use Euler's formula to calculate the unknowns for each planar graph described below.

- a** $v = ?$, $e = 10$ and $f = 4$
- b** $e = ?$, $v = 12$ and $f = 6$
- c** $f = ?$, $e = 8$ and $v = 5$

Solution

- a** State Euler's formula.

$$v + f - e = 2$$

Substitute for e and f .

$$v + 4 - 10 = 2$$

Solve for v .

$$v - 6 = 2$$

$$v = 8$$

State the result.

The graph has 8 vertices.

- b** Substitute into Euler's formula.

$$12 + 6 - e = 2$$

Solve for e .

$$18 - e = 2$$

$$e = 16$$

State the result.

The graph has 16 edges.

- c** Substitute into Euler's formula.

$$5 + f - 8 = 2$$

Solve for f .

$$f - 3 = 2$$

$$f = 5$$

State the result.

The graph has 5 faces.

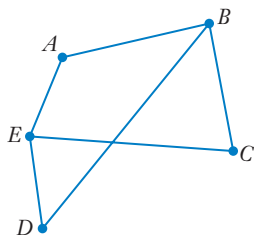
Exercise 8.04 Planar graphs and Euler's formula

Example

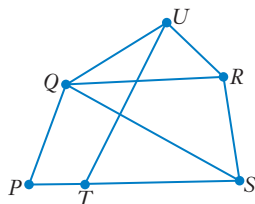
9

1 Draw the following as planar graphs if possible.

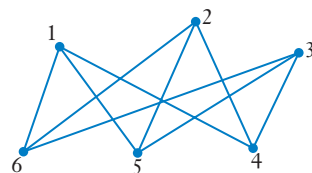
a



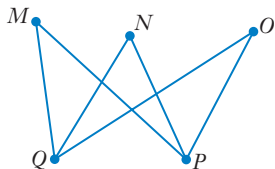
b



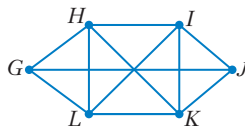
c



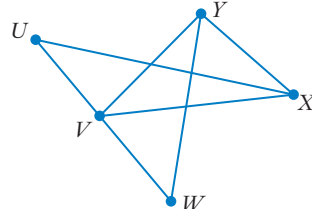
d



e



f

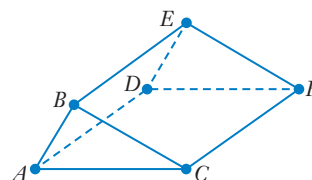


Example

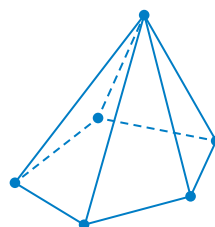
10

2 a Represent this triangular prism as a planar graph.

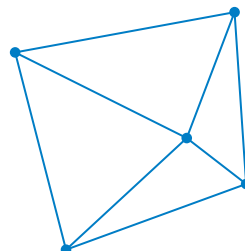
b Verify Euler's formula using the graph.



3 Represent this pentagonal pyramid as a planar graph.



4 What three-dimensional shape could this planar graph represent?



5 This is the adjacency matrix for a planar graph with 4 vertices. Without drawing the graph, the number of faces on the graph is:

A 2

B 3

C 4

D 5

E 6

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- 6 This is the adjacency matrix for a planar graph with 5 vertices.
Without drawing the graph, the number of faces on the graph is:

A 2
D 5

B 3
E 6

C 4

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

- 7 For each planar graph, use Euler's formula to calculate the number of vertices.

a $E = 3$ and $F = 3$

b $E = 5$ and $F = 3$

c $E = 7$ and $F = 4$

d $E = 6$ and $F = 5$

- 8 Use Euler's formula to calculate the number of edges for each planar graph.

a $V = 2$ and $F = 4$

b $V = 3$ and $F = 4$

c $V = 4$ and $F = 6$

d $V = 6$ and $F = 6$

- 9 How many faces would there be for each planar graph described below?

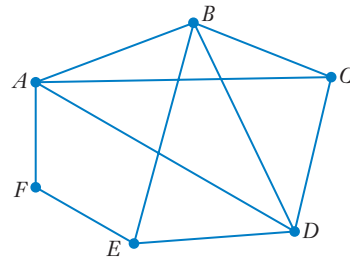
a $E = 4$ and $V = 3$

b $E = 5$ and $V = 3$

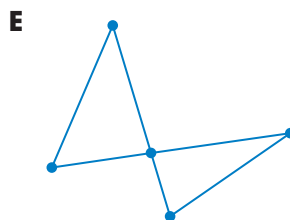
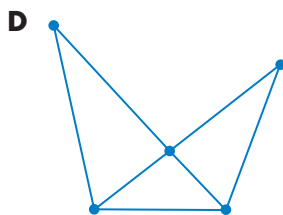
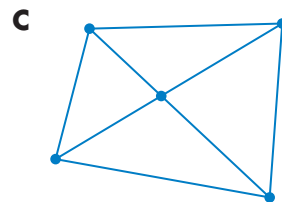
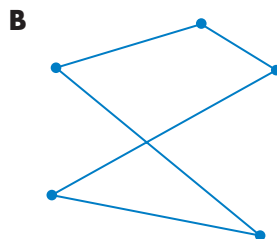
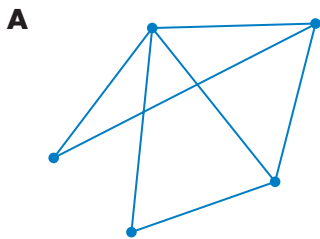
c $E = 7$ and $V = 5$

d $E = 12$ and $V = 6$

- 10 **a** Redraw this graph to show it is planar.
b How many faces does the graph have?



- 11 Which of these graphs is a planar graph with 5 vertices and 2 faces?



Example
11

Problem solving

12 Draw a planar graph that is represented by each adjacency matrix.

a

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

b

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

13 Construct a planar graph with:

a 6 vertices and 4 faces

b 13 edges and 8 faces.

14 Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

15 A graph has 6 vertices with degrees 2, 2, 3, 4, 4, 5.

a How many edges does the graph have?

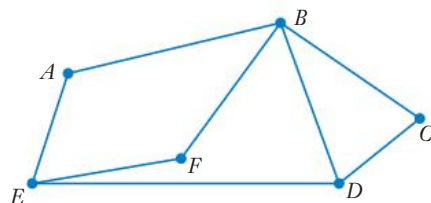
b Could the graph be planar?

c If so, how many faces would it have? If not, explain.



8.05 Eulerian graphs

Graphs can help you analyse movement around a network. This is especially the case for transport and communication. Movement in a network starts and finishes at specified vertices along the edges. You can describe movement through a connected graph by listing the vertices visited in the correct order.



There are special names to describe particular types of movements across a network. Consider this network.

A **walk** is any **route** (sequence of edges) taken through a network, including routes that repeat edges and vertices. For example, *AEFB* and *ABDBF* are both walks. If there is a walk between all possible pairs of vertices, then it is a connected graph.

An **open walk** starts and finishes at a different vertex (e.g. *AEFB*), while a **closed walk** starts and finishes at the same vertex (e.g. *AEFEA*).

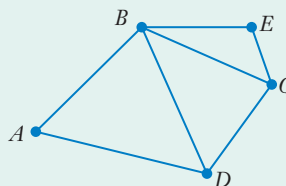
A **trail** is a walk in which you travel along an edge once only (e.g. *EFBDC*). *EFBDC* is an **open trail** because it starts and finishes at a different vertex (e.g. *AEFBD*). A **closed trail** starts and finishes at the same vertex (e.g. *AEDBA*). A closed trail is also called a **circuit**.

A **path** is a walk in which no vertices are repeated, except possibly the start and finish. So *AEFBD* and *AEDCBDA* are both paths. A path beginning and ending at the same vertex is also called a **cycle** (e.g. *AEDCBA*).

EXAMPLE 12

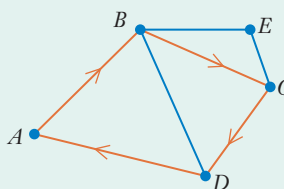
Use the graph to identify each type of route given.

- a $ABCD A$
- b $ABDCBA$
- c $ABECD$
- d $BADBCD$
- e $BADBECB$



Solution

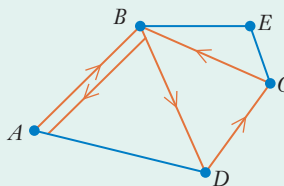
- a Use a highlighter or pencil to trace out the movement $ABCD A$.



Each vertex is visited only once and the trace begins and ends at A .

$ABCD A$ is a cycle.

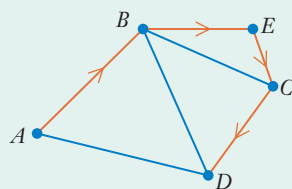
- b Use a highlighter or pencil to trace out the movement $ABDCBA$.



This is a sequence of edges that begins and ends at A . Vertex B is visited twice.

$ABDCBA$ is a closed walk.

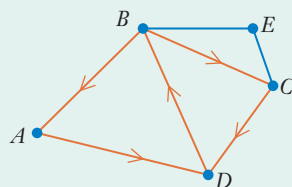
- c Use a highlighter or pencil to trace out the movement $ABECD$.



This is a sequence of edges in which no vertex is repeated.

$ABECD$ is a path.

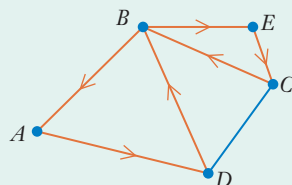
- d Use a highlighter or pencil to trace out the movement $BADBCD$.



This is a sequence of edges in which each edge is only travelled along once.

$BADBCD$ is an open trail.

- e Use a highlighter or pencil to trace out the movement $BADBECEB$.

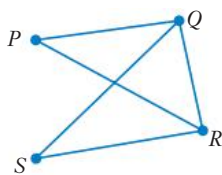
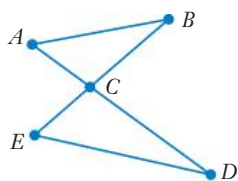


This is a sequence of edges beginning and ending at the same vertex and each edge is only travelled on once.

$BADBECEB$ is a circuit (or closed trail).

In some cases, the most efficient route needs to be found. In terms of distance, the most efficient route follows a path that travels along each edge only once. For example, a courier would want to make deliveries to locations without backtracking if possible.

You should remember that a trail is a walk in which no edges are repeated. An **Eulerian cycle** (or **Eulerian circuit**) is a trail that traverses every edge of a graph exactly once. An **Eulerian path** (or **Eulerian trail**) is an open trail that traverses each edge only once.



The first graph above has an Eulerian cycle, $ACEDCBA$, and the second graph has an Eulerian trail $QPRSQR$.

A graph is called an **Eulerian graph** if it contains an Eulerian circuit and a **semi-Eulerian graph** if it contains an Eulerian path. You can also use the degree of the vertices of a connected graph to determine if it is Eulerian, semi-Eulerian or neither.

Eulerian graphs

For an Eulerian graph:

- all vertices must be of even degree
- a closed trail or **Eulerian cycle** exists

For a semi-Eulerian graph:

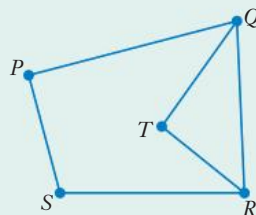
- only 2 vertices can be of odd degree and every other vertex is of even degree
- an open trail or **Eulerian path** exists
- the Eulerian path must start and finish at a vertex of odd degree



Eulerian trails
and circuits

EXAMPLE 13

- Use the degree of the vertices to show that this graph is semi-Eulerian.
- Describe an Eulerian path for this graph.



Solution

- Write the degrees of the vertices.

$$\begin{aligned} \deg(P) &= 2, & \deg(Q) &= 3 \\ \deg(R) &= 3, & \deg(S) &= 2 \\ \deg(T) &= 2 \end{aligned}$$

State the result.

The graph is semi-Eulerian because it has 2 vertices of odd degree and the remaining vertices are of even degree.

- b** The graph is semi-Eulerian.

The Eulerian path must start and end at a vertex of odd degree.

The trail must use each edge exactly once. You are allowed to visit a vertex more than once.

It is possible that other Eulerian trails exist.

An Eulerian path exists.

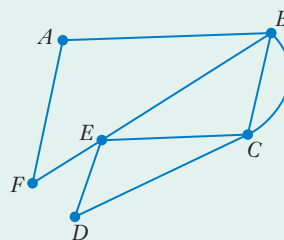
The Eulerian path must start at Q and finish at R or start at R and finish at Q .

$QPSRTQR$ is a Eulerian path.

$QTRSPQR$ and $RSPQTRQ$ are also Eulerian paths.

EXAMPLE 14

- a** Use the degree of the vertices to show that this graph is Eulerian.
- b** Describe an Eulerian cycle for this graph.



Solution

- a** Write the degrees of the vertices.

$$\deg(A) = 2, \quad \deg(B) = 4$$

$$\deg(C) = 4, \quad \deg(D) = 2$$

$$\deg(E) = 4, \quad \deg(F) = 2$$

State the result.

The graph is Eulerian because all vertices are of even degree.

- b** Select any vertex to start and trace out a trail that uses each edge once and ends at the vertex where you started.

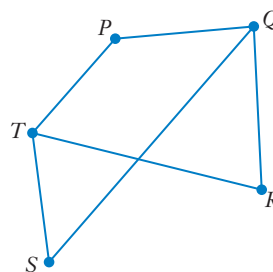
$AFEDCEBCBA$ is an Eulerian cycle.

It is possible that other Eulerian cycles exist.

$BEFABCEDCB$ and $CDEFABECBC$ are also Eulerian cycles.

Exercise 8.05 Eulerian graphs

Questions 1–3 refer to this graph.



- 1 The route $PTSTR$ is best described as:

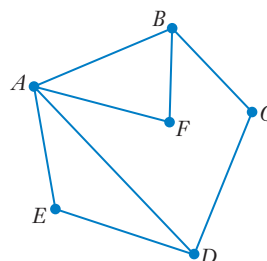
A a closed walk	B an open walk
C a cycle	D an open trail
E a circuit.	
- 2 The route $SQ RTP$ is best described as:

A a closed walk	B an open walk
C a cycle	D an open trail
E a circuit.	
- 3 The route $ST PQS$ is best described as:

A a closed walk	B an open walk
C a cycle	D an open trail
E a circuit.	

- 4 Use this graph to identify the type of each route.

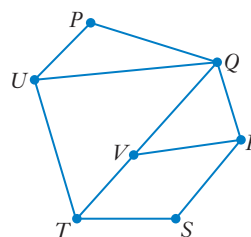
a $CDAEDC$	b $AEDCB$
c $ABCD A$	d $CDAFBC$
e $FBAD$	f $DEAFAD$
g $EABCDE$	h $BAFBC$
i $CDAED$	j $DAFBC$
k $BFAEDAB$	



Example
12

- 5 For this network, name one example of:

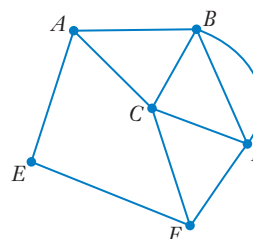
a a path	
b an open walk	
c a cycle	
d an open trail	
e a circuit.	



- 6 **a** State the degree of each vertex for the graph in question 4.
- b** Is the graph in question 4 Eulerian, semi-Eulerian or neither?

- 7 Examine the graph shown on the right.

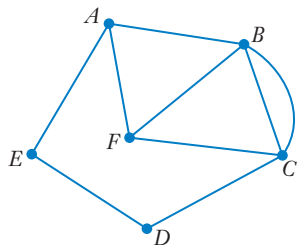
a Use the degree of the vertices to show that the graph is semi-Eulerian.	
b From which vertex must an Eulerian trail begin for this graph?	
c Describe an Eulerian trail for this graph.	



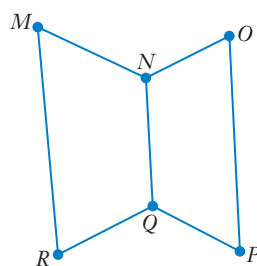
Example
13

8 State whether each graph is semi-Eulerian.

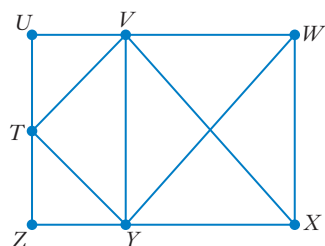
a



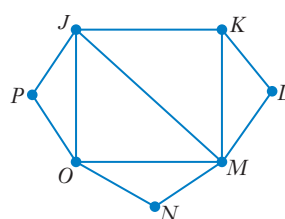
b



c



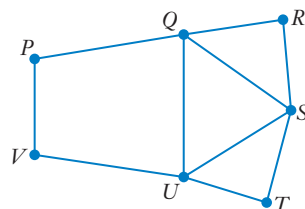
d



9 For each semi-Eulerian graph identified in question 8, name an Eulerian path.

10 a Use the degree of the vertices to show that this graph is Eulerian.

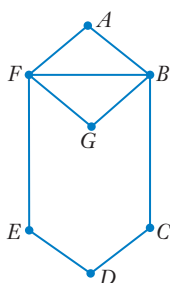
b Describe an Eulerian circuit for this graph.



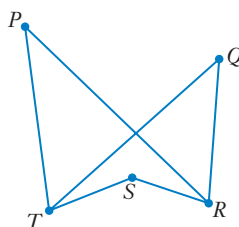
11 i Classify each of the following graphs as Eulerian or semi-Eulerian.

ii Identify the closed trail (Eulerian cycle) or open trail (Eulerian path) that exists in each case.

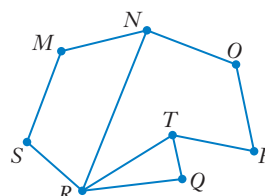
a



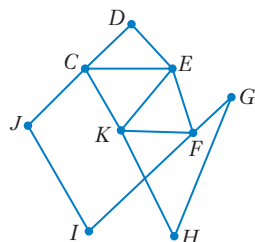
b



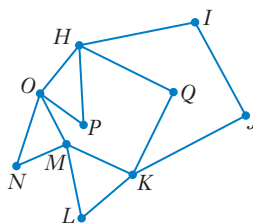
c



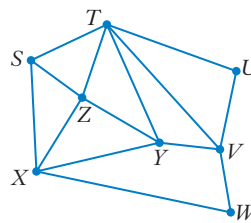
d



e



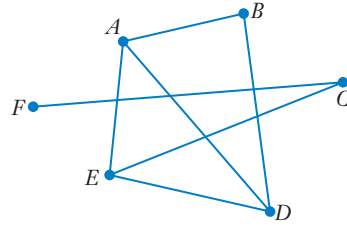
f



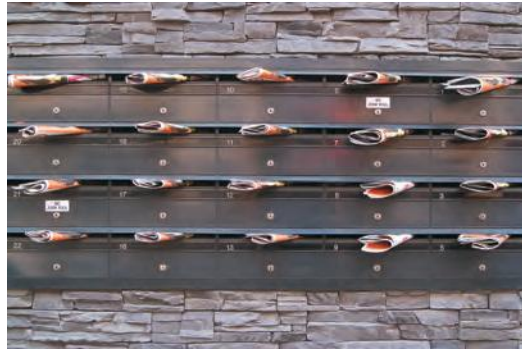
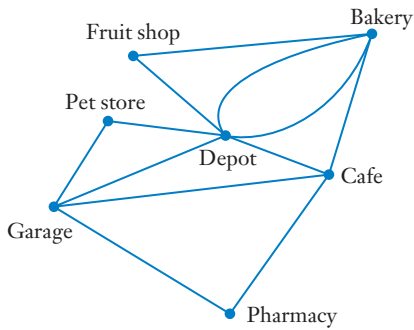
Example
14

Problem solving

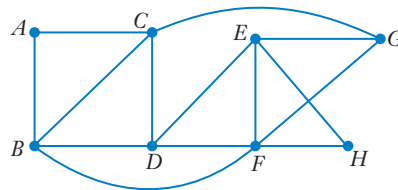
- 12 a** Add 2 edges to transform this graph into an Eulerian graph.
b Use the transformed graph to describe an Eulerian cycle.



- 13** Emily delivers pamphlets along the streets that connect the various locations around a central depot. She must deliver pamphlets along each street and start and end each delivery run at the depot.



- a** What type of graph is formed by the street map?
b What is the most efficient route that Emily could take? Explain your answer.
- 14** The streets of a town are represented by this map. The various intersections of streets are indicated by the letters A, B, C , etc. A security firm is employed to drive along a route that includes each street. The owner of the security company wants the route to cover each street once only.
- a** Is it possible to select a route that will start and finish at the same intersection and still satisfy the owner's request? Explain your answer.
b In order to satisfy the owner's request, at which intersection should the route commence and finish?
c Describe the route that should be taken if it commences at intersection G .





8.06 The travelling salesperson problem

You saw in the previous section that Eulerian paths and cycles are routes in which each edge of a graph is used exactly once, while vertices can be revisited. These types of routes are useful when an efficient course is required that does not involve backtracking.

In some cases, a route is required that visits each vertex exactly once. A classic example of this situation is called the **travelling salesperson problem**. A travelling salesperson typically wishes to visit all towns on a route but wants to visit each town once only.



Alamy Stock Photo/Carol and Mike Werner

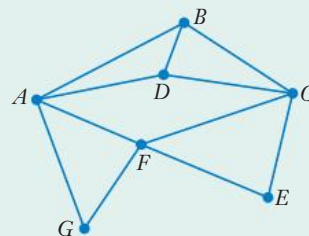
Hamiltonian cycles and paths

- A **Hamiltonian cycle** is a route that includes each vertex in a graph exactly once and starts and finishes at the same vertex.
- A **Hamiltonian path** is a route that includes each vertex in a graph exactly once and starts and finishes at different vertices.

Unlike Eulerian cycles and paths, not all edges of a graph need to be used in finding a Hamiltonian cycle or path. It should be noted that there is no simple method for determining if a Hamilton path or cycle exists, so you need to use trial and error.

EXAMPLE 15

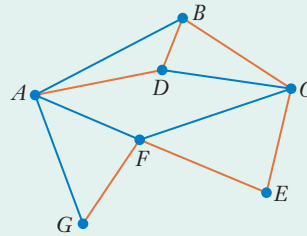
Name a Hamiltonian path for the graph shown here.



Solution

Start at any vertex, such as A .

Move from A to an adjacent vertex, say D .
Now move to another adjacent vertex (but not A). Continue this process until every vertex has been visited. No vertex can be revisited.



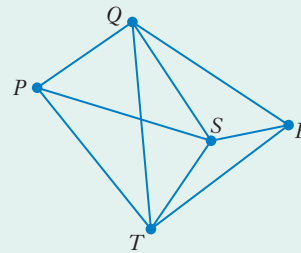
Name the route.

A Hamiltonian path is $ADBCEFG$.

If you can't find a Hamiltonian path starting at a particular vertex, select a different one and try again. There may be more than one Hamiltonian path for any graph. $ABDCEFG$ is also a Hamiltonian path for the previous example.

EXAMPLE 16

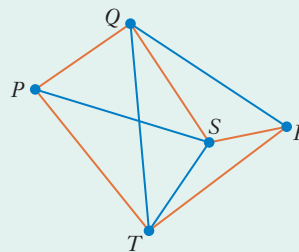
Name a Hamiltonian cycle for this graph.



Solution

Start at any vertex, such as P .

Use the process shown in the previous example to trace out a route that visits every vertex only once and returns to the start.

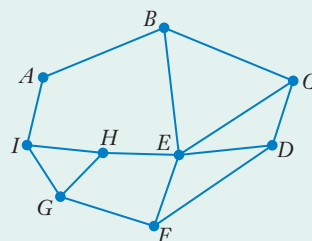


Name the route.

A Hamiltonian cycle is $PQSRTP$.
 $PSTRQP$ is also a Hamiltonian cycle.

EXAMPLE 17

Name a Hamiltonian cycle for this graph that begins at C and does not include edge FG .

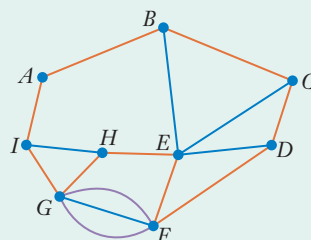


Solution

Edge FG can't be used, so mark it to make sure that you don't forget.

Start at C .

Move to an adjacent vertex, say D . Select another adjacent vertex, say F . You can't use FG , so move to E . Continue in this way until all vertices have been visited.



Name the route.

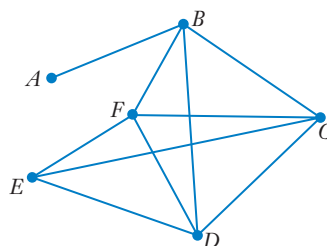
A Hamiltonian cycle is $CDFEHGLABC$.



Shutterstock.com/1000 Words

Exercise 8.06 The travelling salesperson problem

Questions 1 and 2 refer to this diagram.



1 Which one of the following is a Hamiltonian path?

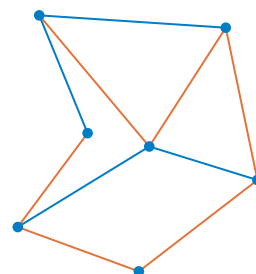
- A** ABFEDC
- B** BDCF EA
- C** ABCFECD
- D** BAF CDE
- E** DEFCFBA

2 Which one of the following statements is true?

- A** CDEFBA is a Hamiltonian cycle.
- B** More than one Hamiltonian path exists.
- C** A Hamiltonian path cannot commence at A.
- D** No Hamiltonian cycle exists.
- E** A Hamiltonian path cannot commence at D.

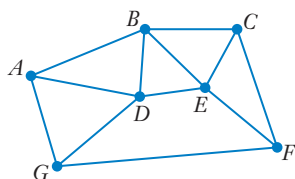
3 The highlighted route shown in the graph on the right is best described as

- A** an Eulerian path
- B** a Hamiltonian path
- C** an Eulerian cycle
- D** a Hamiltonian cycle
- E** none of the above.

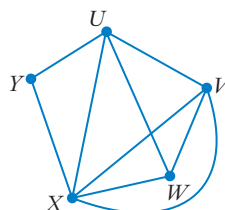


4 List a Hamiltonian path for each graph.

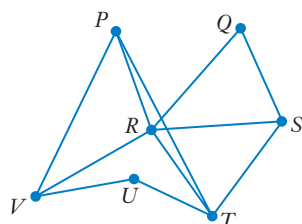
a Graph 1



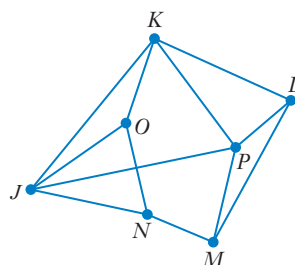
b Graph 2



c Graph 3



d Graph 4



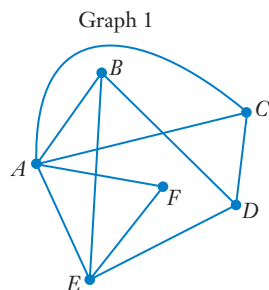
Example
15

- 5** Use the graphs in question 4 to answer the following questions.
- a** For Graph 1, list a Hamiltonian path commencing at B and finishing at C .
 - b** For Graph 2, list a Hamiltonian path commencing at V and finishing at X .
 - c** For Graph 3, list a Hamiltonian path commencing at R and finishing at Q .
 - d** For Graph 4, list a Hamiltonian path commencing at P and finishing at K .

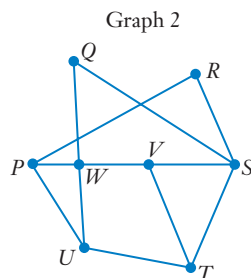
Example
16

- 6** List a Hamiltonian cycle for each graph.

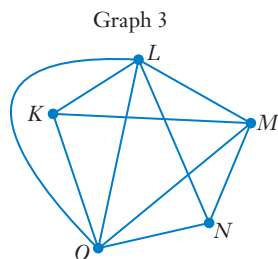
a



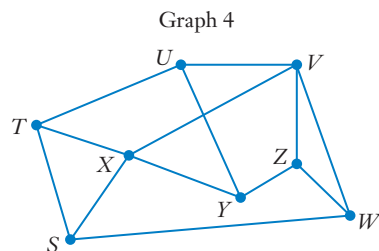
b



c



d

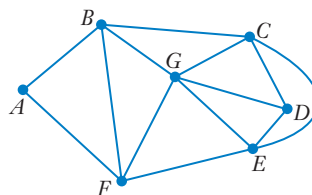


- 7** Use the graphs in question 6 to answer the following questions.
- a** For Graph 1, list a Hamiltonian cycle commencing at B and finishing at C .
 - b** For Graph 2, list a Hamiltonian cycle commencing at V and finishing at X .
 - c** For Graph 3, list a Hamiltonian cycle commencing at R and finishing at Q .
 - d** For Graph 4, list a Hamiltonian cycle commencing at P and finishing at K .

Example
17

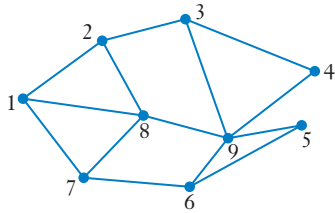
- 8** For this graph, name

- a** a Hamiltonian cycle that begins at G and does not include edge CD
- b** a Hamiltonian path that ends at A and does not include edge FE
- c** a Hamiltonian cycle that begins at E and does not include edge ED or BC
- d** a Hamiltonian path that begins at G and does not include edge CD or AF
- e** a Hamiltonian cycle that begins at A and does not include edge BF or FE .



Problem solving

- 9** Sarena is a salesperson who travels to different suburbs to sell supplies to health food stores. A map of the suburbs and roads with direct linkages between the suburbs is shown below.



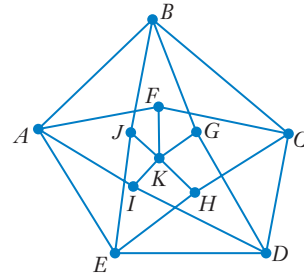
Key

- 1: Bellbowrie
- 2: Mount Ommaney
- 3: Indooroopilly
- 4: Rocklea
- 5: Archerfield
- 6: Richlands
- 7: Wacol
- 8: Jindalee
- 9: Corinda

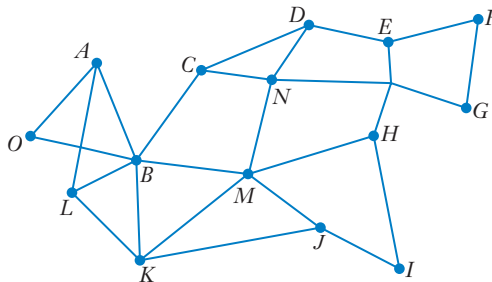
Sarena wants to travel from Indooroopilly and visit each of the other suburbs once before returning to Indooroopilly.

- a** What type of route is this called?
- b** Describe a route that Sarena could take.
- c** List the connections that Sarena does not use.

- 10** Name a Hamiltonian cycle for this graph.



- 11** An orienteering course is set out according to the map shown below.

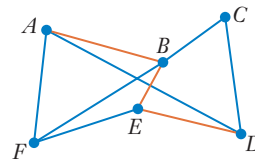


Competitors start and finish at point *A* and need to visit each and every one of the other checkpoints. Find a route that does not involve visiting any checkpoint more than once.

8.07 Spanning trees

A **tree** is a connected graph that does not contain any cycles, loops or multiple edges.

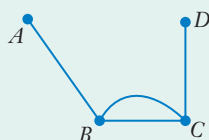
For the graph on the right, $ABED$ is a tree.



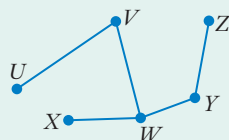
EXAMPLE 18

Determine whether each graph is a tree.

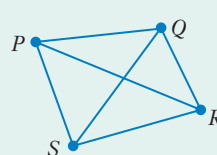
a



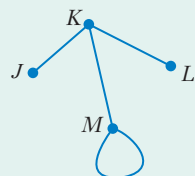
b



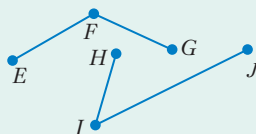
c



d



e



Solution

a Vertices B and C are joined by multiple edges.

This graph is not a tree.

b This graph is connected and contains no loops, cycles or multiple edges.

This graph is a tree.

c This graph contains the cycle $PQRSP$.

This graph is not a tree.

d This graph contains a loop at M .

This graph is not a tree.

e This graph is not connected.

This graph is not a tree.

EXAMPLE 19

For each adjacency matrix, draw the graph that corresponds to the matrix and determine if it is a tree.

a

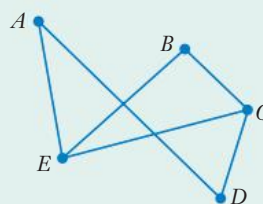
	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>A</i>	0	0	0	1	1
<i>B</i>	0	0	1	0	1
<i>C</i>	0	1	0	1	1
<i>D</i>	1	0	1	0	0
<i>E</i>	1	1	1	0	0

b

	<i>P</i>	<i>Q</i>	<i>R</i>	<i>S</i>	<i>T</i>
<i>P</i>	0	1	0	0	0
<i>Q</i>	1	0	1	0	1
<i>R</i>	0	1	0	1	0
<i>S</i>	0	0	1	0	0
<i>T</i>	0	1	0	0	0

Solution

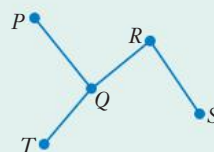
- a** Draw the 5 vertices and corresponding edges.



The graph contains the cycle *ADCBEA*.

The graph is not a tree.

- b** Draw the 5 vertices and corresponding edges.



This graph is connected and contains no loops, cycles or multiple edges.

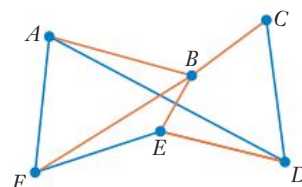
This graph is a tree.

A **spanning tree** is a subgraph that includes all the vertices of the original graph.

The edges marked in orange form a spanning tree for the graph on the right.

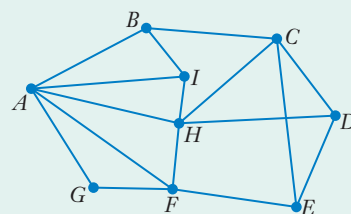
Because it is a tree, a spanning tree cannot have any loops, multiple edges or cycles.

There may be many possible spanning trees for any graph.



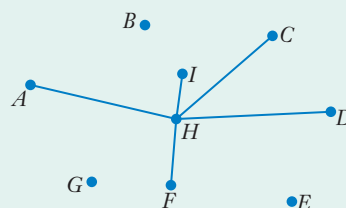
EXAMPLE 20

- Draw a spanning tree for this graph.
- Use the spanning tree to name the edges in the original graph that are not required to connect all vertices.

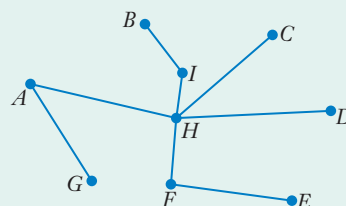


Solution

- Start at any vertex, say H .
Now connect every vertex that is adjacent to H .



Continue connecting adjacent vertices until all vertices are connected. Make sure that there are no cycles.



- Compare the spanning tree with the original graph.

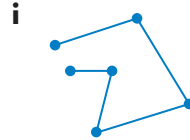
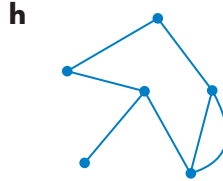
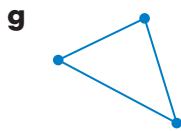
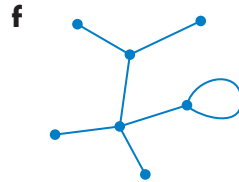
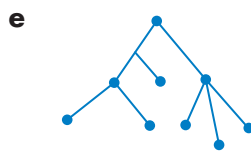
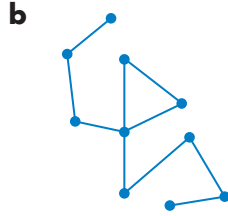
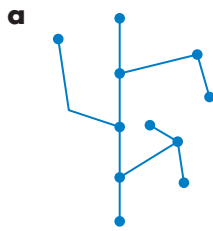
The edges that are not required to connect all vertices are AB , BC , CD , CE , DE , GF , FA and AI .

Another way of identifying a spanning tree is to remove any edges that form a cycle, one at a time.

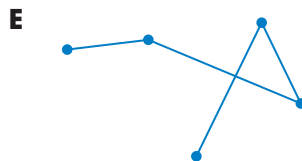
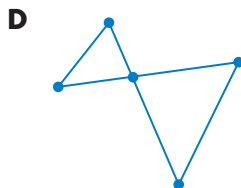
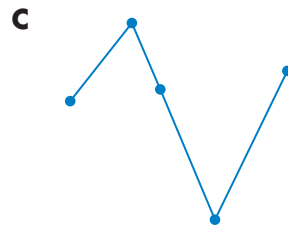
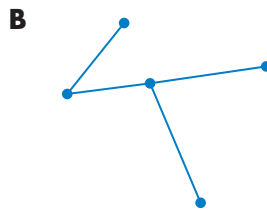
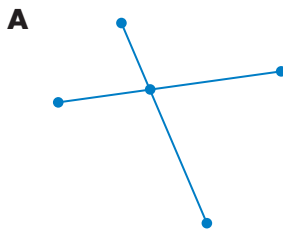
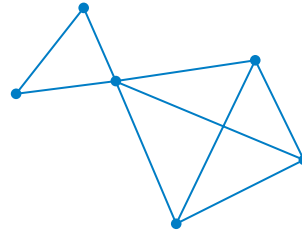
Exercise 8.07 Spanning trees

Example
18

1 Determine whether each graph is a tree, giving reasons.



2 Which of these is not a tree for the graph shown on the right?



- 3 For each adjacency matrix shown, draw its corresponding graph and determine if the graph is a tree.

a

	A	B	C	D
A	0	0	0	1
B	0	0	0	1
C	0	0	0	1
D	1	1	1	0

b

	P	Q	R	S
P	0	1	1	1
Q	1	0	0	1
R	1	0	0	0
S	1	1	0	0

c

	E	F	G	H
E	0	1	1	0
F	1	0	0	1
G	1	0	0	0
H	0	1	0	0

d

	E	F	G	H	I
E	0	0	1	0	0
F	0	0	1	1	0
G	1	1	0	0	0
H	0	1	0	0	1
I	0	0	0	1	0

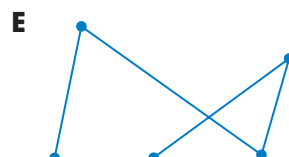
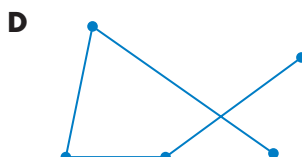
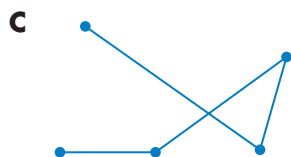
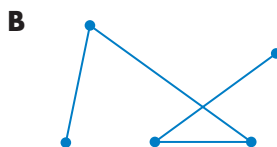
e

	U	V	W	X	Y
U	0	1	0	0	0
V	1	0	1	1	0
W	0	1	0	1	0
X	0	1	1	0	1
Y	0	0	0	1	0

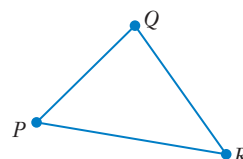
f

	K	L	M	N	O	P
K	0	1	0	0	0	1
L	1	0	0	1	0	0
M	0	0	0	0	0	1
N	0	1	0	0	1	0
O	0	0	0	1	0	1
P	1	0	1	0	1	0

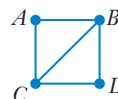
- 4 Which of the following is not a spanning tree for the graph shown on the right?



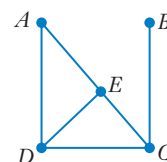
- 5 Many graphs have a number of different possible spanning trees. Sketch all of the distinct spanning trees for this graph.



- 6 The graph shown here has 8 distinct spanning trees. Sketch each one.



- 7 Draw 8 different spanning trees for this graph.

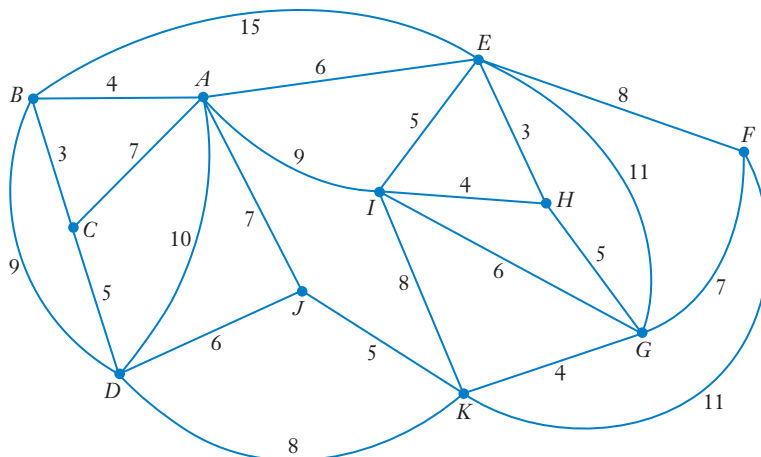


INVESTIGATION

SHORTEST PATHS

Work in groups of 3 or 4 for this investigation.

- 1 Use pieces of string tied to small keyrings to model the network shown below. Label each keyring with the letter of the vertex it represents and join the rings using pieces of string cut to the lengths of the edges.



- 2 To find the shortest path between C and H , simply pull the 2 keyrings tight. The shortest path is shown by the tight strings.
- 3 In your group, discuss the limitations of this type of physical model. Is it sensible to use for small networks? Is it practical to use for large networks? When might it be the most useful?
- 4 Now work in your group to find the shortest path from G to A in the network, without using your string model. Compare the paths from different groups and decide which is the shortest.
- 5 Find the shortest paths from B to F , C to E and G to B in the network. As well as trying to find the shortest paths, your group has to work out a strategy for finding the shortest path.
- 6 Discuss the strategies used by different groups.

In some cases, we will want to make sure that a particular vertex is on the path taken, or is not on the path taken.

Shortest paths with/without a particular vertex

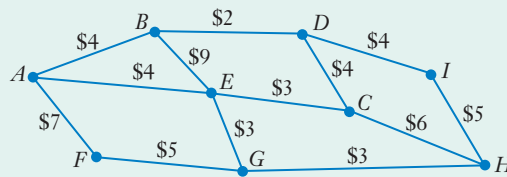
- To find a shortest path including a particular vertex, find the shortest paths from the starting point to that vertex, and from that vertex to the end point.
- To find the shortest path not including a particular vertex, remove that vertex and its connecting edges from the network, then find the shortest path using the remaining network.

EXAMPLE 21

Use the network below to find the shortest path

a from D to H , including E

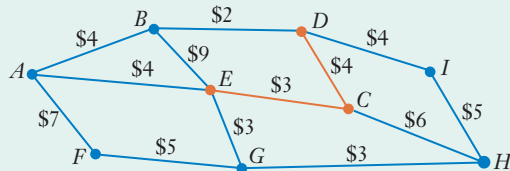
b from B to C , not including D .



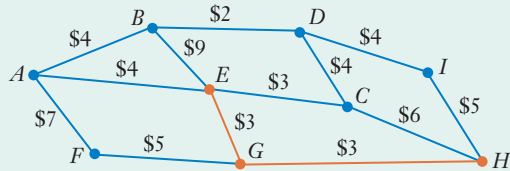
Solution

In this case, the graph shows the costs associated with the connections between adjacent vertices. The shortest path will give the least cost in this case.

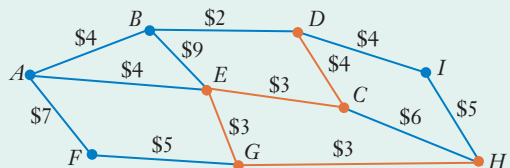
a Find the shortest path from D to E .



Now find the shortest path from E to H .



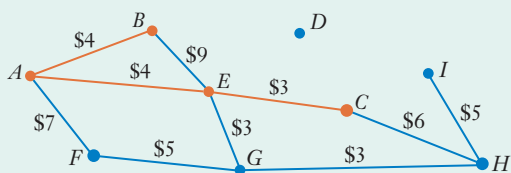
Combine to get the shortest path including E .



Write the answer.

The shortest path from D to H , including E , is $DCEGH$ at \$13.

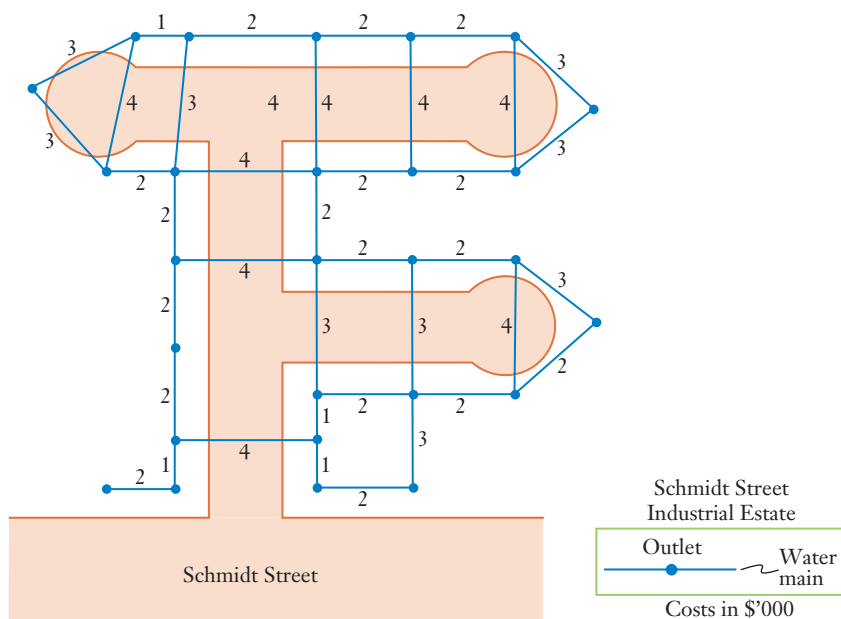
- Write the answer.



The shortest path from B to C , not including D , is $BAEC$ at \$11.

Services need to reach all the points of a real-world network. It is often desirable to do so in the shortest time, shortest distance or at the lowest cost. Telephone connections, rail networks and electricity distribution systems are examples of this type of problem. Each path that is kept open will impose extra cost on the service. The best method often uses the smallest possible spanning tree of the network. It may also be desirable to have some redundancy (extra connections).

The following network shows the possible water mains layouts for an industrial estate. The cost of laying each part is shown next to the section. Work in groups of 3 or 4 to find the cheapest way to connect the pipes so that every location has water.



Discuss your findings as a class group.

Can you work out a strategy for finding the cheapest method?

Minimum spanning trees

A **minimum spanning tree** for a network is a spanning tree with the total length of its edges at a minimum. Every vertex is connected to every other vertex by the network, but the total length of the reduced network is a minimum.

One way of finding a minimum spanning tree is by trial and error.

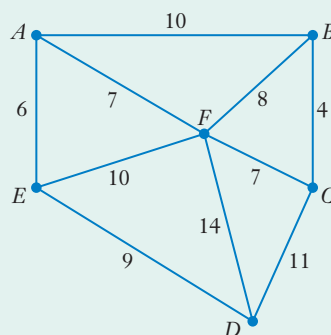
EXAMPLE 22

The water pipes between 6 regional towns need to be replaced. The graph on the right shows the distances in kilometres between the towns A, B, C, D, E and F .

The graph has not been drawn to scale.

The replacement water pipes need to link each town.

- What is the minimum spanning tree for this graph?
- What is the minimum length of water pipe required to connect all the towns?



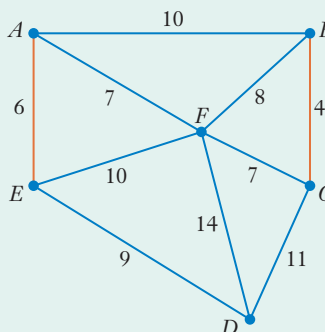
Solution

- List the edges in ascending order of weight (length).

$BC = 4, \quad AE = 6, \quad AF = 7, \quad FC = 7,$
 $FB = 8, \quad ED = 9, \quad AB = 10, \quad EF = 10,$
 $CD = 11, \quad FD = 14$

Select the shortest edge, $BC = 4$.

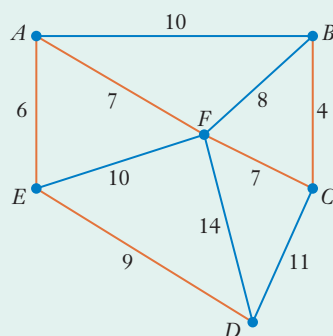
Then select the next shortest leg that does not form a cycle. $AE = 6$.



Now select the next shortest leg that does not form a cycle. $AF = 7$.

Repeat the procedure to select FC and finally ED .

Check that all vertices have been included.



The highlighted tree is the minimum spanning tree.

The minimum spanning tree is BC, CF, FA, AE, ED .

- b** Calculate the distance of the minimum spanning tree.

$$\begin{aligned} \text{Distance of minimum spanning tree} &= BC + CF + FA + AE + ED \\ &= 4 + 7 + 7 + 6 + 9 \\ &= 33 \end{aligned}$$

State the result.

The minimum length of water pipe required to connect the towns is 33 km.

There are different methods (called **algorithms**) for finding minimum spanning trees. One of them is called **Prim's algorithm**.



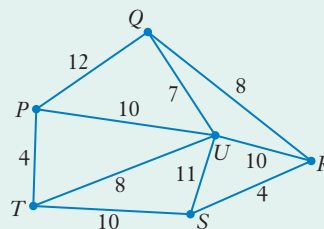
Prim's algorithm

Prim's algorithm

- 1 Choose any starting vertex.
- 2 Look at all the edges connected to this vertex and choose the one with the lowest weight and add this to form a tree.
- 3 Look at all the edges connected to the tree that do not have both vertices in the tree. Choose the edge with the lowest weight and add this edge to the tree.
- 4 Repeat step 3 until all the vertices are included in the tree.

EXAMPLE 23

Use Prim's algorithm to find the minimum spanning tree for this graph.



Solution

Choose any vertex, say P .

Choose an edge with lowest weight, PT .

Mark PT (the tree) and circle the vertices P and T .

Now look at all the edges connected to P and T : PQ , PU , TU and TS . Choose the one with the lowest weight, TU , and add it to the tree.

Mark TU and circle the vertex U .

Look at all the edges connected to the tree: PQ , UQ , US , TS and UR . You don't need to consider edge PU because both P and U are already in the tree.

Choose the edge with the lowest weight, UQ , and add it to the tree.

Mark UQ and circle the vertex Q .

Look at all the edges connected to P , T , U and Q .

You still need to connect vertices R and S to the tree. Look at the edges connected to R and S and choose the one with the lowest weight, QR .

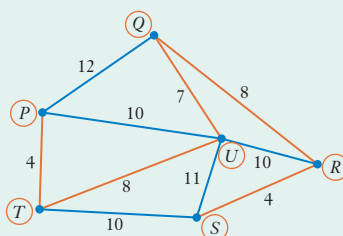
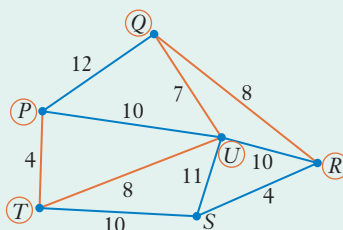
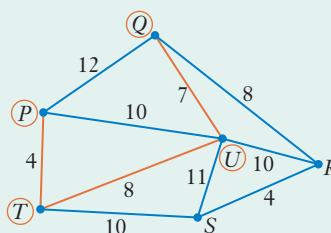
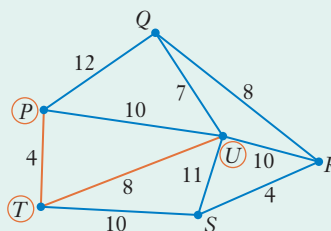
Mark QR and circle the vertex R .

There is only one vertex, S , to add to form the minimum spanning tree.

Choose edge SR and add it to the tree. Mark SR and circle the vertex S .

You can see that all vertices are circled, so the minimum spanning tree is complete.

State the result.

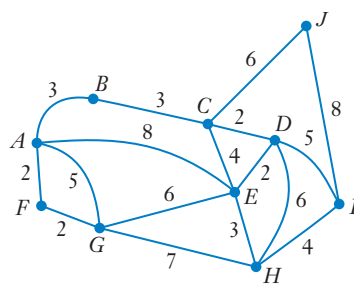


The minimum spanning tree is PT , TU , UQ , QR , RS .

Exercise 8.08 Minimum spanning trees

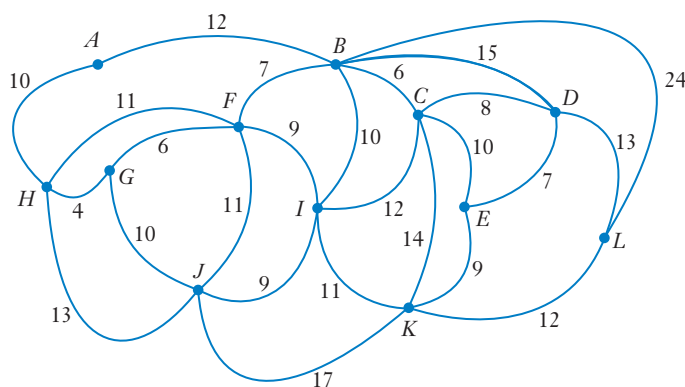
Example
21

- 1 Use trial and error to find the shortest path between A and H in the network shown on the right, where all distances are in kilometres.



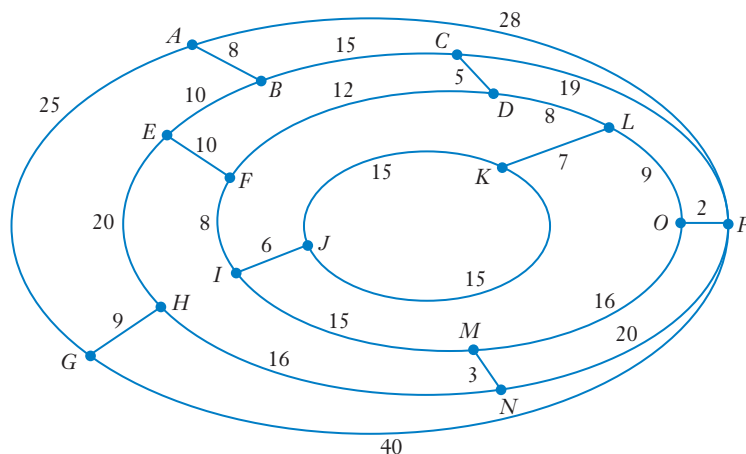
- 2 In the network shown below, find the shortest path from:

- a H to C b D to J c A to L
d G to B , including K e H to B , not including F .

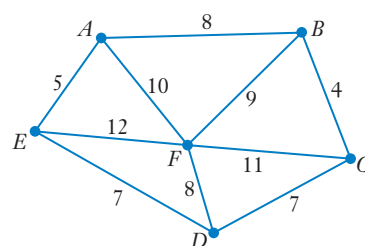


- 3 In the network shown below, find the shortest path from:

- a P to G b J to A c C to N
d E to O , including A e I to L , not including D .



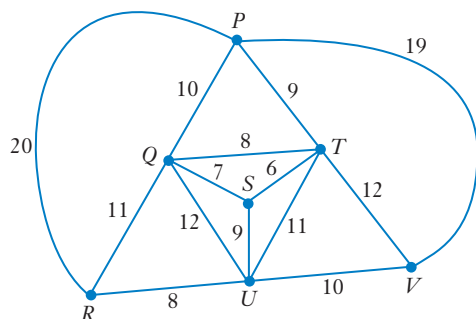
- 4 This graph shows the distances in kilometres between 6 towns, designated by the letters A, B, C, D, E and F , in a regional area. Fibre optic cable is to be laid so that each town is linked.



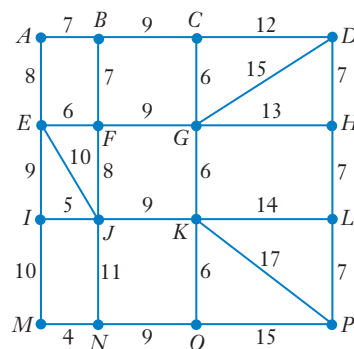
- Use trial-and-error to identify the minimum spanning tree for this graph.
- What is the minimum length of fibre optic cable required to connect all the towns?

- 5 Use trial and error to find the minimum spanning tree for each graph.

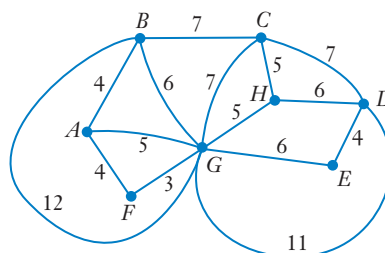
a



b



- 6 Use Prim's algorithm to find a minimum spanning tree for this network and state its length. All distances are in metres.



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9 A telephone network in a remote area has microwave, landline and radio links between sites. The cost to the telephone company of a link varies according to the cost of link maintenance and the volume of calls. The costs, in cents, of direct links are shown in the table.

Site	A	B	C	D	E	F	G	H	I
A	0	12			5	15	18		
B	12	0					10		7
C			0	12		8		18	
D			12	0	8	11	15		9
E	5			8	0	14			12
F	15		8	11	14	0	8	5	
G	18	10		15		8	0		
H			18			5		0	
I		7		9	12				0

a A and H **b** D and B **c** F and I **d** H and E .

- 10** In a research laboratory, there are 1 senior researcher, 4 associate researchers, 6 assistant researchers and 5 junior researchers. The senior researcher decides the assignments each day and can either tell people directly, or have the information passed on through other staff. The seniority of the researchers is senior > associate > assistant > junior.

The times taken to explain tasks to another person, including the tasks of their juniors, are:

Senior to associate: 5 minutes

Senior to junior: 11 minutes

Associate to junior: 8 minutes

Senior to assistant: 9 minutes

Associate to assistant: 4 minutes

Assistant to junior: 4 minutes

- a** Draw a directed network with 16 vertices to show how the information may be passed on.
- b** Find the best way to pass on the information. Note that in finding the shortest spanning tree the connections are directed.



Shutterstock.com/Garodenkoff

- 11** A school needs to circulate important policies around all the staffrooms for reading and signing. The head of department in each staffroom signs the policy before sending it on to the next staffroom, then it returns to the administration for filing. The distances in metres between the staffrooms are shown.

Staffroom	Admin	Maths/ Science	English/ Soc Ed	Art/ Music	Manual Arts	Home Ec/ PE
Admin	0	90	80	70	60	100
Maths/Science	90	0	100	90	150	200
English/Soc Ed	80	100	0	50	150	250
Art/Music	70	90	50	0	200	50
Manual Arts	60	150	150	200	0	200
Home Ec/PE	100	200	250	50	200	0

- a** Draw a network to show the distances between the staffrooms.
- b** Find the best order to send the policy document between staffrooms so that the distance it travels between staffrooms is minimised.
- 12** A government department has offices in a number of buildings throughout the city. The department has an interoffice courier system to ensure that memos and sensitive documents are securely and efficiently passed between offices. The time in hours taken for documents to travel between the offices is as shown in the table below.

Office	P	L	C	I	R	M
Planning	0	3	2	6	4	5
Legal	3	0	3	4	5	4
Contract letting	2	3	0	3	5	6
Inspection	6	4	3	0	4	3
Resources	4	5	5	4	0	2
Management	5	4	6	3	2	0

A highly confidential memo from the management that cannot be copied must be seen and initialled by each of the other department heads. What is the most efficient way for the memo to be passed around the offices and then returned to management?

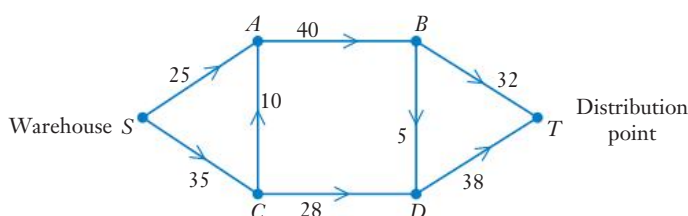
8.09 Flow networks

A **directed graph** is one where each edge has a direction, usually shown by an arrow. A directed graph is also called a **digraph**. Directed graphs are used to represent networks where something is moved from one point to another.

Directed networks involve flow rates. Examples are, traffic flow on highways, data flow across the Internet and water flow in the water grid. All these and other similar networks have common features. In this section, you will examine **network flow**.

In most networks, the objective is to maximise the flow rate. The flow rate in a network always depends on constraints.

The flow network below represents the transportation of widgets from a warehouse to a distribution point via a number of other nodes (vertices). The greatest number of widgets that can be transported along each edge is written on the edge.



A **flow network** is a directed graph where each arc (edge) has a **capacity** or **flow capacity** – its maximum possible flow. In the example above, the capacity is shown by the number of widgets written on each edge.

Each edge also has an **actual flow** – the flow through that edge. The **residual flow capacity** of an edge is the difference between the flow capacity and the actual flow.

The starting node of the network is where all flows commence. It is called the **source** (**S**). In the example above, the warehouse is the source.

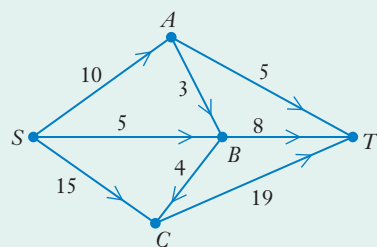
The flow goes through the network to the end node, called the **sink** (**T**). The distribution point is the sink in this example.

The **inflow capacity** of a node is the total of the flow capacities of all edges leading into the node.

The **outflow capacity** of a node is the sum of the flow capacities of the edges directed away from the node.

EXAMPLE 24

The network shown here represents pipes carrying water from a source (S) to a sink (T) via a number of nodes. The flow capacity for each pipe is shown in litres per second.



Determine:

- the outflow capacity from the source
- the inflow capacity into the sink
- the maximum flow into B
- the maximum flow out of B
- the maximum flow out of C

Solution

- a** The outflow from the source is the sum of all outflows from it.

State the result.

$$\begin{aligned}\text{Outflow from } S &= 10 + 5 + 15 \\ &= 30\end{aligned}$$

The outflow capacity is 30 L/s.

- b** The inflow to the sink is the sum of all inflows to it.

State the result.

$$\begin{aligned}\text{Inflow to } T &= 5 + 8 + 19 \\ &= 32\end{aligned}$$

The inflow capacity is 32 L/s.

- c** The maximum flow into node B is the sum of all inflows to it.

State the result.

$$\begin{aligned}B_{\text{In}} &= 3 + 5 \\ &= 8\end{aligned}$$

The maximum inflow to B is 8 L/s.

- d** The maximum flow out of node B is the sum of all outflows leaving it.

State the result.

$$\begin{aligned}B_{\text{Out}} &= 8 + 4 \\ &= 12\end{aligned}$$

The maximum outflow from B is 12 L/s.

- e** The maximum flow out of node C is the sum of all outflows leaving it.

State the result.

$$C_{\text{Out}} = 19$$

The maximum outflow from C is 19 L/s.

When a commodity flows through a network, the maximum flow is the same as the flow capacity. There may be situations where the actual flow is less than the capacity. In these situations, it is convenient to show the actual flow as well as the flow capacity on an edge. This can be done using two numbers separated by an oblique slash or comma. In this way, 4/8 would indicate that the actual flow through an edge is 4 while the capacity of the edge is 8.

When commodities flow through a network, it is important to remember the following:

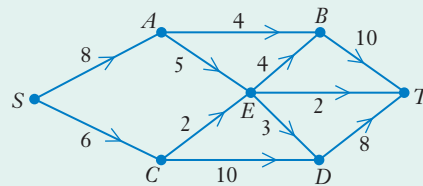
- all flows start from the source
- all outflows end at the sink
- the flow along an edge must be less than or equal to the capacity of the edge
- the flow into any node must be the same as the flow out of the node

EXAMPLE 25

In this network of pipes carrying water, there is no limit to the maximum outflow of the source.

The capacity of each edge is shown in L/min.

Show the flow through the network from source to sink.



Solution

Start by pushing 8 L/min from S to A and follow this through the network.

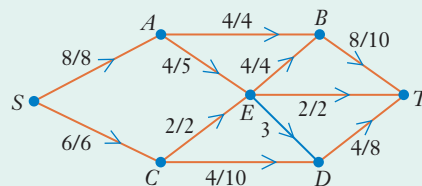
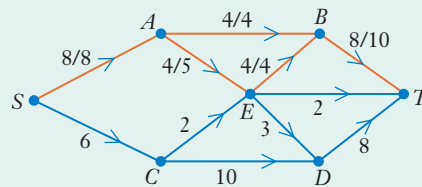
From A , there is capacity for 4 L/min to move along AB to B and also 4 L/min to move along AE to E and then on to B .

This means that 8 L/min flows into B . There is capacity for 8 L/min to flow along BT to T .

Now push 6 L/min from S to C .

From C , there is capacity for 2 L/min to move along CE and then along ET to T . This means that the remaining 4 L/min can move along CD and along DT to T .

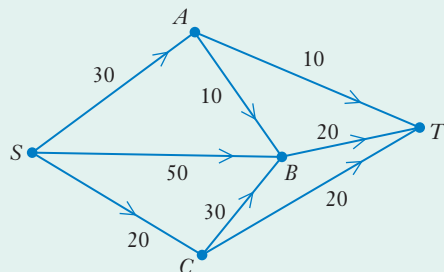
This means that a total of 6 L/min flows into T from E and D .



By calculating the flow through a network, you can find the maximum flow.

EXAMPLE 26

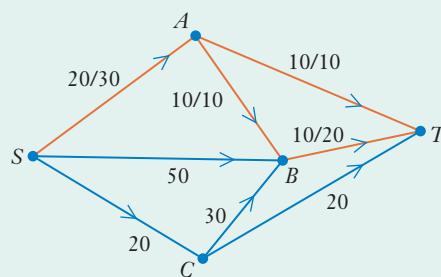
Find the maximum flow through this network.



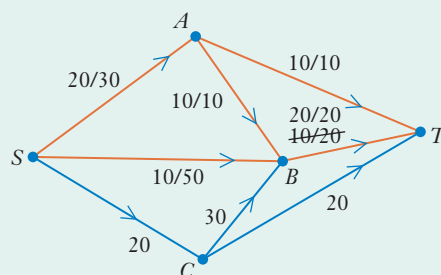
Solution

The outflow from A is 20, so start by pushing 20 from S to A .

From A , 10 flows along AT to T and 10 flows along AB to B and then on to T .

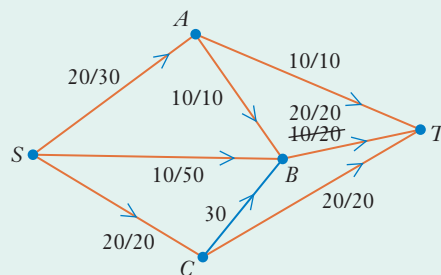


The available capacity of BT is now 10, so this is the maximum outflow from B . So push 10 from S to B and on to T .



The edge CB can't be used as there is no spare capacity in BT .

The capacity of CT is 20, so push 20 from S to C and on to T .



There is no unused capacity in the edges that feed into the sink, so the maximum flow has been achieved. Total the flows into T to find the maximum flow.

$$\begin{aligned}\text{Maximum flow} &= 10 + 20 + 20 \\ &= 50\end{aligned}$$

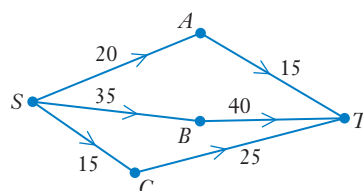
State the result.

The maximum flow for this network is 50.

Exercise 8.09 Flow networks

- 1 a Use the flow network to copy and complete this table.

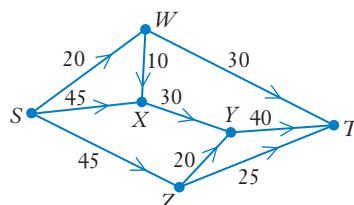
Node	Inflow capacity	Outflow capacity
A		
B		
C		



- b What is the maximum flow through each node?

- 2 a Use the flow network to copy and complete this table.

Node	Inflow capacity	Outflow capacity
W		
X		
Y		
Z		



- b What is the maximum flow through each node?

Questions 3–5 refer to this flow network.

- 3 The maximum flow into C is:

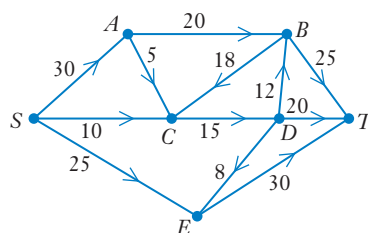
A 5 **B** 15 **C** 33
D 20 **E** 10

- 4 The maximum flow out of D is:

A 20 **B** 12 **C** 15 **D** 30 **E** 8

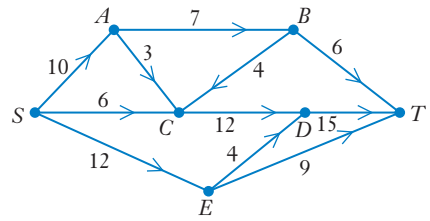
- 5 The maximum flow out of B is:

A 43 **B** 25 **C** 12 **D** 18 **E** 20



Example
24

- 6** Use this flow network to determine:
- a** the outflow capacity from the source
 - b** the inflow capacity into the sink
 - c** the maximum flow into C
 - d** the maximum flow out of C
 - e** the maximum flow into D
 - f** the maximum flow out of D
 - g** the maximum flow into B
 - h** the maximum flow out of B

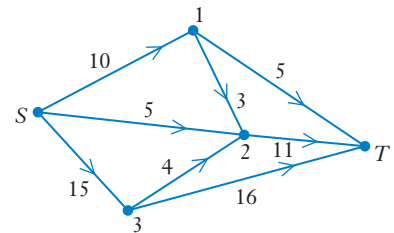
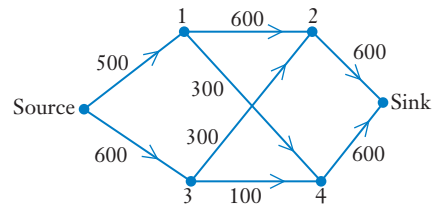


Example
25

- 7** This is the flow network for pipes carrying oil from a refinery (the source) to a storage facility (the sink). The source has a maximum outflow of 900 litres/second (L/s).

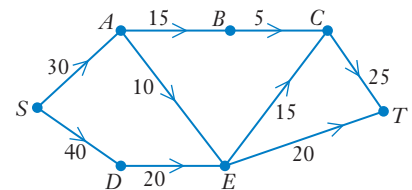
The capacity of each edge is shown in L/s. Show the flow through the network from source to sink by pushing:

- a** 300 L/s from the source to node 1
 - b** 300 L/s from the source to node 2
 - c** 100 L/s from the source to node 2
 - d** 200 L/s from the source to node 1
- 8** Show the flow through this network from source (S) to sink (T).

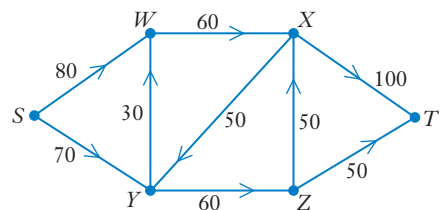


Example
26

- 9** Find the maximum flow through this network.

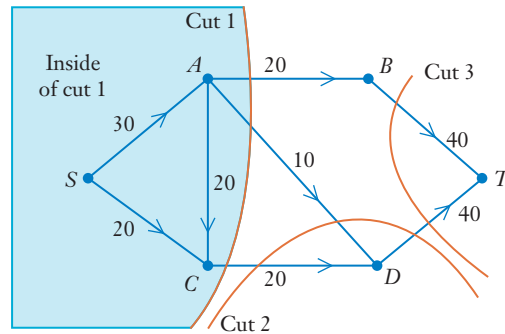


- 10** Find the maximum flow through this network.



8.10 The maximum flow–minimum cut theorem

A network can be **cut** with a line that separates the source from the sink. For a cut to be **valid**, it must stop *all* flow from the source to the sink. The side of the cut that includes the source is called the *inside* of the cut and the other side (including the sink) is called the *outside* of the cut. The inside of cut 1 is shaded in the diagram below. Cut 1 and cut 3 are valid, while cut 2 is not a valid cut.

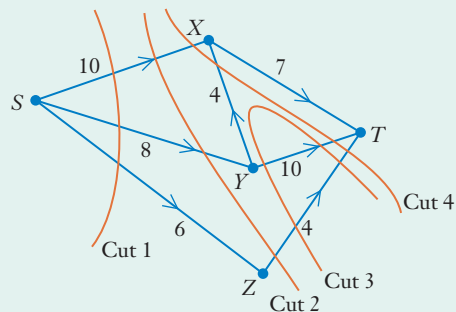


The capacity of a cut is the sum of the flows that it passes across. Only flows that pass from the inside to the outside of a cut are included in the calculation of the capacity of the cut.

EXAMPLE 27

Examine this flow network.

- Identify any cuts that are not valid.
- Calculate the capacity of all valid cuts.



Solution

- Look for a cut that does not completely separate the source from the sink.

State the result.

It is possible for flow to go from S through X to T , bypassing cut 3.

Cut 3 is not valid.

- b** Add the capacities of the edges that cross across the cut from left to right.

$$\text{Capacity of cut 1} = 10 + 8 + 6 = 24$$

$$\text{Capacity of cut 2} = 10 + 8 + 4 = 22$$

$$\begin{aligned}\text{Capacity of cut 4} &= 10 + 4 + 10 + 4 \\ &= 28\end{aligned}$$

From the previous example, you can see that for any given network the capacities of different cuts may vary. The **minimum cut** is the cut with the minimum capacity. The capacity of the minimum cut is related to the flow in a network as follows.

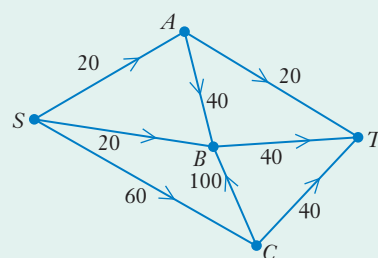
Maximum flow-minimum cut theorem

- 1 Identify valid cuts through the network.
- 2 Find the capacity of each cut.
- 3 The maximum flow of the network = capacity of the minimum cut.

The maximum flow of a network can be found by using the **maximum flow-minimum cut theorem**.

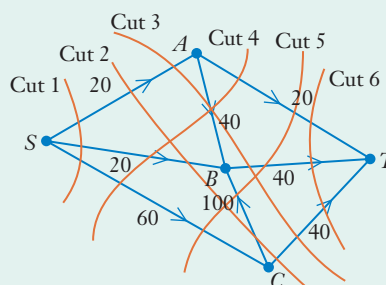
EXAMPLE 28

Calculate the maximum flow for this network.



Solution

Identify the valid cuts.



Calculate the capacity of each cut.

Do not count AB as its flow is in the opposite direction, from outside to inside of the cut.

Do not count CB .

$$\text{Capacity of cut 1} = 20 + 20 + 60 = 100$$

$$\text{Capacity of cut 2} = 20 + 20 + 100 + 40 = 180$$

$$\text{Capacity of cut 3} = 20 + 40 + 40 = 100$$

$$\text{Capacity of cut 4} = 20 + 40 + 20 + 60 = 140$$

$$\text{Capacity of cut 5} = 20 + 40 + 60 = 120$$

$$\text{Capacity of cut 6} = 20 + 40 + 40 = 100$$

The capacity of the minimum cut is 100.

The maximum flow for this network is 100.

You can convert a table of flow information into a flow network.

EXAMPLE 29

- a** Water flows from a dam (source) to a reservoir (sink) via a number of nodes as shown in the following table. The flow capacity of the pipes between each node is shown. Convert the information in the table into a flow network diagram.

From	To	Capacity (L/min)
S	A	400
A	T	150
A	B	50
S	B	250
B	T	350
S	C	500
C	T	350

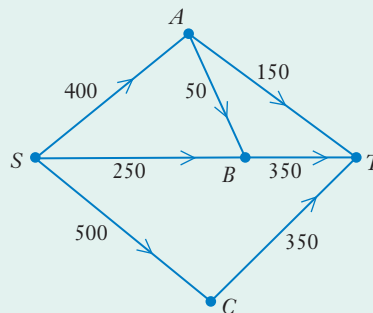
- b** Determine the maximum flow for the network.

Solution

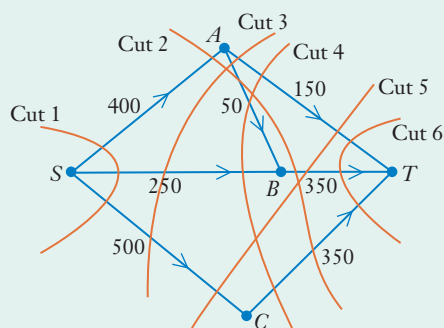
- a** There is a source (S), a sink (T) and 3 other nodes (A , B and C).

Plot the nodes and link the nodes with edges as indicated in the table.

Indicate the direction and capacity of flow on the edges.



- b** Make the valid cuts on the diagram.



Calculate the capacity of each cut.

Do not count AB.

$$\text{Capacity of cut 1} = 400 + 250 + 500 = 1150$$

$$\text{Capacity of cut 2} = 400 + 350 + 350 = 1100$$

$$\text{Capacity of cut 3} = 150 + 50 + 250 + 500 = 950$$

$$\text{Capacity of cut 4} = 150 + 50 + 250 + 350 = 800$$

$$\text{Capacity of cut 5} = 150 + 350 + 500 = 1000$$

$$\text{Capacity of cut 6} = 150 + 350 + 350 = 850$$

Identify the minimum cut.

Cut 4 is the minimum cut.

State the maximum flow.

$$\begin{aligned} \text{Maximum flow} &= \text{minimum cut} \\ &= 800 \end{aligned}$$

State the result.

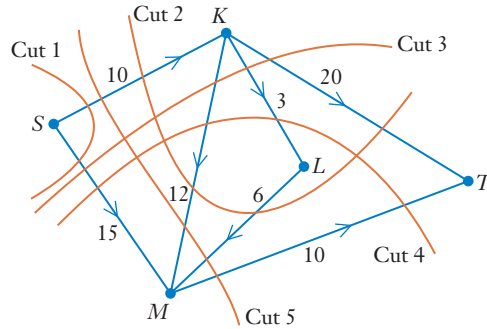
The maximum flow for the network is 800 L/min.



Getty Images/Federica Grossi

Exercise 8.10 The maximum flow–minimum cut theorem

This diagram shows a flow from the source (S) to the sink (T). Use the diagram to answer questions 1–4.



1 Which of the cuts is not valid?

- A** Cuts 1 and 2
- B** Cuts 2 and 4
- C** Cuts 1, 2 and 4
- D** Cuts 3 and 5
- E** Cuts 1, 3 and 5

2 The capacity of cut 3 is:

- A** 50 **B** 25 **C** 35 **D** 48 **E** 33

3 The capacity of cut 5 is:

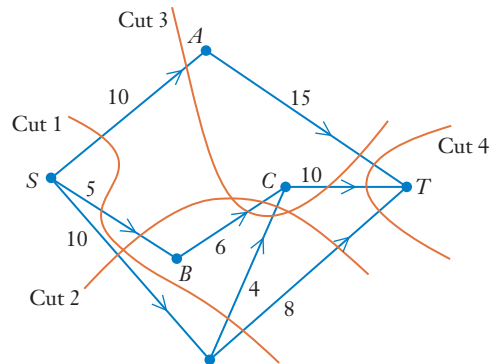
- A** 30 **B** 32 **C** 25 **D** 38 **E** 20

4 The maximum flow from source to sink is:

- A** 50 **B** 30 **C** 20 **D** 40 **E** 35

5 Examine this flow network.

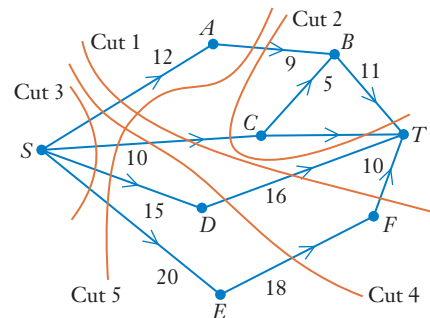
- a** Identify any cuts that are not valid.
- b** Calculate the capacity of all valid cuts.



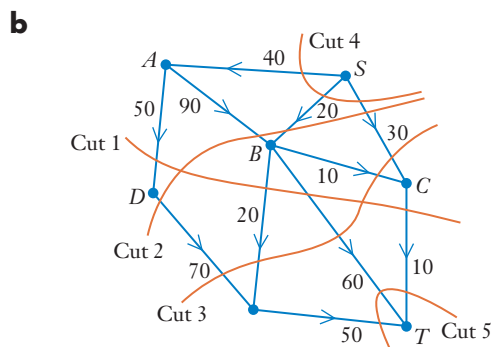
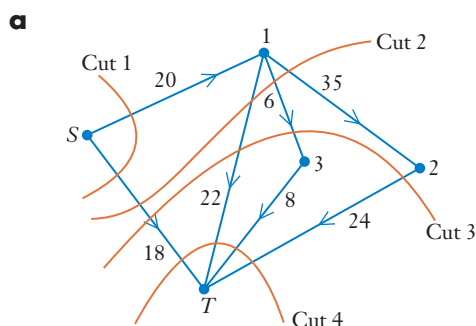
Example
27

6 **a** Identify any cuts in this flow network that are not valid.

b Calculate the capacity of all valid cuts.

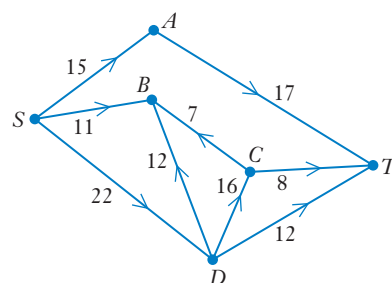


7 For each flow network, which of the cuts shown is the minimum cut?

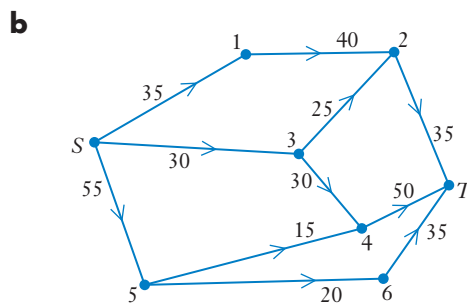
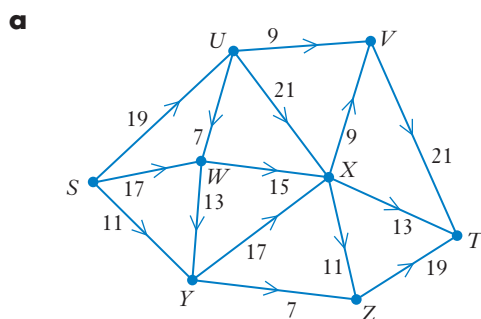


Example
28

8 Calculate the maximum flow for this network.



9 What is the maximum flow for each network?



Example
29

10 a Electricity moves from a generator (source) to a substation (sink) via a number of nodes, as shown in the table. The flow capacity of the wires, measured in megawatts (MW), between each node is shown. Convert the information in the table to a flow network diagram.

b Determine the maximum flow for the network.

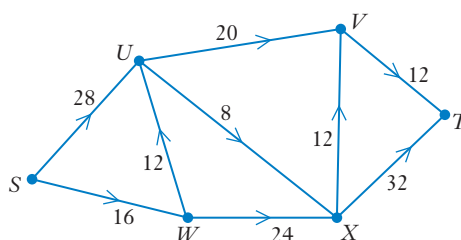
From	To	Capacity (MW)
S	A	1100
A	B	1500
B	T	1200
S	C	800
C	E	200
C	T	300
S	D	500
D	E	900
E	T	1000

- 11 a** A widget manufacturer transports widgets from the factory (source) to a distribution centre (sink) via a number of handling centres numbered 1 to 4. The capacity to transfer widgets from the source to the sink via the nodes is shown in the table. Convert the information in the table to a flow network diagram.
- b** Determine the maximum flow for the network.

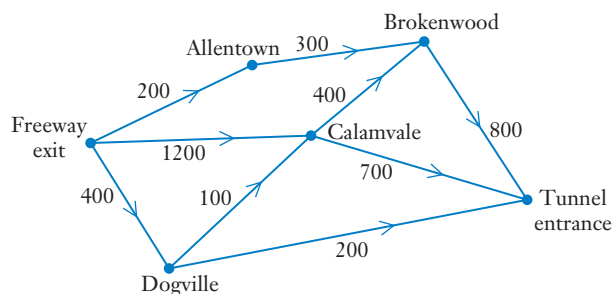
From	To	Capacity (widgets)
S	1	63
1	T	33
1	2	33
2	3	58
S	2	51
S	4	53
4	2	16
4	3	15
4	T	26
3	T	76

Problem solving

- 12** Examine this network.
- a** Identify a cut with a capacity of 36.
- b** Show the flow through the network when the flow capacity is 36.



- 13** The capacity of a busy road network to move traffic from a freeway exit to a tunnel entrance through four towns is shown in the diagram below. The flow rate in vehicles per minute is shown on each edge as is the direction of the traffic.



- a** Calculate the greatest traffic flow from the freeway exit to the tunnel entrance.
- b** What are the actual traffic flow rates between each of the towns when the greatest flow rate is achieved?
- c** When the greatest flow is achieved, how many vehicles per hour pass through Calamvale?
- d** The Department of Main Roads determines that the traffic flow through Calamvale needs to be reduced to a maximum of 1000 vehicles per minute. It is also determined that the traffic flow on only one other edge is to be increased. If the same traffic flow that was calculated in part a is to be maintained, which edge should be chosen, and by how much must its capacity be increased?

8. CHAPTER SUMMARY



Networks
crossword



Networks
find-a-word

Networks, paths and trees

- A **graph** is a set of points called **vertices** (or **nodes**) connected by lines called **edges** (or **arcs**) joining a pair of vertices. **Adjacent vertices** are connected by an edge. A **loop** is an edge that connects a vertex to itself. A graph has **multiple edges** if two or more edges connect the same pair of vertices. An **isolated vertex** is one that is not connected to any other vertex
- There are many different types of graphs: a **simple graph**, a **complete graph**, a **connected graph**, a **disconnected graph**, a **subgraph**, a **directed graph** and a **bipartite graph**
- A **network** is a type of graph. There are many types of networks: a **finite network**, a **directed network**, an **undirected network** and a **weighted network**
- For a graph or network, a **branch** is a vertex that has more than one exit. A **path** or **chain** joins a starting vertex to a finishing vertex via a continuous series of edges. A **circuit** or **cycle** is a path that returns to its starting point. A **simple** chain, path, cycle or circuit does not cross over itself, so does not repeat vertices
- The **degree** of a vertex, written $\deg(V)$, is the number of edges that are directly connected to the vertex
- Graphs that represent the same information are called **isomorphic graphs**
- An **adjacency matrix** can be used to represent a graph or network. It shows how adjacent edges are joined
- A **planar graph** is one that can be drawn without edges that cross. A planar graph is defined by **Euler's formula**:

$$\text{Vertices} + \text{faces} - \text{edges} = 2$$

$$v + f - e = 2$$

- A **walk** is any route (sequence of edges) taken through a network, including routes that repeat edges and vertices. An **open walk** starts and finishes at a different vertex while a **closed walk** starts and finishes at the same vertex
- A **trail** is a walk in which you travel along an edge once only. An **open trail** starts and finishes at a different vertex while a **closed trail (circuit)** starts and finishes at the same vertex
- A **path** is a walk in which no vertices are repeated, except possibly the start and finish. A path beginning and ending at the same vertex is also called a **cycle**

- An **Eulerian cycle** (or **Eulerian circuit**) is a trail that traverses every edge of a graph exactly once. An **Eulerian path** (or **Eulerian trail**) is an open trail that traverses each edge only once
- An **Eulerian graph** contains an Eulerian circuit and a **semi-Eulerian graph** contains an Eulerian path. An Eulerian graph has all vertices of even degree and a closed trail or **Eulerian cycle** exists. A semi-Eulerian graph has only two vertices of odd degree, an open trail or **Eulerian path** exists and it must start and finish at a vertex of odd degree
- A **Hamiltonian cycle** is a route that includes each vertex in a graph exactly once and starts and finishes at the same vertex. A **Hamiltonian path** is a route that includes each vertex in a graph exactly once and starts and finishes at different vertices
- A **tree** is a connected graph that does not contain any loops, multiple edges or cycles, while a **spanning tree** is a subgraph that includes all the vertices of the original graph
- A **minimum spanning tree** for a network has the total length of its edges at a minimum. **Prim's algorithm** can be used to identify a minimum spanning tree
- A **directed graph (digraph)** is one where direction is indicated along each edge
- A **flow network** is a directed graph where each edge has a **capacity** or **flow capacity**. The movement of commodities across the network is known as **network flow**
- Each edge also has an **actual flow** and a **residual flow capacity** – the difference between the flow capacity and the actual flow
- The **source** (**S**) is the node from which all flows commence. The **sink** (**T**) is the node towards which all flows move – the end node
- The **inflow capacity** is the total of the flow capacities of all edges leading into the node and the **outflow capacity** is the sum of the flow capacities of the edges directed away from the node
- A network **cut** is a line that separates the source from the sink. A **valid** cut must stop *all* flow from the source to the sink. The side of the cut that includes the source is called the *inside* of the cut and the other side (including the sink) is called the *outside* of the cut. The capacity of a cut is the sum of the flows that it passes across. Only flows that pass from the inside to the outside of a cut are included in the calculation of the capacity of the cut.
- The **minimum cut** is the cut with the minimum capacity. For any flow network, the **maximum-flow minimum-cut theorem** states:

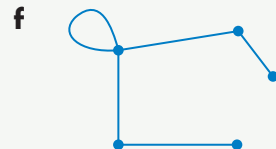
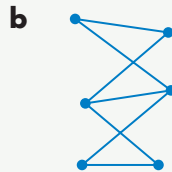
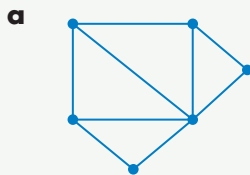
$$\text{Maximum flow} = \text{capacity of the minimum cut}$$

8. CHAPTER REVIEW

Networks, paths and trees

Example
1

1 For each graph, identify:



- i** the number of vertices and edges
- ii** any connected graphs
- iii** any simple graphs
- iv** any complete graphs
- v** any bipartite graphs.

Example
2

2 State whether each graph is a chain, path, cycle or circuit and whether it is simple.

a $ABDG$

b $EFGDBAE$

c $ABCDBEA$

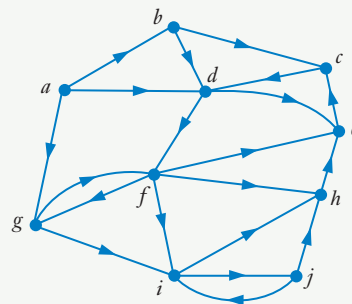
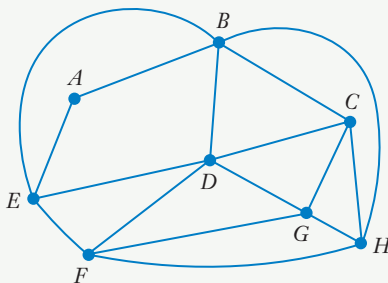
d $DGCHFDB$

e $adba$

f $abcde$

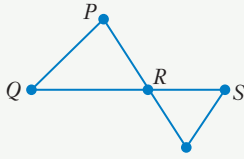
g $fhecdf$

h $gfhjihec$

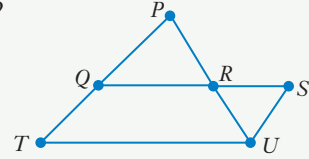
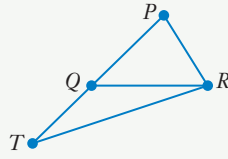


3 Which graph is not a subgraph of the graph shown on the right?

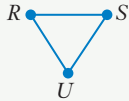
A



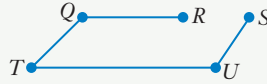
B



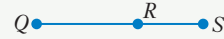
C



D



E



Examples
1-3

4 Classify each network as directed or undirected; connected or disconnected; and cycle, circuit, tree or mixed type.

a



b



c



d



e



f



Example
3

5 The graph shown is:

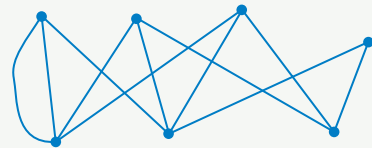
A complete

B disconnected

C bipartite

D Eulerian

E simple



Examples
1-3

6 A complete graph with 8 vertices has:

A 28 edges

B 40 edges

C 56 edges

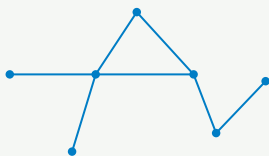
D 25 edges

E 32 edges

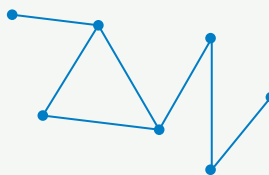
Examples
1-3

7 Determine if any of the graphs shown below are isomorphic.

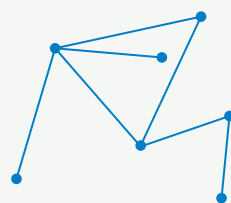
a



b



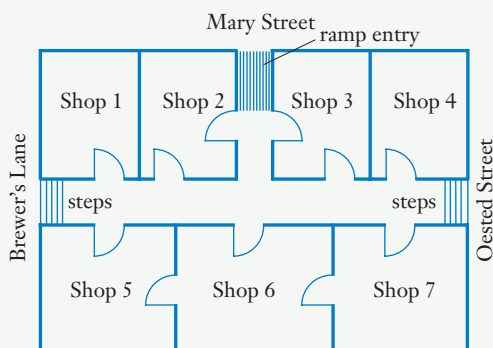
c



Example
4

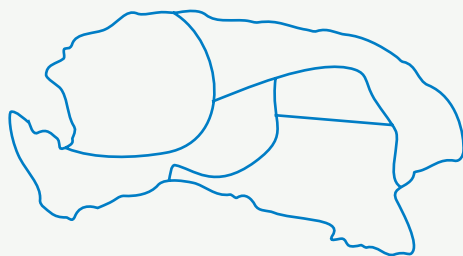
Example
5

- 8** Model the following plan of a shopping arcade as a network, showing the outside as one vertex and the doors and stairs as connections between shops.



Example
6

- 9** How many colours must be used to colour this map so that every border has a different colour on each side?



Example
7

- 10** Represent this graph as an adjacency matrix.

Questions **11–13** refer to these adjacency matrices.

Matrix 1

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

Matrix 2

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Matrix 3

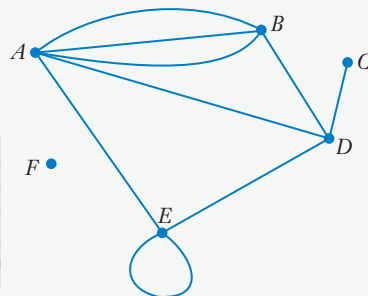
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix 4

$$\begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Matrix 5

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$



Examples
7,8

- 11** Which matrix corresponds to a complete graph?

A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4 **E** Matrix 5

12 Which matrix corresponds to a disconnected graph?

- A** Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4 **E** Matrix 5

Examples
7,8

13 Which matrix corresponds to a graph that has a loop?

- A** Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4 **E** Matrix 5

Examples
7,8

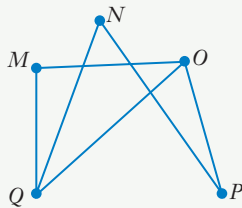
14 Construct a graph that has this adjacency matrix.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>A</i>	0	1	1	1	0
<i>B</i>	1	0	2	0	3
<i>C</i>	1	2	0	1	0
<i>D</i>	1	0	1	1	1
<i>E</i>	0	3	0	1	0

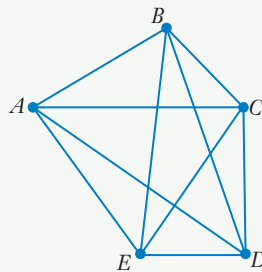
Example
8

15 Redraw each graph as a planar graph if possible.

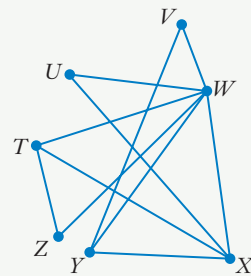
a



b



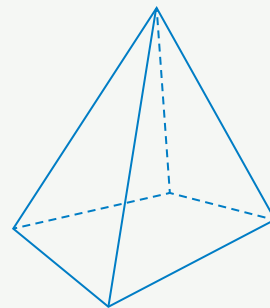
c



Example
9

16 a Represent the tetrahedron on the right as a planar graph.

b Verify Euler's formula using the graph.



Example
10

17 Use Euler's formula to calculate the unknowns for each planar graph described below.

a $v = ?$, $e = 8$ and $f = 5$

b $e = ?$, $v = 8$ and $f = 6$

c $f = ?$, $e = 9$ and $v = 7$

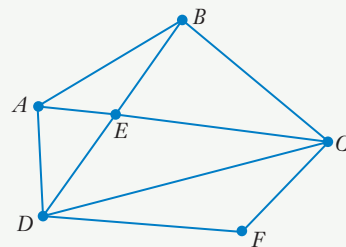
Example
11

Example
12

Questions 18–20 refer to this diagram.

18 The route $ABEBC$ is best described as:

- A** a closed walk
- B** an open walk
- C** a cycle
- D** an open trail
- E** a circuit



Example
12

19 The route $ADEFC$ is best described as:

- A** a closed walk
- B** an open walk
- C** a cycle
- D** an open trail
- E** a circuit

Example
12

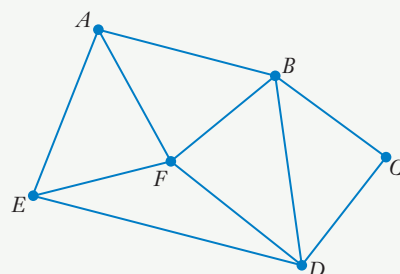
20 The route $AECFDA$ is best described as:

- A** a closed walk
- B** an open walk
- C** a cycle
- D** an open trail
- E** a circuit

Examples
12, 13

21 a Use the graph on the right to describe each of the following as a path, walk, trail, cycle or circuit.

- i** $ABFDC$
- ii** $AEFDBFA$
- iii** $AEFABD$
- iv** $ABFED$
- v** $AFDBFA$

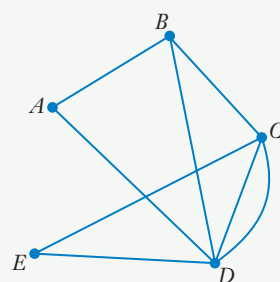


- b** Use the degree of the vertices to show that the graph is semi-Eulerian.
- c** Describe an Eulerian path for this graph.

Example
14

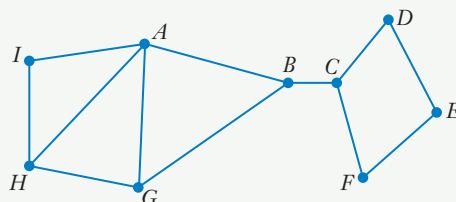
22 a Use the degree of the vertices to show that this graph is Eulerian.

- b** Describe an Eulerian cycle for this graph.

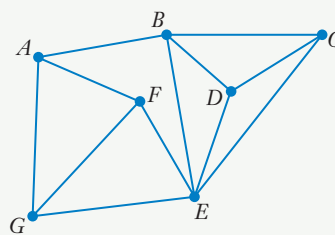


Example
15

23 Name a Hamiltonian path for the graph shown here.



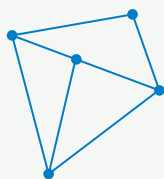
- 24 a** Name a Hamiltonian cycle for the graph shown.
b Name a Hamiltonian cycle that begins at A and does not include edge EC .



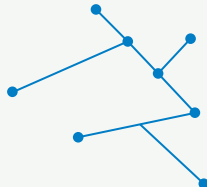
Examples
16, 17

- 25** Determine whether each graph is a tree.

a



b

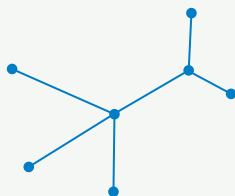


c



Example
18

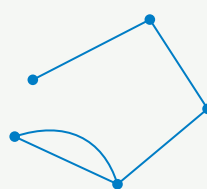
d



e



f



- 26** Draw the graph that corresponds to each adjacency matrix and determine if the graph is a tree.

a

	A	B	C	D	E
A	0	1	0	0	0
B	1	0	1	0	0
C	0	1	0	1	1
D	0	0	1	0	0
E	0	0	1	0	0

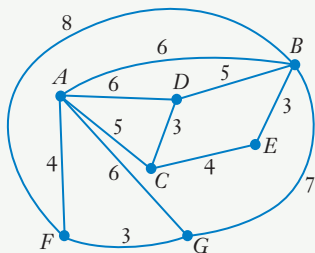
b

	P	Q	R	S	T
P	0	1	0	0	0
Q	1	0	1	0	1
R	0	1	0	1	0
S	0	0	1	0	0
T	0	1	0	0	0

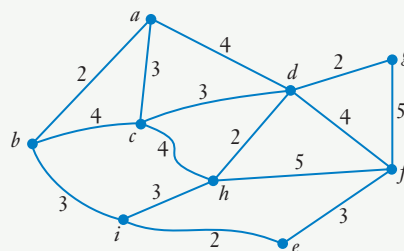
Example
19

- 27 i** Find a minimum spanning tree for each network.
ii In each case, use the spanning tree to name the edges in the original graph that are not required to connect all the vertices.

a



b



Example
20

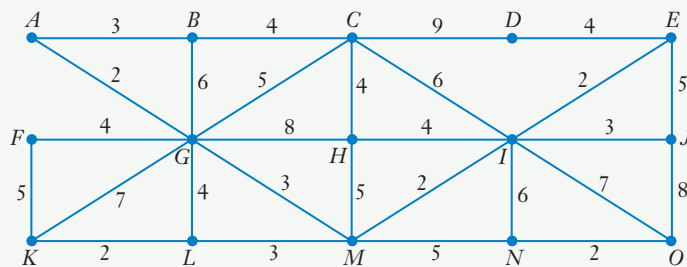
Example
21

28 Use the network shown below to find the shortest path from:

a A to O

b K to J

c E to F .



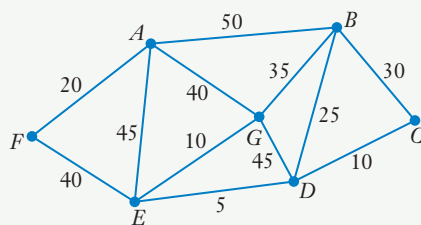
Example
22

29 The sewerage pipes between 7 houses in an estate need to be replaced. The graph on the right shows the distances in metres between the houses marked A, B, C, D, E and F .

The replacement sewerage pipes need to link each house.

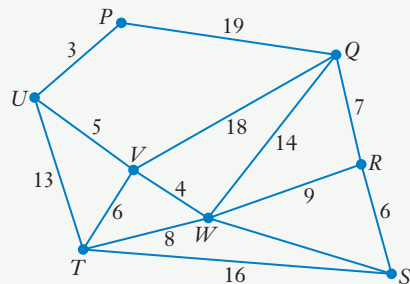
a What is the minimum spanning tree for this graph?

b What is the minimum length of sewerage pipe required to connect all the houses?



Example
23

30 Use Prim's algorithm to find the minimum spanning tree for this graph.



Examples
24-26

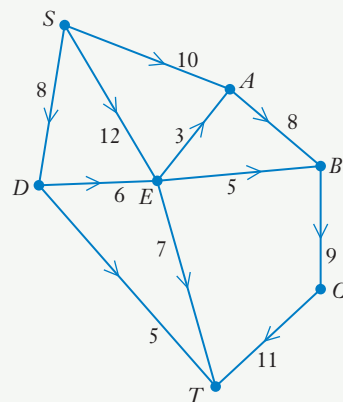
31 The network shown here represents pipes carrying water from a source (S) to a sink (T) via a number of nodes. The flow capacity for each pipe is shown in litres per second.

a Determine:

- i** the outflow capacity from the source
- ii** the inflow capacity into the sink
- iii** the maximum flow into E
- iv** the maximum flow out of E

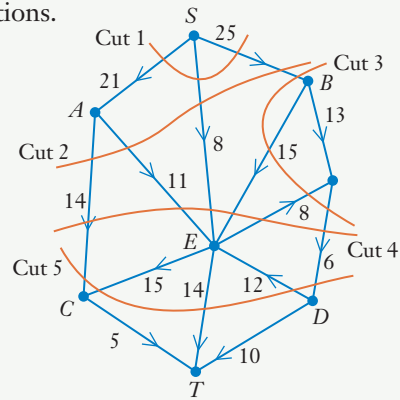
b Show the flow through the network from source to sink.

c What is the maximum flow through the network?



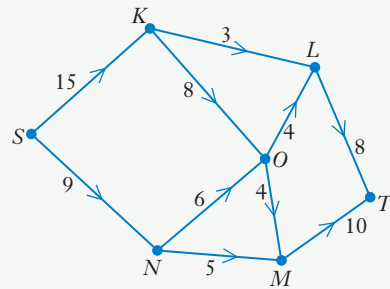
32 Use this flow network to answer the following questions.

- a** Identify any cuts that are not valid.
- b** Calculate the capacity of all valid cuts.



Example
27

33 Use the maximum flow–minimum cut theorem to calculate the maximum flow for this network.



Example
28

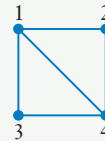
- 34 a** Widgets are moved from a factory (source) to a storage facility (sink) via a number of nodes as shown in the table. The transportation capacity to move the widgets between each node is shown. Convert the information in the table into a flow network diagram.
- b** Determine the maximum flow for the network.

From	To	Capacity
S	A	40
A	B	32
A	C	15
C	B	28
C	T	27
B	T	45
S	C	35
S	D	30
D	T	36

Example
29

Problem solving

35 Draw a sketch of every distinct spanning tree for this graph.



- 36** An overnight freight express firm operating in the eastern states has tabulated the times and costs of sending standard parcels between its depots as follows.

Depots/route	Time (h)	Cost (\$)
Townsville–Mt Isa	10	60
Townsville–Cairns	4	12
Townsville–Rockhampton	7	34
Rockhampton–Brisbane	5	20
Rockhampton–Moree	7	40
Brisbane–Moree	5	24
Brisbane–Coolangatta	1	14
Coolangatta–Sydney	6	24
Dubbo–Sydney	4	20
Moree–Dubbo	4	20
Moree–Broken Hill	7	50
Broken Hill–Adelaide	5	30
Sydney–Canberra	5	20
Sydney–Melbourne	8	32
Canberra–Melbourne	7	28
Melbourne–Adelaide	8	34
Dubbo–Melbourne	8	34
Dubbo–Broken Hill	6	40

- Draw up a combined network showing the times and costs of sending parcels between the depots.
- Find the shortest time it will take to send a parcel from Brisbane to Adelaide.
- Find the lowest cost for sending a parcel from Adelaide to Mt Isa.



Practice quiz

Practice examination 3 ●●●○

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved scientific calculator.
- Access to spreadsheet software would be useful for some questions.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (4 marks)

Calculate the final amount and interest earned for each investment.

- a** \$15 000 at 9.6% p.a. compounding monthly for 2 years
- b** \$4500 at 4.5% p.a. compounding quarterly for 3 years

Question 2 (1 mark)

How much should Felix invest to get back \$5000 in 3 years time at 8.1% p.a. interest, calculated monthly?

Question 3 (2 marks)

How long would it take \$4000 to grow to \$5000 at 8% p.a. interest, calculated quarterly?

Question 4 (2 marks)

What is the effective interest rate for 7.5% p.a. compounding weekly?

Question 5 (2 marks)

Which investment would provide the better return: 8.04% p.a. compounding monthly or 7.95% p.a. compounding daily?

Question 6 (1 mark)

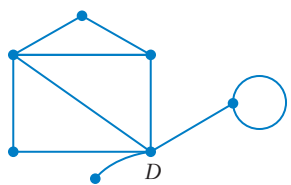
Nami has \$4000 invested at 7.8% p.a. interest calculated fortnightly. What will be the value of her investment in 18 months?

Question 7 (1 mark)

How many vertices and edges does this graph have?

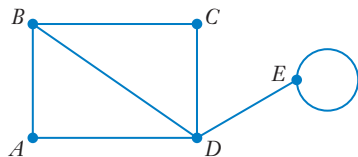
**Question 8** (1 mark)

How many edges are directly connected to vertex D ?



Question 9 (1 mark)

Which point(s) are adjacent to point C ?

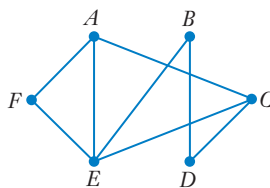
**Question 10** (1 mark)

How many edges would a complete graph have if it contained 11 vertices?

- A** 55
- B** 66
- C** 110
- D** 121
- E** 132

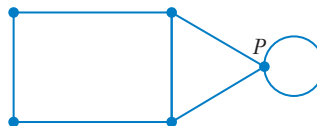
Question 11 (1 mark)

This graph represents tennis games that were played over one day. Which opponent(s) did person A play?

**Question 12** (2 marks)

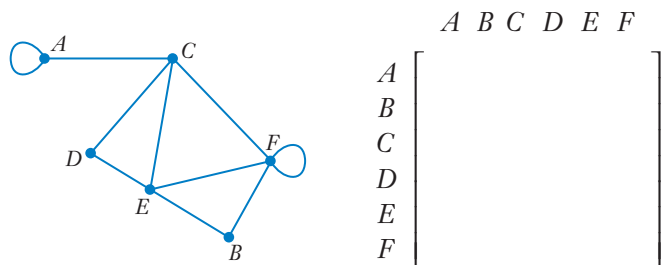
For this graph, calculate:

- a** the degree of vertex P
- b** its degree sum



Question 13 (2 marks)

Copy and complete the adjacency matrix for this graph.



Question 14 (1 mark)

A planar graph has 7 vertices and 9 edges. How many faces must it have?

Question 15 (1 mark)

Which one of these is an Eulerian graph?



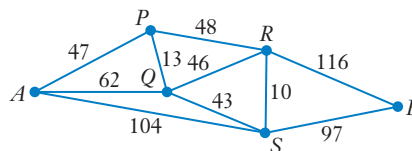
Question 16 (1 mark)

Which one of these is a semi-Eulerian graph?



Question 17 (2 marks)

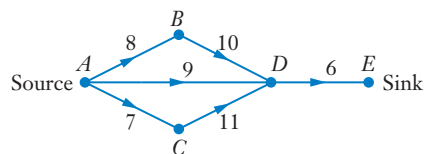
In this graph of a road map, distances are in kilometres. Find the shortest path from A to B .

**Question 18** (1 mark)

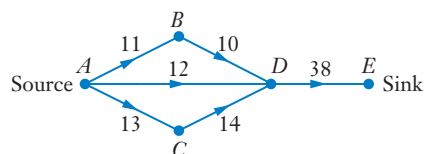
For the vertex D below, what is the inflow capacity, outflow capacity, and maximum flow? The units are L/s.

**Question 19** (1 mark)

State the capacity of a cut through BD .

**Question 20** (2 marks)

State the flow capacity of this network.



Question 21 (2 marks)

Without performing any calculation, identify which of the following investment options will have the greatest return.

A: 7.3% p.a. compounded monthly

B: 7.3% p.a. compounded quarterly

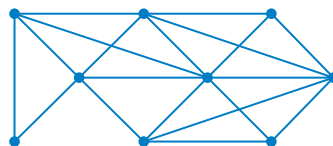
Explain your reasoning.

Question 22 (2 marks)

A work of art grows in value by 15% each year. If it was insured for \$3500 in 2009, what should it be insured for in 2019?

Question 23 (4 marks)

How many edges must be removed from the graph at right to form a spanning tree? Draw the resulting graph.

**Question 24** (2 marks)

An antique Victorian writing desk was valued at \$1500. Four years later, the value had increased to \$2100.

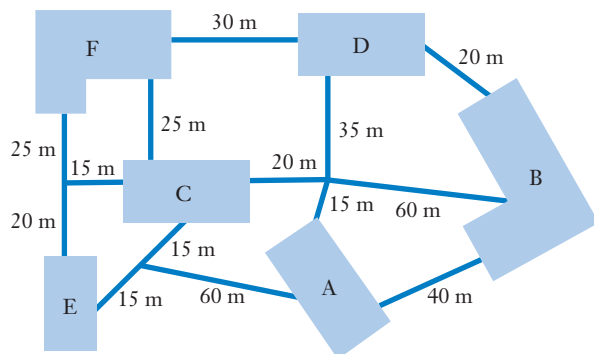
Use the annual percentage growth rate to find what it will be worth in a further 10 years time.

Question 25 (2 marks)

An oil slick is spreading at 12% per hour. At 8 a.m., the slick is estimated to cover 135 m^2 . What area will be covered at 1 p.m.?

Question 26 (4 marks)

This map shows the possible locations of covered walkways in a new school, with the lengths of each section shown.



It costs \$120 per metre to erect the walkways, plus \$850 for each section of walkway roof to be connected to the stormwater system. Each section of joined walkway needs one separate stormwater connection (for example, the walkways that join in the middle of A, B, C and D blocks would each need a connection). It is necessary to construct the walkways so that students can walk to any building via walkways and other buildings without getting wet.

Find the shortest and the cheapest walkway system.

END OF EXAMINATION

9

ANNUITIES AND PERPETUITIES

You already know compound interest depends on the amount borrowed, the interest rate, the interest compounding period and the amount repaid each compounding period.

Compound interest can help you to build wealth to pay for a car, a holiday or even your retirement.

There's a well-known claim often quoted by financial investors: 'Compound interest is the eighth wonder of the world. Those who understand it, earn it. Those who don't, pay it.'

In this chapter, you will examine annuities and perpetuities, which are investments involving regular contributions to an account and interest that compounds at the end of each investment period.

- 9.01 Annuities
- 9.02 Ordinary annuities
- 9.03 Superannuation
- 9.04 Perpetuities
- 9.05 Applications of annuities
- Chapter summary
- Chapter review



SYLLABUS SUBJECT MATTER

Annuities and perpetuities (compound interest investments with periodic payments made from the investment)

- use a recurrence relation $A_{n+1} = rA_n + d$ to model an annuity and investigate (numerically or graphically) the effect of the amount invested, the interest rate, and the payment amount on the duration of the annuity
- solve problems involving annuities, including perpetuities as a special case, e.g. determining the amount to be invested in an annuity to provide a regular monthly income of a certain amount



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

annuity
ordinary annuity
sinking fund

investment annuity
payment
superannuation

deposit
perpetuity
withdrawal

9.01 Annuities

A Christmas Club bank account is a savings account with regular **deposits**. The interest is calculated at regular intervals on the current amount of the investment. This means that the interest paid increases with each regular deposit. This type of investment is an example of an **annuity** because it involves regular payments.

An **annuity** is an investment that has regular deposits made over a period of time. An annuity that involves an initial investment amount with regular deposits and no withdrawal of funds is called an **investment annuity**.



Progress of
an annuity

EXAMPLE 1

An initial deposit of \$10 000 was made on an investment taken out over 5 years at a rate of 6% p.a. (interest calculated monthly), and an additional deposit of \$500 is made each month.

- a** Complete the table below for the first 5 deposits.

n	Value at start of month	Interest	Deposit (d)	Value at end of month
1	10 000			
2				
3				
4				
5				

- b** Calculate how much interest was earned over this time.
c What is the value of the annuity at the end of 5 months?

Solution

- a** State the value at the start of the first month. **Value at the start of the month = 10 000**

Calculate i , the interest rate per compounding period as a decimal. $i = \frac{0.06}{12} = 0.005$

Calculate the interest for the first month. **Interest = $10\,000 \times 0.005$**
= 50

State the value of the deposit. **$d = 500$**

Calculate the value at the end of the first month. **Value at the end of the month**
= $10\,000 + 50 + 500$
= 10 550

Repeat these calculations for each month to complete the table.

Month	Value at start of month	Interest	Deposit (d)	Value at end of month
1	10 000	50	500	10 550
2	10 550	52.75	500	11 102.75
3	11 102.75	55.51	500	11 658.26
4	11 658.26	58.29	500	12 216.55
5	12 216.55	61.08	500	12 777.63

- b** Find the total interest paid. **Interest**
= $50 + 52.75 + 55.51 + 58.29 + 61.08$
= 277.63

- c** Read the value from the table. **At the end of 5 months, the annuity is worth \$12 777.63.**

Reducing balance loans that you studied in Chapter 7 are also examples of annuities because they have regular payments. The difference is that for an investment annuity the principal is growing and it is your own money, while for a reducing balance loan, the principal is reducing and it is the bank's money.

You have previously used a recurrence relation to find the value of a reducing balance loan. The same can be done for an investment annuity.

Value of an investment annuity

$$A_{n+1} = rA_n + d$$

where: A_0 = the initial value of the investment

A_n = the value of the investment at the end of n compounding periods

$$r = 1 + i$$

i = the periodic interest rate as a decimal

d = the deposit per compounding period

EXAMPLE 2

Chelsea receives an inheritance of \$40 000. She wants to save a deposit for a home, so she decides to invest the money at 7.5% p.a. compounded monthly. In addition, Chelsea also makes a monthly deposit of \$800.

- a Write a recurrence relation for this investment annuity.
- b What is the value of the annuity at the end of 2 years?
- c How long will it be before Chelsea has a deposit of \$80 000?

Solution

- a Write the general form of the recurrence relation. $A_{n+1} = rA_n + d$

The initial amount invested is \$40 000. $A_0 = 40\,000$

The interest rate is 7.5% p.a. compounded monthly. Calculate i . $i = \frac{0.075}{12} = 0.006\,25$

State any known values. $A_0 = 40\,000, d = 800$

Use these values to write the recurrence relation. $A_{n+1} = 1.006\,25A_n + 800$, where $A_0 = 40\,000$



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You have already used a formula to calculate the balance of a reducing balance loan.

You can find the value of an investment annuity using a similar formula. The difference is that the value of the investment (A_n) *increases* with the *addition* of a regular deposit (d).

Value of an investment annuity (A_n)

$$A_n = A_0 r^n + d \left(\frac{r^n - 1}{r - 1} \right)$$

where d = the deposit amount per period

A_0 = the initial amount invested

A_n = the balance of the investment after n deposits

$r = 1 + i$

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

This formula can also be written in terms of the interest rate i if we substitute $r = 1 + i$:

$$A_n = A_0 (1 + i)^n + d \left[\frac{(1 + i)^n - 1}{i} \right]$$

For deposits at the *end* of the compounding periods with no initial investment, the value is:

$$A_n = d \left(\frac{r^n - 1}{r - 1} \right) = d \left[\frac{(1 + i)^n - 1}{i} \right]$$

EXAMPLE 3

Amelia currently has \$6000. She plans to go on an overseas trip in 5 years time so she invests her money in an investment annuity that is averaging an interest rate of 7.2% p.a. compounded annually. If she deposits \$3500 each year, find the amount available for her overseas trip.

Solution

State the formula for the value of an investment annuity.

$$A_n = A_0 r^n + d \left(\frac{r^n - 1}{r - 1} \right)$$

Calculate i .

$$i = 0.072$$

State known values.

$$A_0 = 6000, r = 1 + i = 1.072, n = 5, d = \$3500$$

Substitute.

$$\begin{aligned} A_5 &= 6000 \times 1.072^5 + 3500 \left(\frac{1.072^5 - 1}{1.072 - 1} \right) \\ &= 8494.252... + 20\,208.065... \\ &= 28\,702.318... \end{aligned}$$

State the result.

Amelia will have about \$28 702.31 for her overseas trip.

Exercise 9.01 Annuities

- 1** An initial deposit of \$8000 was made on an investment taken out over 6 years at a rate of 5.5% p.a. with interest calculated annually. An additional deposit of \$1200 is made each year.
- a** Copy and complete the table below for the first 5 deposits.

Year	Value at start of year	Interest	Deposit (d)	Value at end of year
1	8000		1200	
2				
3				
4				
5				

- b** Calculate how much interest had been earned after 5 years.
- c** What is the value of the annuity at the end of this period?

Example
1

- 2** Mason inherited \$15 000 and deposits it in an account that pays interest at the rate of 8% p.a. with interest calculated monthly. He decides to make regular monthly deposits of \$350 at the end of each month.

a Copy and complete the table below for the first 7 months.

Month	Value at start of month	Interest	Deposit (d)	Value at end of month
1	15 000		350	
2				
3				
4				
5				
6				
7				

b What is the value of the annuity at the end of this period?

- 3** \$25 000 is placed in an investment annuity paying 5.8% p.a. compounded quarterly and quarterly deposits of \$4500 are made. Use a table or spreadsheet to find the value of the annuity at the end of 2 years.
- 4** Matilda deposits \$5000 in an investment annuity paying 6.8% p.a. compounded annually and also deposits \$1000 at the end of each year. If A_n is the value of the investment after n years, a recurrence relation for A_n is:

A $A_{n+1} = 1.068(A_n + 1000)$, where $A_0 = 5000$

B $A_{n+1} = 1.068A_n + 1000$, where $A_0 = 5000$

C $A_{n+1} = 1.068A_n - 5000$, where $A_0 = 1000$

D $A_{n+1} = 1.068A_n + 1000$, where $A_0 = 5000$

E $A_{n+1} = 1.068(A_n - 1000)$, where $A_0 = 5000$

Example

2

- 5** Benjamin receives a tax refund of \$7500. He wants to save to buy a car, so he decides to invest the money at 6.9% p.a. compounded monthly. In addition, Benjamin also makes a monthly deposit of \$650.

a Write a recurrence relation for this investment annuity.

b Use a table or spreadsheet to calculate the value of the annuity at the end of 3 years.

c How long will it be before Benjamin will be able to use this investment to buy a car worth \$40 000?

- 6** Lucy makes regular quarterly deposits of \$1960 in an account with an interest rate of 6.5% p.a. She wants to use her investment to provide a deposit for a townhouse.
- Write a recurrence relation for this investment.
 - Use a table or spreadsheet to calculate the value of the investment at the end of 4 years.
 - How long will it be before Lucy has a deposit of \$64 000?

- 7** Oscar currently has \$5000. He plans to use this money to buy some replacement equipment for his landscaping business. He invests this money in an investment account that is averaging an interest rate of 8.6% p.a. compounded monthly. He also deposits \$450 each month into the account. Use the formula for the value of an investment annuity to work out how much he will have available to spend on equipment when it needs replacement in 4 years time.



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- 8** An account pays interest at 5.3% p.a. compounded weekly. \$120 is deposited into this account each week. How much will be in the account at the end of 3 years?
- 9** Anastacia deposits \$4000 into an account that pays interest at 4.9% p.a. compounded monthly. She also makes a deposit of \$250 at the end of each month. How much will Anastacia have in this account at the end of 5 years?

Problem solving

- 10** At the *start* of each year, Liam deposits \$2400 into an account earning 7% p.a. interest compounded annually. Interest is paid at the end of the year before the next deposit is made.
- Use a table or spreadsheet to calculate the balance of the account at the end of year 5.
 - Write a recurrence relation for this investment.
- 11** Georgia makes a deposit of \$1200 into an account paying 7.8% p.a. compounded monthly. She also deposits \$400 at the end of each month. What single sum could she have invested at the start to achieve the same financial result at the end of 3 years?



9.02 Ordinary annuities

An annuity is often used to provide a source of income. An initial amount is invested and this amount earns interest. Regular **withdrawals** or **payments** are then made for a fixed period of time until the value of the investment is \$0. This is called an **ordinary annuity**.

You can use a recurrence relation for an ordinary annuity.

Value of an ordinary annuity

$$A_{n+1} = rA_n - d$$

where: A_1 = the initial balance of the investment (the cost of the annuity)

A_n = the value of the investment at the start of the n th compounding period

$$r = 1 + i$$

i = the periodic interest rate as a decimal

d = the withdrawal (payment) per compounding period

You can see that the recurrence relation for an ordinary annuity is different from the recurrence relation for an investment annuity in two important ways. For an ordinary annuity, A_n is the value of the investment at the *start* of the n th compounding period, while for an investment annuity A_n is the value of the investment at the *end* of the n th compounding period. The value of d is *subtracted* in the ordinary annuity recurrence relation because it is a withdrawal, while d is *added* in the case of an investment annuity because it is a deposit.

EXAMPLE 4

Mackenzie receives \$120 000 as a settlement for an insurance claim. She invests the money at 5.8% p.a. compounded annually. She is paid \$10 300 each year after the annual interest is calculated. The amount remaining is then reinvested for the following year.

- a Write a recurrence relation for the value of the investment.
- b What is the value of the investment at the start of the 12th year?
- c How long will the annuity last?
- d How much will the last payment be?

Solution

- a** Write the general form of the recurrence relation. $A_{n+1} = rA_n - d$

The initial amount invested is \$120 000. $A_1 = 120\,000$

The interest rate is 5.8% p.a. compounded annually. $i = 0.058$

Calculate r . $r = 1 + i = 1.058$

State the withdrawal amount. $d = 10\,300$

Use these values to write the recurrence relation. $A_{n+1} = 1.058A_n - 10\,300$, where $A_1 = 120\,000$.

You will need to use a table or spreadsheet to answer the remaining parts of this example.

- b** Open a spreadsheet and enter the headings 'n' in Cell A1, 'Start value' in B1 and 'End value' in C1.

n = the number of years for the investment

Start value = the value of the investment at the start of the year

End value = the value of the investment at the end of the year.

Enter the first 12 years as follows

Enter 1 in cell A2 and 2 in cell A3. Block cells A2 and A3 and then copy this down to A13.

Start value

The initial value of the investment (A_1) is 120 000, so enter this amount in cell B2.

The value of the investment at the start of year 2 is the same as the value at the end of year 1, so enter '=C2' in cell B3.

End value

The value of the investment at the end of year 1 is $A_2 = rA_1 - d$, so enter '=1.058*B2 - 10 300' in cell C2. Now copy cells B3 and C3 down to row 13.

Read the value from the table and state the result.

	A	B	C
1	n	Start value	End value
2	1	120000.00	116660.00
3	2	116660.00	113126.28
4	3	113126.28	109387.60
5	4	109387.60	105432.09
6	5	105432.09	101247.15
7	6	101247.15	96819.48
8	7	96819.48	92135.01
9	8	92135.01	87178.84
10	9	87178.84	81935.21
11	10	81935.21	76387.46
12	11	76387.46	70517.93
13	12	70517.93	64307.97

The value of the investment at the start of the 12th year is \$70 517.93.

- c** Copy the values in each column down until the value in column C changes from positive to negative.

18	17	35651.32	27419.10
19	18	27419.10	18709.40
20	19	18709.40	9494.55
21	20	9494.55	-254.77
22	21	-254.77	-10569.54
23	22	-10569.54	-21482.58
24	23	-21482.58	-33028.57
25			

State the result.

The investment will be worth \$0 at the end of 20 years.

- d** State the value at the start of the 20th year.

$$A_{20} = 9494.55$$

The last payment will be the same as A_{20} with the interest for the final year.

$$\begin{aligned}\text{Last payment} &= A_{20} \times 1.058 \\ &= 9494.55 \times 1.058 \\ &= 10\,045.23\end{aligned}$$

State the result.

The last payment will be \$10 045.23.

The value of an ordinary annuity depends on the initial amount invested, the interest rate and the value of the regular withdrawal.

EXAMPLE 5

Flynn invests a lump sum of \$180 000 in an account paying 6.4% p.a. compounded annually. At the end of each year after interest has been added, he withdraws \$14 600.

- a** How long will the annuity last?
- b** If Flynn had invested the money at 4.3% p.a. compounded annually and made the same withdrawals, how long would the annuity last?
- c** If Flynn made regular annual withdrawals of \$13 000 instead, how long would the original annuity last?

Solution

- a** Write the general form of the recurrence relation.

$$A_{n+1} = rA_n - d$$

\$180 000 is invested initially.

$$A_1 = 180\,000$$

The interest rate is 6.4% p.a. compounded annually.

$$i = 0.064$$

(405

Change the formula you entered into cell C2 to be $=1.043*B2 - 14\ 600$ and then copy C2 down until the value changes from positive to negative.

	A	B	C
15	14	63761.93	51903.69
16	15	51903.69	39535.55
17	16	39535.55	26635.58
18	17	26635.58	13180.91
19	18	13180.91	-852.31
20	19	-852.31	-15488.96
21	20	-15488.96	-30754.98
22	21	-30754.98	-46677.45

Write the answer.

The annuity will last for 18 years.

- The withdrawal amount is now \$13 000 and the original interest rate is applied.

$$A_1 = 180\ 000, i = 1.064, d = 13\ 000$$

Write the new recurrence relation.

$$A_{n+1} = 1.064A_n - 13\ 000, \text{ where } A_1 = 180\ 000$$

Change the formula you entered into cell C2 to be $=1.064*B2 - 13\ 000$ and then copy C2 down until the value changes from positive to negative.

	A	B	C
33	32	44901.06	34774.72
34	33	34774.72	24000.30
35	34	24000.30	12536.32
36	35	12536.32	338.65
37	36	338.65	-12639.68
38	37	-12639.68	-26448.62
39	38	-26448.62	-41141.33
40	39	-41141.33	-56774.37

State the result.

The annuity will run out after 36 years.

From the previous example, you can see that:

- higher interest rates result in a longer duration for an annuity
- higher regular withdrawals result in a shorter duration for an annuity

There is a formula for the withdrawal amount for an ordinary annuity if the initial deposit, interest rate and duration are known.

Withdrawal (payment) value for an ordinary annuity

$$d = \frac{iA}{1 - (1+i)^{-n}} = A \left[\frac{i(1+i)^n}{(1+i)^n - 1} \right]$$

where: d = the withdrawal (payment) amount per period

A = the initial value (cost) of the annuity

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

EXAMPLE 6

Ellie invests \$220 000 in an annuity that pays 6.9% p.a. interest compounded monthly. After interest is added each month, she receives a payment. How much will each payment be if the annuity lasts for 20 years?

Solution

Calculate i .
$$i = \frac{0.069}{12} = 0.005\ 75$$

Calculate n .
$$n = 12 \times 20 = 240$$

The initial investment is \$220 000.
$$A = 220\ 000$$

State the formula for the payment of an ordinary annuity.
$$d = \frac{iA}{1 - (1 + i)^{-n}}$$

Substitute known values into the formula.
$$= \frac{0.00575 \times 220\ 000}{1 - 1.00575^{-240}}$$

Evaluate.
$$= 1692.4771\dots$$

State the result.
$$\text{Ellie will receive monthly payments of } \$1692.47 \text{ for 20 years.}$$

Exercise 9.02 Ordinary annuities

- 1** \$12 000 is invested at the start of the year at 6.5% p.a. compounded annually. At the end of the year, interest is added and then a withdrawal of \$1400 is made. This process is repeated until the investment reaches \$0. If the value of the investment at the start of the n th year is A_n , which of the following, where $A_1 = 12\ 000$, represents a recurrence relation for the value of the investment?

A $A_n = 1.065A_{n+1} - 1400$

B $A_{n+1} = 1.065(A_n - 1400)$

C $A_{n-1} = 1.065A_n + 1400$

D $A_{n+1} = 1.065A_n - 1400$

E $A_{n+1} = 1.065A_n - 1400$

- 2** At the start of the year, Alexis places \$27 000 in an account paying 9% p.a. compounded monthly. At the end of the first month, interest is added and then a withdrawal of \$340 is made. This process is repeated until the investment reaches \$0. If the value of the investment at the start of month n is A_n , which of the following, where $A_1 = 27\ 000$, represents a recurrence relation for the value of the investment?

A $A_{n+1} = 1.09A_n - 340$

B $A_{n+1} = 1.0075A_n - 340$

C $A_{n-1} = 1.0075A_n + 340$

D $A_n = 1.0075A_{n+1} - 340$

E $A_{n+1} = 1.09A_n + 340$

Example
4

- 3** Penelope invests \$90 000 in an account that pays 5.7% p.a. compounded annually. After the annual interest is calculated, she is paid \$9000 and the amount remaining is then reinvested for the following year. This process continues until the investment is all paid out.
 - a** Write a recurrence relation for the value of the investment.
 - b** What is the value of the investment at the start of the 9th year?
 - c** How long will the annuity last?
 - d** How much will the last payment be?
- 4** Sebastian invests \$150 000 at an interest rate of 7.2% p.a. compounded monthly. Each month, after interest is calculated, he withdraws \$1800 and the amount remaining is then reinvested. The value of Sebastian's investment at the start of the n th month is given by the recurrence relation $A_{n+1} = rA_n - d$, where $A_1 = a$.
 - a** Find the values of a , r and d .
 - b** How much is the investment worth at the start of the 14th month?
 - c** For how many months will Sebastian be able to make monthly withdrawals?
 - d** What is the value of the last withdrawal that Sebastian will make?
- 5** An annuity pays regular monthly amounts of \$970. The amount invested is \$110 000 and the interest earned is 6.8% p.a. compounded monthly. How long will the annuity last and what is the value of the final withdrawal?

Example
5

- 6** Tyler invests \$140 000 in an account paying 8.2% p.a. compounded annually. At the end of each year after interest has been added, he withdraws \$14 000.
 - a** How long will the annuity last?
 - b** For how long would Tyler be able to make these withdrawals if the money was invested at 5.5% p.a. compounded annually?
 - c** How long would the original annuity last if Tyler made regular annual withdrawals of \$12 500 instead?
- 7** Annabelle wins a contract worth \$220 000. She decides to invest the money in an account paying 4.7% p.a. compounded quarterly. At the end of each quarter, after interest has been added, she is paid \$4500 from the annuity.
 - a** How long will the annuity last?
 - b** How long would Annabelle be able to make these withdrawals if the money was invested at 6.7% p.a. compounded quarterly instead?
 - c** How long would the original annuity last if Annabelle made regular quarterly withdrawals of \$8000 instead?

Example
6

- 8** Hugo invests \$109 000 in an annuity that pays 5.5% p.a. interest compounded annually. After interest is added each year, he receives a payment. Use the formula for the payment of an ordinary annuity to calculate how much each payment will be if the annuity lasts for 10 years.

- 9** Suppose you inherited \$100 000 and invested it at 6.2% p.a. compounded annually. How much should you withdraw at the end of each of the next 10 years to end up with zero?
- 10** Joshua invests \$80 000 at 7.3% p.a. compounded monthly.
- a** What is the greatest amount that he can withdraw at the end of each month for the investment to last 12 years?
 - b** If he wants the investment to last 15 years, what is the greatest monthly amount he can withdraw?
 - c** If he needs the investment to last only 7 years, what is the greatest monthly amount he can withdraw?

Problem solving

- 11** Priya won the lottery. He will receive \$65 000 per month for the next 20 years.
- a** How much will Priya receive in total?
 - b** What single amount would need to be deposited now in an annuity earning 7.4% p.a. compounded monthly by the lottery owner to fund Priya's payments?
- 12** Assume that you just inherited an annuity that will pay you \$10 000 at the end of each year for 10 years. An acquaintance offers to give you \$60 000 right now for the annuity. If you think a 'fair' rate of return for this investment would be 6% p.a., should you accept the offer?
- 13** Yasmin is involved in a car accident that was someone else's fault. She wins a settlement of \$25 000 each year for the next 25 years, for a total of \$625 000. The insurance company offers Yasmin a one-payment settlement of \$300 000 now instead of the annual payments. If the money can be invested at 7% p.a. compounded annually, should she accept the lump sum offer?



Superannuation

9.03 Superannuation

When people retire from the workforce, they still need an income to pay for their living expenses. In Australia, retirement income is funded through a mix of personal savings, a government pension and superannuation (super). All Australian workers have a personal **superannuation** fund into which their employers contribute money. Personal superannuation is money that is put aside and saved while you're working, so you can have a regular income later in life when you retire.

In practice, the amount of money accumulated depends on the size of any initial deposit, the value of the regular payments into the super fund, the interest earned, inflation and taxation that is paid. To simplify calculations in this section of the book, it is assumed that the interest rate remains the same and the effects of inflation and taxation will not be considered.

Consider a simple superannuation account where regular monthly deposits are made. This situation is really an investment annuity.

Balance of a simple superannuation account

$$A_{n+1} = rA_n + d$$

where: A_1 = the initial balance of the superannuation account

A_n = the value of the investment at the start of n th compounding period

$$r = 1 + i$$

i = the periodic interest rate as a decimal

d = the deposit per compounding period



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EXAMPLE 7

Ashton starts work in a new job. His employer sets up a superannuation account for him by depositing \$450 at the start of each month. The account earns 8.1% p.a. compounded monthly. What will be the balance of Ashton's account after 20 years?

Solution

Write a recurrence relation for the superannuation account.

$A_{n+1} = rA_n - d$, where A_1 is the initial balance.

Calculate i .

$$i = \frac{0.081}{12} = 0.00675$$

Calculate r .

$$r = 1 + i = 1.00675$$

Monthly deposits are \$450.

$$d = 450$$

The initial balance is the value of the first deposit.

$$A_1 = 450$$

Write the recurrence relation for this investment.

$$A_{n+1} = 1.00675A_n + 450, \text{ where } A_1 = 450$$

Open a spreadsheet and enter the headings 'n' (number of months) in Cell A1, 'Start value' in B1 and 'End value' in C1.

C2 : fx = 1.00675*B2+450			
	A	B	C
1	n	Start value	End value
2	1	450.00	903.04
3	2	903.04	
4			
5			
6			

Enter 1 in cell A2 and 2 in cell A3.

The initial value of the investment (A_1) is 450, so enter this amount in cell B2.

The value of the investment at the start of month 2 is the same as the value at the end of month 1, so enter '=C2' in cell B3.

C241 : fx = 1.00675*B241+450			
	A	B	C
236	235	257295.08	259481.83
237	236	259481.83	261683.33
238	237	261683.33	263899.69
239	238	263899.69	266131.01
240	239	266131.01	268377.40
241	240	268377.40	270638.94
242			

The value of the investment at the end of month 1 is $rA_n + d = rA_1 + d$, so enter '=1.00675*B2 + 450' in cell C2.

Copy the columns down until the value in column A reaches 240.

State the result.

After 20 years, Ashton's superannuation account will have a balance of \$270 638.94.

EXAMPLE 8

Angelique currently has a superannuation account with a balance of \$36 000. She changes jobs and decides to transfer her superannuation balance to her new employer's fund. This fund is paying 9.3% p.a. compounded monthly. Each month, her employer contributes \$525 to the account on her behalf.

- a** What is the balance of Angelique's account after 2 years?
- b** How long will it take for Angelique's account to reach a balance of \$300 000?

Solution

a Calculate i .
$$i = \frac{0.093}{12} = 0.007\ 75$$

Calculate r .
$$r = 1 + i = 1.007\ 75$$

Monthly deposits are \$450.
$$d = 525$$

The initial balance is the value of the first deposit.
$$A_1 = 36\ 000$$

Write the recurrence relation for this investment.
$$A_{n+1} = 1.007\ 75A_n + 525, \text{ where } A_1 = 36\ 000.$$

Open a spreadsheet and enter the headings 'n' (number of months) in Cell A1, 'Start value' in B1 and 'End value' in C1.

Enter 1 in cell A2 and 2 in cell A3.

The initial value of the investment (A_1) is 36 000, so enter this amount in cell B2. The value of the investment at the start of month 2 is the same as the value at the end of month 1, so enter '=C2' in cell B3.

The value of the investment at the end of month 1 is $A_2 = rA_1 + d$, so enter '=1.00775*B2 + 525' in cell C2.

	A	B	C
1	n	Start value	End value
2	1	36000.00	36804.00
3	2	36804.00	
4			
5			
6			
7			

Now copy the columns down until the value in column A reaches 24.

	A	B	C
19	18	50549.16	51465.92
20	19	51465.92	52389.78
21	20	52389.78	53320.80
22	21	53320.80	54259.04
23	22	54259.04	55204.54
24	23	55204.54	56157.38
25	24	56157.38	57117.60
26			

State the result.

The balance of Angelique's account after 2 years is \$57 117.60.

- b** Now copy the columns down until the value in column C reaches 300 000.

	A	B	C
161	160	286296.68	289040.48
162	161	289040.48	291805.54
163	162	291805.54	294592.03
164	163	294592.03	297400.12
165	164	297400.12	300229.97
166	165	300229.97	303081.76
167	166	303081.76	305955.64
168	167	305955.64	308851.80

State the result.

The balance of Angelique's account reaches \$300 000 after 164 months or 13 years and 8 months.

You can use the value of an annuity formula given on page 398:

$$A_n = A_0 r^n + d \left(\frac{r^n - 1}{r - 1} \right) = A_0 (1 + i)^n + d \left[\frac{(1 + i)^n - 1}{i} \right]$$

where: d = the deposit amount per period

A_0 = the initial amount invested in the superannuation account

A_n = the balance of the superannuation account after n deposits

$$r = 1 + i$$

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

EXAMPLE 9

Imogen is aged 41 and is planning to retire at 65 years of age. She currently has a superannuation fund that has a balance of \$40 000 and is earning 6.9% p.a. compounded monthly.

- a If her employer makes regular monthly contributions of \$320, how much will be in her account after 10 years?
- b She estimates that she will need a lump sum of \$520 000 to provide for her retirement. Find the regular monthly contributions she will need to make to achieve her lump sum target.

Solution

- a The initial value of the annuity is \$40 000. $A_0 = 40\,000$

Calculate i . $i = \frac{0.069}{12} = 0.005\,75$

Calculate n . $n = 10 \times 12 = 120$

Calculate r . $r = 1 + i = 1.005\,75$

The monthly deposit is \$320. $d = 320$

State the annuities formula. $A_n = A_0 r^n + d \left(\frac{r^n - 1}{r - 1} \right)$

Substitute known values. $A_n = 40\,000 \times 1.005\,75^{120} + \frac{320(1.005\,75^{120} - 1)}{1.005\,75 - 1}$

Evaluate. $\approx 79\,591.17 + 55\,083.38$
 $= 134\,674.55$

State the result. After 10 years, Imogen's account will have a balance of \$134 674.55.
- b The final value of the annuity is \$520 000. $A_n = 520\,000$

Calculate the number of years before she retires. Years before retirement = $65 - 41 = 24$

Calculate n . $n = 24 \times 12 = 288$

State other known values. $r = 1.005\,75$ and $A_0 = 40\,000$

State the annuities formula.	$A_n = A_0 r^n + d \left(\frac{r^n - 1}{r - 1} \right)$
Substitute known values.	$520\,000 = 40\,000 \times 1.005\,75^{288} + d \left(\frac{1.005\,75^{288} - 1}{1.005\,75 - 1} \right)$
Evaluate.	$= 208\,541.20 + d \times \left(\frac{4.213\,53...}{0.005\,75} \right)$
	$311\,458.79 = d \times 732.787...$
	$d = 425.032...$
State the result.	The monthly contribution to achieve a retirement lump sum of \$520 000 is \$425.03. Her employer contributes \$320, so she must contribute \$425.03 – \$320 = \$105.03.

Exercise 9.03 Superannuation

- 1 \$400 is invested at the start of each month in an account that earns 6.6% p.a. compounded monthly. At the end of each month, interest is added. This process is repeated each month. If the value of the investment at the start of the n th month is A_n , which of the following represents a recurrence relation for the value of the investment?
 - A $A_n = 1.0055(A_{n+1} - 400)$, where $A_1 = 400$
 - B $A_n = 1.066A_{n+1} + 400$, where $A_1 = 400$
 - C $A_n = 1.0055A_{n+1} + 400$, where $A_1 = 400$
 - D $A_{n+1} = 1.055A_n + 400$, where $A_1 = 400$
 - E $A_{n+1} = 1.055A_n - 400$, where $A_1 = 400$
- 2 Asam's employer establishes a superannuation account for him by depositing \$380 at the start of each month. The account earns 8.2% p.a. compounded monthly.
 - a If A_n is the value of the account at the start of the n th month, write a recurrence relation for the superannuation account.
 - b What will be the balance of Asam's account after 2 years?
 - c What will be the balance of the account after 10 years?
 - d What will be the balance of the account after 25 years?
- 3 Sofia is self-employed and she establishes a superannuation account by making regular monthly deposits of \$560 at the start of each month. The account earns 7.8% p.a. compounded monthly.
 - a Given that A_n is the value of the account at the start of month n , write a recurrence relation for the superannuation account.
 - b What will be the balance of Sofia's account after 6 years?
 - c What will be the balance of the account after 30 years?

Example
7

- 4** A superannuation account is established by making regular monthly deposits of \$310 at the start of each month. If the account earns 9.2% p.a. compounded monthly, what will be the balance of the account after 15 years?
- 5** Jordan deposits \$25 000 in an account paying 5.4% p.a. compounded monthly. At the end of each month, interest is added and then a deposit of \$440 is made. If the value of the investment at the start of month n is A_n , which of the following represents a recurrence relation for the value of the investment?
- A** $A_n = 1.0045A_{n+1} + 440$, where $A_1 = 25\ 000$
B $A_{n+1} = 1.0045(A_n + 440)$, where $A_1 = 25\ 000$
C $A_n = 1.0045A_{n+1} - 440$, where $A_1 = 25\ 000$
D $A_{n+1} = 1.054A_n + 440$, where $A_1 = 25\ 000$
E $A_{n+1} = 1.0045A_n + 440$, where $A_1 = 25\ 000$

Example
8

- 6** Poppy decides to move her superannuation account to a new fund. Her current balance is \$67 000 and the new fund is paying 10.2% p.a. compounded monthly. Each month, her employer contributes \$480 to the account on her behalf.
- a** If A_n is the value of the account at the start of month n , write a recurrence relation for the new superannuation account.
b What is the balance of Poppy's account after 5 years?
c How long will it take for Poppy's account to reach a balance of \$400 000?
- 7** Chase currently has a balance of \$86 000 in his superannuation account. The fund pays 5.9% p.a. compounded monthly. Chase has just been given a pay rise and now each month his employer contributes \$540 to the account on his behalf.
- a** What is the balance of Chase's account after 7 years?
b How long will it take for Chase's account to reach a balance of \$500 000?
- 8** Indiana has \$50 000 in a superannuation account earning an interest rate of 9% p.a. with interest compounded monthly. Her employer makes monthly contributions of \$250. Indiana is aged 42 and wishes to retire at 60 years of age. Which of the following equations should she use to find the final value of her superannuation account?

- A** $A_{18} = 50\ 000 \times 1.09^{18} + 250 \left(\frac{1.09^{18} - 1}{1.09 - 1} \right)$
B $A_{216} = 50\ 000 \times 1.0075^{216} + 250 \left(\frac{1.0075^{216} - 1}{1.0075 - 1} \right)$
C $A_{216} = 50\ 000 \times 1.0075^{216} - 250 \left(\frac{1.0075^{216} + 1}{1.0075 + 1} \right)$
D $A_{216} = 50\ 000 \times 1.09^{216} + 250 \left(\frac{1.09^{216} - 1}{1.09 - 1} \right)$
E $A_{18} = 50\ 000 \times 1.0075^{18} - 250 \left(\frac{1.09^{18} + 1}{1.09 + 1} \right)$

- 9** Arani currently has a balance of \$22 000 in her superannuation account. Each month, her employer contributes \$230 to the account and the fund pays 7.7% p.a. compounded monthly. How long will it take for Arani's account to reach a balance of \$250 000?
- 10** Luca is aged 37 and is planning to retire at 60 years of age. The current balance of his superannuation account is \$38 000 and the fund earns 8.3% p.a. compounded monthly.
- a** How many more years will Luca need to work before he can retire?
 - b** If Luca's employer makes regular monthly contributions of \$320, how much will be in his account after 10 years?
 - c** He estimates that he will need a lump sum of \$550 000 to provide for his retirement. Find the regular monthly contributions he will need to make to reach his target.

Problem solving

- 11** Paige is 53 and is planning to retire at 65 years of age. She estimates that she needs \$640 000 to fund her retirement. Her current superannuation fund has a balance of \$143 000 and is earning 8.5% p.a. compounded monthly. Find the monthly contributions needed to meet Paige's retirement lump sum target.
- 12** Jake is 52 and is planning to retire at 65 years of age. His current superannuation fund has a balance of \$85 000 and is earning 6.5% p.a. compounded monthly. He recently reviewed his superannuation arrangements and estimates that he needs \$450 000 to retire at age 65. Find the monthly contributions needed to meet his retirement target.
- 13** Alice is 46 and is planning to retire at 65 years of age. She estimates that she needs \$480 000 to fund her retirement. Her current superannuation fund has a balance of \$65 000 and is earning 7% p.a. compounded monthly.
- a** Find the monthly contributions needed to reach her retirement target.
 - b** In the final ten years before retirement, Alice doubles her monthly contribution (calculated from part **a**). How much (to the nearest \$100) will she now have for her retirement?
 - c** If she maintains the contributions in part **b** for the final 10 years and the interest rate also increases to 11% p.a., how much extra (to the nearest \$100) will be available for Alice's retirement?
- 14** A superannuation account earns 7.9% p.a. compounded monthly. The monthly contributions are \$540 and the account balance grew from \$36 000 to \$280 000. How long did it take for this fund to achieve this?



9.04 Perpetuities

Consider an ordinary annuity with an initial value of \$30 000 that earns 8% p.a. compounded annually and has regular annual withdrawals of \$3000. The annuity will be used up in about 21 years. However, if the withdrawals are reduced to \$2400 per annum, the annuity will continue forever.

This is an example of a special annuity called a **perpetuity** or **perpetual annuity**.

A **perpetuity** is an annuity that provides regular payments that continue indefinitely.

Many wealthy people use perpetuities to provide the funds for scholarships or grants offered to students.

You can see that the amount invested does not change and is the same for an indefinite period.

Perpetuities

$$d = Ai$$

where: d = the regular withdrawal (payment) per period

A = the amount invested

i = interest rate per period as a decimal

The number of payments each year must be the same as the compounding period of the interest rate.

EXAMPLE 10

A wealthy entrepreneur agrees to provide \$800 000 to establish a perpetuity to provide an annual grant for cancer research. If the money is invested in government bonds that pay a return of 6.5% p.a. compounded annually, what is the value of the grant that could be awarded?



Getty Images/DragonImage

Solution

State the perpetuities formula.

$$d = Ai$$

Identify the known values.

$$A = 800\,000$$

$$i = 0.065$$

Substitute.

$$d = Ai$$

$$= 800\,000 \times 0.065$$

Evaluate.

$$= 52\,000$$

State the result.

A grant of \$52 000 can be provided annually.

You need to rearrange the perpetuities formula to calculate the amount invested or the interest rate required to provide a particular withdrawal.

EXAMPLE 11

The owner of a large mining company establishes a perpetuity to provide a university scholarship for a student to study geology. The owner is able to secure an investment that has a guaranteed interest rate of 7.5% p.a. compounded annually. If the annual scholarship is \$12 000, how much needs to be invested?

Solution

Make A the subject of the perpetuities formula.

$$d = Ai$$

$$A = \frac{d}{i}$$

Identify known values.

$$d = 12\,000$$

$$i = 0.075$$

Substitute.

$$A = \frac{12\,000}{0.075}$$

$$= 160\,000$$

Write the result.

A perpetuity of \$160 000 would be required.

EXAMPLE 12

A successful basketballer uses \$150 000 to establish a perpetuity for the junior club where she played as a child. The money is invested in a managed fund that returns 3.5% p.a. compounded quarterly.

- a Calculate the amount of the grant.
- b What interest rate (compounded quarterly) would be required for the perpetuity to provide a quarterly grant of \$2400?

Solution

- a State the perpetuities formula. $d = Ai$

Identify the initial investment. $A = 150\,000$

Calculate i . $i = \frac{0.035}{4} = 0.008\,75$

Substitute. $d = Ai$
 $= 150\,000 \times 0.008\,75$

Evaluate. $= 1312.5$

State the result. A grant of \$1312.5 can be provided quarterly.

- b Make i the subject of the perpetuities formula. $d = Ai$
 $i = \frac{d}{A}$

Identify known values. $d = 2400$
 $A = 150\,000$

Substitute. $i = \frac{2400}{150\,000}$
 $= 0.016$

Write as a percentage and include the period. $= 1.6\%$ per quarter

Write as a per annum value. $i = 1.6\% \times 4$ per annum
 $= 6.4\%$ p.a.

Write the result. A grant of \$2400 per quarter would require an interest rate of 6.4% p.a. compounded quarterly.

Exercise 9.04 Perpetuities

Example
10

- 1 A sum of \$180 000 is used to establish a perpetuity that funds an annual study grant. If the money is invested with a long-term return of 5.5% p.a. compounded annually, the value of the grant is:
A \$99 000 **B** \$170 100 **C** \$19 800 **D** \$5500 **E** \$9900
- 2 A famous actor invests \$200 000 to establish a drama scholarship to provide a monthly allowance for a student to study acting. If the money is invested in a bond that returns 6.9% p.a. compounded monthly, then the monthly allowance is:
A \$13 800 **B** \$1150 **C** \$3450 **D** \$11 500 **E** \$6900
- 3 A telecommunications company puts \$300 000 in an investment that pays a return of 7.2% p.a. compounded annually to set up a perpetuity. The perpetuity provides an annual grant for the study of artificial intelligence. What is the value of the grant?
- 4 Aaliyah puts her retirement savings of \$540 000 into a perpetuity that pays 6.2% p.a. compounded monthly. How much does she receive each month?

Example
11

- 5 A perpetuity provides an annual scholarship of \$27 000. If the money is invested in a bond that returns 6% p.a. compounded annually, then the amount invested is:
A \$162 000 **B** \$900 000 **C** \$324 000
D \$450 000 **E** \$600 000
- 6 The grandparents of Holly and Luke want to establish a perpetuity to provide university scholarships for their grandchildren. They would like the perpetuity to provide an annual amount of \$6000 for each grandchild. They are able to invest their money in bonds that have a long-term interest rate of 8% p.a. compounded annually. How much will the grandparents need to invest?
- 7 A perpetuity is established to provide a scholarship that consists of a monthly payment of \$1240. The money is invested in a managed fund that pays a guaranteed return of 6.5% p.a. compounded monthly. How much (to the nearest dollar) needs to be invested to provide the scholarship?

Example
12

- 8 Hamish is a famous blogger who uses \$220 000 to establish a perpetuity. The perpetuity is used to fund a monthly grant to study ways to reduce cyber bullying. The money is invested in utility bonds that return 7.5% p.a. compounded monthly.
a Calculate the amount of the monthly grant.
b What interest rate (compounded monthly) would be required for the perpetuity to provide a grant of \$2250 a month?
- 9 A benefactor has \$150 000 to set up a perpetuity as a grant to help people with diabetes. The benefactor invests in government bonds that return 6.9% p.a. compounded annually.
a Find the amount of the annual grant.
b What interest rate (compounded annually) would be required if the perpetuity is to provide \$12 500 each year?

- 10** The benefactor mentioned in question 9 decides to look at alternative investment options for the perpetuity.
- a** How much extra would the annual grant amount to if the original interest rate was compounded monthly?
 - b** What interest rate (compounded quarterly) would be required to provide quarterly payments of \$3200?

Problem solving

- 11** A philanthropist wants to establish a fund that pays out a scholarship of \$10 000 every year in perpetuity. The first scholarship will be paid out in 5 years time. Assuming an interest rate of 8% p.a. compounded annually, how much will the philanthropist need to pay into the fund?

9.05 Applications of annuities

Some people like to replace their car every 5 years or so. A fund set up to receive periodic payments to pay for an item in the future is called a **sinking fund**.

The periodic payments, together with the interest earned by the payments, are designed to produce a given sum in the future, called the **future value**.

Future value and payments for a sinking fund

If you make a payment of p at the *end* of each period, then the future value after n periods is:

$$A_n = p \left[\frac{(1+i)^n - 1}{i} \right]$$

If you want to accumulate a future value of V_n after n periods, by making payments of p at the *end* of each period, then each payment is:

$$p = A_n \left[\frac{i}{(1+i)^n - 1} \right]$$

where: p = the payment amount per period

A_n = the future value after n payments

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

EXAMPLE 13

- a** A sinking fund is established to buy a piece of equipment that needs to be replaced in a year. A deposit of \$900 per month is placed into an account that pays 9.6% interest, compounded monthly. How much money will be in the sinking fund after 1 year?
- b** Jayden's parents set up a sinking fund to pay for his education. They would like to have \$80 000 after 15 years. The account they choose pays 7.8% p.a. compounded quarterly. How much must each quarterly deposit be in order to reach their goal?

Solution

- a** Write the formula for the future value of a sinking fund.
- $$A_n = p \left[\frac{(1+i)^n - 1}{i} \right]$$
- Calculate i .
- $$i = \frac{0.096}{12} = 0.008$$
- Calculate the number of periods.
- $$n = 1 \times 12 = 12$$
- The payment is \$900.
- $$p = 900$$
- Substitute known values.
- $$A_n = 900 \times \frac{1.008^{12} - 1}{0.008}$$
- Evaluate.
- $$= 11\,288.103\dots$$
- State the result.
- The sinking fund will be worth \$11 288.10 after 1 year.
- b** Write the formula for the payment value of a sinking fund.
- $$p = A_n \left[\frac{i}{(1+i)^n - 1} \right]$$
- Calculate i .
- $$i = \frac{0.078}{4} = 0.0195$$
- Calculate the number of periods.
- $$n = 15 \times 4 = 60$$
- Substitute known values.
- $$p = 80\,000 \times \frac{0.0195}{1.0195^{60} - 1}$$
- Evaluate.
- $$= 713.6608\dots$$
- State the result.
- The quarterly deposit required is \$713.66.

The previous example could have been solved using a spreadsheet in a similar way to that used in previous examples.

Sometimes you need to determine the type of annuity based on the circumstances of the situation.

In these cases, you work out if withdrawals are made or not, whether deposits (payments) are made at the start or the end of a period, and so on.

EXAMPLE 14

Hayley has just received an inheritance of \$300 000. She invests the money in an account that pays 6.4% p.a. compounded monthly, and would like to be able to make monthly withdrawals of \$5000 for a number of years.

- a** What type of annuity does this represent?
- b** How much will be in her account after 5 years?
- c** For how long will she be able to make monthly withdrawals?
- d** What is the value of the last withdrawal?

Solution

- a** This case involves an initial deposit and regular withdrawals.

This is an example of an ordinary annuity.

- b** Write the general form of the recurrence relation.

$$A_{n+1} = rA_n - d$$

\$300 000 is invested initially.

$$A_1 = 300\,000$$

Calculate i .

$$i = \frac{0.064}{12} = 0.007$$

Calculate r .

$$r = 1 + i = 1.007$$

State the withdrawal amount.

$$d = 5000$$

Use these values to write the recurrence relation.

$$A_{n+1} = 1.007A_n - 5000, \text{ where } A_1 = 300\,000.$$

Open a spreadsheet and enter the headings 'n' (number of months) in Cell A1, 'Start value' in B1 and 'End value' in C1.

Enter 1 in cell A2 and 2 in cell A3.

The initial value of the investment (A_1) is 300 000, so enter this amount in cell B2.

The value of the investment at the start of year 2 is the same as the value at the end of year 1, so enter '=C2' in cell B3.

The value of the investment at the end of year 1 is $A_2 = rA_1 - d$, so enter '=1.007*B2 - 3000' in cell C2.

Now copy each column down until the value in column A reaches 60.

State the result.

- c** Copy the cells down until the value in column C changes from positive to negative.

Read the value.

Convert to years.

State the result.

- d** State the value at the start of the 173rd month.

The last payment will be the same as A_{173} with the interest for the final year.

State the result.

C2 : fx =1.007*B2-3000			
	A	B	C
1	n	Start value	End value
2	1	300000.00	299100.00
3	2	299100.00	
4			
5			
6			

C61 : fx =1.007*B61-3000			
	A	B	C
57	56	239874.27	238553.39
58	57	238553.39	237223.27
59	58	237223.27	235883.83
60	59	235883.83	234535.02
61	60	234535.02	233176.76
62	61	233176.76	231809.00
63	62	231809.00	230431.66
64	63	230431.66	229044.69

The balance of the account at the end of 5 years is \$233 176.76.

C174 : fx =1.007*B174-3000			
	A	B	C
171	170	10620.79	7695.14
172	171	7695.14	4749.00
173	172	4749.00	1782.24
174	173	1782.24	-1205.28
175	174	-1205.28	-4213.72
176	175	-4213.72	-7243.21
177	176	-7243.21	-10293.92
178	177	-10293.92	-13365.97

$$n = 173$$

$$\frac{173}{12} = 14.4166\dots$$

$$= 14 \text{ years } 5 \text{ months}$$

The annuity will run out after 14 years and 5 months.

$$A_{173} = 1782.24$$

$$\begin{aligned} \text{Last payment} &= A_{173} \times 1.007 \\ &= 1782.24 \times 1.007 \\ &= 1794.61 \end{aligned}$$

The last payment will be \$1794.61.

EXAMPLE 15

Zac currently has \$5000. He wants to buy a car in 3 years time, so he invests this money in an account that is averaging an interest rate of 8.1% p.a. compounded monthly. Zac also deposits \$900 each month into the account.

- a What type of annuity does this represent?
- b How much will Zac have to spend on a car?

Solution

- a This case involves an initial deposit and regular deposits are made. There are no withdrawals.

This is an example of an investment annuity.

- b State the value of an investment annuity formula.

$$V_n = Ar^n + d \left(\frac{r^n - 1}{r - 1} \right)$$

Calculate i .

$$i = \frac{0.081}{12} = 0.006\ 75$$

Calculate the number of periods.

$$n = 3 \times 12 = 36$$

State the known values.

$$A = 5000$$

$$r = 1 + i = 1.006\ 75$$

Substitute.

$$V_{36} = 5000 \times 1.006\ 75^{36} + 900 \left(\frac{1.006\ 75^{36} - 1}{1.006\ 75 - 1} \right)$$

Evaluate.

$$= 6370.140... + 36\ 537.068...$$

$$= 42\ 907.208...$$

State the result.

Zac will have about \$42 907.20 to buy a car.

You could also solve Example 15 using a spreadsheet.

Exercise 9.05 Applications of annuities

- 1** Harriet has decided to deposit \$600 each month for 20 years in an account that pays interest of 7.2% p.a. compounded monthly.

Example
13

 - a** What type of annuity is this?
 - b** How much will be in the account at the end of 20 years?
 - c** Harriet believes she needs to accumulate \$450 000 in the 20-year period to have enough for retirement. If she can maintain the initial interest rate for the entire period, what monthly contribution should she make to achieve her goal?
- 2** Milan wants to set up an account to supplement his current income. He wants to deposit a sum of money in an account that pays interest at the rate of 8.5% p.a. compounded monthly so that he can draw payments of \$1250 a month for the next 18 years.

Example
14

 - a** What type of annuity is this arrangement?
 - b** How much should he deposit now?
- 3** Aaliyah wants to take an extended holiday in 5 years time. She estimates she will need \$45 000 for the trip. She currently has savings of \$4000 and she puts this in an account that pays 7.6% p.a. interest, compounded monthly. She intends to make regular monthly deposits to the account for the next 5 years.

Example
15

 - a** What type of annuity is represented by this situation?
 - b** If Aaliyah deposits \$400 a month, will she reach her savings goal?
 - c** Calculate the value of the regular deposit required for Aaliyah to reach her target.

Problem solving

- 4** Bella wants to build a nest egg. She deposits \$1200 in an account that returns 8.8% p.a. compounded monthly at the start of the month and continues to make monthly investments of \$1200 at the end of each month after interest has been added. How long should she continue to make monthly payments if her aim is to accumulate \$200 000?
- 5** A services club wants to set up an account to provide an annual grant for local improvement programs. The club has raised \$180 000 and the members of the club want to ensure that the amount invested remains the same. The club is able to invest in energy bonds that return 6.4% p.a. compounded annually.
 - a** What is the value of the annual grant?
 - b** What interest rate (compounded annually) would be required if the investment is to provide \$13 500 each year?
- 6** An initial deposit of \$12 000 was made in an investment account over 5 years at a rate of 9.25% p.a. with interest calculated monthly. An additional deposit of \$1500 is made at the end of each month after interest has been added.
 - a** What is the investment worth at the end of 3 years?
 - b** How much interest did the investment earn in the last month of the investment?

- 7** Jessica places \$200 000 in an account earning 6.4% p.a. compounded quarterly. She wants to make quarterly withdrawals so the account has a zero balance at the end of 6 years. What is the value of the withdrawal she should make, rounded to the nearest \$10?
- 8** Anh is currently aged 43 and he wants to be able to retire at 62 years of age. Right now, he has an account with a balance of \$55 000 and it is earning 5.8% p.a. compounded monthly.
- a** If Anh makes regular monthly contributions of \$450, how much will be in his account after 10 years?
 - b** If Anh needs a lump sum of \$490 000 to fund his retirement, what is the value of the regular monthly contribution he will need to make?
- 9** Thea contributed \$2500 per quarter into an account earning 7.25% p.a. compounded quarterly for a period of 10 years. After 10 years, she increased her quarterly payments to \$3200 for another 10 years. What is the final value of her account?
- 10** As a settlement of an insurance claim, Beau is offered regular payments of \$15 000 each year for the next 15 years. Beau would prefer a one-time payment, now. Assuming the lump sum could be invested at 7.7% p.a. compounded annually, what would be a fair amount that should be offered to Beau?
- 11** A couple are saving for a deposit to buy a house. They have \$15 000 saved already, and can afford to save an additional \$800 per month at the end of each month. If they earn 8.4% p.a. compounded monthly on their savings, how long will it take them to save \$80 000?
- 12** Heidi is a professional athlete who believes that her playing career will last 9 years.
- a** To prepare for her future, she deposits \$48 000 at the end of each year for 9 years in an account paying 8% p.a. compounded annually. How much will she have in her account after 9 years?
 - b** Instead of investing \$48 000 at the end of each year, Heidi decides to deposit \$4000 at the end of each month for 9 years in an account paying 8% compounded monthly. How much will she have after 9 years?
- 13** Elle is considering a number of investment options for the next 20 years. All of the options have an interest rate of 8.4% p.a. compounded annually. Examine each of the following options and identify the one that is best, and why.
- A** Deposit \$5000 at the *start* of each year for 10 years, and then let the accumulated amount grow for 10 years
 - B** Deposit \$2500 at the start of each year for 20 years
 - C** One payment of \$5000 invested for 10 years, and then add \$5000 at the start of each year for 10 years
 - D** One payment of \$16 000 now and no other additional deposits.

9. CHAPTER SUMMARY

Annuities and perpetuities

- An **annuity** is an investment that has regular deposits made over a period of time
- An **investment annuity** has an initial balance with regular deposits and no withdrawals
- The recurrence relation for the value of an investment annuity is

$$A_{n+1} = rA_n + d$$

where: A_0 = the initial value of the investment

A_n = the value of the investment at the end of n compounding periods

$$r = 1 + i$$

i = the periodic interest rate as a decimal

d = the deposit per compounding period

- The formula for the value of an investment annuity (A_n) is:

$$A_n = A_0 r^n + d \left(\frac{r^n - 1}{r - 1} \right) = A_0 (1 + i)^n + d \left[\frac{(1 + i)^n - 1}{i} \right]$$

where: d = the deposit amount per period

A_0 = the initial amount invested

A_n = the balance of the investment after n deposits

$$r = 1 + i$$

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

- For deposits at the end of the compounding periods with no initial investment, the value is

$$A_n = d \left(\frac{r^n - 1}{r - 1} \right) = d \left[\frac{(1 + i)^n - 1}{i} \right]$$

- An **ordinary annuity** is an annuity from which regular **withdrawals** or **payments** are made for a fixed period of time until the value of the investment is \$0
- The recurrence relation for the value of an ordinary annuity is:

$$A_{n+1} = rA_n - d$$

where: A_1 = the initial balance of the investment

A_n = the value of the investment at the start of the n th compounding period

$$r = 1 + i$$

i = the periodic interest rate as a decimal

d = the withdrawal (payment) per compounding period

- The formula for the withdrawal (payment) for an ordinary annuity is:

$$d = \frac{iA}{1 - (1+i)^{-n}}$$

where: d = the withdrawal (payment) amount per period

A = the initial value (cost) of the annuity

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

- Personal **superannuation** is money that is saved during an individual's working life to provide a regular income in retirement
- The recurrence relation for the balance of a simple superannuation account is:

$$A_{n+1} = rA_n + d$$

where: A_1 = the initial balance of the superannuation account

A_n = the value of the investment at the start of n th compounding period

$$r = 1 + i$$

i = the periodic interest rate as a decimal

d = the deposit per compounding period

- A **perpetuity** is an annuity that provides regular payments that continue indefinitely
- The formula for the withdrawal (payment) amount for a perpetuity is:

$$d = Ai$$

where: d = the regular withdrawal (payment) per period

A = the amount invested

i = interest rate per period as a decimal

- A **sinking fund** is an annuity established to receive periodic payments to pay for an item at some future time
- Formulas for the future value and payment of a sinking fund are

$$A_n = p \left[\frac{(1+i)^n - 1}{i} \right] \quad p = A_n \left[\frac{i}{(1+i)^n - 1} \right]$$

where: p = the payment amount (\$) per period

A_n = the future value (\$) after n payments

i = periodic interest rate as a decimal

n = the number of compounding periods in the term

9. CHAPTER REVIEW

Annuities and perpetuities

- 1** An initial deposit of \$20 000 was made on an investment taken out over 6 years at a rate of 5% p.a. compounded monthly, and an additional deposit of \$800 is made each month.
- a** Complete the table below for the first 6 deposits.

Example
1

n	Value at start of month	Interest	Deposit (d)	Value at end of month
1	20 000.00			
2				
3				
4				
5				
6				

- b** Calculate how much interest had been earned over this time.
- c** What is the value of the annuity at the end of 6 months?
- 2** \$8000 is deposited in an investment annuity paying 7% p.a. compounded annually with annual deposits of \$500 at the end of each year. If A_n is the value of the investment after n years, a recurrence relation for A_n is:
- A** $A_{n+1} = 1.07A_n + 500$, where $A_0 = 8000$
- B** $A_{n+1} = 1.07(A_n - 500)$, where $A_0 = 8000$
- C** $A_n = 1.07A_{n+1} + 500$, where $A_0 = 8000$
- D** $A_{n+1} = 1.07(A_n + 500)$, where $A_1 = 8000$
- E** $A_{n+1} = 1.07A_n - 500$, where $A_0 = 8000$
- 3** Ruby deposits \$35 000 into an investment annuity that pays 6.8% p.a. interest compounded quarterly. She also makes regular quarterly deposits of \$1500.
- a** Write a recurrence relation for this investment annuity.
- b** What is the value of the annuity at the end of 3 years?
- c** How long will it be before Ruby has accumulated \$100 000?
- 4** Dylan needs to access a lump sum of money in 7 years time to buy a stake in a restaurant. He places \$8000 in an investment annuity that has an interest rate of 9.5% p.a. compounded annually. He also makes regular annual deposits of \$2200 each year. Use the investment annuity formula to find the amount available for his stake.

Example
2

Example
2

Example
3

Example
4

- 5** \$10 000 is invested at the start of the year at 5.8% p.a. compounded annually. At the end of each year, interest is added and then a withdrawal of \$800 is made. If the value of the investment at the start of the n th year is A_n , which of the following represents a recurrence relation for the value of the investment?

A $A_{n+1} = 1.058(A_n - 800)$, where $A_0 = 12\ 000$
B $A_{n+1} = 1.058(A_n + 800)$, where $A_0 = 10\ 000$
C $A_{n-1} = 1.058A_n + 800$, where $A_1 = 10\ 000$
D $A_{n+1} = 1.058A_n - 800$, where $A_1 = 10\ 000$
E $A_n = 1.058A_{n+1} - 800$, where $A_1 = 10\ 000$

Example
4

- 6** At the start of the year, \$46 000 is placed in an account paying 8.1% p.a. compounded monthly. At the end of each month, interest is added and then a withdrawal of \$950 is made. If A_n is the value of the investment at the start of month n , which of the following represents a recurrence relation for the value of the investment?

A $A_{n+1} = 1.081A_n + 950$, where $A_1 = 46\ 000$
B $A_n = 1.006\ 75A_{n+1} - 950$, where $A_1 = 46\ 000$
C $A_{n-1} = 1.006\ 75A_n - 950$, where $A_1 = 46\ 000$
D $A_{n+1} = 1.081A_n - 950$, where $A_1 = 46\ 000$
E $A_{n+1} = 1.006\ 75A_n + 950$, where $A_1 = 46\ 000$

Example
4

- 7** Larissa invests the \$85 000 she received as a redundancy payment into an account that pays 6.6% p.a. compounded annually. Each year, after the annual interest is added, she withdraws \$7000. The amount remaining is then reinvested for the following year.

a Write a recurrence relation for the value of the investment.
b What is the value of the investment at the start of the 9th year?
c How long will the annuity last?
d How much will the last payment be?

Example
5

- 8** An amount of \$140 000 is invested in an account paying 9.2% p.a. compounded annually. At the end of each year, after interest has been added, \$14 000 is withdrawn.

a How long will the annuity last?
b If the money had been invested at 5.7% p.a. compounded annually and the same withdrawals made, how long would the annuity last?
c If the regular annual withdrawals were \$25 000 instead, how long would the original annuity last?

Example
6

- 9** Elijah invested \$150 000 in an annuity that pays 7.9% p.a. interest compounded monthly. After interest is added each month, he receives a payment. How much will each payment be if the annuity lasts for 25 years?

- 10** \$500 is invested at the start of each month in an account that earns 7.2% p.a. compounded monthly. Interest is added at the end of each month. This process is repeated each month. If the value of the investment at the start of the n th month is A_n , where $A_1 = 500$, which of the following represents a recurrence relation for the value of the investment?

- A** $A_n = 1.006(A_{n+1} - 500)$ **B** $A_{n+1} = 1.006A_n + 500$
C $A_n = 1.072A_{n+1} + 500$ **D** $A_{n+1} = 1.006A_n - 500$
E $A_n = 1.006A_{n+1} + 500$

- 11** Maya starts work in a new job. Her employer establishes a superannuation account for her by depositing \$320 at the start of each month. The account earns 6.5% p.a. compounded monthly. What will be the balance of Maya's account after 20 years?

- 12** \$66 000 is deposited in an account paying 7.8% p.a. compounded monthly. At the end of each month, interest is added and then a deposit of \$590 is made. If the value of the investment at the start of month n is A_n , where $A_1 = 66\ 000$, which of the following represents a recurrence relation for the value of the investment?

- A** $A_{n+1} = 1.078A_n + 590$ **B** $A_n = 1.0065A_{n+1} - 590$
C $A_n = 1.0065A_{n+1} + 590$ **D** $A_{n+1} = 1.0065(A_n + 590)$
E $A_{n+1} = 1.0065A_n + 590$

- 13** Caleb has a balance of \$45 000 in his superannuation account. He decides to transfer his superannuation balance to a different fund that pays 8.8% p.a. compounded monthly. Each month, his employer contributes \$630 to this account.

- a** What is the balance of Caleb's account after 5 years?
b How long will it take for Caleb's account to reach a balance of \$400 000?

- 14** Ivy deposits \$38 000 in a new superannuation account earning an interest rate of 6.9% p.a. with interest compounded monthly. Monthly contributions of \$440 are made to the account. Ivy is currently 45 years of age and would like to retire when she turns 62. Which of the following equations should she use to find the final value of her superannuation account?

- A** $V_{17} = 38\ 000 \times 1.00575^{17} - 440 \left(\frac{1.069^{17} + 1}{1.069 + 1} \right)$
B $V_{17} = 38\ 000 \times 1.069^{17} + 440 \left(\frac{1.069^{17} - 1}{1.069 - 1} \right)$
C $V_{204} = 38\ 000 \times 1.069^{204} + 440 \left(\frac{1.069^{204} - 1}{1.069 - 1} \right)$
D $V_{204} = 38\ 000 \times 1.00575^{204} + 440 \left(\frac{1.00575^{204} - 1}{1.00575 - 1} \right)$
E $V_{204} = 38\ 000 \times 1.00575^{204} - 440 \left(\frac{1.00575^{204} + 1}{1.00575 + 1} \right)$

Example
9

- 15** Lucas is 39 years of age and is planning to retire at 64 years of age. His current superannuation fund balance is \$62 000 and is earning 7.9% p.a. compounded monthly.
- a** If his employer makes regular monthly contributions of \$490, how much will be in his account after 12 years?
 - b** Lucas estimates that he will need a lump sum of \$670 000 to fund his retirement. What regular monthly contributions will he need to make to achieve his target?

Example
10

- 16** A benefactor agrees to provide \$250 000 to establish a perpetuity to provide an annual grant for the homeless. If the money is invested in utility bonds that pay a return of 7.3% p.a. compounded annually, what is the value of the grant that could be awarded?

Example
10

- 17** A bequest of \$150 000 is used to establish a perpetuity that funds an annual school fete. If the money is invested with a long-term return of 8.25% p.a. compounded annually, the value of the grant is:
- A** \$12 375 **B** \$15 000 **C** \$123 750
D \$1237.50 **E** \$24 750

Example
11

- 18** The House Builders Association (HBA) establishes a perpetuity to provide a building industry apprenticeship. The HBA is able to secure an investment that earns an interest rate of 5.5% p.a. compounded annually. If the annual scholarship is \$27 000, how much needs to be invested?

Example
12

- 19** \$240 000 is used to establish a perpetuity to provide a monthly payment to a community health centre. The fund in which the money is invested earns 5.5% p.a. compounded monthly.
- a** Calculate the amount of the grant.
 - b** What interest rate (compounded quarterly) would be required for the perpetuity to provide a monthly grant of \$3600?

Example
12

- 20** A successful artist uses \$360 000 to establish a perpetuity. The perpetuity provides the funds for a monthly grant to stage exhibitions of landscape paintings. If the money is invested in bonds that return 5.5% p.a. compounded monthly, then the monthly allowance is:
- A** \$18 700 **B** \$3300 **C** \$1650
D \$1870 **E** \$16 500

Example
13

- 21** An earthworks company must replace a front-end loader in 7 years time, so a sinking fund is established to buy the replacement. \$1900 per month is placed into an account that pays 7.8% interest, compounded monthly. How much money will be in the sinking fund after 7 years?

22 A sinking fund is set up to pay for the replacement of office equipment. The equipment will need to be replaced in 5 years time and will cost \$65 000. The sinking fund is invested in an account that pays 5.2% p.a. compounded monthly. How much must each monthly deposit be in order to reach their goal?

Example
13

23 Sienna just won \$125 000 in the lottery. She invests the money in an account that pays 5.1% p.a. compounded monthly and would like to be able to make monthly withdrawals of \$1000 for a number of years.

Example
14

- a** What type of annuity is this?
- b** How much will be in her account after 4 years?
- c** For how long will she be able to make monthly withdrawals?
- d** What is the value of the last withdrawal?

24 Ethan wants to travel overseas in 5 years time. He currently has \$3000 that he invests in an account that pays 7.5% p.a. compounded monthly. In addition to his initial investment, he also deposits \$1200 each month.

Example
15

- a** What type of annuity is this?
- b** How much will Ethan have to spend on his trip?

Problem solving

25 Harvey owns a delicatessen. He knows that he will have to replace a meat slicer, coffee roaster and other pieces of equipment in 5 years time. The total cost of the machinery will be \$38 000. Harvey decides to deposit a sum of money into an account at the end of each month to build up enough money to pay for the equipment. If the account where he deposits his money pays 7.3% p.a. compounded monthly, what is the minimum amount that each deposit must be?

26 Molly is a 48-year-old woman. At the end of each quarter, she puts \$4000 in a retirement account that pays 5.4% p.a. interest compounded quarterly. Molly withdraws the entire amount when she turns 60, and deposits it in a different account that pays 7.2% interest compounded monthly. From then on she deposits \$450 into the account at the end of each month. How much is in the account when Molly reaches age 65?

27 An investment account currently has a balance of \$191 653. Annual contributions of \$4800 are made and the account earns 5.5% p.a. compounded annually. What was the initial amount in the account 9 years ago?



10.

NETWORKS, PROJECTS AND ASSIGNMENTS

Completing a project on time and within budget is often difficult. Projects usually involve multiple tasks or steps. Some tasks can't begin before other tasks have been completed, while other tasks may be done independently.

How do project managers know which things should be done first? How do they know how long a large project will take to complete? How do they plan the order of contracts? How do they know the sequence in which the materials should be ordered?

These questions are answered by looking at the sequences of project tasks from activity tables.

In this chapter, you will see how the mathematics of project design is applied to these types of problems.

- 10.01 Project networks
- 10.02 Earliest and latest starting times
- 10.03 Critical paths
- 10.04 Float times
- 10.05 Applications of project networks
- 10.06 Bipartite graphs
- 10.07 Optimum assignment problems
- Chapter summary
- Chapter review

SYLLABUS SUBJECT MATTER

Project planning and scheduling using critical path analysis (CPA)

- construct a network diagram to represent the durations and interdependencies of activities that must be completed during the project, e.g. preparing a meal
- use forward and backward scanning to determine the earliest starting time (EST) and latest starting times (LST) for each activity in the project
- use ESTs and LSTs to locate the critical path/s for the project
- use the critical path to determine the minimum time for a project to be completed
- calculate float times for non-critical activities

Assigning order and the Hungarian algorithm

- use a bipartite graph and its tabular or matrix form to represent an assignment/allocation problem, e.g. assigning four swimmers to the four places in a medley relay team to maximise the team's chances of winning
- determine the optimum assignment/s for small-scale problems by inspection, or by use of the Hungarian algorithm (3×3) for larger problems



Prior learning

General Mathematics 2019 v1.2 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website <https://www.qcaa.qld.edu.au/>.

TERMINOLOGY

activity
assignment
complete matching
critical path
float time
immediate predecessor
optimum allocation
row and column reduction

activity table
backward scanning
concurrent activities
dummy activity
forward scanning
latest starting time (LST)
prerequisite
slack time

allocation
bipartite graph
critical activity
earliest starting time (EST)
Hungarian algorithm
matching
project network



Project networks

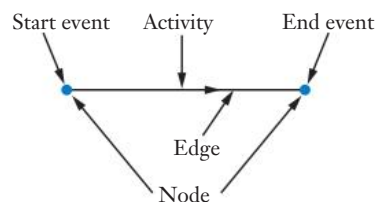
10.01 Project networks

The tasks that are required to complete a project are called **activities**. For a large project, you can draw a directed network to show the sequence of activities and the time required for their completion.



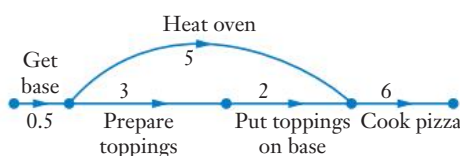
Drawing directed graphs

For a **project network**, each activity is represented by an edge with an arrow to show direction. It is usual to label the activity and show the length of time required to complete the activity on the edge. The nodes (vertices) at the start and end of an activity show the events at the commencement and finish of the activity as shown here.



In some cases, activities can be performed at the same time, and in other cases, they must be done one after the other. Some activities can only commence after another activity has been completed.

Look at the project network below for preparing and cooking a homemade pizza using a pre-prepared base.

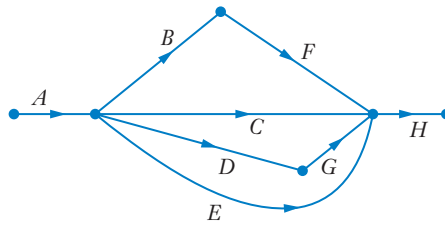


The project network shows that you need to get the base out first. The oven needs to be heated before the pizza can be cooked, but the toppings can be prepared and put on the pizza base while the oven is being heated. The toppings need to be prepared before they can go on the base.

An activity that must be completed immediately before another activity can start is called an **immediate predecessor** or **prerequisite**. In the diagram below, activity *A* is the immediate predecessor of *B*.



Look at the project network below.



In this network activity, *A* precedes multiple activities *B*, *C*, *D* and *E*. Activity *H* is preceded by multiple activities *F*, *C*, *G* and *E*. Activities *E* and *C* can occur at the same time, so they are called **concurrent activities**.

Project network diagrams

- The network must start at a vertex.
- Activities that do not have predecessors are starting activities.
- Multiple predecessors of an activity will end at the same vertex.
- An activity must only be represented by one edge.
- There must be a vertex showing the completion of the project.

An **activity table** shows the order and estimated completion time of each activity required to complete a project. An activity table normally includes headings for the activity, estimated time taken and prerequisites. Each activity is usually coded by a letter.

You can use an activity table to construct a project network.



Getty Images/Portland Press Herald



Project
networks

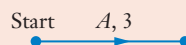
EXAMPLE 1

Draw a project network for the activity table shown.

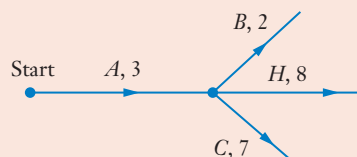
Activity	Time (min)	Prerequisites
<i>A</i>	3	None
<i>B</i>	2	<i>A</i>
<i>C</i>	7	<i>A</i>
<i>D</i>	5	<i>B</i>
<i>E</i>	4	<i>B</i>
<i>F</i>	1	<i>D</i>
<i>G</i>	3	<i>E</i>
<i>H</i>	8	<i>A</i>
<i>I</i>	2	<i>C, F, G, H</i>

Solution

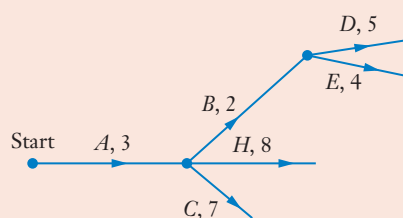
A is the only activity with no prerequisites. Draw a directed edge for *A* and label it '*A, 3*'. Add vertices at each end. Label the first vertex 'Start'.



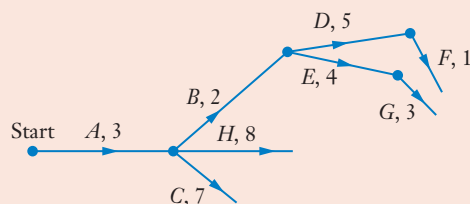
Activities *B*, *C* and *H* all have *A* as a prerequisite. Draw 3 directed edges from the end of *A* and label them '*B, 2*', '*H, 8*' and '*C, 7*'.



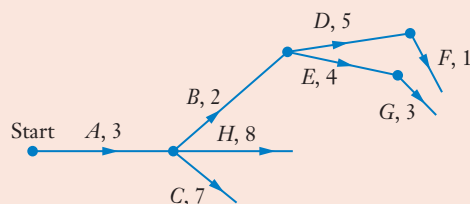
Activities *D* and *E* have *B* as a prerequisite. End activity *B* with a vertex. Then draw 2 directed edges from the end of *B* and label them '*D, 5*' and '*E, 4*'.



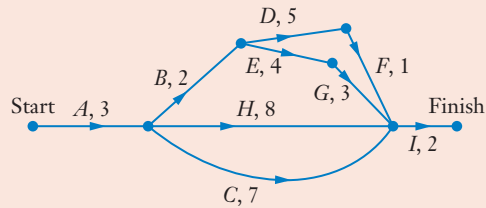
Activity *F* has *D* as a prerequisite. End activity *D* with a vertex. Then draw a directed edge from the end of *D* and label it '*F, 1*'.



Activity *G* has *E* as a prerequisite. End activity *E* with a vertex. Then draw a directed edge from the end of *E* and label it '*G, 3*'.



Activity *I* has *F*, *G*, *H* and *C* as prerequisites. End activities *F*, *G*, *H* and *C* at the same vertex. Draw a directed edge from this vertex and label it '*I*, 2'. Add a vertex at the end and label it 'Finish'.



In some cases, it is difficult to show a prerequisite for an activity. You can use a **dummy activity** to solve this problem. It is shown as a directed edge with a dashed line and a time of zero.

Consider the activity table on the right.

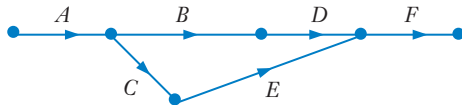
You can see that the start is before *A* and the finish is after *F*.

B and *C* must both follow *A* and *E* must follow *C*.

D and *E* join to a vertex before *F*.

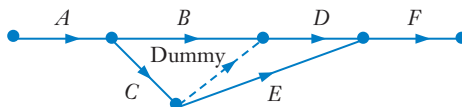
The partially completed project network is shown below.

Activity	Prerequisites
<i>A</i>	None
<i>B</i>	<i>A</i>
<i>C</i>	<i>A</i>
<i>D</i>	<i>B</i> , <i>C</i>
<i>E</i>	<i>C</i>
<i>F</i>	<i>D</i> , <i>E</i>



You can insert a dummy activity to show that *C* is a prerequisite for *D*.

The completed network is shown below.



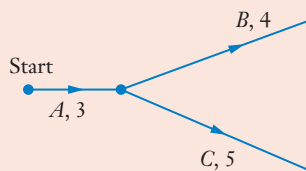
EXAMPLE 2

Draw a project network for this activity table.

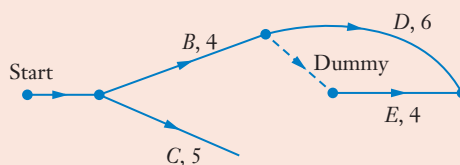
Activity	Time (days)	Prerequisites
<i>A</i>	3	None
<i>B</i>	4	<i>A</i>
<i>C</i>	5	<i>A</i>
<i>D</i>	6	<i>B</i>
<i>E</i>	4	<i>B</i> , <i>C</i>
<i>F</i>	5	<i>C</i>
<i>G</i>	8	<i>D</i> , <i>E</i> , <i>F</i>

Solution

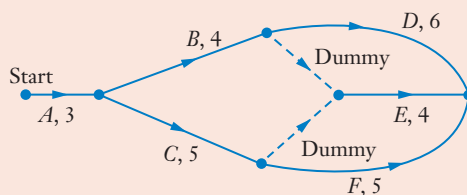
A is the only activity with no prerequisites. Draw a directed edge for A and label it ' $A, 3$ '. Add vertices at each end. Label the first vertex 'Start'. Activities B and C each have A as a prerequisite. Draw two directed edges from the end of A and label them ' $B, 4$ ', and ' $C, 5$ '.



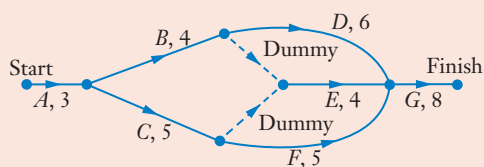
Activities D and E are both preceded by B . D and E are also prerequisites for G , so they both finish at the same vertex. Draw a directed edge starting at the end of B and label it ' $D, 6$ '. Draw another directed edge finishing at the end of D and label it ' $E, 4$ '. Add a dummy activity from the end of B to the start of E to show B as a prerequisite.



Activities E and F are both preceded by C . E and F are also prerequisites for G , so they both finish at the same vertex. Draw a directed edge starting at the end of C and ending at the end of E (and D) and label it ' $F, 5$ '. Add a dummy activity from the end of C to the start of E to show C as a prerequisite.



Activity G is preceded by D , E and F . Draw a directed edge from the end of D , E and F and label it ' $G, 8$ '. Add a vertex at the end and label it 'Finish'.



INVESTIGATION

CONSTRUCTING PROJECT NETWORKS

Work in groups of 3 or 4 people for this activity.

- 1 Select one of the following projects or an alternative agreed to by your teacher.
- 2 Prepare an activity table with estimates of times and prerequisites.
- 3 Draw a project network.
- 4 Make a wall display to show the results of your work.

Projects

- Building a garden wall
- Staging a child's birthday party
- Painting a mural on a blank wall at school
- Taking a school group on a ski trip to Thredbo
- Baking a cake

Discuss the results of your project designs as a class group. Each group should be able to justify the time estimates they have used.

EXAMPLE 3

Draw a network to show the task of publishing a book using this activity table.

Code	Activity	Time (weeks)	Prerequisites
<i>A</i>	Write manuscript	20	None
<i>B</i>	Edit manuscript	6	<i>A</i>
<i>C</i>	Typeset manuscript	4	<i>B</i>
<i>D</i>	Perform rough pagination	3	<i>C</i>
<i>E</i>	Perform final pagination	3	<i>D</i>
<i>F</i>	Do artwork	10	<i>D</i>
<i>G</i>	Mount artwork in final pages	5	<i>E, F</i>
<i>H</i>	Promote book	4	<i>A</i>
<i>I</i>	Print book	6	<i>G</i>
<i>J</i>	Distribute book for sale	2	<i>H, I</i>

Solution

Only activity A has no prerequisites. Draw a directed edge for A and label it ' $A, 20$ '. Add vertices at each end. Label the first vertex 'Start'.

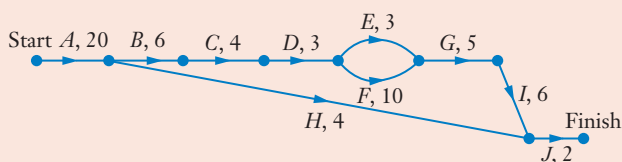
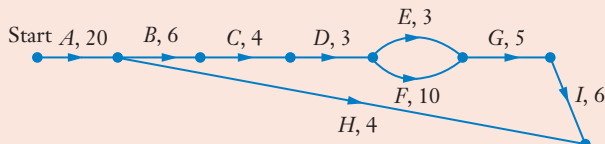
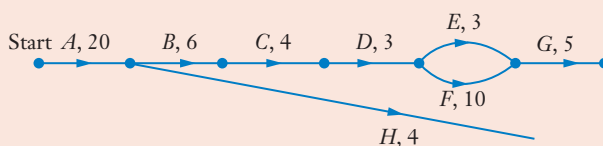
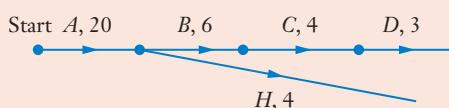
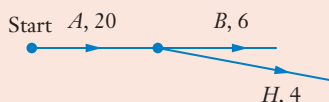
Activities B and H each have A as a prerequisite. Draw two directed edges from the end of A and label them ' $B, 6$ ', and ' $H, 4$ '.

Activity B is a prerequisite for C and C is a prerequisite for D .

Activities E and F are both preceded by D . E and F are also prerequisites for G , so they both finish at the same vertex.

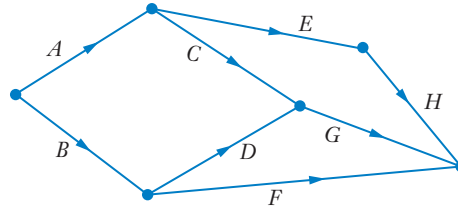
G is a prerequisite for I but H and I are both prerequisites for J . This means that H and I must finish at the same vertex. Draw a directed edge from G and label it ' $I, 6$ '. Extend H so that it ends at the same vertex as I .

The final activity is J and it is preceded by H and I . Draw a directed edge from the end of H and I and label it ' $J, 2$ '. Label the end vertex 'Finish'.



Exercise 10.01 Project networks

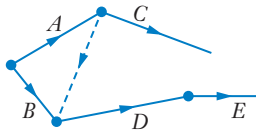
Questions 1–2 refer to this diagram.



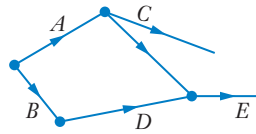
- 1 Which of the following statements is true?
 - A Activity *A* is a prerequisite of *H*.
 - B Activity *C* is a prerequisite of *G*.
 - C Activity *G* must be done before activity *D*.
 - D Activity *H* must be done before activity *A*.
 - E Activity *D* is a prerequisite of *F*.
- 2 Which of the following statements is true?
 - A Activities *A* and *C* are both prerequisites of *G*.
 - B Activity *D* is a dummy activity.
 - C Activities *A* and *B* must be done before activity *C*.
 - D Activities *C* and *E* must be done before activity *H*.
 - E Activities *C* and *D* are both prerequisites of *G*.

- 3 The project network for this activity table is:

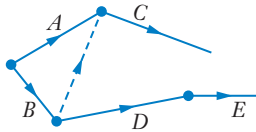
A



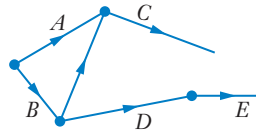
B



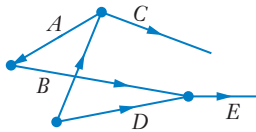
C



D



E



Activity	Prerequisite
<i>A</i>	None
<i>B</i>	None
<i>C</i>	<i>A, B</i>
<i>D</i>	<i>B</i>
<i>E</i>	<i>D</i>

- 4 Draw a project network for each activity table.

a

Activity	Time	Prerequisites
<i>A</i>	3	None
<i>B</i>	2	<i>A</i>
<i>C</i>	5	None
<i>D</i>	6	<i>B, C</i>

b

Activity	Time	Prerequisites
<i>A</i>	4	None
<i>B</i>	3	None
<i>C</i>	6	<i>B</i>
<i>D</i>	5	<i>A</i>
<i>E</i>	2	<i>D, C</i>

Example
1

5 Draw a project network for each activity table.

a

Activity	Time	Prerequisites
<i>A</i>	5	None
<i>B</i>	4	<i>A</i>
<i>C</i>	6	<i>B</i>
<i>D</i>	7	<i>A</i>
<i>E</i>	5	<i>D</i>
<i>F</i>	2	<i>D</i>
<i>G</i>	3	<i>F</i>
<i>H</i>	2	<i>C</i>
<i>I</i>	4	<i>H, E, G</i>

b

Activity	Time	Prerequisites
<i>A</i>	2	None
<i>B</i>	1	<i>A</i>
<i>C</i>	5	<i>A</i>
<i>D</i>	3	<i>A</i>
<i>E</i>	8	<i>B</i>
<i>F</i>	3	<i>D</i>
<i>G</i>	6	<i>F</i>
<i>H</i>	7	<i>C</i>
<i>I</i>	4	<i>E</i>
<i>J</i>	3	<i>G, H, I</i>

Example
2

6 Draw a project network for each activity table.

a

Activity	Time	Prerequisites
<i>A</i>	4	None
<i>B</i>	5	None
<i>C</i>	7	<i>B</i>
<i>D</i>	8	<i>A, B</i>
<i>E</i>	3	<i>C, D</i>

b

Activity	Time	Prerequisites
<i>A</i>	6	None
<i>B</i>	3	<i>A</i>
<i>C</i>	5	None
<i>D</i>	7	None
<i>E</i>	3	<i>B, C, D</i>
<i>F</i>	4	<i>D</i>

7 Draw a project network for each activity table.

a

Activity	Time	Prerequisites
<i>A</i>	6	None
<i>B</i>	8	<i>A</i>
<i>C</i>	7	<i>A</i>
<i>D</i>	10	<i>B</i>
<i>E</i>	7	<i>C</i>
<i>F</i>	6	<i>C</i>
<i>G</i>	4	<i>D, E, F</i>
<i>H</i>	3	<i>G</i>

b

Activity	Time	Prerequisites
<i>A</i>	5	None
<i>B</i>	7	<i>A</i>
<i>C</i>	3	<i>B</i>
<i>D</i>	3	<i>B</i>
<i>E</i>	6	<i>A</i>
<i>F</i>	10	<i>E</i>
<i>G</i>	4	<i>D, E</i>
<i>H</i>	2	<i>C</i>
<i>I</i>	4	<i>F, G, H</i>

8 Draw a project network for children preparing and serving breakfast in bed.

Code	Activity	Time (min)	Prerequisites
<i>A</i>	Cut bread for toast	1	None
<i>B</i>	Make toast	3	<i>A</i>
<i>C</i>	Spread toast	1	<i>B</i>
<i>D</i>	Fetch newspaper	2	None
<i>E</i>	Boil water in jug	5	None
<i>F</i>	Make tea	1	<i>E</i>
<i>G</i>	Cook sausages and eggs	15	None
<i>H</i>	Put crockery, cutlery and newspaper on tray	2	<i>D</i>
<i>I</i>	Put food on tray	2	<i>C, F, G, H</i>
<i>J</i>	Take tray to bedroom	1	<i>I</i>

9 Draw a project network for the task of making a dressing table.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Order mirror	10	None
<i>B</i>	Order drawer pull handles	10	None
<i>C</i>	Make frame	5	None
<i>D</i>	Make top	5	<i>C</i>
<i>E</i>	Make drawers	7	<i>C</i>
<i>F</i>	Fit top	1	<i>D</i>
<i>G</i>	Fit drawers	1	<i>E</i>
<i>H</i>	Finish and polish surfaces	3	<i>F, G</i>
<i>I</i>	Fit drawer pull handles	1	<i>B, H</i>
<i>J</i>	Fit mirror	1	<i>A, H</i>

Problem solving

- 10** Work out an activity table for the project of organising a barbecue picnic and trophy presentation for the end-of-season breakup of a local netball club.
- 11** Work out an activity table for the council project of building a children's playground in a local park.

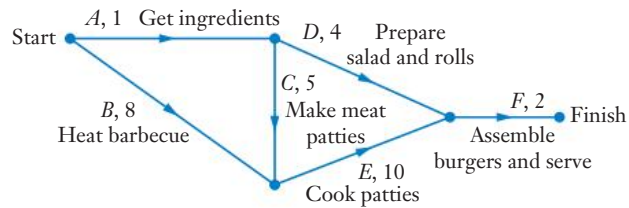


10.02 Earliest and latest starting times

All projects need to be completed within a certain period of time. The activities involved in preparing, barbecuing and serving a meal of hamburgers are given in the activity table below.

Code	Activity	Time (min)	Prerequisites
A	Get ingredients	1	None
B	Heat barbecue	8	None
C	Make meat patties	5	A
D	Prepare salad and rolls	4	A
E	Cook patties	10	B, C
F	Assemble burgers and serve	2	D, E

The project network for this project is shown here.



There is no point in commencing all the activities in a project at the same time. The timing of various activities depends on the time taken for prerequisites to be completed. A process called **forward scanning** allows you to determine the **earliest starting time** (EST) for each activity in a network. The earliest starting time for any activity is the earliest time it is possible to start the activity based on prerequisites.

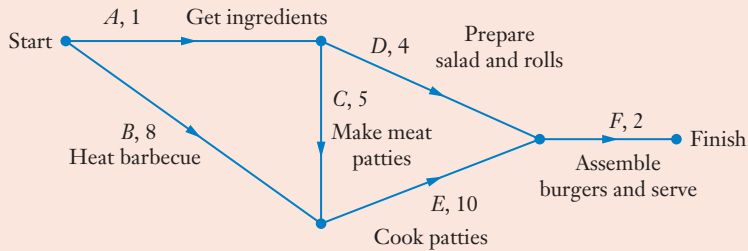


Forward scanning

- 1 Change the vertices to circles so information can be written in them.
- 2 The vertex at the start of a network has an earliest starting time of 0. Write 0 in the circle for the first vertex.
- 3 Add the time taken for an activity to its EST to find the EST for adjacent activities.
- 4 Wherever 2 or more activities join at a vertex, choose the highest total time as the EST.
- 5 Continue until the finish of the network is reached.

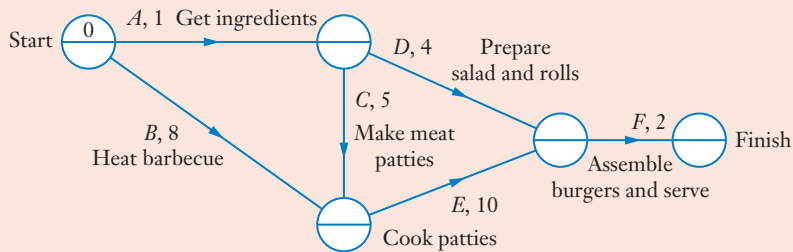
EXAMPLE 4

Determine the earliest starting time for the activities in this network.

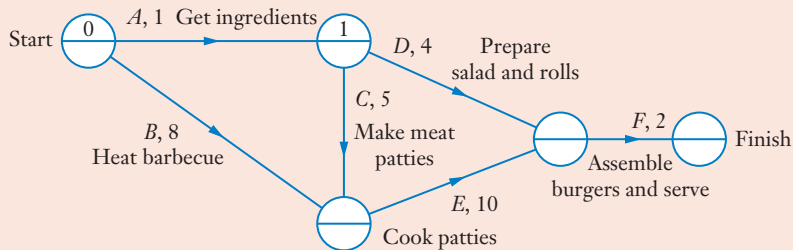


Solution

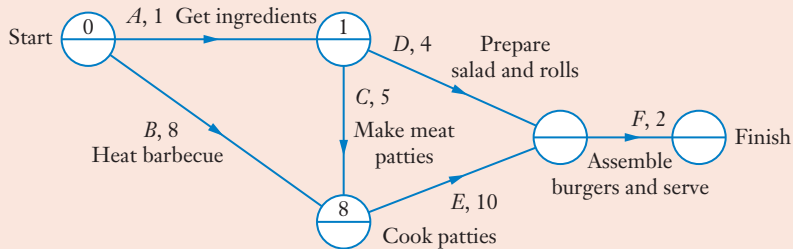
Change the vertices to circles and make the EST for the first vertex 0.



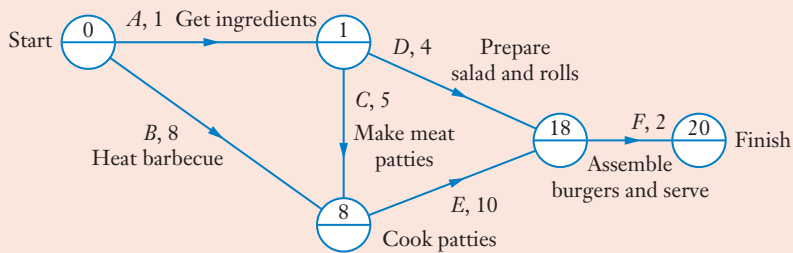
Add the activity time for *A* to the EST for the start vertex to calculate the EST for *C* and *D*.



Activity *E* has two prerequisites. Adding the activity time for *C* gives an EST for activity *E* as $1 + 5 = 6$. Adding the activity time for *B* gives an EST for activity *E* of 8. Use the higher value.



Continue this process with all other vertices.



Getty Images/vgajic

A process called **backward scanning** allows you to determine the **latest starting time (LST)** for each activity in a network. The latest starting time for any activity is the latest time it is possible to start the activity without delaying the completion of the project.

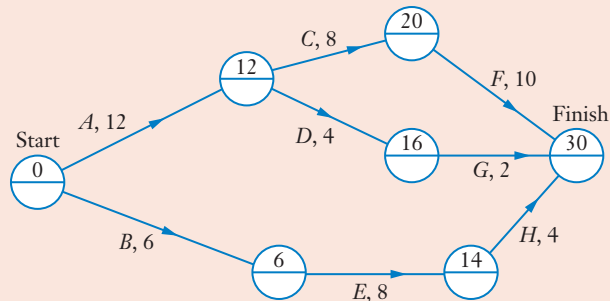
Backward scanning

- 1 Begin with a network in which the EST for each activity has been calculated.
- 2 The vertex at the end of a network has a latest starting time equal to the EST of that vertex. Write this value in the bottom part of the circle for the last vertex.
- 3 Subtract the time taken for an activity from its LST to find the LST for adjacent activities.
- 4 Wherever 2 or more activities join at a vertex, choose the lowest total time as the LST.
- 5 Continue until the start of the network is reached.

EXAMPLE 5

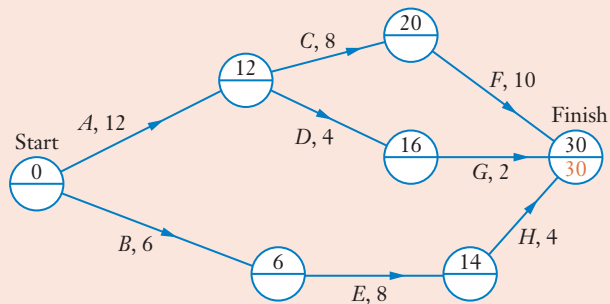
This network has the earliest starting times calculated.

Determine the latest starting time for each activity.



Solution

Begin at the last vertex in the network. The EST is 30, so enter 30 in the bottom part of the circle in the last vertex.



Latest starting times

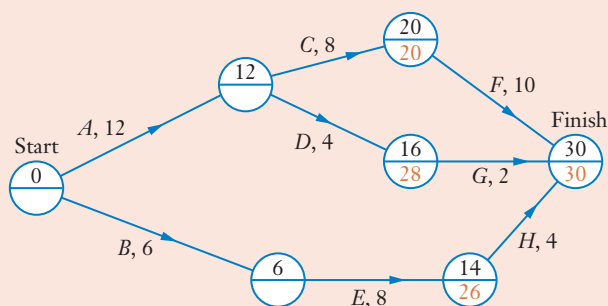
There is only one path backwards to the start of activities *F*, *G* and *H*. The LST for each is calculated by working backwards and subtracting the activity times for each.

$$\text{LST for } F = 30 - 10 = 20$$

$$\text{LST for } G = 30 - 2 = 28$$

$$\text{LST for } H = 30 - 4 = 26$$

Enter the LST in the bottom part of the vertices for the relevant activities.

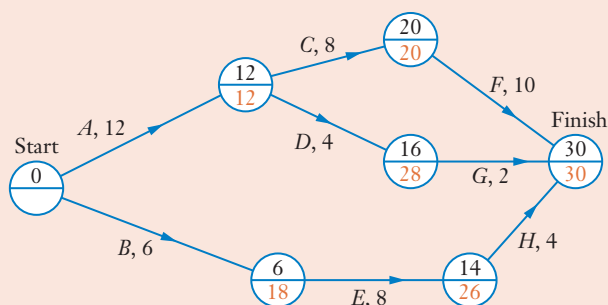


There are two paths backwards to the start of activities *C* and *D*. The LST for each is calculated by working backwards.

$$\text{LST for } C = 20 - 8 = 12$$

$$\text{LST for } D = 28 - 4 = 24$$

$12 < 24$, so enter the LST of 12 in the bottom part of the vertex at the end of activities *D* and *C*.



There is only one path backwards to the start of activity *E*.

$$\text{LST for } E = 26 - 8 = 18$$

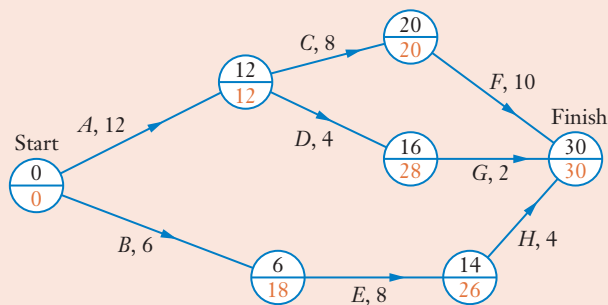
Enter 18 as the LST in the vertex at the end of activity *E*.

There are two paths backwards to the start of activities *A* and *B*.

$$\text{LST for } A = 12 - 12 = 0$$

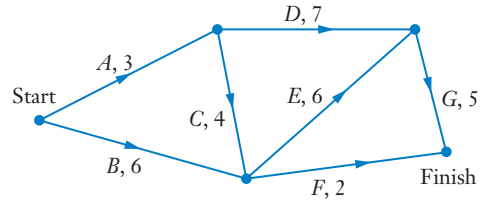
$$\text{LST for } B = 18 - 6 = 12$$

$0 < 12$, so enter the LST of 0 in the bottom part of the vertex at the end of activities *A* and *B*.



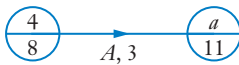
Exercise 10.02 Earliest and latest starting times

This project network shows activities A, B, C, D, E, F and G and the times for each activity in minutes. Use the project network to answer questions 1–5.



- What is the earliest starting time for activity G ?
A 10 min **B** 12 min **C** 13 min
D 15 min **E** 7 min
- What is the latest starting time for activity B ?
A 7 min **B** 6 min **C** 10 min
D 8 min **E** 12 min
- What is the earliest starting time for the last vertex?
A 12 min **B** 8 min **C** 15 min
D 23 min **E** 18 min
- If the time taken to complete activity A is increased to 5 minutes, by how much will the earliest starting time for the last vertex be increased?
A 5 min **B** 1 min **C** 2 min
D 0 min **E** 3 min
- If the time taken to complete activity E is increased to 10 minutes, by how much will the earliest starting time for the last vertex be increased?
A 6 min **B** 4 min **C** 2 min
D 10 min **E** 8 min
- Find the value of each pronumeral in each project network.

a



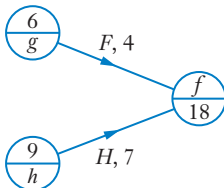
b



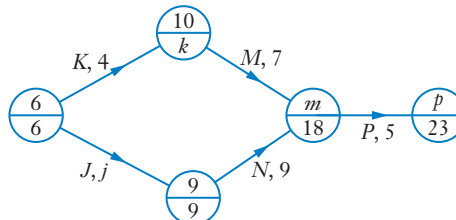
c



d

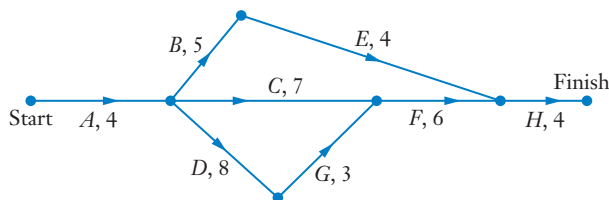


e



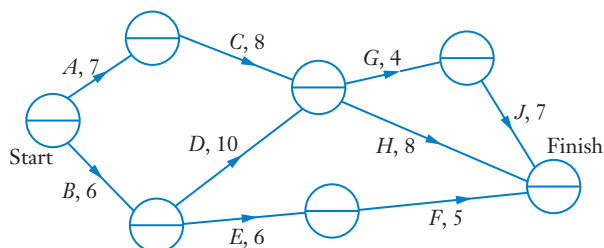
Example
4

7 Use this project network diagram to find the earliest starting time for the node at:



- a** the start
b the end of activity *D*
c the start of activity *F*
d the end of activities *E* and *F*
e the end of activity *C*

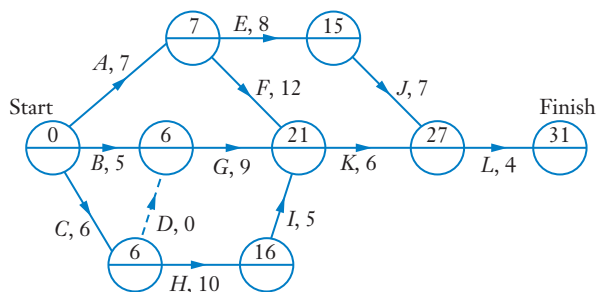
8 Use forward scanning to determine the earliest starting time for each activity in this network.



Example
5

9 This network has the earliest starting times already calculated.

- a** What type of activity is *D*?
b Determine the latest starting times for each activity.

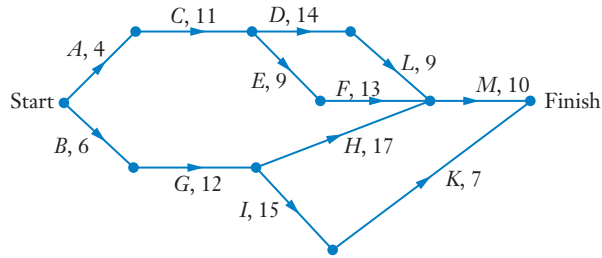


10 The activity table for a project is shown.

- a** Draw the project that corresponds to this table.
b Use forward scanning to calculate the earliest starting time for each activity.
c Use backward scanning to calculate the latest starting time for each activity.

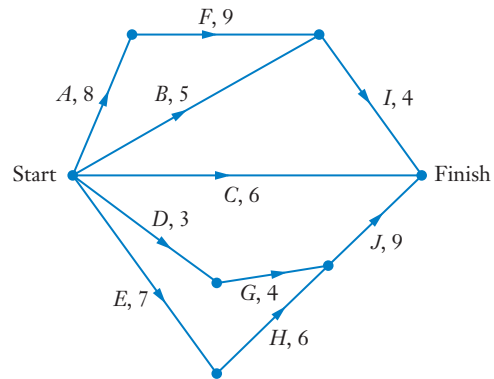
Activity	Time (hrs)	Prerequisites
<i>A</i>	4	None
<i>B</i>	10	None
<i>C</i>	7	<i>A</i>
<i>D</i>	12	None
<i>E</i>	8	<i>B</i>
<i>F</i>	4	<i>C</i>
<i>G</i>	5	<i>F</i> , <i>D</i> , <i>E</i>
<i>H</i>	7	<i>F</i> , <i>D</i> , <i>E</i>
<i>I</i>	3	<i>G</i>
<i>J</i>	6	<i>H</i> , <i>I</i>

11 This project network shows the time taken, in days, for a number of activities.



- Use forward scanning to calculate the earliest starting time for each activity.
- Use backward scanning to calculate the latest starting time for each activity.
- If the time taken for activity *G* increased to 15 days, how would this affect the earliest starting time for the last vertex?
- If the time taken for activity *H* increased to 20 days, how would this affect the earliest starting time for the last vertex?

12 This project network shows the time taken in hours for a number of activities.



- Use forward scanning to calculate the earliest starting time for each activity.
- Use backward scanning to calculate the latest starting time for each activity.
- If the time taken for activity *D* increased to 8 hours, how would this affect the earliest starting time for the last vertex?
- If the time taken for activity *H* increased to 8 hours, how would this affect the time taken to complete the project?



Shutterstock.com/David Tadevosian

Problem solving

13 This activity table lists the steps for making a raised garden bed using sleepers.

Code	Activity	Time (h)	Prerequisites
<i>A</i>	Take the trailer to the dump	2	<i>F</i>
<i>B</i>	Put the new plants into the bed	3	<i>D, H</i>
<i>C</i>	Mark out the site of the garden bed	1	None
<i>D</i>	Put the new soil into the bed	3	<i>G, M</i>
<i>E</i>	Water the plants in	0.5	<i>B</i>
<i>F</i>	Remove the existing vegetation and put in trailer	4	<i>C</i>
<i>G</i>	Put the side sleepers in and bolt into place	3	<i>J</i>
<i>H</i>	Buy the plants to put in the bed	2	<i>L</i>
<i>I</i>	Concrete corner posts into place	1.5	<i>K</i>
<i>J</i>	Allow concrete to dry	10	<i>I</i>
<i>K</i>	Dig holes for the corner posts	3	<i>C</i>
<i>L</i>	Plan the layout of the plants to be placed in the bed	2	None
<i>M</i>	Use the trailer to get the new soil	2.5	None

- Draw a project network for this project. Note that the use of the trailer makes several networks possible. Justify your choice.
- Find the earliest starting time and latest starting time for each activity.
- If the plant nursery experiences an 8 hour delay in the supply of the plants for the project, how will this affect the earliest starting time for the last vertex?



Getty Images Plus / iStock / Karen Massier

- 14** The various activities and associated times required for the task of putting on a local art exhibition are shown in this table.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Invite artists to submit photographs of work	10	None
<i>B</i>	Select paintings for exhibition	5	<i>A</i>
<i>C</i>	Book hall	1	None
<i>D</i>	Have catalogue printed	30	<i>B, C</i>
<i>E</i>	Invite local celebrity to open exhibition	3	<i>C</i>
<i>F</i>	Arrange catering for opening of exhibition	2	<i>C</i>
<i>G</i>	Arrange rosters for volunteers to manage exhibition	1	<i>C</i>
<i>H</i>	Set up hall with display screens for paintings	2	<i>C</i>
<i>I</i>	Collect paintings	3	<i>B, C</i>
<i>J</i>	Mount paintings	2	<i>H, I</i>
<i>K</i>	Hold exhibition	10	<i>D, E, F, G, J</i>
<i>L</i>	Return unsold paintings to artists	5	<i>K</i>
<i>M</i>	Pay artists for work sold	6	<i>K</i>

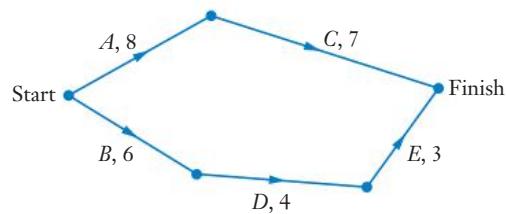
- Draw a project network for this project.
- Find the earliest starting time and latest starting time for each activity.
- The organisers have set aside 8 weeks to run the project. Will this be enough time?

10.03 Critical paths



For most projects there are a number of different possible paths from the start to the finish. Some are shorter than others. You usually want to know which path is the longest because that path will determine the time taken to complete the project. The path with the longest time is called the **critical path**.

This project network shows five activities (*A, B, C, D* and *E*) and the time in hours required to complete each activity. There are two paths from the start to the finish of the project.



The time take for each path is:

$$AC = 8 + 7 = 15$$

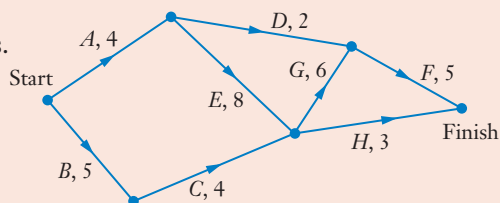
$$BDE = 6 + 4 + 3 = 13$$

This means that path *AC* is the critical path because it takes longer than path *BDE* and the time required to complete the project is 15 hours.

EXAMPLE 6

The project network for a particular task is shown here. The times are shown in days.

- List all possible paths and the time taken to complete each one.
- Identify the critical path and hence the shortest time in which the project can be completed.



Solution

- List each path from Start to Finish.

ADF

AEGF

AEH

BCGF

BCH

Calculate the time taken for each path.

$$ADF = 4 + 2 + 5 = 11 \text{ days}$$

$$AEGF = 4 + 8 + 6 + 5 = 23 \text{ days}$$

$$AEH = 4 + 8 + 3 = 15 \text{ days}$$

$$BCGF = 5 + 4 + 6 + 5 = 20 \text{ days}$$

$$BCH = 5 + 4 + 3 = 12 \text{ days}$$

- The critical path is the path with the longest time.

The critical path = *AEGF*

The shortest time required to complete the project is the same as the longest time for any path, i.e. the critical path.

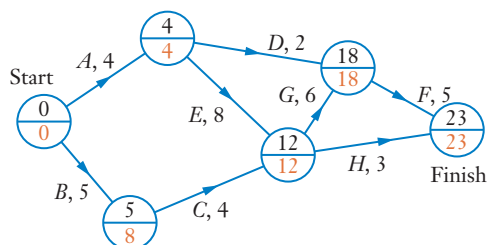
The shortest time to complete this project is 23 days.

When you performed forward and backward scanning in the previous section, you would have noticed that for some nodes, the EST and LST are the same. The EST and LST values for the network in the previous example are shown here.

The activities with nodes where $EST = LST$ are *A*, *E*, *G* and *F*.

This is also the critical path for the network.

This fact is useful when finding the **critical path** on more complex networks.



Critical path

For any network, the **critical path** is the path with the longest total time.

For a critical path, the EST and LST values will be equal at each vertex.

You can combine the forward and backward scanning processes with your knowledge of the critical path to determine a process for identifying the critical path for a network.

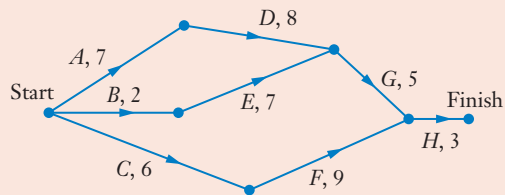
Identifying the critical path for a network

- 1 Use forward scanning to determine the EST for each activity.
- 2 Use backward scanning to determine the LST for each activity.
- 3 Identify activities for which $EST = LST$.
- 4 The activities for which $EST = LST$ form the critical path.

EXAMPLE 7

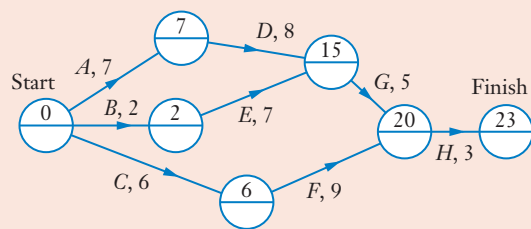
This project network shows a number of activities and the time taken to complete them in hours.

- a Find the EST and LST for each activity.
- b Identify the critical path.
- c Determine the shortest time taken to complete the project.

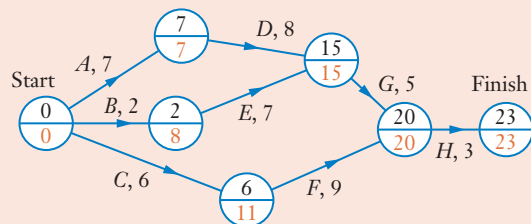


Solution

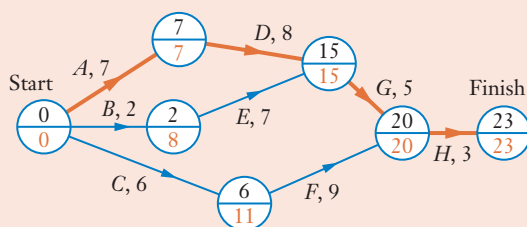
- a Use the forward scanning process to determine the EST for each activity.



Use the backward scanning process to determine the LST for each activity.



- b** Identify activities where $EST = LST$.



Identify the critical path.

Critical path = $ADGH$

- c** State the time taken for the critical path.

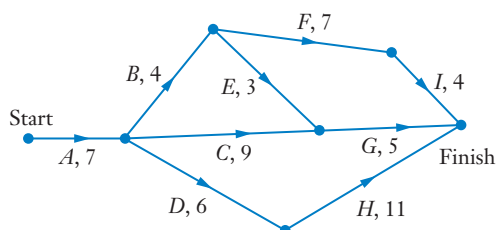
Critical path time = 23 hours

The shortest time for the project is the same as the time for the critical path.

The shortest time to complete this project is 23 hours.

Exercise 10.03 Critical paths

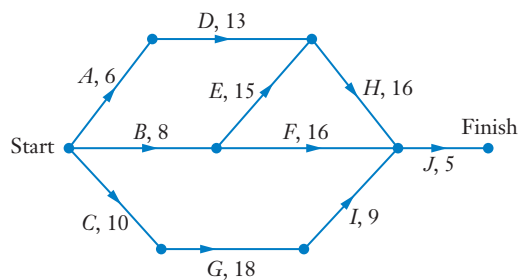
Use the project network to answer questions 1–2. The times given are in days.



- 1 The length of path $ABEG$ is:
A 19 days **B** 12 days **C** 21 days **D** 24 days **E** 22 days
- 2 The path with the longest duration is:
A $ABFI$ **B** ACG **C** $ABEG$ **D** ADH **E** $ABEFI$

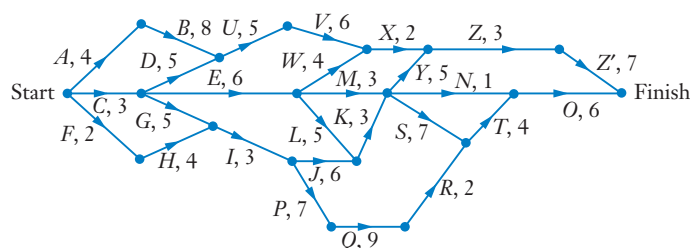
Example
6

- 3 The project network for a particular task is shown here. The times are shown in hours.
 - a** List all possible paths and the time taken to complete each one.
 - b** Identify the critical path and hence the shortest time in which the project can be completed.

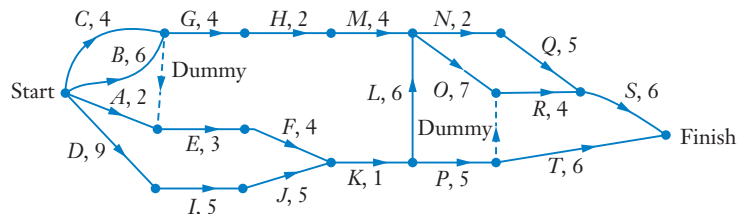


- 7** For each project network, find the critical path and shortest time taken to complete the project.

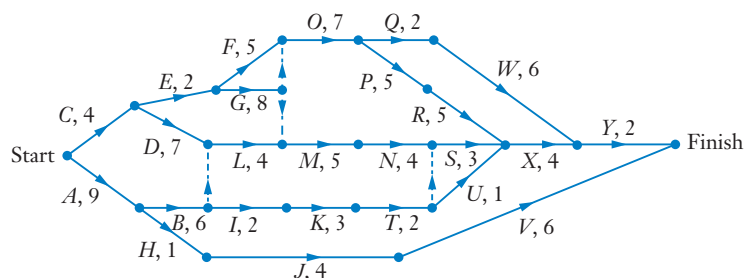
a



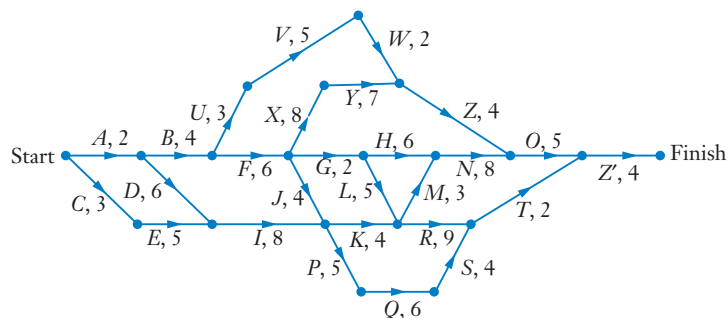
b



c



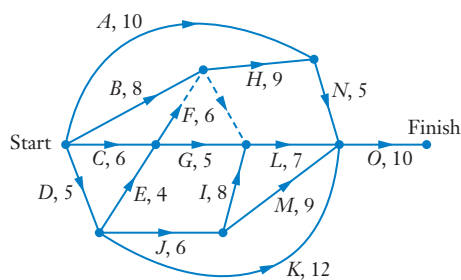
d



- 8** The activities that make up a project and the time taken to complete each in hours is shown on the right.

a Determine the critical path and the shortest time taken to complete the project.

b Additional resources are allocated to the project so that task C is completed in 2 hours, task F in 3 hours and task H in 4 hours. Determine the new critical path and shortest completion time for the project.





Critical paths
and activity
float times

10.04 Float times

The activities on the critical path are called **critical activities**. The various activities in a project are often subject to delays. You have seen that delays in critical activities will delay completion of a project. However, it is possible to experience a delay in non-critical activities without affecting the time to complete a project. This is because non-critical activities have an amount of **float time** or **slack time**.

Float time

The float time (or slack time) of an activity is the maximum delay possible in starting the activity that does not affect the time of completion of the project. For critical steps, the float time is zero.

For this project network, the activity times are measured in hours and the forward scan and backward scans have been done.

Examine activity *E*.

The earliest starting time at the start of *E* is 3.

The latest starting time at the end of *E* is 12.

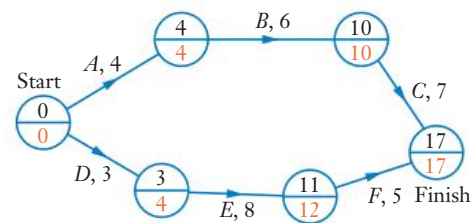
The activity time for *E* is 8.

The float time for an activity is the maximum delay possible that does not affect the completion time for the project. It is the difference between the latest starting time at the end of an activity and the earliest starting time at the start of an activity *less* the activity time.

Float time for *E* = $LST_{End} - EST_{Start} - \text{activity time}$

$$= 12 - 3 - 8$$

$$= 1 \text{ hour}$$



Float time

The float time for an activity = $LST_{End} - EST_{Start} - \text{activity time}$.

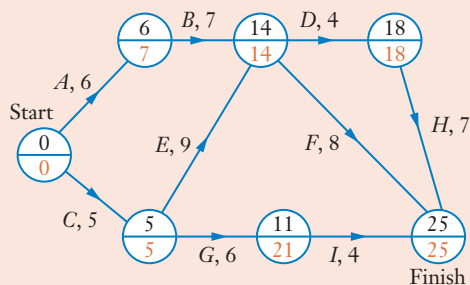
The critical path has nodes where float time = 0.

EXAMPLE 8

The forward scan and backward scan for this project network have been completed. The activity times are shown in days.

Determine:

- the critical path
- the float time for D
- the float time for G
- the float time for A
- the float time for I



Solution

- Identify nodes for which the difference between LST and EST is 0.

Critical path = $CEDH$

- Activities on the critical path have float times of 0. D is part of the critical path.

Float time for $D = 0$ days

- State the parameters for G .

$$LST_{\text{End}} = 21$$

$$EST_{\text{Start}} = 5$$

$$\text{Activity time} = 6$$

Calculate the float time for G .

Float time for G

$$= LST_{\text{End}} - EST_{\text{Start}} - \text{activity time}$$

$$= 21 - 5 - 6$$

$$= 10$$

- State the parameters for A .

$$LST_{\text{End}} = 7$$

$$EST_{\text{Start}} = 0$$

$$\text{Activity time} = 6$$

Calculate the float time for A .

Float time for A

$$= LST_{\text{End}} - EST_{\text{Start}} - \text{activity time}$$

$$= 7 - 0 - 6$$

$$= 1 \text{ day}$$

- e State the parameters for *I*.

$$LST_{\text{End}} = 25$$

$$EST_{\text{Start}} = 11$$

$$\text{Activity time} = 4 \text{ days}$$

Calculate the float time for *I*.

Float time for *I*

$$= LST_{\text{End}} - EST_{\text{Start}} - \text{activity time}$$

$$= 25 - 11 - 4$$

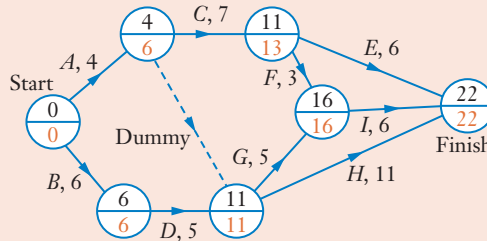
$$= 10 \text{ days}$$

Knowing the float times for the activities in a project allows a project manager to better organise the activities in the project. For example, it may save money to delay non-critical activities. Using float times, it is also possible for a project manager to determine where any additional resources should be directed so that they produce the greatest effect.

EXAMPLE 9

The activity times in the project network shown here are in hours.

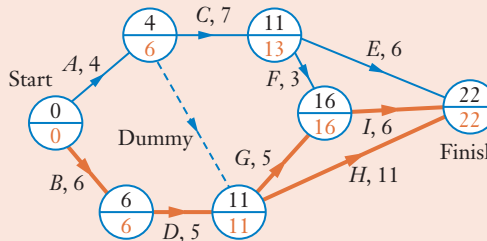
- Calculate the float time for each of the non-critical activities.
- If *G* is delayed by 3 hours, what effect does this have on the duration of the project and the critical activities?
- Extra resources can be used to shorten the duration of one of the activities *B*, *F* or *I* (on the original network) by 1 hour. Which of these activities should be shortened, and why?



Solution

- Identify the critical path by finding nodes for which the difference between LST and EST is 0.

There are two critical paths.



BDGI and *BDH* are both critical paths.

Non-critical activities are not part of a critical path. Identify the non-critical activities.

Use the formula to calculate the float times for the non-critical activities.

b Activity *G* is part of a critical path. Any delay in a critical activity will delay the project.

c *F* is a non-critical activity.

B is a critical activity.

I is a critical activity.

State the conclusion.

Activities *A*, *C*, *F* and *E* are non-critical.

Float time for *A* = $6 - 0 - 4 = 2$ hours

Float time for *C* = $13 - 4 - 7 = 2$ hours

Float time for *E* = $22 - 11 - 6 = 5$ hours

Float time for *F* = $16 - 11 - 3 = 2$ hours

A delay of 3 hours in activity *G* will cause a delay of 3 hours in the project completion time.

Reducing the duration of activity *F* will not shorten the project.

Reducing the duration of activity *B* by 1 hour will shorten the project by 1 hour.

Even though *I* is a critical activity, there are two critical paths (*BDGI* and *BDH*), so reducing *I* would not have any effect unless *H* was reduced as well.

Shorten the duration of *B* because this will reduce the duration of the project.

Exercise 10.04 Float times

Use the project activity table shown on the right to answer questions 1–2.

1 If there is a delay of 3 minutes in activity *C*, the time taken to complete the project is:

- A** 24 min **B** 44 min
C 20 min **D** 22 min
E 18 min

2 If the duration of activity *D* is reduced by 3 minutes, the time taken to complete the project is:

- A** 41 min **B** 27 min **C** 19 min **D** 21 min **E** 24 min

Activity	Time (min)	Prerequisites
<i>A</i>	8	None
<i>B</i>	10	None
<i>C</i>	12	<i>A</i>
<i>D</i>	6	<i>B</i>
<i>E</i>	8	<i>C</i> , <i>E</i>

-

B 12 min

D 4 min

Figure 1: A directed graph with 5 nodes and 6 edges. The nodes are labeled with their initial and final values. The edges are labeled with their names and weights. The graph shows a sequence of operations: A (0 to 6, weight 6), B (0 to 10, weight 9), C (0 to 22, weight 15), D (6 to 11, weight 5), E (6 to 10, weight 4), and F (11 to 22, weight 9). The final values are shown in orange.

- B** *AEG*

D *AEDF*

- B** A and B

D A , E and G

- B** 15 weeks

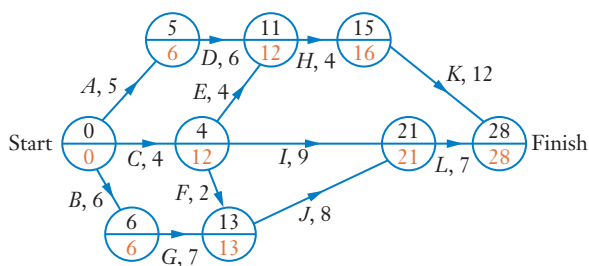
D 22 weeks

- B** 18 weeks

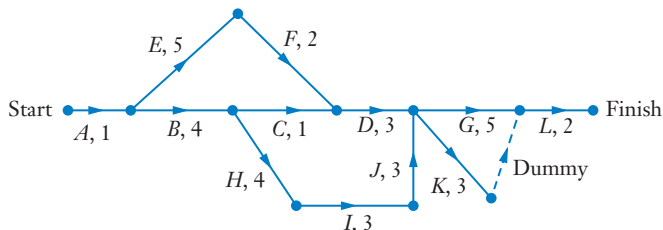
D 22 weeks

- 8** This project network shows project activities and associated times in days.

- a** Identify the critical path.
b Calculate the float time for:
 i *K* ii *D*
 iii *C* iv *I*



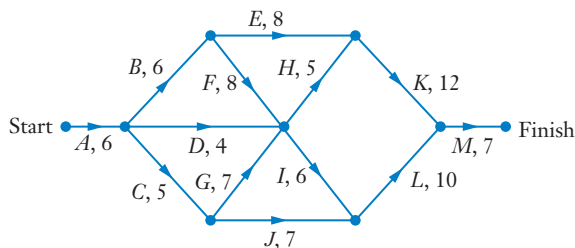
- 9** The project network below shows project activities and associated times in hours.



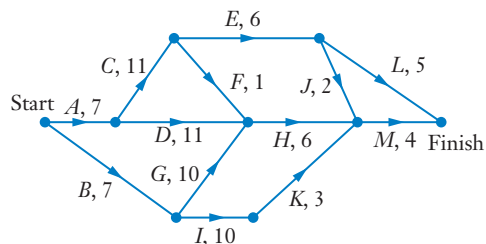
- a** Complete a forward and backward scan.
b Determine the critical path.
c Calculate the float time for:
 i *D* ii *F* iii *C* iv *E*

- 10** Use float times to find the critical path of each of the following project networks.

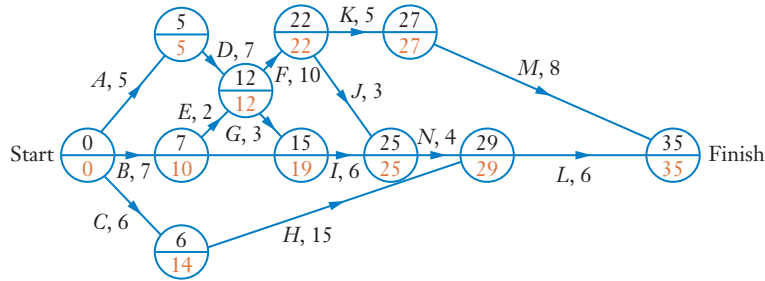
a



b

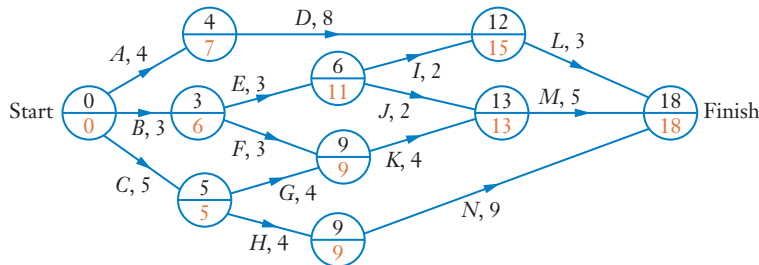


11 The activity times in this project network are shown in hours.



- Calculate the float time for each of the non-critical activities.
- If *F* is delayed by 5 hours, what effect does this have on the duration of the project and the critical activities?
- If *I* is delayed by 3 hours, what effect does this have on the duration of the project and the critical activities?
- Extra resources can be used to shorten the duration of one of the activities *H*, *F* or *M* (on the original network) by 1 hour. Which of these activities should be shortened, and why?

12 The activity times in this project network are shown in weeks.



- Calculate the float time for each of the non-critical activities.
- D* is delayed by 5 weeks. Calculate the critical path and the shortest completion time for the project.
- E* is delayed by 6 weeks. Calculate the critical path and the shortest completion time for the project.
- Additional resources are provided for the project. Due to scheduling, only one of the activities *C*, *J* or *K* (on the original network) can be shortened by 1 week. Which of these activities should be shortened, and why?

Problem solving

- 13** The activity table for a project is shown on the right.
- a** Use the activity table to construct a project network.
 - b** Find the critical path and the shortest completion time for the project.
 - c** Which activities can experience a small delay without affecting the completion time of the project?
 - d** Due to scheduling issues, activity *G* is delayed by 2 days and activity *J* is able to be completed in 2 less days. What is the new completion time for the project?

Activity	Time (days)	Prerequisites
<i>A</i>	7	None
<i>B</i>	2	None
<i>C</i>	15	None
<i>D</i>	10	<i>A, B</i>
<i>E</i>	10	<i>A, B</i>
<i>F</i>	5	<i>D</i>
<i>G</i>	8	<i>E</i>
<i>H</i>	5	<i>D</i>
<i>I</i>	2	<i>H, G</i>
<i>J</i>	8	<i>F</i>
<i>K</i>	2	<i>C, I</i>
<i>L</i>	3	<i>K</i>
<i>M</i>	4	<i>J, L</i>

10.05 Applications of project networks

Critical path analysis helps project managers plan the implementation of complex projects involving many tasks so that they are completed in the minimum amount of time and at the minimum cost. Critical path analysis also allow project managers to understand how changes to times required for individual project activities affect the completion time for a project.

EXAMPLE 10

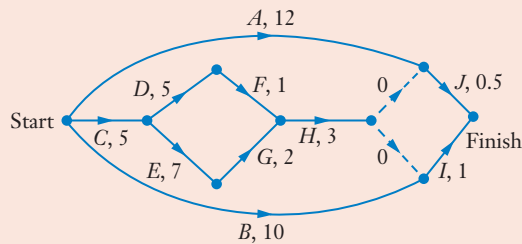
The activities involved in making a dressing table are shown.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Order mirror	12	None
<i>B</i>	Order drawer handles	10	None
<i>C</i>	Make frame	5	None
<i>D</i>	Make top	5	<i>C</i>
<i>E</i>	Make drawers	7	<i>C</i>
<i>F</i>	Fit top	1	<i>D</i>
<i>G</i>	Fit drawers	2	<i>E</i>
<i>H</i>	Finish and polish surfaces	3	<i>F, G</i>
<i>I</i>	Fit drawer handles	1	<i>B, H</i>
<i>J</i>	Fit mirror	0.5	<i>A, H</i>

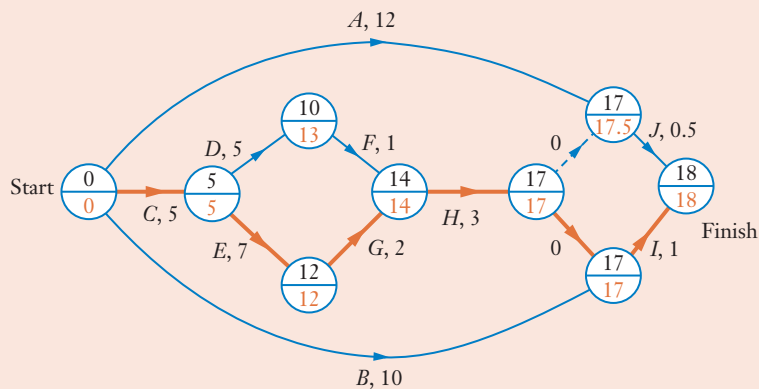
- a** Construct a project network for this task.
- b** Determine the critical path and the shortest time to complete the project.
- c** The delivery of the drawer handles is delayed. For how long can the delivery be delayed before it affects the completion time for the project?
- d** The weather is unsuitable for the polishing of surfaces to be undertaken. If polishing is delayed by 2 days, how does this affect the project completion time?

Solution

- a** Use the process described in Section 10.1 of this chapter to construct the project network. Start with the activities that do not have prerequisites, i.e. *A*, *B* and *C*. *H* is a prerequisite for both *I* and *J*, but two edges can't be connected by multiple edges, so dummy activities need to be added.



- b** Perform the forward and backward scan and identify the critical path as described previously.



State the critical path.

Critical path = *CEGHI*

State the shortest time to complete the project.

The shortest time to complete the project is 18 days.

- c** The ordering (and delivery) of the drawer handles (activity *B*) is a non-critical activity. Calculate the float time.

State the result.

- d** The polishing of the surfaces (activity *H*) is a critical activity.

Float time for *B*

$$= LST_{\text{End}} - EST_{\text{Start}} - \text{activity time}$$

$$= 17 - 0 - 10$$

$$= 7 \text{ days}$$

The delivery of the drawer handles can be delayed by up to 7 days before the completion time for the project is affected.

A delay of 2 days for surface polishing will increase the project completion time to 20 days.

Exercise 10.05 Applications of project networks

Example
10

- 1** The table below describes the steps involved in producing a video clip for YouTube.

Activity	Description	Duration (days)	Predecessor
<i>A</i>	Decide on the theme and content	2	None
<i>B</i>	Prepare storyboard and write script	10	<i>A</i>
<i>C</i>	Obtain equipment and props	5	<i>A</i>
<i>D</i>	Cast and hire actors	7	<i>B</i>
<i>E</i>	Hire production workers	3	<i>B, C</i>
<i>F</i>	Prepare scenes and wardrobe	5	<i>E</i>
<i>G</i>	Record scenes	12	<i>D, F</i>
<i>H</i>	Edit recorded scenes	8	<i>G</i>
<i>I</i>	Combine scenes and finalise video	2	<i>H</i>
<i>J</i>	Prepare website and upload video	1	<i>I</i>

- a** Construct a project network for this task.
- b** Determine the critical path and the shortest time to complete the project.
- c** How will it affect the completion time for the project if it takes 3 extra days to hire the production workers? Explain your answer.
- d** It is decided that the editing of scenes and finalisation of the video will be done as a single activity that will be completed in 6 days. How does this affect the project completion time?

Problem solving

2 A rock-fill dam is being constructed. The activity table for the project is shown.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Preliminary works	85	None
<i>B</i>	River diversion	150	<i>A</i>
<i>C</i>	Site excavation	200	<i>B</i>
<i>D</i>	Bank construction	600	<i>C</i>
<i>E</i>	Permanent roads	50	<i>B</i>
<i>F</i>	Bank facing	200	<i>D</i>
<i>G</i>	Spillway construction	250	<i>D</i>
<i>H</i>	Inlet tower construction	100	<i>D</i>
<i>I</i>	Outlet valve house	100	<i>D</i>
<i>J</i>	Outlet pipeline	60	<i>I</i>
<i>K</i>	Remove river diversion	50	<i>E, F, G, H, J</i>
<i>L</i>	Clean up and move out	40	<i>K</i>

- a** Use a project network to find the critical path and the total time taken.
- b** What is the effect of a delay of 60 days in permanent road construction?
- c** What is the effect of a delay of 60 days in facing the bank?
- d** What is the effect of a delay of 60 days in spillway construction?

3 A local publisher decides to print calendars that advertise local businesses. The publisher will arrange the printing and distribution of the calendars and will charge businesses to advertise on the calendar. The activity table for the calendar's production is given below.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Publicise calendar	30	None
<i>B</i>	Sell advertising space	40	None
<i>C</i>	Design calendar layout	30	None
<i>D</i>	Typeset calendar with advertisements	10	<i>C</i>
<i>E</i>	Let contract for delivery of calendars	10	None
<i>F</i>	Print calendars	30	<i>B, D</i>
<i>G</i>	Send calendars to distributors	5	<i>F</i>
<i>H</i>	Send 1 copy to each advertiser	1	<i>F</i>

- a** Use a project network to find the critical path and the total time taken.
- b** What is the effect of a delay of 20 days in publicising the calendar?
- c** What is the effect of a delay of 20 days in selling advertising space?
- d** What is the effect of a delay of 20 days in designing the calendar layout?

4 Draw a project network for making a lemon meringue pie.

Code	Activity	Time (min)	Prerequisites
<i>A</i>	Fold 5 tbsp lemon juice, zest of 1 lemon and 125 g butter into egg yolk mixture	3	<i>L</i>
<i>B</i>	Roll pastry out	3	<i>F</i>
<i>C</i>	Cool pie and serve	20	<i>H</i>
<i>D</i>	Place 4 tbsp sugar and 4 tbsp cornflour into saucepan and blend in 1 cup of hot water	1	None
<i>E</i>	Heat and stir in saucepan on low until mixture thickens	5	<i>D</i>
<i>F</i>	Make pastry	15	None
<i>G</i>	Put baking paper and rice on top of pastry	2	<i>K</i>
<i>H</i>	Top pie with meringue and place in oven pre-heated to 200°C. Turn off oven and bake for 10 minutes	11	<i>N, P</i>
<i>I</i>	Whip egg white until fluffy	2	<i>J</i>
<i>J</i>	Separate 2 eggs	1	None
<i>K</i>	Chill pastry before baking	10	<i>O</i>
<i>L</i>	Fold egg yolks into thickened mixture and heat for 1 minute	2	<i>E, J</i>
<i>M</i>	Bake pastry in oven pre-heated to 180°C	12	<i>G, K</i>
<i>N</i>	Gradually add cup of sugar to egg whites and whip to stiff peaks to make meringue mixture	2	<i>I</i>
<i>O</i>	Put pastry into flan dish	1	<i>B</i>
<i>P</i>	Remove paper/rice and pour lemon mixture into baked pie shell	2	<i>A, M</i>



Getty Images/OJO Images/Tom Meriton

- 5** A family has to move from Gladstone to Mt Isa because the mother has a new job as a project manager at the mine. The activity table for the move is shown below.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Engage real estate agent to sell house in Gladstone	1	None
<i>B</i>	Sell Gladstone house	20	<i>A</i>
<i>C</i>	Find a suitable house in Mt Isa	10	None
<i>D</i>	Negotiate purchase of Mt Isa house	5	<i>C</i>
<i>E</i>	Buy house at Mt Isa	1	<i>B, D</i>
<i>F</i>	Arrange temporary accommodation in Mt Isa until new house is ready	3	None
<i>G</i>	Cancel deliveries to Gladstone house	2	<i>F</i>
<i>H</i>	Get estimates from movers	5	None
<i>I</i>	Sign moving and storage contract	1	<i>H</i>
<i>J</i>	Find school for children in Mt Isa	2	None
<i>K</i>	Travel with family to Mt Isa	1	<i>G</i>
<i>L</i>	Have furniture moved/stored	1	<i>I, K</i>
<i>M</i>	Move into temporary accommodation	1	<i>K</i>
<i>N</i>	Move to new house	5	<i>E, L, M</i>
<i>O</i>	Send change-of-address notes to relevant people	1	<i>N</i>
<i>P</i>	Change insurances to new house	1	<i>N</i>
<i>Q</i>	Arrange deliveries to new house	1	<i>N</i>

- Use a project network to find the critical path and the total time taken.
- What is the effect of a delay of 10 days in selling the house in Gladstone?
- What is the effect of a delay of 10 days in getting estimates from the movers?
- What is the effect of a 10-day delay in finding a suitable house in Mt Isa?



Getty Images/kali9

- 6** A microwave transmission station is to be built. The activity table is shown below.

Code	Activity	Time (weeks)	Prerequisites
<i>A</i>	Purchase land	10	None
<i>B</i>	Obtain planning permission	6	<i>A</i>
<i>C</i>	Erect building	12	<i>B</i>
<i>D</i>	Design transmitter	8	None
<i>E</i>	Build transmitter	10	<i>B, D</i>
<i>F</i>	Test and correct transmitter	2	<i>E</i>
<i>G</i>	Design aerial	6	None
<i>H</i>	Build aerial	4	<i>B, G</i>
<i>I</i>	Test and correct aerial	1	<i>H</i>
<i>J</i>	Install transmitter on building	2	<i>C, F</i>
<i>K</i>	Install aerial on building	4	<i>I, J</i>
<i>L</i>	Open and begin transmission	1	<i>K</i>

- Use a project network to find the critical path and the total time taken.
 - What is the effect of a delay of 3 weeks in building the transmitter?
 - What is the effect of a delay of 3 weeks in building the aerial?
 - What is the effect of a 3 week delay in installing the transmitter?
- 7** A sports ground is being constructed in a new suburb. The activity table for the construction is shown below.

Code	Activity	Time (weeks)	Prerequisites
<i>A</i>	Design grounds layout	6	None
<i>B</i>	Clear land	4	<i>A</i>
<i>C</i>	Roughly level area	2	<i>B</i>
<i>D</i>	Design drainage of grounds	3	<i>A</i>
<i>E</i>	Install drainage	2	<i>C, D</i>
<i>F</i>	Cover with topsoil	1	<i>E</i>
<i>G</i>	Plant turf	6	<i>F</i>
<i>H</i>	Fence the grounds	2	<i>B</i>
<i>I</i>	Design clubhouse	4	<i>A</i>
<i>J</i>	Build clubhouse	10	<i>C, I</i>

- Use a project network to find the critical path and the total time taken.
- What is the effect of a delay of 3 weeks in levelling the area?
- What is the effect of a delay of 3 weeks in installing drainage?
- What is the effect of a delay of 3 weeks in designing the clubhouse?



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- 8 a** Draw a project network for assembling and using a ‘flat-pack’ barbecue.

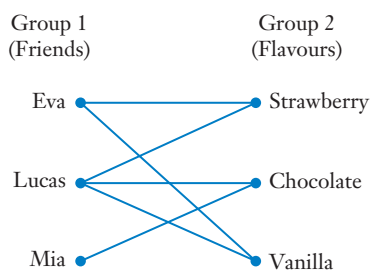
Code	Activity	Time (min)	Prerequisites
<i>A</i>	Read the instructions	5	None
<i>B</i>	Unpack the pieces from the box	6	None
<i>C</i>	Screw the metal parts together	10	<i>A, B</i>
<i>D</i>	Put the wheels on	5	<i>C</i>
<i>E</i>	Screw the wooden parts to the frame	10	<i>D</i>
<i>F</i>	Place the heat beads in the tray	3	<i>E</i>
<i>G</i>	Fill the gas bottle	20	<i>E</i>
<i>H</i>	Connect the gas bottle	2	<i>G</i>
<i>I</i>	Light the barbecue and heat the beads	15	<i>F, H</i>
<i>J</i>	Prepare the food	20	None
<i>K</i>	Cook the food	15	<i>I, J</i>

- b** Find the greatest delay possible in each activity below without affecting the completion time of the project.

- i** Reading the instructions
- ii** Putting the wheels on
- iii** Placing the heat beads in the tray
- iv** Preparing the food

10.06 Bipartite graphs

Three friends, Eva, Lucas and Mia, decide to buy some gelato. Eva likes strawberry and vanilla, Lucas likes strawberry, chocolate and vanilla, while Mia only likes chocolate. This information can be shown as a graph.



This graph consists of 2 groups of distinct vertices. Group 1 consists of the friends and group 2 consists of the flavours that each friend prefers. You can see that the vertices that are in the same group have no edges between them.

Bipartite graphs

A **bipartite graph** is one in which the vertices can be put into 2 distinct (disjoint) groups so that the only edges are between those 2 groups and there are no edges between vertices within the same group.

Bipartite graphs are useful when you are trying to find the best way to match the elements of 2 distinct groups.

EXAMPLE 11

A matchmaking website receives applications from 5 men (Alex, Ben, Carter, Dylan and Eli) and 5 women (Georgia, Hannah, Ivy, Julia and Kate). After filling out their details, it is found that:

- Alex is compatible with Georgia and Julia
- Ben is compatible with Ivy
- Carter is compatible with all 5 women
- Dylan is not compatible with any of the women
- Eli is compatible with Julia

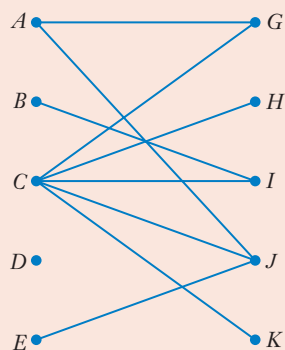
Draw a bipartite graph to represent this information.

Solution

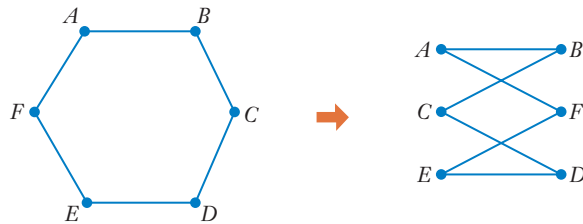
Use the first letters of the names to represent the vertices of the graph. Place the vertices for the men one under each other on the one side and the vertices for the women on the other side.

$A \bullet$	$\bullet G$
$B \bullet$	$\bullet H$
$C \bullet$	$\bullet I$
$D \bullet$	$\bullet J$
$E \bullet$	$\bullet K$

Draw in the edges from the men to the women according to their compatibilities.



It is not always obvious if a graph is bipartite. It is possible to redraw a graph so that links are preserved but the vertices are in 2 distinct groups. For example, examine the graph on the left below. Just by looking at the graph it isn't clear whether or not it is bipartite.



Vertex F can be moved to the right with its links intact and vertex C can be moved to the left with its links intact to form the graph on the right. The graph on the right is clearly bipartite.

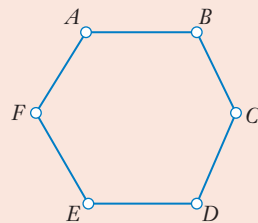
It is also possible to use colouring to determine if a graph is bipartite. For a bipartite graph, it is possible to use 2 colours such that no 2 adjacent vertices share the same colour.

This can be checked using the following vertex colouring algorithm.

- 1 Colour any one of the vertices red.
- 2 Identify all uncoloured vertices that are adjacent to a red vertex. Colour them blue.
- 3 Identify all uncoloured vertices that are adjacent to a blue vertex. Colour them red.
- 4 Repeat steps 2 and 3 until all the vertices are coloured red or blue.
- 5 If there are any 2 adjacent vertices of the same colour, then your graph is not bipartite; otherwise it is bipartite.
- 6 If the graph is bipartite, the colouring algorithm gives the 2 required sets of points (one red and one blue).

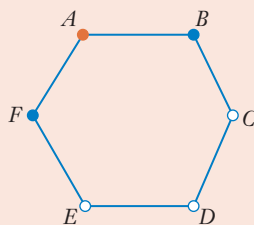
EXAMPLE 12

- a Use the vertex-colouring algorithm to determine if this graph is bipartite.
- b If the graph is bipartite, state the vertices in each group.



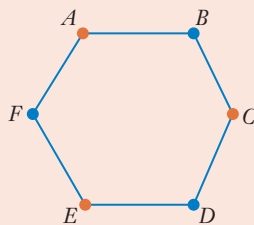
Solution

- a** Begin with any vertex, say A , and colour it red. Locate the adjacent vertices (B and F) and colour them blue.



Locate the vertices adjacent to the blue vertices (E and C) and colour them red.

Locate the vertices adjacent to the red vertices (D) and colour them blue.



State the result.

- b** The vertices in a group will share the same colour.

The graph is bipartite.

Group 1 vertices = A, C and E

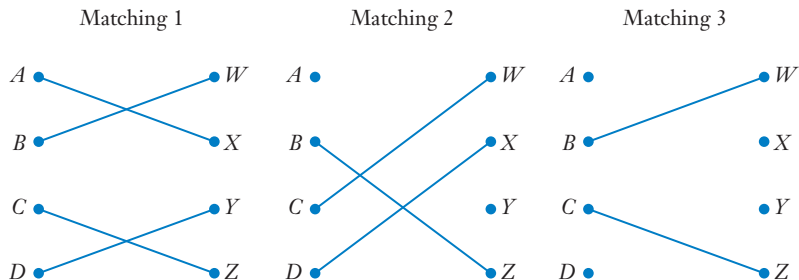
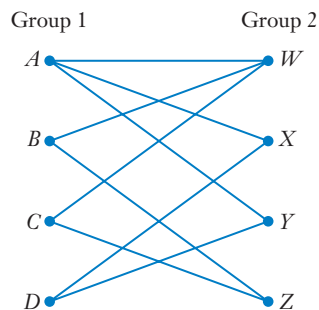
Group 2 vertices = B, D and F

Look at this bipartite graph.

A **matching** is a one-to-one pairing of some or all of the vertices in group 1 with the vertices in group 2.

One-to-one means that an element of group 1 may be paired with only one element of group 2, and an element of group 2 may be paired with only one element of group 1.

There are many possible matchings. Here are three.

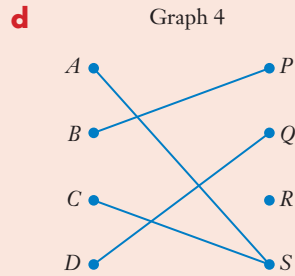
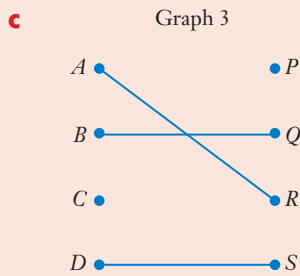
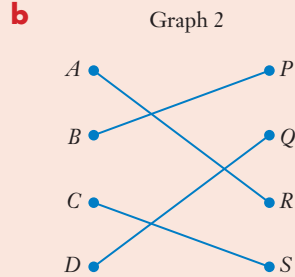
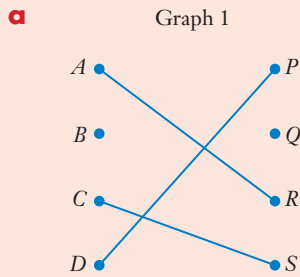
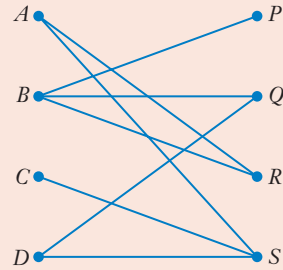


Matching 1 is called a **complete matching**, since every vertex in the first group is matched to a vertex in the second group. Matchings 2 and 3 are examples of matchings that are not complete.

EXAMPLE 13

Look at this bipartite graph.

Determine if each of the following is a matching for this graph. If it is a matching, classify it as a complete or incomplete matching.



Solution

- a** Compare graph 1 with the original graph.

State the conclusion.

There is no link between D and P on the original graph.

Graph 1 is not a matching for the original graph.

- b** Compare graph 2 with the original graph.

State the conclusion.

All the links in graph 2 are present in the original graph.

All 4 vertices in the left-hand group are linked to one and only one vertex in the right-hand group.

Graph 2 is a complete matching for the original graph.

- c** Compare graph 3 with the original graph.

State the conclusion.

All the links in graph 3 are present in the original graph.

Only 3 vertices in the left-hand group are linked to a unique vertex in the right-hand group.

Graph 3 is an incomplete matching for the original graph.

- d** Compare graph 4 with the original graph.

State the conclusion.

All the links in graph 4 are present in the original graph.

Each of the 4 vertices in the left-hand group is linked to a vertex in the right-hand group, but *A* and *C* are both linked to the same vertex, *S*.

Graph 4 is not a matching for the original graph.

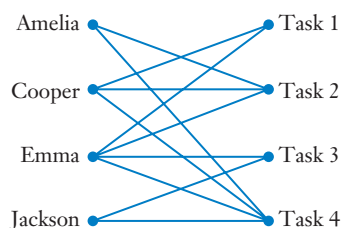
Exercise 10.06 Bipartite graphs

- 1** Four trainees are at different stages of their training.

The tasks that each trainee is qualified to perform in a production process are shown in the graph.

Their supervisor wants to assign each trainee a task for the next production run.

Which of the following tables shows an allocation that is not valid?



A

Amelia	Task 2
Cooper	Task 1
Emma	Task 3
Jackson	Task 4

B

Amelia	Task 4
Cooper	Task 2
Emma	Task 1
Jackson	Task 3

C

Amelia	Task 2
Cooper	Task 4
Emma	Task 1
Jackson	Task 3

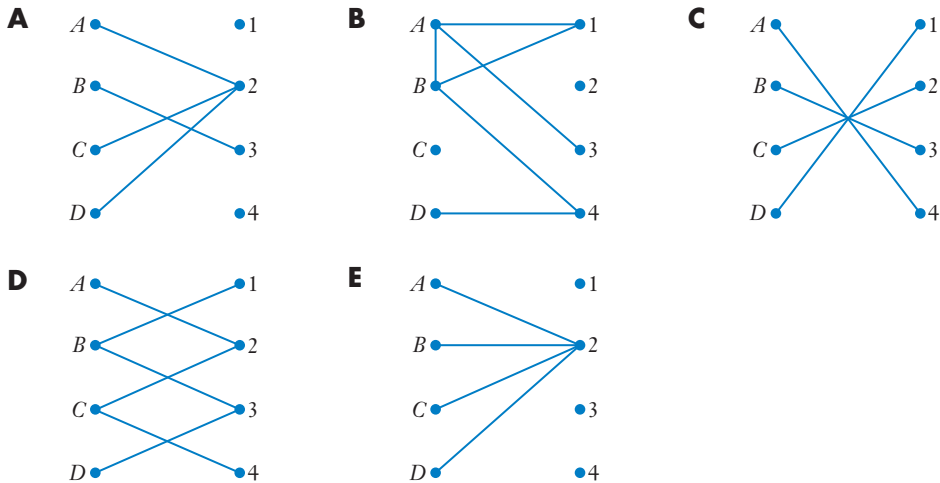
D

Amelia	Task 2
Cooper	Task 3
Emma	Task 1
Jackson	Task 4

E

Amelia	Task 4
Cooper	Task 1
Emma	Task 2
Jackson	Task 3

2 Which of these is not a bipartite graph?



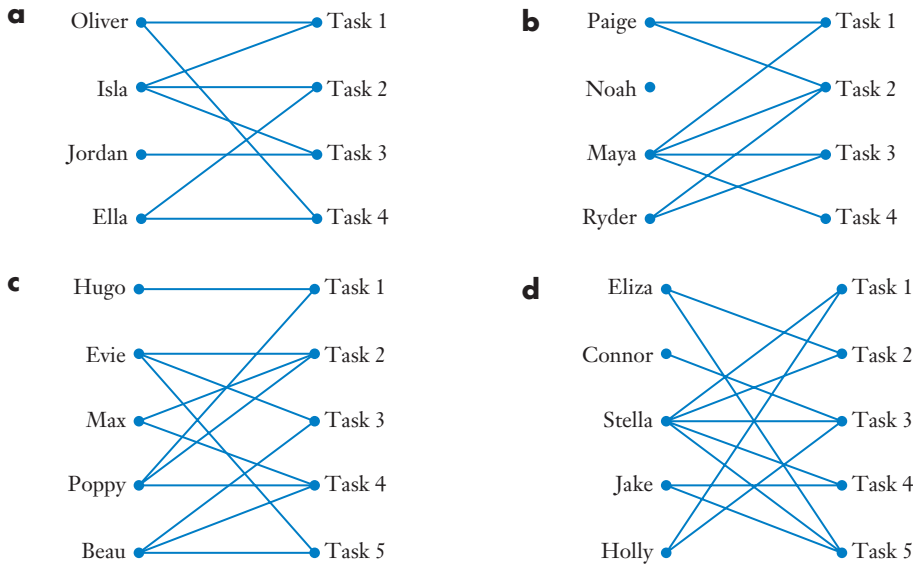
3 Five trainees, Chloe, Sam, Harry, Ryan and Zara, work on the production line of a car manufacturer. Their training covers the areas of instrumentation (I), brake systems (B), lighting (L), glasswork (G) and exterior finishing (E). Each apprentice is at a different stage of training and requires experience in the tasks as shown in the table.

Trainee	Task
Chloe	I, L, E
Sam	B, L, G
Harry	I, L
Ryan	E
Zara	G, B

Example
11

Draw a bipartite graph to represent this information.

4 Determine which tasks could be allocated to each person according to each bipartite graph.



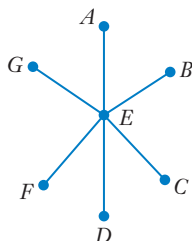
- 5 A dispatcher for a taxi company has 6 drivers and various jobs that each is suitably located to do. This information is shown in the table. Draw a bipartite graph to represent this information.

Driver	Job
A	2, 4, 6
B	1, 4
C	3, 4
D	6
E	4, 5
F	1, 3, 5

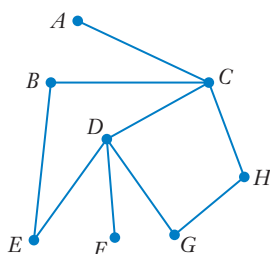
Example
12

- 6 Use the vertex-colouring algorithm to determine if each graph is bipartite. If the graph is bipartite, state the vertices in each group.

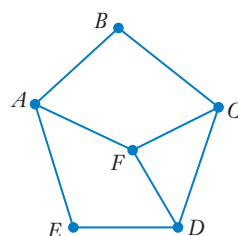
a



b

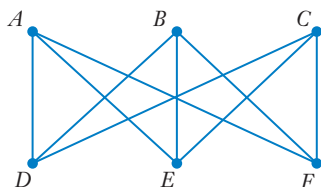


c

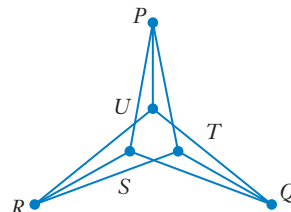


- 7 Determine if each graph is bipartite. If a graph is bipartite, state the vertices in each group.

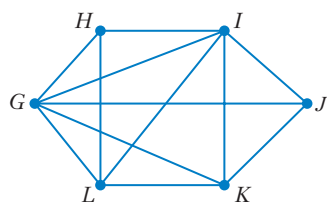
a



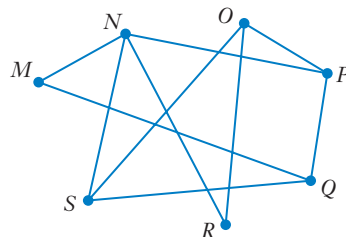
b



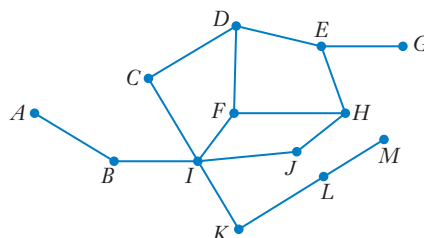
c



d

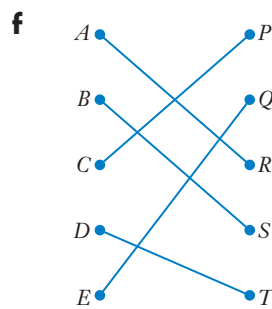
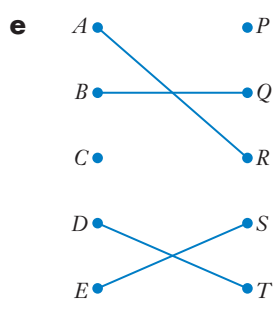
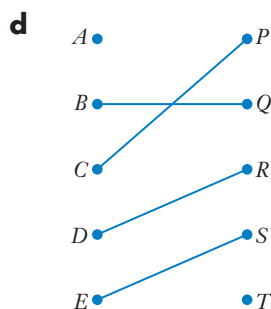
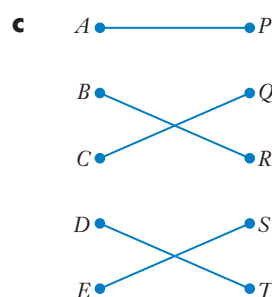
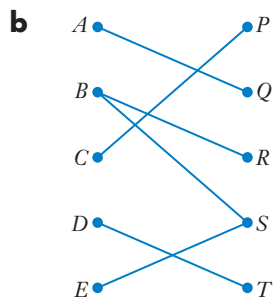
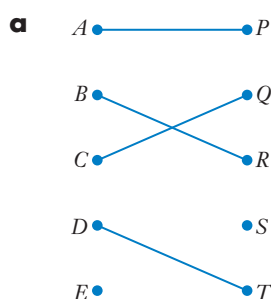
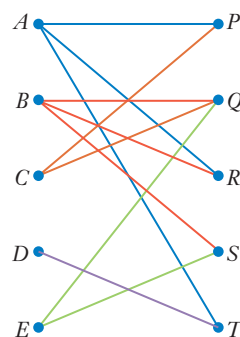


- 8 Determine if this graph is bipartite. If it is, identify the vertices in each group.

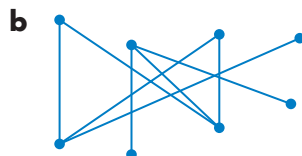


9 Look at this bipartite graph.

Determine if each diagram is a matching for the graph shown above. If it is a matching, classify it as a complete or incomplete matching.



10 Find a complete matching for each graph, if possible.



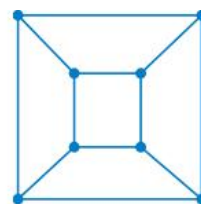
Problem solving

11 There are 6 applicants for 6 positions advertised by a local employer. The table indicates the positions for which each applicant is qualified. Each position can only accept one applicant and each applicant can only be appointed to one job.

- Draw a bipartite graph to represent this information.
- Work out an assignment of jobs to applicants that results in the greatest number of applicants being placed in positions.

Applicant	Position
A	2, 3
B	None
C	1, 4
D	3
E	3, 4
F	6

12 Draw at least 5 complete matchings for this graph.



Allocation
and the
Hungarian
algorithm

10.07 Optimum assignment problems

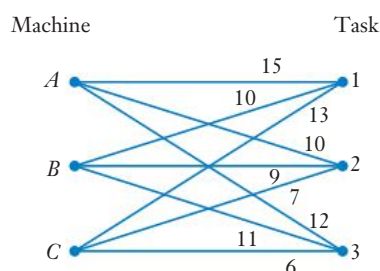
A situation where activities or tasks have to be matched to agents such as individuals and machines is called an **allocation** (or **assignment**) problem. A key feature of assignment problems is that one agent is assigned to *one and only one* task.

Consider the situation where a manufacturing plant has 3 machines capable of carrying out 3 different tasks. The time taken for each task to be performed by each machine is shown below.

- Machine *A* can perform task 1 in 15 min, task 2 in 10 min and task 3 in 12 min
- Machine *B* can perform task 1 in 10 min, task 2 in 9 min and task 3 in 11 min
- Machine *C* can perform task 1 in 13 min, task 2 in 7 min and task 3 in 6 min

This information can be represented in a bipartite graph as shown.

This graph is a bit cluttered with information.



It is also possible to show this information more clearly as a table.

	Task 1	Task 2	Task 3
<i>A</i>	15	10	12
<i>B</i>	10	9	11
<i>C</i>	13	7	6

An even clearer way to show this information is by using a matrix.

$$\begin{bmatrix} 15 & 10 & 12 \\ 10 & 9 & 11 \\ 13 & 7 & 6 \end{bmatrix}$$

A bipartite graph is useful for seeing which tasks can be assigned to individual machines, but in many cases you need to decide on the optimal way to allocate tasks. The **optimum allocation** could be the most efficient in terms of time, cost, energy consumption, etc. It depends on the situation. The simplest allocation problems can be solved by inspection – by examining the information.

EXAMPLE 14

A company uses 3 equally qualified contractors to complete different maintenance tasks. The costs for each task to be performed by the different contractors are shown below.

- Contractor *A* charges \$40 to perform task 1, \$70 for task 2 and \$30 for task 3
- Contractor *B* charges \$20 to perform task 1, \$80 for task 2 and \$40 for task 3
- Contractor *C* charges \$35 to perform task 1, \$65 for task 2 and \$35 for task 3

- Show this information as a table.
- Match the task to the contractors to minimise the cost.
- Find the minimum cost to complete the 3 tasks.

Solution

- Use a 4×4 table.

	Task 1	Task 2	Task 3
<i>A</i>	40	70	30
<i>B</i>	20	80	40
<i>C</i>	35	65	35

- Find the cheapest contractor to perform task 1.

Contractor *B* completes task 1.

This leaves contractors *A* and *C* available to perform tasks 2 and 3.

Contractor *C* completes task 2.

This leaves contractor *A* for task 3.

Contractor *A* completes task 3.

- Add the costs for all 3 tasks.

$$\begin{aligned}\text{Total cost} &= 20 + 65 + 30 \\ &= 115\end{aligned}$$

State the result.

The total cost to complete the three tasks is \$115.

The best approach for more complex allocation problems is to use matrices. By representing the information as a matrix, it is possible to use the method of **row and column reduction**.

Allocation using row and column reduction

- 1 Choose the smallest number in a row and subtract it from every element in the row. Repeat this process for each row.
- 2 Choose a column that does not have a zero value. Subtract the smallest number in the column from every element in the column. Repeat this process until there are no columns without a zero value.
- 3 Cover all the zeros using the smallest number of horizontal or vertical lines. The process is complete if the number of lines is equal to the number of rows.
- 4 The zeros in the reduced matrix represent the allocations. Draw a bipartite graph showing all the allocations.
- 5 Choose the allocations that produce the optimum total.

EXAMPLE 15

Four trainees A , B , C and D are available to perform 4 different tasks. Each task is in sequence, so task 1 must be completed before task 2 can commence and so on. Each trainee can only perform one task. The trainees have different skill sets and this means that each can perform the tasks in different periods of time (minutes) as shown in the table below.

	Task 1	Task 2	Task 3	Task 4
A	17	24	27	20
B	29	28	23	26
C	21	18	17	14
D	28	26	22	19

- a Convert the table to a matrix and then undertake row and column reductions.
- b Find the optimum allocation and show it as a bipartite graph.
- c Find the minimum time required for the trainees to perform the tasks.

Solution

- a Convert the table to a matrix.

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \left[\begin{array}{cccc} 17 & 24 & 27 & 20 \\ 29 & 28 & 23 & 26 \\ 21 & 18 & 17 & 14 \\ 28 & 26 & 22 & 19 \end{array} \right] \end{matrix}$$

Complete the row reductions.

Row 1: Smallest number is 17, so subtract 17 from each element in the row.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 10 & 3 \\ 29 & 28 & 23 & 26 \\ 21 & 18 & 17 & 14 \\ 28 & 26 & 22 & 19 \end{bmatrix}$$

Complete the row reductions for the remaining rows.

Smallest number for row 2 = 23.

Smallest number for row 3 = 14.

Smallest number for row 4 = 19.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 10 & 3 \\ 6 & 5 & 0 & 3 \\ 7 & 4 & 3 & 0 \\ 9 & 7 & 3 & 0 \end{bmatrix}$$

Column 2 requires reduction.

Smallest number is 4, so subtract 4 from all elements.

No other columns require column reduction.

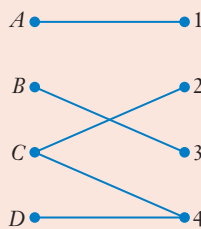
$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 10 & 3 \\ 6 & 1 & 0 & 3 \\ 7 & 0 & 3 & 0 \\ 9 & 3 & 3 & 0 \end{bmatrix}$$

Cover the zeros with the smallest number of lines.

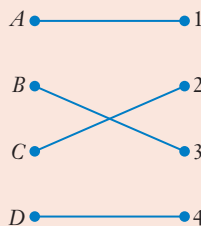
The number of lines is equal to the number of rows, so the reduction process is complete.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 10 & 3 \\ 6 & 1 & 0 & 3 \\ 7 & 0 & 3 & 0 \\ 9 & 3 & 3 & 0 \end{bmatrix}$$

- b** The zeros in the row and column reduced matrix represent the allocation of tasks to trainees. Draw a bipartite graph to show the connections.



Look at the tasks. They all have single connections except task 4. Task 4 is connected to trainees C and D, but D is only connected to task 3, so eliminate the connection between C and task 4.



State the allocation.

Trainee *A* must complete task 1.

Trainee *B* must complete task 3.

Trainee *C* must complete task 2.

Trainee *D* must complete task 4.

c Calculate the time taken.

$$\begin{aligned}\text{Time taken} &= 17 + 23 + 18 + 19 \\ &= 77\end{aligned}$$

State the result.

The minimum time required for the trainees to perform the tasks is 77 minutes.

There are cases where the row and column reduction method does not give the same number of lines covering the zeros as there are rows. The **Hungarian algorithm** may be used to find an optimum solution.

Hungarian algorithm

- 1 Perform row and column reductions and cover the zeros with the minimum number of horizontal or vertical lines.
- 2 If the number of lines is less than the number of rows, locate the smallest uncovered number and add it to every covered number. Add it twice if the number is covered twice.
- 3 Locate the smallest number and subtract it from all of the numbers in the matrix.
- 4 Cover all the zeros using the smallest number of horizontal or vertical lines. The process is complete if the number of lines is equal to the number of rows. If this is not the case, repeat steps 1–3 until a solution is found.
- 5 The zeros in the reduced matrix represent the allocations. Draw a bipartite graph showing all the allocations.
- 6 Choose the allocations that produce the optimum total.

EXAMPLE 16

Four courier drivers are available to pick up packages at 4 different locations. The distance (in km) each courier is located from the parcels is given in the table.

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
A	9	17	10	4
B	7	22	12	7
C	10	17	17	9
D	12	20	14	4



Shutterstock.com/Dima Sidelnikov

- Determine the allocation of couriers to parcels that minimises the distance travelled.
- Find the minimum distance that needs to be travelled to pick up all 4 parcels.

Solution

- Convert the table to a matrix.

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 9 & 17 & 10 & 4 \\ 7 & 22 & 12 & 7 \\ 10 & 17 & 17 & 9 \\ 12 & 20 & 14 & 4 \end{bmatrix} \end{matrix}$$

Complete the row reductions.

Smallest number for row 1 = 4

Smallest number for row 2 = 7

Smallest number for row 3 = 9

Smallest number for row 4 = 4

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 5 & 13 & 6 & 0 \\ 0 & 15 & 5 & 0 \\ 1 & 8 & 8 & 0 \\ 8 & 16 & 10 & 0 \end{bmatrix} \end{matrix}$$

Columns 2 and 3 require reduction.

Smallest number for column 2 = 8

Smallest number for column 3 = 5

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 5 & 5 & 1 & 0 \\ 0 & 7 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 8 & 8 & 5 & 0 \end{bmatrix} \end{matrix}$$

Cover the zeros with the minimum number of horizontal or vertical lines.

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 5 & 5 & 1 & 0 \\ 0 & 7 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 8 & 8 & 5 & 0 \end{bmatrix} \end{matrix}$$

The minimum number of lines is 3 and there are 4 rows, so continue with the Hungarian algorithm.

The smallest uncovered number is 1, so add 1 to all numbers covered once and add 2 to all numbers covered twice.

The smallest number is 1, so subtract it from all numbers in the matrix.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 1 & 1 \\ 1 & 8 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 8 & 8 & 5 & 1 \end{bmatrix}$$

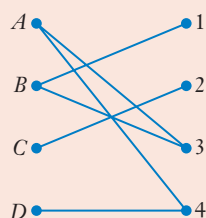
$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 0 & 0 \\ 0 & 7 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 7 & 7 & 4 & 0 \end{bmatrix}$$

Cover the zeros with the minimum number of horizontal or vertical lines.

A minimum of 4 lines is needed and there are 4 rows, so the allocation is complete.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 0 & 0 \\ 0 & 7 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 7 & 7 & 4 & 0 \end{bmatrix}$$

Use the zeros in the row and column reduced matrix to draw a bipartite graph to show the allocations.

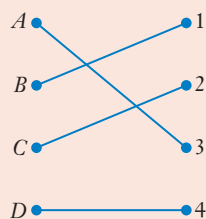


Look at the tasks.

C is the only courier connected with parcel 2, so C must be allocated parcel 2.

B is the only courier connected with parcel 1, so B must be allocated parcel 1 and the connection between B and parcel 3 must be eliminated.

Now, A is the only courier connected with parcel 3, so A must be allocated parcel 3 and the connection between A and parcel 4 must be eliminated.



Now *D* is the only courier connected to parcel 4.

Write the allocation.

A3, B1, C2, D4

- b** Calculate the distance travelled.

$$\begin{aligned}\text{Distance} &= 10 + 7 + 17 + 4 \\ &= 38\end{aligned}$$

State the result.

The minimum distance travelled for the couriers to collect the parcels is 38 km.

Exercise 10.07 Optimum assignment problems

- 1** A supervisor needs to allocate 3 different tasks to 3 workers. From experience, the supervisor knows the time taken for the workers to complete the various tasks.
 - Emily takes 11 min to perform task 1, 8 min for task 2 and 12 min for task 3
 - Troy takes 13 min to perform task 1, 12 min for task 2 and 10 min for task 3
 - Ruby takes 10 min to perform task 1, 13 min for task 2 and 11 min for task 3
 - a** Show this information as a table.
 - b** Match the tasks to the workers to minimise the time.
 - c** Find the minimum time to complete the 3 tasks.
- 2** The personnel manager of a company with branches across Queensland must assign 4 new graduates to 4 regional branches. The graduates are equally well qualified, so the decision is based on the costs of relocating the graduates to the regions. The relocation costs (in thousands of dollars) for each graduate are:

	Cairns	Mackay	Toowoomba	Ipswich
<i>A</i>	9	10	7	7
<i>B</i>	5	8	7	8
<i>C</i>	9	8	11	10
<i>D</i>	10	12	9	7

- a** Match the graduates to the regions to minimise relocation costs.
- b** Find the total cost of relocating the graduates.

Example
14

- 3 Given the matrix shown, the total value of the optimal allocation is:

A 17 **B** 16 **C** 19 **D** 18 **E** 15

$$\begin{matrix} & 1 & 2 & 3 \\ A & \begin{bmatrix} 8 & 6 & 6 \end{bmatrix} \\ B & \begin{bmatrix} 4 & 5 & 7 \end{bmatrix} \\ C & \begin{bmatrix} 9 & 5 & 8 \end{bmatrix} \end{matrix}$$

- 4 Four taxis (A , B , C and D) must be assigned to pick up 4 different passengers. The distance (km) that each taxi is from each of the customers is shown in the matrix.

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ A & \begin{bmatrix} 10 & 4 & 9 & 5 \end{bmatrix} \\ B & \begin{bmatrix} 8 & 11 & 10 & 7 \end{bmatrix} \\ C & \begin{bmatrix} 9 & 8 & 6 & 7 \end{bmatrix} \\ D & \begin{bmatrix} 6 & 11 & 8 & 8 \end{bmatrix} \end{matrix}$$

This data is converted into a matrix and row and column reductions are performed. The resulting matrix is:

A $\begin{bmatrix} 4 & 0 & 3 & 0 \\ 2 & 7 & 4 & 2 \\ 3 & 4 & 0 & 2 \\ 0 & 7 & 2 & 3 \end{bmatrix}$

B $\begin{bmatrix} 6 & 3 & 0 & 1 \\ 1 & 4 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 5 & 0 & 2 \end{bmatrix}$

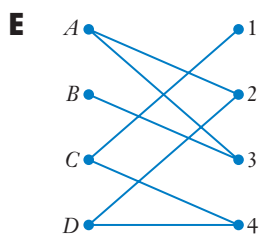
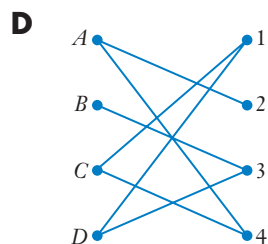
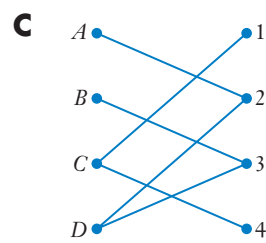
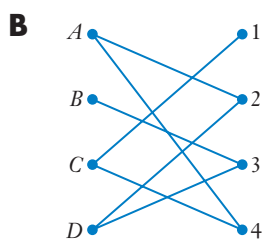
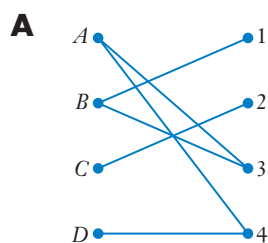
C $\begin{bmatrix} 6 & 0 & 5 & 1 \\ 1 & 4 & 3 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 5 & 2 & 2 \end{bmatrix}$

D $\begin{bmatrix} 5 & -1 & 4 & 0 \\ 0 & 3 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ -2 & 3 & 0 & 0 \end{bmatrix}$

E $\begin{bmatrix} 5 & -1 & 4 & 0 \\ 0 & 3 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ -2 & 3 & 0 & 0 \end{bmatrix}$

- 5 Four trainees (A , B , C and D) must be assigned one of 4 different tasks. The time taken for each trainee to perform each of the tasks is tabulated and converted into a matrix. The matrix, following row and column reduction, is shown. The bipartite graph representing the allocation of trainees to tasks is:

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ A & \begin{bmatrix} 4 & 0 & 2 & 0 \end{bmatrix} \\ B & \begin{bmatrix} 3 & 1 & 0 & 5 \end{bmatrix} \\ C & \begin{bmatrix} 0 & 6 & 2 & 0 \end{bmatrix} \\ D & \begin{bmatrix} 4 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$



- 6 A purchasing officer for a major company needs to purchase 4 different pieces of equipment. There are only 4 suppliers who are able to offer all of the pieces of equipment. Company policy states that only one piece of equipment may be purchased from any supplier. The cost of each piece of equipment from the suppliers is shown in the table.

	Item A	Item B	Item C	Item D
Supplier 1	11	8	13	14
Supplier 2	12	18	9	18
Supplier 3	14	15	13	8
Supplier 4	15	16	5	7

- Convert the table into a matrix and then undertake row and column reductions.
 - Find the optimum allocation and show it as a bipartite graph.
 - Find the minimum cost of the equipment.
- 7 The following tables show the times taken, in minutes, for workers *A*, *B*, *C* and *D* to perform tasks 1, 2, 3 and 4. Use row and column reductions to find the optimum allocation of tasks and the minimum time taken in each case.

a

	Task 1	Task 2	Task 3
<i>A</i>	6	8	9
<i>B</i>	7	9	6
<i>C</i>	10	7	11

b

	Task 1	Task 2	Task 3	Task 4
<i>A</i>	15	10	12	11
<i>B</i>	10	14	9	11
<i>C</i>	10	9	13	9
<i>D</i>	11	12	13	14

- 8 Four taxis are available to pick up passengers at 4 different locations. The distance (km) that each taxi has to travel to reach each passenger is given in the table.

	Fare 1	Fare 2	Fare 3	Fare 4
Taxi A	20	27	22	21
Taxi B	26	19	23	44
Taxi C	22	13	16	19
Taxi D	16	31	24	33

- Use the Hungarian algorithm to determine the allocation of taxis to passengers that minimises the distance travelled.
- Find the minimum distance that needs to be travelled to pick up all 4 fares.

- 9 The following tables show the costs, in thousands of dollars, for contractors A , B , C and D to complete jobs 1, 2, 3 and 4. Use the Hungarian algorithm to find the optimum allocation of tasks and the minimum cost to complete all 4 jobs in each case.

a

	Job 1	Job 2	Job 3	Job 4
A	16	12	15	18
B	16	17	11	16
C	16	13	14	17
D	15	14	13	15

b

	Job 1	Job 2	Job 3	Job 4
A	20	28	45	24
B	28	21	24	23
C	32	17	34	25
D	18	23	20	17

- 10 The delivery times, in minutes, for 4 couriers (A , B , C and D) to deliver parcels to 4 different locations (1, 2, 3 and 4) is shown in the table below.

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
Courier A	30	22	24	25
Courier B	26	22	36	23
Courier C	43	21	23	39
Courier D	40	22	23	39

- a** Determine the allocation of couriers to parcels that minimises the time taken.
b Find the minimum time needed to deliver all 4 parcels.
- 11 The matrices below show the times (in minutes) taken for 4 workers to complete 4 different tasks. In each case, find the minimum time required to complete the 4 tasks.

a

$$\begin{bmatrix} 12 & 15 & 20 & 18 \\ 22 & 17 & 14 & 20 \\ 19 & 22 & 27 & 15 \\ 35 & 28 & 12 & 18 \end{bmatrix}$$

b

$$\begin{bmatrix} 41 & 39 & 27 & 33 \\ 52 & 47 & 40 & 46 \\ 30 & 28 & 25 & 29 \\ 44 & 38 & 48 & 39 \end{bmatrix}$$



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Problem solving

- 12** An RACQ dispatcher has 4 service vehicles available to respond to 4 separate calls for assistance in a Southside service area. The times, in minutes, for each of the service vehicles to reach the callers is shown in the table below.

	Caller 1	Caller 2	Caller 3	Caller 4
Service vehicle 1	42	36	52	26
Service vehicle 2	36	30	48	35
Service vehicle 3	31	28	50	21
Service vehicle 4	44	38	49	39

- a** Convert this table to a matrix and perform row and column reductions.
 - b** Is it possible to make an optimum allocation using the row/column reduced matrix? Why?
 - c** Find the minimum time needed for the service vehicles to reach the callers.
- 13** A display home village will have 5 different display homes. Five builders are approached and asked to submit quotations to complete the build for each of the homes. The quotes (in thousands of dollars) submitted by each builder are shown in the table below.

	Endeavour	Tropicana	Beachside	Forrest Grove	Alpine
Masters Homes	220	310	460	320	400
Custom Build	280	330	420	450	380
Savannah Homes	280	370	510	440	290
Deakin Homes	460	440	560	470	380
Stylemaster	340	480	390	280	330

The builders are equally qualified and have quoted on the same specifications for each house. The developers who are planning the display home village want to complete the project for the minimum cost.

Find the optimum allocation of builders to construct the houses and the total cost of construction.

10. CHAPTER SUMMARY



Year 12
formulas

Networks, projects and assignments

- **Activities** are the tasks required to complete a project
- The sequence of activities that make up a project is shown as a directed network called a **project network**. Each edge of the network represents an activity, while the nodes represent the **events** at the start and finish of the activity. An edge includes an arrow to show direction and a number to indicate the time required for completion
- An **immediate predecessor** or **prerequisite** is an activity that must be completed immediately before another activity can commence
- **Concurrent** activities are activities that can occur at the same time
- Project network diagrams
 - The network must start at a vertex
 - Activities that do not have predecessors are starting activities
 - Multiple predecessors of an activity will end at the same vertex
 - An activity must only be represented by one edge
 - There must be a vertex showing the completion of the project
- An **activity table** shows the order and estimated completion time for each activity required to complete a project
- An activity with zero time allocated is called a **dummy activity** and is shown as a directed edge with a dashed line
- **Forward scanning** is used to determine the **earliest starting time** (EST) for each activity in a network
 - 1 Change the vertices to circles so information can be written in them.
 - 2 The vertex at the start of a network has an earliest starting time of 0. Write 0 in the circle for the first vertex.
 - 3 Add the time taken for an activity to its EST to find the EST for adjacent activities.
 - 4 Wherever 2 or more activities join at a vertex, choose the highest total time as the EST.
 - 5 Continue until the finish of the network is reached.

- **Backward scanning** is used to determine the **latest starting time (LST)** for each activity in a network. The latest starting time for any activity is the latest time it is possible to start the activity without delaying the completion of the project
 - 1 Begin with a network in which the EST for each activity has been calculated
 - 2 The vertex at the end of a network has a latest starting time equal to the EST of that vertex. Write this value in the bottom part of the circle for the last vertex
 - 3 Subtract the time taken for an activity from its LST to find the LST for adjacent activities
 - 4 Wherever 2 or more activities join at a vertex, choose the lowest total time as the LST
 - 5 Continue until the start of the network is reached
- The **critical path** in a network is the path with the longest total time. For a critical path, the EST and LST values are equal at each vertex
- **Critical activities** lie on the critical path
- The **float time** (or **slack time**) of an activity is the maximum delay possible in starting the activity that does not affect the completion time of the project. For critical steps, the float time is zero

The float time for an activity = $LST_{\text{End}} - EST_{\text{Start}} - \text{activity time}$

- A **bipartite graph** is one in which the vertices can be put into two distinct (disjoint) groups so that the only edges are between those two groups and there are no edges between vertices within the same group
- A **matching** is a one-to-one pairing of some or all of the vertices in one group of a bipartite graph with the vertices in the other group
- A **complete matching** is one in which every vertex in the first group is matched to a vertex in the second group
- In an **allocation** (or **assignment**) problem, one agent is assigned to one and only one task
- The **optimum allocation** is the most efficient in terms of time, cost, energy consumption, etc.

- **Allocation using row and column reduction**

- 1 Choose the smallest number in a row and subtract it from every element in the row. Repeat this process for each row
- 2 Choose a column that does not have a zero value. Subtract the smallest number in the column from every element in the column. Repeat this process until there are no columns without a zero value
- 3 Cover all the zeros using the smallest number of horizontal or vertical lines. The process is complete if the number of lines is equal to the number of rows
- 4 The zeros in the reduced matrix represent the allocations. Draw a bipartite graph showing all of the allocations
- 5 Choose the allocations that produce the optimum total

- **The Hungarian algorithm**

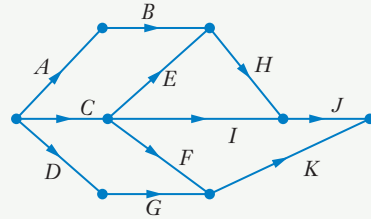
- 1 Perform row and column reductions and cover the zeros with the minimum number of horizontal or vertical lines
- 2 If the number of lines is less than the number of rows, locate the smallest uncovered number and add it to every covered number. Add it twice if the number is covered twice
- 3 Locate the smallest number and subtract it from all of the numbers in the matrix
- 4 Cover all the zeros using the smallest number of horizontal or vertical lines. The process is complete if the number of lines is equal to the number of rows. If this is not the case, repeat steps 1–3 until a solution is found
- 5 The zeros in the reduced matrix represent the allocations. Draw a bipartite graph showing all the allocations
- 6 Choose the allocations that produce the optimum total

10. CHAPTER REVIEW

Networks, projects and assignments

1 Which of the following statements is true?

- A** Activity *C* is a prerequisite of *J*.
- B** Activity *E* is a prerequisite of *H*.
- C** Activity *H* must be done before activity *B*.
- D** Activity *C* must be done before activity *B*.
- E** Activity *G* is a prerequisite of *I*.



Example
1

2 Which of the following statements is true about the network in question 1?

- A** Activity *J* is a dummy activity.
- B** Activities *E* and *F* are both prerequisites of *K*.
- C** Activities *A* and *D* must be done before activity *G*.
- D** Activities *B* and *E* are both prerequisites of *H*.
- E** Activities *C* and *E* must be done before activity *H*.

Example
1

3 Draw a project network for the activity table shown below.

Activity	Time (min)	Prerequisites
<i>A</i>	3	None
<i>B</i>	4	None
<i>C</i>	2	<i>A</i>
<i>D</i>	5	<i>A</i>
<i>E</i>	4	<i>B, D</i>
<i>F</i>	3	<i>B, D</i>
<i>G</i>	2	<i>C, F</i>
<i>H</i>	6	<i>E</i>
<i>I</i>	4	<i>E</i>
<i>J</i>	5	<i>G, H</i>
<i>K</i>	3	<i>I, J</i>

Example
1

Example
2

4 Draw a project network for this activity table.

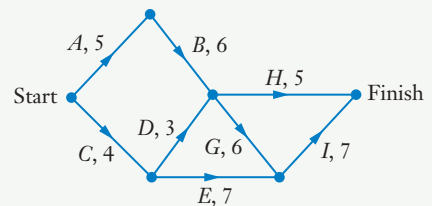
Activity	Time (days)	Prerequisites
<i>A</i>	3	None
<i>B</i>	6	None
<i>C</i>	5	None
<i>D</i>	4	<i>A, B</i>
<i>E</i>	7	<i>C</i>
<i>F</i>	3	<i>C</i>
<i>G</i>	2	<i>D, E</i>

Example
3

5 Draw a network to show the task of baking and decorating a birthday cake using the activities table below.

Code	Activity	Time (min)	Prerequisites
<i>A</i>	Find cake recipe	10	None
<i>B</i>	Find icing recipe	8	None
<i>C</i>	Walk to shops	15	None
<i>D</i>	Measure out and mix icing ingredients	12	<i>B</i>
<i>E</i>	Measure out and mix cake ingredients	15	<i>A</i>
<i>F</i>	Buy cake decorations	20	<i>C</i>
<i>G</i>	Bake cake	40	<i>E</i>
<i>H</i>	Walk back from the shops	15	<i>F</i>
<i>I</i>	Cool cake	30	<i>G</i>
<i>J</i>	Ice cake	10	<i>D, I</i>
<i>K</i>	Decorate cake	5	<i>H, J</i>

This project network shows activities *A* through *I*, and the times for each activity in hours. Use it to answer questions 6–10.



Examples
4, 5

6 The earliest starting time for activity *H* is:

- A** 12 hours **B** 7 hours **C** 11 hours **D** 10 hours **E** 15 hours

Examples
4, 5

7 The latest starting time for activity *E* is:

- A** 10 hours **B** 13 hours **C** 11 hours **D** 15 hours **E** 8 hours

Examples
4, 5

8 The earliest starting time for the last vertex is:

- A** 24 hours **B** 16 hours **C** 21 hours **D** 15 hours **E** 12 hours

9 If the time taken to complete activity *E* is increased to 10 hours, by how much will the earliest starting time for the last vertex be increased?

- A** 6 hours **B** 10 hours **C** 3 hours **D** 0 hours **E** 2 hours

Examples
4, 5

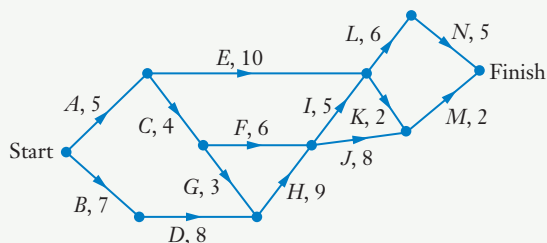
10 If the time taken to complete activity *G* is increased to 10 hours, by how much will the earliest starting time for the last vertex be increased?

- A** 10 hours **B** 6 hours **C** 4 hours **D** 0 hours **E** 5 hours

Examples
4, 5

11 a Determine the earliest starting time for the activities in this network.

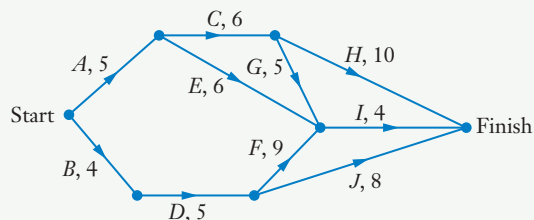
Examples
4, 5



b Determine the latest starting times for each activity.

12 The project network for a particular task is shown here, with times in hours.

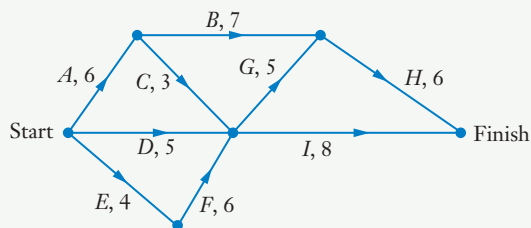
Example
6



a List all possible paths and the time taken to complete each one.

b Identify the critical path and hence the shortest time in which the project can be completed.

Use this project network to answer questions 13–14. The times given are in hours.



13 The length of path *EFI* is:

- A** 20 hours **B** 15 hours **C** 18 hours **D** 13 hours **E** 19 hours

Examples
6

14 The path with the longest duration is:

- A** *ABH* **B** *EFI* **C** *ACGH* **D** *EFGH* **E** *DI*

Examples
6

Use this project activity table to answer questions 15–16.

Activity	Time (hours)	Prerequisites
<i>A</i>	4	None
<i>B</i>	6	None
<i>C</i>	5	<i>A</i>
<i>D</i>	7	<i>A</i>
<i>E</i>	8	<i>B</i>
<i>F</i>	6	<i>D, E</i>
<i>G</i>	6	<i>C</i>
<i>H</i>	3	<i>G, F</i>

Example
7

- 15** If there is a delay of 3 hours in activity *E*, the time taken to complete the project is:
A 26 hours **B** 24 hours **C** 23 hours **D** 30 hours **E** 19 hours

Example
7

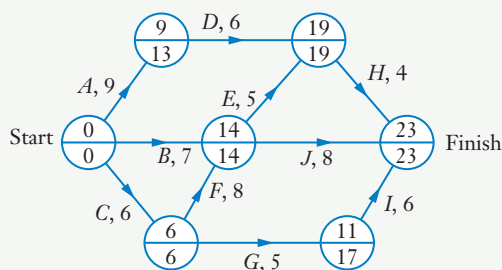
- 16** If the duration of activity *D* is increased to 10 hours, the time taken to complete the project is:
A 27 hours **B** 20 hours **C** 19 hours **D** 23 hours **E** 28 hours

Example
8

- 17** Part of a project network is shown here. Times are in hours. The float time for activity *D* is:
A 11 hours **B** 14 hours **C** 7 hours
D 25 hours **E** 5 hours



The results of a forward scan and backward scan are displayed on this project network. Activity durations are measured in days. Use it to answer questions 18–21.



Example
7

- 18** The critical path is:
A *BEH* **B** *ADEJ* **C** *CFEH* **D** *CGI* **E** *CFJ*

Example
7

- 19** The number of non-critical activities is:
A 6 **B** 4 **C** 1 **D** 5 **E** 3

Example
7

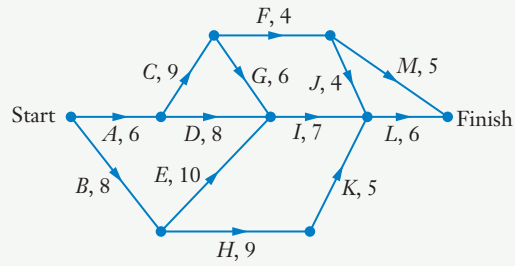
- 20** If activity *J* is by delayed by 3 days, what is the shortest completion time for the project?
A 26 days **B** 20 days **C** 18 days **D** 25 days **E** 21 days

Example
7

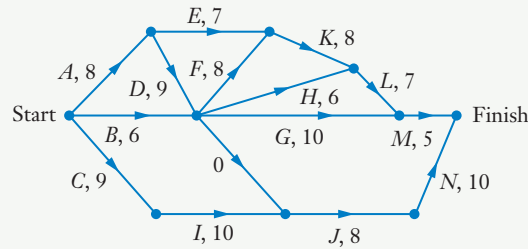
- 21** If activity *E* is by delayed by 2 days, what is the shortest completion time for the project?
A 23 days **B** 25 days **C** 21 days **D** 19 days **E** 27 days

22 This project network shows a number of activities and the time taken to complete them in minutes.

- Find the EST and LST for each activity.
- Identify the critical path.
- Determine the shortest time taken to complete the project.
- Determine the float time for:
 - D*
 - F*
 - K*
 - H*



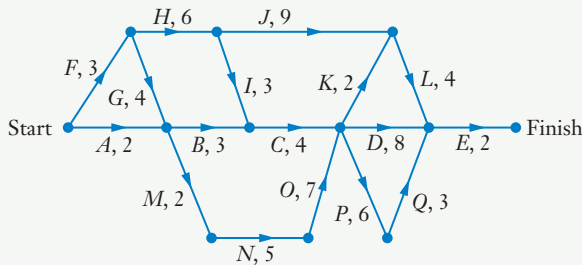
23 The activity times in the project network shown here are in hours.



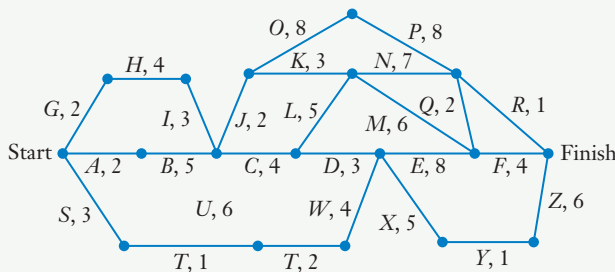
- Calculate the float time for each of the non-critical activities.
- If *G* is delayed by 8 hours, what effect does this have on the duration of the project and the critical activities?
- Extra resources can be used to shorten the duration of one of the activities *E*, *F* or *K* (on the original network) by 2 hours. Which of these activities should be shortened, and why?

24 Find the critical path and length for each project network.

a



b



Example
10

25 The activity table for a school charity fashion parade is shown.

Code	Activity	Time (days)	Prerequisites
<i>A</i>	Invite celebrity to open the parade	5	None
<i>B</i>	Invite local shops to participate	3	None
<i>C</i>	Choose the models	5	<i>B</i>
<i>D</i>	Book the venue	1	<i>B</i>
<i>E</i>	Confirm clothes with shops taking part in the parade	3	<i>B</i>
<i>F</i>	Fit the models with clothes	3	<i>C, E</i>
<i>G</i>	Advertise the fashion parade	10	<i>A, D</i>
<i>H</i>	Check prices with shops and prepare catalogue	10	<i>E</i>
<i>I</i>	Choose the music and master of ceremonies	3	None
<i>J</i>	Set up the catwalk	2	<i>D</i>
<i>K</i>	Hold a rehearsal in the afternoon before the fashion parade	1	<i>F, I, J</i>
<i>L</i>	Hold the fashion parade	1	<i>G, H, K</i>

- Draw the project network.
- Determine the critical path and the shortest time to complete the project.
- What is the effect of a delay in preparing the catalogue?
- What is the effect of a delay in choosing the music?

Example
11

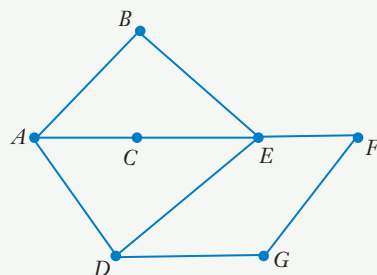
26 The favourite subjects studied by 5 students are listed below.

- Hannah – Maths, Chemistry, Technology
- Nate – Maths, Economics
- Ella – Legal Studies, Chemistry
- Riley – Economics
- Mila – Maths, Chemistry

Draw a bipartite graph to represent this information.

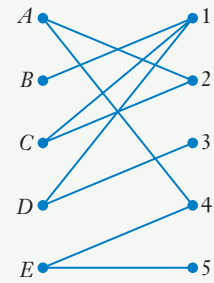
Example
12

- Use the vertex-colouring algorithm to determine if this graph is bipartite.
- If the graph is bipartite, state the vertices in each group.

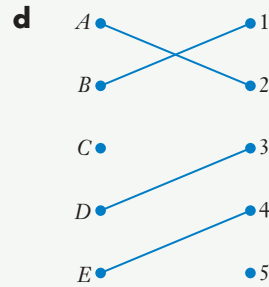
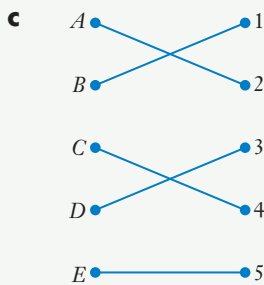
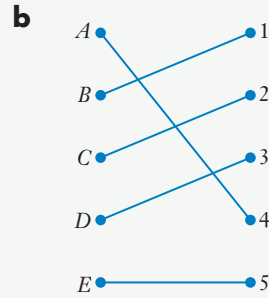
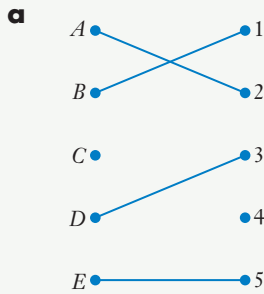


28 Look at the bipartite graph on the right.

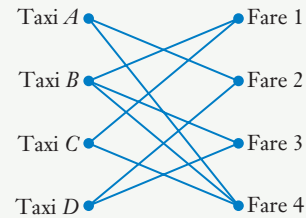
Determine if each graph below is a matching for this graph. If it is a matching, classify it as a complete or incomplete matching.



Example
13



29 A taxi dispatcher has 4 taxis available to pick up four passengers in different locations. Company policy states that a taxi must reach a fare within 10 minutes. This bipartite graph shows which passenger each taxi can reach within 10 minutes. If each taxi is to collect 1 passenger, a valid allocation is:



Example
14

A

Taxi A	Fare 2
Taxi B	Fare 3
Taxi C	Fare 1
Taxi D	Fare 4

B

Taxi A	Fare 2
Taxi B	Fare 4
Taxi C	Fare 1
Taxi D	Fare 3

C

Taxi A	Fare 4
Taxi B	Fare 2
Taxi C	Fare 1
Taxi D	Fare 3

D

Taxi A	Fare 2
Taxi B	Fare 4
Taxi C	Fare 3
Taxi D	Fare 1

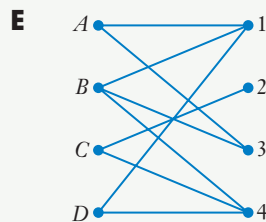
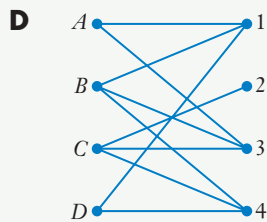
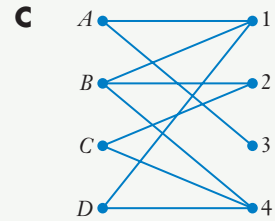
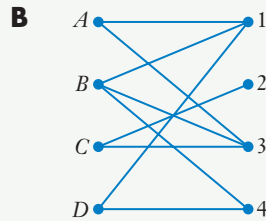
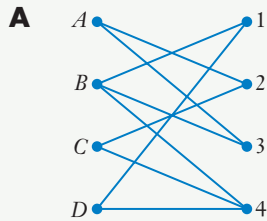
E

Taxi A	Fare 4
Taxi B	Fare 2
Taxi C	Fare 1
Taxi D	Fare 3

- 30** Four trainees are required to complete 4 tasks. The times in which each trainee can complete each task are recorded in a table and then row and column reduction is completed. This process gives the matrix on the right.

The bipartite graph for this network is:

	1	2	3	4
Trainee A	0	3	0	2
Trainee B	0	5	0	0
Trainee C	6	0	4	0
Trainee D	0	1	2	0



- 31** Three equally qualified workers are chosen to complete different tasks in a manufacturing process.
- Worker A takes 25 min to perform task 1, 20 min for task 2 and 22 min for task 3
 - Worker B takes 18 min to perform task 1, 17 min for task 2 and 15 min for task 3
 - Worker C takes 16 min to perform task 1, 18 min for task 2 and 17 min for task 3
- Show this information as a table.
 - Match the tasks to the workers to minimise the time.
 - Find the minimum time to complete the 3 tasks.

- 32** Four couriers *A*, *B*, *C* and *D* are available to collect 4 different parcels. The dispatcher must ensure that each parcel is collected by a different courier. The couriers are located at different distances in kilometres from each parcel and these distances are shown in the table below.

	Task 1	Task 2	Task 3	Task 4
<i>A</i>	17	25	26	20
<i>B</i>	28	27	23	25
<i>C</i>	20	18	17	14
<i>D</i>	28	25	23	19

- Convert the table into a matrix and then undertake row and column reductions.
- Find the optimum allocation and show it as a bipartite graph.
- Find the minimum amount of travel required for the couriers to collect the parcels.

- 33** Four equally qualified building contractors are available to undertake 4 different construction projects. The quotes, in thousands of dollars, provided by each contractor to complete the different projects are given by the table.

	Project 1	Project 2	Project 3	Project 4
<i>A</i>	32	46	28	52
<i>B</i>	41	39	32	56
<i>C</i>	38	47	23	49
<i>D</i>	44	32	22	47

Company policy dictates that each of the construction projects must be given to a different contractor.

- Determine the allocation of building contractors to construction projects that minimises the total cost of the projects.
- Find the minimum cost of completing the 4 projects.

Problem solving

- 34** The times (in minutes) taken for 5 different trainees to complete 5 separate processes are shown in the table below.

	Process 1	Process 2	Process 3	Process 4	Process 5
Trainee <i>A</i>	28	32	37	44	35
Trainee <i>B</i>	33	41	37	52	39
Trainee <i>C</i>	41	36	35	38	41
Trainee <i>D</i>	45	38	53	49	39
Trainee <i>E</i>	35	44	51	42	55

The 5 processes must be performed to complete a manufacturing project. The project manager wants to complete the project in the shortest possible time. Find the optimum allocation of trainees to processes and the total time taken.

- 35** Work out an activity table for the project of preparing a school yearbook. Remember that it should include class photographs, class notes and articles about the major events of the year.
- 36** A developer is subdividing some land for residential development. An activity table for the project is shown below.

Code	Activity	Time (weeks)	Prerequisites
<i>A</i>	Purchase land	12	None
<i>B</i>	Prepare plans	8	<i>A</i>
<i>C</i>	Submit plans for council approval	12	<i>B</i>
<i>D</i>	Arrange finance for main works	3	<i>C</i>
<i>E</i>	Do final survey	2	<i>D</i>
<i>F</i>	Do roadworks	10	<i>E</i>
<i>G</i>	Submit subdivision plans for preparation of land titles	6	<i>B</i>
<i>H</i>	Install sewer, stormwater mains and connection points	6	<i>E</i>
<i>I</i>	Have electricity and phone main lines connected	3	<i>E</i>
<i>J</i>	Advertise blocks for sale	12	<i>B</i>
<i>K</i>	Sell blocks	12	<i>E, G, J</i>
<i>L</i>	Repay loans	2	<i>F, H, I, K</i>

- a** Draw the project network, find the critical path and the minimum time for completion.
- b** What is the maximum delay possible in installing the sewer and stormwater mains without delaying completion of the project?



Practice quiz

Practice Examination 4 ●●●●

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved scientific calculator.
- Access to spreadsheet software would be useful for some questions.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (3 marks)

An initial deposit of \$15 000 was made on an investment taken out over 4 years at a rate of 8% p.a. compounded annually, and an additional deposit of \$1000 is made each year.

- a** Complete the table below for the first 4 deposits.

n	Value at start of year	Interest	Deposit (d)	Value at end of year
1	15 000.00			
2				
3				
4				

- b** Calculate how much interest had been earned over the 4 years.
c What is the value of the annuity at the end of 4 years?

Question 2 (4 marks)

Evelyn invests \$65 000 that she inherited into an ordinary annuity that pays 5.6% p.a. compounded annually. Each year, after the annual interest is added, she withdraws \$7000. The amount remaining is then reinvested for the following year.

- a Write a recurrence relation for the value of the investment.
- b What is the value of the investment at the start of the 7th year?
- c How long will the annuity last?
- d How much will the last payment be?

Question 3 (1 mark)

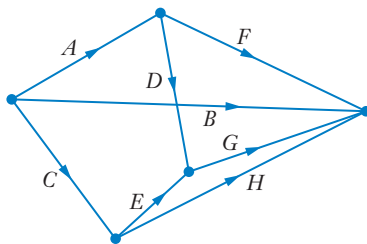
Levi starts work in a new job. His employer sets up a superannuation account for him by depositing \$350 at the start of each month. The account earns 7.5% p.a. compounded monthly. What will be the balance of Levi's account after 25 years?

Question 4 (2 marks)

A company will need to replace a piece of equipment in 6 years time. The company deposits \$2400 at the start of each month into a sinking fund account that pays 5.1% interest, compounded monthly. How much money will be in the sinking fund when the piece of equipment needs to be purchased?

Question 5 (1 mark)

Which one of the following statements is true?



- A Activity A must be done before activity B.
- B Activity C is a prerequisite of G.
- C Activity E is a prerequisite of G.
- D Activity F has two prerequisites.
- E Activity D is a prerequisite of H.

Question 6 (1 mark)

Draw a project network for the activity table shown below.

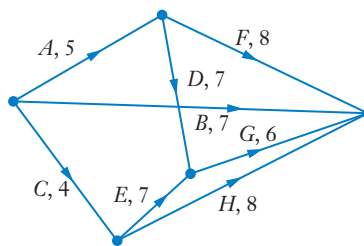
Activity	Time (days)	Prerequisites
<i>A</i>	3	None
<i>B</i>	2	None
<i>C</i>	7	None
<i>D</i>	5	<i>A</i>
<i>E</i>	4	<i>C</i>
<i>F</i>	1	<i>B, C, D</i>
<i>G</i>	3	<i>B, C, D</i>
<i>H</i>		<i>F, E</i>

Question 7 (1 mark)

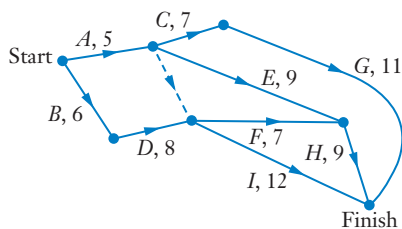
This project network shows activities *A* through *H*, and the times for each activity in days.

The earliest starting time for activity *G* is:

- A** 11 days **B** 10 days **C** 13 days
D 12 days **E** 9 days

**Question 8** (2 marks)

Use the network below to answer the following questions.



- Determine the earliest starting time for the activities in this network.
- Determine the latest starting times for each activity.

Questions 9 and 10 refer to this project activity table.

Activity	Time (hours)	Prerequisites
<i>A</i>	5	None
<i>B</i>	6	None
<i>C</i>	4	None
<i>D</i>	8	<i>A</i>
<i>E</i>	7	<i>A</i>
<i>F</i>	6	<i>B, E</i>
<i>G</i>	3	<i>C</i>
<i>H</i>	5	<i>D, F</i>

Question 9 (1 mark)

If there is a delay of 3 hours in activity *F*, the time taken to complete the project is:

- A** 26 hours **B** 23 hours **C** 15 hours
D 20 hours **E** 14 hours

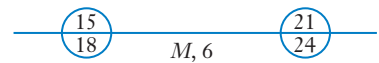
Question 10 (1 mark)

If there is a delay of 3 hours in activity *E*, the time taken to complete the project is:

- A** 27 hours **B** 20 hours **C** 26 hours
D 21 hours **E** 18 hours

Question 11 (1 mark)

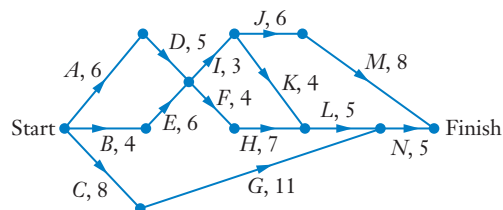
Part of a project network is shown here. Times are in hours. The float time for activity *M* is:



- A** 6 hours **B** 15 hours **C** 21 hours
D 3 hours **E** 8 hours

Question 12 (3 marks)

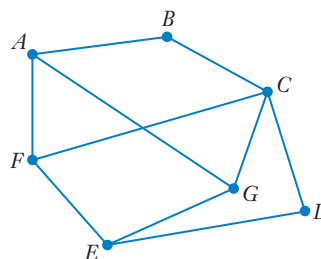
The activity times in the project network shown here are in hours.



- Calculate the float time for each of the non-critical activities.
- If K is delayed by 3 hours, what effect does this have on the duration of the project and the critical activities?
- Extra resources can be used to shorten the duration of one of the activities F , G or J (on the original network) by 2 hours. Which of these activities should be shortened, and why?

Question 13 (2 marks)

- Use the vertex-colouring algorithm to determine if this graph is bipartite.
- If the graph is bipartite, state the vertices in each group.



Question 14 (3 marks)

Three different components need to be purchased for the maintenance of a machine. The components are available from 3 different suppliers. The cost of each part from each supplier is shown below.

- Component A costs \$54 from supplier 1, \$48 from supplier 2 and \$48 from supplier 3
- Component B costs \$56 from supplier 1, \$51 from supplier 2 and \$49 from supplier 3
- Component C costs \$55 from supplier 1, \$51 from supplier 2 and \$50 from supplier 3

The purchasing policy of the company states that any supplier can only provide one component.

- Show this information as a table.
- Match each component to a contractor in order to minimise the cost.
- Find the minimum cost to obtain all 3 components.

Question 15 (4 marks)

A mobile phone company wishes to hire 4 applications developers (A , B , C and D) to write 4 different apps (applications). The apps need to be developed in the same time frame, so each developer will only be able to develop at most 1 app. The estimates of the times (in weeks) given by the 4 developers are shown in the table.

	App 1	App 2	App 3	App 4
Developer A	38	42	37	41
Developer B	36	29	51	43
Developer C	31	47	26	39
Developer D	40	35	33	42

- a Perform row (and if necessary, column) reductions.
- b Perform the Hungarian algorithm (if necessary).
- c Determine the optimal allocation and the total time required to complete the 4 tasks.

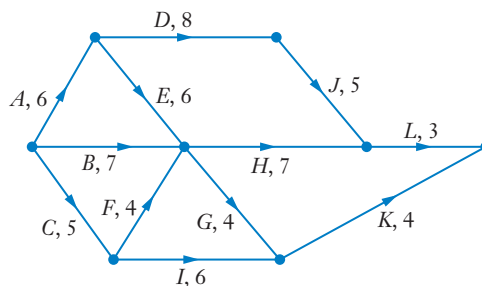
Question 16 (3 marks)

Scarlett places \$120 000 in an account earning 8.6% p.a. compounded monthly. She wants to make monthly withdrawals so the account has a zero balance at the end of 7 years. What is the value of the withdrawal she should make?

Question 17 (7 marks)

The project network below shows the activities, along with the times (in hours), needed to complete a particular project.

Activity	Prerequisite	EST
<i>A</i>	None	0
<i>B</i>	None	0
<i>C</i>	None	0
<i>D</i>		6
<i>E</i>	<i>A</i>	6
<i>F</i>	<i>C</i>	
<i>G</i>		12
<i>H</i>	<i>E, B, F</i>	12
<i>I</i>	<i>C</i>	5
<i>J</i>	<i>D</i>	
<i>K</i>		16
<i>L</i>	<i>J, H</i>	25
<i>X</i>	<i>I, G</i>	16



The corresponding incomplete activity table is also shown. It displays the earliest start times rather than activity times for each activity. The project network is incomplete because an activity, *X*, needs to be added.

Use a completed network to find the critical path, shortest completion time and the latest start time for activity *J*.

Question 18 (3 marks)

An investment account currently has a balance of \$506 740. If monthly contributions of \$1500 are made and the account earns 5.7% p.a. compounded monthly, what was the initial amount in the account 10 years ago?

Question 19 (7 marks)

A new plant for a major manufacturer is being constructed. The activity table for the construction is shown.

Activity	Description	Time (weeks)	Prerequisites
<i>A</i>	Excavate site	2	None
<i>B</i>	Lay the foundation and slab	4	<i>A</i>
<i>C</i>	Erect the wall and roof trusses	10	<i>B</i>
<i>D</i>	Install the roof	6	<i>C</i>
<i>E</i>	Complete the exterior plumbing	4	<i>C</i>
<i>F</i>	Complete the interior plumbing	5	<i>E</i>
<i>G</i>	Fix the exterior cladding	7	<i>D</i>
<i>H</i>	Complete the exterior painting	9	<i>E, G</i>
<i>I</i>	Complete the electrical work	7	<i>C</i>
<i>J</i>	Install interior cladding	8	<i>F, I</i>
<i>K</i>	Lay the internal flooring	4	<i>J</i>
<i>L</i>	Complete the interior painting	5	<i>J</i>
<i>M</i>	Install the exterior fixtures	2	<i>H</i>
<i>N</i>	Install the interior fixtures	6	<i>K, L</i>

Find the shortest completion time for the project, and the changes to the time if the external cladding is delivered 3 weeks late or the electrical work is completed 2 weeks early.

END OF EXAMINATION

ANSWERS

Chapter 1

Exercise 1.01

1 a

	Surf	Swim	Lie on beach	
Staying locally	12	15	4	31
Visiting	6	7	6	19
	18	22	10	50

b 7

2

	Logging support		
	Yes	No	
NSW	20	56	76
Victoria	18	63	81
Queensland	25	43	68
NT	8	23	31
SA	12	48	60
WA	24	39	63
Tasmania	15	32	47
ACT	2	25	27
	124	329	453

3

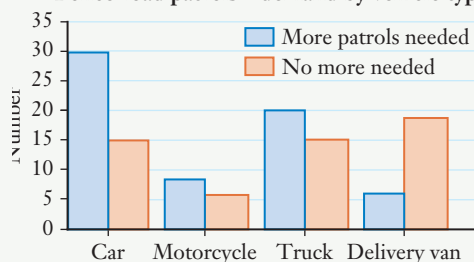
		Flat	Unit	House	
Refrigerators and freezers	1	40	35	40	115
	2	10	20	70	100
	3	5	5	30	40
	4	0	0	20	20
	> 4	0	0	5	5
		55	60	165	280

4 a 23

b 47

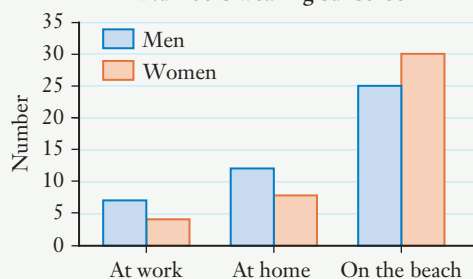
5

Police road patrols – demand by vehicle type



6

Numbers wearing sunscreen



7

Music type		Age		
		≤ 20 (student)	>20 (teacher)	
	C	5	6	11
	J	5	7	12
	E	9	5	14
	CW	5	6	11
	R	12	8	20
		36	32	68

It looks like more students have a greater preference for easy-listening and rock than teachers.

8

		Years of education									
		11	12	13	14	15	16	17	18	19	
Worker	B	4	6	9	2	9	0	0	0	0	30
	W	0	10	11	8	11	10	0	0	0	50
	M	0	0	0	2	3	1	6	4	4	20
		4	16	20	12	23	11	6	4	4	100

Blue-collar workers have the lowest number of years of education, and management workers have the highest.

Exercise 1.02

- 1 **a** Single parents **b** Men
c Accidental injuries **d** Sports games
e Low-income families
- 2 **a** The people interviewed
b The women interviewed
c Those in favour of daylight saving
d Teenagers **e** Pet-owners
- 3 **a** 34% **b** 60% **c** 21% **d** 37%
e 85% **f** 21% **g** 10%
- 4 **a** 26% **b** 22% **c** 15% **d** 5% **e** 15%
- 5 **a** 27% **b** 15% **c** 26% **d** 26% **e** 12%
- 6 **a**

	Republic			
	Yes	Don't care	No	
Labor	28	4	17	49
Liberal/ National	25	3	27	55
Minor parties	8	7	6	21
	61	14	50	125

b

	Republic			
	Yes	Don't care	No	
Labor	57.1	8.2	34.7	100
Liberal/ National	45.5	5.5	49.1	100
Minor parties	38.1	33.3	28.6	100
	48.8	11.2	40	100

Labor voters favour a republic, but Liberal/National voters oppose it. Minor party voters are equivocal.

7

	Oil exploration support		
	Yes	No	
NSW, Qld	35	65	100
ACT, NT, Vic, SA	27.6	72.4	100
Tas, WA	35.7	64.3	100
	31.7	68.3	100

Australians are generally opposed to oil exploration on the reef.

8

	Age group		
	40 or less	More than 40	
Less than 2	38.5	61.5	100
2	50	50	100
More than 2	70.6	29.4	100
	55	45	100

People under 40 order more takeaways than people over 40.

- 9 Neither age groups nor number of siblings can be easily divided to make more than 5.

The combinations shown are the best that can be achieved.

		Age group			
		25 or less	26 to 35	35+	
Siblings	2 or less	9	8	5	22
	More than 2	4	4	3	11
		13	12	8	33

		Age group			
		25 or less	26 to 35	35+	
Siblings	2 or less	69	67	63	67
	More than 2	31	33	38	33
		100	100	100	100

There is a tendency for those who are older to have more siblings, but this is only slight and could be a random variation, as some of the group sizes are less than 5.

It could be that families were larger in the past.

Exercise 1.03

1 a

	Heart attack		
	Yes	No	
Obese	3020	2820	5840
Non-obese	6380	41 780	48 160
	9400	44 600	54 000

- b Obese 52%, non-obese 13%
 c Obese people are 4 times more likely to die of heart attack than non-obese people.

2

		Pregnant		
		Yes	No	
Test	Positive	79.4%	20.6%	100%
	Negative	4.5%	95.5%	100%
		30%	70%	100%

- a About 21% b About 4.5%
 About 79% of positives are true and about 21% are false.

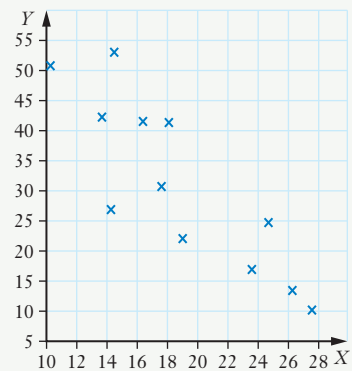
3 a

	Full-time	Part-time	Not returning	
Professional	3%	6%	3%	12%
Administration	5%	7%	13%	25%
Clerical	17%	18%	28%	63%
	25%	31%	44%	100%

- b 6% c 50% d 19%
 e 22 f Less than 3
 4 a 33 b 49 or 50 c Less than 3
 d 17 e 19
 5 About 63% of positive results are correct and about 37% are false. The ratio is 12 : 7.
 6 The student confused percentage by row with percentage by table total. Recalculating by row shows that 60% of women would vote for paid maternity leave.
 7 a No, 47% b Correct c No, 12%

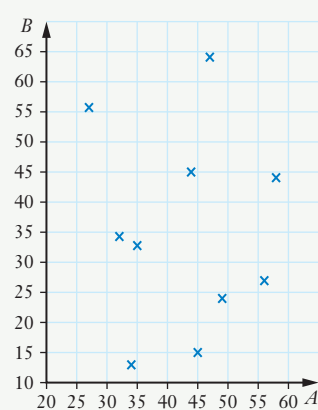
Exercise 1.04

1 a



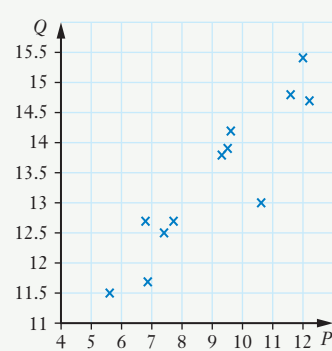
- b X and Y have a negative association.

2 a

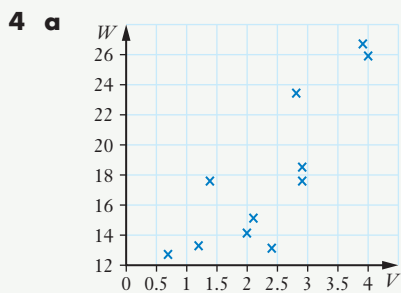


- b A and B have no apparent association.

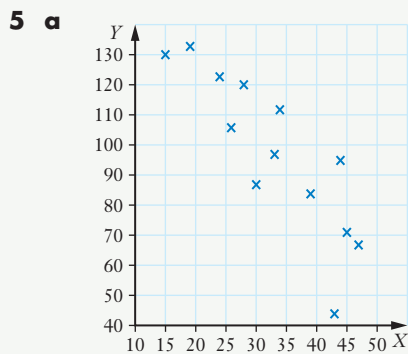
3 a



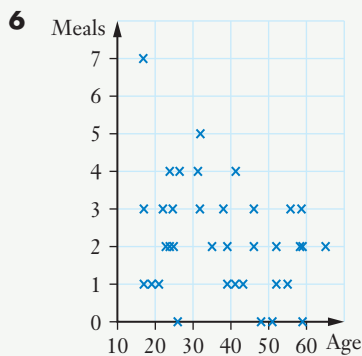
- b P and Q have a positive association.



b V and W have a positive association.



b X and Y have a negative association.

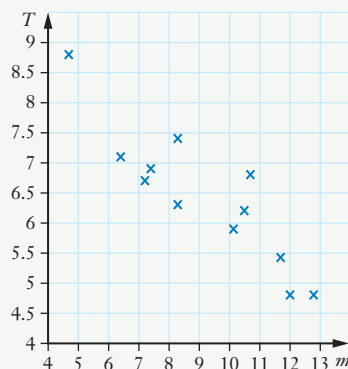


The values go downward from left to right, so there is a negative association between age and the number of takeaways.

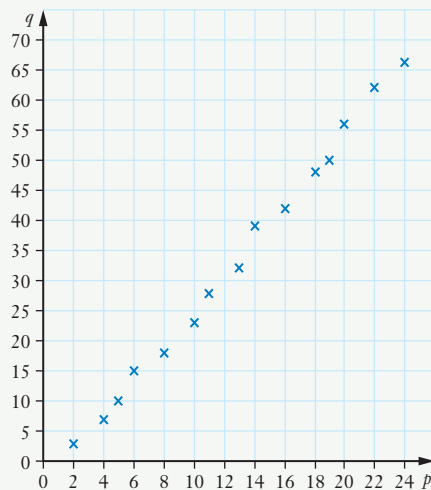
Exercise 1.05

- 1 a** Moderate negative linear association
- b** No linear association
- c** Strong positive linear association
- d** Weak negative linear association
- e** No apparent association
- f** Weak positive linear association
- g** Moderate positive linear association

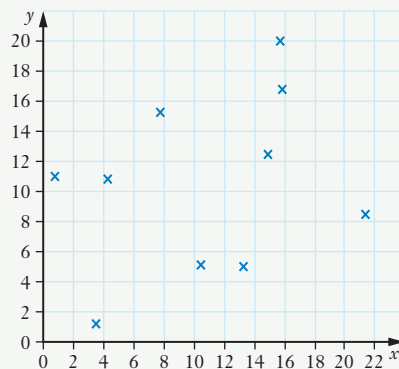
- 2** There is a moderate negative linear association of m and T .



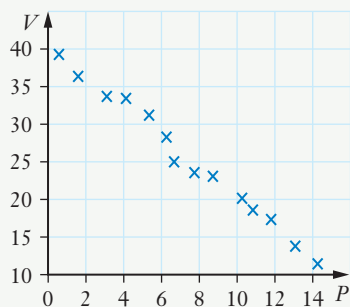
- 3** There is a strong positive linear association of p and q .



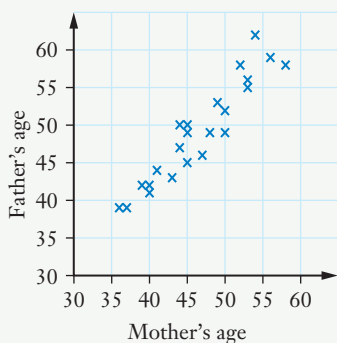
- 4** There is a weak positive linear association of x and y .



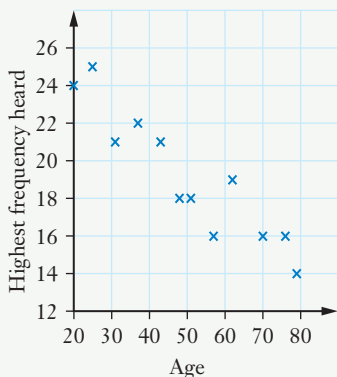
- 5** There is a strong negative linear association of P and V .



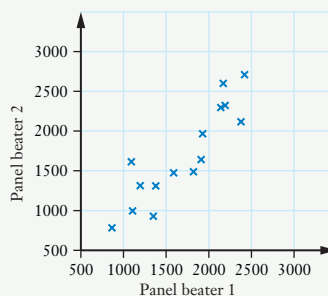
- 6** Parents' ages have a moderate positive linear association. They have similar ages.



- 7** There is a moderate negative linear association between age and the highest frequency heard. People lose hearing as they get older.



8



There is a moderate positive linear association of their quotes.

Although their quotes follow the same trend, either could be the cheaper for particular jobs.

Exercise 1.06

- 1 a** 0.92
b There is a strong positive association.
- 2 a** -0.28
b There is a weak negative association.
- 3 a** 0.61
b There is a moderate positive association.
- 4 a** -0.85
b There is a strong negative association.
- 5 a** 0.15
b There is a little or no association.
- 6** 38.2, 6.3, 9.2, 0.66, a moderate positive association
- 7** -96.9 , 11.7, 19.8, -0.42 , a moderate negative association
- 8** -0.88 , a strong negative association
- 9** 0.33, a weak positive association
- 10** -0.32 , a weak negative association
- 11** 0.82, a strong positive association
- 12** 0.62, a moderate positive association
- 13** $r = 0.98$, a very strong positive association
- 14** Using $1 = E-$, up to $15 = A+$, $r = 0.67$, a moderate positive association

Chapter review

1 a

	Southside	Northside	Total
Female	320	320	640
Male	210	330	540
Total	530	650	1180

b 330

c 530

2 a

	E	G	M	Total
Apprenticeship	2	2	1	5
TAFE	3	3	2	8
Work	3	3	2	8
Uni	1	5	6	12
Total	9	13	11	33

b 13, including apprentices

3 E

4 A

5

	Caffeine	Sugar	Total
None	3	0	3
Low	4	4	8
Medium	1	1	2
High	2	4	6
Total	10	9	19

6 a Number of people at the game

b Number of students at Kalkadoon SHS

c Number of Brisbane supporters on the train

7 a About 38%

b About 0.014

8

Time	Morning shower	Night shower	Totals
Before 6:00	7.2	0	7.2
Before 6:30	10.1	8.7	18.8
Before 7:00	13	7.2	20.3
Before 7:30	10.1	10.1	20.3
Before 8:00	7.2	13	20.3
Before 8:30	0	13	13
Totals	47.8	52.2	100

Students who prefer a night shower tend to get up later than those preferring a morning shower. Slightly over half prefer a night shower.

9 a

	Osteoporosis		
	Yes	No	
Very active	10	50	60
Sedentary	60	120	180
	70	170	240

b 60

c Very active 17%, sedentary 33%

d Those with sedentary lifestyles were twice as likely to develop osteoporosis as those with very active lifestyles.

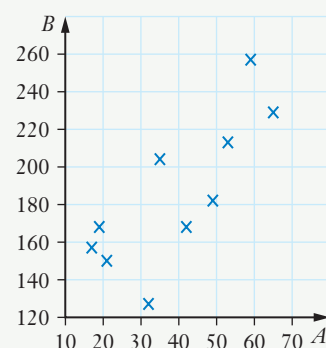
10

	Blood disorder		
	Yes	No	
Positive	0.392%	1.992%	2.384%
Negative	0.008%	97.608%	97.616%
	0.400%	99.600%	100.000%

a 84%

b 0.008%

11 a



b There is a moderate positive linear association.

12 a A strong negative linear association

b A non-linear association

c A weak positive linear association

d A moderate negative linear association

13 D

14 a 0.49, moderate positive linear association

b -0.30, weak negative linear association

c 0.87, strong positive association

15 a -0.89 , strong negative association

b 0.10 , little or no association

16

	7	8	9	10	11	12	Total
10–30 min	17	11	5	6	1	2	42
40–60 min	0	5	2	2	4	3	16
70–90 min	0	0	10	4	2	4	20
100–120 min	0	0	0	0	4	4	8
> 120 min	0	0	0	0	5	6	11
Total	17	16	17	12	16	19	97

Students in later years say they spend more time than younger students.

17 About 18% and 38%.

18 The probabilities are 0.435% for true positives and 12.935% for false positives. There are about 28 times as many false positives as true positives. A single blood test is not much use.

19 $r = 0.95$, strong positive association. Higher loss is associated with a higher mouse population.

Chapter 2

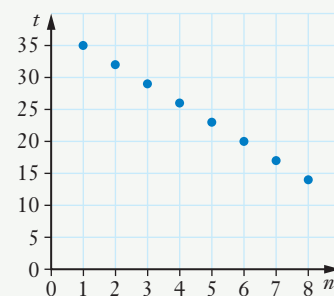
Exercise 2.01

- 1 a 9.6, AP with $d = 1.2$ b 384, not an AP
 c 6, AP with $d = -2$ d 31, not an AP
 e $6\frac{3}{4}$, AP with $d = \frac{1}{4}$ f 6.2, AP with $d = -0.4$
 g $7\frac{31}{32}$, not an AP h 58, AP with $d = 6$
 i -21 , AP with $d = -4$ j 683.4375, not an AP
- 2 a 11, not an AP b 42, not an AP
 c 32, AP with $d = 3$ d 77, AP with $d = -3$
 e 4.8, AP with $d = -0.6$ f $\frac{1}{81}$, not an AP
 g $1\frac{1}{2}$, AP with $d = \frac{1}{6}$ h 3.4, not an AP
 i 2.222222, not an AP j $2\frac{1}{4}$, AP with $d = -\frac{1}{12}$

3 a

n	1	2	3	4	5	6	7	8
t_n	35	32	29	26	23	20	17	14

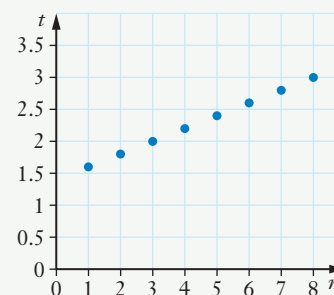
b



4 a

n	1	2	3	4	5	6	7	8
t_n	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0

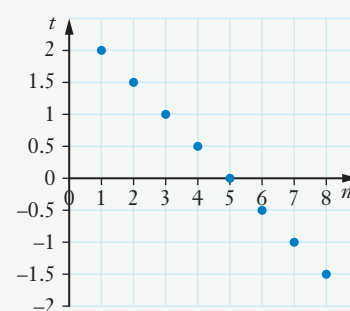
b



5 a

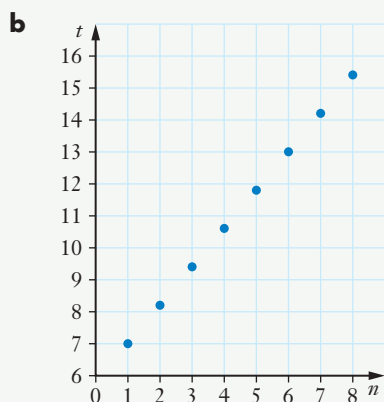
n	1	2	3	4	5	6	7	8
t_n	2	1.5	1	0.5	0	-0.5	-1	-1.5

b



6 a

n	1	2	3	4	5	6	7	8
t_n	7	8.2	9.4	10.6	11.8	13.0	14.2	15.4



7 a

Year	1	2	3	4	5	6	7	8
Diameter	4.26	4.22	4.18	4.14	4.10	4.06	4.02	3.98

b 10 years

8 Nearly 6 months (25 weeks)

9 Assuming 3 months is 13 weeks, she will have \$98.

10 About $11\frac{1}{2}$ years (11.42...)

Exercise 2.02

- 1 a** -25 **b** 15.6 **c** 38.2
d -63 **e** 4.24
2 a 64 **b** -9.6 **c** -208
d 31.9 **e** $4\frac{1}{3}$
3 $t_{58} = 402$
4 $t_{20} = 27$
5 $t_{22} = 1.98$
6 $t_{39} = 14.4$
7 $a = -2.4, d = 1.8$
8 $a = 46, d = -2$
9 $a = 44.4, d = -1.1$
10 20.6
11 $t_{95} = -3.2$
12 28

Exercise 2.03

- 1 a** \$121.90 **b** $\$(2300 + 121.9n)$
c \$3153.30

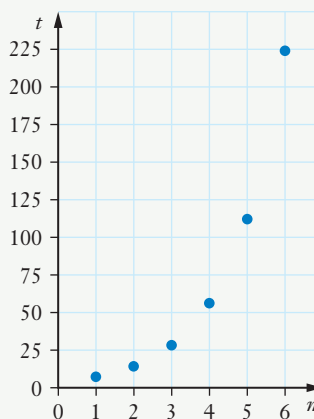
- 2 a** \$3478.95 **b** \$429.50 **c** \$12 498.45
d \$347.18 **e** $\$(12\,498.45 - 347.18n)$
f \$4166.13
3 a $\$(6.3 + 2.17n)$ **b** \$62.72
4 a \$1015 **b** \$406
c $\$(6890 - 1015n)$ **d** $\$(6890 - 0.0404n)$
e \$3845 **f** \$3276.60
5 a 16.125 s **b** 16 s
6 28 weeks; a little over 6 months
7 About 6:23 p.m., assuming it keeps raining at the same rate and no water is taken out of the tank.
8 $\$(3200 + 153.6n)$, assuming there are no deposits or withdrawals.
9 $\$(10\,735.11 - 223.65n)$, \$2683.71
10 $\$(42\,800 - 4960n)$, \$27 920

Exercise 2.04

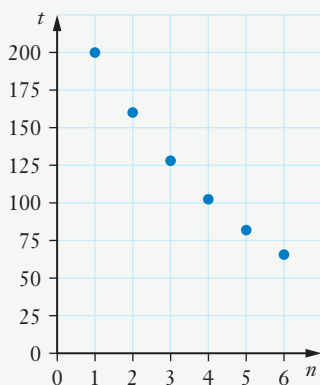
- 1 a** GP, $r = 2$ **b** Not a GP **c** GP, $r = -\frac{1}{2}$
d Not a GP **e** GP, $r = -\frac{1}{3}$ **f** GP, $r = \frac{2}{3}$
2 a Not a GP **b** GP, $r = -5$ **c** Not a GP
d GP, $r = -1\frac{1}{2}$ **e** GP, $r = 3$ **f** GP, $r = \frac{3}{4}$

3

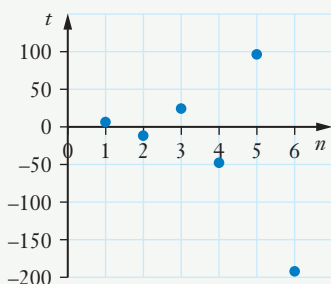
n	1	2	3	4	5	6
t_n	7	14	28	56	112	224



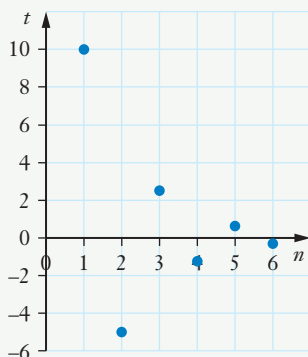
4	n	1	2	3	4	5	6
	t_n	200	160	128	102.4	81.92	65.536



5	n	1	2	3	4	5	6
	t_n	6	-12	24	-48	96	-192



6	n	1	2	3	4	5	6
	t_n	10	-5	2.5	-1.25	0.625	-0.3125



The terms of a GP with a negative common ratio alternate between curved lines that are symmetrical across the x -axis.

- 7** 1458, 486, 162, 54, 18, 6
8 0.3125, 1.25, 5, 20, 80, 320
9 37 500, 7500, 1500, 300, 60, 12
10 $\frac{4}{a}$
11 \$5400, \$5751, \$6124.82, \$6522.93, $r = 1.065$
12 For each \$100 of value, the value goes \$100, \$106, \$112.36, ... with $r = 1.06$

Exercise 2.05

- 1** **a** 1458 **b** -640 **c** 7.0875
d -12 288 **e** -0.0486 **f** 20.736
2 **a** -3937.024 **b** 70 589.4 **c** $\frac{2}{27}$
d $-\frac{13}{256}$ **e** $-\frac{7}{512}$ **f** 607.140 781 25
3 **a** 6144 **b** $-\frac{1}{225}$ **c** 562.5
d $\frac{32}{81}$ **e** $-\frac{5}{64}$ **f** $\frac{1}{5}$
4 **a** 768 **b** 36 **c** $\frac{3}{32}$
d 1250 **e** 12 **f** 933.12
5 $a = 400, r = -\frac{1}{5}$
6 $a = 5, r = 2$
7 $a = 9, r = \pm \frac{1}{2}$
8 $a = 0.08, r = 5$
9 $t_{12} \approx 307.9$
10 $t_6 = 1458$
11 $t_8 = \frac{5}{8}$
12 $t_8 \approx -133.5$
13 $t_9 \approx 205$
14 About 6 weeks ($\approx 37 \text{ cm}^2$)
15 About another 8 weeks ($t_{11} \approx 111 \text{ mice/ha}$)

Exercise 2.06

- 1** **a** About \$6333.62 **b** About \$7414.92
c About \$12 873.51
2 **a** About \$8801.03 **b** About \$10 138.47
c About \$16 634.47
3 About \$5282.50
4 About \$11 129.84
5 **a** About 1.736 m **b** About 1.206 m
c About 0.485 m

- 6 **a** About 12.6% **b** About 26.7%
c About 80.6%
- 7 **a** About \$84 503 **b** About \$65 439
c About \$34 534
- 8 **a** \$4165 **b** \$2915.50
c About \$1428.60
- 9 About 15 years
- 10 Assuming the distance from the original height changed in the same way each time, it would take 5 cycles.
- 11 5 years
- 12 Assuming that it is 'stopped' when the rebound is less than 1 mm, it will stop after 9 bounces. It also assumes that it keeps bouncing in the same way.

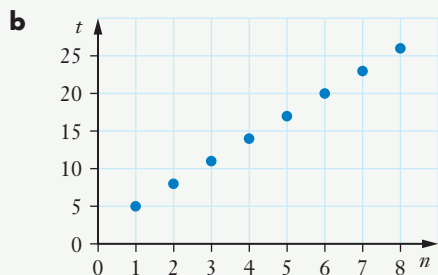
Chapter review

- 1 **a** 45, not an AP **b** $-55, d = -8$
c 3, $d = 0.4$

2 C

3 **a**

n	1	2	3	4	5	6	7	8
t_n	5	8	11	14	17	20	23	26

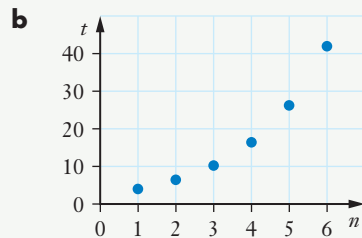


- 4 **a** About 5 or 6 years **b** About 12.2 m
c About another 33 or 34 years
- 5 D
- 6 10.1
- 7 $t_{159} = 400.2$
- 8 $a = 5, d = 1.5$
- 9 **a** \$34.80 **b** $$(600 + 34.80n)$
c 9 years
- 10 **a** \$17 199.40 **b** \$13 799.60
c \$10399.80
- 11 \$3920
- 12 A

- 13 **a** $r = 2$ **b** Not a GP **c** Not a GP

14 **a**

n	1	2	3	4	5	6
t_n	4	6.4	10.24	16.384	26.2144	41.943 04

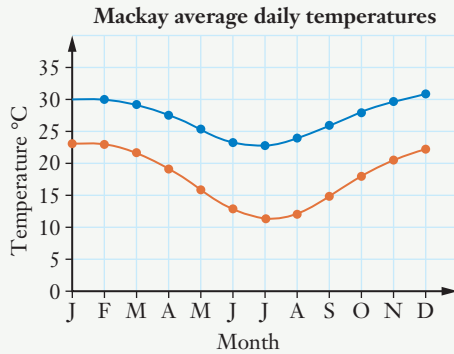


- 15 200, 160, 128, 102.4, 81.92, 65.536
- 16 **a** 1.5552 **b** 437.4
- 17 **a** 2 or -2 **b** 0.6
c 76.8 or -76.8
- 18 $t_{15} \approx 115.553$
- 19 **a** About \$985.54 **b** About \$1132.57
c About \$1603.39
- 20 **a** About 4.3 cm
b It would be bouncing about 8.9 cm, possibly causing damage.
- 21 **a** About \$3339 **b** About \$2586
c About \$1672
- 22 Assuming that it keeps reducing in the same way, it will take about 60 years.
- 23 35.6
- 24 \$6000
- 25 7 years
- 26 Assuming that it keeps increasing by the same percentage each week, it will cover 3200 m^2 .
- 27 12 years
- 28 \$1959.96

Chapter 3

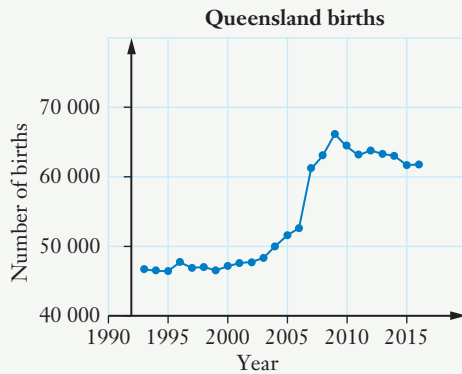
Exercise 3.01

1 a



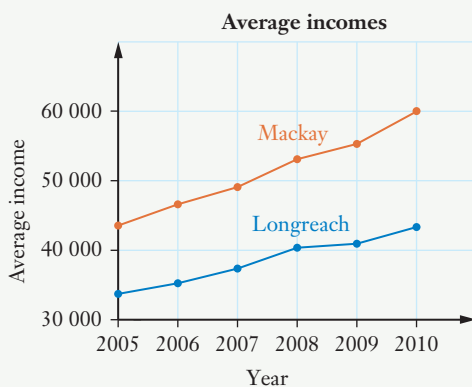
b The temperature is lowest in winter.

2 a



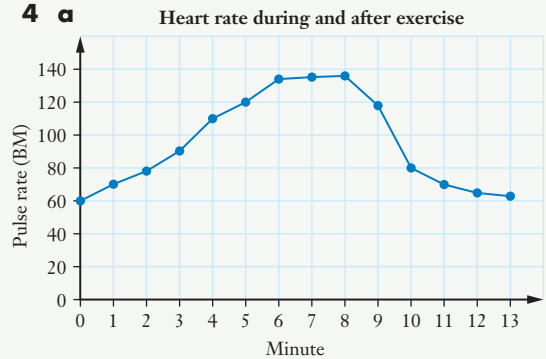
b There was a marked increase in births between 2005–2010, and then it stabilised at the higher level.

3 a



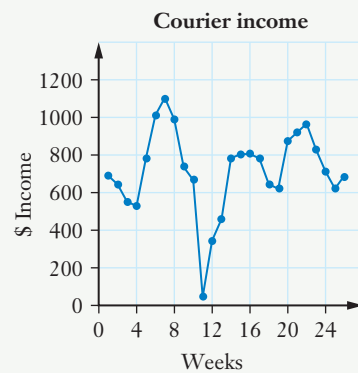
b There is a steady increase of income, but the incomes in Mackay are higher than those in Longreach.

4 a



b Her heart rate goes up during exercise and recovers about 5 minutes after exercising.

5 a



b The income is not regular.

Exercise 3.02

- 1 Increasing trend
- 2 Seasonal
- 3 Decreasing trend
- 4 Seasonal
- 5 Cyclic
- 6 Irregular
- 7 Seasonal
- 8 Seasonal with an increasing trend
- 9 Seasonal with a decreasing trend
- 10 Seasonal with an increasing trend

Exercise 3.03

- 1 On Saturday of week 3, the sales appear low.
- 2 There are outliers in December 2009 and March 2015.
- 3 Although visitor numbers were low in 2010, it is not an outlier.

- 4 October 2016 is outside the general range so is an outlier of the downward trend.
- 5 The cost of feed in May 2016 is outside the general range of the prices over this period.

Exercise 3.04

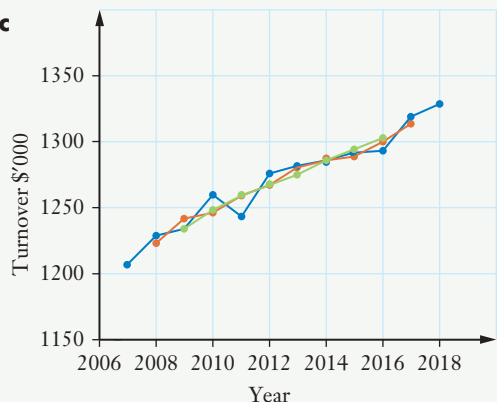
1 a

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
3-y ave	1223	1241	1245	1259	1267	1280	1286	1289	1301	1314

b

Year	2009	2010	2011	2012	2013	2014	2015	2016
5-y ave	1234.2	1248	1258.6	1268.4	1275	1285	1293.8	1303.2

c



- d The smoothed averages straighten the line, the 5-year a bit more than the 3-year.

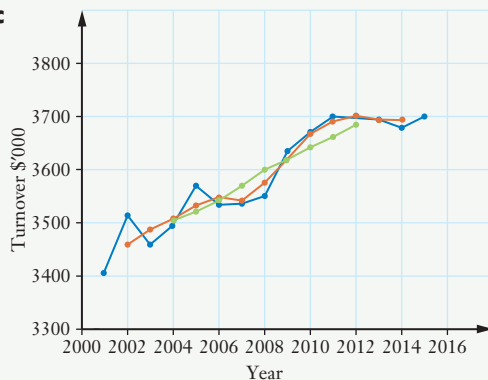
2 a

Year	2002	2003	2004	2005	2006	2007	2008
3-y ave	3460	3489	3508	3533	3547	3541	3574
Year	2009	2010	2011	2012	2013	2014	
3-y ave	3620	3669	3692	3700	3693	3694	

b

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012
7-y ave	3502	3523	3540	3571	3600	3619	3642	3663	3685

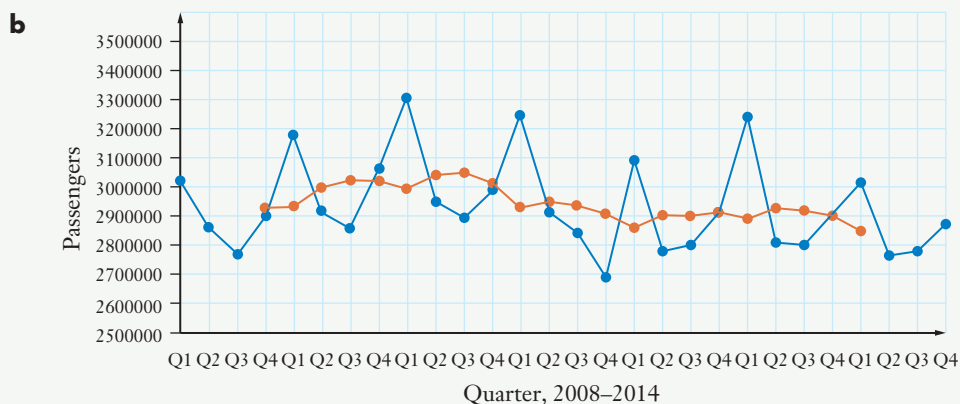
c



- d The smoothed averages straighten the line, the 7-year much more than the 3-year.

3 a

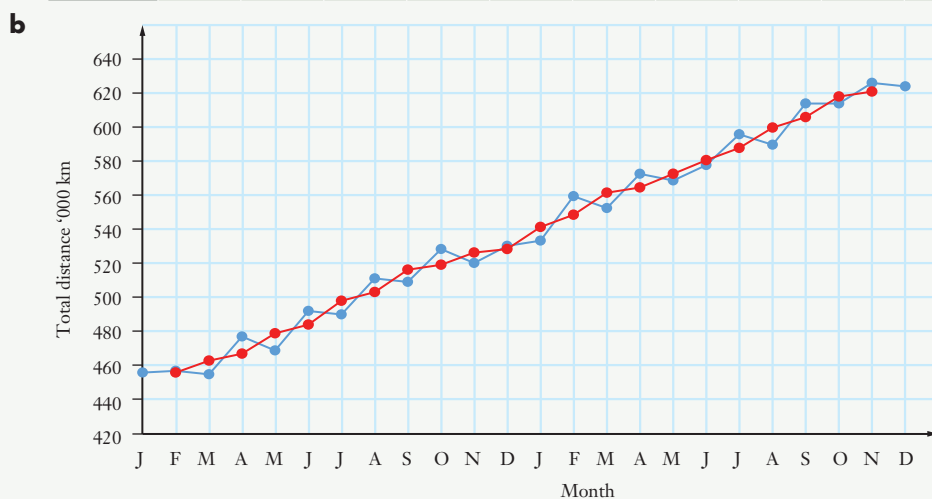
Time	Q4	Q1	Q2	Q3	Q4	Q1	Q2
7-Q ave	2926895	2932305	2996195	3021891	3020727	2992715	3039531
Time	Q3	Q4	Q1	Q2	Q3	Q4	Q1
7-Q ave	3048387	3017308	2928784	2948456	2933076	2906357	2858523
Time	Q2	Q3	Q4	Q1	Q2	Q3	Q4
7-Q ave	2904149	2899868	2915925	2890422	2924604	2919860	2901831
Time	Q1						
7-Q ave	2849875						



c The original data is seasonal and the smoothing does not remove the seasonality.

4 a

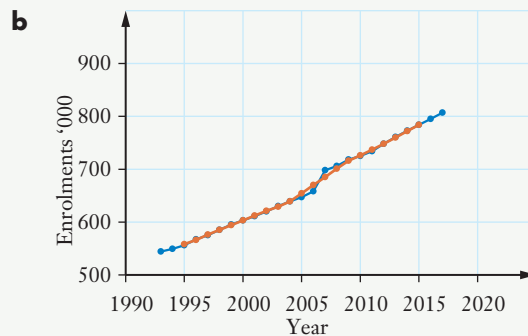
Month	F	M	A	M	J	J	A	S	O	N	D
3-m ave	456	463	467	479	484	498	503	516	519	526	528
Month	J	F	M	A	M	J	J	A	S	O	N
3-m ave	541	548	561	564	572	580	587	599	605	617	620



The trend is much clearer on the graph of moving averages.

5 a

Year	1995	1996	1997	1998	1999	2000	2001
5-y ave	558	566	576	585	594	603	612
Year	2002	2003	2004	2005	2006	2007	2008
5-y ave	620	629	639	654	670	685	701
Year	2009	2010	2011	2012	2013	2014	2015
5-y ave	716	726	737	748	760	772	784

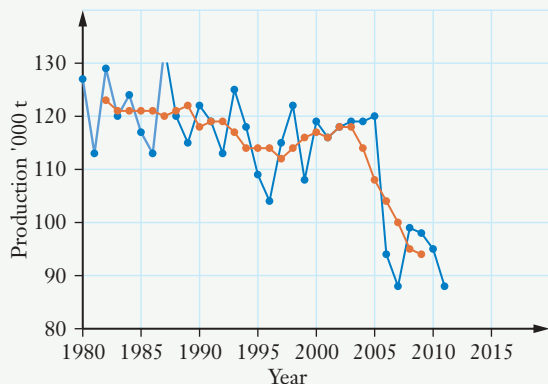


c Both graphs show an unusual increase in enrolments in 2006–2007, when Prep was introduced.

6 a

Year	82	83	84	85	86	87	88
Ave	122.6	120.6	120.6	121.4	121.4	119.6	120.6
Year	89	90	91	92	93	94	95
Ave	121.8	117.8	118.8	119.4	116.8	113.8	114.2
Year	96	97	98	99	00	01	02
Ave	113.6	111.6	113.6	116	116.6	116	118.2
Year	03	04	05	06	07	08	09
Ave	118.4	114	108	104	99.8	94.8	93.6

b



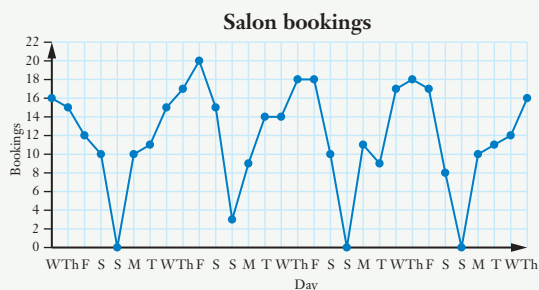
c The 5-year moving average is less uneven, but it still shows a significant drop around 2005.

7 2020, by extending the smoothed line. The prediction could be wrong if production methods changed or more coal was discovered in the mine.

8 It looks like there is a cycle of about 12–13 years between about 4000 and 5000 kt. In this case, production in 2025 is likely to be low, about 4000 kt. If the cycle length varies or there is some other change, the prediction could be incorrect.

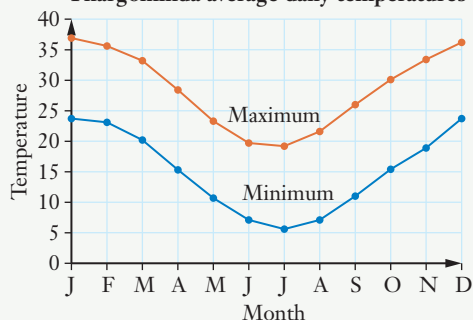
Chapter review

1 a



b She is not usually open on Sunday, and has more bookings later in the week than at the beginning of the week.

2 a Thargominda average daily temperatures



b The temperature varies about 15° from day to night and summer to winter.

3 B

4 There is a seasonal variation with the high in summer.

5 There is a rising trend.

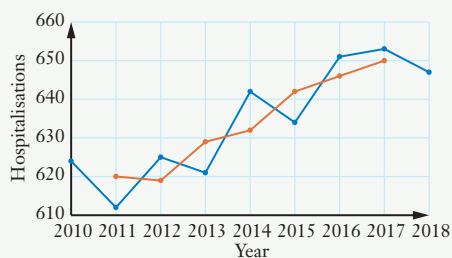
6 There are outliers in 1997 and 2006.

7 There are outliers in Sept.–Nov. 2015.

8 a

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018
3-y ave		620	619	629	632	642	646	650	

b

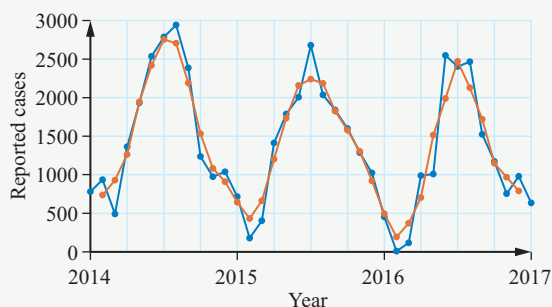


c The moving averages make a much smoother line than the original data. There is an upward trend.

9 a

2014	J	F	M	A	M	J	J	A	S	O	N	D
3-m ave		735	929	1261	1942	2416	2753	2704	2188	1532	1082	908
2015	J	F	M	A	M	J	J	A	S	O	N	D
3-m ave	642	431	663	1200	1734	2156	2239	2184	1824	1574	1302	920
2016	J	F	M	A	M	J	J	A	S	O	N	D
3-m ave	493	192	370	704	1514	1986	2471	2129	1719	1148	967	789

b



c The seasonal variation is not removed by 3-month smoothing.

10 It shows seasonal variation and a rising trend.

11 It will probably be about \$190–200/ha, but the trend could change.

Practice examination 1

1

	TI	Casio	Sharp	Total
Sandy Hills SHS	23	20	4	47
Mt Flute SHS	47	11	16	74

2 C

3 a

	Dark	Medium	Light	Total
Male	9	6	4	19
Female	2	5	4	11
Total	11	11	8	30

b

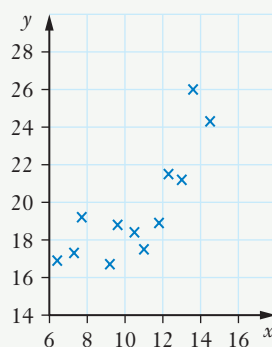
	Dark	Medium	Light	Total
Male	47%	32%	21%	100%
Female	18%	45%	36%	99%

Females don't add to 100% because of rounding errors.

c In this group, females prefer lighter toast than males.

4 B

5 a



b There is a moderate positive association.

6 a $r \approx -0.85$

b There is a strong negative association.

7 a Arithmetic sequence, $d = -8$

b Neither

c Neither

d Geometric sequence, $r = 1.5$

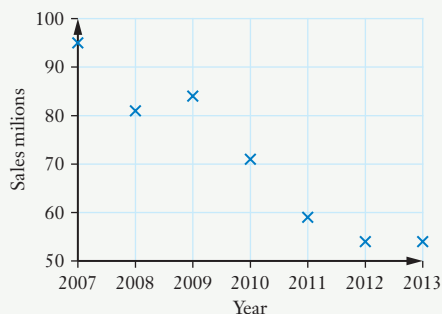
8 53

9 It takes 58 terms for the sequence to exceed 400.

10 $1\frac{1}{4}$

11 $t_2 = 0.0878$

12 a



b There is a negative association, sales of DVDs decreased.

13 a Irregular b Decreasing trend

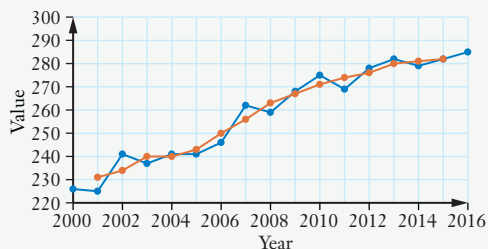
c Increasing trend d Irregular

14 E

15 a Averaged data

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Value		230.7	234.3	239.7	239.7	242.7	249.7	255.7	263
Year	2009	2010	2011	2012	2013	2014	2015	2016	
Value	267.3	270.7	274	276.3	279.7	281	282		

b



c The graph with the smoothed data is smoother.

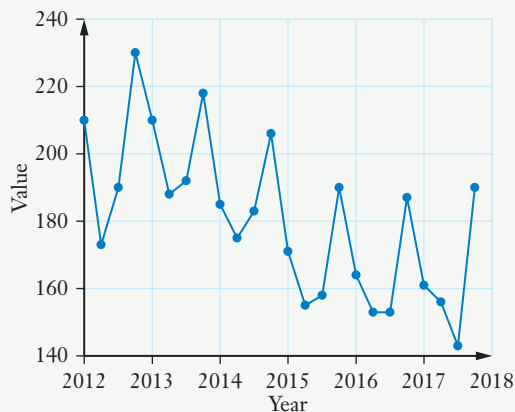
16

	r	h	g
F	7	5	2
H	2	15	13
U	0	8	10

The probabilities of being mostly red are 50% in full sun, about 7% in half sun and 0% in full shade, so the best chance of getting mostly red is in full sun.

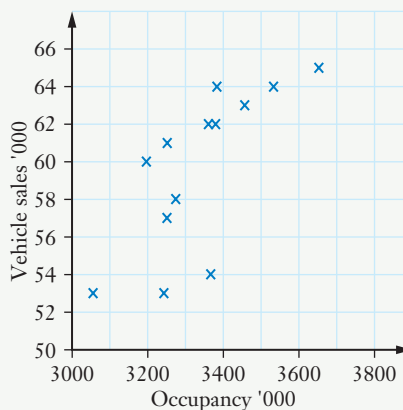
17 Areas each day are about 450, 518, 595, 684, 787, 905, ..., so it takes 6 days.

18



It is seasonal with a negative trend.

19



There is a strong positive association with $r = 0.93$, but there is no relationship because there is no mechanism that could explain the association.

20 Straight-line method

$$\text{depreciation} = \frac{140\,000 - 45\,000}{6} \approx \$15\,833/\text{year}.$$

$$\begin{aligned} \text{Book value after 3 years} \\ = \$140\,000 - 3 \times \$15\,833 = \$92\,500 \end{aligned}$$

Double declining-balance depreciation

$$\text{rate} = \frac{2 \times 15\,833}{140\,000} \times 100\% \approx 22.6\%/\text{year}$$

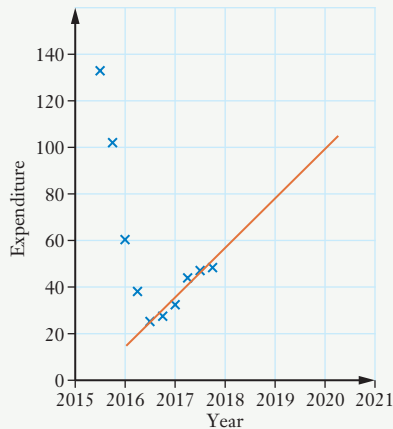
$$\text{Value after 3 years} = \$140\,000 \times 0.774^3 \approx \$56\,731$$

The double declining-balance method gives a book value of \$56 731 compared to \$92 500 for the straight-line method.

21 The 3-quarter centred averages:

2015	Mar	Jun	Sep	Dec	2016	Mar	Jun	Sep	Dec
Expenditure (\$millions)		132.8	102.0	60.4		38.1	25.1	27.5	32.4
2017	Mar	Jun	Sep	Dec					
Expenditure (\$millions)	43.9	47.1	48.3						

The graph is:

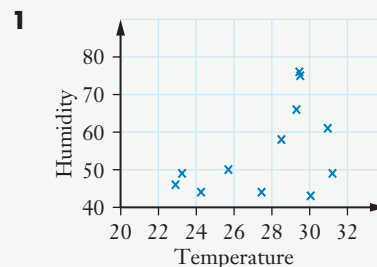


If expenditure kept increasing as it was in 2016–2017, the amount spent in the March quarter of 2020 would be about \$114.6 million. However, this is well outside the data range, so it may be incorrect.

Chapter 4**Exercise 4.01**

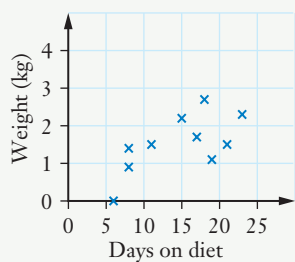
- 1**
 - a** Temperature, evaporation rate
 - b** Time, mass of ice melted
 - c** Mass of flour, cost
 - d** Number of hens, number of eggs laid
 - e** Number of storms, erosion of beach

- 2**
 - a** Rainfall, river flow
 - b** Number of birds, seedlings eaten
 - c** Time spent training, 100 m time
 - d** Logs milled, sawdust
 - e** Sawdust, time to sweep
- 3**
 - a** Wrong way round
 - b** Correct
 - c** Correct
 - d** Wrong way round
- 4** Age of student, height of student; Weight of vegetable, cost of vegetable; Sail area of yacht, speed of yacht; Number of tiles available, floor area tiled; Size of army, cost of army; Length of yacht, sail area of yacht
- 5** The following are not numerical, so cannot be explanatory or response. Type of vegetable; Hair colour of student

Exercise 4.02

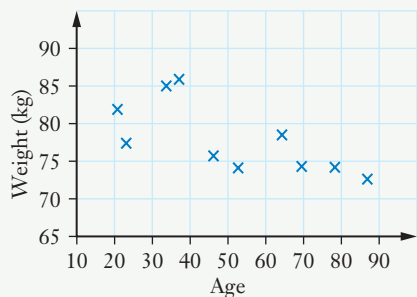
There seems to be a weak positive relationship.

2



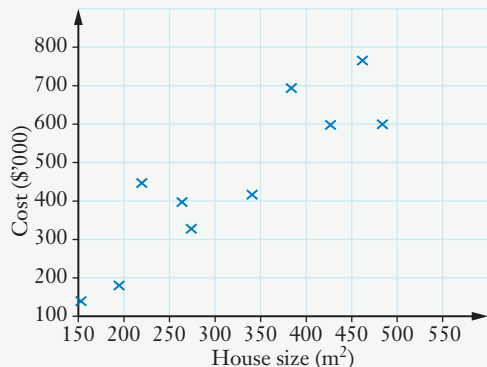
There is a weak positive relationship.

3



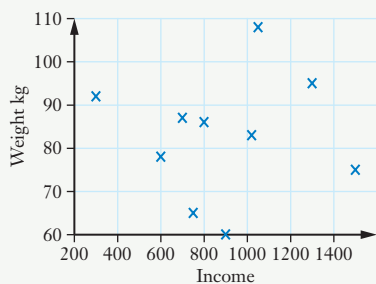
There is a weak negative relationship.

4



There is a moderate positive relationship.

5

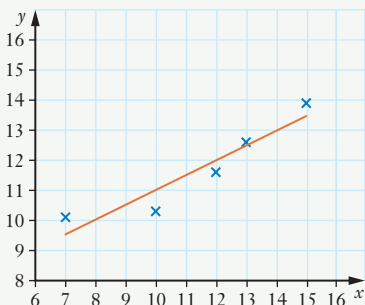


There is no relationship.

Exercise 4.03

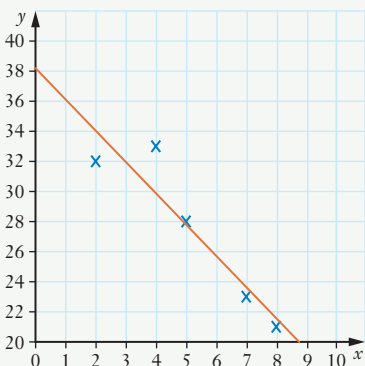
1 a $y \approx 0.49x + 6.09$

b



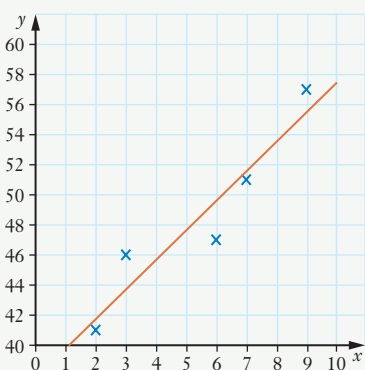
2 a $y \approx -2.08x + 38.21$

b



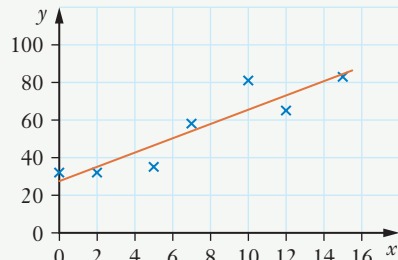
3 a $y \approx 1.96x + 37.80$

b



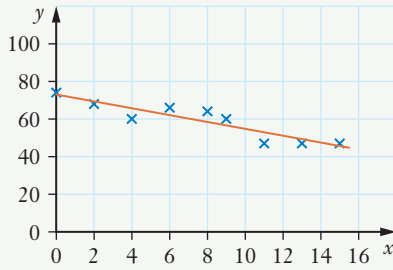
4 a $y \approx 3.81x + 27.41$

b



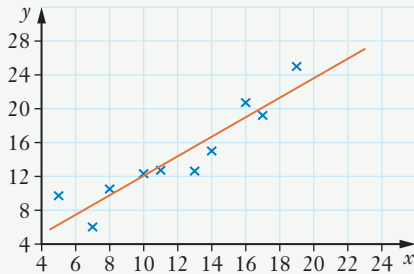
5 a $y \approx 73.05 - 1.83x$

b



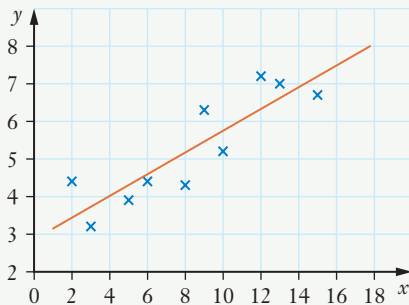
6 a $y \approx 1.15x + 0.53$

b



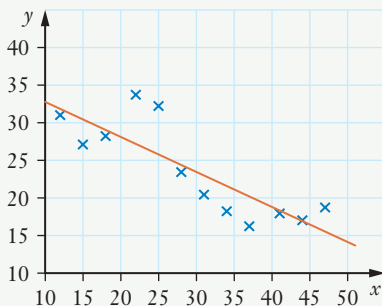
7 a $y \approx 0.29x + 2.85$

b



8 a $y \approx 37.43 - 0.47x$

b



9 $t \approx 0.212h + 0.904$. Perhaps the longer the shop is open, the more customers come in and spend money. Perhaps the shop opens for the hours that it knows will make money on any particular day.

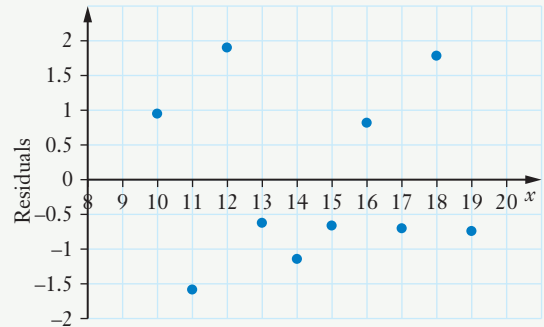
10 $t \approx 140.16 - 8.64m$, about 36 months

Exercise 4.04

1 a

x	10	11	12	13	14	15	16	17	18	19
Residuals	0.95	-1.58	1.9	-0.62	-1.14	-0.66	0.82	-0.7	1.78	-0.74

b

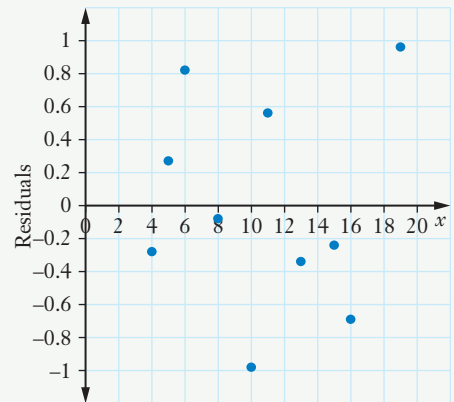


c The residuals are evenly scattered on both sides of the axis.

2 a

x	4	5	6	8	10	11	13	15	16	19
Residuals	-0.28	0.27	0.82	-0.08	-0.98	0.56	-0.34	-0.24	-0.69	0.96

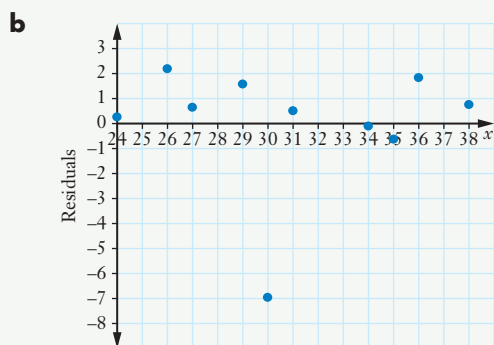
b



c The residuals are evenly scattered on both sides of the axis.

3 a

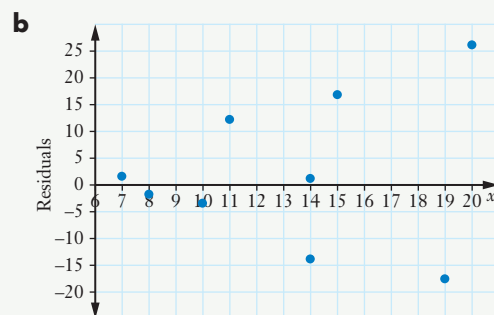
x	24	26	27	29	30	31	34	35	36	38
Residuals	0.25	2.18	0.64	1.57	-6.96	0.5	-0.11	-0.64	1.82	0.75



c The residuals have an outlier at $x = 30$.

4 a

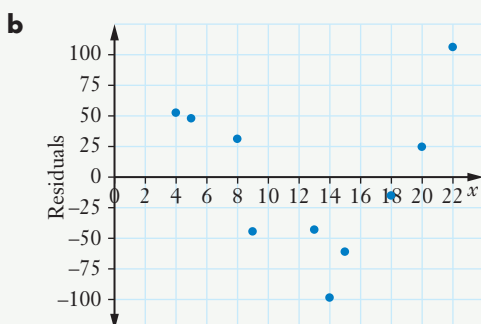
x	7	8	10	11	14	14	15	17	19	20
Residuals	1.54	-1.81	-3.49	12.16	-13.87	1.13	16.78	-20.91	-17.59	26.06



c The residuals spread out to the right.

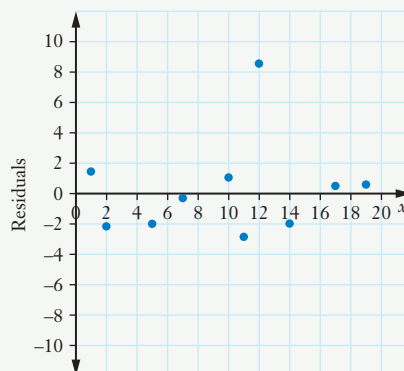
5 a

x	4	5	8	9	13	14	15	18	20	22
Residuals	52.54	47.9	31.07	-44.55	-43.03	-98.7	-61.3	-15.12	24.64	106.4

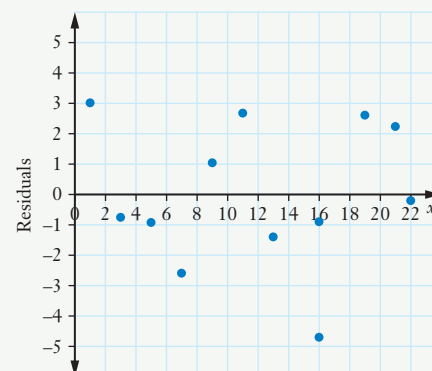


c The residuals start as positive, go negative and end as positive again; possible curve.

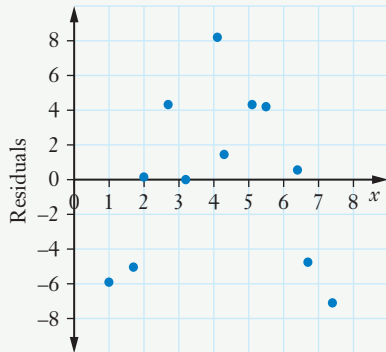
- 6 a** Inappropriate, outliers
b Inappropriate, spread to the right
c Appropriate, evenly spread to both sides with no pattern
d Inappropriate, has a pattern that could be a curve
e Appropriate, evenly spread to both sides with no pattern
f Inappropriate, outliers
7 a Inappropriate, spread to the right
b Inappropriate, has a pattern that could be a curve
c Inappropriate, outliers
d Appropriate, evenly spread to both sides with no pattern
e Inappropriate, a curve
f Inappropriate, spread to the right
8 a $y = 24.656 - 0.991x$
b Inappropriate, outliers



- 9 a** $y = 1.534x + 6.266$
b Appropriate



- 10 **a** $y = 51.781 - 5.956x$
b Inappropriate, perhaps a curve as it starts below the axis, goes above, and then goes under again



Exercise 4.05

- a** Slope is 0.7 and y -intercept is 0.85.
b The price increases by 70 cents for each kg, but there are handling charges of 85 cents, regardless of the weight.
- a** Slope is 0.24 and y -intercept is 64.
b On average, the weight of men between the ages of 20 and 60 increases by 0.24 kg every year. The y -intercept means that at that rate, they would have a birth weight of 64 kg, but the relationship is only for ages 20–60 so this is not valid.
- a** Gradient ≈ 0.00097 , y -intercept ≈ 0.03
b $h \approx 0.00097r + 0.03$
c The river level increases by about 0.00097 m for every mm of rainfall. The y -intercept means that there is a minimum level of about 3 cm.
- a** Gradient ≈ 6.52 , y -intercept ≈ -50
b $y = 6.52r - 50$
c The yield of peanuts increases by about 6.52 kg/ha for every increase of 1 mm in the rainfall. The y intercept does not have any meaning in this context as it is not possible to have a negative rainfall.
- a** There is no relationship between the variables.
b There is a 20% commonality between the variables.
c Half of the response variable is determined by the explanatory variable. There is a moderate relationship.

- The response variable has a definite relationship with the explanatory variable. 80% of the variation in the response variable is dependent on the explanatory variable.
- The variation in the variables is perfect.
- That larger countries have a slight tendency to have larger populations, but that only 26% of the population variation is due to the area. Most of the variation of population is not related to the area.

Exercise 4.06

- a** 20 million **b** 23.1 million
c 24.8 million **d** 27.2 million
e 32 million
- a** 111% **b** 135% **c** 144% **d** 32%
e -79%, but this is not possible. The extrapolation is too far outside the data range.
- a** 73% **b** 52.5%
c 114%, but humidity cannot be this far over 100%. The temperature is outside the range of the average for a North Queensland town.
d -50%, but this is not possible. The temperature is outside the range of the average for a North Queensland town.
- a** \$240 000 **b** \$580 000
c \$70 000, but an average modern house would be bigger than 100 m², so it is outside the data range.
d \$1 430 000, but an average house would not be 900 m², so it is outside the data range
- $f = 27.92 - 0.18a$, so the frequencies would be 25.2 kHz, 19.8 kHz and 11.7 kHz, but 90 could be extrapolated too far outside the data range.

Exercise 4.07

- a** It seems unlikely.
b The decrease in Devils is due to an infectious cancer, so it is likely to be coincidence.
- It is likely that they have a confounding variable, the population.
- Coal mining is a dangerous occupation, so they could be related by a confounding variable, coal production. Otherwise, it could be coincidence.
- It is most likely to be coincidence as there seems to be no possible mechanism, unless they are related to better health care.

- 5** It is possible that land-clearing is associated with both as a confounding variable, or that both are related to increased population. Otherwise it could be coincidence.
- 6** It is most likely to be coincidence as there seems to be no possible mechanism.

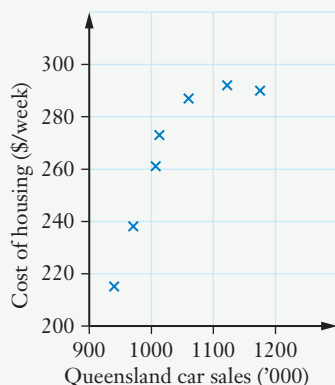
Exercise 4.08

- 1** Males: 9.7%, Females: 12.1%; Women are 24.7% times more likely to have asthma than men.
- 2** Overweight males: 60.2%, Overweight females: 52%; Men are 15.6% times more likely to be overweight or obese than women.
- 3** 236 000
- 4** Not really, it is likely to be the same as the increase in population.
- 5** 25.2 million, but 2030 is a long time ahead of the data range.
- 6** For every increase of 1 million in population there is an increase of 1.1618 million phones. This does suggest some of the phones purchased are replacement phones.

Chapter review

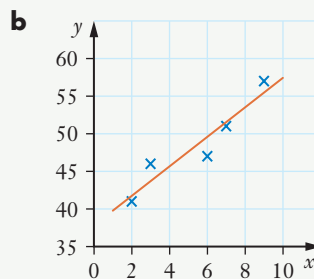
- 1 a** Dryness = response, time = explanatory
- b** Flow = response, diameter = explanatory
- c** Speed = response, motor = explanatory
- 2** B
- 3 a** Correct
- b** The wrong way around, days at sea is the explanatory variable.

4

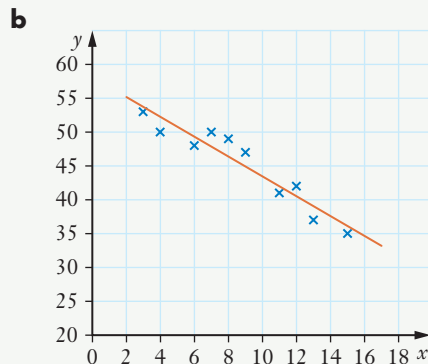


It doesn't seem likely that car sales and housing costs would have a relationship. Up until about 2010, perhaps they had a confounding variable.

5 a $y = 13.2x + 16.8$



6 a $y = 58.1 - 1.47x$

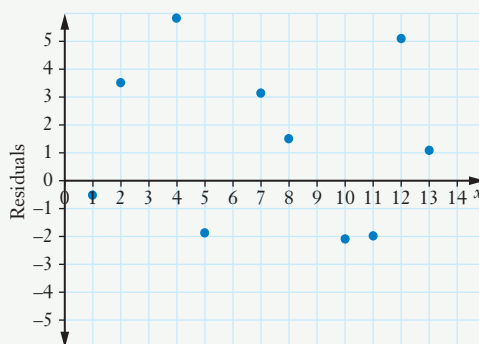


7 E

8 a

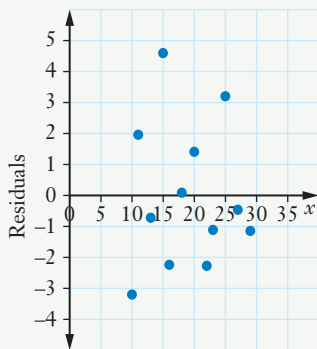
x	1	2	4	5	7	8	10	11	12	13
Residuals	-0.56	3.6	5.91	-1.93	3.38	1.54	-2.15	-1.99	5.17	1.32

b



- c** The residuals are scattered evenly on both sides of the axis.
- 9 a** Inappropriate, there is a possible curved pattern.
- b** Appropriate, points are evenly scattered above and below the axis.
- c** Inappropriate, the points spread out to the right.

- 10 a** Slope is 1.4 and the intercept is 125.
b It costs \$1.40 every minute the repairer is there and \$125 for the call.
- 11 a** Slope ≈ 17 , y -intercept ≈ 475
b The slope means the farm price increased about \$17/tonne each year. The y -intercept means that it was probably about \$475/tonne in 2000.
- 12** It means that 75% of the changes in the response variable change in the same way as the explanatory variable.
- 13** Interpolation predicts a value within the range of the original data. Extrapolation predicts a value outside that range.
- 14 a** 16.5° **b** 27.1°
- 15 a** It is very unlikely.
b It could be coincidence, but it is more likely that the population is a confounding variable.
- 16** Response and explanatory variables are both numeric. Eye colour and type of metal are both categorical variables, so they cannot be compared.
- 17** $y \approx 1.839x - 1.175$, $R^2 = 0.96$. The residuals below are evenly scattered above and below the axis. This means it is appropriate.



- 18** When the fisheries catch has gone down by \$100 000, the amount spent on gambling has gone up by \$1.169 million. 60% of the change is common to both variables. It is likely that the catch is decreasing over time and the amount of gambling is increasing and the variables are confounded by time.

3	Month	J	F	M	A	M	J	J	A	S	O	N	D
	Index	1.004	0.918	0.944	0.888	0.952	0.974	0.926	1.025	1.064	1.085	1.128	1.094

- 19** The table can be written as percentages horizontally and vertically.

	Blue eyes	Dark eyes
Fair hair	66.3%	39.0%
Dark hair	33.7%	61.0%
	100%	100%

or

	Blue eyes	Dark eyes	
Fair hair	53.4%	46.6%	100%
Dark hair	27.1%	72.9%	100%

Someone with blue eyes is nearly twice as likely to have fair hair as dark hair, while those with fair hair are only a little more likely to have blue eyes than dark eyes.

Someone with dark eyes is about 50% more likely to have dark hair than light hair, while those with dark hair are about 2.7 times as likely to have dark eyes than blue eyes.

Fair hair goes with blue eyes, and dark eyes go with dark hair.

- 20** The scatterplot shows a weak negative relationship. $M \approx 1\,460\,000 - 665y$, with $R^2 \approx 0.306$. Only 30% of the decline in marriages is common to the year, so it is likely that something else is more important. It may be that the attitude to marriage in Australian society is changing.

Chapter 5

Exercise 5.01

1	Quarter	Q1	Q2	Q3	Q4
	Index	0.868	1.06	1.1	0.972

2	Quarter	Q1	Q2	Q3	Q4
	Index	1.04	0.994	0.943	1.023

4	Month	J	F	M	A	M	J	J	A	S	O	N	D
	Index	0.837	0.857	0.971	0.991	1.105	1.172	1.092	1.138	1.031	1.011	0.871	0.924

5	Quarter	Q1	Q2	Q3	Q4
	Index	1.053	0.972	0.934	1.041

6	Quarter	Q1	Q2	Q3	Q4
	Index	1.057	0.98	0.955	1.008

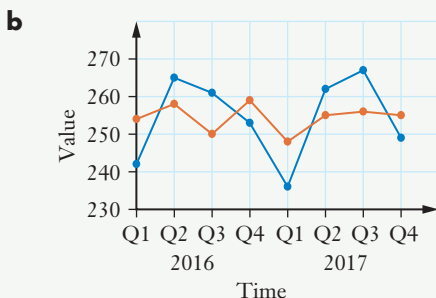
7	Quarter	Q1	Q2	Q3	Q4
	Index	0.914	1.043	1.031	1.013

8	Month	J	F	M	A	M	J	J	A	S	O	N	D
	Index	1.06	1.11	1.054	1.006	0.959	0.972	0.945	0.91	0.956	0.988	1.013	1.026

Exercise 5.02

1 a

2016	Q1	Q2	Q3	Q4
Original	242	265	261	253
Deseasonalised value	254	258	250	259
2017	Q1	Q2	Q3	Q4
Original	236	262	267	249
Deseasonalised value	248	255	256	255

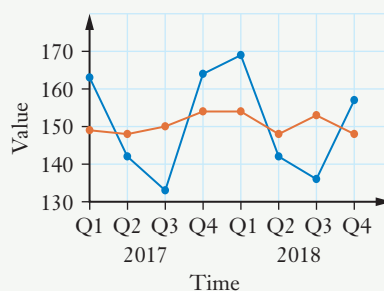


c The smoothed series does not vary as much as the original.

2 a

2017	Q1	Q2	Q3	Q4
Original	163	142	133	164
Deseasonalised value	149	148	150	154
2018	Q1	Q2	Q3	Q4
Original	169	142	136	157
Deseasonalised value	154	148	153	148

b

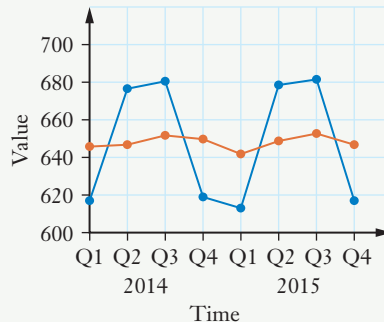


c The smoothed series is much smoother than the original.

3 a

2014	Q1	Q2	Q3	Q4
Original	617	677	681	619
Deseasonalised value	646	647	652	650
2015	Q1	Q2	Q3	Q4
Original	613	679	682	617
Deseasonalised value	642	649	653	647

b

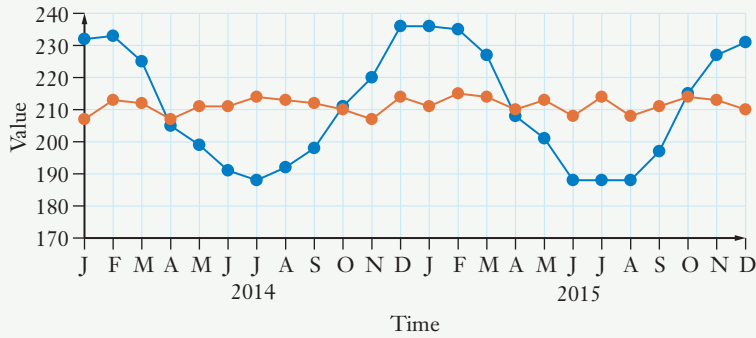


The indexed series is much smoother than the original.

4 a

2014	J	F	M	A	M	J	J	A	S	O	N	D
Original	232	233	225	205	199	191	188	192	198	211	220	236
Indexed	207	213	212	207	211	211	214	213	212	210	207	214
2015	J	F	M	A	M	J	J	A	S	O	N	D
Original	236	235	227	208	201	188	188	188	197	215	227	231
Indexed	211	215	214	210	213	208	214	208	211	214	213	210

b

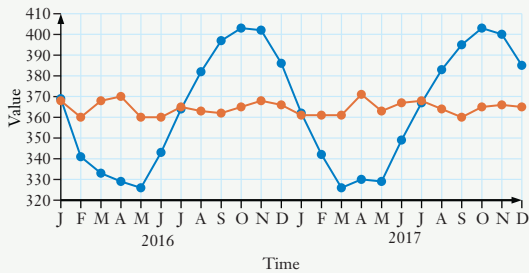


c The indexed series is much smoother than the original.

5 a

2016	J	F	M	A	M	J	J	A	S	O	N	D
Original	369	341	333	329	326	343	364	382	397	403	402	386
Indexed	368	360	368	370	360	360	365	363	362	365	368	366
2017	J	F	M	A	M	J	J	A	S	O	N	D
Original	362	342	326	330	329	349	367	383	395	403	400	385
Indexed	361	361	361	371	363	367	368	364	360	365	366	365

b

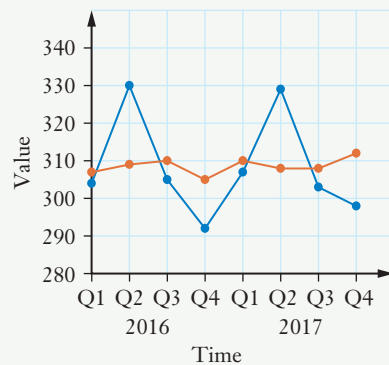


c The indexed series is much smoother than the original.

6

Quarter	Q1	Q2	Q3	Q4
Index	0.99	1.068	0.985	0.956

Plot of original and indexed data:



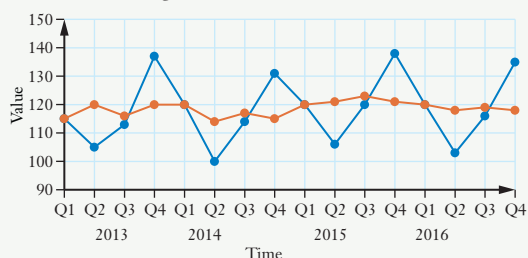
The indexed plot is much smoother than the original.

7 Index is:

Quarter	Q1	Q2	Q3	Q4
Index	1.004	0.875	0.978	1.143

The indexed plot is smoother than the original.

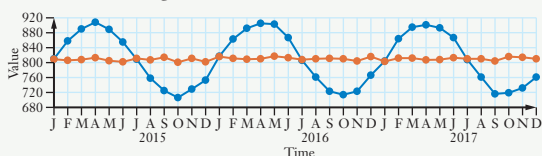
Plot of original and indexed data:



8 Index is:

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	1.001	1.064	1.102	1.117	1.105	1.066	0.997	0.939	0.891	0.881	0.899	0.939

Plot of original and indexed data:



The indexed plot is much smoother than the original.

Exercise 5.03

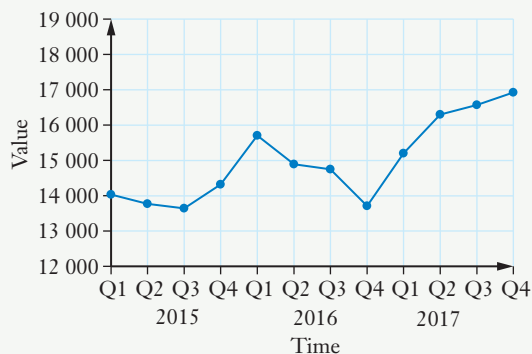
1 a Index:

Quarter	Q1	Q2	Q3	Q4
Index	0.915	1.13	0.935	1.019

b Deseasonalised data:

2015	Q1	Q2	Q3	Q4
Value	14 001	13 770	13 650	14 331
2016	Q1	Q2	Q3	Q4
Value	15 667	14 890	14 760	13 720
2017	Q1	Q2	Q3	Q4
Value	15 169	16 297	16 580	16 934

c



d There is a positive trend.

e Increasing population or more transport or more farming

2 There is a positive trend. Profits have been more increasing.

3 There is no apparent trend. Wages and salaries have not changed.

4 There is a positive trend.

5 There is no apparent trend.

Exercise 5.04

1 a $V \approx 4.1Y - 8146.5$ b 135.5

c The data is over 5 years and this is 3 years later, so it is probably accurate.

2 a $V \approx 6.786Y - 13\,445$ b 330.58

c The data is over 8 years, but this is 14 years later, so it may not be accurate.

3 a $V \approx 25.771Y - 51\,245$ b 863.962

c The data is over 6 years and this is 6 years later, so it may not be accurate.

4 a $V \approx 8.16Y - 16\,170$ b 329.52

c The data is over 10 years but this is 15 years later, so it may not be accurate.

5 a $V \approx 3.94Y - 7763$ b 235.2

c The data is over 20 years and this is 14 years later, so it may be accurate.

6 a $V \approx 5.875Y - 11\,544$ b 353

c The data is over 15 years and this is 8 years later, so it is probably accurate.

Exercise 5.05

- 1 **a** About 344 **b** About 313
c About 350 **d** About 382
2 **a** About 135 **b** About 120
3 **a** About 429 **b** About 440

4 **a**

Quarter	Q1	Q2	Q3	Q4
Index	0.993	1.021	1.005	0.981

b $V \approx 5.45Y - 10\,699$ **c** About 333

d About 245

5 **a**

Quarter	Q1	Q2	Q3	Q4
Index	0.993	0.964	0.993	1.050

b $V \approx 9.675Y - 19\,082$

c About 477

d About 324

6 About 256

Chapter review

1

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	1.135	1.145	1.056	1.01	0.914	0.848	0.848	0.851	0.937	1.013	1.099	1.142

2

Quarter	Q1	Q2	Q3	Q4
Index	0.922	0.989	1.098	0.991

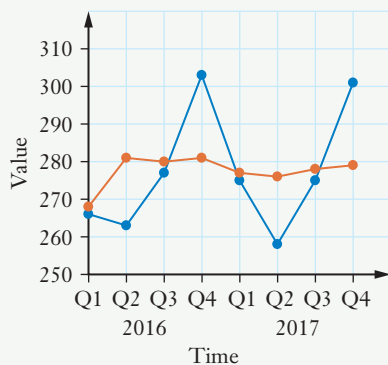
3 B

4 A

5 **a**

2016	Q1	Q2	Q3	Q4	2017	Q1	Q2	Q3	Q4
Value	268	281	280	281	Value	277	276	278	279

b



6

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	164	162	161	165	162	163	165	165	163	179	178	183

- 7 a** $Y \approx 6.9Y - 13\,768$ **b** About 204.5
c The data is over 5 years and 2025 is 8 years later, so it may not be accurate.
- 8 a** $Y \approx 5.11Y - 9917$ **b** About 431
c The data is over 15 years and 2025 is 10 years later, so it is probably accurate.

9 a

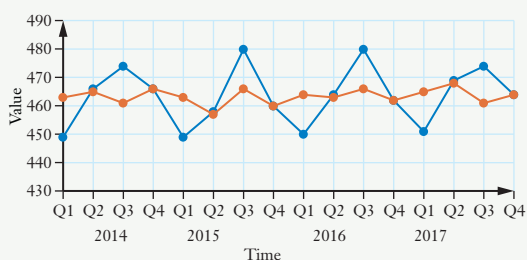
Quarter	Q1	Q2	Q3	Q4
Index	1.023	0.992	0.987	0.998

- b** $Y \approx 6.05Y - 11\,841$ **c** About 375

10 Index:

Quarter	Q1	Q2	Q3	Q4
Index	0.97	1.002	1.029	0.999

Plots



The indexed values are much smoother than the original values.

- 11** The seasonally adjusted data show an upward trend, but the values are all the same for each year. This suggests that the ABS has obtained the data for each quarter from annual data.
- 12** $W \approx 26.482Y - 52\,215$, giving \$1278.64 in 2020 and \$1543.46 in 2030.

The figure for 2020 is probably close, but the one for 2030 may be incorrect because it is 13 years into the future and the data only covers 7 years.

Chapter 6

Exercise 6.01

- 1 a** No **b** Equator
c South Pole **d** Yes
- 2** East

- 3 a** 20°S **b** 60°N **c** 20°N
d 0°N **e** 40°N **f** 0°N
g 20°S **h** 40°S **i** 60°N
j 40°S
- 4 a** 80°E **b** 20°W **c** 140°E
d 40°E **e** 80°W **f** 100°E
g 20°E **h** 60°E **i** 80°W
j 20°E

- 5 a** 0°N 82°E **b** 41°S 0°E
c 48°S 110°W **d** 70°N 25°W
e 53°N 67°E **f** 15°N 37°W
g 84°N 46°W **h** 37°S 102°E
- 6 a** 30°N 31°E **b** 60°N 25°E
c 27°S 153°E **d** 36°N 5°W
e 53°N 6°W **f** 22°N 114°E
g 12°S 131°E **h** 18°N 66°W
i 19°N 99°W **j** 35°S 56°W

- 7 a** Alice Springs **b** Brussels
c Rio de Janeiro **d** Yangon
e Ho Chi Minh City **f** Santiago
g Shanghai **h** Havana
i Prague **j** Copenhagen
- 8 a** 27.5°S **b** 41°
9 a 4° **b** 6.5°
10 a 51° **b** 40.5°

Exercise 6.02

- 1 a** 27° 30' 0"S 153° 21' 0"E
b 27° 28' 12"S 153° 21' 0"E
c 27° 30' 12"S 153° 15' 36"E
- 2 a** West Peel Artificial Reef
b Raby Bay
- 3 a** 33° 51' 8"S 151° 12' 38"E
b 41° 53' 25"N 12° 29' 32"E
c 38° 53' 52"N 77° 2' 11"W

- 4 **a** Table Mountain, Cape Town
b All England Tennis Club, Wimbledon
c Niagara Falls, Canada/USA
5 1' 0" latitude and 3' 12" longitude
6 Between latitudes 27° 30' 48"S and 27° 31' 24"S and between longitudes 153° 15' 36"E and 153° 17' 12"E.

Exercise 6.03

- 1 3781 km
2 8340 km
3 10 342 km, about 17 h 14 min
4 **a** 5517 km, 34 667 km, 1444 km/h
b 1649 km, 10 361 km, 432 km/h
c 555 km, 3489 km, 145 km/h
5 9314 km
6 1149 km
7 14 119 km
8 About 7 h 12 min
9 19°S 147°E (Townsville)
10 About 26 days and 7 hours
11 **a** 5677 km, 35 667 km, 1486 km/h
b 6024 km, 37 849 km, 1577 km/h
c 5625 km, 35 345 km, 1473 km/h
d 6254 km, 39 295 km, 1637 km/h

Exercise 6.04

- 1 **a** 6 hours **b** 4 hours 56 minutes
c 2 hours 8 minutes **d** 10 hours 12 minutes
e 24 minutes
2 **a** 2 hours 28 minutes **b** 9 hours 12 minutes
c 7 hours 48 minutes **d** 14 hours 44 minutes
e 8 hours 24 minutes
3 **a** 2 hours **b** 9 hours **c** 8 hours
d 15 hours **e** 8 hours 30 minutes

Exercise 6.05

- 1 **a** 4:00 a.m. the next day
b 1:20 p.m.
c 12:45 a.m. the next day
d 7:15 p.m.
e 12:15 p.m.

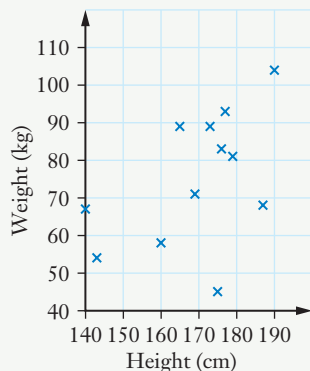
- 2 **a** 1:00 p.m. Wednesday
b 8:00 a.m. Wednesday
c 8:00 p.m. Tuesday
d 3:00 a.m. Wednesday
e 2:00 a.m. Wednesday
f 1:00 a.m. Wednesday
g 8:00 p.m. Tuesday
h 9:00 a.m. Wednesday
3 **a** 4:30 p.m. Saturday **b** 3:30 p.m. Saturday
4 **a** 3:05 p.m. Wednesday **b** 5:05 a.m. Wednesday
5 2:00 a.m. Tuesday
6 3:15 p.m. Wednesday
7 7:10 a.m. Tuesday
8 4:25 p.m. the next day
9 4:45 p.m. Thursday
10 7:15 a.m. Wednesday

Chapter review

- 1 **A** 40°N 112°E **B** 70°N 36°W
C 42°S 75°E **D** 37°S 0°E
2 **a** 34°S 151°E **b** 42°N 12°E
c 3°N 102°E **d** 35°N 136°E
3 9118 km
4 **a** 5219 km **b** 3654 km
5 **a** 32 792 km, 1366 km/h
b 22 960 km, 957 km/h
6 3395 km
7 **a** 25° 15'S 152° 49'E **b** 25° 10'S 152° 39'E
8 **a** Toogoom **b** Airfield
9 **a** 5 hours **b** 2 hours 24 minutes
c 32 minutes
10 **a** 7:45 a.m. **b** 11:25 p.m.
c 9:30 a.m. the next day
11 **a** 12 midnight **b** 1 a.m.
c 8 p.m. the previous day
d 2 a.m.
12 12:05 p.m. the same day
13 1:40 a.m. Tuesday
14 11:44 p.m. Wednesday the next week
15 2:50 a.m. Thursday

Practice examination 2

- 1 C
2 D
3



If the person with height 175 cm is considered underweight, there is a moderate positive relationship, so taller people tend to weigh more.

4

x	y	x^2	xy
3	4	9	12
4	10	16	40
7	7	49	49
8	17	64	136
10	21	100	210
32	59	238	447

9

Month	J	F	M	A	M	J	J	A	S	O	N	D
Value	502	494	509	499	490	499	496	495	510	535	549	545

- 10 **a** Least squares model is $V \approx 5.96Y - 11687$.
b In 2025, the model predicts a value of 382.
c The prediction may not be accurate because the model is based on data from 2010–2017 and 2025 is a further 8 years later.
- 11 **a** 1 hour **b** 5 hours 44 minutes
- 12 7562 km
- 13 **a** 9 a.m. on Sunday **b** 3 a.m. on Monday
- 14 The number of hairdressers and the production of carrots have 65% of their variation in common. The commonality may be population.

$$\text{Average of } x = \frac{32}{5} = 6.4$$

$$\text{Average of } y = \frac{59}{5} = 11.8$$

$$\text{Slope of least squares line} = \frac{n \sum xy - \sum x \sum y}{n \sum (x^2) - (\sum x)^2}$$

$$a = \frac{5 \times 447 - 32 \times 59}{5 \times 238 - 32^2} = \frac{347}{166} = 2.090\dots$$

$$y\text{-intercept} = 11.8 - 2.090\dots \times 6.4 = -1.578\dots$$

The equation of the least-squares line is $y \approx 2.1x - 1.6$

- 5 Residuals are:

x	3	5	6	7	9	10
y	4	5	10	13	11	16
Residuals	0.07	-2.15	1.24	2.63	-2.59	0.8

- 6 C

- 7 **a** The y -intercept means that the seagrass grows about $60 \text{ cm}^2/\text{y}$ when there is no pollution and the slope means that the growth decreases by $5 \text{ cm}^2/\text{y}$ for every increase of 1 ppm in water pollution.

- b** $40 \text{ cm}^2/\text{year}$

8

Q1	Q2	Q3	Q4
1.007	0.95	1.009	1.033

- 15 The model gives a yield of $Y = 0.6 \times 12 + 6.7 = 13.9 \text{ t/ha}$. However, 2020 could have different rainfall from 2018 and 12 hours is well outside the time used to develop the model, so it may not be an accurate prediction.
- 16 For 2022, the long term trend gives $V = 3.5 \times 2022 - 6896 = 181$. For March, the value is $1.037 \times 181 \approx 188$. The value at the end of March is predicted to be 188.
- 17 **a** 3186 km **b** 20 015 km **c** 834 km/h
- 18 The least-squares line is $P = 1.18A + 372$, with $R^2 = 0.77$. It predicts prices of \$429 000 and \$1 100 000, but these will probably be accurate because the R^2 value is high.

- 19** Least-squares line is $I = 2.143Y - 4186$ with $R^2 = 0.95$.
The base year is most likely to be when $Y = 100$.
 $100 = 2.143Y - 4186$
 $2.143Y = 4286$
 $Y = 2000$.
The base year is probably 2000.

20 2

21 **a** 6 days **b** 9 days

Chapter 7

Exercise 7.01

- 1** C
2 D
3 B
4 **a** \$1140 **b** \$20 140 **c** \$1208.40
d \$21 348.40 **e** \$23 987.06 **f** \$4987.06
5 **a** \$525 **b** \$15 525 **c** \$562.39
d \$17 815.30 **e** \$2815.30
6 $A_{n+1} = 1.068A_n$, where $A_0 = 10\ 000$
7 **a** $A_{n+1} = 1.082A_n$, where $A_0 = 46\ 000$
b $A_{n+1} = 1.053A_n$, where $A_0 = 32\ 800$
c $A_{n+1} = 1.0725A_n$, where $A_0 = 112\ 000$
d $A_{n+1} = 1.046A_n$, where $A_0 = 18\ 500$
e $A_{n+1} = 1.0375A_n$, where $A_0 = 26\ 950$
8 **a** \$31 815.56 **b** \$1558.15
9 **a** \$380 900.57 **b** \$18 565.44
10 **a** \$14 038.30 **b** \$2038.30
11 \$1816.09

Exercise 7.02

- 1** A
2 D
3 E
4 **a** \$2676.45, \$676.45 **b** \$9540.15, \$1540.15
c \$10 906.81, \$3906.81 **d** \$1007.86, \$507.86
e \$4237.05, \$2097.05 **f** \$4461.45, \$4401.45
5 **a** \$1724.71, \$224.71 **b** \$3318.63, \$478.63
c \$21 441.79, \$12 441.79 **d** \$1331.40, \$531.40
e \$5844.03, \$844.03
6 **a** \$9338.08, \$1838.08 **b** \$9323.31, \$1823.31
c \$9389.38, \$1889.38 **d** \$9379.33, \$1879.33
Investment **c** is best.

- 7** 18.92% p.a.
8 14.72% p.a.
9 6.79% p.a.
10 \$8101.72
11 \$15 223.41
12 4 years
13 17 years
14 The investment with weekly rests is better as it returns \$29 237.58 while the other investment returns \$29 115.47.
15 12 months
16 10 years
17 The investment with annual rests is better. It will return about 1.4937 times the amount invested while the other investment only returns about 1.4390 times the amount invested.
18 5.79% p.a.
19 \$8263.97
20 14 years 9 months

Exercise 7.03

- 1** A
2 **a** 9.38% **b** 11.61% **c** 12.36% **d** 6.18%
3 **a** 7.76% **b** 14.34% **c** 8.67% **d** 8.32%
4 8.54%
5 **a** 8.38% **b** 10.61%
c 15.26% **d** 18.66%
6 7.82%, 7.76%, 7.68%; 7.6% compounded quarterly is best.
7 The insurance bonds are best with an effective rate of 7.98%.
8 The interest rate compounding quarterly is better (6.22% effective).
9 Because the interest rates are the same, the one that is compounded more often has the higher effective interest rate.

Exercise 7.04

- 1** C
2 D
3 **a** \$9835.76 **b** \$12 762.32
c \$28 704.93 **d** \$38 982.18
e \$23 592.52

- 4** \$762.79
5 \$9814.47
6 \$13 854.47
7 a \$7462.15 **b** \$5638.83
c \$11 969.36 **d** \$11 824.14
e \$12 606.09
8 \$3986.39
9 \$2541.60
10 \$7747.10
11 \$55 256.18
12 \$30 000.43
13 \$15 493.47
14 \$8703.96
15 a \$13 299.19 **b** \$14 075.18
16 14.87%
17 a 18.18% **b** 708 m²
c 51 days

Exercise 7.05

1 E

2 a

Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
1	25 000	145.83	306	24 839.83
2	24 839.83	144.90	306	24 678.73
3	24 678.73	143.96	306	24 516.69
4	24 516.69	143.01	306	24 353.70

b

Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
1	64 000	480	720	63 760
2	63 760	478.20	720	63 518.20
3	63 518.20	476.39	720	63 274.59
4	63 274.59	474.56	720	63 029.15
5	63 029.15	472.72	720	62 781.87
6	62 781.87	470.86	720	62 532.73

c

Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
1	100 000	650	1000	99 650
2	99 650	647.73	1000	99 297.73
3	99 297.73	645.44	1000	98 943.17
4	98 943.17	643.13	1000	98 586.30
5	98 586.30	640.81	1000	98 227.11
6	98 227.11	638.48	1000	97 865.59

d	Fortnight	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
	1	100 000	300	500	99 800
	2	99 800	299.40	500	99 599.40
	3	99 599.40	298.80	500	99 398.20
	4	99 398.20	298.19	500	99 196.39
	5	99 196.39	297.59	500	98 993.98
	6	98 993.98	296.98	500	98 790.96
	7	98 790.96	296.37	500	98 587.33
	8	98 587.33	295.76	500	98 383.09
	9	98 383.09	295.15	500	98 178.24
	10	98 178.24	294.53	500	97 972.77
	11	97 972.77	293.92	500	97 766.69
	12	97 766.69	293.30	500	97 559.99
	13	97 559.99	292.68	500	97 352.67

e	Fortnight	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
	1	85 000	392.31	800	84 592.31
	2	84 592.31	390.43	800	84 182.74
	3	84 182.74	388.54	800	83 771.28
	4	83 771.28	386.64	800	83 357.92
	5	83 357.92	384.73	800	82 942.65
	6	82 942.65	382.81	800	82 525.46
	7	82 525.46	380.89	800	82 106.35
	8	82 106.35	378.95	800	81 685.30
	9	81 685.30	377.01	800	81 262.31
	10	81 262.31	375.06	800	80 837.37
	11	80 837.37	373.10	800	80 410.47
	12	80 410.47	371.13	800	79 981.60
	13	79 981.60	369.15	800	79 550.75

f	Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
	1	70 000	875	980	69 895
	2	69 895	873.69	980	69 788.69
	3	69 788.69	872.36	980	69 681.05
	4	69 681.05	871.01	980	69 572.06
	5	69 572.06	869.65	980	69 461.71
	6	69 461.71	868.27	980	69 349.98

g	Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
	1	70 000	875	1500	69 375
	2	69 375	867.19	1500	68 742.19
	3	68 742.19	859.28	1500	68 101.47
	4	68 101.47	851.27	1500	67 452.74
	5	67 452.74	843.16	1500	66 795.90
	6	66 795.90	834.95	1500	66 130.85

h	Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
	1	70 000	875	900	69 975
	2	69 975	874.69	900	69 949.69
	3	69 949.69	874.37	900	69 924.06
	4	69 924.06	874.05	900	69 898.11
	5	69 898.11	873.73	900	69 871.84
	6	69 871.84	873.40	900	69 845.24

3 a \$28 455.14 **b** \$1555.14

4 a \$21 946.74 **b** \$2053.26

c \$946.74

5 a \$4652.34 **b** \$466.55

6 a \$686.34 **b** \$17.07

7 a $A_n = 1.0105A_{n-1} - 180$

b \$5282.87

8 a $A_n = \left(1 + \frac{0.137}{12}\right)A_{n-1} - 450$

b \$14 650.36

9 a $A_n = \left(1 + \frac{0.067}{26}\right)A_{n-1} - 70.84$

b \$4413.68

10 a $A_n = \left(1 + \frac{0.089}{26}\right)A_{n-1} - 194.56$

b 4 years

11 a $A_n = \left(1 + \frac{0.097}{12}\right)A_{n-1} - 527.50$

b 5 years

c It will take an extra year and a quarter (15 months) with the final payment of \$21.87.

Exercise 7.06

1 a \$207.58 **b** \$40.76

c \$4751.14 **d** \$528.42

e \$2579.40

2 a \$10 491.16 **b** \$97 915.31

c \$69 949.77 **d** \$2228.86

e \$105 448.34

3 a 3 years and 9 months

b 16 weeks (4 months)

- c 17 years and 10 months
d 3 years and 48 weeks
e 14 years
- 4 a \$454.04 b \$27 242.40 c \$7242.40
- 5 \$1688.86
- 6 5 years and 7 months
- 7 a \$152 375.65 b \$128 632.58
c \$83 978.13
- 8 a i \$644.74/month, \$1473.76
ii \$450.43/month, \$2215.48
iii \$353.73/month, \$2979.04
iv \$321.65/month, \$3369.10
- b As the time taken to repay the loan increases, the monthly repayment value decreases but the total amount of interest paid increases.
- 9 a \$433.80
b i Principal repaid = \$279.89
Interest repaid = \$153.91
ii Principal repaid = \$338.08
Interest repaid = \$95.72
c As the number of payments increases, the amount paid off the principal increases and the amount of interest paid decreases.
- 10 1 year and 8 months ago

Exercise 7.07

- 1 E 2 B 3 C
- 4 a \$370.03 b \$31 082.52 c \$6082.52
d \$380.92 e \$522.72
- 5 a \$3665.82 b \$3401.37 c \$7404.60

13 a

Month	Principal (\$)	Interest (\$)	Payment (\$)	Balance (\$)
1	19 000	142.50	251	18 891.50
2	18 891.50	141.69	251	18 782.19
3	18 782.19	140.87	251	18 672.06
4	18 672.06	140.04	251	18 561.10
5	18 561.10	139.21	251	18 449.31
6	18 449.31	138.37	251	18 336.68

- b Total interest = \$842.68,
Total repayments = \$1506
Reduction of the principal = \$663.32, Loan balance = \$18 336.68

- 6 a 9 years (234 payments)
b 102 payments c \$3268.60
- 7 a \$51 164.21 b 14 years c \$8418
- 8 a \$151 519.40
b The loan will take another 143 months (11 years 11 months) to pay.
c \$19 911.71
- 9 a \$83 463.58 b \$27 091.68
- 10 a \$1373.80 b \$634.06 c 1 fortnight
- 11 a 7 months b \$6273.61
- 12 There are 26 fortnights in a year, so this is effectively the same as making 13 monthly payments in a year. You can work out the correct fortnightly rate by multiplying the monthly payment by $\frac{6}{13}$.

Chapter review

- 1 a \$9528.13 b \$1528.13
- 2 D
- 3 a \$37 571.42 b \$1643.95
- 4 B
- 5 The investment is worth \$34 911.23 when interest is compounded yearly and \$35 497.04 when interest is compounded monthly – \$585.81 more.
- 6 a \$16 485.95 b 9.18% p.a.
- 7 6.14% p.a. 8 A
- 9 Option 3, 7.59% p.a. compounded monthly provides the best return.
- 10 \$18 280.75
- 11 \$10 314.91
- 12 \$32 540.06

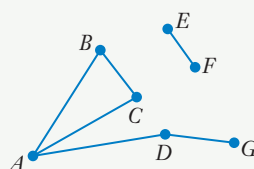
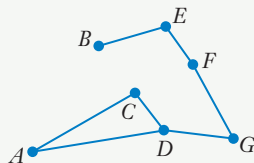
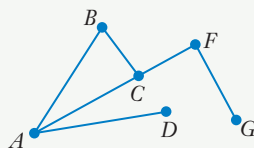
- 14 a** \$218 176.03 **b** \$7583.28
15 a $A_n = 1.009\,416\ldots \times A_{n-1} - 919$
b \$19 689.51 **c** \$17 464.60
16 a \$938.84 **b** \$78 862.56
c \$23 862.56
17 \$13 226.76
18 7.5 years
19 a \$899.56 **b** \$1557
20 a 11 years
b Reduces the time to repay by 23 months.
c Reduces the interest by \$11 247.52.
21 \$1134.10
22 \$8320
23 \$43 551.28
24 \$797.17

Chapter 8

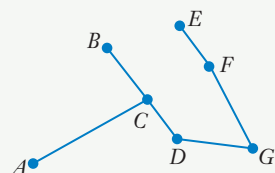
Exercise 8.01

- 1** C
2 D
3 a 5 vertices, 4 edges **b** 6 vertices, 10 edges
c 6 vertices, 5 edges **d** 5 vertices, 4 edges
e 5 vertices, 6 edges **f** 6 vertices, 9 edges
4 a a, b, d, e and f
b No graphs are complete.
c d **d** c and d
e f **f** a and c
5 a M **b** R
c S **d** T and V
e P or T
6 a Simple path **b** Circuit
c Simple circuit **d** Path
e Simple cycle **f** Simple path
g Simple circuit **h** Chain
i Simple path **j** Cycle
7 a E or G
b A, B, C, D, E, F and G are all branches.
c Many possible answers e.g. $BEFG, BCDG, BADG, BEFCDG, BACDG$, etc.
d Many possible answers e.g. $ABEFCA, ABCFGDA$, etc.

- e** Many possible answers e.g.



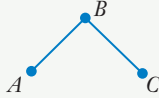
- f** Many possible answers e.g.



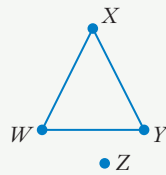
- 8 a** Path **b** Simple circuit
c Circuit **d** Simple path
e Simple cycle **f** Cycle
g Circuit **h** Simple path
i Path **j** Chain
9 a Path **b** Simple chain
c Cycle **d** Simple circuit
e Simple path **f** Path
g Circuit **h** Simple circuit
i Simple path **j** Simple circuit
10 a Unconnected undirected trees
b Connected undirected
c Connected directed cycle
d Connected directed
e Connected undirected tree
f Connected undirected

11 Other answers are possible.

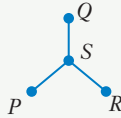
a



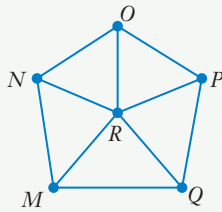
b



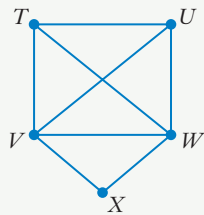
c



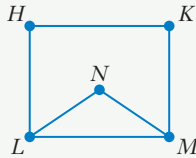
d



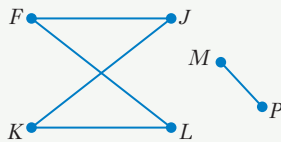
e



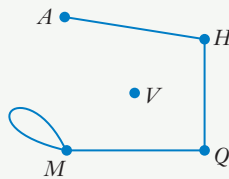
f



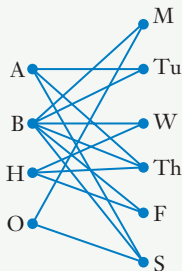
g



h



12



13 a Yes: $C-P-B-S-H-T-M$

b $M-H-S$ and $M-H-T-B-S$

c No

d Two: $B-S-T-M-H$ and $B-C-B-B-S-T-M-H$

Exercise 8.02

1 D

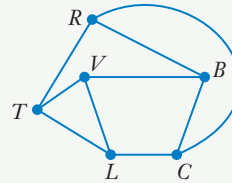
2 B

3 True in all cases.

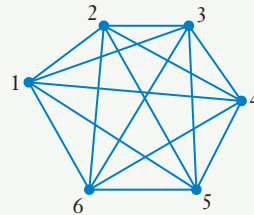
4 In all cases, the pairs of graphs have the same number of vertices and edges, and the same degree and connections for corresponding vertices.

5 In each case, the pairs of graphs do not have the same degree and connections for corresponding vertices.

6



7 a



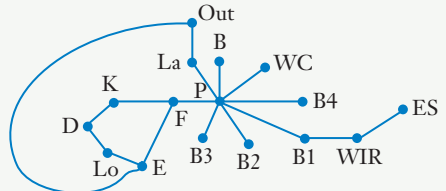
b Complete

c 12

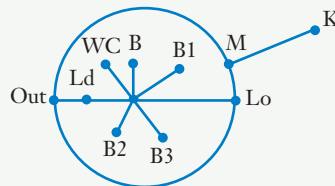
d 10

8 105

9 a



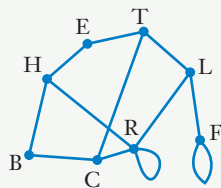
b



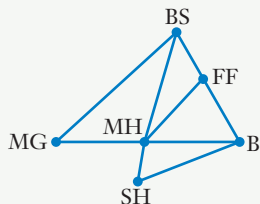
10 a 3 colours

b 4 colours

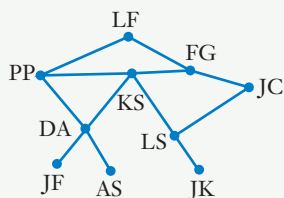
11



12



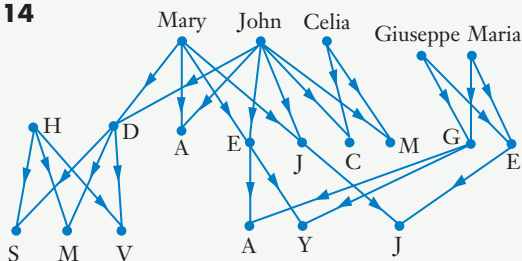
13 a



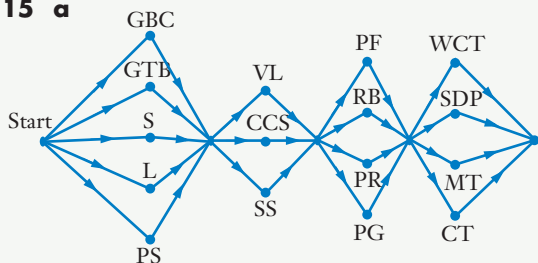
b Luke Fellowes, Kerry Stubbs, David Alsop

c Felix Goldschmeid (known to LF and KS)

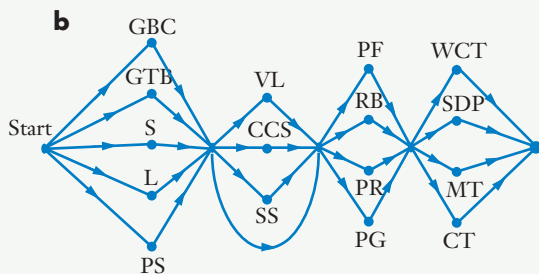
14



15 a



b



c 240

Exercise 8.03

For answers where a graph needs to be drawn, many different graphs are possible. Any graph that is isomorphic with the answer supplied is correct.

1 B

2 C

3 E

4 D

5 B

6 i All adjacency matrices are correct.

ii a 7

b 12

c 12

d 20

iii The answers are the same.

7 a

$$P \begin{matrix} Q & R & S & T \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

b

$$A \begin{matrix} S & D & F & G \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

c

$$L \begin{matrix} M & N & O & P \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

d

	A	B	C	D	E	F
A	0	0	0	1	1	1
B	0	0	0	1	1	1
C	0	0	0	1	1	1
D	1	1	1	0	0	0
E	1	1	1	0	0	0
F	1	1	1	0	0	0

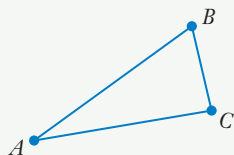
e

	U	V	W	X	Y	Z
U	0	1	2	1	1	1
V	1	0	0	0	0	1
W	2	0	0	1	0	0
X	1	0	1	0	1	0
Y	1	0	0	1	0	0
Z	1	1	0	0	0	1

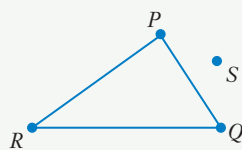
f

	Q	W	E	R	T	Y
Q	0	1	0	0	0	0
W	1	0	1	0	2	1
E	0	1	0	1	1	0
R	0	0	1	0	0	1
T	0	2	1	0	0	1
Y	0	1	0	1	1	2

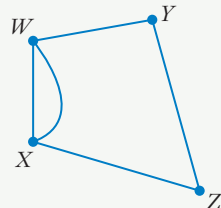
8 a



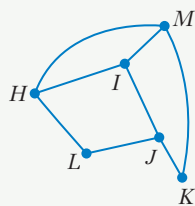
b



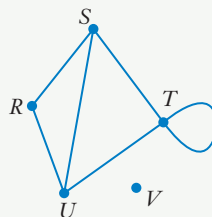
c



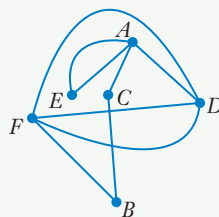
d



e



f



9 a

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

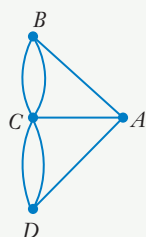
c

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

d

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

10 a



b

$$\begin{matrix} & A & B & C & D \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{matrix}$$

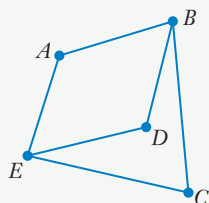
c Use a pencil to try to draw a path on the graph that passes through each vertex exactly once without lifting the pencil.

d There is no solution.

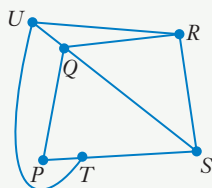
Exercise 8.04

For answers where a graph needs to be drawn, many different graphs are possible. Any graph that is isomorphic with the answer supplied is correct.

1 a

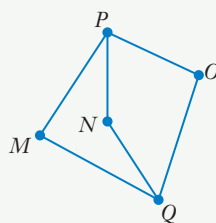


b



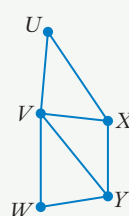
c Non-planar

d

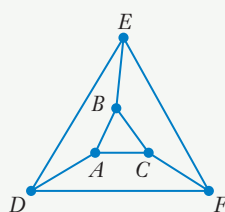


e Non-planar

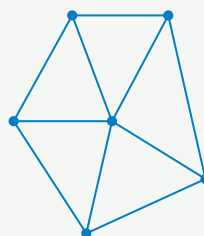
f



2



3



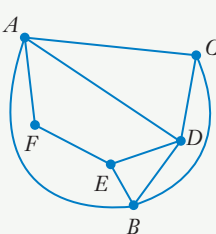
4 A quadrilateral-based pyramid.

5 B

6 E

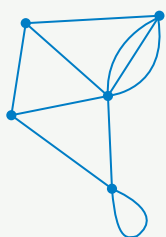
7 a	2	b	4	c	5	d	3
8 a	4	b	5	c	8	d	10
9 a	3	b	4	c	4	d	8

10 a



11 B

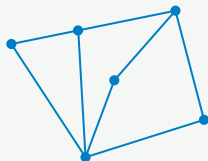
12 a



b



13 a



b



14 No. The values do not satisfy Euler's formula.

15 a 10

- b Yes. Using trial and error a graph with those vertex degrees can be drawn without any crossing edges.
c Using Euler's formula: $6 + F - 10 = 2$ so $F = 6$.

Exercise 8.05

For questions where answers involve identifying a route, many different answers are possible. The routes provided are examples.

1 B

2 D

3 C

4 a Closed walk

b Open walk

c Cycle

d Circuit (closed trail)

e Open walk

f Closed walk

g Cycle

h Open trail

i Open walk

j Open trail

k Circuit

5 a *PUQVRS*

b *PUQVQU*

c *UQVIRSTU*

d *TVRQUP*

e *TVRQUT*

6 a $\deg(A) = 4$, $\deg(B) = 3$, $\deg(C) = 2$, $\deg(D) = 3$, $\deg(E) = 2$, $\deg(F) = 2$

b Semi-Eulerian, because it has 2 vertices of odd degree and the remaining vertices are of even degree.

7 a The graph is semi-Eulerian because it has 2 vertices of odd degree and the remaining vertices are of even degree.

b Either *A* or *F*

c One Euler trail is *FEACBDCFDDBA*, but others exist.

8 a, b and d are semi-Eulerian.

9 a *FAEDCFBCBA* b *QRMNOPQN*

c Not semi-Eulerian d *KLMNOPJOMJKM*

10 a The graph is Eulerian because all vertices are of even degree.

b One Eulerian circuit is *SUTSQPVUQRS*, but others exist.

11 i Semi-Eulerian: b, c and f; Eulerian: a, d and e

ii a *ABCDEFGBFA* (Eulerian cycle)

b *TPRSTQR* (Eulerian path)

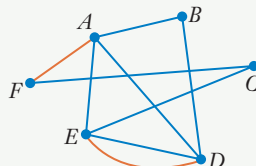
c *NMSRNOPTQRT* (Eulerian path)

d *DEFKECKHGFJIJCD* (Eulerian cycle)

e *IJKLMNOPMKQHPOHI* (Eulerian cycle)

f *STUVWXYZVTYZXSZT* (Eulerian path)

12 a



b *BAFCEDEADB*

13 a Eulerian graph

b *DFBDBCPCGDCPD* is the most efficient route because it only travels on each road once. The route is an Eulerian cycle.

14 a No, it not possible because the graph is not Eulerian and so cannot have an Eulerian cycle (closed trail).

b Either *G* or *F*.

c *GCBACDFBDEFHEGF*

Exercise 8.06

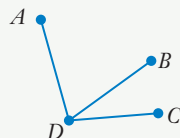
For questions where answers involve identifying a route, many different answers are possible. The routes provided are examples.

- 1 A
- 2 D
- 3 B
- 4 a *ABCDFEG* b *YUVXX*
c *PRQSTUV* d *JKONMPL*
- 5 a *BADGFEC* b *VWUYX*
c *RPVUTSQ* d *PLMNOJK*
- 6 a *ABCEFA* b *UPRSQWVTU*
c *LKMNOL* d *UVWZYXSTU*
- 7 a *FACDEF* b *VTUPRSQWV*
c *OLKMNO* d *WZYXSTUVW*
- 8 a *GFABCEDG* b *FGEDCBA*
c *ECDGBAFE* d *GFEDCBA*
e *AFGEDCBA*
- 9 a A Hamiltonian cycle
b 3–4–9–5–6–7–8–1–2–3
c 2–8, 3–9, 8–9, 9–6 and 7–1
- 10 *DIKHCFAEJBGD*
- 11 *ALKJIHOGFEDCNMBOA*

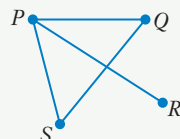
Exercise 8.07

- 1 a Tree – the graph is connected and contains no loops, cycles or multiple edges.
b Not a tree – the graph contains a cycle.
c Not a tree – the graph is disconnected.
d Tree – the graph is connected and contains no loops, cycles or multiple edges.
e Tree – the graph is connected and contains no loops, cycles or multiple edges.
f Not a tree – the graph contains a loop.
g Not a tree – the graph contains a cycle.
h Not a tree – the graph contains cycles and multiple edges.
i Tree – the graph is connected and contains no loops, cycles or multiple edges.
- 2 D

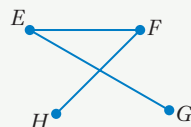
3 a Tree



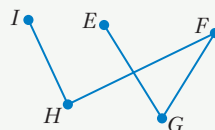
b Not a tree



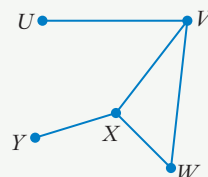
c Tree



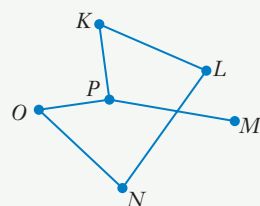
d Tree



e Not a tree

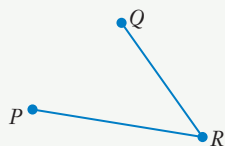
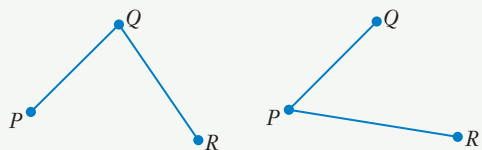


f Not a tree

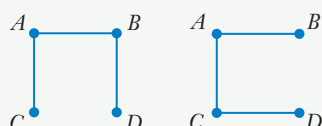
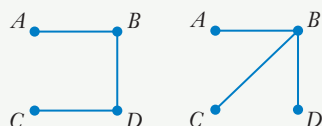
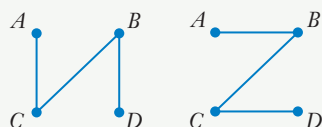
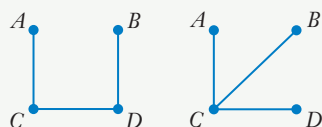


4 B

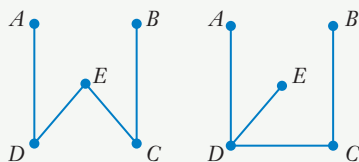
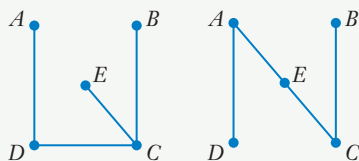
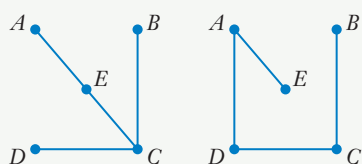
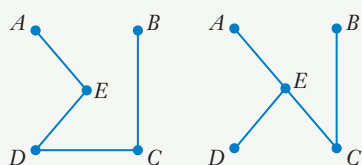
5



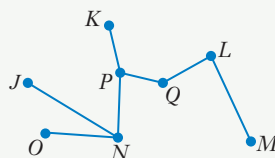
6



7



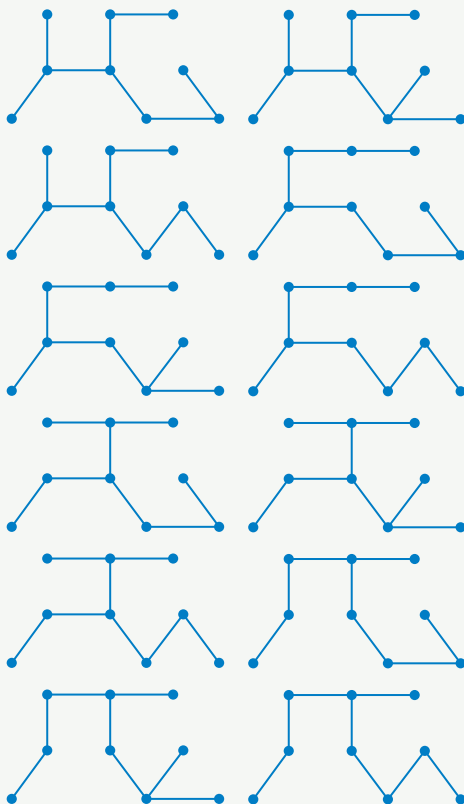
8 a Many possible answers.



b Answers will vary depending on the spanning tree. JK, JO, NQ, NM, KQ and KL .

9 AB, BC, FE and ED .

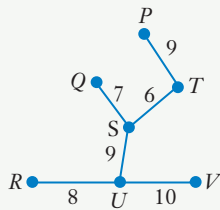
10 There are 12 distinct spanning trees.



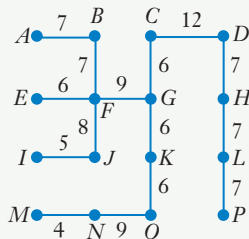
Exercise 8.08

- 1 A shortest path from A to H is $AFGH$, which is 11 km long.
- 2 **a** $HGFBC = 23$ **b** $DCIJ = 29$
c $ABL = 36$ **d** $GFIKCB = 46$
e $HAB = 22$
- 3 **a** $PG = 40$ **b** $JIFEBA = 42$
c $CPN = 39$ **d** $EBAPO = 48$
e $IJKL = 28$
- 4 **a** The minimum spanning tree is BC, CD, DE, AE, AF .
b The minimum length of fibre optic cable required to connect the towns is 33 km.

5 **a**

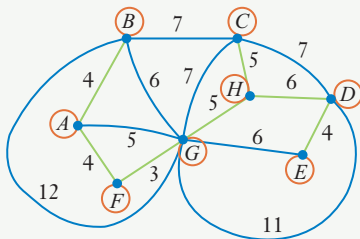


b

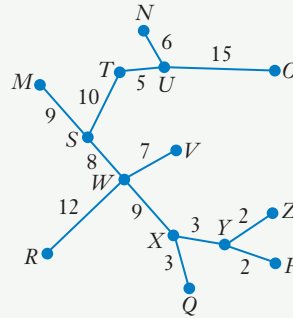


For **b**, you could use BC or JK instead of FG .

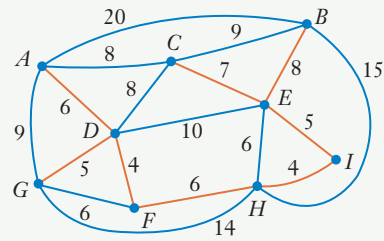
- 6 Length = 31 m



- 7 You can use MT instead of ST .

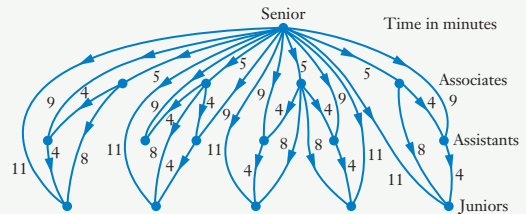


8



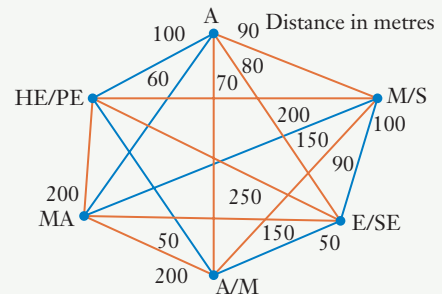
- 9 **a** $AFH = 20c$ **b** $DIB = 16c$
c $FDI = 20c$ **d** $HFE = 19c$

10 **a**



- b** The best method is 'down-the-line', where each person's task is explained by their immediate superior. This method takes 64 minutes of work time.

11 **a**



- b** A, MA, M/S, E/SE, A/M, HE/PE, A = 510 m

- 12 M, R, P, C, L, I, M = 18 h

Exercise 8.09

1 a

Node	Inflow capacity	Outflow capacity
A	20	15
B	35	40
C	15	25

b $A = 15, B = 35, C = 15$

2 a

Node	Inflow capacity	Outflow capacity
W	20	40
X	55	30
Y	50	40
Z	45	45

b $W = 20, X = 30, Y = 45, Z = 40$

3 C

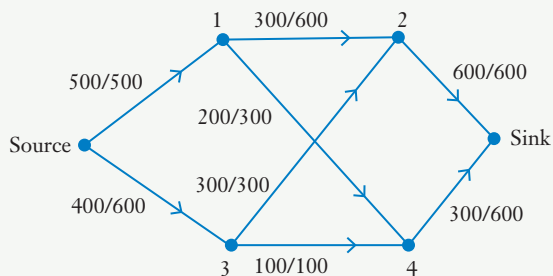
4 D

5 A

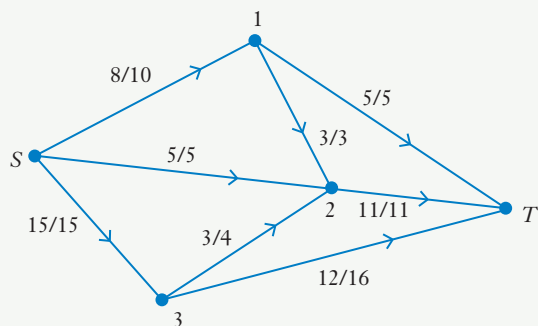
6 a 28 **b** 30 **c** 13 **d** 12

e 16 **f** 15 **g** 7 **h** 10

7



8



9 Maximum flow = 35

10 Maximum flow = 120

Exercise 8.10

1 B

2 A

3 E

4 C

5 a Cuts 2 and 3 are not valid.

b Cut 1 = 27, cut 4 = 33

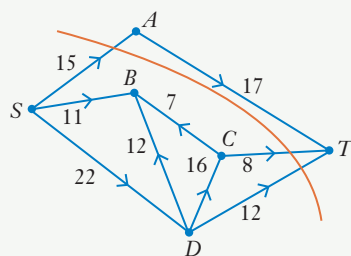
6 a Cut 2 is not valid.

b Cut 1 = 48, cut 3 = 57, cut 4 = 56, cut 5 = 54

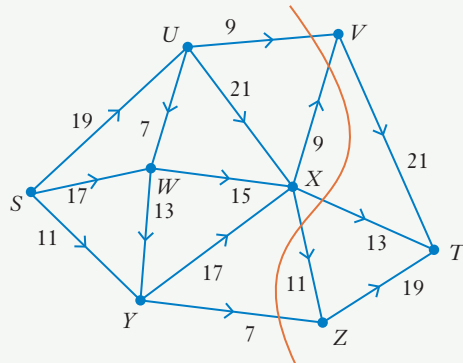
7 a Cut 1

b Cut 4

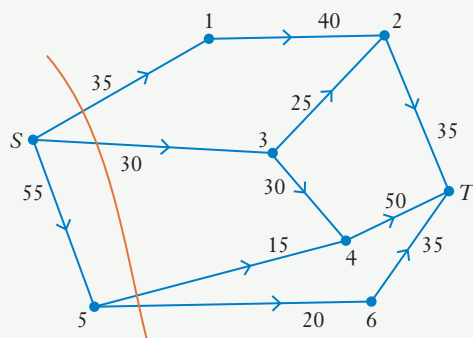
8 Maximum flow = 35



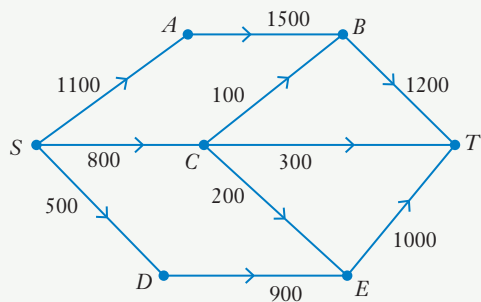
9 a Maximum flow = 49



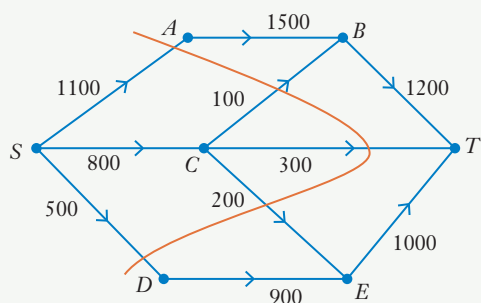
b Maximum flow = 100



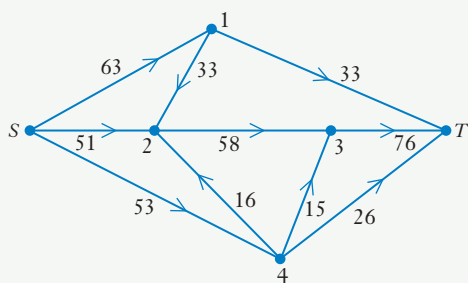
10 a



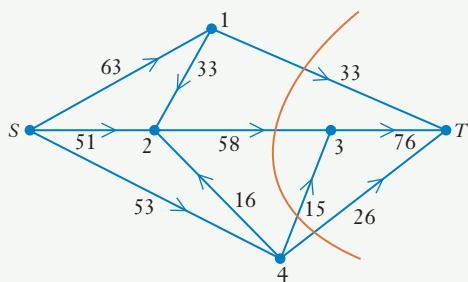
b Maximum flow = 2200 MW



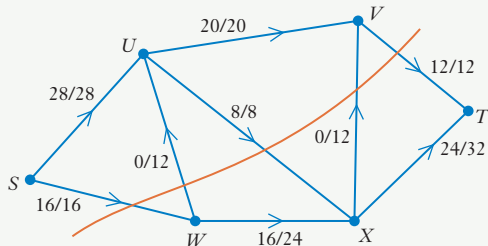
11 a



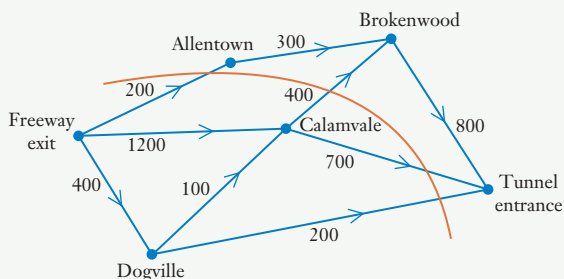
b Maximum flow = 132 widgets



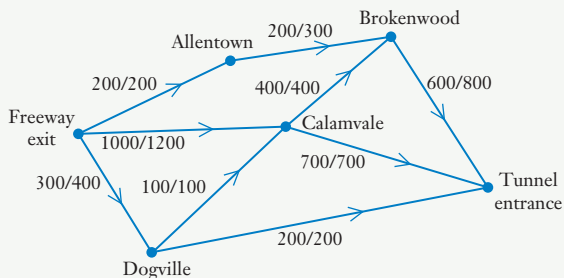
12 a and b



13 a 1500 vehicles/minute



b



c 1100 vehicles/min

d The edge from the freeway exit to Allentown needs to be increased by 100 vehicles/minute.

Chapter review

1 i a $v = 6, e = 9$

c $v = 5, e = 7$

e $v = 7, e = 6$

ii a, b, c, d and f

iv d

2 a Simple path

c Circuit

e Simple cycle

g Simple circuit

b $v = 6, e = 7$

d $v = 4, e = 6$

f $v = 5, e = 5$

iii a, b, d, e

v b

b Simple circuit

d Path

f Simple path

h Chain

3 B

4 a Connected undirected circuit

b Connected undirected tree

c Connected directed mixed

d Disconnected directed (mixed)

e Connected undirected mixed

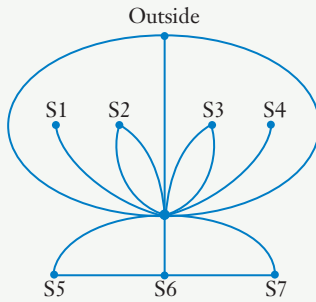
f Connected directed cycle

5 C

6 A

7 a and c are isomorphic.

8



9 4 colours

10

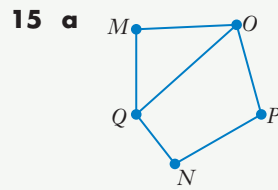
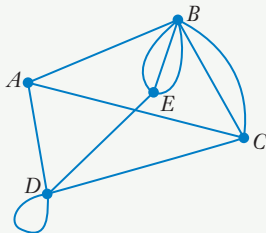
	A	B	C	D	E	F
A	0	3	0	1	1	0
B	3	0	0	1	0	0
C	0	0	0	1	0	0
D	1	1	1	0	1	0
E	1	0	0	1	1	0
F	0	0	0	0	0	0

11 C

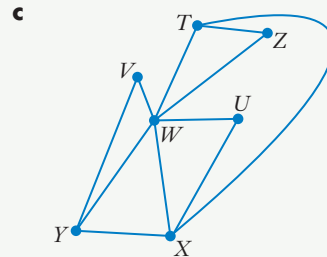
12 D

13 B

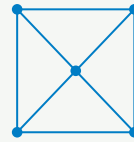
14



b Not possible



16 a



b $v + f - e = 5 + 5 - 8 = 2$

17 a $v = 5$

b $e = 12$

c $f = 4$

18 B

19 D

20 E

21 a i Path ii Circuit iii Trail
iv Cycle v Walk

b The graph is semi-Eulerian because it has 2 vertices of odd degree and the remaining vertices are of even degree.

c ABCDBFDEFAE

22 a The graph is Eulerian because all vertices are of even degree.

b BADECDCB

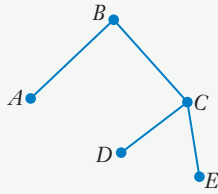
23 IHGABCDEF (others exist)

24 a GFECDBAG

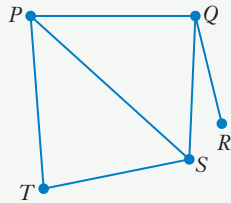
b AGFEDCBA

25 b and d are trees.

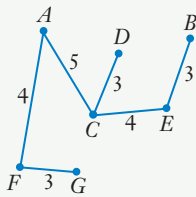
26 a Tree



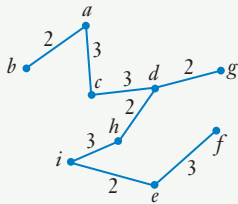
b Not a tree



27 i a



b



ii a FB, AB, AD, DB, AG, GB

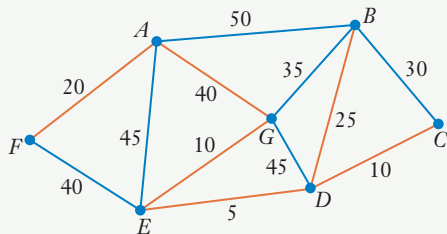
b $ad, bc, ch, gf, df, hf, bi$

28 a $AGMNO = 12$

b $KLMIJ = 10$

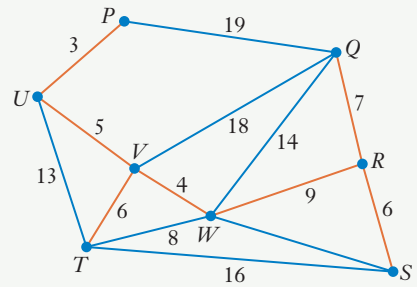
c $EIMGF = 11$

29 a



b 110 m

30



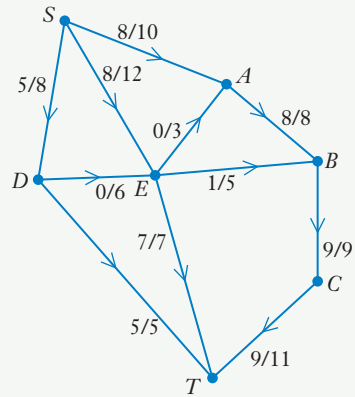
31 a i 31 L/s

ii 23 L/s

iii 18 L/s

iv 15 L/s

b

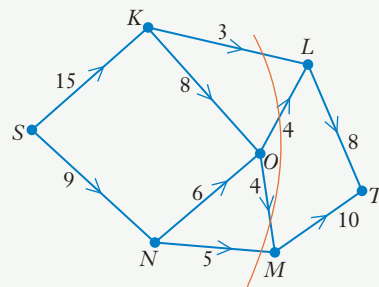


c 21 L/s

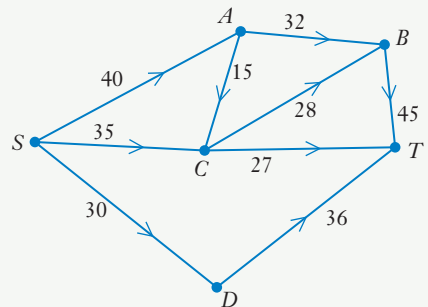
32 a Cut 3

b Cut 1 = 54, cut 2 = 58, cut 4 = 54, cut 5 = 49

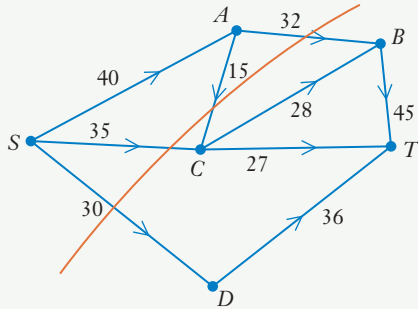
33 Maximum flow = 16



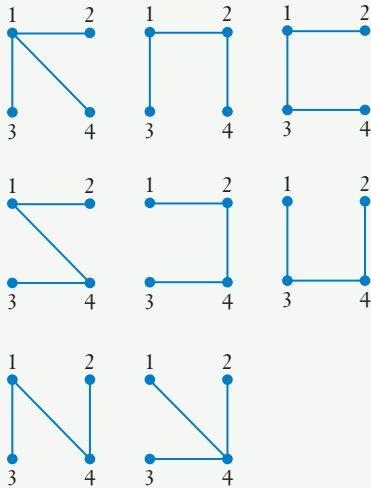
34 a



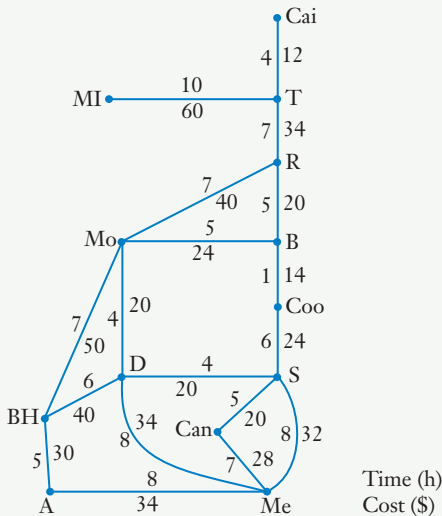
b Maximum flow = 112 widgets



35 There are 8 distinct spanning trees.



36 a



b B-Mo-BH-A = 17 h

c A-BH-Mo-R-T-MI = \$214

Practice examination 3

1 a \$18 161.18, \$3161.18

b \$5146.53, \$646.53

2 \$3924.56

3 3 years

4 7.78%

5 8.04% compounded monthly is better.

6 \$4495.69

7 6 vertices; 7 edges

8 5

9 B and D

10 A

11 C, E and F

12 a 4

b 14

13

	A	B	C	D	E	F
A	1	0	1	0	0	0
B	0	0	0	0	1	1
C	1	0	0	1	1	1
D	0	0	1	0	1	0
E	0	1	1	1	0	1
F	0	1	1	0	1	1

14 4

15 E

16 A

17 APQSB, 200 km

18 50 L/s, 45 L/s, 45 L/s

19 10

20 35

21 Option A. The investment with the shorter compounding period will pay more interest.

22 \$14 200

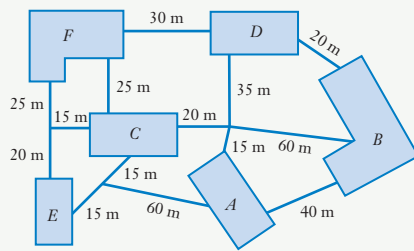
23 11



24 About \$4900

25 238 m^2

26



140 m and \$22 750

Chapter 9

Exercise 9.01

1 a

Year	Value at start of year	Interest	Deposit (d)	Value at end of year
1	8000.00	440.00	1200.00	9640.00
2	9640.00	530.20	1200.00	11 370.20
3	11 370.20	625.36	1200.00	13 195.56
4	13 195.56	725.76	1200.00	15 121.32
5	15 121.32	831.67	1200.00	17 152.99

b \$3152.99

c \$17 152.99

2 a

Month	Value at start of month	Interest	Deposit (d)	Value at end of month
1	15 000.00	100.00	350.00	15 450.00
2	15 450.00	103.00	350.00	15 903.00
3	15 903.00	106.02	350.00	16 359.02
4	16 359.02	109.06	350.00	16 818.08
5	16 818.08	112.12	350.00	17 280.20
6	17 280.20	115.20	350.00	17 745.40
7	17 745.40	118.30	350.00	18 213.70

b \$18 213.70

3 \$65 932.48

4 B

5 a $A_{n+1} = 1.00575A_n + 650$, where $A_0 = 7500$

b \$35 135.02

c After 42 months (3 years 6 months)

6 a $A_{n+1} = 1.01625A_n + 1960$, where $A_0 = 0$

b \$35 487.76

c After 6 years 9 months (27 quarters)

7 \$32 716.17

8 \$41 379.55

9 \$22 066.15

10 a \$14 767.90

b $A_{n+1} = 1.07(A_n + 2400)$, where $A_0 = 2400$

11 \$14 002.38

Exercise 9.02

1 D

2 B

3 a $A_{n+1} = 1.057A_n - 9000$, where $A_1 = 90\ 000$

b \$52 106.83

c 16 years

d \$2064.87

4 a $a = \$150\ 000$, $r = 1.006$, $d = 1800$

b \$137 869.39

c 9 years and 8 months

d \$1567.97

5 The annuity lasts for 15 years and 3 months and the last withdrawal is \$87.79.

6 a 22 years **b** 15 years **c** 32 years

7 a 18 years and 6 months

b 25 years and 9 months

c 8 years and 6 months

8 \$14 460.78

9 \$13 715.82

10 a \$835.55

b \$732.54

c \$1219.18

11 a \$15 600 000

b \$8 130 186.87

12 No, the annuity pays an extra \$24 357.08.

13 Yes, the lump sum will earn an extra \$47 003.85.

Exercise 9.03

- 1 C
- 2 **a** $A_{n+1} = \left(1 + \frac{0.082}{12}\right) A_n + 380$, where $A_1 = 380$
b \$10 321.39 **c** \$71 160.92
d \$376 291.21
- 3 **a** $A_{n+1} = 1.0065A_n + 560$, where $A_1 = 560$
b \$52 101.06 **c** \$807 252.37
- 4 \$120 671.56
- 5 E
- 6 **a** $A_{n+1} = 1.0085A_n + 480$, where $A_1 = 67\ 000$
b \$148 701.21
c 12 years and 11 months
- 7 **a** \$185 837.60
b 19 years and 4 months
- 8 B
- 9 20 years and 10 months
- 10 **a** 23 years **b** \$146 430.22
c \$358.22
- 11 \$983.64
- 12 \$1034.34
- 13 **a** \$495.89 **b** \$565 800
c \$235 700
- 14 About 14 years and 3 months

Exercise 9.04

- 1 E
- 2 B
- 3 \$21 600
- 4 \$2790
- 5 D
- 6 \$150 000
- 7 \$289 923
- 8 **a** \$1375 **b** About 12.3% p.a.
- 9 **a** \$10 350 **b** About 8.3% p.a.
- 10 **a** \$333.67 **b** About 8.5% p.a.
- 11 About \$85 073

Exercise 9.05

- 1 **a** This is an example of a sinking fund or an investment annuity.
b \$320 257.40 **c** \$843.07
- 2 **a** This is an example of an ordinary annuity.
b \$138 051.76

- 3 **a** This is an example of an investment annuity.
b No, she will fall about \$10 071 short.
c \$538.51
- 4 9 years and 1 month
- 5 **a** \$11 520 **b** 7.5%
- 6 **a** \$77 784.36 **b** \$1005.16
- 7 \$10 100
- 8 **a** \$171 045.68 **b** \$784.26
- 9 \$483 103.97
- 10 \$73 951.50
- 11 4 years 11 and months
- 12 **a** \$599 402.78 **b** \$629 718.14
- 13 Option A has the greatest return (\$179 273.24).
The other options return \$129 648.80 (B),
\$136 239 (C) and \$80 298.17 (D)

Chapter review

1 **a**

n	Value at start of month	Interest	Deposit (d)	Value at end of month
1	20 000.00	83.33	800.00	20 883.33
2	20 883.33	87.01	800.00	21 770.34
3	21 770.34	90.71	800.00	22 661.05
4	22 661.05	94.42	800.00	23 555.47
5	23 555.47	98.15	800.00	24 453.62
6	24 453.62	101.89	800.00	25 355.51

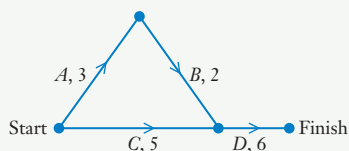
- b** \$555.51 **c** \$25 355.51
- 2 A
- 3 **a** $A_{n+1} = 1.017A_n + 1500$, where $A_0 = 35\ 000$
b \$62 629.03 **c** 6 years and 6 months
- 4 \$35 654.24
- 5 D
- 6 C
- 7 **a** $A_{n-1} = 1.066A_n - 7000$, where $A_1 = 85\ 000$
b \$70 942.71 **c** 26 years **d** \$2102.72
- 8 **a** 29 years **b** 16 years **c** 9 years
- 9 \$1147.80
- 10 B
- 11 \$158 104.76
- 12 E
- 13 **a** \$117 028.32 **b** 15 years
- 14 D

- 15 a** \$276.542.02
b \$241.60
16 \$18 250
17 A
18 \$490 909.09
19 a \$1100 **b** 18% p.a.
20 C
21 \$211 422.17
22 \$950.93
23 a This is an example of an ordinary annuity.
b \$100 099.10 **c** 14 years 11 months
d \$658.04
24 a This is an example of an investment annuity.
b \$91 392.41
25 \$526.67
26 \$415 655.51
27 \$84 999.96

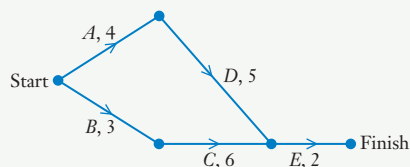
Chapter 10

Exercise 10.01

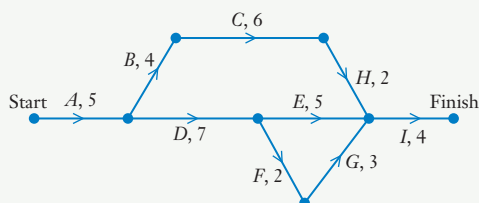
- 1** B
2 E
3 C
4 a



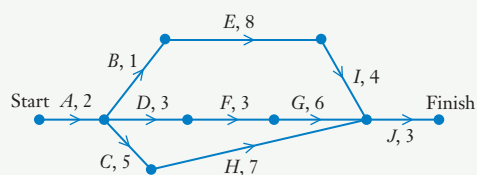
b



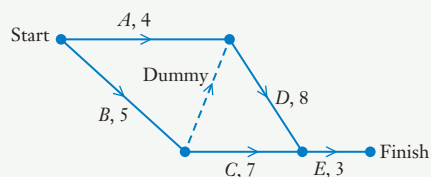
5 a



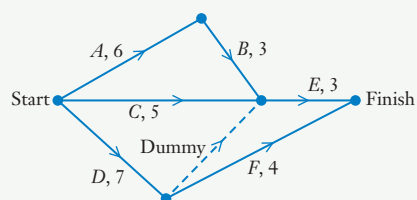
b



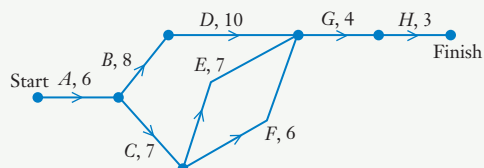
6 a



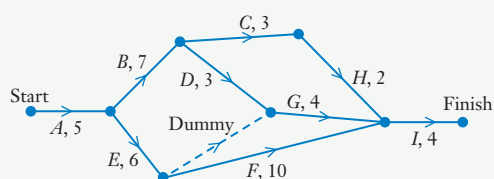
b



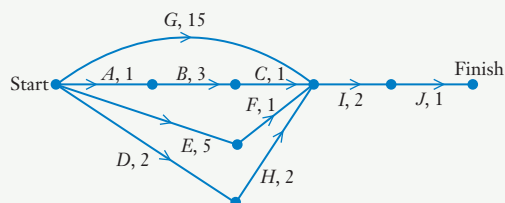
7 a



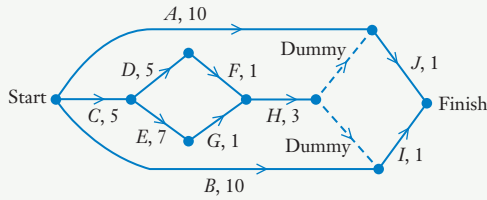
b



8



9



Other tables are possible for questions 10 and 11.

10

Code	Activity	Time (days)	Prerequisites
A	Set up committee	1	None
B	Decide trophies	5	A
C	Decide site	2	A
D	Decide date	2	A
E	Publicity	10	B, C, D
F	Order meat and drinks	2	E
G	Hold trophy day	1	F

11

Code	Activity	Time (days)	Prerequisites
A	Decide site	1	None
B	Design playground	1	A
C	Order materials	5	B
D	Build playground	2	C
E	Hold opening ceremony	1	D
F	Perform maintenance checks	2	E

Exercise 10.02

1 C

2 A

3 E

4 C

5 B

6 **a** $a = 7$ **b** $b = 2, c = 2$
c $d = 6, e = 13$ **d** $f = 16, g = 14, h = 11$
e $j = 3, k = 11, m = 18, p = 23$

7 **a** 0

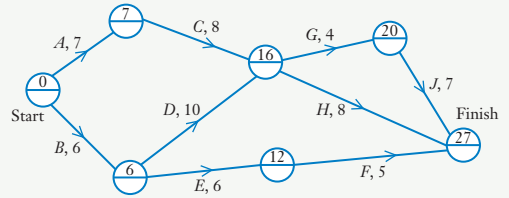
b 12

c 15

d 21

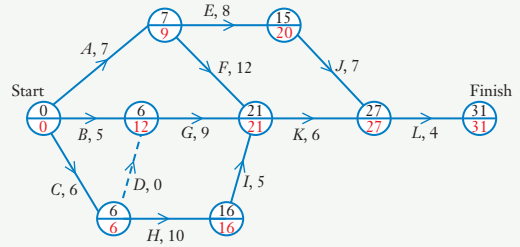
e 15

8

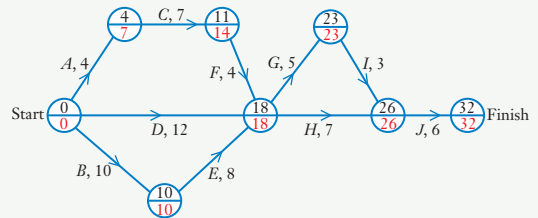


9 **a** A dummy activity.

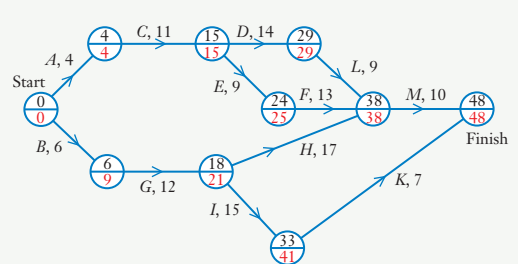
b



10 **a, b, c**



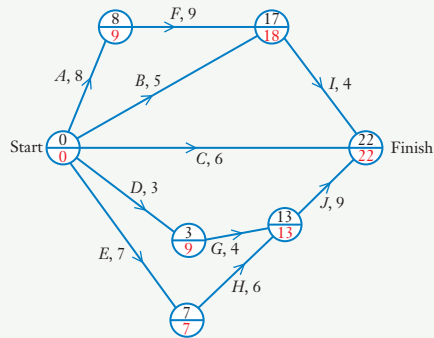
11 **a, b**



c It would have no effect.

d It would have no effect.

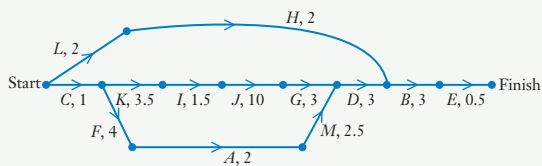
12 a, b



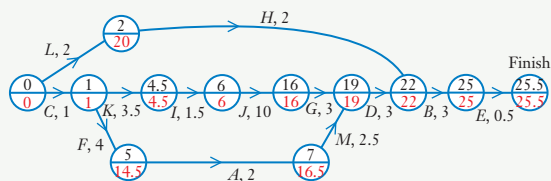
c No effect

d It would increase the time by 2 hours.

13 a Getting rid of old vegetation before getting new soil makes less clutter on site.

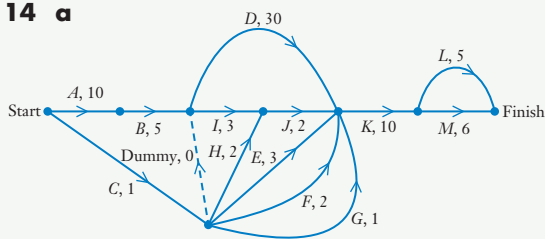


b

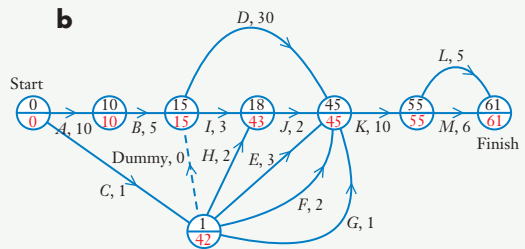


c It will have no effect.

14 a



b



c No, they will need 5 more days.

Exercise 10.03

1 A

2 D

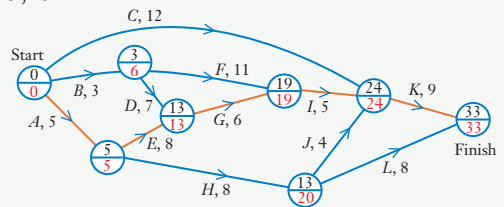
3 a $ADHJ = 40$, $BEHJ = 44$, $BFJ = 29$, $CGIJ = 42$

b Critical path = $BEHJ$, shortest time for project = 44 hours

4 a $ADI = 30$, $BEI = 27$, $BGJ = 23$, $CFEI = 35$, $CFGJ = 31$, $CHJ = 29$

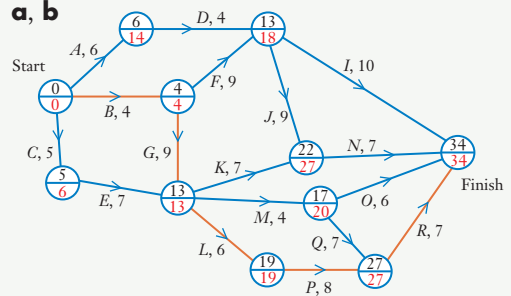
b Critical path = $CFEI$, shortest time for project = 35 minutes

5 a, b



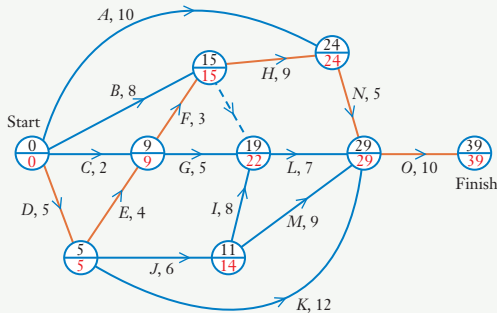
c 33 weeks

6 a, b

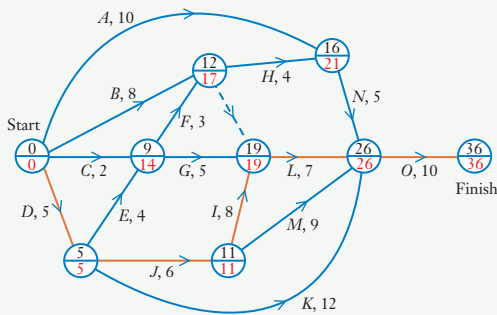


c 34 days

- 7 a** $CGIPQRTQ = 39$ **b** $DIJKLORS = 43$
c $ABLMNSXY$ or $CEGOPRXY = 37$
d $ADIKMNOZ'$ or $ABFJKMNOZ'$ or $ABFXYZOZ'$ or $ACEIKMNOZ' = 40$
- 8 a** 39 hours



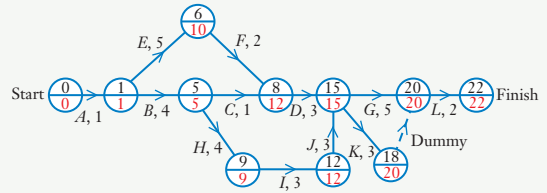
b 36 hours



Exercise 10.04

- 1** A
2 D
3 C
4 B
5 C
6 E
7 D
8 a $BGJL$
b i 1 day **ii** 1 day
iii 8 days **iv** 8 days

9 a



b $ABHIJGL$

- c i** 4 hours **ii** 4 hours
iii 6 hours **iv** 4 hours

10 a $ABFHKM$

b $ACEJM$

- 11 a** Float time for $B = 3$ hours;
 Float time for $C = 8$ hours;
 Float time for $E = 3$ hours;
 Float time for $G = 4$ hours;
 Float time for $H = 8$ hours;
 Float time for $I = 4$ hours

b Completion time extended by 5 hours.

c No effect.

d Reduce the duration of F by 1 hour. H is non-critical and reducing M will have no effect unless you can also reduce J , N or L .

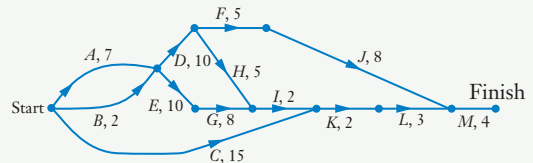
- 12 a** Float time for A , B , D , F and $L = 3$ weeks;
 Float time for E and $J = 5$ weeks;
 Float time for $I = 7$ weeks

b Critical path = ADL ;
 Completion time = 20 weeks

c Critical path = $BEJM$;
 Completion time = 19 weeks

d Reduce the duration of C by 1 week.

13 a



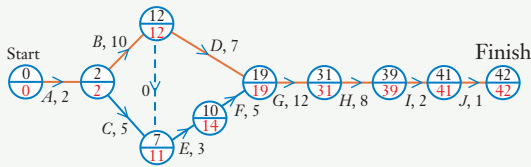
b Critical path = $AEGIKLM$;
 Completion time = 36 days

c B , C , D , F , H , and J

d 38 days

Exercise 10.05

1 a, b Project completion time = 42 days



- c** It will have no effect. The activity is non-critical and has a float time of 4 days.
- d** It reduces the project completion time to 38 days.

2 a $ABCDGKL = 1375$ days

b No change

c A new critical path $ABCDFKL$ and 10 days delay.

d 60 days delay

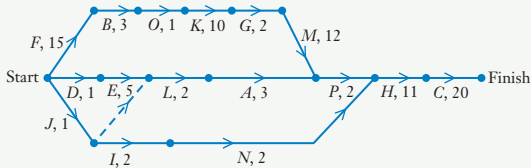
3 a $CDFG$ or $BFG = 75$ days

b No change

c Critical path BFG only and 20 days delay.

d Critical path $CDFG$ only and 20 days delay.

4 In minutes:



5 a $ABENO$ or $ABENP$ or $ABENQ = 28$ days

b 10 days delay

c No change

d A new critical path $CDENO$ or $CDENP$ or $CDENQ$ and 4 days delay.

6 a $ABCJKL$ or $ABEFJKL = 35$ weeks

b Critical path $ABEFJKL$ only and 3 weeks delay.

c No change

d 3 weeks delay

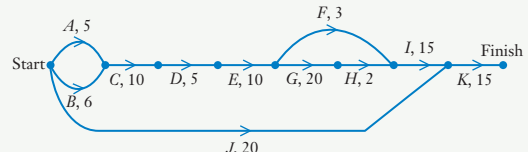
7 a $ABCJ = 22$ weeks

b 3 weeks delay

c A new critical path $ABCEFG$ and 2 weeks delay.

d A new critical path AIJ and 1 week delay.

8 a In minutes:



b i 1 min

ii 0 min

iii 19 min

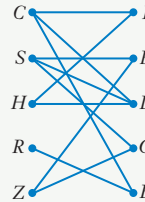
iv 48 min

Exercise 10.06

1 D

2 B

3



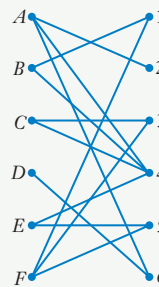
4 a Oliver – Tasks 1 and 4; Isla – Tasks 1, 2 and 3; Jordan – Task 3; Ella – Tasks 2 and 4

b Paige – Tasks 1 and 2; Noah – No task; Maya – Tasks 1, 2, 3 and 4; Ryder – Tasks 2 and 3

c Hugo – Task 1; Evie – Tasks 2, 3 and 5; Max – Tasks 2 and 4; Poppy – Tasks 1, 2 and 4; Beau – Tasks 3, 4 and 5

d Eliza – Tasks 2 and 5; Connor – Task 3; Stella – Tasks 1, 2, 3, 4 and 5; Jake – Tasks 4 and 5; Holly – Tasks 1 and 3

5



6 a Bipartite; Group 1 = $\{A, B, C, D, F, G\}$, Group 2 = $\{E\}$

b Bipartite; Group 1 = $\{A, B, D, H\}$, Group 2 = $\{C, E, F, G\}$

c Not bipartite

7 a Bipartite; Group 1 = $\{A, B, C\}$, Group 2 = $\{D, E, F\}$

b Bipartite; Group 1 = $\{P, Q, R\}$, Group 2 = $\{U, S, T\}$

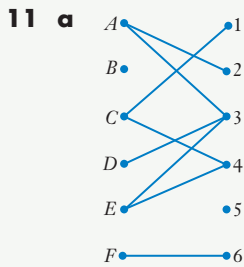
- c Not bipartite
 d Bipartite; Group 1 = {N, O, Q},
 Group 2 = {M, P, R, S}

8 The graph is bipartite;
 Group 1 = {A, I, D, G, H, L},
 Group 2 = {B, C, F, E, J, K, M}

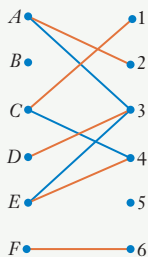
- 9 a Incomplete matching
 b Not a matching
 c Complete matching
 d Not a matching
 e Incomplete matching
 f Complete matching



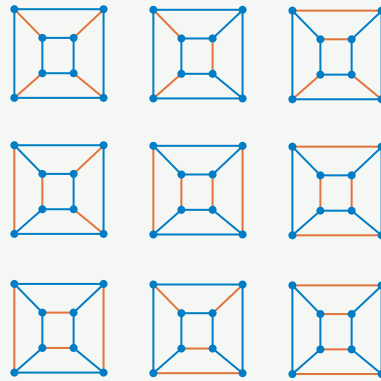
b No complete matching.



b The matching shown in orange is the maximum number of applicants placed in positions.



12 Matchings shown in orange.



Exercise 10.07

1 a

	Task 1	Task 2	Task 3
Emily	11	8	12
Troy	13	12	10
Ruby	10	13	11

b Emily – task 2, Troy – task 3, Ruby – task 1

c 28 min

2 a A – Toowoomba, B – Cairns, C – Mackay,
 D – Ipswich

b \$27 000

3 E

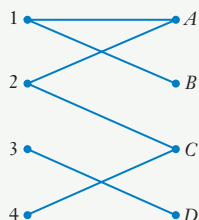
4 C

5 B

6 a

$$\begin{matrix} A & B & C & D \\ 1 & \begin{bmatrix} 0 & 0 & 5 & 6 \\ 0 & 9 & 0 & 9 \\ 3 & 7 & 5 & 0 \\ 7 & 11 & 0 & 2 \end{bmatrix} \end{matrix}$$

b



c \$33

7 a A-1, B-3, C-2; 19 min

b A-2, B-3, C-4, D-1; 39 min

8 a A-4, B-2, C-3, D-1 b 72 km

9 a A-2, B-3, C-1, D-4; \$54 000

b A-1, B-3, C-2, D-4; \$78 000

10 a A-4, B-1, C-2, D-3 b 95 min

11 a A-1, B-2, C-4, D-3; 56 minutes

b A-3, B-4, C-1, D-2, 141 minutes

12 a
$$\begin{bmatrix} 10 & 10 & 15 & 0 \\ 0 & 0 & 7 & 5 \\ 4 & 7 & 18 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b No. The zeros can be covered by 3 lines and there are 4 rows.

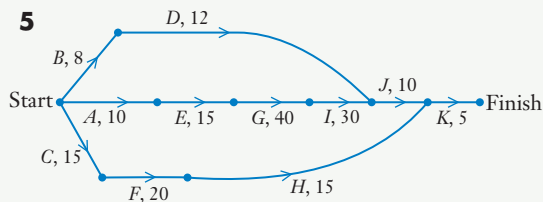
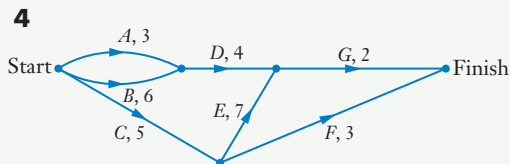
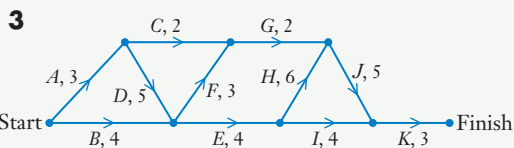
c A-4, B-2, C-1, D-3; 136 min

13 Masters Homes – Endeavour, Custom Build – Beachside, Savannah Homes – Alpine, Deakin Homes – Tropicana, Stylemaster – Forrest Grove. Total cost = \$1 650 000.

Chapter review

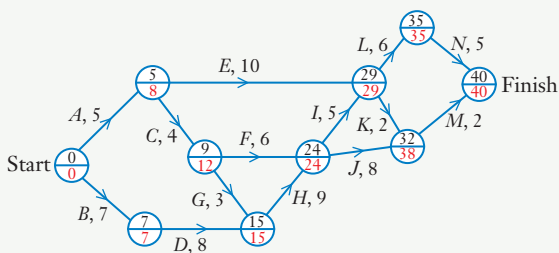
1 B

2 D



6 C 7 E 8 A 9 D 10 C

11 a, b



12 a ACH = 21 hours; ACGI = 20 hours;
AEI = 15 hours; BDJ = 17 hours;
BDFI = 22 hours

b Critical path = BDFI,
shortest time = 22 hours

13 C

14 D

15 A

16 D

17 C

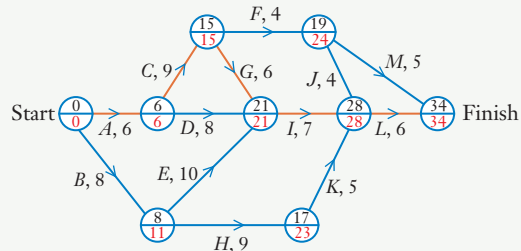
18 C

19 A

20 D

21 B

22 a, b



c 34 min

d i 7 min ii 5 min iii 6 min iv 6 min

23 a C, I, J, N = 8 hours; E, H = 10 hours;
B = 11 hours; G = 13 hours

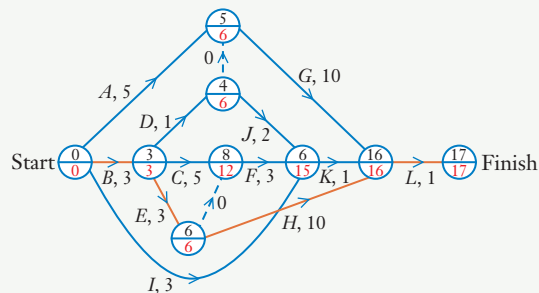
b G is non-critical and has a float time of 13 hours, so a delay of 8 hours has no effect on the duration of the project.

c Shorten either F or K. Both are critical activities and any reduction in either will shorten the duration of the project.

24 a FGMNOPQE = 32

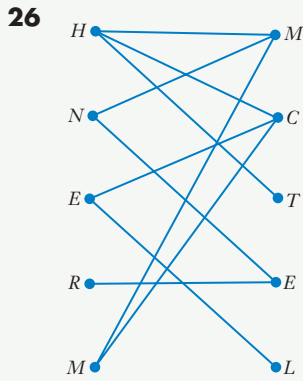
b GHIUVWXYZ or GHIUVWEF = 33

25 a, b Shortest completion time = 17 days



c It is a critical step, so it will delay the parade.

d A delay of up to 12 days has no effect.



- 27** **a** Bipartite
b Group 1 = A, E, G ; Group 2 = B, C, D, F
28 **a** Incomplete **b** Complete
c Not a matching **d** Incomplete

29 B

30 E

31 **a**

	Task 1	Task 2	Task 2
A	25	20	22
B	18	17	15
C	16	18	17

b A – Task 2, B – Task 3, C – Task 1

c 51 min

32 **a**

$$\begin{bmatrix} 17 & 25 & 26 & 20 \\ 28 & 27 & 23 & 25 \\ 20 & 18 & 17 & 14 \\ 28 & 25 & 23 & 19 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 4 & 9 & 3 \\ 5 & 0 & 0 & 2 \\ 6 & 0 & 3 & 0 \\ 9 & 2 & 4 & 0 \end{bmatrix}$$

b A — 1

B — 2

C — 3

D — 4

c 77 km

- 33** **a** Contractor A – Project 1; Contractor B – Project 2; Contractor C – Project 3; Contractor D – Project 4

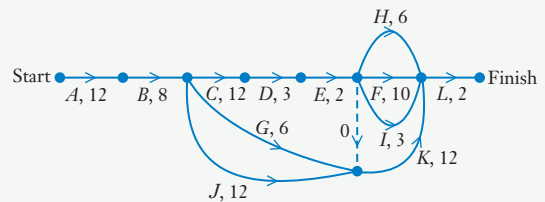
b \$141 000

- 34** Trainee A – Process 2; Trainee B – Process 3; Trainee C – Process 4; Trainee D – Process 5; Trainee E – Process 1. Total time = 181 min

- 35** Other tables are possible.

Code	Activity	Time (days)	Prerequisite
A	Arrange team photos	5	None
B	Gather photos covering the major events in the year	10	None
C	Obtain class reports	15	None
D	Get principal's report	5	None
E	Get team reports	15	None
F	Design cover	10	None
G	Get reports for the major events in the year	30	None
H	Type articles	15	C, D, E, G
I	Design layout	10	A, B, H
J	Layout pages and cover	5	F, I
K	Have printing done	10	J
L	Distribute	2	K

- 36** **a**



b 6 weeks

Practice examination 4

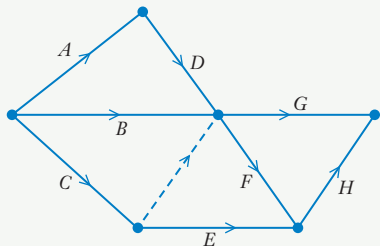
- 1** **a**

n	Value at start of year	Interest	Deposit (d)	Value at end of year
1	15 000.00	1200.00	1000.00	17 200.00
2	17 200.00	1376.00	1000.00	19 576.00
3	19 576.00	1566.08	1000.00	22 142.08
4	22 142.08	1771.37	1000.00	24 913.45

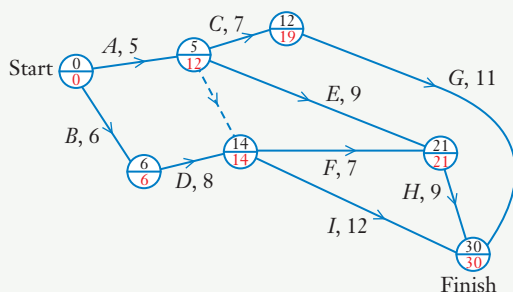
b \$5913.45

c \$24 913.45

- 2 **a** $A_{n+1} = 1.056A_n - 7000$, where $A_1 = 65\ 000$
b \$41 797.81 **c** 14 years
d \$3339.24
3 \$307 041.31
4 \$204 914.68
5 C
6



- 7 D
8 **a, b**



- 9 A
10 C
11 D
12 **a** $B, E = 1$ hour; $I, J, K, M = 4$ hours;
 $C, G = 8$ hours
b K is non-critical and has a float time of 4 hours, so a delay of 3 hours has no effect on the duration of the project.
c Shorten F . It is a critical activity and any reduction will shorten the duration of the project.
13 **a** Bipartite
b Group 1 = A, C, E ; Group 2 = B, D, F, G

14 **a**

	Supplier 1	Supplier 2	Supplier 3
Component A	54	48	48
Component B	56	51	49
Component C	55	51	50

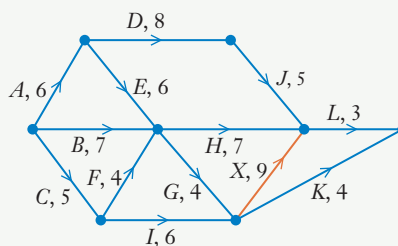
- b** A – Supplier 2, B – Supplier 3, C – Supplier 1
c \$152

15 **a** $\begin{bmatrix} 0 & 5 & 0 & 0 \\ 6 & 0 & 22 & 10 \\ 4 & 21 & 0 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 7 & 4 & 0 \\ 4 & 0 & 24 & 8 \\ 0 & 19 & 0 & 5 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

- c** Developer A – App 4; Developer B – App 2;
Developer C – App 1; Developer D – App 3;
134 weeks

16 \$1906.41

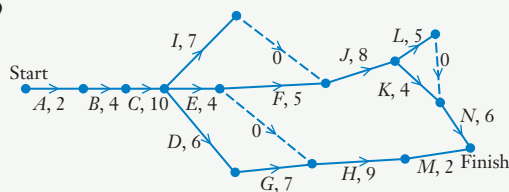
17



Critical path = $AEGXL$;
Shortest time = 28 hours;
Latest start time for J is 20 hours.

18 \$150 000

19



Completion time is 44 weeks whether or not the external cladding is 3 weeks late or the electrical work is finished 2 weeks earlier.

GLOSSARY AND INDEX

accelerated depreciation method: *See* **double declining balance method.**

activity: A task required to complete a project. (p. 438)

activity table: A table that shows the order and estimated completion time of each activity required to complete a project. (p. 439)

actual flow: The actual quantity of a commodity that moves through an edge of a network. (p. 361)

adjacent vertices: Vertices in a graph or network that are connected by an edge. (p. 294)

adjacency matrix: A matrix that shows how adjacent edges are joined. (p. 316)

allocation: A situation where each agent is assigned to one and only one task. (p. 486)

annuity: An investment that has regular deposits made over a period of time. (p. 394)

arc: An edge with a direction (shown by an arrow). (p. 294)

arithmetic sequence, arithmetic progression (AP): A sequence whose successive terms differ by the same amount. (p. 49) *See also* **common difference.**

assignment: *See* **allocation.**

association: A pattern in the ordered pairs of bivariate data, often seen in a scatterplot. (p. 22)

Australian Central Standard Time (ACST): UTC + 9.5 h; the time zone used in South Australia and the Northern Territory. (p. 214)

Australian Eastern Standard Time (AEST): UTC + 10 h; the time zone in Queensland, New South Wales, the Australian Capital Territory, Victoria and Tasmania. (p. 214)

Australian Western Standard Time (AWST): UTC + 8 h; the time zone used in Western Australia. (p. 214)

backward scanning: A process used to determine the latest starting time for each activity in a network. (p. 451)

bipartite graph: A graph in which the vertices can be divided into 2 distinct groups so that every edge of the graph connects a vertex in one group to a vertex in the other. (p. 295, 477)

bivariate: Bivariate data has 2 variables. It is collected in pairs. (p. 4)

book value: The current value of an asset, its value after depreciation. (p. 60) *See also* **prime cost method, declining balance method, unit cost method.**

branch: A vertex that has more than one exit. (p. 298)

capacity: The maximum quantity of a commodity that moves through an edge of a network. (p. 361)

categorical: Data with values that are names. *See also* **numerical.** (p. 4)

Central Standard Time: *see* **Australian Central Standard Time.**

centred moving average: Moving average with the result plotted at the middle time of the average. (p. 104)

chain: *See* **path.**

chord: Line segment that joins 2 points on a circle. (p. 206)

circle of illumination: The division between day and night on the Earth when viewed on a globe. (p. 213)

circuit: A path that returns to its starting point. (p. 298, 331)

closed trail: A trail that starts and finishes at the same vertex. (p. 331)

closed walk: A walk that starts and finishes at the same vertex. (p. 330)

coefficient of determination, R^2 : The square of the correlation coefficient of bivariate data that shows the proportion of the response variable that is determined by the least-squares line of best fit. (p. 144)

coincidence: A random relationship between variables of bivariate data that depends on the particular sample and does not show a real relationship. (p. 151)

column percentages: Percentages in a contingency table based on totals down the column. (p. 11)

See also **row percentages, table percentages.**

common difference: The difference between successive terms of an arithmetic sequence. (p. 49)

common ratio: The ratio of successive terms of a geometric sequence. (p. 64)

complete graph: A graph that has edges connecting each vertex to every other vertex in the graph. (p. 295)

complete matching: A matching in which every vertex in the first group of a bipartite graph is matched to a vertex in the second group. (p. 481)

compound interest: The interest paid on the original principal and on the accumulated past interest. (p. 77, 234)

compounding period: The length of time between interest calculations in compound interest. (p. 236)

concurrent activities: Activities that can occur at the same time. (p. 439)

confounding variable: A third variable that is the explanatory variable of both variables of bivariate data; a variable that causes the changes in both the variables of the data. (p. 151)

connected graph: A graph in which it is possible to reach every vertex of a graph by moving along the edges. (p. 295)

contingency table: A table of bivariate data with one variable across columns and the other down rows. (p. 11)

continuous: Data that can have any value between its limits. (p. 4) *See also* **discrete.**

Coordinated Universal Time: Abbreviated as UTC, reference for all time zones. For example, Queensland is in a UTC + 10 h time zone, meaning it is 10 hours ahead of the reference. (p. 214)

correlation coefficient: A number between -1 and 1 that measures how well the least-squares line fits a set of data. It is given by $r = \frac{s_{xy}}{s_x s_y}$, where s_{xy} is the

covariance and s_x, s_y are the standard deviations of the variables x and y . A value of 0 means there is no association. -1 and 1 show perfect negative and positive associations respectively. (p. 31)

covariance: s_{xy}, σ_{xy} , a measure of how well a straight line fits bivariate data. It is the average of the moments of the ordered pairs and their means. (p. 31)

critical activity: An activity that lies on the critical path. (p. 463)

critical path: The path with the longest total time in a network. (p. 457)

cut: A line that separates a flow network source from the sink. (p. 367)

cycle: A path beginning and ending at the same vertex. (p. 298, 331)

daylight saving: adjusting clocks forward in summer to have an extra hour of daylight in the evening. (p. 214)

December quarter, 4th quarter, Q4: The period from October to December inclusive. (p. 165)

declining balance method, reducing balance method: Depreciation method that reduces the value of an asset by the same ratio or percentage each year; successive values form a sequence with common ratio $(1 - r)$, where r is the depreciation rate as a decimal. (p. 80) *See also* **prime cost method, unit cost method.**

degree (of a vertex): The number of edges that are directly connected to the vertex, written $\deg(V)$. (p. 305)

dependent variable: *See* **response variable.**

deposit: The value of the regular payment into an annuity. (p. 394)

depreciation: Loss of value of an asset (item) over its working life. (p. 60) *See also* **appreciation.**

deseasonalisation: Smoothing of a time series by dividing the values for each quarter or month by the corresponding value of the seasonal index. (p. 169)

directed graph, directed network, digraph: A graph in which each edge has a direction. All the edges are arcs. (p. 295, 361)

disconnected graph: A graph in which it is not possible to reach every vertex of a graph by moving along the edges. (p. 295)

discrete: Data that has only some possible values, like integers. (p. 4) *See also* **continuous.**

double declining balance method, accelerated depreciation method: Declining balance method of depreciation with the percentage depreciation set as twice the straight-line percentage depreciation. (p. 80)

dummy activity: An activity with zero time allocated; shown as a directed edge with a dashed line in a project network. (p. 441)

earliest starting time (EST): The earliest possible time that an activity can begin. (p. 448)

Eastern Standard Time: *See* **Australian Eastern Standard Time.**

edge: A line that joins 2 vertices. (p. 294) *See also* **arc, graph.**

effective interest rate: The interest rate that is actually earned or paid in one year by compound interest with a shorter compounding period. (p. 251)

equator: The centremost and longest parallel of latitude that divides the Earth equally into the northern and southern hemispheres. (p. 196)

Euler's formula: For any planar graph: Vertices + faces – edges = 2 or $v + f - e = 2$. (p. 325)

Eulerian circuit: *See* **Eulerian cycle**.

Eulerian cycle: A trail that traverses every edge of a graph exactly once. (p. 332)

Eulerian graph: Contains an Eulerian circuit. It has all vertices of even degree and a closed trail. (p. 333)

Eulerian path: An open trail that traverses each edge only once. (p. 332)

Eulerian trail: *See* **Eulerian path**.

explanatory variable, independent variable: A statistical variable not changed by another statistical variable. (p. 124) *See also* **response variable**.

exponential decay, exponential growth, exponential relationship: Reduction or growth of a sequence of values by the same ratio in each period; relationship with successive values that form a geometric sequence; relationship such that $A_n = A_0 r^n$, where $r > 0$: decay for $r < 1$, growth for $r > 1$. (p. 76)

extrapolation: Prediction of values of the response variable of bivariate data that is outside the range of the data. (p. 146) *See also* **interpolation**.

false negative: An incorrect negative result on a pathology test. (p. 17)

false positive: An incorrect positive result on a pathology test. (p. 17)

finite network: A network that has a limited number of vertices and edges. (p. 297)

fixed-term loan: A loan in which the interest rate, instalment value and the frequency of payments remain unchanged. (p. 275)

flexible-term loan: A loan in which the interest rate, instalment value or the frequency of payments may vary. (p. 275)

float time: The maximum delay possible in starting an activity that does not affect the completion time of a project. $= LST_{End} - EST_{Start} - \text{activity time}$. (p. 463)

flow capacity: *See* **capacity**.

flow network: A directed graph where each arc has a capacity or flow capacity. (p. 361)

forward scanning: A process used to determine the earliest starting time for each activity in a network. (p. 448)

future value of money: The value of a sum of money at a specific future date, taking interest earned into account. (p. 255)

general term: A term in a sequence, usually written as a formula involving the term number, n . (p. 54, 69) *See also* **n th term**.

geometric sequence, geometric progression, GP: A sequence whose successive terms are in the same ratio. (p. 64)

gradient: *See* **slope**.

graph, network: A set of points called vertices (or nodes) connected by lines called edges (or arcs) joining pairs of vertices. (p. 294)

great circles: circles on a sphere (e.g. the Earth) that have their centres at the same place as the sphere's centre. The shortest distance on the surface between 2 points on a sphere is along a great circle. (p. 196)

Greenwich Mean Time: Abbreviated as GMT; the predecessor to UTC and the same as UTC. (p. 214)

Hamiltonian cycle: A route that includes each vertex in a graph exactly once and starts and finishes at the same vertex. (p. 338)

Hamiltonian path: A route that includes each vertex in a graph exactly once and starts and finishes at different vertices. (p. 338)

Hungarian algorithm: A process for determining optimum allocation. (p. 490)

immediate predecessor: An activity that must be completed immediately before another activity can commence. (p. 439)

independent variable: *See* **explanatory variable**.

inflow capacity: The total of the flow capacities of all edges leading into a node. (p. 361)

instalment: One of a number of successive payments (usually equal) made to settle a debt. (p. 260)

intercept, y-intercept: The value of y where a straight-line model of bivariate data crosses the vertical axis, particularly the least-squares model. (p. 130) *See also* **slope**.

interest: The price of borrowing or lending money. (p. 234)

International Date Line: The imaginary boundary line on the Earth that separates the current day from the next day, according to standard time zones. (p. 215)

interpolation: Prediction of the response variable values for bivariate data that is within the range of the data. (p. 146) *See also* **extrapolation**.

investment annuity: An annuity that has an initial balance with regular deposits and no withdrawals. (p. 394)

irregular time series: A time series with values that do not follow a pattern. (p. 94)

isolated vertex: A vertex of a graph/network that is not connected to any other vertex. (p. 294)

isomorphic graphs: Graphs that represent the same information. (p. 306)

June quarter, 2nd quarter, Q2: The period from April to June inclusive. (p. 165)

latest starting time (LST): The latest time it is possible to start an activity without delaying the completion of the project. (p. 451)

latitude: Position, in degrees north or south, with reference to the equator. (p. 196)

least-squares model: Straight-line model of bivariate data with the smallest possible total of the squares of the distances of the points from the line. (p. 130)
See also **intercept, straight-line relationship.**

linear decay, linear growth: Reduction or growth of a sequence of values by the same amount in each period; successive values for an arithmetic sequence. (p. 52)
See also **exponential growth, exponential decay.**

linear: The ordered pairs of linear association tend to be in a straight line. (p. 25)

loan schedule: A table of the interest, payments and amount owing for a loan over time. (p. 260)

longitude: Position, in degrees east or west, with reference to the prime meridian. (p. 196)

loop: An edge that connects a vertex to itself in a graph or network. (p. 294)

March quarter, 1st quarter, Q1: The period from January to March inclusive. (p. 165)

marginal frequency: The total of frequencies across or down a contingency table. (p. 5)

marginal percentage: The total of percentages across or down a contingency table. (p. 11)

matching: A one-to-one pairing of some or all of the vertices in one group of a bipartite graph with the vertices in the other group. (p. 480)

maximum flow-minimum cut theorem: Maximum flow = capacity of the minimum cut. (p. 368)

mechanism: A set of specific steps that show how changes in one of the variables of bivariate data causes changes in the other. (p. 151)

meridian: A line of longitude, connecting the north and south poles. (p. 196)

minimum cut: The cut with the minimum capacity. (p. 368)

minimum spanning tree: A spanning tree with the minimum total length of its edges. (p. 353)

moderate association: A linear association with points that are scattered around a straight line, but mostly close to the line. (p. 25)

moment: The product of the distances of a point from a central point (a, b) . For the point (x, y) and the central point $(\bar{x}, \bar{y})$ it is given by $(x - \bar{x})(y - \bar{y})$. (p. 32)

moving average: Averages the same number of successive values to smooth a time series. (p. 103)

multiple edges: Where 2 or more edges connect the same pair of vertices. (p. 294)

negative association: A pattern in numeric bivariate data where the low values of a variable have a tendency to occur with the high values of the other. (p. 22)

network flow: The movement of commodities across a network. (p. 361)

network: *See* **graph.**

node: *See* **vertex.**

nominal: Categorical data with no natural order, like colours. (p. 4) *See also* **ordinal.**

nominal interest rate: The stated annual interest rate for a loan or investment. (p. 250)

non-linear: An association that is not linear. (p. 25)

n th term: Term number n of a sequence, the general term of a sequence, shown as t_n . (p. 48) *See also* **general term.**

numerical: Data with values that are numbers. (p. 4)
See also **categorical.**

open trail: A trail that starts and finishes at different vertices. (p. 331)

open walk: A walk that starts and finishes at different vertices. (p. 330)

optimum allocation: The most efficient allocation in terms of time, cost, energy consumption, etc. (p. 487)

ordinal: Categorical data with a natural order, like preferences. (p. 4) *See also* **nominal.**

ordinary annuity: An annuity that has an initial balance with regular deposits and regular withdrawals. (p. 402)

outflow capacity: The sum of the flow capacities of the edges directed away from a node in a flow network. (p. 361)

outlier: A time series value that is very different from the rest of the values. (p. 98)

parallels of latitude: Lines, parallel to the equator, which circle the world east-west. (p. 196)

path, chain: A trail with no repeated vertices. (p. 298, 331)

payment: The regular amount deposited/withdrawn from an investment annuity, ordinary annuity, superannuation account or perpetuity. (p. 402)

Pearson's correlation coefficient: *See* **correlation coefficient.** (p. 32)

periodic interest rate: The annual (nominal) interest rate divided by the number of compounding periods per year. (p. 236)

perpetuity: An annuity that has regular payments that continue indefinitely. (p. 418)

planar graph: A graph that can be drawn without edges that cross. (p. 323)

positive association: A pattern in numeric bivariate data where the variables have a tendency to decrease or increase together. (p. 22)

prerequisite: *See immediate predecessor.*

present value of money: The current value of a sum of money that must be invested to achieve a future value, taking interest earned into account. (p. 256)

Prim's algorithm: An algorithm used to identify a minimum spanning tree. (p. 354)

prime cost: The cost of an asset when it is first purchased. (p. 60)

prime cost method: *See straight-line method*

Prime Meridian: The line of longitude at 0° used as the reference for time zones. The time zone on the Prime Meridian is UTC (or GMT). (p. 196)

principal: The amount of money borrowed or invested. (p. 234)

project network: The sequence of activities that make up a project, shown as a directed network (or directed graph). (p. 438)

Q1, Q2, Q3, Q4: Respectively, the March, June, September and December quarters of the calendar year. (p. 165)

quarter: A 3-month period of the year, particularly one of Q1, Q2, Q3, Q4. (p. 165)

R^2 : *See coefficient of determination.*

recursive: A definition of a sequence that shows how each term is calculated from the previous term. (p. 49)

reducing balance loan: A loan in which the interest is calculated at each compounding period. (p. 260)

reducing balance method: *See declining balance method.*

reducing interest loan: *See reducing balance loan.*

relationship: An association with an equation that models the data, particularly a linear association. (p. 127)

repayment (payment): *See instalment.*

residual: The vertical distance between a data point of bivariate data and the line of best fit (least-squares line). (p. 137)

residual flow capacity: The difference between the flow capacity and the actual flow. (p. 361)

residual plot: A plot of all the residuals for a straight-line model of bivariate data against the explanatory variable x . (p. 137)

response variable, dependent variable: A statistical variable that could be changed by another statistical variable. *See also explanatory variable.* (p. 124)

rest: *See compounding period.*

route: A sequence of edges on a graph or network. (p. 330)

row and column reduction: A process for determining optimum allocation. (p. 487)

row percentage: Percentages in a contingency table based on totals across the rows. (p. 11) *See also column percentages, table percentages.*

rule of 5: The statement that probabilities based on frequencies of less than 5 are unreliable. (p. 12)

salvage value, scrap value: The value of an asset at the end of its working life. (p. 60)

scatterplot, scatter graph, scattergram: A graph of the ordered pairs of numerical bivariate data. (p. 21) *See also time series plot.*

scrap value: *See salvage value.*

seasonal index: The values that show the variation of a time series across the quarters or months of a year. (p. 164)

seasonal time series: A time series with values that rise and fall in the same way each year, often linked to the seasons. (p. 94)

semi-Eulerian graph: Contains an Eulerian path. It has only two vertices of odd degree, an open trail or Eulerian path exists and it must start and finish at a vertex of odd degree. (p. 333)

September quarter, 3rd quarter, Q3: The period from July to September inclusive. (p. 165)

sequence: A list of numbers in a particular order, usually written as $t_1, t_2, t_3, \dots$ (p. 48)

simple (path or circuit): A path or circuit that does not cross over itself, so does not repeat nodes. (p. 298)

simple graph: A graph where the vertices are connected by no more than one edge. (p. 295)

simple interest: Interest that is calculated using the formula $I = PRT$. (p. 234)

sink (T): The node in a flow network towards which all flows move – the end node. (p. 361)

sinking fund: An annuity set up to receive periodic payments to pay for an item at some future time. (p. 422)

slack time: *See float time.*

slope: The vertical distance between two points of a least-squares model of bivariate data, divided by the horizontal distance. It measures the change of the response variable per unit change in the explanatory variable. (p. 130) *See also intercept.*

small circles: Circles on the Earth that *do not* have their centre at the centre of the Earth. (p. 196)

See also great circles.

smoothing: Removal of variation from a time series (to make important trends more obvious). (p. 103)

See also moving average, centred moving average.

source: The node in a flow network from which all flows commence. (p. 361)

spanning tree: A connected graph/network that includes all the vertices of the original.

(p. 298, 345)

standard deviation: The standard deviation of a variable measures its spread from the mean. It is

defined as $\sigma_x = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$ for a population of

size n or $s_x = \sqrt{\frac{\sum (x - \bar{x})^2}{n-1}}$ for a sample of size n .

(p. 30)

standard time zones: Time zones that are internationally recognised as standard. (p. 214)

straight-line method, prime cost method: The method of depreciation that reduces the value of an asset by the same amount each year using the formula

$$\text{Depreciation} = \frac{\text{Prime cost} - \text{Salvage value}}{\text{Life in years}} / \text{year.}$$

(p. 60) *See also declining balance method, salvage value, unit cost method.*

straight-line relationship, straight-line model:

Model of the relationship between the explanatory variable (x) and response variable (y) of bivariate data of the form $y = ax + b$ or $y = a + bx$. (p. 128) *See also least-squares model.*

strong association: A linear association with points that are almost all on a straight line. (p. 25)

subgraph: A graph that is all or part of another graph without any new edges or vertices. (p. 295)

superannuation: Money that is put aside and saved while you're working, so you can have a regular income later in life when you retire. (p. 409)

table percentages: Percentages based on the grand total of all frequencies in a contingency table. (p. 11) *See also column percentages, row percentages.*

term: Any of the members of a sequence. (p. 48)

See also n th term.

time series: A set of (numeric) data collected at regular periods of time. (p. 90)

time series plot: A graph of a time series with time on the horizontal axis. Points are often joined to make a line graph. (p. 90) *See also scatterplot.*

time zone: An area of a country or state that has its clocks set to a common time. (p. 214)

trail: A walk with no repeated edges. (p. 331)

tree: A connected graph/network that does not contain any loops, multiple edges or cycles. (p. 298)

trend: A gradual increase or decrease in the values of a time series. (p. 94)

true negative: A correct negative result on a pathology test. (p. 17)

true positive: A correct positive result on a pathology test. (p. 17)

two-way table: *See contingency table.*

undirected network: A network that has edges with no directions. (p. 297)

unit cost method, unit production cost method:

The method of depreciation that reduces the value of the asset by the same amount each year using the formula

$$\text{Depreciation} = \frac{\text{Prime cost} - \text{Salvage value}}{\text{Units of production}} / \text{unit.}$$

(p. 60) *See also reducing balance method, prime cost method, scrap value.*

univariate: Data with only one variable. (p. 4)

valid (cut): A cut in a flow network that stops all flow from the source to the sink. (p. 367)

vertex, node: The points joined by edges in a graph or network. (p. 294)

walk: Any route between 2 vertices of a network, perhaps including repeated edges and vertices. (p. 330)

weak association: A linear association with points that are mostly diagonally across a scatterplot, but do not have a clear straight line. (p. 25)

weighted network: A network in which a number is given for the 'length' of each edge. (p. 297)

Western Standard Time: *See Australian Western Standard Time.*

y-intercept: *See intercept.*

OVERVIEW

Nelson QMaths 11 & 12 has been rewritten and updated to cover the new Queensland senior courses: *Essential Mathematics, General Mathematics, Mathematical Methods* and *Specialist Mathematics*.



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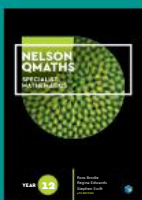
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- Syllabus grid, answers and glossary/index included

