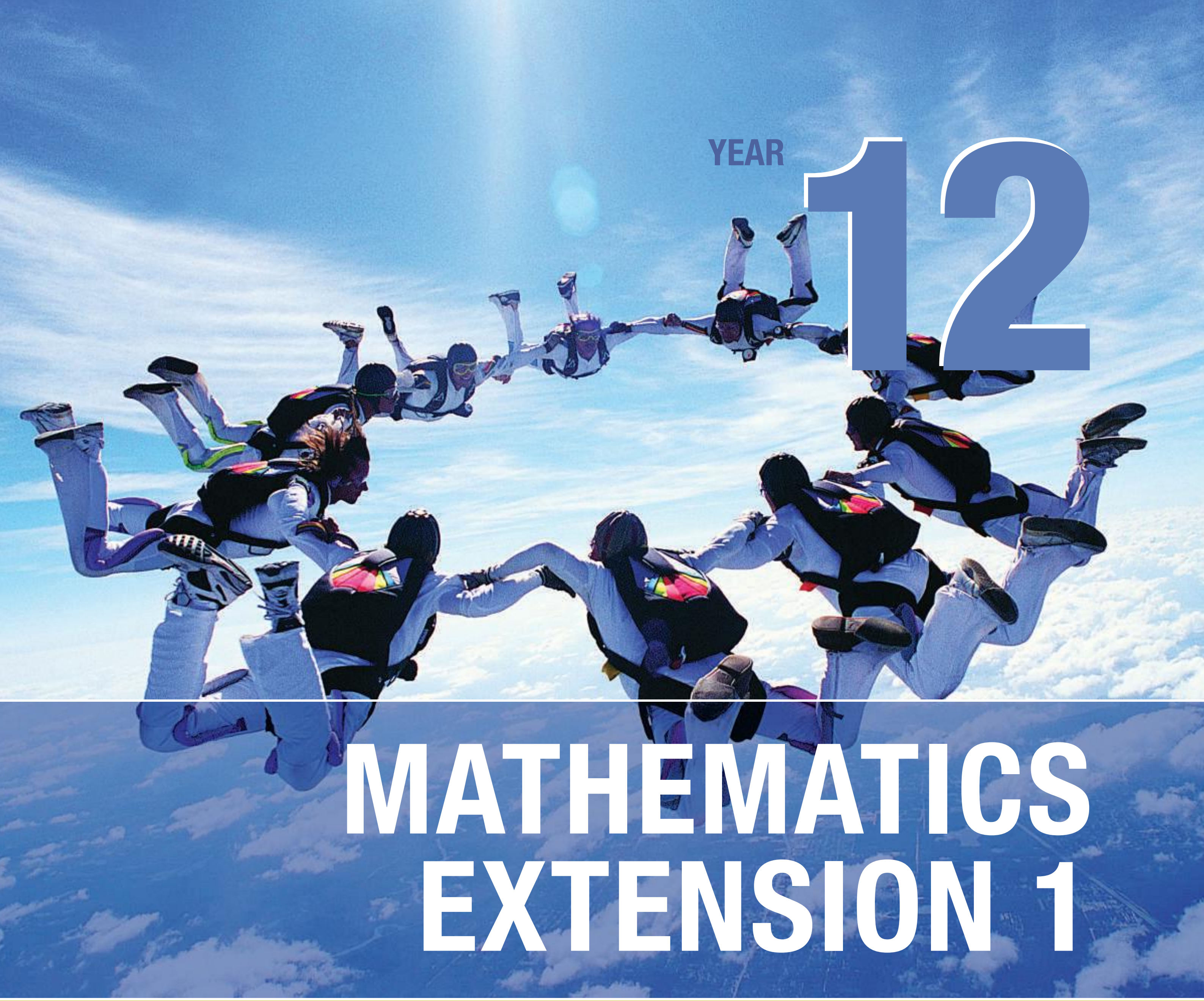


YEAR 12

MATHEMATICS EXTENSION 1

CambridgeMATHS

STAGE 6

BILL PENDER

DAVID SADLER | DEREK WARD | BRIAN DOROFAEFF | JULIA SHEA

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

CAMBRIDGE UNIVERSITY PRESS

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

First published 2020
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3

Cover and text designed by Sardine Design
Typeset by diacriTech
Printed in China by C & C Offset Printing Co., Ltd.

A catalogue record for this book is available from the National Library of Australia at www.nla.gov.au

ISBN 978-1-108-76630-2 Paperback

Additional resources for this publication at www.cambridge.edu.au/GO

Cambridge University Press acknowledges the Australian Aboriginal and Torres Strait Islander peoples of this nation. We acknowledge the traditional custodians of the lands on which our company is located and where we conduct our business. We pay our respects to ancestors and Elders, past and present. Cambridge University Press is committed to honouring Australian Aboriginal and Torres Strait Islander peoples' unique cultural and spiritual relationships to the land, waters and seas and their rich contribution to society.

Table of contents

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Online appendices

See the interactive textbook for guides to spreadsheets, the Desmos graphing calculator, and links to scientific calculator guides.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Rationale — Extension 1 Year 12

CambridgeMATHS Extension 1 Year 12 covers all syllabus dotpoints for Year 12 of both new courses being implemented in Term IV 2019 — the Mathematics Advanced course and the Mathematics Extension 1 course. This rationale serves as a guide to how the book covers the dotpoints of the two syllabuses. Further documents are available in the teacher resources.

The exercises

No-one should try to do all the questions! We have written long exercises with a great variety of questions so that everyone will find enough questions of a suitable standard — they cater for differentiated teaching to a wide range of students. The division of almost all exercises into Foundation, Development and Enrichment sections helps with this. Each student will need to tackle a selection of questions, and there should be plenty left for revision.

- Compared to our previous Extension 1 textbooks, the **Foundation** section in each exercise provides a gentler start. There are more questions, and they are straightforward. Students need encouragement to assimilate comfortably the new ideas and methods presented in the text so that they are prepared and confident before tackling later problems.
- The **Development** sections are graded from reasonably straightforward questions to harder problems. They are intended to bring students up to Extension 1 standard. The examinations have traditionally included questions that students find difficult, and we assume that this will continue with the HSC examinations on the new syllabuses — there are questions at the end of the Development sections intended to prepare students for these demands.
- The **Enrichment** sections contain 'interesting' questions that are probably beyond the Extension 1 examination standard (although one can never be sure of this) and often beyond the Extension 1 syllabus. Some are algebraically or conceptually challenging, some establish significant results beyond the theory of the course, perhaps linking with Extension 2, some make difficult connections between topics or give an alternative approach, some deal with logical problems unsuitable for the text of an Extension 1 book. Students taking the Extension 2 course should attempt some of these.

The Extension 1 material

This book presents the Year 12 Advanced course and Year 12 Extension 1 course within the one book. Despite the tight integration of the two courses, there are various practical reasons why Advanced material and Extension 1 material need to be easily identified and separated. In this book:

- The ten chapters from the Advanced textbook have been retained, and so have their sections, with a handful of additional sections needed in the Extension 1 book. Some of the text and exercises in those chapters has been made slightly more demanding so that it is suitable for the higher standards customary in Extension 1.
- The Extension 1 material has been placed into seven separate chapters distributed appropriately through the volume, and clearly marked as Extension 1 content.

The circumstances of individual schools, and particularly the need to coordinate with the Extension 2 course, may require the order of the chapters to be varied.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The structure of Advanced calculus

Calculus is the centre of these two courses, and the structure of calculus has determined the order in which the topics are presented in the book. Explanation is particularly needed for how the ten Advanced-content chapters organise Year 12 calculus and group the other topics around it.

CHAPTER 1: Sequences and series are the discrete analogue of infinitesimal calculus. In particular, a definite integral consists of ‘an infinite sum of infinitesimally thin strips’, so that sums of series, which are conceptually much easier, should precede integration. Also, linear and exponential functions are the continuous analogues of APs and GPs.

CHAPTER 3: Systematic curve-sketching using differentiation is traditionally the first and most straightforward application of calculus. But before reaching for the derivative, it is important to give a systematic account of the various non-calculus approaches to curve-sketching.

CHAPTERS 4–5: Differentiation, curve-sketching with calculus, and integration, are three dramatic new ideas in school that change students’ perceptions of the nature of mathematics, making it far more imaginative and speculative, and allowing easy solutions of problems that may otherwise seem impossible. The first encounter with these three ideas should involve only algebraic functions, because the complexities of the special functions associated with e^x and $\sin x$ are unnecessary here, and they cause confusion if introduced too early.

1 Differentiation was introduced in Year 11, so curve-sketching and integration are the next two topics in the book. The derivative of e^x was established last year because of the interest in rates, but we have briefly left e^x aside in thesetwo chapters so that the story is not confused.

CHAPTER 6: With the basic methods of calculus now established, the significance of e^x as a function that is its own derivative, and of $\log_e x$ as a primitive of x^{-1}, can be explained, and the importance of the special number e made clear. This chapter is necessarily long — the ideas are unfamiliar and unsettling, their far-reaching significance is difficult to explain, and students need time to assimilate them.

CHAPTER 7: The other group of special functions are the trigonometric functions, with their special number π. The standard forms now become more numerous, but the basic methods of calculus remain the same, and by the end of Chapter 7 these methods will have been reviewed three times — for exponential functions, for logarithmic functions, and for trigonometric functions.

CHAPTERS 9 & 14: These are application chapters. Chapter 9 applies calculus to motion and rates. Chapter 14 is the discrete analogue of Chapter 9, applying series to practical situations, particularly finance.The discussion of motion and rates in Chapter 9 allows a review of all the functions introduced in calculus. Motion, in particular, allows the derivatives to be perceived by the senses — the first derivative is velocity, which we see, and the secondderivative is acceleration, which we feel.

1 Learning to handle applications of calculus is surprisingly difficult, and while a few motion questions have been included in earlier exercises, motion needs a sustained account if it is to be mastered — it requires confusing contrasts between displacement, distance and distance travelled, and between velocity and speed. Rates were addressed in Year 11 (Chapter 17, an Extension 1 chapter), but still require further experiences of translating physical events into the abstractions of calculus.

CHAPTERS 15–16: Integration and the exponential function together make possible a coherent presentation of statistics and the normal distribution in these two chapters. Probability becomes an area, and most of the mysteries of $e^{-\frac{1}{2}x^2}$ can now be rigorously expounded.

ISBN 978-1-108-76630-2

Syllabus coverage of the chapters

Chapter 1: Sequences and series (Advanced)

1A Sequences and how to specify them
1B Arithmetic sequences
1C Geometric sequences
1D Solving problems involving APs and GPs
1E Adding up the terms of a sequence
1F Summing an arithmetic series
1G Summing a geometric series
1H The limiting sum of a geometric series
1I Recurring decimals and geometric series
Chapter 1 Review

Syllabus References: MA-M1.2
MA-M1.3

Chapter 1 presents the theory of arithmetic and geometric sequences and series. Its purpose at the start of the book is to give a wider mathematical context for linear and exponential functions in Chapters 3 and 6, for the derivative used in Chapter 4, and for the definite integral in Chapter 5. There are some practical examples throughout, but there are many more in Chapter 14, and it may be appropriate to bring forward some questions from Sections 14A–14C.

Arithmetic and geometric sequences are closely related, and are explained together as the theory progresses through sequences in Sections 1A–1C, problems in Section 1D, and the sums of series in Sections 1E–1G. The chapter concludes in Sections 1H–1I with limiting sums of GPs and the explanation — finally — of what a recurring decimal actually is.

Sigma notation is introduced for several reasons. It allows a more concise notation for series, it prepares the ground for the continuous sum $\int$ of integration, it makes more precise the rather vague use of Σ in the statistics chapters, it was badly needed in Year11 with Chapter 15 on the binomial expansion, and the notation is needed in later mathematics courses. Nevertheless, neither the text nor the exercises rely on it, and the few questions that use it can easily be avoided or adapted.

Chapter 2: Mathematical induction (Extension 1)

2A Using mathematical induction for series
2B Proving divisibility by mathematical induction
Chapter 2 Review

Syllabus References: ME-P1

Mathematical induction is a very general method of proof, and is actually is one of the axioms of mathematics itself, so that learning how to use mathematical induction is an essential part of learning the structure of mathematics.

In this course, mathematical induction is only applied to two tasks — proving formulae for the partial sums of series, and proving divisibility (the Extension 2 course has more). We have nevertheless placed some further well-known and straightforward proofs using mathematical induction into the Enrichment sections so that the general nature of the approach can be understood.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Because the method is so fundamental, it is also important to give the odd question with an invalid argument asking for an explanation of how the method was misused.

Chapter 3: Graphs and equations (Advanced)

Syllabus References: MA-F1.2 dotpoint 2 (An alternative interval notation)

MA-F1.2 dotpoint 7 (An alternative composite function notation)

MA-F2

MA-T3

The syllabus item F1 stresses ‘any function within the scope of this syllabus’. As explained in the introduction, the methods used here will be extended to other functions as they are introduced in Chapters 6, 7 and 16, apart from the trigonometric graphs that are covered in Section 3J. It also stresses ‘real-life contexts’, which will be further developed in Chapter 9.

Regions in the coordinate plane are not mentioned in the syllabus, but they receive sustained attention in the current NESA Extension 1 study guides. Section 3F deals with the sketching of regions specified by a set of inequalities in x and y. Some idea of regions would be very helpful when studying the Extension 1 topic of areas of regions and volumes of rotation (Section 12F).

F1.2 dotpoints 2 & 7 & Sections 3A: The chapter begins with two pieces of notation that were flagged last year as being a distraction at the start of Year 11. Bracket interval notation such as $[3, 6]$ or $(2, \infty)$ is an alternative for inequality interval notation such as $3 \le x \le 6$ or $x > 2$. It requires more sophistication than was appropriate at the start of Year 11, and in particular it requires the symbol $\cup$ for the union of sets, which was introduced for probability in Sections 12C–12D late in the Year 11 book. A composite function $g(f(x))$ can also be written as $g \circ f(x)$, which was unnecessarily abstract for the start of Year 11.

F2 dotpoint 2, sub-dotpoints 1–2 & Sections 3A–3C: These sections consolidate and extend the curve-sketching methods from Year 11, particularly the sign of the function and asymptotes, and organise them into a curve-sketching menu, which gives asystematic approach to sketching an unknown function (this organisation of approaches to curve-sketching is only our suggestion).

F2 dotpoint 2, sub-dotpoints 3–4 & Sections 3D–3E: Graphing is closely related to the solving of equations and inequations. These sections formalise several important connections and methods.

F2 dotpoint 1 & Sections 3G–3I: After reviewing the translations, reflections and rotations introduced last year, these sections introduce dilations, then investigate which transformations do not commute with other transformations — a difficult question with a surprisingly simple answer. Replacement is our preferred method of relating transformations to the equations of the function or relation, but the formulae approach is also presented. The sections conclude with a complicated formula that transforms a function successively by four separate transformations.

T3 & Section 3J: Radians were introduced last year, so transformations of trigonometric graphs can be covered here. They lead to the four ideas of amplitude, period, phase and mean value, and they can be dealt with by the same complicated formula given in Section 3I.

Chapter 4: Curve-sketching using the derivative (Advanced)

4A Increasing, decreasing and stationary at a point
4B Stationary points and turning points
4C Some less familiar curves
4D Second and higher derivatives
4E Concavity and points of inflection
4F Systematic curve sketching with the derivative
4G Global maximum and minimum
4H Applications of maximisation and minimisation
4I Maximisation and minimisation in geometry
4J Primitive functions
Chapter 4 Review

Syllabus References: MA-C3.1
MA-C3.2
MA-C4.1 dotpoints 1, 2, 3, 11 (without the use of the integral sign)

The first and most straightforward application of the derivative is to assist in the sketch of a function by examining its gradient, stationary points, concavity and inflections. This chapter explains those procedures, which will be reviewed and extended progressively to exponential functions (Section 6E), logarithmic functions (Section 6H) and trigonometric functions (Section 7E), and used in various applications throughout the book, particularly in Chapter 9 on motion and rates.

C3.1, 3.2 dotpoints 1–2 & Sections 4A–4F: What is needed here for curve-sketching are *pointwise* definitions of increasing and decreasing, and of concave up and concave down. The *interval-wise* definitions are delayed until Section 9D, where they are needed for the discussion of rates in Sections 9D–9F. Section 4F concludes the discussion by extending the curve-sketching menu of Chapter 3 with two more steps involving the first and second derivatives.

C3.2 dotpoint 3 & Sections 4G–4I: The global maximum of a function is now introduced, followed by two exercises giving diverse examples of the use of these procedures to find the maxima or minima of functions in various practical situations.

C4.1 & Section 4J: Primitive functions have a strong graphical interpretation, and in preparation for the following integration chapter, it is useful to review and extend here the discussion of primitives in Section 9D of the Year 11 book.

More is required in Extension 1 than in Advanced. Section 3C has more detail about cusps and vertical tangents, and Section 4I has more detail about optimisation in geometric situations.

Chapter 5: Integration (Advanced)

5A Areas and the definite integral
5B The fundamental theorem of calculus
5C The definite integral and its properties

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5D Proving the fundamental theorem
5E The indefinite integral
5F Finding areas by integration
5G Areas of compound regions
5H The trapezoidal rule
5I The reverse chain rule
Chapter 5 Review

Syllabus References: MA-C4.1
MA-C4.2

Having briefly reviewed the primitive in Section 4J, this chapter now begins and develops integration in the traditional way by asking questions about areas of regions where some of the boundaries are neither lines nor arcs of circles. The procedures in this chapter will be reviewed and extended progressively to exponential functions (Sections 6D–6E), reciprocal functions (Sections 6I–6J) and trigonometric functions (Sections 7D–7E), and used in various applications throughout the book.

C4.2 dotpoints 1–3, 5–8 & Sections 5A–5D: These sections introduce the definite integral in terms of area, and first calculate integrals using area formulae. There is also some guided work on limiting sums of areas. The fundamental theorem isstated and used in Section 5B, but not proven until Section 5D, when its significance will have become clear. Using geometric arguments, Section 5C explores the behaviour of the definite integral when the curve is below the x-axis or the integral runs backwards, and identities are developed that can simplify the evaluation of a definite integral.

C4.2 dotpoints 4, 9–11 & Sections 5F–5H: Areas between curves are introduced, including areas between a curve and the y-axis. Then the trapezoidal rule is developed, first geometrically, then algebraically, then using spreadsheets to aid the computation. Some practical examples are given, and others occur throughout the book, particularly with motion in Chapter 9.

C4.1 & Sections 5E, 5I: The basic ideas of primitives were discussed in Section 4H (*C4.1 dotpoints 1–3, 9–11*), and the remaining standard forms are dealt with in Chapters 6–7 as the new special functions are developed. The reverse chain rule is introduced here with powers of functions (*C4.1 dotpoint 4*), in preparation for its further use with special functions.

Chapter 6: The exponential and logarithmic functions (Advanced)

6A Review of exponential functions base e
6B Differentiation of exponential functions
6C Applications of differentiation
6D Integration of exponential functions
6E Applications of integration
6F Review of logarithmic functions
6G Differentiation of logarithmic functions
6H Applications of differentiation of $\log_e X$
6I Integration of the reciprocal function
6J Applications of integration of $1/x$
6K Calculus with other bases
Chapter 6 Review

Syllabus References: MA-F2, as relevant
MA-C2.1 dotpoints 3–5

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

MA-C2.2, as relevant

MA-C3.1, as relevant

MA-C3.2, as relevant

MA-C4.1, as relevant

MA-C4.2, as relevant

As stated in the introduction to these notes, the exponential and logarithmic functions are unfamiliar and difficult for students, and need their own sustained development rather than being interlocked with powers of x and trigonometric functions. All curve-sketching, equation-solving and calculus relevant to these functions is completed in these sections, in preparation for its further use, particularly in Chapter 9 on motion and rates, in Chapter 13 on differential equations, and Chapter 16 on the normal distribution.

The topic was begun in Chapter 11 of the Year 11 book with the differentiation of the special number e and the exponential function e^x. The results, but not all the explanations, are reviewed here in Sections 6A and 6F. Some further review of that earlier chapter may be appropriate.

We draw attention to what are possibly unfamiliar standard forms in Section 6I (*C4.1 dotpoint 7*).

$$\int \frac{1}{x}dx = \log|x| + C \quad \text{and} \quad \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C.$$

Students will now need to cope with absolute values in their integrals, and the further consequences of this standard form have made the topic more difficult than it used to be.

Section 6K on calculus with other bases is important because in ordinary language, half-lives and doubling times are so often used. The standard forms are to be learnt, but students should be aware of the other possibility — convert everything to base e before applying calculus.

Chapter 7: The trigonometric functions (Advanced)

7A The behaviour of $\sin x$ near the origin
7B Differentiating the trigonometric functions
7C Applications of differentiation
7D Integrating the trigonometric functions
7E Applications of integration
Chapter 7 Review

Syllabus References: MA-C2.1, as relevant

MA-C2.2, as relevant

MA-C3.1, as relevant

MA-C3.2, as relevant

MA-C4.1, as relevant

MA-C4.2, as relevant

As in the previous chapter, the trigonometric functions are difficult for students, who learn best when these functions are developed separately before being combined with other functions. All curve-sketching, equation-solving and calculus relevant to these functions is completed in these sections, in preparation for its further use, particularly in Chapter 9 on motion and rates, Chapter 11 on trigonometric equations, and Chapter 12 on further calculus.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Radian measure, based on the special number π, was introduced in Chapter 11 of the Year 11 book, and the geometric formulae established there are important tools in this chapter. The graphs in radians were also discussed there at length, then subjected to four separate transformations in Section 3I.

The fact that the derivative of $\sin x$ is $\cos x$ should be informally clear by now, but the informal graphical proof is presented again. Using the geometric formulae, we are now able to prove in Section 7A using limits that $\sin x$ has gradient exactly 1 at the origin. The rest of the proof uses compound angle formulae from Chapter 17 of the Year 11 book to move this result along the curve.

The standard forms for integration on the current formulae sheet seem to go significantly beyond the standard forms in C4.1. There may be future adjustment here.

Chapter 8: Vectors (Extension 1)

8A Directed intervals and vectors
8B Components and column vectors
8C The dot product (or scalar product)
8D Geometric problems
8E Projections
8F Applications to physical situations
Chapter 8 Review

Syllabus References: ME-V1.1
ME-V1.2

Vectors and linear algebra are fundamental in all mathematics, but difficult to motivate at school. The most obvious motivation is that vectors, understood as 'quantities with directions', are needed in physics and many other sciences — Section 8F gives examples of this.

Application in science, however, cannot be used for the exposition of the topic in mathematics. On the other hand, the abstract axiomatic approach in many university courses is quite out of the question. The presentation here in terms of Euclidean geometry (Section 8A) using trigonometry (Section 8C) is the best approach. Our presentation is as rigorous as is reasonable at school, and we ignore some of the tricky machinery required to set up vectors rigorously this way.

An origin and coordinate axes are introduced early (Section 8B) so that the topic can be linked quickly to the familiar coordinate axes, but care should be taken to distinguish between questions where and origin and coordinate axes are involved, and questions that refer to the Euclidean plane without an origin or axes.

Proving Euclidean theorems is a large part of this topic (Section 8D particularly), but readers may miss the systematic rigour of Euclidean geometry. We have chosen to be unclear, for example, about the distinction between the definition and the properties of a parallelogram, judging that too much rigour in a preliminary course kills the subject. Some classes would benefit from more care and detail in the exposition of Section 8A. Projections are difficult, whereas components are not. Explanations in terms of components seem to be the best way to approach Section 8E.

Chapter 9: Motion and rates (Advanced)

9A Average velocity and speed
9B Velocity and acceleration as derivatives
9C Integrating with respect to time
9D Rates and differentiation

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9E Review of related rates
9F Rates and integration
Chapter 9 Review

Syllabus References:

Motion: MA-C1.2 dotpoint 3, MA-C1,3 dotpoints 6–7, MA-C3.1 dotpoint 2, MA-C3.2 dotpoint 3

Rates: MA-C1.2 dotpoints 1–2, MA-C1.3 dotpoint 8, MA-C1.4 dotpoint 4, MA-C3.2 dotpoint 3,

MA-M1.2 dotpoint 5, MA-M1.4 dotpoint xtension 1 rates: ME-C1.1, ME-C1.3

The syllabus references are scattered through the topics (the references above include both Year 11 and Year 12), but there is a constant concern that calculus be modelled by rates of various types, and in particular applied to motion and to exponential growth and decay. Rates appear in everyday things that we can see and feel, such as velocity, acceleration, the rise and fall of the tides and the temperature. Then they appear in important contemporary concerns such as populations, radioactive decay and inflation.

As explained in the introduction to this Rationale, learning to apply calculus to practical situations is surprisingly difficult. What is needed is sustained and contrasting work to gain experience in reinterpreting the abstract objects of calculus. A few motion questions have been asked already, and rates were addressed in Chapter 17 of the Year 11 book.

Motion is particularly tricky and needs its own treatment in Section 9A–9C separate from other rates. Related Rates are reviewed from Section 16A of the Year 11 book (Section 9E), but exponential growth and decay is not reviewed here (as it is in the Advanced book) because it was well covered in Year 11 (Chapter 17), and it is reviewed in Chapter 13: Differential equations.

There are several common ideas through the whole chapter. First, there are the correspondences, radient of a chord $\leftrightarrow$ average rate $\leftrightarrow$ average velocity, radient of a tangent $\leftrightarrow$ instantaneous rate $\leftrightarrow$ instantaneous velocity.

Secondly, concavity can be interpreted as indicating whether the rate of increase or decrease is increasing or decreasing, and in motion as the direction in which the particle is accelerating and whether the speed is increasing or decreasing. Thirdly, differentiation moves from the quantity to the rate (Section 9D), and integration from the rate to the quantity (Section 9F).

Section 9D gives at last a precise definition of *increasing in an interval* and of *concave up in an interval*. Curve-sketching with calculus (Chapter 3) was not an appropriate time for these ideas, but they are part of the languageof rates.

Chapter 10: Projectile motion (Extension 1)

10A Projectile motion — the time equations
10B Projectile motion — the equation of path
Chapter 10 Review

Syllabus References: ME-V1.3

Projectile motion is the only motion in two dimensions in this course. The new idea of vectors (Chapter 8) is obviously involved here, but it turns out that the main use of vectors here is first, to allow more concise descriptions of the motion, and secondly, to make clearer the role of components in the analysis of the motion.

The derivation of the equation of path is demanding, but standard. Some of the applications of the equation of path, however, can be rather difficult.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 11: Trigonometric equations (Extension 1)

11A Equations involving compound angles
11B The sum of sine and cosine functions
11C Using the t-formula to solve equations
Chapter 11 Review

Syllabus References: ME-T3

Compound angle formulae were introduced in Chapter 17 of the Year 11 book. Here they are applied to the solution of trigonometric equations.

Expressing the sum of two sine or cosine functions of the same period as a single sine or cosine function is particularly important because it is the beginning of Fourier analysis, which lies behind all our electronic communications and so manynatural phenomena.

Chapter 12: Further calculus (Extension 1)

12A Inverse trigonometric functions — differentiating
12B Inverse trigonometric functions — integrating
12C Further trigonometric integrals
12D Integration by substitution
12E Further integration by substitution
12F Volumes of rotation
Chapter 12 Review

Syllabus References: ME-C2
ME-C3.1

This chapter develops calculus further in several directions, using the methods introduced earlier dealing with inverse trigonometric functions and trigonometric identities (Chapter 17 of the Year 11 book), trigonometric equations (Chapter 11) and calculus with trigonometric functions (Chapter 7).

ME-C2, dotpoint 4: Differentiation of inverse functions was in introduced in Year 11 as an example of the chain rule in Section 9E, and is used particularly when differentiating the logarithmic function in Section 6G, and with differential equations in Chapter 13.

ME-C2 dotpoints 5–6 & Sections 12A–12B: Chapter 17 of the Year 11 book developed inverse trigonometric functions. We can now develop the calculus of the inverse trigonometric functions, which leads to two important standard forms allowing the integration of $\dfrac{1}{\sqrt{1 - x^2}}$ and $\dfrac{1}{1 + x^2}$.

ME-C2 dotpoints 2–3 & Section 12C: This section develops further trigonometric integrals.

ME-C2 dotpoint 1 & Section 12D–12E: The reverse chain rule is further developed to integration by substitution.

ME-C3.1 & Section 12F: Volumes of rotation are found by returning to the idea of a definite integral as an ‘infinite sum of infinitesimals’ and slicing the solid into thin slices. It will benefit from at least some prior work on regions in Section 3F.

Areas between curves, and areas between a curve and the x-axis are mostly covered in Chapters 5–7 on integration and the special functions.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 13: Differential equations (Extension 1)

13A Differential equations
13B Slope fields
13C Separable differentiable equations
13D $y' = g(y)$ and the logistic equation
13E Applications of differential equations
Chapter 13 Review

Syllabus References: ME-C3.2

This is a difficult topic, with many new ideas and new words, and with many pathological curves that are best avoided at school. We have done our best to expound the ideas in a logical order, where students' experience of differential equations will grow as each section develops. Some difficult concepts are unavoidable, but we have quietly avoided many others.

It may seem that the logistic equation and its solutions and applications should dominate the discussions, but a great many more DEs are accessible by these methods, and we have given a variety of them.

Sections 13A–13E have been written theoretically, keeping the standard variables x and y, because this seems the best approach while learning the methods and ideas. Then Section 13F passes over more seriously to applications.

WolframAlpha seems currently the most satisfactory online software for slope fields, and some details of its use are given in Exercise 13B. We have preferred the word 'slope field' over 'direction field' because the line elements have no direction or length, only gradient.

It is required that first-order linear DEs be identified, but it is difficult to explain the word 'linear' at school in the absence of linear algebra. Yet solving them does not seem to be required — or is it intended that integrating factorsbe taught? Some adjustment may be necessary here.

Chapter 14: Series and finance (Advanced)

14A Applications of APs and GPs
14B The use of logarithms with GPs
14C Simple and compound interest
14D Investing money by regular instalments
14E Paying off a loan
14F Annuities and tables of interest factors
14G The present value of an annuity
Chapter 14 Review

Syllabus References: MA-M1.1
MA-M1.2 (review with applications)
MA-M1.3 (review with applications)
MA-M1.4

M1.2–M1.3 & Sections 14A–14B: Arithmetic and geometric sequences and series were presented in Chapter 1 to give a wider context to the intense calculus of the next few chapters. Those formulae are reviewed here, but the emphasis is on applications of series to practical situations.

Section 14B uses logarithms with problems involving GPs. These methods should be contrasted with trial-and-error, which is attractive for occasional use, but clumsy in comparison.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

M1.1, M1.4 & Sections 14C–14E: These sections are still about APs and GPs, but specifically directed towards interest, annuities (or superannuation) and paying off loans. The algebra involved in these last two computations is difficult and tedious. Nevertheless, learning formulae should be discouraged, because the question only has to start or finish the transaction slightly differently and any formula will fail.

Sections 14F–14G deal with the use of future value tables and present value tables to assist in questions involving annuities, thus avoiding calculations with GPs. The Advanced syllabus in section M1.1 requires familiarity with these tables, and these two sections address this requirement.

The basic approach presented here is to treat each payment as a separate investment. Another widely-used approach is recursion, where the monthly bank statement becomes the goal of the computations. Opportunity has been given to use recursion instead — later questions do not specify which approach is to be used, and earlier questions can easily be adapted.

Series are discrete objects, and their applications in Chapter 14 contrast with the applications of continuous functions in Chapter 9. This contrast has an interesting companion — the discrete probability distributions of Chapter 13 last year contrast with the continuous probability distributions of Chapter 16 this year. Datasets, on the other hand, are always discrete.

Chapter 15: Displaying and interpreting data (Advanced)

15A Displaying data
15B Grouped data and histograms
15C Quartiles and interquartile range
15D Bivariate data
15E Formulae for correlation and regression
15F Using technology with bivariate data
Chapter 15 Review

Syllabus References: MA-S2.1
MA-S2.2

This chapter is in sharp contrast to Chapters 12–13 in the Year 11 book on probability and discrete probability distributions. Those chapters were mostly about theoretical probabilities, with a little sampling and data-gathering at the end. This chapter, however, is all about data — its display in tables and charts, and the calculations of summary statistics — in order to gain a global view of a dataset that could otherwise be just a meaningless jumble. After this, we seek interpretation (requiring common sense and judgement) and prediction (risky), and we find a scientist to ask about causation.

Sections 15A–15C concern univariate data — some of this may be familiar from earlier years.

Sections 15D–15F mostly concern bivariate data, which may be less familiar. Pearson's correlation coefficient and the least-squares line of best fit are a difficult part of this chapter. The intention is that they be calculated by technology, which can be rather complicated whatever technology is used. The formulae for them are particularly difficult, but because we dislike black boxes, Section 15E gives the formulae and a short exercise to drill them. The rest of the chapter is quite independent of these formulae, which could be regarded as Enrichment.

The technology used may be a calculator, or a spreadsheet, or special statistics software, and all these things may be online. Technology in schools is still very variable, with no agreed equipment, rules or procedures. We have given, here and elsewhere in the book, detailed instructions about using Excel, which is probably the most widely-used spreadsheet, but even this is a highly contentious decision because the Mac versions are different, the Windows versions keep changing, and many people prefer other technologies.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The final exercise is an investigation. It contains links to large datasets on the online interactive textbook, activities such as surveys to generate data for analysis, and investigations allowing the reader to search out raw data from the internet. Many things here could become projects.

No guidance has been given about using a correction factor when calculating the variance of a sample that is not a population. We do not want to make difficult things more complicated, so we have used division by n, which is simpler, particularly when the formulae are developed for relative frequencies. In Section 15B and in Chapter 13 last year, we have warned about the technology issues that arise when computing variance, and we have written Challenge subsections, where we have explainedthe correction as multiplying the variance by $\frac{n}{n-1}$. Adjustment may be needed here.

Chapter 16: Continuous probability distributions (Advanced)

16A Relative frequency
16B Continuous distributions
16C Mean and variance of a distribution
16D The standard normal distribution
16E General normal distributions
16F Applications of the normal distribution
16G Investigations using the normal distribution
Chapter 16 Review
Appendix — The standard normal distribution

Syllabus References: MA-S3.1
MA-S3.2

Chapter 16 moves via Section 16A from the data of Chapter 15 back to probability theory, but with the constant accompaniment of data, and this time the theory is about continuous random variables. Many things about data lead into the continuous theory — the relative frequencies, the histograms and their associated areas, the grouping that is so often necessary, the appearance of the frequency polygon — besides of course the nature of what is being measured.

Various examples of continuous distributions are given in Sections 16B–16C. We have included some distributions where the domain of the values is unbounded, because the domain of the normal distribution is the whole real number line $(-\infty, \infty)$. We have done any calculations over unbounded domains without the usual complicated machinery, and the authors hope that such things will remain a side issue to the important statistical ideas going on. More machinery may be appropriate in some classes.

The standard normal distribution is discussed in Section 16D, then stretched and shifted in Section 16E to become the general normal distribution, where z-scores become a good example of the transformations studied in Chapter 3.

We have used a short table of values of the normal in our calculations so that everything is accessible to students whose only technology is a standard non-statistics scientific calculator. Some practice doing this is useful for everyone, but as technology develops, such tables will fall out of use just as log tables have done. Reading a printed table backwards is a particular nuisance because interpolation is clumsy and time-consuming.

ISBN 978-1-108-76630-2

The inflections and variance of the standard normal are difficult, but should be accessible to most students. On the other hand, the proof that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$ will unfortunately have to wait for far more sophisticated methods of calculus at some later time.

As in Chapter 15, Exercise 16G concludes the chapter with an investigation exercise using large datasets, and projects can easily be developed from these questions.

Chapter 17: Binomial distributions Extension 1

17A Binomial probability
17B Binomial distributions
17C Normal approximations to a binomial
17D Sample proportions
Chapter 17 Review

Syllabus References: ME-S1.1
ME-S1.2

All the machinery to calculate individual binomial probabilities is already at hand from the Year 11 book in Chapter 12: Probability, Chapter 14: Combinatorics, and Chapter 15: Binomial expansions. Section 17A develops these individual binomial probabilities, then Section 17B uses them to deal with binomial distributions using the methods from the Year 11 book, Chapter 13: Discrete probability distributions.

The final task of the chapter, and the book, is to approximate using the normal distribution. The reference *ME-S1.2* only requires normal approximations to the sample proportions, but we have made the judgement that approximating only sample proportions by the normal is rather forbidding, and that it is better to introduce first the more approachable idea of approximating binomial distributions by the normal. The calculations turn out to be almost the same, but with sample proportions the numbers are fiddly and less clear. Once the idea is mastered with normal approximations to the binomial in Section 17C, then normal approximations to the sample proportions in Section 17D seem clearer.

The authors were unsure what attitude to take to continuity corrections. Depending on where in the distribution they are being used, and how wide the interval is, a continuity correction may be negligible or quite dramatic. We have chosen to use them in Section 17C because the correction is more obvious there where it only involves half-integers. We have, however, mostly avoided them in Section 17D because the fractions become tedious and in any case the situations involving sample proportions tend to involve larger numbers. Again, future adjustment may be needed.

Bill Pender, June 2019

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Overview

As part of the *CambridgeMATHS* series, this resource is part of a continuum from Year 7 through to 12. The four components of *Mathematics Extension 1 Year 12* — the print book, downloadable PDF textbook, online Interactive Textbook and Online Teaching Resource — contain a range of resources available to schools in a single package at a convenient low price. There are no extra subscriptions or per-student charges to pay.

Features of the print textbook

1 Refer to the *Rationale* for details of question categories in the exercises and syllabus coverage.
2 Each section begins at the top of the page to make them easy to find and access.
3 Plenty of numbered worked examples are provided, with video versions for most of them.
4 Important concepts are formatted in numbered boxes for easy reference.
5 Investigation exercises and suggestions for projects are included.
6 Proofs for important results are provided in certain chapters.
7 Chapter review exercises assess learning in the chapter.

Downloadable PDF textbook

8 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online. PDF search and commenting tools are enabled.

Digital resources in the Interactive Textbook powered by the HOTmaths platform *(shown on the next page opposite)*

The Interactive Textbook is an online HTML version of the print textbook powered by the HOTmaths platform, completely designed and reformatted for on-screen use, with easy navigation. It is included with the print book, or available as digital-only purchase. Its features include:

9 Video versions of the examples to encourage independent learning.
10 All exercises including chapter reviews have the option of being done interactively on line, using **workspaces** and **self-assessment tools**. Students rate their level of confidence in answering the question and can flag the ones that gave them difficulty. Working and answers, whether typed or handwritten, and drawings, can be saved and submitted to the teacher electronically. Answers displayed on screen if selected and worked solutions (if enabled by the teacher) open in pop-up windows.
11 Teachers can give feedback to students on their self-assessment and flagged answers.

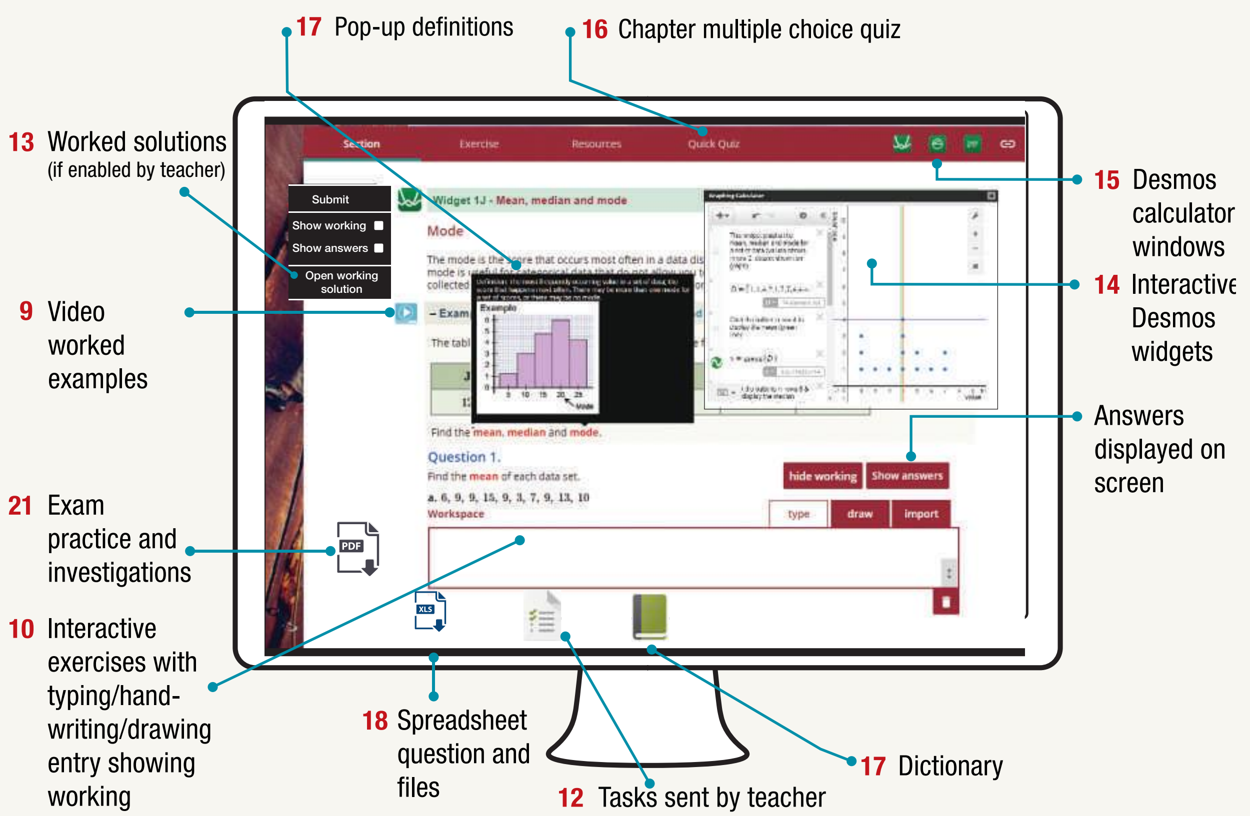
Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 The full suite of the HOTmaths learning management system and communication tools are included in the platform, with similar interfaces and controls.
13 Worked solutions are included and can be enabled or disabled in the student accounts by the teacher.
14 Interactive widgets and activities based on embedded Desmos windows demonstrate key concepts and enable students to visualise the mathematics.
15 Desmos scientific and graphics calculator windows are also included.
16 Chapter Quizzes of automatically marked multiple-choice questions are provided for students to test their progress.
17 Definitions pop up for key terms in the text, and are also provided in a dictionary.
18 Spreadsheet files are provided to support questions and examples based on the use of such technology.
19 Online guides are provided to spreadsheets and the Desmos graphing calculator, while links to scientific calculator guides on the internet are provided.
20 Users who had Year 11 digital accounts may access Year 11 material for revision of prior knowledge.
21 Examination and assessment practice items are available

INTERACTIVE TEXTBOOK POWERED BY THE *HOTmaths* PLATFORM

Numbers refer to the descriptions on the opposite page. HOTmaths platform features are updated regularly. Content shown is from Mathematics Standard.



Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Online Teaching Suite powered by the HOTmaths platform

22 The Online Teaching Suite is automatically enabled with a teacher account and appears in the teacher's copy of the Interactive Textbook. All the assets and resources are in Teacher Resources panel for easy access.

23 Teacher support documents include editable teaching programs with a scope and sequence document and curriculum grid.

24 Chapter tests are provided as printable PDFs or editable Word documents.

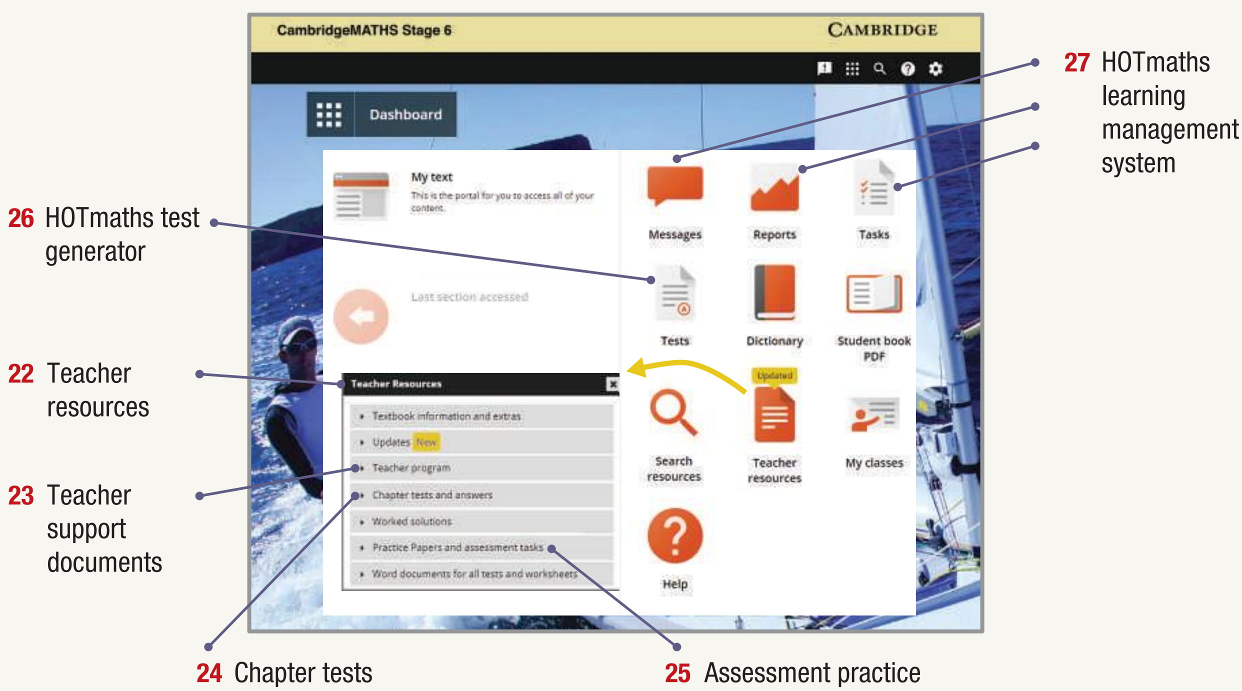
25 Assessment practice items (unseen by students) are included in the teacher resources.

26 The HOTmaths test generator is included.

27 The HOTmaths learning management system with class and student reports and communication tools is included.

ONLINE TEACHING SUITE POWERED BY THE *HOTmaths* PLATFORM

Numbers refer to the descriptions on the previous page. HOTmaths platform features are updated regularly.



Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Acknowledgements

The author and publisher wish to thank the following sources for permission to reproduce material:

Cover: © Getty Images / Digital Vision

Images: © Getty Images / Daniel Lange, Chapter 1 Opener / oxygen, 1A (1) / MirageC 1C (1) / GeorgePeters, 1H (1) / Westend61, 1I (1) / fotog, 1F (1), Ch1 review / kevinjeon00, Ch1 review (2) / Comstock, Chapter 2 Opener / Graiki, 2B (1) / Busà Photography, 2B (2) / R M STEELE, Chapter 3 Opener / KT Design/Science Photo Library, 3A (1), Chapter 4 Opener / Yifei Fang, 3D (1) / Thorsten Frisch, 3E (1) / Gremlin, 3G (1) / Mangiwau, 3H (1) / Instants, 4A (1) / MirageC, 4C (1), 4F (1) / Xvision, 4F (2) / Alexander Spatari, 4G (1) / filo, 4H (1) / abzee, 4E (1) / TheDman, 4E (2) / Farm Images, 4E (3) / Patrycja Polechonskam, 4E (4) / Kryssia Campos, 4I (2) / Franklin Kappa, Chapter 5 Opener / Xvision, 5A (1) / Rosley Majid, 5B (1) / PhotoviewPlus, 5C (1) / Xvision, 5E (1) / Nicole Janke, 5F (1) / piranka, 5G (1) / Smith Collection, 5I (1) / RunPhoto, Chapter 6 Opener / Sebastian-Alexander Stamatis, 6C (1) / Westend61, 6D (1) / Simon McGill, 6E (1) / Yuichiro Chino, 6G (1) / Zak Whiteman, 6I (1) / Julien Fourniol/Baloulumix, Chapter 7 Opener / Orbon Alija, 7A (1) / Pascal Fraboul, 7A (2) / Yaorusheng, 7C (1) / Tatsiana Volskaya, 7D (1) / aaaaimages, 7E (1) / Philippe LEJEANVRE, Ch7 review / Louise LeGresley, Chapter 8 Opener / Mark Garlick/Science Photo Library, 8A (1) /Abstract Aerial Art, 8C (1) / Jasmin Merdan, 8D (1) / Os Tartarouchos, 8D (2) / cwogofoto / 500px, 8E (1) / Zoonar RF, Ch 8 review / Enrique Díaz / 7cero, 9A (1) / Instants, 9B (1) / VladartDesign, 9D (1) / traffic_analyzer, 9F (1) / Westend61, Chapter 10 Opener / Tom and Steve, 10B (1) / Manuela Schewe-Behnisch, Ch 10 review / oxygen, Chapter 11 Opener / Schon, 11B (1) / Alexander Spatari, 11C (1) / Christoph Hetzmannseder, Chapter 12 Opener / Julien Fourniol/Baloulumix, 12C (1) / Konstantinos Zouganelis, 12D (1) / Ivan, Chapter 13 Opener / Sirachai Arunrugstichai, 13A (1) / Alfred Paieda/Science Photo Library, 13E (1) / Chris Clor, Chapter 14 Opener / visualspace, 14E (1) / Sean Gladwell, Ch14 review / Sean Gladwell, Chapter 15 Opener / shan.shihan, 15A (1) / Xvision, 15D (1), 16C (1) / Perry Wunderlich, Chapter 16 Opener / Peter Hermes Furian, 16A (1) / Vesna Jovanovic, 16F (1) / Straculencu Mihai, Chapter 17 Opener / malerapaso, 17A (1) / Fuse, 17A (2) / Rob Webb, 17B (1) / nycshooter, 17C (1) / Hal Bergman Photography, 17D (1).

Every effort has been made to trace and acknowledge copyright. The publisher apologises for any accidental infringement and welcomes information that would redress this situation

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

About the authors

Dr Bill Pender is retired Subject Master in Mathematics at Sydney Grammar School, where he taught from 1975 to 2009. He has an MSc and PhD in Pure Mathematics from Sydney University, with theses in geometry and group theory, and a BA(Hons) in Early English from Macquarie University. He spent a year at Bonn University in Germany, and he has lectured and tutored at Sydney University, and at the University of NSW where he was a Visiting Fellow in 1989. He was a member of the NSW Syllabus Committee in Mathematics for two years and subsequently of the Review Committee for the Years 9–10 Advanced Syllabus, and has contributed extensively to syllabus discussions over the last 30 years. He was a member of the AMSI Mathematics Education Committee for several years, and an author of the ICE-EM Years 7–10 textbooks for the National Curriculum.

Dr Brian Dorofaeff is currently Subject Master in Mathematics and Second Master in Charge of Statistics at Sydney Grammar School. He is an experienced classroom teacher, having taught for over 20 years in New South Wales. He has previous experience in HSC marking and has been on the HSC Mathematics Examination Committee. Brian holds a Ph.D. in Mathematics from the University of New South Wales.

David Sadler is currently teaching senior mathematics part-time at Talent 100. He taught for 36 years at Sydney Grammar School and was Head of Mathematics for 7 years. He also taught at UNSW for one year. He was an HSC marker for many years and has been a presenter at various conferences and professional development courses. He has a strong passion for excellence in mathematics education and has previously co-authored several senior texts for Cambridge University press.

Derek Ward has taught Mathematics at Sydney Grammar School since 1991 and is Master in Charge of Database Administration. He has an MSc in Applied Mathematics and a BScDipEd, both from the University of NSW, where he was subsequently Senior Tutor for three years. He has an AMusA in Flute, and sings in the Choir of Christ Church St Laurence.

Julia Shea is currently Principal of St Peter's Girls' School in Adelaide. She was formerly Head of Mathematics and Director of Curriculum at Newington College, Sydney. She has a BSc and DipEd from the University of Tasmania, she taught for six years at Rosny College, a State Senior College in Hobart, and was a member of the Executive Committee of the Mathematics Association of Tasmania for five years. She then taught for five years at Sydney Grammar School before moving to Newington College.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The Book of Nature is written in the language of Mathematics.

— The seventeenth-century Italian scientist Galileo

It is more important to have beauty in one's equations than to have them fit experiment.

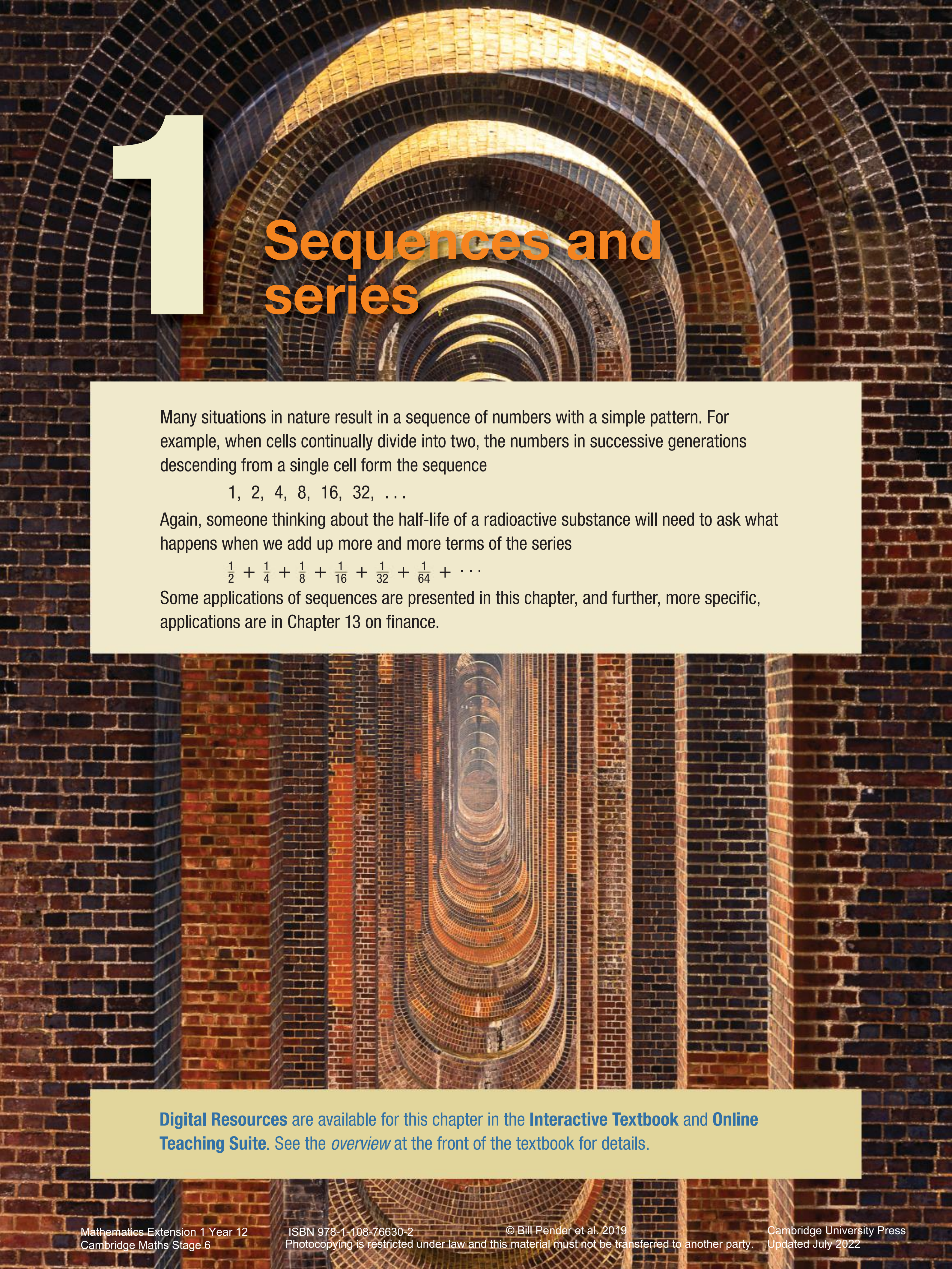
— The twentieth-century English physicist Dirac

Even if there is only one possible unified theory, it is just a set of rules and equations. What is it that breathes fire into the equations and makes a universe for them to describe? The usual approach of science of constructing a mathematical model cannot answer the questions of why there should be a universe for the model to describe.

— Steven Hawking, A Brief History of Time

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



1 Sequences and series

Many situations in nature result in a sequence of numbers with a simple pattern. For example, when cells continually divide into two, the numbers in successive generations descending from a single cell form the sequence

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \ldots$$

Again, someone thinking about the half-life of a radioactive substance will need to ask what happens when we add up more and more terms of the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \cdots$$

Some applications of sequences are presented in this chapter, and further, more specific, applications are in Chapter 13 on finance.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1A Sequences and how to specify them

A typical *infinite sequence* is formed by arranging the positive odd integers in increasing order:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

The three dots . . . indicate that the sequence goes on forever, with no last term. The sequence starts with the first term 1, then has second term 3, third term 5, and so on. The symbol T_n will usually be used to stand for the nth term, thus

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 5, \quad T_4 = 7, \quad T_5 = 9, \quad \ldots$$

The two-digit odd numbers less than 100 form a *finite sequence*:

$$1,\ 3,\ 5,\ 7,\ \ldots,\ 99$$

where the dots . . . stand for the 45 terms that have been omitted.

There are three different ways to specify a sequence, and it is important to be able to display a given sequence in each of these different ways.

Write out the first few terms

The easiest way is to write out the first few terms until the pattern is clear to the reader. Continuing with our example of the positive odd integers, we could write the sequence as

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots$$

This sequence clearly continues as $11,\ 13,\ 15,\ 17,\ 19,\ \ldots,$ and with a few more calculations, it becomes clear that $T_{11} = 21$, $T_{14} = 27$, and $T_{16} = 31$.

Give a formula for the *n*th term

The formula for the nth term of this sequence is

$$T_n = 2n - 1,$$

because the nth term is always 1 less than $2n$. Giving the formula does not rely on the reader recognising a pattern, and any particular term of the sequence can now be calculated quickly:

$$\begin{aligned} T_{30} &= 60 - 1 \\ &= 59 \end{aligned} \qquad \begin{aligned} T_{100} &= 200 - 1 \\ &= 199 \end{aligned} \qquad \begin{aligned} T_{244} &= 488 - 1 \\ &= 487 \end{aligned}$$

Say where to start and how to proceed

The sequence of odd positive integers starts with 1, then each term is 2 more than the previous one. Thus the sequence is completely specified by writing down these two statements:

$$\begin{aligned} &T_1 = 1, && \text{(start the sequence with 1)} \\ &T_n = T_{n-1} + 2, \quad \text{for } n \ge 2. && \text{(every term is 2 more than the previous term)} \end{aligned}$$

Such a specification is called a *recursive* formula of a sequence. Most of the sequences studied in this chapter are based on this idea.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 1 1A

a Write down the first five terms of the sequence given by $T_n = 7n - 3$.
b Describe how each term T_n can be obtained from the previous term T_{n-1}.

SOLUTION

a $T_1 = 7 - 3 = 4$ $\quad T_2 = 14 - 3 = 11$ $\quad T_3 = 21 - 3 = 18$ $\quad T_4 = 28 - 3 = 25$ $\quad T_5 = 35 - 3 = 32$

b Each term is 7 more than the previous term. That is, $T_n = T_{n-1} + 7$.

Example 2 1A

a Find the first five terms of the sequence given by $T_1 = 14$ and $T_n = T_{n-1} + 10$.
b Write down a formula for the nth term T_n.

SOLUTION

a $T_1 = 14$ $\quad T_2 = T_1 + 10 = 24$ $\quad T_3 = T_2 + 10 = 34$ $\quad T_4 = T_3 + 10 = 44$ $\quad T_5 = T_4 + 10 = 54$

b From this pattern, the formula for the nth term is clearly $T_n = 10n + 4$.

1 THREE WAYS TO SPECIFY A SEQUENCE

- Write out the first few terms until the pattern is clear to the reader.
- Give a formula for the nth term T_n.
- Say where to start and how to proceed. That is:
  - Say what the value of T_1 is.
  - Then for $n \geq 2$, give a formula for T_n in terms of the preceding terms.

Using the formula for T_n to solve problems

Many problems about sequences can be solved by forming an equation using the formula for T_n.

Example 3 1A

Find whether 300 and 400 are terms of the sequence $T_n = 7n + 20$.

SOLUTION

Put $T_n = 300$.
Then $7n + 20 = 300$
$7n = 280$
$n = 40$.
Hence 300 is the 40th term.

Put $T_n = 400$.
Then $7n + 20 = 400$
$7n = 380$
$n = 54\frac{2}{7}$.
Hence 400 is not a term of the sequence.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Example 4 1A

a Find how many negative terms there are in the sequence $T_n = 12n - 100$.
b Find the first positive term of the sequence $T_n = 7n - 60$.

SOLUTION

a Put $T_n < 0$.
Then $12n - 100 < 0$
$n < 8\frac{1}{3}$,
so there are eight negative terms.

b Put $T_n > 0$.
Then $7n - 60 > 0$
$7n > 60$
$n > 8\frac{4}{7}$.
Thus the first positive term is $T_9 = 3$.

Note: The question, 'Find the first positive term' requires two answers:
- Which number term is it?
- What is its value?

Thus the correct answer is, 'The first positive term is $T_9 = 3$'.

Exercise 1A

FOUNDATION

1 Alex collects stamps. He found a collection of 700 stamps in the attic a few years ago, and every month since then he has been buying 150 interesting stamps to add to his collection. Thus the numbers of stamps at the end of each month after his discovery form a sequence

850, 1000, . . .

a Copy and continue the sequence to at least 12 terms followed by dots . . . .
b After how many months did his collection first exceed 2000 stamps?

2 Write down the next four terms of each sequence.
a 6, 16, 26, . . . **b** 3, 6, 12, . . . **c** 38, 34, 30, . . . **d** 24, 12, 6, . . .
e $-1, 1, -1, \ldots$ **f** 1, 4, 9, . . . **g** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ **h** $16, -8, 4, \ldots$

3 Find the first four terms of the sequence whose nth term T_n is:
a $T_n = 5n - 2$ **b** $T_n = 5^n$ **c** $T_n = 6 - 2n$ **d** $T_n = 7 \times 10^n$
e $T_n = n^3$ **f** $T_n = n(n + 1)$ **g** $T_n = (-1)^n$ **h** $T_n = (-3)^n$

4 Write down the first four terms of each sequence described below.
a The first term is 11, and every term after that is 50 more than the previous term.
b The first term is 15, and every term after that is 3 less than the previous term.
c The first term is 5, and every term after that is twice the previous term.
d The first term is -100, and every term after that is one fifth of the previous term.

5 Write out the first twelve terms of the sequence 7, 12, 17, 22, . . .
a How many terms are less than 30?
b How many terms lie between 20 and 40?
c What is the 10th term?
d What number term is 37?
e Is 87 a term in the sequence?
f Is 201 a term in the sequence?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 Write out the first twelve terms of the sequence $\frac{3}{4},\ 1\frac{1}{2},\ 3,\ 6,\ \ldots$

a How many terms are less than 400?

b How many terms lie between 20 and 100?

c What is the 10th term?

d What number term is 192?

e Is 96 a term in the sequence?

f Is 100 a term in the sequence?

DEVELOPMENT

7 For each sequence, write out the first five terms. Then explain how each term is obtained from the previous term.

a $T_n = 12 + n$

b $T_n = 4 + 5n$

c $T_n = 15 - 5n$

d $T_n = 3 \times 2^n$

e $T_n = 7 \times (-1)^n$

f $T_n = 80 \times \left(\frac{1}{2}\right)^n$

8 The nth term of a sequence is given by $T_n = 3n + 1$.

a Put $T_n = 40$, and hence find which term of the sequence 40 is.

b Put $T_n = 30$, and hence show that 30 is not a term of the sequence.

c Similarly, find whether 100, 200 and 1000 are terms of the sequence.

9 Answer each question by forming an equation and solving it.

a Find whether 44, 200 and 306 are terms of the sequence $T_n = 10n - 6$.

b Find whether 40, 72 and 200 are terms of the sequence $T_n = 2n^2$.

c Find whether 8, 96 and 128 are terms of the sequence $T_n = 2^n$.

10 The nth term of a sequence is given by $T_n = 10n + 4$.

a Put $T_n < 100$, and hence find how many terms are less than 100.

b Put $T_n > 56$, and find the first term greater than 56. State its number and its value.

11 Answer each question by forming an inequation and solving it.

a How many terms of the sequence $T_n = 2n - 5$ are less than 100?

b What is the first term of the sequence $T_n = 7n - 44$ greater than 100?

12 In each part, the two lines define a sequence T_n. The first line gives the first term T_1. The second line defines how each subsequent term T_n is obtained from the previous term T_{n-1}. Write down the first four terms of each sequence.

a $T_1 = 5,$
$T_n = T_{n-1} + 12,$ for $n \geq 2$.

b $T_1 = 12,$
$T_n = T_{n-1} - 10,$ for $n \geq 2$.

c $T_1 = 20,$
$T_n = \frac{1}{2}T_{n-1},$ for $n \geq 2$.

d $T_1 = 1,$
$T_n = -T_{n-1},$ for $n \geq 2$.

13 Give a recursive formula for the nth term T_n of each sequence in terms of the $(n - 1)$th term T_{n-1}.

a $16,\ 21,\ 26, \ldots$

b $7,\ 14,\ 28, \ldots$

c $9,\ 2,\ -5,\ \ldots$

d $4,\ -4,\ 4,\ \ldots$

14 Write down the first four terms of each sequence. Then state which terms of the whole sequence are zero.

a $T_n = \sin 90n°$

b $T_n = \cos 90n°$

c $T_n = \cos 180n°$

d $T_n = \sin 180n°$

15 **a** Which terms of the sequence $T_n = n^2 - 3n$ are 28 and 70?

b How many terms of this sequence are less than 18?

16 **a** Which terms of the sequence $T_n = \frac{3}{32} \times 2^n$ are $1\frac{1}{2}$ and 96?

b Find the first term in this sequence which is greater than 10.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 The correct definition of a sequence is: 'A *sequence* is a function whose domain is the set of positive integers'. Graph the sequences in question 2, with n on the horizontal axis and T_n on the vertical axis. If a simple curve joins the points, draw it and give its equation.

18 A sequence is defined by $T_n = \dfrac{1}{n} - \dfrac{1}{n+1}$.

a Find $T_1 + T_2 + T_3 + T_4$, and give a formula for $T_1 + T_2 + \cdots + T_n$.

b Show that $T_n = \dfrac{1}{n(n+1)}$, and find which term of the sequence $\frac{1}{30}$ is.

19 **a** Which terms of the sequence $T_n = \dfrac{n-1}{n}$ are 0.9 and 0.99?

b Find $T_{n+1} : T_n$, and prove that $\dfrac{T_n}{T_{n+1}} + \dfrac{1}{n^2} = 1$.

c Find $T_2 \times T_3 \times \cdots \times T_n$.

d Prove that $T_{n+1} - T_{n-1} = \dfrac{2}{n^2 - 1}$.

ENRICHMENT

20 **a** Write out the first 12 terms of the *Fibonacci sequence*, which is defined by

$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \ge 3.$

b Write out the first 12 terms of the *Lucas sequence*, which is defined by

$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2}, \quad \text{for } n \ge 3.$

c Explain why every third term of each sequence is even and the rest are odd.

d Write out the first twelve terms of the sequences

$L_1 + F_1,\ L_2 + F_2,\ L_3 + F_3,\ \ldots$ and $L_1 - F_1,\ L_2 - F_2,\ L_3 - F_3,\ \ldots$

How are these two new sequences related to the Fibonacci sequence, and why?

A possible project:

21 This open-ended investigation could be developed into a project.

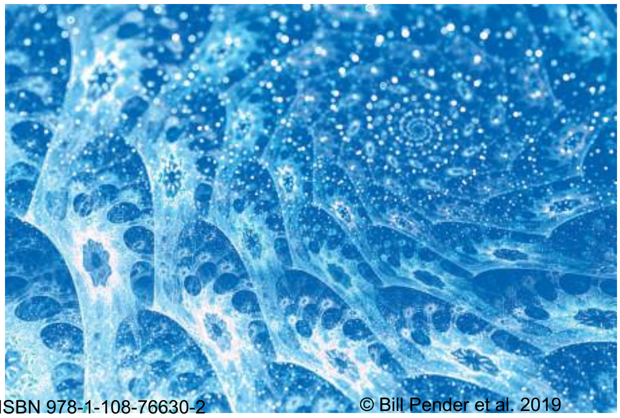
a Generate the Fibonacci and Lucas sequences on a spreadsheet such as Excel.

b Use the spreadsheet to generate the successive ratios $\dfrac{F_n}{F_{n-1}}$ and $\dfrac{L_n}{L_{n-1}}$.

c Investigate the golden mean and its relationship with these sequences.

d Investigate how and why these sequences occur in the natural world.

e Investigate other sequences generated in similar ways.



ISBN 978-1-108-76630-2

An introduction to countably infinite sets:

22 This difficult question introduces some of the extraordinary ideas about infinity associated with Georg Cantor in the late 19th century. Sequences are all about listing, and as discussed in Chapter 13 (Year 11), an infinite set S is called *countably infinite* if its members can be *listed*, meaning that they can be written in a sequence $T_1, T_2, T_3, \ldots$. Cantor, using a Hebrew letter, assigned the symbol $\aleph_0$, 'aleph nul', to this infinity.

a The set of integers is countably infinite because it can be listed as

$0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4,\ \ldots$.

What is the 20th number on the list, and what is the position of the integer -20?

1	2	3	4	5	6	7	8	
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	$\ldots$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\ldots$
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\ldots$
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\ldots$
$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\ldots$
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	$\ldots$
$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

b The table to the right contains all the positive rational numbers. In fact, every positive rational number appears infinitely many times in the table. By taking successive diagonals, show that the positive rational numbers can be listed.

c Copy and complete the following proof by contradiction that *the set of real numbers cannot be listed*. This means that set of real numbers is not countably infinite, and that the infinity of real numbers is 'greater' that the infinity of whole numbers.

Suppose that there were a listing of all the real numbers in the interval $0 \le x < 1$,

$T_1,\ T_2,\ T_3,\ T_4,\ T_5,\ T_6,\ \ldots$

Imagine that each real number T_n in the sequence is written as an infinite decimal string of digits 0.dddddd . . . (where each d represents a digit).

Define a real number x in the interval $(0, 1)$ by specifying each decimal place in turn,

$$n\text{th decimal place of } x = \begin{cases} 1, & \text{if the } nth \text{ decimal place of } T_n \text{ is zero,} \\ 0, & \text{otherwise.} \end{cases}$$

Then x is not on the list because . . .

(A minor qualification is needed — see Question **8** of Exercise 1I.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1B Arithmetic sequences

A simple type of sequence is an *arithmetic sequence*. This is a sequence such as

$$3,\ 13,\ 23,\ 33,\ 43,\ 53,\ 63,\ 73,\ 83,\ 93,\ \dots,$$

in which the difference between successive terms is constant — in this example each term is 10 more than the previous term. Notice that all the terms can be generated from the *first term* 3 by repeated addition of this *common difference* 10.

In the context of successive terms of sequences, the word *difference* will always mean some term minus the previous term.

Definition of an arithmetic sequence

Arithmetic sequences are called APs for short. The initials stand for 'arithmetic progression' — an old name for the same thing.

2 ARITHMETIC SEQUENCES

- The *difference* between successive terms in a sequence T_n always means some term minus the previous term, that is,

  $$\text{difference} = T_n - T_{n-1}, \quad \text{where } n \geq 2.$$

- A sequence T_n is called an *arithmetic sequence* or AP if

  $$T_n - T_{n-1} = d, \quad \text{for all } n \geq 2,$$

  where d is a constant, called the *common difference*.
- The terms of an arithmetic sequence can be generated from the first term by repeated addition of this common difference:

  $$T_n = T_{n-1} + d, \quad \text{for all } n \geq 2.$$

Example 5 1B

Test whether each sequence is an AP. If the sequence is an AP, find its first term a and its common difference d.

a $46,\ 43,\ 40,\ 37,\ \dots$ **b** $1,\ 4,\ 9,\ 16,\ \dots$ **c** $\log_e 6,\ \log_e 12,\ \log_e 24,\ \log_e 48,\ \dots$

SOLUTION

a $T_2 - T_1 = 43 - 46 = -3$ $\quad T_3 - T_2 = 40 - 43 = -3$ $\quad T_4 - T_3 = 37 - 40 = -3$

Hence the sequence is an AP with $a = 46$ and $d = -3$.

b $T_2 - T_1 = 4 - 1 = 3$ $\quad T_3 - T_2 = 9 - 4 = 5$ $\quad T_4 - T_3 = 16 - 9 = 7$

The differences are not all the same, so the sequence is not an AP.

c $T_1 = \log_e 6 = \log_e 2 + \log_e 3$ $\quad T_2 = \log_e 12 = 2\log_e 2 + \log_e 3$ $\quad T_3 = \log_e 24 = 3\log_e 2 + \log_e 3$ $\quad T_4 = \log_e 48 = 4\log_e 2 + \log_e 3$

Hence the sequence is an AP with $a = \log_e 6$ and $d = \log_e 2$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A formula for the nth term of an AP

Let the first term of an AP be a and the common difference be d. Then the first few terms of the sequence are

$$T_1 = a, \qquad T_2 = a + d, \qquad T_3 = a + 2d, \qquad T_4 = a + 3d, \qquad \ldots$$

From this pattern, the general formula for the nth term is clear:

3 THE nTH TERM OF AN AP

$$T_n = a + (n - 1)d$$

where a is the first term and d is the common difference.

Example 6 — 1B

a Write out the first five terms of the AP with $a = 130$ and $d = -3$.
b Find the 20th term and a formula for the nth term.
c Find the first negative term.

SOLUTION

a $130, 127, 124, 121, 118, \ldots$

b
$$\begin{aligned} T_{20} &= a + 19d \\ &= 130 + 19 \times (-3) \\ &= 73 \\ T_n &= 130 - 3(n - 1) \\ &= 133 - 3n \end{aligned}$$

c Put
$$\begin{aligned} T_n &< 0 \\ 133 - 3n &< 0 \\ 3n &> 133 \\ n &> 44\tfrac{1}{3}, \end{aligned}$$

so the first negative term is $T_{45} = -2$.

Example 7 — 1B

a Find a formula for the nth term of the sequence 26, 35, 44, 53, ...
b How many terms are there in the sequence 26, 35, 44, 53, ..., 917?

SOLUTION

a This is an AP with $a = 26$ and $d = 9$.

Hence
$$\begin{aligned} T_n &= a + (n - 1)d \\ &= 26 + 9(n - 1) \\ &= 26 + 9n - 9 \\ &= 17 + 9n. \end{aligned}$$

b Put
$$\begin{aligned} T_n &= 917. \\ \text{Then } 17 + 9n &= 917 \\ 9n &= 900 \\ n &= 100, \end{aligned}$$

so there are 100 terms in the sequence.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 8 1B

The first term of an AP is 105 and the 10th term is 6. Find the common difference and write out the first five terms.

SOLUTION

First, we know that $T_1 = 105$,
that is, $a = 105.$ (1)
Secondly, we know that $T_{10} = 6$,
so using the formula for the 10th term, $a + 9d = 6.$ (2)
Substituting (1) into (2),

$$
\begin{aligned}
105 + 9d &= 6 \\
9d &= -99 \\
d &= -11,
\end{aligned}
$$

so the common difference is $d = -11$ and the sequence is 105, 94, 83, 72, 61, . . .

Arithmetic sequences and linear functions

Take a linear function such as $f(x) = 30 - 8x$, and substitute the positive integers. The result is an arithmetic sequence

$$22,\ 14,\ 6,\ -2,\ -10,\ \ldots .$$

x	1	2	3	4	5
$f(x)$	22	14	6	-2	-10

The formula for the nth term of this AP is $T_n = 22 - 8(n - 1) = 30 - 8n$. This is a function whose domain is the set of positive integers, and its equation is the same as the linear function above, with only a change of pronumeral from x to n.

Every arithmetic sequence can be generated in this way.

Exercise 1B FOUNDATION

1 Write out the next three terms of these sequences. They are all APs.
a 3, 8, 13, . . . **b** 35, 25, 15, . . . **c** $4\frac{1}{2}, 6, 7\frac{1}{2}, \ldots$

2 Write out the first four terms of the APs whose first terms and common differences are:
a $a = 3$ and $d = 2$ **b** $a = 7$ and $d = -4$ **c** $a = 30$ and $d = -11$
d $a = -9$ and $d = 4$ **e** $a = 3\frac{1}{2}$ and $d = -2$ **f** $a = 0.9$ and $d = 0.7$

3 Find the differences $T_2 - T_1$ and $T_3 - T_2$ for each sequence to test whether it is an AP. If the sequence is an AP, state the values of the first term a and the common difference d.
a 3, 7, 11, . . . **b** 11, 7, 3, . . . **c** 23, 34, 45, . . .
d $-12, -7, -2, \ldots$ **e** $-40, 20, -10, \ldots$ **f** 1, 11, 111, . . .
g $8, -2, -12, \ldots$ **h** $-17, 0, 17, \ldots$ **i** $10, 7\frac{1}{2}, 5, \ldots$

4 Use the formula $T_n = a + (n - 1)d$ to find the 11th term T_{11} of the APs in which:
a $a = 7$ and $d = 6$ **b** $a = 15$ and $d = -7$ **c** $a = 10\frac{1}{2}$ and $d = 4$

5 Use the formula $T_n = a + (n - 1)d$ to find the nth term T_n of the APs in which:
a $a = 1$ and $d = 4$ **b** $a = 100$ and $d = -7$ **c** $a = -13$ and $d = 6$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 **a** Find the first term a and the common difference d of the AP $6,\ 16,\ 26,\ \dots$

b Find the ninth term T_9, the 21st term T_{21} and the 100th term T_{100}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

7 Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If the sequence is an AP, use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

a $8,\ 11,\ 14,\ \dots$ **b** $21,\ 15,\ 9,\ \dots$ **c** $8,\ 4,\ 2,\ \dots$

d $-3,\ 1,\ 5,\ \dots$ **e** $1\frac{3}{4},\ 3,\ 4\frac{1}{4},\ \dots$ **f** $12, -5, -22, \dots$

g $\sqrt{2},\ 2\sqrt{2},\ 3\sqrt{2},\ \dots$ **h** $1,\ 4,\ 9,\ 16,\ \dots$ **i** $-2\frac{1}{2},\ 1,\ 4\frac{1}{2},\ \dots$

DEVELOPMENT

8 **a** Use the formula $T_n = a + (n - 1)d$ to find the nth term T_n of $165,\ 160,\ 155,\ \dots$

b Solve $T_n = 40$ to find the number of terms in the finite sequence $165,\ 160,\ 155,\ \dots,\ 40$.

c Solve $T_n < 0$ to find the first negative term of the sequence $165,\ 160,\ 155,\ \dots$

9 Find T_n for each AP. Then solve $T_n < 0$ to find the first negative term.

a $20,\ 17,\ 14,\ \dots$ **b** $82,\ 79,\ 76,\ \dots$ **c** $24\frac{1}{2},\ 24,\ 23\frac{1}{2},\ \dots$

10 Find the number of terms in each finite sequence.

a $10,\ 12,\ 14,\ \dots,\ 30$ **b** $1,\ 4,\ 7,\ \dots,\ 100$ **c** $105,\ 100,\ 95,\ \dots,\ 30$

d $100,\ 92,\ 84,\ \dots,\ 4$ **e** $-12,\ -10\frac{1}{2},\ -9,\ \dots,\ 0$ **f** $2,\ 5,\ 8,\ \dots,\ 2000$

11 The nth term of an arithmetic sequence is $T_n = 7 + 4n$.

a Write out the first four terms, and hence find the values of a and d.

b Find the sum and the difference of the 50th and the 25th terms.

c Prove that $5T_1 + 4T_2 = T_{27}$.

d Which term of the sequence is 815?

e Find the last term less than 1000 and the first term greater than 1000.

f Find which terms are between 200 and 300, and how many of them there are.

12 **a** Let T_n be the sequence $8,\ 16,\ 24,\ \dots$ of positive multiples of 8.

i Show that the sequence is an AP, and find a formula for T_n.

ii Find the first term of the sequence greater than 500 and the last term less than 850.

iii Hence find the number of multiples of 8 between 500 and 850.

b Use the same steps to find the number of multiples of 11 between 1000 and 2000.

c Use the same steps to find the number of multiples of 7 between 800 and 2000.

13 **a** The first term of an AP is $a = 7$ and the fourth term is $T_4 = 16$. Use the formula $T_n = a + (n - 1)d$ to find the common difference d. Then write down the first four terms.

b Find the 20th term of an AP with first term 28 and 11th term 108.

c Find the 100th term of an AP with first term 32 and 20th term -6.

14 Ionian Windows charges \$500 for the first window, then \$300 for each additional window.

a Write down the cost of 1 window, 2 windows, 3 windows, 4 windows, . . .

b Use the formula $T_n = a + (n - 1)d$ to find the cost of 15 windows.

c Find the cost of n windows.

d Find the maximum number of windows whose total cost is less than \$10 000.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 Many years ago, 160 km of a railway line from Nevermore to Gindarinda was built. On 1st January 2001, work was resumed, with 20 km of new track completed each month.

a Write down the lengths of track 1 month later, 2 months later, 3 months later, ...

b Use the formula $T_n = a + (n - 1)d$ to find how much track there was after 12 months.

c Find a formula for the length after n months.

d The distance from Nevermore to Gindarinda is 540 km. Form an equation and solve it to find how many months it took to complete the track.

16 [Simple interest and APs]
A principal of \$2000 is invested at 6% per annum simple interest. Let $\$A_n$ be the total amount (principal plus interest) at the end of n years.

a Write out the values of A_1, A_2, A_3 and A_4.

b Use the formula $T_n = a + (n - 1)d$ to find a formula for A_n, and evaluate A_{12}.

c How many years will it take before the total amount exceeds \$6000?

17 **a** Write down the first few terms of the AP generated by substituting the positive integers into the linear function $f(x) = 12 - 3x$. Then write down a formula for the nth term.

b **i** Find the formula of the nth term T_n of the AP $-3,\ -1,\ 1,\ 3,\ 5\ldots$. Then write down the linear function $f(x)$ that generates this AP when the positive integers are substituted into it.

ii Graph the function and mark the points $(1, -3)$, $(2, -1)$, $(3, 1)$, $(4, 3)$, $(5, 5)$.

18 Find the common difference of each AP. Then find x if $T_{11} = 36$.

a $5x - 9,\ 5x - 5,\ 5x - 1,\ \ldots$

b $16,\ 16 + 6x,\ 16 + 12x,\ \ldots$

19 Find the common difference of each AP. Then find a formula for the nth term T_n.

a $\log_3 2,\ \log_3 4,\ \log_3 8,\ \ldots$

b $\log_a 54,\ \log_a 18,\ \log_a 6,\ \ldots$

c $x - 3y,\ 2x + y,\ 3x + 5y,\ \ldots$

d $5 - 6\sqrt{5},\ 1 + \sqrt{5},\ -3 + 8\sqrt{5},\ \ldots$

e $1.36,\ -0.52,\ -2.4,\ \ldots$

f $\log_a 3x^2,\ \log_a 3x,\ \log_a 3,\ \ldots$

20 How many terms of the sequence $100,\ 97,\ 94, \ldots$ have squares less than 400?

ENRICHMENT

21 **a** What are the first term and difference of the AP generated by substituting the positive integers into the linear function with gradient m and y-intercept b?

b What are the gradient and y-intercept of the linear function that generates an AP with first term a and difference d when the positive integers are substituted into it?

22 [The set of all APs forms a two-dimensional space.]
Let $\mathcal{A}(a, d)$ represent the AP whose first term is a and difference is d.

a The *sum* of two sequences T_n and U_n is defined to be the sequence whose nth term is $T_n + U_n$. Show that for all constants λ and μ, and for all values of a_1, a_2, d_1 and d_2, the sequence $\lambda\mathcal{A}(a_1, d_1) + \mu\mathcal{A}(a_2, d_2)$ is an AP, and find its first term and common difference.

b Write out the sequences $\mathcal{A}(1, 0)$ and $\mathcal{A}(0, 1)$. Show that any AP $\mathcal{A}(a, d)$ with first term a and difference d can be written in the form $\lambda\mathcal{A}(1, 0) + \mu\mathcal{A}(0, 1)$, and find λ and μ.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1C Geometric sequences

A *geometric sequence* is a sequence like this:

$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ 1458,\ \ldots$$

in which the *ratio* of successive terms is constant — in this example, each term is 3 times the previous term. Because the ratio is constant, all the terms can be generated from the *first term* 2 by repeated multiplication by this *common ratio* 3.

In the context of successive terms of sequences, the word *ratio* will always mean some term divided by the previous term.

Definition of a geometric sequence

The old name was ‘geometric progression’ and geometric sequences are called GPs for short.

4 GEOMETRIC SEQUENCES

- The *ratio* of successive terms in a sequence T_n always means some term divided by the previous term, that is,
  $$\text{ratio} = \frac{T_n}{T_{n-1}}, \quad \text{where } n \geq 2.$$
- A sequence T_n is called a *geometric sequence* if
  $$\frac{T_n}{T_{n-1}} = r, \quad \text{for all } n \geq 2,$$
  where r is a non-zero constant, called the *common ratio*.
- The terms of a geometric sequence can be generated from the first term by repeated multiplication by this common ratio:
  $$T_n = T_{n-1} \times r, \quad \text{for all } n \geq 2.$$

Thus arithmetic sequences have a common difference and geometric sequences have a common ratio, so the methods of dealing with them are quite similar.

Example 9 — 1C

Test whether each sequence is a GP. If the sequence is a GP, find its first term a and its ratio r.

a $40,\ 20,\ 10,\ 5,\ \ldots$ **b** $5,\ 10,\ 100,\ 200,\ \ldots$ **c** $e^2, e^5, e^8, e^{11}, \ldots$

SOLUTION

a Here $\dfrac{T_2}{T_1} = \dfrac{20}{40} = \tfrac{1}{2}$ and $\dfrac{T_3}{T_2} = \dfrac{10}{20} = \tfrac{1}{2}$ and $\dfrac{T_4}{T_3} = \dfrac{5}{10} = \tfrac{1}{2}$,

so the sequence is a GP with $a = 40$ and $r = \tfrac{1}{2}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Here $\dfrac{T_2}{T_1} = \dfrac{10}{5} = 2$ and $\dfrac{T_3}{T_2} = \dfrac{100}{10} = 10$ and $\dfrac{T_4}{T_3} = \dfrac{200}{100} = 2.$

The ratios are not all the same, so the sequence is not a GP.

c Here $\dfrac{T_2}{T_1} = \dfrac{e^5}{e^2} = e^3$ and $\dfrac{T_3}{T_2} = \dfrac{e^8}{e^5} = e^3$ and $\dfrac{T_4}{T_3} = \dfrac{e^{11}}{e^8} = e^3,$

so the sequence is a GP with $a = e^2$ and $r = e^3$.

A formula for the *n*th term of a GP

Let the first term of a GP be a and the common ratio be r. Then the first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad T_5 = ar^4, \quad \dots$$

From this pattern, the general formula for the nth term is clear:

5 THE *n*TH TERM OF A GP

$$T_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

Example 10 1C

Write out the first five terms, and calculate the 10th term, of the GP with:

a $a = 3$ and $r = 2$,

b $a = 45$ and $r = \frac{1}{3}$.

SOLUTION

a $3,\ 6,\ 12,\ 24,\ 48,\ \dots$

$$\begin{aligned} T_{10} &= ar^9 \\ &= 3 \times 2^9 \\ &= 1536 \end{aligned}$$

b $45,\ 15,\ 5,\ 1\tfrac{2}{3},\ \tfrac{5}{9},\ \dots$

$$\begin{aligned} T_{10} &= a \times r^9 \\ &= 45 \times \left(\tfrac{1}{3}\right)^9 \\ &= 5 \times 3^{-7} \end{aligned}$$

Zeroes and GPs don't mix

No term of a GP can be zero. For example, if $T_2 = 0$, then $\dfrac{T_3}{T_2}$ would be undefined, contradicting the definition that $\dfrac{T_3}{T_2} = r$.

Similarly, the ratio of a GP cannot be zero. Otherwise $T_2 = ar$ would be zero, which is impossible, as we have explained above.

Negative ratios and alternating signs

The sequence $2,\ -6,\ 18,\ -54,\ \dots$ is an important type of GP. Its ratio is $r = -3$, which is negative, so the terms are alternately positive and negative.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 11 1C

a Show that $2,\ -6,\ 18,\ -54,\ \dots$ is a GP, and find its first term a and ratio r.
b Find a formula for the nth term, and hence find T_6 and T_{15}.

SOLUTION

a Here $\dfrac{T_2}{T_1} = \dfrac{-6}{2} = -3$ and $\dfrac{T_3}{T_2} = \dfrac{18}{-6} = -3$ and $\dfrac{T_4}{T_3} = \dfrac{-54}{18} = -3$,

so the sequence is a GP with $a = 2$ and $r = -3$.

b Using the formula for the nth term, $T_n = ar^{n-1}$
$= 2 \times (-3)^{n-1}$.

Hence $T_6 = 2 \times (-3)^5 = -486$, because 5 is odd. and $T_{15} = 2 \times (-3)^{14} = 2 \times 3^{14}$, because 14 is even.

Using a switch to alternate the sign

Here are two classic GPs with ratio -1:

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \dots \qquad \text{and} \qquad 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$

The first has formula $T_n = (-1)^n$, and the second has formula $T_n = (-1)^{n-1}$.

These sequences provide a way of writing any GP that alternates in sign using a *switch*. For example, the sequence $2,\ -6,\ 18,\ -54,\ \dots$ in the previous worked example has formula $T_n = 2 \times (-3)^{n-1}$, which can also be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^{n-1}$$

to emphasise the alternating sign, and $-2,\ 6,\ -18,\ 54,\ \dots$ can be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^n.$$

Solving problems involving GPs

As with APs, the formula for the nth term allows many problems to be solved by forming an equation and solving it.

Example 12 1C

a Find a formula for the nth term of the geometric sequence $5,\ 10,\ 20,\ \dots$.
b Hence find whether 320 and 720 are terms of this sequence.

SOLUTION

a The sequence is a GP with $a = 5$ and $r = 2$.
Hence $T_n = ar^{n-1}$
$= 5 \times 2^{n-1}$.

b Put $T_n = 320$.
Then $5 \times 2^{n-1} = 320$
$2^{n-1} = 64$
$n - 1 = 6$
$n = 7$,
so 320 is the seventh term T_7.

Put $T_n = 720$.
Then $5 \times 2^{n-1} = 720$
$2^{n-1} = 144$.
But 144 is not a power of 2,
so 720 is not a term of the sequence.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 13 1C

The first term of a GP is 448 and the seventh term is 7. Find the common ratio and write out the first seven terms.

SOLUTION

First, we know that $T_1 = 448$

that is, $a = 448.$ (1)

Secondly, we know that $T_7 = 7$

so using the formula for the 7th term, $ar^6 = 7.$ (2)

Substituting (1) into (2),

$$448r^6 = 7$$
$$r^6 = \tfrac{1}{64}$$
$$r = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}.$$

Thus either the ratio is $r = \frac{1}{2}$, and the sequence is

$$448,\ 224,\ 112,\ 56,\ 28,\ 14,\ 7,\ \ldots$$

or the ratio is $r = -\frac{1}{2}$, and the sequence is

$$448,\ -224,\ 112,\ -56,\ 28,\ -14, 7,\ \ldots$$

Geometric sequences and exponential functions

Take the exponential function $f(x) = 54 \times 3^{-x}$, and substitute the positive integers. The result is a geometric sequence

$$18,\ 6,\ 2,\ \tfrac{2}{3},\ \tfrac{2}{9},\ \ldots$$

x	1	2	3	4	5
$f(x)$	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$

The formula for the nth term of this GP is $T_n = 18 \times \left(\frac{1}{3}\right)^{n-1} = 54 \times 3^{-n}$. This is a function whose domain is the set of positive integers, and its equation is the same as the exponential function, with only a change of pronumeral from x to n.

Thus the graph of an arithmetic sequence is the positive integer points on the graph of a linear function, and the graph of a geometric sequence is the positive integer points on the graph of an exponential function.

Exercise 1C FOUNDATION

1 Write out the next three terms of each sequence. They are all GPs.

a $1,\ 2,\ 4,\ \ldots$
b $81,\ 27,\ 9,\ \ldots$
c $-7,\ -14,\ -28,\ \ldots$
d $-2500,\ -500,\ -100,\ \ldots$
e $3,\ -6,\ 12,\ \ldots$
f $-25,\ 50,\ -100,\ \ldots$
g $5,\ -5,\ 5,\ \ldots$
h $-1000,\ 100,\ -10,\ \ldots$
i $0.04,\ 0.4,\ 4,\ \ldots$

2 Write out the first four terms of the GPs whose first terms and common ratios are:

a $a = 12$ and $r = 2$
b $a = 5$ and $r = -2$
c $a = 18$ and $r = \frac{1}{3}$
d $a = 18$ and $r = -\frac{1}{3}$
e $a = 6$ and $r = -\frac{1}{2}$
f $a = -7$ and $r = -1$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Find the ratios $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ for each sequence to test whether it is a GP. If the sequence is a GP, write down the first term a and the common ratio r.

a $4, 8, 16, \ldots$ **b** $16, 8, 4, \ldots$ **c** $2, 4, 6, \ldots$
d $-1000, -100, -10, \ldots$ **e** $-80, 40, -20, \ldots$ **f** $29, 29, 29, \ldots$
g $1, 4, 9, \ldots$ **h** $-14, 14, -14, \ldots$ **i** $6, 1, \frac{1}{6}, \ldots$

4 Use the formula $T_n = ar^{n-1}$ to find the fourth term of the GP with:

a $a = 5$ and $r = 2$ **b** $a = 300$ and $r = \frac{1}{10}$ **c** $a = -7$ and $r = 2$
d $a = -64$ and $r = \frac{1}{2}$ **e** $a = 11$ and $r = -2$ **f** $a = -15$ and $r = -2$

5 Use the formula $T_n = ar^{n-1}$ to find the nth term T_n of the GP with:

a $a = 1$ and $r = 3$ **b** $a = 5$ and $r = 7$ **c** $a = 8$ and $r = -\frac{1}{3}$

6 **a** Find the first term a and the common ratio r of the GP $7, 14, 28, \ldots$
b Find the sixth term T_6 and an expression for the 50th term T_{50}.
c Find a formula for the nth term T_n.

7 **a** Find the first term a and the common ratio r of the GP $10, -30, 90, \ldots$
b Find the sixth term T_6 and an expression for the 25th term T_{25}.
c Find a formula for the nth term T_n.

8 Find $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ to test whether each sequence is a GP. If the sequence is a GP, use the formula $T_n = ar^{n-1}$ to find a formula for the nth term, then find T_6.

a $10, 20, 40, \ldots$ **b** $180, 60, 20, \ldots$ **c** $64, 81, 100, \ldots$
d $35, 50, 65, \ldots$ **e** $\frac{3}{4}, 3, 12, \ldots$ **f** $-48, -24, -12, \ldots$

9 Find the common ratio of each GP, find a formula for T_n, and find T_6.

a $1, -1, 1, \ldots$ **b** $-2, 4, -8, \ldots$ **c** $-8, 24, -72, \ldots$
d $60, -30, 15, \ldots$ **e** $-1024, 512, -256, \ldots$ **f** $\frac{1}{16}, -\frac{3}{8}, \frac{9}{4}, \ldots$

DEVELOPMENT

10 Use the formula $T_n = ar^{n-1}$ to find how many terms there are in each finite sequence.

a $1, 2, 4, \ldots, 64$ **b** $-1, -3, -9, \ldots, -81$ **c** $8, 40, 200, \ldots, 125\,000$
d $7, 14, 28, \ldots, 224$ **e** $2, 14, 98, \ldots, 4802$ **f** $\frac{1}{25}, \frac{1}{5}, 1, \ldots, 625$

11 **a** The first term of a GP is $a = 25$ and the fourth term is $T_4 = 200$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.
b Find the common ratio r of a GP for which:
i $a = 3$ and $T_6 = 96$ **ii** $a = 1000$ and $T_7 = 0.001$
iii $a = 32$ and $T_6 = -243$ **iv** $a = 5$ and $T_7 = 40$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 The nth term of a geometric sequence is $T_n = 25 \times 2^n$.

a Write out the first six terms and hence find the values of a and r.

b Which term of the sequence is 6400?

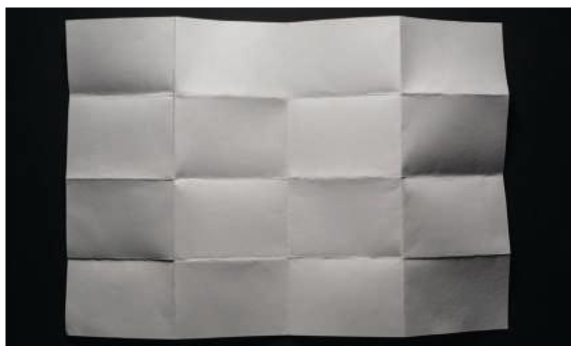
c Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.

d Prove that $T_9 \times T_{11} = 25 \times T_{20}$.

e Write out the terms between 1000 and 100 000. How many of them are there?

f Verify by calculations that $T_{11} = 51\,200$ is the last term less than 100 000 and that $T_{12} = 102\,400$ is the first term greater than 100 000.

13 A piece of paper 0.1 mm thick is folded successively 100 times. How thick is it now?



14 [Compound interest and GPs]

A principal $\$P$ is invested at 7% per annum compound interest. Let A_n be the total amount at the end of n years.

a Write down A_1, A_2 and A_3.

b Show that the total amount at the end of n years forms a GP with first term $1.07 \times P$ and ratio 1.07, and find the nth term A_n.

c Use trial-and-error on the calculator to find how many full years it will take for the amount to double, and how many years it will take for it to become ten times the original principal.

15 [Depreciation and GPs]

A car originally costs \$20 000, then at the end of every year, it is worth only 80% of what it was worth a year before. Let W_n be its worth at the end of n years.

a Write down expressions for W_1, W_2 and W_3, and find a formula for W_n.

b Use trial-and-error on the calculator to find how many complete years it takes for the value to fall below \$2000.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 Find the nth term of each GP.

a $\sqrt{6}, 2\sqrt{3}, 2\sqrt{6}, \ldots$ **b** $ax, a^2x^3, a^3x^5, \ldots$ **c** $-\frac{x}{y}, -1, -\frac{y}{x}, \ldots$

17 a Find a formula for T_n in $2x, 2x^2, 2x^3, \ldots$ Then find x if $T_6 = 2$.

b Find a formula for T_n in $x^4, x^2, 1, \ldots$ Then find x if $T_6 = 3^6$.

c Find a formula for T_n in $2^{-16}x, 2^{-12}x, 2^{-8}x, \ldots$ Then find x if $T_6 = 96$.

18 a Show that $2^5, 2^2, 2^{-1}, 2^{-4}, \ldots$ is a GP, and find its nth term.

b Show that $\log_2 96,\ \log_2 24,\ \log_2 6, \ldots$ is an AP, and show that $T_n = 7 - 2n + \log_2 3$.

19 a Write down the first few terms of the GP generated by substituting the positive integers into the exponential function $f(x) = \frac{4}{25} \times 5^x$. Then write down a formula for the nth term.

b i Find the formula of the nth term T_n of the GP $5,\ 10,\ 20,\ 40,\ 80, \ldots$. Then write down the exponential function $f(x)$ that generates this GP when the positive integers are substituted into it.

ii Graph the function (without the same scale on both axes) and mark the points $(1, 5),\ (2, 10),\ (3, 20),\ (4, 40),\ (5, 80)$.

20 [GPs are essentially exponential functions.]

a Show that if $f(x) = kb^x$ is any exponential function, then the sequence $T_n = kb^n$ is a GP, and find its first term and common ratio.

b Conversely, if T_n is a GP with first term a and ratio r, find the exponential function $f(x)$ such that $T_n = f(n)$.

c Plot on the same axes the points of the GP $T_n = 2^{4-n}$ and the graph of the continuous function $y = 2^{4-x}$.

ENRICHMENT

21 a What are the first term and common ratio of the GP generated by substituting the positive integers into the exponential function $f(x) = cb^x$?

b What is the equation of the exponential function that generates a GP with first term a and ratio r when the positive integers are substituted into it?

22 [Products and sums of GPs]

Suppose that $T_n = ar^{n-1}$ and $U_n = AR^{n-1}$ are two GPs.

a Show that the sequence $V_n = T_nU_n$ is a GP, and find its first term and common ratio.

b Show that the sequence $W_n = T_n + U_n$ is a GP if and only if $r = R$ and $a + A \neq 0$, and find the formula for W_n in this case. (Hint: if W_n is a GP, then $W_nW_{n+2} = {W_{n+1}}^2$. Substitute into this condition, and deduce that $(R - r)^2 = 0$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1D Solving problems involving APs and GPs

This section deals with APs and GPs together and presents some further approaches to problems about the terms of APs and GPs.

A condition for three numbers to be in AP or GP

The three numbers $10,\ 25,\ 40$ form an AP because the differences $25 - 10 = 15$ and $40 - 25 = 15$ are equal.

Similarly, $10,\ 20,\ 40$ form a GP because the ratios $\frac{20}{10} = 2$ and $\frac{40}{20} = 2$ are equal.

These situations occur quite often and a formal statement is worthwhile:

6 THREE NUMBERS IN AP OR GP

- Three numbers a, m and b form an AP if
  $$m - a = b - m, \qquad \text{that is,} \qquad m = \tfrac{1}{2}(a + b).$$
- Three non-zero numbers a, g and b form a GP if
  $$\frac{g}{a} = \frac{b}{g}, \qquad \text{that is,} \qquad g^2 = ab.$$

The number m is already familiar because it is the mean of a and b. In the context of sequences, the numbers m and g are called *the arithmetic mean* and *a geometric mean* of a and b, but these terms are not required in the course.

Example 14 — 1D

a Find the value of m if $3,\ m,\ 12$ form an AP.

b Find the value of g if $3,\ g,\ 12$ form a GP.

SOLUTION

a Because $3,\ m,\ 12$ form an AP,
$$\begin{aligned} m - 3 &= 12 - m \\ 2m &= 15 \\ m &= 7\tfrac{1}{2}. \end{aligned}$$

b Because $3,\ g,\ 12$ form a GP,
$$\begin{aligned} \frac{g}{3} &= \frac{12}{g} \\ g^2 &= 36 \\ g &= 6 \text{ or } -6. \end{aligned}$$

Solving problems leading to simultaneous equations

Many problems about APs and GPs lead to simultaneous equations. These are best solved by elimination.

7 PROBLEMS ON APS AND GPS LEADING TO SIMULTANEOUS EQUATIONS

- With APs, eliminate a by subtracting one equation from the other.
- With GPs, eliminate a by dividing one equation by the other.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 15 1D

The third term of an AP is 16 and the 12th term is 79. Find the 41st term.

SOLUTION

Let the first term be a and the common difference be d.

Because $T_3 = 16$, $\quad a + 2d = 16, \quad (1)$

and because $T_{12} = 79$, $\quad a + 11d = 79. \quad (2)$

Subtracting (1) from (2), $\quad 9d = 63$ (the key step that eliminates a)

$d = 7.$

Substituting into (1), $\quad a + 14 = 16$

$a = 2.$

Hence $\quad T_{41} = a + 40d$

$= 282.$

Example 16 1D

Find the first term a and the common ratio r of a GP in which the fourth term is 6 and the seventh term is 162.

SOLUTION

Because $T_4 = 6$, $\quad ar^3 = 6, \quad (1)$

and because $T_7 = 162$, $\quad ar^6 = 162. \quad (2)$

Dividing (2) by (1), $\quad r^3 = 27$ (the key step that eliminates a)

$r = 3.$

Substituting into (1), $\quad a \times 27 = 6$

$a = \frac{2}{9}.$

Solving GP problems involving trial-and-error or logarithms

Equations and inequations involving the terms of a GP are index equations, so logarithms are needed for a systematic approach.

Trial-and-error, however, is quite satisfactory for simpler problems, and the reader may prefer to leave the application of logarithms until Chapter 13.

Example 17 1D

a Find a formula for the nth term of the geometric sequence $2, 6, 18, \ldots$.

b Use trial-and-error to find the first term greater than 1 000 000.

c Use logarithms to find the first term greater than 1 000 000.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a This is a GP with $a = 2$ and $r = 3$,
so $T_n = ar^{n-1}$
$= 2 \times 3^{n-1}$.

b Put $T_n > 1000000$.
Using the calculator, $T_{12} = 354294$
and $T_{13} = 1062882$.
Hence the first term over 1000000 is $T_{13} = 1062882$.

c Put $T_n > 1000000$.

$$\begin{aligned} \text{Then } 2 \times 3^{n-1} &> 1000000 \\ 3^{n-1} &> 500000 \\ n - 1 &> \log_3 500000 \quad \text{(remembering that } 2^3 = 8 \text{ means } 3 = \log_2 8\text{)} \\ n - 1 &> \frac{\log_{10} 500000}{\log_{10} 3} \quad \text{(the change-of-base formula)} \\ n - 1 &> 11.94\ldots \\ n &> 12.94\ldots \end{aligned}$$

Hence the first term over 1000000 is $T_{13} = 1062882$.

Exercise 1D

FOUNDATION

1 Find the value of m if each set of numbers below forms an arithmetic sequence.
(Hint: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find m.)

a $5, m, 17$ **b** $32, m, 14$ **c** $-12, m, -50$
d $-23, m, 7$ **e** $m, 22, 32$ **f** $-20, -5, m$

2 Each triple of number forms a geometric sequence. Find the value of g. (Hint: Form an equation using the identity $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$, then solve it to find g.)

a $2, g, 18$ **b** $48, g, 3$ **c** $-10, g, -90$
d $-98, g, -2$ **e** $g, 20, 80$ **f** $-1, 4, g$

3 Find x if each triple of three numbers forms: **i** an AP, **ii** a GP.

a $4,\ x,\ 16$ **b** $1,\ x,\ 49$ **c** $16,\ x,\ 25$ **d** $-5,\ x,\ -20$
e $x,\ 10,\ 50$ **f** $x,\ 12,\ 24$ **g** $x,\ -1,\ 1$ **h** $x,\ 6,\ -12$
i $20,\ 30,\ x$ **j** $-36,\ 24,\ x$ **k** $-\frac{1}{4},\ -3,\ x$ **l** $7,\ -7,\ x$

DEVELOPMENT

4 In these questions, substitute into $T_n = a + (n - 1)d$ or $T_n = ar^{n-1}$.

a Find the first six terms of the AP with first term $a = 7$ and sixth term $T_6 = 42$.
b Find the first four terms of the GP with first term $a = 27$ and fourth term $T_4 = 8$.
c Find the first five terms of the AP with $a = 48$ and $T_5 = 3$.
d Find the first five terms of the GP with $a = 48$ and $T_5 = 3$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

5 Use simultaneous equations and the formula $T_n = a + (n - 1)d$ to solve these problems.

a Find the first term and common difference of the AP with $T_{10} = 18$ and $T_{20} = 48$.

b Find the first term and common difference of the AP with $T_5 = 24$ and $T_9 = -12$.

c Find the first term and common difference of the AP with $T_4 = 6$ and $T_{12} = 34$.

6 Use simultaneous equations and the formula $T_n = ar^{n-1}$ to solve these problems.

a Find the first term and common ratio of the GP with $T_3 = 16$ and $T_6 = 128$.

b Find the first term and common ratio of the GP with $T_2 = \frac{1}{3}$ and $T_6 = 27$.

c Find the first term and common ratio of the GP with $T_5 = 6$ and $T_9 = 24$.

7 a The third term of an AP is 7 and the seventh term is 31. Find the eighth term.

b The common difference of an AP is -7 and the 10th term is 3. Find the second term.

c The common ratio of a GP is 2 and the sixth term is 6. Find the second term.

8 Use either trial-and-error or logarithms to solve these problems.

a Find the smallest value of n such that $3^n > 1\,000\,000$.

b Find the largest value of n such that $5^n < 1\,000\,000$.

c Find the smallest value of n such that $7^n > 1\,000\,000\,000$.

d Find the largest value of n such that $12^n < 1\,000\,000\,000$.

9 Let T_n be the sequence $2,\ 4,\ 8,\ \dots$ of powers of 2.

a Show that the sequence is a GP, and show that the nth term is $T_n = 2^n$.

b Find how many terms are less than $1\,000\,000$. (You will need to solve the inequation $T_n < 1\,000\,000$ using trial-and-error or logarithms.)

c Use the same method to find how many terms are less than $1\,000\,000\,000$.

d Use the same method to find how many terms are less than 10^{20}.

e How many terms are between $1\,000\,000$ and $1\,000\,000\,000$?

f How many terms are between $1\,000\,000\,000$ and 10^{20}?

10 Find a formula for T_n for these GPs. Then find how many terms exceed 10^{-6}. (You will need to solve the inequation $T_n > 10^{-6}$ using trial-and-error or logarithms.)

a $98,\ 14,\ 2,\ \dots$ **b** $25,\ 5,\ 1,\ \dots$ **c** $1,\ 0.9,\ 0.81,\ \dots$

11 When light passes through one sheet of very thin glass, its intensity is reduced by 3%. (Hint: 97% of the light gets though each sheet.)

a If the light passes through 50 sheets of this glass, find by what percentage (correct to the nearest 1%) the intensity will be reduced.

b What is the minimum number of sheets that will reduce the intensity below 1%?

12 a Find a and d for the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.

b Find a and r for the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.

c The fourth, sixth and eighth terms of an AP add to -6. Find the sixth term.

13 Each set of three numbers forms an AP. Find x and write out the numbers.

a $x - 1,\ 17,\ x + 15$ **b** $2x + 2,\ x - 4,\ 5x$

c $x - 3,\ 5,\ 2x + 7$ **d** $3x - 2,\ x,\ x + 10$

14 Each set of three numbers forms a GP. Find x and write out the numbers.

a $x,\ x + 1,\ x$ **b** $2 - x,\ 2,\ 5 - x$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 Find x and write out the three numbers if they form: **i** an AP, **ii** a GP.

a $x,\ 24,\ 96$
b $0.2,\ x,\ 0.00002.$
c $x,\ 0.2,\ 0.002.$
d $x - 4,\ x + 1,\ x + 11$
e $x - 2,\ x + 2,\ 5x - 2$
f $\sqrt{5} + 1,\ x,\ \sqrt{5} - 1$
g $\sqrt{2},\ x,\ \sqrt{8}$
h $2^4,\ x,\ 2^6$
i $7,\ x,\ -7$

16 **a** Find a and b if a, b, 1 forms a GP, and b, a, 10 forms an AP.
b Find a and b if $a,\ 1,\ a + b$ forms a GP, and $b,\ \frac{1}{2},\ a - b$ forms an AP.

17 **a** Three non-zero numbers form both an AP and a GP. Prove that they are all equal. (Hint: Let the numbers be $x - d$, x and $x + d$, and prove that $d = 0$.)
b Show that in an AP, the first, fourth and seventh terms form another AP.
c Show that in a GP, the first, fourth and seventh terms form another GP.

18 **a** Show that if the first, second and fourth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the positive integer multiples of the first term.
b Show that if the first, second and fifth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the odd positive integer multiples of the first term.
c Find the common ratio of the GP in which the first, third and fourth terms form an arithmetic sequence. $\left(\text{Hint: } r^3 - 2r^2 + 1 = (r - 1)(r^2 - r - 1)\right)$
d Find the GP in which each term is one more than the sum of all the previous terms.

ENRICHMENT

19 Let a and b be positive real numbers with $a \le b$. Let $m = \frac{1}{2}(a + b)$ and $g = \sqrt{ab}$, so that the sequence a, m, b forms an AP and the sequence a, g, b forms a GP.
a Explain why $(a - b)^2 \ge 0$.
b Expand this to prove that $(a + b)^2 \ge 4ab$, and hence show that $g \le m$.
c Find a trivial, and a non-trivial, example of numbers a and b so that $g = \frac{1}{2}(a + m)$.

20 [Geometric sequences in musical instruments]
The pipe lengths in a rank of organ pipes decrease from left to right. The lengths form a GP, and the 13th pipe along is exactly half the length of the first pipe (making an interval called an *octave*).
a Show that the ratio of the GP is $r = \left(\frac{1}{2}\right)^{\frac{1}{12}}$.
b Show that the eighth pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).
c Show that the fifth pipe along is just under four-fifths the length of the first pipe (a *major third*).
d Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.
e What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

21 Construction of the positive geometric mean of two positive numbers a and b:
a Construct a number line OAB with O at zero, $OA = a$, and $OB = b$.
b Construct the midpoint M of AB, and construct the circle with diameter AB.
c Construct the midpoint N of OM, and construct the circle with diameter OM.
d Let the circles meet at S and T.
e Prove that $OS = OT$ is the positive geometric mean of a and b.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1E Adding up the terms of a sequence

Adding the terms of a sequence is often important. For example, a boulder falling from the top of a high cliff falls 5 metres in the first second, 15 metres in the second second, 25 metres in the third second, and so on. The distance that it falls in the first 10 seconds is the sum of the 10 numbers

$$5 + 15 + 25 + 35 + \cdots + 95 = 500.$$

A notation for the sums of terms of a sequence

For any sequence $T_1, T_2, T_3, \ldots$, define S_n to be the sum of the first n terms of the sequence.

8 THE SUM OF THE FIRST *n* TERMS OF A SEQUENCE

Given a sequence $T_1, T_2, T_3, \ldots$, define

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

The sum S_n is variously called:

- the *sum of the first n terms* of the sequence,
- the *sum to n terms* of the sequence,
- the *nth partial sum* of the sequence ('partial' meaning 'part of the sequence').

For example, the sum of the first 10 terms of the sequence $5, 15, 25, 35, \ldots$ is

$$\begin{aligned} S_{10} &= 5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85 + 95 \\ &= 500 \end{aligned}$$

which is also called the 10th partial sum of the sequence.

The sequence $S_1, S_2, S_3, S_4, \ldots$ of sums

The partial sums $S_1, S_2, S_3, S_4, \ldots$ form another sequence. For example, with the sequence $5, 15, 25, 35, \ldots$,

$$S_1 = 5 \qquad \begin{aligned} S_2 &= 5 + 15 \\ &= 20 \end{aligned} \qquad \begin{aligned} S_3 &= 5 + 15 + 25 \\ &= 45 \end{aligned} \qquad \begin{aligned} S_4 &= 5 + 15 + 25 + 35 \\ &= 80 \end{aligned}$$

Example 18 — 1E

Copy and complete this table for the successive sums of a sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n										

SOLUTION

Each entry for S_n is the sum of all the terms T_n up to that point.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n	5	20	45	80	125	180	245	320	405	500

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Recovering the sequence from the partial sums

Suppose we know that the partial sums S_n of a sequence are the successive squares,

S_n: 1, 4, 9, 16, 25, 36, 49, 64, ...

and we want to recover the terms T_n. The first term is $T_1 = S_1 = 1$, and then we can take successive differences, giving the sequence

T_n: 1, 3, 5, 7, 9, 11, 13, 15, ...

9 RECOVERING THE TERMS FROM THE PARTIAL SUMS

The original sequence T_n can be recovered from the sequence S_n of partial sums by taking successive differences,

$$T_1 = S_1$$
$$T_n = S_n - S_{n-1}, \quad \text{for } n \geq 2.$$

Example 19 1E

By taking successive differences, list the terms of the original sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n										
S_n	1	5	12	22	35	51	70	92	117	145

SOLUTION

Each entry for T_n is the difference between two successive sums S_n.

n	1	2	3	4	5	6	7	8	9	10
T_n	1	4	7	10	13	16	19	22	25	28
S_n	1	5	12	22	35	51	70	92	117	145

Example 20 1E

Confirm the example given above by proving algebraically that if the partial sums S_n of a sequence are the successive squares, then the sequence T_n is the sequence of odd numbers.

SOLUTION

We are given that $S_n = n^2$.

Hence $T_1 = S_1$
$= 1$, which is the first odd number,

and for $n \geq 2$, $T_n = S_n - S_{n-1}$
$= n^2 - (n-1)^2$
$= 2n - 1$, which is the nth odd number.

Note: Taking successive differences in a sequence is analogous to differentiation in calculus, and the results have many similarities to differentiation. For example, in the worked example above, taking finite differences of a quadratic function yields a linear function. Questions **13–14** in Exercise 1E have further analogies, which are not pursued in this course.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Sigma notation

This is a concise notation for the sums of a sequence. For example:

$$\begin{aligned}\sum_{n=2}^{5} n^2 &= 2^2 + 3^2 + 4^2 + 5^2 \\ &= 4 + 9 + 16 + 25 \\ &= 54\end{aligned} \qquad \begin{aligned}\sum_{n=6}^{10} n^2 &= 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\ &= 36 + 49 + 64 + 81 + 100 \\ &= 330\end{aligned}$$

The first sum says 'evaluate the function n^2 for all the integers from $n = 2$ to $n = 5$, then add up the resulting values'. There are 4 terms, and their sum is 54.

10 SIGMA NOTATION

Suppose that $T_1,\ T_2,\ T_3,\ \dots$ is a sequence. Then

$$\sum_{n=5}^{20} T_n = T_5 + T_6 + T_7 + T_8 + \cdots + T_{20}$$

(Any two integers can obviously be substituted for the numbers 5 and 20.)

We used the symbol Σ before in Chapter 13 of the Year 11 book. It stands for the word 'sum', and is a large version of the Greek capital letter Σ called 'sigma' and pronounced 's'. The superscripts and subscripts on the sigma sign, however, are used for the first time in this chapter.

Example 21 1E

Evaluate these sums.

a $\displaystyle\sum_{n=4}^{7} (5n + 1)$

b $\displaystyle\sum_{n=1}^{5} (-2)^n$

SOLUTION

a $$\begin{aligned}\sum_{n=4}^{7} (5n + 1) &= 21 + 26 + 31 + 36 \\ &= 114\end{aligned}$$

b $$\begin{aligned}\sum_{n=1}^{5} (-2)^n &= -2 + 4 - 8 + 16 - 32 \\ &= -22\end{aligned}$$

Series

The word *series* is often used imprecisely, but it always refers to the activity of adding up terms of a sequence. For example, the phrase

'the series $1 + 4 + 9 + \cdots$'

means that one is considering the successive partial sums $S_1 = 1,\ S_2 = 1 + 4,\ S_3 = 1 + 4 + 9,\ \dots$ of the sequence of positive squares.

The precise definition is that a *series* is the sequence of partial sums of a sequence. That is, given a sequence $T_1,\ T_2,\ T_3,\ \dots,$ the corresponding series is the sequence

$$S_1 = T_1,\quad S_2 = T_1 + T_2,\quad S_3 = T_1 + T_2 + T_3,\quad S_4 = T_1 + T_2 + T_3 + T_4,\quad \dots$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 1E

FOUNDATION

1 Find the sum S_4 of the first four terms of each sequence.

a $3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots$

b $2,\ 6,\ 18,\ 54,\ 162,\ 486,\ \ldots$

c $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$

d $32 - 16 + 8 - 4 + 2 - 1 + \cdots$

2 Find the partial sums S_4, S_5 and S_6 for each series (which you will need to continue).

a $1 - 2 + 3 - 4 + \cdots$

b $81 + 27 + 9 + 3 + \cdots$

c $30 + 20 + 10 + \cdots$

d $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

3 Copy and complete these tables of a sequence and its partial sums.

a

T_n	2	5	8	11	14	17	20
S_n							

b

T_n	40	38	36	34	32	30	28
S_n							

c

T_n	2	−4	6	−8	10	−12	14
S_n							

d

T_n	7	−7	7	−7	7	−7	7
S_n							

4 Rewrite each partial sum without sigma notation, then evaluate it.

a $\sum_{n=1}^{6} 2n$

b $\sum_{n=1}^{6} (3n + 2)$

c $\sum_{k=3}^{7} (18 - 3n)$

d $\sum_{n=5}^{8} n^2$

e $\sum_{n=1}^{4} n^3$

f $\sum_{n=0}^{5} 2^n$

g $\sum_{n=2}^{4} 3^n$

h $\sum_{\ell=1}^{31} (-1)^{\ell}$

i $\sum_{\ell=1}^{40} (-1)^{\ell-1}$

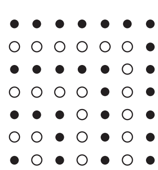
j $\sum_{n=5}^{105} 4$

k $\sum_{n=0}^{4} (-1)^n (n + 5)$

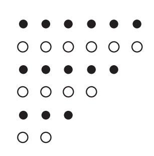
l $\sum_{n=0}^{4} (-1)^{n+1} (n + 5)$

DEVELOPMENT

5 **a** Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2.



b Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$.



c Part **b** shows why the sums $1,\ 1 + 2 = 3,\ 1 + 2 + 3 = 6,\ 1 + 2 + 3 + 4 = 10,\ \ldots$ are called the *triangular numbers*. Write out the first 15 triangular numbers.

6 Each table below gives the successive sums $S_1,\ S_2,\ S_3,\ \ldots$ of a sequence. By taking successive differences, write out the terms of the original sequence.

a

T_n							
S_n	1	4	9	16	25	36	49

b

T_n							
S_n	2	6	14	30	62	126	254

c

T_n							
S_n	−3	−8	−15	−24	−35	−48	−63

d

T_n							
S_n	8	0	8	0	8	0	8

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 [The Fibonacci and Lucas sequences] Each table below gives the successive sums S_n of a sequence. By taking successive differences, write out the terms of the original sequence.

a

T_n								
S_n	1	2	3	5	8	13	21	34

b

T_n								
S_n	3	4	7	11	18	29	47	76

8 The nth partial sum of a series is $S_n = 3^n - 1$.

a Write out the first five partial sums.

b Take differences to find the first five terms of the original sequence.

c Write down S_{n-1}, then use the result $T_n = S_n - S_{n-1}$ to find a formula for T_n. (Hint: This will need the factorisation $3^n - 3^{n-1} = 3^{n-1}(3 - 1) = 2 \times 3^{n-1}$.)

9 Repeat the steps of the previous question for the sequence whose nth partial sum is:

a $S_n = 10(2^n - 1)$ **b** $S_n = 4(5^n - 1)$ **c** $S_n = \frac{1}{4}(4^n - 1)$

(Hint: You will need factorisations such as $2^n - 2^{n-1} = 2^{n-1}(2 - 1)$.)

10 Find the nth term and the first three terms of the sequence for which S_n is:

a $S_n = 3n(n + 1)$ **b** $S_n = 5n - n^2$ **c** $S_n = 4n$

d $S_n = n^3$ **e** $S_n = 1 - 3^{-n}$ **f** $S_n = \left(\frac{1}{7}\right)^n - 1$

11 Rewrite each sum in sigma notation, starting each sum at $n = 1$. Do not evaluate it.

a $1^3 + 2^3 + 3^3 + \cdots + 40^3$ **b** $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40}$ **c** $3 + 4 + 5 + \cdots + 22$

d $2 + 2^2 + 2^3 + \cdots + 2^{12}$ **e** $-1 + 2 - 3 + \cdots + 10$ **f** $1 - 2 + 3 - \cdots - 10$

ENRICHMENT

12 In these sequences, the first term will not necessarily obey the same rule as the succeeding terms, in which case the formula for the sequence will need to be given piecewise:

a $S_n = n^2 + 4n + 3$ **b** $S_n = 7(3^n - 4)$ **c** $S_n = \dfrac{1}{n}$ **d** $S_n = n^3 + n^2 + n$

Find T_1 and a formula for T_n for each sequence. How could you have predicted whether or not the general formula would hold for T_1?

13 a The partial sums of a sequence T_n are given by $S_n = 2^n$. Use the formula in Box 9 to find a formula for T_n.

b Confirm your answer by writing out the calculation in table form, as in Question **6**.

c In Chapter 11 of the Year 11 book, you differentiated $y = e^x$. What is the analogy to this result?

14 a Prove that $n^3 - (n - 1)^3 = 3n^2 - 3n + 1$.

b The partial sums of a sequence T_n are given by $S_n = n^3$. Use the formula in Box 9 to find a formula for T_n.

c The terms of the sequence T_n are the partial sums of a third sequence U_n. Use the formula in Box 9 to find a formula for T_n.

d Confirm your answer by writing out in table form the successive taking of differences in parts **b** and **c**.

e In Chapter 8 of the Year 11 volume, you differentiated powers of x. What is the analogy to these results?

15 a Write out the terms of $\displaystyle\sum_{r=1}^{10}\left(\frac{1}{r} - \frac{1}{r + 1}\right)$ and hence show that the sum is $\frac{10}{11}$.

b Rationalise the denominator of $\dfrac{1}{\sqrt{k + 1} + \sqrt{k}}$ and hence evaluate $\displaystyle\sum_{n=1}^{15}\frac{1}{\sqrt{k + 1} + \sqrt{k}}$.

c Evaluate $\displaystyle\sum_{r=1}^{4}\left(\sum_{s=1}^{4}\left(\sum_{t=1}^{4} rst\right)\right)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1F Summing an arithmetic series

There are two formulae for adding up the first n terms of an AP.

Adding the terms of an AP

Consider adding the first six terms of the AP

$$5 + 15 + 25 + 35 + 45 + 55 + \cdots$$

Writing out the sum, $S_6 = 5 + 15 + 25 + 35 + 45 + 55.$
Reversing the sum, $S_6 = 55 + 45 + 35 + 25 + 15 + 5,$
and adding the two, $2S_6 = 60 + 60 + 60 + 60 + 60 + 60$
$= 6 \times 60,$ because there are 6 terms in the series.
Dividing by 2, $S_6 = \frac{1}{2} \times 6 \times 60$
$= 180.$

Notice that 60 is the sum of the first term $T_1 = 5$ and the last term $T_6 = 55$.

In general, let $\ell = T_n$ be the last term of an AP with first term a and difference d.

Then $S_n = a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.$
Reversing the sum, $S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a,$
and adding, $2S_n = (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)$
$= n \times (a + \ell),$ because there are n terms in the series.
Dividing by 2, $S_n = \frac{1}{2}n(a + \ell).$

Example 22 1F

Add up all the integers from 100 to 200 inclusive.

SOLUTION

The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms.

The first term is $a = 100$ and the last term is $\ell = 200$.

Using $S_n = \frac{1}{2}n(a + \ell),$

$$\begin{aligned} S_{101} &= \tfrac{1}{2} \times 101 \times (100 + 200) \\ &= \tfrac{1}{2} \times 101 \times 300 \\ &= 15\,150. \end{aligned}$$

An alternative formula for summing an AP

This alternative form is equally important.

The previous formula is $S_n = \frac{1}{2}n(a + \ell),$ where $\ell = T_n = a + (n - 1)d.$

Substituting $\ell = a + (n - 1)d,$ $S_n = \frac{1}{2}n\big(a + a + (n - 1)d\big)$

so $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big).$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

11 TWO FORMULAE FOR SUMMING AN AP

Suppose that the first term a of an AP, and the number n of terms, are known.

- When the last term $\ell = T_n$ is known, use $S_n = \frac{1}{2}n(a + \ell)$.
- When the difference d is known, use $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$.

If you have a choice, use the first because it is simpler.

Example 23 1F

Consider the arithmetic series $100 + 94 + 88 + 82 + \cdots$.

a Find S_{10}.

b Find S_{41}.

SOLUTION

The series is an AP with $a = 100$ and $d = -6$.

a Using $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$,

$$\begin{aligned} S_{10} &= \tfrac{1}{2} \times 10 \times (2a + 9d) \\ &= 5 \times (200 - 54) \\ &= 730. \end{aligned}$$

b Similarly,

$$\begin{aligned} S_{41} &= \tfrac{1}{2} \times 41 \times (2a + 40d) \\ &= \tfrac{1}{2} \times 41 \times (200 - 240) \\ &= \tfrac{1}{2} \times 41 \times (-40) \\ &= -820. \end{aligned}$$

Example 24 1F

a Find how many terms are in the sum $41 + 45 + 49 + \cdots + 401$.

b Hence evaluate the sum $41 + 45 + 49 + \cdots + 401$.

SOLUTION

a The series is an AP with first term $a = 41$ and difference $d = 4$.

To find the numbers of terms, put $T_n = 401$

$$\begin{aligned} a + (n-1)d &= 401 \\ 41 + 4(n-1) &= 401 \\ 4(n-1) &= 360 \\ n - 1 &= 90 \\ n &= 91. \end{aligned}$$

Thus there are 91 terms in the series.

b Because we now know both the difference d and the last term $\ell = T_{91}$, either formula can be used. It's always easier to use $S_n = \frac{1}{2}n(a + \ell)$ if you can.

Using $S_n = \frac{1}{2}n(a + \ell)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (41 + 401) \\ &= \tfrac{1}{2} \times 91 \times 442 \\ &= 20111. \end{aligned}$$

OR

Using $S_n = \frac{1}{2}n(2a + (n-1)d)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (2a + 90d) \\ &= \tfrac{1}{2} \times 91 \times (82 + 360) \\ &= 20111. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Solving problems involving the sums of APs

Problems involving sums of APs are solved using the formulae developed for the nth term T_n and the sum S_n of the first n terms.

Example 25 1F

a Find an expression for the sum S_n of n terms of the series $40 + 37 + 34 + \cdots$.

b Hence find the least value of n for which the partial sum S_n is negative.

SOLUTION

The sequence is an AP with $a = 40$ and $d = -3$.

a
$$\begin{aligned} S_n &= \tfrac{1}{2}n(2a + (n-1)d) \\ &= \tfrac{1}{2} \times n \times \big(80 - 3(n-1)\big) \\ &= \tfrac{1}{2} \times n \times (80 - 3n + 3) \\ &= \frac{n(83-3n)}{2} \end{aligned}$$

b
$$\begin{aligned} &\text{Put} & S_n &< 0. \\ &\text{Then} & \frac{n(83-3n)}{2} &< 0 \\ &\boxed{\times 2} & n(83-3n) &< 0 \\ &\boxed{\div n} & 83 - 3n &< 0, \quad \text{because } n \text{ is positive,} \\ && 83 &< 3n \\ && n &> 27\tfrac{2}{3}. \end{aligned}$$

Hence S_{28} is the first sum that is negative.

Example 26 1F

The sum of the first 10 terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION

$$\begin{aligned} &\text{The first piece of information given is} & S_{10} &= 0 \\ && 5(2a + 9d) &= 0 \\ &\boxed{\div 5} & 2a + 9d &= 0. & (1) \\ &\text{The second piece of information given is} & T_1 + T_2 &= 24 \\ && a + (a + d) &= 24 \\ && 2a + d &= 24. & (2) \\ &\text{Subtracting (2) from (1),} & 8d &= -24 \\ && d &= -3, \\ &\text{and substituting this into (2),} & 2a - 3 &= 24 \\ && a &= 13\tfrac{1}{2}. \end{aligned}$$

Hence the AP is $13\tfrac{1}{2} + 10\tfrac{1}{2} + 7\tfrac{1}{2} + \cdots$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 1F

FOUNDATION

1 Let $S_7 = 2 + 5 + 8 + 11 + 14 + 17 + 20$. By reversing the sum and adding in columns, evaluate S_7.

2 State how many terms each sum has, then find the sum using $S_n = \frac{1}{2}n(a + \ell)$.

a $1 + 2 + 3 + 4 + \cdots + 100$
b $1 + 3 + 5 + 7 + \cdots + 99$
c $2 + 4 + 6 + 8 + \cdots + 100$
d $3 + 6 + 9 + 12 + \cdots + 300$
e $101 + 103 + 105 + \cdots + 199$
f $1001 + 1002 + 1003 + \cdots + 10000$

3 Use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_6 of the first 6 terms of the series with:

a $a = 5$ and $d = 10$
b $a = 8$ and $d = 2$
c $a = -3$ and $d = -9$
d $a = -7$ and $d = -12$

4 Use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum of the stated number of terms.

a $2 + 5 + 8 + \cdots$ (12 terms)
b $40 + 33 + 26 + \cdots$ (21 terms)
c $-6 - 2 + 2 + \cdots$ (200 terms)
d $33 + 30 + 27 + \cdots$ (23 terms)
e $-10 - 7\frac{1}{2} - 5 + \cdots$ (13 terms)
f $10\frac{1}{2} + 10 + 9\frac{1}{2} + \cdots$ (40 terms)

5 First use the formula $T_n = a + (n - 1)d$ to find how many terms there are in each sum. Then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum, where ℓ is the last term T_n.

a $50 + 51 + 52 + \cdots + 150$
b $8 + 15 + 22 + \cdots + 92$
c $-10 - 3 + 4 + \cdots + 60$
d $4 + 7 + 10 + \cdots + 301$
e $6\frac{1}{2} + 11 + 15\frac{1}{2} + \cdots + 51\frac{1}{2}$
f $-1\frac{1}{3} + \frac{1}{3} + 2 + \cdots + 13\frac{2}{3}$

6 Find these sums by any appropriate method.

a $2 + 4 + 6 + \cdots + 1000$
b $1000 + 1001 + \cdots + 3000$
c $1 + 5 + 9 + \cdots$ (40 terms)
d $10 + 30 + 50 + \cdots$ (12 terms)

7 Find and simplify the sum of the first n terms of each series.

a $5 + 10 + 15 + \cdots$
b $10 + 13 + 16 + \cdots$
c $3 + 7 + 11 + \cdots$
d $-9 - 4 + 1 + \cdots$
e $5 + 4\frac{1}{2} + 4 + \cdots$
f $\big(1 - \sqrt{2}\big) + 1 + \big(1 + \sqrt{2}\big) + \cdots$

8 Use either standard formula for S_n to find a formula for the sum of the first n:

a positive integers,
b odd positive integers,
c positive integers divisible by 3,
d odd positive multiples of 100.

DEVELOPMENT

9 a How many legs are there on 15 fish, 15 ducks, 15 dogs, 15 beetles, 15 spiders, and 15 ten-legged grubs? How many of these creatures have the mean number of legs?

b Matthew Flinders High School has 1200 pupils, with equal numbers of each age from 6 to 17 years inclusive. It also has 100 teachers and ancillary staff, all aged 30 years, and one Principal aged 60 years. What is the total of the ages of everyone in the school?

c An advertising graduate earns \$28000 per annum in her first year, then each successive year her salary rises by \$1600. What are her total earnings over 10 years?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 By substituting appropriate values of k, find the first term a and last term ℓ of each sum. Then evaluate the sum using $S_n = \frac{1}{2}n(a + \ell)$. (Note that all four series are APs.)

a $\sum_{k=1}^{200} (600 - 2k)$ **b** $\sum_{k=1}^{61} (93 - 3k)$

c $\sum_{k=1}^{40} (3k - 50)$ **d** $\sum_{k=10}^{30} (5k + 3)$

11 Find the sums of these APs, whose terms are logarithms.

a $\log_a 2 + \log_a 4 + \log_a 8 + \cdots + \log_a 1024$

b $\log_5 243 + \log_5 81 + \log_5 27 + \cdots + \log_5 \frac{1}{243}$

c $\log_b 36 + \log_b 18 + \log_b 9 + \cdots + \log_b \frac{9}{8}$

d $\log_x \frac{27}{8} + \log_x \frac{9}{4} + \log_x \frac{3}{2} + \cdots$ (10 terms)

12 Solve these questions using the formula $S_n = \frac{1}{2}n(a + \ell)$ whenever possible — otherwise use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$.

a Find the last term if a series with 10 terms and first term -23 has sum -5.

b Find the first term if a series with 40 terms and last term $8\frac{1}{2}$ has sum 28.

c Find the common difference if a series with 8 terms and first term 5 has sum 348.

d Find the first term if a series with 15 terms and difference $\frac{2}{7}$ has sum -15.

13 **a** Show that the sum to n terms of the AP $60 + 52 + 44 + 36 + \cdots$ is $S_n = 4n(16 - n)$.

b Hence find how many terms must be taken to make the sum: **i** zero, **ii** negative.

c Find the two values of n for which the sum S_n is 220.

d Show that $S_n = -144$ has two integer solutions, but that only one has meaning.

e For what values of n does the sum S_n exceed 156?

f Prove that no sum S_n can exceed 256.

14 First use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_n for each arithmetic series. Then use quadratic equations to find the number of terms if S_n has the given value.

a $42 + 40 + 38 + \cdots$ where $S_n = 0$ **b** $60 + 57 + 54 + \cdots$ where $S_n = 0$

c $45 + 51 + 57 + \cdots$ where $S_n = 153$ **d** $2\frac{1}{2} + 3 + 3\frac{1}{2} + \cdots$ where $S_n = 22\frac{1}{2}$

15 Find the first term and the number of terms if a series has:

a $d = 4$, $\ell = 32$ and $S_n = 0$ **b** $d = -3$, $\ell = -10$ and $S_n = 55$

16 **a** Logs of wood are stacked with 10 on the top row, 11 on the next, and so on. If there are 390 logs, find the number of rows, and the number of logs on the bottom row.

b A stone dropped from the top of a 245-metre cliff falls 5 metres in the first second, 15 metres in the second second, and so on in arithmetic sequence. Find a formula for the distance after n seconds, and find how long the stone takes to fall to the ground.

c A truck spends several days depositing truckloads of gravel from a quarry at equally spaced intervals along a straight road. The first load is deposited 20 km from the quarry, the last is 10 km further along the road. If the truck travels 550 km during these deliveries, including its return to the quarry after the last delivery, how many trips does it make, and how far apart are the deposits?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 **a** The sum of the first and fourth terms of an AP is 16, and the sum of the third and eighth terms is 4. Find the sum of the first 10 terms.

b The sum of the first 10 terms of an AP is zero, and the 10th term is -9. Find the first and second terms.

c The sum to 16 terms of an AP is 96, and the sum of the second and fourth terms is 45. Find the fourth term, and show that the sum to four terms is also 96.

ENRICHMENT

18 **a** Find $1 + 2 + \cdots + 24$.

b Show that $\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n} = \frac{n+1}{2}$.

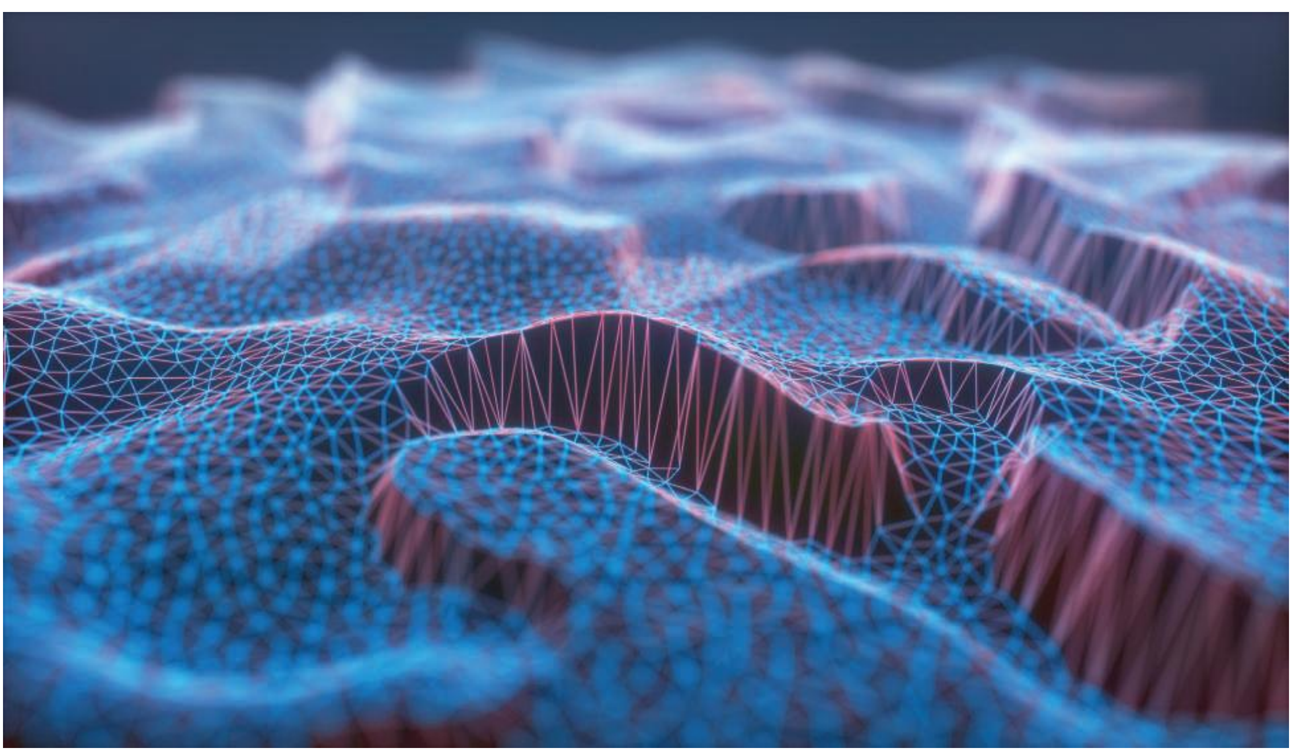
c Hence find the sum of the first 300 terms of
$\frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \cdots$.

19 **a** Find a formula for the nth triangular number $S_n = 1 + 2 + 3 + \cdots + n$.

b For what values of n is S_n: **i** divisible by 5, **ii** even?

c What is the smallest value of n for which S_n is divisible by:

i 29, **ii** 35, **iii** 26, **iv** 38,
v two distinct primes, **vi** three distinct primes, **vii** four distinct primes?



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1G Summing a geometric series

There is also a simple formula for finding the sum of the first n terms of a GP. The approach, however, is quite different from the approach used for APs.

Adding up the terms of a GP

This method is easier to understand with a general GP. Let us find the sum S_n of the first n terms of the GP $a + ar + ar^2 + \cdots$

Writing out the sum, $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \quad (1)$

Multiplying both sides by r, $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$

Subtracting (1) from (2), $\quad (r-1)\,S_n = ar^n - a.$

Then provided that $r \neq 1$, $\quad S_n = \dfrac{a(r^n - 1)}{r - 1}.$

If $r < 1$, there is a more convenient form. Taking opposites of top and bottom,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Method for summing a GP

Thus again there are two forms to remember.

12 TWO FORMULAE FOR SUMMING A GP

Suppose that the first term a, the ratio r, and the number n of terms are known.

- When $r > 1$, use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$.
- When $r < 1$, use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

Example 27 1G

a Find the sum of all the powers of 5 from 5^0 to 5^7.

b Find the sum of the first six terms of the geometric series $2 - 6 + 18 - \cdots$.

SOLUTION

a The sum $5^0 + 5^1 + \cdots + 5^7$ is a GP with $a = 1$ and $r = 5$.

Using $S_n = \dfrac{a(r^n - 1)}{r - 1}$, $\quad$ (in this case $r > 1$)

$$\begin{aligned} S_8 &= \frac{a(r^8 - 1)}{r - 1} \quad \text{(there are 8 terms)} \\ &= \frac{1 \times (5^8 - 1)}{5 - 1} \\ &= 97\,656. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b The series $2 - 6 + 18 - \cdots$ is a GP with $a = 2$ and $r = -3$.

$$\text{Using } S_n = \frac{a(1 - r^n)}{1 - r}, \qquad \text{(in this case } r < 1)$$

$$\begin{aligned} S_6 &= \frac{a(1 - r^6)}{1 - r} \\ &= \frac{2 \times (1 - (-3)^6)}{1 + 3} \\ &= -364. \end{aligned}$$

Solving problems about the sums of GPs

As always, read the question very carefully and write down all the information in symbolic form.

Example 28 — 1G

The sum of the first four terms of a GP with ratio 3 is 200. Find the four terms.

SOLUTION

$$\begin{aligned} \text{It is known that} \qquad S_4 &= 200. \\ \text{Using the formula,} \quad \frac{a(3^4 - 1)}{3 - 1} &= 200 \\ \frac{80a}{2} &= 200 \\ 40a &= 200 \\ a &= 5. \end{aligned}$$

So the series is $5 + 15 + 45 + 135 + \cdots$

Solving problems involving trial-and-error or logarithms

As remarked already in Section 1D, logarithms are needed for solving GP problems systematically, but trial-and-error is quite satisfactory for simpler problems.

Example 29 — 1G

a Find a formula for the sum of the first n terms of the GP $2 + 6 + 18 + \cdots$.

b How many terms of this GP must be taken for the sum to exceed one billion?

SOLUTION

a The sequence is a GP with $a = 2$ and $r = 3$,

$$\begin{aligned} \text{so } S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{2(3^n - 1)}{3 - 1} \\ &= 3^n - 1. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Put $S_n > 1000000000.$
Then $3^n - 1 > 1000000000$
$3^n > 1000000001.$
Using trial-and-error on the calculator,
$3^{18} = 387420489$
and $3^{19} = 1162261467,$
so S_{19} is the first sum over one billion.

OR

Put $S_n > 1000000000.$
Then $3^n - 1 > 1000000000$
$3^n > 1000000001$
$$n > \frac{\log_{10} 1000000001}{\log_{10} 3}$$
$n > 18.86\ldots,$
so S_{19} is the first sum over one billion.

An exceptional case

If the ratio of a GP is 1, then the formula for S_n doesn't work, because the denominator $r - 1$ would be zero. All the terms, however, are equal to the first term a, so the formula for the partial sum S_n is just

$$S_n = an.$$

This series is also an AP with first term a and difference 0. The last term is also a, so

$$S_n = \tfrac{1}{2}n(a + \ell) = \tfrac{1}{2}n(a + a) = an.$$

Exercise 1G

FOUNDATION

1 Let $S_6 = 2 + 6 + 18 + 54 + 162 + 486$. By taking $3S_6$ and subtracting S_6 in columns, evaluate S_6.

2 'As I was going to St Ives, I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits. Kits, cats, sacks and wives, how many were going to St Ives?'

Only the speaker was going *to* St Ives, but how many were going the other way?

3 **a** Use the formula $S_7 = \dfrac{a(r^7 - 1)}{r - 1}$ to find $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6$.

b Use the formula $S_7 = \dfrac{a(1 - r^7)}{1 - r}$ to find $1 - 3 + 3^2 - 3^3 + 3^4 - 3^5 + 3^6$.

4 Find these sums using $S_n = \dfrac{a(r^n - 1)}{r - 1}$ when $r > 1$, or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ when $r < 1$.
Then find a formula for the sum S_n of the first n terms of each series.

a $1 + 2 + 4 + 8 + \cdots$ (10 terms)
b $2 + 6 + 18 + \cdots$ (5 terms)
c $-1 - 10 - 100 - \cdots$ (5 terms)
d $-1 - 5 - 25 - \cdots$ (5 terms)
e $1 - 2 + 4 - 8 + \cdots$ (10 terms)
f $2 - 6 + 18 - \cdots$ (5 terms)
g $-1 + 10 - 100 + \cdots$ (5 terms)
h $-1 + 5 - 25 + \cdots$ (5 terms)

5 Find these sums. Then find a formula for the sum S_n of the first n terms of each series. Be careful when dividing by $1 - r$, because $1 - r$ is a fraction in each case.

a $8 + 4 + 2 + \cdots$ (10 terms)
b $9 + 3 + 1 + \cdots$ (6 terms)
c $45 + 15 + 5 + \cdots$ (5 terms)
d $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$
e $8 - 4 + 2 - \cdots$ (10 terms)
f $9 - 3 + 1 - \cdots$ (6 terms)
g $-45 + 15 - 5 + \cdots$ (5 terms)
h $\frac{2}{3} - 1 + \frac{3}{2} - \frac{9}{4} + \frac{27}{8}$

6 Find an expression for S_n. Hence approximate S_{10} correct to four significant figures.

a $1 + 1.2 + (1.2)^2 + \cdots$
b $1 + 0.95 + (0.95)^2 + \cdots$
c $1 + 1.01 + (1.01)^2 + \cdots$
d $1 + 0.99 + (0.99)^2 + \cdots$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

7 The King takes a chessboard of 64 squares, and places 1 grain of wheat on the first square, 2 grains on the next square, 4 grains on the next square, and so on.

a How many grains are on: **i** the last square, **ii** the whole chessboard?

b Given that 1 litre of wheat contains about 30 000 grains, how many cubic kilometres of wheat are there on the chessboard?

8 Find S_n and S_{10} for each series, rationalising the denominators in your answers.

a $1 + \sqrt{2} + 2 + \cdots$ **b** $2 - 2\sqrt{5} + 10 - \cdots$

9 Find these sums. First write out some terms and identify a and r.

a $\displaystyle\sum_{n=1}^{7} 3 \times 2^n$ **b** $\displaystyle\sum_{n=3}^{8} 3^{n-1}$ **c** $\displaystyle\sum_{n=1}^{8} 3 \times 2^{3-n}$

10 **a** The first term of a GP is $\frac{1}{8}$ and the fifth term is 162. Find the first five terms of the GP, then find their sum.

b The first term of a GP is $-\frac{3}{4}$ and the fourth term is 6. Find the sum of the first six terms.

c The second term of GP is 0.08 and the third term is 0.4. Find the sum to eight terms.

d The ratio of a GP is $r = 2$ and the sum to eight terms is 1785. Find the first term.

e A GP has ratio $r = -\frac{1}{2}$ and the sum to eight terms is 425. Find the first term.

11 **a** Each year when the sunflower paddock is weeded, only half the previous weight of weed is dug out. In the first year, 6 tonnes of weed is dug out.

i How much is dug out in the 10th year?

ii What is the total dug out over 10 years (correct to four significant figures)?

b Every two hours, half of a particular medical isotope decays. If there was originally 20 grams, how much remains after a day (correct to two significant figures)?

c The price of Victoria shoes is increasing over a 10-year period by 10% per annum, so that the price in each of those 10 years is P, $1.1 \times P$, $(1.1)^2 \times P$, ... I buy one pair of these shoes each year.

i Find an expression for the total paid over 10 years (correct to the nearest cent).

ii Hence find the initial price P if the total paid is \$900.

12 The number of people attending the yearly Abletown Show is rising by 5% per annum, and the number attending the yearly Bush Creek Show is falling by 5% per annum. In the first year under consideration, 5000 people attended both shows.

a Find the total number attending each show during the first six years.

b Show that the number attending the Abletown Show first exceeds ten times the number attending the Bush Creek Show in the 25th year.

c What is the ratio (correct to three significant figures) of the total number attending the Abletown Show over these 25 years to the total attending the Bush Creek Show?

13 **a** Show that the sum S_n of the first n terms of $7 + 14 + 28 + \cdots$ is $S_n = 7(2^n - 1)$.

b For what value of n is S_n equal to 1785?

c Show that $T_n = 7 \times 2^{n-1}$, and find how many terms are less than 70 000.

d Use trial-and-error to find the first sum S_n that is greater than 70 000.

e Prove that the sum S_n of the first n terms is always 7 less than the $(n + 1)$th term.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 The powers of 3 that are greater than 1 form a GP 3, 9, 27, . . .

a Find using logarithms how many powers of 3 there are between 2 and 10^{20}.

b Show that $S_n = \frac{3}{2}(3^n - 1)$, and find the smallest value of n for which $S_n > 10^{20}$.

15 Find a formula for S_n, and hence find n for the given value of S_n.

a $5 + 10 + 20 + \cdots$ where $S_n = 315$

b $5 - 10 + 20 - \cdots$ where $S_n = -425$

c $18 + 6 + 2 + \cdots$ where $S_n = 26\frac{8}{9}$

d $48 - 24 + 12 - \cdots$ where $S_n = 32\frac{1}{4}$

16 Find the nth terms of the sequences:

a $\frac{2}{1}, \frac{2+4}{1+3}, \frac{2+4+6}{1+3+5}, \ldots$

b $\frac{1}{1}, \frac{1+2}{1+4}, \frac{1+2+4}{1+4+16}, \ldots$

17 **a** Show that in any GP, $S_{2n} : S_n = (r^n + 1) : 1$. Hence find the common ratio of the GP if $S_{12} : S_6 = 65 : 1$.

b Show that if S_n and Σ_n are the sums to n terms of GPs with ratios r and r^2 respectively, but the same first term, then $\Sigma_n : S_n = (r^n + 1) : (r + 1)$.

c In any GP, let $R_n = T_{n+1} + T_{n+2} + \cdots + T_{2n}$. Show that $R_n : S_n = r^n : 1$, and hence find r if $R_8 : S_8 = 1 : 81$.

ENRICHMENT

18 Given a GP in which $T_1 + T_2 + \cdots + T_{10} = 2$ and $T_{11} + T_{12} + \cdots + T_{30} = 12$, find $T_{31} + T_{32} + \cdots + T_{60}$.

19 Show that the formula for the nth partial sum of a GP can also be written independently of n, in terms only of a, r and the last term $\ell = T_n = ar^{n-1}$, as

$$S_n = \frac{r\ell - a}{r - 1} \quad \text{or} \quad S_n = \frac{a - r\ell}{1 - r}.$$

a Hence find:

i $1 + 2 + 4 + \cdots + 1\,048\,576$

ii $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{2187}$

b Find n and r if $a = 1$, $\ell = 64$, $S_n = 85$.

c Find ℓ and n if $a = 5$, $r = -3$, $S_n = -910$.

20 **a** The sequence $T_n = 2 \times 3^n + 3 \times 2^n$ is the sum of two GPs. Find S_n.

b The sequence $T_n = 2n + 3 + 2^n$ is the sum of an AP and a GP. Use a combination of AP and GP formulae to find S_n.

c It is given that the sequence $10, 19, 34, 61, \ldots$ has the form $T_n = a + nd + b2^n$, for some values of a, d and b. Find these values, and hence find S_n.

21 **a** Show that $1 + 2 + 2^2 + \cdots + 2^n = 2^n - 1$, for all whole numbers n.

b Show that if $2^n - 1$ is prime (called a *Mersenne prime*), then n is prime.

c A whole number is called *perfect* if it equals the sum of all its divisors less than itself.

i Show that $6 = 2 \times 3$ and $28 = 2^2 \times 7$ are perfect numbers.

ii Show that if $p = 2^n - 1$ is prime, where n is a whole number, then $2^{n-1}p$ is perfect.

iii Write down the first five Mersenne primes and the corresponding perfect numbers.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1H The limiting sum of a geometric series

There is a sad story of a perishing frog, dying of thirst only 8 metres from the edge of a waterhole. He first jumps 4 metres towards it, his second jump is 2 metres, then each successive jump is half the previous jump. Does the frog perish?

The jumps form a GP, whose terms T_n and sums S_n are as follows:

T_n	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...
S_n	4	6	7	$7\frac{1}{2}$	$7\frac{3}{4}$	$7\frac{7}{8}$	$7\frac{15}{16}$	...

The successive jumps $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots$ have limit zero, because they are halving each time. It seems too that the successive sums S_n have limit 8, meaning that the frog's total distance gets 'as close as we like' to 8 metres. So provided that the frog can stick his tongue out even the merest fraction of a millimetre, eventually he will get some water to drink and be saved.

The limiting sum of a GP

We can describe all this more precisely by looking at the sum S_n of the first n terms and examining what happens as $n \to \infty$.

The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ is a GP with $a = 4$ and $r = \frac{1}{2}$.
Using the formula for the sum to n terms of the series,

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \qquad \text{(using this formula because } r < 1\text{)} \\ &= \frac{4\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \\ &= 4 \times \left(1 - \left(\tfrac{1}{2}\right)^n\right) \div \tfrac{1}{2} \\ &= 8\left(1 - \left(\tfrac{1}{2}\right)^n\right). \end{aligned}$$

As n increases, the term $\left(\frac{1}{2}\right)^n$ gets progressively closer to zero:

$$\left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}, \quad \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8}, \quad \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}, \quad \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32}, \quad \left(\tfrac{1}{2}\right)^6 = \tfrac{1}{64}, \quad \ldots$$

so that $\left(\frac{1}{2}\right)^n$ has limit zero as $n \to \infty$.

Hence S_n does indeed have limit $8(1 - 0) = 8$, as the table of values suggested.

There are several different common notations and words for this situation:

13 NOTATIONS FOR THE LIMITING SUM

Take as an example the series $4 + 2 + 1 + \frac{1}{2} + \cdots$.

- $S_n \to 8$ as $n \to \infty$. ('S_n has limit 8 as n increases without bound.')
- $\lim\limits_{n \to \infty} S_n = 8$ ('The limit of S_n, as n increases without bound, is 8.')
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ has *limiting sum* $S_\infty = 8$.
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ *converges to the limit* $S_\infty = 8$.
- $4 + 2 + 1 + \frac{1}{2} + \cdots = 8$.

The symbols S_∞ and S are both commonly used for the limiting sum.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The general case

Suppose now that T_n is a GP with first term a and ratio r, so that

$$T_n = ar^{n-1} \qquad \text{and} \qquad S_n = \frac{a(1 - r^n)}{1 - r}.$$

Suppose also that the ratio r lies in the interval $-1 < r < 1$.

Then as $n \to \infty$, the successive powers $r^1, r^2, r^3, r^4, \ldots$ get smaller and smaller,

that is, $\qquad r^n \to 0 \qquad$ and $\qquad 1 - r^n \to 1$.

Thus both the nth term T_n and the sum S_n *converge* to a limit,

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} ar^{n-1} = 0, \qquad \text{and} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}.$$

14 THE LIMITING SUM OF A GEOMETRIC SERIES

- Suppose that $|r| < 1$, that is, $-1 < r < 1$.
  Then $\qquad r^n \to 0$ as $n \to \infty$,
  so the terms and the partial sums of the GP both *converge to a limit*,
  $$\lim_{n\to\infty} T_n = 0 \qquad \text{and} \qquad S_\infty = \lim_{n\to\infty} S_n = \frac{a}{1 - r}.$$
- If $|r| \ge 1$, then the partial sums S_n do not converge to a limit.

Example 30 — 1H

Explain why these series have limiting sums, and find them.

a $18 + 6 + 2 + \cdots$

b $18 - 6 + 2 - \cdots$

SOLUTION

a Here $a = 18$ and $r = \frac{1}{3}$.
Because $-1 < r < 1$, the series converges.

$$S_\infty = \frac{18}{1 - \frac{1}{3}} = 18 \times \tfrac{3}{2} = 27$$

b Here $a = 18$ and $r = -\frac{1}{3}$.
Because $-1 < r < 1$, the series converges.

$$S_\infty = \frac{18}{1 + \frac{1}{3}} = 18 \times \tfrac{3}{4} = 13\tfrac{1}{2}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 31 1H

a For what values of x does the series $1 + (x - 2) + (x - 2)^2 + \cdots$ converge?

b When the series does converge, what is its limiting sum?

SOLUTION

The sequence is a GP with first term $a = 1$ and ratio $r = x - 2$.

a The GP converges when
$$\begin{aligned} -1 &< r < 1 \\ -1 &< x - 2 < 1 \\ \boxed{+\,2} \qquad 1 &< x < 3. \end{aligned}$$

b The limiting sum is then
$$\begin{aligned} S_\infty &= \frac{1}{1 - (x - 2)} \\ &= \frac{1}{3 - x}. \end{aligned}$$

Solving problems involving limiting sums

As always, the first step is to write down in symbolic form everything that is given in the question.

Example 32 1H

Find the ratio of a GP whose first term is 10 and whose limiting sum is 40.

SOLUTION

We know that $S_\infty = 40.$

Using the formula, $\dfrac{a}{1 - r} = 40,$

and substituting $a = 10$ gives
$$\begin{aligned} \frac{10}{1 - r} &= 40 \\ 10 &= 40(1 - r) \\ 1 &= 4 - 4r \\ 4r &= 3 \\ r &= \tfrac{3}{4}. \end{aligned}$$

Sigma notation for infinite sums

When $-1 < r < 1$ and the GP converges, the limiting sum S_∞ can also be written as an infinite sum, either using sigma notation or using dots, so that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad \text{or} \qquad a + ar + ar^2 + \cdots = \frac{a}{1 - r},$$

and we say that 'the series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\dfrac{a}{1 - r}$'.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 1H

FOUNDATION

1 **a** Copy and complete the table of values opposite for the GP with $a = 18$ and $r = \frac{1}{3}$.

n	1	2	3	4	5	6
T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n						

b Find the limiting sum using $S_\infty = \dfrac{a}{1 - r}$.

c Find the difference $S_\infty - S_6$.

2 **a** Copy and complete the table of values opposite for the GP with $a = 24$ and $r = -\frac{1}{2}$.

n	1	2	3	4	5	6
T_n	24	-12	6	-3	$1\frac{1}{2}$	$-\frac{3}{4}$
S_n						

b Find the limiting sum using $S_\infty = \dfrac{a}{1 - r}$.

c Find the difference $S_\infty - S_6$.

3 Identify the first term a and ratio r of each GP and hence find S_∞.

a $8 + 4 + 2 + \cdots$ **b** $-4 - 2 - 1 - \cdots$ **c** $1 - \frac{1}{3} + \frac{1}{9} - \cdots$

d $36 - 12 + 4 - \cdots$ **e** $60 - 30 + 15 - \cdots$ **f** $60 - 12 + 2\frac{2}{5} - \cdots$

4 Find each ratio r to test whether there is a limiting sum. Find the limiting sum if it exists.

a $1 - \frac{1}{2} + \frac{1}{4} - \cdots$ **b** $4 - 6 + 9 - \cdots$ **c** $12 + 4 + \frac{4}{3} + \cdots$

d $1000 + 100 + 10 + \cdots$ **e** $-2 + \frac{2}{5} - \frac{2}{25} + \cdots$ **f** $-\frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \cdots$

5 Bevin dropped the Nelson Bros Bouncy Ball from a height of 8 metres. It bounced continually, each successive height being half of the previous height.

a Show that the first distance travelled down-and-up is 12 metres, and explain why the successive down-and-up distances form a GP with $r = \frac{1}{2}$.

b Through what distance did the ball ‘eventually’ travel?

6 These examples will show that a GP does not have a limiting sum when $r \geq 1$ or $r \leq -1$. Copy and complete the tables for these GPs, then describe the behaviour of S_n as $n \to \infty$.

a $r = 1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

b $r = -1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

c $r = 2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

d $r = -2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

7 For each series, find S_∞ and S_4, then find the difference $S_\infty - S_4$.

a $80 + 40 + 20 + \cdots$ **b** $100 + 10 + 1 + \cdots$ **c** $100 - 80 + 64 - \cdots$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

8 When Brownleigh Council began offering free reflective house numbers to its 10000 home owners, 20% installed them in the first month. The number installing them in the second month was only 20% of those in the first month, and so on.

a Show that the numbers installing them each month form a GP.

b How many home owners will 'eventually' install them? ('Eventually' means take S_∞.)

c How many eventual installations were not done in the first four months?

9 The Wellington Widget Factory has been advertising its unbreakable widgets every month. The first advertisement brought in 1000 sales, but every successive advertisement is only bringing in 90% of the previous month's sales.

a How many widget sales will the advertisements 'eventually' bring in?

b About how many eventual sales were not brought in by the first 10 advertisements?

10 Find the limiting sums if they exist, rationalising denominators where necessary.

a $1 + (1.01) + (1.01)^2 + \cdots$

b $1 + (1.01)^{-1} + (1.01)^{-2} + \cdots$

c $16\sqrt{5} + 4\sqrt{5} + \sqrt{5} + \cdots$

d $7 + \sqrt{7} + 1 + \cdots$

e $4 - 2\sqrt{2} + 2 - \cdots$

f $5 - 2\sqrt{5} + 4 - \cdots$

g $9 + 3\sqrt{10} + 10 + \cdots$

h $1 + (1 - \sqrt{3}) + (1 - \sqrt{3})^2 + \cdots$

11 Expand each series for a few terms. Then write down a and r, and find the limiting sum.

a $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{1}{3}\right)^n$

b $\displaystyle\sum_{n=1}^{\infty} 7 \times \left(\tfrac{1}{2}\right)^n$

c $\displaystyle\sum_{n=1}^{\infty} 40 \times \left(-\tfrac{3}{5}\right)^n$

12 Find, in terms of x, an expression for the limiting sum of the series on the LHS of each equation. Then solve the equation to find x.

a $5 + 5x + 5x^2 + \cdots = 10$

b $5 - 5x + 5x^2 - \cdots = 15$

c $x + \dfrac{x}{3} + \dfrac{x}{9} + \cdots = 2$

d $x - \dfrac{x}{3} + \dfrac{x}{9} - \cdots = 2$

13 **a** Suppose that $a + ar + ar^2 + \cdots$ is a GP with limiting sum. Show that the four sequences $a + ar + ar^2 + \cdots,\ a - ar + ar^2 + \cdots,\ a + ar^2 + ar^4 + \cdots,\ ar + ar^3 + ar^5 + \cdots,$ are all GPs, and that their limiting sums are in the ratio $1 + r : 1 - r : 1 : r$.

b Find the limiting sums of these four GPs, and verify the ratio proven above:

i $48 + 24 + 12 + \cdots$

ii $48 - 24 + 12 + \cdots$

iii $48 + 12 + \cdots$

iv $24 + 6 + \cdots$

14 Find the condition for each GP to have a limiting sum, then find that limiting sum.

a $7 + 7x + 7x^2 + \cdots$

b $2x + 6x^2 + 18x^3 + \cdots$

c $1 + (x - 1) + (x - 1)^2 + \cdots$

d $1 + (1 + x) + (1 + x)^2 + \cdots$

15 Find the condition for each GP to have a limiting sum, then find that limiting sum.

a $1 + (x^2 - 1) + (x^2 - 1)^2 + \cdots$

b $1 + \dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots$

16 **a** Show that a GP has a limiting sum if $0 < 1 - r < 2$.

b By calculating the common ratio, show that there is no GP with first term 8 and limiting sum 2.

c A GP has positive first term a, and has a limiting sum S_∞. Show that $S_\infty > \tfrac{1}{2}a$.

d Find the range of values of the limiting sum of a GP with:

i $a = 6$

ii $a = -8$

iii $a > 0$

iv $a < 0$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 The series $v + v^2 + v^3 + \cdots$ has a limiting sum w.

a Write w in terms of v.

b Find v in terms of w.

c Hence find the limiting sum of the series $w - w^2 + w^3 - \cdots$, assuming that $|w| < 1$.

d Test your results with $v = \frac{1}{3}$.

ENRICHMENT

18 Suppose that $T_n = ar^{n-1}$ is a GP with a limiting sum.

a Find the ratio r if the limiting sum equals 5 times the first term.

b Find the first three terms if the second term is 6 and the limiting sum is 27.

c Find the ratio if the sum of all terms except the first equals 5 times the first term.

d Show that the sum S of all terms from the third on is $\dfrac{ar^2}{1 - r}$.

i Hence find r if S equals the first term.

ii Find r if S equals the second term.

iii Find r if S equals the sum of the first and second terms.

19 The series $4 + 12 + 36 + \cdots$ has no limiting sum because $r > 1$. Nevertheless, substitution into the formula for the limiting sum gives

$$S_\infty = \frac{4}{1 - 3} = -2.$$

Can any meaning be given to this calculation and its result? (Hint: Look at the extension of the series to the *left* of the first term.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1I Recurring decimals and geometric series

It would not have been easy in Chapter 2 of the Year 11 book to convert a recurring decimal back to a fraction. Now, however, we can express a recurring decimal as an infinite GP — its value is the limiting sum of that GP, which is easily expressed as a fraction.

Example 33 1I

Express these recurring decimals as infinite GPs. Then use the formula for the limiting sum to find their values as fractions reduced to lowest terms.

a $0.\dot{2}\dot{7}$ **b** $2.6\dot{4}\dot{5}$

SOLUTION

a Expanding the decimal, $0.\dot{2}\dot{7} = 0.272727\ldots$

$$= 0.27 + 0.0027 + 0.000027 + \cdots$$

This is an infinite GP with first term $a = 0.27$ and ratio $r = 0.01$.

Hence
$$0.\dot{2}\dot{7} = \frac{a}{1 - r}$$
$$= \tfrac{0.27}{0.99}$$
$$= \tfrac{27}{99}$$
$$= \tfrac{3}{11}.$$

b This example is a little more complicated, because the first part is not recurring.

Expanding the decimal, $2.6\dot{4}\dot{5} = 2.645454545\ldots$

$$= 2.6 + (0.045 + 0.00045 + \cdots)$$

This is 2.6 plus an infinite GP with first term $a = 0.045$ and ratio $r = 0.01$.

Hence
$$2.6\dot{4}\dot{5} = 2.6 + \tfrac{0.045}{0.99}$$
$$= \tfrac{26}{10} + \tfrac{45}{990}$$
$$= \tfrac{286}{110} + \tfrac{5}{110}$$
$$= \tfrac{291}{110}.$$

15 EXPRESSING A RECURRING DECIMAL AS A FRACTION

- To convert a recurring decimal as a fraction, write the recurring part as a GP.
- The ratio will be between 0 and 1, so the series will have an infinite sum.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 1I

FOUNDATION

Note: These prime factorisations will be useful in this exercise:

$9 = 3^2$ $999 = 3^3 \times 37$ $99999 = 3^2 \times 41 \times 271$

$99 = 3^2 \times 11$ $9999 = 3^2 \times 11 \times 101$ $999999 = 3^3 \times 7 \times 11 \times 13 \times 37$

1 Write each recurring decimal as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

a $0.\dot{3}$ **b** $0.\dot{1}$ **c** $0.\dot{7}$ **d** $0.\dot{6}$

2 Write each recurring decimal as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

a $0.\dot{2}\dot{7}$ **b** $0.\dot{8}\dot{1}$ **c** $0.\dot{0}\dot{9}$ **d** $0.\dot{1}\dot{2}$

e $0.\dot{7}\dot{8}$ **f** $0.\dot{0}2\dot{7}$ **g** $0.\dot{1}3\dot{5}$ **h** $0.\dot{1}8\dot{5}$

DEVELOPMENT

3 Write each recurring decimal as the sum of an integer or terminating decimal and an infinite GP. Then express it as a fraction in lowest terms.

a $12.\dot{4}$ **b** $7.\dot{8}\dot{1}$ **c** $8.4\dot{6}$ **d** $0.2\dot{3}\dot{6}$

4 **a** Express the repeating decimal $0.\dot{9}$ as an infinite GP, and hence show that it equals 1.

b Express $2.7\dot{9}$ as 2.7 plus an infinite GP, and hence show that it equals 2.8.

5 Use GPs to express these as fractions in lowest terms.

a $0.\dot{0}95\dot{7}$ **b** $0.\dot{2}47\dot{5}$ **c** $0.\dot{2}3076\dot{9}$ **d** $0.\dot{4}2857\dot{1}$

e $0.25\dot{5}\dot{7}$ **f** $1.1\dot{0}3\dot{7}$ **g** $0.0\dot{0}027\dot{1}$ **h** $7.7\dot{7}1428\dot{5}$

ENRICHMENT

6 Last year we proved in Section 2B of the Year 11 book that $\sqrt{2}$ is irrational. Why can we now conclude that when $\sqrt{2}$ is written as a decimal, it is not a recurring decimal?

7 **a** [The periods of recurring decimals]

Let p be any prime other than 2 or 5. Explain why the cycle length of the recurring decimal equal to $1/p$ is n digits, where n is the least power of 10 that has remainder 1 when divided by p.

b Use the factorisations of $10^k - 1$ given at the start of this exercise to predict the periods of the decimal representations of $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{27}$, $\frac{1}{37}$, $\frac{1}{41}$, $\frac{1}{101}$ and $\frac{1}{271}$, then write each as a recurring decimal.

8 **a** Use limiting sums of GPs to prove that $0.46\dot{9} = 0.47$.

b Explain how every recurring decimal with an infinite string of 9s can be written as a terminating decimal.

c Explain how every terminating decimal can be written as a recurring decimal with an infinite string of 9s.

d What qualification was needed in Question **22c** of Exercise 1A?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Two techniques in mental arithmetic: You would have used quite a bit of mental arithmetic in this chapter. Here are some techniques that are well worth knowing and practising to make life easier (and this course does not require them).

9 **Doubling and halving** are easy. This means that when multiplying and dividing with even numbers, we can break down the calculation into smaller pieces that can be done mentally.

a To multiply by an even number, take out the factors of 2, then multiply the resulting odd numbers together, then use doubling to get the final answer.

$$14 \times 24 = 2^4 \times (7 \times 3) = 2^4 \times 21 = 2^3 \times 42 = 2^2 \times 84 = 2 \times 168 = 336$$

b To multiply by a multiple of 5, combine each 5 with a 2 using doubling and halving

$$15 \times 26 = 30 \times 13 = 390$$

$$125 \times 108 = 250 \times 54 = 500 \times 27 = 1000 \times 13\tfrac{1}{2} = 13\,500$$

c To divide by 5 or a multiple of 5, double top and bottom.

$$\frac{62}{5} = \frac{124}{10} = 12.4 \qquad \frac{48}{15} = \frac{96}{30} = 3.2$$

d Some practice — make up your own, and use a calculator to check.

$$11 \times 44,\ 12 \times 77,\ 18 \times 14,\ 14 \times 35,\ 15 \times 21,\ 75 \times 16,\ \frac{85}{5},\ \frac{42}{5},\ \frac{36}{15}$$

10 **The difference of squares** makes multiplying two odd numbers straightforward as long as you know your squares.

$$13 \times 17 = (15 - 2)(15 + 2) = 15^2 - 2^2 = 225 - 4 = 221$$

a To find a square, add and subtract to give a product that can be done easily, then use the difference of squares in reverse. In this example, multiplying by 20 is simple.

$$23^2 = (20 \times 26) + 3^2 = (10 \times 52) + 3^2 = 520 + 9 = 529$$

b Half-integers can easily be squared in this way.

$$\left(8\tfrac{1}{2}\right)^2 = (8 \times 9) + \left(\tfrac{1}{2}\right)^2 = 72\tfrac{1}{4}$$

c Some practice — it is worth learning by heart the squares up to 20^2.

$$9 \times 13,\ 17 \times 23,\ 23 \times 37,\ 17^2,\ 13 \times 21,\ 18^2,\ 17 \times 19,\ 19^2,\ 17 \times 21,\ 41^2,\ 28^2$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 1 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 1 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Write out the first 12 terms of the sequence $50,\ 41,\ 32,\ 23,\ \dots$

- **a** How many positive terms are there?
- **b** How many terms lie between 0 and 40?
- **c** What is the 10th term?
- **d** What number term is -13?
- **e** Is -100 a term in the sequence?
- **f** What is the first term less than -35?

2 The nth term of a sequence is given by $T_n = 58 - 6n$.

- **a** Find the first, 20th, 100th and the 1 000 000th terms.
- **b** Find whether 20, 10, -56 and -100 are terms of the sequence.
- **c** Find the first term less than -200, giving its number and its value.
- **d** Find the last term greater than -600, giving its number and its value.

3 Find the original sequence T_n if its partial sums S_n are:

- **a** the sequence $4,\ 11,\ 18,\ 25,\ 32,\ 39,\ \dots,$
- **b** the sequence $0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \dots,$
- **c** given by $S_n = n^2 + 5$,
- **d** given by $S_n = 3^n$,

4 Evaluate these expressions:

- **a** $\displaystyle\sum_{n=3}^{6} (n^2 - 1)$
- **b** $\displaystyle\sum_{n=-2}^{2} (5n - 3)$
- **c** $\displaystyle\sum_{n=0}^{6} (-1)^n$
- **d** $\displaystyle\sum_{n=1}^{6} \left(\tfrac{1}{2}\right)^n$

5
- **a** Write out the first eight terms of the sequence $T_n = 5 \times (-1)^n$.
- **b** Find the sum of the first seven terms and the sum of the first eight terms.
- **c** How is each term obtained from the previous term?
- **d** What are the 20th, 75th and 111th terms?

6 Test each sequence to see whether it is an AP, a GP or neither. State the common difference of any AP and the common ratio of any GP.

- **a** $76,\ 83,\ 90,\ \dots$
- **b** $100,\ -21,\ -142,\ \dots$
- **c** $1,\ 4,\ 9,\ \dots$
- **d** $6,\ 18,\ 54,\ \dots$
- **e** $6,\ 10,\ 15,\ \dots$
- **f** $48,\ -24,\ 12,\ \dots$

7
- **a** State the first term and common difference of the AP $23,\ 35,\ 47,\ \dots$
- **b** Use the formula $T_n = a + (n - 1)d$ to find the 20th term and the 600th term.
- **c** Show that the formula for the nth term is $T_n = 11 + 12n$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d Hence find whether 143 and 173 are terms of the sequence.

e Hence find the first term greater than 1000 and the last term less than 2000.

f Hence find how many terms there are between 1000 and 2000.

8 A shop charges \$20 for one case of soft drink and \$16 for every subsequent case.

a Show that the costs of 1 case, 2 cases, 3 cases, . . . form an AP and state its first term and common difference.

b Hence find a formula for the cost of n cases.

c What is the largest number of cases that I can buy with \$200, and what is my change?

d My neighbour paid \$292 for some cases. How many did he buy?

9 **a** Find the first term and common ratio of the GP $50, \ 100, \ 200, \ \ldots$

b Use the formula $T_n = ar^{n-1}$ to find a formula for the nth term.

c Hence find the eighth term and the twelfth term.

d Find whether 1600 and 4800 are terms of the sequence.

e Find the product of the fourth and fifth terms.

f Use logarithms, or trial-and-error on the calculator, to find how many terms are less than 10 000 000.

10 On the first day that Barry exhibited his paintings, there were 486 visitors. On each subsequent day, there were only a third as many visitors as on the previous day.

a Show that the number of visitors on successive days forms a GP and state the first term and common ratio.

b Write out the terms of the GP until the numbers become absurd.

c For how many days were there at least 10 visitors?

d What was the total number of visitors while the formula was still valid?

e Use the formula $S_\infty = \dfrac{a}{1-r}$ to find the 'eventual' number of visitors if the absurdity of fractional numbers of people were ignored.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

11 Find the second term x of the sequence $15,\ x,\ 135$:

a if the sequence is an AP,

b if the sequence is a GP.

12 Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum of the first 41 terms of each AP.

a $51 + 62 + 73 + \cdots$

b $100 + 75 + 50 + \cdots$

c $-35 - 32 - 29 - \cdots$

13 Use the formula $T_n = a + (n-1)d$ to find the number of terms in each AP, then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum of the series.

a $23 + 27 + 31 + \cdots + 199$

b $200 + 197 + 194 + \cdots - 100$

c $12 + 12\frac{1}{2} + 13 + \cdots + 50$

14 Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ to find the sum of the first 6 terms of each GP.

a $3 + 6 + 12 + \cdots$

b $6 - 18 + 54 - \cdots$

c $-80 - 40 - 20 - \cdots$

15 Find the limiting sum of each GP, if it exists.

a $240 + 48 + 9\frac{3}{5} + \cdots$

b $-6 + 9 - 13\frac{1}{2} + \cdots$

c $-405 + 135 - 45 + \cdots$

16 **a** For what values of x does the GP $(2 + x) + (2 + x)^2 + (2 + x)^3 + \cdots$ have a limiting sum?

b Find a formula for the value of this limiting sum when it does exist.

17 Use the formula for the limiting sum of a GP to express as a fraction:

a $0.\dot{3}\dot{9}$

b $0.\dot{4}6\dot{8}$

c $12.30\dot{4}\dot{5}$

18 **a** The second term of an AP is 21 and the ninth term is 56. Find the 100th term.

b Find the sum of the first 20 terms of an AP with third term 10 and 12th term -89.

c The third term of a GP is 3 and the eighth term is -96. Find the sixth term.

d Find the difference of the AP with first term 1 if the sum of the first 10 terms is -215.

e Find how many terms there are in an AP with first term $4\frac{1}{2}$ and difference -1 if the sum of all the terms is 8.

f Find the common ratio of a GP with first term 60 and limiting sum 45.

g The sum of the first 10 terms of a GP with ratio -2 is 682. Find the fourth term.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 Mathematical induction

Mathematical induction is a method of proof quite different from other methods of proof. It is used for proving theorems that claim that a certain statement is true for integer values of some variable. It is based on recursion, which is why it has been placed immediately after Chapter 1 on series.

Section 2A uses mathematical induction to prove formulae for the sums of series. Section 2B then extends the method to theorems about divisibility. Mathematical induction, however, is used throughout mathematics, and some extra questions at the end of Exercise 2B contain some further examples where it can be applied.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2A Using mathematical induction for series

The logic of mathematical induction is most easily understood when it is applied to sums of series.

In this course, proof by mathematical induction can only be applied after we already have a clear statement of the theorem to be proven. The example below examines a typical situation in which a clear pattern is easily generated, but no obvious explanation emerges for why that pattern occurs.

An example proving a formula for the sum of a series

Find a formula for the sum of the first n cubes, and prove it by mathematical induction.

Some calculations for low values of n: Here is a table of values of the first 10 positive cubes and their partial sums.

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
n^3	1	8	27	64	125	216	343	512	729	1000	$\cdots$
$1^3 + 2^3 + \cdots + n^3$	1	9	36	100	225	441	784	1296	2025	3025	$\cdots$
Form	1^2	3^2	6^2	10^2	15^2	21^2	28^2	36^2	45^2	55^2	$\cdots$

The surprising thing here is that the last row consists of the squares of the *triangular numbers*, where the nth triangular number is the sum of all the positive integers up to n,

$$1 + 2 = 3, \quad 1 + 2 + 3 = 6, \quad 1 + 2 + 3 + 4 = 10, \quad 1 + 2 + 3 + 4 + 5 = 15.$$

Using the formula for the sum of an AP (the number of terms times the average of first and last term), the formula for the nth triangular number is $\frac{1}{2}n(n + 1)$.

Hence the sum of the first n cubes seems to be $\frac{1}{4}n^2(n + 1)^2$.

We have now arrived at a *conjecture*, meaning that we appear to have a true theorem, but we have no clear idea why it is true. We cannot even be sure yet that is true, because showing that a statement is true for the first 10 positive integers is most definitely not a proof that it is true for all integers.

The following worked exercise gives a precise statement and proof of the conjectured result.

Example 1 2A

Prove by mathematical induction that for all integers $n \geq 1$,

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n + 1)^2.$$

The proof below is a typical proof by mathematical induction. Read it carefully, then read the explanation of the proof in the notes below.

SOLUTION

Proof: By mathematical induction.

A When $n = 1$, RHS $= \frac{1}{4} \times 1 \times 2^2$
$= 1$
$=$ LHS,

so the statement is true for $n = 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose $\quad 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2. \qquad (**)$
We prove the statement for $n = k + 1$.
That is, we prove $1^3 + 2^3 + 3^3 + 4^3 + \cdots + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$.

$$\begin{aligned}
\text{LHS} &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 \\
&= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3, \text{ by the induction hypothesis } (**), \\
&= \tfrac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right) \\
&= \tfrac{1}{4}(k+1)^2(k^2 + 4k + 4) \\
&= \tfrac{1}{4}(k+1)^2(k+2)^2 \\
&= \text{RHS}.
\end{aligned}$$

C It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

Notes on the proof

There are three clear parts.

- Part A proves the statement for the starting value, which in this case is $n = 1$.
- Part B is the most complicated, and proves that whenever the statement is true for some integer $k \geq 1$, then it is also true for the next integer $k + 1$.
- Part C concludes by appealing to the principle of mathematical induction.

Any question on proof by mathematical induction is testing your ability to write a coherent account of the proof — you are advised to follow the structure given here.

The first four lines of Part B are particularly important, and these four sentences should be repeated strictly in all proofs. The first and second sentences of Part B set up what is assumed about k, writing down the specific statement for $n = k$, a statement later referred to as ‘the induction hypothesis’. The third and fourth sentences set up specifically what it is that we intend to prove.

Statement of the principle of mathematical induction

With this proof as an example, here is a formal statement of the principle of mathematical induction.

1 MATHEMATICAL INDUCTION

Suppose that a statement is to be proven for all integers n greater than or equal to some starting value n_1. Suppose also that two things have been proven:

A The statement is true for $n = n_1$.

B Whenever the statement is true for some positive integer $k \geq n_1$. then it is also true for the next integer $k + 1$.

Then (part **C**) the statement is true for all integers $n \geq n_1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Mathematical induction is an axiom of mathematics. In formal mathematical logic, even the whole numbers 0, 1, 2, 3, . . . cannot be defined without it, and it allows sentences such as, 'The whole numbers continue for ever' to be made absolutely precise. The final statement **C** in the proof above must never be omitted.

An example proving the formula for the sum of a GP

The next worked example proves the formula for the sum of a GP. We used dots notation . . . when developing the formula, which was perfectly acceptable, but a proof that requires dots depends on mathematical induction for its validity.

Example 2 — 2A

Prove that for all real numbers a and $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, \text{ for all integers } n \geq 1.$$

SOLUTION

The setting-out is almost identical to the previous example.

Proof: By mathematical induction.

A When $n = 1$,
$$\begin{aligned} \text{RHS} &= \frac{a(r^1 - 1)}{r - 1} \\ &= a \\ &= \text{LHS}, \end{aligned}$$
so the statement is true for $n = 1$.

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\quad a + ar + ar^2 + \cdots + ar^{k-1} = \dfrac{a(r^k - 1)}{r - 1}. \qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $\quad a + ar + ar^2 + \cdots + ar^k = \dfrac{a(r^{k+1} - 1)}{r - 1}.$

$$\begin{aligned} \text{LHS} &= a + ar + ar^2 + \cdots + ar^{k-1} + ar^k \\ &= \frac{a(r^k - 1)}{r - 1} + ar^k, \text{ by the induction hypothesis } (**), \\ &= \frac{ar^k - a}{r - 1} + \frac{ar^{k+1} - ar^k}{r - 1} \\ &= \frac{ar^{k+1} - a}{r - 1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} \\ &= \text{RHS}. \end{aligned}$$

C It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

Note: The original proof of the formula in Section 1G using the dots . . . was much clearer intuitively. It is very common that the proof by mathematical induction does not display the intuitive idea nearly as well.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 2A

FOUNDATION

1 Copy and complete the proof by mathematical induction that for all integers $n \geq 1$,
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

A When $n = 1$, RHS $= \ldots.$
$= \text{LHS},$
so the statement is true for . . . .

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose . . . $(**)$
We prove the statement for $n = k + 1$.
That is, we prove . . .
LHS $= \ldots$, by the induction hypothesis $(**)$,
$= \ldots$
$= \text{RHS}.$

C It follows from parts **A** and **B** by mathematical induction that . . . .

2 Prove by mathematical induction that for all positive integer values of n:

a $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$

b $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

c $1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$

d $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

e $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$

f $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

g $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1)$

h $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

i $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$

3 What are the limiting sums of the series in parts **h** and **i** of the previous question?

DEVELOPMENT

4 Prove by mathematical induction that for all positive integers $n \geq 1$:

a $1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2(n + 1) = \frac{1}{12}n(n + 1)(n + 2)(3n + 1)$

b $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n + 1)^2 = \frac{1}{12}n(n + 1)(n + 2)(3n + 5)$

c $2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (n + 1) \times 2^{n-1} = n \times 2^n$

5 Prove by mathematical induction that for all positive integer values of n:

a $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n + 1)! - 1$

b $2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (n^2 + 1)n! = n(n + 1)!$

c $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n + 1)!} = 1 - \dfrac{1}{(n + 1)!}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 **a** Suppose that the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2 + 2$ is true for the positive integer $n = k$. Prove that it is also true for $n = k + 1$.

b Explain why we cannot conclude that the statement is true for all integers $n \ge 1$.

7 **a** Attempt to prove by mathematical induction that $3 + 6 + 9 + \cdots + 3n = n(n + 1) + 1$ for all positive integer values of n.

b Where does the proof break down?

8 **a** Use the factor theorem to show that $n + 1$ is a factor of $P(n) = 4n^3 + 18n^2 + 23n + 9$, and hence factor $P(n)$.

b Hence prove by mathematical induction that for all integers $n \ge 1$,

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n - 1)(2n + 1) = \tfrac{1}{3}n(4n^2 + 6n - 1).$$

9 Prove by mathematical induction that for all integers $n \ge 1$,

$$1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + n) = \tfrac{1}{6}n(n + 1)(n + 2).$$

(Hint: Use Question **2** part **a**.)

10 Prove by mathematical induction that for all positive integers n,

$$(n + 1)(n + 2)(n + 3) \times \cdots \times 2n = 2^n\big(1 \times 3 \times 5 \times \cdots \times (2n - 1)\big).$$

11 Prove by mathematical induction that for all positive integer values of n:

a $\displaystyle\sum_{r=1}^{n}(r^3 - r) = \tfrac{1}{4}(n - 1)(n)(n + 1)(n + 2)$

b $\displaystyle\sum_{r=1}^{n}\left(3r^5 + r^3\right) = \tfrac{1}{2}n^3(n + 1)^3$

c $\displaystyle\sum_{r=1}^{n} r^2 \times 2^r = (n^2 - 2n + 3) \times 2^{n+1} - 6$

ENRICHMENT

12 Let $H(n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Use mathematical induction to prove that for all positive integers $n \ge 1$,

$$n + H(1) + H(2) + H(3) + \cdots + H(n - 1) = nH(n).$$

13 **a** Prove the trigonometric identity $\dfrac{\cos\alpha - \cos(\alpha + 2\beta)}{2\sin\beta} = \sin(\alpha + \beta)$.

b Hence prove by mathematical induction that for all integers $n \ge 1$,

$$\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n - 1)\theta = \frac{1 - \cos 2n\theta}{2\sin\theta}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2B Proving divisibility by mathematical induction

Divisibility is another standard situation where mathematical induction can be used to prove a result. The example below again uses low values of n to establish a hypothesis, and then proves that hypothesis using mathematical induction.

An example of proving divisibility

Find the largest integer that is a divisor of $3^{4n} - 1$ for all integers $n \geq 0$. Then prove the result by mathematical induction.

Some calculations for low values of n: Here is a table using just the first four values of n, starting this time at $n = 0$,

n	0	1	2	3	$\cdots$
$3^{4n} - 1$	0	80	6560	531440	$\cdots$

It seems likely from this that 80 is a divisor of all the numbers — remember that every number is a divisor of zero. Certainly no number greater than 80 can be a divisor of all of them. So we write down the likely theorem and try to provide a proof. Various proofs are available, but here is the proof by mathematical induction:

Example 3 — 2B

Prove by mathematical induction that for all integers $n \geq 0$,
$3^{4n} - 1$ is divisible by 80.

SOLUTION

The key step in all divisibility proofs is the introduction of an extra pronumeral in the second line of part **B**. In this case we have used the letter m.

Proof: By mathematical induction

A The starting value here is $n = 0$, not $n = 1$ as before.
When $n = 0$, $3^{4n} - 1 = 0$, which is divisible by 80 (and by every number), so the statement is true for $n = 0$.

B Suppose that $k \geq 0$ is an integer for which the statement is true.
That is, suppose $3^{4k} - 1 = 80m$, for some integer m. $(**)$
We prove the statement for $n = k + 1$.
That is, we prove $3^{4k+4} - 1$ is divisible by 80.

$$\begin{aligned} 3^{4k+4} - 1 &= 3^{4k} \times 3^4 - 1 \\ &= (80m + 1) \times 81 - 1, \text{ by the induction hypothesis } (**), \\ &= 80 \times 81m + 81 - 1 \\ &= 80m \times 81 + 80 \\ &= 80(81m + 1), \text{ which is divisible by 80, as required.} \end{aligned}$$

C It follows from parts **A** and **B** by mathematical induction that the statement is true for all whole numbers n.

Note: In the second sentence of part **B**, the induction hypothesis $(**)$ has interpreted divisibility by 80 as being $80m$ where m is an integer. In the fourth sentence of Part **B**, however, the statement of what is to be proven does not interpret divisibility at all. Proofs of divisibility work more easily this way.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Further remarks on mathematical induction

Divisibility and summing a series and are classic places where proof by mathematical induction is used. The method, however, is used routinely throughout all branches of mathematics. Some extra questions at the end of Exercise 2B give some applications in geometry, combinatorics and calculus. In each case, the structure and words given in the examples above should be followed.

Exercise 2B

FOUNDATION

1 Copy and complete this proof that $7^n - 1$ is divisible by 6, for all positive integers n.

A When $n = 1$, $7^n - 1 = \ldots$
so the statement is true for $n = 1$.

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose . . . $(**)$
We prove the statement for $n = k + 1$.
That is, we prove . . .

$$\begin{aligned} 7^{k+1} - 1 &= \ldots \\ &= \ldots, \text{ by the induction hypothesis } (**), \\ &= \ldots, \text{ which is divisible by 6, as required.} \end{aligned}$$

C It follows from parts **A** and **B** by mathematical induction that the statement is true for . . .

2 Prove by mathematical induction that for all integers $n \geq 1$:

a $5^n - 1$ is divisible by 4,
b $9^n + 3$ is divisible by 6,
c $3^{2n} + 7$ is divisible by 8,
d $5^{2n} - 1$ is divisible by 24.

DEVELOPMENT

3 **a** Copy and complete the table of values to the right. Then make a conjecture about the largest number that $11^n - 1$ is divisible by, for all integers $n \geq 0$.

n	0	1	2	3	4
$11^n - 1$					

b Prove your conjecture by mathematical induction.

4 Prove by mathematical induction that for all integers $n \geq 0$:

a $n^3 + 2n$ is divisible by 3,
b $8^n - 7n + 6$ is divisible by 7,
c $9(9^n - 1) - 8n$ is divisible by 64.

5 Prove by mathematical induction that for all integers $n \geq 0$:

a $5^n + 2 \times 11^n$ is divisible by 3,
b $3^{3n} + 2^{n+2}$ is divisible by 5,
c $11^{n+2} + 12^{2n+1}$ is divisible by 133.

6 Prove by mathematical induction that $x - 1$ is a factor of $x^n - 1$ for all integers $n \geq 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 Show that for all whole numbers k, if $8k^2 + 14$ is divisible by 4, then $8(k+1)^2 + 14$ is also divisible by 4. Show, however, that $8n^2 + 14$ is never divisible by 4 if n is a whole number. Which step of proof by induction does this counter-example show is necessary?

8 **a** Show that $f(n) = n^2 - n + 17$ is prime for $n = 0, 1, 2, \ldots, 16$. Show, however, that $f(17)$ is not prime. Which step of proof by induction does this counter-example show is necessary?

b Begin to show that $f(n) = n^2 + n + 41$ is prime for $n = 0, 1, 2, \ldots, 39$ but not for 40.

Note: There is no formula for generating prime numbers — these two quadratics are interesting because of the long unbroken sequences of primes that they produce.

ENRICHMENT

9 Prove by mathematical induction that $3^{2^n} - 1$ is divisible by 2^{n+1} for all integers $n \ge 0$. (Note that 3^{2^n} means 3 to the power of 2^n.)

Further examples of theorems proven using mathematical induction

10 [Geometry]
Prove by mathematical induction that the sum of the angles of a convex polygon with $n \ge 3$ sides is $n - 2$ straight angles.
(Hint: In step B, dissect the $(k+1)$-gon into a k-gon and a triangle.)

11 [Combinatorics]
Prove by mathematical induction that every n-member set has 2^n subsets.
(Hint: In step B, when a new member is added to a k-member set, every subset of the resulting $(k+1)$-member set either contains the new member, or does not contain it.)

12 [Calculus]
Use mathematical induction, combined with the product rule, to prove that

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ for all integers } n \ge 1.$$

(Hint: In step B, write x^{k+1} as $x \times x^k$, then apply the product rule.)

13 [The induction step can skip through the integers]
Prove these divisibility results:
a $n^2 + 2n$ is a multiple of 8, for all even integers $n \ge 0$.
b $3^n + 7^n$ is divisible by 10, for all odd integers $n \ge 1$.
(Hint: In step B of each proof, advance from k to $k + 2$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 2 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 2 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Prove by mathematical induction that for all positive integer values of n:

a $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

b $1 + 7 + 7^2 + \cdots + 7^{n-1} = \frac{1}{6}(7^n - 1)$

c $1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n + 4) = \frac{1}{6}n(n + 1)(2n + 13)$

d $\dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \dfrac{1}{4 \times 5} + \cdots + \dfrac{1}{(n + 1)(n + 2)} = \dfrac{n}{2(n + 2)}$

e $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{n + 2}{2^n}$

2 Prove these results by mathematical induction:

a $7^{2n-1} + 5$ is divisible by 12, for all integers $n \geq 1$,

b $2^{2n} + 6n - 1$ is divisible by 9, for all integers $n \geq 0$,

c $2^{2n+2} + 5^{2n-1}$ is divisible by 21, for all integers $n \geq 1$,

d $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9, for all integers $n \geq 0$.

3 **a** Copy and complete the table of values to the right. Then make a conjecture about the largest number that $2^{3n} - 3^n$ is divisible by, for all whole numbers $n \geq 0$.

n	0	1	2	3
$2^{3n} - 3^n$				

b Prove your conjecture by mathematical induction.

4 Prove by mathematical induction that for all positive integer values of n:

a $\displaystyle\sum_{r=1}^{n} r \times r! = (n + 1)! - 1$ **b** $\displaystyle\sum_{r=1}^{n} \frac{r - 1}{r!} = 1 - \frac{1}{n!}$

What is the limiting sum of the series in part **b**?

5 Prove that $1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$, for all integers $n \geq 1$.

(Hint: Use the factorisation $6k^3 + 15k^2 + 11k + 2 = (k + 1)(6k^2 + 9k + 2)$.)

6 Prove that $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$, for all integers $n \geq 1$.
(Hint: Use the factorisation $2k^4 + 8k^3 + 11k^2 + 6k + 1 = (k + 1)^2(2k^2 + 4k + 1)$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Graphs and equations

This chapter completes non-calculus curve-sketching and the use of graphs to solve equations and inequations. Some material in the early sections and at the start of transformations is review, particularly of material covered in Chapter 5 of the Year 11 book, although the context is different. Readers who are confident of the earlier material may choose to be selective about the questions they attempt.

Sections 3A–3C discuss four approaches to curve-sketching — domain, odd and even symmetry, zeroes and sign, and asymptotes — and combine them into an informal menu for sketching any curve given its equation. This prepares the ground for Chapter 4, where the methods of calculus will be added to gain more information about the graph. Some necessary alternative notation for intervals and for composition of functions is introduced in Section 3A.

Section 3A also emphasises the role of the graph in solving inequations, and Sections 3D–3E develop graphical methods further to help solve various equations and inequations. Section 3F uses inequalities in two variables to specify regions in the coordinate plane.

Sections 3G–3I review transformations, and add stretching (or dilation) to the list of available transformations. The problem of combining two or more transformations is addressed, including a crucial and difficult question, 'Does it matter in which order the transformations are applied?'

Trigonometric graphs, in particular, benefit very greatly from a systematic approach using translations and dilations, because the amplitude, period, phase and mean value all depend on them. The final Section 3J deals with these graphs.

Whether explicitly suggested in an exercise or not, graphing software in any form is always very useful in confirming results and improving approximations. It is particularly suited to investigating what happens when changes are made to a function's equation.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3A The sign of a function

The main purpose of this section is to review briefly the method of finding the sign of a function by using a table of signs with test values that dodge around zeroes and discontinuities.

This algorithm has the same two purposes as the whole chapter. First, when sketching a curve, we usually want to know very early where the curve is above the x-axis and where it is below. Secondly, solving an inequation is equivalent to finding the sign of a function, because we can always put all the terms on the left. For example,

$$x^3 + 1 \ge x^2 + x \qquad \text{can be written as} \qquad x^3 - x^2 - x + 1 \ge 0.$$

Before all this, however, two extra pieces of notation need to be introduced.

- Bracket interval notation is an alternative to inequality interval notation.
- Composition of functions has an alternative notation $f \circ g$.

Bracket interval notation

There is an alternative notation for intervals that will make the notation in this section and later a little more concise. The notation encloses the endpoints of the interval in brackets, using a square bracket if the endpoint is included and a round bracket if the endpoint is not included.

Here are the five examples from the Year 11 book written in both notations:

Diagram	Using inequalities	Using brackets
	$\frac{1}{3} \le x \le 3$	$\left[\frac{1}{3}, 3\right]$
Read this as, 'The closed interval from $\frac{1}{3}$ to 3'.		
	$-1 < x < 5$	$(-1, 5)$
Read this as, 'The open interval from -1 to 5'.		
	$-2 \le x < 3$	$[-2, 3)$
Read this as, 'The interval from -2 to 3, including -2 but excluding 3'.		
	$x \ge -5$	$[-5, \infty)$
Read this as, 'The closed ray from -5 to the right'.		
	$x < 2$	$(-\infty, 2)$
Read this as, 'The open ray from 2 to the left'.		

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The first interval $\left[\frac{1}{3}, 3\right]$ is *closed*, meaning that it contains all its endpoints.

The second interval $(-1, 5)$ is *open*, meaning that it does not contain any of its endpoints.

The third interval $[-2, 3)$ is neither open nor closed — it contains one of its endpoints, but does not contain the other endpoint.

The fourth interval $[-5, \infty)$ is *unbounded on the right*, meaning that it continues towards infinity. It only has one endpoint -5, which it contains, so it is closed.

The fifth interval $(-\infty, 2)$ is *unbounded on the left*, meaning that it continues towards negative infinity. It only has one endpoint 2, which it does not contain, so it is open.

'Infinity' and 'negative infinity', with their symbols ∞ and $-\infty$, are not numbers. They are ideas used in specific situations and phrases to make language and notation more concise. Here, they indicate that an interval is unbounded on the left or right, and the symbol '$(-\infty, 2)$' means 'all real numbers less than 2'.

Bracket interval notation has some details that need attention.

- The variable x or y or whatever is missing. This can be confusing when we are talking about domain and range, or solving an inequation for some variable. When, however, we are just thinking about 'all real numbers greater than 100', no variable is involved, so the notation $(100, \infty)$ is more satisfactory than $x > 100$.
- The notation can be dangerously ambiguous. For example, the open interval $(-1, 5)$ can easily be confused with the point $(-1, 5)$ in the coordinate plane.
- Infinity and negative infinity are not numbers, as remarked above.
- The set $\mathbb{R}$ of all real numbers can be written as $(-\infty, \infty)$.
- The notation $[4, 4]$ is the one-member set $\{4\}$, called a *degenerate interval* because it has length zero.
- Notations such as $(4, 4)$, $(4, 4]$, $[7, 3]$ and $[7, 3)$ all suggest the empty set, if they mean anything at all, and should be avoided in this course.

1 BRACKET INTERVAL NOTATION

- A square bracket means that the endpoint is included, and a round bracket means that the endpoint is not included.
- For $a < b$, we can form the four *bounded intervals* below. The first is closed, the last is open, and the other two are neither open nor closed.

$$[a, b] \qquad \text{and} \qquad [a, b) \qquad \text{and} \qquad (a, b] \qquad \text{and} \qquad (a, b).$$

- For any real number a, we can form the four *unbounded intervals* below. The first two are closed, and the last two are open.

$$[a, \infty) \qquad \text{and} \qquad (-\infty, a] \qquad \text{and} \qquad (a, \infty) \qquad \text{and} \qquad (-\infty, a).$$

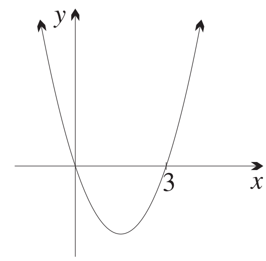
- The notation $(-\infty, \infty)$ means the whole real number line $\mathbb{R}$.
- The notation $[a, a]$ is the one-member set $\{a\}$, called a *degenerate interval*.
- An interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

For those who enjoy precision, the interval $(-\infty, \infty)$ is both open and closed (it has no endpoints), and a degenerate interval $[a, a]$ is closed.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The union of intervals



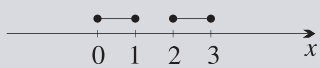
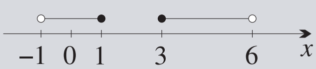
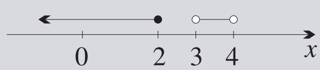
The graph to the right is the quadratic $y = x(x-3)$. From the graph, we can see that the inequation

$$x(x-3) \ge 0 \qquad \text{has solution} \qquad x \le 0 \ \text{ or } \ x \ge 3.$$

This set is the *union* of the two intervals $(-\infty, 0]$ and $[3, \infty)$, so when using bracket interval notation, we write the set as

$$(-\infty, 0] \cup [3, \infty).$$

Here are some further examples using both types of interval notation. The close association between the word 'or' and the union of sets was discussed in Sections 12C and 12D of the Year 11 book in the context of probability.

Diagram	Using inequalities	Using brackets
	$0 \le x \le 1$ or $2 \le x \le 3$	$[0, 1] \cup [2, 3]$
	$-1 < x \le 1$ or $3 \le x < 6$	$(-1, 1] \cup [3, 6)$
	$x \le 2$ or $3 < x < 4$	$(-\infty, 2] \cup (3, 4)$

Some alternative notation for composite functions:

If $f(x)$ and $g(x)$ are two functions, the composite $g\big(f(x)\big)$ of two function $f(x)$ and $g(x)$ can also be written as $g \circ f(x)$. Thus

$$g \circ f(x) = g\big(f(x)\big) \text{ for all } x \text{ for which } f(x) \text{ and } g\big(f(x)\big) \text{ are defined.}$$

The advantage of this notation is that the composite function $g\big(f(x)\big)$ has a clear symbol $g \circ f$ that displays the composition of functions as a binary operator $\circ$ on the set of functions, with notation analogous to addition $a + b$, which is a binary operator on the set of numbers.

Be careful, however, when calculating $g \circ f(2)$, to apply the function f before the function g, because $g \circ f(x)$ means $g\big(f(x)\big)$. Section 4E of the Year 11 book developed composite functions in some detail, and Exercise 3A contains only a few mostly computational questions as practice of the new notation.

The composite $g \circ f(x)$ is often written with extra brackets as $(g \circ f)(x)$, and readers may prefer to add these extra brackets.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 1 — 3A

If $f(x) = x + 3$ and $g(x) = x^2$, find:

a **i** $g \circ f(5)$ **ii** $f \circ g(5)$ **iii** $g \circ g(5)$ **iv** $f \circ f(5)$

b **i** $g \circ f(x)$ **ii** $f \circ g(x)$ **iii** $g \circ g(x)$ **iv** $f \circ f(x)$

SOLUTION

a **i** $g \circ f(5) = g(8) = 64$

ii $f \circ g(5) = f(25) = 28$

iii $g \circ g(5) = g(25) = 625$

iv $f \circ f(5) = f(8) = 11$

b **i** $g \circ f(x) = g(x + 3) = (x + 3)^2$

ii $f \circ g(x) = f(x^2) = x^2 + 3$

iii $g \circ g(x) = g(x^2) = x^4$

iv $f \circ f(x) = f(x + 3) = x + 6$

Finding the sign of a function using a table of signs

This algorithm was introduced in the Year 11 book — in the context of polynomials in Sections 3G and 10B, and for more general functions in Sections 5B — and need only be summarised here.

2 FINDING THE SIGN OF A FUNCTION USING A TABLE OF SIGNS

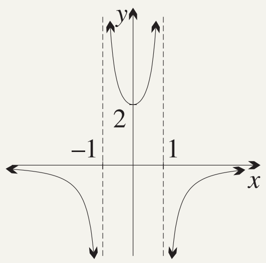
- The sign of a function tells us where its graph is above and below the x-axis.
- A function can only change sign (but might not) at a zero or a discontinuity.
- To examine the sign of a function, draw up a *table of signs*. This is a table of test values that dodge around any zeroes and discontinuities.
- When all the terms of an inequation are moved to the left, solving the inequation is equivalent to finding the sign of the LHS.

The worked examples below do not include polynomials, which have already been covered extensively.

Example 2 — 3A

Solve $f(x) < 0$ using the graph of $y = f(x)$ drawn to the right.

Write the answer using both interval notations.



SOLUTION

The solutions are where the curve is below the x-axis.

That is $x < -1$ or $x > 1$,

or alternatively, $(-\infty, -1) \cup (1, \infty)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 3 — 3A

Solve the inequality $\dfrac{3}{2^x - 1} \le 1$ by moving everything to the left and constructing a table of signs. Give the answer in both notations.

SOLUTION

Moving 1 to the left, $\dfrac{3}{2^x - 1} - \dfrac{2^x - 1}{2^x - 1} \le 0$

$$\frac{4 - 2^x}{2^x - 1} \le 0$$

We now draw up a table of signs for $y = \dfrac{4 - 2^x}{2^x - 1}$.

This function has a zero at $x = 2$ and a discontinuity at $x = 0$.

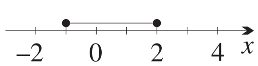
x	-1	0	1	2	3
y	-7	*	2	0	$-\frac{4}{7}$
sign	$-$	*	$+$	0	$-$

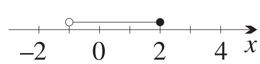
Hence the solution is $x < 0$ or $x \ge 2$, or alternatively $(-\infty, 0) \cup [2, \infty)$.

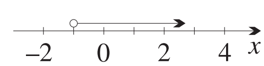
Exercise 3A — FOUNDATION

1 For each number line, write the graphed interval using:

i inequality interval notation, **ii** bracket interval notation.

a 

b 

c 

2 For each interval given by inequality interval notation:

i draw the interval on a number line, **ii** write it using bracket interval notation.

a $-1 \le x < 2$ **b** $x \le 2$ **c** $x < 2$

3 For each interval given by bracket interval notation:

i draw the interval on a number line, **ii** write it using inequality interval notation.

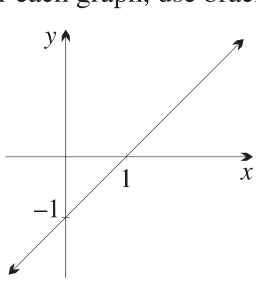
a $[-1, \infty)$ **b** $(-1, 2)$ **c** $(-\infty, \infty)$

4 If $f(x) = x + 1$ and $g(x) = 2^x$, find:

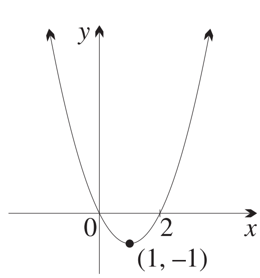
a **i** $g \circ f(3)$ **ii** $f \circ g(3)$ **iii** $g \circ g(3)$ **iv** $f \circ f(3)$

b **i** $g \circ f(x)$ **ii** $f \circ g(x)$ **iii** $g \circ g(x)$ **iv** $f \circ f(x)$

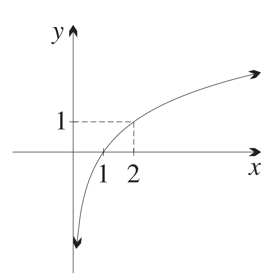
5 For each graph, use bracket interval notation to state where the function is negative.

a 

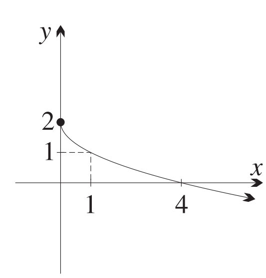
$f(x) = x - 1$

b 

$f(x) = x^2 - 2x$

c 

$f(x) = \log_2 x$

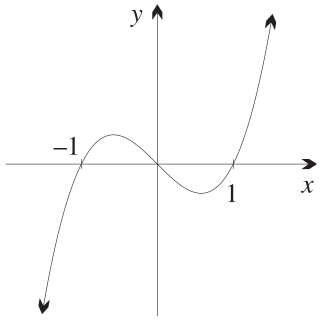
d 

$f(x) = 2 - \sqrt{x}$

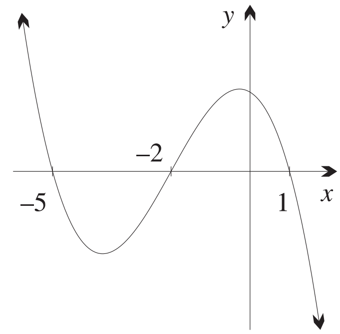
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

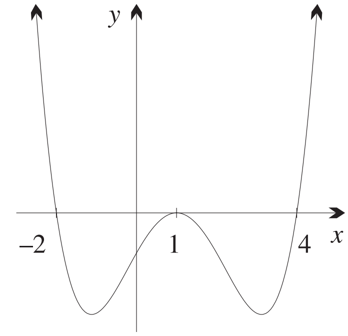
6 Use the given graph of the LHS to help solve each inequation.

a


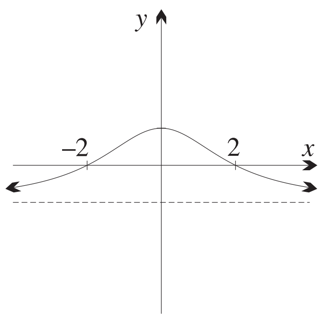
$x(x-1)(x+1) \geq 0$

b


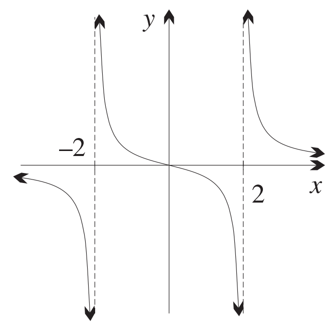
$(1-x)(x+2)(x+5) \leq 0$

c


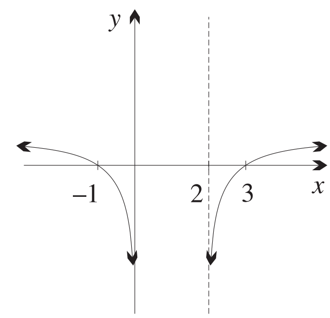
$(x-1)^2(x-4)(x+2) > 0$

d


$\dfrac{4-x^2}{4+x^2} \geq 0$

e


$\dfrac{x}{x^2-4} < 0$

f


$\log_e\left(\dfrac{2x(x-2)}{x^2-2x+3}\right) \leq 0$

7 Find the natural domain of each function.

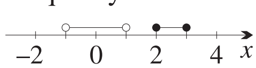
a $f(x) = \dfrac{1}{2x+3}$ **b** $g(x) = \sqrt{2-x}$ **c** $h(x) = \dfrac{e^x + e^{-x}}{2}$

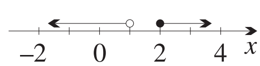
d $a(x) = \log_e(x+1)$ **e** $b(x) = \dfrac{1}{\sqrt{x+3}}$ **f** $c(x) = \log_e(x^2+2x+3)$

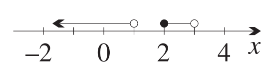
DEVELOPMENT

8 For each number line, write the graphed compound interval using:

i inequality interval notation, **ii** bracket interval notation.

a


b


c


9 For each compound interval given by inequality interval notation:

i draw the number line graph, **ii** write it using bracket interval notation.

a $x = -1$ or $x \geq 2$ **b** $x \leq -1$ or $2 < x \leq 3$ **c** $-1 < x \leq 1$ or $x > 2$

10 For each compound interval given by bracket interval notation:

i draw the number line graph, **ii** write it using inequality interval notation.

a $[-1, 1] \cup [2, \infty)$ **b** $[-1, 1) \cup (2, 3]$ **c** $(-1, 1] \cup [3, 3]$

11 Re-write the solutions to Question **6** using bracket interval notation.

12 For each inequation, find the zeroes and discontinuities of the function on the left-hand side. Then use a table of signs to solve the inequation.

a $\dfrac{x-2}{x+1} \geq 0$ **b** $\dfrac{x-1}{x^2-2x-3} \geq 0$ **c** $\dfrac{x^2+2x+1}{x-2} < 0$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 Write the natural domain of each function using bracket interval notation.

a $f(x) = \dfrac{\sqrt{x}}{x^2 - 1}$

b $f(x) = \dfrac{1}{\sqrt{x^2 - 5x - 6}}$

c $f(x) = \dfrac{1}{\sqrt{3 + 2x - x^2}}$

d $f(x) = \dfrac{x - 1}{\sqrt{x^2 - 2x + 3}}$

14 Let $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$

a Carefully sketch $y = f(x)$.

b Hence solve $f(x) \geq 0$.

15 Let $h(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

a Find the domain of $h(x)$. Write your answer using inequality interval notation.

b Determine $h'(x)$ and hence show that $h'(x) < 0$ for all values of x in its domain.

16 **a** Consider the function $y = \dfrac{|x|}{x - 1}$.

i Determine the natural domain.

ii Find any intercepts with the axes.

iii Determine where the function is positive or zero.

iv Use appropriate graphing software or applications to confirm your answers.

b Repeat the steps of part **a** for $y = \dfrac{|x|}{x^2 - 1}$.

17 **a** Let $f(x) = \sin\left(x + \dfrac{\pi}{3}\right)$, $g(x) = e^x$ and $h(x) = 1 - x^2$. Show that the composition of these three functions is associative. That is, show that

$$\big((f \circ g) \circ h\big)(x) = \big(f \circ (g \circ h)\big)(x)$$

b Prove that composition of functions is always associative. That is, prove that

$$\big((f \circ g) \circ h\big)(x) = \big(f \circ (g \circ h)\big)(x)$$

for all x where both sides are defined, regardless of the choice of functions.

18 An interval is *closed* if it contains all of its endpoints, and *open* if it does not contain any of its endpoints.

a Explain why the degenerate interval $[5, 5]$ is closed.

b Explain why the interval $(-\infty, \infty)$ is open.

c Explain why the interval $(-\infty, \infty)$ is closed.

ENRICHMENT

19 Let $f(x) = 1 + x + x^2 + x^3 + \cdots + x^{2n+1}$.

a Show that the only solution of $f(x) = 0$ is $x = -1$.

b Show that $(x + 1)^2$ is a factor of $f'(x) - (n + 1)x^{2n}$.

c Hence show that a tangent to $y = f(x)$ is never horizontal.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3B Vertical and horizontal asymptotes

So far in this course we have discussed three steps in sketching an unknown function (leaving transformations aside for the moment). After factoring:

1. Identify the domain.
2. Test whether the function has even or odd symmetry or neither.
3. Identify the zeroes and discontinuities and draw up a table of signs.

This section introduces a fourth step:

4. Identify a curve's vertical and horizontal asymptotes.

This may also involve describing the curve's behaviour near them.

Vertical asymptotes were discussed at length in Section 5C of the Year 11 book, and are only summarised here. Finding horizontal asymptotes, however, needs some further techniques.

Vertical asymptotes

It is usually best to draw up a table of signs first so that the behaviour near any vertical asymptote can be quickly identified.

3 TESTING FOR VERTICAL ASYMPTOTES

Always factor the function first as far as possible.

- If the denominator has a zero at $x = a$, and the numerator is *not* zero at $x = a$, then the vertical line $x = a$ is an asymptote.
- The choice between $y \to \infty$ and $y \to -\infty$ can be made by looking at a table of signs.

Once vertical asymptotes have been identified, the behaviour of the curve near them can then be seen from the table of signs and described using the notation $x \to a^+$ and $x \to a^-$.

Be careful. The function $y = \dfrac{x^2 - 4}{x - 2}$ is $y = x + 2$, for $x \neq 2$. Its graph is the line $y = x - 2$ with the point $(2, 4)$ removed. There is a discontinuity at $x = 2$, but no asymptote there — the equation has a zero at $x = 2$ in the numerator as well as in the denominator.

Example 4 3B

Consider the function $f(x) = \dfrac{-2}{x + 2}$.

a Write down the domain, using both interval notations.
b Test whether the function is odd or even or neither.
c Find any zeroes and discontinuities and construct a table of signs.
d Find the vertical asymptote, and describe the behaviour near it.
e Find the horizontal asymptote, and describe the behaviour near it.
f Sketch the curve.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a The domain is $x \neq -2$. In bracket interval notation, this is $(-\infty, -2) \cup (-2, \infty)$.

b $f(-x) = \dfrac{-2}{-x + 2}$, which is neither $f(x)$ nor $-f(x)$, so $f(x)$ is neither even nor odd.

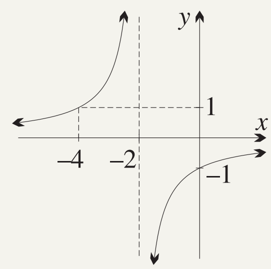
c There are no zeroes, and there is a discontinuity at $x = -2$.

x	-4	-2	0
$f(x)$	1	$*$	-1
sign	$+$	$*$	$-$

d At $x = -2$, the bottom is zero, but the top is non-zero,
so $x = 2$ is a vertical asymptote.
From the table of signs, $f(x) \to \infty$ as $x \to (-2)^-$,
and $f(x) \to -\infty$ as $x \to (-2)^+$.

e $f(x) \to 0$ as $x \to -\infty$,
and $f(x) \to 0$ as $x \to \infty$,
so the x-axis is a horizontal asymptote in both directions.

f



Horizontal asymptotes, and the behaviour as $x \to \infty$ and as $x \to -\infty$

It was very straightforward in the previous worked example to see that the x-axis is an asymptote to each curve. But it is not so straightforward to find the horizontal asymptotes, if indeed they exist, for curves such as

$$y = \frac{x-1}{x-4} \qquad \text{or} \qquad y = \frac{x-1}{x^2-4}.$$

Such curves are called *rational functions* because they are the ratio of two polynomials. For rational functions, *dividing top and bottom by the highest power of x in the denominator* makes the situation clear.

4 BEHAVIOUR FOR LARGE *X*

- Divide top and bottom by the highest power of x in the denominator.
- Then use the fact that $\dfrac{1}{x} \to 0$ as $x \to \infty$ and as $x \to -\infty$.
- If $f(x)$ tends to a definite limit b as $x \to \infty$ or as $x \to -\infty$, then the horizontal line $y = b$ is a horizontal asymptote on the right or on the left.

Example 5 **3B**

a Examine the behaviour of the function $y = \dfrac{x-1}{x-4}$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

b Find any vertical asymptotes of the function $y = \dfrac{x-1}{x-4}$. Then use a table of signs to describe the behaviour of the curve near them.

c Sketch the curve.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Dividing top and bottom by x gives

$$y = \frac{1-\frac{1}{x}}{1-\frac{4}{x}},$$

so $y \to \dfrac{1-0}{1-0} = 1$ as $x \to \infty$ and as $x \to -\infty$.

Hence $y = 1$ is a horizontal asymptote.

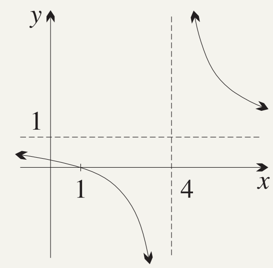
b When $x = 4$, the denominator vanishes, but the numerator does not, so $x = 4$ is an asymptote.

From the table of signs to the right, dodging around the zero at $x = 1$ and the discontinuity at $x = 4$:

As $x \to 4^-$, $y \to -\infty$, and as $x \to 4^+$, $y \to +\infty$,

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	−	*	+

c



Example 6 3B

Examine the behaviour of these functions as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

a $y = \dfrac{x-1}{x^2-4}$

b $y = \dfrac{x^2-1}{x-4}$

c $y = \dfrac{3-5x-4x^2}{4-5x-3x^2}$

SOLUTION

a Dividing top and bottom by x^2, $y = \dfrac{\frac{1}{x}-\frac{1}{x^2}}{1-\frac{4}{x^2}}$.

Hence as $x \to \infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$, and the x-axis $y = 0$ is a horizontal asymptote.

b Dividing top and bottom by x, $y = \dfrac{x-\frac{1}{x}}{1-\frac{4}{x}}$.

Hence as $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$, and there are no horizontal asymptotes.

c Dividing top and bottom by x^2, $y = \dfrac{\frac{3}{x^2}-\frac{5}{x}-4}{\frac{4}{x^2}-\frac{5}{x}-3}$.

Hence as $x \to \infty$, $y \to \frac{4}{3}$, and as $x \to -\infty$, $y \to \frac{4}{3}$, and $y = \frac{4}{3}$ is a horizontal asymptote.

Horizontal asymptotes of functions with exponentials

In Chapter 13, when working with the logistic equation, we will need functions with exponentials in the top and bottom. The form suitable for the limit as $x \to \infty$ may be different from the form suitable for the limit as $x \to -\infty$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 7 3B

a Find any horizontal asymptotes of $y = \dfrac{2^x + 1}{2^x - 1}$.

SOLUTION

As $x \to -\infty$, $2^x + 1 \to 1$ and $2^x - 1 \to -1$, so $y \to -1$.

For the limit as $x \to \infty$, divide through by 2^x to give $y = \dfrac{1 + 2^{-x}}{1 - 2^{-x}}$.

Then as $x \to \infty$, $1 + 2^{-x} \to 1$ and $1 - 2^x \to 1$, so $y \to 1$.

Hence $y = -1$ is a horizontal asymptote on the left, and $y = 1$ on the right.

Exercise 3B

FOUNDATION

1 Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator and taking the limit as $x \to \infty$ and as $x \to -\infty$:

a $f(x) = \dfrac{1}{x + 1}$ **b** $f(x) = \dfrac{x - 3}{x + 4}$ **c** $f(x) = \dfrac{2x + 1}{3 - x}$

d $f(x) = \dfrac{5 - x}{4 - 2x}$ **e** $\dfrac{1}{x^2 + 1}$ **f** $\dfrac{x}{x^2 + 4}$

2 Sketch the curve $y = \dfrac{x}{x - 2}$ after performing the following steps.

a Write down the natural domain.
b Find the intercepts and examine the sign.
c Show that $y = 1$ is the horizontal asymptote.
d Investigate the behaviour near the vertical asymptote.

3 Consider $y = \dfrac{x - 1}{x + 3}$.

a Where is the function undefined?
b Find the intercepts and examine the sign of the function.
c Identify and investigate the vertical and horizontal asymptotes.
d Hence sketch the curve.
e Is this function one-to-one or many-to-one?

4 Investigate the domain, intercepts, sign and asymptotes of the function $y = -\dfrac{1}{(x - 2)^2}$ and hence sketch its graph.

5 Let $y = \dfrac{2}{x^2 + 1}$.

a Determine the horizontal asymptote.
b Explain why there are no vertical asymptotes.
c Show that the tangent to the curve is horizontal at the y-intercept.
d Sketch $y = \dfrac{2}{x^2 + 1}$. Use a table of values if needed.
e What is the range of the function?
f Is this function one-to-one or many-to-one?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 Let $y = \dfrac{3}{(x + 1)(x - 3)}$.

a State the natural domain.
b Find the y-intercept.
c Show that $y = 0$ is a horizontal asymptote.
d Draw up a table of values.
e Identify the vertical asymptotes, and use the table of values to write down its behaviour near them.
f Sketch the graph of the function and state its range.

7 **a** Follow steps similar to those in Question 6 in order to sketch $y = \dfrac{4}{4 - x^2}$.
b What is the range of $y = \dfrac{4}{4 - x^2}$?

DEVELOPMENT

8 Consider the function $y = \dfrac{3x}{x^2 + 1}$.

a Show that it is an odd function.
b Show that it has only one intercept with the axes at the origin.
c Show that the x-axis is a horizontal asymptote.
d Hence sketch the curve.

9 **a** Show that $y = \dfrac{4 - x^2}{4 + x^2}$ is even.
b Find its three intercepts with the axes.
c Determine the equation of the horizontal asymptote.
d Sketch the curve.

10 In each case a rational function $f(x)$ has been given. Factor where needed, and hence find any vertical and horizontal asymptotes of the graph of $y = f(x)$.

a $\dfrac{x^2 + 5x + 6}{x^2 - 4x + 3}$ **b** $\dfrac{x^2 - 2x + 1}{x^2 + 5x + 4}$ **c** $\dfrac{x - 5}{x^2 + 3x - 10}$ **d** $\dfrac{1 - 4x^2}{1 - 9x^2}$

(Computer sketches of these curves may be useful to put these features in context.)

11 Consider the function $f(x) = \dfrac{x}{x^2 - 4}$.

a Determine whether the function is even or odd.
b State the domain of the function and the equations of any vertical asymptotes.
c Use a table of test points of $f(x)$ to analyse the sign of the function.
d Find the equation of the horizontal asymptote.
e Find the derivative, and hence explain why the curve $y = f(x)$ is always decreasing.
f Sketch the graph of $y = f(x)$, showing all important features.
g Use the graph to state the range of the function.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 This question looks at graphs that have holes rather than vertical asymptotes.

a Show that $\dfrac{x^2 - 4}{x - 2} = x + 2$, provided $x \neq 2$. Hence sketch the graph of $y = \dfrac{x^2 - 4}{x - 2}$.

b Similarly sketch graphs of:

i $y = \dfrac{(x + 1)(3 - x)}{x + 1}$ **ii** $y = \dfrac{x^3 - 1}{x - 1}$ **iii** $y = \dfrac{(x + 2)(x - 2)}{(x - 2)(x + 1)}$

13 Consider the function $y = x + \dfrac{1}{x}$.

a Show that the function is odd. What symmetry does its graph have?

b State the domain of the function and the equation of the vertical asymptote.

c Use a table of values of y to analyse the sign of the function.

d Show that the derivative is $y' = \dfrac{x^2 - 1}{x^2}$.

e Find any points where the tangent is horizontal.

f Show that $\lim\limits_{|x| \to \infty} (y - x) = 0$. (This means that $y = x$ is an asymptote to the curve.)

g Sketch the graph of the function.

h Write down the range of the function.

14 In each question above, the horizontal asymptote was the same on the left as $x \to -\infty$, and on the right as $x \to \infty$. This is not always the case. Consider the function $y = \dfrac{1 - e^x}{1 + e^x}$.

a Find $\lim\limits_{x \to -\infty} y$.

b Multiply the numerator and denominator by e^{-x}, then find $\lim\limits_{x \to \infty} y$.

c Determine any intercepts with the axes.

d Using no other information, sketch this curve.

e Test algebraically whether the function is even, odd or neither.

f Similarly sketch $y = \dfrac{1 + e^x}{1 - e^x}$, taking care with the vertical asymptote.

ENRICHMENT

15 **a** Show that $\dfrac{x^2}{x - 1} = x + 1 + \dfrac{1}{x - 1}$, and deduce that $y = \dfrac{x^2}{x - 1}$ has an oblique asymptote $y = x + 1$. Then sketch the graph.

b Likewise sketch $y = \dfrac{x^2 - 4}{x + 1} = x - 1 - \dfrac{3}{x + 1}$, showing the oblique asymptote.

16 Investigate the asymptotic behaviour of the following functions, and graph them:

a $y = \dfrac{x^3 - 1}{x}$ **b** $y = \dfrac{1}{x} + \sqrt{x}$ **c** $y = |x| + \dfrac{1}{x}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3C A curve-sketching menu

We can now combine four approaches to curve-sketching into an informal four-step approach for sketching an unknown graph. This simple menu cannot possibly deal with every possible graph. Nevertheless, it will allow the main features of a surprising number of functions to be found. Two further steps involving calculus will be added in Chapter 4.

A 'sketch' of a graph is not an accurate plot. It is a neat diagram showing the main features of the curve.

5 SKETCHES

- A sketch should show any x- and y-intercepts if they are accessible, any vertical or horizontal asymptotes, and any other significant points on the curve.
- There should always be some indication of scale on both axes, and both axes should be labelled.

A curve-sketching menu

Here is our informal four-step approach to sketching an unknown function.

6 A CURVE SKETCHING MENU

0 **Preparation:** Combine any fractions using a common denominator, then factor top and bottom as far as possible.
1 **Domain:** What is the domain? (*Always* do this first.)
2 **Symmetry:** Is the function odd, or even, or neither?
3 **A** **Intercepts:** What are the y-intercept and the x-intercepts (zeroes)?
B **Sign:** Where is the function positive, and where is it negative?
4 **A** **Vertical asymptotes:** Examine the behaviour near any discontinuities, noting any vertical asymptotes.
B **Horizontal asymptotes:** Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

Finding the domain and finding the zeroes may both require factoring, which is the reason why the preparatory Step 0 is useful. Factoring, however, may not always be possible, even with the formula for the roots of a quadratic, and in such cases approximation methods may be useful.

Questions will normally give guidance as to what is required. Our menu is not an explicit part of the course, but rather a suggested way to organise the approaches presented in the course.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

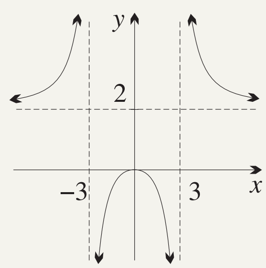
Putting it all together — first example

All that remains is to give two examples of the whole process.

Example 8 3C

Apply the steps in Box 6 to sketch $f(x) = \dfrac{2x^2}{x^2 - 9}$.

SOLUTION



0 Preparation: $f(x) = \dfrac{2x^2}{(x-3)(x+3)}$.

1 Domain: $x \neq 3$ and $x \neq -3$.

2 Symmetry: $f(-x) = \dfrac{2(-x)^2}{(-x)^2 - 9}$

$= \dfrac{2x^2}{x^2 - 9}$

$= f(x)$.

so $f(x)$ is even, with line symmetry in the$y$-axis.

3 Intercepts and sign: When $x = 0$, $y = 0$.
There is a zero at $x = 0$, and discontinuities at $x = 3$ and $x = -3$.

x	-4	-3	-1	0	1	3	4
$f(x)$	$\frac{32}{7}$	*	$-\frac{1}{4}$	0	$-\frac{1}{4}$	*	$\frac{32}{7}$
sign	$+$	*	$-$	0	$-$	*	$+$

4 Vertical asymptotes: At $x = 3$ and $x = -3$, the denominator vanishes, but the numerator does not, so $x = 3$ and $x = -3$ are vertical asymptotes. To make this more precise, it follows from the table of signs that

$$f(x) \to \infty \text{ as } x \to 3^+ \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to 3^-,$$

$$f(x) \to -\infty \text{ as } x \to (-3)^+ \quad \text{and} \quad f(x) \to \infty \text{ as } x \to (-3)^-$$

Horizontal asymptotes: Dividing through by x^2, $f(x) = \dfrac{2}{1 - \frac{9}{x^2}}$,

so $f(x) \to 2$ as $x \to \infty$ and as $x \to -\infty$, and $y = 2$ is a horizontal asymptote.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Putting it all together — second example

The second example requires a common denominator. The calculations involving intercepts and sign have been done with an alternative approach using signs rather than numbers.

Example 9 3C

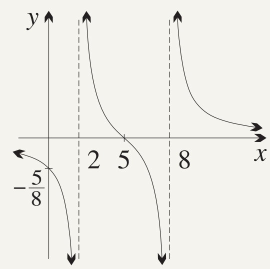
Apply the steps in Box 6 to sketch $f(x) = \dfrac{1}{x-2} + \dfrac{1}{x-8}$.

SOLUTION

0 $f(x) = \dfrac{(x-8)+(x-2)}{(x-2)(x-8)}$

$= \dfrac{2x-10}{(x-2)(x-8)}$

$= \dfrac{2(x-5)}{(x-2)(x-8)}$.



1 The domain is $x \neq 2$ and $x \neq 8$.

2 $f(x)$ is neither even nor odd.

3 When $x = 0$, $y = -\dfrac{5}{8}$.

There is a zero at $x = 5$, and discontinuities at $x = 2$ and $x = 8$.

x	0	2	3	5	7	8	10
$x-2$	−	0	+	+	+	+	+
$x-5$	−	−	−	0	+	+	+
$x-8$	−	−	−	−	−	0	+
$f(x)$	−	*	+	0	−	*	+

If only the signs are calculated, at least these three lines of working should be shown.

4 At $x = 2$ and $x = 8$ the denominator vanishes, but the numerator does not, so $x = 2$ and $x = 8$ are vertical asymptotes.

From the original form of the given equation, $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$, so $y = 0$ is a horizontal asymptote.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 3C

FOUNDATION

1 Complete the following steps in order to sketch the graph of $y = \dfrac{9}{x^2 - 9}$.

a Factor the function as far as possible.
b State the domain using bracket interval notation.
c Show that the function is even. What symmetry does the graph have?
d Write down the coordinates of any intercepts with the axes.
e Investigate the sign of the function using a table of values. Where is $y \le 0$?
f Write down the equation of any vertical asymptote.
g What value does y approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
h Sketch the graph of the function showing these features.
i The graph appears to be horizontal at its y-intercept. Find y' and hence confirm that the graph is horizontal there.

2 Complete the following steps in order to sketch the graph of $y = \dfrac{x}{4 - x^2}$.

a Factor the function as far as possible.
b State the domain using bracket interval notation.
c Show that the function is odd. What symmetry does the graph have?
d Write down the coordinates of any intercepts with the axes.
e Investigate the sign of the function using a table of values. Where is $y \ge 0$?
f Write down the equation of any vertical asymptote.
g What value does y approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
h Sketch the graph of the function showing these features.
i Use the quotient rule to show that $y' = \dfrac{(x^2 + 4)}{(4 - x^2)^2}$, and hence explain why the graph always has a positive gradient.

3 Follow these steps to graph $y = f(x)$ where $f(x) = \dfrac{1}{x - 1} + \dfrac{1}{x - 4}$.

a Combine the two fractions using a common denominator, then factor the numerator and denominator as far as possible.
b State the domain.
c Determine whether $f(x)$ is even, odd or neither (the answers to part **a** may help).
d Write down the coordinates of any intercepts with the axes.
e Investigate the sign of the function using a table of values. Where is $y > 0$?
f Write down the equation of any vertical asymptote.
g What value does $f(x)$ approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
h Sketch the graph of the function showing these features.

4 Let $y = x^3 - 4x$.

a Factor this function.
b State the domain using bracket interval notation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Write down the coordinates of any intercepts with the axes.
d Show that the function is odd. What symmetry does the graph have?
e Does this function have any asymptotes?
f Use this information and a table of values to sketch the curve.
g The graph seems to have a peak somewhere in the interval $-2 < x < 0$ and a trough in the interval $0 < x < 2$. Use calculus to find the x-coordinates of these points and add them to the diagram.

5 Let $y = \dfrac{3x - 3}{x^2 - 2x - 3}$.
a State the domain and any intercepts with the axes.
b Explain why the function is neither even nor odd.
(Hint: The answers to part **a** may help.)
c Write down the equations of the asymptotes.
d Sketch the graph of this curve.

6 Let $y = -x^3 + 6x^2 - 8x$.
a State the domain using inequality interval notation, and write down the coordinates of any intercepts with the axes.
b Use this information and a table of values to sketch the curve.
c The graph seems to be horizontal somewhere in the interval $0 < x < 2$, and again in the interval $2 < x < 4$. Use calculus to find the x-coordinates of these points and add them to the diagram.

DEVELOPMENT

7 Let $y = \dfrac{x^2 + 2x + 1}{x^2 + 2x - 3}$.
a State the domain and any intercepts with the axes.
b Explain why the function is neither even nor odd.
c Write down the equations of the asymptotes.
d Sketch the graph of this curve.
e What is the range of this function?

8 Let $f(x) = \dfrac{x^2 - 4}{x^2 - 4x}$. You may assume that $f(x)$ is neither even nor odd.
a State the domain of $f(x)$ and write down the intercepts of $y = f(x)$.
b Write down the equations of the asymptotes.
c Sketch the graph of $y = f(x)$.
d What is the range of this function?
e The graph crosses its horizontal asymptote in the interval $0 < x < 4$. Find the coordinates of this point and add it to the graph.

9 **a** Show that $y = \dfrac{1}{x + 1} - \dfrac{1}{x}$ can be written as $y = -\dfrac{1}{x(x + 1)}$. Then identify the domain and any zeroes, examine the asymptotes and sign, and hence sketch the graph.
b Likewise express $y = \dfrac{1}{x + 3} + \dfrac{1}{x - 3}$ with a common denominator and sketch it.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 **a** Examine the sign and asymptotes of $y = \dfrac{1}{x(x-2)}$ and hence sketch the curve.

b Likewise sketch $y = \dfrac{2}{x^2 - 4}$.

11 Use the curve-sketching menu as appropriate to obtain the graphs of:

a $y = \dfrac{1 + x^2}{1 - x^2}$

b $y = \dfrac{x + 1}{x(x - 3)}$

c $y = \dfrac{x - 1}{(x + 1)(x - 2)}$

d $y = \dfrac{x^2 - 2x}{x^2 - 2x + 2}$

e $y = \dfrac{x^2 - 4}{(x + 2)(x - 1)}$

f $y = \dfrac{x^2 - 2}{x}$

12 The curve $y = e^{-\frac{1}{2}x^2}$ is essential in statistics because it is related to the normal distribution, studied later in this course.

a Determine the domain and any intercepts of this function.

b Determine whether the function is even or odd, and investigate any asymptotes.

c By considering the maximum value of $-x^2$, find the highest point on this curve.

d Confirm your answer by showing that the tangent is horizontal there.

e Sketch the curve, and hence state its range.

f Which is higher, $y = e^{-\frac{1}{2}x^2}$ or $y = 2^{-\frac{1}{2}x^2}$?

ENRICHMENT

13 **a** **i** By considering the graphs of $y = e^x$ and $y = x$, or otherwise, determine which function is greater for $x \geq 0$.

ii Use part **i** to explain why $e^{-\frac{1}{2}x^2} < \dfrac{2}{x^2}$ for $x \neq 0$.

iii Hence, or otherwise, determine $\lim\limits_{x \to \infty} xe^{-\frac{1}{2}x^2}$.

b The graph of $y = e^{-\frac{1}{2}x^2}$, drawn in Question 12 appears to be steepest at $x = -1$ where it has positive gradient, and at $x = 1$ where it has negative gradient. In order to confirm this, the range of $y' = -xe^{-\frac{1}{2}x^2}$ is needed.

i Use the curve-sketching menu to sketch the graph of $f(x) = -xe^{-\frac{1}{2}x^2}$. You will need your answer to part **a** to determine the horizontal asymptote. Include on the sketch the x-coordinates of any points where the tangent is horizontal.

ii Hence prove the claim that $y = e^{-\frac{1}{2}x^2}$ is steepest at $x = 1$ or -1.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3D Solving inequations

We can now turn our attention to equations, and more importantly to inequations (or 'inequalities' as they are often called). Again, the material in this section was covered in Section 5A of the Year 11 book, but it combines many approaches and needs review.

The basic approaches to solving inequations

Here is a summary of the basic methods that we have used in solving inequations.

7 SOLVING INEQUATIONS — A BASIC SUMMARY

- The graphical interpretation of an inequation $f(x) < g(x)$ is,
  'Where is the graph of $y = g(x)$ above the graph of $y = f(x)$?'
- Anything can be added to or subtracted from both sides of an inequation.
- When multiplying or dividing both sides by a negative number, the inequality symbol is reversed.
- As with equations, never multiply or divide by 0.
- A denominator can be removed by multiplying by its square, which is always positive or zero — the zeroes then become special cases.
- If the corresponding equation has been solved, and the discontinuities can be found, a table of signs will solve the inequation.

We will now review how these six dotpoints apply to various types of inequations.

Linear inequations: Linear inequations can be solved completely by the second and third dotpoints, and this needs no further review. Three considerations:

- If the graph is drawn, then the first dotpoint can be used.
- If the equation has been solved, then a table of signs can be used.
- If the linear equation has unknown constants, such as $ax > c$, then division by a could be disastrous, because a could be negative or even zero.

Quadratic and polynomial inequations: The standard method is to sketch the graph and read the solution off the graph. This was well covered in Year 11.

If the polynomial can be factored, as for example $(x - 2)(x - 5) < 0$, then an algebraic argument using the signs of the two factors can be used, but this easily leads to confusion. In this case, the two factors must have opposite signs, so $2 < x < 5$.

Inequations with a variable in the denominator: The standard approach is to multiply through by the denominator's square, which is positive, so that we don't generate cases depending on whether the inequality symbol is reversed.

- Keep expressions factored where possible — re-factoring may be difficult.
- If the denominator has zeroes, these must be treated as special cases.

The other approach, which is usually longer, is to sketch the graph and read the solution off the graph. The table of signs alone, however, may be sufficient.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 10 3D

a Solve $\dfrac{3}{x-4} \geq -1$ by multiplying through by the square of the denominator.

b Use a sketch of $y = 1 + \dfrac{3}{x-4}$ to solve $\dfrac{3}{x-4} \geq -1$.

SOLUTION

a $\dfrac{3}{x-4} \geq -1$

We multiply both sides by $(x-4)^2$, which is positive, or zero if $x = 4$.

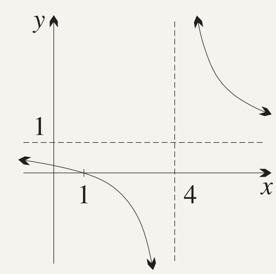
$$\boxed{\times\ (x-4)^2} \qquad 3(x-4) \geq -(x-4)^2, \text{ where } x \neq 4,$$
$$(x-4)^2 + 3(x-4) \geq 0$$
$$(x-4)(x-4+3) \geq 0$$
$$(x-4)(x-1) \geq 0.$$

Hence after sketching the parabola $y = (x-4)(x-1)$,

$x \leq 1$ or $x \geq 4$.

But $x = 4$ was excluded in the first step, so the solution is

$x \leq 1$ or $x > 4$.



b Moving everything to the LHS, $1 + \dfrac{3}{x-4} \geq 0$

$$\frac{x-1}{x-4} \geq 0.$$

From the graph, the solution is $x \leq 1$ or $x > 4$.

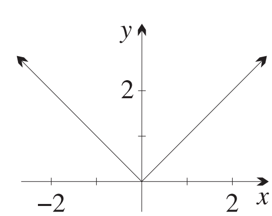
The LHS was graphed in worked Example 5, but the table of signs below is also sufficient working.

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	−	*	+

From the graph or from the table, the solution is $x \leq 1$ or $x > 4$.

Solving absolute value equations and inequations

The six basic methods in Box 7 apply just as much to absolute value inequations, and many can be solved that way, particularly if a reasonable graph can be drawn quickly.



$y = |x|$.

But absolute value equations and inequations are usually best solved using their own particular approaches. Absolute value was discussed in detail in Sections 4D, 5A and 5E of the Year 11 book, and these summaries do not do those sections justice.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 ABSOLUTE VALUE

The meaning of absolute value:

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.
- $|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$

Solving absolute value equations and inequations:

Let $a \geq 0$ be a real number.

- Rewrite an equation $|f(x)| = a$ as $f(x) = -a$ or $f(x) = a$.
- Rewrite an inequation $|f(x)| < a$ as $-a < f(x) < a$.
- Rewrite an inequation $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$.

Example 11 — 3D

a Solve $|10 - x^2| = 6$. **b** Solve $|10 - x^2| < 6$. **c** Solve $|10 - x^2| > 6$.

SOLUTION

a $10 - x^2 = 6$ or $10 - x^2 = -6$
$x^2 = 4$ or $x^2 = 16$
$x = 2$ or -2 or 4 or -4.

b $-6 < 10 - x^2 < 6$ OR Use part **a** and a table of signs.
$\boxed{-10}$ $-16 < -x^2 < -4$
$\boxed{\times(-1)}$ $4 < x^2 < 16$
$-4 < x < -2$ or $2 < x < 4$.

c $10 - x^2 < -6$ or $10 - x^2 > 6$ OR Use part **a** and a table of signs.
$x^2 > 16$ or $x^2 < 4$
$x < -4$ or $-2 < x < 2$ or $x > 4$.

Exercise 3D

FOUNDATION

1 Solve each inequation, and graph your solution on the number line.

a $x - 2 < 3$ **b** $3x \geq -6$ **c** $4x - 3 \leq -7$
d $6x - 5 < 3x - 17$ **e** $\frac{1}{5}x - \frac{1}{2}x < 3$ **f** $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) \geq 2$

2 Solve each inequation, then write your answer using bracket interval notation.

a $3 - 2x > 7$ **b** $3 - 3x \leq 19 + x$ **c** $12 - 7x > -2x - 18$

3 Write down and solve a suitable inequation to find the values of x for which the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.

4 Solve each double inequation, then write your answer in bracket interval notation.

a $-1 \leq 2x \leq 3$ **b** $-4 < -2x < 8$ **c** $-7 \leq 5 - 3x < 4$
d $-5 < x - 3 \leq 4$ **e** $-7 \leq 5x + 3 < 3$ **f** $-4 < 1 - \frac{1}{3}x \leq 3$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 **a** Sketch the lines $y = 1 - x$, $y = 2$ and $y = -1$ on a number plane and find the points of intersection.
b Solve the inequation $-1 < 1 - x \le 2$ and relate the answer to the graph.

6 In each case, solve for x by sketching an appropriate parabola.
a $x^2 + 2x - 8 < 0$ **b** $x^2 - 2x - 3 > 0$ **c** $-x^2 + 7x - 10 \ge 0$
d $x^2 + 4x + 3 \ge 0$ **e** $2x^2 - 11x + 5 > 0$ **f** $2x^2 + 13x + 20 \le 0$

7 Solve the following equations and inequations, and graph each solution on a number line.
a $|x - 4| = 1$ **b** $|2x - 3| = 7$ **c** $|x + 3| > 4$
d $|-x - 10| \le 6$ **e** $|3 - 2x| \le 1$ **f** $|3x + 4| > 2$

DEVELOPMENT

8 Multiply through by the square of the denominator and hence solve:
a $\dfrac{1}{x} \le 2$ **b** $\dfrac{3}{2 - x} > 1$ **c** $\dfrac{4}{3 - 2x} < 1$ **d** $\dfrac{5}{4x - 3} \ge -2$

9 Write down the equations of the two branches of the function, then sketch its graph:
a $y = |x - 2| + x + 1$ **b** $y = |2x + 4| - x + 5$ **c** $y = 3|x - 1| + x - 1$

10 Solve the following inequations involving logarithms and exponentials:
a $3^x \ge 27$ **b** $1 < 5^x \le 125$ **c** $\frac{1}{16} \le 2^x \le 16$
d $2^{-x} > 16$ **e** $\log_2 x < 3$ **f** $-2 \le \log_5 x \le 4$

11 In each case, use an appropriate inequation to solve the problem.
a Where is the parabola $y = x^2 - 2x$ below the line $y = x$?
b Where is the parabola $y = x^2 - 2x - 3$ above the line $y = 3 - 3x$?
c Where is the hyperbola $y = \dfrac{1}{x}$ below the line $y = -x$?
d Where is the hyperbola $y = \dfrac{2}{x - 1}$ above the line $y = \frac{1}{2}x - 2$?
e Where is the parabola $y = x^2 + 2x - 3$ below the parabola $y = 5 - 4x - x^2$?

12 **a** **i** Write down the equations of the two branches of the function $y = |2x| + x$.
ii Hence solve $|2x| + x > 1$.
b Likewise, solve these inequations.
i $3|x - 2| + x - 2 \le 2$. **ii** $|x + 1| - \frac{1}{2}x < 3$

13 Solve for x:
a $\dfrac{2x + 1}{x - 3} > 1$ **b** $\dfrac{x - 1}{x + 1} \le 2$ **c** $\dfrac{3x}{2x - 1} \ge 4$

14 **a** Where is $\cos x > 1 - \sin^2 x$ in the domain $0 \le x \le 2\pi$?
b Where is $\tan x \le \sec^2 x - 1$ in the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$?

15 Solve:
a $1 < |x + 2| \le 3$ **b** $1 \le |2x - 3| < 4$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 Determine whether these statements are true or false. If false, give a counter-example. If true, provide examples with:

i $x > 0$ and $y > 0$, **ii** $x > 0$ and $y < 0$, **iii** $x < 0$ and $y > 0$, **iv** $x < 0$ and $y < 0$.

a $|x + y| = |x| + |y|$ **b** $|x + y| \le |x| + |y|$ **c** $|x - y| \le |x| - |y|$

d $|x - y| \le |x| + |y|$ **e** $|x - y| \ge \big||x| - |y|\big|$ **f** $2^{|x|} = 2^x$

ENRICHMENT

17 A student was asked to solve the inequation $\sqrt{5 - x} > x + 1$.

a The student decided to square both sides to get $5 - x > x^2 + 2x + 1$, and then solved this inequation. Explain why this gives the wrong answer.

b Find the correct solution.

18 **a** Draw the graph of $f(x) = |5 - 2x^2|$ by considering the equations of its branches.

b Hence or otherwise solve the inequation $|5 - 2x^2| \ge 3$.

c More generally, prove that if $|g(x)| \ge k$ then either $g(x) \le -k$ or $g(x) \ge k$.

19 Prove that $x^2 + xy + y^2 > 0$ for any non-zero values of x and y.

20 **a** Prove that $(x + y)^2 \ge 4xy$.

b Hence prove that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \ge \dfrac{4}{x^2 + y^2}$.

21 **a** Expand $(a - b)^2 + (b - c)^2 + (a - c)^2$, and hence prove that $a^2 + b^2 + c^2 \ge ab + bc + ac$.

b Expand $(a + b + c)\left((a - b)^2 + (b - c)^2 + (a - c)^2\right)$, and hence prove the identity $a^3 + b^3 + c^3 \ge 3abc$, for positive a, b and c.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3E Using graphs to solve equations and inequations

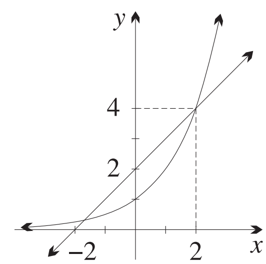
In this section, graphs are used to solve more general equations and inequations. The advantage of such an approach is that once the graphs are drawn, it is usually obvious from the picture how many solutions there are, and indeed if there are any solutions at all, as well as their approximate values. Exact solutions can sometimes then be calculated once the situation has been sorted out from the picture.

Constructing two functions from a given equation

Here is an equation that cannot be solved algebraically, so that a graphical approach is appropriate:

$$2^x = x + 2.$$

To the right, $y = \text{LHS}$ and $y = \text{RHS}$ are graphed together. (In other situations, some rearrangement of the equation first may be appropriate.)



The first thing to notice is that there are two solutions, because the graphs intersect twice.

The second thing is to examine what the values of the two solutions are. One solution is exactly $x = 2$, because

$$2^2 = 4 = 2 + 2, \qquad \text{so } (2,\ 4) \text{ lies on both graphs.}$$

The other solution is just to the right of $x = -2$. From the graph, we might guess $x \doteqdot -1.7$, and if necessary we can refine this solution in several ways:

- Plot the graphs carefully on graph paper (an old method that works).
- Use trial and error on a calculator (see Question 12 in Exercise 3E).
- Use graphing software that can generate approximations (if you have it).

Counting the number of solutions of an equation

Often, however, we only want to know how many solutions an equation has, and roughly where they are.

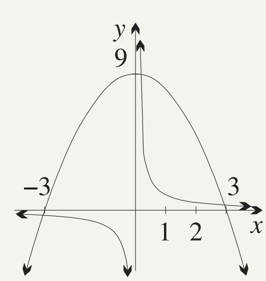
Example 12 3E

a Graph $y = 9 - x^2$ and $y = \dfrac{1}{x}$ on the one set of axes.

b Use your graph to investigate the equation $9 - x^2 = \dfrac{1}{x}$. How many solutions does the equation have, and approximately where are they?

SOLUTION

a The two functions are sketched to the right.



b There are three points of intersection of the two graphs.
Thus there are three solutions:
- one just to the left of $x = -3$,
- one just to the right of $x = 0$,
- and one just to the left of $x = 3$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 GRAPHICAL SOLUTION OF AN EQUATION

- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
  — It may be appropriate to rearrange the original equation first.
- The solutions are the x-coordinates of any points of intersection.
- You may be interested only in the number of solutions.
- The graph will give a rough idea where any solution lies.

Solving an inequation using graphs

Now consider the inequation

$$2^x < x + 2.$$

From the sketch at the start of the section, the curve $y = 2^x$ is only below the curve $y = x + 2$ between the two points of intersection. Hence the solution of the inequation is approximately $-1.7 < x < 2$.

10 GRAPHICAL SOLUTION OF AN INEQUATION

- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
- Then examine which curve lies above the other at each value of x.

Absolute value equations and inequations — graphical solutions

The most straightforward approach to absolute value equations and inequations is to draw a sketch to sort out the situation. Then the exact values can usually be found algebraically.

The next worked example benefits greatly from the diagram that makes the situation so clear.

Example 13 3E

a Draw the graph of $y = |2x - 5|$.
b Write down the equations of the right-hand and left-hand branches.
c One the same diagram, draw the graph of $y = x + 2$.
d Hence find the points P and Q of intersection.
e Solve $|2x - 5| = x + 2$.
f Solve $|2x - 5| \ge x + 2$.

SOLUTION

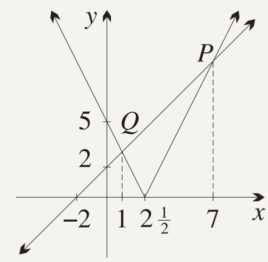
a To find the x-intercept, put $y = 0$, then $2x - 5 = 0$
$$x = 2\tfrac{1}{2}$$
To find the y-intercept, put $x = 0$, then
$$\begin{aligned} y &= |0 - 5| \\ &= 5. \end{aligned}$$
We can now sketch the right-hand branch by symmetry (or use a table of values).

b For $x \ge 2\frac{1}{2}$, $y = 2x - 5$.
For $x < 2\frac{1}{2}$, $y = -2x + 5$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c The two graphs intersect at two points P and Q as shown.



d The points P and at Q can be now found algebraically.
P is the intersection of $y = x + 2$ with $y = 2x - 5$,

$$x + 2 = 2x - 5$$
$$x = 7, \quad \text{so } P = (7, 9),$$

and Q is the intersection of $y = x + 2$ with $y = -2x + 5$,

$$x + 2 = -2x + 5$$
$$x = 1, \quad \text{so } Q = (1, 3).$$

e Hence the solutions are $x = 7$ or $x = 1$.

f Look at where $y = |2x - 5|$ is on or above $y = x + 2$.
From the graph, $x \le 1$ or $x \ge 7$.

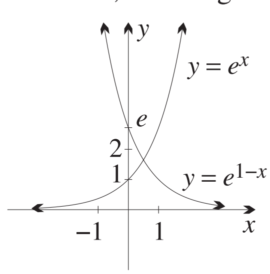
Exercise 3E

FOUNDATION

Note: Graphing software would be particularly useful in this exercise.

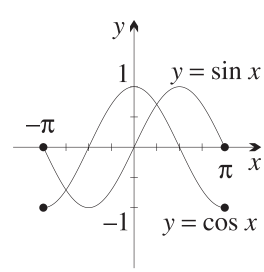
1 In each case, use the given graph to determine the number of solutions of the equation.

a



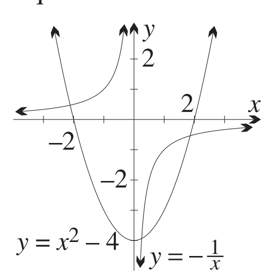
$e^x = e^{1-x}$

b



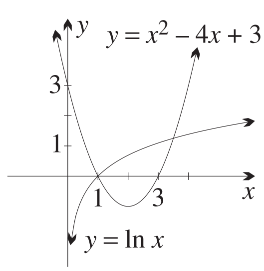
$\cos x = \sin x, \quad -\pi \le x \le \pi$

c



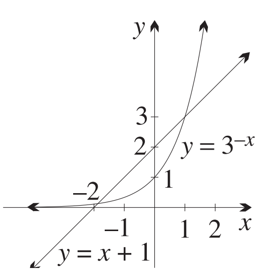
$x^2 - 4 = -\dfrac{1}{x}$

d



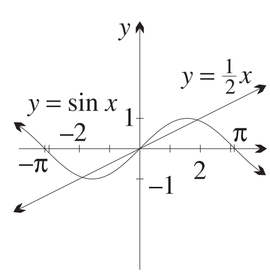
$\ln x = x^2 - 4x + 3$

e



$3^x = x + 2$

f



$\sin x = \dfrac{1}{2}x$

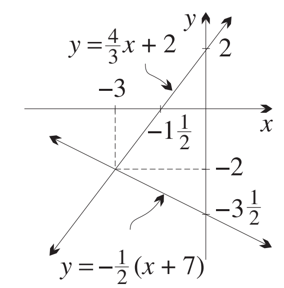
2 For each equation in Question 1, read the solutions from the graph, approximated correct to one decimal place where appropriate.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

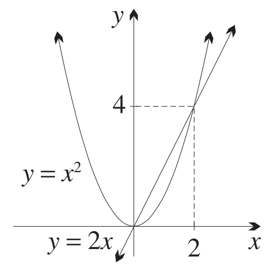
3 Use the given graphs to help solve each inequation.

a



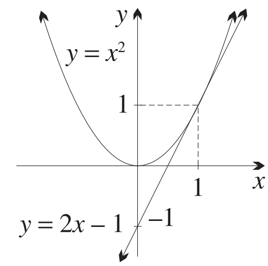
$\frac{4}{3}x + 2 \leq -\frac{1}{2}(x + 7)$

b



$x^2 \leq 2x$

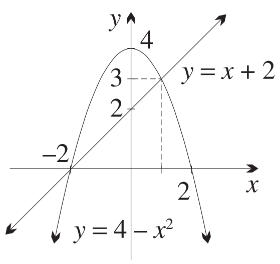
c



$x^2 \leq 2x - 1$

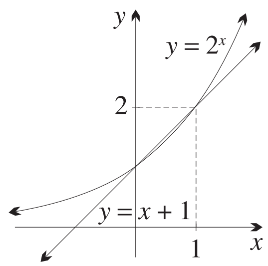
4 Solve these inequations using the given graphs.

a



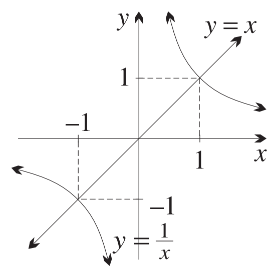
$4 - x^2 < x + 2$

b



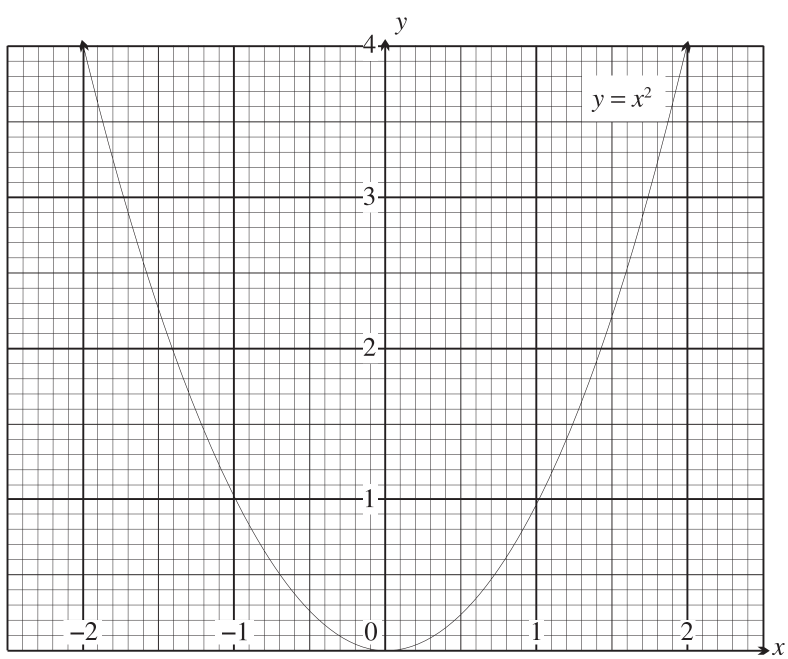
$2^x \leq x + 1$

c



$\frac{1}{x} < x$

5



In preparation for the following questions, photocopy the sketch above, which shows the graph of the function

$$y = x^2, \text{ for } -2 \leq x \leq 2.$$

a Read $\sqrt{2}$ and $\sqrt{3}$ off the graph, correct to one decimal place, by locating 2 and 3 on the y-axis and reading the answer off the x-axis. Check your approximations using a calculator.

b Draw on the graph the line $y = x + 2$, and hence read off the graph the solutions of $x^2 = x + 2$. Then check your solution by solving $x^2 = x + 2$ algebraically.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c From the graph, write down the solution of $x^2 > x + 2$.
d Draw a suitable line to solve $x^2 = 2 - x$ and $x^2 \le 2 - x$. Check your results by solving $x^2 = 2 - x$ algebraically.
e Draw $y = x + 1$, and hence solve $x^2 = x + 1$ approximately. Check your result algebraically.
f Find approximate solutions for these quadratic equations by rearranging each with x^2 as subject, and drawing a suitable line on the graph.
i $x^2 + x = 0$ **ii** $x^2 - x - \frac{1}{2} = 0$ **iii** $2x^2 - x - 1 = 0$

6 In each part:
i Carefully sketch each pair of equations. **ii** Read off the points of intersection.
iii Write down the equation satisfied by the x-coordinates of the points of intersection.
a $y = x - 2$ and $y = 3 - \frac{1}{4}x$ **b** $y = x$ and $y = 2x - x^2$
c $y = \dfrac{2}{x}$ and $y = x - 1$ **d** $y = x^3$ and $y = x$

7 Use your graphs from the previous question to solve the following inequations.
a $x - 2 \ge 3 - \frac{1}{4}x$ **b** $x < 2x - x^2$ **c** $\dfrac{2}{x} > x - 1$ **d** $x^3 > x$

8 **a** **i** Sketch on the same number plane the functions $y = |x + 1|$ and $y = \frac{1}{2}x - 1$.
ii Hence explain why all real numbers are solutions of the inequation $|x + 1| > \frac{1}{2}x - 1$.
b Draw a sketch of the curve $y = 2^x$ and the line $y = -1$. Hence explain why the inequation $2^x \le -1$ has no solutions.
c Draw $y = 2x - 1$ and $y = 2x + 3$ on the same number plane, and hence explain why the inequation $2x - 1 \le 2x + 3$ is true for all real values of x.

9 Sketch each pair of equations, and hence find the points of intersection.
a $y = |x + 1|$ and $y = 3$ **b** $y = |x - 2|$ and $y = x$
c $y = |2x|$ and $2x - 3y + 8 = 0$ **d** $y = |x| - 1$ and $y = 2x + 2$

10 Use your answers to the previous question to help solve:
a $|x + 1| \le 3$ **b** $|x - 2| > x$ **c** $|2x| \ge \dfrac{2x + 8}{3}$ **d** $|x| > 2x + 3$

DEVELOPMENT

11 Explain how the graphs of Question 1 parts **a**, **b** and **c** could be used to solve:
a $e^{2x} = e$ **b** $\tan x = 1$ **c** $x^3 - 4x + 1 = 0$.

12 [Solving an equation by trial and error]
a In Section 3E we sketched $y = 2^x$ and $y = x + 2$ on one set of axes, and saw that there is a solution a little to the right of $x = -2$. Fill in the table of values below, and hence find the negative solution of $2^x = x + 2$ correct to three decimal places.

x	-2	-1.7	-1.6	-1.68	-1.69	-1.691	-1.6905
2^x							
$x + 2$							

b For parts **c** and **e** of Question 1, use trial and error on the calculator to find the negative solution of the equation, correct to three decimal places.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 Sketch graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them.

a $1 - \frac{1}{2}x = x^2 - 2x$ **b** $|2x| = 2^x$ **c** $x^3 - x = \frac{1}{2}(x + 1)$

d $4x - x^2 = \dfrac{1}{x}$ **e** $2^x = 2x - x^2$ **f** $2^{-x} - 1 = \dfrac{1}{x}$

14 [Break-even point]

A certain business has fixed costs of \$900 plus costs of \$30 per item sold. The sale price of each item is \$50. If enough items are sold then the company is able to exactly pay its total costs. That point is called the *break-even point*. Companies may use several different methods to graph this information. Here are two such methods. In each case, let n be the number of items sold.

a **i** The gross profit per item is $\$50 - \$30 = \$20$. Sketch the graph of the gross profit for n items $y = \$20 \times n$.

ii On the same graph sketch the fixed costs $y = \$900$.

iii The point where these two lines cross is the break-even point. How many items need to be sold to break even?

b **i** On a new graph, draw $y = \$50 \times n$, the total sales for n items.

ii On the same graph, draw $y = \$900 + \$30 \times n$, the total cost for n items.

iii Does the break-even point for this graph agree with part **a**?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 **a** Sketch $y = x^2 - 2$, $y = x$ and $y = -x$ on the same number plane, and find all points of intersection of the three functions.
b Hence find the solutions of $x^2 - 2 = |x|$.
c Hence solve $x^2 - 2 > |x|$.
d Finally solve $x^2 - 2 \le -|x|$.

16 **a** Sketch $y = |2x + 1|$.
b Draw on the same number plane $y = x + c$ for $c = -1$, $c = 0$ and $c = 1$.
c For what values of c does $|2x + 1| = x + c$ have two solutions?

17 **a** Use a diagram and Pythagoras' theorem to show that for $b > 0$, the perpendicular distance from the line $x + y = 2b$ to the origin is $b\sqrt{2}$.
b Hence find the range of values of b for which the line intersects the circle $x^2 + y^2 = 9$ twice.

18 **a** Sketch $y = |7x - 4|$ and $y = 4x + 3$ on the same number plane to find the number of solutions of $|7x - 4| = 4x + 3$.
b Why is it inappropriate to use the graph to find the exact solutions?
c Find the solutions by separately considering the two branches of the absolute value.

19 Draw appropriate graphs, using a computer or graphics calculator, in order to find the solutions of these equations correct to one decimal place.
a $x^3 = 2(x - 2)^2$
b $x^3 = \sqrt{4 - x^2}$
c $2^x = -x(x + 2)$
d $2^{-x} = 2x - x^2$

ENRICHMENT

20 **a** Carefully sketch the graph of $y = |2x + 4| + |x - 1| - 5$ and write down the equation of each branch.
b On the same number plane draw the lines $y = -1$ and $y = 2$. Hence solve the inequation $-1 \le |2x + 4| + |x - 1| - 5 \le 2$.

21 **a** Show that $y = mx + b$ intersects $y = |x + 1|$ if $m > 1$ or $m < -1$.
b Given that $-1 \le m \le 1$, find the relationship between b and m so that the two graphs do not intersect.
c Generalise these results for $y = |px - q|$.

22 **a** Draw $y = a^x$ and $y = \log_a x$ for:
i $a = 3$
ii $a = 2$
iii $a = \sqrt{2}$
b Conclude how many solutions $a^x = \log_a x$ may have.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3F Regions in the coordinate plane

The circle $x^2 + y^2 = 25$ divides the plane into two *regions* — inside the circle and outside the circle. The graph of the inequation $x^2 + y^2 > 25$ will be one of these regions. It remains to work out which of these regions should be shaded.

Graphing regions

Here is how to sketch the region described by an inequation.

11 GRAPHING THE REGION CORRESPONDING TO AN INEQUATION

1. **The boundary:** Replace the inequality symbol by the equal sign, and graph the curve. This will normally be the boundary of the region, and should be drawn broken if it is excluded, and unbroken if it is included.
2. **Shading:** Determine which parts are included and which are excluded, and shade the parts that are included. This can be done in two ways:
   - *Always possible:* Take one or more test points not on any boundary, and substitute into the LHS and RHS of the original inequation. The origin is the easiest test point, otherwise try to choose points on the axes.
   - *Quicker, but not always possible:* Alternatively, solve the inequation for y if possible, and shade the region above or below the curve. Or solve for x, and sketch the region to the right or left of the curve.
3. **Check boundaries and corners:**
   - Check that boundaries are correctly broken or unbroken.
   - Corner points must be marked clearly with a closed circle if they are included, or an open circle if excluded.

Example 14 3F

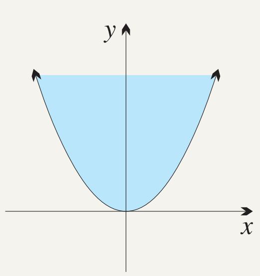
Sketch $y \ge x^2$.

SOLUTION

The boundary is $y = x^2$, and is included.
Because the inequation is $y \ge x^2$,
the region involved is the region *above* the curve.
Alternatively, take a test point $(0, 1)$, then

$$\text{LHS} = 1 \qquad \text{and} \qquad \text{RHS} = 0,$$

so $\text{LHS} \ge \text{RHS}$, meaning that $(0, 1)$ is in the region.



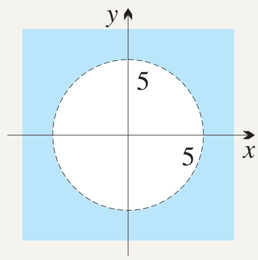
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 15 3F

Sketch the region $x^2 + y^2 > 25$.

SOLUTION



The boundary is $x^2 + y^2 = 25$, and is excluded.
The inequation cannot be solved for x or y without complicated cases.
Instead, take a test point $(0, 0)$, then

$$\text{LHS} = 0 \qquad \text{and} \qquad \text{RHS} = 25,$$

so $LHS \not> RHS$, meaning that $(0, 0)$ is not in the region.

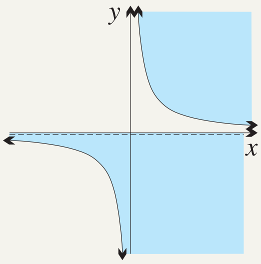
Points where the LHS or the RHS is undefined

There may be points in the plane where the LHS or the RHS of the inequation is undefined, as in the next worked example. If so, the set of all these points is a possible boundary of the region too, and will be excluded.

Example 16 3F

Sketch the region $x \geq \dfrac{1}{y}$.

SOLUTION



The boundary is $x = \dfrac{1}{y}$, and is included.

Also, the x-axis $y = 0$ is a boundary, because the RHS is undefined when $y = 0$. This boundary is excluded.

Because the inequation is $x \geq \dfrac{1}{y}$, the region to be shaded is the region to the right of the curve.

Alternatively, take test points, but this is more elaborate.

The point $(0, 1)$ is out because $0 \ngeq \dfrac{1}{1}$.

The point $(2, 2)$ is in because $2 \geq \dfrac{1}{2}$.

The point $(0, -1)$ is in because $0 \geq \dfrac{1}{-1}$.

The point $(-2, -2)$ is out because $-2 \ngeq \dfrac{1}{-2}$.

Troops on both sides of the border

Some dictators like to control their borders by having troops on both sides of the border. You cannot always assume that if one side of a boundary is in, then the other side is out, or vice versa.

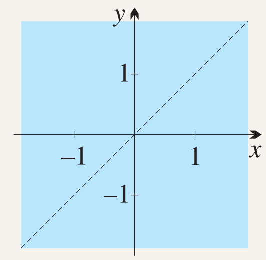
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 17 3F

a Sketch $(y - x)^2 > 0$. **b** Describe $(y - x)^2 \le 0$.

SOLUTION



a The boundary is $(y - x)^2 = 0$, that is, $y = x$, and is excluded. But if $P(x, y)$ is any point off the line, then $x \ne y$, so $(x - y)^2 > 0$, and the points is included. Hence every point in the plane is in the region except for the points on the line $y = x$.

b Similarly, the region $(y - x)^2 \le 0$ is just the line $y = x$.

12 TWO QUALIFICATIONS OF THE METHOD FOR FINDING REGIONS

- Points where the LHS or the RHS is undefined form a possible boundary.
- Both sides of a boundary may be in, or both sides may be out.

Intersections and unions of regions

Some questions will ask explicitly for the intersection or union of two regions. Other questions will implicitly ask for intersections. For example,

$$|2x + 3y| < 6$$

means $-6 < 2x + 3y < 6$, so is the intersection of $2x + 3y > -6$ and $2x + 3y < 6$.

Or there may be a restriction on x or on y, as in

$$x^2 + y^2, \text{ where } x \le 3 \text{ and } y > -4,$$

which means the intersection of three different regions.

13 INTERSECTIONS AND UNIONS OF REGIONS

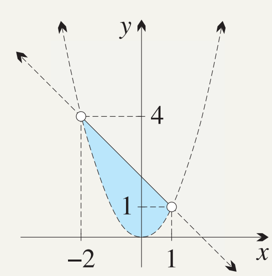
- Draw each region, then sketch the intersection or union.
- Pay particular attention to whether corner points are included or excluded.

Example 18 3F

Graph the intersection and union of the regions

$$y > x^2 \text{ and } x + y \le 2.$$

SOLUTION



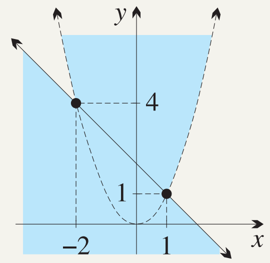
The boundary of the first region is $y = x^2$, and the region lies above the curve (with the boundary excluded).

The boundary of the second region is $x + y = 2$. Solving for y gives $y \le 2 - x$, so the region lies below the curve (with the boundary included).

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

By inspection, or by simultaneous equations, the parabola and the line meet at $(1, 1)$ and $(-2, 4)$. These points are excluded from the intersection because they are not in the region $y > x^2$, but included in the union because they are in the region $x + y \le 2$.

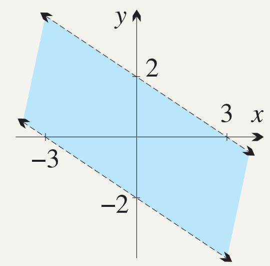


Example 19 3F

Graph the region $|2x + 3y| < 6$.

SOLUTION

This is the region $-6 < 2x + 3y < 6$.
The boundaries are the parallel lines
$2x + 3y = 6$ and $2x + 3y = -6$, both of which are excluded.
The required region is the region between these two lines.

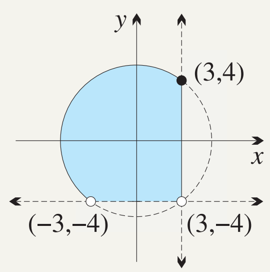


Example 20 3F

Graph the region
$$x^2 + y^2 \le 25, \quad \text{where } x \le 3 \text{ and } y > -4,$$
giving the coordinates of each corner point.

SOLUTION

The boundaries are $x^2 + y^2 = 25$ (included), and the vertical and horizontal lines $x = 3$ (included) and $y = -4$ excluded.
The points of intersection are $(3, 4)$ (included), $(3, -4)$ excluded and $(-3, -4)$ excluded.



Exercise 3F

FOUNDATION

1 For each inequation:

i sketch the boundary,

ii shade the region above or below the boundary, as required.

a $y < 1$ **b** $y > x - 1$ **c** $y \le 3 - x$ **d** $y < \frac{1}{2}x - 1$

2 For each inequation:

i sketch the boundary,

ii shade the region to the left or right of the boundary, as required.

a $x > 1$ **b** $x \ge y + 2$ **c** $x < 2y - 1$ **d** $x > 3 - y$

3 For each inequation, sketch the boundary line, then use a suitable test point to decide which side of the line to shade.

a $2x + 3y - 6 > 0$ **b** $x - y + 4 \ge 0$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

4 For each inequation, sketch the boundary circle, then use a suitable test point to decide which region to shade.

a $x^2 + y^2 < 4$ **b** $x^2 + y^2 \geq 1$

c $(x - 2)^2 + y^2 \leq 4$ **d** $(x + 1)^2 + (y - 2)^2 > 9$

5 Sketch the following regions (some of the quadratics need factoring):

a $y < x^2 - 2x - 3$ **b** $y \geq x^2 + 2x + 1$

c $y > 4 - x^2$ **d** $y < (5 - x)(1 + x)$

6 Draw the following regions of the number plane:

a $y > 2^x$ **b** $y \leq |x + 1|$ **c** $y > x^3$ **d** $y \leq \log_2 x$

7 a Find the intersection point of the lines $x = -1$ and $y = 2x - 1$.

b Hence sketch the intersection of the regions $x > -1$ and $y \leq 2x - 1$, paying careful attention to the boundaries and their point of intersection.

c Likewise sketch the union of the two regions.

8 a Sketch on separate number planes the two regions $y < x$ and $y \geq -x$. Hence sketch:

i the union of these two regions, **ii** the intersection of the regions.

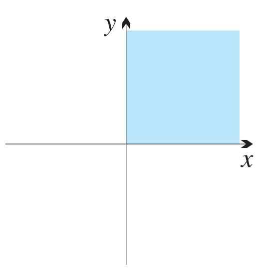
Pay careful attention to the boundaries and their points of intersection.

b Similarly, graph the union and intersection of $y > \frac{1}{2}x + 1$ and $y \leq -x - 2$.

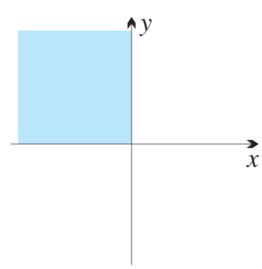
DEVELOPMENT

9 Identify the inequations that correspond to the following regions:

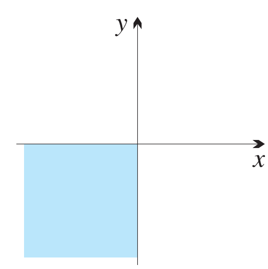
a



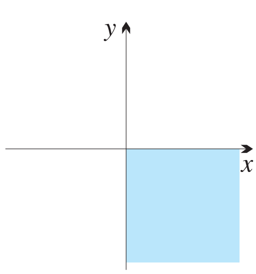
b



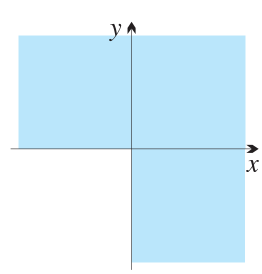
c



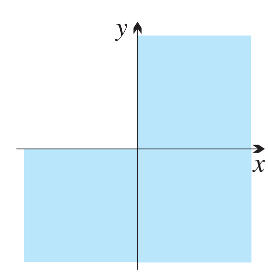
d



e

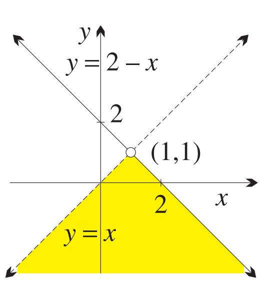


f

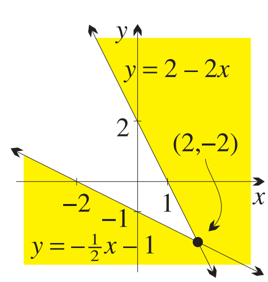


10 Write down intersections or unions that correspond to the following regions:

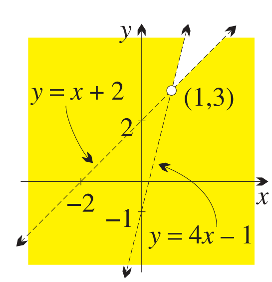
a



b



c



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 **a** Show that the lines $y = x + 1$, $y = -\frac{1}{2}x - 2$, and $y = 4x - 2$ intersect at $(-2, -1)$, $(0, -2)$ and $(1, 2)$. Then sketch all three on the same number plane.

b Hence sketch the regions indicated by:

i $y < x + 1$ and $y \geq -\frac{1}{2}x - 2$

ii $y < x + 1$ and $y \geq -\frac{1}{2}x - 2$ and $y < 4x - 2$

iii $y > x + 1$ or $y < -\frac{1}{2}x - 2$ or $y < 4x - 2$

12 **a** Sketch the intersection of $x^2 + y^2 > 1$ and $x^2 + y^2 \leq 9$.

b What is the union of these two regions?

13 **a** Sketch the union of $x^2 + y^2 \leq 1$ and $y > 2 - x$.

b What is the intersection of these two regions?

14 **a** Find the intersection points of the line $y = 4 - x$ and the circle $x^2 + y^2 = 16$.

b Hence sketch

i the intersection of, and

ii the union, of $y \geq 4 - x$ and $x^2 + y^2 < 16$.

15 **a** The inequation $|x| < 2$ implies the intersection of two regions in the number plane. Write down the equations of these two regions. Hence sketch the region $|x| < 2$.

b The inequation $|x - y| \leq 2$ implies the intersection of two regions. Write down the inequations of these two regions. Hence sketch $|x - y| \leq 2$.

16 Sketch the region $x^2 + y^2 \geq 5$ for the domain $x > -1$ and range $y < 2$, and give the coordinates of each corner.

17 Sketch the region $y \leq x^2 - 2x + 2$, where $y \geq 0$ and $0 \leq x \leq 2$.

18 **a** Draw the curve $y = \sqrt{x}$.

b Explain why the y-axis $x = 0$ is a boundary of the region $y < \sqrt{x}$.

c Hence sketch the region $y < \sqrt{x}$.

19 **a** Explain why $x = 0$ is a boundary of the region $y \geq \dfrac{1}{x}$.

b Hence sketch the region $y \geq \dfrac{1}{x}$.

20 Carefully sketch $y < \sqrt{4 - x^2}$, paying attention to implied boundaries.

21 Sketch the region defined by $x > |y + 1|$. (Hint: $x = |y + 1|$ is the inverse of what function?)

ENRICHMENT

22 **a** How many regions do the coordinate axes and the hyperbola $y = \dfrac{1}{x}$ divide the number plane into?

b Carefully sketch the following regions.(Hint: It may help to take test points in each of the regions found in the previous part.)

i $xy < 1$ **ii** $1 > \dfrac{1}{xy}$

23 Graph the regions:

a $|y| > |x|$ **b** $|xy| \geq 1$ **c** $\dfrac{1}{x} > \dfrac{1}{y}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3G Review of translations and reflections

This short section reviews translations and reflections in preparation for dilations.

Translations and reflections

Here are the rules from Chapter 4 of the Year 11 book.

14 A SUMMARY OF SHIFTING AND REFLECTING

Transformation	By replacement	By function rule
Shift horizontally h right	Replace x by $x - h$	$y = f(x) \to y = f(x - h)$
Shift vertically k up	Replace y by $y - k$	$y = f(x) \to y = f(x) + k$
Reflect in the y-axis	Replace x by $-x$	$y = f(x) \to y = f(-x)$
Reflect in the x-axis	Replace y by $-y$	$y = f(x) \to y = -f(x)$
Rotate 180° about O	Replace x by $-x$, y by $-y$	$y = f(x) \to y = -f(-x)$

The equation of a transformed relation can always be obtained *by replacement*, whether or not the relation is a function. The second method, *by function rule*, can only be used when the relation is a function.

Example 21 — 3G

Write down the equation of the resulting graph when each transformation below is applied to the circle $(x - 1)^2 + (y + 2)^2 = 1$.

a shift left 3 units **b** reflect in the x-axis **c** rotate 180° about O

Then sketch all four circles on one set of axes.

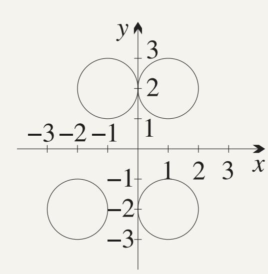
SOLUTION

The original circle is $(x - 1)^2 + (y + 2)^2 = 1$.

a Shifting left 3 units gives $((x + 3) - 1)^2 + (y + 2)^2 = 1$
$(x + 2)^2 + (y + 2)^2 = 1$.

b Reflecting in the x-axis gives $(x - 1)^2 + (-y + 2)^2 = 1$
$(x - 1)^2 + (y - 2)^2 = 1$.

c Rotating 180° about O gives $(-x - 1)^2 + (-y + 2)^2 = 1$
$(x + 1)^2 + (y - 2)^2 = 1$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 22 3G

Write down the equation of the resulting graph when each transformation below is applied to the exponential curve $y = 2^x$.

a shift down 1 unit **b** reflect in the y-axis **c** rotate 180° about O

Then sketch all four curves on one set of axes.

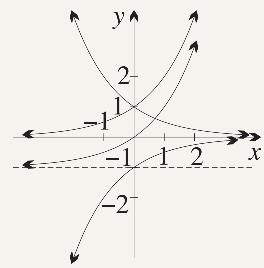
SOLUTION

The original curve is $y = 2^x$.

a Shifting down 1 unit gives $y = 2^x - 1$.

b Reflecting in the y-axis gives $y = 2^{-x}$.

c Rotating 180° about O gives $y = -2^{-x}$.



Exercise 3G

FOUNDATION

1 Write down the new equation for each function or relation after the given shift has been applied. Draw a graph of the image after the shift.

a $y = x^2$: right 2 units
b $y = 2^x$: down 1 unit
c $y = x^2 - 1$: down 3 units
d $y = \frac{1}{x}$: right 3 units
e $x^2 + y^2 = 4$: up 1 unit
f $y = \log_2 x$: left 1 units
g $y = \sin x$: left $\frac{\pi}{2}$ units
h $y = \sqrt{x}$: up 2 units

2 Repeat the previous question for the given reflection line or rotation.

a x-axis **b** y-axis **c** x-axis **d** x-axis
e y-axis **f** rotate 180° **g** rotate 180° **h** y-axis

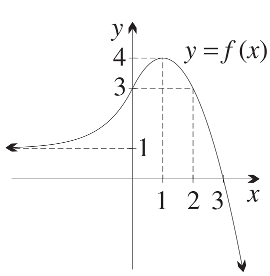
3 In which parts of Question 2 was the result the same as the original function? In each case, explain geometrically why that happened.

4 Use the shifting results and completion of the square where necessary to determine the centre and radius of each circle.

a $(x + 1)^2 + y^2 = 4$
b $(x - 1)^2 + (y - 2)^2 = 1$
c $x^2 - 4x + y^2 = 0$
d $x^2 + y^2 - 6y = 16$

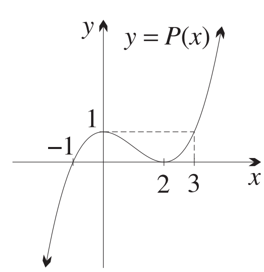
5 In each case an unknown function has been drawn. Draw the functions specified below each.

a



i $y = f(x - 2)$
ii $y = f(x) - 2$

b



i $y = P(x + 1)$
ii $y = P(x) + 1$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 The composition of functions can sometimes result in translations.

a Let $h(x) = x - 3$. Draw the following using the graph of $f(x)$ given in Question **5a**:

i $y = f \circ h(x)$ **ii** $y = h \circ f(x)$

b Let $k(x) = x + 2$. Draw the following using the graph of $P(x)$ given in Question **5b**:

i $y = P \circ k(x)$ **ii** $y = k \circ P(x)$

DEVELOPMENT

7 Write down the equation for each function after the given translation has been applied.

a $y = x^2$: left 1 unit, up 2 units

b $y = \frac{1}{x}$: right 2 units, up 3 units

c $y = \cos x$: right $\frac{\pi}{3}$ units, down 2 units

d $y = e^x$: left 2 units, down 1 unit

8 In each part explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each curve:

a From $y = -x$: **i** $y = -x + 2$ **ii** $y = -x - 2$ **iii** $y = x + 4$

b From $y = x^2$: **i** $y = (x + 1)^2$ **ii** $y = -(x + 1)^2$ **iii** $y = (x + 1)^2 - 1$

c From $y = \sqrt{x}$: **i** $y = \sqrt{x + 4}$ **ii** $y = -\sqrt{x}$ **iii** $y = -\sqrt{x + 4}$

d From $y = \dfrac{2}{x}$: **i** $y = \dfrac{2}{x} - 1$ **ii** $y = \dfrac{2}{x + 2} - 1$ **iii** $y = -\dfrac{2}{x}$

e From $y = \sin x$: **i** $y = \sin x - 1$ **ii** $y = \sin\left(x - \frac{\pi}{4}\right) - 1$ **iii** $y = -\sin x$

9 Answer the following questions about the cubic $y = x^3 - 3x$.

a Find the coordinates of the two points where the tangent is horizontal.

b The cubic is shifted 1 unit up.

i Write down the equation of this new cubic.

ii Show that the x-coordinates where the tangent is horizontal have not changed.

c The original cubic is shifted 1 unit left.

i Write down the equation of this third cubic, expanding any brackets.

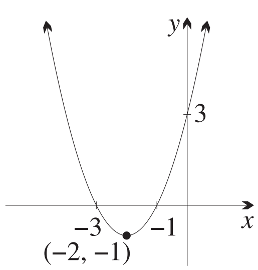
ii Show that the y-coordinates where the tangent is horizontal have not changed.

10 Complete the square then sketch each circle, stating the centre and radius. Find any intercepts with the axes by substituting $x = 0$ and $y = 0$.

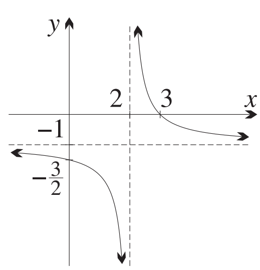
a $x^2 + 4x + y^2 - 8y = 0$ **b** $x^2 - 2x + y^2 + 4y = -1$

11 Describe each graph below as a standard curve transformed either by two shifts, or by a reflection followed by a shift. Hence write down its equation.

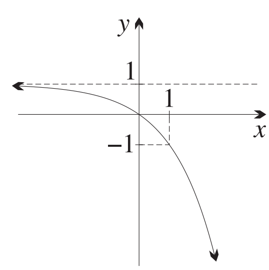
a



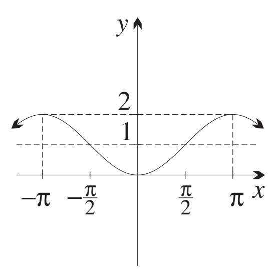
b



c

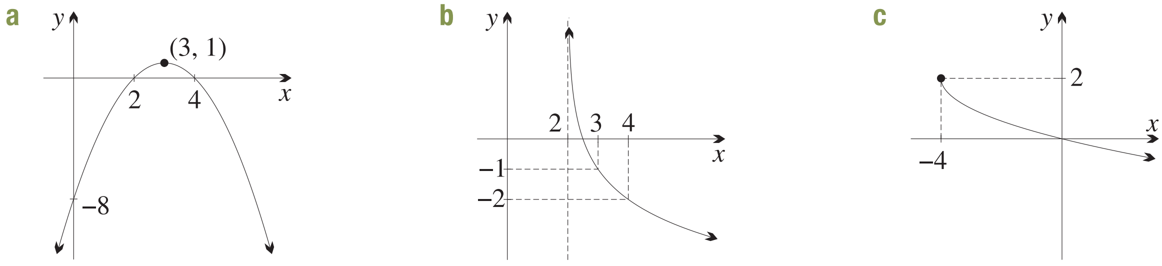


d



ISBN 978-1-108-76630-2

12 Describe each graph below as the given standard curve transformed by a reflection followed by two shifts, and hence write down its equation.



a Start with $y = x^2$.

b Start with $y = \log_2 x$.

c Start with $y = \sqrt{x}$.

ENRICHMENT

13 **a** Let $\mathcal{I}$ be a reflection in the line $y = x$ and $\mathcal{H}$ be a reflection in the line $y = 0$.

i Which functions are unchanged by applying $\mathcal{I}$, then $\mathcal{H}$, then $\mathcal{I}$, then $\mathcal{H}$? It may help to use a square piece of paper with something written on it.

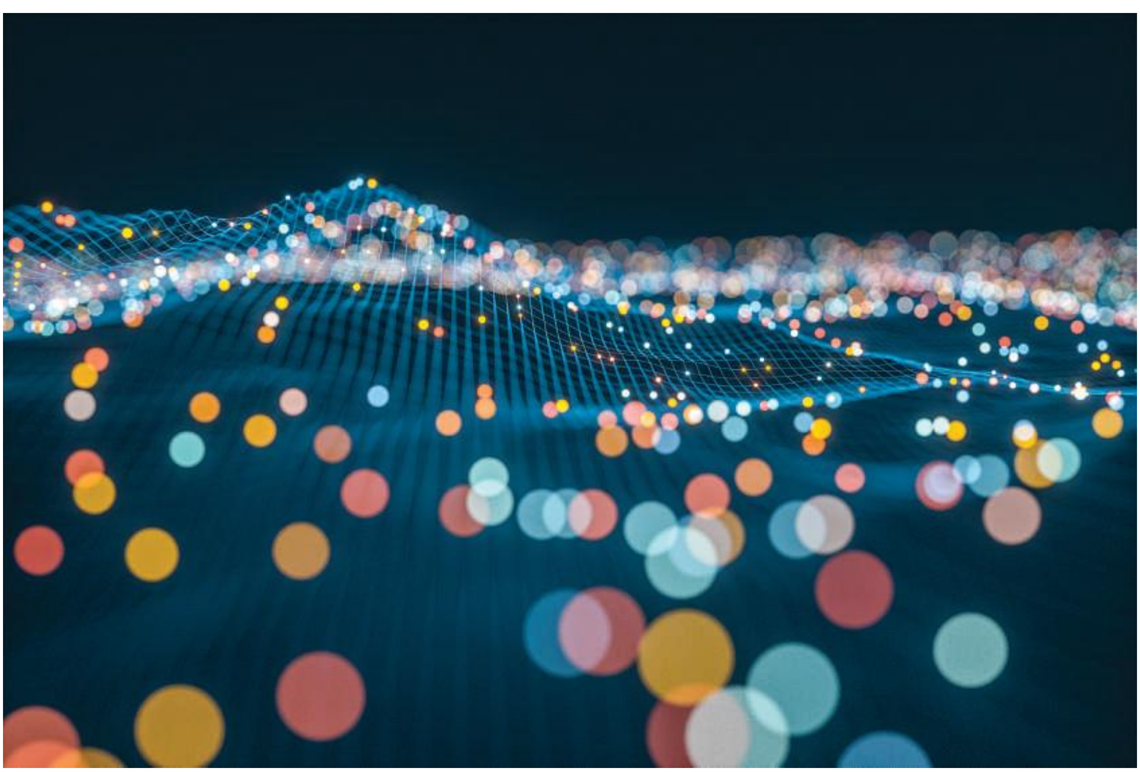
ii Do the transformations $\mathcal{I}$ and $\mathcal{H}$ commute? That is, if $\mathcal{I}$ is applied first and then $\mathcal{H}$, is the result the same as when $\mathcal{H}$ is applied first followed by $\mathcal{I}$?

b The graph of $y = f(x)$ is shifted left by a, reflected in the y-axis and finally shifted right by a.

i What is the equation of the new graph?

ii The new graph could instead be achieved by a single reflection in which line?

iii Another function $y = g(x)$ undergoes the same transformation and it is noted that there is no change. What symmetry must $g(x)$ possess? Prove the result by substituting $x = a + t$.



ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

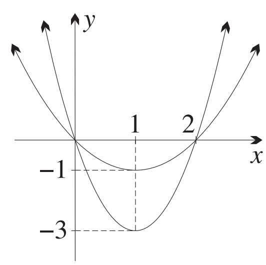
3H Dilations

A dilation is a stretch of a curve in one direction. For example, a dilation distorts a circle into an ellipse. Dilations are another kind of transformation of curves, and they belong naturally with the translations and reflections that were reviewed in the previous section. Most of the functions in the course can be reduced to very simple functions using a combination of translations, reflections and dilations.

Stretching a graph vertically

Compare the graphs of

$$y = x(x-2) \qquad \text{and} \qquad y = 3x(x-2).$$



Each value in the table below for $y = 3x(x-2)$ is three times the corresponding value in the table for $y = x(x-2)$. This means that the graph of $y = 3x(x-2)$ is obtained from the graph of $y = x(x-2)$ by *stretching away from the x-axis in the vertical* direction by a factor of 3:

x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8
$3x(x-2)$	24	9	0	-3	0	9	24

We can rewrite the equation $y = 3x(x-2)$ as $\frac{y}{3} = x(x-2)$. This makes it clear that the stretching has been obtained by replacing y by $\frac{y}{3}$.

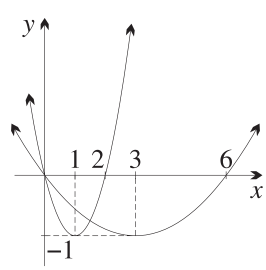
The x-axis is the *axis of dilation*. Points on the x-axis do not move, and all other points on the graph triple their distance from the x-axis.

15 VERTICAL DILATIONS — STRETCHING A GRAPH VERTICALLY

- To stretch a graph in a vertical direction by a factor of a, replace y by $\dfrac{y}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = a\,f(x)$.
- The *axis of dilation* for these transformations is the x-axis.

Stretching a graph horizontally

By analogy with the previous example, the graph of $y = x(x-2)$ can be *stretched horizontally away from the y-axis* by a factor of 3 by replacing x by $\dfrac{x}{3}$, giving the new function



$$y = \frac{x}{3}\left(\frac{x}{3} - 2\right) = \tfrac{1}{9}x(x-6).$$

Two table of values should make this clear. The first table is the original graph, the second is the new function.

x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8

x	-6	-3	0	3	6	9	12
$\frac{x}{3}\left(\frac{x}{3}-2\right)$	8	3	0	-1	0	3	8

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The y-coordinates in each table are the same, but we needed to treble the x-coordinates to produce those same y-coordinates. Thus the y-axis is the *axis of dilation*, and the *dilation factor* is 3, because the point $(0, 0)$ on the y-axis does not move, and all other points on the graph triple their distance from the y-axis.

16 HORIZONTAL DILATIONS — STRETCHING A GRAPH HORIZONTALLY

- To stretch the graph in a horizontal direction by a factor of a, replace x by $\frac{x}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = f\left(\frac{x}{a}\right)$.
- The *axis of dilation* for these transformations is the y-axis.

Example 23 **3H**

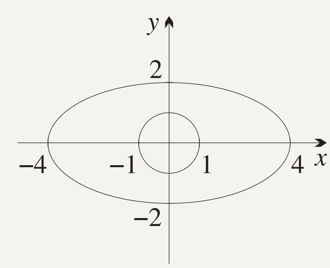
Obtain the graph of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$.

SOLUTION

The equation can be rewritten as

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1,$$

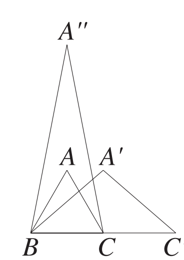
which is the unit circle stretched vertically by a factor of 2 and horizontally by a factor of 4.



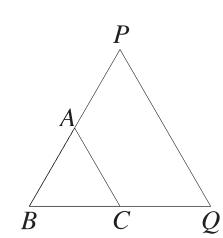
Note: Any curve of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called an *ellipse*. It can be obtained from the unit circle $x^2 + y^2 = 1$ by stretching horizontally by a factor of a and vertically by a factor of b, so that its x-intercepts are a and $-a$ and its y-intercepts are b and $-b$.

Enlargements

The dilation of a figure is usually not similar to the original. For example, the equilateral triangle ABC in the figure to the right with its base on the x-axis is stretched to the squat isosceles triangle $A'BC'$ by a horizontal dilation with factor 2, and it is stretched to the skinny isosceles triangle $A''BC$ by a vertical dilation with factor 3.



But if two dilations with the same factor 2, one horizontal and the other vertical, are applied in order to the equilateral triangle ABC — the order does not matter — the result is the similar equilateral triangle PBQ. Such a combined transformation is called an *enlargement*, and the factor 2 is called the *enlargement factor* or *similarity factor*.



In the coordinate plane, the *centre* of an enlargement is normally taken as the origin.

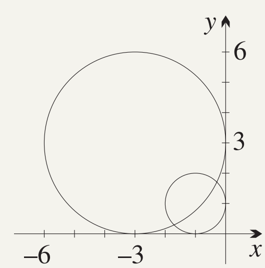
ISBN 978-1-108-76630-2

17 ENLARGMENTS

- An enlargement of a figure is similar to the original. In particular, matching angles are equal, and the ratios of matching lengths are equal.
- The composition of two dilations with the same factor, one horizontal and one vertical, is an *enlargement* with centre the origin.
- To apply an enlargement with factor a, replace x by $\dfrac{x}{a}$ and y by $\dfrac{y}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = af\left(\dfrac{x}{a}\right)$.

Example 24 — 3H

Apply an enlargement with centre the origin and factor 3 to the circle $(x + 1)^2 + (y - 1)^2 = 1$. Write down the new function, then sketch both curves.



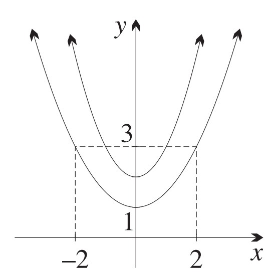
SOLUTION

The new function is $\left(\dfrac{x}{3} + 1\right)^2 + \left(\dfrac{y}{3} - 1\right)^2 = 1$

$\boxed{\times\ 3^2 = 9}$ $\quad (x + 3)^2 + (y - 3)^2 = 9.$

The two circles are sketched to the right.

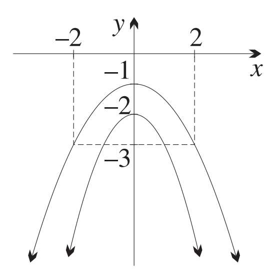
Stretching with a fractional or negative factor



In the upper diagram to the right, a vertical dilation with factor $\frac{1}{2}$ has been applied to the parabola $y = x^2 + 2$ to yield the parabola

$$\frac{y}{\frac{1}{2}} = x^2 + 2, \qquad \text{that is} \qquad y = \tfrac{1}{2}x^2 + 1.$$

The result is a compression, but we still call it a dilation.



In the lower diagram to the right, vertical dilations with factors -1 and $-\frac{1}{2}$ have been applied to the same parabola $y = x^2 + 2$. The results are the parabolas

$$y = -x^2 - 2 \qquad \text{and} \qquad y = -\tfrac{1}{2}x^2 - 1.$$

The first parabola is the reflection of the original in the x-axis. The second parabola is the reflection in the x-axis of the compressed image.

When the dilation factor is negative, the dilation can be thought of as a dilation with positive factor followed by a reflection.

In particular, a reflection is a dilation with factor -1.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 DILATIONS WITH A FRACTIONAL OR NEGATIVE FACTOR

- If the dilation factor is between 0 and 1, the graph is compressed.
- If the dilation factor is negative, the dilation is the composition of a dilation with positive factor and a reflection — the order does not matter.
- In particular:
  — A reflection is a dilation with factor -1.
  — A rotation of 180° about the origin is an enlargement with factor -1, and is often called a *reflection in the origin.*

Example 25 3H

Write down the new functions when each dilation is applied to the parabola $y = (x - 3)(x - 5)$. Then sketch the four curves on one set of axes.

a A horizontal dilation with factor -2.
b A horizontal dilation with factor $-\frac{1}{2}$.
c A vertical dilation with factor -1.

SOLUTION

a Replacing x by $\dfrac{x}{-2}$,

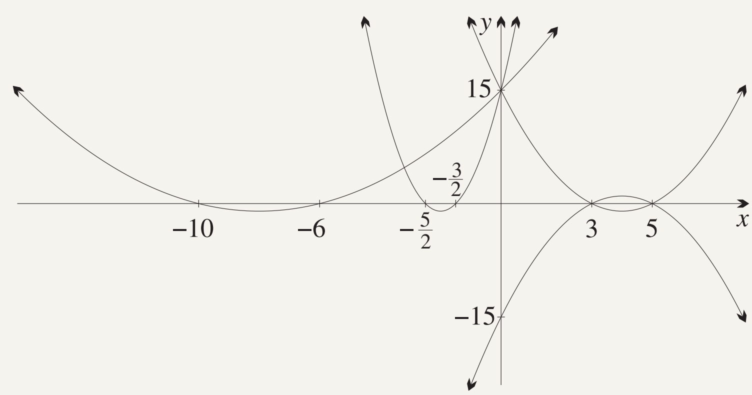
$y = (-\frac{x}{2} - 3)(-\frac{x}{2} - 5)$
$y = \frac{1}{4}(x + 6)(x + 10)$

b Replacing x by $\dfrac{x}{-1/2} = -2x$,

$y = (-2x - 3)(-2x - 5)$
$y = 4(x + 1\frac{1}{2})(x + 2\frac{1}{2})$

c Replacing y by $\dfrac{y}{-1} = -y$,

$-y = (x - 3)(x - 5)$
$y = -(x - 3)(x - 5)$



Note: The word 'dilation' is often used to mean 'enlargement', but in this course, it means a stretching in just one direction. Be careful when looking at other sources.

Exercise 3H FOUNDATION

1 Write down the new equation for each function or relation after the given dilation has been applied. Draw a graph of the image after the shift.

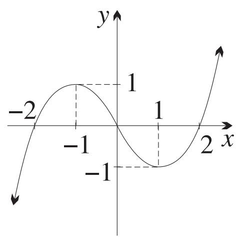
a $y = x^2$: horizontally by $\frac{1}{2}$
b $y = 2^x$: vertically by 2
c $y = x^2 - 1$: vertically by -1
d $y = \frac{1}{x}$: horizontally by 2
e $x^2 + y^2 = 4$: vertically by $\frac{1}{3}$
f $y = \log_2 x$: horizontally by -1
g $y = \sin x$: horizontally by $\frac{1}{2}$
h $y = -\sqrt{x}$: horizontally by -2

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

2 In each case an unknown function has been drawn. Use dilations to draw the new functions indicated beneath each.

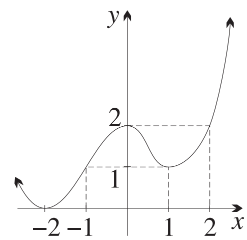
a $y = f(x)$



i $y = f(2x)$

ii $y = 2f(x)$

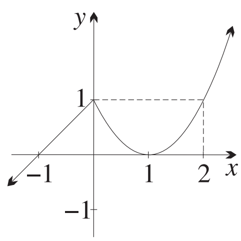
b $y = P(x)$



i $y = P\left(\frac{x}{2}\right)$

ii $y = \frac{1}{2}P(x)$

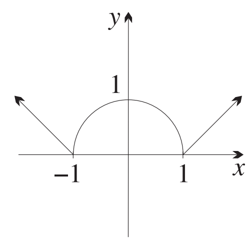
c $y = h(x)$



i $\frac{y}{2} = h(x)$

ii $y = h\left(\frac{x}{2}\right)$

d $y = g(x)$



i $2y = g(x)$

ii $y = g(2x)$

3 Sketch $x + y = 1$. Then explain how each graph below may be obtained by dilations of the first graph (there may be more than one answer), and sketch it.

a $\frac{x}{2} + y = 1$

b $\frac{x}{2} + \frac{y}{4} = 1$

c $2x + y = 1$

4 **a** The circle $(x - 3)^2 + y^2 = 4$ is enlarged by factor $\frac{1}{3}$ with centre the origin. Write down the new equation and draw both circles on the one set of axes.

b The hyperbola $y = \dfrac{1}{x}$ is enlarged by factor $\sqrt{3}$ with centre the origin. Write down the new equation and draw both hyperbolae on the one set of axes.

5 In each case graph the three given equations on one set of axes by using dilations:

a $y = x(4 + x)$, $y = 2x(4 + x)$, and $y = \frac{x}{2}\left(4 + \frac{x}{2}\right)$.

b $x^2 + y^2 = 36$, $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 36$, and $(2x)^2 + (3y)^2 = 36$.

6 The composition of functions can sometimes result in dilations.

a Let $k(x) = 3x$. Draw the following using the graph of $h(x)$ given in Question 2c:

i $y = h \circ k(x)$ **ii** $y = k \circ h(x)$

b Let $\ell(x) = \frac{1}{3}x$. Draw the following using the graph of $g(x)$ given in Question 2d:

i $y = g \circ \ell(x)$ **ii** $y = \ell \circ g(x)$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

7 Sketch each group of three trigonometric functions on the one set of axes.

a $y = \sin x$, $y = 3\sin x$, $y = 3\sin 2x$ **b** $y = \cos x$, $y = \cos\frac{x}{2}$, $y = 2\cos\frac{x}{2}$

8 Answer the following questions about the cubic $y = x^3 - 3x$.

a Find the coordinates of the two points where the tangent is horizontal.

b The cubic is dilated vertically by factor 2.

i Write down the equation of this new cubic.

ii Show that the x-coordinates where the tangent is horizontal have not changed.

c The original cubic is dilated horizontally by factor 3.

i Write down the equation of this third cubic.

ii Show that the y-coordinates where the tangent is horizontal have not changed.

9 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = 3x^2 - 6x$ **b** $y = \dfrac{1}{x-4}$ and $y = \dfrac{1}{2x-4}$

c $y = \cos x$ and $y = \cos\frac{x}{4}$ **d** $y = \dfrac{1}{x+1}$ and $y = \dfrac{2}{x+1}$

10 Consider the hyperbola $y = \dfrac{1}{x}$.

a The hyperbola is stretched horizontally by factor 2. Write down its equation.

b The original hyperbola is stretched vertically by factor 2. Write down its equation.

c What do you notice about the answers to parts **a** and **b**?

d Can the hyperbolae in parts a or b be achieved by an enlargement?

e Investigate whether there are any other functions which exhibit similar behaviour.

11 Consider the parabola $y = x^2$.

a The parabola is dilated horizontally by factor $\frac{1}{2}$. Write down its equation.

b The original parabola is dilated vertically by factor 4. Write down its equation.

c What do you notice about the answers to parts **a** and **b**?

d Can the parabolas in parts a or b be achieved by an enlargement?

e Investigate whether there are any other functions which exhibit similar behaviour.

12 The mass M grams of a certain radioactive substance after t years is modeled by the formula $M = 3 \times 2^{-\frac{1}{53}t}$.

a Find the initial mass.

b Find the time taken for the mass to halve, called the *half-life*.

c Suppose now that the initial mass is doubled.

i Explain this in terms of a dilation and hence write down the new equation for M.

ii Show that the dilation does not change the value of the half-life.

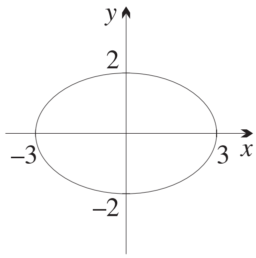
13 Show that the equation $y = mx$ of a straight line through the origin is unchanged by any enlargement with centre the origin.

ISBN 978-1-108-76630-2

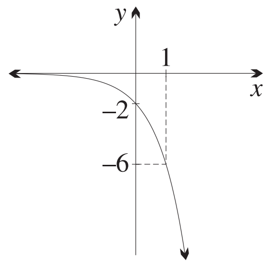
Cambridge University Press
Updated July 2022

14 Describe each graph below as a standard curve transformed by dilations, and hence write down its equation.

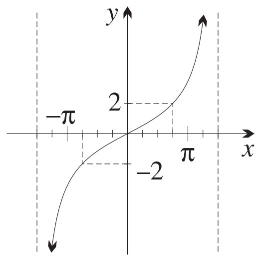
a



b



c



ENRICHMENT

15 **a** For each pair of curves, suggest two simple and distinct transformations by which the second equation may be obtained from the first:

i $y = 2^x,\ y = 2^{x+1}$ **ii** $y = \dfrac{1}{x},\ y = \dfrac{k^2}{x}$ **iii** $y = 3^x,\ y = 3^{-x}$

b Investigate other combinations of curves and transformations with similar ambiguity.

16 The parabola $y = x^2$ is stretched horizontally by factor a. Clearly a horizontal stretch by factor $\frac{1}{a}$ will restore the original parabola. What other stretch will produce a new parabola which appears identical to the original parabola $y = x^2$?

17 Determine how the curve $y = x^3 - x$ must be transformed in order to obtain the graph of $y = x^3 - 3x$. (Hint: Only stretchings are involved.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3I Combinations of transformations

We will now apply several transformations to a graph one after the other.

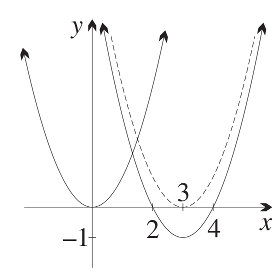
In this section we will mostly regard reflections in the axes as dilations with factor -1, and of course a rotation of 180° about the origin is the composition of two reflections. This reduces the types of transformations to just four — two translations and two dilations:

19 A SUMMARY OF TRANSFORMATIONS

Transformation	By replacement	By function rule
Shift horizontally right h	Replace x by $x - h$	$y = f(x) \rightarrow y = f(x - h)$
Shift vertically up k	Replace y by $y - k$	$y = f(x) \rightarrow y = f(x) + k$
Stretch horizontally factor a	Replace x by $\dfrac{x}{a}$	$y = f(x) \rightarrow y = f\left(\dfrac{x}{a}\right)$
Stretch vertically factor b	Replace y by $\dfrac{y}{b}$	$y = f(x) \rightarrow y = b\,f(x)$

We shall see that it sometimes matters in which order the two transformations are applied. Two transformations are said to *commute* if the order in which they are applied does not matter, whatever graph they are applied to.

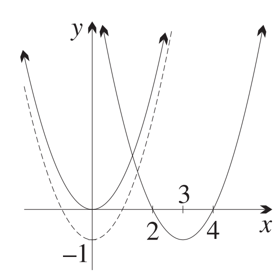
Two translations always commute



Suppose that the parabolic graph $y = x^2$ is shifted right 3 and then down 1.

$$\begin{aligned} &\text{Shifting right 3,} & y &= (x-3)^2, \\ &\text{and shifting down 1,} & y + 1 &= (x-3)^2 \\ & & y &= (x-3)^2 - 1. \end{aligned}$$

The result is exactly the same if the graph $y = x^2$ is shifted down 1 and then right 3.



$$\begin{aligned} &\text{Shifting down 1,} & y + 1 &= x^2 \\ & & y &= x^2 - 1, \\ &\text{and shifting right 3,} & y &= (x-3)^2 - 1. \end{aligned}$$

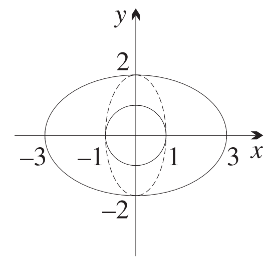
Thus the two translations commute.
In general, any two translations commute.

Two dilations always commute

Suppose that the circle graph $x^2 + y^2 = 1$ is stretched vertically with factor 2 and then horizontally with factor 3.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



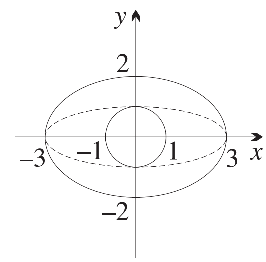
Stretching vertically factor 2, $x^2 + \left(\dfrac{y}{2}\right)^2 = 1$

$$x^2 + \frac{y^2}{4} = 1,$$

and stretching horizontally factor 3, $\left(\dfrac{x}{3}\right)^2 + \dfrac{y^2}{4} = 1$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The result is the same if the graph is stretched horizontally with factor 3 and then vertically with factor 2.



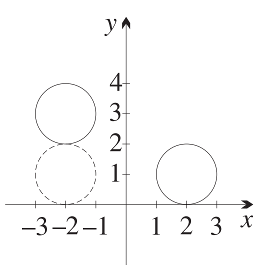
Stretching horizontally factor 3, $\dfrac{x^2}{9} + y^2 = 1,$

and stretching vertically factor 2, $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

Thus the two dilations commute, and in general, any two dilations commute.

A horizontal dilation and a vertical translation commute

Apply a reflection in the y-axis (horizontal dilation with factor -1), then shift up 2, successively to the circle $(x - 2)^2 + (y - 1)^2 = 1$.



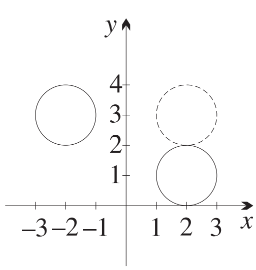
Reflecting in the y-axis, $(-x - 2)^2 + (y - 1)^2 = 1$

$$(x + 2)^2 + (y - 1)^2 = 1,$$

and shifting up 2, $(x + 2)^2 + (y - 2 - 1)^2 = 1$

$$(x + 2)^2 + (y - 3)^2 = 1.$$

The resulting circle $(x + 2)^2 + (y - 3)^2 = 1$ is the same if the transformations are done in the reverse order.



Shifting up 2, $(x - 2)^2 + (y - 3)^2 = 1,$

and reflecting in the y-axis, $(-x - 2)^2 + (y - 3)^2 = 1$

$$(x + 2)^2 + (y - 3)^2 = 1.$$

Thus the two transformations commute. In general, any horizontal dilation and any vertical translation commute. Similarly, any vertical dilation and any horizontal translation commute.

20 COMMUTING TRANSFORMATIONS

- Any two translations commute.
- Any two dilations commute (including reflections).
- A translation and a dilation commute if one is vertical and the other horizontal.

Transformations that do not commute

In the remaining case, the transformations do not commute. That is, a translation and a dilation do not commute when they are both horizontal or both vertical. The next worked examples give two examples of this.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 26 3I

a The reflection in the y-axis (horizontal dilation with factor -1) and the translation left 2 are applied successively to the circle $(x+3)^2+y^2=1$. Find the equation of the resulting graph, and sketch it.

b Repeat when the transformations are done in the reverse order.

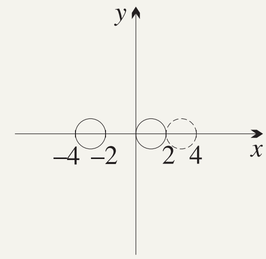
SOLUTION

a Applying the reflection, $(-x+3)^2+y^2=1$

$(x-3)^2+y^2=1,$

and applying the translation, $(x+2-3)^2+y^2=1$

$(x-1)^2+y^2=1.$

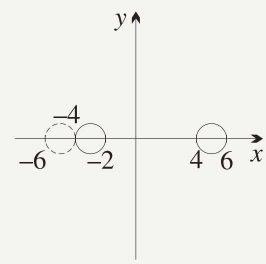


b Applying the translation, $(x+2+3)^2+y^2=1$

$(x+5)^2+y^2=1,$

and applying the reflection, $(-x+5)^2+y^2=1$

$(x-5)^2+y^2=1.$



Example 27 3I

a The vertical dilation with factor $\frac{1}{2}$ and the translation down 3 units are applied successively to the parabola $y=x^2+4$. Find the equation of the resulting graph, and sketch it.

b Repeat when the transformations are done in the reverse order.

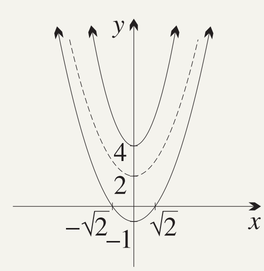
SOLUTION

a Applying the dilation, $y=\frac{1}{2}(x^2+4)$

$y=\frac{1}{2}x^2+2,$

and applying the translation, $y=\frac{1}{2}x^2+2-3$

$y=\frac{1}{2}x^2-1.$

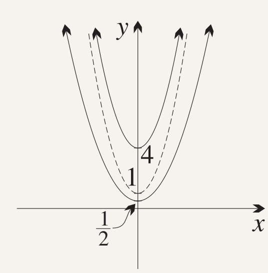


b Applying the translation, $y=x^2+4-3$

$y=x^2+1,$

and applying the dilation, $y=\frac{1}{2}(x^2+1)$

$y=\frac{1}{2}x^2+\frac{1}{2}.$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

21 TRANSFORMATIONS THAT DO NOT COMMUTE

- A vertical translation and a vertical dilation do not commute.
- A horizontal translation and a horizontal dilation do not commute.

A universal formula involving all four transformations

When the graph is a function, there is a universal formula that allows the four transformations to be applied to any function $y = f(x)$. The formula is:

$$y = kf\big(a(x + b)\big) + c.$$

This formula is useful because it applies all four transformations to any function $y = f(x)$. It is useful for trigonometric functions, and for computer programs. The formula is tricky to use, however, and although readers must know the formula, most problems should be done using the methods already presented.

Here is how to analyse the successive transformations involved in this formula.

Start with	$y = f(x)$.
Stretching horizontally with factor $\dfrac{1}{a}$ gives	$y = f(ax)$.
Shifting left b gives	$y = f\big(a(x + b)\big)$
Stretching vertically with factor k gives	$y = kf\big(a(x + b)\big)$
Shifting up c gives	$y = kf\big(a(x + b)\big) + c$

Another way to analyse this formula is to rewrite it progressively so that the four successive transformations can be seen:

$$y = kf\big(a(x + b)\big) + c$$

$$\boxed{-c} \quad y - c = kf\big(a(x + b)\big)$$

$$\boxed{\div k} \quad \frac{y - c}{k} = f\big(a(x + b)\big)$$

$$\frac{y - c}{k} = f\left(\frac{x + b}{1/a}\right)$$

$$\frac{y - c}{k} = f\left(\frac{x - (-b)}{1/a}\right)$$

22 A UNIVERSAL FORMULA INVOLVING ALL FOUR TRANSFORMATIONS

The following sequence of transformations transforms the function $y = f(x)$ to

$$y = kf\big(a(x + b)\big) + c.$$

1. Stretch horizontally with factor $1/a$.
2. Shift left b.
3. Stretch vertically with factor k.
4. Shift up c.

Alternatively, the vertical dilation and translation (step 3 then step 4) could be done before the horizontal dilation and translation (step 1 then step 2).

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 3I

FOUNDATION

1 Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.

a i The parabola is shifted right 1 unit. Sketch the situation and find its equation, expanding any brackets.

ii The parabola in part i is then shifted up 2 units. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is translated up 2 units. Sketch the result and find its equation.

ii The parabola in part i is then translated 1 unit right. Sketch the situation and find its equation, expanding any brackets.

c Parts a and b used the same translations, 1 unit right and 2 units up, but in a different order. Do these transformations commute?

2 As in Question 1, start with the parabola $y = x^2 - 2x$.

a i The parabola is dilated by factor 2 horizontally. Sketch the situation and find its equation.

ii The parabola in part i is then dilated by factor 3 vertically. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is stretched by 3 vertically. Sketch the result and find its equation.

ii The parabola in part i is then stretched by 2 horizontally. Sketch the situation and find its equation.

c Parts a and b used the same dilations, factor 2 horizontally and factor 3 vertically, but in a different order. Do these transformations commute?

3 Once again, start with the parabola $y = x^2 - 2x$.

a i The parabola is dilated by factor 2 horizontally. Sketch the situation and find its equation.

ii The parabola in part i is then translated 1 unit up. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is shifted 1 unit up. Sketch the result and find its equation.

ii The parabola in part i is then stretched by 2 horizontally. Sketch the situation and find its equation.

c Parts a and b used the same transformations, stretch by factor 2 horizontally and shift 1 unit up, but in a different order. Do these transformations commute?

4 Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.

a i The parabola is shifted right by 1 unit. Sketch the situation and find its equation, expanding any brackets.

ii The shifted parabola is then reflected in the y-axis. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is reflected in the y-axis. Sketch the result and find its equation.

ii The reflected parabola is then shifted 1 unit right. Sketch the situation and find its equation, expanding any brackets.

c Parts a and b used a shift 1 unit right and reflection in the y-axis, but in a different order. Do these transformations commute?

DEVELOPMENT

5 Which of the following pairs of transformations commute?

a reflection in the y-axis and horizontal translation

b vertical dilation and vertical translation

c vertical dilation and reflection in the x-axis

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d horizontal translation and vertical translation
e horizontal dilation and horizontal translation
f reflection in the x-axis and horizontal translation

6 Write down the new equation for each function or relation after the given transformations have been applied. Draw a graph of the image.
a $y = x^2$: right 2 units, then dilate by factor $\frac{1}{2}$ horizontally
b $y = 2^x$: down 1 unit then reflect in the y-axis
c $y = x^2 - 1$: down 3 units, then dilate by factor -1 vertically
d $y = \frac{1}{x}$: right 3 units then dilate by factor 2 vertically
e $x^2 + y^2 = 4$: up 2 units then dilate by factor $\frac{1}{2}$ vertically
f $y = \log_2 x$: left 1 units then dilate by factor 2 horizontally
g $y = \sin x$: left π units then reflect in the x-axis
h $y = \sqrt{x}$: up 2 units then dilate by factor -1 horizontally

7 Identify the various transformations to help graph these trigonometric functions. Make sure the transformations are applied in the correct order when they do not commute.
a $y = \sin 2x + 1$ **b** $y = 2 \sin x + 1$ **c** $y = 2 \sin(x + \frac{\pi}{4})$ **d** $y = \sin(2x + \frac{\pi}{4})$

8 Determine the equation of the curve after the given transformations have been applied in the order stated.
a $y = x^2$: left 1, down 4, dilate horizontally by 2,
b $y = x^2$: down 4, dilate horizontally by 2, left 1,
c $y = 2^x$: down 1, right 1, dilate vertically by -2,
d $y = \dfrac{1}{x}$: right 2, dilate by 2 vertically, up 1.

9 The parabola $y = (x - 1)^2$ is shifted 2 left and then reflected in the y-axis.
a Show that the new parabola has the same equation.
b Investigate why this has happened.

ENRICHMENT

10 Identify the transformations of these trigonometric functions and hence sketch them.
a $y = 3 \cos 2x + 1$ **b** $y = 2 \cos\left(x - \frac{\pi}{3}\right) + 2$
c $y = \cos\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 1$ **d** $y = \cos\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 1$

11 Sketch these transformed trigonometric functions.
a $y = 2 \sin\left(\frac{1}{2}x + \frac{\pi}{6}\right) + 1$ **b** $y = 3 \cos\left(2x + \frac{\pi}{4}\right) - 1$ **c** $y = 1 - \cos\left(2x - \frac{\pi}{3}\right)$

12 **a** Let H be a horizontal translation by a units and V be a vertical translation by b units. Show that H and V commute. That is, show that for all functions $y = f(x)$ the result of applying H followed by V is the same as applying V followed by H.
b Let E be a horizontal dilation by factor a and U be a vertical dilation by factor b. Show that E and U commute.
c Let F be a reflection in the y-axis and L be a reflection in the x-axis. Show that F and L commute.
d Now suppose that any two of the above transformations are applied in succession. Which pairs of transformations commute and why?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

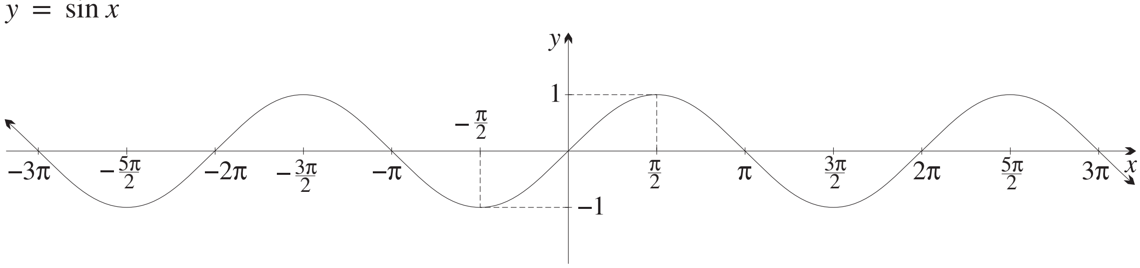
3J Trigonometric graphs

In Sections 11G–11J of the Year 11 book, we developed *radians* as a way of measuring angles. An angle measured in radians is a pure number, without units, and some important conversions are:

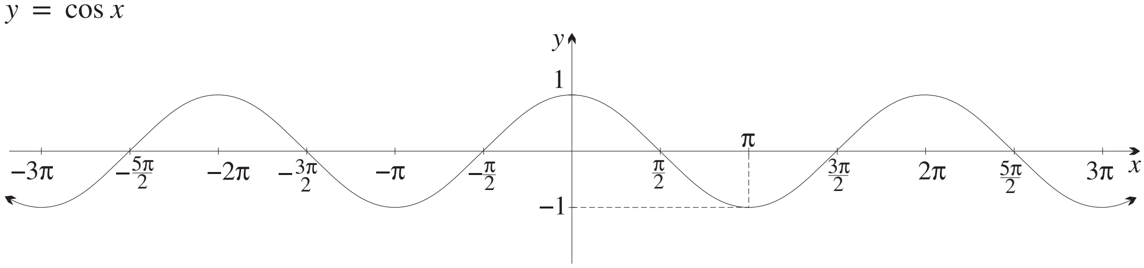
$$2\pi = 360°, \quad \pi = 180°, \quad \tfrac{\pi}{2} = 90°, \quad \tfrac{\pi}{3} = 60°, \quad \tfrac{\pi}{4} = 45°, \quad \tfrac{\pi}{6} = 30°.$$

In Section 11J we drew all six trigonometric functions. Here are $\sin x$, $\cos x$ and $\tan x$.

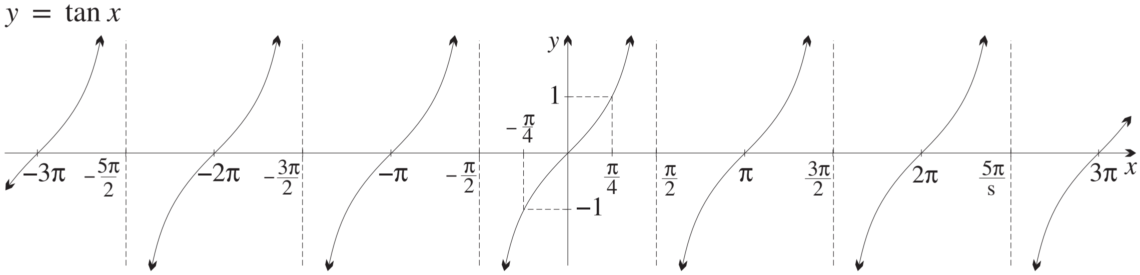
$y = \sin x$



$y = \cos x$



$y = \tan x$



The investigation Exercise 11J in the Year 11 book dealt thoroughly with the symmetries of these three graphs under translations, reflections in the axes, and rotations about the origin. We now have dilations, and this section shows how to generate any basic wave graph by combinations of translations and dilations.

Reflections and rotations do not need review, so all the dilations in this section have positive factors. We will consider separately, then in combination:

- Vertical dilations with positive factors — this leads to the *amplitude*.
- Horizontal dilations with positive factors — this leads to the *period*.
- Translations left and right — this leads to the *phase*.
- Translations up and down — this leads to the *mean value*.

Vertical dilations and amplitude

The *amplitude* of a wave is the maximum height of the wave above its mean position. The graphs on the previous page both show that $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0 (the average of 1 and -1). Thus both have amplitude 1.

Now let us apply a vertical dilation with factor a to $y = \sin x$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Replacing y by $\frac{y}{a}$ gives $\frac{y}{a} = \sin x$, and then multiplying by a gives $y = a \sin x$.

This function is also a wave, but its amplitude is now a, because it has maximum value $y = a$, minimum value $y = -a$, and mean value $y = 0$. Exactly the same argument applies to $y = \cos x$.

23 VERTICAL DILATIONS AND AMPLITUDE

- The *amplitude* of a wave is the maximum height of the wave above its mean position.
- $y = \sin x$ and $y = \cos x$ both have amplitude 1.
- $y = a \sin x$ and $y = a \cos x$ both have amplitude a.
- $y = a \sin x$ and $y = a \cos x$ are $y = \sin x$ or $y = \cos x$ stretched vertically with factor a.

We can stretch the function $y = \tan x$ vertically to $y = a \tan x$ in the usual way. But the function increases without bound near its asymptotes, so the idea of amplitude makes no sense. Instead, we can conveniently tie down the vertical scale of $y = a \tan x$ by using the fact that $\tan \frac{\pi}{4} = 1$, so when $x = \frac{\pi}{4}$, $y = a$.

Horizontal dilations and period

The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the previous page are waves, with a pattern that repeats every revolution. Thus they both have period 2π.

The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π.

24 THE PERIODS OF THE TRIGONOMETRIC FUNCTIONS

- The *period* of a function that repeats is the length of the smallest repeating unit.
- $y = \sin x$ and $y = \cos x$ have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

Now consider the function $y = \sin nx$, which we can also write as $y = \sin \dfrac{x}{1/n}$.

This shows that it is a horizontal dilation of $y = \sin x$ with factor $\dfrac{1}{n}$.

Because $y = \sin x$ has period 2π, the dilation $y = \sin nx$ therefore has period $\dfrac{2\pi}{n}$.

The same arguments apply to $y = \cos nx$ and $y = \tan nx$.

25 HORIZONTAL DILATIONS AND PERIOD

- $y = \sin nx$ and $y = \cos nx$ have period $\dfrac{2\pi}{n}$.
- $y = \tan nx$ has period $\dfrac{\pi}{n}$.
- These functions are $y = \sin x$, $y = \cos x$ and $y = \tan x$ stretched horizontally with factor $\dfrac{1}{n}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 28 3J

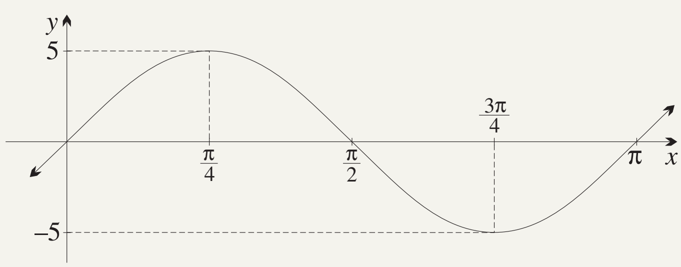
Find the period and amplitude of:

a $y = 5 \sin 2x$ **b** $y = 2 \tan \frac{1}{3}x$

Then sketch one period of the function, showing all intercepts, turning points and asymptotes.

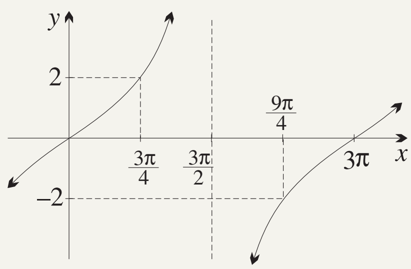
SOLUTION

a $y = 5 \sin 2x$ has an amplitude of 5, and a period of $\dfrac{2\pi}{2} = \pi$.



b $y = 2 \tan \frac{1}{3}x$ has period $\dfrac{\pi}{1/3} = 3\pi$.

It has no amplitude, but when $x = \frac{3\pi}{4}$, $y = 2 \tan \frac{\pi}{4} = 2$.



Horizontal translations and phase

There are two steps in understanding phase. First, look at horizontal translations of the three basic trigonometric functions $y = \sin x$ and $y = \cos x$ and $y = \tan x$.

When one of these is shifted horizontally, the *phase* or *phase shift* is defined to be the displacement of the curve *to the left*. For example:

- $y = \sin\,(x + \frac{\pi}{3})$ is $y = \sin x$ translated $\frac{\pi}{3}$ to the left, so its phase is $\frac{\pi}{3}$.
- $y = \cos\,(x - \frac{\pi}{2})$ is a translation of $-\frac{\pi}{2}$ to the left, so its phase is $-\frac{\pi}{2}$.
- $y = \tan\,(x + \frac{\pi}{4})$ is a translation of $\frac{\pi}{4}$ to the left, so its phase is $\frac{\pi}{4}$.

In general, $y = \sin\,(x + b)$ is $y = \sin x$ translated left by b, so its phase is b.

The same argument applies to $y = \cos\,(x + b)$ and $y = \tan\,(x + b)$.

26 HORIZONTAL TRANSLATIONS AND PHASE

- When $y = \sin x$ or $y = \cos x$ or $y = \tan x$ is translated horizontally, the *phase* or *phase shift* is the displacement of the curve *to the left*.
- Thus $y = \sin\,(x + b)$, $y = \cos\,(x + b)$ and $y = \tan\,(x + b)$ all have phase b.

Example 29 3J

Use horizontal translations to sketch these functions, and state their phase.

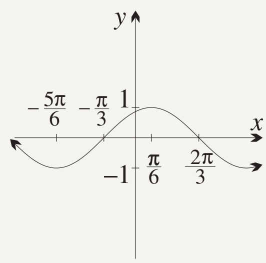
a $y = \sin\,(x + \frac{\pi}{3})$ **b** $y = \cos\,(x - \frac{5\pi}{3})$ **c** $y = \tan\,(x + \frac{\pi}{4})$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

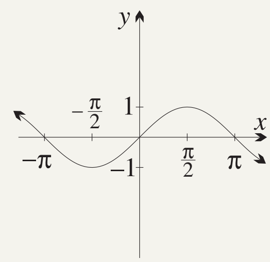
SOLUTION

a



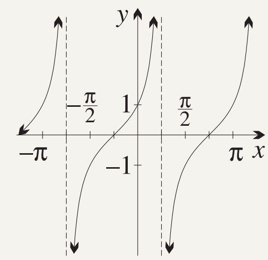
Shift $y = \sin x$ left $\frac{\pi}{3}$.
The phase is $\frac{\pi}{3}$.

b



Shift $y = \cos x$ left $-\frac{5\pi}{2}$.
The phase is $-\frac{5\pi}{2}$.

c



Shift $y = \tan x$ left $\frac{\pi}{4}$.
The phase is $\frac{\pi}{4}$.

Note: The phase is not uniquely defined, because we can add and subtract multiples of the period.

- In part **b**, we could also write the phase as $-\frac{5\pi}{4} + 2\pi = -\frac{\pi}{2}$ or as $-\frac{5\pi}{2} + 4\pi = \frac{3\pi}{2}$.
- In part **a**, we could also say that the phase is $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$.
- In part **c**, we could also say that the phase is $\frac{\pi}{4} - \pi = -\frac{3\pi}{4}$, or as $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Combining period and phase

The second step in understanding phase is to combine a horizontal translation with a horizontal dilation. Two such transformations do not commute, so we need to be careful of the order in which they are applied. Here is an example where $y = \sin x$ has undergone such a compound transformation:

$$y = \sin 2\left(x + \tfrac{\pi}{6}\right) \qquad \text{or equivalently} \qquad y = \sin\left(2x + \tfrac{\pi}{3}\right).$$

The rule for finding the phase is: *Express the compound transformation as a horizontal dilation followed by a horizontal translation. The phase, or phase shift, is the shift to the left of the final horizontal translation.*

The first form of the function above displays dilation-followed-by-translation.

- First dilate $y = \sin x$ horizontally by a factor of $\frac{1}{2}$ to obtain $y = \sin 2x$,
- then shift left by $\frac{\pi}{6}$ to obtain $y = \sin 2\left(x + \frac{\pi}{6}\right)$.

We can therefore conclude that the phase of the function is $\frac{\pi}{6}$.

The second form of the function above displays translation-followed-by-dilation.

- First translate $y = \sin x$ left $\frac{\pi}{3}$ to obtain $y = \sin\left(x + \frac{\pi}{3}\right)$,
- then dilate horizontally by a factor of $\frac{1}{2}$ to obtain $y = \sin\left(2x + \frac{\pi}{3}\right)$.

This does not give us the phase since the transformations are in the wrong order.

The function, $y = \cos 3\left(x - \frac{\pi}{4}\right)$ has phase $-\frac{\pi}{4}$, because to obtain it from $y = \cos x$:

- First dilate $y = \cos x$ horizontally by a factor of $\frac{1}{3}$ to obtain $y = \cos 3x$,
- then shift left by $-\frac{\pi}{4}$ to obtain $y = \cos 3\left(x - \frac{\pi}{4}\right)$.

Again, $y = \tan \frac{1}{3}\left(x + \frac{\pi}{2}\right)$ has phase $\frac{\pi}{2}$, because to obtain it from $y = \tan x$:

- First dilate $y = \tan x$ horizontally by a factor of 3 to obtain $y = \tan \frac{1}{3}x$,
- then shift left by $\frac{\pi}{2}$ to obtain $y = \tan \frac{1}{3}\left(x + \frac{\pi}{2}\right)$.

27 COMBINING PERIOD AND PHASE

- Suppose that $y = \sin x$ or $y = \cos x$ or $y = \tan x$ is dilated horizontally and then translated horizontally. The *phase*, or *phase shift*, is the shift to the left of the final horizontal translation.
- Thus $y = \sin a(x + b)$, $y = \cos a(x + b)$ and $y = \tan a(x + b)$ all have phase b.
- **Method:** Take any coefficient of x outside the bracket, rewriting, for example,

$$y = \sin\left(2x + \tfrac{\pi}{3}\right) \qquad \text{as} \qquad y = \sin 2\left(x + \tfrac{\pi}{6}\right).$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 30 3J

Find the phase of each function, then describe how its graph can be obtained by a horizontal dilation followed by a horizontal translation.

a $y = \sin \frac{1}{2}(x - \frac{\pi}{4})$ **b** $y = \cos(5x + \pi)$ **c** $y = \tan 2(3x + \frac{\pi}{4})$

SOLUTION

a The phase is $-\frac{\pi}{4}$. The graph is $y = \sin x$ dilated horizontally with factor 2, then shifted left $-\frac{\pi}{4}$.

b Rewrite the function as $y = \cos 5(x + \frac{\pi}{5})$. Hence the phase is $\frac{\pi}{5}$, and the graph is $y = \cos x$ dilated horizontally with factor $\frac{1}{5}$, then shifted left $\frac{\pi}{5}$.

c Rewrite the function as $y = \tan 6(x + \frac{\pi}{12})$. Hence the phase is $\frac{\pi}{12}$, and the graph is $y = \tan x$ dilated horizontally with factor $\frac{1}{6}$, then shifted left $\frac{\pi}{12}$.

Vertical translations and the mean value

A vertical translation shifts the mean value of the wave from 0 to some other value. Perform any vertical translation last after any vertical stretch since the two transformations do not commute.

28 VERTICAL TRANSLATIONS AND THE MEAN VALUE:

- The *mean value* of a wave is the mean of its maximum and minimum values.
- $y = \sin x + c$ and $y = \cos x + c$ both have mean value c.
- $y = \sin x + c$ is the result of shifting $y = \sin x$ up c.
- In a combination of transformations, do any vertical translation last.

The function $y = \tan x + c$ is not a wave. It has no amplitude, and it has no mean value (although it does have points of inflection on the line $y = c$).

Putting it all together

The two worked examples below put all four transformations together. The clearest order to list the four transformations (which may not all be involved) is:

1. any horizontal dilation — this gives the period,
2. any horizontal translation — this gives the phase,
3. any vertical dilation — this gives the amplitude (sine and cosine only),
4. any vertical translation — this gives the mean value (sine and cosine only).

Rather than formulae, we recommend thinking through the four successive transformations above, because this usually leads to a better understanding of the situation. For the sine and cosine functions

$$y = k \sin a(x + b) + c \qquad \text{and} \qquad y = k \cos a(x + b) + c,$$

the period is $\frac{2\pi}{a}$, the phase is b, the amplitude is k, and the mean value is c.
For the tangent function $y = k \sin a(x + b) + c$, the period is $\frac{\pi}{a}$ and the phase is b.

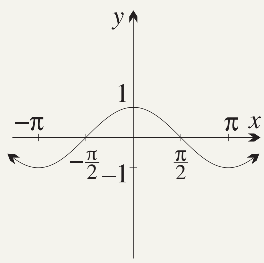
ISBN 978-1-108-76630-2

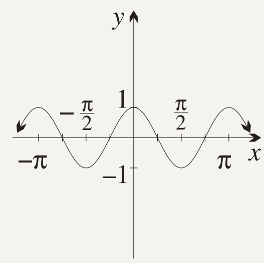
Cambridge University Press
Updated July 2022

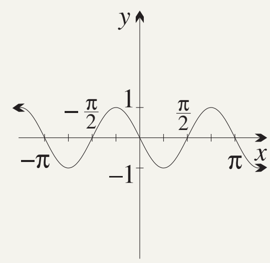
Example 31 — 3J

Use four successive transformations to sketch $y = 3\cos 2\left(x + \frac{\pi}{4}\right) - 2$, and state the period, phase, amplitude and mean value (ignoring x-intercepts in the final diagram).

SOLUTION



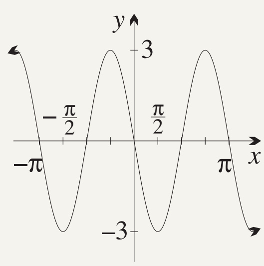


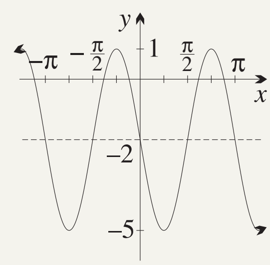


1 Start with $y = \cos x$.

2 Stretch horizontally factor $\frac{1}{2}$, giving $y = \cos 2x$. The period is now $\frac{2\pi}{2} = \pi$.

3 Shift $y = \cos 2x$ left $\frac{\pi}{4}$, giving $y = \cos 2\left(x + \frac{\pi}{4}\right)$. Phase is now $\frac{\pi}{4}$.





4 Then stretch vertically with factor 3, giving $y = 3\cos 2\left(x + \frac{\pi}{4}\right)$. The amplitude is now 3.

5 Shift the whole thing down 2 units, giving $y = 3\cos 2\left(x + \frac{\pi}{4}\right) - 2$. The mean value is now -2.

Example 32 — 3J

Find the period, phase, amplitude and mean value of:

a $y = \frac{1}{5}\sin\left(4x - \frac{\pi}{2}\right) - 2$

b $y = 3\tan 7\left(2x + \frac{3\pi}{8}\right) - 1$

SOLUTION

a Rewrite the function with 4 outside the bracket as $y = \frac{1}{5}\sin 4\left(x - \frac{\pi}{8}\right) - 2$.
Hence the period is $\frac{2\pi}{4} = \frac{\pi}{2}$, the phase is $-\frac{\pi}{8}$, the amplitude is $\frac{1}{5}$, and the mean value is -2.

b Rewrite the function with 2 outside the bracket as $y = 3\tan 14\left(x + \frac{3\pi}{16}\right) - 1$.
Hence the period is $\frac{\pi}{14}$, the phase is $\frac{3\pi}{16}$, there is no amplitude, and there is no mean value.

Oddness and evenness of the trigonometric functions

Box 29 quickly reviews the oddness and evenness of the sine, cosine and tangent functions. These are crucially important properties of the three functions.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

29 ODDNESS AND EVENNESS OF THE TRIGONOMETRIC FUNCTIONS

- The functions $y = \sin x$ and $y = \tan x$ are odd functions and they both have point symmetry in the origin.
  Thus: $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$
- The function $y = \cos x$ is an even function and it has line symmetry in the y-axis.
  Thus: $\cos(-x) = \cos x$

Graphical solutions of trigonometric equations

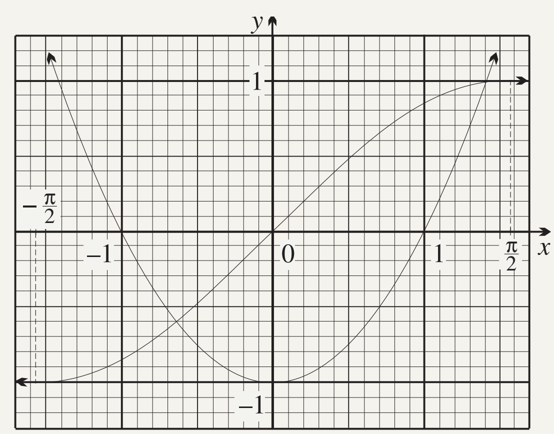
Many trigonometric equations cannot be solved by algebraic methods. Approximation methods using the graphs can usually be used instead and a graph-paper sketch will show:

- how many solutions there are
- the approximate values of the solutions.

Example 33 3J

a Find, by drawing a graph, the number of solutions of $\sin x = x^2 - 1$.

b Then use the graph to find approximations correct to one decimal place.



SOLUTION

a Here are $y = \sin x$ and $y = x^2 - 1$.
Clearly the equation has two solutions.

b The positive solution is $x \doteqdot 1.4$,
and the negative solution is $x \doteqdot -0.6$.

Note: Technology is particularly useful here. It allows sketches to be drawn quickly, and many programs will give the approximate coordinates of the intersections.

Exercise 3J FOUNDATION

1 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, stating the amplitude in each case.
 i $y = \frac{1}{2}\sin x$ **ii** $y = 2\sin x$ **iii** $y = 3\sin x$
 b Describe the transformation from $y = \sin x$ to $y = k\sin x$. (Assume that k is positive.)
 c How does the graph of $y = k\sin x$ change as k increases?

2 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the period in each case.
 i $y = \cos\frac{1}{2}x$ **ii** $y = \cos 2x$ **iii** $y = \cos 3x$
 b Describe the transformation from $y = \cos x$ to $y = \cos nx$. (Assume that n is positive.)
 c How does the graph of $y = \cos nx$ change as n increases?

3 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the period in each case.
 i $y = \tan x$ **ii** $y = \tan\frac{1}{2}x$ **iii** $y = \tan 2x$
 b Describe the transformation from $y = \tan x$ to $y = \tan ax$. (Assume that a is positive.)
 c How does the graph of $y = \tan ax$ change as a increases?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the phase in each case.
  i $y = \sin\left(x + \frac{\pi}{2}\right)$ **ii** $y = \sin(x + \pi)$ **iii** $y = \sin(x + 2\pi)$
b Describe the transformation from $y = \sin x$ to $y = \sin(x + \alpha)$. (Assume that α is positive.)
c Describe the transformation when α is a multiple of 2π.

5 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the mean value and the range in each case.
  i $y = \cos x + 1$ **ii** $y = \cos x + 2$ **iii** $y = \cos x + \frac{1}{2}$
b Describe the transformation from $y = \cos x$ to $y = \cos x + c$. (Assume that c is positive.)
c How does the graph of $y = \cos x + c$ change as c increases?

DEVELOPMENT

---

6 State the amplitude and period of each function, then sketch its graph for $-\pi \le x \le \pi$.
a $y = 3\cos 2x$ **b** $y = 2\sin\frac{1}{2}x$ **c** $y = \tan\frac{3x}{2}$ **d** $y = 2\cos 3x$

7 Write down a sequence of transformations that will transform $y = \sin x$ to the given function, and hence sketch the given function for $0 \le x \le 2\pi$.
a $y = 3\sin 3x$ **b** $y = -2\sin\frac{x}{2}$ **c** $y = 3\sin\left(x - \frac{\pi}{2}\right) + 2$

8 Write down a sequence of transformations that will transform $y = \cos x$ to the given function, and hence sketch the given function for $-\pi \le x \le \pi$.
a $y = 5\cos\frac{1}{2}x$ **b** $y = -2\cos 2x - 2$ **c** $y = \cos\left(2\left(x - \frac{\pi}{2}\right)\right)$

9 **a** **i** What horizontal dilation followed by what horizontal translation together transform $y = \sin x$ into $y = \sin 5(x + \frac{\pi}{3})$?
  ii State the period and the phase.
b Repeat for the transformation from $y = \cos x$ to $y = \cos\frac{1}{3}(x + \pi)$.
c Repeat for the transformation from $y = \tan x$ to $y = \tan\frac{3}{5}(x - \frac{2\pi}{5})$.

10 **a** What are the period, phase, amplitude and mean value of each function?
  i $y = 3\sin 2(x - \frac{\pi}{3})$ **ii** $y = \frac{5}{2}\cos\frac{1}{3}(x + \pi) + 5$ **iii** $y = 2\tan 3(x + \frac{\pi}{8}) - 3$

11 **a** **i** Rewrite $y = \sin(6x - \frac{\pi}{2})$ to display it as $y = \sin x$ transformed by a horizontal dilation followed by a horizontal translation.
  ii Hence write down the period and phase of the function.
  iii How can $y = \sin(6x - \frac{\pi}{2})$ be obtained from $y = \sin x$ by a horizontal translation followed by a horizontal dilation?
b Repeat part **a** for the function $y = \cos(\frac{1}{4}x + \frac{\pi}{2})$ as a transformation of $y = \cos x$.
c Repeat part **a** for the function $y = \tan(\frac{3}{5}x - \frac{4\pi}{5})$ as a transformation of $y = \tan x$.

12 **a** Write down, in the order given below Box 22, a sequence of dilations and translations that will transform $y = \sin x$ to the given function.
  i $y = \sin(3x + \frac{\pi}{2})$ **ii** $y = \frac{1}{4}\sin(4x - \pi) - 4$ **iii** $y = 6\sin(\frac{x}{2} + \frac{\pi}{4}) + 5$
b What are the period, phase, amplitude and mean value of each function in part **a**?

13 Solve each equation, for $0 \le x \le 2\pi$. Then indicate the solutions on a diagram showing sketches of the functions on the LHS and RHS of the equation.
a $2\sin\left(x - \frac{\pi}{3}\right) = 1$ **b** $2\cos 2x = -1$

14 Solve each equation, for $0 \le x \le \pi$, giving solutions correct to 3 decimal places.
a $\cos(x + 0.2) = -0.3$ **b** $\tan 2x = 0.5$

15 **a** Find the vertex of the parabola $y = x^2 - 2x + 4$.
b Hence show graphically that $x^2 - 2x + 4 > 3\sin x$ for all real values of x.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 **a** Sketch the graph of $y = 2\cos x$ for $-2\pi \le x \le 2\pi$.

b On the same diagram, carefully sketch the line $y = 1 - \frac{1}{2}x$, showing its x- and y-intercepts.

c How many solutions does the equation $2\cos x = 1 - \frac{1}{2}x$ have?

d Mark with the letter P the point on the diagram from which the negative solution of the equation in part **c** is obtained.

e Prove algebraically that if n is a solution of the equation in part **c**, then $-2 \le n \le 6$.

17 **a** What is the period of the function $y = \sin\frac{\pi}{2}x$?

b Sketch the curve $y = 1 + \sin\frac{\pi}{2}x$, for $0 \le x \le 4$.

c Through what fixed point does the line $y = mx$ always pass for varying values of m?

d By considering possible points of intersection of the graphs of $y = 1 + \sin\frac{\pi}{2}x$ and $y = mx$, find the values of m for which the equation $\sin\frac{\pi}{2}x = mx - 1$ has exactly one real solution in the domain $0 \le x \le 4$.

18 The depth of water in Dolphin Bay varies according to the tides. The depth is modelled by the equation $x = 2\cos\left(\frac{\pi}{7}t\right) + 8$, where x metres is the depth and t hours is the time since the last high tide. Last Saturday, it was high tide at 7 am.

a How deep is the bay at high tide?

b How deep is the bay at low tide?

c When did the first low tide after 7 am occur?

d At what time last Saturday morning was the depth 9 metres?

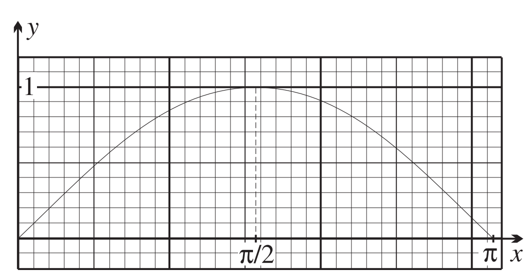
19 **a** Sketch $y = 3\sin 2x$ and $y = 4\cos 2x$ on the same diagram, for $-\pi \le x \le \pi$.

b Hence sketch the graph of $y = 3\sin 2x - 4\cos 2x$ on the same diagram, for $-\pi \le x \le \pi$.

c Estimate the amplitude of the graph sketched in part **b**.

ENRICHMENT

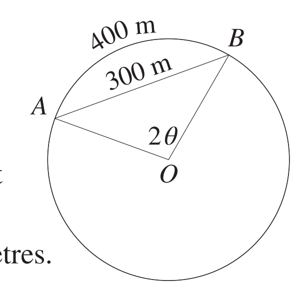
20 **a** **i** Photocopy this graph of $y = \sin x$, for $0 \le x \le \pi$, and on it graph the line $y = \frac{3}{4}x$.



ii Measure the gradient of $y = \sin x$ at the origin.

iii For what values of k does $\sin x = kx$ have a solution, for $0 < x < \pi$?

b The diagram shows points A and B on a circle with centre O, where $\angle AOB = 2\theta$, chord AB has length 300 metres, and the minor arc AB has length 400 metres.

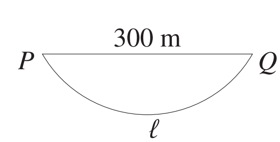


i Show that $\sin\theta = \dfrac{3}{4}\theta$.

ii Use the graph from part **a i** to determine θ, correct to one decimal place.

iii Hence find $\angle AOB$ in radians, correct to one decimal place, and show that the radius of the circle is about 154 metres.

c P and Q are two points 300 metres apart. The circular arc PQ has length ℓ metres.



i If C is the centre of the arc and $\angle PCQ = 2\alpha$, show that $\sin\alpha = \dfrac{300\alpha}{\ell}$.

ii Use your answer to part **a iii** to find the possible range of values of ℓ.

21 Consider the equation $\dfrac{1}{1 + \cos x} = \dfrac{2x}{\pi}$.

a Show that $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$ satisfy the equation.

b On the same diagram, sketch $y = \dfrac{2x}{\pi}$ and $y = \dfrac{1}{1 + \cos x}$ for $0 \le x < \pi$.

c Deduce that $4x\cos^2\frac{1}{2}x > \pi$, for $\frac{\pi}{3} < x < \frac{\pi}{2}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 3 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

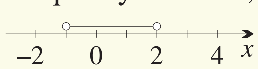
Chapter 3 Multiple-choice quiz

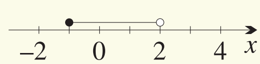
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

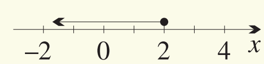
Chapter review exercise

1 For each number line, write the graphed interval using:

i inequality notation, **ii** bracket notation.

a 

b 

c 

2 If $f(x) = x^2 - 1$ and $g(x) = x + 1$, find:

a **i** $f \circ g(-2)$ **ii** $g \circ f(-2)$ **iii** $f \circ f(-2)$ **iv** $g \circ g(-2)$

b **i** $f \circ g(x)$ **ii** $g \circ f(x)$ **iii** $f \circ f(x)$ **iv** $g \circ g(x)$

3 Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator, and taking the limit as $x \to \infty$ and as $x \to -\infty$.

a $f(x) = \dfrac{1}{x + 2}$ **b** $f(x) = \dfrac{x - 3}{2x + 5}$ **c** $f(x) = \dfrac{x}{x^2 + 1}$

4 Let $y = x^3 - 9x^2 + 18x$.

a State the domain using inequality interval notation.

b Write down the coordinates of any intercepts with the axes.

c Does this function have any asymptotes?

d Use this information and a table of values to sketch the curve.

e The graph seems to be horizontal somewhere in the interval $0 < x < 3$, and again in the interval $3 < x < 6$. Use calculus to find the x-coordinates of these points, and add them to the diagram.

5 Solve each double inequation, then write your answer in bracket interval notation.

a $-6 < -3x \le 12$ **b** $-2 < 2x + 1 < 1$

c $-7 \le 5 + 4x < 7$ **d** $-4 \le 1 - \frac{1}{2}x \le 3$

6 Carefully draw the graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them.

a $x - 2 = \log_2 x$

b $\cos x = 1 - x^2$

c $x(x - 2)(x + 2) = 2 - |x|$

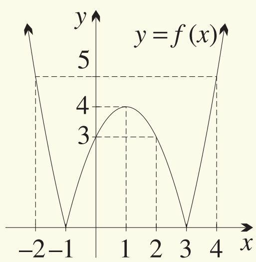
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

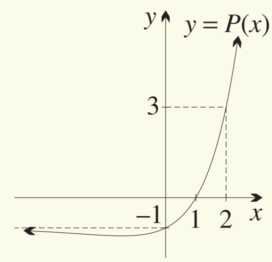
Review

7 In each case an unknown function has been drawn. Draw the functions specified below it.

a



b



a i $y = f(x - 1)$ **ii** $y = f(x) + 1$

b i $y = P(x + 1)$ **ii** $y = P(x) - 1$

8 In each case apply the indicated dilation to the corresponding function in Question **7** and draw the resulting graph.

a i $y = f\left(\frac{1}{2}x\right)$ **ii** $y = \frac{1}{2}f(x)$

b i $y = P(2x)$ **ii** $y = 2P(x)$

9 In each case, completely factor the given polynomial where necessary and hence sketch its graph. A table of values may also help. Then use the graph to solve $f(x) \le 0$.

a $f(x) = (x + 1)(x - 3)$ **b** $f(x) = x(x - 2)(x + 1)$ **c** $f(x) = x^2 - 4x - 5$

d $f(x) = 3 - 2x - x^2$ **e** $f(x) = 2x - x^2 - x^3$ **f** $f(x) = x^3 + 4x^2 + 4x$

10 Let $y = \dfrac{4}{(x + 2)(2 - x)}$.

a State the natural domain.

b Find the y-intercept.

c Show that $y = 0$ is a horizontal asymptote.

d Draw up a table of values.

e Identify the vertical asymptotes, and use the table of values to describe its behaviour near them.

f Sketch the graph of the function and state its range using bracket interval notation.

11 a Factor the right-hand side of $y = \dfrac{3x + 3}{x^2 + 2x - 3}$.

b State the domain and any intercepts with the axes.

c Explain why the function is neither even nor odd.
(Hint: The answers to part **a** may help.)

d Find the equations of the asymptotes.

e Sketch the graph of this curve.

12 Solve these absolute value equations and inequations algebraically.

a $|2x| = 7$ **b** $|3x - 2| = 1$ **c** $|3x + 5| \le 4$ **d** $|6x + 7| > 5$

13 Carefully sketch the functions on the LHS and RHS of each inequation on the same number plane. Then use the graph to solve the inequations.

a $x - 1 \ge 1 + \frac{1}{2}x$

b $\dfrac{1}{1 - x} > 1 - 2x$

c $|2x| \le x + 3$

d $\left|\frac{1}{2}x + 1\right| > \frac{1}{4}(x + 5)$

14 a Find the points where $y = x^2 - 2x + 1$ intersects $y = 1 + 4x - x^2$.

b Hence sketch the region where both $y \ge x^2 + 2x + 1$ and $y \le 1 + 4x - x^2$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 Write down the equation for each function after the given translations have been applied.

a $y = x^2$: right 2 units, up 1 unit
b $y = \frac{1}{x}$: left 2 units, down 3 units
c $y = \sin x$: left $\frac{\pi}{6}$ units, down 1 unit
d $y = e^x$: right 2 units, up 1 unit

16 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = \frac{1}{4}x^2 - x$
b $y = \dfrac{1}{x-4}$ and $y = \dfrac{1}{2x-8}$
c $y = \cos x$ and $y = \frac{1}{3}\cos x$
d $y = \dfrac{1}{x+1}$ and $y = \dfrac{2}{x+2}$

17 Which of the following pairs of transformations commute?

a reflection in the y-axis and reflection in the x-axis,
b vertical reflection and vertical translation,
c horizontal translation and horizontal dilation,
d vertical translation and horizontal dilation.

18 Identify the various transformations of the standard functions and hence graph each. Make sure the transformations are applied in the correct order when they do not commute.

a $y = 4 - 2^x$
b $y = \dfrac{1}{2}(x-2)^2 - 1$
c $y = 2\sin\left(x + \frac{\pi}{6}\right) + 1$

19 Write down the amplitude and period, then sketch the graph for $-\pi \le x \le \pi$.

a $y = 4\sin 2x$
b $y = \frac{3}{2}\cos\frac{1}{2}x$

20 **a** Explain how the graph of $y = \tan x$ can be transformed into the graph of $y = 1 - \tan x$

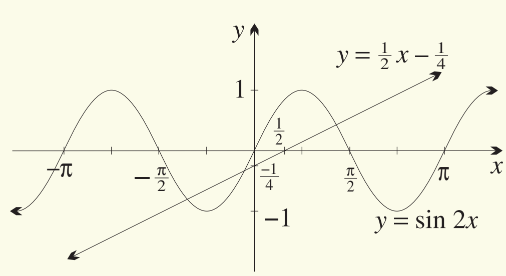
b Hence sketch $y = 1 - \tan x$ for $-\pi \le x \le \pi$.

21 **a** Rewrite the function if necessary, then write down a sequence of dilations and translations, in the order given below Box 29, that will transform $y = \cos x$ into:

i $y = 3\cos 5x - 2$
ii $y = 4\cos 4\left(x + \frac{\pi}{6}\right)$
iii $y = \cos\left(2x - \frac{\pi}{3}\right) + 1$

b What are the period, phase, amplitude and mean value of each function in part **b**?

22 In the given diagram, the curve $y = \sin 2x$ is graphed for $-\pi \le x \le \pi$, and the line $y = \frac{1}{2}x - \frac{1}{4}$ is graphed.



a In how many points does the line $y = \frac{1}{2}x - \frac{1}{4}$ meet the curve $y = \sin 2x$?

b State the number of solutions of the equation $\sin 2x = \frac{1}{2}x - \frac{1}{4}$. How many of these solutions are positive?

c Briefly explain why the line $y = \frac{1}{2}x - \frac{1}{4}$ will not meet the curve $y = \sin 2x$ outside the domain $-\pi \le x \le \pi$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 Curve-sketching using the derivative

This chapter will use the derivative to extend the systematic approach to sketching curves developed in Chapter 3 by asking two further questions:

1 Where is the curve sloping upwards, where is it sloping downwards, and where does it have any maximum or minimum values?

2 Where is the curve concave up, where is it concave down, and are there points of inflection where the curve changes from one concavity to the other?

These are standard procedures for investigating unfamiliar curves. In particular, the algorithm for finding the maximum and minimum values of a function can be applied to all sorts of practical and theoretical questions.

The chapter concludes with a fuller account of primitives than was appropriate in Year 11, in preparation for integration in Chapter 5.

Curve-sketching software is very useful when studying this chapter, because it can easily show the effect on the graph of changing the equation of the curve.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4A Increasing, decreasing and stationary at a point

We have used the terms increasing and decreasing freely so far without precise definitions. This section uses tangents to formalise the ideas of increasing and decreasing at a point. Later, in Chapter 9, we will use chords to formalise the ideas of increasing and decreasing over an interval.

Increasing, decreasing and stationary at a point

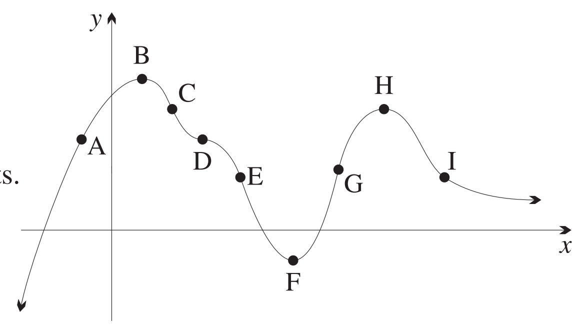
At a point where a curve is sloping upwards, the tangent has positive gradient, and y is increasing as x increases. At a point where it is sloping downwards, the tangent has negative gradient, and y is decreasing as x increases.

1 INCREASING, DECREASING AND STATIONARY AT A POINT

Let $f(x)$ be a function that can be differentiated at $x = a$.
- If $f'(a) > 0$, then $f(x)$ is called *increasing* at $x = a$.
- If $f'(a) < 0$, then $f(x)$ is called *decreasing* at $x = a$.
- If $f'(a) = 0$, then $f(x)$ is called *stationary* at $x = a$.

For example, the curve in the diagram to the right is:
- increasing at A and G,
- decreasing at C, E and I,
- stationary at B, D, F and H.

Think about the tangent to the curve at each of the nine points.



Example 1 4A

a Differentiate $y = (x - 2)(x - 4)$.

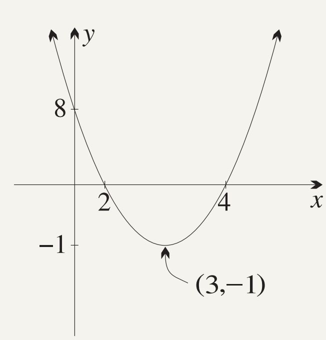
b Hence find the values of x where the curve is stationary, where it is increasing, and where it is decreasing. Then sketch the curve.

SOLUTION

a Expanding, $y = x^2 - 6x + 8$,
and differentiating, $y' = 2x - 6$
$= 2(x - 3)$.

b When $x = 3$, $y' = 0$, so the curve is stationary at $x = 3$.
When $x > 3$, $y' > 0$, so the curve is increasing for $x > 3$.
When $x < 3$, $y' < 0$, so the curve is decreasing for $x < 3$.

c



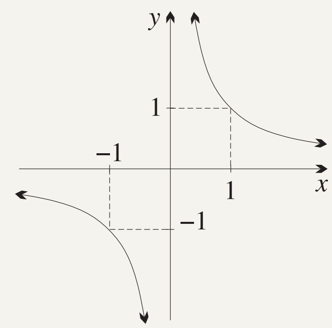
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 2 4A

a Differentiate $y = \frac{1}{x}$.

b Hence show that the graph has no stationary points, and is decreasing for all values of x in its domain.



SOLUTION

a Differentiating, $\frac{dy}{dx} = -\frac{1}{x^2}$.

b The domain is $x \neq 0$, so for all x in the domain, x^2 is positive, the derivative is negative, and the curve is decreasing.

Note: The value $y = \frac{1}{2}$ at $x = 2$ is greater than the value $y = -\frac{1}{2}$ at $x = -2$, despite the fact that the curve is decreasing for all x in the domain. This sort of thing can of course only happen because of the break in the curve at $x = 0$.

Example 3 4A

Find where $y = x^3 - 4x$ is decreasing, and sketch the curve.

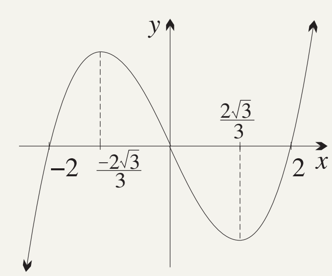
SOLUTION

$y' = 3x^2 - 4$,

so y' has zeroes at $x = \frac{2}{3}\sqrt{3}$ and at $x = -\frac{2}{3}\sqrt{3}$, and is negative between the two zeroes.

So the curve is decreasing for $-\frac{2}{3}\sqrt{3} < x < \frac{2}{3}\sqrt{3}$.

To sketch the curve, notice also that the function is odd, and that it factors as $y = x(x-2)(x+2)$, so there are zeroes at $x = 0$, $x = 2$ and $x = -2$.



Example 4 4A

a Show that $f(x) = x^3 + x - 1$ is always increasing.

b Find $f(0)$ and $f(1)$, and hence explain why the curve has exactly one x-intercept.

SOLUTION

a Differentiating, $f'(x) = 3x^2 + 1$.

Because squares can never be negative, $f'(x)$ can never be less than 1, so the function is increasing for every value of x.

b Substituting, $f(0) = -1$ and $f(1) = 1$.

Because $f(0)$ is negative and $f(1)$ is positive, and the curve is continuous, the curve crosses the x-axis somewhere between 0 and 1.

Because the function is increasing for every value of x, it can never go back and cross the x-axis at a second point.

ISBN 978-1-108-76630-2

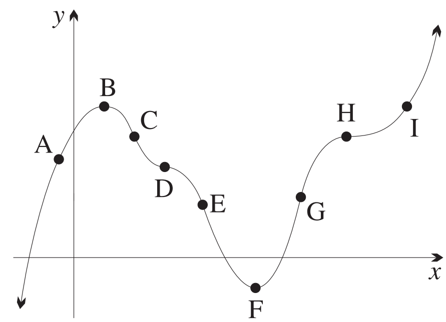
Cambridge University Press
Updated July 2022

Exercise 4A

FOUNDATION

1 In the diagram to the right, name the points where:

a $f'(x) > 0$
b $f'(x) < 0$
c $f'(x) = 0$



2 a Show that $y = -5x + 2$ is decreasing for all x.
b Show that $y = x + 7$ is increasing for all x.
c Show that $y = x^3$ is increasing for all values of x, apart from $x = 0$ where it is stationary.
d Show that $y = 3x^2$ is increasing for $x > 0$ and decreasing for $x < 0$. What happens at $x = 0$?
e Show that the function $y = \sqrt{x}$ is increasing for all values of $x > 0$.
f Show that $y = \dfrac{1}{x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$.

3 a Find $f'(x)$ for the function $f(x) = 4x - x^2$.
b For what values of x is:
i $f'(x) > 0$, ii $f'(x) < 0$, iii $f'(x) = 0$?
c Find $f(2)$, then, by interpreting these results geometrically, sketch a graph of $f(x)$.

4 a Find $f'(x)$ for the function $f(x) = x^3 - 3x^2 + 5$.
b For what values of x is:
i $f'(x) > 0$, ii $f'(x) < 0$, iii $f'(x) = 0$?
c Evaluate $f(0)$ and $f(2)$, then, by interpreting these results geometrically, sketch a graph of $y = f(x)$.

5 a Differentiate $f(x) = -\dfrac{3}{x}$, and hence prove that $f(x)$ increases for all x in its domain.
b Explain why $f(-1) > f(2)$ despite this fact.

DEVELOPMENT

6 Find the derivative of each function. By solving $y' > 0$, find the values of x for which the function is increasing.
a $y = x^2 - 4x + 1$ b $y = 7 - 6x - x^2$ c $y = 2x^3 - 6x$ d $y = x^3 - 3x^2 + 7$

7 a Find the values of x for which $y = x^3 - x^2 - x - 1$ is decreasing.
b Find the values of x for which $y = x^3 - 3x^2 - 24x + 15$ is increasing.

8 a Find $f'(x)$ for the function $f(x) = \frac{1}{3}x^3 + x^2 + 5x + 7$.
b By completing the square, show that $f'(x)$ is always positive, and hence that $f(x)$ is increasing for all x.
c Explain why the equation $f(x) = 0$ has exactly one real solution.

9 Use the quotient rule to find $f'(x)$, and hence show that:
a $f(x) = \dfrac{2x}{x - 3}$ is decreasing for all $x \neq 3$,
b $f(x) = \dfrac{x^3}{x^2 + 1}$ is increasing for all x, apart from $x = 0$ where it is stationary.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

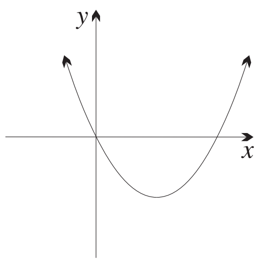
10 Sketch graphs of continuous curves satisfying the properties below:

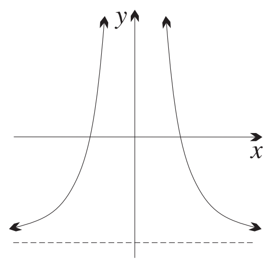
a $f(1) = f(-3) = 0$,
$f'(-1) = 0$,
$f'(x) > 0$ when $x < -1$,
$f'(x) < 0$ when $x > -1$.

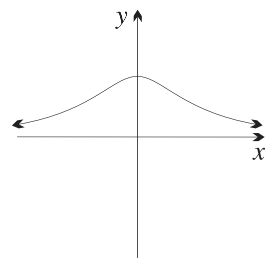
b $f(x) > 0$ for all x,
$f'(0) = 0$,
$f'(x) < 0$ for $x < 0$,
$f'(x) > 0$ for $x > 0$.

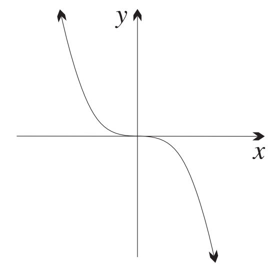
c $f(x)$ is odd,
$f(3) = 0$ and $f'(1) = 0$,
$f'(x) > 0$ for $x > 1$,
$f'(x) < 0$ for $0 \le x < 1$.

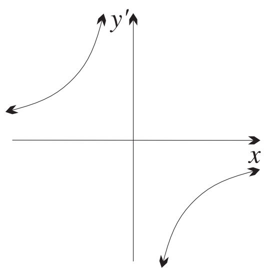
11 The graphs of four functions **a**, **b**, **c** and **d** are shown below. The graphs of the derivatives of these functions, in scrambled order, are shown in **I**, **II**, **III** and **IV**. Match the graph of each function with the graph of its derivative.

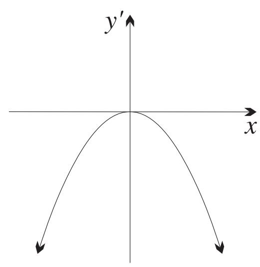
a


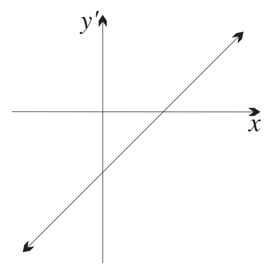
b


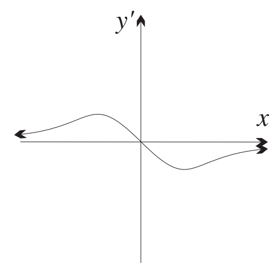
c


d


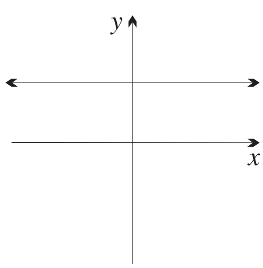
I


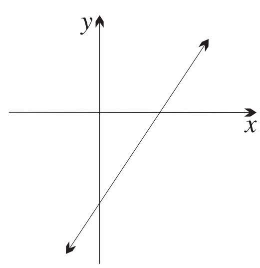
II


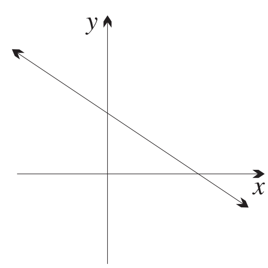
III


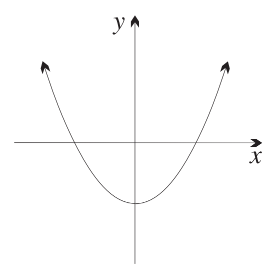
IV


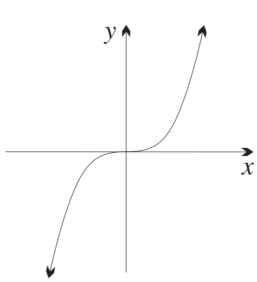
12 Look carefully at each of the functions graphed below to establish where they are increasing, decreasing and stationary. Hence draw a graph of the derivative of each function.

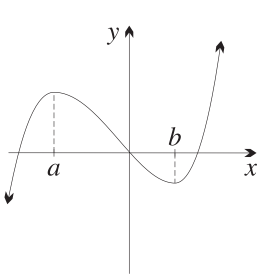
a


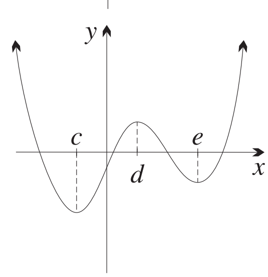
b


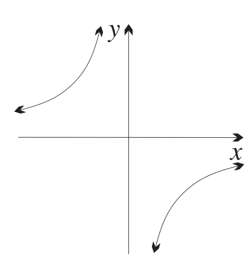
c


d


e


f


g


h


ISBN 978-1-108-76630-2

ENRICHMENT

13 For what values of x is $y = \dfrac{x^2}{2x^2 + x + 1}$ decreasing?

14 a For $f(x) = \dfrac{1 - x^2}{x^2 + 1}$:

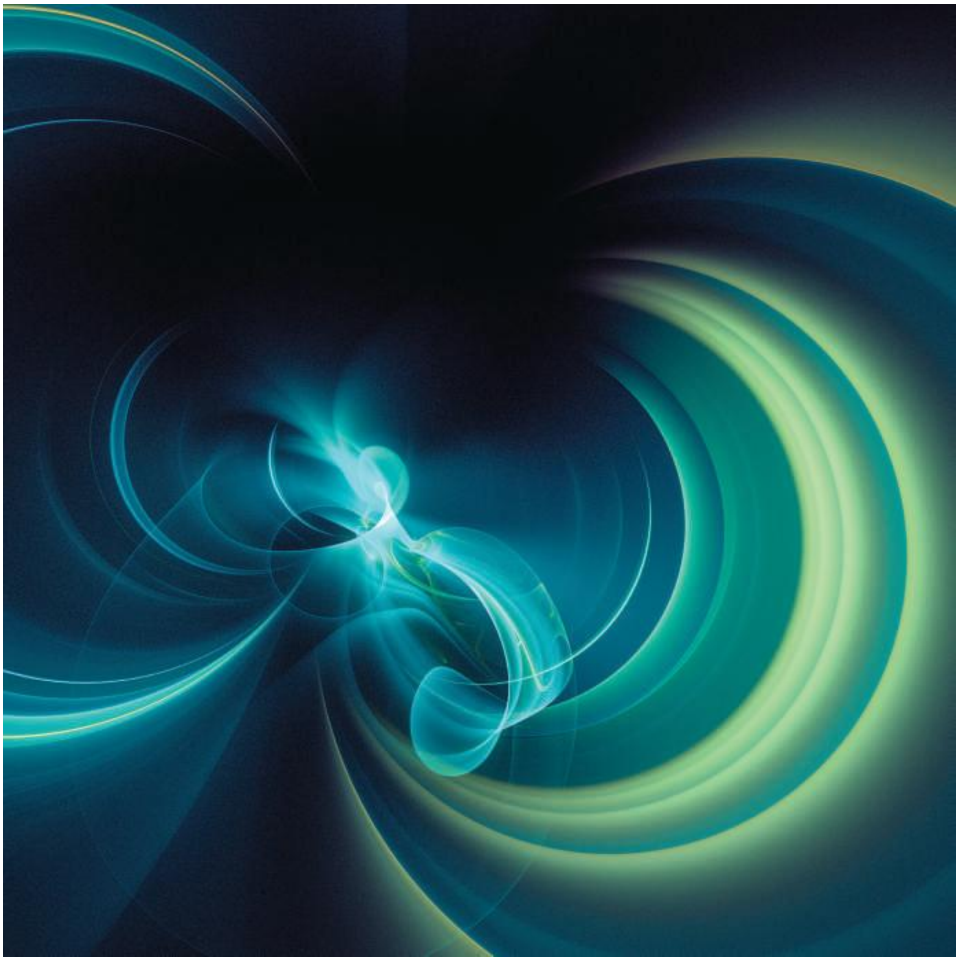
i find $f'(x)$, ii evaluate $f(0)$, iii show that $f(x)$ is even.

b Hence explain why $f(x) \leq 1$ for all x.

15 A function $f(x)$ has derivative $f'(x) = -x(x + 2)(x - 1)$.

a Sketch the graph of $y = f'(x)$, and hence establish where $f(x)$ is increasing, decreasing and stationary.

b Sketch a possible graph of $y = f(x)$, given that $f(0) = 2$.

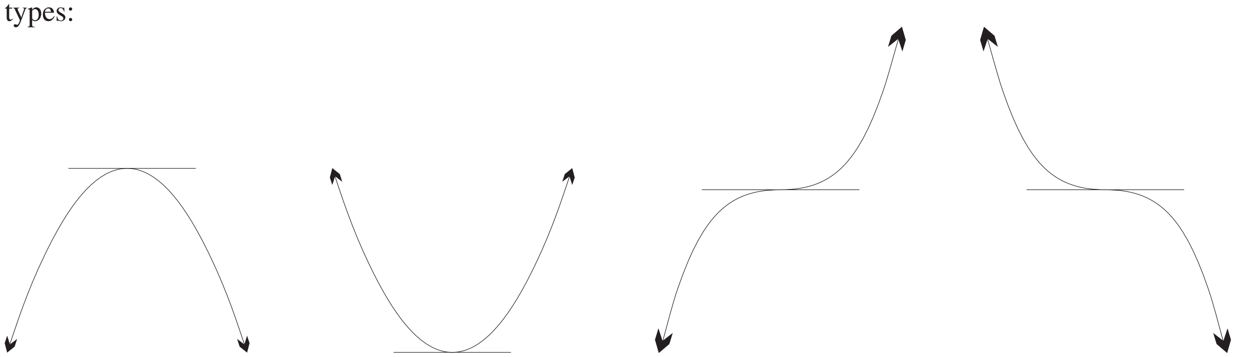


ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4B Stationary points and turning points

Stationary points on a curve that is not a constant function near that point can be classified into four different types:



Maximum turning point | Minimum turning point | Stationary point of inflection | Stationary point of inflection

Turning points

The first stationary point is a *maximum turning point* — the curve turns smoothly from increasing to decreasing, with a maximum value at the point.

The second stationary point is a *minimum turning point* — the curve turns smoothly from decreasing to increasing, with a minimum value at the point.

2 TURNING POINTS

A stationary point is called a *turning point* if the derivative changes sign around the point.
- At a *maximum turning point*, the curve changes from increasing to decreasing.
- At a *minimum turning point*, the curve changes from decreasing to increasing.

Stationary points of inflection

In the third and fourth diagrams above, there is no turning point. In the third diagram, the curve is increasing on both sides of the stationary point, and in the fourth, the curve is decreasing on both sides.

Instead, the curve *flexes* around the stationary point, changing *concavity* from downwards to upwards, or from upwards to downwards. The surprising effect is that *the tangent at this type of stationary point actually crosses the curve.*

3 POINTS OF INFLECTION

- A *point of inflection* is a point on the curve where the tangent crosses the curve. This means that the concavity changes from upwards to downwards, or from downwards to upwards, around the point.
- A *stationary point of inflection* is a point of inflection where the tangent is horizontal. Thus it is both a point of inflection and a stationary point.

Local or relative maximum and minimum

A *local* or *relative maximum* is a point where the curve reaches a maximum in its immediate neighbourhood. Sometimes there is no tangent at the point — look at points C and I in the diagram below.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

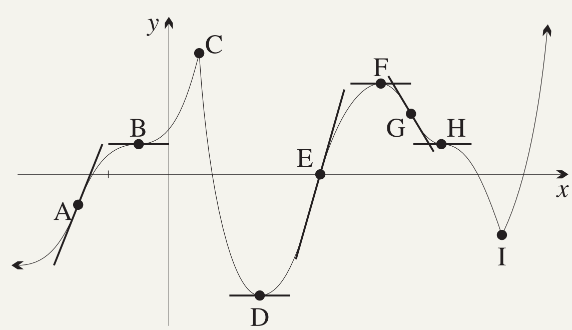
4 LOCAL MAXIMA AND MINIMA

Let $A(a, f(a))$ be a point on $y = f(x)$. There may or may not be a tangent at A.

- The point A is called a *local* or *relative maximum* if
  $f(x) \le f(a)$, for all x in some small interval around a.
- Similarly, A is called a *local* or *relative minimum* if
  $f(x) \ge f(a)$, for all x in some small interval around a.

Example 5 — 4B

Classify the points labelled A–I in the diagram below.



SOLUTION

C and F are local maxima, but only F is a maximum turning point.

D and I are local minima, but only D is a minimum turning point.

B and H are stationary points of inflection.

A, E and G are also points of inflection, but are not stationary points.

Note: The point F is called a *maximum turning point* rather than a 'local maximum turning point'. This is because when we are classifying turning points, we are only ever interested in the immediate neighbourhood of the point.

Analysing stationary points with a table of slopes

Section 3A explained how a function can only change sign at a zero or a discontinuity. Similarly, the derivative $f'(x)$ can only change sign at a zero or discontinuity of $f'(x)$, meaning a stationary point of $f(x)$ or a point where $f(x)$ is not differentiable.

This gives a straightforward method for analysing the stationary points. The method also gives an overall picture of the shape of the function.

5 USING THE DERIVATIVE $f'(x)$ TO ANALYSE STATIONARY POINTS AND SLOPE

1. Find the zeroes and discontinuities of the derivative $f'(x)$.
2. Then draw up a table of test values of the derivative $f'(x)$ dodging around its zeroes and discontinuities, with the slopes underneath, to see where the gradient changes sign.

The resulting *table of slopes* shows not only the nature of each stationary point, but also where the function is increasing and decreasing across its whole domain. This gives an outline of the shape of the curve, in preparation for a proper sketch.

ISBN 978-1-108-76630-2

Example 6 4B

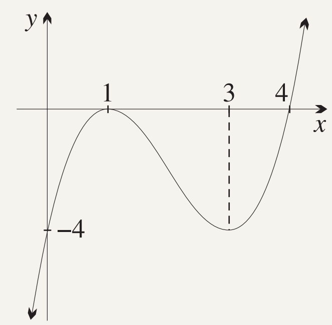
Find the stationary points of the cubic $y = x^3 - 6x^2 + 9x - 4$, use a table of slopes to determine their nature, and sketch the curve.

SOLUTION

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 12x + 9\\ &= 3(x^2 - 4x + 3)\\ &= 3(x-1)(x-3),\end{aligned}$$

so y' has zeroes at $x = 1$ and 3, and no discontinuities.

x	0	1	2	3	4
y'	9	0	-3	0	9
slope	/	—	\	—	/



When $x = 1$, $y = 1 - 6 + 9 - 4$
$= 0$,
and when $x = 3$, $y = 27 - 54 + 27 - 4$
$= -4$.

Hence $(1, 0)$ is a maximum turning point, and $(3, -4)$ is a minimum turning point.

Note: Only the signs of y' are relevant, but if the actual values of y' are not calculated, some other argument should be given as to how the signs were obtained.

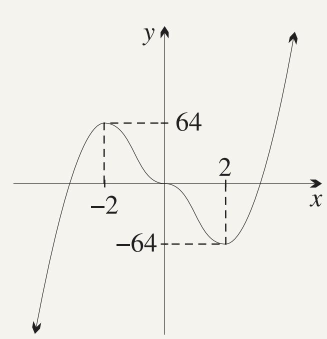
Example 7 4B

Find the stationary points of the quintic $f(x) = 3x^5 - 20x^3$, use a table of slopes to determine their nature, and sketch the curve.

SOLUTION

$$\begin{aligned}f'(x) &= 15x^4 - 60x^2\\ &= 15x^2(x^2 - 4)\\ &= 15x^2(x-2)(x+2),\end{aligned}$$

so $f'(x)$ has zeroes at $x = -2$, $x = 0$ and $x = 2$, and has no discontinuities.



x	-3	-2	-1	0	1	2	3
$f'(x)$	675	0	-45	0	-45	0	675
slope	/	—	\	—	\	—	/

When $x = 0$, $y = 0 - 0 = 0$,
when $x = 2$, $y = 96 - 160 = -64$,
and when $x = -2$, $y = -96 + 160 = 64$.

Hence $(-2, 64)$ is a maximum turning point, $(2, -64)$ is a minimum turning point, and $(0, 0)$ is a stationary point of inflection.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Note: This function $f(x) = 3x^5 - 20x^3$ is odd, and it has as its derivative $f'(x) = 15x^4 - 60x^2$, which is even. In general, the derivative of an even function is odd, and the derivative of an odd function is even — see Question 20 of Exercise 4B. This provides a useful check.

Finding pronumerals in a function

In this worked example, the pronumerals in a function are found using information about a stationary point of the curve.

Example 8 — 4B

The graph of the cubic $f(x) = x^3 + ax^2 + bx$ has a stationary point at $A(2, 2)$. Find a and b.

SOLUTION

To find the two unknown constants, we need two independent equations.

Because $f(2) = 2$, $\quad 2 = 8 + 4a + 2b$

$$2a + b = -3. \qquad (1)$$

Differentiating, $\quad f'(x) = 3x^2 + 2ax + b,$

and because $f'(2) = 0$, $\quad 0 = 12 + 4a + b$

$$4a + b = -12. \qquad (2)$$

Subtracting (1) from (2), $\quad 2a = -9$

$$a = -4\tfrac{1}{2},$$

and substituting into (1), $\quad -9 + b = -3$

$$b = 6.$$

Exercise 4B — FOUNDATION

1 By finding where the derivative equals zero, determine the x-coordinates of any stationary points of each function.

a $y = x^2 - 6x + 8$ b $y = x^2 + 4x + 3$ c $y = x^3 - 3x$

2 By finding where the derivative equals zero, determine the coordinates of any stationary points of each function. (Hint: Remember that you find the y-coordinate by substituting the x-coordinate into the original function.)

a $y = x^2 - 4x + 7$ b $y = x^2 - 8x + 16$ c $y = 3x^2 - 6x + 1$
d $y = -x^2 + 2x - 1$ e $y = x^3 - 3x^2$ f $y = x^4 - 4x + 1$

3 Find the derivative of each function and complete the given table to determine the nature of the stationary point. Sketch each graph, indicating all important features.

a $y = x^2 - 4x + 3$:

x	1	2	3
y'			

b $y = 12 + 4x - x^2$:

x	1	2	3
y'			

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

c $y = x^2 + 6x + 8$:

x	-4	-3	-2
y'			

d $y = 15 - 2x - x^2$:

x	-2	-1	-0
y'			

4 Differentiate each function and show that there is a stationary point at $x = 1$. Then use a table of values of $f'(x)$ to determine the nature of the stationary point at $x = 1$.

a $f(x) = x^2 - 2x - 3$
b $f(x) = 15 + 2x - x^2$
c $f(x) = x^3 + 3x^2 - 9x + 2$
d $f(x) = x^3 - 3x^2 + 3x + 1$

5 Find the stationary point of each function and use a table of values of $\frac{dy}{dx}$ to determine its nature. Sketch each graph, indicating all intercepts with the axes.

a $y = x^2 + 4x - 12$
b $y = 5 - 4x - x^2$

6 a Show that the derivative of $y = x^3 - 3x^2$ is $\frac{dy}{dx} = 3x(x - 2)$.

b Use a table of values of $\frac{dy}{dx}$ to show that there is a maximum turning point at $(0, 0)$ and a minimum turning point at $(2, -4)$.

c Sketch the graph of the function, showing all important features.

7 a Show that the derivative of $y = 12x - x^3$ is $y' = 3(2 - x)(2 + x)$.

b Use a table of values of y' to show that there is a maximum turning point at $(2, 16)$ and a minimum turning point at $(-2, -16)$.

c Sketch the graph of the function, showing all important features.

DEVELOPMENT

8 Find the stationary points of each function, then determine their nature using a table of values of $\frac{dy}{dx}$. Sketch each graph. (You need not find the x-intercepts.)

a $y = 2x^3 + 3x^2 - 36x + 15$
b $y = x^3 + 4x^2 + 4x$
c $y = 16 + 4x^3 - x^4$
d $y = 3x^4 - 16x^3 + 24x^2 + 11$

9 a Use the product rule to show that if $y = x(x - 2)^3$, then $y' = 2(2x - 1)(x - 2)^2$.

b Find any stationary points and use a table of gradients to classify them.

c Sketch the graph of the function, indicating all important features.

10 a Use the product rule to show that if $y = x^2(x - 4)^2$, then $\frac{dy}{dx} = 4x(x - 4)(x - 2)$.

b Find any stationary points and use a table of gradients to classify them.

c Sketch the graph of the function, indicating all important features.

11 a Use the product rule to show that if $y = (x - 5)^2(2x + 1)$, then $y' = 2(x - 5)(3x - 4)$.

b Find any stationary points and use a table of gradients to classify them.

c Sketch the graph of the function, indicating all important features.

12 a The tangent to the curve $y = x^2 + ax - 15$ is horizontal at the point where $x = 4$. Find the value of a.

b The curve $y = x^2 + ax + 7$ has a turning point at $x = -1$. Find the value of a.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 a The curve $f(x) = ax^2 + 4x + c$ has a turning point at $(-1, 1)$. Find a and c.

b Find b and c if $y = x^3 + bx^2 + cx + 5$ has stationary points at $x = -2$ and $x = 4$.

14 The curve $y = ax^2 + bx + c$ passes through the points $(1, 4)$ and $(-1, 6)$, and there is a maximum turning point at $x = -\frac{1}{2}$.

a Show that $a + b + c = 4$, $a - b + c = 6$ and $-a + b = 0$.

b Hence find the values of a, b and c.

15 The line $y = 2x$ is the tangent to the curve $y = ax^2 + bx + c$ at the origin, and there is a maximum turning point at $x = 1$.

a Explain why $c = 0$.

b Explain why $\dfrac{dy}{dx} = 2$ when $x = 0$ and use this fact to deduce that $b = 2$.

c Show that $2a + b = 0$ and hence find the value of a.

16 The function $y = ax^3 + bx^2 + cx + d$ has a maximum turning point at $(-2, 27)$ and a minimum turning point at $(1, 0)$. Find the values of a, b, c and d.

ENRICHMENT

17 a If $f(x) = \dfrac{3x}{x^2 + 1}$, show that $f'(x) = \dfrac{3(1 - x)(1 + x)}{(x^2 + 1)^2}$.

b Hence find any stationary points and determine their nature.

c Sketch the graph of $y = f(x)$, indicating all important features.

d Hence state how many roots the equation $\dfrac{3x}{x^2 + 1} = c$ has for:

i $c > \frac{3}{2}$ **ii** $c = \frac{3}{2}$ **iii** $0 < c < \frac{3}{2}$ **iv** $c = 0$

(Hint: Sketch the horizontal line $y = c$ on the same number plane and see how many times the graphs intersect.)

18 Show that the curve $y = x^a(1 - x)^b$ has a turning point whose x-coordinate divides the interval between the points $(0, 0)$ and $(1, 0)$ in the ratio $a : b$.

19 Consider the polynomial function $P(x) = (x - p)(x - q)(x - r)$, where p, q and r are distinct real numbers.

a Sketch a possible graph of $y = P(x)$. (Do not attempt to find the stationary or inflection points.)

b Expand $P(x)$, writing it in the form $ax^3 + bx^2 + cx + d$.

c Hence, or otherwise, prove that $(p + q + r)^2 > 3(pq + qr + rp)$.

20 a It was claimed just after worked Example 7 that the derivative of an even function is odd. Draw graphs of some even functions to explain why this is so.

b Similarly, draw graphs to explain why the derivative of an odd function is even.

c Explain how this works when differentiating powers of x.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4C Some less familiar curves

Our table of slopes in the previous section relied on one principle:

The derivative y' of a function can only change sign at a zero or discontinuity of the derivative.

Most of the examples in Exercise 4B, however, had no discontinuities in y', because such things can cause trouble. This section deals specifically with the sketching of such functions. Readers may prefer to work through the more straightforward sections of this chapter first, and return later to this section.

A discontinuity of y' can occur at a vertical asymptote

When a function $y = f(x)$ has an asymptote at $x = a$, then the derivative y' is undefined there. We have dealt with these situations before, particularly in Section 5C in the Year 11 book and in Section 3B of the previous chapter. The next worked example shows how to work with the derivative and the table of slopes in this situation.

Example 9 **4C**

a Differentiate $y = \dfrac{1}{x(x-4)}$, find the zeroes and discontinuities of y', then use a table of slopes to analyse the slope of the function.

b Analyse the sign of the function in its domain, find and describe any vertical and horizontal asymptotes, then sketch the curve.

SOLUTION

a Differentiating using the chain rule,

$$\frac{dy}{dx} = \frac{-1}{x^2(x-4)^2} \times (2x-4)$$
$$= \frac{2(2-x)}{x^2(x-4)^2},$$

so y' has a zero at $x = 2$
and discontinuities at $x = 0$ and $x = 4$.

Let $u = x^2 - 4x$,
then $y = \dfrac{1}{u}$.
Hence $\dfrac{du}{dx} = 2x - 4$
and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$.

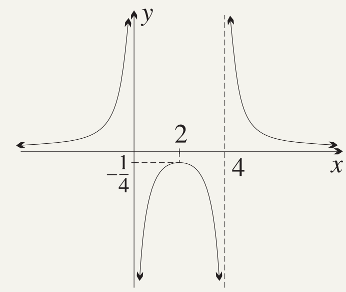
x	-1	0	1	2	3	4	5
y'	$\frac{6}{25}$	*	$\frac{2}{9}$	0	$-\frac{2}{9}$	*	$-\frac{6}{25}$
	/	*	/	–	\	*	\

So the function has a maximum turning point at $(2, -\frac{1}{4})$, it is increasing for $x < 2$ (except at $x = 0$), and it is decreasing for $x > 2$ (except at $x = 4$).

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b The function has domain $x \neq 0$ and $x \neq 4$, and is never zero, and it has discontinuities at $x = 0$ and $x = 4$.



x	-1	0	2	4	5
y'	$\frac{1}{5}$	*	$-\frac{1}{4}$	*	$\frac{1}{5}$

so $y > 0$ for $x < 0$ or $x > 4$, and $y < 0$ for $0 < x < 4$.
As $x \to 4^+$, $y \to \infty$, and as $x \to 4^-$, $y \to -\infty$, as $x \to 0^+$, $y \to -\infty$, and as $x \to 0^-$, $y \to \infty$, so $x = 0$ and $x = 4$ are vertical asymptotes.
Also, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$, so the x-axis is a horizontal asymptote.

Rational functions

A *rational function* is a function such as $y = \dfrac{x - 2}{x^2 - 4x}$, which is the ratio of two polynomials. These functions can be very complicated to sketch. Besides the difficult algebra of the quotient rule, there may be asymptotes, zeroes, turning points and inflections. The curve $y = \dfrac{1}{x(x-4)}$ above is a rational function, but was a little simpler to handle because of the constant numerator.

The function may be defined at a point where y' is undefined

In the next three worked examples, a function is defined at some value $x = a$, but the derivative y' is not defined there. The most obvious examples of this are absolute value functions, which have already been sketched in Section 5E in Year 11 and Sections 3D–3E in the previous chapter.

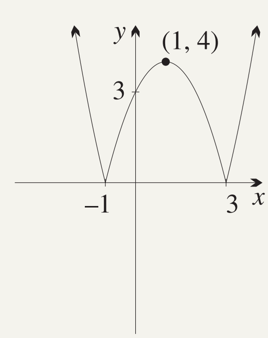
The first graph is sketched without calculus, as it should be. The sketch then helps to understand the differentiation and the table of slopes.

Example 10 4C

a Sketch $y = |(x+1)(x-3)|$ using transformations.
b Hence find the derivative y'.
c Draw up a table of slopes, and describe the curve.

SOLUTION

a Proceeding as in Section 5E (Year 11), draw $y = (x+1)(x-3)$, then reflect the part below the x-axis up above the x-axis.



b There is no tangent at $x = -1$ or $x = 3$, so y' is undefined there.

$$\text{For } x < -1 \text{ or } > 3,\quad y = x^2 - 2x - 3$$
$$y' = 2x - 2,$$
$$\text{and for } -1 < x < 3,\quad y = -x^2 + 2x + 3$$
$$y' = -2x + 2,$$

giving a piecewise definition of y'.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Hence y' has discontinuities at $x = -1$ and $x = 3$, and a single zero at $x = 1$.

x	-2	-1	0	1	2	3	4
y'	-6	$*$	2	0	-2	$*$	6
slope	\	$*$	/	0	\	$*$	/

The table tells us what we already know — there are global minima at $(-1, 0)$ and $(3, 0)$, and a (local) maximum turning point at $(1, 4)$.

Vertical tangents

The function $y = x^{\frac{1}{3}}$ in the next worked example can be sketched as the inverse function of $y = x^3$, using reflection in the diagonal line $y = x$. This time, however, the working is done without reference to any transformation.

Example 11 — 4C

a Differentiate $y = x^{\frac{1}{3}}$, find the zeroes and discontinuities of y', and draw up a table of slopes.
b Examine the behaviour of y' near $x = 0$, and hence of the curve.
c Examine the behaviour of y' and y as $x \to \infty$ and as $x \to -\infty$, then sketch the curve.

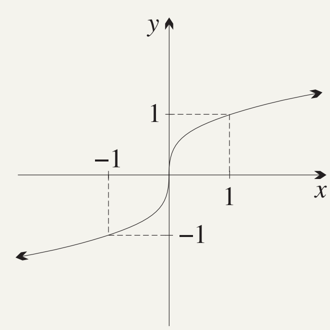
SOLUTION

a The curve passes through the origin and is continuous there.

Differentiating, $y' = \frac{1}{3}x^{-\frac{2}{3}}$,

so y' has no zeroes, and has a discontinuity at $x = 0$.

x	-1	0	1
y'	$\frac{1}{3}$	$*$	$\frac{1}{3}$
slope	/	$*$	/



b Because $y' \to \infty$ as $x \to 0^+$ and as $x \to 0^-$, there is a vertical tangent at $(0, 0)$.

c Here $y' \to 0$ as $x \to \infty$ and as $x \to -\infty$, so the curve flattens out away from the origin, but $y \to \infty$ as $x \to \infty$, so there is no horizontal asymptote.

Cusps

In the last worked example, the gradient became infinite near the point, but the gradient was positive on both sides, so there was still a well-defined vertical tangent at the origin. The curve has a cusp, where it becomes vertical on both sides of the point, but there is no vertical tangent there because the gradients have opposite signs on either side.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 12 4C

a Differentiate $y = x^{\frac{2}{3}}$, find the zeroes and discontinuities of y', and draw up a table of slopes.

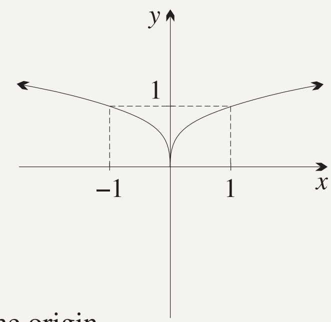
b Examine the behaviour of y' near $x = 0$, and hence of the curve.

c Examine the behaviour of y' and y as $x \to \infty$ and as $x \to -\infty$, then sketch the curve.

SOLUTION

a The curve passes through the origin and is continuous there.

Differentiating, $y' = \frac{2}{3}x^{-\frac{1}{3}}$,
so y' has no zeroes, and has a discontinuity at $x = 0$.



x	-1	0	1
y'	$-\frac{2}{3}$	*	$\frac{2}{3}$
	\	*	/

b As $x \to 0^+$, $y' \to \infty$, and as $x \to 0^-$, $y' \to -\infty$, so there is a cusp at the origin.

c Again $y' \to 0$ as $x \to \infty$ and as $x \to -\infty$, so the curve flattens out away from the origin. Again $y \to \infty$ as $|x| \to \infty$, so there is no horizontal asymptote.

6 SOME OF THE POSSIBILITIES WHEN THE DERIVATIVE y' IS UNDEFINED AT $x = a$

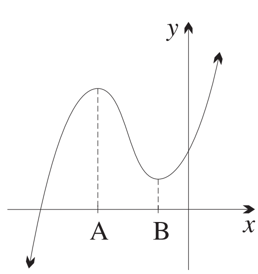
- There may be a vertical asymptote at $x = a$.
- Suppose also that the function is defined at $x = a$.
  - The two sides may meet at an acute or obtuse angle at the point.
  - There may be a *vertical tangent* at $x = a$, where the gradient has the same sign on both sides, and the curve passes smoothly through the point.
  - There may be a *cusp* at $x = a$, where the curve becomes vertical on both sides of the point, but the gradient has opposite sign around the point.

Exercise 4C

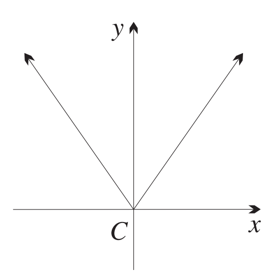
FOUNDATION

1 All the zeroes and discontinuities of y' have been labelled on each graph drawn below. State which of these are local maxima or minima, mentioning if they are turning points. Also state which of these are horizontal points of inflection.

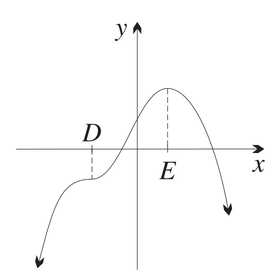
a



b



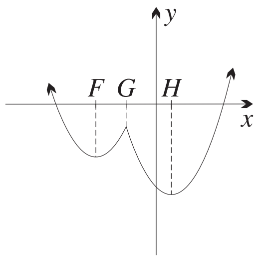
c



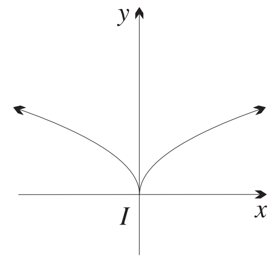
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

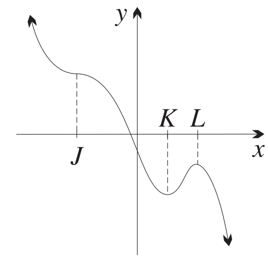
d



e



f



2 The derivatives of various functions have been given below. In each part, find the zeroes and discontinuities of y'. Use a table of test points of y' to find which zeroes and discontinuities of y' give turning points or horizontal points of inflection.

a $y' = x(x-3)^2$ **b** $y' = (x+2)^3(x-4)$ **c** $y' = \dfrac{x}{x-1}$

d $y' = \dfrac{x^2}{x-1}$ **e** $y' = \dfrac{x}{(x-1)^2}$ **f** $y' = \dfrac{x^2}{(x-1)^3}$

g $y' = x - \dfrac{1}{x}$ **h** $y' = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ **i** $y' = \dfrac{2-x}{\sqrt{2+x}(1-x)^3}$

DEVELOPMENT

3 **a** Graph $y = |x| + 3$ using transformations. **b** Find y' when $x > 0$ and when $x < 0$.
c For what points on the graph is y' is undefined?

4 **a** Graph $y = |x-2|$ using transformations. **b** Find y' when $x > 2$ and when $x < 2$.
c For what points on the graph is y' is undefined?

5 Differentiate each function using the quotient rule, then find any zeroes and discontinuities of y' and draw up a table of slopes to analyse them. Find any horizontal and vertical asymptotes, then sketch the function.

a $y = \dfrac{x}{x^2-1}$ **b** $y = \dfrac{x^2}{1+x^2}$ **c** $y = \dfrac{x^2-4}{x^2-1}$ **d** $y = \dfrac{x^2+1}{(x-1)^2}$

6 **a** Differentiate $f(x) = (x-2)^{\frac{1}{5}}$.
b Show that there are no stationary points, but that at $x = 2$, y is defined, but y' is not defined. What happens at $x = 2$?
c Draw up a table of slopes, and sketch the curve.

7 **a** Differentiate $f(x) = (x-1)^{\frac{2}{3}}$. Then show that there are no stationary points, but at $x = 1$, y is defined, but y' is not defined.
b By considering the sign of $f'(x)$ when $x < 1$ and when $x > 1$, determine what is happening at $x = 1$. Then sketch the curve.
c Which of the curves in Questions 6–7 has a global maximum or minimum?

8 **a** State the domain of the function $y = x + \dfrac{1}{x}$.
b Show that $\dfrac{dy}{dx} = \dfrac{x^2-1}{x^2}$, and write down any zeroes and discontinuities of y'.
c Analyse them using a table of slopes.
d Describe what happens to $(y-x)$ as $x \to \infty$ and $x \to -\infty$, and hence find the oblique asymptote. Find also any vertical asymptotes.
e Sketch a graph of the function indicating all important features.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 **a** State the domain of the function $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$.

b Show that $f'(x) = \dfrac{x - 1}{2x\sqrt{x}}$, and write down any zeroes of $f'(x)$.

c Use a table of slopes to determine their nature.

d Describe what happens to $f(x)$ and to $f'(x)$ as $x \to \infty$.

e Sketch a graph of the function, indicating all important features.

10 Use the steps outlined in the previous two questions to graph these functions.

a $y = x - \dfrac{1}{x}$ **b** $y = x^2 + \dfrac{1}{x^2}$

ENRICHMENT

11 **a** Find the domain and any asymptotes of $y = \dfrac{\sqrt{x}}{\sqrt{9 + x^2}}$.

b Show that its derivative is $\dfrac{dy}{dx} = \dfrac{(3 - x)(3 + x)}{2(9 + x^2)^{\frac{3}{2}}\sqrt{x}}$.

c Find the zeroes and discontinuities of y', and analyse them with a table of slopes.

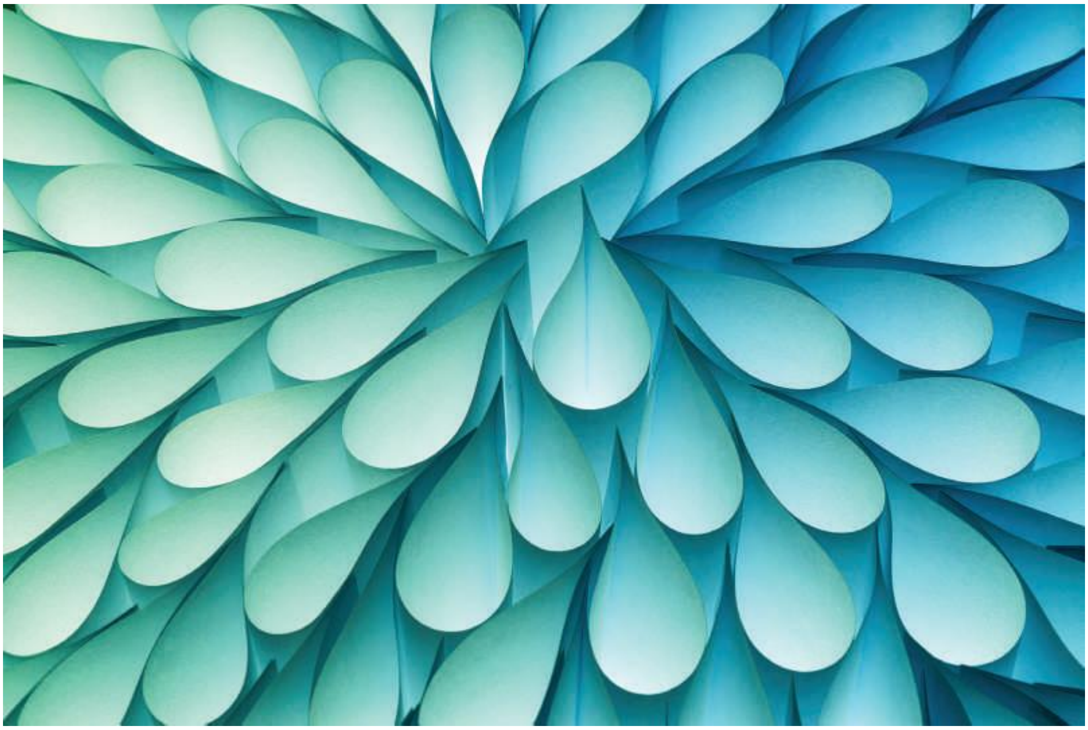
d By examining the limit of the derivative as $x \to 0^+$, determine the shape of the curve near the origin, then sketch the curve.

12 **a** Find the coordinates of any points on $y = x^{\frac{1}{2}} - x^{\frac{3}{2}}$ where y' is zero or undefined.

b Hence sketch a graph of $y^2 = x(1 - x)^2$.

13 Sketch graphs of these functions.

a $y = |(x - 1)(x - 3)|$ **b** $y = |x - 2| + |x + 1|$ **c** $y = x^2 + |x|$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4D Second and higher derivatives

The derivative of the derivative of a function is called the *second derivative* of the function. There are various notations, including

$$\frac{d^2y}{dx^2} \quad \text{and} \quad f''(x) \quad \text{and} \quad f^{(2)}(x) \quad \text{and} \quad y'' \quad \text{and} \quad y^{(2)}.$$

This section reviews the algebra of higher derivatives from Section 9D of the Year 11 book in preparation for the geometric implications of the second derivative in the next section.

Example 13 4D

Find the successive derivatives of $y = x^4 + x^3 + x^2 + x + 1$.

SOLUTION

$$y = x^4 + x^3 + x^2 + x + 1 \qquad \frac{d^2y}{dx^2} = 12x^2 + 6x + 2 \qquad \frac{d^4y}{dx^4} = 24$$

$$\frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1 \qquad \frac{d^3y}{dx^3} = 24x + 6 \qquad \frac{d^5y}{dx^5} = 0$$

Because the fifth derivative is zero, all the higher derivatives are also zero.

Example 14 4D

Find the first four derivatives of $f(x) = x^{-1}$, giving each answer as a fraction.

SOLUTION

$$f'(x) = -x^{-2} = -\frac{1}{x^2} \qquad f''(x) = 2x^{-3} = \frac{2}{x^3} \qquad f^{(3)}(x) = -6x^{-4} = -\frac{6}{x^4} \qquad f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$$

Exercise 4D

FOUNDATION

1 Find the first, second and third derivatives of each function.

a $y = x^3$ **b** $y = x^{10}$ **c** $y = x^7$ **d** $y = x^2$ **e** $y = 2x^4$
f $y = 3x^5$ **g** $y = 4 - 3x$ **h** $y = x^2 - 3x$ **i** $y = 4x^3 - x^2$ **j** $y = 4x^5 + 2x^3$

2 Expand each product, then find the first and second derivatives.

a $y = x(x + 3)$ **b** $y = x^2(x - 4)$ **c** $y = (x - 2)(x + 1)$
d $y = (3x + 2)(x - 5)$ **e** $y = 3x^2(2x^3 - 3x^2)$ **f** $y = 4x^3(x^5 + 2x^2)$

DEVELOPMENT

3 Find the first, second and third derivatives of each function.

a $y = x^{0.3}$ **b** $y = x^{-1}$ **c** $y = x^{-2}$ **d** $y = 5x^{-3}$ **e** $y = x^2 + x^{-1}$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

4 By writing each function with a negative index, find its first and second derivatives.

a $f(x) = \dfrac{1}{x^3}$ **b** $f(x) = \dfrac{1}{x^4}$ **c** $f(x) = \dfrac{3}{x^2}$ **d** $f(x) = \dfrac{2}{x^3}$

5 Use the chain rule to find the first and second derivatives of each function.

a $y = (x + 1)^2$ **b** $y = (3x - 5)^3$ **c** $y = (1 - 4x)^2$ **d** $y = (8 - x)^{11}$

6 By writing each function with a negative index, find its first and second derivatives.

a $y = \dfrac{1}{x + 2}$ **b** $y = \dfrac{1}{(3 - x)^2}$ **c** $y = \dfrac{1}{(5x + 4)^3}$ **d** $y = \dfrac{2}{(4 - 3x)^2}$

7 By writing each function with a fractional index, find its first and second derivatives.

a $f(x) = \sqrt{x}$ **b** $f(x) = \sqrt[3]{x}$ **c** $f(x) = x\sqrt{x}$

d $f(x) = \dfrac{1}{\sqrt{x}}$ **e** $f(x) = \sqrt{x + 2}$ **f** $f(x) = \sqrt{1 - 4x}$

8 **a** Find $f'(x)$ and $f''(x)$ for the function $f(x) = x^3 + 3x^2 + 5x - 6$.

b Hence evaluate:

i $f'(0)$ **ii** $f'(1)$ **iii** $f''(0)$ **iv** $f''(1)$

9 **a** If $f(x) = 3x + x^3$, find:

i $f'(2)$ **ii** $f''(2)$ **iii** $f'''(2)$ **iv** $f''''(2)$

b If $f(x) = (2x - 3)^4$, find:

i $f'(1)$ **ii** $f''(1)$ **iii** $f'''(1)$ **iv** $f''''(1)$

10 Use the quotient rule to find the first derivative of each function. Then use the chain rule to find the second derivative.

a $y = \dfrac{x}{x + 1}$ **b** $y = \dfrac{x - 1}{2x + 5}$

11 If $f(x) = x(x - 1)^4$, use the product rule to find $f'(x)$ and $f''(x)$.

12 Find the values of x for which $y'' = 0$ if:

a $y = x^4 - 6x^2 + 11$ **b** $y = x^3 + x^2 - 5x + 7$

13 **a** Find the first, second and third derivatives of x^n.

b Find the nth and $(n + 1)$th derivatives of x^n.

14 **a** If $y = 3x^2 + 7x + 5$, prove that $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}$.

b If $y = (2x - 1)^4$, prove that $\dfrac{d}{dx}\left(y\dfrac{dy}{dx}\right) = y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2$.

c If $y = 2x^2 - \dfrac{3}{\sqrt{x}}$, prove that $2x^2\dfrac{d^2y}{dx^2} = x\dfrac{dy}{dx} + 2y$.

ENRICHMENT

15 Given $y = x^a + x^{-b}$, find positive integers a and b such that $x^2y'' + 2xy' = 12y$.

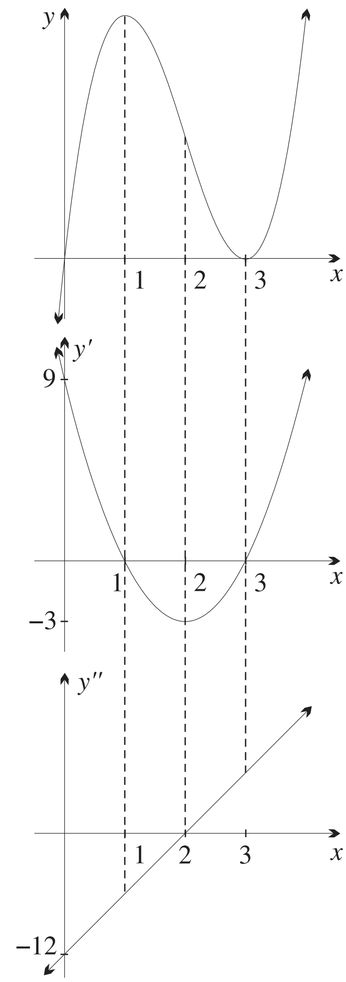
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4E Concavity and points of inflection

Sketched to the right are a cubic function and its first and second derivatives. These sketches will show how the *concavity* of the original graph can be determined from the sign of the second derivative.

$$\begin{aligned} y &= x^3 - 6x^2 + 9x = x(x-3)^2 \\ y' &= 3x^2 - 12x + 9 = 3(x-1)(x-3) \\ y'' &= 6x - 12 \quad\quad\;\; = 6(x-2) \end{aligned}$$



The sign of each derivative tells us whether the function above it is increasing or decreasing. Thus the second graph describes the gradient of the first, and the third graph describes the gradient of the second.

To the right of $x = 2$, the top graph is concave up. This means that as one moves along the curve to the right from $x = 2$, the tangent gets steeper, with its gradient steadily increasing. Thus for $x > 2$, the gradient function y' is increasing as x increases, as can be seen in the middle graph. The bottom graph is the gradient of the middle graph, and accordingly y'' is positive for $x > 2$.

Similarly, to the left of $x = 2$ the top graph is concave down. This means that its gradient function y' is steadily decreasing as x increases. The bottom graph is the derivative of the middle graph, so y'' is negative for $x < 2$.

This example demonstrates that the concavity of a graph $y = f(x)$ at any value $x = a$ is determined by the sign of its second derivative at $x = a$.

7 CONCAVITY AND THE SECOND DERIVATIVE

- If $f''(a)$ is negative, the curve is concave down at $x = a$.
- If $f''(a)$ is positive, the curve is concave up at $x = a$.

Points of inflection

As foreshadowed in Section 4B, a *point of inflection* is a point where the tangent crosses the curve. This means that the curve curls away from the tangent on opposite sides of the tangent, and this in turn means that the concavity changes sign around the point.

The three diagrams above show how the point of inflection at $x = 2$ results in a minimum turning point at $x = 2$ in the middle graph of y'. Hence the bottom graph of y'' has a zero at $x = 2$ and changes sign around $x = 2$.

This discussion gives us a method of analysing concavity and finding points of inflection. Once again, we use the fact that y'' can only change sign at a zero or a discontinuity of y''.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 USING $f''(x)$ TO ANALYSE CONCAVITY AND FIND POINTS OF INFLECTION

A *point of inflection* is a point where the tangent crosses the curve.

1. Find the zeroes and discontinuities of the second derivative $f''(x)$.
2. Then use a table of test values of the second derivative $f''(x)$ dodging around its zeroes and discontinuities, with the concavities underneath, to see where the concavity changes sign.

The table of concavities will show not only any points of inflection, but also the concavity of the graph across its whole domain.

Before drawing the sketch, it is often useful to find the gradient of the tangent at each point of inflection. Such tangents are called *inflectional tangents*.

Example 15 4E

a Find any turning points of $f(x) = x^5 - 5x^4$.

b Draw up a table of concavities. Find any points of inflection and the gradients of the inflectional tangents, and describe the concavity. Then sketch.

SOLUTION

Here
$$f(x) = x^5 - 5x^4 \quad = x^4(x - 5)$$
$$f'(x) = 5x^4 - 20x^3 \quad = 5x^3(x - 4)$$
$$f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3).$$

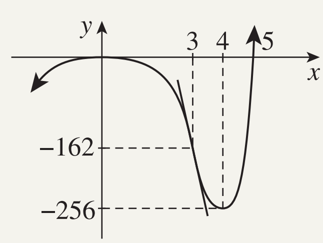
a $f'(x)$ has zeroes at $x = 0$ and $x = 4$, and no discontinuities:

x	-1	0	1	4	5
$f'(x)$	25	0	-15	0	625
slope	/	—	\	—	/

so $(0, 0)$ is a maximum turning point, and $(4, -256)$ is a minimum turning point.

b $f''(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities:

x	-1	0	1	3	4
$f''(x)$	-80	0	-40	0	320
concavity	⌢	·	⌢	·	⌣



so $(3, -162)$ is a point of inflection, but $(0, 0)$ is not.

Because $f'(3) = -135$, the inflectional tangent has gradient -135.

The graph is concave down for $x < 0$ and $0 < x < 3$, and concave up for $x > 3$.

Note: The example given above is intended to show that $f''(x) = 0$ is *not a sufficient condition* for a point of inflection. *The sign of $f''(x)$ must also change around the point* — this happened at $x = 3$, but not at $x = 0$.

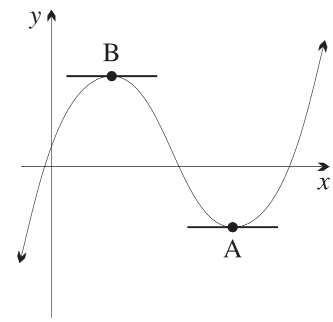
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the second derivative to test stationary points

If a curve is concave up at a stationary point, then the point is a minimum turning point, as at the point A.

Similarly, if a curve is concave down at a stationary point, then the point is a maximum turning point, as at the point B. This gives an alternative test of a stationary point.



9 USING THE SECOND DERIVATIVE TO TEST A STATIONARY POINT

Suppose that the curve $y = f(x)$ has a stationary point at $x = a$.

- If $f''(a) > 0$, the curve is concave up at $x = a$, and there is a minimum turning point there.
- If $f''(a) < 0$, the curve is concave down at $x = a$, and there is a maximum turning point there.
- If $f''(a) = 0$, *more work is needed. Go back to the table of values of* $f'(x)$.

The third dotpoint is most important — all four cases shown on page 137 are possible for the shape of the curve at $x = a$ when the second derivative is zero there.

The previous example of the point $(0, 0)$ on $y = x^5 - 5x^4$ shows that a stationary point where $f''(x) = 0$ can be a turning point. The next worked example is an example where a stationary point turns out to be a point of inflection.

Example 16 4E

Use the second derivative, if possible, to determine the nature of the stationary points of the graph of $f(x) = x^4 - 4x^3$. Find also any points of inflection, examine the concavity over the whole domain, and sketch the curve.

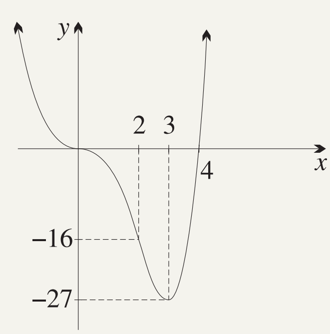
SOLUTION

Here,
$$\begin{aligned} f(x) &= x^4 - 4x^3 &&= x^3(x-4) \\ f'(x) &= 4x^3 - 12x^2 &&= 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x &&= 12x(x-2), \end{aligned}$$

so $f'(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities.
Because $f''(3) = 36$ is positive, $(3, -27)$ is a minimum turning point,
but $f''(0) = 0$, *so no conclusion can be drawn about* $x = 0$.

x	-1	0	1	3	4
$f'(x)$	-16	0	-8	0	64
slope	\	—	\	—	/

So $(0, 0)$ is a stationary point of inflection.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$f''(x)$ has zeroes at $x = 0$ and $x = 2$, and no discontinuities,

x	-1	0	1	2	3
$f''(x)$	36	0	-12	0	36
concavity	$\smile$	$\cdot$	$\frown$	$\cdot$	$\smile$

So, besides the horizontal inflection at $(0, 0)$,
there is a non-stationary inflection at $(2, -16)$,
and the inflectional tangent at $(2, -16)$ has gradient -16.
The graph is concave down for $0 < x < 2$, and concave up for $x < 0$ and for $x > 2$.

Finding pronumerals in a function

In this worked example, a pronumeral in a function is found using information about the concavity of the graph.

Example 17 4E

For what values of b is $y = x^4 - bx^3 + 5x^2 + 6x - 8$ concave down when $x = 2$?

SOLUTION

Differentiating, $y' = 4x^3 - 3bx^2 + 10x + 6$,
and differentiating again, $y'' = 12x^2 - 6bx + 10$,
so when $x = 2$, $y'' = 48 - 12b + 10$
$= 58 - 12b$.

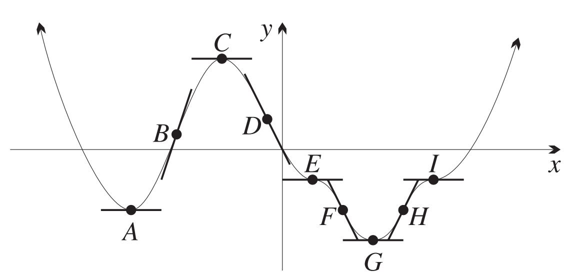
In order for the curve to be concave down at $x = 2$,

$$58 - 12b < 0$$
$$12b > 58$$
$$b > 4\tfrac{5}{6}.$$

Exercise 4E

FOUNDATION

1 Complete the table below for the function to the right. At each point, state whether the first and second derivatives are positive, negative or zero.



Point	A	B	C	D	E	F	G	H	I
y'									
y''									

2 Find $f''(x)$ for each function. By evaluating $f''(0)$, state whether the curve is concave up ($f''(x) > 0$) or concave down ($f''(x) < 0$) at $x = 0$.

a $f(x) = x^3 - 3x^2$

b $f(x) = x^3 + 4x^2 - 5x + 7$

c $f(x) = x^4 + 2x^2 - 3$

d $f(x) = 6x - 7x^2 - 8x^4$

3 By showing that $f'(2) = 0$, prove that each curve has a stationary point at $x = 2$. Then evaluate $f''(2)$ to determine the nature of the stationary point.

a $f(x) = x^2 - 4x + 4$

b $f(x) = 5 + 4x - x^2$

c $f(x) = x^3 - 12x$

d $f(x) = 2x^3 - 3x^2 - 12x + 5$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 A curve is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$.

a Explain why $y = x^2 - 3x + 7$ is concave up for all values of x.
b Explain why $y = -3x^2 + 2x - 4$ is concave down for all values of x.

5 **a** Find the second derivative $\frac{d^2y}{dx^2}$ of $y = x^3 - 3x^2 - 5x + 2$.

b Hence find the values of x for which the curve is:
i concave up,
ii concave down.

6 **a** Find the second derivative $\frac{d^2y}{dx^2}$ of $y = x^3 - x^2 - 5x + 1$.

b Hence find the values of x for which the curve is:
i concave up,
ii concave down.

DEVELOPMENT

7 A function has second derivative $y'' = 3x^3(x + 3)^2(x - 2)$. Determine the x-coordinates of the points of inflection on the graph of the function.

8 **a** If $f(x) = x^3 - 3x$, show that $f'(x) = 3(x - 1)(x + 1)$ and $f''(x) = 6x$.
b By solving $f'(x) = 0$, find the coordinates of any stationary points.
c Examine the sign of $f''(1)$ and $f''(-1)$ to determine their nature.
d Find the coordinates of the point of inflection. Remember that you must show that the sign of $f''(x)$ changes about this point.
e Sketch the graph of $f(x)$, indicating all important features.

9 **a** If $f(x) = x^3 - 6x^2 - 15x + 1$, show that $f'(x) = 3(x - 5)(x + 1)$ and $f''(x) = 6(x - 2)$.
b Find any stationary points and use the sign of $f''(x)$ to determine their nature.
c Find the coordinates of any points of inflection.
d Sketch the graph of $f(x)$, indicating all important features.

10 **a** If $y = x^3 - 3x^2 - 9x + 11$, show that $y' = 3(x - 3)(x + 1)$ and $y'' = 6(x - 1)$.
b Find any stationary points and use the sign of y'' to determine their nature.
c Find the coordinates of any points of inflection.
d Sketch the graph of the function, indicating all important features.

11 **a** If $y = 3 + 4x^3 - x^4$, show that $y' = 4x^2(3 - x)$ and $y'' = 12x(2 - x)$.
b Find any stationary points and use a table of test values of y' to determine their nature.
c Find the coordinates of any points of inflection.
d Sketch the graph of the function, indicating all important features.

12 Find the range of values of x for which the curve $y = 2x^3 - 3x^2 - 12x + 8$ is:
a increasing, that is $y' > 0$,
b decreasing, that is $y' < 0$,
c concave up, that is $y'' > 0$,
d concave down, that is $y'' < 0$.

ISBN 978-1-108-76630-2

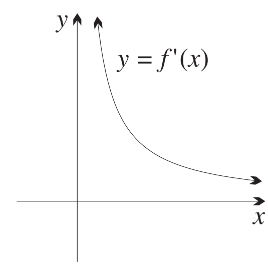
Cambridge University Press
Updated July 2022

13 **a** If $y = x^3 + 3x^2 - 72x + 14$, find y' and y''.
b Show that the curve has a point of inflection at $(-1, 88)$.
c Show that the gradient of the tangent at the point of inflection is -75.
d Hence find the equation of the tangent at the point of inflection.

14 **a** If $f(x) = x^3$ and $g(x) = x^4$, find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$.
b Both $f(x)$ and $g(x)$ have a stationary point at $(0, 0)$. Evaluate $f''(x)$ and $g''(x)$ when $x = 0$. Can you determine the nature of the stationary points from this calculation?
c Use tables of values of $f'(x)$ and $g'(x)$ to determine the nature of the stationary points.

15 **a** Find a if the curve $y = x^3 - ax^2 + 3x - 4$ has an inflection at the point where $x = 2$.
b For what values of a is $y = x^3 + 2ax^2 + 3x - 4$ concave up at the point where $x = -1$?
c Find a and b if the curve $y = x^4 + ax^3 + bx^2$ has an inflection at $(2, 0)$.
d For what values of a is $y = x^4 + ax^3 - x^2$ concave up and increasing when $x = 1$?

16 The diagram to the right shows the graph of the derivative $y = f'(x)$ of the function $y = f(x)$, with domain $x > 0$.



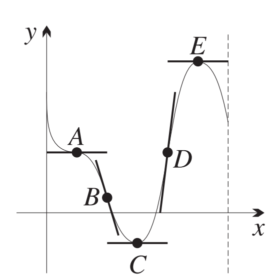
a State whether the graph of $y = f(x)$ is increasing or decreasing throughout its domain.
b State whether the graph of $y = f(x)$ is concave up or concave down throughout its domain.

17 Sketch a small section of the graph of the continuous function $f(x)$ about $x = a$ if:
a $f'(a) > 0$ and $f''(a) > 0$,
b $f'(a) > 0$ and $f''(a) < 0$,
c $f'(a) < 0$ and $f''(a) > 0$,
d $f'(a) < 0$ and $f''(a) < 0$.

18 A function has equation $y = \frac{1}{3}x^3 - 3x^2 + 11x - 9$.
a Show that the function has no stationary points.
b Show that there is a point of inflection.
c How many x-intercepts does the graph of the function have? Justify your answer.

19 **a** Show that if $y = \dfrac{x + 2}{x - 3}$, then $y'' = \dfrac{10}{(x - 3)^3}$.
b By examining the sign of $(x - 3)^3$, determine when the curve is concave up, and when it is concave down.
c Sketch the graph of $y = \dfrac{x + 2}{x - 3}$.

20 Given the graph of $y = f(x)$ drawn to the right, on separate axes sketch graphs of:



a $y = f'(x)$
b $y = f''(x)$

21 **a** Show that if $f(x) = x(x - 1)^3$, then $f'(x) = (x - 1)^2(4x - 1)$ and $f''(x) = 6(x - 1)(2x - 1)$.
b Sketch $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ on the same axes and compare them.

22 A curve has equation $y = ax^3 + bx^2 + cx + d$ and crosses the x-axis at $x = -1$. It has a turning point at $(0, 5)$ and a point of inflection at $x = \frac{1}{2}$. Find the values of a, b, c and d.

ISBN 978-1-108-76630-2

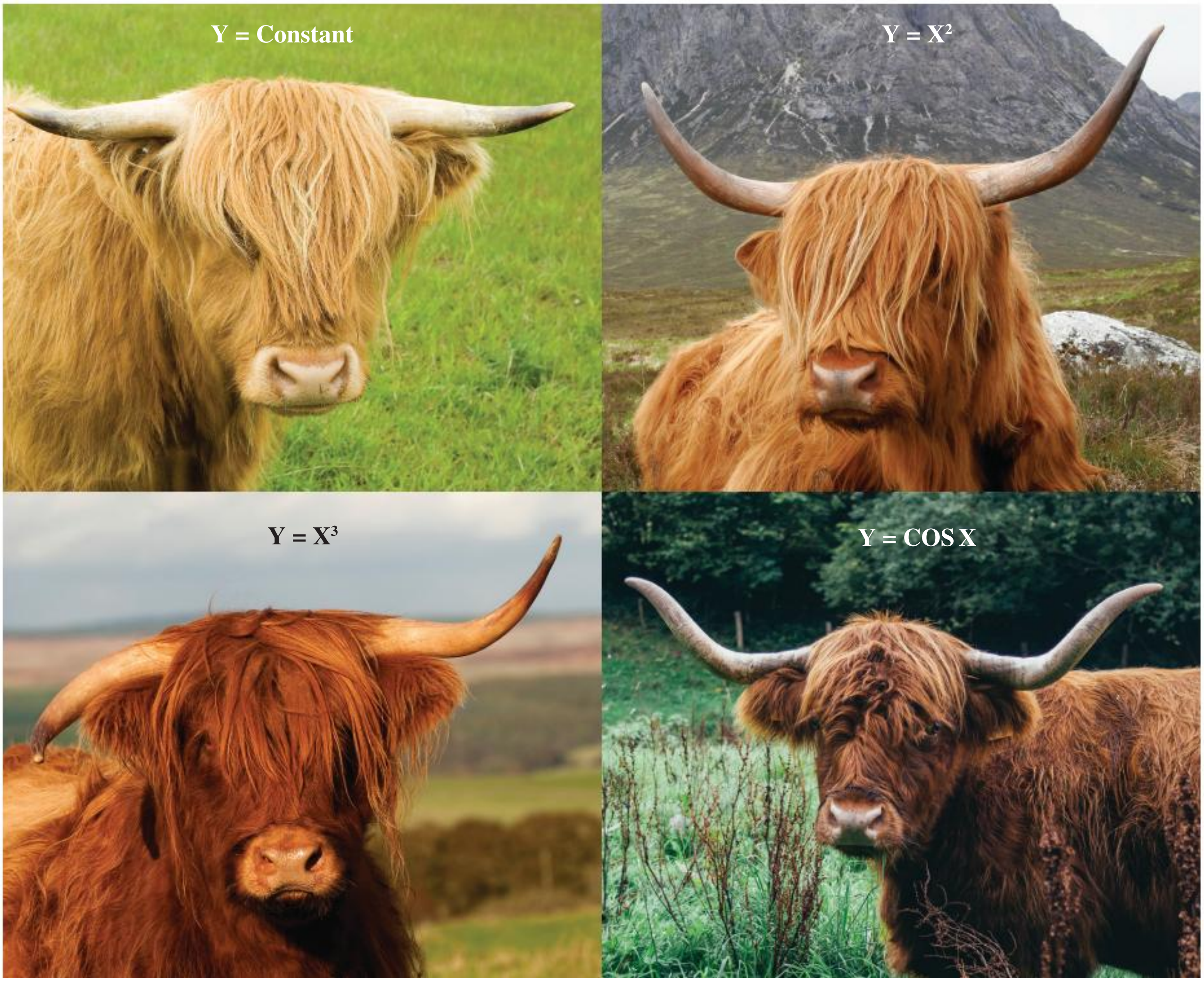
Cambridge University Press
Updated July 2022

ENRICHMENT

23 **a** Find the values of x for which the function $y = x^{\frac{2}{3}}$ is:

i increasing, **ii** decreasing, **iii** concave up, **iv** concave down.

b Hence sketch the graph of the function.



ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

4F Systematic curve sketching with the derivative

In Section 3C of the last chapter, we developed a systematic four-step approach to sketching an unfamiliar curve:

1 domain, **2** symmetry, **3** intercepts and sign, **4** asymptotes.

This chapter has used the derivative to examine the gradient and concavity of curves and to find their turning points and inflections. We can now add these methods to the menu as steps 5 and 6.

Few curves in this course would require consideration of all the points in the summary below. Questions almost always give some guidance as to which methods to use for any particular function.

10 A SUMMARY OF CURVE-SKETCHING METHODS

1 **Domain:** Find the domain of $f(x)$. (*Always* do this first.)

2 **Symmetry:** Find whether the function is even or odd, or neither.

3 **A** **Intercepts:** Find the y-intercept and all x-intercepts (zeroes).

B **Sign:** Use a table of test values of $f(x)$, that is, a table of signs, to find where the function is positive, and where it is negative.

4 **A** **Vertical asymptotes:** Examine any discontinuities to see whether there are vertical asymptotes there.

B **Horizontal asymptotes:** Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

5 **The first derivative:**

A Find the zeroes and discontinuities of $f'(x)$.

B Use a table of test values of $f'(x)$, that is, a table of slopes, to determine the nature of the stationary points and the slope of the function throughout its domain.

6 **The second derivative:**

A Find the zeroes and discontinuities of $f''(x)$.

B Use a table of test values of $f''(x)$, that is, a table of concavities, to find any points of inflection and the concavity of the function throughout its domain.

7 **Any other features:**

A routine warning of incompleteness.

The final Step 7 is a routine warning that many important features of functions will not be picked up using this menu. For example, every parabola has an axis of symmetry, but the even-and-odd test only picks up that axis of symmetry when it is the y-axis. Even more importantly, the trigonometric functions repeat periodically, and tests for periodicity are not mentioned.

An example of a curve with turning points and asymptotes

The curve in the worked example below has three asymptotes and a turning point. Such curves are never easy to analyse, but it is worth having one such example that combines the calculus approaches of the present chapter with the previous non-calculus approaches.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 18 4F

Consider the curve $y = \dfrac{1}{x(x - 4)}$.

a Write down the domain of the function.

b Use a table of test values to analyse the sign of the function.

c Find any vertical and horizontal asymptotes.

d Show that the derivative is $y' = \dfrac{2(2 - x)}{x^2(x - 4)^2}$.

e Find all the zeroes and discontinuities of $f'(x)$. Then use a table of test values of $f'(x)$ to analyse stationary points and find where the function is increasing and decreasing.

f Sketch the curve and hence write down the range of the function.

SOLUTION

a The domain of the function is $x \neq 0$ and $x \neq 4$.

b The function is never zero, and it has discontinuities at $x = 0$ and $x = 4$.

x	-1	0	2	4	5
y	$\frac{1}{5}$	*	$-\frac{1}{4}$	*	$\frac{1}{5}$

Hence y is positive for $x < 0$ or $x > 4$,
and y is negative for $0 < x < 4$.

c The lines $x = 0$ and $x = 4$ are vertical asymptotes.
Also, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$,
so the x-axis is a horizontal asymptote.

d Differentiating using the chain rule,

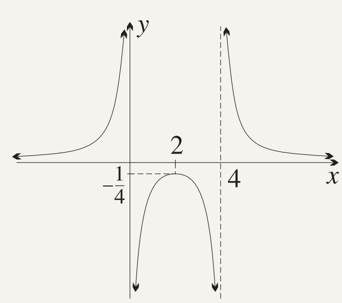
$$y' = \frac{-1}{x^2(x - 4)^2} \times\ )2x - 4)$$
$$= \frac{2(2 - x)}{x^2(x - 4)^2}.$$

Let $u = x^2 - 4x$.
Then $y = \dfrac{1}{u}$.
Hence $\dfrac{du}{dx} = 2x - 4$
and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$.

e Hence y' has a zero at $x = 2$ and discontinuities at $x = 0$ and $x = 4$.

x	-1	0	1	2	3	4	5
y'	$\frac{6}{25}$	*	$\frac{2}{9}$	0	$-\frac{2}{9}$	*	$-\frac{6}{25}$
slope	/	*	/	—	\	*	\

Thus there is a maximum turning point at $\left(2, -\frac{1}{4}\right)$, the curve is increasing for $x < 2$ (except at $x = 0$), and it is decreasing for $x > 2$ (except at $x = 4$).



f The graph is sketched to the right.
From the graph, the range is $y > 0$ or $y \leq -\frac{1}{4}$.

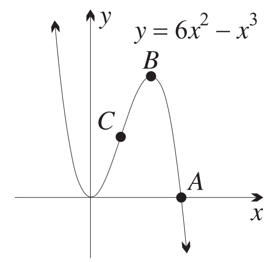
ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 4F

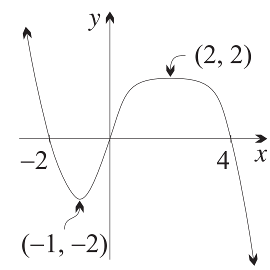
FOUNDATION

1 The diagram to the right shows a sketch of $y = 6x^2 - x^3$. The curve cuts the x-axis at A, and it has a maximum turning point at B and a point of inflection at C.



a Find the coordinates of A.
b Find the coordinates of B.
c Find the coordinates of C.

2 The diagram to the right shows a curve $y = f(x)$. From the sketch, find the values of x for which:



a $f'(x) = 0$,
b $f''(x) = 0$,
c $f(x)$ is increasing,
d $f''(x) > 0$.

3 a Show that $y = 27x - x^3$ is an odd function. What symmetry does its graph display?
b Show that $y' = 3(9 - x^2)$ and $y'' = -6x$.
c Find the coordinates of the stationary points. Then determine their nature, either by examining the sign of $f''(3)$ and $f''(-3)$, or by means of a table of test values of y'.
d Show, using a table of test values of y'', that $x = 0$ is a point of inflection.
e By substituting into the gradient function y', find the gradient at the inflection.
f Sketch the graph of the function, indicating all important features.

4 a If $f(x) = 2x^3 - 3x^2 + 5$, show that $f'(x) = 6x(x - 1)$ and $f''(x) = 6(2x - 1)$.
b Find the coordinates of the stationary points. Then determine their nature, either by examining the sign of $f''(0)$ and $f''(1)$, or by means of a table of test values of y'.
c Explain why there is a point of inflection at $x = \frac{1}{2}$, and find the gradient there.
d Sketch the graph of $f(x)$, indicating all important features.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

5 Find the first and second derivatives of each function below. Hence find the coordinates of any stationary points and determine their nature. Then find any points of inflection. Sketch the graph of each function. You do not need to find the x-intercepts in part **b**.

a $y = x(x - 6)^2$ **b** $y = x^3 - 3x^2 - 24x + 5$

6 **a** If $y = 12x^3 - 3x^4 + 11$, show that $y' = 12x^2(3 - x)$ and $y'' = 36x(2 - x)$.

b By solving $y' = 0$, find the coordinates of any stationary points.

c By examining the sign of y'', establish the nature of the stationary point at $x = 3$. Why does this method fail for the stationary point at $x = 0$?

d Use a table of test values of y' to show that there is a stationary point of inflection at $x = 0$.

e Show that there is a change in concavity at $x = 2$.

f Sketch the graph of the function, showing all important features.

7 Using the method outlined in the previous question, sketch $y = x^4 - 16x^3 + 72x^2 + 10$.

8 **a** Show that the derivative of $f(x) = \dfrac{1}{x^2 - 4}$ is $f'(x) = -\dfrac{2x}{(x^2 - 4)^2}$.

b Show that $y = f(x)$ has a stationary point at $x = 0$. Then determine its nature, using a table of values of $f'(x)$.

c Show that the function is even. What symmetry does its graph exhibit?

d State the domain of the function and the equations of any vertical asymptotes.

e What value does $f(x)$ approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.

f Sketch the graph of $y = f(x)$, showing all important features.

g Use the graph to state the range of the function.

9 **a** Show that the derivative of $f(x) = \dfrac{x}{x^2 - 4}$ is $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$.

b Explain why the curve $y = f(x)$ has no stationary points, and why the curve is always decreasing.

c Given that $f''(x) = \dfrac{2x^3 + 24x}{(x^2 - 4)^3}$, show that $(0, 0)$ is a point of inflection. Then find the gradient of the tangent at this point.

d State the domain of the function and the equations of any vertical asymptotes.

e What value does $f(x)$ approach as x becomes large? Hence write down the equation of the horizontal asymptote.

f Show that the function is odd. What symmetry does its graph have?

g Use a table of test values of y to analyse the sign of the function.

h Sketch the graph of $y = f(x)$, showing all important features.

i Use the graph to state the range of the function.

10 **a** If $f(x) = \dfrac{x^2}{1 + x^2}$, show that $f'(x) = \dfrac{2x}{(1 + x^2)^2}$ and $f''(x) = \dfrac{2 - 6x^2}{(1 + x^2)^3}$.

b Hence find the coordinates of any stationary points and determine their nature.

c Find the coordinates of any points of inflection.

d State the equation of the horizontal asymptote.

e Sketch the graph of the function, indicating all important features.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 **a** If $f(x) = \dfrac{4x}{x^2 + 9}$, show that $f'(x) = \dfrac{36 - 4x^2}{(x^2 + 9)^2}$ and $f''(x) = \dfrac{8x^3 - 216x}{(x^2 + 9)^3}$.

b Hence find the coordinates of any stationary points and determine their nature.

c Find the coordinates of any points of inflection.

d State the equation of the horizontal asymptote.

e Sketch the graph of the function, indicating all important features.

ENRICHMENT

12 A curve has equation $y = x^5 - 15x^3$. Sketch the curve showing the x-intercepts, the stationary points and the points of inflection.

13 Sketch graphs of these two rational functions, indicating all stationary points, points of inflection and intercepts with the axes. For each question solve the equation $y = 1$ to see where the graph cuts the horizontal asymptote:

a $y = \dfrac{x^2 - x - 2}{x^2}$ **b** $y = \dfrac{x^2 - 2x}{(x + 2)^2}$

14 **a** Sketch $f(x) = (x + 5)(x - 1)$, clearly indicating the turning point.

b Where do the graphs of the functions $y = f(x)$ and $y = \dfrac{1}{f(x)}$ intersect?

c Differentiate $y = \dfrac{1}{f(x)}$, and explain why $\dfrac{1}{f(x)}$ increases as $f(x)$ decreases and vice versa.

d Using part **a**, find and analyse the stationary point of $y = \dfrac{1}{(x + 5)(x - 1)}$.

e Hence sketch a graph of the reciprocal function on the same diagram as part **a**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4G Global maximum and minimum

Australia has many high mountain peaks, each of which is a *local* or *relative maximum*, because each is the highest point relative to other peaks in its immediate locality. Mount Kosciuszko is the highest of these, but it is still not a *global* or *absolute maximum*, because there are higher peaks on other continents of the globe. Mount Everest in Asia is the global maximum over the whole world.

11 GLOBAL MAXIMUM AND MINIMUM

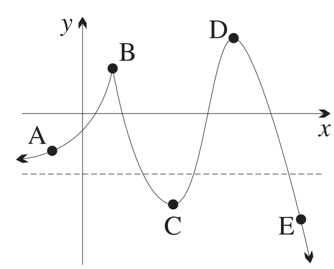
Let $A\big(a, f(a)\big)$ be a point on a curve $y = f(x)$.

- The point A is a *global* or *absolute maximum* if

  $f(x) \le f(a)$, for all x in the domain.
- Similarly, A is a *global* or *absolute minimum* if

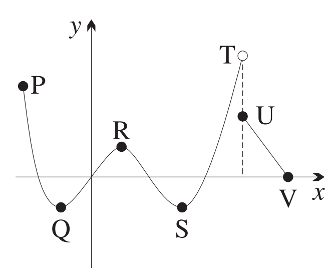
  $f(x) \ge f(a)$, for all x in the domain.

The following diagrams illustrate what has to be considered in the general case.



In the upper diagram, the domain is all real numbers.

- There are local maxima at the point B, where $f'(x)$ is undefined, and at the turning point D. This point D is also the global maximum.
- There is a local minimum at the turning point C, which is lower than all points on the curve to the left past A. There is no global minimum, however, because the curve goes infinitely far downwards to the right of E.



In the lower diagram, the domain is the closed interval on the x-axis from P to V.

- There are local maxima at the turning point R and at the endpoint P. There is no global maximum, however, because the point T has been omitted from the curve.
- There are local minima at the two turning points Q and S, and at the endpoint V. These points Q and S have equal heights and are thus both global minima.

Testing for global maximum and minimum

These examples show that there are three types of points that must be considered and compared when finding the global maximum and minimum of a function $f(x)$ defined on some domain.

12 TESTING FOR GLOBAL MAXIMUM AND MINIMUM

Examine and compare:

- turning points,
- boundaries of the domain (or the behaviour for large x),
- discontinuities of $f'(x)$ (to pick up sharp corners or discontinuities).

More simply, examine and compare turning points and boundary points — and discontinuities if there are any.

ISBN 978-1-108-76630-2

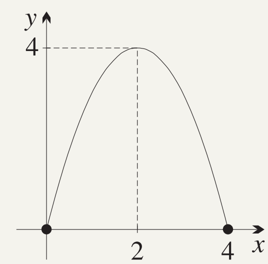
Cambridge University Press
Updated July 2022

Example 19 4G

Find the absolute maximum and minimum of $f(x) = 4x - x^2$ over the domain $0 \le x \le 4$.

Note: Calculus is not needed here because the function is a quadratic.

SOLUTION

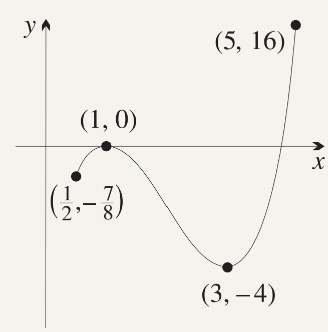


The graph is a concave-down quadratic.
Factoring, $f(x) = x(4 - x)$, so the x-intercepts are $x = 0$ and $x = 4$.
Taking their average, the axis of symmetry is $x = 2$,
and substituting, the vertex is $(2, 4)$.
Hence, from the sketch, the absolute maximum is 4 at $x = 2$,
and the absolute minimum is 0 at the endpoints where $x = 0$ or 4.

Example 20 4G

Find the global maximum and minimum of the function $f(x) = x^3 - 6x^2 + 9x - 4$, where $\frac{1}{2} \le x \le 5$.

SOLUTION



The unrestricted curve was sketched in Section 4B (worked Example 6) and substituting the boundaries,

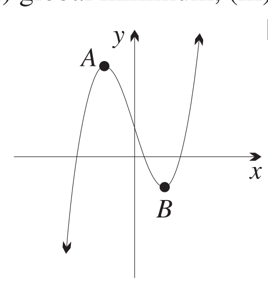
$$f\left(\tfrac{1}{2}\right) = -\tfrac{7}{8} \qquad \text{and} \qquad f(5) = 16.$$

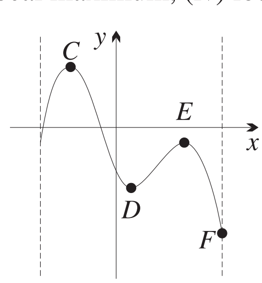
Hence the global maximum is 16, at $x = 5$,
and the global minimum is -4, at $x = 3$.

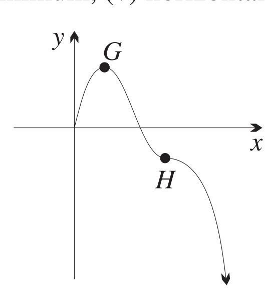
Exercise 4G

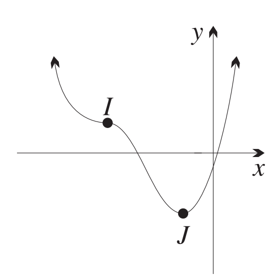
FOUNDATION

1 In the diagrams below, classify each labelled point as one of the following: (i) global maximum, (ii) global minimum, (iii) local maximum, (iv) local minimum, (v) horizontal point of inflection.

a


b


c


d


ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 Sketch each function and state its global minimum and maximum in the specified domain.

a $y = x^2, -2 \le x \le 2$

b $y = 5 - x, 0 \le x \le 3$

c $y = \sqrt{16 - x^2}, -4 \le x \le 4$

d $y = |x|, -5 \le x \le 1$

e $y = \sqrt{x}, 0 \le x \le 8$

f $y = 1/x, -4 \le x \le -1$

g $y = \begin{cases} -1, & \text{for } x < -2, \\ x + 1, & \text{for } -2 \le x < 1, \\ 2, & \text{for } x \ge 1. \end{cases}$

DEVELOPMENT

3 Sketch the graph of each function, clearly indicating any stationary points. Determine the global minimum and maximum of the function in the specified domain.

a $y = x^2 - 4x + 3, 0 \le x \le 5$

b $y = x^3 - 3x^2 + 5, -3 \le x \le 2$

c $y = 3x^3 - x + 2, -1 \le x \le 1$

d $y = x^3 - 6x^2 + 12x, 0 \le x \le 3$

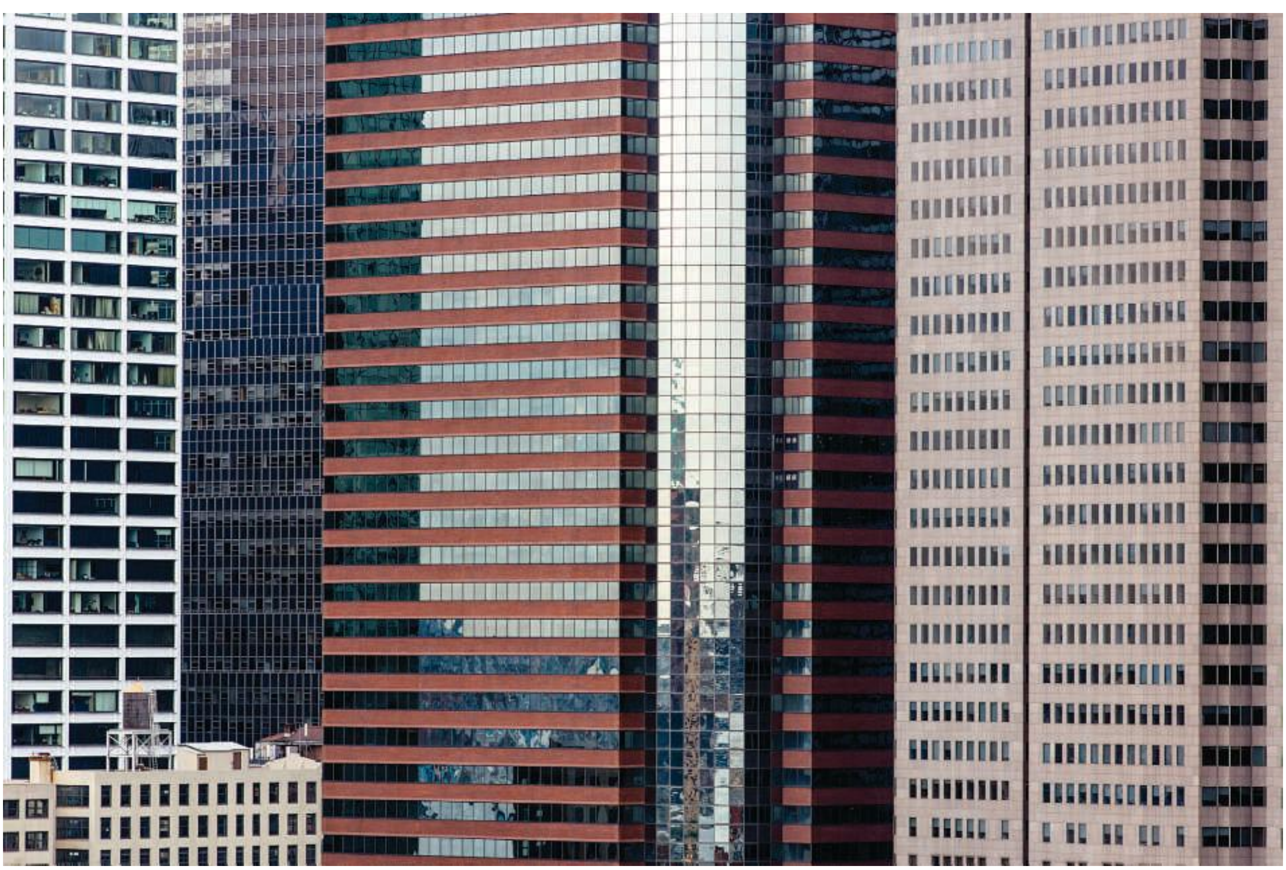
ENRICHMENT

4 Find (i) any local maxima or minima, and (ii) the global maximum and minimum of the function $y = x^4 - 8x^2 + 11$ for each domain.

a $1 \le x \le 3$

b $-4 \le x \le 1$

c $-1 \le x \le 0$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4H Applications of maximisation and minimisation

Here are some of many practical applications of maximisation and minimisation.

- Maximise the volume of a box built from a rectangular sheet of cardboard.
- Minimise the fuel used in a flight.
- Maximise the profits from manufacturing and selling an article.
- Minimise the amount of metal used in a can of soft drink.

Such problems can be solved using calculus, provided that a clear functional relationship can first be established.

13 MAXIMISATION AND MINIMISATION PROBLEMS

Usually a diagram should be drawn. Then:

1. Introduce the two variables from which the function is to be formed.
   'Let y (or whatever) be the quantity that is to be maximised,
   and let x (or whatever) be the quantity that can be varied.'
2. Form an equation in the two variables, noting any restrictions.
3. Find the global maximum or minimum.
4. Write a careful conclusion.

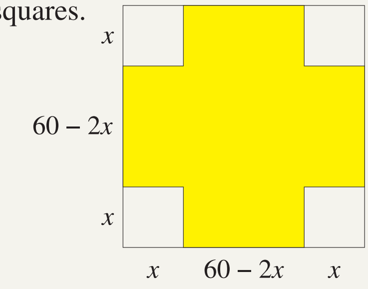
Note: A claim that a stationary point is a maximum or minimum must always be justified by a proper analysis of the nature of the stationary point.

Example 21 — 4H

An open rectangular box is to be made by cutting square corners out of a square piece of cardboard measuring 60 cm × 60 cm, and folding up the sides. What is the maximum volume of the box, and what are its dimensions then?

SOLUTION

Let V be the volume of the box, and let x be the side lengths of the cut-out squares. Then the box is x cm high, with base a square of side length $60 - 2x$,



$$\text{so}\quad V = x(60 - 2x)^2$$
$$= 3600x - 240x^2 + 4x^3, \quad \text{where } 0 \le x \le 30.$$
$$\text{Differentiating,}\quad V' = 3600 - 480x + 12x^2$$
$$= 12(x - 30)(x - 10),$$

so V' has zeroes at $x = 10$ and $x = 30$, and no discontinuities.

Also, $V'' = -480 + 24x$,

so $V''(10) = -240 < 0$ and $V''(30) = 240 > 0$.

Hence $(10, 16000)$ is the global maximum in the domain $0 \le x \le 30$,

and the maximum volume is 16000 cm^3 when the box is 10 cm × 40 cm × 40 cm.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 22 4H

A certain cylindrical soft drink can is required to have a volume of 250 cm^3.

a Show that the height of the can is $\dfrac{250}{\pi r^2}$, where r is the base radius.

b Show that the total surface area is $S = 2\pi r^2 + \dfrac{500}{r}$.

c Show that $r = \dfrac{5}{\pi^{\frac{1}{3}}}$ gives a global minimum of S in the domain $r > 0$.

d Show that to minimise the surface area of the can, the diameter of its base should equal its height.

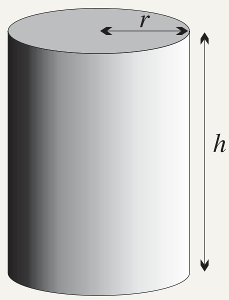
SOLUTION

a Let the height of the can be h cm.

Then volume $= \pi r^2 h$

$$250 = \pi r^2 h$$

$$h = \frac{250}{\pi r^2}.$$



b Each end has area πr^2 and the curved side has area $2\pi rh$, so

$$\begin{aligned} S &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r \times \frac{250}{\pi r^2} \\ &= 2\pi r^2 + \frac{500}{r}, \quad \text{where } r > 0. \end{aligned}$$

c Differentiating,

$$\begin{aligned} \frac{dS}{dr} &= 4\pi r - \frac{500}{r^2} \\ &= \frac{4\pi r^3 - 500}{r^2}. \end{aligned}$$

To find stationary points, put

$$\begin{aligned} \frac{dS}{dr} &= 0 \\ 4\pi r^3 &= 500 \\ r^3 &= \frac{125}{\pi} \\ r &= \frac{5}{\pi^{\frac{1}{3}}}. \end{aligned}$$

Differentiating again,

$$\frac{d^2S}{dr^2} = 4\pi + \frac{1000}{r^3},$$

which is positive for all $r > 0$.

Hence the stationary point is a global minimum in the domain $r > 0$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d When $r = \dfrac{5}{\pi^{\frac{1}{3}}}$, $\quad h = \dfrac{250}{\pi r^2}$

$$= \frac{250}{\pi} \times \frac{\pi^{\frac{2}{3}}}{25}$$

$$= \frac{10}{\pi^{\frac{1}{3}}}$$

$$= 2r.$$

Hence the minimum surface area occurs when the diameter equals the height.

Cost and time problems

There is often an optimum speed at which the costs of running a boat or truck are minimised.

- At slow speeds, wages and fixed costs rise.
- At high speeds, the costs of fuel and wear rise.

If some formula for these costs can be found, calculus can find the best speed.

Example 23 — 4H

The cost C (in dollars per hour) of running a boat depends on the speed v km/h of the boat according to the formula $C = 500 + 40v + 5v^2$.

a Show that the total cost for a trip of 100 km is $T = \dfrac{50\,000}{v} + 4000 + 500v$.

b What speed will minimise the total cost of the trip?

SOLUTION

a Because time $= \dfrac{\text{distance}}{\text{speed}}$, the time for the trip is $\dfrac{100}{v}$ hours.

Hence the total cost is $\quad T = (\text{cost per hour}) \times (\text{time for the trip})$

$$= (500 + 40v + 5v^2) \times \frac{100}{v}$$

$$= \frac{50\,000}{v} + 4000 + 500v, \quad \text{where } v > 0.$$

b Differentiating, $\quad \dfrac{dT}{dv} = -\dfrac{50\,000}{v^2} + 500$

$$= \frac{500(-100 + v^2)}{v^2}$$

$$= \frac{500(v - 10)(v + 10)}{v^2},$$

so $\dfrac{dT}{dv}$ has a single zero at $v = 10$ in the domain $v > 0$, and no discontinuities.

Differentiating again, $\quad \dfrac{d^2T}{dv^2} = \dfrac{100\,000}{v^3}$, which is positive for all $v > 0$,

so $v = 10$ gives a global minimum in the domain $v > 0$.

Thus a speed of 10 km/h will minimise the cost of the trip.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 4H

FOUNDATION

Note: You must always prove that any stationary point is a maximum or minimum, either by creating a table of values of the derivative, or by substituting into the second derivative. It is never acceptable to assume this from the wording of a question.

1 a Given that $P = xy$ and $2x + y = 12$, write P as a function of x.

b Find $\dfrac{dP}{dx}$, and hence determine the value of x that maximises P.

c Hence find the maximum value of P.

2 a Given that $Q = x^2 + y^2$ and $x + y = 8$, write Q as a function of x.

b Find $\dfrac{dQ}{dx}$ and hence determine the value of x that minimises Q.

c Hence find the minimum value of Q.

3 The quantity V of vitamins present in a patient's bloodstream t hours after taking the vitamin tablets is given by $V = 4t^2 - t^3$, for $0 \le t \le 3$. Find $\dfrac{dV}{dt}$ and hence determine when the quantity of vitamins in the patient's bloodstream is at its maximum.

4 A landscaper is constructing a rectangular garden bed. Three of the sides are to be fenced using 40 metres of fencing, while an existing wall will form the fourth side of the rectangle.

a Let x be the length of each of the two sides perpendicular to the wall. Show that the side parallel to the wall has length $40 - 2x$.

b Show that the area of the garden bed is given by $A = 40x - 2x^2$.

c Find $\dfrac{dA}{dx}$ and hence find the value of x that maximises A.

d Find the maximum possible area of the garden bed.

5 A rectangle has a constant area of 36 cm^2.

a If x is the length of the rectangle, show that the width is $\dfrac{36}{x}$.

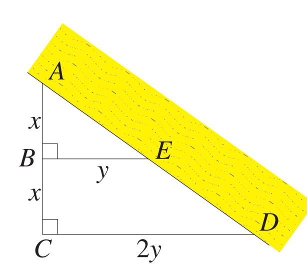
b Show that the perimeter of the rectangle is given by $P = 2x + \dfrac{72}{x}$.

c Show that $\dfrac{dP}{dx} = 2 - \dfrac{72}{x^2}$ and hence that the minimum value of P occurs at $x = 6$.

d Find the minimum possible perimeter of the rectangle.

DEVELOPMENT

6 A farmer has a field of total area 1200 m^2. To keep his animals separate, he sets up his field with fences at AC, CD and BE, as shown in the diagram. The side AD is beside a river, so no fence is needed there. The point B is the midpoint of AC, and CD is twice the length of BE. Let $AB = x$ and $BE = y$.



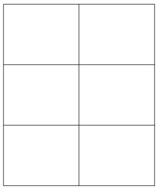
a Show that the total length of fencing is $L = 2x + \dfrac{1800}{x}$.

b Hence find the values of x and y that allow the farmer to use the least possible length of fencing.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 A window frame consisting of six equal rectangles is illustrated to the right. Only 12 metres of frame is available for its construction.



a If the entire frame has height h metres and width w metres, show that $w = \frac{1}{4}(12 - 3h)$.

b Show that the area of the window is $A = 3h - \frac{3}{4}h^2$.

c Find $\dfrac{dA}{dh}$ and hence find the dimensions of the frame for which the area of the window is maximised.

8 A 10 cm length of wire is cut into two pieces from which two squares are formed.

a If one piece has length x, find the side length of each square.

b Show that the combined area of the two squares is $A = \frac{1}{8}(x^2 - 10x + 50)$.

c Find $\dfrac{dA}{dx}$ and hence find the value of x that minimises A.

d Find the least possible combined area.

9 The total cost of producing x telescopes per day is given by $C = \left(\frac{1}{5}x^2 + 15x + 10\right)$ dollars, and each telescope is sold for a price of $\left(47 - \frac{1}{3}x\right)$ dollars.

a Find an expression for the revenue R raised from the sale of x telescopes per day.

b Find an expression for the daily profit $P = R - C$ made if x telescopes are sold.

c How many telescopes should be made daily in order to maximise the profit?

10 A box with volume 32 cm^3 has a square base and no lid. Let the square base have length x and the box have height h.

a Show that the surface area of the box is $S = x^2 + 4xh$.

b Show that $h = \dfrac{32}{x^2}$ and hence that $S = x^2 + \dfrac{128}{x}$.

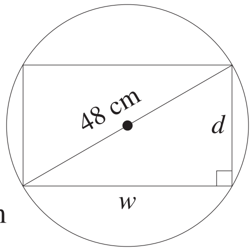
c Find the dimensions of the box that minimise its surface area.

11 **a** An open rectangular box is to be formed by cutting squares of side length x cm from the corners of a rectangular sheet of metal that has length 40 cm and width 15 cm. Express the width, length and height of the box in terms of x.

b Show that the volume of the box is given by $V = 600x - 110x^2 + 4x^3$.

c Find $\dfrac{dV}{dx}$ and hence find the value of x that maximises the volume of the box.

12 Engineers have determined that the strength s of a rectangular beam varies as the product of the width w and the square of the depth d of the beam, that is, $s = kwd^2$ for some constant $k > 0$.



a A particular cylindrical log has a diameter of 48 cm. Use Pythagoras' theorem to show that $s = kw(2304 - w^2)$.

b Hence find the dimensions of the strongest rectangular beam that can be cut from the log.

13 A closed rectangular box has length x cm, width y cm and height h cm. It is to be made from 300 cm^2 of thin sheet metal, and the perimeter of the base is to be 40 cm.

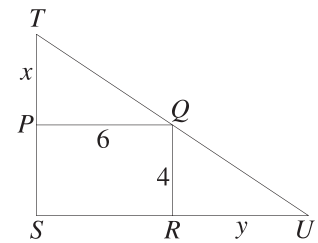
a Show that the volume of the box is given by $V = 150h - 20h^2$.

b Hence find the dimensions of the box that meets all the requirements and has the maximum possible volume.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 In the diagram to the right, $PQRS$ is a rectangle with sides $PQ = 6$ cm and $QR = 4$ cm. The points T and U lie on the lines SP and SR respectively so that T, Q and U are collinear. Let $PT = x$ and $RU = y$.



a Show that $xy = 24$.

b Show that the area of $\triangle TSU$ is given by $A = 24 + 3x + \dfrac{48}{x}$.

c Hence find the minimum possible area of $\triangle TSU$.

15 An open cylindrical water tank has base radius x metres and height h metres. Each square metre of the base costs a dollars to manufacture and each square metre of the curved surface costs b dollars. The combined cost of the base and curved surface is c dollars.

a Find c in terms of a, b, x and h.

b Show that the volume of the tank is given by $V = \dfrac{x}{2b}(c - \pi a x^2)$.

c As x varies, prove that V is at its maximum when the cost of the base is $\dfrac{c}{3}$ dollars.

16 A page of a book is to have 80 cm^2 of printed material. There is to be a 2 cm margin at the top and bottom and a 1 cm margin on each side of the page. Let the page have width x and height y.

a Show that $(y - 4)(x - 2) = 80$ and hence that $y = 4 + \dfrac{80}{x - 2}$.

b Show that the area of the page is $A = \dfrac{4x^2 + 72x}{x - 2}$.

c Use the quotient rule to show that $\dfrac{dA}{dx} = \dfrac{4(x^2 - 4x - 36)}{(x - 2)^2}$.

d What should be the dimensions of the page in order to use the least amount of paper?

17 A transport company runs a truck from Hobart to Launceston, a distance of 250 km, at a constant speed of v km/h. For a given speed v, the cost per hour is $6400 + v^2$ cents.

a Show that the cost of the trip, in cents, is $C = 250\left(\dfrac{6400}{v} + v\right)$.

b Find the speed at which the cost of the journey is minimised.

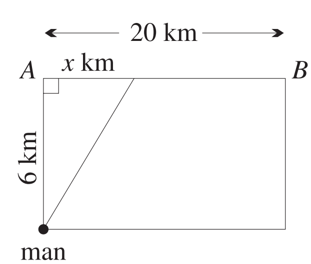
c Find the minimum cost of the journey.

18 The intensity I produced by a light of power W at a distance x metres from the light is given by $I = \dfrac{W}{x^2}$. Two lights L_1 and L_2, of power W and $2W$ respectively, are positioned 30 metres apart.

a Write down an expression for the combined intensity I_c of L_1 and L_2 at a point P which is x metres from L_1 on the interval L_1L_2.

b Hence find the distance PL_1, correct to the nearest centimetre, so that the combined intensity of L_1 and L_2 at P is minimised.

19 A man in a rowing boat is presently 6 km from the nearest point A on the shore. He wants to reach, as soon as possible, a point B that is a further 20 km along the shore from A.



a He can row at 8 km/h and he can run at 10 km/h. He rows to a point on the shore x km from A, and then he runs to B. Show that the time taken for the journey is $T = \frac{1}{8}\sqrt{36 + x^2} + \frac{1}{10}(20 - x)$.
(Hint: Recall that time = distance/speed.)

b Show that the time for the journey is minimised if he lands 8 km from A.

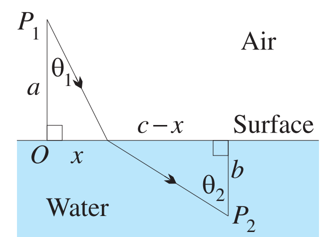
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

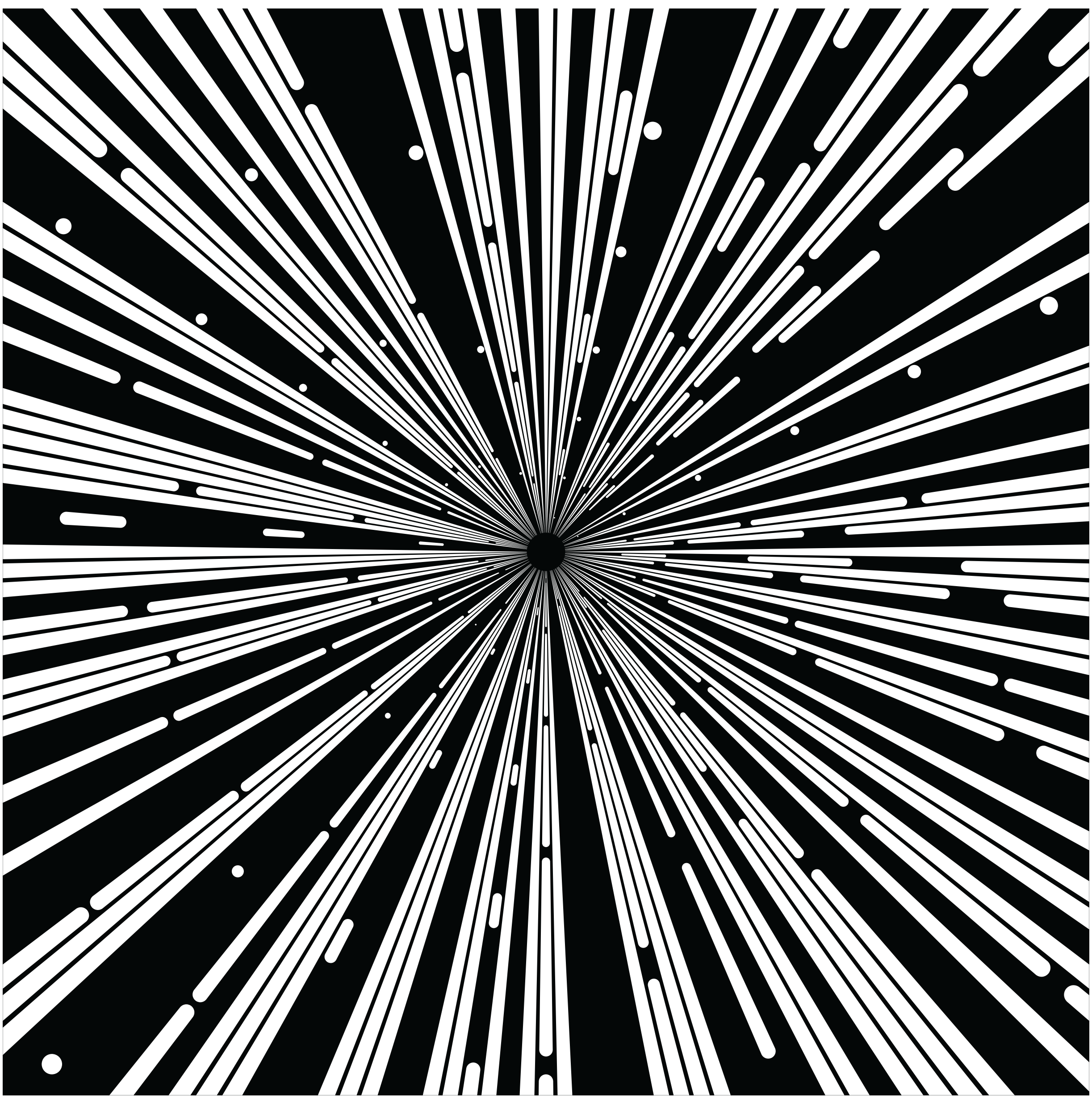
ENRICHMENT

20 If an object is placed u cm in front of a lens of focal length f cm, then the image appears v cm behind the lens, where $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$. Show that the minimum distance between the image and the object is $4f$ cm.

21 Snell's law states that light travels through a homogeneous medium in a straight line at a constant velocity dependent upon the medium. Let the velocity of light in air be v_1 and the velocity of light in water be v_2.



Show that light will travel from a point P_1 in air to a point P_2 in water in the shortest possible time if $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$, where the light ray makes angles of θ_1 and θ_2 in air and water respectively with a normal to the surface.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4I Maximisation and minimisation in geometry

Geometrical problems provided the classic situations where maximisation and minimisation problems were first clearly stated and solved. In many of these problems, the answer and the solution both have considerable elegance and clarity, and they make the effectiveness of calculus very obvious.

Maximisation and Minimisation in Geometry

Geometrical problems are the classic situation where maximisation and minimisation problems were first clearly stated and solved. In many of these problems, the answer and the solution both have considerable elegance, and they make the effectiveness of the methods of calculus very obvious.

Example 24 [A harder problem] 4I

A cylinder of radius r cm, height h cm and volume V is inscribed in a cone with base radius 6 cm and height 20 cm.

a Use similarity to show that $h = \frac{10}{3}(6 - r)$, and hence that $V = \frac{10}{3}\pi r^2(6 - r)$.

b Find the dimensions of the cylinder that has maximum volume, and show that this maximum volume is $\frac{4}{9}$ of the volume of the cone.

Note: Many geometrical problems use similarity in the way used in part **a** to establish the dimensions of a figure.

SOLUTION

a From the diagram, $\triangle AMP \,|||\, \triangle AOB$,

so, using matching sides, $\dfrac{AM}{MP} = \dfrac{AO}{OB}$

$$\frac{20 - h}{r} = \frac{20}{6}$$

$\boxed{\times 3r}$ $\quad 60 - 3h = 10r$

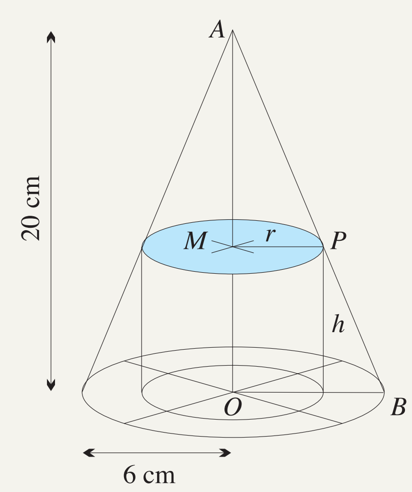
$$-3h = 10r - 60$$

$$h = \tfrac{10}{3}(6 - r).$$

Hence, using $V = \pi r^2 h$, $\quad V = \pi \times r^2 \times \frac{10}{3}(6 - r)$

$$= \tfrac{10}{3}\pi r^2(6 - r),$$

$$= \tfrac{10}{3}\pi(6r^2 - r^3).$$



b Differentiating, $\quad V' = \frac{10}{3}\pi(12r - 3r^2)$

$= 10\pi r(4 - r)$, which is zero when $r = 0$ or $r = 4$.

Differentiating again, $V'' = 10\pi(4 - 2r)$, which is negative when $r = 4$,

so, because $r \geq 0$, the cylinder has maximum volume when $r = 4$.

Substituting, this maximum volume is $\frac{320\pi}{3}$ cm^3.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using $V = \frac{1}{3}\pi r^2 h$, volume of cone $= \frac{1}{3} \times \pi \times 6^2 \times 20$

$= \frac{720\pi}{3}\ \text{cm}^2$,

so $\dfrac{\text{maximum volume}}{\text{volume of cone}} = \frac{320}{720}$

$= \frac{4}{9}$, as required.

[This ratio is actually independent of the dimensions of the original cone.]

Example 25 [A difficult example] 4I

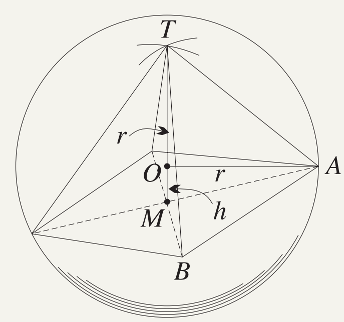
A square pyramid is inscribed in a sphere (the word 'inscribed' means that all five vertices of the pyramid lie on the surface of the sphere). What is the maximum ratio of the volumes of the pyramid and the sphere, and what are the corresponding proportions of the pyramid?

SOLUTION

Let the volume of the pyramid be V.
Let the height MT of the pyramid be $r + h$, where $-r \le h \le r$,
r being the constant radius of the sphere.

Then from the diagram, $MA^2 = r^2 - h^2$ (Pythagoras' theorem)

and because $MB = MA$, $AB^2 = 2(r^2 - h^2)$.



So $V = \frac{1}{3} \times AB^2 \times MT$

$= \frac{2}{3}(r^2 - h^2)(r + h)$

and the required function is $V = \frac{2}{3}(r^3 + r^2h - rh^2 - h^3)$.

Differentiating, $\dfrac{dV}{dh} = \frac{2}{3}(r^2 - 2rh - 3h^2)$

$= \frac{2}{3}(r - 3h)(r + h)$,

so dV/dh has zeroes at $h = \frac{1}{3}r$ or $-r$, and no discontinuities.

Also, $\dfrac{d^2V}{dh^2} = \frac{2}{3}(-2r - 6h)$,

which is negative for $h = \frac{1}{3}r$, giving a maximum (it is positive for $h = -r$).
So the volume is maximum when $h = \frac{1}{3}r$, that is, when $MT = \frac{4}{3}r$,

and then $AB^2 = 2(r^2 - \frac{1}{9}r^2)$

$= \frac{16}{9}r^2$.

This means that $AB = MT = \frac{4}{3}r$,

so that the height and base side length of the pyramid are equal.

Then ratio of volumes $= \frac{1}{3} \times \frac{16}{9}r^2 \times \frac{4}{3}r : \frac{4}{3}\pi r^3$

$= 16 : 27\pi$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 4I

FOUNDATION

Note: You must always prove that any stationary point is a maximum or minimum, either by creating a table of values of the derivative, or by substituting into the second derivative. It is never acceptable to assume this from the wording of a question.

1 The sum of the height h of a cylinder and the circumference of its base is 10 metres.

a Show that $h = 10 - 2\pi r$, where r is the radius of the cylinder.

b Show that the volume of the cylinder is $V = \pi r^2(10 - 2\pi r)$.

c Find $\dfrac{dV}{dr}$ and hence find the value of r at which the volume is a maximum.

d Hence find the maximum possible volume of the cylinder.

2 A closed cylindrical can is to have a surface area of 60π cm^2.

a Let the cylinder have height h and radius r. Show that $h = \dfrac{30 - r^2}{r}$.

b Show that the volume of the can is $V = \pi r(30 - r^2)$.

c Find $\dfrac{dV}{dr}$ and hence find the maximum possible volume of the can.

3 The sum of the radii r_1 and r_2 of two circles is constant, so that $r_1 + r_2 = k$, where k is a constant.

a Find an expression for the sum S of the areas of the circles in terms of r_1.

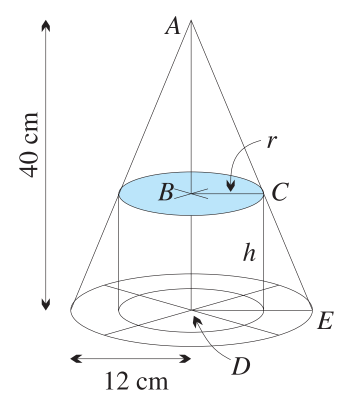
b Hence show that the sum of the areas is least when $r_1 = r_2$.

4 A piece of wire of length L is bent to form a sector of a circle of radius r.

a If the sector subtends an angle of θ radians at the centre, show that $\theta = \dfrac{L}{r} - 2$.

b Show that the area of the sector is maximised when $r = \frac{1}{4}L$.

5 A cylinder of height h cm and radius r cm is enclosed in a cone of height 40 cm and radius 12 cm.



a Explain why $\triangle ABC \,|||\, \triangle ADE$.

b By using ratios of corresponding sides, show that $h = 40 - \frac{10}{3}r$.

c Show that the volume of the cylinder is given by $V = 40\pi r^2 - \frac{10}{3}\pi r^3$.

d Find $\dfrac{dV}{dr}$ and hence find the value of r for which the volume of the cylinder is maximised.

DEVELOPMENT

6 Prove that of all the rectangles with a certain fixed perimeter, the one with the greatest area is a square.

7 A cylinder of height h metres is inscribed in a sphere of constant radius R metres.

a If the cylinder has radius r metres, show that $r^2 = R^2 - \frac{1}{4}h^2$.

b Show that the volume of the cylinder is given by $V = \frac{\pi}{4}h(4R^2 - h^2)$.

c Show that the volume of the cylinder is maximised when $h = \frac{2}{3}\sqrt{3}R$.

d Hence show that the ratio of the volume of the sphere to the maximum volume of the cylinder is $\sqrt{3} : 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 A cylindrical can, open at one end, is to have a fixed outside surface area S.

a Show that if the can has height h and radius r, then $h = \dfrac{S - \pi r^2}{2\pi r}$.

b Find an expression for the volume of the cylinder in terms of r, and hence show that the maximum possible volume is attained when the height of the can equals its radius.

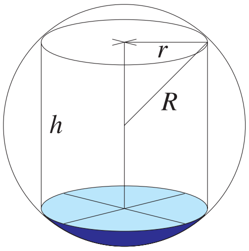
9 A cone of height h is inscribed in a sphere of constant radius R. Find the ratio $h : R$ when the volume of the cone is maximised.

10 A rectangle is inscribed in a quadrant of a circle of radius r so that two of its sides are along the bounding radii of the quadrant.

a If the rectangle has length x and width y, show that its area is given by $A = y\sqrt{r^2 - y^2}$.

b Show that the maximum possible area of the rectangle is $\frac{1}{2}r^2$.

11 A cylinder, open at both ends, is inscribed in a sphere of constant radius R.



a Let the cylinder have height h and radius r as illustrated in the diagram. Show that $h = 2\sqrt{R^2 - r^2}$.

b Show that the surface area of the cylinder is given by $S = 4\pi r\sqrt{R^2 - r^2}$.

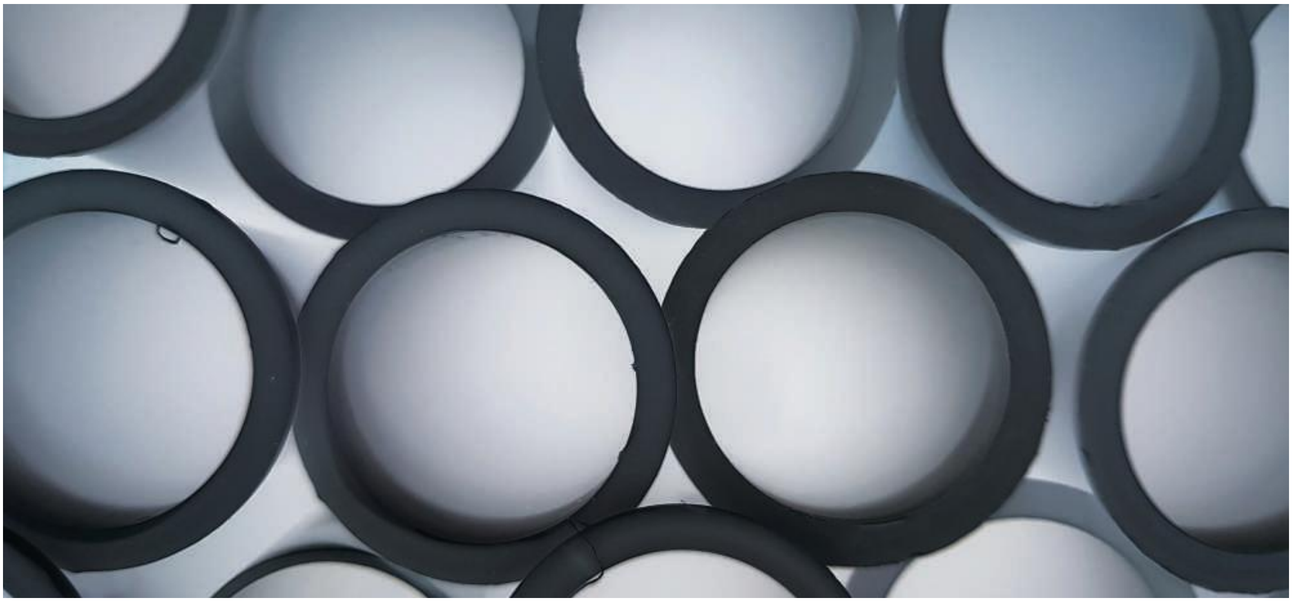
c Hence find the maximum surface area of the cylinder in terms of R.

12 A canned fruit producer wishes to minimise the area of sheet metal used in manufacturing cylindrical cans of a given volume. Find the ratio of radius to height for the desired can.

ENRICHMENT

13 A cone has base radius r and height h. As r and h vary, the curved surface area $S = \pi rs$, where s is the slant height, is kept constant. Prove that the maximum volume occurs when $h : r = \sqrt{2} : 1$.

14 An isosceles triangle is to circumscribe a circle of constant radius r. (So the 3 sides of the triangle are tangents to the circle.) Prove that the minimum area of such a triangle is $3\sqrt{3}\,r^2$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4J Primitive functions

This section reverses the process of differentiation and asks, 'What can we say about a function if we know its derivative?' The results of this section will be needed when integration is introduced in Chapter 5.

This topic was briefly mentioned in Section 9D of the Year 11 book, but without any terminology or formulae, and we begin the discussion again here.

Functions with the same derivative

Many different functions may all have the same derivative. For example, all these functions have the same derivative $2x$:

$$x^2, \qquad x^2 + 3, \qquad x^2 - 2, \qquad x^2 + 4\tfrac{1}{2}, \qquad x^2 - \pi.$$

These functions are all the same apart from a constant term. This is true generally — *any two functions with the same derivative differ only by a constant.*

14 FUNCTIONS WITH THE SAME DERIVATIVE

A If a function $f(x)$ has derivative zero in an interval $a < x < b$, then $f(x)$ is a constant function in $a < x < b$.

B If $f'(x) = g'(x)$ for all x in an interval $a < x < b$, then $f(x)$ and $g(x)$ differ by a constant in $a < x < b$.

Proof

A Because the derivative is zero, the gradient of the curve is zero throughout the interval. The curve is therefore a horizontal straight line, and $f(x)$ is a constant function.

B Take the difference between $f(x)$ and $g(x)$ and apply part **A**.

Let $\qquad h(x) = f(x) - g(x)$.

Then $\qquad h'(x) = f'(x) - g'(x)$

$\qquad\qquad = 0,\quad$ for all x in the interval $a < x < b$.

Hence by part **a**, $\quad h(x) = C,\quad$ where C is a constant,

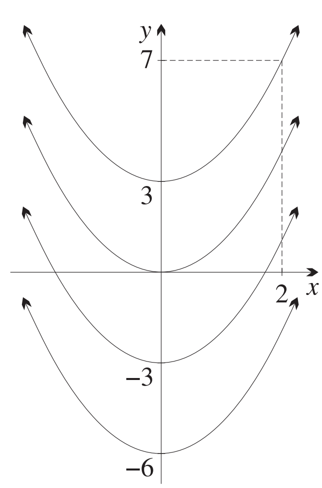
so $\qquad f(x) - g(x) = C,\quad$ as required.

The family of curves with the same derivative

Continuing with the very first example, the various functions whose derivatives are $2x$ are all of the form

$$f(x) = x^2 + C, \qquad \text{where } C \text{ is a constant.}$$

By taking different values of the constant C, these functions form an infinite family of curves, each consisting of the parabola $y = x^2$ translated upwards or downwards.



Initial or boundary conditions

If we know also that the graph of the function passes through a particular point, say $(2, 7)$, then we can evaluate the constant C by substituting the point into $f(x) = x^2 + C$,

$$7 = 4 + C.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Thus $C = 3$ and hence $f(x) = x^2 + 3$ — in place of the infinite family of functions, there is now a single function.

Such an extra condition is called an *initial condition* or *boundary condition*.

Primitives

We need a suitable name for the result of this reverse process. The words *primitive* and *anti-derivative* are both used.

15 A PRIMITIVE OF A FUNCTION

- A function $F(x)$ is called *a primitive* or *an anti-derivative* of $f(x)$ if the derivative of $F(x)$ is $f(x)$,
  $F'(x) = f(x)$.
- If $F(x)$ is any primitive of $f(x)$ then *the general primitive* of $f(x)$ is
  $F(x) + C$, where C is a constant.

The general primitive of $f(x)$ is also called just *the primitive* of $f(x)$. For example, each these functions is *a primitive* of $x^2 + 1$,

$$\tfrac{1}{3}x^3 + x, \qquad \tfrac{1}{3}x^3 + x + 7, \qquad \tfrac{1}{3}x^3 + x - 13, \qquad \tfrac{1}{3}x^3 + x + 4\pi,$$

whereas *the primitive* of $x^2 + 1$ is $\tfrac{1}{3}x^3 + x + C$, where C is a constant.

A rule for finding primitives

We have seen that a primitive of x is $\tfrac{1}{2}x^2$, and a primitive of x^2 is $\tfrac{1}{3}x^3$. Reversing the formula $\dfrac{d}{dx}(x^{n+1}) = (n+1)x^n$ gives the general rule:

16 FINDING PRIMITIVES

If $\dfrac{dy}{dx} = x^n$, where $n \neq -1$,

then $y = \dfrac{x^{n+1}}{n+1} + C$, for some constant C.

'Increase the index by 1 and divide by the new index.'

Example 26 — 4J

Find the primitives (or anti-derivatives) of:

a $x^3 + x^2 + x + 1$

b $5x^3 + 7$

SOLUTION

a Let $\dfrac{dy}{dx} = x^3 + x^2 + x + 1$.

Then $y = \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x + C$,
for some constant C.

b Let $f'(x) = 5x^3 + 7$.

Then $f(x) = \tfrac{5}{4}x^4 + 7x + C$,
for some constant C.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 27 4J

Rewrite each function with negative or fractional indices, and find the primitive.

a $\dfrac{1}{x^2}$ **b** $\sqrt{x}$

SOLUTION

a Let $f'(x) = \dfrac{1}{x^2}$

$= x^{-2}$.

Then $f(x) = -x^{-1} + C$, where C is a constant,

$= -\dfrac{1}{x} + C$.

b Let $\dfrac{dy}{dx} = \sqrt{x}$

$= x^{\frac{1}{2}}$.

Then $y = \frac{2}{3}x^{\frac{3}{2}} + C$, where C is a constant.

Linear extension

Reversing the formula $\dfrac{d}{dx}(ax + b)^{n+1} = a(n + 1)(ax + b)^n$ gives:

17 EXTENSION TO POWERS OF LINEAR FUNCTIONS

If $\dfrac{dy}{dx} = (ax + b)^n$, where $n \neq -1$,

then $y = \dfrac{(ax + b)^{n+1}}{a(n + 1)} + C$, for some constant C.

'Increase the index by 1, then divide by the new index and by the coefficient of x.'

Example 28 4J

Find the primitives of:

a $(3x + 1)^4$ **b** $(1 - 3x)^6$ **c** $\dfrac{1}{(x + 1)^2}$ **d** $\sqrt{x + 1}$

SOLUTION

a Let $\dfrac{dy}{dx} = (3x + 1)^4$.

Then $y = \dfrac{(3x + 1)^5}{5 \times 3} + C$

where C is a constant,

$= \dfrac{(3x + 1)^5}{15} + C$.

b Let $\dfrac{dy}{dx} = (1 - 3x)^6$.

Then $y = \dfrac{(1 - 3x)^7}{7 \times (-3)} + C$,

where C is a constant,

$y = -\dfrac{(1 - 3x)^7}{21} + C$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Let $\dfrac{dy}{dx} = \dfrac{1}{(x+1)^2}$
$= (x+1)^{-2}$.

Then $y = \dfrac{(x+1)^{-1}}{-1} + C$,
where C is a constant,
$= -\dfrac{1}{x+1} + C$.

d Let $\dfrac{dy}{dx} = \sqrt{x+1}$
$= (x+1)^{\frac{1}{2}}$.

Then $y = \dfrac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$,
where C is a constant,
$y = \frac{2}{3}(x+1)^{\frac{3}{2}} + C$.

Finding the primitive, given an initial condition

If the derivative of a function is known, plus an initial condition (or boundary condition), then substitute the condition into the general primitive to find the constant, and hence the original function.

18 FINDING A FUNCTION, GIVEN ITS DERIVATIVE AND AN INITIAL CONDITION:

- First find the primitive, taking care to include the constant of integration.
- Then substitute the known value of the function to work out the constant.

Example 29 — 4J

Given that $\dfrac{dy}{dx} = 6x^2 + 1$, and $y = 12$ when $x = 2$, find y as a function of x.

SOLUTION

We know that $\dfrac{dy}{dx} = 6x^2 + 1$,
So $y = 2x^3 + x + C$, for some constant C.
substituting $x = 2$ and $y = 12$, $12 = 16 + 2 + C$,
so $C = -6$, and hence $y = 2x^3 + x - 6$.

Example 30 — 4J

Given that $f''(x) = 12(x-1)^2$, and $f(0) = f(1) = 0$, find $f(4)$.

SOLUTION

We know that $f''(x) = 12(x-1)^2$.
Taking the primitive, $f'(x) = 4(x-1)^3 + C$, for some constant C,
and taking the primitive again, $f(x) = (x-1)^4 + Cx + D$, for some constant D.
Because $f(0) = 0$, $0 = 1 + 0 + D$
$D = -1$.
Because $f(1) = 0$, $0 = 0 + C - 1$.
Hence $C = 1$, so $f(4) = 81 + 4 - 1$
$= 84$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 4J

FOUNDATION

1 Find the primitive of each function.

a x^6 b x^3 c x^{10} d $3x$
e 5 f $5x^9$ g $21x^6$ h 0

2 Find the primitive of each function.

a $x^2 + x^4$ b $4x^3 - 5x^4$ c $2x^2 + 5x^7$
d $x^2 - x + 1$ e $3 - 4x + 16x^7$ f $3x^2 - 4x^3 - 5x^4$

3 Find the primitive of each function, after first expanding the product.

a $x(x - 3)$ b $(x + 1)(x - 2)$ c $(3x - 1)(x + 4)$
d $x^2(5x^3 - 4x)$ e $2x^3(4x^4 + 1)$ f $(x - 3)(1 + x^2)$

4 Find y as a function of x if:

a $y' = 2x + 3$ and: i $y = 3$ when $x = 0$, ii $y = 8$ when $x = 1$.
b $y' = 9x^2 + 4$ and: i $y = 1$ when $x = 0$, ii $y = 5$ when $x = 1$.
c $y' = 3x^2 - 4x + 7$ and: i $y = 0$ when $x = 0$, ii $y = -1$ when $x = 1$.

DEVELOPMENT

5 Write each function using a negative power of x. Then find the primitive function, writing it as a fraction without a negative index.

a $\dfrac{1}{x^2}$ b $\dfrac{1}{x^3}$ c $-\dfrac{2}{x^3}$ d $-\dfrac{3}{x^4}$ e $\dfrac{1}{x^2} - \dfrac{1}{x^3}$

6 Write each function using a fractional index, and hence find the primitive.

a $\sqrt{x}$ b $\dfrac{1}{\sqrt{x}}$ c $\sqrt[3]{x}$ d $\dfrac{2}{\sqrt{x}}$ e $\sqrt[5]{x^3}$

7 Find y as a function of x if $\dfrac{dy}{dx} = \sqrt{x}$ and:

a $y = 1$ when $x = 0$, b $y = 2$ when $x = 9$.

8 Find each family of curves whose gradient function is given below. Then sketch the family, and find the member of the family passing through $A(1, 2)$.

a $\dfrac{dy}{dx} = -4x$ b $\dfrac{dy}{dx} = 3$ c $\dfrac{dy}{dx} = 3x^2$ d $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

9 Recall that if $\dfrac{dy}{dx} = (ax + b)^n$, then $y = \dfrac{(ax + b)^{n+1}}{a(n + 1)} + C$, for some constant C. Use this result to find the primitive of each function.

a $(x + 1)^3$ b $(x - 2)^5$ c $(x + 5)^2$ d $(2x + 3)^4$ e $(3x - 4)^6$
f $(5x - 1)^3$ g $(1 - x)^3$ h $(1 - 7x)^3$ i $\dfrac{1}{(x - 2)^4}$ j $\dfrac{1}{(1 - x)^{10}}$

10 Find the primitive of each function. Use the rule given in the previous question and the fact that $\sqrt{u} = u^{\frac{1}{2}}$.

a $\sqrt{x + 1}$ b $\sqrt{x - 5}$ c $\sqrt{1 - x}$ d $\sqrt{2x - 7}$ e $\sqrt{3x - 4}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 a Find y if $y' = (x - 1)^4$, given that $y = 0$ when $x = 1$.

b Find y if $y' = (2x + 1)^3$, given that $y = -1$ when $x = 0$.

c Find y if $y' = \sqrt{2x + 1}$, given that $y = \frac{1}{3}$ when $x = 0$.

12 a Find the equation of the curve through the origin whose gradient is $\dfrac{dy}{dx} = 3x^4 - x^3 + 1$.

b Find the curve passing through $(2, 6)$ with gradient function $\dfrac{dy}{dx} = 2 + 3x^2 - x^3$.

c Find the curve through the point $\left(\frac{1}{5}, 1\right)$ with gradient function $y' = (2 - 5x)^3$.

13 Find y if $\dfrac{dy}{dt} = 8t^3 - 6t^2 + 5$, and $y = 4$ when $t = 0$. Hence find y when $t = 2$.

14 The primitive of x^n is $\dfrac{x^{n+1}}{n + 1}$, provided that $n \neq -1$. Why can't this rule be used when $n = -1$?

15 Find y if $y'' = 6x + 4$, given that when $x = 1$, $y' = 2$ and $y = 4$.
(Hint: Find y' and use the condition $y' = 2$ when $x = 1$ to find the constant of integration. Then find y and use the condition $y = 4$ when $x = 1$ to find the second constant).

16 A function $f(x)$ has second derivative $f''(x) = 2x - 10$. Its graph passes through the point $(3, -34)$, and at this point the tangent has a gradient of 20.

a Show that $f'(x) = x^2 - 10x + 41$.

b Hence find $f(x)$, and show that its graph cuts the y-axis at $(0, -121)$.

17 If $y'' = 8 - 6x$, show that $y = 4x^2 - x^3 + Cx + D$, for some constants C and D. Hence find the equation of the curve given that it passes through the points $(1, 6)$ and $(-1, 8)$.

ENRICHMENT

18 The gradient function of a curve is given by $f'(x) = -\dfrac{1}{x^2}$. Find the equation of the curve, given that $f(1) = f(-1) = 2$. Sketch a graph of the function.

19 a Prove that for a polynomial of degree n, the $(n + 1)$th and higher derivatives vanish, but the nth does not.

b Prove that if the $(n + 1)$th derivative of a polynomial vanishes but the nth does not, then the polynomial has degree n.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 4 Review

Review activity

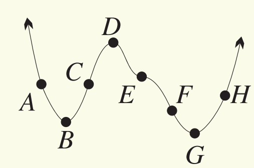
- Create your own summary of this chapter on paper or in a digital document.

Chapter 4 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise



1 In the diagram to the right, name the points where:

a $f'(x) > 0$ **b** $f'(x) < 0$ **c** $f'(x) = 0$
d $f''(x) > 0$ **e** $f''(x) < 0$ **f** $f''(x) = 0$

2 **a** Find the derivative $f'(x)$ of the function $f(x) = x^3 - x^2 - x - 7$.

b Hence find whether $f(x)$ is increasing, decreasing or stationary at:

i $x = 0$ **ii** $x = 1$ **iii** $x = -1$ **iv** $x = 3$

3 **a** Find the derivative $f'(x)$ of the function $f(x) = x^2 - 4x + 3$.

b Find the values of x for which $f(x)$ is:

i increasing, **ii** decreasing, **iii** stationary.

4 Differentiate each function, then evaluate $f'(1)$ to determine whether the function is increasing, decreasing or stationary at $x = 1$.

a $f(x) = x^3$ **b** $f(x) = (x + 2)(x - 3)$ **c** $f(x) = (x - 1)^5$ **d** $f(x) = \dfrac{x + 1}{x - 3}$

5 Find the first and second derivatives of:

a $y = x^7$ **b** $y = x^3 - 4x^2$ **c** $y = (x - 2)^5$ **d** $y = \dfrac{1}{x}$

6 Find $f''(x)$ for each function. By evaluating $f''(1)$, state whether the curve is concave up or concave down at $x = 1$.

a $f(x) = x^3 - 2x^2 + 4x - 5$ **b** $f(x) = 6 - 2x^3 - x^4$

7 **a** Find the second derivative $f''(x)$ of the function $f(x) = 2x^3 - 3x^2 + 6x - 1$.

b Find the values of x for which $f(x)$ is:

i concave up, **ii** concave down.

8 Find the values of x for which the curve $y = x^3 - 6x^2 + 9x - 11$ is:

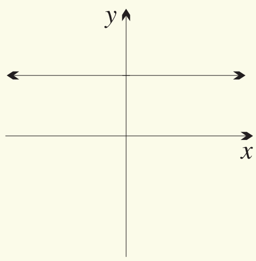
a increasing, **b** decreasing, **c** concave up, **d** concave down.

ISBN 978-1-108-76630-2

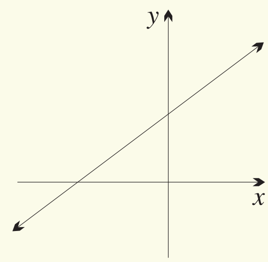
Cambridge University Press
Updated July 2022

9 Look carefully at each function graphed below to establish where it is increasing, decreasing and stationary. Hence sketch the graph of the derivative of each function.

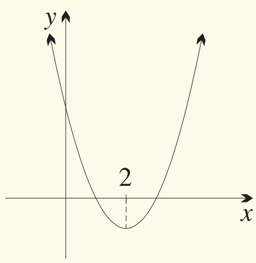
a



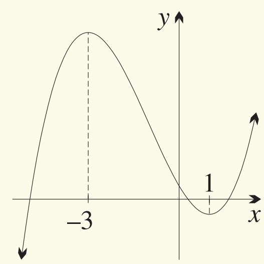
b



c

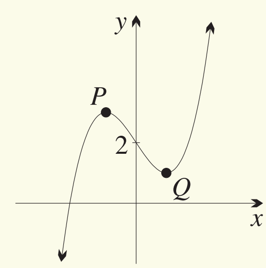


d



10 The curve $y = x^3 + x^2 - x + 2$ is graphed to the right.

The points P and Q are stationary points.



a Find the coordinates of P and Q.

b For what values of x is the curve concave up?

c For what values of k are there three distinct solutions of the equation $x^3 + x^2 - x + 2 = k$?

11 Sketch the graph of each function, indicating all stationary points and points of inflection.

a $y = x^2 - 6x - 7$ **b** $y = x^3 - 6x^2 + 8$ **c** $y = 2x^3 - 3x^2 - 12x + 1$

12 a Sketch the graph of the function $y = x^3 - 3x^2 - 9x + 11$, indicating all stationary points.

b Hence determine the global maximum and minimum values of the function in the domain $-2 \le x \le 6$.

13 a The tangent to $y = x^2 - ax + 9$ is horizontal at $x = -1$. Find the value of a.

b The curve $y = ax^2 + bx + 3$ has a turning point at $(-1, 0)$. Find the values of a and b.

14 a Show that the curve $y = x^4 - 4x^3 + 7$ has a point of inflection at $(2, -9)$.

b Find the gradient of the curve at this point of inflection.

c Hence show that the equation of the tangent at the point of inflection is $16x + y - 23 = 0$.

15 The number S of students logged onto a particular website over a five-hour period is given by the formula $S = 175 + 18t^2 - t^4$, for $0 \le t \le 5$.

a What is the initial number of students that are logged on?

b How many students are logged on at the end of the five hours?

c What was the maximum number of students logged onto the website during the five-hour period?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

16 A rectangular sheet of cardboard measures 16 cm by 6 cm. Equal squares of side length x cm are cut out of the corners and the sides are turned up to form an open rectangular box.

a Show that the volume V of the box is given by $V = 4x^3 - 44x^2 + 96x$.

b Find, in exact form, the maximum volume of the box.

17 A right-angled triangle has base 60 cm and height 80 cm. A rectangle is inscribed in the triangle, so that one of its sides lies along the base of the triangle.

a Let the rectangle have length y cm and height x cm. By using similar triangles, show that $y = \frac{3}{4}(80 - x)$.

b Hence find the dimensions of the rectangle of maximum area.

18 A coal chute is built in the shape of an upturned cone. The sum of the base radius r and the height h is 12 metres.

a Show that the volume V of the coal chute is given by $V = 4\pi r^2 - \frac{1}{3}\pi r^3$. (Recall that the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.)

b Find the radius of the cone that yields the maximum volume.

19 Find the primitive of each function.

a x^7 **b** $2x$ **c** 4 **d** $10x^4$ **e** $8x + 3x^2 - 4x^3$

20 Find the primitive of each function after first expanding the brackets.

a $3x(x - 2)$ **b** $(x + 1)(x - 5)$ **c** $(2x - 3)^2$

21 Find the primitive of each function.

a $(x + 1)^5$ **b** $(x - 4)^7$ **c** $(2x - 1)^3$

22 Find the primitive of each function after writing the function as a power of x.

a $\dfrac{1}{x^2}$ **b** $\sqrt{x}$

23 Find the equation of the curve passing through the point $(2, 5)$ with gradient function $f'(x) = 3x^2 - 4x + 1$.

24 If $f'(x) = 4x - 3$ and $f(2) = 7$, find $f(4)$.

The remaining questions are more difficult

25 a Show that the function $f(x) = \dfrac{1}{x^2 - x - 2}$ is neither even nor odd.

b Show that $f'(x) = \dfrac{1 - 2x}{(x^2 - x - 2)^2}$.

c Show that there is a stationary point at $\left(\frac{1}{2}, -\frac{4}{9}\right)$ and determine its nature.

d Solve $x^2 - x - 2 = 0$, and hence state the equations of any vertical asymptotes.

e What value does $f(x)$ approach as x becomes large?

f Sketch the graph of $y = f(x)$, showing all important features.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

26 a Find where the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$ cuts the x-axis and the y-axis.

b Explain why the lines $x = 2$ and $x = -2$ are vertical asymptotes.

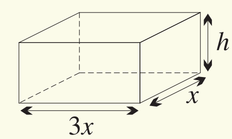
c Use the quotient rule to show that $y' = -\dfrac{6x}{(x^2 - 4)^2}$.

d Hence show that there is a maximum turning point at $\left(0, \frac{1}{4}\right)$.

e Show that the function is even.

f Sketch a graph of the function, showing all important features.

27 The steel frame of a rectangular prism, as illustrated in the diagram, is three times as long as it is wide.

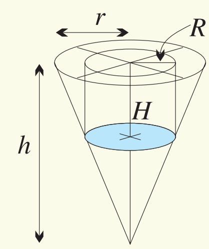


a Find an expression in terms of x and h for S, the length of steel required to construct the frame.

b The prism has a volume of 4374 m^3. Show that $S = 16x + \dfrac{5832}{x^2}$.

c Find the dimensions of the frame so that the minimum amount of steel is used.

28 A cylinder of height H and radius R is inscribed in a cone of constant height h and constant radius r.



a Use similar triangles to show that $H = \dfrac{h(r - R)}{r}$.

b Find an expression for the volume of the cylinder in terms of the variable R.

c Find the maximum possible volume of the cylinder in terms of h and r.

29 a The perimeter of an isosceles triangle is 12 cm. Find its maximum area.

b Prove the general result that for all isosceles triangles of constant perimeter, the one with maximum area is equilateral.

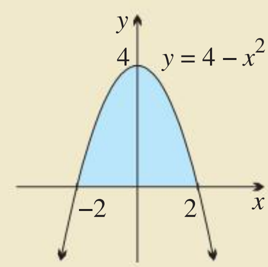
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 Integration

The calculation of areas has so far been restricted to regions bounded by straight lines or parts of circles. This chapter will extend the study of areas to regions bounded by more general curves.

For example, it will be possible to calculate the area of the shaded region in the diagram to the right, bounded by the parabola $y = 4 - x^2$ and the x-axis.



The method developed in this chapter is called *integration*. We will soon show that finding tangents and finding areas are inverse processes, so that integration is the inverse process of differentiation. This surprising result is called the *fundamental theorem of calculus* — the word ‘fundamental’ is well chosen because the theorem is the basis of the way in which calculus is used throughout mathematics and science.

Graphing software that can also estimate selected areas is useful in the chapter to illustrate how answers change as the curves and boundaries are varied.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5A Areas and the definite integral

All area formulae and calculations of area are based on two principles:

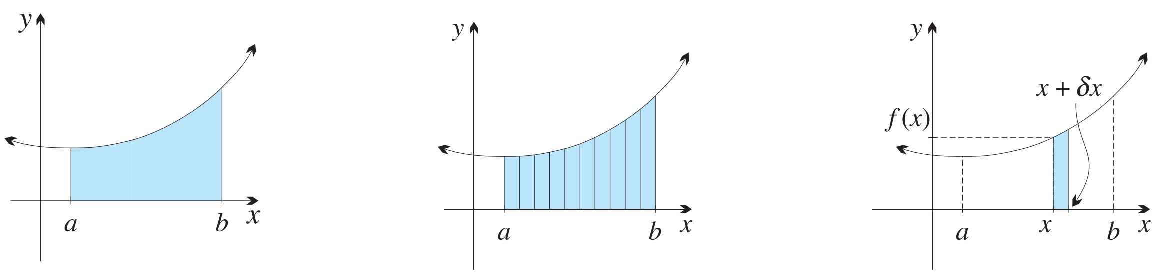
1. Area of a rectangle $=$ length $\times$ breadth.
2. When a region is dissected, the area is unchanged.

A region bounded by straight lines, such as a triangle or a trapezium, can be cut up and rearranged into rectangles with a few well-chosen cuts. Dissecting a curved region into rectangles, however, requires an infinite number of rectangles, and so must be a limiting process, just as differentiation is.

A new symbol — the definite integral

Some new notation is needed to reflect this process of infinite dissection as it applies to functions and their graphs.

The diagram on the left below shows the region contained between a given curve $y = f(x)$ and the x-axis, from $x = a$ to $x = b$, where $a \le b$. The curve must be continuous and, for the moment, never go below the x-axis.



In the middle diagram, the region has been dissected into a number of thin strips. Each strip is approximately a rectangle, but only roughly so, because the upper boundary is curved. The area of the region is the sum of the areas of all the strips.

The third diagram shows just one of the strips, above the value x on the x-axis. Its height at the left-hand end is $f(x)$, and provided the strip is very thin, the height is still about $f(x)$ at the right-hand end. Let the width of the strip be δx, where δx is, as usual in calculus, thought of as being very small. Then, roughly,

$$\begin{aligned}\text{area of strip} &\doteqdot \text{height} \times \text{width}\\ &\doteqdot f(x)\,\delta x.\end{aligned}$$

Adding up all the strips, using sigma notation for the sum,

$$\begin{aligned}\text{area of shaded region} &\doteqdot \sum_{x=a}^{b} (\text{area of each strip})\\ &\doteqdot \sum_{x=a}^{b} f(x)\,\delta x.\end{aligned}$$

Now imagine that there are infinitely many of these strips, each infinitesimally thin, so that the inaccuracy disappears. This involves taking the limit, and we might expect to see something like this

$$\text{area of shaded region} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x,$$

but instead, we use the brilliant and flexible notation introduced by Leibnitz.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The width δx is replaced by the symbol dx, which suggests an infinitesimal width, and an old form $\int$ of the letter S is used to suggest an infinite sum under a smooth curve. The result is the strange-looking symbol $\int_a^b f(x)\,dx$. We now *define* this symbol to be the shaded area,

$$\int_a^b f(x)\,dx = \text{area of shaded region.}$$

The definite integral

This new object $\int_a^b f(x)\,dx$ is called a *definite integral*. The rest of the chapter is concerned with evaluating definite integrals and applying them.

1 THE DEFINITE INTEGRAL

Let $f(x)$ be a function that is continuous in a closed interval $[a, b]$, where $a \le b$.
For the moment, suppose that $f(x)$ is never negative in the interval.

- The *definite integral* $\int_a^b f(x)\,dx$ is defined to be the area of the region between the curve and the x-axis, from $x = a$ to $x = b$.
- The function $f(x)$ is called the *integrand*, and the values $x = a$ and $x = b$ are called the *lower* and *upper limits* (or *bounds*) of the integral.

The name 'integration' suggests putting many parts together to make a whole. The notation arises from building up the region from an infinitely large number of infinitesimally thin strips. Integration is 'making a whole' from these thin slices.

Evaluating definite integrals using area formulae

When the function is linear or circular, the definite integral can be calculated from the graph using well-known area formulae, although a quicker method will be developed later for linear functions.

Here are the relevant area formulae:

2 AREA FORMULAE FOR TRIANGLE, TRAPEZIUM AND CIRCLE

Triangle:	Area $= \frac{1}{2}bh$	$= \frac{1}{2} \times$ base $\times$ height
Trapezium:	Area $= \frac{1}{2}(a + b)h$	$=$ average of parallel sides $\times$ width
Circle:	Area $= \pi r^2$	$= \pi \times$ square of the radius

For a trapezium, h is the perpendicular distance between the parallel sides. Depending on the orientation, the word 'height' or 'width' may be more appropriate. Similarly, any side of a triangle may be taken as its 'base'.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

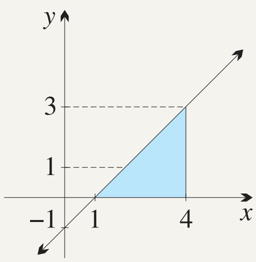
Example 1 5A

Evaluate using a graph and area formulae:

a $\displaystyle\int_1^4 (x-1)\,dx$ **b** $\displaystyle\int_2^4 (x-1)\,dx$ **c** $\displaystyle\int_{-2}^2 |x|\,dx$ **d** $\displaystyle\int_{-5}^5 \sqrt{25-x^2}\,dx$

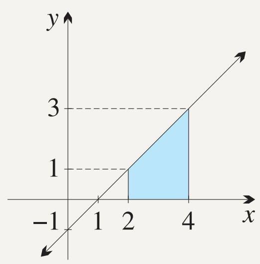
SOLUTION

a The area represented by the integral is the shaded triangle, with base $4-1=3$ and height 3.



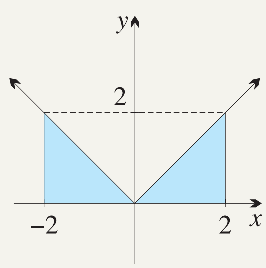
$$\text{Hence} \int_1^4 (x-1)\,dx = \tfrac{1}{2} \times \text{base} \times \text{height}$$
$$= \tfrac{1}{2} \times 3 \times 3$$
$$= 4\tfrac{1}{2}.$$

b The area represented by the integral is the shaded trapezium, with width $4-2=2$ and parallel sides of length 1 and 3.



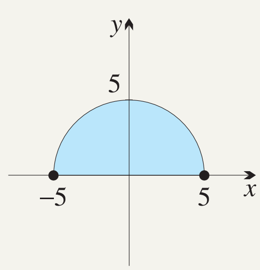
$$\text{Hence} \int_2^4 (x-1)\,dx = \text{average of parallel sides} \times \text{width}$$
$$= \frac{1+3}{2} \times 2$$
$$= 4.$$

c Each shaded triangle has base 2 and height 2.



$$\text{Hence} \int_{-2}^2 |x|\,dx = 2 \times \left(\tfrac{1}{2} \times \text{base} \times \text{height}\right)$$
$$= 2 \times \left(\tfrac{1}{2} \times 2 \times 2\right)$$
$$= 4.$$

d The shaded region is a semi-circle with radius 5.

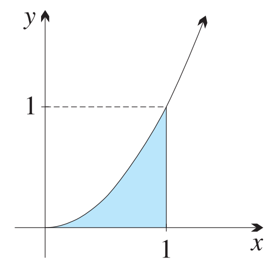


$$\text{Hence} \int_{-5}^5 \sqrt{25-x^2}\,dx = \frac{1}{2} \times \pi r^2$$
$$= \frac{1}{2} \times 5^2 \times \pi$$
$$= \frac{25\pi}{2}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using upper and lower rectangles to trap an integral



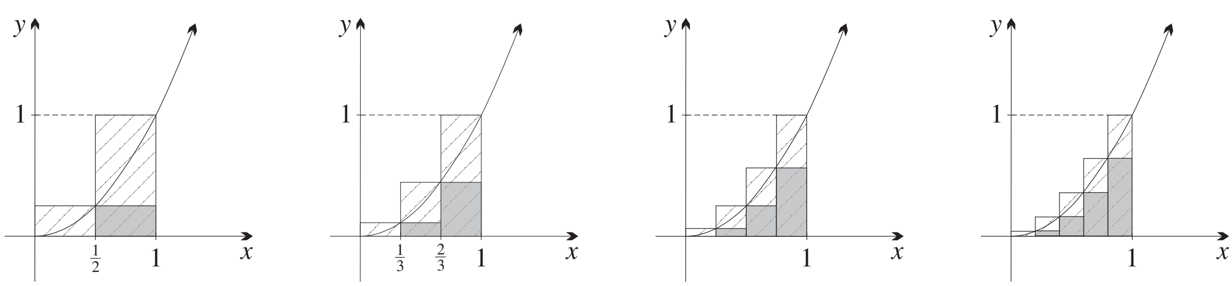
First-principles integration calculations are more elaborate than those in first-principles differentiation. But they are used when proving the fundamental theorem of calculus, which will soon make calculations of integrals straightforward. One should therefore carry out a very few such calculations in order to understand what is happening. The technique, in geometric forms, was already highly developed by the Greeks.

We shall find the integral $A = \int_0^1 x^2\,dx$ sketched above.

The first step in the process is to trap the integral using *upper rectangles* (or *outer rectangles*) that the shaded region lies inside, and *lower rectangles* (or *inner rectangles*) that lie inside the shaded region. To begin the process, notice that the shaded area is completely contained within the unit square, so

$$0 < A < 1.$$

In the four pictures below, the region has been sliced successively into two, three, four and five strips, then upper and lower rectangles have been constructed so that the region is trapped, or sandwiched, between the upper and lower rectangles. Calculating the areas of these upper and lower rectangles provides tighter and tighter bounds on the area A.



In the first picture, $\frac{1}{2} \times \left(\frac{1}{2}\right)^2 < A < \frac{1}{2} \times \left(\frac{1}{2}\right)^2 + \frac{1}{2} \times 1^2$

$$\tfrac{1}{8} < A < \tfrac{5}{8}.$$

In the second picture, $\frac{1}{3} \times \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \left(\frac{2}{3}\right)^2 < A < \frac{1}{3} \times \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \left(\frac{2}{3}\right)^2 + \frac{1}{3} \times 1^2$

$$\tfrac{5}{27} < A < \tfrac{14}{27}.$$

In the third picture, $\frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) < A < \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2\right)$

$$\tfrac{14}{64} < A < \tfrac{30}{64}.$$

In the fourth picture, $\frac{1}{5}\left(\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right) < A < \frac{1}{5}\left(\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{5}{5}\right)^2\right)$

$$\tfrac{30}{125} < A < \tfrac{55}{125}.$$

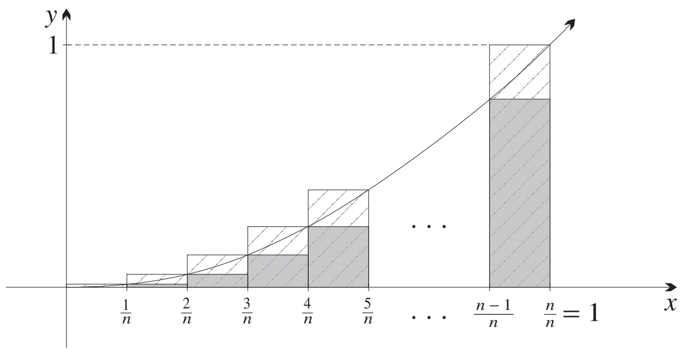
The limiting process

The bounds on the area are getting tighter, but the exact value of the area can only be obtained if this sandwiching process can be turned into a limiting process. The calculations below will need the formula for the sum of the first n squares proven in Question **2f** of Exercise 2A by induction:

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



A Divide the interval $0 \le x \le 1$ into n subintervals, each of width $\frac{1}{n}$.

On each subinterval form the upper rectangle and the lower rectangle.
Then the required region is entirely contained within the upper rectangles, and, in turn, the lower rectangles are entirely contained within the required region. Thus however many strips the region has been dissected into,

$$(\text{sum of lower rectangles}) \le A \le (\text{sum of upper rectangles}).$$

B The heights of the successive upper rectangles are $\frac{1^2}{n^2}, \frac{2^2}{n^2}, \dots, \frac{n^2}{n^2}$, and so, using the formula quoted above,

$$\begin{aligned}
\text{sum of upper rectangles} &= \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right) \\
&= \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2) \\
&= \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6}, \quad \text{(using the formula above)} \\
&= \frac{1}{3} \times \frac{n}{n} \times \frac{n+1}{n} \times \frac{2n+1}{2n} \\
&= \frac{1}{3} \times \left(1 + \frac{1}{n}\right) \times \left(1 + \frac{1}{2n}\right),
\end{aligned}$$

hence the sum of the upper rectangles has limit $\frac{1}{3}$ as $n \to \infty$.

C The heights of the successive lower rectangles are $0, \frac{1^2}{n^2}, \frac{2^2}{n^2}, \dots, \frac{(n-1)^2}{n^2}$, so substituting $n - 1$ for n into the quoted formula,

$$\begin{aligned}
\text{sum of lower rectangles} &= \frac{1}{n}\left(0 + \frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right) \\
&= \frac{1}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2) \\
&= \frac{1}{n^3} \times \frac{(n-1)n(2n-1)}{6} \\
&= \frac{1}{3} \times \frac{n}{n} \times \frac{n-1}{n} \times \frac{2n-1}{2n} \\
&= \frac{1}{3} \times \left(1 - \frac{1}{n}\right) \times \left(1 - \frac{1}{2n}\right),
\end{aligned}$$

hence the sum of the lower rectangles also has limit $\frac{1}{3}$ as $n \to \infty$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

D Finally, (sum of lower rectangles) $\le A \le$ (sum of upper rectangles), and both these sums have the same limit $\frac{1}{3}$, so it follows that $A = \frac{1}{3}$.

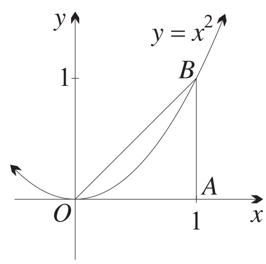
The conclusion of all this heavy machinery of limits is simply that $\int_0^1 x^2\,dx = \frac{1}{3}$.

You will be relieved to know that we will soon have much quicker methods.

Exercise 5A

FOUNDATION

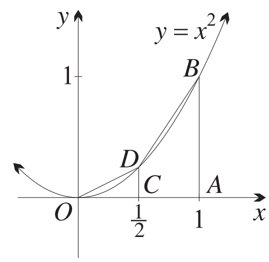
1



a Find the area of $\triangle OAB$ in the diagram above.

b Hence explain why $\int_0^1 x^2\,dx < \frac{1}{2}$.

2

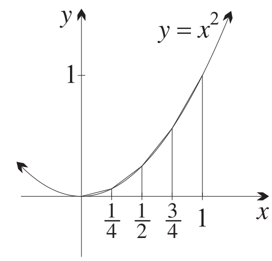


a Find the area of $\triangle OCD$ in the diagram above.

b Find the area of the trapezium $CABD$.

c Hence explain why $\int_0^1 x^2\,dx < \frac{3}{8}$.

3

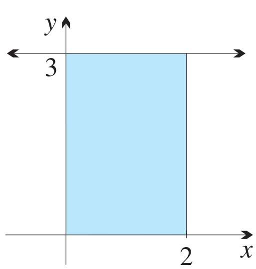


a Use the diagram above to show that $\int_0^1 x^2\,dx < \frac{11}{32}$.

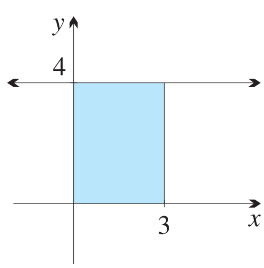
b Explain why $\frac{11}{32}$ is a better approximation to $\int_0^1 x^2\,dx$ than $\frac{3}{8}$ is.

4 Use area formulae to calculate these definite integrals (sketches are given below).

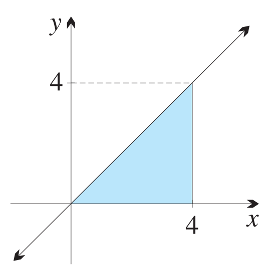
a $\int_0^2 3\,dx$



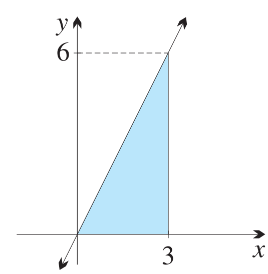
b $\int_0^3 4\,dx$



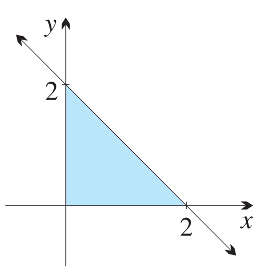
c $\int_0^4 x\,dx$



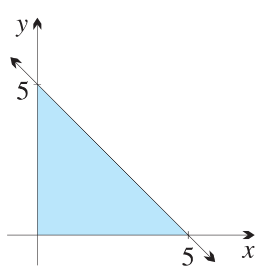
d $\int_0^3 2x\,dx$



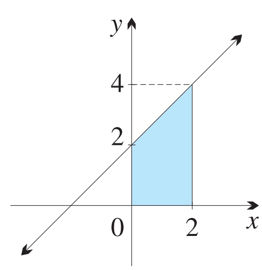
e $\int_0^2 (2 - x)\,dx$



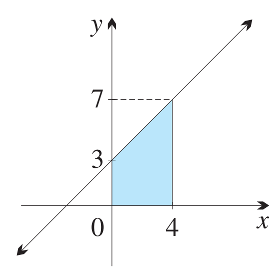
f $\int_0^5 (5 - x)\,dx$



g $\int_0^2 (x + 2)\,dx$



h $\int_0^4 (x + 3)\,dx$

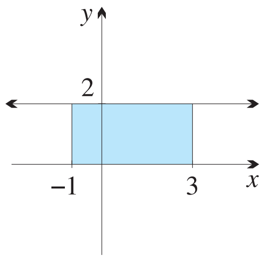


ISBN 978-1-108-76630-2

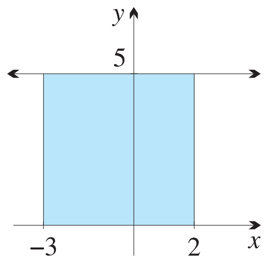
Cambridge University Press
Updated July 2022

5 Use area formulae to calculate the sketched definite integrals.

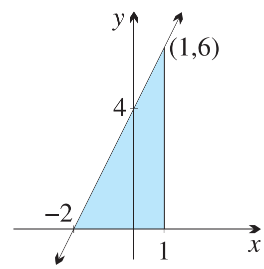
a $\int_{-1}^{3} 2\,dx$



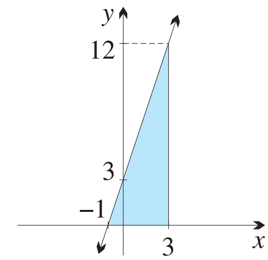
b $\int_{-3}^{2} 5\,dx$



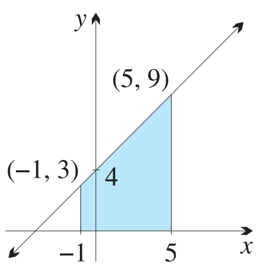
c $\int_{-2}^{1} (2x + 4)\,dx$



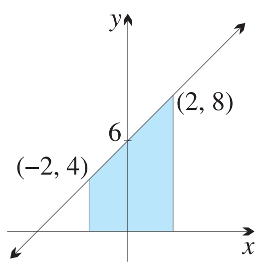
d $\int_{-1}^{3} (3x + 3)\,dx$



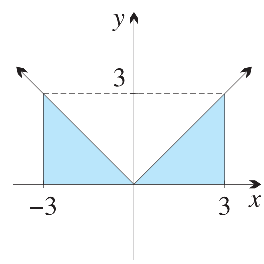
e $\int_{-1}^{5} (x + 4)\,dx$



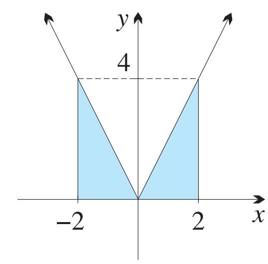
f $\int_{-2}^{2} (x + 6)\,dx$



g $\int_{-3}^{3} |x|\,dx$



h $\int_{-2}^{2} |2x|\,dx$



6 Sketch a graph of each definite integral, then use an area formula to calculate it.

a $\int_{0}^{3} 5\,dx$ **b** $\int_{-3}^{0} 5\,dx$ **c** $\int_{-1}^{4} 5\,dx$ **d** $\int_{-2}^{6} 5\,dx$

e $\int_{-5}^{0} (x + 5)\,dx$ **f** $\int_{0}^{2} (x + 5)\,dx$ **g** $\int_{2}^{4} (x + 5)\,dx$ **h** $\int_{-1}^{3} (x + 5)\,dx$

i $\int_{4}^{8} (x - 4)\,dx$ **j** $\int_{4}^{10} (x - 4)\,dx$ **k** $\int_{5}^{7} (x - 4)\,dx$ **l** $\int_{6}^{10} (x - 4)\,dx$

m $\int_{-2}^{2} |x|\,dx$ **n** $\int_{-4}^{4} |x|\,dx$ **o** $\int_{0}^{5} |x - 5|\,dx$ **p** $\int_{5}^{10} |x - 5|\,dx$

7 Sketch a graph of each definite integral, then use an area formula to calculate it.

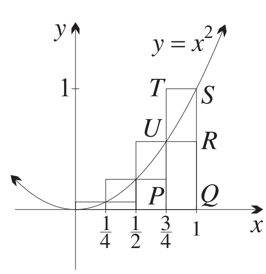
a $\int_{-4}^{4} \sqrt{16 - x^2}\,dx$ **b** $\int_{-5}^{0} \sqrt{25 - x^2}\,dx$

DEVELOPMENT

8 **a** In the diagram to the right, add the areas of the lower rectangles. (For example, $PQRU$ is a lower rectangle.)

b Add the areas of the upper rectangles. (For example, $PQST$ is an upper rectangle.)

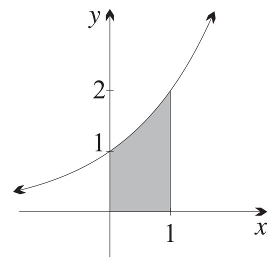
c Hence explain why $\frac{7}{32} < \int_{0}^{1} x^2\,dx < \frac{15}{32}$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 The area of the region in the diagram to the right is given by $\int_0^1 2^x\,dx$.



a Use one lower and one upper rectangle to show that $1 < \int_0^1 2^x\,dx < 2$.

b Use 2 lower and 2 upper rectangles of equal width to show that (with decimals rounded to one place)

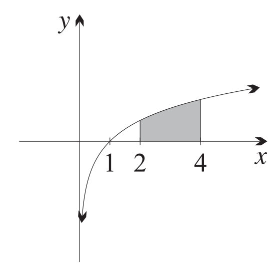
$$1.2 < \int_0^1 2^x\,dx < 1.7.$$

c Use 4 lower and 4 upper rectangles of equal width to show that

$$1.3 < \int_0^1 2^x\,dx < 1.6.$$

d What trend can be identified in the parts above?

10 The area of the region in the diagram to the right is given by $\int_2^4 \ln x\,dx$.



a Use 2 lower and 2 upper rectangles to show that (with decimals rounded to 2 places) $1.79 < \int_2^4 \ln x\,dx < 2.48$.

b Use 4 lower and 4 upper rectangles of equal width to show that

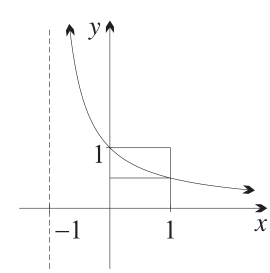
$$1.98 < \int_2^4 \ln x\,dx < 2.33.$$

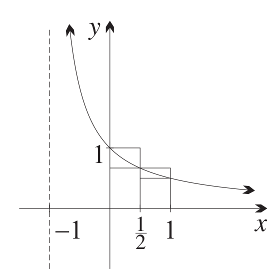
c Use 8 lower and 8 upper rectangles of equal width to show that

$$2.07 < \int_2^4 \ln x\,dx < 2.24.$$

d What trend can be identified in the parts above?

11 Let $A = \int_0^1 \frac{1}{x+1}\,dx$.





a Use the areas of the lower and upper rectangles in the top diagram to show that $\frac{1}{2} < A < 1$.

b Use the areas of the 2 lower and 2 upper rectangles in the bottom diagram to show that $\frac{7}{12} < A < \frac{5}{6}$. (That is, $0.58 < A < 0.83$, correct to 2 decimal places.)

c Use 3 lower and 3 upper rectangles of equal width to show that $\frac{37}{60} < A < \frac{47}{60}$. (That is, $0.62 < A < 0.78$, correct to 2 decimal places.)

d Finally, use 4 lower and 4 upper rectangles of equal width to show that $\frac{533}{840} < A < \frac{319}{420}$. (That is, correct to 2 decimal places, $0.63 < A < 0.76$.)

e As the number of rectangles increases, is the interval within which A lies getting bigger or smaller?

f The exact value of A is $\ln 2 = 0.693147\ldots$. Do the lower and upper limits of the intervals in parts **a** to **d** seem to be approaching the exact value?

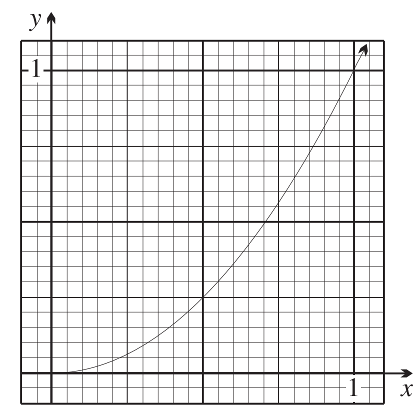
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 [Technology] Some of the previous questions involve summing the areas of lower and upper rectangles to approximate a definite integral. Many software programs can do this automatically, using any prescribed number of rectangles. Steadily increasing the number of rectangles will show the sums of the lower and upper rectangles converging to the exact area, which can be checked either using area formulae or using the exact value of the definite integral as calculated later in the course.

Investigate some of the definite integrals from Questions 1–3 and 8–13 in this way.

13 The diagram to the right shows the graph of $y = x^2$ from $x = 0$ to $x = 1$, drawn on graph paper.
The scale is 20 little divisions to 1 unit. This means that 400 little squares make up 1 square unit.



a Count how many little squares there are under the graph from $x = 0$ to $x = 1$ (keeping reasonable track of fragments of squares), then divide by 400 to approximate $\int_0^1 x^2\,dx$.
Write your answer correct to 2 decimal places.

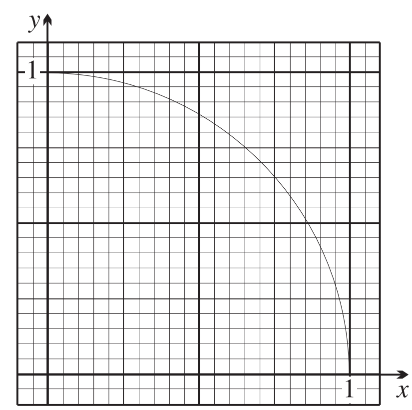
b By counting the appropriate squares, approximate:

i $\int_0^{\frac{1}{2}} x^2\,dx$ **ii** $\int_{\frac{1}{2}}^1 x^2\,dx$

Confirm that the sum of the answers to parts **i** and **ii** is the answer to part **a**.

14 The diagram to the right shows the quadrant

$$y = \sqrt{1 - x^2}, \text{ from } x = 0 \text{ to } x = 1.$$



As before, the scale is 20 little divisions to 1 unit.

a Count how many little squares there are under the graph from $x = 0$ to $x = 1$.

b Divide by 400 to approximate $\int_0^1 \sqrt{1 - x^2}\,dx$.
Write your answer correct to 2 decimal places.

c Hence, using the fact that a quadrant has area $\frac{1}{4}\pi r^2$, find an approximation for π. Give your answer correct to 2 decimal places.

ENRICHMENT

15 Using exactly the same setting out as in the example in the notes above this exercise, prove from first principles that $\int_0^1 x^3\,dx = \frac{1}{4}$. Use upper and lower rectangles, and take the limit as the number of rectangular strips approaches infinity.

Note: This calculation will need the formula $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$ for the sum of the first n cubes, which was proven by mathematical induction in worked Example 1 of Section 2A.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 Prove the following two definite integrals from first principles. Use upper and lower rectangles, and take the limit as the number of rectangular strips approaches infinity.

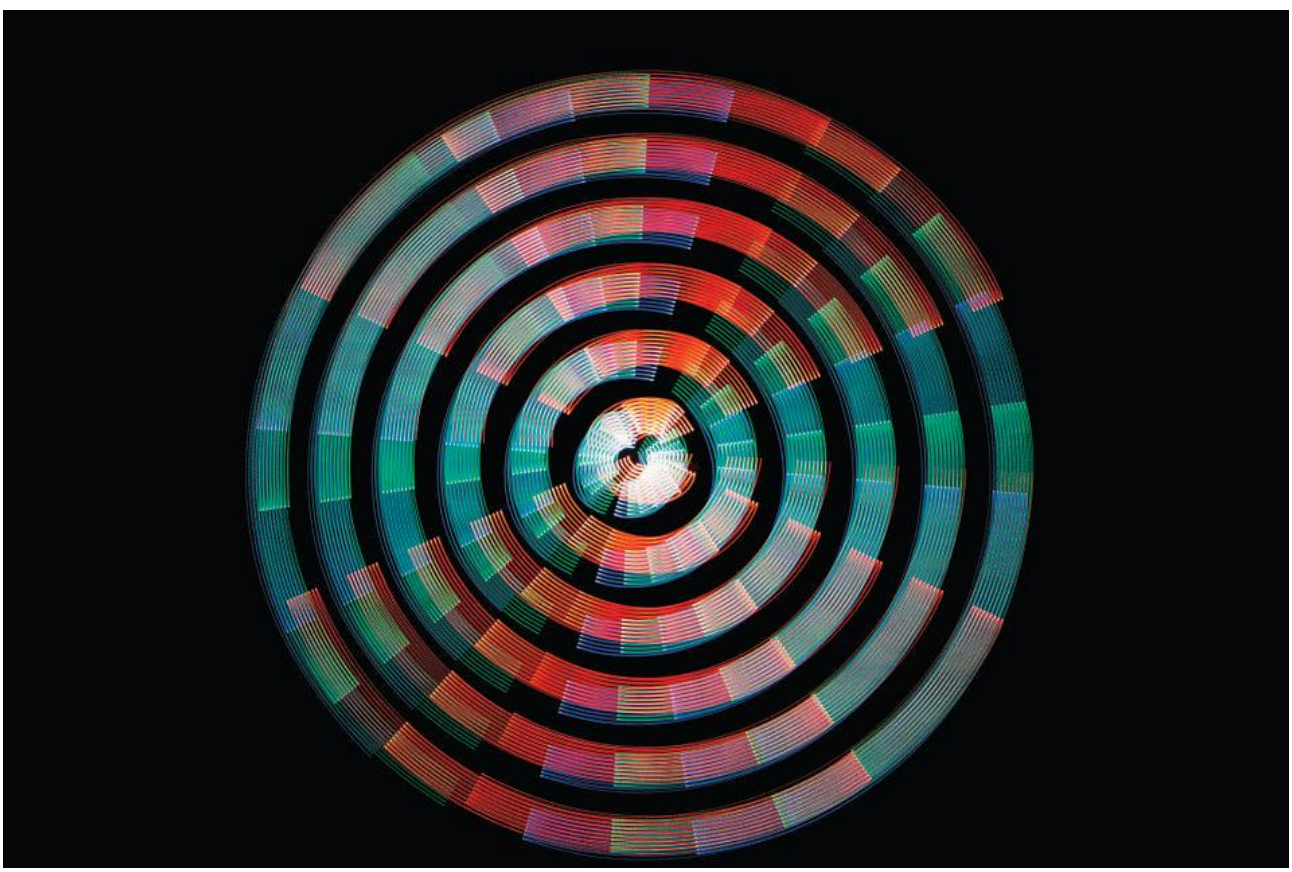
a $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}$ **b** $\displaystyle\int_0^a x^3\,dx = \frac{a^4}{4}$

17 Draw a large sketch of $y = x^2$ for $0 \le x \le 1$, and let U be the point $(1, 0)$. For some positive integer n, let $P_0 (= O), P_1, P_2, \ldots, P_n$ be the points on the curve with x-coordinates $x = 0$, $x = \frac{1}{n}$, $x = \frac{2}{n}, \ldots,$ $x = \frac{n}{n} = 1$. Join the chords $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$, and join P_nU.

a Use the area formula for a trapezium to find the area of the polygon $P_0P_1P_2 \cdots P_nU$.

b Explain geometrically why this area is always greater than $\displaystyle\int_0^1 x^2\,dx$.

c Show that its limit as $n \to \infty$ is $\frac{1}{3}$. This is half of the proof that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5B The fundamental theorem of calculus

There is a remarkably simple formula for evaluating definite integrals, based on taking the primitive of the function. The formula is called the *fundamental theorem of calculus* because the whole of calculus depends on it. We have delayed its challenging proof until Section 5D, when its usefulness will have been established.

Primitives

Let us first review from the last section of Chapter 4 what primitives are, and the first step in finding them.

3 PRIMITIVES

- A function $F(x)$ is called a *primitive* or *anti-derivative* of a function $f(x)$ if its derivative is $f(x)$:
  $F(x)$ is a primitive of $f(x)$ if $F'(x) = f(x)$.
- To find the general primitive of a power x^n, where $n \neq -1$:
  $$\text{If } \frac{dy}{dx} = x^n, \text{ then } y = \frac{x^{n+1}}{n+1} + C, \text{ for some constant } C.$$

'Increase the index by 1 and divide by the new index.'

Statement of the fundamental theorem

The fundamental theorem says that a definite integral can be evaluated by writing down any primitive $F(x)$ of $f(x)$, then substituting the upper and lower limits into it and subtracting.

4 THE FUNDAMENTAL THEOREM OF CALCULUS

Let $f(x)$ be a function that is continuous in a closed interval $[a, b]$. Then
$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) \text{ is any primitive of } f(x).$$

This result is extraordinary because it says that taking areas and taking tangents are inverse processes, which is not obvious.

Using the fundamental theorem to evaluate an integral

The conventional way to set out these calculations is to enclose the primitive in square brackets, writing the lower and upper limits as subscript and superscript respectively.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 2 5B

Evaluate these definite integrals. Then draw diagrams to show the regions that they represent, and check the answers using area formulae.

a $\displaystyle\int_0^2 2x\,dx$ **b** $\displaystyle\int_2^4 (2x-3)\,dx$

SOLUTION

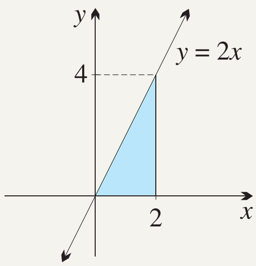
a $\displaystyle\int_0^2 2x\,dx = \Big[x^2\Big]_0^2$ (x^2 is a primitive of $2x$)

$= 2^2 - 0^2$ (substitute 2, then substitute 0 and subtract)

$= 4$

Using areas, area of triangle $= \dfrac{1}{2} \times 2 \times 4$

$= 4.$



b $\displaystyle\int_2^4 (2x-3)\,dx = \Big[x^2 - 3x\Big]_2^4$ (take the primitive of each term)

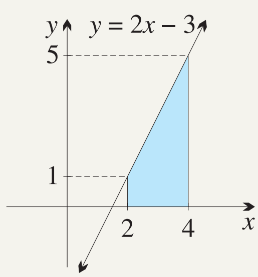
$= (16-12) - (4-6)$ (substitute 4, then substitute 2)

$= 4 - (-2)$

$= 6$

Using areas, area of trapezium $= \dfrac{1+5}{2} \times 2$

$= 6.$



Note: Whenever the primitive has two or more terms, brackets are needed when substituting the upper and lower limits of integration.

Example 3 5B

Evaluate these definite integrals.

a $\displaystyle\int_0^1 5x^2\,dx$ **b** $\displaystyle\int_0^4 (25-x^2)\,dx$

SOLUTION

a $\displaystyle\int_0^1 5x^2\,dx = \left[\frac{5}{3}x^3\right]_0^1$ (increase the index from 2 to 3, then divide by 3)

$= \frac{5}{3} - 0$ (substitute 1, then substitute 0 and subtract)

$= \frac{5}{3}$

b $\displaystyle\int_0^4 (25-x^2)\,dx = \Big[25x - \tfrac{1}{3}x^3\Big]_0^4$

$= (100 - \tfrac{1}{3} \times 64) - (0-0)$ (never omit a substitution of 0)

$= 78\tfrac{2}{3}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Expanding brackets in the integrand

As with differentiation, it is often necessary to expand the brackets in the integrand before finding a primitive. With integration, there is no ‘product rule’ that could avoid the expansion.

Example 4 5B

Expand the brackets in each integral, then evaluate it.

a $\displaystyle\int_1^6 x(x+1)\,dx$

b $\displaystyle\int_0^3 (x-4)(x-6)\,dx$

SOLUTION

a $\displaystyle\int_1^6 x(x+1)\,dx$

$$= \int_1^6 (x^2+x)\,dx$$
$$= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^6$$
$$= (72+18)-\left(\frac{1}{3}+\frac{1}{2}\right)$$
$$= 90-\tfrac{5}{6} \quad \text{(care with the fractions)}$$
$$= 89\tfrac{1}{6}$$

b $\displaystyle\int_0^3 (x-4)(x-6)\,dx$

$$= \int_0^3 (x^2-10x+24)\,dx$$
$$= \left[\frac{x^3}{3}-5x^2+24x\right]_0^3$$
$$= (9-45+72)-(0-0+0)$$
$$= 36$$

Writing the integrand as separate fractions

If the integrand is a fraction with two or more terms in the numerator, it should normally be written as separate fractions, as with differentiation. With integration, there is no ‘quotient rule’ that could avoid this.

Example 5 5B

Write each integrand as two separate fractions, then evaluate the integral.

a $\displaystyle\int_1^2 \frac{3x^4-2x^2}{x^2}\,dx$

b $\displaystyle\int_{-3}^{-2} \frac{x^3-2x^4}{x^3}\,dx$

SOLUTION

a $\displaystyle\int_1^2 \frac{3x^4-2x^2}{x^2}\,dx$

$$= \int_1^2 (3x^2-2)\,dx$$
$$= \left[x^3-2x\right]_1^2$$
$$= (8-4)-(1-2)$$
$$= 4-(-1)$$
$$= 5$$

b $\displaystyle\int_{-3}^{-2} \frac{x^3-2x^4}{x^3}\,dx$

$$= \int_{-3}^{-2} (1-2x)\,dx$$
$$= \left[x-x^2\right]_{-3}^{-2}$$
$$= (-2-4)-(-3-9)$$
$$= -6-(-12)$$
$$= 6$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Negative indices

The fundamental theorem also works when the indices are negative. Care is needed when converting between negative powers of x and fractions.

Example 6

Use negative indices to evaluate these definite integrals.

a $\displaystyle\int_1^5 x^{-2}\,dx$

b $\displaystyle\int_1^2 \frac{1}{x^4}\,dx$

SOLUTION

a Increase the index from -2 to -1, and divide by -1.

$$\begin{aligned}\int_1^5 x^{-2}\,dx &= \left[\frac{x^{-1}}{-1}\right]_1^5\\ &= \left[-\frac{1}{x}\right]_1^5\\ &= -\frac{1}{5} - (-1)\\ &= -\frac{1}{5} + 1\\ &= \frac{4}{5}\end{aligned}$$

b Increase the index from -4 to -3, and divide by -3.

$$\begin{aligned}\int_1^2 \frac{1}{x^4}\,dx &= \int_1^2 x^{-4}\,dx\\ &= \left[\frac{x^{-3}}{-3}\right]_1^2\\ &= \left[-\frac{1}{3x^3}\right]_1^2\\ &= -\frac{1}{24} - \left(-\frac{1}{3}\right)\\ &= \frac{7}{24}\end{aligned}$$

Note: The negative index -1 cannot be handled by this rule, because it would generate division by 0, which is nonsense:

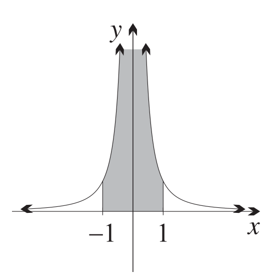
$$\int_1^2 \frac{1}{x}\,dx = \int_1^2 x^{-1}\,dx = \left[\frac{x^0}{0}\right]_2^1 = \text{nonsense}.$$

Chapter 6 on exponential and logarithmic functions will handle this integral.

Warning: Do not integrate across an asymptote

The following calculation seems just as valid as part (b) above:

$$\begin{aligned}\int_{-1}^1 x^{-4}\,dx &= \left[\frac{x^{-3}}{-3}\right]_{-1}^1\\ &= -\frac{1}{3} - \frac{1}{3}\\ &= -\frac{2}{3}.\end{aligned}$$



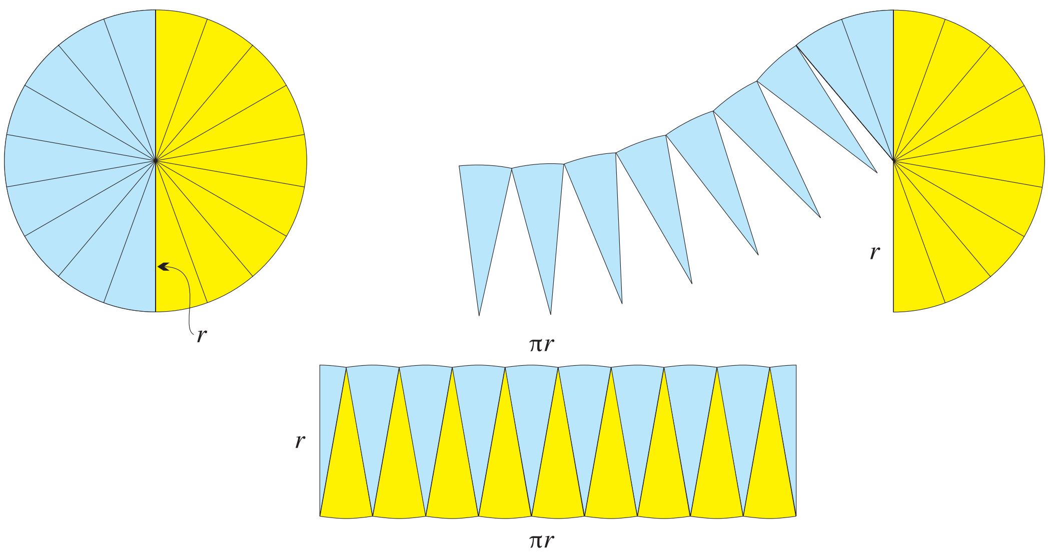
But in fact the calculation is nonsense — the function has an asymptote at $x = 0$, so it is not even defined there. (And the function $y = x^{-4}$ is always positive, so how could it give a negative answer for an integral?) You cannot integrate across an asymptote, and you always need to be on the lookout for such meaningless integrals.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The area of a circle

In earlier years, the formula $A = \pi r^2$ for the area of a circle was developed. Because the boundary is a curve, some limiting process had to be used in its proof. For comparison with the notation for the definite integral explained at the start of this section, here is the most common version of that argument — a little rough in its logic, but very quick. It involves dissecting the circle into infinitesimally thin sectors and then rearranging them into a rectangle.



The height of the rectangle in the lower diagram is r. Because the circumference $2\pi r$ is divided equally between the top and bottom sides, the length of the rectangle is πr. Hence the rectangle has area πr^2, which is therefore the area of the circle.

Exercise 5B

FOUNDATION

Technology: Many programs allow definite integrals to be calculated automatically. This allows not just quick checking of the answers, but experimentation with further definite integrals. It would be helpful to generate screen sketches of the graphs and the regions associated with the integrals.

1 Evaluate these definite integrals using the fundamental theorem.

a $\int_0^1 2x\,dx$ **b** $\int_1^4 2x\,dx$ **c** $\int_1^3 4x\,dx$

d $\int_2^5 8x\,dx$ **e** $\int_2^3 3x^2\,dx$ **f** $\int_0^3 5x^4\,dx$

g $\int_1^2 10x^4\,dx$ **h** $\int_0^1 12x^5\,dx$ **i** $\int_0^1 11x^{10}\,dx$

2 **a** Evaluate these definite integrals using the fundamental theorem.

i $\int_0^1 4\,dx$ **ii** $\int_2^7 5\,dx$ **iii** $\int_4^5 dx$

b Check your answers by sketching the graph of the region involved.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Evaluate these definite integrals using the fundamental theorem.

a $\int_3^6 (2x + 1)\,dx$ **b** $\int_2^4 (2x - 3)\,dx$ **c** $\int_0^3 (4x + 5)\,dx$

d $\int_2^3 (3x^2 - 1)\,dx$ **e** $\int_1^4 (6x^2 + 2)\,dx$ **f** $\int_0^1 (3x^2 + 2x)\,dx$

g $\int_1^2 (4x^3 + 3x^2 + 1)\,dx$ **h** $\int_0^2 (2x + 3x^2 + 8x^3)\,dx$ **i** $\int_3^5 (3x^2 - 6x + 5)\,dx$

4 Evaluate these definite integrals using the fundamental theorem. You will need to take care when finding powers of negative numbers.

a $\int_{-1}^0 (1 - 2x)\,dx$ **b** $\int_{-1}^0 (2x + 3)\,dx$ **c** $\int_{-2}^1 3x^2\,dx$

d $\int_{-1}^2 (4x^3 + 5)\,dx$ **e** $\int_{-2}^2 (5x^4 + 6x^2)\,dx$ **f** $\int_{-2}^{-1} (4x^3 + 12x^2 - 3)\,dx$

5 Evaluate these definite integrals using the fundamental theorem. You will need to take care when adding and subtracting fractions.

a $\int_1^4 (x + 2)\,dx$ **b** $\int_0^2 (x^2 + x)\,dx$ **c** $\int_0^3 (x^3 + x^2)\,dx$

d $\int_{-1}^1 (x^3 - x + 1)\,dx$ **e** $\int_{-2}^3 (2x^2 - 3x + 1)\,dx$ **f** $\int_{-4}^{-2} (16 - x^3 - x)\,dx$

DEVELOPMENT

6 By expanding the brackets first, evaluate these definite integrals.

a $\int_2^3 x(2 + 3x)\,dx$ **b** $\int_0^2 (x + 1)(3x + 1)\,dx$ **c** $\int_{-1}^1 x^2(5x^2 + 1)\,dx$

d $\int_{-1}^2 (x - 3)^2\,dx$ **e** $\int_{-1}^0 x(x - 1)(x + 1)\,dx$ **f** $\int_{-1}^0 (1 - x^2)^2\,dx$

7 Write each integrand as separate fractions, then evaluate the integral.

a $\int_1^3 \frac{3x^3 + 4x^2}{x}\,dx$ **b** $\int_1^2 \frac{4x^4 - x}{x}\,dx$ **c** $\int_2^3 \frac{5x^2 + 9x^4}{x^2}\,dx$

d $\int_1^2 \frac{x^3 + 4x^2}{x}\,dx$ **e** $\int_1^3 \frac{x^3 - x^2 + x}{x}\,dx$ **f** $\int_{-2}^{-1} \frac{x^3 - 2x^5}{x^2}\,dx$

8 Evaluate these definite integrals using the fundamental theorem. You will need to take care when finding powers of fractions.

a $\int_0^{\frac{1}{2}} x^2\,dx$ **b** $\int_0^{\frac{2}{3}} (2x + 3x^2)\,dx$ **c** $\int_{\frac{1}{2}}^{\frac{3}{4}} (6 - 4x)\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 **a** Evaluate these definite integrals.

i $\displaystyle\int_5^{10} x^{-2}\,dx$ **ii** $\displaystyle\int_2^{3} 2x^{-3}\,dx$ **iii** $\displaystyle\int_{\frac{1}{2}}^{1} 4x^{-5}\,dx$

b By writing them with negative indices, evaluate these definite integrals.

i $\displaystyle\int_1^{2} \frac{1}{x^2}\,dx$ **ii** $\displaystyle\int_1^{4} \frac{1}{x^3}\,dx$ **iii** $\displaystyle\int_{\frac{1}{2}}^{1} \frac{3}{x^4}dx$

10 **a** **i** Show that $\displaystyle\int_2^{k} 3\,dx = 3k - 6$.

ii Hence find the value of k, given that $\displaystyle\int_2^{k} 3\,dx = 18$.

b **i** Show that $\displaystyle\int_0^{k} x\,dx = \frac{1}{2}k^2$.

ii Hence find the positive value of k, given that $\displaystyle\int_0^{k} x\,dx = 18$.

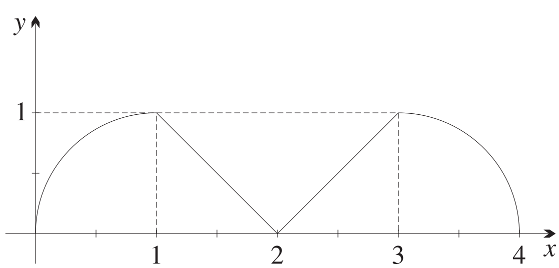
11 Find the value of k if $k > 0$ and.

a $\displaystyle\int_k^{3} 2\,dx = 4$ **b** $\displaystyle\int_k^{8} 3\,dx = 12$ **c** $\displaystyle\int_2^{3} (k-3)\,dx = 5$

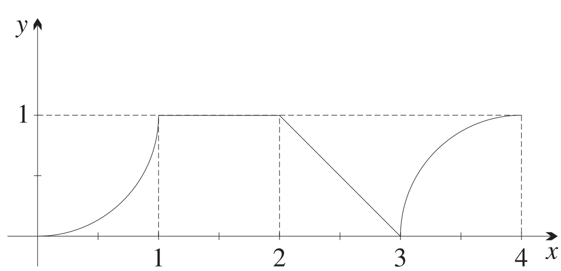
d $\displaystyle\int_3^{k} (x-3)\,dx = 0$ **e** $\displaystyle\int_1^{k} (x+1)\,dx = 6$ **f** $\displaystyle\int_1^{k} (k+3x)\,dx = \frac{13}{2}$

12 Use area formulae to find $\displaystyle\int_0^{4} f(x)\,dx$ in each sketch of $f(x)$.

a



b



13 Evaluate each definite integral.

a $\displaystyle\int_1^{2} \frac{1+x^2}{x^2}\,dx$ **b** $\displaystyle\int_{-2}^{-1} \frac{1+2x}{x^3}\,dx$ **c** $\displaystyle\int_{-3}^{-1} \frac{1-x^3-4x^5}{2x^2}\,dx$

14 Evaluate these definite integrals.

a $\displaystyle\int_1^{3} \left(x+\frac{1}{x}\right)^2 dx$ **b** $\displaystyle\int_1^{2} \left(x^2+\frac{1}{x^2}\right)^2 dx$ **c** $\displaystyle\int_{-2}^{-1} \left(\frac{1}{x^2}+\frac{1}{x}\right)^2 dx$

15 **a** Explain why the function $y = \dfrac{1}{x^2}$ is never negative.

b Sketch the integrand and explain why the argument below is invalid.

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

c Without evaluating any integrals, say which of these integrals are meaningless.

i $\displaystyle\int_0^{2} \frac{1}{(3-x)^2}\,dx$ **ii** $\displaystyle\int_2^{4} \frac{1}{(3-x)^2}\,dx$ **iii** $\displaystyle\int_4^{6} \frac{1}{(3-x)^2}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 Question **16** should only be attempted in Exercise 5D. It appears as Question **6** in Exercise 5D.

a Use the differential form of the fundamental theorem $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ to find:

i $\dfrac{d}{dx}\displaystyle\int_1^x t^2\,dt$

ii $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + 3t)\,dt$

iii $\dfrac{d}{dx}\displaystyle\int_a^x \frac{1}{t}\,dt$

iv $\dfrac{d}{dx}\displaystyle\int_a^x (t^3 - 3)^4\,dt$

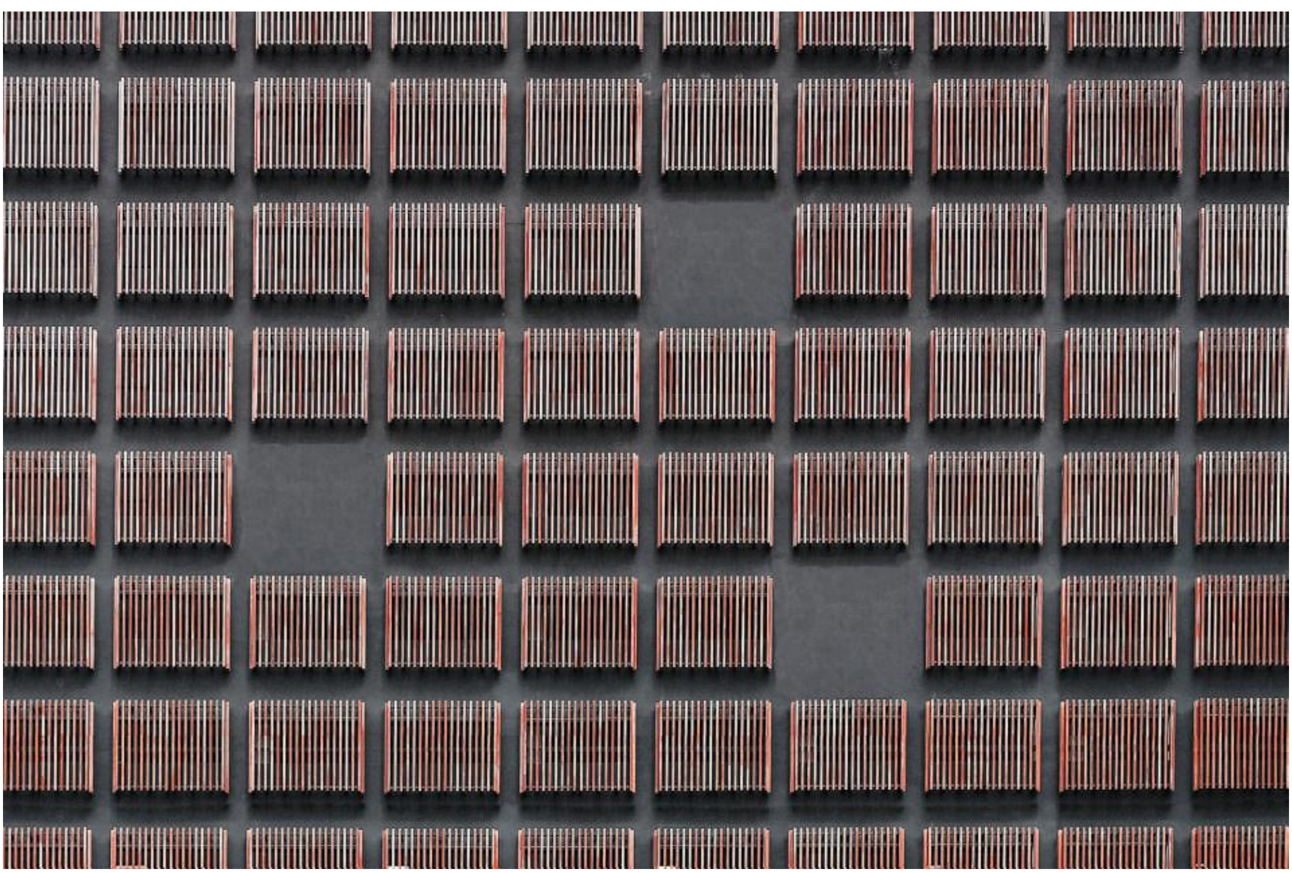
b Confirm your answers to parts (i) and (ii) above by evaluating the definite integral and then differentiating with respect to x.

ENRICHMENT

17 The derivative of the function $U(x)$ is $u(x)$.

a Find $V'(x)$, where $V(x) = (a - x)U(x) + \displaystyle\int_0^x U(t)\,dt$ and a is a constant.

b Hence prove that $\displaystyle\int_0^a U(x)\,dx = aU(0) + \int_0^a (a - x)\,u(x)\,dx$.



ISBN 978-1-108-76630-2

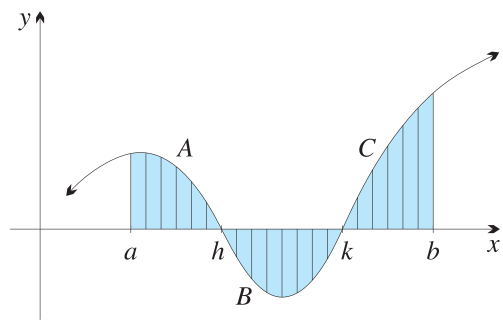
Cambridge University Press
Updated July 2022

5C The definite integral and its properties

This section will first extend the theory to functions with negative values. Then some simple properties of the definite integral will be established using arguments about the dissection of regions.

Integrating functions with negative values

When a function has negative values, its graph is below the x-axis, so the 'heights' of the little rectangles in the dissection are negative numbers. This means that any areas below the x-axis should contribute negative values to the value of the final integral.



For example, in the diagram above, where $a < h < k < b$, the region B is below the x-axis because the function $f(x)$ is negative in $[h, k]$. So the heights of all the infinitesimal strips making up B are negative, and B therefore contributes a negative number to the definite integral,

$$\int_a^b f(x)\, dx = +\text{ area } A - \text{ area } B + \text{ area } C.$$

Thus we attach the sign + or − to each area, depending on whether the curve, and therefore the region, is above or below the x-axis. For this reason, the three terms in the sum above are often referred to as *signed areas under the curve*, because a sign has been attached to the area of each region.

5 THE DEFINITE INTEGRAL AS THE SUM OF SIGNED AREAS

Let $f(x)$ be a function continuous in the closed interval $[a, b]$, where $a \le b$, and suppose that we are taking the definite integral over $[a, b]$.

- For regions where the curve is above the x-axis, we attach + to the area.
  For regions where the curve is below the x-axis, we attach − to the area.
  These areas, with signs attached, are called *signed areas under the curve*.
- The *definite integral* $\displaystyle\int_a^b f(x)\, dx$ is the sum of these signed areas under the curve in the interval $[a, b]$.

The whole definite integral $\displaystyle\int_a^b f(x)\, dx$ is often also referred to as *the signed area under the curve*.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

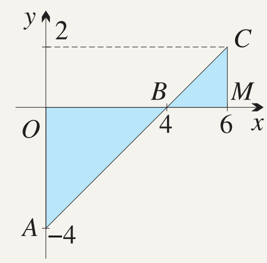
Example 7 5C

Evaluate these definite integrals.

a $\displaystyle\int_0^4 (x - 4)\,dx$ **b** $\displaystyle\int_4^6 (x - 4)\,dx$ **c** $\displaystyle\int_0^6 (x - 4)\,dx$

Sketch the graph of $y = x - 4$ and shade the regions associated with these integrals. Then explain how each result is related to the shaded regions.

SOLUTION



a $$\int_0^4 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^4$$
$$= (8 - 16) - (0 - 0)$$
$$= -8$$

b $$\int_4^6 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_4^6$$
$$= (18 - 24) - (8 - 16)$$
$$= -6 - (-8)$$
$$= 2$$

c $$\int_0^6 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^6$$
$$= (18 - 24) - (0 - 0)$$
$$= -6$$

Triangle OAB is below the x-axis and has area 8, and triangle BMC is above the x-axis, and has area 2, so the answers -8 in part **a** and $+2$ in part **b** are expected. This integral in part **c** represents the area of $\triangle BMC$ minus the area of $\triangle OAB$, so the value of the integral is $2 - 8 = -6$ is expected.

Odd and even functions

In the first example below, the function $y = x^3 - 4x$ is an odd function, with point symmetry in the origin. Thus the area of each shaded hump is the same. Hence the whole integral from $x = -2$ to $x = 2$ is zero, because the equal humps above and below the x-axis cancel out.

In the second diagram, the function $y = x^2 + 1$ is even, with line symmetry in the y-axis. Thus the areas to the left and right of the y-axis are equal, so there is a doubling instead of a cancelling.

6 INTEGRATING ODD AND EVEN FUNCTIONS

- If $f(x)$ is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.
- If $f(x)$ is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

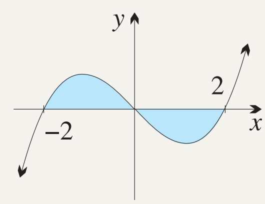
Example 8 5C

Sketch these integrals, then evaluate them using symmetry.

a $\int_{-2}^{2} (x^3 - 4x)\,dx$ **b** $\int_{-2}^{2} (x^2 + 1)\,dx$

SOLUTION

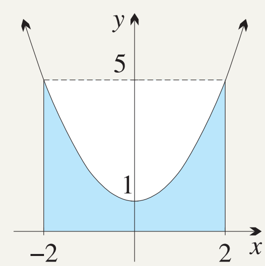
a $\int_{-2}^{2} (x^3 - 4x)\,dx = 0,$ because the integrand is odd.



Without this simplification, the calculation is:

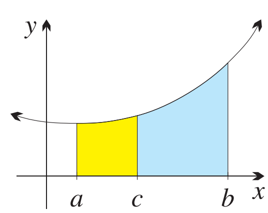
$$\int_{-2}^{2} (x^3 - 4x)\,dx = \Big[\tfrac{1}{4}x^4 - 2x^2\Big]_{-2}^{2}$$
$$= (4 - 8) - (4 - 8)$$
$$= 0, \text{ as before.}$$

b Because the integrand is even,



$$\int_{-2}^{2} (x^2 + 1)\,dx = 2\int_{0}^{2} (x^2 + 1)\,dx$$
$$= 2\Big[\tfrac{1}{3}x^3 + x\Big]_{0}^{2}$$
$$= 2\Big((2\tfrac{2}{3} + 2) - (0 + 0)\Big)$$
$$= 9\tfrac{1}{3}.$$

Dissection of the interval



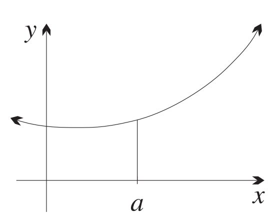
When a region is dissected, its area remains the same. In particular, we can always dissect the region by dissecting the interval $a \le x \le b$ over which we are integrating.

Thus if $f(x)$ is being integrated over the closed interval $[a, b]$ and the number c lies in this interval, then:

7 DISSECTION OF THE INTERVAL

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

Intervals of zero width



Suppose that a function is integrated over an interval $a \le x \le a$ of width zero, and that the function is defined at $x = a$. In this situation, the region also has width zero, so the integral is zero.

8 INTERVALS OF ZERO WIDTH

$$\int_{a}^{a} f(x)\,dx = 0$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Running an integral backwards from right to left

A further small qualification must be made to the definition of the definite integral. Suppose that the limits of the integral are reversed, so that the integral 'runs backwards' from right to left over the interval. Then its value reverses in sign:

9 REVERSING THE INTERVAL

Let $f(x)$ be continuous in the closed interval $[a, b]$, where $a \le b$. Then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

This agrees perfectly with calculations using the fundamental theorem, because

$$F(a) - F(b) = -\big(F(b) - F(a)\big).$$

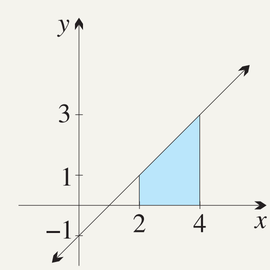
Example 9 5C

Evaluate and compare these two definite integrals using the fundamental theorem.

a $\displaystyle\int_2^4 (x - 1)\,dx$

b $\displaystyle\int_4^2 (x - 1)\,dx$

SOLUTION



a $$\int_2^4 (x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_2^4$$
$$= (8 - 4) - (2 - 2)$$
$$= 4,$$

which is positive, because the region is above the x-axis.

b $$\int_4^2 (x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_4^2$$
$$= (2 - 2) - (8 - 4)$$
$$= -4,$$

which is the opposite of part **a**, because the integral runs backwards from right to left, from $x = 4$ to $x = 2$.

Sums of functions

When two functions are added, the two regions are piled on top of each other, so that:

10 INTEGRAL OF A SUM

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 10 5C

Evaluate these two expressions and show that they are equal.

a $\int_0^1 (x^2 + x + 1)\, dx$

b $\int_0^1 x^2\, dx + \int_0^1 x\, dx + \int_0^1 1\, dx$

SOLUTION

a
$$\begin{aligned}\int_0^1 (x^2 + x + 1)\, dx &= \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_0^1 \\ &= \left(\tfrac{1}{3} + \tfrac{1}{2} + 1\right) - (0 + 0 + 0) \\ &= 1\tfrac{5}{6}.\end{aligned}$$

b
$$\begin{aligned}\int_0^1 x^2\, dx + \int_0^1 x\, dx + \int_0^1 1\, dx &= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 + \Big[x\Big]_0^1 \\ &= \left(\tfrac{1}{3} - 0\right) + \left(\tfrac{1}{2} - 0\right) + (1 - 0) \\ &= 1\tfrac{5}{6}, \quad \text{the same as in part } \mathbf{a}.\end{aligned}$$

Multiples of functions

Similarly, when a function is multiplied by a constant, the region is stretched vertically by that constant, so that:

11 INTEGRAL OF A MULTIPLE

$$\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$$

Example 11 5C

Evaluate these two expressions and show that they are equal.

a $\int_1^3 10x^3\, dx$

b $10\int_1^3 x^3\, dx$

SOLUTION

a
$$\begin{aligned}\int_1^3 10x^3\, dx &= \left[\frac{10x^4}{4}\right]_1^3 \\ &= \frac{810}{4} - \frac{10}{4} \\ &= \frac{800}{4} \\ &= 200\end{aligned}$$

b
$$\begin{aligned}10\int_1^3 x^3\, dx &= 10 \times \left[\frac{x^4}{4}\right]_1^3 \\ &= 10 \times \left(\frac{81}{4} - \frac{1}{4}\right) \\ &= 10 \times \frac{80}{4} \\ &= 200\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Inequalities with definite integrals

Suppose that a curve $y = f(x)$ is always underneath another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is less than the area under the curve $y = g(x)$.

In the language of definite integrals:

12 INEQUALITY

If $f(x) \le g(x)$ in the closed interval $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

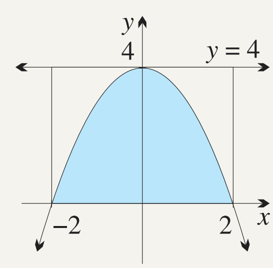
Example 12 5C

a Sketch the graph of $f(x) = 4 - x^2$, for $-2 \le x \le 2$.

b Explain why $0 \le \int_{-2}^{2} (4 - x^2)\,dx \le 16$.

SOLUTION

a The parabola and line are sketched opposite.



b Clearly $0 \le 4 - x^2 \le 4$ over the interval $-2 \le x \le 2$.
Hence the region associated with the integral is inside the square of side length 4 in the diagram opposite.

Exercise 5C FOUNDATION

Technology: All the properties of the definite integral discussed in this section have been justified visually from sketches of the graphs. Computer sketches of the graphs in this exercises would be helpful in reinforcing these explanations. The simplification of integrals of even and odd functions is particularly important and is easily demonstrated visually by graphing programs.

1 Evaluate $\int_4^5 (2x - 3)\,dx$ and $\int_5^4 (2x - 3)\,dx$. What do you notice?

2 Show, by evaluating the definite integrals, that:

a $\int_0^1 6x^2\,dx = 6\int_0^1 x^2\,dx$

b $\int_{-1}^2 (x^3 + x^2)\,dx = \int_{-1}^2 x^3\,dx + \int_{-1}^2 x^2\,dx$

c $\int_0^3 (x^2 - 4x + 3)\,dx = \int_0^2 (x^2 - 4x + 3)\,dx + \int_2^3 (x^2 - 4x + 3)\,dx$

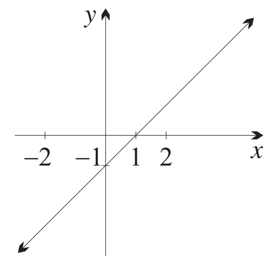
3 Without evaluating the definite integrals, explain why:

a $\int_2^2 (x^2 - 3x)\,dx = 0$

b $\int_{-2}^2 x\,dx = 0$

ISBN 978-1-108-76630-2

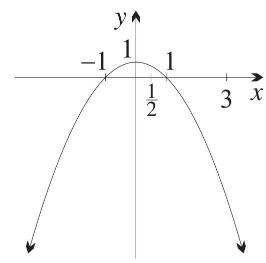
Cambridge University Press
Updated July 2022



4 The diagram to the right shows the line $y = x - 1$. Without evaluating the definite integrals, explain why:

a $\displaystyle\int_0^1 (x - 1)\,dx$ is negative,
b $\displaystyle\int_1^2 (x - 1)\,dx$ is positive,
c $\displaystyle\int_0^2 (x - 1)\,dx$ is zero,
d $\displaystyle\int_{-2}^2 (x - 1)\,dx$ is negative.

5 The diagram to the right shows the parabola $y = 1 - x^2$. Without evaluating the definite integrals, explain why:



a $\displaystyle\int_{-1}^1 (1 - x^2)\,dx$ is positive,
b $\displaystyle\int_1^3 (1 - x^2)\,dx$ is negative,
c $\displaystyle\int_{-1}^0 (1 - x^2)\,dx = \int_0^1 (1 - x^2)\,dx$,
d $\displaystyle\int_0^{\frac{1}{2}} (1 - x^2)\,dx > \int_{\frac{1}{2}}^1 (1 - x^2)\,dx$.

6 **a** If $\displaystyle\int_1^3 f(x)\,dx = 7$, then what is the value of $\displaystyle\int_3^1 f(x)\,dx$?
b If $\displaystyle\int_2^1 g(x)\,dx = -5$, then what is the value of $\displaystyle\int_1^2 g(x)\,dx$?

7 Use a diagram to explain why $\displaystyle\int_0^1 2x\,dx > \int_0^1 x\,dx$.

8 Write $\displaystyle\int_{-2}^0 x^3\,dx + \int_0^1 x^3\,dx$ as a single integral, and then use a diagram to explain why this definite integral is negative.

DEVELOPMENT

9 In each part, evaluate the definite integrals. Then use the properties of the definite integral to explain the relationships amongst the integrals within that part.

a **i** $\displaystyle\int_0^2 (3x^2 - 1)\,dx$ **ii** $\displaystyle\int_2^0 (3x^2 - 1)\,dx$
b **i** $\displaystyle\int_0^1 20x^3\,dx$ **ii** $\displaystyle 20\int_0^1 x^3\,dx$
c **i** $\displaystyle\int_1^4 (4x + 5)\,dx$ **ii** $\displaystyle\int_1^4 4x\,dx$ **iii** $\displaystyle\int_1^4 5\,dx$
d **i** $\displaystyle\int_0^2 12x^3\,dx$ **ii** $\displaystyle\int_0^1 12x^3\,dx$ **iii** $\displaystyle\int_1^2 12x^3\,dx$
e **i** $\displaystyle\int_3^3 (4 - 3x^2)\,dx$ **ii** $\displaystyle\int_{-2}^{-2} (4 - 3x^2)\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 Use the properties of the definite integral to evaluate each integral without using a primitive function. Give reasons.

a $\int_3^3 \sqrt{9 - x^2}\, dx$ **b** $\int_4^4 (x^3 - 3x^2 + 5x - 7)\, dx$ **c** $\int_{-1}^1 x^3\, dx$

d $\int_{-5}^5 (x^3 - 25x)\, dx$ **e** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx$ **f** $\int_{-2}^2 \frac{x}{1 + x^2}\, dx$

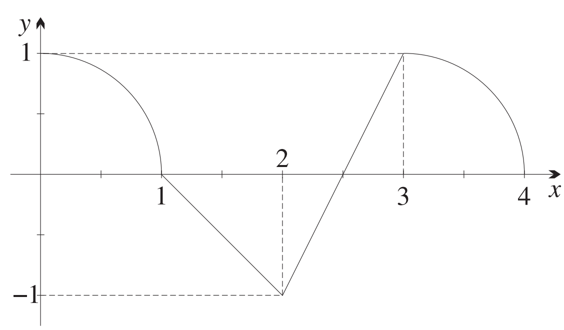
11 a On one set of axes sketch $y = x^2$ and $y = x^3$, clearly showing the points of intersection.

b Hence explain why $0 < \int_0^1 x^3\, dx < \int_0^1 x^2\, dx < 1$.

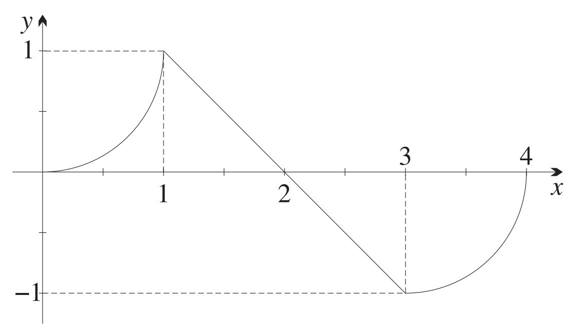
c Check the inequality in part **b** by evaluating each integral.

12 Use area formulae to find $\int_0^4 f(x)\, dx$, given the following sketches of $f(x)$.

a



b



13 a Calculate using a graph and an area formula.

i $\int_0^5 1\, dx$ **ii** $\int_0^5 x\, dx$

b Using these results, and the properties of integrals of sums and multiples, evaluate:

i $\int_0^5 2x\, dx$ **ii** $\int_0^5 (x + 1)\, dx$ **iii** $\int_0^5 (3x - 2)\, dx$

14 a Calculate these definite integrals using graphs and area formulae.

i $\int_{-\frac{1}{4}}^{\frac{1}{4}} (1 - 4x)\, dx$ **ii** $\int_{-5}^1 |x + 5|\, dx$ **iii** $\int_0^2 (|x| + 3)\, dx$

b Hence write down the value of:

i $\int_{\frac{1}{4}}^{-\frac{1}{4}} (1 - 4x)\, dx$ **ii** $\int_1^{-5} |x + 5|\, dx$ **iii** $\int_2^0 (|x| + 3)\, dx$

15 Using the properties of the definite integral, explain why:

a $\int_{-k}^k (ax^5 + cx^3 + ex)\, dx = 0$

b $\int_{-k}^k (bx^4 + dx^2 + f)\, dx = 2\int_0^k (bx^4 + dx^2 + f)\, dx$

16 Sketch a graph of each integral and hence determine whether each statement is true or false.

a $\int_{-1}^1 2^x\, dx = 0$ **b** $\int_0^2 3^x > 0$ **c** $\int_{-2}^{-1} \frac{1}{x}\, dx > 0$ **d** $\int_2^1 \frac{1}{x}\, dx > 0$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 a i Show that $\int_3^4 dx = \int_2^3 dx = \int_1^2 dx = 1$.

ii Show that $\frac{2}{7}\int_3^4 x\,dx = \frac{2}{5}\int_2^3 x\,dx = \frac{2}{3}\int_1^2 x\,dx = 1$.

iii Show that $\frac{3}{37}\int_3^4 x^2\,dx = \frac{3}{19}\int_2^3 x^2\,dx = \frac{3}{7}\int_1^2 x^2\,dx = 1$.

b Use the results above and the theorems on the definite integral to calculate:

i $\int_1^4 dx$ **ii** $\int_1^3 x\,dx$ **iii** $\int_2^1 x^2\,dx$

iv $\int_1^2 (x^2 + 1)\,dx$ **v** $\int_1^3 7x^2\,dx$ **vi** $\int_1^4 (3x^2 - 6x + 5)\,dx$

ENRICHMENT

18 State with reasons whether each statement is true or false.

a $\int_{-90}^{90} \sin^3 x°\,dx = 0$ **b** $\int_{-30}^{30} \sin\ 4x°\cos\ 2x°\,dx = 0$

c $\int_{-1}^{1} 2^{-x^2}\,dx = 0$ **d** $\int_0^1 2^x\,dx < \int_0^1 3^x\,dx$

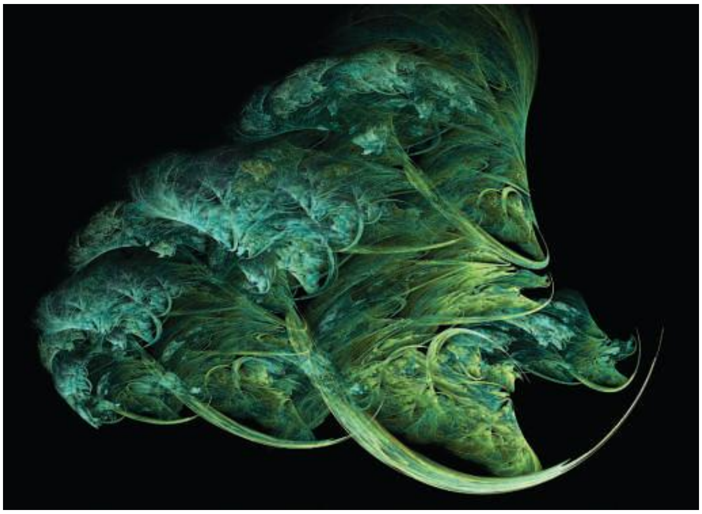
e $\int_{-1}^{0} 2^x dx < \int_{-1}^{0} 3^x\,dx$

f $\int_0^1 \frac{dt}{1 + t^n} \le \int_0^1 \frac{dt}{1 + t^{n+1}}$, where $n = 1, 2, 3, \ldots$.

19 Evaluate each definite integral, then establish the result that follows.
Provide a sketch of each situation.

a $\int_1^N \frac{1}{x^2}\,dx$ converges to 1 as $N \to \infty$. **b** $\int_\varepsilon^1 \frac{1}{x^2}\,dx$ diverges to ∞ as $\varepsilon \to 0^+$.

c $\int_1^N \frac{1}{\sqrt{x}}\,dx$ diverges to ∞ as $N \to \infty$. **d** $\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$ converges to 2 as $\varepsilon \to 0^+$.



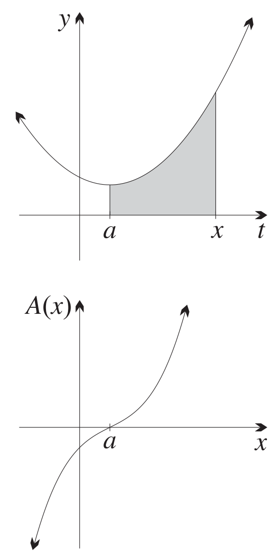
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5D Proving the fundamental theorem

This section develops a proof of the fundamental theorem of calculus, as stated and used in Sections 5B and 5C. This would be the time to review the account of first-principles integration in Section 5A, because the approach used there of trap-ping an integral between upper and lower rectangles is used again in this proof.

The definite integral as a function of its upper limit



The value of a definite integral $\int_a^b f(x)\,dx$ changes when the value of b changes. This means that it is a function of its upper limit b. When we want to emphasise the functional relationship with the upper limit, we usually replace the letter b by the letter x, which is conventionally the variable of a function.

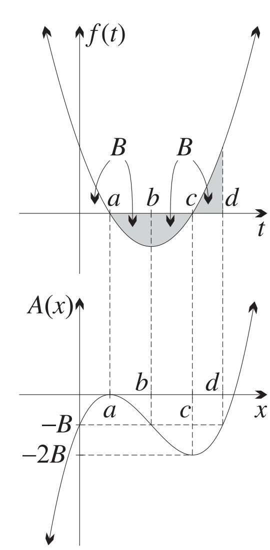
In turn, the original letter x needs to be replaced by some other letter, usually t. Then the definite integral is clearly represented as a function of its upper limit x, as in the first diagram above. This function is called the *signed area function* for $f(x)$ starting at $x = a$, and is defined by

$$A(x) = \int_a^x f(t)\,dt$$

The function $A(x)$ is always zero at $x = a$. For the function sketched above, $A(x)$ increases at an increasing rate — see the second sketch. $A(x)$ is also defined to the left of a, where it is negative because the integrals run backwards.

The signed area function

The function in the sketch above was never negative. But the definite integral is the *signed area* under the curve, meaning that a negative sign is attached to areas of regions below the x-axis (provided that the integral is not running backwards).



The upper graph of $f(t)$ to the right is a parabola with axis of symmetry $x = b$. The four areas marked B are all equal. Here are some properties of the signed area function

$$A(x) = \int_a^x f(t)\,dt.$$

- $A(a) = 0$, as always.
- In the interval $[a, b]$, $f(t)$ is negative and decreasing, so $A(x)$ decreases at an increasing rate.
- In the interval $[b, c]$, $f(t)$ is negative and increasing, so $A(x)$ decreases at a decreasing rate.
- In the interval $[c, \infty)$, $f(t)$ is positive and increasing, so $A(x)$ increases at an increasing rate.
- $A(b) = -B$ and $A(c) = -2B$ and $A(d) = -B$.
- $A(x)$ is also defined in the interval $(-\infty, a]$ to the left of a, where it is negative because the integrals run backwards and the curve is above the x-axis.

The lower diagram above is a sketch of the *signed area function* $A(x)$ of $f(x)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 THE SIGNED AREA FUNCTION

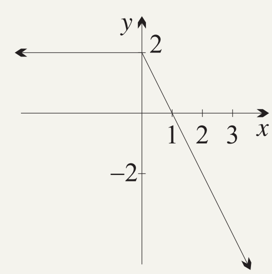
Suppose that $f(x)$ is a function defined in some interval I containing a. The *signed area function* for $f(x)$ starting at a is the function defined by the definite integral

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for all } x \text{ in the interval } I.$$

In Chapter 16 on continuous probability distributions, the cumulative distribution function of a continuous distribution will be defined in this way.

Example 13 — 5D

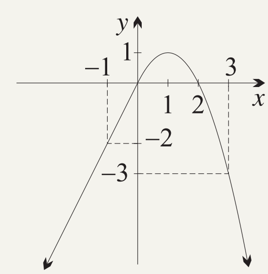
Let $A(x) = \int_0^x f(t)\,dt$ be the signed area function starting at $t = 0$ for the graph sketched to the right. Use area formulae to draw up a table of values for $y = A(x)$ in the interval $[-3, 3]$, then sketch $y = A(x)$.



SOLUTION

Use triangles for $x > 0$ and rectangles for $x < 0$.

x	-3	-2	-1	0	1	2	3
$A(x)$	-6	-4	-2	0	1	0	-3



For $x = -2$ and $x = -1$, $A(x)$ is negative because the integrals run backwards and the curve is above the x-axis. The area function $A(x)$ is increasing for $t < 1$ because $y > 0$, and is decreasing for $x > 1$ because $y < 0$.

The fundamental theorem — differential form

We can now state and prove the *differential form* of the fundamental theorem of calculus, from which we will derive the *integral form* used already in Sections 5B and 5C.

Theorem: If $f(x)$ is continuous, then the signed area function for $f(x)$ is a primitive of $f(x)$. That is,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

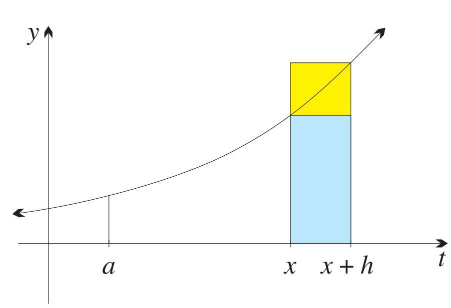
Proof: Because the theorem is so fundamental, its proof must begin with the definition of the derivative as a limit,

$$A'(x) = \lim_{h \to 0}\frac{A(x+h) - A(x)}{h}.$$

Subtracting areas in the diagram to the right,

$$A(x+h) - A(x) = \int_x^{x+h} f(t)\,dt,$$

so $$A'(x) = \lim_{h \to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt.$$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

This limit is handled by means of a clever sandwiching technique.
Suppose that $f(t)$ is increasing in the interval $[x, x+h]$, as in the diagram above.
Then the lower rectangle on the interval $[x, x+h]$ has height $f(x)$,
and the upper rectangle on the interval $[x, x+h]$ has height $f(x+h)$,

so using areas, $h \times f(x) \le \int_x^{x+h} f(t)\,dt \le h \times f(x+h)$

$\boxed{\div h}$ $$f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x+h). \qquad (1)$$

Thus the middle expression is 'sandwiched' between $f(x)$ and $f(x+h)$.

Because $f(x)$ is continuous, $f(x+h) \to f(x)$ as $h \to 0$,

so by (1), $\lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$, meaning that $A'(x) = f(x)$, as required.

If $f(x)$ is decreasing in the interval $[x, x+h]$, the same argument applies, but with the inequalities reversed.

Note: This theorem shows that the signed area function $A(x) = \int_a^x f(t)\,dt$ is a primitive of $f(x)$. It is therefore often written as $F(x)$ rather than $A(x)$.

Example 14 5D

Use the differential form of the fundamental theorem to simplify these expressions. Do not try to evaluate the integral and then differentiate it.

a $\frac{d}{dx}\int_0^x (t^2+1)^3\,dt$ **b** $\frac{d}{dx}\int_4^x (\log_e t)\,dt$ **c** $\frac{d}{dx}\int_{-3}^x e^{-t^2}\,dt$

SOLUTION

The differential form says that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Hence:

a $\frac{d}{dx}\int_0^x (t^2+1)^3\,dt = (x^2+1)^3$

b $\frac{d}{dx}\int_4^x (\log_e t)\,dt = \log_e x$

c $\frac{d}{dx}\int_{-3}^x e^{-t^2}\,dt = e^{-x^2}$

The fundamental theorem — integral form

The integral form of the fundamental theorem is the familiar form that we have been using in Sections 4B and 4C.

Theorem: Suppose that $f(x)$ is continuous in the closed interval $[a, b]$, and that $F(x)$ is a primitive of $f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Proof: We now know that $F(x)$ and $\int_a^x f(t)\,dt$ are both primitives of $f(x)$.

Because any two primitives differ only by a constant,

$$\int_a^x f(t)\,dt = F(x) + C, \quad \text{for some constant } C.$$

Substituting $x = a$, $\int_a^a f(t)\,dt = F(a) + C$,

but $\int_a^a f(t)\,dt = 0$, because the area in this definite integral has zero width,

so
$$\begin{aligned} 0 &= F(a) + C \\ C &= -F(a) \end{aligned}$$

Thus
$$\int_a^x f(t)\,dt = F(x) - F(a).$$

and changing letters from x to b and from t to x gives

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{as required.}$$

Example 15 5D

Use the integral form of the fundamental theorem to evaluate each integral. Then differentiate your result, thus confirming the consistency of the discussion above.

a $\dfrac{d}{dx}\displaystyle\int_1^x 6t^2\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_{-2}^x (t^3 - 9t^2 + 5)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_4^x \frac{1}{t^2}\,dt$

SOLUTION

a
$$\begin{aligned} \int_1^x 6t^2\,dt &= \Big[2t^3\Big]_1^x \\ &= 2x^3 - 2, \end{aligned}$$

so
$$\begin{aligned} \frac{d}{dx}\int_1^x 6t^2\,dt &= \frac{d}{dx}(2x^3 - 2) \\ &= 6x^2, \quad \text{consistent with the differential form.} \end{aligned}$$

b
$$\begin{aligned} \int_{-2}^x (t^3 - 9t^2 + 5)\,dt &= \Big[\tfrac{1}{4}t^4 - 3t^3 + 5t\Big]_{-2}^x \\ &= \left(\tfrac{1}{4}x^4 - 3x^3 + 5x\right) - (4 + 24 - 10) \\ &= \tfrac{1}{4}x^4 - 3x^3 + 5x - 18, \end{aligned}$$

so
$$\begin{aligned} \frac{d}{dx}\int_{-2}^x (t^3 - 9t^2 + 5)\,dt &= \frac{d}{dx}\left(\tfrac{1}{4}x^4 - 3x^3 + 5x - 18\right) \\ &= x^3 - 9x^2 + 5, \quad \text{consistent with the differential form.} \end{aligned}$$

c
$$\begin{aligned} \int_4^x \frac{1}{t^2}\,dt &= \int_4^x t^{-2}\,dt \\ &= \Big[-t^{-1}\Big]_4^x \\ &= -x^{-1} + \tfrac{1}{4}, \end{aligned}$$

so
$$\begin{aligned} \frac{d}{dx}\int_4^x \frac{1}{t^2}\,dt &= \frac{d}{dx}\left(-x^{-1} + \tfrac{1}{4}\right) \\ &= x^{-2}, \quad \text{consistent with the differential form.} \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

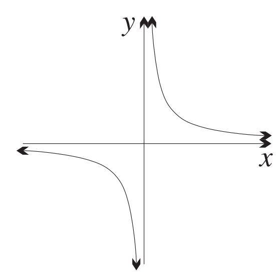
Super challenge — continuous functions

In Section 9K of the Year 11 volume, we defined continuity at a point — you can draw the curve through the point without lifting the pencil off the paper.

Then throughout this chapter, we have been using the phrase, 'continuous in the closed interval $[a, b]$'. This idea is also straightforward, and informally means that you can place the pencil on the left-hand endpoint $\big(a, f(a)\big)$ and draw the curve to the right-hand endpoint $\big(b, f(b)\big)$ without lifting the pencil off the paper.

There is, however, a global notion of a *continuous function*:

A continuous function is a function continuous at every value in its domain.



This may look obvious, like so many definitions in mathematics, but it is not.

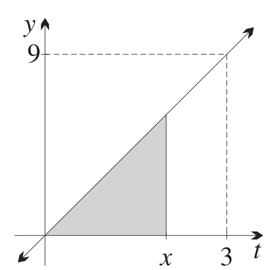
It means, for example that the reciprocal function $y = \dfrac{1}{x}$ is a continuous function. This is because $x = 0$ is not in its domain, so the function is continuous at every value of x in its domain.

Thus $y = \dfrac{1}{x}$ is a 'continuous function with a discontinuity'.

(Or perhaps the word 'discontinuity' is redefined). We recommend avoiding the concept completely unless some question specifically requires it.

Exercise 5D

FOUNDATION



1 The graph to the right shows $y = 3t$, for $0 \le t \le 3$.

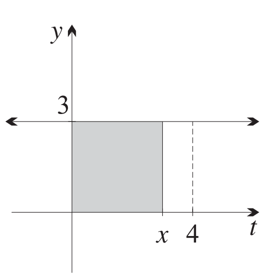
a Use the triangle area formula to find the signed area function

$$A(x) = \int_0^x 3t\,dt, \text{ for } 0 \le x \le 3.$$

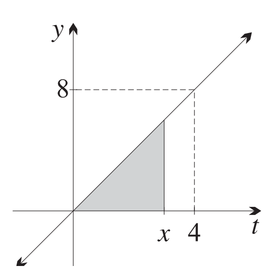
b Differentiate $A(x)$ to show that $A'(x)$ is the original function, apart from a change of letter.

2 Write down the equation of each function, then use area formulae, not integration, to calculate the signed area function $A(x) = \displaystyle\int_0^x f(t)\,dt$, for $0 \le t \le 4$. Then differentiate $A(x)$ to confirm that $A'(x)$ is the original function, apart from a change of letter.

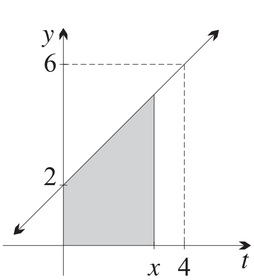
a



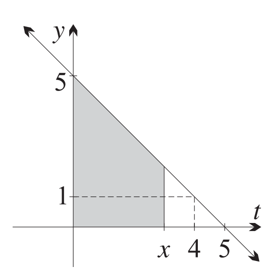
b



c



d



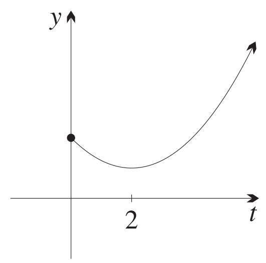
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

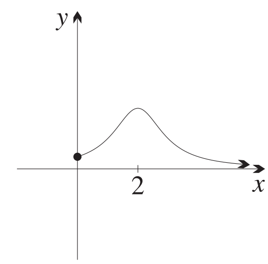
DEVELOPMENT

3 For each function sketched below, describe the behaviour of the signed area function $A(x) = \int_0^x f(t)\,dt$, for all $x \ge 0$, in the interval $[0, 2]$ and in the interval $[2, \infty)$. Then draw a freehand sketch of $y = A(x)$.

a



b



4 The differential form $\dfrac{d}{dx}\int_a^x f(t)\,dt$ of the fundamental theorem tells us that the derivative of the integral is the original function, with a change of letter. Use this to simplify these expressions. Do not attempt to find primitives.

a $\dfrac{d}{dx}\displaystyle\int_1^x \frac{1}{t}\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^3}\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_0^x e^{-\frac{1}{2}t^2}\,dt$

5 Use the differential form of the fundamental theorem to simplify these expressions. Then confirm the consistency of the discussion in this section by performing the integration and then differentiating.

a $\dfrac{d}{dx}\displaystyle\int_1^x (3t^2 - 12)\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + 4t)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_2^x \frac{1}{t^2}\,dt$

6 **a** Use the differential form of the fundamental theorem $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ to find:

i $\dfrac{d}{dx}\displaystyle\int_1^x t^2\,dt$ **ii** $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + 3t)\,dt$

iii $\dfrac{d}{dx}\displaystyle\int_a^x \frac{1}{t}\,dt$ **iv** $\dfrac{d}{dx}\displaystyle\int_a^x (t^3 - 3)^4\,dt$

b Confirm your answers to parts (i) and (ii) above by evaluating the definite integral and then differentiating with respect to x.

7 Do not use primitives in this question. The intention in each part is to sketch the signed area function $A(x)$ freehand from the sketch of the integrand, then describe its behaviour. Pay attention to the restricted domain of x.

a Sketch $y = e^t$, then sketch the signed area function $A(x) = \int_0^x e^t\,dt$ for $x \ge 0$. How would you describe the behaviour of $y = A(x)$?

b Sketch $y = \log_e t$, then sketch the signed area function $A(x) = \int_1^x \log_e t\,dt$ for $x \ge 1$. How would you describe the behaviour of $y = A(x)$?

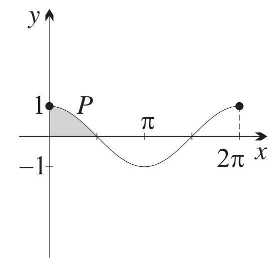
c Sketch $y = \dfrac{1}{t}$, then sketch the signed area function $A(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$ for $x \ge 1$. How would you describe the behaviour of $y = A(x)$?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

8 **a** Sketched to the right is $y = \cos x$, for $0 \le x \le 2\pi$. Copy and complete the table of values for the signed area function $A(x) = \int_0^x \cos t\, dt$, for $0 \le x \le 2\pi$, given that the region marked P has area exactly 1 (this is proven in Chapter 7). Then sketch $y = A(x)$.



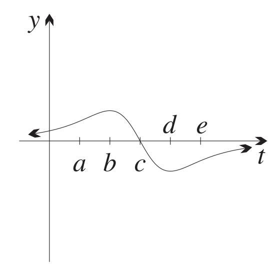
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$A(x)$					

What is your guess for the equation of $A(x)$, and what does this suggest the derivative of $\sin x$ is?

b Sketch $y = \sin x$, for $0 \le x \le 2\pi$, and repeat the procedures in part **a**.

9 The function $y = f(t)$ sketched to the right has point symmetry in $(c, 0)$.

Let $A(x) = \int_a^x f(t)\, dt$.



a Where is $A(x)$ increasing, and when it is decreasing?

b Where does $A(x)$ have a maximum turning point, and where does $A(x)$ have a minimum turning point?

c Where does $A(x)$ have inflections?

d Where are the zeroes of $A(x)$?

e Where is $A(x)$ positive, and where it is negative?

f Sketch $y = A(x)$.

10 This 'super challenge' question may illuminate the definition of a continuous function:

A continuous function is a function that is continuous at every number in its domain.

Classify these functions as continuous or not continuous according to the definition above.

a $y = x - 2$

b $y = \dfrac{1}{x - 2}$

c $y = \begin{cases} \dfrac{1}{x-2}, & \text{for } x \ne 2, \\ 0, & \text{for } x = 2. \end{cases}$

d $y = \sqrt{x}$

e $y = \dfrac{1}{\sqrt{x}}$

f $y = \begin{cases} \dfrac{1}{\sqrt{x}}, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5E The indefinite integral

Now that primitives have been established as the key to calculating definite integrals, this section turns again to the task of finding primitives. First, a new and convenient notation for the primitive is introduced.

The indefinite integral

Because of the close connection established by the fundamental theorem between primitives and definite integrals, the term *indefinite integral* is often used for the general primitive. The usual notation for the indefinite integral of a function $f(x)$ is an integral sign without any upper or lower limits. For example, *the primitive* or *the indefinite integral* of $x^2 + 1$ is

$$\int (x^2 + 1)\,dx = \frac{x^3}{3} + x + C, \quad \text{for some constant } C.$$

The word 'indefinite' suggests that the integral cannot be evaluated further because no limits for the integral have yet been specified.

The constant of integration

A definite integral ends up as a pure number. An indefinite integral, on the other hand, is a function of x — the pronumeral x is carried across to the answer.

It also contains an unknown constant C (or c, as it is often written) and the indefinite integral can also be regarded as a function of C (or of c). The constant is called a 'constant of integration' and is an important part of the answer — it must always be included.

The only exception to including the constant of integration is when calculating definite integrals, because in that situation any primitive can be used.

Note: Strictly speaking, the words 'for some constant C' or 'where C is a constant' should follow the first mention of the new pronumeral C, because no pronumeral should be used without having been formally introduced. There is a limit to one's patience, however (in this book there is often no room), and usually it is quite clear that C is the constant of integration. If another pronumeral such as D is used, it would be wise to introduce it formally.

Standard forms for integration

The two rules for finding primitives given in the last section of Chapter 4 can now be restated in this new notation.

14 STANDARD FORMS FOR INTEGRATION

Suppose that $n \neq -1$. Then

- $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for some constant } C.$$
- $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad \text{for some constant } C.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The word ‘integration’ is commonly used to refer both to the finding of a primitive, and to the evaluating of a definite integral. Similarly, the unqualified term ‘integral’ is used to refer both to the indefinite integral and to the definite integral.

Example 16 5E

Use the standard form $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ to find:

a $\int 9\,dx$

b $\int 12x^3\,dx$

SOLUTION

a $\int 9\,dx = 9x + C$, for some constant C

Note: We know that $9x$ is the primitive of 9, because $\dfrac{d}{dx}(9x) = 9$.

But the formula still gives the correct answer, because $9 = 9x^0$,

so increasing the index to 1 and dividing by this new index 1,

$$\begin{aligned}\int 9x^0\,dx &= \frac{9x^1}{1} + C, \quad \text{for some constant } C\\ &= 9x + C.\end{aligned}$$

b $$\begin{aligned}\int 12x^3\,dx &= 12 \times \frac{x^4}{4} + C, \quad \text{for some constant } C\\ &= 3x^4 + C\end{aligned}$$

Example 17 5E

Use the standard form $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$ to find:

a $\int (3x+1)^5\,dx$

b $\int (5-2x)^2\,dx$

SOLUTION

a $$\begin{aligned}\int (3x+1)^5\,dx &= \frac{(3x+1)^6}{3 \times 6} + C \qquad (\text{here } n = 5 \text{ and } a = 3 \text{ and } b = 1)\\ &= \tfrac{1}{18}(3x+1)^6 + C, \quad \text{for some content } C\end{aligned}$$

b $$\begin{aligned}\int (5-2x)^2\,dx &= \frac{(5-2x)^3}{(-2) \times 3} + C \qquad (\text{here } n = 2 \text{ and } a = -2 \text{ and } b = 5)\\ &= -\tfrac{1}{6}(5-2x)^3 + C, \quad \text{for some content } C\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Negative indices

Both standard forms apply with negative indices as well as positive indices, as in the next worked example. The exception is the index -1, where the rule is nonsense because it results in division by zero. We shall deal with the integration of x^{-1} in Chapter 6.

Example 18 5E

Use negative indices to find these indefinite integrals.

a $\displaystyle\int \frac{12}{x^3}\,dx$

b $\displaystyle\int \frac{dx}{(3x+4)^2}$

SOLUTION

a
$$\begin{aligned}\int \frac{12}{x^3}\,dx &= \int 12x^{-3}\,dx\\ &= 12 \times \frac{x^{-2}}{-2} + C\\ &= -\frac{6}{x^2} + C\end{aligned}$$

b
$$\begin{aligned}\int \frac{dx}{(3x+4)^2} &= \int (3x+4)^{-2}\,dx\\ &= \frac{(3x+4)^{-1}}{3 \times (-1)} + C\\ &= -\frac{1}{3(3x+4)} + C\end{aligned}$$

Special expansions

In many integrals, brackets must be expanded before the indefinite integral can be found. The next worked example uses the special expansions. Part **b** also requires negative indices.

Example 19 5E

Find these indefinite integrals.

a $\displaystyle\int (x^3-1)^2\,dx$

b $\displaystyle\int \left(3-\frac{1}{x^2}\right)\left(3+\frac{1}{x^2}\right)dx$

SOLUTION

a
$$\begin{aligned}\int (x^3-1)^2\,dx &= \int (x^6 - 2x^3 + 1)\,dx \qquad \text{(using } (A+B)^2 = A^2 + 2AB + B^2\text{)}\\ &= \frac{x^7}{7} - \frac{x^4}{2} + x + C\end{aligned}$$

b
$$\begin{aligned}\int \left(3-\frac{1}{x^2}\right)\left(3+\frac{1}{x^2}\right)dx &= \int \left(9 - \frac{1}{x^4}\right)dx \qquad \text{(using } (A-B)(A+B) = A^2 - B^2\text{)}\\ &= \int (9 - x^{-4})\,dx\\ &= 9x - \frac{x^{-3}}{-3} + C\\ &= 9x + \frac{1}{3x^3} + C\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Fractional indices

The standard forms for finding primitives of powers also apply to fractional indices. These calculations require quick conversions between fractional indices and surds. The next worked example finds each indefinite integral and then uses it to evaluate a definite integral.

Example 20 5E

Use fractional and negative indices to evaluate:

a $\displaystyle\int_1^4 \sqrt{x}\,dx$

b $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

SOLUTION

a Increase the index to $\frac{3}{2}$, and divide by $\frac{3}{2}$.

$$\begin{aligned}\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{\frac{1}{2}}\,dx\\ &= \tfrac{2}{3} - \Big[x^{\frac{3}{2}}\Big]_1^4\\ &= \tfrac{2}{3} \times (8 - 1)\\ &= 4\tfrac{2}{3}\end{aligned}$$

b Increase the index to $\frac{1}{2}$, and divide by $\frac{1}{2}$.

$$\begin{aligned}\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{-\frac{1}{2}}\,dx\\ &= \tfrac{2}{1} - \Big[x^{\frac{1}{2}}\Big]_1^4\\ &= 2 \times (2 - 1)\\ &= 2\end{aligned}$$

Example 21 5E

a Use index notation to express $\dfrac{1}{\sqrt{9 - 2x}}$ as a power of $9 - 2x$.

b Hence find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{9 - 2x}}$.

SOLUTION

a $\dfrac{1}{\sqrt{9 - 2x}} = (9 - 2x)^{-\frac{1}{2}}$.

b Hence $\displaystyle\int \frac{1}{\sqrt{9 - 2x}} = \int (9 - 2x)^{-\frac{1}{2}}\,dx$

$$\begin{aligned}&= \frac{(9 - 2x)^{\frac{1}{2}}}{-2 \times \frac{1}{2}} + C, \quad \text{using} \int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)},\\ &= -\sqrt{9 - 2x} + C.\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 5E

FOUNDATION

Technology: Many programs that can perform algebraic manipulation are also able to deal with indefinite integrals. They can be used to check the questions in this exercise and to investigate the patterns arising in such calculations.

1 Find these indefinite integrals.

a $\int 4\,dx$ **b** $\int 1\,dx$ **c** $\int 0\,dx$ **d** $\int (-2)\,dx$

e $\int x\,dx$ **f** $\int x^2\,dx$ **g** $\int x^3\,dx$ **h** $\int x^7\,dx$

2 Find the indefinite integral of each function. Use the notation of the previous question.

a $2x$ **b** $4x$ **c** $3x^2$ **d** $4x^3$

e $10x^9$ **f** $2x^3$ **g** $4x^5$ **h** $3x^8$

3 Find these indefinite integrals.

a $\int (x + x^2)\,dx$ **b** $\int (x^4 - x^3)\,dx$ **c** $\int (x^7 + x^{10})\,dx$

d $\int (2x + 5x^4)\,dx$ **e** $\int (9x^8 - 11)\,dx$ **f** $\int (7x^{13} + 3x^8)\,dx$

g $\int (4 - 3x)\,dx$ **h** $\int (1 - x^2 + x^4)\,dx$ **i** $\int (3x^2 - 8x^3 + 7x^4)\,dx$

4 Find the indefinite integral of each function. (Leave negative indices in your answers.)

a x^{-2} **b** x^{-3} **c** x^{-8}

d $3x^{-4}$ **e** $9x^{-10}$ **f** $10x^{-6}$

5 Find these indefinite integrals. (Leave fractional indices in your answers.)

a $\int x^{\frac{1}{2}}\,dx$ **b** $\int x^{\frac{1}{3}}\,dx$ **c** $\int x^{\frac{1}{4}}\,dx$

d $\int x^{\frac{2}{3}}\,dx$ **e** $\int x^{-\frac{1}{2}}\,dx$ **f** $\int 4x^{\frac{1}{2}}\,dx$

6 By first expanding the brackets, find these indefinite integrals.

a $\int x(x + 2)\,dx$ **b** $\int x(4 - x^2)\,dx$ **c** $\int x^2(5 - 3x)\,dx$

d $\int x^3(x - 5)\,dx$ **e** $\int (x - 3)^2\,dx$ **f** $\int (2x + 1)^2\,dx$

g $\int (1 - x^2)^2\,dx$ **h** $\int (2 - 3x)(2 + 3x)\,dx$ **i** $\int (x^2 - 3)(1 - 2x)\,dx$

7 Write each integrand as separate fractions, then perform the integration.

a $\int \dfrac{x^2 + 2x}{x}\,dx$ **b** $\int \dfrac{x^7 + x^8}{x^6}\,dx$ **c** $\int \dfrac{2x^3 - x^4}{4x}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

8 Write these functions with negative indices and hence find their indefinite integrals.

a $\frac{1}{x^2}$ **b** $\frac{1}{x^3}$ **c** $\frac{1}{x^5}$ **d** $\frac{1}{x^{10}}$

e $\frac{3}{x^4}$ **f** $\frac{5}{x^6}$ **g** $\frac{7}{x^8}$ **h** $\frac{1}{3x^2}$

i $\frac{1}{7x^5}$ **j** $-\frac{1}{5x^3}$ **k** $\frac{1}{x^2} - \frac{1}{x^5}$ **l** $\frac{1}{x^3} + \frac{1}{x^4}$

9 Write these functions with fractional indices and hence find their indefinite integrals.

a $\sqrt{x}$ **b** $\sqrt[3]{x}$ **c** $\frac{1}{\sqrt{x}}$ **d** $\sqrt[3]{x^2}$

10 Use the indefinite integrals of the previous question to evaluate:

a $\int_0^9 \sqrt{x}\, dx$ **b** $\int_0^8 \sqrt[3]{x}\, dx$ **c** $\int_{25}^{49} \frac{1}{\sqrt{x}}\, dx$ **d** $\int_0^1 \sqrt[3]{x^2}\, dx$

11 By using the rule $\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C$, find:

a $\int (x + 1)^5\, dx$ **b** $\int (x + 2)^3\, dx$ **c** $\int (4 - x)^4\, dx$

d $\int (3 - x)^2\, dx$ **e** $\int (3x + 1)^4\, dx$ **f** $\int (4x - 3)^7\, dx$

g $\int (5 - 2x)^6\, dx$ **h** $\int (1 - 5x)^7\, dx$ **i** $\int (2x + 9)^{11}\, dx$

j $\int 3(2x - 1)^{10}\, dx$ **k** $\int 4(5x - 4)^6\, dx$ **l** $\int 7(3 - 2x)^3\, dx$

12 Find:

a $\int \left(\frac{1}{3}x - 7\right)^4 dx$ **b** $\int \left(\frac{1}{4}x - 7\right)^6 dx$ **c** $\int \left(1 - \frac{1}{5}x\right)^3 dx$

13 Find:

a $\int \frac{1}{(x + 1)^3}\, dx$ **b** $\int \frac{1}{(x - 5)^4}\, dx$ **c** $\int \frac{1}{(3x - 4)^2}\, dx$

d $\int \frac{1}{(2 - x)^5}\, dx$ **e** $\int \frac{3}{(x - 7)^6}\, dx$ **f** $\int \frac{8}{(4x + 1)^5}\, dx$

g $\int \frac{2}{(3 - 5x)^4}\, dx$ **h** $\int \frac{4}{5(1 - 4x)^2}\, dx$ **i** $\int \frac{7}{8(3x + 2)^5}\, dx$

14 By expanding the brackets, find:

a $\int \sqrt{x}(3\sqrt{x} - x)\, dx$ **b** $\int (\sqrt{x} - 2)(\sqrt{x} + 2)\, dx$ **c** $\int (2\sqrt{x} - 1)^2\, dx$

15 **a** Evaluate these definite integrals.

i $\int_0^1 x^{\frac{1}{2}}\, dx$ **ii** $\int_1^4 x^{-\frac{1}{2}}\, dx$ **iii** $\int_0^8 x^{\frac{1}{3}}\, dx$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

b By writing them with fractional indices, evaluate these definite integrals.

i $\int_0^4 \sqrt{x}\,dx$ **ii** $\int_1^9 x\sqrt{x}\,dx$ **iii** $\int_1^9 \frac{dx}{\sqrt{x}}$

16 Expand the brackets and hence find:

a $\int_2^4 (2 - \sqrt{x})(2 + \sqrt{x})\,dx$ **b** $\int_0^1 \sqrt{x}\left(\sqrt{x} - 4\right)dx$ **c** $\int_4^9 \left(\sqrt{x} - 1\right)^2 dx$

17 Explain why the indefinite integral $\int \frac{1}{x}\,dx$ cannot be found in the usual way using the standard form $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.

18 Find each indefinite integral.

a $\int \sqrt{2x - 1}\,dx$ **b** $\int \sqrt{7 - 4x}\,dx$

c $\int \sqrt[3]{4x - 1}\,dx$ **d** $\int \frac{1}{\sqrt{3x + 5}}\,dx$

19 Evaluate these definite integrals.

a $\int_0^2 (x + 1)^4\,dx$ **b** $\int_2^3 (2x - 5)^3\,dx$ **c** $\int_{-2}^2 (1 - x)^5\,dx$

d $\int_0^5 \left(1 - \frac{x}{5}\right)^4 dx$ **e** $\int_0^1 \sqrt{9 - 8x}\,dx$ **f** $\int_2^7 \frac{1}{\sqrt{x + 2}}\,dx$

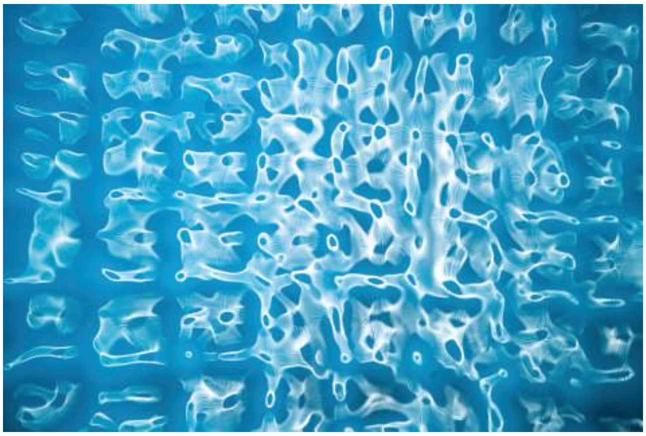
g $\int_{-2}^0 \sqrt[3]{x + 1}\,dx$ **h** $\int_1^5 \sqrt{3x + 1}\,dx$ **i** $\int_{-3}^0 \sqrt{1 - 5x}\,dx$

ENRICHMENT

20 **a** If u and v are differentiable functions of x, prove that $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

b Hence find:

i $\int x(x - 1)^4\,dx$ **ii** $\int x\sqrt{1 + x}\,dx$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5F Finding areas by integration

The aim of this section and the next is to use definite integrals to find the areas of regions bounded by curves, lines and the coordinate axes.

Sections 5F–5G ignore integrals that run backwards. Running an integral backwards reverses its sign, which would confuse the discussion of areas in these sections. When finding areas, we decide what integrals to create, and we naturally avoid integrals that run backwards.

Areas and definite integrals

Areas and definite integrals are closely related, but they are not the same thing.

- An area is always positive, whereas a definite integral may be positive or negative, depending on whether the curve is above or below the x-axis.

Problems on areas require care when finding the required integral or combination of integrals. Some particular techniques are listed below, but the general rule is to draw a diagram first to see which pieces need to be added or subtracted.

15 FINDING AN AREA

When using integrals to find the area of a region:

1. Draw a sketch of the curves, showing relevant intercepts and intersections.
2. Create and evaluate the necessary definite integral or integrals.
3. Write a conclusion, giving the required area in square units.

Regions above the *x*-axis

When a curve lies entirely above the x-axis, the relevant integral will be positive, and the area of the region will be equal to the integral, apart from needing units.

Example 22 **5F**

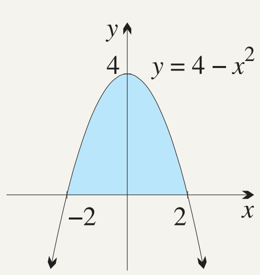
Find the area of the region bounded by the curve $y = 4 - x^2$ and the x-axis. (This was the example sketched in the introduction to this chapter.)

SOLUTION

The curve meets the x-axis at $(2, 0)$ and $(-2, 0)$.

The region lies entirely above the x-axis and the relevant integral is

$$\begin{aligned}\int_{-2}^{2}(4 - x^2)\,dx &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) \\ &= 5\tfrac{1}{3} - \left(-5\tfrac{1}{3}\right) \\ &= 10\tfrac{2}{3},\end{aligned}$$



which is positive because the curve lies entirely above the x-axis.

Hence the required area is $10\frac{2}{3}$ square units.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Regions below the x-axis

When a curve lies entirely below the x-axis, the relevant integral will be negative, and the area will be the opposite of this.

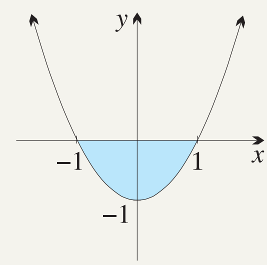
Example 23 5F

Find the area of the region bounded by the curve $y = x^2 - 1$ and the x-axis.

SOLUTION

The curve meets the x-axis at $(1, 0)$ and $(-1, 0)$.
The region lies entirely below the x-axis, and the relevant integral is

$$\begin{aligned}\int_{-1}^{1} (x^2 - 1)\,dx &= \left[\frac{x^3}{3} - x\right]_{-1}^{1} \\ &= \left(\tfrac{1}{3} - 1\right) - \left(-\tfrac{1}{3} + 1\right) \\ &= -\tfrac{2}{3} - \tfrac{2}{3} \\ &= -1\tfrac{1}{3},\end{aligned}$$



which is negative, because the curve lies entirely below the x-axis.
Hence the required area is $1\tfrac{1}{3}$ square units.

Curves that cross the x-axis

When a curve crosses the x-axis, the area of the region between the curve and the x-axis cannot usually be found by means of a single integral. This is because integrals representing regions below the x-axis have negative values.

Example 24 5F

a Sketch the cubic curve $y = x(x + 1)(x - 2)$, showing the x-intercepts.

b Shade the region enclosed between the x-axis and the curve, and find its area.

c Find $\displaystyle\int_{-1}^{2} x(x + 1)(x - 2)\,dx$ and explain why this integral does not represent the area of the region described in part **b**.

SOLUTION

a The curve has x-intercepts $x = -1$, $x = 0$ and $x = 2$, and is graphed below.

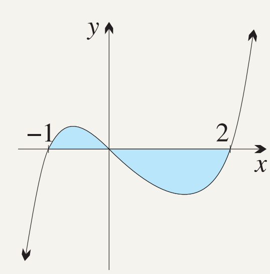
b Expanding the cubic,

$$\begin{aligned} y &= x(x + 1)(x - 2) \\ &= x(x^2 - x - 2) \\ &= x^3 - x^2 - 2x.\end{aligned}$$

For the region above the x-axis,

$$\begin{aligned}\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right) \\ &= \tfrac{5}{12},\end{aligned}$$

so area above $= \tfrac{5}{12}$ square units.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

For the region below the x-axis,

$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2\\ &= \left(4 - 2\tfrac{2}{3} - 4\right) - (0 - 0 - 0)\\ &= -2\tfrac{2}{3},\end{aligned}$$

so $\qquad$ area below $= 2\frac{2}{3}$ square units.

Adding these, $\qquad$ total area $= \frac{5}{12} + 2\frac{2}{3}$

$\qquad\qquad\qquad\qquad\qquad = 3\frac{1}{12}$ square units.

c

$$\begin{aligned}\int_{-1}^2 x(x+1)(x-2)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^2\\ &= \left(4 - 2\tfrac{2}{3} - 4\right) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right)\\ &= -2\tfrac{2}{3} + \tfrac{5}{12}\\ &= -2\tfrac{1}{4}.\end{aligned}$$

This integral represents the difference $2\frac{2}{3} - \frac{5}{12} = 2\frac{1}{4}$ of the two areas, and is negative because the area below is larger than the area above.

Areas associated with odd and even functions

As always in mathematics, calculations are often much easier if symmetries can be recognised.

Example 25 5F

a Show that $y = x^3 - x$ is an odd function.

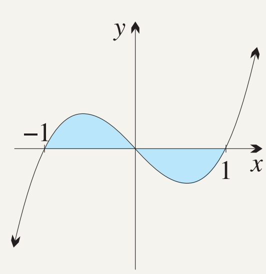
b Using part **a**, find the area between the curve $y = x^3 - x$ and the x-axis.

SOLUTION

a Let $\quad f(x) = x^3 - x$.

Then

$$\begin{aligned}f(-x) &= (-x)^3 - (-x)\\ &= -x^3 + x\\ &= -f(x), \text{ so } f(x) \text{ is odd.}\end{aligned}$$



b Factoring,

$$\begin{aligned}y &= x(x^2 - 1)\\ &= x(x-1)(x+1),\end{aligned}$$

so the x-intercepts are $x = -1$, $x = 0$ and $x = 1$.

The two shaded regions have equal areas because the function is odd.

First,

$$\begin{aligned}\int_0^1 (x^3 - x)\,dx &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1\\ &= \left(\tfrac{1}{4} - \tfrac{1}{2}\right) - (0 - 0)\\ &= -\tfrac{1}{4},\end{aligned}$$

so area below the x-axis $= \frac{1}{4}$ square units.

Doubling, $\qquad$ total area $= \frac{1}{2}$ square units.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Area between a graph and the y-axis

Integration with respect to y rather than x can often give a result quickly, and avoid subtraction of areas.

Suppose then that we can find the inverse function, and write x is a function of y.

- A definite integral with respect to y represents the signed area of the region between the curve and the y-axis.
- This means that the definite integral is the sum of areas of regions to the right of the x-axis, minus the sum of areas of regions to the left of the x-axis.
- The limits of integration are values of y rather than values of x.

16 THE DEFINITE INTEGRAL AND INTEGRATION WITH RESPECT TO y

Let x be a continuous function of y in some closed interval $a \le y \le b$.

Then the definite integral $\int_a^b x\,dy$ is the sum of the areas of regions to the right of the y-axis, from $y = a$ to $y = b$, minus the sum of the areas of regions to the left of the y-axis.

When the function is given with y as a function of x, we first need to solve the equation for x. We are thus working with the inverse function.

Example 26 — 5F

a Sketch the lines $y = x + 1$ and $y = 5$, and shade the region between these lines to the right of the y-axis.

b Write the equation of the line so that x is a function of y.

c Use integration with respect to y to find the area of this region.

d Confirm the result by area formulae.

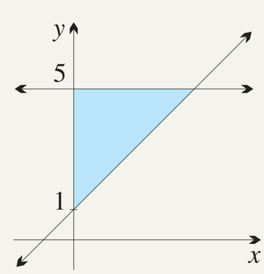
SOLUTION

a The lines are sketched below. They meet at $(4, 5)$.

b The given equation is $y = x + 1$.
Solving for x, $x = y - 1$.

c The required integral is

$$\int_1^5 (y - 1)\,dy = \left[\frac{y^2}{2} - y\right]_1^5$$
$$= \left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)$$
$$= 7\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)$$
$$= 8,$$



which is positive, because the region is to the right of the y-axis.
Hence the required area is $8\,\text{u}^2$.

d Area of triangle $= \tfrac{1}{2} \times \text{base} \times \text{height}$
$= \tfrac{1}{2} \times 4 \times 4$
$= 8\,\text{u}^2$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

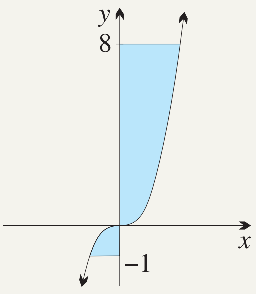
Example 27 5F

The curve in the diagram is the cubic $y = x^3$.

a Write the equation of the cubic so that x is a function of y.

b Use integration with respect to y to find the areas of the shaded regions to the right and left of the y-axis.

c Find the total area of the two shaded regions.



SOLUTION

a The given equation is $y = x^3$.

Solving for x,

$$x^3 = y$$
$$x = y^{\frac{1}{3}}.$$

b For the region to the right of the y-axis,

$$\int_0^8 y^{\frac{1}{3}}\,dy = \tfrac{3}{4}\Big[y^{\frac{4}{3}}\Big]_0^8$$
$$= \tfrac{3}{4} \times (16 - 0) \qquad (\text{because } 8^{\frac{4}{3}} = 2^4 = 16)$$
$$= 12,$$

so area $= 12$ square units.

For the region to the left of the y-axis,

$$\int_{-1}^0 y^{\frac{1}{3}}\,dy = \tfrac{3}{4}\Big[y^{\frac{4}{3}}\Big]_{-1}^0$$
$$= \tfrac{3}{4} \times (0 - 1) \qquad (\text{because } (-1)^{\frac{4}{3}} = (-1)^4 = 1)$$
$$= -\tfrac{3}{4},$$

so area $= \frac{3}{4}$ square units.

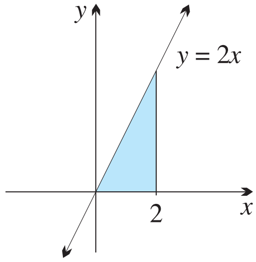
c Adding these, total area $= 12\frac{3}{4}$ square units.

Exercise 5F FOUNDATION

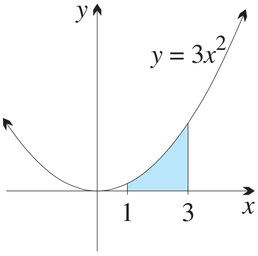
Technology: Graphing software would help in identifying the definite integrals that need to be evaluated to find the area of a given region.

1 Find the area of each shaded region below by evaluating the appropriate integral.

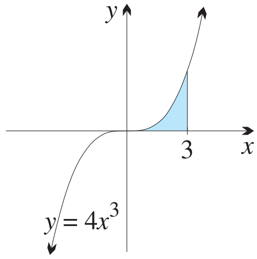
a



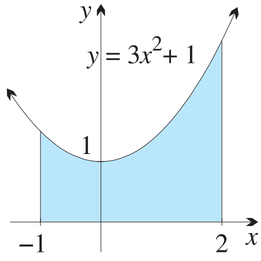
b



c

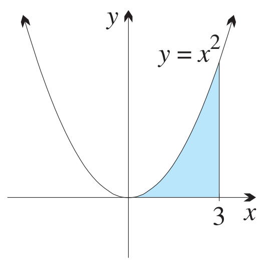


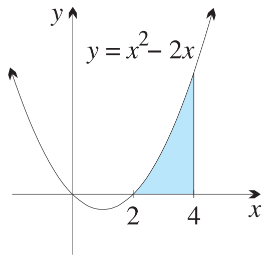
d

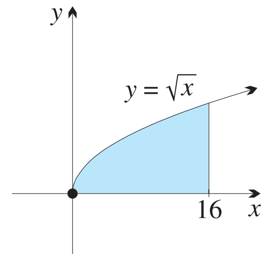


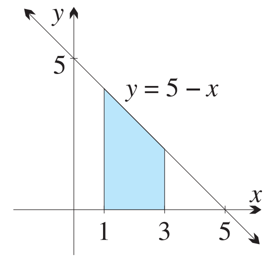
ISBN 978-1-108-76630-2

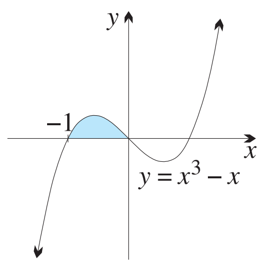
Cambridge University Press
Updated July 2022

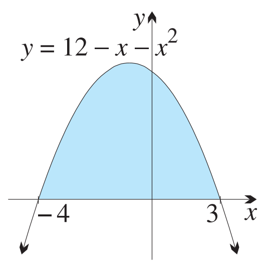
e


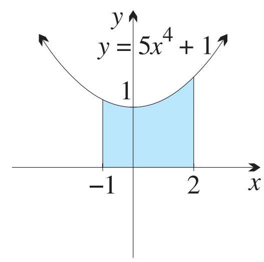
f


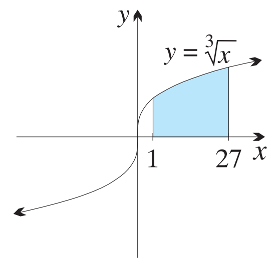
g


h


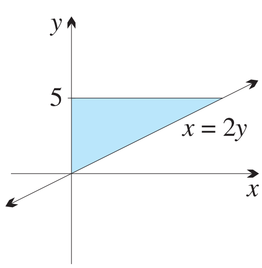
i


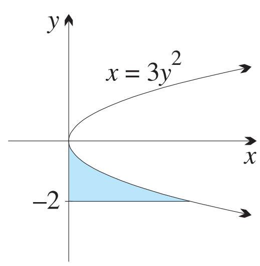
j


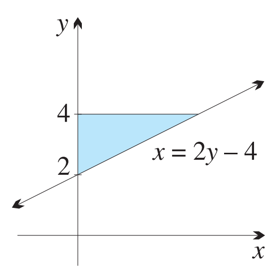
k


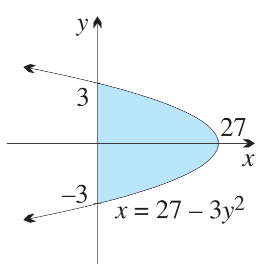
l


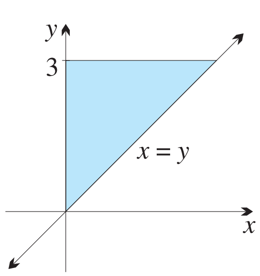
2 Find the area of each shaded region below by evaluating an integral of the form $\displaystyle\int_{y_1}^{y_2} g(y)\,dy$.

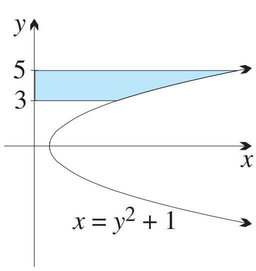
a


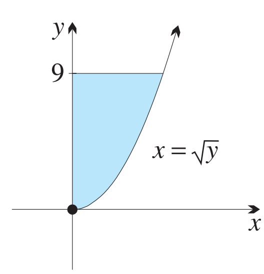
b


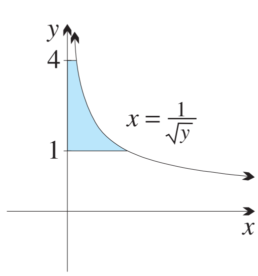
c


d


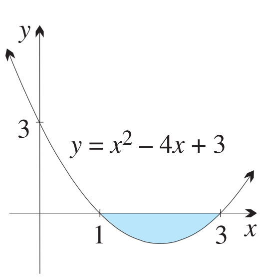
e


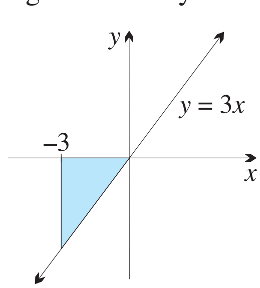
f


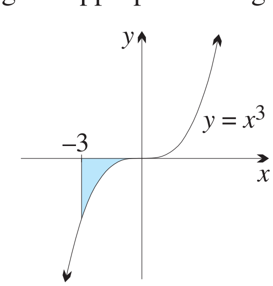
g


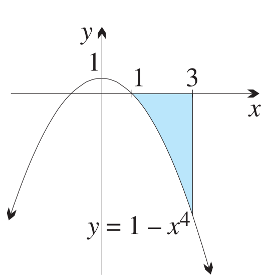
h


3 Find the area of each shaded region below by evaluating the appropriate integral.

a


b


c


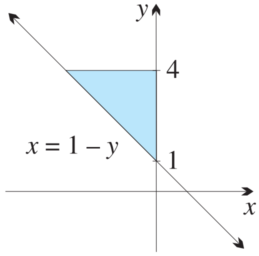
d


ISBN 978-1-108-76630-2

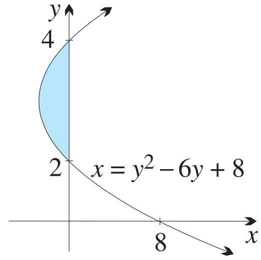
Cambridge University Press
Updated July 2022

4 Find the area of each shaded region below by evaluating the appropriate integral.

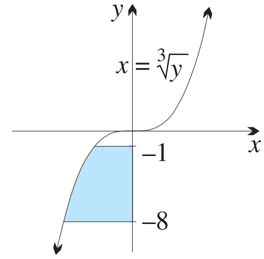
a



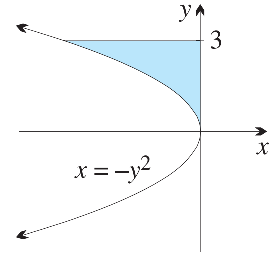
b



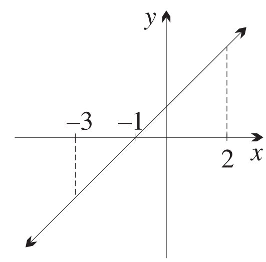
c



d



5 The line $y = x + 1$ is graphed on the right.



a Copy the diagram, and shade the region between the line $y = x + 1$ and the x-axis from $x = -3$ to $x = 2$.

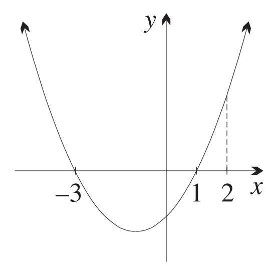
b By evaluating $\int_{-1}^{2} (x + 1)\, dx$, find the area of the shaded region above the x-axis.

c By evaluating $\int_{-3}^{-1} (x + 1)\, dx$, find the area of the shaded region below the x-axis.

d Hence find the area of the entire shaded region.

e Find $\int_{-3}^{2} (x + 1)\, dx$, and explain why this integral does not give the area of the shaded region.

6 The curve $y = (x - 1)(x + 3) = x^2 + 2x - 3$ is graphed.



a Copy the diagram, and shade the region between the curve $y = (x - 1)(x + 3)$ and the x-axis from $x = -3$ to $x = 2$.

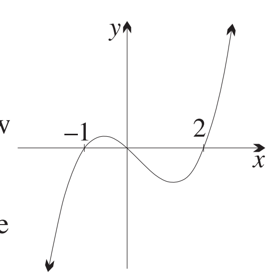
b By evaluating $\int_{-3}^{1} (x^2 + 2x - 3)\, dx$, find the area of the shaded region below the x-axis.

c By evaluating $\int_{1}^{2} (x^2 + 2x - 3)\, dx$, find the area of the shaded region above the x-axis.

d Hence find the area of the entire shaded region.

e Find $\int_{-3}^{2} (x^2 + 2x - 3)\, dx$, and explain why this integral does not give the area of the shaded region.

7 The curve $y = x(x + 1)(x - 2) = x^3 - x^2 - 2x$ is graphed.



a Copy the diagram, and shade the region bounded by the curve and the x-axis.

b By evaluating $\int_{0}^{2} (x^3 - x^2 - 2x)\, dx$, find the area of the shaded region below the x-axis.

c By evaluating $\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx$, find the area of the shaded region above the x-axis.

d Hence find the area of the entire region you have shaded.

e Find $\int_{-1}^{2} (x^3 - x^2 - 2x)\, dx$, and explain why this integral does not give the area of the shaded region.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

8 In each part below, find the area of the region bounded by the graph of the given function and the x-axis between the specified values. Remember that areas above and below the x-axis must be calculated separately.

a $y = x^2$, between $x = -3$ and $x = 2$,
b $y = 2x^3$, between $x = -4$ and $x = 1$,
c $y = 3x(x - 2)$, between $x = 0$ and $x = 2$,
d $y = x - 3$, between $x = -1$ and $x = 4$,
e $y = (x - 1)(x + 3)(x - 2)$, between $x = -3$, and $x = 2$
f $y = -2x(x + 1)$, between $x = -2$ and $x = 2$.

9 In each part below, find the area of the region bounded by the graph of the given function and the y-axis between the specified values. Remember that areas to the right and to the left of the y-axis must be calculated separately.

a $x = y - 5$, between $y = 0$ and $y = 6$,
b $x = 3 - y$, between $y = 2$ and $y = 5$,
c $x = y^2$, between $y = -1$ and $y = 3$,
d $x = (y - 1)(y + 1)$, between $y = 3$ and $y = 0$.

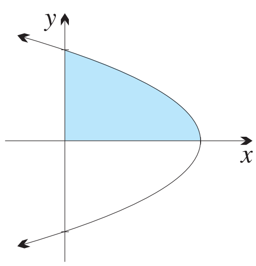
10 In each part below you should sketch the curve and look carefully for any symmetries that will simplify the calculation.

a Find the area of the region bounded by the given curve and the x-axis.
  i $y = x^7$, for $-2 \le x \le 2$,
  ii $y = x^3 - 16x = x(x - 4)(x + 4)$, for $-4 \le x \le 4$,
  iii $y = x^4 - 9x^2 = x^2(x - 3)(x + 3)$, for $-3 \le x \le 3$.

b Find the area of the region bounded by the given curve and the y-axis.
  i $x = 2y$, for $-5 \le y \le 5$,
  ii $x = y^2$, for $-3 \le y \le 3$,
  iii $x = 4 - y^2 = (2 - y)(2 + y)$, for $-2 \le y \le 2$.

11 Find the area of the region bounded by $y = |x + 2|$ and the x-axis, for $-2 \le x \le 2$.

12 The diagram shows the parabola $y^2 = 16(2 - x)$.



a Find the x-intercept and the y-intercepts.
b Find the exact area of the shaded region:
  i by integrating $y = 4\sqrt{2 - x}$ with respect to x,
  ii by integrating with respect to y. (You will need to make x the subject of the equation.)

13 The gradient of a curve is $y' = x^2 - 4x + 3$, and the curve passes through the origin.

a Find the equation of the curve.
b Show that the curve has turning points at $\left(1, 1\frac{1}{3}\right)$ and $(3, 0)$, and sketch its graph.
c Find the area of the region bounded by the curve and the x-axis between the two turning points.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 Sketch $y = x^2$ and mark the points $A(a, a^2)$, $B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.

a Show that $\int_0^a x^2\,dx = \frac{2}{3}(\text{area } \triangle OAP)$.

b Show that $\int_{-a}^a x^2\,dx = \frac{1}{3}(\text{area of rectangle } ABQP)$.

15 Given positive real numbers a and n, let A, P and Q be the points (a, a^n), $(a, 0)$ and $(0, a^n)$ respectively. Find the ratios:

a $\int_0^a x^n\,dx : (\text{area of } \triangle AOP)$

b $\int_0^a x^n\,dx : (\text{area of rectangle } OPAQ)$

16 a Show that $x^4 - 2x^3 + x = x(x - 1)(x^2 - x - 1)$. Then sketch a graph of the function $y = x^4 - 2x^3 + x$ and shade the three regions bounded by the graph and the x-axis.

b If $a = \frac{1}{2}\left(1 + \sqrt{5}\right)$, evaluate a^2, a^4 and a^5.

c Show that the area of one shaded region equals the sum of the areas of the other two.

ENRICHMENT

17 Consider the function $G(x) = \int_0^x g(u)\,du$, where $g(u) = \begin{cases} 4 - \frac{4}{3}u, & \text{for } 0 \le u < 6, \\ u - 10, & \text{for } 6 \le u \le 12. \end{cases}$

a Sketch a graph of $g(u)$.

b Find the stationary points of the function $y = G(x)$ and determine their nature.

c Find the values of x for which $G(x) = 0$.

d Sketch the curve $y = G(x)$, indicating all important features.

e Find the area bounded by the curve $y = G(x)$ and the x-axis for $0 \le x \le 6$.

18 a Show that for $n < -1$, $\int_1^N x^n\,dx$ converges as $N \to \infty$, and find the limit.

b Show that for $n > -1$, $\int_\varepsilon^1 x^n\,dx$ converges as $\varepsilon \to 0^+$, and find the limit.

c Interpret these two results as areas.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5G Areas of compound regions

When a region is formed by two or more different curves, some dissection process is usually needed before integrals can be used to calculate its area. Thus a preliminary sketch of the region becomes all the more important.

Areas of regions under a combination of curves

Some regions are bounded by different curves in different parts of the x-axis.

Example 28 — 5G

a Sketch the curves $y = x^2$ and $y = (x - 2)^2$ on one set of axes.
b Shade the region bounded by $y = x^2$, $y = (x - 2)^2$ and the x-axis.
c Find the area of this shaded region.

SOLUTION

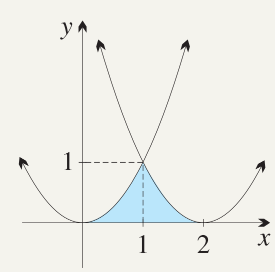
a The two curves intersect at $(1, 1)$, because it is easily checked by substitution that this point lies on both curves.

b The whole region is above the x-axis, but it will be necessary to find separately the areas of the regions to the left and right of $x = 1$.

c First,
$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \tfrac{1}{3}.$$

Secondly,
$$\int_1^2 (x - 2)^2\,dx = \left[\frac{(x - 2)^3}{3}\right]_1^2 = 0 - \left(-\tfrac{1}{3}\right) = \tfrac{1}{3}.$$

Combining these,
$$\text{area} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3} \text{ square units.}$$



Note: In this worked example, the second parabola is the first shifted right 2, and a parabola is symmetric about its axis of symmetry. This is why the two pieces have the same area.

Areas of regions between curves

Suppose that one curve $y = f(x)$ is always below another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area of the region between the curves from $x = a$ to $x = b$ can be found by subtraction.

17 AREA BETWEEN CURVES

If $f(x) \le g(x)$ in the interval $a \le x \le b$, then

$$\text{area between the curves} = \int_a^b \big(g(x) - f(x)\big)\,dx.$$

That is, take the integral of the top curve minus the bottom curve.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The assumption that $f(x) \le g(x)$ is important. If the curves cross each other, then separate integrals will need to be taken or else the areas of regions where different curves are on top will begin to cancel each other out.

Example 29 5G

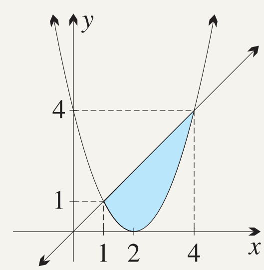
a Find the two points where the curve $y = (x-2)^2$ meets the line $y = x$.
b Draw a sketch and shade the area of the region between these two graphs.
c Find the shaded area.

SOLUTION

a Substituting $y = x$ into $y = (x-2)^2$ gives

$$\begin{aligned}(x-2)^2 &= x\\ x^2 - 4x + 4 &= x\\ x^2 - 5x + 4 &= 0\\ (x-1)(x-4) &= 0,\\ x &= 1 \text{ or } 4,\end{aligned}$$

so the two graphs intersect at $(1, 1)$ and $(4, 4)$.



b The sketch is drawn to the right.

c In the shaded region, the line is above the parabola.

$$\begin{aligned}\text{Hence area } &= \int_1^4 \left(x - (x-2)^2\right) dx\\ &= \int_1^4 \left(x - (x^2 - 4x + 4)\right) dx\\ &= \int_1^4 (-x^2 + 5x - 4)\, dx\\ &= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4\\ &= \left(-21\tfrac{1}{3} + 40 - 16\right) - \left(-\tfrac{1}{3} + 2\tfrac{1}{2} - 4\right)\\ &= 2\tfrac{2}{3} + 1\tfrac{5}{6}\\ &= 4\tfrac{1}{2} \text{ square units.}\end{aligned}$$

Note: The formula given in Box 17 for the area of the region between two curves holds even if the region crosses the x-axis.

To illustrate this point, the next example is the previous example shifted down 2 units so that the region between the line and the parabola crosses the x-axis. The area of course remains the same — and notice how the formula still gives the correct answer.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 30 — 5G

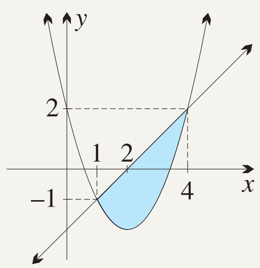
a Find the two points where the curves $y = x^2 - 4x + 2$ and $y = x - 2$ meet.
b Draw a sketch and find the area of the region between these two curves.

SOLUTION

a Substituting $y = x - 2$ into $y = x^2 - 4x + 2$ gives

$$\begin{aligned} x^2 - 4x + 2 &= x - 2 \\ x^2 - 5x + 4 &= 0 \\ (x - 1)(x - 4) &= 0 \\ x &= 1 \text{ or } 4. \end{aligned}$$

so the two graphs intersect at $(1, -1)$ and $(4, 2)$.



b Again, the line is above the parabola,

$$\begin{aligned} \text{so area} &= \int_1^4 \Big((x - 2) - (x^2 - 4x + 2) \Big)\, dx \\ &= \int_1^4 (-x^2 + 5x - 4)\, dx \\ &= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x \right]_1^4 \\ &= \Big(-21\tfrac{1}{3} + 40 - 16 \Big) - \Big(-\tfrac{1}{3} + 2\tfrac{1}{2} - 4 \Big) \\ &= 4\tfrac{1}{2} \text{ square units.} \end{aligned}$$

Areas of regions between curves that cross

Now suppose that one curve $y = f(x)$ is sometimes above and sometimes below another curve $y = g(x)$ in the relevant interval. In this case, the conditions of Box 17 no longer hold, and separate integrals will need to be calculated.

Example 31 — 5G

The diagram below shows the curves

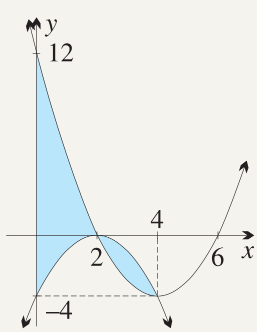
$$y = -x^2 + 4x - 4 \qquad \text{and} \qquad y = x^2 - 8x + 12$$

meeting at the points $(2, 0)$ and $(4, -4)$. Find the area of the shaded region.

SOLUTION

In the left-hand region, the second curve is above the first,

$$\begin{aligned} \text{So} \qquad \text{area} &= \int_0^2 \Big((x^2 - 8x + 12) - (-x^2 + 4x - 4) \Big)\, dx \\ &= \int_0^2 (2x^2 - 12x + 16)\, dx \\ &= \left[\frac{2x^3}{3} - 6x^2 + 16x \right]_0^2 \\ &= 5\tfrac{1}{3} - 24 + 32 \\ &= 13\tfrac{1}{3}\,\text{u}^2. \end{aligned}$$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

In the right-hand region, the first curve is above the second,

$$\text{so} \qquad \text{area} = \int_2^4 \big((-x^2 + 4x - 4) - (x^2 - 8x + 12)\big)\,dx$$

$$= \int_2^4 (-2x^2 + 12x - 16)\,dx$$

$$= \left[-\frac{2x^3}{3} + 6x^2 - 16x\right]_2^4$$

$$= \left(-42\tfrac{2}{3} + 96 - 64\right) - \left(-5\tfrac{1}{3} + 24 - 32\right)$$

$$= -10\tfrac{2}{3} + 13\tfrac{1}{3}$$

$$= 2\tfrac{2}{3}\,\text{u}^2.$$

$$\text{Hence total area} = 13\tfrac{1}{3} + 2\tfrac{2}{3}$$

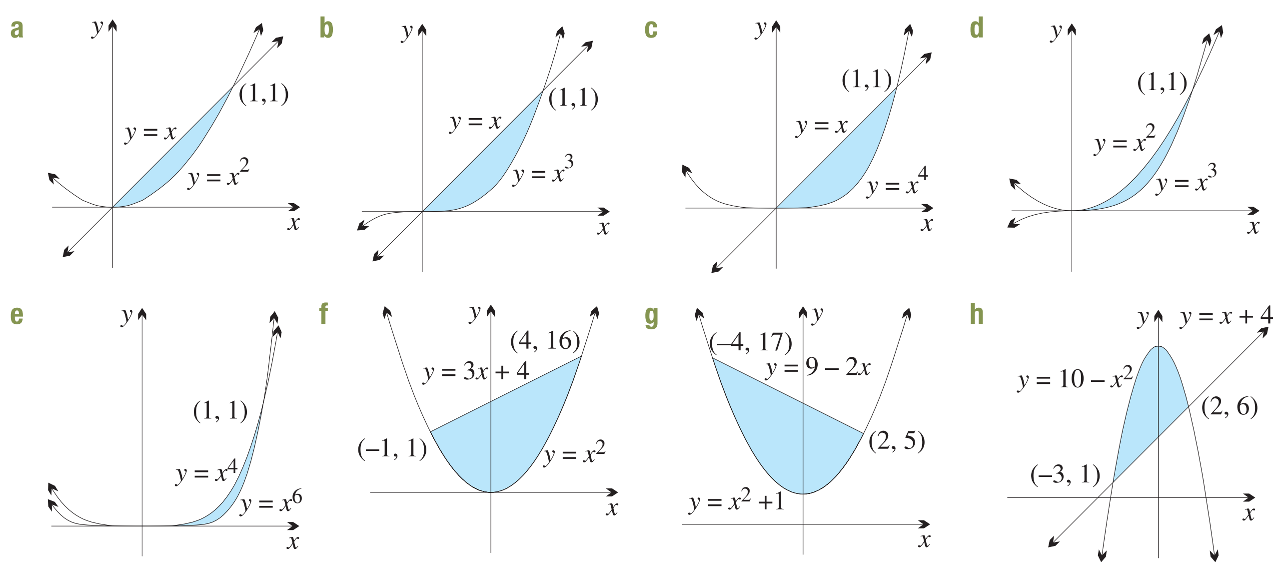
$$= 16\,\text{u}^2.$$

Exercise 5G

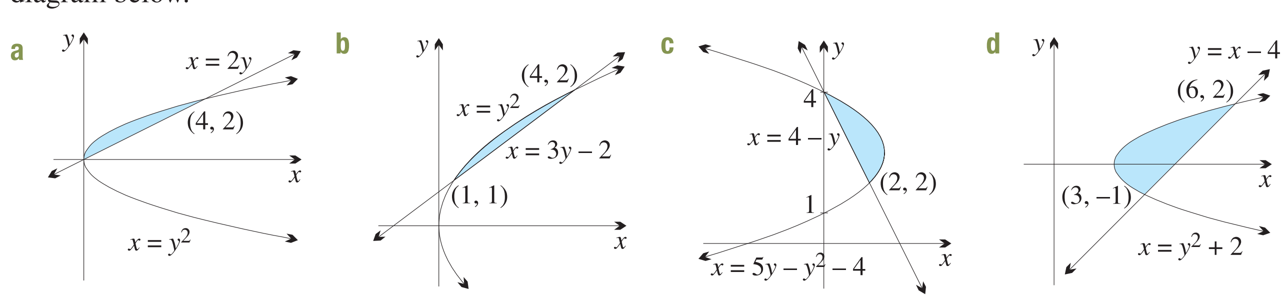
FOUNDATION

Technology: Graphing programs are particularly useful with compound regions because they allow the separate parts of the region to be identified clearly.

1 Find the area of the shaded region in each diagram below.



2 By considering regions between the curves and the y–axis, find the area of the shaded region in each diagram below.

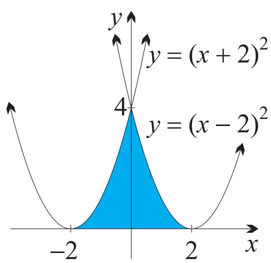


ISBN 978-1-108-76630-2

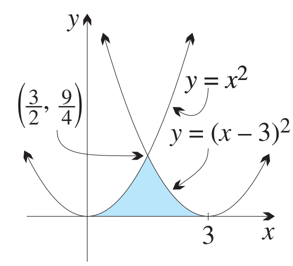
Cambridge University Press
Updated July 2022

3 Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and add them.

a

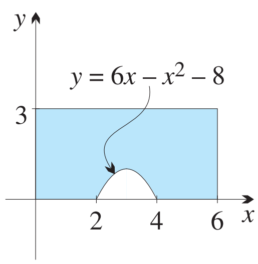


b

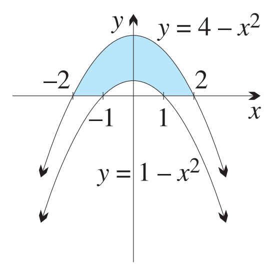


4 Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and subtract one from the other.

a



b



DEVELOPMENT

5 **a** By solving the equations simultaneously, show that the parabola $y = x^2 + 4$ and the line $y = x + 6$ intersect at the points $(-1, 5)$ and $(2, 8)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{-1}^{2} \Big((x + 6) - (x^2 + 4) \Big)\, dx = \int_{-1}^{2} (x - x^2 + 2)\, dx,$$

and evaluate the integral.

6 **a** By solving the equations simultaneously, show that the parabola $y = 3x - x^2 = x(3 - x)$ and the line $y = x$ intersect at the points $(0, 0)$ and $(2, 2)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{0}^{2} (3x - x^2 - x)\, dx = \int_{0}^{2} (2x - x^2)\, dx,$$

and evaluate the integral.

7 **a** By solving the equations simultaneously, show that the parabola $y = (x - 3)^2$ and the line $y = 14 - 2x$ intersect at the points $(-1, 16)$ and $(5, 4)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{-1}^{5} \Big((14 - 2x) - (x - 3)^2 \Big)\, dx = \int_{-1}^{5} (4x + 5 - x^2)\, dx,$$

and evaluate the integral.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Solve simultaneously to find the points of intersection of each pair of graphs. Then sketch the graphs on the same diagram and shade the region enclosed between them. By evaluating the appropriate definite integral, find the area of the shaded region in each case.

a $y = x + 3$ and $y = x^2 + 1$

b $y = 9 - x^2$ and $y = 3 - x$

c $y = x^2 - x + 4$ and $y = -x^2 + 3x + 4$

9 **a** By solving the equations simultaneously, show that the parabola $y = x^2 + 2x - 8$ and the line $y = 2x + 1$ intersect at the points $(3, 7)$ and $(-3, -5)$.

b Sketch both graphs on the same diagram, and shade the region enclosed between them.

c Despite the fact that it crosses the x-axis, the region has area given by

$$\int_{-3}^{3} \Big((2x + 1) - (x^2 + 2x - 8) \Big)\, dx = \int_{-3}^{3} (9 - x^2)\, dx.$$

Evaluate the integral and hence find the area of the region enclosed between the curves.

10 **a** By solving the equations simultaneously, show that the parabola $y = x^2 - x - 2$ and the line $y = x - 2$ intersect at the points $(0, -2)$ and $(2, 0)$.

b Sketch both graphs on the same diagram, and shade the region enclosed between them.

c Despite the fact that it is below the x–axis, the region has area given by

$$\int_{0}^{2} \Big((x - 2) - (x^2 - x - 2) \Big)\, dx = \int_{0}^{2} (2x - x^2)\, dx.$$

Evaluate this integral and hence find the area of the region between the curves.

11 Solve simultaneously to find the points of intersection of each pair of graphs. Then sketch the graphs on the same diagram, and shade the region enclosed between them. By evaluating the appropriate definite integral, find the area of the shaded region in each case.

a $y = x^2 - 6x + 5$ and $y = x - 5$

b $y = -3x$ and $y = 4 - x^2$

c $y = x^2 - 1$ and $y = 7 - x^2$

12 **a** On the same number plane, sketch the graphs of the parabolas $y = x^2$ and $x = y^2$, clearly indicating their points of intersection. Shade the region enclosed between them.

b Explain why the area of this region is given by $\displaystyle\int_{0}^{1} \left(\sqrt{x} - x^2 \right) dx$.

c Find the area of the region bounded by the two curves.

13 Tangents are drawn to the parabola $x^2 = 8y$ at the points $A(4, 2)$ and $B(-4, 2)$.

a Draw a diagram of the situation and note the symmetry about the y–axis.

b Find the equation of the tangent at the point A.

c Find the area of the region bounded by the curve and the tangents.

14 **a** Show that the tangent to the curve $y = x^3$ at the point where $x = 2$ has equation $y = 12x - 16$.

b Show by substitution that the tangent and the curve intersect again at the point $(-4, -64)$.

c Find the area of the region enclosed between the curve and the tangent.

15 Consider the curves $y = x^3 - 3$ and $y = -x^2 + 10x - 11$.

a Show by substitution that the curves intersect at three points whose x-values are -4, 1 and 2.

b Sketch the curves showing clearly their intersection points.

c Find the area of the region enclosed by the two curves.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 **a** Find the points of intersection of the curves $y = x^2(1 - x)$ and $y = x(1 - x)^2$.
b Hence find the area bounded by the two curves.

17 **a** Given the two functions $f(x) = (x + 1)(x - 1)(x - 3)$ and $g(x) = (x + 1)(x - 1)$, for what values of x is $f(x) > g(x)$?
b Sketch a graph of the two functions on the same number plane, and find the area enclosed between them.

18 **a** Sketch a graph of the function $y = 12x - 32 - x^2$, clearly indicating the x-intercepts.
b Find the equation of the tangent to the curve at the point A where $x = 5$.
c If the tangent meets the x-axis at B, and C is the x-intercept of the parabola closer to the origin, find the area of the region bounded by AB, BC and the arc CA.

ENRICHMENT

19 Find the value of k for which the line $y = kx$ bisects the area enclosed by the curve $4y = 4x - x^2$ and the x-axis.

20 The *average value* of a continuous function $f(x)$ over an interval $a \le x \le b$ is defined to be $\frac{1}{b - a}\int_a^b f(x)\,dx$ if $a \ne b$, or $f(a)$ if $a = b$. If k is the average value of $f(x)$ on the interval $a \le x \le b$, show that the area of the region bounded by $f(x)$ above the line $y = k$ is equal to the area of the region bounded by $f(x)$ below the line $y = k$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

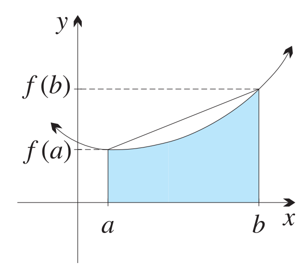
5H The trapezoidal rule

Methods of approximating definite integrals become necessary when exact calculations using primitives are not possible. This can happen for two reasons.

- The primitives of many important functions cannot be written down in a formula suitable for calculation — this is the case for the important normal distribution in Chapter 16.
- Some values of a function may be known only from experiments, and the function formula may be unknown.

The trapezoidal rule

Besides taking upper and lower rectangles, the most obvious way to approximate an integral is to replace the curve by a straight line, that is, by a chord joining $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. The resulting region is a trapezium, so this approximation method is called the *trapezoidal rule*.



Consider the trapezium in the diagram to the right.

Here width $= b - a$,

and average of parallel sides $= \dfrac{f(a) + f(b)}{2}$.

Hence area of trapezium $=$ width $\times$ average of parallel sides

$$= \frac{b-a}{2}\big(f(a) + f(b)\big).$$

The area of this trapezium is taken as an approximation of the integral.

18 THE TRAPEZOIDAL RULE USING ONE SUBINTERVAL

Let $f(x)$ be a function that is continuous in the closed interval $[a, b]$.

- Approximating the curve from $x = 1$ to $x = b$ by a chord allows the region under the curve to be approximated by a trapezium, giving

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}\big(f(a) + f(b)\big).$$

- If the function is linear, then the chord coincides with the curve and the formula is exact.
- Always start a trapezoidal-rule calculation by constructing a table of values.

Subdividing the interval

Given an integral over an interval $[a, b]$, we can split that interval $[a, b]$ up into a number of subintervals and apply the trapezoidal rule to each subinterval in turn. This will usually improve the accuracy of the approximation.

Here is the method applied to the reciprocal function $y = \dfrac{1}{x}$, whose primitive we will establish in Chapter 6.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 32 5H

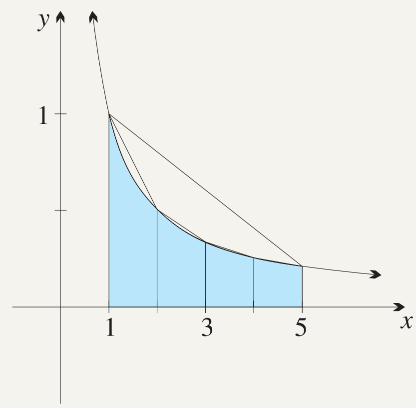
Find approximations of $\int_1^5 \frac{1}{x}\,dx$ using the trapezoidal rule with:

a one subinterval, **b** four subintervals.

SOLUTION

Always begin with a table of values of the function.

x	1	2	3	4	5
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



a One application of the trapezoidal rule, using the whole interval as the subinterval, requires just two values of the function.

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \frac{5-1}{2} \times \big(f(1) + f(5)\big)\\ &\doteqdot 2 \times \big(1 + \tfrac{1}{5}\big)\\ &\doteqdot 2\tfrac{2}{5}\end{aligned}$$

b Four applications of the trapezoidal rule require five values of the function.

Dividing the interval $1 \le x \le 5$ into four equal subintervals,

$$\int_1^5 \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \int_3^4 \frac{1}{x}\,dx + \int_4^5 \frac{1}{x}\,dx$$

Each subinterval has width 1, so applying the trapezoidal rule to each integral,

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \tfrac{1}{2}\big(f(1) + f(2)\big) + \tfrac{1}{2}\big(f(2) + f(3)\big) + \tfrac{1}{2}\big(f(3) + f(4)\big) + \tfrac{1}{2}\big(f(4) + f(5)\big)\\ &\doteqdot \tfrac{1}{2}\big(\tfrac{1}{1} + \tfrac{1}{2}\big) + \tfrac{1}{2}\big(\tfrac{1}{2} + \tfrac{1}{3}\big) + \tfrac{1}{2}\big(\tfrac{1}{3} + \tfrac{1}{4}\big) + \tfrac{1}{2}\big(\tfrac{1}{4} + \tfrac{1}{5}\big)\\ &\doteqdot 1\tfrac{41}{60}.\end{aligned}$$

Note: Always take subintervals of equal width unless otherwise indicated.

Concavity and the trapezoidal rule

The curve in the example above is concave up, so every chord is above the curve, and every approximation found using the trapezoidal rule is therefore greater than the integral.

Similarly, if a curve is concave down, then every chord is below the curve, and every trapezoidal-rule approximation is less than the integral. The second derivative can be used to test concavity.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

19 CONCAVITY AND THE TRAPEZOIDAL RULE

- If the curve is concave up, the trapezoidal rule overestimates the integral.
- If the curve is concave down, the trapezoidal rule underestimates the integral.
- If the curve is linear, the trapezoidal rule gives the exact value of the integral.

The second derivative $\dfrac{d^2y}{dx^2}$ can be used to test the concavity.

Example 33 5H

a Use the trapezoidal rule with one subinterval (that is, two function values) to approximate $\displaystyle\int_1^5 (200x - x^4)\,dx$.

b Use the second derivative to explain why the approximation underestimates the integral.

SOLUTION

a Construct a table of values for $y = 200x - x^4$.

x	1	5
y	199	375

$$\begin{aligned}\int_1^5 (200x - x^4)\,dx &\doteqdot \frac{5-1}{2} \times \big(f(1) + f(5)\big)\\ &\doteqdot 2 \times (199 + 375)\\ &\doteqdot 1148\end{aligned}$$

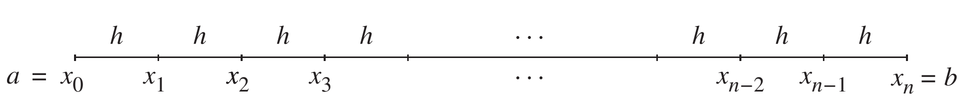
b The function is $y = 200x - x^4$.
Differentiating, $y' = 200 - 4x^3$
and $y'' = -12x^2$.
Because $y'' = -12x^2$ is negative throughout the interval $1 \le x \le 5$,
the curve is concave down throughout this interval.
Hence the trapezoidal rule underestimates the integral.

A formula for multiple applications of the trapezoidal rule

When the trapezoidal rule is being applied two or three times, it is easier to perform the two or three calculations required. These separate calculations also reinforce the meaning of the approximation, and help to gain an intuitive understanding of the accuracy of the estimates.

But increasing accuracy with the trapezoidal rule requires larger numbers of applications of the rule, and this can quickly become tedious. Let us then develop a single formula that splits an integral into n subintervals of equal width and applies the trapezoidal rule to each — anyone writing a program or using a spreadsheet to estimate integrals would want to do this.

The first step is to divide the interval $[a, b]$ into n equal subintervals, each of width h, like this:



ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

There are $n + 1$ points altogether, and they divide the interval into n equal subintervals. The endpoints are $a = x_0$ and $b = x_{n+1}$, and the $n - 1$ division points in between are $x_1, x_2, \ldots, x_{n-1}$.

There are n subintervals, so $nh = b - a$, and the width h of each subinterval is

$$h = \frac{b - a}{n}.$$

Thus starting with $a = x_0$, the successive values of the division points are

$$\begin{array}{lcl} x_0 = a & & x_{n-2} = a + (n-2)h \\ x_1 = a + h & \ldots & x_{n-1} = a + (n-1)h \\ x_2 = a + 2h & & x_n = a + nh = a + (b - a) = b \end{array}$$

That is, $x_r = a + rh$, for $r = 0, 1, 2, \ldots, n - 1, n$.
Now we can apply the trapezoidal rule to each subinterval in turn,

$$\begin{aligned} \int_a^b f(x)\,dx &= \int_a^{x_1} f(x)\,dx + \int_{x_2}^{x_1} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)\,dx + \int_{x_{n-1}}^{b} f(x)\,dx \\ &\doteqdot \frac{h}{2}\Big(f(a) + f(x_1)\Big) + \frac{h}{2}\Big(f(x_1) + f(x_2)\Big) + \cdots \\ &\quad + \frac{h}{2}\Big(f(x_{n-2}) + f(x_{n-1})\Big) + \frac{h}{2}\Big(f(x_{n-1}) + f(b)\Big) \\ &\doteqdot \frac{h}{2}\Big(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\Big). \end{aligned}$$

20 TRAPEZOIDAL-RULE FORMULA USING *n* SUBINTERVALS

Let $f(x)$ be a function that is continuous in the closed interval $[a, b]$. Then

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}\Big(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\Big)$$

where $h = \dfrac{b - a}{n}$ and $x_r = a + rh$, for $r = 1, 2, \ldots, n - 1$.

A common rearrangement of this formula, using nested brackets, is

$$\int_a^b f(x)\,dx \doteqdot \frac{b - a}{2n}\Big(f(a) + f(b) + 2\big(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big)\Big).$$

Using the formula for the trapezoidal rule

The formula may look complicated at first sight, but it is actually quite straightforward to use, provided that:

- We begin with a sensible value of the width h of each subinterval.
- We construct a clear table of values to work from.

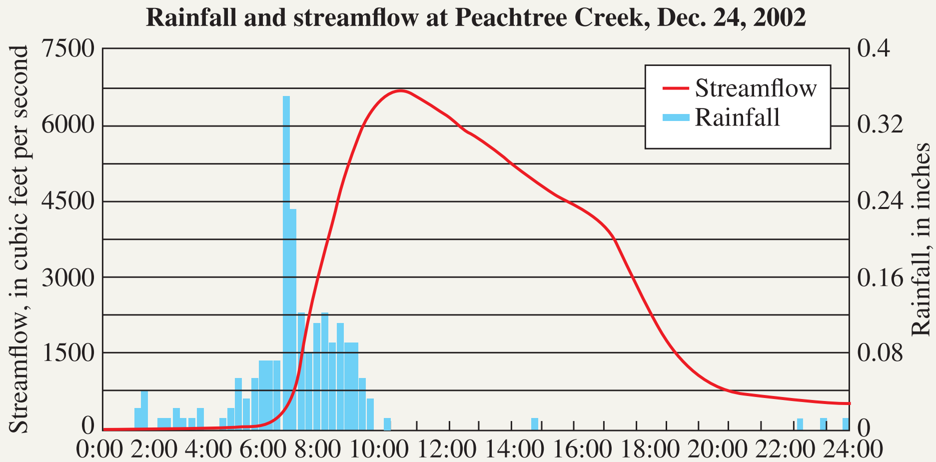
Here is an example where there is no equation of the function, but simply a set of experimental results gathered by recording equipment.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 34 5H

The flow at Peachtree Creek on 4th December 2002 after a storm is shown in the graph below. The flow rate in cubic feet per second is sketched as a function of the time t in hours.



We can estimate the total amount of water that flowed down the creek after the storm that day by integrating from $t = 4$ to $t = 24$. Use the trapezoidal rule with two-hour subintervals to approximate the total amount of water.

SOLUTION

The graph is very inaccurate, like so much internet data, but here is a rough table of values of the flow rate in cubic feet per second as a function of time t in hours.

t	4	6	8	10	12	14	16	18	20	22	24
Flow rate	100	600	5500	6700	5800	4800	4100	1800	800	600	500

The units need attention. The time is in hours, so the flow rates must be converted to cubic feet per hour by multiplying by $60 \times 60 = 3600$.

To avoid zeroes, let R be the flow rate in millions of cubic feet per hour.

t	4	6	8	10	12	14	16	18	20	22	24
R	0.36	2.16	19.8	24.12	20.88	17.28	14.76	6.48	2.88	2.16	1.8

Here $h = 2$ and $n = 10$. Also $a = x_0 = 4, x_1 = 6, \ldots, x_{n-1} = 22, x_n = b = 24$.

$$\text{Hence} \int_4^{20} R\,dt \doteqdot \frac{2}{2}\Big(f(4) + 2f(6) + 2f(8) + \cdots + 2f(22) + f(24)\Big)$$
$$\doteqdot 0.36 + 4.32 + 39.6 + 48.24 + 34.56 + 29.52 + 12.96 + 5.76 + 4.32 + 1.8$$
$$\doteqdot 223.2.$$

Alternatively, using the second formula,

$$\int_4^{20} R\,dt \doteqdot \frac{24 - 4}{20}\Big(f(4) + f(24) + 2\big(f(6) + f(8) + \cdots + f(22)\big)\Big)$$
$$\doteqdot 0.36 + 1.8 + 2(2.16 + 19.8 + 24.12 + 17.28 + 14.76 + 62.48 + 1.44 + 1.08)$$
$$= 2.16 + 2 \times 110.52$$
$$= 223.2.$$

Thus about 223 million cubic feet of water flowed down the creek from 4:00 am to midnight.

ISBN 978-1-108-76630-2

Using a spreadsheet for calculations

The authors used a spreadsheet for all the calculations above — the trapezoidal-rule formula is well suited for machine computation. The next worked example shows how to use an Excel spreadsheet to carry out such a calculation, but any spreadsheet can be used. Note that:

- Excel commands and procedures have been changing over successive versions.
- Mac users will need some adjustments, particularly when implementing the 'fill down' and 'fill right' commands.

The calculation involves the integration of $\dfrac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. We will see in Chapter 16 that this function is the probability density function of the normal distribution, and is the most important function in statistics. There is no simple equation for its primitive, so approximations are always necessary.

Example 35 — 5H

Let $\phi(x) = \dfrac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. Approximate the integral $\displaystyle\int_0^1 \phi(x)\,dx$ using the trapezoidal rule with 10 subintervals.

(The symbol ϕ is the lower case Greek letter 'phi', corresponding to Latin f.)

SOLUTION

In these instructions, we enter a formula into a cell by typing the $=$ sign as the first character. The cell in column D and row is labelled D3, and in formulae, this refers to its contents. Normally leave the top row clear for later titles.

1 On a new sheet in Excel, enter 0 into Cell `A2` and press `Enter`.
— Select Cell `B2` and type `=0.1+A2` and press `Enter`.
— Cell `B2` should now show 0.1.

2 Select Cells `B2:K2` and press `Ctrl+R` to 'fill right'.
— Cells `C2:K2` should now show 0.2, 0.3, . . . , 1.

3 Type `=EXP(-A2*A2/2)/SQRT(2*PI())` into Cell `A3`.
— Cell `A3` should now show $\phi(0) = 0.398942$.

4 Select Cells `A3:K3` and press `Ctrl+R` to 'fill right'.
— Cells `B3:K3` should now show $\phi(0.1) = 0.396953, \dots, \phi(1) = 0.241971$.

We now have the table of values for the function $\phi(x)$, and we need to add

$$\phi(0) + 2\phi(0.1) + 2\phi(0.2) + \cdots + 2\phi(0.9) + \phi(1)$$

5 Select Cell `A4` and enter `=A3`. This should duplicate the value in `A3`.
— Select Cell `K4` and enter `=K3`. Again, this duplicates the value in `K3`.
— Select Cell `B4` and enter `=2*B3`. This should double the value in `B3`.
— Select Cells `B4:J4` and press `Ctrl+R` to 'fill right'.
— Add the row by selecting Cell `L4` and typing `=SUM(A4:K4)`.

In this case, $h = 0.1$ so we multiply by $\frac{1}{2}h = 0.05$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 Select Cell `L5` and type `=L4*0.05` — this shows the final answer.

Hence $\int_0^1 \phi(x)\,dx \doteqdot 0.341$. We shall find in Chapter 16 that this is approximately the probability that a score in a normal distribution lies between the mean 0 and one standard deviations above the mean. The correct approximation to three decimal places is 0.398 — we will see in Chapter 16 that the curve is concave down in the interval $[0, 1]$, which explains why our estimate is a little smaller than it should be.

Readers may like to repeat the calculations above using 100 subintervals and see how close the approximation is then.

Exercise 5H

FOUNDATION

Technology: It is not difficult to write (or download) a program that will allow the calculations of the trapezoidal rule to be automated. It can then be applied to many examples from this exercise. The number of subintervals used can be steadily increased, and the approximations may then converge to the exact value of the integral. An accompanying screen sketch showing the curve and the chords would be helpful in giving a visual impression of the size and the sign of the error.

1 Approximate $\int_2^6 f(x)\,dx$ using the formula $\frac{1}{2}(a + b)h$ for the area of a trapezium.

a

x	2	6
$f(x)$	8	12

b

x	2	6
$f(x)$	6.2	4.8

c

x	2	6
$f(x)$	−4	−9

2 Three function values are given in the table to the right.

x	2	6	10
$f(x)$	12	20	30

a Approximate $\int_2^{10} f(x)\,dx$ by calculating the areas of two trapezia and then adding.

b Check your answer to part a by using the formula for the trapezoidal rule.

3 Use the trapezoidal rule with the three given function values to approximate $\int_{-5}^{5} f(x)\,dx$.

x	−5	0	5
$f(x)$	2.4	2.6	4.4

4 Show, by means of a diagram, that the trapezoidal rule will.

a overestimate $\int_a^b f(x)\,dx$, if $f''(x) > 0$ for $a \le x \le b$,

b underestimate $\int_a^b f(x)\,dx$, if $f''(x) < 0$ for $a \le x \le b$.

5 a Complete this table for the function $y = x(4 - x)$.

x	0	1	2	3	4
y					

b Hence use the trapezoidal rule with five function values to approximate $\int_0^4 x(4 - x)\,dx$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c What is the exact value of $\int_0^4 x(4 - x)\,dx$, and why does it exceed the approximation? Sketch the curve and the four chords involved.

d Calculate the percentage error in the approximation (that is, divide the error by the exact answer and convert to a percentage).

6 **a** Complete this table for the function $y = \dfrac{6}{x}$.

x	1	2	3	4	5
y					

b Use the trapezoidal rule with the five function values above, that is with four subintervals, to approximate $\int_1^5 \dfrac{6}{x}\,dx$.

c Show that the second derivative of $y = \dfrac{6}{x}$ is $y'' = 12x^{-3}$, and use this result to explain why the approximation will exceed the exact value of the integral.

7 **a** Complete this table correct to four decimal places for the function $y = \sqrt{x}$.

x	9	10	11	12	13	14	15	16
y								

b Approximate $\int_9^{16} \sqrt{x}\,dx$, using the trapezoidal rule with the eight function values above. Give your answer correct to three significant figures.

c What is the exact value of $\int_9^{16} \sqrt{x}\,dx$? Show that the second derivative of $y = x^{\frac{1}{2}}$ is $y'' = -\frac{1}{4}x^{-\frac{3}{2}}$, and use this result to explain why the approximation is less than the value of the definite integral.

DEVELOPMENT

8 Use the trapezoidal rule with three function values to approximate each definite integral, writing your answer correct to two significant figures where necessary.

a $\int_0^1 2^{-x}\,dx$ **b** $\int_{-2}^0 2^{-x}\,dx$ **c** $\int_1^3 \sqrt[3]{9 - 2x}\,dx$ **d** $\int_{-13}^{-1} \sqrt{3 - x}\,dx$

9 Use the trapezoidal rule with five function values to approximate each definite integral, writing your answer correct to three significant figures where necessary.

a $\int_2^6 \dfrac{1}{x}\,dx$ **b** $\int_0^2 \dfrac{1}{2 + \sqrt{x}}\,dx$ **c** $\int_4^8 \sqrt{x^2 - 3}\,dx$ **d** $\int_1^2 \log_{10} x\,dx$

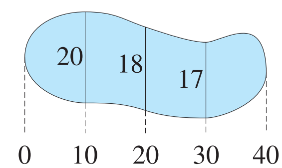
10 An object is moving along the x-axis with values of the velocity v in m/s at various times t given in the table to the right. Given that the distance travelled may be found by calculating the area under the velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the particle in the first 5 seconds.

t	0	1	2	3	4	5
v	1.5	1.3	1.4	2.0	2.4	2.7

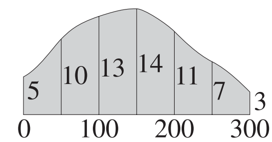
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 The diagram to the right shows the width of a lake at 10-metre intervals. Use the trapezoidal rule to estimate the surface area of the water.



12 The diagram to the right shows a vertical rock cutting of length 300 metres alongside a straight horizontal section of highway. The heights of the cutting are measured at 50-metre intervals. Use the trapezoidal rule to estimate the area of the vertical rock cutting.



13 **a** Use the trapezoidal rule with five function values to approximate $\int_0^1 \sqrt{1 - x^2}\, dx$, giving your answer correct to four decimal places.

a Use part **a** and the fact that $y = \sqrt{1 - x^2}$ is a semi-circle to approximate π. Give your answer correct to one decimal place, and explain why your approximation is less than π.

14 Use the trapezoidal rule with five function values, together with appropriate log laws, to show that $\int_1^5 \ln x\, dx \doteqdot \ln 54$.

ENRICHMENT

15 **a** Show that the function $y = \sqrt{x}$ is increasing for all $x > 0$.

a By dividing the area under the curve $y = \sqrt{x}$ into n equal subintervals, show that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \geq \int_0^n \sqrt{x} dx = \frac{2n\sqrt{n}}{3}.$$

b Use the trapezoidal rule to show that for all integers $n \geq 1$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leq \frac{\sqrt{n}(4n + 3)}{6}.$$

c Hence estimate $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{12\,000}$, correct to the nearest hundred.

An investigation using a spreadsheet for trapezoidal-rule calculations

16 Work through the spreadsheet example just above this exercise. Then use a spreadsheet estimate these integrals using the trapezoidal rule with 5, 10, 20 and perhaps more subintervals.

a $\int_1^{11} \frac{1}{x} dx$ **b** $\int_1^{11} \log_e x\, dx$ **c** $\int_0^{10} e^{-x^2} dx$

You will need to look at the results and perhaps vary the number of decimal places that you are using in the calculations and recording in your answers.

Possible spreadsheet projects

It is possible to program a spreadsheet so that the number of subintervals can be entered as a single variable. The construction of such a program and similar programs could be incorporated into a longer project examining the usefulness and accuracy of the trapezoidal rule, or examining some physical phenomena.

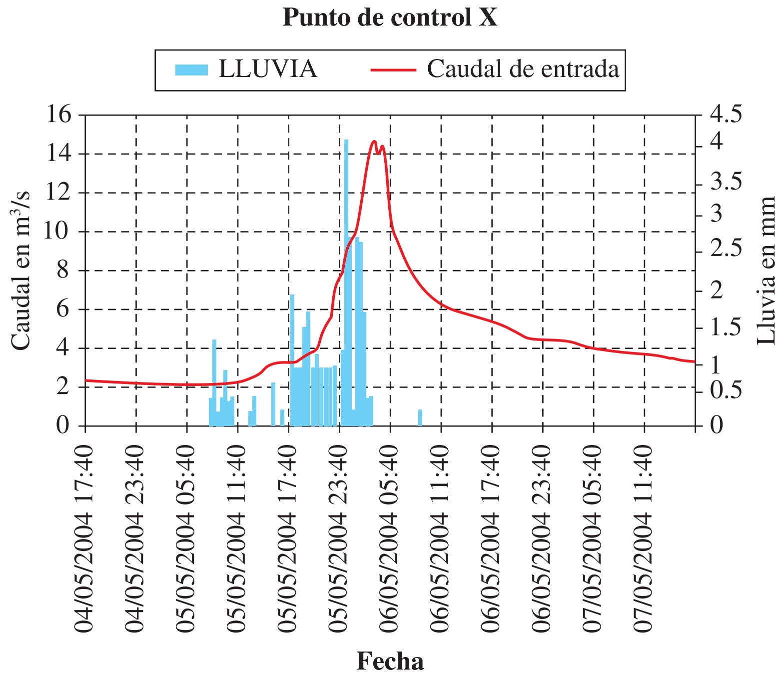
In the next diagram, it is clear that in parts of the graph where there is a lot of activity, the subintervals should be quite narrow, whereas in other calmer parts they can be far wider. Such variability could also be incorporated into the spreadsheet and its formulae.

ISBN 978-1-108-76630-2

An investigation (and possible project) integrating a graph from the web by the trapezoidal rule

There is a great deal of data available on the web for a sustained investigation of river flow. The following question suggests some interesting questions about one such situation, but there are many more situations and questions. Integrating graphs of all kinds from the web using the trapezoidal rule could be the basis of various different projects.

17



The hydrograph above shows the rate of flow through Control Point X on the Turia River in Spain over a three-day period in May 2004. The rate of flow ('Caudal') is given as a function of the date–times ('Fecha') — notice that the successive date–times on the horizontal axis are separated by exactly 6 hours. The rainfall ('Lluvia') is given by the vertical bars.

The units of time are hours, and the units of the flow rate are 'cubic metres per second'. The flow rate R should be converted to units of 'thousands of cubic meters per hour' so that time is in hours and there are fewer zeroes — multiply by $\dfrac{60 \times 60}{1000} = 3.6$.

a From the graph, copy and complete the table of values of the flow rate R at the first four date–times, 04/05/2004 17:40 to 05/05/2004 11:40.

t	17:40	23:40	05:40	11:40
R				

Then use the trapezoidal rule to estimate the total volume of water that flowed through the control point in those 18 hours.

b Draw up a similar table for the 18 hours of heavy flow from 05/05/2004 17:40 to 06/05/2004 11:40, but use 3 hours as the separation between successive times.

t	17:40	20:40	23:40	02:40	05:40	08:40	11:40
R							

Then use the trapezoidal rule to estimate the total volume of water that flowed through the Control Point in those 18 hours. Why are 3 hours suggested here in part **b** for the width of the subintervals, where 6 hours was used in part **a**?

c How many times more water flowed down the river in the second 18-hour period? Look at the rainfall record, and discuss how the river flow responded to the rainfall.

ISBN 978-1-108-76630-2

5I The reverse chain rule

When we use the chain rule to differentiate a composite function, the result is a product of two terms. For example, in the first worked example below,

$$\frac{d}{dx}(x^3+5)^4 = 4(x^3+5)^3 \times 3x^2.$$

This section deals with the problem of reversing a chain-rule differentiation.

Reversing a chain-rule differentiation

Finding primitives is the reverse process of differentiation. Thus once any differentiation has been performed, the process can then be reversed to give a primitive.

Example 36 5I

a Differentiate $(x^3+5)^4$ with full setting-out of the chain rule.
b Hence find a primitive of $12x^2(x^3+5)^3$.
c Hence find the primitive of $x^2(x^3+5)^3$.

SOLUTION

a Let $y = (x^3+5)^4$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 4(x^3+5)^3 \times 3x^2$

$= 12x^2(x^3+5)^3$.

Let $u = x^3+5$.
Then $y = u^4$.
Hence $\dfrac{du}{dx} = 3x^2$
and $\dfrac{dy}{du} = 4u^3$.

b By part **a**, $\dfrac{d}{dx}(x^3+5)^4 = 12x^2(x^3+5)^3$.

Reversing this, $\displaystyle\int 12x^2(x^3+5)^3\,dx = (x^3+5)^4$.

c Dividing by 12, $\displaystyle\int x^2(x^3+5)^3\,dx = \tfrac{1}{12}(x^3+5)^4 + C$, for some constant C.

Note: It is best not to add the arbitrary constant until the last line, because it would be pointless to divide C by 12 as well.

Example 37 5I

a Differentiate $\dfrac{1}{1+x^2}$ with full setting-out of the chain rule.

b Hence find the primitive of $\dfrac{x}{(1+x^2)^2}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Let $y = \dfrac{1}{1+x^2}$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -(1+x^2)^{-2} \times 2x$

$= \dfrac{-2x}{(1+x^2)^2}$.

Let $u = 1 + x^2$.
Then $y = u^{-1}$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = -u^{-2}$.

b By part **a**, $\dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\right) = \dfrac{-2x}{(1+x^2)^2}$.

Reversing this, $\displaystyle\int \frac{-2x}{(1+x^2)^2}\,dx = \frac{1}{1+x^2}, + C,$ for some constant C,

$\boxed{\div\ (-2)}$ $\displaystyle\int \frac{x}{(1+x^2)^2}\,dx = \frac{-1}{2\,(1+x^2)} + C.$

21 REVERSING A CHAIN-RULE DIFFERENTIATION

Once a chain-rule differentiation, or any differentiation, has been performed, the result can be written down in reverse as an indefinite integral.

A formula for the reverse chain rule

There is a formula for the reverse chain rule. Start with the formula for differentiating a function using the chain rule — we gave the formula in two forms:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad\qquad \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}f'(x)$$

and we can reverse both forms of the formula,

$$\int u^{n-1}\frac{du}{dx}\,dx = \frac{u^n}{n} \qquad\qquad \int \big(f(x)\big)^{n-1} f'(x)\,dx = \frac{\big(f(x)\big)^n}{n}.$$

Then replacing $n - 1$ by n and n by $n + 1$,

$$\int u^n\frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} \qquad \text{OR} \qquad \int \big(f(x)\big)^n f'(x)\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1}.$$

Take your pick which formula you prefer to use. The difficult part is recognising what u or $f(x)$ should be in the function that you are integrating.

We shall do worked Example 36 **c** again using the formula.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 38 5I

Use the formula for the reverse chain rule to find $\int x^2(x^3+5)^3\,dx$.

SOLUTION

The key to finding this integral is to realise that x^2 is a multiple of the derivative of x^3+5. At this point, we can put $u = x^3+5$ or $f(x) = x^3+5$, and everything works smoothly after that. Here are two strong recommendations:

- Show working identifying u or $f(x)$ and its derivative on the right-hand side.
- Write down the standard form above substituting the particular value of n.

Using the first formula,

$$\begin{aligned}\int x^2(x^3+5)^3\,dx &= \tfrac{1}{3}\int 3x^2(x^3+5)^3\,dx \\ &= \tfrac{1}{3}\times\tfrac{1}{4}(x^3+5)^4 + C, \\ &\quad\text{for some constant } C, \\ &= \tfrac{1}{12}(x^3+5)^4 + C.\end{aligned}$$

Let $u = x^3+5$.

Then $\dfrac{du}{dx} = 3x^2$.

Here $\displaystyle\int u^3\frac{du}{dx}\,dx = \tfrac{1}{4}u^4$.

Using the second formula,

$$\begin{aligned}\int x^2(x^3+5)^3\,dx &= \tfrac{1}{3}\int 3x^2(x^3+5)^3\,dx \\ &= \tfrac{1}{3}\times\tfrac{1}{4}(x^3+5)^4 + C, \\ &\quad\text{for some constant } C, \\ &= \tfrac{1}{12}(x^3+5)^4 + C.\end{aligned}$$

Let $f(x) = x^3+5$.

Then $f'(x) = 3x^2$.

Here $\displaystyle\int \big(f(x)\big)^3 f'(x)\,dx = \tfrac{1}{4}\big(f(x)\big)^4$.

Notice that the two notations differ only in the working in the right-hand column.

Example 39 5I

Use the formula for the reverse chain rule to find:

a $\displaystyle\int x\sqrt{1-x^2}\,dx$

b $\displaystyle\int_0^2 x\sqrt{1-x^2}\,dx$

SOLUTION

a This integral is based on the recognition that $\dfrac{d}{dx}(1-x^2) = -2x$.

Using the first formula,

$$\begin{aligned}\int x\sqrt{1-x^2}\,dx &= -\tfrac{1}{2}\int (-2x)\times(1-x^2)^{\frac{1}{2}}\,dx \\ &= -\tfrac{1}{2}\times\tfrac{2}{3}(1-x^2)^{\frac{3}{2}}, \\ &\quad\text{for some constant } C, \\ &= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

Let $u = 1-x^2$.

Then $\dfrac{du}{dx} = -2x$.

Here $\displaystyle\int u^{\frac{1}{2}}\frac{du}{dx}\,dx = \tfrac{2}{3}u^{\frac{3}{2}}$.

ISBN 978-1-108-76630-2

Using the second formula,

$$\int x\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int (-2x)\times(1-x^2)^{\frac{1}{2}}\,dx$$
$$= -\tfrac{1}{2}\times\tfrac{2}{3}(1-x^2)^{\frac{3}{2}}$$
for some constant C,
$$= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + C,$$

Let $f(x) = 1 - x^2$.
Then $f'(x) = -2x$.
Here $\int \left(f(x)^{\frac{1}{2}}\right) f'(x)\,dx = \tfrac{2}{3}\left(f(x)\right)^{\frac{3}{2}}$.

b The definite integral is meaningless because $\sqrt{1-x^2}$ is undefined for $x > 1$.

22 A FORMULA FOR THE REVERSE CHAIN RULE

- The reversing of the chain rule can be written as a formula in two ways:

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C \qquad \text{OR} \qquad \int \left(f(x)\right)^n f'(x)\,dx = \frac{\left(f(x)\right)^{n+1}}{n+1} + C.$$

- The vital step in using this formula is to identity u or $f(x)$ and its derivative.

To use the formula, we have to write the integrand as a product. One factor is a power of a function $f(x)$ or u. The other factor is the derivative of that function.

Exercise 5I

FOUNDATION

1 a Find $\frac{d}{dx}(2x+3)^4$.

b Hence find:

i $\int 8(2x+3)^3\,dx$ **ii** $\int 16(2x+3)^3\,dx$

2 a Find $\frac{d}{dx}(3x-5)^3$.

b Hence find:

i $\int 9(3x-5)^2\,dx$ **ii** $\int 27(3x-5)^2\,dx$

3 a Find $\frac{d}{dx}(1+4x)^5$.

b Hence find:

i $\int 20(1+4x)^4\,dx$ **ii** $\int 10(1+4x)^4\,dx$

4 a Find $\frac{d}{dx}(1-2x)^4$.

b Hence find:

i $\int -8(1-2x)^3\,dx$ **ii** $\int -2(1-2x)^3\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 a Find $\frac{d}{dx}(4x + 3)^{-1}$.

b Hence find:

i $\int -4(4x + 3)^{-2}\,dx$

ii $\int (4x + 3)^{-2}\,dx$

6 a Find $\frac{d}{dx}(2x - 5)^{\frac{1}{2}}$.

b Hence find:

i $\int (2x - 5)^{-\frac{1}{2}}\,dx$

ii $\int \frac{1}{3}(2x - 5)^{-\frac{1}{2}}\,dx$

7 a Find $\frac{d}{dx}(x^2 + 3)^4$.

b Hence find:

i $\int 8x(x^2 + 3)^3\,dx$

ii $\int 40x(x^2 + 3)^3\,dx$

8 a Find $\frac{d}{dx}(x^3 - 1)^5$.

b Hence find:

i $\int 15x^2(x^3 - 1)^4\,dx$

ii $\int 3x^2(x^3 - 1)^4\,dx$

9 a Find $\frac{d}{dx}\sqrt{2x^2 + 3}$.

b Hence find:

i $\int \frac{2x}{\sqrt{2x^2 + 3}}\,dx$

ii $\int \frac{x}{\sqrt{2x^2 + 3}}\,dx$

10 a Find $\frac{d}{dx}(\sqrt{x} + 1)^3$.

b Hence find:

i $\int \frac{3\left(\sqrt{x} + 1\right)^2}{2\sqrt{x}}\,dx$

ii $\int \frac{\left(\sqrt{x} + 1\right)^2}{\sqrt{x}}\,dx$

11 a Find $\frac{d}{dx}(x^3 + 3x^2 + 5)^4$.

b Hence find:

i $\int 12(x^2 + 2x)(x^3 + 3x^2 + 5)^3\,dx$

ii $\int (x^2 + 2x)(x^3 + 3x^2 + 5)^3\,dx$

12 a Find $\frac{d}{dx}(5 - x^2 - x)^7$.

b Hence find:

i $\int (-14x - 7)(5 - x^2 - x)^6\,dx$

ii $\int (2x + 1)(5 - x^2 - x)^6\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

13 Find these indefinite integrals using the reverse chain rule in either form

$$\int f'(x)\,\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + C \qquad \text{OR} \qquad \int u^n\,\frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$$

a $\int 5(5x+4)^3\,dx$ (Let $f(x) = 5x+4$ or $u = 5x+4$.)

b $\int -3(1-3x)^5\,dx$ (Let $f(x) = 1-3x$ or $u = 1-3x$.)

c $\int 2x(x^2-5)^7\,dx$ (Let $f(x) = x^2-5$ or $u = x^2-5$.)

d $\int 3x^2(x^3+7)^4\,dx$ (Let $f(x) = x^3+7$ or $u = x^3+7$.)

e $\int \frac{6x}{(3x^2+2)^2}\,dx$ (Let $f(x) = 3x^2+2$ or $u = 3x^2+2$.)

f $\int \frac{-6x^2}{\sqrt{9-2x^3}}\,dx$ (Let $f(x) = 9-2x^3$ or $u = 9-2x^3$.)

14 Find these indefinite integrals using the reverse chain rule.

a $\int 10x(5x^2+3)^2\,dx$

b $\int 2x(x^2+1)^3\,dx$

c $\int 12x^2(1+4x^3)^5\,dx$

d $\int x(1+3x^2)^4\,dx$

e $\int x^3(1-x^4)^7\,dx$

f $\int 3x^2\sqrt{x^3-1}\,dx$

g $\int x\sqrt{5x^2+1}\,dx$

h $\int \frac{2x}{\sqrt{x^2+3}}\,dx$

i $\int \frac{x+1}{\sqrt{4x^2+8x+1}}\,dx$

j $\int \frac{x}{(x^2+5)^3}\,dx$

15 Evaluate these definite integrals using the reverse chain rule.

a $\int_{-1}^{1} x^2(x^3+1)^4\,dx$

b $\int_0^1 \frac{x}{(5x^2+1)^3}\,dx$

c $\int_0^{\frac{1}{2}} x\sqrt{1-4x^2}\,dx$

d $\int_{-3}^{-1} (x+5)(x^2+10x+3)^2\,dx$

ENRICHMENT

16 Use the reverse chain rule to find:

a $\int \frac{\left(1-\frac{1}{x}\right)^5}{x^2}\,dx$

b $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 **a** What is the domain of the function $f(x) = x\sqrt{x^2 - 1}$?

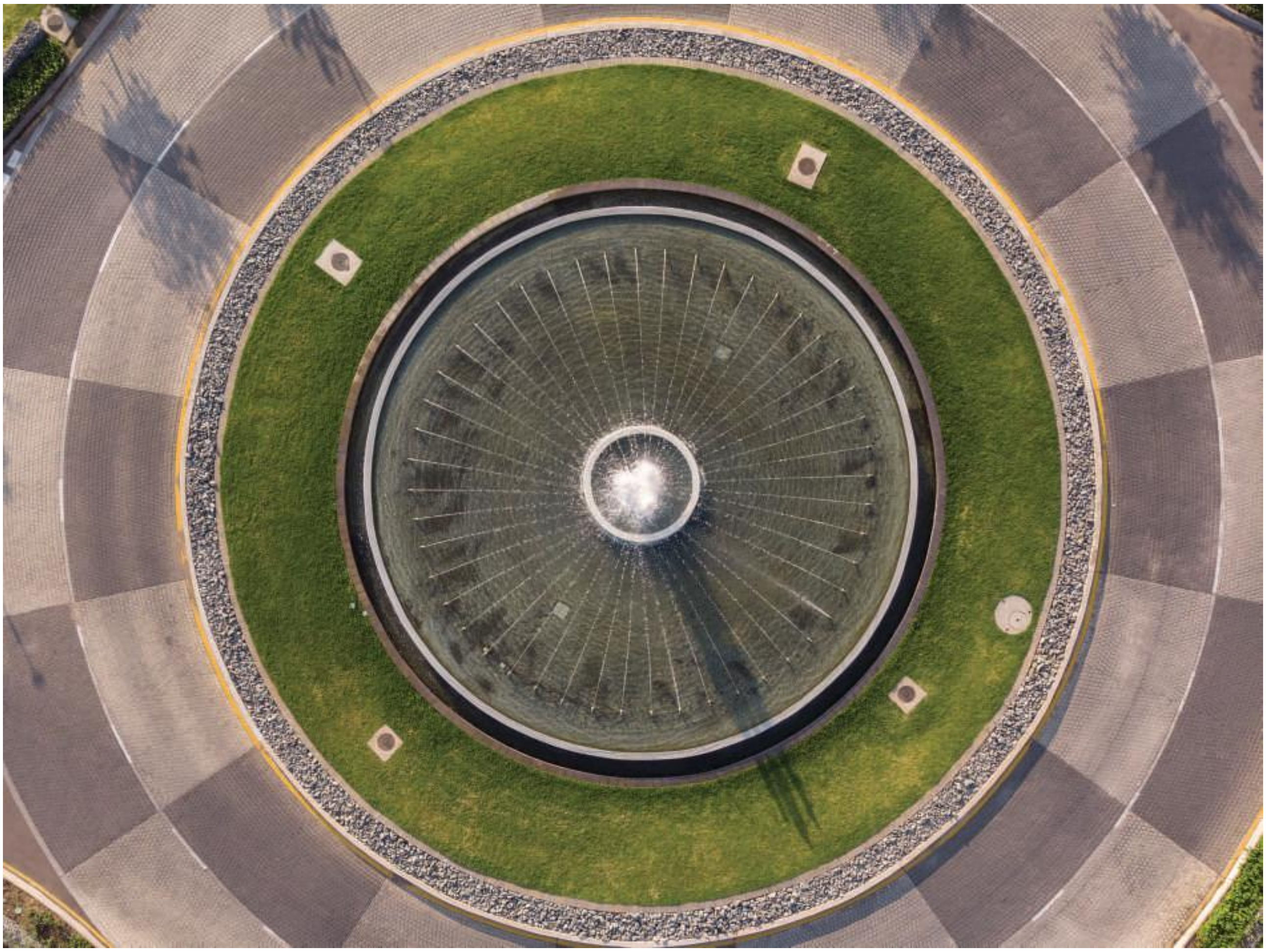
b Find $f'(x)$, and hence show that the function has no stationary points in its domain.

c Show that the function is odd, and hence sketch its graph.

d By evaluating the appropriate definite integral, find the area of the region bounded by the curve and the x-axis from $x = 1$ to $x = 3$.

18 **a** Sketch $y = x(7 - x^2)^3$, indicating all stationary points and intercepts with the axes.

b Find the area of the region enclosed between the curve and the x-axis.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 5 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 5 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Review

1 Evaluate these definite integrals, using the fundamental theorem.

a $\int_0^1 3x^2\,dx$ **b** $\int_1^2 x\,dx$ **c** $\int_2^5 4x^3\,dx$

d $\int_{-1}^1 x^4\,dx$ **e** $\int_{-4}^{-2} 2x\,dx$ **f** $\int_{-3}^{-1} x^2\,dx$

g $\int_0^2 (x+3)\,dx$ **h** $\int_{-1}^4 (2x-5)\,dx$ **i** $\int_{-3}^1 (x^2-2x+1)\,dx$

2 By expanding the brackets where necessary, evaluate these definite integrals.

a $\int_1^3 x(x-1)\,dx$ **b** $\int_{-1}^0 (x+1)(x-3)\,dx$ **c** $\int_0^1 (2x-1)^2\,dx$

3 Write each integrand as separate fractions, then evaluate the integral.

a $\int_1^2 \frac{x^2-3x}{x}\,dx$ **b** $\int_2^3 \frac{3x^4-4x^2}{x^2}\,dx$ **c** $\int_{-2}^{-1} \frac{x^3-2x^4}{x^2}\,dx$

4 **a** **i** Show that $\int_4^k 5\,dx = 5k-20$.

ii Hence find the value of k if $\int_4^k 5\,dx = 10$.

b **i** Show that $\int_0^k (2x-1)\,dx = k^2-k$.

ii Hence find the positive value of k for which $\int_0^k (2x-1)\,dx = 6$.

5 Without finding a primitive, use the properties of the definite integral to evaluate these integrals, stating reasons.

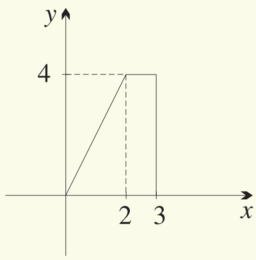
a $\int_3^3 (x^3-5x+4)\,dx$ **b** $\int_{-2}^2 x^3\,dx$ **c** $\int_{-3}^3 (x^3-9x)\,dx$

ISBN 978-1-108-76630-2

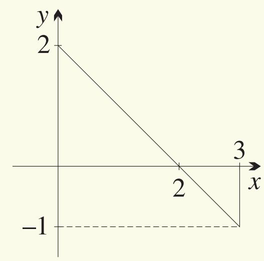
Cambridge University Press
Updated July 2022

6 Use area formulae to find $\int_0^3 f(x)\,dx$, given the following sketches of $f(x)$.

a



b



7 **a** Find each signed area function.

i $A(x) = \int_{-2}^{x} (4 - t)\,dt$

ii $A(x) = \int_{2}^{x} t^{-2}\,dt$

b Differentiate the results in part **a** to find:

i $\dfrac{d}{dx}\int_{-2}^{x} (4 - t)\,dt$

ii $\dfrac{d}{dx}\int_{2}^{x} t^{-2}\,dt$

c Without first performing the integration, use the fundamental theorem of calculus to find these functions.

i $\dfrac{d}{dx}\int_{7}^{x} (t^5 - 5t^3 + 1)\,dt$

ii $\dfrac{d}{dx}\int_{3}^{x} \dfrac{t^2 + 4}{t^2 - 1}\,dt$

8 Find these indefinite integrals.

a $\int (x + 2)\,dx$

b $\int (x^3 + 3x^2 - 5x + 1)\,dx$

c $\int x(x - 1)\,dx$

d $\int (x - 3)(2 - x)\,dx$

e $\int x^{-2}\,dx$

f $\int \dfrac{1}{x^7}\,dx$

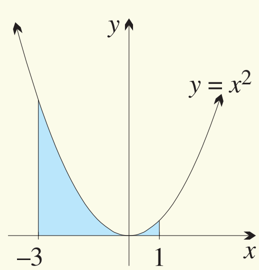
g $\int \sqrt{x}\,dx$

h $\int (x + 1)^4\,dx$

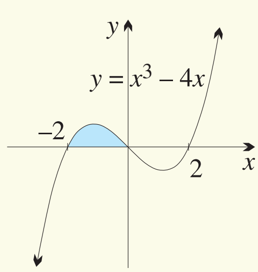
i $\int (2x - 3)^5\,dx$

9 Find the area of each shaded region below by evaluating a definite integral.

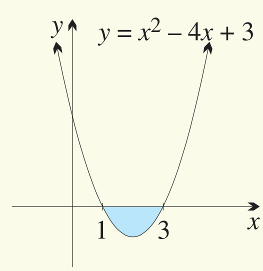
a



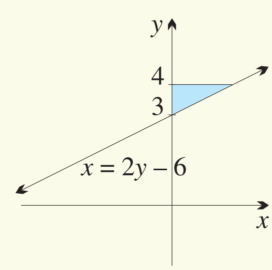
b



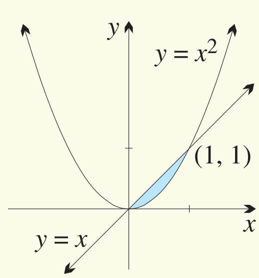
c



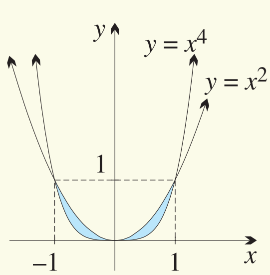
d



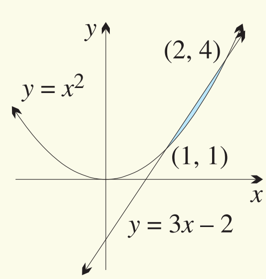
e



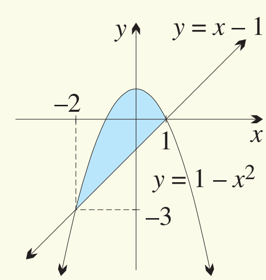
f



g



h



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

10 a By solving the equations simultaneously, show that the curves $y = x^2 - 3x + 5$ and $y = x + 2$ intersect at the points $(1, 3)$ and $(3, 5)$.

b Sketch both curves on the same diagram and find the area of the region enclosed between them.

11 a Use the trapezoidal rule with three function values to approximate $\int_1^3 2^x\,dx$.

b Use the trapezoidal rule with five function values to approximate $\int_1^3 \log_{10} x\,dx$. Give your answer correct to two significant figures.

12 a Find $\frac{d}{dx}(3x + 4)^6$.

b Hence find:

i $\int 18(3x + 4)^5\,dx$ **ii** $\int 9(3x + 4)^5\,dx$

13 a Find $\frac{d}{dx}(x^2 - 1)^3$.

b Hence find:

i $\int 6x(x^2 - 1)^2\,dx$ **ii** $\int x(x^2 - 1)^2\,dx$

14 Find these indefinite integrals using the reverse chain rule.

a $\int 3x^2(x^3 + 1)^4\,dx$ **b** $\int \frac{2x}{(x^2 - 5)^3}\,dx$

15 Use the reverse chain rule to show that $\int_0^1 \frac{x}{\sqrt{x^2 + 3}}\,dx = 2 - \sqrt{3}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 The exponential and logarithmic functions

Chapter 11 of the Year 11 book began to extend calculus beyond algebraic functions to exponential functions and trigonometric functions. This chapter completes what is needed of the calculus of exponential functions, and introduces the calculus of the logarithmic functions. Chapter 7 will then bring the trigonometric functions into calculus as well.

The special number $e \doteqdot 2.7183$ was introduced as the most satisfactory base to use for the powers and logarithms discussed in Chapter 8 last year, and we established the two standard derivatives,

$$\frac{d}{dx}\,e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}\,e^{ax+b} = ae^{ax+b}.$$

We sketched the graphs of $y = e^x$ and its inverse function $y = \log_e x$, transformed them in various ways, and developed exponential growth and decay further in Chapter 16.

All this is assumed knowledge in the present chapter and is quickly reviewed in Section 6A and 6F, apart from exponential growth and decay, which will be reviewed in Chapter 9 on motion and rates, then generalised in Chapter 13 on differential equations.

Sections 6A–6E deal mostly with exponential functions base e, Sections 6F–6J deal mostly with logarithmic functions base e, and the final Section 6K uses the change-of-base formula to extend the topic to exponential and logarithmic functions with bases other than e.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6A Review of exponential functions base e

Section 6A and Section 6F will review the ideas in Sections 11A–11F in the Year 11 book. Two small topics, however, are new in these two review sections.

- Dilations of exponential (Section 6A) and logarithmic (Section 6F) functions.
- Exponential and logarithmic equations reducible to quadratics (Section 6F).

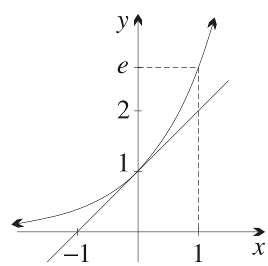
We will not list again the index laws and the logarithmic laws that were covered in Chapter 8 of that book and revisited in Chapter 11, but some early exercises will review them.

The exponential function $y = e^x$ is the subject of Sections 6A–6E. Section 6F then brings the logarithmic function $y = \log_e x$ into the discussion.

The number e and the function $y = e^x$

The fundamental result established in Chapter 11 of the Year 11 book is that the function $y = e^x$ is its own derivative,

$$\frac{d}{dx}e^x = e^x, \qquad \text{that is,} \qquad \text{gradient equals height.}$$



The number $e \doteqdot 2.7183$ is defined to be the base so that the exponential graph $y = e^x$ has gradient exactly 1 at the y-intercept. It is an irrational number, and it plays a role in exponential functions similar to the role that π plays in trigonometric functions.

To the right is a sketch of $y = e^x$. Its most significant properties are listed in Box 1.

1 THE FUNCTION $y = e^x$

- There is only one exponential function $y = e^x$ that is its own derivative, and the number $e \doteqdot 2.7183$ is defined to be the base of this function. Thus

$$\frac{d}{dx}e^x = e^x, \qquad \text{that is,} \qquad \text{at each point, gradient equals height.}$$

- The gradient at the y-intercept is 1.
- The domain is all real numbers, and the range is $y > 0$.
- The line $y = 0$ is a horizontal asymptote.
- The function is one-to-one, that is, the inverse relation is a function.
- Differentiating again, $\dfrac{d^2}{dx^2}e^x = e^x$,

  so the function is always concave up, increasing at an increasing rate.

Sections 6A–6E occasionally require the inverse function $\log_e x$ of e^x, and we need the two inverse function identities:

$$\log_e e^x = x \quad \text{for all real } x \qquad \text{and} \qquad e^{\log_e x} = x \quad \text{for } x > 0.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the calculator

On the calculator, $\boxed{\text{ln}}$ means $\log_e x$ and $\boxed{\text{log}}$ means $\log_{10} x$. The function e^x is usually on the same button as $\log_e x$, and is accessed using $\boxed{\text{shift}}$ followed by $\boxed{\text{ln}}$, or by some similar sequence.

Transformations of $y = e^x$

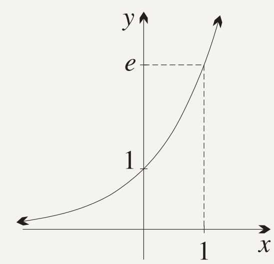
We applied translations and reflections to the curves, as in the next worked example. The first part shows the graph of $y = e^{-x}$, which is just as important in science as $y = e^x$ because $y = e^x$ governs exponential growth, and $y = e^{-x}$ governs exponential decay.

Example 1 — 6A

Sketch each function using a transformation of the graph of $y = e^x$ sketched to the right. Describe the transformation, show and state the y-intercept and the horizontal asymptote, and write down the range.

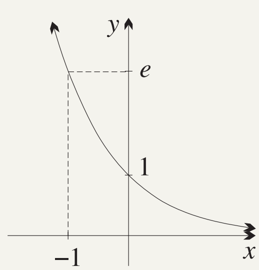
a $y = e^{-x}$ **b** $y = e^x + 3$ **c** $y = e^{x-2}$

Which transformations can also be done using a dilation?



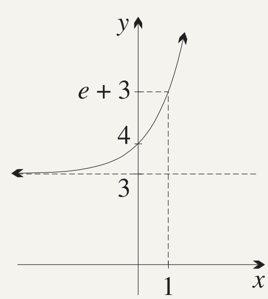
SOLUTION

a



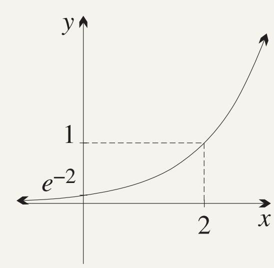
To graph $y = e^{-x}$,
reflect $y = e^x$ in y-axis.
y-intercept: $(0, 1)$
asymptote: $y = 0$
range: $y > 0$

b



To graph $y = e^x + 3$,
shift $y = e^x$ up 3.
y-intercept: $(0, 4)$
asymptote: $y = 3$
range: $y > 3$

c



To graph $y = e^{x-2}$,
shift $y = e^x$ right 2.
y-intercept: $(0, e^{-2})$
asymptote: $y = 0$
range: $y > 0$

- The equation $y = e^{-x}$ in part **a** is a reflection in the y-axis, and any reflection in the y-axis can be regarded as a horizontal dilation with factor -1.
- The equation $y = e^{x-2}$ in part **c** can be written as $y = e^{-2} \times e^x$, so it is also a vertical dilation of $y = e^x$ with factor $e^{-2} \doteqdot 0.135$.

Dilations of $y = e^x$

Dilations were only introduced in Section 3H of this book. In the context of exponential and logarithmic functions, dilations need further attention because some of them have an interesting property — they can be done with a shift in the other direction, as we have already seen in part **c** above.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

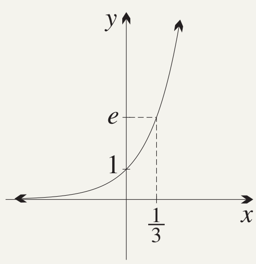
Example 2 6A

Use dilations of $y = e^x$ to generate a sketch of each function. Identify which dilation is also a shift in the other direction.

a $y = e^{3x}$ **b** $y = 3e^x$

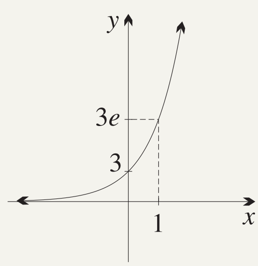
SOLUTION

a



Dilate $y = e^x$ vertically with factor $\frac{1}{3}$.

b



Dilate $y = e^x$ vertically with factor 3.

- $y = 3e^x$ can be written as $y = e^{\log_e 3} \times e^x = e^{x + \log_e 3}$, so it can also be regarded as a shift left by $\log_e 3$.

Tangents and normals to the exponential function

We applied the derivative to sketches of exponential functions. Here is a shortened form of the worked example given in Section 11D of the Year 11 book.

Example 3 6A

Let A be the point on the curve $y = 2e^x$ where $x = 1$.

a Find the equations of the tangent and normal at the point A.

b Show that the tangent at A passes through the origin, and find the point B where the normal meets the x-axis.

c Sketch the situation and find the area of $\triangle AOB$.

SOLUTION

a Substituting into $y = 2e^x$ shows that $A = (1, 2e)$.

Differentiating $y = 2e^x$ gives $\quad y' = 2e^x$,

so at $A(1, 2e)$, where $x = 1$, $\quad y' = 2e \quad$ (which we know because gradient = height).

Hence, using point–gradient form, the tangent at A is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2e &= 2e(x - 1) \\ y &= 2ex. \end{aligned}$$

The normal at A has gradient $-\dfrac{1}{2e}$ (it is perpendicular to the tangent),

so its equation is $\quad y - 2e = -\dfrac{1}{2e}(x - 1)$

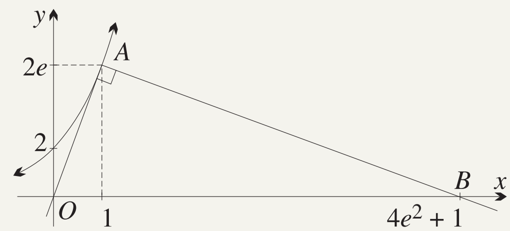
$\boxed{\times\ 2e}$ $\quad 2ey - 4e^2 = -x + 1$

$$x + 2ey = 4e^2 + 1.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b The tangent passes through the origin O because its y-intercept is zero.
To find the x-intercept B of the normal,
put $y = 0$,
thus $x = 4e^2 + 1$,
so B has coordinates $(4e^2 + 1, 0)$.



c Hence area $\triangle AOB = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times (4e^2 + 1) \times 2e$
$= e(4e^2 + 1)$ square units.

Exercise 6A

FOUNDATION

Note: You will need the $\boxed{e^x}$ function on your calculator. This will require $\boxed{\text{shift}}$ followed by $\boxed{\ln}$, or some similar sequence of keys.

1 Simplify these expressions using the index laws.
a $2^3 \times 2^7$ **b** $e^4 \times e^3$ **c** $2^6 \div 2^2$ **d** $e^8 \div e^5$ **e** $(2^3)^4$ **f** $(e^5)^6$

2 Simplify these expressions using the index laws.
a $e^{2x} \times e^{5x}$ **b** $e^{10x} \div e^{8x}$ **c** $(e^{2x})^5$ **d** $e^{2x} \times e^{-7x}$ **e** $e^x \div e^{-4x}$ **f** $(e^{-3x})^4$

3 Write each expression as a power of e, then use your calculator to approximate it correct to four significant figures.
a e^2 **b** e^{-3} **c** e **d** $\dfrac{1}{e}$ **e** $\sqrt{e}$ **f** $\dfrac{1}{\sqrt{e}}$

4 a Write down the first and second derivatives of $y = e^x$.
b Hence copy and complete the sentence, 'The curve $y = e^x$ is always concave . . . , and is always . . . at . . . rate.'

5 a Find the gradient of the tangent to $y = e^x$ at $P(1, e)$, then find the equation of the tangent at P and show that it has x-intercept 0.
b Similarly find the equation of the tangent at $Q(0, 1)$, and show that its x-intercept is -1.
c Find the equation of the tangent at $R(-1, \frac{1}{e})$, and show that its x-intercept is -2.

6 a What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?
b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.
c Hence find the equations of the tangent and normal at P (in general form).

7 Sketch each curve using a single transformation of $y = e^x$, and describe the transformation.
a $y = e^x + 1$ **b** $y = e^x - 2$ **c** $y = \frac{1}{3}e^x$ **d** $y = e^{\frac{1}{2}x}$

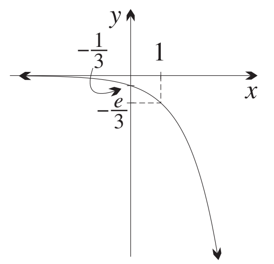
8 Sketch each curve using a single transformation of $y = e^{-x}$, and describe the transformation.
a $y = e^{-x} - 1$ **b** $y = -e^{-x}$ **c** $y = e^{-2x}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

9 The graph to the right is a dilation of $y = e^x$. Describe the dilation, and write down the equation of the curve.



10 Expand and simplify:

a $(e^x + 1)(e^x - 1)$
b $(e^{4x} + 3)(e^{2x} + 3)$
c $(e^{-3x} - 2)e^{3x}$
d $(e^{-2x} + e^{2x})^2$

11 Write as a sum of powers of e:

a $\dfrac{e^{4x} + e^{3x}}{e^{2x}}$
b $\dfrac{e^{2x} - e^{3x}}{e^{4x}}$
c $\dfrac{e^{10x} + 5e^{20x}}{e^{-10x}}$
d $\dfrac{6e^{-x} + 9e^{-2x}}{3e^{3x}}$

12 a What is the gradient of the tangent to $y = e^x$ at its y-intercept?
b What transformation maps $y = e^x$ to $y = e^{-x}$?
c Use this transformation to find the gradient of $y = e^{-x}$ at its y-intercept.
d Sketch $y = e^x$ and $y = e^{-x}$ on one set of axes.
e How can the transformation be interpreted as a dilation?

13 Write down the first four derivatives of each function. For which curves is it true that at each point on the curve, the gradient equals the height?

a $y = e^x + 5$
b $y = e^x + x^3$
c $y = 4e^x$
d $y = 5e^x + 5x^2$

14 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = e^x$ at the points where:

a $x = 0$
b $x = 1$
c $x = -2$
d $x = 5$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

15 a What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?
b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.
c Hence find the equation of the tangent at the point P found in part **a**.

ENRICHMENT

16 a Use, and describe, a dilation to sketch $y = e^{2x}$.
b Use, and describe, a subsequent translation to sketch $y = e^{2(x-1)}$.
c Use, and describe, a subsequent dilation to sketch $y = \frac{1}{2}e^{2(x-1)}$.
d Use, and describe, a subsequent translation to sketch $y = \frac{1}{2}e^{2(x-1)} - 2$.

17 a Interpret the transformation from $y = e^x$ to $y = e^{x+2}$ as a translation. Then interpret it as a dilation.
b Interpret the transformation from $y = e^x$ to $y = 2e^x$ as a dilation. Then interpret it as a translation by first writing the coefficient 2 as $e^{\log_e 2}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6B Differentiation of exponential functions

We can now develop the calculus of functions involving e^x, picking up the story at differentiation, where two standard forms were established in Chapter 11 (Year 11),

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}e^{ax+b} = ae^{ax+b}.$$

Using the two standard forms

The second standard form above requires the chain rule with $u = ax + b$. It is proven again in Question **6a** of Exercise 6B.

Example 4 — 6B

Differentiate:

a $y = e^x + e^{-x}$ **b** $y = 5e^{4x-3}$ **c** $y = e^{2-\frac{1}{2}x}$ **d** $y = \sqrt{e^x} + \dfrac{1}{\sqrt{e^x}}$

SOLUTION

a Given $y = e^x + e^{-x}$.
For e^{-x}, $a = -1$ and $b = 0$,
so $y' = e^x - e^{-x}$.

b Given $y = 5e^{4x-3}$.
Here $a = 4$ and $b = -3$,
so $y' = 20e^{4x-3}$.

c Given $y = e^{2-\frac{1}{2}x}$.
Here $a = -\frac{1}{2}$ and $b = 2$,
so $y' = -\frac{1}{2}e^{2-\frac{1}{2}x}$.

d Here $y = \sqrt{e^x} + \dfrac{1}{\sqrt{e^x}}$
$y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$,
so $y' = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x}$.

Differentiating using the chain rule

The chain rule can be applied in the usual way. As always, the full setting out should continue to be used until readers are very confident with missing some of the steps.

Example 5 — 6B

Use the chain rule to differentiate:

a $y = e^{1-x^2}$ **b** $y = (e^{2x} - 3)^4$

SOLUTION

a Here $y = e^{1-x^2}$.
Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -2xe^{1-x^2}.$$

Let $u = 1 - x^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = -2x$
and $\dfrac{dy}{du} = e^u$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Here $y = (e^{2x} - 3)^4$.

Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4(e^{2x} - 3)^3 \times 2e^{2x} \\ &= 8e^{2x}(e^{2x} - 3)^3.\end{aligned}$$

Let $u = e^{2x} - 3$.

Then $y = u^4$.

Hence $\dfrac{du}{dx} = 2e^{2x}$

and $\dfrac{dy}{du} = 4u^3$.

A formula for the chain rule

Some people prefer to learn a formula for chain rule differentiation that can be used for part **a** above. The formula can be written in two ways, using u and using $f(x)$,

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \qquad \text{OR} \qquad \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x).$$

If you use the formula, choose which form you prefer. In the next worked example, part **a** of the previous example is done again using both forms of the formula. Make sure that you are using the right formula, and that you show at least u or $f(x)$ on the right.

Notice that part **b** requires the formula for differentiating powers of functions of x, as reviewed in Section 5I,

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \text{OR} \qquad \frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}f'(x).$$

Example 6 — 6B

Use the chain rule, with a shorter setting out, to differentiate:

a $y = e^{1-x^2}$

b $y = (e^{2x} - 3)^4$

SOLUTION

a $y = e^{1-x^2}$

$y' = -2xe^{1-x^2}$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$

$\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$.

OR

Let $f(x) = 1 - x^2$.

Then $f'(x) = -2x$

$\dfrac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$.

b $y = (e^{2x} - 3)^4$

$y' = 8e^{2x} \times (e^{2x} - 3)^3$

Let $u = e^{2x} - 3$.

Then $\dfrac{du}{dx} = 2e^{2x}$

$\dfrac{d}{dx}u^4 = 4u^3\dfrac{du}{dx}$.

OR

Let $f(x) = e^{2x} - 3$.

Then $f'(x) = 2e^{2x}$

$\dfrac{d}{dx}\big(f(x)\big)^4 = 4\big(f(x)\big)^3 f'(x)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 THREE STANDARD DERIVATIVES FOR EXPONENTIAL FUNCTIONS

- $\dfrac{d}{dx} e^x = e^x$
- $\dfrac{d}{dx} e^{ax+b} = ae^{ax+b}$
- $\dfrac{d}{dx} e^u = e^u \dfrac{du}{dx}$ OR $\dfrac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$

Using the product rule

A function such as $y = x^3 e^x$ is the product of the two functions $u = x^3$ and $v = e^x$. Thus it can be differentiated by the product rule.

Often the result can be factored, allowing any stationary points to be found.

Example 7 — 6B

Find the derivatives of these functions. Then factor the derivative and write down all the stationary points.

a $y = x^3 e^x$ **b** $y = xe^{5x-2}$

SOLUTION

a Here $y = x^3 e^x$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= e^x \times 3x^2 + x^3 \times e^x,$$

and taking out the common factor $x^2 e^x$,

$$y' = x^2 e^x (3 + x).$$

Hence y' has zeroes at $x = 0$ and $x = -3$,

and the stationary points are $(0, 0)$ and $(-3, -27e^{-3})$.

Let $u = x^3$
and $v = e^x$.
Then $u' = 3x^2$
and $v' = e^x$.

b Here $y = xe^{5x-2}$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= e^{5x-2} \times 1 + x \times 5e^{5x-2}$$
$$= e^{5x-2}(1 + 5x).$$

Hence y' has a stationary point at $\left(-\frac{1}{5}, -\frac{1}{5}e^{-3}\right)$.

Let $u = x$
and $v = e^{5x-2}$.
Then $u' = 1$
and $v' = 5e^{5x-2}$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Using the quotient rule

A function such as $y = \dfrac{e^{5x}}{x}$ is the quotient of the two functions $u = e^{5x}$ and $v = x$. Thus it can be differentiated by the quotient rule.

Example 8 — 6B

Differentiate these functions, then find the x-values of all stationary points.

a $\dfrac{e^{5x}}{x}$

b $\dfrac{e^x}{1 - x^2}$

SOLUTION

a Let $y = \dfrac{e^{5x}}{x}$. Then applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2} = \frac{5xe^{5x} - e^{5x}}{x^2} = \frac{e^{5x}(5x - 1)}{x^2}.$$

Let $u = e^{5x}$
and $v = x$.
Then $u' = 5e^{5x}$
and $v' = 1$.

Hence there is a stationary point where $x = \frac{1}{5}$.

b Let $y = \dfrac{e^x}{1 - x^2}$. Then applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2} = \frac{(1 - x^2)e^x + 2xe^x}{(1 - x^2)^2} = \frac{e^x(1 + 2x - x^2)}{(1 - x^2)^2}.$$

Let $u = e^x$
and $v = 1 - x^2$.
Then $u' = e^x$
and $v' = -2x$.

Hence there is a stationary point where $x^2 - 2x - 1 = 0$,
and calculating $\Delta = 8$ first, $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$.

Exercise 6B

FOUNDATION

Technology: Programs that perform algebraic differentiation can be used to confirm the answers to many of these questions.

1 Use the standard form $\dfrac{d}{dx} e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{7x}$ **b** $y = 4e^{3x}$ **c** $y = 6e^{\frac{1}{3}x}$ **d** $y = -\frac{1}{2}e^{-2x}$
e $y = e^{3x+4}$ **f** $y = e^{4x-3}$ **g** $y = e^{-3x+4}$ **h** $y = e^{-2x-7}$

2 Differentiate:

a $y = e^x + e^{-x}$ **b** $y = e^{2x} - e^{-3x}$ **c** $y = \dfrac{e^x - e^{-x}}{2}$
d $y = \dfrac{e^x + e^{-x}}{3}$ **e** $y = \dfrac{e^{2x}}{2} + \dfrac{e^{3x}}{3}$ **f** $y = \dfrac{e^{4x}}{4} + \dfrac{e^{5x}}{5}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Use the index laws to write each expression as a single power of e, then differentiate it.

a $y = e^x \times e^{2x}$ **b** $y = e^{3x} \times e^{-x}$ **c** $y = (e^x)^2$ **d** $y = (e^{2x})^3$

e $y = \dfrac{e^{4x}}{e^x}$ **f** $y = \dfrac{e^x}{e^{2x}}$ **g** $y = \dfrac{1}{e^{3x}}$ **h** $y = \dfrac{1}{e^{5x}}$

4 a i For the function $f(x) = e^{-x}$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

ii What is the pattern in these derivatives?

b i For the function $f(x) = e^{2x}$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

ii What is the pattern in these derivatives?

5 Expand the brackets and then differentiate:

a $e^x(e^x + 1)$ **b** $e^{-x}(2e^{-x} - 1)$ **c** $(e^x + 1)^2$ **d** $(e^x + 3)^2$

e $(e^x - 1)^2$ **f** $(e^x - 2)^2$ **g** $(e^x + e^{-x})(e^x - e^{-x})$ **h** $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

DEVELOPMENT

6 Use the chain rule with full setting-out to differentiate:

a $y = e^{ax+b}$ **b** $y = e^{x^2}$ **c** $y = e^{-\frac{1}{2}x^2}$ **d** $y = e^{x^2+1}$

e $y = e^{1-x^2}$ **f** $y = e^{x^2+2x}$ **g** $y = e^{6+x-x^2}$ **h** $y = \frac{1}{2}e^{3x^2-2x+1}$

7 Use the product rule to differentiate:

a $y = xe^x$ **b** $y = xe^{-x}$ **c** $y = (x - 1)e^x$ **d** $y = (x + 1)e^{3x-4}$

e $y = x^2e^{-x}$ **f** $y = (2x - 1)e^{2x}$ **g** $y = (x^2 - 5)e^x$ **h** $y = x^3e^{2x}$

8 Use the quotient rule to differentiate:

a $y = \dfrac{e^x}{x}$ **b** $y = \dfrac{x}{e^x}$ **c** $y = \dfrac{e^x}{x^2}$ **d** $y = \dfrac{x^2}{e^x}$

e $y = \dfrac{e^x}{x + 1}$ **f** $y = \dfrac{x + 1}{e^x}$ **g** $y = \dfrac{x - 3}{e^{2x}}$ **h** $y = \dfrac{1 - x^2}{e^x}$

9 Expand and simplify each expression, then differentiate it.

a $(e^x + 1)(e^x + 2)$ **b** $(e^{2x} + 3)(e^{2x} - 2)$ **c** $(e^{-x} + 2)(e^{-x} + 4)$

d $(e^{-3x} - 1)(e^{-3x} - 5)$ **e** $(e^{2x} + 1)(e^x + 1)$ **f** $(e^{3x} - 1)(e^{-x} + 4)$

10 Use the chain rule to differentiate:

a $y = (1 - e^x)^5$ **b** $y = (e^{4x} - 9)^4$ **c** $y = \dfrac{1}{e^x - 1}$ **d** $y = \dfrac{1}{(e^{3x} + 4)^2}$

11 a Show by substitution that the function $y = 3e^{2x}$ satisfies the equation $\dfrac{dy}{dx} = 2y$.

b Show by substitution that the function $y = 5e^{-4x}$ satisfies the equation $\dfrac{dy}{dx} = -4y$.

12 Find the first and second derivatives of each function below. Then evaluate both derivatives at the value given.

a $f(x) = e^{2x+1}$ at $x = 0$ **b** $f(x) = e^{-3x}$ at $x = 1$

c $f(x) = xe^{-x}$ at $x = 2$ **d** $f(x) = e^{-x^2}$ at $x = 0$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{ax}$ **b** $y = e^{-kx}$ **c** $y = Ae^{kx}$ **d** $y = Be^{-\ell x}$

e $y = e^{px+q}$ **f** $y = Ce^{px+q}$ **g** $y = \dfrac{e^{px} + e^{-qx}}{r}$ **h** $\dfrac{e^{ax}}{a} + \dfrac{e^{-px}}{p}$

14 Use the product, quotient and chain rules as appropriate to differentiate:

a $y = (e^x + 1)^3$ **b** $y = (e^x + e^{-x})^4$ **c** $y = (1 + x^2)e^{1+x}$

d $y = (x^2 - x)e^{2x-1}$ **e** $y = \dfrac{e^x}{e^x + 1}$ **f** $y = \dfrac{e^x + 1}{e^x - 1}$

15 Write each expression as the sum of simple powers of e, then differentiate it.

a $y = \dfrac{e^x + 1}{e^x}$ **b** $y = \dfrac{e^{2x} + e^x}{e^x}$ **c** $y = \dfrac{2 - e^x}{e^{2x}}$

d $y = \dfrac{3 + e^x}{e^{4x}}$ **e** $y = \dfrac{e^x + e^{2x} - 3e^{4x}}{e^x}$ **f** $y = \dfrac{e^{2x} + 2e^x + 1}{e^{2x}}$

16 **a** Show that $y = 2e^{3x}$ is a solution of each equation by substituting separately into the LHS and RHS:

i $y' = 3y$ **ii** $y'' - 9y = 0$

b Show that $y = \frac{1}{2}e^{-3x} + 4$ is a solution of $\dfrac{dy}{dx} = -3(y - 4)$ by substituting y into each side of the equation.

c Show that $y = e^{-3x} + x - 1$ is a solution of the differential equation $y'' + 2y' - 3y = 5 - 3x$.

d Show by substitution that each function is a solution of the equation $y'' + 2y' + y = 0$.

i $y = e^{-x}$ **ii** $y = xe^{-x}$

ENRICHMENT

17 Differentiate these functions.

a $y = \sqrt{e^x}$ **b** $y = \sqrt[3]{e^x}$ **c** $y = \dfrac{1}{\sqrt{e^x}}$ **d** $y = \dfrac{1}{\sqrt[3]{e^x}}$

e $e^{\sqrt{x}}$ **f** $e^{-\sqrt{x}}$ **g** $e^{\frac{1}{x}}$ **h** $e^{-\frac{1}{x}}$

i $e^{x-\frac{1}{x}}$ **j** e^{e^x}

18 Define the two functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

a Show that $\dfrac{d}{dx}\cosh x = \sinh x$ and $\dfrac{d}{dx}\sinh x = \cosh x$.

b Find the second derivative of each function, and show that they both satisfy $y'' = y$.

c Show that $\cosh^2 x - \sinh^2 x = 1$.

19 **a** Show that $y = Ae^{kx}$ is a solution of:

i $y' = ky$ **ii** $y'' - k^2 y = 0$

b Show that $y = Ae^{kx} + C$ is a solution of $\dfrac{dy}{dx} = k(y - C)$.

c Show that $y = (Ax + B)e^{3x}$ is a solution of $y'' - 6y' + 9y = 0$.

20 Find the values of λ that make $y = e^{\lambda x}$ a solution of:

a $y'' + 3y' - 10y = 0$ **b** $y'' + y' - y = 0$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6C Applications of differentiation

Differentiation can now be applied in the usual ways to examine functions involving e^x. Sketching of such curves is an important application. Some of these sketches require some subtle limits that would normally be given if a question needed them.

The graphs of e^x and e^{-x}

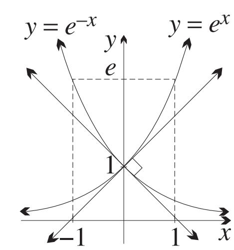
The graphs of $y = e^x$ and $y = e^{-x}$ are the fundamental graphs of this chapter. Because x is replaced by $-x$ in the second equation, the two graphs are reflections of each other in the y-axis.

For $y = e^x$:

x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$. The gradient of $y = e^x$ at $(0, 1)$ is 1, so by reflection, the gradient of $y = e^{-x}$ at $(0, 1)$ is -1. This means that the curves are perpendicular at their point of intersection.

As remarked earlier, the function $y = e^{-x}$ is just as important as $y = e^x$ in applications. It describes a great many physical situations where a quantity 'dies away exponentially', like the dying away of the sound of a plucked string.

An example of curve sketching

The following curve-sketching example illustrates the use of the six steps of our informal curve-sketching menu in the context of exponential functions. One special limit is given in part **d** so that the sketch may be completed.

Example 9 6C

Sketch the graph of $y = xe^{-x}$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function, and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. (You may assume that as $x \to \infty$, $xe^{-x} \to 0$.)

e Find any stationary points and examine their nature.

f Find any points of inflection, and examine the concavity.

ISBN 978-1-108-76630-2

SOLUTION

a The domain of $y = xe^{-x}$ is the whole real number line.

b $f(-x) = -xe^{x}$, which is neither $f(x)$ nor $-f(x)$, so the function is neither even nor odd.

c The only zero is $x = 0$. From the table of signs, y is positive for $x > 0$ and negative for $x < 0$.

x	-1	0	1
y	$-e$	0	e^{-1}
sign	$-$	0	$+$

d As given in the question, $y \to 0$ as $x \to \infty$, so the x-axis is a horizontal asymptote on the right. Also, $y \to -\infty$ as $x \to -\infty$.

e Differentiating using the product rule,

$$\begin{aligned} f'(x) &= vu' + uv' \\ &= e^{-x} - xe^{-x} \\ &= e^{-x}(1 - x). \end{aligned}$$

Let $u = x$
and $v = e^{-x}$.
Then $u' = 1$
and $v' = -e^{-x}$.

Hence $f'(x) = 0$ when $x = 1$ (notice that e^{-x} can never be zero), so $\left(1, \frac{1}{e}\right)$ is the only stationary point.

Differentiating again by the product rule,

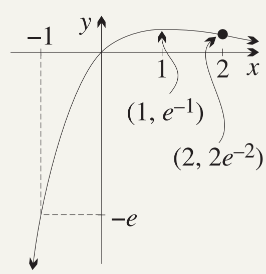
$$\begin{aligned} f''(x) &= vu' + uv' \\ &= -e^{-x} - (1 - x)e^{-x} \\ &= e^{-x}(x - 2), \end{aligned}$$

Let $u = 1 - x$
and $v = e^{-x}$.
Then $u' = -1$
and $v' = -e^{-x}$.

so $f''(1) = -e^{-1} < 0$, and $(1, e^{-1})$ is thus a maximum turning point.

f $f''(x) = e^{-x}(x - 2)$ has a zero at $x = 2$, and taking test values around $x = 2$,

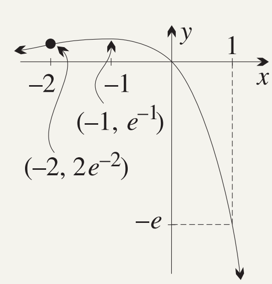
x	0	2	3
$f''(x)$	-2	0	e^{-3}
	$\frown$	$\cdot$	$\smile$



Thus there is an inflection at $(2, 2e^{-2}) \doteqdot (2, 0.27)$.
The curve is concave down for $x < 2$ and concave up for $x > 2$.

Example 10 6C

[Transforming graphs]
Use a suitable transformation of the graph sketched in the previous worked example to sketch $y = -xe^{x}$.



SOLUTION

$y = xe^{-x}$ becomes $y = -xe^{x}$ when x is replaced by $-x$. Graphically, this transformation is a reflection in the y-axis, hence the new graph is as sketched to the right.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A difficulty with the limits of xe^x and xe^{-x}

Sketching the graph of $y = xe^{-x}$ above required knowing the behaviour of xe^{-x} as $x \to \infty$. This limit is puzzling, because when x is a large number, e^{-x} is a small positive number, and the product of a large number and a small number could be large, small, or anything in between.

In fact, e^{-x} gets small as $x \to \infty$ much more quickly than x gets large, and the product xe^{-x} gets small. The technical term for this is that e^{-x} *dominates* x. A table of values should make it reasonably clear that $\lim_{x\to\infty} xe^{-x} = 0$.

x	0	1	2	3	4	5	6	7	$\cdots$
xe^{-x}	0	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	$\cdots$
approx	0	0.37	0.27	0.15	0.073	0.034	0.015	0.006	$\cdots$

Similarly, when x is a large negative number, e^x is a very small number, so it is unclear whether xe^x is large or small. Again, e^x dominates x, meaning that $xe^x \to 0$ as $x \to -\infty$. A similar table should make this reasonably obvious.

x	0	-1	-2	-3	-4	-5	-6	-7	$\cdots$
xe^x	0	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	$\cdots$
approx	0	-0.37	-0.27	-0.15	-0.073	-0.034	-0.015	-0.006	$\cdots$

These limits would normally be given in any question where they are needed, but we will nevertheless box the general results for completeness.

3 DOMINANCE

- The function e^x *dominates* the function x, that is

$$\lim_{x\to\infty} xe^{-x} = 0 \quad\text{and}\quad \lim_{x\to-\infty} xe^x = 0.$$

- More generally, the function e^x *dominates* the function x^k, for all $k > 0$,

$$\lim_{x\to\infty} x^k e^{-x} = 0 \quad\text{and}\quad \lim_{x\to-\infty} x^k e^x = 0.$$

More colourfully, in a battle between x and e^x, e^x always wins. We will provide a proof in Question 23 of Exercise 6J.

Exercise 6C

FOUNDATION

Technology: Graphing programs can be used in this exercise to sketch the curves and then investigate the effects on the curve of making small changes in the equations. It is advisable, however, to puzzle out most of the graphs first using the standard methods of the curve-sketching menu.

The results about dominance would normally be given in questions where they are needed, but this exercise is an exception because so many questions require them.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

1 **a** Find the y-coordinate of the point A on the curve $y = e^{2x-1}$ where $x = \frac{1}{2}$.
b Find the derivative of $y = e^{2x-1}$, and show that the gradient of the tangent at A is 2.
c Hence find the equation of the tangent at A, and prove that it passes through O.

2 **a** Write down the coordinates of the point R on the curve $y = e^{3x+1}$ where $x = -\frac{1}{3}$.
b Find $\dfrac{dy}{dx}$ and hence show that the gradient of the tangent at R is 3.
c What is the gradient of the normal at R?
d Hence find the equation of the normal at R in general form.

3 **a** Find the equation of the normal to $y = e^{-x}$ at the point $P(-1, e)$.
b Find the x- and y-intercepts of the normal.
c Find the area of the triangle whose vertices lie at the intercepts and the origin.

4 **a** Find the equation of the tangent to $y = e^{x}$ at its y-intercept $B(0, 1)$.
b Find the equation of the tangent to $y = e^{-x}$ at its y-intercept $B(0, 1)$.
c Find the points F and G where the tangents in parts **a** and **b** meet the x-axis.
d Sketch $y = e^{x}$ and $y = e^{-x}$ on the same set of axes, showing the two tangents.
e What sort of triangle is $\triangle BFG$, and what is its area?

5 **a** Show that the equation of the tangent to $y = 1 - e^{-x}$ at the origin is $y = x$.
b Deduce the equation of the normal at the origin without further use of calculus.
c What is the equation of the asymptote of this curve?
d Sketch the curve, showing the points T and N where the tangent and normal respectively cut the asymptote.
e Find the area of $\triangle OTN$.

6 **a** Find the first and second derivatives for the curve $y = x - e^{x}$.
b Deduce that the curve is concave down for all values of x.
c Find any stationary points, then determine their nature using the second derivative.
d Sketch the curve and write down its range.
e Finally, sketch $y = e^{x} - x$ by recognising the simple transformation.

DEVELOPMENT

7 **a** Show that the tangent to $y = e^{x}$ at $T(t, e^{t})$ has gradient e^{t}.
b Find the equation of the tangent at $x = t$, and show that its x-intercept is $t - 1$.
c Compare this result with Question **4** parts **a** and **c** above, and explain geometrically what is happening.

8 Consider the curve $y = x\,e^{x}$.
a Where is the function zero, positive and negative? Is it even, odd or neither?
b Show that $y' = (1 + x)\,e^{x}$ and $y'' = (2 + x)\,e^{x}$.
c Show that there is one stationary point, and determine its nature.
d Find the coordinates of the lone point of inflection.
e What happens to y, y' and y'' as $x \to \infty$?
f Given that $y \to 0$ as $x \to -\infty$, sketch the curve, then write down its range.
g Hence also sketch $y = -x\,e^{-x}$ by recognising the simple transformation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 Consider the function $y = (1 - x)\, e^x$.

- **a** Find the zero and draw up a table of signs.
- **b** Find y' and y''.
- **c** Show that the curve has a maximum turning point at its y-intercept, and a point of inflection at $(-1, 2\, e^{-1})$.
- **d** What happens to y, y' and y'' as $x \to \infty$?
- **e** Given that $y \to 0$ as $x \to -\infty$, sketch the graph and write down its range.

10
- **a** Given that $y = x^2\, e^{-x}$, show that $y' = x(2 - x)\, e^{-x}$ and $y'' = (2 - 4x + x^2)\, e^{-x}$.
- **b** Show that the function has a minimum turning point at the origin and a maximum turning point at $(2, 4\, e^{-2})$.
- **c** **i** Show that $y'' = 0$ at $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$.
  - **ii** Use a table of values for y'' to show that there are inflection points at these values.
- **d** Given that $y \to 0$ as $x \to \infty$, sketch the graph and write down its range.

11
- **a** Find the intercepts of $y = (1 + x)^2\, e^{-x}$.
- **b** Show that the curve has two turning points, and classify them.
- **c** Examine the behaviour of y as $x \to \infty$ and hence deduce that it also has two inflections.
- **d** Sketch the curve and write down its range.

12 Show that $y = (x^2 + 3x + 2)\, e^x$ has an inflection point at one of its x-intercepts. Sketch the curve and label all important features (ignore the y-coordinates of the stationary points).

13
- **a** What is the natural domain of $y = \dfrac{e^x}{x}$?
- **b** Show that the curve has a local minimum at $(1, e)$ but no inflection points.
- **c** Sketch the curve and state its range.

14
- **a** Given that $y = e^{-\frac{1}{2}x^2}$, find y' and y''.
- **b** Show that this curve has a maximum turning point at its y-intercept, and has two points of inflection.
- **c** Examine the behaviour of y as $x \to -\infty$ and $x \to \infty$.
- **d** Sketch the graph and write down its range.

15
- **a** Classify the stationary points of $y = x\, e^{-x^2}$.
- **b** Locate the three inflection points, sketch the curve and write down its range.

16 [Technology] This question extends the remarks made in the text about *dominance*. Using a calculator or a spreadsheet, draw up a table of values to examine:

- **a** the behaviour of $y = \dfrac{e^x}{x}$: **i** as $x \to -\infty$, **ii** as $x \to \infty$.
- **b** the behaviour of $y = \dfrac{e^{-x}}{x}$: **i** as $x \to -\infty$, **ii** as $x \to \infty$.
- **c** the behaviour of $x^k e^k$ as $x \to \infty$ and as $x \to -\infty$, for various real values of k.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

17 Find the x-coordinates of the stationary points of $y = x\,e^{-|x|}$.

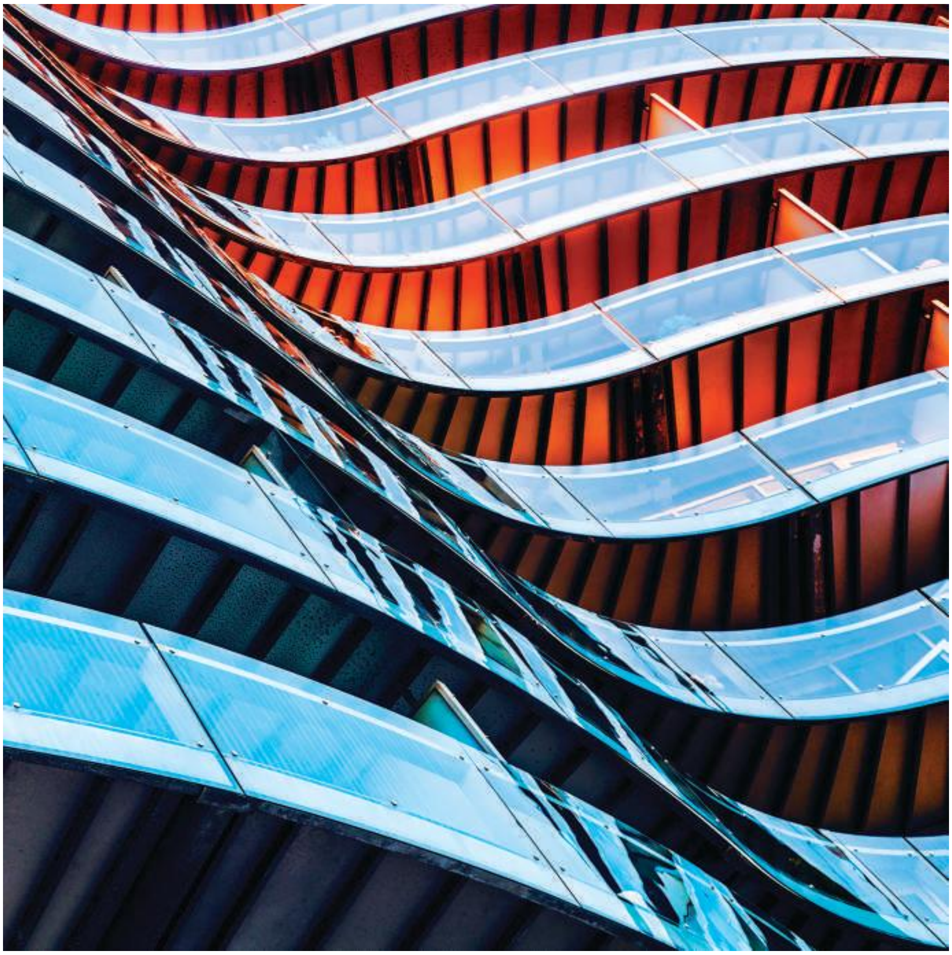
18 **a** What is the natural domain of $y = e^{\frac{1}{x}}$?

b Carefully determine the behaviour of y and y' as $x \to -\infty$, $x \to 0$ and $x \to \infty$.

c Deduce that there must be an inflection point and find it.

d Sketch the curve and give its range.

e Follow steps **a–d** to sketch the graph of $y = x\,e^{\frac{1}{x}}$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6D Integration of exponential functions

Finding primitives is the reverse of differentiation. Thus the new standard forms for differentiation can now be reversed to provide standard forms for integration.

Standard forms for integration

Reversing the standard forms for differentiating exponential functions gives the standard forms for integrating them.

Reversing $\dfrac{d}{dx}e^x = e^x$ gives $\displaystyle\int e^x\,dx = e^x + C$, for some constant C.

Reversing $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$ gives $\displaystyle\int ae^{ax+b}\,dx = e^{ax+b}$,

and dividing through by a, $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$, for some constant C.

4 STANDARD FORMS FOR INTEGRATION

- $\displaystyle\int e^x\,dx = e^x + C$, for some constant C
- $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$, for some constant C

There is also an associated formula for the reverse chain rule, but it is not included in the course. For reference, this formula has the usual two forms:

$$\int e^u\,\frac{du}{dx}\,dx = e^u + C \qquad \text{OR} \qquad \int e^{f(x)}f'(x)\,dx = e^{f(x)} + C.$$

Example 11 **6D**

Find these indefinite integrals.

a $\displaystyle\int e^{3x+2}\,dx$

b $\displaystyle\int (1 - x + e^x)\,dx$

SOLUTION

a $\displaystyle\int e^{3x+2}\,dx = \tfrac{1}{3}e^{3x+2} + C$ $\quad$ ($a = 3$ and $b = 2$)

b $\displaystyle\int (1 - x + e^x)\,dx = x - \tfrac{1}{2}x^2 + e^x + C$ $\quad$ (integrating each term separately)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Definite integrals

Definite integrals are evaluated in the usual way by finding the primitive and substituting.

Example 12 6D

Evaluate these definite integrals.

a $\int_0^2 e^x\,dx$

b $\int_2^3 e^{5-2x}\,dx$

SOLUTION

a $$\begin{aligned}\int_0^2 e^x\,dx &= \Big[e^x\Big]_0^2\\ &= e^2 - e^0\\ &= e^2 - 1\end{aligned}$$

b $$\begin{aligned}\int_2^3 e^{5-2x}\,dx &= -\tfrac{1}{2}\Big[e^{5-2x}\Big]_2^3 \qquad (a = -2 \text{ and } b = 5)\\ &= -\tfrac{1}{2}(e^{-1} - e)\\ &= -\frac{1}{2}\left(\frac{1}{e} - e\right)\\ &= \frac{e^2 - 1}{2e}\end{aligned}$$

Given the derivative, find the function

As before, if the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be determined.

Example 13 6D

It is known that $f'(x) = e^x$ and that $f(1) = 0$.

a Find the original function $f(x)$.

b Hence find $f(0)$.

SOLUTION

a It is given that $f'(x) = e^x$.

Taking the primitive, $f(x) = e^x + C$, for some constant C.

Substituting $f(1) = 0$, $0 = e^1 + C$

$C = -e$.

Hence $f(x) = e^x - e$.

b Substituting $x = 0$, $f(0) = e^0 - e$

$= 1 - e$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 14 6D

a If $y' = 1 + 2e^{-x}$ and $y(0) = 1$, find y.

b Hence find $y(1)$.

SOLUTION

a It is given that $y' = 1 + 2e^{-x}$.

Taking the primitive, $y = x - 2e^{-x} + C$, for some constant C.

Substituting $y(0) = 1$,
$$\begin{aligned} 1 &= 0 - 2e^{0} + C \\ 1 &= 0 - 2 + C \\ C &= 3. \end{aligned}$$

Hence $y = x - 2e^{-x} + 3$.

b Substituting $x = 1$,
$$\begin{aligned} y(1) &= 1 - 2e^{-1} + 3 \\ &= 4 - 2e^{-1}. \end{aligned}$$

Given a derivative, find an integral

The result of any differentiation can be reversed. This often allows a new primitive to be found.

Example 15 6D

a Use the chain rule to differentiate e^{x^2}.

b Hence find $\displaystyle\int_{-1}^{1} 2xe^{x^2}\,dx$.

c Why is this result obvious without any calculation?

SOLUTION

a Let $y = e^{x^2}$.

Applying the chain rule,
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2x\,e^{x^2}. \end{aligned}$$

Let $u = x^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = e^u$.

b Reversing part **a**, $\displaystyle\int 2xe^{x^2}\,dx = e^{x^2}$.

Hence
$$\begin{aligned} \int_{-1}^{1} 2xe^{x^2}\,dx &= \Big[e^{x^2}\Big]_{-1}^{1} \\ &= e^1 - e^1 \\ &= 0. \end{aligned}$$

c The function $2x\,e^{x^2}\,dx$ is odd, and the interval $[-1, 1]$ in the definite integral is symmetric in the y-axis, so the integral is zero.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using a formula for the reverse chain rule

There are some situations where the reverse chain rule formula from Section 5I can be used.

Example 16 — 6D

Use the reverse chain rule formula to find $\displaystyle\int \frac{e^{2x}}{(1 - e^{2x})^3}\,dx$.

SOLUTION

$$\int \frac{e^{2x}}{(1 - e^{2x})^3}\,dx$$
$$= -\tfrac{1}{2}\int \frac{-2e^{2x}}{(1 - e^{2x})^3}\,dx$$
$$= -\tfrac{1}{2} \times \left(-\tfrac{1}{2}\right) \times (1 - e^{2x})^{-2}$$

Let $u = 1 - e^{2x}$.
Then $u' = -2e^{2x}$.

$$\int u^{-3}\frac{du}{dx}\,dx = \frac{u^{-2}}{-2}$$

OR

Let $f(x) = 1 - e^{2x}$.
Then $f'(x) = -2e^{2x}$.

$$\int \big(f(x)\big)^{-3} f'(x)\,dx = \frac{\big(f(x)\big)^{-2}}{-2}$$

$$= \frac{1}{4(1 - e^{2x})^2} + C, \quad \text{for some constant } C.$$

Exercise 6D

FOUNDATION

Technology: Some algebraic programs can display the primitive and evaluate the exact value of an integral. These can be used to check the questions in this exercise and also to investigate the effect of making small changes to the function or to the limits of integration.

1 Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find each indefinite integral.

a $\displaystyle\int e^{2x}\,dx$ **b** $\displaystyle\int e^{3x}\,dx$ **c** $\displaystyle\int e^{\frac{1}{3}x}\,dx$

d $\displaystyle\int e^{\frac{1}{2}x}\,dx$ **e** $\displaystyle\int 10e^{2x}\,dx$ **f** $\displaystyle\int 12e^{3x}\,dx$

g $\displaystyle\int e^{4x+5}\,dx$ **h** $\displaystyle\int e^{4x-2}\,dx$ **i** $\displaystyle\int 6\,e^{3x+2}\,dx$

j $\displaystyle\int 4\,e^{4x+3}\,dx$ **k** $\displaystyle\int e^{7-2x}\,dx$ **l** $\displaystyle\int \tfrac{1}{2}e^{1-3x}\,dx$

2 Evaluate these definite integrals.

a $\displaystyle\int_0^1 e^x\,dx$ **b** $\displaystyle\int_1^2 e^x\,dx$ **c** $\displaystyle\int_{-1}^3 e^{-x}\,dx$ **d** $\displaystyle\int_{-2}^0 e^{-x}\,dx$

e $\displaystyle\int_0^2 e^{2x}\,dx$ **f** $\displaystyle\int_{-1}^2 20e^{-5x}\,dx$ **g** $\displaystyle\int_{-3}^1 8e^{-4x}\,dx$ **h** $\displaystyle\int_{-1}^3 9e^{6x}\,dx$

i $\displaystyle\int_{-1}^1 e^{2x+1}\,dx$ **j** $\displaystyle\int_{-2}^0 e^{4x-3}\,dx$ **k** $\displaystyle\int_{-2}^{-1} e^{3x+2}\,dx$ **l** $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x}\,dx$

m $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} e^{2+3x}\,dx$ **n** $\displaystyle\int_1^2 6e^{3x+1}\,dx$ **o** $\displaystyle\int_2^3 12e^{4x-5}\,dx$ **p** $\displaystyle\int_1^2 12e^{8-3x}\,dx$

ISBN 978-1-108-76630-2

3 Express each function using negative indices, and hence find its primitive.

a $\dfrac{1}{e^x}$ **b** $\dfrac{1}{e^{2x}}$ **c** $\dfrac{1}{e^{3x}}$ **d** $-\dfrac{3}{e^{3x}}$ **e** $\dfrac{6}{e^{2x}}$ **f** $\dfrac{8}{e^{-2x}}$

4 **a** A function $f(x)$ has derivative $f'(x) = e^{2x}$. Find the equation of $f(x)$, which will involve an arbitrary constant.

b It is also known that $f(0) = -2$. Find the arbitrary constant and hence write down the equation of $f(x)$.

c Find $f(1)$ and $f(2)$.

DEVELOPMENT

5 Find $f(x)$ and then find $f(1)$, given that:

a $f'(x) = 1 + 2e^x$ and $f(0) = 1$ **b** $f'(x) = 1 - 3e^x$ and $f(0) = -1$

c $f'(x) = 2 + e^{-x}$ and $f(0) = 0$ **d** $f'(x) = 4 - e^{-x}$ and $f(0) = 2$

e $f'(x) = e^{2x-1}$ and $f\left(\frac{1}{2}\right) = 3$ **f** $f'(x) = e^{1-3x}$ and $f\left(\frac{1}{3}\right) = \frac{2}{3}$

g $f'(x) = e^{\frac{1}{2}x+1}$ and $f(-2) = -4$ **h** $f'(x) = e^{\frac{1}{3}x+2}$ and $f(-6) = 2$

6 Expand the brackets and then find primitives of:

a $e^x(e^x + 1)$ **b** $e^x(e^x - 1)$ **c** $e^{-x}(2e^{-x} - 1)$

d $(e^x + 1)^2$ **e** $(e^x - 1)^2$ **f** $(e^x - 2)^2$

g $(e^x + e^{-x})(e^x - e^{-x})$ **h** $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

7 Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find these indefinite integrals.

a $\displaystyle\int e^{7x+q}\,dx$ **b** $\displaystyle\int e^{3x-k}\,dx$ **c** $\displaystyle\int e^{sx+1}\,dx$ **d** $\displaystyle\int e^{kx-1}\,dx$

e $\displaystyle\int p e^{px+q}\,dx$ **f** $\displaystyle\int m e^{mx+k}\,dx$ **g** $\displaystyle\int A e^{sx-t}\,dx$ **h** $\displaystyle\int B e^{kx-+}\,dx$

8 Express each function below as a power of e or a multiple of a power of e, and hence find its primitive.

a $\dfrac{1}{e^{x-1}}$ **b** $\dfrac{1}{e^{3x-1}}$ **c** $\dfrac{1}{e^{2x+5}}$

d $\dfrac{4}{e^{2x-1}}$ **e** $\dfrac{10}{e^{2-5x}}$ **f** $\dfrac{12}{e^{3x-5}}$

9 By writing each integrand as the sum of multiples of powers of e, find:

a $\displaystyle\int \frac{e^x + 1}{e^x}\,dx$ **b** $\displaystyle\int \frac{e^{2x} + 1}{e^x}\,dx$ **c** $\displaystyle\int \frac{e^x - 1}{e^{2x}}\,dx$

d $\displaystyle\int \frac{e^x - 3}{e^{3x}}\,dx$ **e** $\displaystyle\int \frac{2e^{2x} - 3e^x}{e^{4x}}\,dx$ **f** $\displaystyle\int \frac{2e^x - e^{2x}}{e^{3x}}\,dx$

10 **a** Find y as a function of x if $y' = e^{x-1}$, and $y = 1$ when $x = 1$. What is the y-intercept of this curve?

b The gradient of a curve is given by $y' = e^{2-x}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve? What is its horizontal asymptote?

c It is known that $f'(x) = e^x + \dfrac{1}{e}$ and that $f(-1) = -1$. Find $f(0)$.

d Given that $f''(x) = e^x - e^{-x}$ and that $y = f(x)$ is horizontal as it passes through the origin, find $f(x)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 By first writing each integrand as a sum of powers of e, find:

a $\displaystyle\int_0^1 e^x(2e^x - 1)\,dx$

b $\displaystyle\int_{-1}^1 (e^x + 2)^2\,dx$

c $\displaystyle\int_0^1 (e^x - 1)(e^{-x} + 1)\,dx$

d $\displaystyle\int_{-1}^1 (e^{2x} + e^{-x})(e^{2x} - e^{-x})\,dx$

e $\displaystyle\int_0^1 \frac{e^{3x} + e^x}{e^{2x}}\,dx$

f $\displaystyle\int_{-1}^1 \frac{e^x - 1}{e^{2x}}\,dx$

12 **a** **i** Differentiate e^{x^2+3}. **ii** Hence find $\displaystyle\int 2xe^{x^2+3}\,dx$.

b **i** Differentiate e^{x^2-2x+3}. **ii** Hence find $\displaystyle\int (x - 1)e^{x^2-2x+3}\,dx$.

c **i** Differentiate e^{3x^2+4x+1}. **ii** Hence find $\displaystyle\int (3x + 2)e^{3x^2+4x+1}\,dx$.

d **i** Differentiate $y = e^{x^3}$. **ii** Hence find $\displaystyle\int_{-1}^0 x^2e^{x^3}\,dx$.

13 Write each integrand as a power of e, and hence find the indefinite integral.

a $\displaystyle\int \frac{1}{(e^x)^2}\,dx$

b $\displaystyle\int \frac{1}{(e^x)^3}\,dx$

c $\displaystyle\int \sqrt{e^x}\,dx$

d $\displaystyle\int \sqrt[3]{e^x}\,dx$

e $\displaystyle\int \frac{1}{\sqrt{e^x}}\,dx$

f $\displaystyle\int \frac{1}{\sqrt[3]{e^x}}\,dx$

14 **a** Differentiate $y = x\,e^x$, and hence find $\displaystyle\int_0^2 x\,e^x\,dx$.

b Differentiate $y = x\,e^{-x}$, and hence find $\displaystyle\int_{-2}^0 x\,e^{-x}\,dx$.

15 By first simplifying each integrand, find:

a $\displaystyle\int \frac{e^x - e^{-x}}{\sqrt{e^x}}\,dx$

b $\displaystyle\int \frac{e^x + e^{-x}}{\sqrt[3]{e^x}}\,dx$

16 **a** Show that $f(x) = xe^{-x^2}$ is an odd function.

b Hence evaluate $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} xe^{-x^2}\,dx$ without finding a primitive.

ENRICHMENT

17 Under Box 4, we gave the reverse chain rule for integrals of exponential functions,

$$\int e^u\frac{du}{dx}\,dx = e^u + C \quad \text{OR} \quad \int e^{f(x)}f'(x)\,dx = e^{f(x)} + C.$$

Use one or other of the formulae to find primitives of these functions

a xe^{x^2}

b $4xe^{x^2-7}$

c $(3x + 2)e^{3x^2+4x+1}$

d $(x^2 - 2x)e^{x^3-3x^2}$

e $x^{-2}\,e^{x^{-1}}$

f $-\sqrt{x}\,e^{-x\sqrt{x}}$

18 Show that $\displaystyle\int \frac{e^x + 1}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}}\,dx = 2\,e^{\frac{1}{2}x} + C$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A power series for e^x:

19 The intention of this question is to outline a reasonably rigorous proof of the famous result that e^x can be written as the limit of the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

a For any positive number R, we know that $1 < e^t < e^R$ for $0 < t < R$, because e^x is an increasing function. Integrate this inequality over the interval $t = 0$ to $t = x$, where $0 < x < R$, and hence show that $x < e^x - 1 < e^R x$.

b Change the variable to t, giving $t < e^t - 1 < e^R t$. Then integrate this new inequality from $t = 0$ to $t = x$, and hence show that $\frac{x^2}{2!} < e^x - 1 - x < \frac{e^R x^2}{2!}$.

c Do this process twice more, and prove that:

i $\frac{x^3}{3!} < e^x - 1 - x - \frac{x^2}{2!} < \frac{e^R x^3}{3!}$

ii $\frac{x^4}{4!} < e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} < \frac{e^R x^4}{4!}$

d Now use induction to prove that $\frac{x^{n+1}}{(n+1)!} < e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} < \frac{e^R x^{n+1}}{(n+1)!}$.

e Show that as $n \to \infty$, the left and right expressions converge to zero. Hence prove that the infinite power series converges to e^x for $x > 0$. (Hint: Let k be the smallest integer greater than x and show that for $n > k$, each term in the sequence is less than the corresponding term of a geometric sequence with ratio $\frac{x}{k}$.)

f Prove that the power series also converges to e^x for $x < 0$.

20 a Use the power series in the previous question to show that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots.$$

b Find α, the value of the right-hand side, correct to four decimal places when $x = 0.5$.

c Let $u = e^{0.5}$. Show that $u^2 - 2\alpha u + 1 = 0$.

d Solve this quadratic equation and hence estimate both $e^{0.5}$ and $e^{-0.5}$ correct to two decimal places. Compare your answers with the values obtained directly from the calculator.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6E Applications of integration

The normal methods of finding areas by integration can now be applied to functions involving e^x.

Finding the area between a curve and the *x*-axis

A sketch is essential here, because the definite integral attaches a negative sign to the area of any region below the x-axis (provided that the integral does not run backwards).

Example 17 — 6E

a Use shifting to sketch $y = e^x - e$, showing the intercepts and asymptote.
b Find the area of the region between this curve, the x-axis and the y-axis.

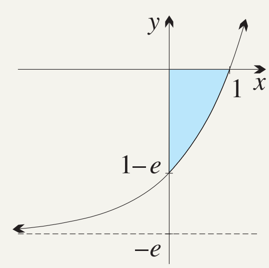
SOLUTION

a Move the graph of $y = e^x$ down e units.

When $x = 0$, $\quad y = e^0 - e$
$\qquad\qquad\qquad = 1 - e.$

When $y = 0$, $\quad e^x = e$
$\qquad\qquad\qquad x = 1.$

The horizontal asymptote moves down to $y = -e$.



b $\displaystyle\int_0^1 (e^x - e)\,dx = \Big[e^x - ex\Big]_0^1$ (the number e is a constant)

$$= (e^1 - e) - (e^0 - 0)$$
$$= -1.$$

This integral is negative because the region is below the x-axis.
Hence the required area is 1 square unit.

Finding areas between curves

If a curve $y = f(x)$ is always above $y = g(x)$ in an interval $a \le x \le b$, then the area of the region between the curves is

$$\text{area between the curves} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

ISBN 978-1-108-76630-2

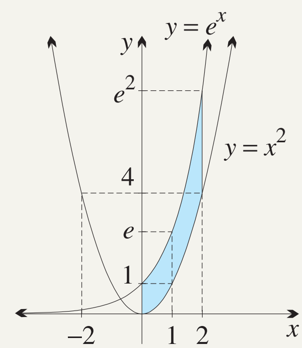
Cambridge University Press
Updated July 2022

Example 18 6E

a Sketch the curves $y = e^x$ and $y = x^2$ in the interval $-2 \le x \le 2$.
b Find the area of the region between the curves, from $x = 0$ to $x = 2$.

SOLUTION

a The graphs are drawn to the right.
Note that for $x > 0$, $y = e^x$ is always above $y = x^2$.



b Using the standard formula above,

$$\begin{aligned} \text{area} &= \left[e^x - \tfrac{1}{3}x^3\right]_0^2 \\ &= \left(e^2 - \tfrac{8}{3}\right) - (e^0 - 0) \\ &= e^2 - 3\tfrac{2}{3} \text{ square units.} \end{aligned}$$

Exercise 6E

FOUNDATION

Technology: Graphing programs that can calculate the areas of specified regions may make the problems in this exercise clearer, particularly when no diagram has been given.

1 **a** Use the standard form $\int e^x dx = e^x + C$ to evaluate each definite integral. Then approximate it correct to two decimal places.

i $\int_0^1 e^x\,dx$ **ii** $\int_{-1}^0 e^x\,dx$ **iii** $\int_{-2}^0 e^x\,dx$ **iv** $\int_{-3}^0 e^x\,dx$

b The graph below shows $y = e^x$ from $x = -5$ to $x = 1$, with a scale of 10 divisions to 1 unit, so that 100 little squares equal 1 square unit. By counting squares under the curve from $x = 0$ to $x = 1$, find an approximation to $\int_0^1 e^x\,dx$, and compare it with the approximation obtained in part **a**.

c Count squares to the left of the y-axis to obtain approximations to:

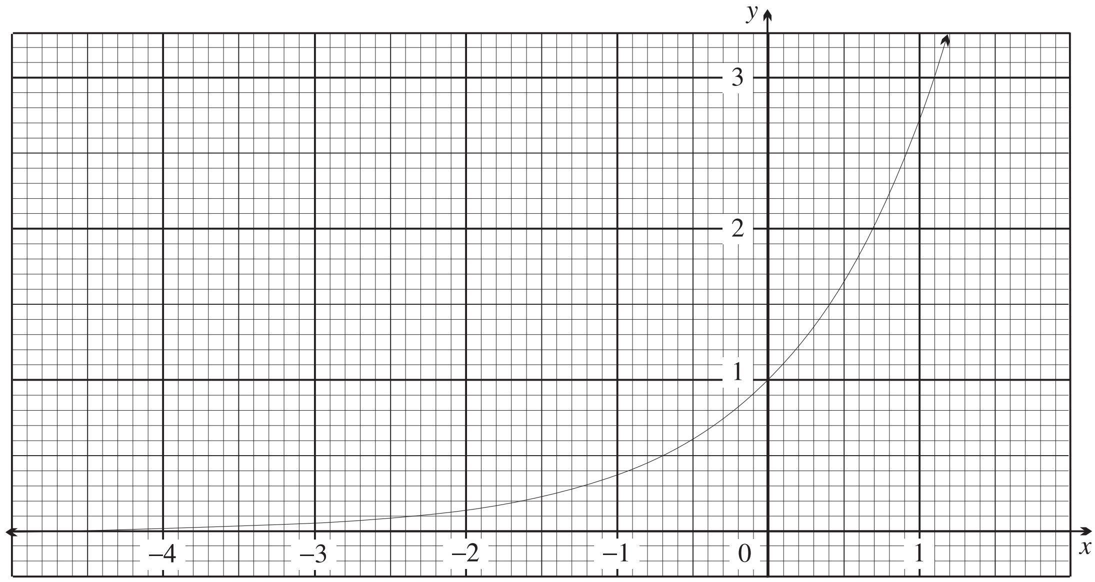
i $\int_{-1}^0 e^x\,dx$, **ii** $\int_{-2}^0 e^x\,dx$, **iii** $\int_{-3}^0 e^x\,dx$,

and compare the results with the approximations obtained in part **a**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d Continue counting squares to the left of $x = -3$, and estimate the total area under the curve to the left of the y-axis.



2 Answer these questions first in exact form, then correct to four significant figures. In each case use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

a Find the area between the curve $y = e^{2x}$ and the x-axis from $x = 0$ to $x = 3$.

b Find the area between the curve $y = e^{-x}$ and the x-axis from $x = 0$ to $x = 1$.

c Find the area between the curve $y = e^{\frac{1}{3}x}$ and the x-axis from $x = -3$ to $x = 0$.

3 In each case find the area between the x-axis and the given curve between the given x-values. Use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

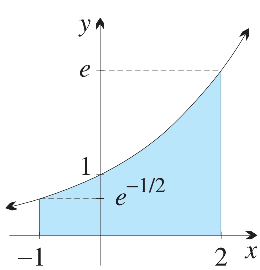
a $y = e^{x+3}$, for $-2 \le x \le 0$

b $y = e^{2x-1}$, for $0 \le x \le 1$

c $y = e^{-2x-1}$, for $-2 \le x \le -1$

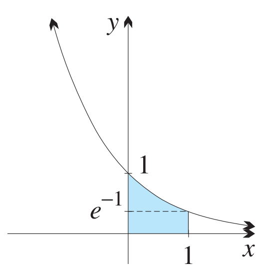
d $y = e^{\frac{1}{3}x+2}$, for $0 \le x \le 3$

4 a



Find the area of the region bounded by the curve $y = e^{\frac{1}{2}x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

b



Find the area of the region bounded by the curve $y = e^{-x}$, the x-axis, the y-axis and the line $x = 1$.

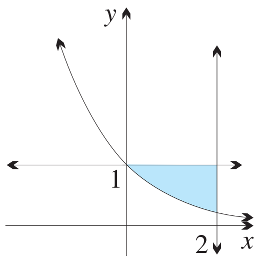
5 a Find the area between the curve $y = e^{x} + e^{-x}$ and the x-axis, from $x = -2$ to $x = 2$.

b Find the area between the curve $y = x^2 + e^{x}$ and the x-axis, from $x = -3$ to $x = 3$.

ISBN 978-1-108-76630-2

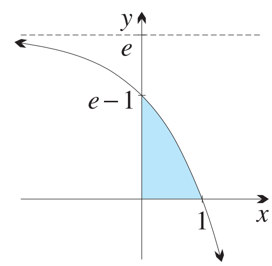
Cambridge University Press
Updated July 2022

6 **a**



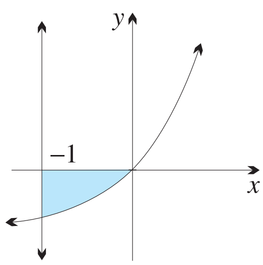
Find the area of the region bounded by the curve $y = e^{-x}$ and the lines $x = 2$ and $y = 1$.

b



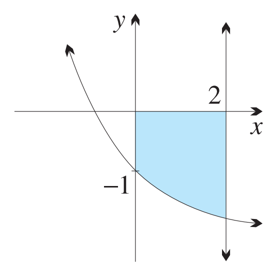
Find the area of the region in the first quadrant bounded by the coordinate axes and the curve $y = e - e^{x}$.

c



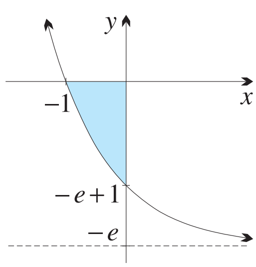
Find the area between the x-axis, the curve $y = e^{x} - 1$ and the line $x = -1$.

d



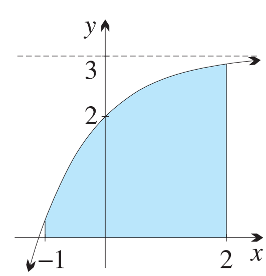
What is the area bounded by $x = 2$, $y = e^{-x} - 2$, the x-axis and the y-axis?

e



Find the area of the region bounded by the curve $y = e^{-x} - e$ and the coordinate axes.

f



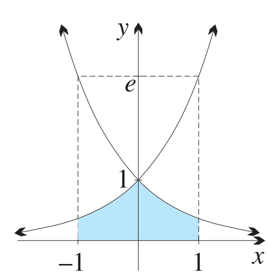
Find the area of the region bounded by the curve $y = 3 - e^{-x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

DEVELOPMENT

7 **a** Sketch the curves $y = e^{x}$ and $y = x + 1$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

b Sketch the curves $y = e^{x}$ and $y = 1 - x$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

8 The diagram to the right shows the region above the x-axis, below both $y = e^{x}$ and $y = e^{-x}$, between $x = -1$ and $x = 1$.



a Explain why the area of this region may be written as

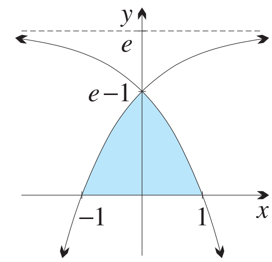
$$\text{area} = 2\int_0^1 e^{-x}\,dx.$$

b Hence find the area of this region.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 The diagram to the right shows the region above the x-axis, below both $y = e - e^{-x}$ and $y = e - e^{x}$.

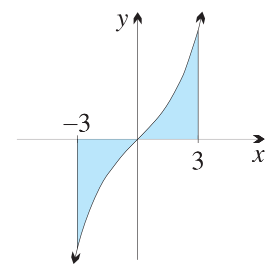


a Explain why the area of this region may be written as

$$\text{area} = 2\int_0^1 (e - e^x)\,dx.$$

b Hence find the area of this region.

10 The diagram to the right shows the region between the curve $y = e^x - e^{-x}$, the x-axis and the lines $x = -3$ and $x = 3$.



a Show that $y = e^x - e^{-x}$ is an odd function.

b Hence write down the value of $\int_{-3}^{3} (e^x - e^{-x})\,dx$ without finding a primitive.

c Explain why the area of this region may be written as

$$\text{area} = 2\int_0^3 (e^x - e^{-x})\,dx.$$

d Hence find the area of this region.

11 a Show that the curves $y = x^2$ and $y = e^{x+1}$ intersect at $x = -1$.

b Hence sketch the region in the second quadrant between these two curves and the y-axis.

c Find its area.

12 a Sketch the region between the graphs of $y = e^x$ and $y = x$, between the y-axis and $x = 2$, then find its area.

b Find the intercepts of the curve $y = 8 - 2^x$ and hence find the area of the region bounded by this curve and the coordinate axes.

13 In this question, give all approximations correct to four decimal places.

a Find the area between the curve $y = e^x$ and the x-axis, for $0 \le x \le 1$, by evaluating an appropriate integral. Then approximate the result.

b Estimate the area using the trapezoidal rule with two subintervals (that is, with three function values).

c Is the trapezoidal-rule approximation greater than or less than the exact value? Give a geometric explanation.

14 a Differentiate e^{-x^2} and hence write down a primitive of xe^{-x^2}.

b Hence find the area between the curve $y = xe^{-x^2}$ and the x-axis from $x = 0$ to $x = 2$, and from $x = -2$ to $x = 2$.

15 a i Evaluate the integral $\int_N^0 e^x\,dx$.

ii What is its limit as $N \to -\infty$?

b i Evaluate the integral $\int_0^N e^{-x}\,dx$.

ii What is its limit as $N \to \infty$?

c Similarly, evaluate $\int_0^N 2xe^{-x^2}\,dx$, and find its limit as $N \to \infty$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

16 **a** Find $\displaystyle\int_{\delta}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

b What happens to the integral as $\delta \to 0^+$?

17 **a** Differentiate $x\,e^{-x}$, and hence find $\displaystyle\int_0^N x\,e^{-x}\,dx$.

b Find the limit of this integral as $N \to \infty$ (use the dominance results).

c Differentiate $x^2\,e^{-x}$, and hence find $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6F Review of logarithmic functions

Section 6A reviewed exponential functions from Sections 11A–11F of the Year 11 book, and this section will complete the review of those sections with a summary of logarithms base e. The two small topics that are new are:

- Dilations of logarithmic functions.
- Exponential and logarithmic equations reducible to quadratics.

The function $y = \log_e x$

As discussed in the Year 11 book, an exponential function to any base is one-to-one, so its inverse relation is also a function, and is called a *logarithmic function* to the same base. Remember that

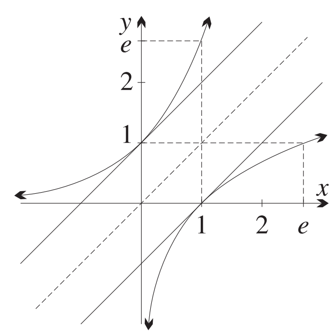
$$3 = \log_2 8 \text{ means } 8 = 2^3 \qquad \text{and} \qquad y = \log_e x \text{ means } x = e^y.$$

'The log is the index, when the number is written as a power of the base.'

Algebraically, the fact that $y = \log_e x$ is the inverse function of $y = e^x$ means that the composite of the two functions, in either order, is the identity function,

$$\log_e e^x = x, \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x, \text{ for all } x > 0.$$

Geometrically, when the functions are sketched on one graph, they are reflections of each other in the diagonal line $y = x$.



- Both graphs have gradient 1 at their intercepts, $y = e^x$ at its y-intercept, and $y = \log_e x$ at its x-intercept.
- Their domains and ranges are reversed, which is more easily seen with bracket interval notation:
  For $y = e^x$, domain $= (-\infty, \infty)$, range $= (0, \infty)$.
  For $y = \log_e x$, domain $= (0, \infty)$, range $= (-\infty, \infty)$.
- $y = e^x$ has a horizontal asymptote $y = 0$.
  $y = \log_e x$ has a vertical asymptote $x = 0$.
- Both are increasing throughout their domain, $y = e^x$ at an increasing rate, $y = \log_e x$ at a decreasing rate.

5 THE FUNCTION $y = \log_e x$ OR $\ln x$

- The function $y = \log_e x$ is the inverse function of $y = e^x$,
  $$y = \log_e x \qquad \text{means that} \qquad x = e^y.$$
- The composition of the functions $y = e^x$ and $y = \log_e x$, in any order, is the identity function,
  $$\log_e e^x = x, \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x, \text{ for all } x > 0.$$
- The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in $y = x$.
- This reflection exchanges the domain and range, exchanges the asymptotes, and exchanges the intercepts with the axes.
- The tangents to both curves at their intercepts have gradient 1.
- $y = e^x$ is always concave up, and $y = \log_e x$ is always concave down.
- Both graphs are one-to-one, and both graphs are increasing, $y = e^x$ at an increasing rate, $y = \log_e x$ at a decreasing rate.

The derivative of $y = \log_e x$ will be obtained in Section 6G.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Notation and the calculator

Write the function as $y = \log_e x$ or as $y = \ln x$ ('logs naperian' or 'logs natural'). We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise to readers that the base is e, but $\ln x$ is also standard notation.

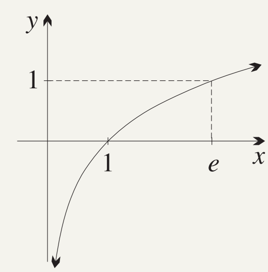
In mathematics, but not elsewhere, interpret $\log x$ as $\log_e x$. Be particularly careful on the calculator, where $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$.

Example 19 — 6F

Sketch each function using a transformation of the graph of $y = \log_e x$ sketched to the right. Describe the transformation, write down the domain, and show and state the x-intercept and the vertical asymptote.

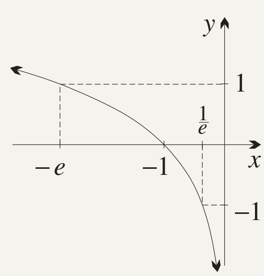
a $y = \log_e(-x)$ **b** $y = \log_e x - 2$ **c** $y = \log_e(x + 3)$

Which transformations can also be done using a dilation?



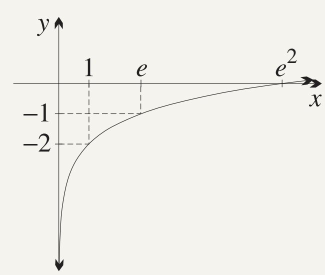
SOLUTION

a



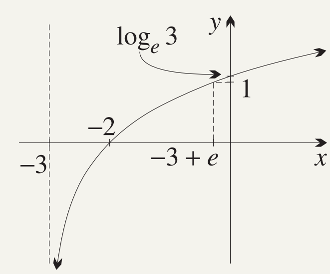
To graph $y = \log_e(-x)$,
reflect $y = \log_e x$ in y-axis.
domain: $x < 0$
x-intercept: $(-1, 0)$
asymptote: $x = 0$

b



To graph $y = \log_e x - 2$,
shift $y = \log_e x$ down 2.
domain: $x > 0$
x-intercept: $(e^2, 0)$
asymptote: $x = 0$

c



To graph $y = \log_e(x + 3)$,
shift $y = \log_e x$ left 3.
domain: $x > -3$
x-intercept: $(-2, 0)$
asymptote: $x = -3$

- The equation $y = \log_e(-x)$ in part **a** is a reflection in the y-axis, and any reflection in the y-axis can be regarded as a horizontal dilation with factor -1.
- In part **b**, the equation $y = \log_e x - 2 = \log_e x - \log_e e^2 = \log_e(e^{-2}x)$ can be regarded as a horizontal dilation with factor $e^2 \doteqdot 7.39$.

Dilations of $y = \log_e x$

Dilations of logarithmic functions share an interesting property with dilations of exponential functions — some of them can be done with a shift in the other direction, as we saw in part **b** above.

Example 20 — 6F

Use dilations of $y = \log_e x$ to generate a sketch of each function. Identify which dilation is also a shift in the other direction.

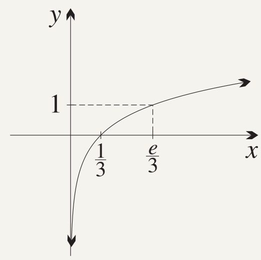
a $y = \log_e 3x$ **b** $y = 3 \log_e x$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

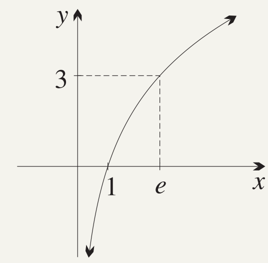
SOLUTION

a $y = \log_e 3x$



Dilate $y = \log_e x$ horizontally factor $\frac{1}{3}$.

b $y = 3\log_e x$



Dilate $y = \log_e x$ vertically factor 3.

- $y = \log_e 3x$ can be written as $y = \log_e x + \log_e 3$, so it is shift up $\log_e 3$.

Using the inverse identities

We conclude with a review of some of the manipulations needed when using logarithms base e. First, some simple examples of using the two inverse identities

$$\log_e e^x = x \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x \text{ for all } x > 0.$$

Example 21 — 6F

Simplify:

a $\log_e e^6$ **b** $\log_e e$ **c** $\log_e \frac{1}{e}$ **d** $\log_e \dfrac{1}{\sqrt{e}}$ **e** $e^{\log_e 10}$ **f** $e^{\log_e 0.1}$

SOLUTION

a $\log_e e^6 = 6$

b $\log_e e = \log_e e^1 = 1$

c $\log_e \frac{1}{e} = \log_e e^{-1} = -1$

d $\log_e \dfrac{1}{\sqrt{e}} = \log_e e^{-\frac{1}{2}} = -\frac{1}{2}$

e $e^{\log_e 10} = 10$

f $e^{\log_e 0.1} = 0.1$

Conversion between exponential statements and logarithm statements

We recommended that the following sentence be committed to memory:

$$\log_2 8 = 3 \qquad \text{because} \qquad 8 = 2^3.$$

- The base of the power is the base of the log.
- The log is the index, when the number is written as a power of the base.

This pattern applies in exactly the same way when the base is e.

$$\log_e x = y \qquad \text{means} \qquad x = e^y.$$

Example 22 — 6F

Convert each statement to the other form.

a $x = e^3$ **b** $\log_e x = -1$ **c** $x = \log_e 10$ **d** $e^x = \frac{1}{2}$

SOLUTION

a $\log_e x = 3$ **b** $x = \frac{1}{e}$ **c** $e^x = 10$ **d** $x = \log_e \frac{1}{2}$

ISBN 978-1-108-76630-2

The change-of-base formula

We developed the general change of base formula. What is needed here is conversion to base e from a base b, which must be a positive number not equal to 1,

$$\log_b x = \frac{\log_e x}{\log_e b}, \quad \text{for all } x > 0.$$

Example 23 — 6F

a Locate $\log_2 100$ and $\log_3 100$ between two whole numbers.
b Use logarithms base e to solve $2^x = 100$ and $3^x = 100$ correct to three decimal places.

SOLUTION

a $2^6 < 100 < 2^7$, so $\log_2 100$ lies between 6 and 7.
$3^4 < 100 < 3^5$, so $\log_3 100$ lies between 4 and 5.

b
$$\begin{aligned} 2^x &= 100 \\ x &= \log_2 100 \\ &= \frac{\log_e 100}{\log_e 2} \\ &\doteqdot 6.644 \end{aligned} \qquad \begin{aligned} 3^x &= 100 \\ x &= \log_3 100 \\ &= \frac{\log_e 100}{\log_e 3} \\ &\doteqdot 4.192 \end{aligned}$$

Alternatively, take logarithms base e of both sides.

6 THE CHANGE-OF-BASE FORMULA

Suppose that the new base b is a positive number not equal to 1. Then

$$\log_b x = \frac{\log_e x}{\log_e b}.$$

'The log of the number over the log of the base.'

Exponential and logarithmic equations reducible to quadratics

Exponential and logarithmic equations can sometimes be reduced to quadratics with a substitution (although the working is sometimes easier without the substitution). This approach can be used whether or not the base is e.

Example 24 — 6F

a Use the substitution $u = 2^x$ to solve the equation $4^x - 7 \times 2^x + 12 = 0$.
b Use the substitution $u = e^x$ to solve the equation $3e^{2x} - 11e^x - 4 = 0$.
c Solve $\log_e x - \dfrac{9}{\log_e x} = 0$ with and without the substitution $u = \log_e x$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Writing $4^x = (2^x)^2$, the equation becomes

$$(2^x)^2 - 7 \times 2^x + 12 = 0.$$

Substituting $u = 2^x$,

$$\begin{aligned} u^2 - 7u + 12 &= 0 \\ (u-4)(u-3) &= 0 \\ u &= 4 \text{ or } 3, \end{aligned}$$

and returning to x,

$$\begin{aligned} 2^x &= 4 \text{ or } 2^x = 3 \\ x &= 2 \text{ or } \log_2 3. \end{aligned}$$

b Writing $e^{2x} = (e^x)^2$, the equation becomes

$$3(e^x)^2 - 11e^x - 4 = 0.$$

Substituting $u = e^x$,

$$\begin{aligned} 3u^2 - 11u - 4 &= 0 \quad (\alpha + \beta = -11, \alpha\beta = 3 \times (-4) = -12) \\ 3u^2 - 12u + u - 4 &= 0 \quad (\alpha \text{ and } \beta \text{ are } -12 \text{ and } 1) \\ 3u(u-4) + (u-4) &= 0 \\ (3u+1)(u-4) &= 0 \\ u &= -\tfrac{1}{3} \text{ or } 4, \end{aligned}$$

and returning to x,

$$e^x = -\tfrac{1}{3} \text{ or } e^x = 4.$$

Because e^x is never negative,

$$\begin{aligned} e^x &= 4 \\ x &= \log_e 4. \end{aligned}$$

c The equation is

$$\log_e x - \frac{9}{\log_e x} = 0.$$

Substituting $u = \log_e x$,

$$u - \frac{9}{u} = 0$$

$\boxed{\times\ u}$

$$\begin{aligned} (u^2 - 9) &= 0 \\ (u-3)(u+3) &= 0 \\ u &= 3 \text{ or } -3, \end{aligned}$$

and returning to x,

$$\log_e x = 3 \text{ or } -3.$$

Hence

$$x = e^3 \text{ or } e^{-3}.$$

Alternatively,

$$\log_e x - \frac{9}{\log_e x} = 0$$

$\boxed{\times\ \log_e x}$

$$\begin{aligned} (\log_e x)^2 - 9 &= 0 \\ (\log_e x)^2 &= 9 \\ \log_e x &= 3 \text{ or } -3 \\ x &= e^3 \text{ or } e^{-3}. \end{aligned}$$

Exercise 6F

FOUNDATION

Remember that on the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise the base.

1 Use the calculator's $\boxed{\ln}$ button to approximate, correct to four significant figures:

a $\log_e 10$ **b** $\log_e 0.1$ **c** $\ln 123\,456$ **d** $\ln 0.000\,006$ **e** $\log_e 50$ **f** $\log_e 0.02$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 Use the laws for logarithms to express as a single logarithm:

a $\ln 5 + \ln 4$ **b** $\ln 30 - \log_e 6$ **c** $\ln 12 - \ln 15 + \ln 10^2$

3 Use the identities $\log_e e^x = x$ for all real x, and $e^{\log_e x} = x$ for $x > 0$, to simplify:

a $\log_e e^3$ **b** $\log_e e^{-1}$ **c** $\log_e \frac{1}{e^2}$ **d** $\log_e \sqrt{e}$

e $e^{\ln 5}$ **f** $e^{\ln 0.05}$ **g** $e^{\ln 1}$ **h** $e^{\ln e}$

4 **a** Use your calculator to confirm that $\log_e 1 = 0$.

b Write 1 as a power of e, then use the identities in Question **3** to explain why $\log_e 1 = 0$.

c Use your calculator to confirm that $\log_e e = 1$. (You will need to find $e = e^1$ first.)

d Write e as a power of e, then use the identities in Question **3** to explain why $\log_e e = 1$.

5 Convert each exponential statement to logarithmic form, and each logarithmic statement to exponential form.

a $x = e^6$ **b** $\log_e x = -2$ **c** $x = \ln 24$ **d** $e^x = \frac{1}{3}$

6 Use the change-of base formula to express each logarithm in terms of logarithms base e. Then approximate it correct to four significant figures.

a $\log_2 7$ **b** $\log_{10} 25$ **c** $\log_3 0.04$

7 **a** What transformation maps $y = e^x$ to $y = \log_e x$, and how can this transformation be used to find the gradient of $y = \log_e x$ at its x-intercept?

b What transformation maps $y = \log_e x$ to $y = \log_e(-x)$, and how can this transformation also be interpreted as a dilation?

c Sketch $y = \log_e x$ and $y = \log_e(-x)$ on one set of axes.

8 Sketch each curve using a single transformation of $y = \log_e x$, and describe the transformation.

a $y = \log_e x + 1$ **b** $y = \log_e x - 2$ **c** $y = \log_e\left(\frac{1}{2}x\right)$ **d** $y = \frac{1}{3}\log_e x$

9 Sketch each curve using a single transformation of $y = \log_e(-x)$, and describe the transformation.

a $y = \log_e(-x) - 1$ **b** $y = -\log_e(-x)$ **c** $y = 3\log_e(-x)$

DEVELOPMENT

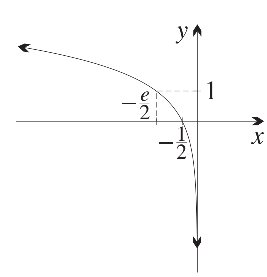
10 Simplify these expressions involving logarithms to the base e:

a $e \log_e e$ **b** $\frac{1}{e}\ln\frac{1}{e}$ **c** $3\log_e e^2$

d $\ln\sqrt{e}$ **e** $e\log_e e^3 - e\log_e e$ **f** $\log_e e + \log_e \frac{1}{e}$

g $\log_e e^e$ **h** $\log_e(\log_e e^e)$ **i** $\log_e\left(\log_e(\log_e e^e)\right)$

11 The graph drawn to the right is a dilation of $y = \log_e(-x)$. Describe the dilation, and write down the equation of the curve.



ISBN 978-1-108-76630-2

12 **a** Use the substitution $u = 2^x$ to solve $4^x - 9 \times 2^x + 14 = 0$.

b Use the substitution $u = 3^x$ to solve $3^{2x} - 8 \times 3^x - 9 = 0$.

c Use similar substitutions, or none, to solve:

i $25^x - 26 \times 5^x + 25 = 0$

ii $9^x - 5 \times 3^x + 4 = 0$

iii $3^{2x} - 3^x - 20 = 0$

iv $7^{2x} + 7^x + 1 = 0$

v $3^{5x} = 9^{x+3}$

vi $4^x - 3 \times 2^{x+1} + 2^3 = 0$

13 Use the substitution $u = e^x$ or $u = e^{2x}$ to reduce these equations to quadratics and solve them. Write your answers as logarithms base e, unless they can be further simplified.

a $e^{2x} - 2e^x + 1 = 0$

b $e^{2x} + e^x - 6 = 0$

c $e^{4x} - 10e^{2x} + 9 = 0$

d $e^{4x} - e^{2x} = 0$

14 Use a substitution such as $u = 4^x$ to solve each equation. Give each solution as a rational number, or approximate correct to three decimal places.

a $2^{4x} - 7 \times 2^{2x} + 12 = 0$

b $100^x - 10^x - 1 = 0$

c $\left(\frac{1}{5}\right)^{2x} - 7 \times \left(\frac{1}{5}\right)^{x} + 10 = 0$

15 Use a substitution, or none, to solve:

a $(\log_e x)^2 - 5\log_e x + 4 = 0$

b $(\log_e x)^2 = 3\log_e x$

16 **a** Solve $\ln(x^2 + 5x) = 2\ln(x + 1)$

b Solve $\log_e(7x - 12) = 2\log_e x$.

ENRICHMENT

17 **a** Use, and describe, a dilation of $y = \log_e x$ to sketch $y = \log_e 2x$.

b Use, and describe, a subsequent translation to sketch $y = \log_e 2(x - 1)$.

c Use, and describe, a subsequent dilation to sketch $y = \frac{1}{2}\log_e 2(x - 1)$.

d Use, and describe, a subsequent translation to sketch $y = \frac{1}{2}\log_e 2(x - 1) - 2$.

18 In Box 6, when stating the change-of-base formula $\log_b x = \dfrac{\log_e x}{\log_e b}$, we required that the base b must be positive and not equal to 1. Why is this restriction on b necessary?

19 **a** Interpret the transformation from $y = \log_e x$ to $y = \log_e(5x)$ as a dilation. Then interpret it as a translation.

b Interpret the transformation from $y = \log_e x$ to $y = \log_e x + 2$ as a translation. Then interpret it as a dilation by writing 2 as $\log_e e^2$.

20 It can be shown (with some considerable difficulty) that the continued fraction to the right approaches the value e–1. With the help of a calculator, use this continued fraction to find a rational approximation for e that is accurate to four significant figures.

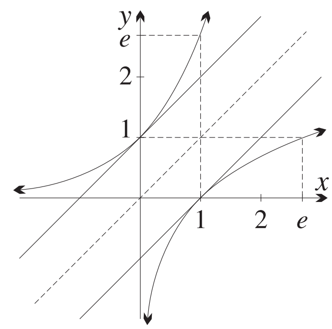
$$1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}}}}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6G Differentiation of logarithmic functions

Calculus with the exponential function $y = e^x$ requires also the calculus of its inverse function $y = \log_e x$.



The diagram to the right shows once again the graphs of both curves drawn on the same set of axes — they are reflections of each other in the diagonal line $y = x$. Using this reflection, the important features of $y = \log_e x$ are:

- The domain is $x > 0$ and the range is all real y.
- The x-intercept is 1, and the gradient there is 1.
- The y-axis is a vertical asymptote.
- As $x \to \infty$, $y \to \infty$ (look at its reflection $y = e^x$ to see this).
- Throughout its whole domain, $\log_e x$ is increasing at a decreasing rate.

Differentiating the logarithmic function

The logarithmic function $= \log_e x$ can be differentiated easily using the known derivative of its inverse function e^x.

Let $y = \log_e x$.

Then $x = e^y$, by the definition of logarithms.

Differentiating, $\dfrac{dx}{dy} = e^y$, because the exponential function is its own derivative,

$= x$, because $e^y = x$,

and taking reciprocals, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Hence the derivative of the logarithmic function is the reciprocal function.

7 THE DERIVATIVE OF THE LOGARITHMIC FUNCTION IS THE RECIPROCAL FUNCTION

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

The next worked example uses the derivative to confirm that $y = \log_e x$ has two properties that were already clear from the reflection in the diagram above.

Example 25 6G

a Find the gradient of the tangent to $y = \log_e x$ at its x-intercept.

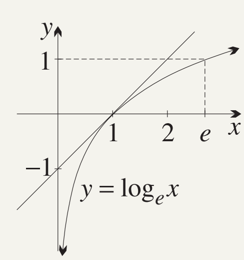
b Prove that $y = \log_e x$ is always increasing, and always concave down.

SOLUTION

a The function is $y = \log_e x$.

Differentiating, $y' = \dfrac{1}{x}$.

The graph crosses the x-axis at $(1, 0)$, and substituting $x = 1$ into y', gradient at x-intercept $= 1$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b The domain is $x > 0$, and $y' = \dfrac{1}{x}$ is positive for all $x > 0$.

Differentiating again, $\quad y'' = -\dfrac{1}{x^2}$, which is negative for all $x > 0$.

Hence $y = e^x$ is always increasing, and always concave down.

Example 26 — 6G

Differentiate these functions using the standard form above.

a $y = x + \log_e x$ **b** $y = 5x^2 - 7\log_e x$

SOLUTION

a
$$y = x + \log_e x$$
$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

b
$$y = 5x^2 - 7\log_e x$$
$$\frac{dy}{dx} = 10x - \frac{7}{x}$$

Further standard forms

The next worked example uses the chain rule to develop two further standard forms for differentiation.

Example 27 — 6G

Differentiate each function using the chain rule. (Part **b** is a standard form.)

a $\log_e(3x + 4)$ **b** $\log_e(ax + b)$ **c** $\log_e(x^2 + 1)$

SOLUTION

a Let $y = \log_e(3x + 4)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{3x + 4} \times 3$$
$$= \frac{3}{3x + 4}.$$

Let $u = 3x + 4$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

b Let $y = \log_e(ax + b)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{ax + b} \times a$$
$$= \frac{a}{ax + b}.$$

Let $u = ax + b$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

c Let $y = \log_e(x^2 + 1)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{x^2 + 1} \times 2x$$
$$= \frac{2x}{x^2 + 1}.$$

Let $u = x^2 + 1$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Standard forms for differentiation

It is convenient to write down two further standard forms for differentiation based on the chain rule, giving three forms altogether.

8 THREE STANDARD FORMS FOR DIFFERENTIATING LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

$$\frac{d}{dx}\log_e(ax + b) = \frac{a}{ax + b}$$

$$\frac{d}{dx}\log_e u = \frac{u'}{u} \qquad \text{OR} \qquad \frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$$

The second of these standard forms was proven in part **b** of the previous worked example. Part **a** was an example of it.

The third standard form is a more general chain-rule extension — part **c** of the previous worked example was a good example of it. This standard form will be needed later for integration. For now, either learn it — in one of its two forms — or apply the chain rule each time.

Example 28 6G

Using the standard forms developed above, differentiate:

a $y = \log_e(4x - 9)$ **b** $y = \log_e(1 - \frac{1}{2}x)$ **c** $y = \log_e(4 + x^2)$

SOLUTION

a For $y = \log_e(4x - 9)$, use the second standard form with $ax + b = 4x - 9$.

Thus $y' = \dfrac{4}{4x - 9}$.

b For $y = \log_e(1 - \frac{1}{2}x)$, use the second standard form with $ax + b = -\frac{1}{2}x + 1$.

$$\text{Thus } y' = \frac{-\frac{1}{2}}{-\frac{1}{2}x + 1}$$

$$= \frac{1}{x - 2}, \text{ after multiplying top and bottom by } -2.$$

c For $y = \log_e(4 + x^2)$,

$$y' = \frac{2x}{4 + x^2}.$$

Let $u = 4 + x^2$.
Then $u' = 2x$.
$$\frac{d}{dx}\log_e u = \frac{u'}{u}$$

OR

Let $f(x) = 4 + x^2$.
Then $f'(x) = 2x$.
$$\frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$$

Alternatively, use the chain rule, as in the previous worked example.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Using the product and quotient rules

These two rules are used in the usual way.

Example 29 — 6G

Differentiate:

a $x^3 \ln x$ by the product rule,

b $\dfrac{\ln(1+x)}{x}$ by the quotient rule.

SOLUTION

a Let $y = x^3 \ln x$.

Then
$$y' = vu' + uv'$$
$$= 3x^2 \ln x + x^3 \times \frac{1}{x}$$
$$= x^2(1 + 3\ln x).$$

Let $u = x^3$
and $v = \ln x$.
Then $u' = 3x^2$
and $v' = \dfrac{1}{x}$.

b Let $y = \dfrac{\ln(1+x)}{x}$.

Then
$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{\dfrac{x}{1+x} - \ln(1+x)}{x^2}$$
$$= \frac{x - (1+x)\ln(1+x)}{x^2(1+x)}.$$

Let $u = \ln(1+x)$
and $v = x$.
Then $u' = \dfrac{1}{1+x}$
and $v' = 1$.

Using the log laws to make differentiation easier

The next worked example shows the use of the log laws to avoid a combination of the chain and quotient rules.

Example 30 — 6G

Use the log laws to simplify each expression, then differentiate it.

a $\log_e 7x^2$ **b** $\log_e(3x-7)^5$ **c** $\log_e \dfrac{1+x}{1-x}$.

SOLUTION

a Let $y = \log_e 7x^2$.

Then
$$y = \log_e 7 + \log_e x^2$$
$$= \log_e 7 + 2\log_e x,$$
so
$$\frac{dy}{dx} = \frac{2}{x}.$$

b Let $y = \log_e(3x-7)^5$.

Then
$$y = 5\log_e(3x-7),$$
so
$$\frac{dy}{dx} = \frac{15}{3x-7}.$$

c Let $y = \log_e \dfrac{1+x}{1-x}$.

Then
$$y = \log_e(1+x) - \log_e(1-x),$$
so
$$\frac{dy}{dx} = \frac{1}{1+x} + \frac{1}{1-x}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 6G

FOUNDATION

Note: Remember that on the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise the base.

1 Use the standard form $\dfrac{d}{dx}\log_e(ax+b)=\dfrac{a}{ax+b}$ to differentiate:

a $y=\log_e(x+2)$ **b** $y=\log_e(x-3)$ **c** $y=\log_e(3x+4)$
d $y=\log_e(2x-1)$ **e** $y=\log_e(-4x+1)$ **f** $y=\log_e(-3x+4)$
g $y=\ln(-2x-7)$ **h** $y=3\ln(2x+4)$ **i** $y=5\ln(3x-2)$

2 Differentiate these functions.

a $y=\log_e 2x$ **b** $y=\log_e 5x$ **c** $y=\log_e 3x$ **d** $y=\log_e 7x$
e $y=4\ln 7x$ **f** $y=3\ln 5x$ **g** $y=4\ln 6x$ **h** $y=3\ln 9x$

3 Find $\dfrac{dy}{dx}$ for each function. Then evaluate $\dfrac{dy}{dx}$ at $x=3$.

a $y=\log_e(x+1)$ **b** $y=\log_e(2x-1)$ **c** $y=\log_e(2x-5)$
d $y=\log_e(4x+3)$ **e** $y=5\ln(x+1)$ **f** $y=6\ln(2x+9)$

4 Differentiate these functions.

a $2+\log_e x$ **b** $5-\log_e(x+1)$ **c** $x+4\log_e x$
d $2x^4+1+3\log_e x$ **e** $\ln(2x-1)+3x^2$ **f** $x^3-3x+4+\ln(5x-7)$

DEVELOPMENT

5 Use the log laws to simplify each function, then differentiate it.

a $y=\ln x^3$ **b** $y=\ln x^2$ **c** $y=\ln x^{-3}$
d $y=\ln x^{-2}$ **e** $y=\ln\sqrt{x}$ **f** $y=\ln\sqrt{x+1}$

6 Differentiate these functions.

a $y=\log_e\frac{1}{2}x$ **b** $y=\log_e\frac{1}{3}x$ **c** $y=3\log_e\frac{1}{5}x$
d $y=-6\log_e\frac{1}{2}x$ **e** $y=x+\log_e\frac{1}{7}x$ **f** $y=4x^3-\log_e\frac{1}{5}x$

7 Use the full setting-out of the chain rule to differentiate:

a $\ln(x^2+1)$ **b** $\ln(2-x^2)$ **c** $\ln(1+e^x)$

8 Use the standard form $\dfrac{d}{dx}\log_e u=\dfrac{u'}{u}$ OR $\dfrac{d}{dx}\log_e f(x)=\dfrac{f'(x)}{f(x)}$ to differentiate:

a $\log_e(x^2+3x+2)$ **b** $\log_e(1+2x^3)$ **c** $\ln(e^x-2)$
d $x+3-\ln(x^2+x)$ **e** $x^2+\ln(x^3-x)$ **f** $4x^3-5x^2+\ln(2x^2-3x+1)$

ISBN 978-1-108-76630-2

9 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = \ln x$ at the points where:

a $x = 1$ **b** $x = 3$ **c** $x = \frac{1}{2}$ **d** $x = 4$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

10 Differentiate these functions using the product rule.

a $x \log_e x$ **b** $x \log_e (2x + 1)$ **c** $(2x + 1) \log_e x$ **d** $x^4 \log_e x$

e $(x + 3) \log_e (x + 3)$ **f** $(x - 1) \log_e (2x + 7)$ **g** $e^x \log_e x$ **h** $e^{-x} \log_e x$

11 Differentiate these functions using the quotient rule.

a $y = \dfrac{\log_e x}{x}$ **b** $y = \dfrac{\log_e x}{x^2}$ **c** $y = \dfrac{x}{\log_e x}$

d $y = \dfrac{x^2}{\log_e x}$ **e** $y = \dfrac{\log_e x}{e^x}$ **f** $y = \dfrac{e^x}{\log_e x}$

12 Use the log laws to simplify each function, then differentiate it.

a $y = \log_e 5x^3$ **b** $y = \log_e \sqrt[3]{x}$ **c** $y = \log_e \dfrac{3}{x}$

d $y = \ln \sqrt{2 - x}$ **e** $y = \ln \dfrac{1 + x}{1 - x}$ **f** $y = \ln x\sqrt{x + 1}$

g $y = \log_e 2^x$ **h** $y = \log_e e^x$ **i** $y = \log_e x^x$

13 Find the first and second derivatives of each function, then evaluate both derivatives at the value given.

a $f(x) = \log_e (x - 1)$ at $x = 3$ **b** $f(x) = \log_e (2x + 1)$ at $x = 0$

c $f(x) = \log_e x^2$ at $x = 2$ **d** $f(x) = x \log_e x$ at $x = e$

14 Differentiate each function using the chain, product or quotient rules. Then find any values of x for which the derivative is zero.

a $y = x \log_e x - x$ **b** $y = x^2 \log_e x$ **c** $y = \dfrac{\log_e x}{x}$

d $y = (\log_e x)^4$ **e** $y = (2 \log_e x - 3)^4$ **f** $y = \dfrac{1}{\log_e x}$

g $y = \log_e (\log_e x)$ **h** $y = x \ln x$ **i** $y = \dfrac{1}{x} + \ln x$

15 **a** Show that $y = \dfrac{x}{\log_e x}$ is a solution of the equation $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2$.

b Show that $y = \log_e (\log_e x)$ is a solution of the equation $x\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0$.

16 This result will be used in Section 6I.

a Copy and complete the statement $\log_e |x| = \begin{cases} \ldots, & \text{for } x > 0, \\ \ldots, & \text{for } x < 0. \end{cases}$

b Use part **a** to sketch the curve $y = \log_e |x|$.

c By differentiating separately the two branches in part **a**, show that

$$\frac{d}{dx} \log_e |x| = \frac{1}{x}, \text{ for all } x \neq 0.$$

d Why was $x = 0$ excluded in this discussion?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

An alternative definition of e^x and e:

17 a If $y = \log_e x$, use differentiation by first principles to show that $y' = \lim_{h \to 0} \log_e \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$.

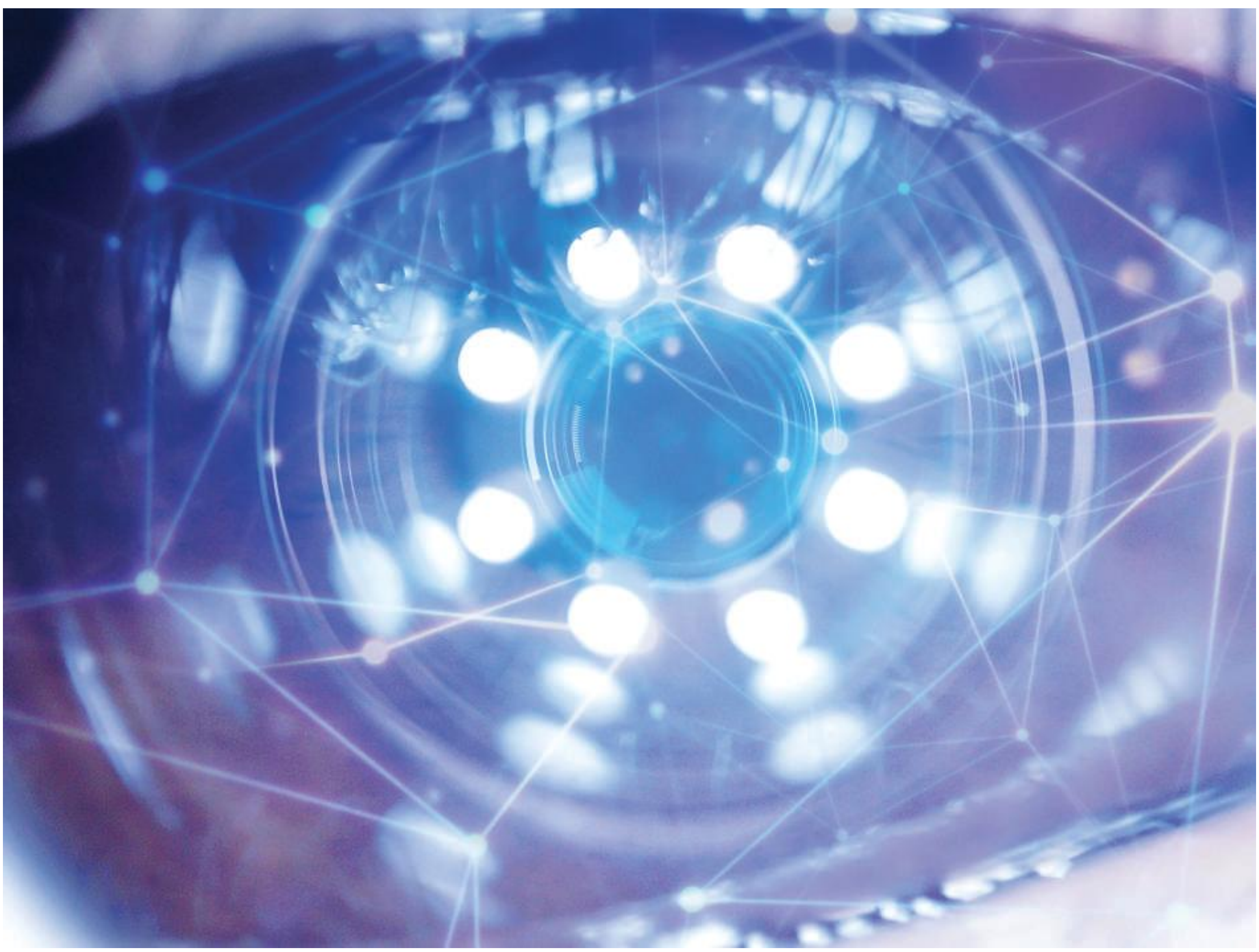
b Use the fact that $y' = \dfrac{1}{x}$ to show that $\lim_{h \to 0} \log_e \left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \frac{1}{x}$.

c Substitute $n = \frac{1}{h}$ and $u = \frac{1}{x}$ to prove these two important limits:

i $\displaystyle \lim_{n \to \infty} \left(1 + \frac{u}{n}\right)^n = e^u$ **ii** $\displaystyle \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

d Investigate how quickly, or slowly, $\left(1 + \frac{1}{n}\right)^n$ converges to e by using your calculator with these values of n.

i 1 **ii** 10 **iii** 100 **iv** 1000 **v** 10 000



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6H Applications of differentiation of $\log_e x$

Differentiation can now be applied in the usual way to study the graphs of functions involving $\log_e x$.

The geometry of tangents and normals

The derivative can be used as usual to investigate the geometry of tangents and normals to a curve.

Example 31 — 6H

a Show that the tangent to $y = \log_e x$ at $T(e, 1)$ has equation $x = ey$.

b Find the equation of the normal to $y = \log_e x$ at $T(e, 1)$.

c Sketch the curve, the tangent and the normal, and find the area of the triangle formed by the y-axis and the tangent and normal at T.

SOLUTION

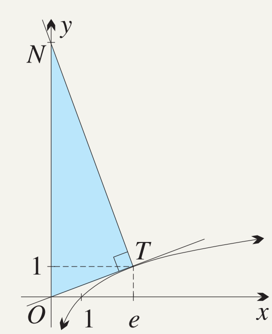
a Differentiating, $\dfrac{dy}{dx} = \dfrac{1}{x}$,

so the tangent at $T(e, 1)$ has gradient $\dfrac{1}{e}$,

and the tangent is $y - 1 = \dfrac{1}{e}(x - e)$

$$ey - e = x - e$$
$$x = ey$$
$$y = \frac{x}{e}.$$

Notice that this tangent has gradient $\frac{1}{e}$ and passes through the origin.



b The tangent at $T(e, 1)$ has gradient $\frac{1}{e}$, so the normal there has gradient $-e$.

Hence the normal has equation $y - 1 = -e(x - e)$

$$y = -ex + (e^2 + 1).$$

c Substituting $x = 0$, the normal has y-intercept $N(0, e^2 + 1)$.
Hence the base ON of $\triangle ONT$ is $(e^2 + 1)$ and its altitude is e.
Thus the triangle $\triangle ONT$ has area $\frac{1}{2}e(e^2 + 1)$ square units.

An example of curve sketching

Here are the six steps of our informal curve-sketching menu applied to the function $y = x\log_e x$.

Example 32 — 6H

Sketch the graph of $y = x\log_e x$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. (You may assume that $x\log_e x \to 0$ as $x \to 0^+$.)

e Find any stationary points and examine their nature.

f Find any points of inflection, and examine the concavity.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a The domain is $x > 0$, because $\log_e x$ is undefined for $x \le 0$.

b The function is undefined when x is negative, so it is neither even nor odd.

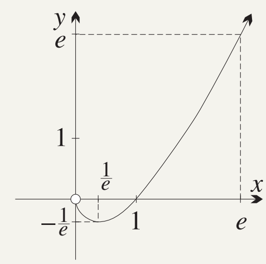
c The only zero is at $x = 1$, and the curve is continuous for $x > 0$.

We take test values at $x = e$ and at $= \frac{1}{e}$.

When $x = e$, $y = e\log_e e$
$= e \times 1$
$= e$.

When $x = e^{-1}$, $y = e^{-1}\log_e e^{-1}$
$= e^{-1} \times (-1)$
$= -e^{-1}$.

x	0	e^{-1}	1	e
y	*	$-e^{-1}$	0	e
sign	*	$-$	0	$+$



Hence y is negative for $0 < x < 1$ and positive for $x > 1$.

d As given in the hint, $y \to 0$ as $x \to 0^+$.
Also, $y \to \infty$ as $x \to \infty$.

e Differentiating by the product rule,

$$f'(x) = vu' + uv'$$
$$= \log_e x + x \times \frac{1}{x}$$
$$= \log_e x + 1,$$
and $f''(x) = \frac{1}{x}$.

Let $u = x$
and $v = \log_e x$.
Then $u' = 1$
and $v' = \frac{1}{x}$.

Putting $f'(x) = 0$ gives $\log_e x = -1$
$x = e^{-1}$.

Substituting, $f'(e^{-1}) = e > 0$
and $f(e^{-1}) = -e^{-1}$, as above,
so $(e^{-1}, -e^{-1})$ is a minimum turning point.
Also $f'(x) \to -\infty$ as $x \to 0^+$, so the curve becomes vertical near the origin.

f Because $f''(x)$ is always positive, there are no inflections, and the curve is always concave up.

A difficulty with the limits of $x\log_e x$ and $\dfrac{\log_e x}{x}$

The curve-sketching example above involved knowing the behaviour of $x\log_e x$ as $x \to 0^+$. When x is a small positive number, $\log_e x$ is a large negative number, so it is not immediately clear whether the product $x\log_e x$ becomes large or small as $x \to 0^+$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

In fact, $x \log_e x \to 0$ as $x \to 0^+$, and x is said to *dominate* $\log_e x$, in the same way that e^x dominated x in Section 6C. Here is a table of values that should make it reasonably clear that $\lim_{x\to 0^+} x \log_e x = 0$:

x	$\frac{1}{e}$	$\frac{1}{e^2}$	$\frac{1}{e^3}$	$\frac{1}{e^4}$	$\frac{1}{e^5}$	$\frac{1}{e^6}$	$\frac{1}{e^7}$	...
$x \log_e x$	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	...
approx.	−0.37	−0.27	−0.15	−0.073	−0.034	−0.015	−0.006	...

Such limits would normally be given in any question where they are needed.

A similar problem arises with the behaviour of $\dfrac{\log_e x}{x}$ as $x \to \infty$, because both top and bottom get large when x is large. Again, x dominates $\log_e x$, meaning that $\lim_{x\to\infty} \dfrac{\log_e x}{x} = 0$, as the following table should make reasonably obvious:

x	e	e^2	e^3	e^4	e^5	e^6	e^7	...
$\frac{\log_e x}{x}$	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	...
approx.	0.37	0.27	0.15	0.073	0.034	0.015	0.006	...

Again, this limit would normally be given if it is needed, but we box the results for completeness. We will provide a proof in Question **23** of Exercise 6J.

9 DOMINANCE

- The function x *dominates* the function $\log_e x$, that is

$$\lim_{x\to 0^+} x \log_e x = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\log_e x}{x} = 0.$$

- More generally, the function x^k *dominates* the function $\log_e x$, for all $k > 0$.

Exercise 6H

FOUNDATION

1 **a** Find the tangent to $y = \log_e x$ at $P(e, 1)$, and prove that it passes through O.

b Find the tangent to $y = \log_e x$ at $Q(1, 0)$, and prove that it passes through $A(0, -1)$.

c Find the tangent to $y = \log_e x$ at $R(\frac{1}{e}, -1)$, and prove that it passes through $B(0, -2)$.

d Find the normal to $y = \log_e x$ at $A(1, 0)$, and its y-intercept.

2 **a** In Question **1a** you showed that the tangent at $P(e, 1)$ on the curve $y = \log_e x$ passes through the origin. Sketch the graph, showing the tangent, and explain graphically why no other tangent passes through the origin.

b Again arguing geometrically from the graph, classify the points in the plane according to whether 0, 1 or 2 tangents pass through them.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Find, giving answers in the form $y = mx + b$, the equations of the tangent and normal to:

a $y = 4\log_e x$ at the point $Q(1, 0)$,
b $y = \log_e x + 3$ at the point $R(1, 3)$,
c $y = 2\log_e x - 2$ at the point $S(1, -2)$,
d $y = 1 - 3\log_e x$ at the point $T(1, 1)$.

4 **a** Show that the point $P(1, 0)$ lies on the curve $y = \log_e(3x - 2)$.
b Find the equations of the tangent and normal at P, and their y-intercepts.
c Find the area of the triangle formed by the tangent, the normal and the y-axis.

5 **a** Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient $\frac{1}{2}$. Then find the equation of the tangent and normal there, in the form $y = mx + b$.
b Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient 2. Then find the equation of the tangent and normal there, in the form $y = mx + b$.

6 **a** Write down the natural domain of $y = x - \log_e x$. What does this answer tell you about whether the function is even, odd or neither?
b Find its first two derivatives.
c Show that the curve is concave up for all values of x in its domain.
d Find the minimum turning point.
e Sketch the curve and write down its range.
f Finally sketch the curve $y = \log_e x - x$ by recognising the simple transformation.

7 **a** Write down the domain of $y = \dfrac{1}{x} + \ln x$.
b Show that the first and second derivatives may be expressed as single fractions as $y' = \dfrac{x - 1}{x^2}$ and $y'' = \dfrac{2 - x}{x^3}$.
c Show that the curve has a minimum at $(1, 1)$ and an inflection at $(2, \frac{1}{2} + \ln 2)$.
d Sketch the graph and write down its range.

8 Consider the curve $y = x\log_e x - x$.
a Write down the domain and x-intercept.
b Draw up a table of signs for the function.
c Show that $y' = \log_e x$ and find y''.
d Hence show that there is one stationary point and determine its nature.
e What does y'' tell you about the curve?
f Given that $y \to 0^-$ as $x \to 0^+$, and that the tangent approaches vertical as $x \to 0^+$, sketch the curve and write down its range.

9 **a** Write down the domain of $y = \log_e(1 + x^2)$.
b Is the curve, even, odd or neither?
c Find where the function is zero, and explain what its sign is otherwise.
d Show that $y' = \dfrac{2x}{1 + x^2}$ and $y'' = \dfrac{2(1 - x^2)}{(1 + x^2)^2}$.
e Hence show that $y = \log_e(1 + x^2)$ has one stationary point, and determine its nature.
f Find the coordinates of the two points of inflection.
g Hence sketch the curve, and then write down its range.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

10 **a** Find the domain of $y = (\ln x)^2$.
b Find where the function is zero, and explain what its sign is otherwise.
c Find y' and show that $y'' = \dfrac{2(1 - \ln x)}{x^2}$.
d Hence show that the curve has an inflection at $x = e$.
e Classify the stationary point at $x = 1$, sketch the curve, and write down the range.

11 **a** Find and classify the lone stationary point of $y = x^2 \log x$ in its natural domain.
b Show that there is an inflection at $x = e^{-\frac{3}{2}}$.
c Examine the behaviour of y and y' as $x \to 0^+$.
d Hence sketch the graph of this function, then write down its range.

12 **a** Write down the domain of $y = \dfrac{\log_e x}{x}$, then find any horizontal or vertical asymptotes.
b Find y' and y''.
c Find any stationary points and determine their nature.
d Find the exact coordinates of the lone point of inflection.
e Sketch the curve, and write down its range.

13 Carefully classify the significant points of $y = \dfrac{x}{\log x}$, and show that there is an inflection at $\left(e^2, \frac{1}{2}e^2\right)$.
Examine the behaviour of y and y' as $x \to 0^+$ and as $x \to \infty$, then sketch the curve and write down its range. Use your techniques of sketching the reciprocal to compare this graph with the graph in the previous question.

14 **a** Write down the domain of $y = \log_e\left(\dfrac{x^2}{x + 1}\right)$.
b Show that $y' = \dfrac{x + 2}{x(x + 1)}$, for x in the domain of the function.
c Show that substituting $x = -2$ into the equation in part **b** gives zero. Explain why there is nevertheless no stationary point at $x = -2$.
d How many inflection points does this curve have?
e Sketch the graph.

15 **a** What is the natural domain of $y = \ln(\ln x)$?
b What is the x-intercept?
c Find y' and y'', and explain why there are no stationary points.
d Explain why there is no inflection at $x = e^{-1}$.
e Sketch the curve.

16 [Technology]
This question confirms the remarks about *dominance* in the text of this section. Use technology to complete the two tables of values below. Do they confirm the values of $\lim\limits_{x \to \infty} \dfrac{\log_e x}{x}$ and $\lim\limits_{x \to 0^+} x \log_e x$ given in the text?

x	2	5	10	20	40	4000
$\dfrac{\log_e x}{x}$						

x	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{4000}$
$x \log_e x$						

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

17 Show that $y = x^{\frac{1}{\ln x}}$ is a constant function and find the value of this constant. What is the natural domain of this function? Sketch its graph.

18 **a** Differentiate $y = x^x$ by taking logs of both sides. Then examine the behaviour of $y = x^x$ near $x = 0$, and show that the curve becomes vertical as $x \to 0^+$.

b Locate and classify any stationary points, and where the curve has gradient 1.

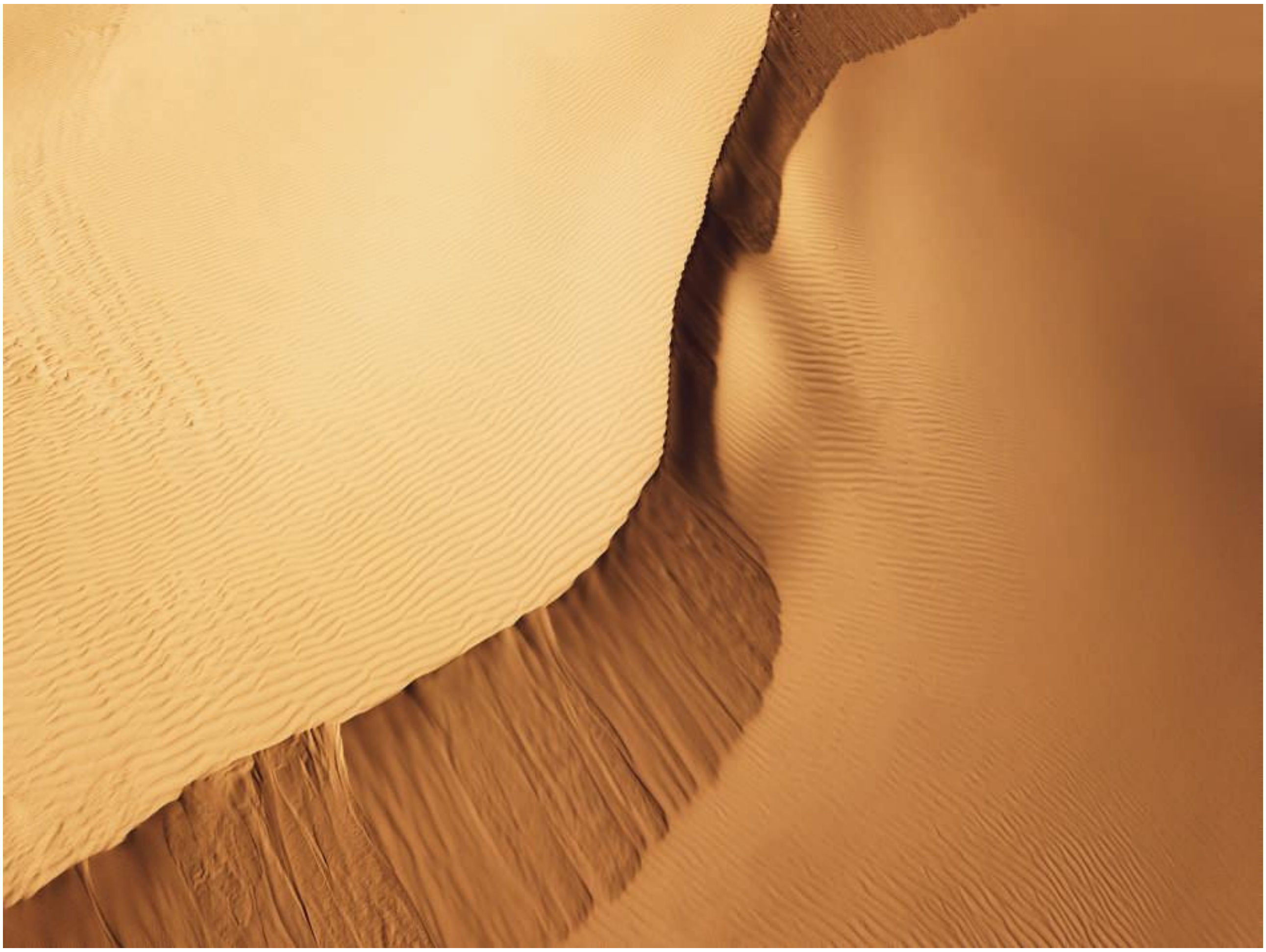
c Sketch the function, and state its domain and range.

19 **a** Find the limits of $y = x^{\frac{1}{x}}$ as $x \to 0^+$ and as $x \to \infty$.

b Show that there is a maximum turning point when $x = e$.

c Show that $y = x^x$ and $y = x^{\frac{1}{x}}$ have a common tangent at $x = 1$.

d Sketch the graph of the function.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

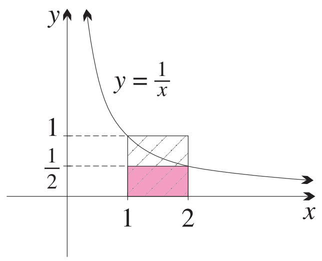
6I Integration of the reciprocal function

The reciprocal function $y = \dfrac{1}{x}$ is an important function — we have seen that it is required whenever two quantities are inversely proportional to each other. So far, however, it has not been possible to integrate the reciprocal function, because the usual rule for integrating powers of x gives nonsense:

$$\text{When } n = -1, \int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{gives} \int x^{-1}\,dx = \frac{x^0}{0},$$

which is nonsense because of the division by zero.

Yet the graph of $y = \dfrac{1}{x}$ to the right shows that there should be no problem with definite integrals involving $\dfrac{1}{x}$, provided that the integral does not cross the discontinuity at $x = 0$. For example, the diagram shows the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$, which the little rectangles show has a value between $\frac{1}{2}$ and 1.



Integration of the reciprocal function

Reversing the standard form for differentiating $\log x$ will now give the necessary standard forms for integrating $\dfrac{1}{x}$.

$$\text{Reversing} \quad \frac{d}{dx}\log_e x = \frac{1}{x} \text{ gives } \int \frac{1}{x}\,dx = \log_e x + C.$$

This is a new standard form for integrating the reciprocal function.
The only qualification is that $x > 0$, otherwise $\log_e x$ is undefined, so we have

$$\int \frac{1}{x}\,dx = \log_e x + C, \text{ for some constant } C, \text{ provided that } x > 0.$$

Example 33 6I

a Find the definite integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$ sketched above.

b Approximate the integral correct to three decimal places and verify that

$$\tfrac{1}{2} < \int_1^2 \frac{1}{x}\,dx < 1.$$

SOLUTION

a $$\int_1^2 \frac{1}{x}\,dx = \Big[\log_e x\Big]_1^2$$
$$= \log_e 2 - \log_e 1$$
$$= \log_e 2, \quad \text{because } \log_e 1 = 0.$$

b Hence $\displaystyle\int_1^2 \frac{1}{x}\,dx \doteqdot 0.693,$

which is indeed between $\frac{1}{2}$ and 1, as the diagram above indicated.

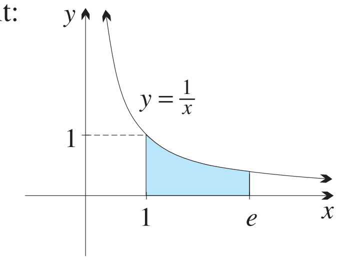
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A characterisation of e

Integrating the reciprocal function from 1 to e gives an amazingly simple result:

$$\begin{aligned}\int_1^e \frac{1}{x}\,dx &= \Big[\log_e x\Big]_1^e \\ &= \log_e e - \log_e 1 \\ &= 1 - 0 \\ &= 1.\end{aligned}$$

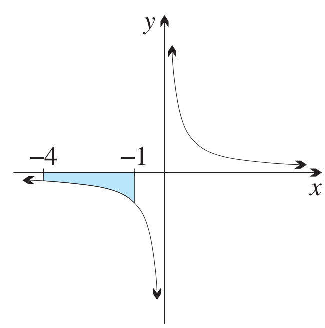


The integral is sketched to the right. The example is very important because it characterises e as the real number satisfying $\int_1^e \frac{1}{x}\,dx = 1$. In other expositions of the theory, this integral is taken as the definition of e.

The primitive of $y = \dfrac{1}{x}$ on both sides of the origin

So far our primitive is restricted by the condition $x > 0$, meaning that we can only deal with definite integrals on the right-hand side of the origin. The full graph of the reciprocal function $y = 1/x$, however, is a hyperbola, with two disconnected branches separated by the discontinuity at $x = 0$.

Clearly there is no reason why we should not integrate over a closed interval such as $-4 \le x \le -1$ on the left-hand side of the origin. We can take any definite integrals of $\frac{1}{x}$, provided only that we do not work across the asymptote at $x = 0$. If x is negative, then $\log(-x)$ is well defined, and using our previous standard forms,



$$\frac{d}{dx}\log(-x) = -\left(\frac{1}{-x}\right) = \frac{1}{x},$$

and reversing this, $\log(-x)$ is a primitive of $\frac{1}{x}$ when x is negative,

$$\int \frac{1}{x}\,dx = \log_e(-x) + C, \quad \text{provided that } x < 0.$$

The absolute value function is designed for just these situations. We can combine the two results into one standard form for the whole reciprocal function,

$$\int \frac{1}{x}\,dx = \log_e |x| + C, \quad \text{provided that } x \ne 0.$$

Question 16 of Exercise 6G gives more detail about this standard form.

Each branch may have its own constant of integration

Careful readers will realise that because $y = \frac{1}{x}$ has two disconnected branches, there can be different constants of integration in the two branches. So the general primitive of $\frac{1}{x}$ is

$$\int \frac{1}{x}\,dx = \begin{cases} \log_e x + A, & \text{for } x > 0, \\ \log(-x) + B, & \text{for } x < 0, \end{cases} \qquad \text{where } A \text{ and } B \text{ are constants.}$$

If a boundary condition is given for one branch, this has no implication at all for the constant of integration in the other branch.

In any physical interpretation, however, the function would normally have meaning in only one of the two branches, so the complication discussed here is rarely needed, and the over-simplified forms in Box 10 below are standard and generally used — the qualification is understood and taken account of when necessary.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Three standard forms

As always, reversing the other standard forms for differentiation gives two more standard forms.

10 STANDARD FORMS FOR INTEGRATING RECIPROCAL FUNCTIONS

- $$\int \frac{1}{x}\,dx = \log_e|x| + C$$
- $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + C$$
- $$\int \frac{u'}{u}\,dx = \log_e|u| + C \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$$

No calculation involving these primitives may cross an asymptote.

The final warning always applies to the primitive of any function, but it is mentioned here because it is such an obvious issue.

Example 34 6I

Evaluate these definite integrals using the first two standard forms above.

a $\displaystyle\int_e^{e^2} \frac{5}{x}\,dx$ **b** $\displaystyle\int_1^4 \frac{1}{1-2x}\,dx$ **c** $\displaystyle\int_1^5 \frac{1}{x-2}\,dx$

SOLUTION

a
$$\begin{aligned}\int_e^{e^2} \frac{5}{x}\,dx &= 5\Big[\log_e|x|\Big]_e^{e^2}\\ &= 5(\log_e e^2 - \log_e e)\\ &= 5(2-1)\\ &= 5\end{aligned}$$

b
$$\begin{aligned}\int_1^4 \frac{1}{1-2x}\,dx &= -\tfrac{1}{2}\Big[\log_e|1-2x|\Big]_1^4 \qquad (\text{here } a = -2 \text{ and } b = 1)\\ &= -\tfrac{1}{2}(\log_e|-7| - \log_e|-1|)\\ &= -\tfrac{1}{2}(\log_e 7 - 0)\\ &= -\tfrac{1}{2}\log_e 7\end{aligned}$$

c This definite integral is meaningless because it crosses the asymptote at $x = 2$.

Using the third standard form

The vital point in using the third standard form,

$$\int \frac{u'}{u}\,dx = \log_e|u| \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)|,$$

is that the top must be the derivative of the bottom. Choose whichever form of the reverse chain rule you are most comfortable with.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 35 6I

Evaluate these definite integrals using the third standard form above.

a $\displaystyle\int_0^1 \frac{2x}{x^2+2}\,dx$ **b** $\displaystyle\int_4^5 \frac{x}{9-x^2}\,dx$ **c** $\displaystyle\int_0^2 \frac{3x}{1-x^3}\,dx$

SOLUTION

a Let $u = x^2 + 2$ OR $f(x) = x^2 + 2$.
Then $u' = 2x$ $f'(x) = 2x$.

Hence in the fraction $\dfrac{2x}{x^2+1}$, the top is the derivative of the bottom.

Thus, using $\displaystyle\int \frac{u'}{u}\,dx = \log_e|f(x)|$ OR $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)|$,

$$\begin{aligned}\int_0^1 \frac{2x}{x^2+2}\,dx &= \Big[\log_e(x^2+2)\Big]_0^1\\ &= \log_e 3 - \log_e 2.\end{aligned}$$

Note: The use of absolute value signs here is unnecessary (but is not wrong) because $x^2 + 2$ is never negative.

b Let $u = 9 - x^2$ OR $f(x) = 9 - x^2$.
Then $u' = -2x$ $f'(x) = -2x$.

The first step is to make the top the derivative of the bottom,

$$\begin{aligned}\int_4^5 \frac{x}{9-x^2}\,dx &= -\tfrac{1}{2}\int_4^5 \frac{-2x}{9-x^2}\,dx, \quad \text{of the form } \int \frac{u'}{u}\,dx \text{ OR } \int \frac{f'(x)}{f(x)}\,dx,\\ &= -\tfrac{1}{2}\Big[\log_e|9-x^2|\Big]_4^5\\ &= -\tfrac{1}{2}(\log_e|-16| - \log_e|-7|)\\ &= -\tfrac{1}{2}(4\log_e 2 - \log_e 7)\\ &= -2\log_e 2 + \tfrac{1}{2}\log_e 7.\end{aligned}$$

c This definite integral is meaningless because it crosses the asymptote at $x = 1$.

Given the derivative, find the function

Finding the function from the derivative involves a constant that can be found if the value of y is known for some value of x.

Example 36 6I

a Find $f(x)$, if $f'(x) = \dfrac{2}{3-x}$ and the graph passes through the origin.

b Hence find $f(2)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Here $\qquad f'(x) = \dfrac{2}{3 - x}$.

Taking the primitive, $\qquad f(x) = -2\ln|3 - x| + C,\quad$ for some constant C.

Because $f(0) = 0$, $\qquad 0 = -2\ln 3 + C$

$\qquad C = 2\ln 3.$

Hence $\qquad f(x) = 2\ln 3 - 2\ln|3 - x|$.

b Substituting $x = 2$ gives $f(2) = 2\ln 3 - 2\ln 1$

$\qquad = 2\ln 3.$

Note: As remarked above, this working is over-simplified, because each branch may have its own constant of integration. But there is no problem in this question because the asymptote is at $x = 3$, so that the given point $(0, 0)$ on the curve, and the value $x = 2$ in part **b**, are both on the same side of the asymptote.

A primitive of $\log_e x$

The next worked example is more difficult, but it is important because it produces a primitive of $\log_e x$, which the theory has not yielded so far. There is no need to memorise the result.

Example 37 — 6I

a Differentiate $x\log_e x$ by the product rule.

b Show by differentiation that $x\log_e x - x$ is a primitive of $\log_e x$.

c Use this result to evaluate $\displaystyle\int_1^e \log_e x\,dx$.

SOLUTION

a Differentiating by the product rule,

$$\frac{d}{dx}(x\log_e x) = vu' + uv'$$
$$= \log_e x + x \times \frac{1}{x},$$
$$= 1 + \log_e x.$$

Let $u = x$
and $v = \log_e x$.
Then $u' = 1$
and $v' = \dfrac{1}{x}$.

b Let $\qquad y = x\log_e x - x.$

Then $\qquad y' = (1 + \log_e x) - 1,\qquad$ using the result of part **a**,

$\qquad = \log_e x.$

Reversing this result gives the primitive of $\log_e x$,

$$\int \log_e x\,dx = x\log_e x - x + C.$$

c Part **b** can now be used to find the definite integral,

$$\int_1^e \log_e x\,dx = \Big[x\log_e x - x\Big]_1^e$$
$$= (e\log_e e - e) - (1\log_e 1 - 1)$$
$$= (e\log_e e - e) - (0 - 1)$$
$$= (e - e) + 1$$
$$= 1.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 6I

FOUNDATION

1 First rewrite each integral using the result $\int \frac{k}{x}\,dx = k\int \frac{1}{x}\,dx$, where k is a constant. Then use the standard form $\int \frac{1}{x}\,dx = \log_e |x| + C$ to integrate it.

a $\int \frac{2}{x}\,dx$ b $\int \frac{1}{3x}\,dx$ c $\int \frac{4}{5x}\,dx$ d $\int \frac{3}{2x}\,dx$

2 Use the standard form $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e |ax+b| + C$ to find these indefinite integrals.

a $\int \frac{1}{4x+1}\,dx$ b $\int \frac{1}{5x-3}\,dx$ c $\int \frac{6}{3x+2}\,dx$

d $\int \frac{15}{5x+1}\,dx$ e $\int \frac{4}{4x+3}\,dx$ f $\int \frac{dx}{3-x}$

g $\int \frac{dx}{7-2x}$ h $\int \frac{4\,dx}{5x-1}$ i $\int \frac{12\,dx}{1-3x}$

3 Evaluate these definite integrals. Simplify your answers where possible.

a $\int_1^5 \frac{1}{x}\,dx$ b $\int_1^3 \frac{1}{x}\,dx$ c $\int_{-8}^{-2} \frac{1}{x}\,dx$

d $\int_{-3}^9 \frac{1}{x}\,dx$ e $\int_1^4 \frac{dx}{2x}$ f $\int_{-15}^{-5} \frac{dx}{5x}$

4 Evaluate these definite integrals, then use the function labelled $\boxed{\ln}$ on your calculator to approximate each integral correct to four significant figures.

a $\int_0^1 \frac{dx}{x+1}$ b $\int_{-7}^{-5} \frac{dx}{x+2}$ c $\int_{-5}^{-2} \frac{dx}{2x+3}$

d $\int_1^2 \frac{3}{5-2x}\,dx$ e $\int_{-1}^1 \frac{3}{7-3x}\,dx$ f $\int_0^{11} \frac{5}{2x-11}\,dx$

5 Evaluate these definite integrals. Simplify your answers where possible.

a $\int_1^e \frac{dx}{x}$ b $\int_1^{e^2} \frac{dx}{x}$ c $\int_e^{e^4} \frac{dx}{x}$ d $\int_{\sqrt{e}}^e \frac{dx}{x}$

6 Find primitives of these function by first writing them as separate fractions.

a $\frac{x+1}{x}$ b $\frac{x+3}{5x}$ c $\frac{1-8x}{9x}$

d $\frac{3x^2-2x}{x^2}$ e $\frac{2x^2+x-4}{x}$ f $\frac{x^4-x+2}{x^2}$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

DEVELOPMENT

7 In each case show that the numerator is the derivative of the denominator. Then use the form $\int \frac{u'}{u}\,dx = \log_e|u| + C$ or $\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$ to integrate the expression.

a $\dfrac{2x}{x^2 - 9}$ **b** $\dfrac{6x + 1}{3x^2 + x}$ **c** $\dfrac{2x + 1}{x^2 + x - 3}$

d $\dfrac{5 - 6x}{2 + 5x - 3x^2}$ **e** $\dfrac{x + 3}{x^2 + 6x - 1}$ **f** $\dfrac{3 - x}{12x - 3 - 2x^2}$

g $\dfrac{e^x}{1 + e^x}$ **h** $\dfrac{e^{-x}}{1 + e^{-x}}$ **i** $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Why is it unnecessary (but not wrong) to use absolute value signs in the answers to parts **g**–**i**?

8 Use the standard form $\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e|ax + b| + C$ to find these indefinite integrals.

a $\int \dfrac{1}{3x - k}\,dx$ **b** $\int \dfrac{1}{mx - 2}\,dx$ **c** $\int \dfrac{p}{px + q}\,dx$ **d** $\int \dfrac{A}{sx - t}\,dx$

9 Find $f(x)$, and then find $f(2)$, given that:

a $f'(x) = 1 + \dfrac{2}{x}$ and $f(1) = 1$ **b** $f'(x) = 2x + \dfrac{1}{3x}$ and $f(1) = 2$

c $f'(x) = 3 + \dfrac{5}{2x - 1}$ and $f(1) = 0$ **d** $f'(x) = 6x^2 + \dfrac{15}{3x + 2}$ and $f(1) = 5\ln 5$

10 a Given that the derivative of $f(x)$ is $\dfrac{x^2 + x + 1}{x}$ and $f(1) = 1\frac{1}{2}$, find $f(x)$.

b Given that the derivative of $g(x)$ is $\dfrac{2x^3 - 3x - 4}{x^2}$ and $g(2) = -3\ln 2$, find $g(x)$.

11 a Find y as a function of x if $y' = \dfrac{1}{4x}$ and $y = 1$ when $x = e^2$. Where does this curve meet the x-axis on the right-hand side of the origin?

b The gradient of a curve is given by $y' = \dfrac{2}{x + 1}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve?

c Find $y(x)$, given that $y' = \dfrac{2x + 5}{x^2 + 5x + 4}$ and $y = 1$ when $x = 1$. Hence evaluate $y(0)$.

d Write down the equation of the family of curves with the property $y' = \dfrac{2 + x}{x}$. Hence find the curve that passes through $(1, 1)$ and evaluate y at $x = 2$ for this curve.

e Given that $f''(x) = \dfrac{1}{x^2}$, $f'(1) = 0$ and $f(1) = 3$, find $f(x)$ and hence evaluate $f(e)$.

12 Use one of the forms $\int \frac{u'}{u}\,dx = \log_e|u| + C$ OR $\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$ to find:

a $\int \dfrac{3x^2}{x^3 - 5}\,dx$ **b** $\int \dfrac{4x^3 + 1}{x^4 + x - 5}\,dx$ **c** $\int \dfrac{x^3 - 3x}{x^4 - 6x^2}\,dx$

d $\int \dfrac{10x^3 - 7x}{5x^4 - 7x^2 + 8}\,dx$ **e** $\int_2^3 \dfrac{3x^2 - 1}{x^3 - x}\,dx$ **f** $\int_e^{2e} \dfrac{2x + 2}{x^2 + 2x}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 a i Differentiate $y = x\log_e x - x$.

ii Hence find $\int \log_e x\,dx$ and $\int_{\sqrt{e}}^{e} \log_e x\,dx$.

b i Find the derivative of $y = 2x^2\log_e x - x^2$.

ii Hence find $\int x\log_e x\,dx$ and $\int_{e}^{2} x\log_e x\,dx$.

c Differentiate $(\log_e x)^2$, and hence find $\int_{\sqrt{e}}^{e} \dfrac{\log_e x}{x}\,dx$.

d Differentiate $\ln(\ln x)$ and hence determine the family of primitives of $\dfrac{1}{x\ln x}$.

14 Find the value of a if:

a $\int_1^a \dfrac{1}{x}\,dx = 5$ **b** $\int_a^e \dfrac{1}{x}\,dx = 5$ **c** $\int_a^{-1} \dfrac{1}{x}\,dx = -2$ **d** $\int_{-e}^{a} \dfrac{1}{x}\,dx = -2$

15 a Find $\int \dfrac{e^x}{e^x + 1}\,dx$.

b Find $\int_1^e \left(x + \dfrac{1}{x^2}\right)^2 dx$.

c Show that $x\,e^x = e^{x + \log_e x}$, and hence differentiate $x\,e^x$ without using the product rule.

16 Stella found the primitive of the function $\dfrac{1}{5x}$ by taking out a factor of $\frac{1}{5}$,

$$\int \frac{1}{5x}\,dx = \frac{1}{5}\int \frac{1}{x}\,dx = \tfrac{1}{5}\log_e|x| + C_1,\ \text{for some constant } C_1.$$

Magar used the second standard form in Box 10 with $a = 5$ and $b = 0$,

$$\int \frac{1}{5x}\,dx = \tfrac{1}{5}\log_e|5x| + C_2,\ \text{for some constant } C_2.$$

Explain what is going on. Will this affect their result when finding a definite integral?

ENRICHMENT

17 A certain curve has gradient $y' = \dfrac{1}{x}$ and its two branches pass through the two points $(-1, 2)$ and $(1, 1)$. Find the equation of the curve.

A power series for $\log_e(1 + x)$:

18 a Use the formula for the partial sum of a GP to prove that for $t \neq -1$,

$$1 - t + t^2 - t^3 + \cdots + t^{2n} = \frac{1}{1 + t} + \frac{t^{2n+1}}{1 + t}.$$

b Integrate both sides of this result from $t = 0$ to $t = x$ to show that for $x > -1$,

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2n+1}}{2n + 1} - \int_0^x \frac{t^{2n+1}}{1 + t}\,dt.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Explain why $\dfrac{t^{2n+1}}{1+t} \le t^{2n+1}$, for $0 \le t \le 1$. Hence prove that for $0 \le x \le 1$, the integral

$$\int_0^x \frac{t^{2n+1}}{1+t}\,dt \text{ converges to 0 as } n \to \infty. \text{ Hence show that}$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \text{ for } 0 \le x \le 1.$$

d **i** Use this series to approximate $\log_e \frac{3}{2}$ correct to two decimal places.

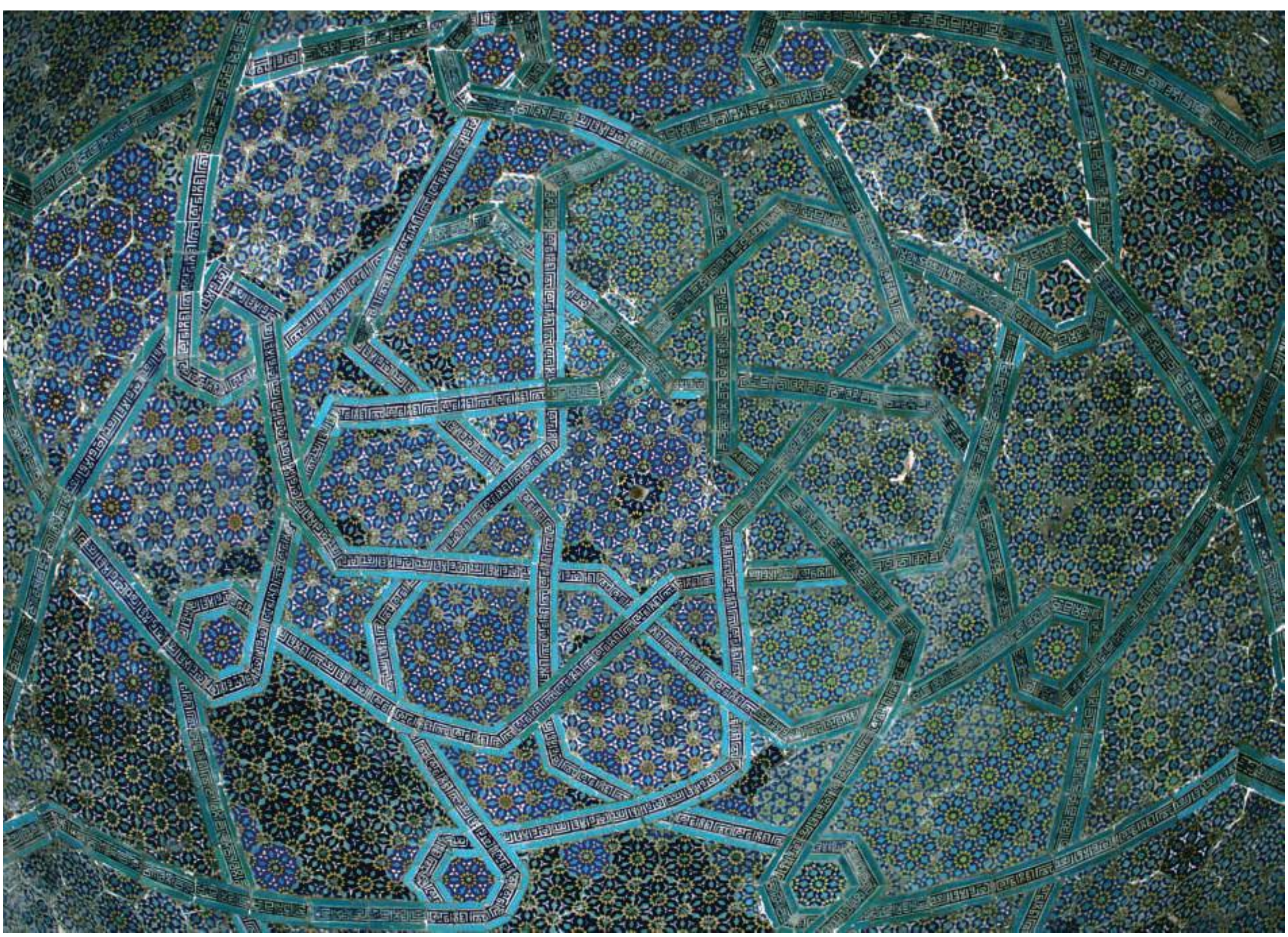
ii Write down the series converging to $\log_e 2$ — called the *alternating harmonic series*.

e With a little more effort, it can be shown that the series in part **c** converges to the given limit for $-1 < x \le 1$ (the proof is a reasonable challenge). Use this to write down the series converging to $\log_e(1-x)$ for $-1 \le x < 1$, and hence approximate $\log_e \frac{1}{2}$ correct to two decimal places.

f Use both series to show that for $-1 < x < 1$,

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Use this result and an appropriate value of x to find $\log_e 3$ correct to five significant figures.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6J Applications of integration of 1/*x*

The usual applications of integration can now be applied to the reciprocal function, whose primitive was previously unavailable.

Finding areas by integration

The next worked example involves finding the area between two given curves.

Example 38 6J

a Find where the hyperbola $xy = 2$ meets the line $x + y = 3$.

b Sketch the situation.

c Find the area of the region between the two curves, in exact form, and then correct to three decimal places.

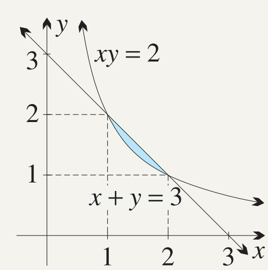
SOLUTION

a Substituting the line $y = 3 - x$ into the hyperbola,

$$\begin{aligned} x(3-x) &= 2 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x &= 1 \text{ or } 2, \end{aligned}$$

so the curves meet at $(1, 2)$ and $(2, 1)$.

b The hyperbola $xy = 2$ has both axes as asymptotes.
The line $x + y = 3$ has x-intercept $(3, 0)$ and y-intercept $(0, 3)$.



c $$\begin{aligned} \text{Area} &= \int_1^2 (\text{top curve} - \text{bottom curve})\,dx \\ &= \int_1^2 \left((3-x) - \frac{2}{x}\right) dx \\ &= \Big[3x - \tfrac{1}{2}x^2 - 2\log_e|x|\Big]_1^2 \\ &= \left(6 - 2 - 2\log_e 2\right) - \left(3 - \tfrac{1}{2} - 2\log_e 1\right) \\ &= \left(4 - 2\log_e 2\right) - \left(2\tfrac{1}{2} - 0\right) \\ &= \left(1\tfrac{1}{2} - 2\log_e 2\right) \text{ square units} \\ &\fallingdotseq 0.114 \text{ square units.} \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 6J

FOUNDATION

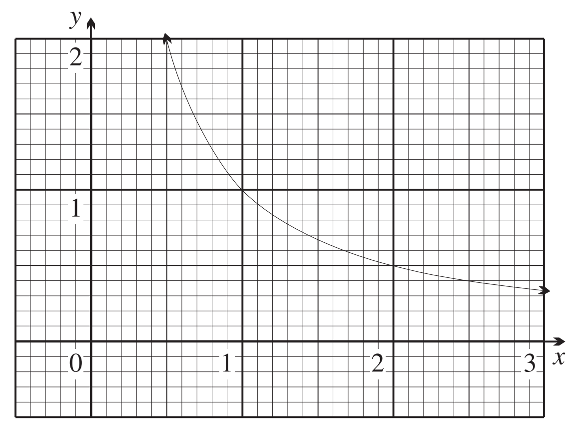
1 a Show that $\displaystyle\int_1^e \frac{1}{x}\,dx = 1$.

b This question uses the result in part **a** to estimate e from a graph of $y = \dfrac{1}{x}$.

The diagram to the right shows the graph of $y = \dfrac{1}{x}$ from $x = 0$ to $x = 3$.

The graph has been drawn on graph paper with a scale of 10 little divisions to 1 unit, so that 100 of the little squares make 1 square unit.

Count the number of squares in the column from $x = 1.0$ to 1.1, then the squares in the column from $x = 1.1$ to 1.2, and so on.



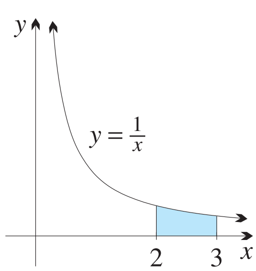
Continue until the number of squares equals 100 — the x-value at this point will be an estimate of e.

2 Find the area between the curve $y = \dfrac{1}{x}$ and the x-axis within the given interval. Answer first in exact form, and then correct to four significant figures.

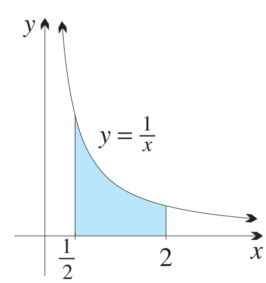
i $1 \le x \le 5$ **ii** $e \le x \le e^2$ **iii** $2 \le x \le 8$

3 Find the area of each shaded region.

a



b



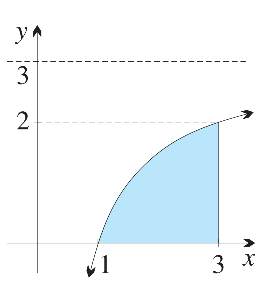
4 a Find the area between $y = \dfrac{1}{3x + 2}$ and the x-axis for $0 \le x \le 1$.

b Find the area between $y = \dfrac{3}{x - 1}$ and the x-axis for $2 \le x \le e^3 + 1$.

c Find the area between $y = \dfrac{1}{x} + x$ and the x-axis, from $x = \frac{1}{2}$ to $x = 2$.

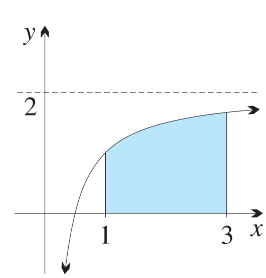
d Find the area between $y = \dfrac{1}{x} + x^2$ and the x-axis, from $x = 1$ to $x = 3$.

5 a



Find the area of the region bounded by $y = 3 - \dfrac{3}{x}$, the x-axis and $x = 3$.

b

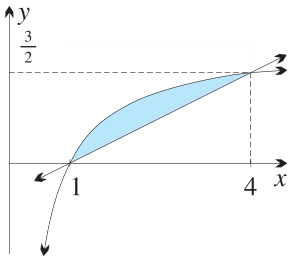


Find the area of the region bounded by $y = 2 - \dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 3$.

ISBN 978-1-108-76630-2

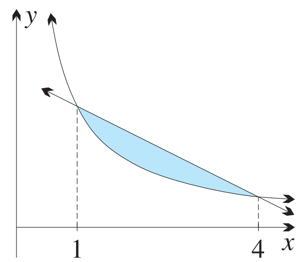
Cambridge University Press
Updated July 2022

6 **a**



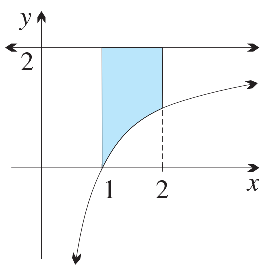
Find the area of the region bounded by $y = 2 - \dfrac{2}{x}$ and the line $y = \frac{1}{2}(x - 1)$.

b



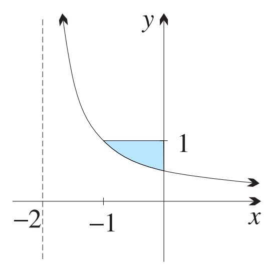
Find the area of the region between $y = \dfrac{2}{x}$ and the line $x + 2y - 5 = 0$.

7 **a**



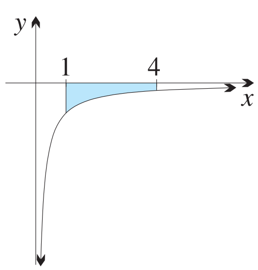
Find the area of the region in the first quadrant bounded by $y = 2 - \dfrac{2}{x}$ and $y = 2$, and lying between $x = 1$ and $x = 2$.

b



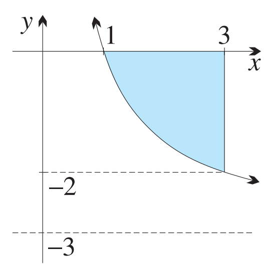
Find the area of the region bounded by the curve $y = \dfrac{1}{x + 2}$, the y-axis and the horizontal line $y = 1$.

8 **a**



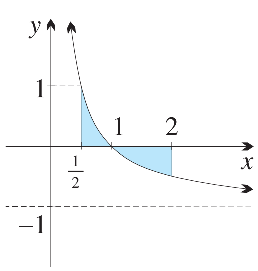
Find the area of the region bounded by $y = -\dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 4$.

b



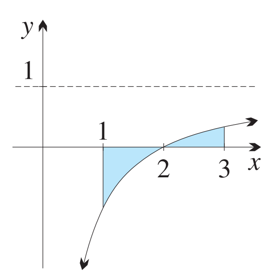
Find the area of the region bounded by $y = \dfrac{3}{x} - 3$, the x-axis and $x = 3$.

9 **a**



Find the area of the region bounded by $y = \dfrac{1}{x} - 1$, the x-axis, $x = \frac{1}{2}$ and $x = 2$.

b



Find the area of the region bounded by $y = 1 - \dfrac{2}{x}$, the x-axis, $x = 1$ and $x = 3$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

DEVELOPMENT

10 a Sketch the region bounded by $y = 1$, $x = 8$ and the curve $y = \frac{4}{x}$.

b Find the area of this region with the aid of an appropriate integral.

11 a Find the two intersection points of the curve $y = \frac{1}{x}$ with the line $y = 4 - 3x$.

b Find the area between these two curves.

12 a Differentiate $x^2 + 1$, and hence find the area under the graph $y = \frac{x}{x^2 + 1}$, between $x = 0$ and $x = 2$.

b Differentiate $x^2 + 2x + 3$, and hence find the area under the graph $y = \frac{x + 1}{x^2 + 2x + 3}$, between $x = 0$ and $x = 1$.

13 a Sketch the region bounded by the x-axis, $y = x$, $y = \frac{1}{x}$ and $x = e$.

b Hence find the area of this region by using two appropriate integrals.

14 a Use the trapezoidal rule with four subintervals to approximate the area between the curve $y = \ln x$ and the x-axis, between $x = 1$ and $x = 5$. Answer correct to four decimal places.

b Differentiate $y = x \log_e x$. Hence find the exact value of the area, and approximate it correct to four decimal places.

c Is the estimate greater than or less than the exact value? Explain how the graph could have predicted this.

15 a Show that $4x = 2(2x + 1) - 2$.

b Hence evaluate the area under $y = \frac{4x}{2x + 1}$ between $x = 0$ and $x = 1$.

16 a Show that the curves $y = \frac{6}{x}$ and $y = x^2 - 6x + 11$ intersect when $x = 1, 2$ and 3.

b Graph these two curves and shade the two areas enclosed by them.

c Find the total area enclosed by the two curves.

17 a Use upper and lower rectangles to prove that $\frac{1}{2} < \int_{2^n}^{2^{n+1}} \frac{1}{x}\,dx < 1$, for $n \geq 0$.

b Hence prove that $\int_1^{2^n} \frac{1}{x}\,dx \to \infty$ as $n \to \infty$.

ENRICHMENT

18 Consider the two curves $y = 6e^{-x}$ and $y = e^x - 1$.

a Let $u = e^x$. Show that the x-coordinate of the point of intersection of these two curves satisfies $u^2 - u - 6 = 0$.

b Hence find the coordinates of the point of intersection.

c Sketch the curves on the same number plane, and shade the region bounded by them and the y-axis.

d Find the area of the shaded region.

19 The hyperbola $y = \frac{1}{x} + 1$ meets the x-axis at $(-1, 0)$. Find the area contained between the x-axis and the curve from:

a $x = -e$ to $x = -1$, **b** $x = -1$ to $x = -e^{-1}$, **c** $x = -e$ to $x = -e^{-1}$.

ISBN 978-1-108-76630-2

20 Find a primitive of $\dfrac{1}{x + \sqrt{x}}$.

21 **a** Show that:

$$\frac{d}{dx}\log_e\left(x + \sqrt{x^2 + a^2}\,\right) = \frac{1}{\sqrt{x^2 + a^2}} \text{ and } \frac{d}{dx}\log_e\left|x + \sqrt{x^2 - a^2}\,\right| = \frac{1}{\sqrt{x^2 - a^2}}.$$

b Use these results to find:

i $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2 + 1}}\,dx$

ii $\displaystyle\int_4^8 \frac{1}{\sqrt{x^2 - 16}}\,dx$

22 Consider the area under $y = \dfrac{1}{x}$ between $x = n$ and $x = n + 1$.

a Show that $\displaystyle\frac{1}{n + 1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$.

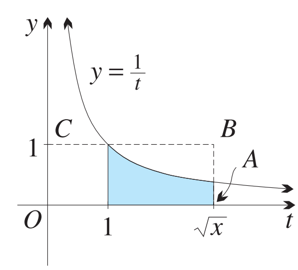
b Hence show that $\dfrac{n}{n + 1} < \log_e(1 + \frac{1}{n})^n < 1$.

c Take the limit as $n \to \infty$ to show that $\lim\limits_{n \to \infty}(1 + \frac{1}{n})^n = e$.

d Repeat the above steps, replacing $n + 1$ with $n + t$ and show that $\lim\limits_{n \to \infty}(1 + \frac{t}{n})^n = e^t$.

Proving the dominance limits:

This question provides proofs of the basic dominance limits used in many of the curve-sketching questions in Exercises 6C and 6H.



23 **a** The shaded region in the diagram to the right shows the definite integral $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\,dt$.

i Explain why $\displaystyle 0 < \int_1^{\sqrt{x}} \frac{1}{t}\,dt <$ area $ABCO$.

ii Evaluate the integral, and prove that

$$0 < \frac{\log_e x}{x} < \frac{2}{\sqrt{x}}.$$

iii Hence prove that $\lim\limits_{x \to \infty} \dfrac{\log_e x}{x} = 0$.

b Substitute $x = \dfrac{1}{u}$ into the limit in part **a** to prove that $\lim\limits_{x \to 0^+} x \log_e x = 0$.

c Substitute $x = e^u$ into the limit in part **b** to prove that $\lim\limits_{x \to -\infty} xe^x = 0$.

d Substitute $x = e^u$ into the limit in part **a** to prove that $\lim\limits_{x \to \infty} xe^{-x} = 0$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6K Calculus with other bases

In applications of exponential functions where calculus is required, the base e can generally be used. For example, the treatment of exponential growth in Chapter 11 of the Year 11 book was done entirely using base e.

The change-of-base formula, however, allows calculus to be applied to exponential and logarithmic functions of any base without conversion to base e. In this section, we develop three further standard forms that allows calculus to be applied straightforwardly to functions such as $y = 2^x$ and $y = 10^x$.

Throughout this section, the other base a must be positive and not equal to 1.

Logarithmic functions to other bases

Any logarithmic function can be expressed easily in terms of $\log_e x$ by using the change-of-base formula. For example,

$$\log_2 x = \frac{\log_e x}{\log_e 2}.$$

Thus every other logarithmic function is just a constant multiple of $\log_e x$. This allows any other logarithmic function to be differentiated easily.

Example 39 6K

a Express the function $y = \log_5 x$ in terms of the function $\log_e x$.

b Hence use the calculator function labelled $\boxed{\ln}$ to approximate, correct to four decimal places:

i $\log_5 30$ **ii** $\log_5 2$ **iii** $\log_5 0.07$

c Check the results of part **b** using the function labelled $\boxed{x^y}$.

SOLUTION

a $\log_5 x = \dfrac{\log_e x}{\log_e 5}$

b **i** $\log_5 30 = \dfrac{\log_e 30}{\log_e 5} \doteqdot 2.1133$

ii $\log_5 2 = \dfrac{\log_e 2}{\log_e 5} \doteqdot 0.4307$

iii $\log_5 0.07 = \dfrac{\log_e 0.07}{\log_e 5} \doteqdot -1.6523$

c Checking these results using the function labelled $\boxed{x^y}$:

i $5^{2.1133} \doteqdot 30$ **ii** $5^{0.4307} \doteqdot 2$ **iii** $5^{-1.6523} \doteqdot 0.07$

11 LOGARITHMIC FUNCTIONS WITH OTHER BASES

Every logarithmic function can be written as a multiple of a logarithmic function base e:

$$\log_a x = \frac{\log_e x}{\log_e a}, \qquad \text{that is} \qquad \log_a x = \frac{1}{\log_e a} \times \log_e x.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Differentiating logarithmic functions with other bases

Once the function is expressed as a multiple of a logarithmic function base e, it can be differentiated using the previous standard forms.

Example 40 6K

Use the change-of-base formula to differentiate:

a $y = \log_2 x$

b $y = \log_a x$

SOLUTION

a Here $y = \log_2 x$.
Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e 2}.$$

Because $\log_e 2$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e 2}$$

$$= \frac{1}{x \log_e 2}.$$

b Here $y = \log_a x$.
Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e a}.$$

Because $\log_e a$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e a}$$

$$= \frac{1}{x \log_e a}.$$

Part **b** above gives the formula in the general case:

12 DIFFERENTIATING LOGARITHMIC FUNCTIONS WITH OTHER BASES

- Either use the change-of-base formula to convert to logarithms base e.
- Or use the standard form $\dfrac{d}{dx} \log_a x = \dfrac{1}{x \log_e a}$.

Example 41 6K

a Differentiate $\log_{10} x$.

b Differentiate $\log_{1.05} x$.

SOLUTION

a $\dfrac{d}{dx} \log_{10} x = \dfrac{1}{x \log_e 10}$

b $\dfrac{d}{dx} \log_{1.05} x = \dfrac{1}{x \log_e 1.05}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A characterisation of the logarithmic function

We have already discussed in Section 6A that the tangent to $y = \log_e x$ at the x-intercept has gradient exactly 1.

The worked example below shows that this property distinguishes the logarithmic function base e from all other logarithmic functions.

Example 42 6K

a Show that the tangent to $y = \log_a x$ at the x-intercept has gradient $\dfrac{1}{\log_e a}$.

b Show that the function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

SOLUTION

a Here $\quad y = \log_a x$.

When $y = 0$, $\quad \log_a x = 0$

$x = 1$,

so the x-intercept is $(1, 0)$.

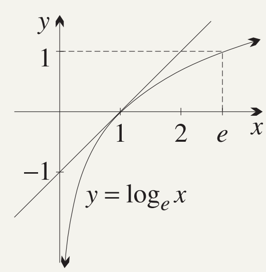
Differentiating, $\quad y' = \dfrac{1}{x \log_e a}$,

so when $x = 1$, $\quad y' = \dfrac{1}{\log_e a}$, as required.

b The gradient at the x-intercept is 1 if and only if

$$\log_e a = 1$$
$$a = e^1$$
$$= e,$$

that is, if and only if the original base a is equal to e.



13 THE GRADIENT AT THE x-INTERCEPT

The function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

Exponential functions with other bases

Before calculus can be applied to an exponential function $y = a^x$ with base a different from e, it must be written as an exponential function with base e. The important identity used to do this is

$$e^{\log_e a} = a,$$

which simply expresses the fact that the functions e^x and $\log_e x$ are inverse functions. Now a^x can be written as

$$a^x = \left(e^{\log_e a}\right)^x, \quad \text{replacing } a \text{ by } e^{\log_e a},$$
$$= e^{x \log_e a}, \quad \text{using the index law } \left(e^k\right)^x = e^{kx}.$$

Thus a^x has been expressed in the form e^{kx}, where $k = \log_e a$ is a constant.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 EXPONENTIAL FUNCTIONS WITH OTHER BASES

- Every positive real number can be written as a power of e:
$$a = e^{\log_e a}$$
- Every exponential function can be written as an exponential function base e:
$$a^x = e^{x \log_e a}$$

Example 43 6K

Express these numbers and functions as powers of e.

a 2 **b** 2^x **c** 5^{-x}

SOLUTION

a $2 = e^{\log_e 2}$

b $2^x = \left(e^{\log_e 2}\right)^x$
$\quad = e^{x \log_e 2}$

c $5^{-x} = \left(e^{\log_e 5}\right)^{-x}$
$\quad = e^{-x \log_e 5}$

Differentiating and integrating exponential functions with other bases

Write the function as a power of e. It can then be differentiated and integrated.

First,
$$\begin{aligned} a^x &= e^{\log_e a^x} \\ &= e^{x \log_e a}. \end{aligned}$$

Differentiating,
$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \log_e a} \\ &= e^{x \log_e a} \times \log_e a, && \text{because } \frac{d}{dx} e^{kx} = k e^{kx}, \\ &= a^x \log_e a, && \text{because } e^{x \log_e a} = a^x. \end{aligned}$$

Integrating,
$$\begin{aligned} \int a^x dx &= \int e^{x \log_e a} dx \\ &= \frac{e^{x \log_e a}}{\log_e a}, && \text{because } \int e^{kx} = \frac{1}{k} e^{kx}, \\ &= \frac{a^x}{\log_e a}, && \text{because } e^{x \log_e a} = a^x. \end{aligned}$$

This process can be carried through every time, or the results can be remembered as standard forms.

15 DIFFERENTIATION AND INTEGRATION WITH OTHER BASES

There are two approaches.

- Write all powers with base e before differentiating or integrating.
- Alternatively, use the standard forms:

$$\frac{d}{dx} a^x = a^x \log_e a \qquad \text{and} \qquad \int a^x dx = \frac{a^x}{\log_e a} + C$$

Note: The formulae for differentiating and integrating a^x both involve the constant $\log_e a$. This constant $\log_e a$ is 1 when $a = e$, so the formulae are simplest when the base is e. Again, this indicates that e is the appropriate base to use for calculus with exponential functions.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 44 6K

Differentiate $y = 2^x$. Hence find the gradient of $y = 2^x$ at the y-intercept, correct to three significant figures.

SOLUTION

Here $\qquad y = 2^x$.

Using the standard form, $\qquad y' = 2^x \log_e 2$.

Hence when $x = 0$, $\qquad y' = 2^0 \times \log_e 2$

$$= \log_e 2$$

$$\doteqdot 0.693.$$

Note: This result may be compared with the results of physically measuring this gradient in Question 1 of Exercise 11A in the Year 11 book.

Example 45 6K

a Show that the line $y = x + 1$ meets the curve $y = 2^x$ at $A(0, 1)$ and $B(1, 2)$.

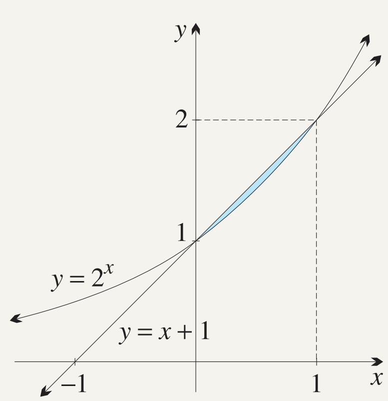
b Sketch the two curves and shade the region contained between them.

c Find the area of this shaded region, correct to four significant figures.

SOLUTION

a Simple substitution of $x = 0$ and $x = 1$ into both functions verifies the result.

b The graph is drawn to the right.



c $$\text{Area} = \int_0^1 (\text{upper curve} - \text{lower curve})\,dx$$

$$= \int_0^1 (x + 1 - 2^x)\,dx$$

$$= \left[\tfrac{1}{2}x^2 + x - \frac{2^x}{\log_e 2}\right]_0^1$$

$$= \left(\tfrac{1}{2} + 1 - \frac{2}{\log_e 2}\right) - \left(0 + 0 - \frac{1}{\log_e 2}\right)$$

$$= 1\tfrac{1}{2} - \frac{1}{\log_e 2} \text{ square units}$$

$$\doteqdot 0.05730 \text{ square units.}$$

Exercise 6K FOUNDATION

1 Use the change-of-base formula $\log_a x = \dfrac{\log_e x}{\log_e a}$ and the function labelled $\boxed{\ln}$ on your calculator to evaluate each expression correct to three significant figures. Then check your answers using the function labelled $\boxed{x^y}$.

a $\log_2 3$ **b** $\log_2 10$ **c** $\log_5 26$ **d** $\log_3 0.0047$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 Use the change-of-base formula to express these with base e, then differentiate them.

a $y = \log_2 x$ **b** $y = \log_{10} x$ **c** $y = 3\log_5 x$

3 Use the standard form $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}$ to differentiate:

a $y = \log_3 x$ **b** $y = \log_7 x$ **c** $y = 5\log_6 x$

4 Express these functions as powers of e, then differentiate them.

a $y = 3^x$ **b** $y = 4^x$ **c** $y = 2^x$

5 Use the standard form $\frac{d}{dx}a^x = a^x\log_e a$ to differentiate:

a $y = 10^x$ **b** $y = 8^x$ **c** $y = 3 \times 5^x$

6 Convert each integrand to a power of e and then integrate.

a $\int 2^x\,dx$ **b** $\int 6^x\,dx$ **c** $\int 7^x\,dx$ **d** $\int 3^x\,dx$

7 Use the result $\int a^x\,dx = \frac{a^x}{\log_e a} + C$ to find each primitive, then evaluate the definite integral correct to four significant figures.

a $\int_0^1 2^x\,dx$ **b** $\int_0^1 3^x\,dx$ **c** $\int_{-1}^1 5^x\,dx$ **d** $\int_0^2 4^x\,dx$

8 **a** Complete the table of values to the right, giving your answers correct to two decimal places where necessary.

b Use this table of values to sketch the three curves $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ on the same set of axes.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$					
$\log_e x$					
$\log_4 x$					

DEVELOPMENT

9 **a** Differentiate $y = \log_2 x$. Hence find the gradient of the tangent to the curve at $x = 1$.

b Hence find the equation of the tangent there.

c Do likewise for:

i $y = \log_3 x$, **ii** $y = \log_5 x$.

10 Give the exact value of each integral, then evaluate it correct to four decimal places.

a $\int_1^3 2^x\,dx$ **b** $\int_{-1}^1 (3^x + 1)\,dx$ **c** $\int_0^2 (10^x - 10x)\,dx$

11 Use the change-of-base formula to express $y = \log_{10} x$ with base e, and hence find y'.

a Find the gradient of the tangent to this curve at the point $(10, 1)$.

b Thus find the equation of this tangent in general form.

c At what value of x will the tangent have gradient 1?

12 **a** Find the equations of the tangents to each of $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ at the points where $x = 3$.

b Show that the three tangents all meet at the same point on the x-axis.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

13 **a** Show that the curves $y = 2^x$ and $y = 1 + 2x - x^2$ intersect at $A(0, 1)$ and $B(1, 2)$.
b Sketch the curves and find the area between them.

14 Find the intercepts of the curve $y = 8 - 2^x$, and hence find the area of the region bounded by this curve and the coordinate axes.

15 **a** Sketch the curve $y = 3 - 3^x$, showing the intercepts and asymptote.
b Find the area contained between the curve and the axes.

16 **a** Show that the curves $y = x + 1$ and $y = 4^x$ intersect at the y-intercept and at $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
b Write the area of the region enclosed between these two curves as an integral.
c Evaluate the integral found in part **b**.

17 **a** Show that the tangent to $y = \log_3 x$ at $x = e$ passes through the origin.
b Show that the tangent to $y = \log_5 x$ at $x = e$ passes through the origin.
c Show that the same is true for $y = \log_a x$, for any base a.

18 **a** Differentiate $x \log_e x - x$, and hence find $\int \log_e x \, dx$.
b Use the change-of-base formula and the integral in part **a** to evaluate $\int_1^{10} \log_{10} x \, dx$.

ENRICHMENT

19 As always, the three standard forms in this section have linear extensions. The pronumeral m is used here instead of the usual a because a is being used for the base.

a Use the standard form $\frac{d}{dx} \log_a (mx + b) = \frac{m}{(mx + b)\log_e a}$ to differentiate:

i $y = \log_3 x$ **ii** $y = \log_7 (2x + 3)$ **iii** $y = 5 \log_6 (4 - 9x)$

b Use the standard form $\frac{d}{dx} a^{mx+b} = ma^{mx+b} \log_e a$ to differentiate:

i $y = 10^x$ **ii** $y = 8^{4x-3}$ **iii** $y = 3 \times 5^{2-7x}$

c Use the standard form $\int a^{mx+b} \, dx = \frac{a^{mx+b}}{m \log_e a} + C$ to find:

i $\int 3^{5x} \, dx$ **ii** $\int 6^{2x+7} \, dx$ **iii** $\int 5 \times 7^{4-9x} \, dx$

20 If the positive base a of $y = a^x$ and $y = \log_a x$ is small enough, then the two curves will intersect. What base must be chosen so that the two are tangent at the point of contact? Proceed as follows:
a Rewrite both equations with base e, and let $k = \log_e a$.
b Explain why the gradient of the tangent at the point of contact must be 1.
c Use the last part to obtain two equations for the gradient.
d Solve these simultaneously to find k, and hence write down the base a.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 **a** Sketch the graphs of $y = e^x$ and $y = e^{-x}$ on the same number plane. Add the line that reflects each graph onto the other graph. Then draw the tangents at the y-intercepts, and mark the angle between them.

b Sketch the graphs of $y = e^x$ and $y = \log_e x$ on the same number plane. Add the line that reflects each graph onto the other graph. Then draw the tangents at the intercepts with the axes.

2 Use your calculator, and in some cases the change-of base formula, to approximate each expression correct to four significant figures.

a e^4 **b** e **c** $e^{-\frac{3}{2}}$ **d** $\log_e 2$

e $\log_{10} \frac{1}{2}$ **f** $\log_2 0.03$ **g** $\log_{1.05} 586$ **h** $\log_8 3\frac{3}{7}$

3 Use logarithms to solve these equations correct to four significant figures. You will need to apply the change-of-base formula before using your calculator.

a $3^x = 14$ **b** $2^x = 51$ **c** $4^x = 1345$ **d** $5^x = 132$

4 Simplify:

a $e^{2x} \times e^{3x}$ **b** $e^{7x} \div e^x$ **c** $\dfrac{e^{2x}}{e^{6x}}$ **d** $(e^{3x})^3$

5 Solve each equation using a suitable substitution to reduce it to a quadratic.

a $9^x - 7 \times 3^x - 18 = 0$ **b** $e^{2x} - 11e^x + 28 = 0$

6 Sketch the graph of each function on a separate number plane, and state its range.

a $y = e^x$ **b** $y = e^{-x}$ **c** $y = e^x + 1$ **d** $y = e^{-x} - 1$

7 **a** **i** Explain how $y = e^{x-3}$ can be obtained by translating $y = e^x$, and sketch it.

ii Explain how $y = e^{x-3}$ can be obtained by dilating $y = e^x$.

b **i** Explain how $y = \log_e 3x$ can be obtained by dilating $y = \log_e x$, and sketch it.

ii Explain how $y = \log_e 3x$ can be obtained by translating $y = \log_e x$.

8 Differentiate:

a $y = e^x$ **b** $y = e^{3x}$ **c** $y = e^{2x+3}$ **d** $y = e^{-x}$

e $y = e^{-3x}$ **f** $y = 3e^{2x+5}$ **g** $y = 4e^{\frac{1}{2}x}$ **h** $y = \frac{2}{3}e^{6x-5}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

9 Write each function as a single power of e, and then differentiate it.

a $y = e^{3x} \times e^{2x}$ **b** $y = \dfrac{e^{7x}}{e^{3x}}$ **c** $y = \dfrac{e^{x}}{e^{4x}}$ **d** $y = (e^{-2x})^3$

10 Differentiate each function using the chain, product and quotient rules as appropriate.

a $y = e^{x^3}$ **b** $y = e^{x^2 - 3x}$ **c** $y = xe^{2x}$ **d** $y = (e^{2x} + 1)^3$

e $y = \dfrac{e^{3x}}{x}$ **f** $y = x^2 e^{x^2}$ **g** $y = (e^{x} - e^{-x})^5$ **h** $y = \dfrac{e^{2x}}{2x + 1}$

11 Find the first and second derivatives of:

a $y = e^{2x+1}$ **b** $y = e^{x^2+1}$

12 Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x-intercept and y-intercept of this tangent.

13 Consider the curve $y = e^{-3x}$.

a Find the gradient of the normal to the curve at the point where $x = 0$.

b Find y'' and hence determine the concavity of the curve at the point where $x = 0$.

14 Consider the curve $y = e^x - x$.

a Find y' and y''.

b Show that there is a stationary point at $(0, 1)$, and determine its nature.

c Explain why the curve is concave up for all values of x.

d Sketch the curve and write down its range.

15 Find the stationary point on the curve $y = xe^{-2x}$ and determine its nature.

16 Find:

a $\displaystyle\int e^{5x}\,dx$ **b** $\displaystyle\int 10e^{2-5x}\,dx$ **c** $\displaystyle\int e^{\frac{1}{5}x}\,dx$ **d** $\displaystyle\int 3e^{5x-4}\,dx$

17 Find the exact value of:

a $\displaystyle\int_0^2 e^{x}\,dx$ **b** $\displaystyle\int_0^1 e^{2x}\,dx$ **c** $\displaystyle\int_{-1}^0 e^{-x}\,dx$

d $\displaystyle\int_{-\frac{2}{3}}^0 e^{3x+2}\,dx$ **e** $\displaystyle\int_0^{\frac{1}{2}} e^{3-2x}\,dx$ **f** $\displaystyle\int_0^2 2e^{\frac{1}{2}x}\,dx$

18 Find the primitive of:

a $\dfrac{1}{e^{5x}}$ **b** $e^{3x} \times e^{x}$ **c** $\dfrac{6}{e^{3x}}$ **d** $(e^{3x})^2$

e $\dfrac{e^{3x}}{e^{5x}}$ **f** $\dfrac{e^{3x} + 1}{e^{2x}}$ **g** $e^{2x}(e^{x} + e^{-x})$ **h** $(1 + e^{-x})^2$

19 Find the exact value of:

a $\displaystyle\int_0^1 (1 + e^{-x})\,dx$ **b** $\displaystyle\int_0^2 (e^{2x} + x)\,dx$ **c** $\displaystyle\int_0^1 \frac{2}{e^{x}}\,dx$

d $\displaystyle\int_0^{\frac{1}{3}} e^{3x}(1 - e^{-3x})\,dx$ **e** $\displaystyle\int_0^1 \frac{e^{2x} + 1}{e^{x}}\,dx$ **f** $\displaystyle\int_0^1 (e^{x} + 1)^2\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

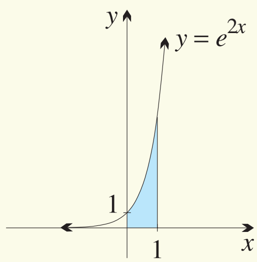
20 If $f'(x) = e^x - e^{-x} - 1$ and $f(0) = 3$, find $f(x)$ and then find $f(1)$.

21 **a** Differentiate e^{x^3}.

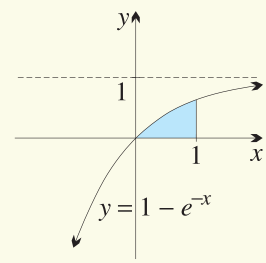
b Hence find $\int_0^1 x^2 e^{x^3}\,dx$.

22 Find the area of each region correct to three significant figures.

a

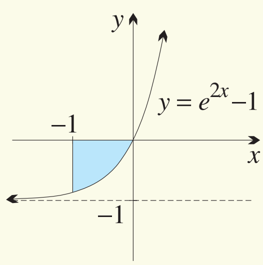


b

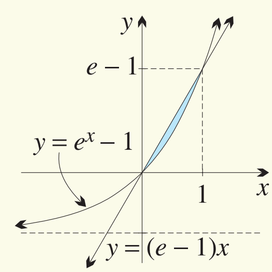


23 Find the exact area of the shaded region.

a



b



24 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_2 x$ **b** $y = -\log_2 x$ **c** $y = \log_2 (x - 1)$ **d** $y = \log_2 (x + 3)$

25 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_e x$ **b** $y = \log_e (-x)$ **c** $y = \log_e (x - 2)$ **d** $y = \log_e x + 1$

26 Use the log laws to simplify:

a $e \log_e e$ **b** $\log_e e^3$ **c** $\ln \frac{1}{e}$ **d** $2e \ln \sqrt{e}$

27 Differentiate these functions.

a $\log_e x$ **b** $\log_e 2x$ **c** $\log_e (x + 4)$

d $\log_e (2x - 5)$ **e** $2 \log_e (5x - 1)$ **f** $x + \log_e x$

g $\ln (x^2 - 5x + 2)$ **h** $\ln (1 + 3x^5)$ **i** $4x^2 - 8x^3 + \ln (x^2 - 2)$

28 Use the log laws to simplify each function and then find its derivative.

a $\log_e x^3$ **b** $\log_e \sqrt{x}$ **c** $\ln x(x + 2)$ **d** $\ln \frac{x}{x - 1}$

29 Differentiate these functions using the product or quotient rule.

a $x \log_e x$ **b** $e^x \log_e x$ **c** $\frac{x}{\ln x}$ **d** $\frac{\ln x}{x^2}$

30 Find the equation of the tangent to the curve $y = 3 \log_e x + 4$ at the point $(1, 4)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

31 Consider the function $y = x - \log_e x$.

a Show that $y' = \dfrac{x-1}{x}$.

b Hence show that the graph of $y = x - \log_e x$ has a minimum turning point at $(1, 1)$.

32 Find these indefinite integrals.

a $\displaystyle\int \frac{1}{x}\,dx$ **b** $\displaystyle\int \frac{3}{x}\,dx$ **c** $\displaystyle\int \frac{1}{5x}\,dx$ **d** $\displaystyle\int \frac{1}{x+7}\,dx$

e $\displaystyle\int \frac{1}{2x-1}\,dx$ **f** $\displaystyle\int \frac{1}{2-3x}\,dx$ **g** $\displaystyle\int \frac{2}{2x+9}\,dx$ **h** $\displaystyle\int \frac{8}{1-4x}\,dx$

33 Evaluate these definite integrals.

a $\displaystyle\int_0^1 \frac{1}{x+2}\,dx$ **b** $\displaystyle\int_1^4 \frac{1}{4x-3}\,dx$ **c** $\displaystyle\int_1^e \frac{1}{x}\,dx$ **d** $\displaystyle\int_{e^2}^{e^3} \frac{1}{x}\,dx$

34 Use the standard form $\displaystyle\int \frac{u'}{u}\,dx = \log_e|u| + C$ OR $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$ to find:

a $\displaystyle\int \frac{2x}{x^2+4}\,dx$ **b** $\displaystyle\int \frac{3x^2-5}{x^3-5x+7}\,dx$ **c** $\displaystyle\int \frac{x}{x^2-3}\,dx$ **d** $\displaystyle\int \frac{x^3-1}{x^4-4x}\,dx$

35 Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 4$.

36 **a** By solving the equations simultaneously, show that the curve $y = \dfrac{5}{x}$ and the line $y = 6 - x$ intersect at the points $(1, 5)$ and $(5, 1)$.

b By sketching both graphs on the same number plane, find the area of the region enclosed between them.

37 Find the derivatives of:

a e^x **b** 2^x **c** 3^x **d** 5^x

38 Find these indefinite integrals.

a $\displaystyle\int e^x\,dx$ **b** $\displaystyle\int 2^x\,dx$ **c** $\displaystyle\int 3^x\,dx$ **d** $\displaystyle\int 5^x\,dx$

39 **a** Differentiate $x\log_e x$, and hence find $\displaystyle\int \log_e x\,dx$.

b Differentiate xe^x, and hence find $\displaystyle\int xe^x\,dx$.

c Hence prove that $\displaystyle\int_1^e \frac{1}{x}\,dx = \int_1^e \log_e x\,dx = \int_0^1 xe^x\,dx = 1$.

40 **a** Find the gradient of $y = 2^x$ at $A(3, 8)$.

b Find the gradient of $y = \log_2 x$ at $B(8, 3)$.

c Explain geometrically why the two gradients are reciprocals of each other.

41 **a** Find $\displaystyle\int_0^3 2^x\,dx$ and $\displaystyle\int_{-3}^0 2^x\,dx$.

b Explain geometrically why the first is 8 times the second.

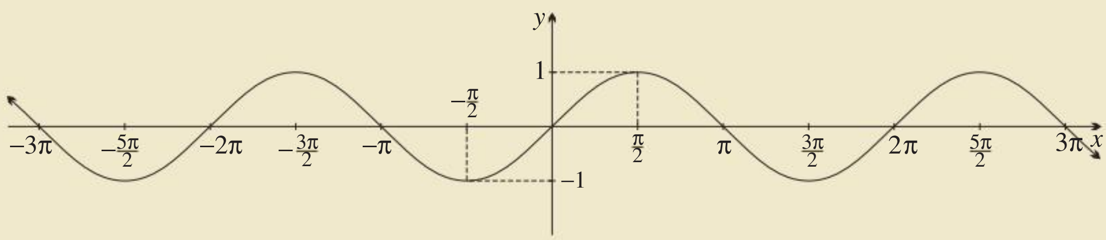
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

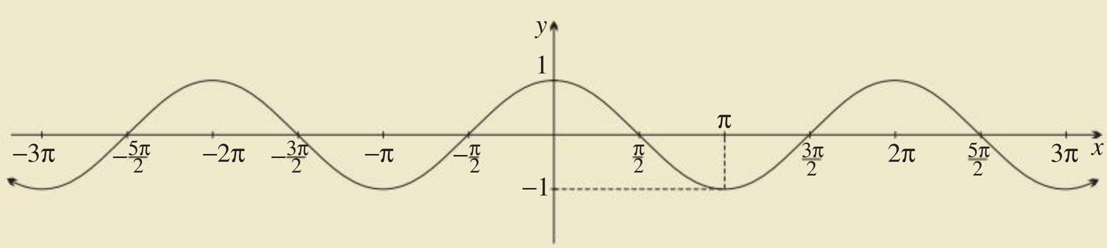
7 The trigonometric functions

This chapter extends calculus to the trigonometric functions. The sine and cosine functions are extremely important because their graphs are waves. They are therefore essential in the modelling of all the many wave-like phenomena such as sound waves, light and radio waves, vibrating strings, tides, and economic cycles. The alternating current that we use in our homes fluctuates in a sine wave. Most of the attention in this chapter is given to these two functions.

$y = \sin x$



$y = \cos x$



In the second half of Chapter 11 of the Year 11 book, we introduced radian measure, promising that it was the correct way to measure angles when doing calculus. We drew the six trigonometric graphs in radians and discussed their symmetries in some detail, and also developed area formula for calculating arc length and the areas of sectors and segments. Then in the last section of Chapter 3, we applied translations, reflections and dilations to the trigonometric graphs and developed the three ideas of amplitude, period and phase. All this previous work is required in the present chapter.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7A The behaviour of sin x near the origin

This section proves an important limit that is the crucial step in finding the derivative of $\sin x$ in the next section. This limit establishes that the curve $y = \sin x$ has gradient 1 when it passes through the origin. Geometrically, this means that the line $y = x$ is the tangent to $y = \sin x$ at the origin.

Note: The limit established in this section provides the geometric basis for differentiating the trigonometric functions on Section 7B. The section could well be left to a second reading of the chapter at a later time.

A fundamental inequality

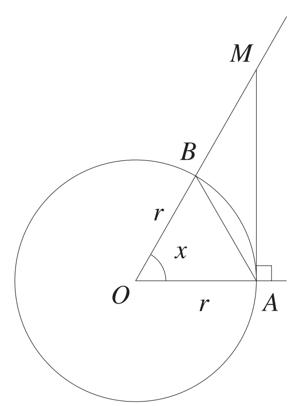
First, an appeal to geometry is needed to establish an inequality concerning x, $\sin x$ and $\tan x$.

1 AN INEQUALITY FOR sin *x* AND tan *x* NEAR THE ORIGIN

- For all acute angles x, $\quad \sin x < x < \tan x$.
- For $-\frac{\pi}{2} < x < 0$, $\quad \sin x > x > \tan x$.

Proof

A Let x be an acute angle.
Construct a circle of centre O and any radius r,
and a sector AOB subtending the angle x at the centre O.
Let the tangent at A meet the radius OB at M
(the radius OB will need to be produced) and join the chord AB.



In $\triangle OAM$, $\quad \dfrac{AM}{r} = \tan x$,

so $\quad AM = r\tan x$.

It is clear from the diagram that

$$\text{area } \triangle OAB < \text{area sector } OAB < \text{area } \triangle OAM,$$

and using area formulae for triangles and sectors,

$$\tfrac{1}{2}r^2\sin x < \tfrac{1}{2}r^2 x < \tfrac{1}{2}r^2\tan x$$

$$\boxed{\div\ \tfrac{1}{2}r^2} \qquad \sin x < x < \tan x.$$

B Because x, $\sin x$ and $\tan x$ are all odd functions,

$$\sin x > x > \tan x, \quad \text{for } -\tfrac{\pi}{2} < x < 0.$$

The main theorem

This inequality now allows two fundamental limits to be proven:

2 TWO FUNDAMENTAL LIMITS

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Proof

When x is acute, $\sin x < x < \tan x$.

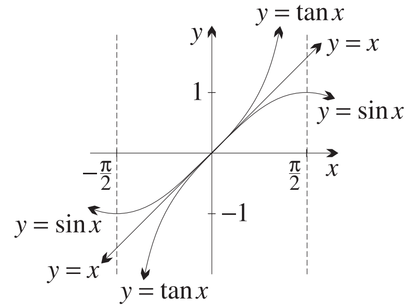
Dividing through by $\sin x$, $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$.

As $x \to 0^+$, $\cos x \to 1$, so $\dfrac{x}{\sin x} \to 1$ as $x \to 0^+$.

But $\dfrac{x}{\sin x}$ is even, so $\dfrac{x}{\sin x} \to 1$ as $x \to 0^-$.

Combining these two limits, $\dfrac{x}{\sin x} \to 1$ as $x \to 0$.

Finally,
$$\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$$
$$\to 1 \times 1, \text{ as } x \to 0.$$



The diagram to the right shows what has been proven about the graphs of $y = x$, $y = \sin x$ and $y = \tan x$ near the origin.

- The line $y = x$ is a common tangent at the origin to both $y = \sin x$ and $y = \tan x$.
- On both sides of the origin, $y = \sin x$ curls away from the tangent towards the x-axis.
- On both sides of the origin, $y = \tan x$ curls away from the tangent in the opposite direction.

3 THE BEHAVIOUR OF $\sin x$ AND $\tan x$ NEAR THE ORIGIN

- The line $y = x$ is a tangent to both $y = \sin x$ and $y = \tan x$ at the origin.
- When $x = 0$, the derivatives of both $\sin x$ and $\tan x$ are exactly 1.

Approximations to the trigonometric functions for small angles

For 'small' angles, positive or negative, the limits above yield good approximations for the three trigonometric functions (the angle must, of course, be expressed in radians).

4 SMALL-ANGLE APPROXIMATIONS

- Suppose that x is a 'small' angle (written in radians). Then

  $\sin x \doteqdot x$ and $\cos x \doteqdot 1$ and $\tan x \doteqdot x$.

In order to use these approximations, one needs to get some idea about how good the approximations are. Two questions in Exercise 7A below ask for tables of values for $\sin x$, $\cos x$ and $\tan x$ for progressively smaller angles.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 1 7A

Use the small-angle approximations in Box 4 to give approximate values of:

a $\sin 1°$ **b** $\cos 1°$ **c** $\tan 1°$

SOLUTION

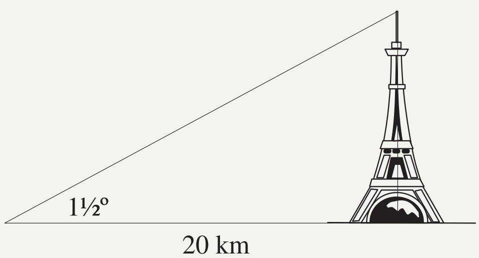
The 'small angle' of $1°$ is $\frac{\pi}{180}$ radians. Hence, using the approximations above:

a $\sin 1° \doteqdot \frac{\pi}{180}$ **b** $\cos 1° \doteqdot 1$ **c** $\tan 1° \doteqdot \frac{\pi}{180}$

Example 2 7A

Approximately how high is a tower that subtends an angle of $1\frac{1}{2}°$ when it is 20 km away?

SOLUTION



Convert 20 km to 20000 metres.

Then from the diagram, using simple trigonometry,

$$\frac{\text{height}}{20000} = \tan 1\tfrac{1}{2}°$$

$$\text{height} = 20000 \times \tan 1\tfrac{1}{2}°.$$

But the 'small' angle $1\frac{1}{2}°$ expressed in radians is $\frac{\pi}{120}$,

so $\tan 1\frac{1}{2}° \doteqdot \frac{\pi}{120}$.

Hence, approximately, $\text{height} \doteqdot 20000 \times \dfrac{\pi}{120}$

$$\doteqdot \frac{500\pi}{3} \text{ metres}$$

$$\doteqdot 524 \text{ metres.}$$

Example 3 7A

The sun subtends an angle of $0°31'$ at the Earth, which is 150000000 km away. What is the sun's approximate diameter?

Note: This problem can be done similarly to the previous problem, but like many small-angle problems, it can also be done by approximating the diameter to an arc of the circle.

SOLUTION

First, $0°31' = \dfrac{31°}{60}$

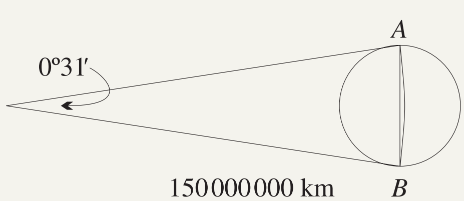
$$= \frac{31}{60} \times \frac{\pi}{180} \text{ radians.}$$

Because the diameter AB is approximately equal to the arc length AB,

$$\text{diameter} \doteqdot r\theta$$

$$\doteqdot 150000000 \times \frac{31}{60} \times \frac{\pi}{180}$$

$$\doteqdot 1353000 \text{ km.}$$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 7A

FOUNDATION

1 **a** Copy and complete the following table of values, giving entries correct to six decimal places. (Your calculator must be in radian mode.)

angle size in radians	1	0.5	0.2	0.1	0.08	0.05	0.02	0.01	0.005	0.002
$\sin x$										
$\dfrac{\sin x}{x}$										
$\tan x$										
$\dfrac{\tan x}{x}$										
$\cos x$										

b What are the limits of $\dfrac{\sin x}{x}$ and $\dfrac{\tan x}{x}$ as $x \to 0$?

2 [Technology]

XLS

The previous question is perfect for a spreadsheet approach. The spreadsheet columns can be identical to the rows above. Various graphs can then be drawn using the data from the spreadsheet.

3 **a** Express $2°$ in radians.

b Explain why $\sin 2° \doteqdot \dfrac{\pi}{90}$.

c Taking π as 3.142, find $\sin 2°$, correct to four decimal places, *without* using a calculator.

4 **a** Copy and complete the following table of values, giving entries correct to four significant figures. For each column, hold x in the calculator's memory until the column is complete:

angle size in degrees	$60°$	$30°$	$10°$	$5°$	$2°$	$1°$	$20'$	$5'$	$1'$	$30''$	$10''$
angle size x in radians											
$\sin x$											
$\dfrac{\sin x}{x}$											
$\tan x$											
$\dfrac{\tan x}{x}$											
$\cos x$											

b Write x, $\sin x$ and $\tan x$ in ascending order, for acute angles x.

c Although $\sin x \to 0$ and $\tan x \to 0$ as $x \to 0$, what are the limits, as $x \to 0$, of:

i $\dfrac{\sin x}{x}$?

ii $\dfrac{\tan x}{x}$?

d Experiment with your calculator, or a spreadsheet, to find how small x must be in order for $\dfrac{\sin x}{x} > 0.999$ to be true.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 [Technology]

A properly prepared spreadsheet makes it easy to ask a sequence of questions like part **d** of the previous question. One can ask how small x must be for each of the following three functions to be closer to 1 than 0.1, 0.001, 0.0001, 0.00001, . . .

$$\frac{\sin x}{x} \quad \text{and} \quad \frac{\tan x}{x} \quad \text{and} \quad \cos x$$

DEVELOPMENT

6 Find the following limits:

a $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

b $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

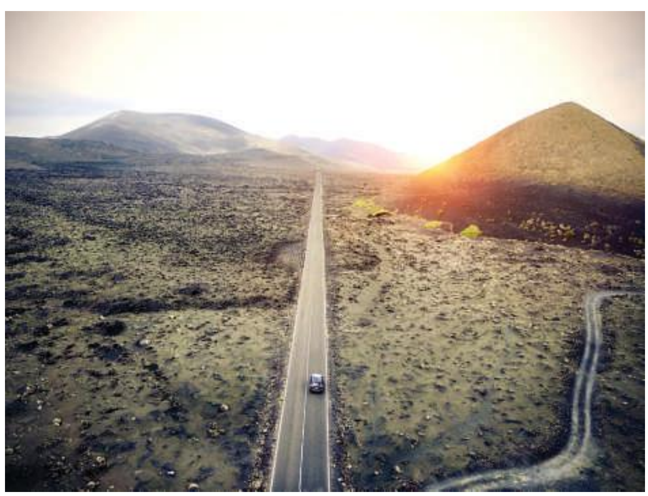
c $\lim\limits_{x \to 0} \dfrac{\sin x}{2x}$

d $\lim\limits_{x \to 0} \dfrac{\sin 3x}{2x}$

e $\lim\limits_{x \to 0} \dfrac{5x}{\sin 3x}$

f $\lim\limits_{x \to 0} \dfrac{\sin 3x + \sin 5x}{x}$

7 A car travels 1 km up a road that is inclined at 5° to the horizontal. Through what vertical distance has the car climbed? (Use the fact that $\sin x \doteqdot x$ for small angles, and give your answer correct to the nearest metre.)



8 A tower is 30 metres high. What angle, correct to the nearest minute, does it subtend at a point 4 km away? (Use the fact that when x is small, $\tan x \doteqdot x$.)

9 [Technology]

Draw on one screen the graphs $y = \sin x$, $y = \tan x$ and $y = x$, noting how the two trigonometric graphs curl away from $y = x$ in opposite directions. Zoom in on the origin until the three graphs are indistinguishable.

10 [Technology]

Draw the graph of $y = \dfrac{\sin x}{x}$. It is undefined at the y-intercept, but the curve around this point is flat, and clearly has limit 1 as $x \to 0$. Other features of the graph can be explained, and the exercise can be repeated with the function $y = \dfrac{\tan x}{x}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 The moon subtends an angle of $31'$ at an observation point on Earth, 400 000 km away. Use the fact that the diameter of the moon is approximately equal to an arc of a circle whose centre is the point of observation to show that the diameter of the moon is approximately 3600 km. (Hint: Use a diagram like that in the last worked exercise in the notes above.)

12 A regular polygon of 300 sides is inscribed in a circle of radius 60 cm. Show that each side is approximately 1.26 cm.

13 [A better approximation for $\cos x$ when x is small]

The chord AB of a circle of radius r subtends an angle x at the centre O.

a Find AB^2 by the cosine rule, and find the length of the arc AB.

b By equating arc and chord, show that for small angles, $\cos x \doteqdot 1 - \dfrac{x^2}{2}$.

Explain whether the approximation is bigger or smaller than $\cos x$.

c Check the accuracy of the approximation for angles of $1°$, $10°$, $20°$ and $30°$.

14 [Technology]

Sketch on one screen the graphs of $y = \cos x$ and $y = 1 - \frac{1}{2}x^2$ as discussed in the previous question. Which one is larger, and why? A spreadsheet may help you to identify the size of the error for different values of x.

ENRICHMENT

15 a Write down the compound-angle formula for $\sin(A - B)$.

b Hence show that, for small x, $\sin(\theta - x) \doteqdot \sin\theta - x\cos\theta$.

c Use the result in **b** to show that $\sin 29°57' \doteqdot \dfrac{3600 - \sqrt{3}\pi}{7200}$.

d To how many decimal places is the approximation in **c** accurate?

e Use similar methods to obtain approximations to $\sin 29°$, $\cos 31°$, $\tan 61°$, $\cot 59°$ and $\sin 46°$, checking the accuracy of your approximations using the calculator.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

7B Differentiating the trigonometric functions

Using the limit from Section 7A, we can now establish the derivatives of the three trigonometric functions $\sin x$, $\cos x$ and $\tan x$.

Standard forms

Here are the formulae, proven below, for these derivatives.

5 STANDARD DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

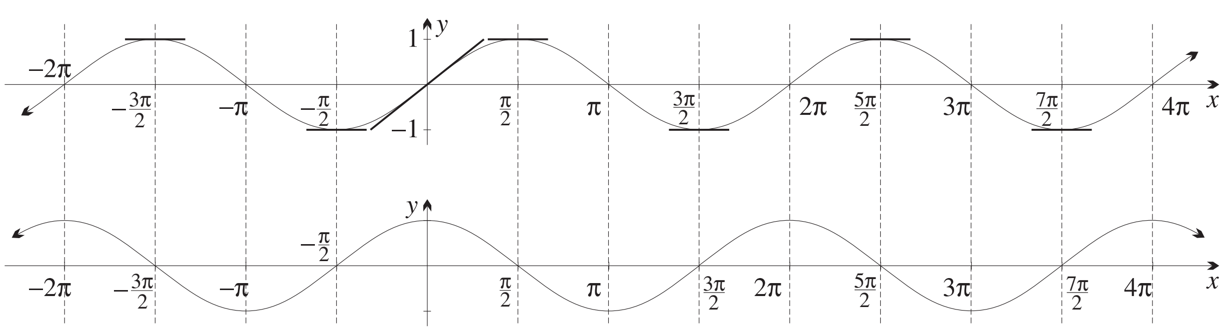
- $\dfrac{d}{dx}\sin x = \cos x$
- $\dfrac{d}{dx}\cos x = -\sin x$
- $\dfrac{d}{dx}\tan x = \sec^2 x$

The exercises ask for derivatives of the secant, cosecant and cotangent functions.

A graphical demonstration that the derivative of sin *x* is cos *x*

The upper graph in the sketch below is $y = \sin x$. The lower graph is a rough sketch of the derivative of $y = \sin x$. This second graph is straightforward to construct simply by paying attention to where the gradients of tangents to $y = \sin x$ are zero, maximum and minimum. The lower graph is periodic, with period 2π, and has a shape unmistakably like a cosine graph.

Moreover, it was proven in the previous section that the gradient of $y = \sin x$ at the origin is exactly 1. This means that the lower graph has a maximum of 1 when $x = 0$. By symmetry, all its maxima are 1 and all its minima are -1. Thus the lower graph not only has the distinctive shape of the cosine curve, but has the correct amplitude as well.



This doesn't prove conclusively that the derivative of $\sin x$ is $\cos x$, but it is very convincing.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Proving that the derivative of sin x is cos x

Completing the proof requires the fundamental limit from Section 7A, together with a trigonometric identity from Exercise 11G of the Year 11 book. The first-principles differentiation formula is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and applying this formula to the function $f(x) = \sin x$ gives

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

We now appeal to the formula for the difference of sines, proven in Question **10** of Exercise 17G in the Year 11 book,

$$\sin P - \sin Q = 2\cos \tfrac{1}{2}(P+Q) \sin \tfrac{1}{2}(P-Q).$$

$$\begin{aligned}
\text{Then } \frac{d}{dx}(\sin x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{2\cos\frac{1}{2}(2x+h)\sin\frac{1}{2}h}{h} \\
&= \lim_{h \to 0} \left(\cos(x + \tfrac{1}{2}h) \times \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \right) \\
&= (\cos x) \times 1, \text{ because } \lim_{h \to 0} \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} = 1 \text{ as proven in Section 7A,} \\
&= \cos x.
\end{aligned}$$

The derivatives of cos x and tan x

These calculations are now straightforward:

$$\frac{d}{dx}\cos x = -\sin x \qquad \text{and} \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Proof:

A Let $y = \cos x$.

Then $y = \sin\left(\frac{\pi}{2} - x\right)$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \quad \text{(chain rule)} \\
&= -\cos\left(\tfrac{\pi}{2} - x\right) \\
&= -\sin x.
\end{aligned}$$

Let $u = \frac{\pi}{2} - x$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = -1$
and $\dfrac{dy}{du} = \cos u$.

B Let $y = \tan x$.

Then $y = \dfrac{\sin x}{\cos x}$.

$$\begin{aligned}
y' &= \frac{vu' - uv'}{v^2} \quad \text{(quotient rule)} \\
&= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}, \quad \text{because } \cos^2 x + \sin^2 x = 1, \\
&= \sec^2 x.
\end{aligned}$$

Let $u = \sin x$
and $v = \cos x$.
Then $u' = \cos x$
and $v' = -\sin x$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Differentiating using the three standard forms

These worked examples use the standard forms to differentiate functions involving $\sin x$, $\cos x$ and $\tan x$.

Example 4 7B

Differentiate these functions.

a $y = \sin x + \cos x$

b $y = x - \tan x$

Hence find the gradient of each curve when $x = \frac{\pi}{4}$.

SOLUTION

a The function is $y = \sin x + \cos x$.

Differentiating, $y' = \cos x - \sin x$.

When $x = \frac{\pi}{4}$, $y' = \cos\frac{\pi}{4} - \sin\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= 0.$$

b The function is $y = x - \tan x$.

Differentiating, $y' = 1 - \sec^2 x$.

When $x = \frac{\pi}{4}$, $y' = 1 - \sec^2\frac{\pi}{4}$

$$= 1 - \left(\sqrt{2}\right)^2$$

$$= -1.$$

Example 5 7B

If $f(x) = \sin x$, find $f'(0)$. Hence find the equation of the tangent to $y = \sin x$ at the origin, then sketch the curve and the tangent.

SOLUTION

Here $f(x) = \sin x$,

and substituting $x = 0$, $f(0) = 0$,

so the curve passes through the origin.

Differentiating, $f'(x) = \cos x$,

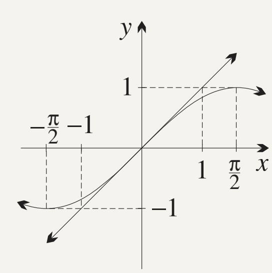
and substituting $x = 0$, $f'(0) = \cos 0$

$= 1$,

so the tangent to $y = \sin x$ at the origin has gradient 1.

Hence its equation is $y - 0 = 1(x - 0)$

$y = x$.



Note: This result was already clear from the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ proven in the previous section. The simplicity of the result confirms that radian measure is the correct measure to use for angles when doing calculus.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the chain rule to generate more standard forms

A simple pattern emerges when the chain rule is used to differentiate functions such as $\cos(3x + 4)$, where the angle $3x + 4$ is a linear function.

Example 6 7B

Use the chain rule to differentiate:

a $y = \cos(3x + 4)$ **b** $y = \tan(5x - 1)$ **c** $y = \sin(ax + b)$

SOLUTION

a Here $y = \cos(3x + 4)$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= -\sin(3x + 4) \times 3\\ &= -3\sin(3x + 4).\end{aligned}$$

Let $u = 3x + 4$.
Then $y = \cos u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = -\sin u$.

b Here $y = \tan(5x - 1)$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \sec^2(5x - 1) \times 5\\ &= 5\sec^2(5x - 1).\end{aligned}$$

Let $u = 5x - 1$.
Then $y = \tan u$.
Hence $\dfrac{du}{dx} = 5$
and $\dfrac{dy}{du} = \sec^2 u$.

c Here $y = \sin(ax + b)$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \cos(ax + b) \times a\\ &= a\cos(ax + b).\end{aligned}$$

Let $u = ax + $ b.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \cos u$.

The last result in the previous worked example can be extended to the other trigonometric functions, giving the following standard forms:

6 STANDARD DERIVATIVES OF FUNCTIONS OF $ax + b$

- $\dfrac{d}{dx}\sin(ax + b) = a\cos(ax + b)$
- $\dfrac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$
- $\dfrac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 7 7B

Use the extended standard forms given in Box 6 above to differentiate:

a $y = \cos 7x$ **b** $y = 4\sin\left(3x - \frac{\pi}{3}\right)$ **c** $y = \tan \frac{3}{2}x$

SOLUTION

a The function is $y = \cos 7x$, so $a = 7$ and $b = 0$,

and $\dfrac{dy}{dx} = -7\sin 7x$.

b The function is $y = 4\sin\left(3x - \frac{\pi}{3}\right)$, so $a = 3$ and $b = -\frac{\pi}{3}$,

and $\dfrac{dy}{dx} = 12\cos\left(3x - \frac{\pi}{3}\right)$.

c The function is $y = \tan\frac{3}{2}x$, so $a = \frac{3}{2}$ and $b = 0$,

and $\dfrac{dy}{dx} = \frac{3}{2}\sec^2\frac{3}{2}x$

Using the chain rule with trigonometric functions

The chain rule can also be applied in the usual way to differentiate compound functions.

Example 8 7B

Use the chain rule to differentiate:

a $y = \tan^2 x$ **b** $y = \sin\left(x^2 - \frac{\pi}{4}\right)$

SOLUTION

a Here $y = \tan^2 x$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2\tan x \sec^2 x.$$

Let $u = \tan x$.
Then $y = u^2$.
Hence $\dfrac{du}{dx} = \sec^2 x$
and $\dfrac{dy}{du} = 2u$.

b Here $y = \sin\left(x^2 - \frac{\pi}{4}\right)$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2x\cos\left(x^2 - \frac{\pi}{4}\right).$$

Let $u = x^2 - \frac{\pi}{4}$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = \cos u$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the product rule with trigonometric functions

A function such as $y = e^x \cos x$ is the product of the two functions $u = e^x$ and $v = \cos x$. It can therefore be differentiated using the product rule.

Example 9 — 7B

Use the product rule to differentiate:

a $y = e^x \cos x$

b $y = 5 \cos 2x \cos \frac{1}{2}x$

SOLUTION

a Here $y = e^x \cos x$.

Applying the product rule,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^x \cos x - e^x \sin x$$
$$= e^x(\cos x - \sin x).$$

Let $u = e^x$
and $v = \cos x$.
Then $\dfrac{du}{dx} = e^x$
and $\dfrac{dv}{dx} = -\sin x$.

b Here $y = 5 \cos 2x \cos \frac{1}{2}x$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= -10 \sin 2x \cos \tfrac{1}{2}x - \tfrac{5}{2}\cos 2x \sin \tfrac{1}{2}x.$$

Let $u = 5 \cos 2x$
and $v = \cos \frac{1}{2}x$.
Then $u' = -10 \sin 2x$
and $v' = -\frac{1}{2} \sin \frac{1}{2}x$.

Using the quotient rule with trigonometric functions

A function such as $y = \dfrac{\sin x}{x}$ is the quotient of the two functions $u = \sin x$ and $v = x$. Thus it can be differentiated using the quotient rule.

Example 10 — 7B

Use the quotient rule to differentiate:

a $y = \dfrac{\sin x}{x}$

b $y = \dfrac{\cos 2x}{\cos 5x}$

SOLUTION

a Here $y = \dfrac{\sin x}{x}$.

Applying the quotient rule,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$
$$= \frac{x \cos x - \sin x}{x^2}.$$

Let $u = \sin x$
and $v = x$.
Then $\dfrac{du}{dx} = \cos x$
and $\dfrac{dv}{dx} = 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Here $y = \dfrac{\cos 2x}{\cos 5x}$.

Applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{-2\sin 2x \cos 5x + 5\cos 2x \sin 5x}{\cos^2 5x}.$$

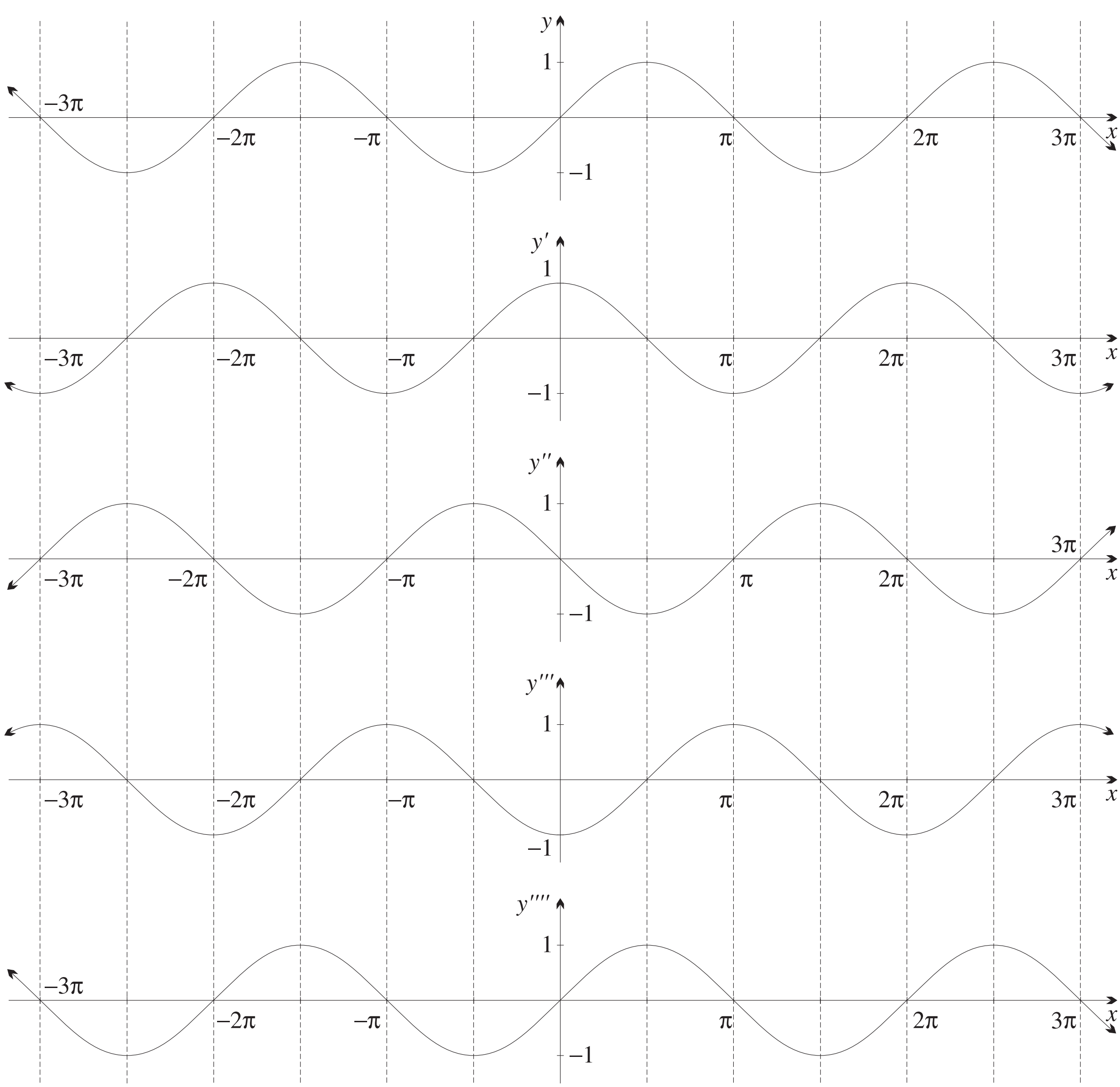
Let $u = \cos 2x$
and $v = \cos 5x$.
Then $u' = -2\sin 2x$
and $v' = -5\sin 5x$.

Successive differentiation of sine and cosine

Differentiating $y = \sin x$ repeatedly,

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} = -\sin x, \qquad \frac{d^3y}{dx^3} = -\cos x, \qquad \frac{d^4y}{dx^4} = \sin x.$$

Thus differentiation is an *order 4 operation* on the sine function, meaning that when differentiation is applied four times, the original function returns. Sketched below are the graphs of $y = \sin x$ and its first four derivatives.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Each application of differentiation shifts the wave left $\frac{\pi}{2}$, which is a quarter of the period 2π. Thus differentiation advances the phase by $\frac{\pi}{2}$, meaning that

$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \tfrac{\pi}{2}\right) \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right)$$

Double differentiation shifts the wave left π, which is a half the period 2π, and thus advances the phase by π. Double differentiation also exchanges $y = \sin x$ with its opposite function $y = -\sin x$, with each graph being the reflection of the other in the x-axis. It has similar effects on the cosine function. Thus both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'' = -y$.

Four differentiations shift the wave left 2π, which is one full period, where it coincides with itself again. Thus the differentiation transformation acting on the sine and cosine functions has *order 4*, and both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'''' = y$.

7 DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS AS PHASE SHIFT

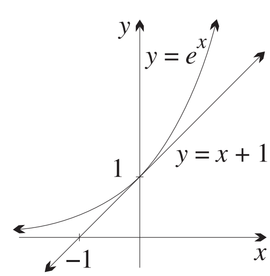
- Differentiation of $y = \sin x$ and $y = \cos x$ shifts each curve left $\frac{\pi}{2}$, advancing the phase $\frac{\pi}{2}$,
$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \tfrac{\pi}{2}\right) \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right)$$
- The second derivatives of $\sin x$ and $\cos x$ reflect each curve in the x-axis,
$$\frac{d^2}{dx^2}\sin x = -\sin x \quad \text{and} \quad \frac{d^2}{dx^2}\cos x = -\cos x$$
- Differentiation of $\sin x$ and $\cos x$ has order 4,
$$\frac{d^4}{dx^4}\sin x = \sin x \quad \text{and} \quad \frac{d^4}{dx^4}\cos x = \cos x$$

The properties of the exponential function $y = e^x$ are quite similar. The first derivative of $y = e^x$ is $y' = e^x$ and the second derivative of $y = e^{-x}$ is $y'' = e^{-x}$. This means there are now four functions whose fourth derivatives are equal to themselves:

$$y = \sin x, \qquad y = \cos x, \qquad y = e^x, \qquad y = e^{-x}.$$

This is one clue amongst many others in the course that the trigonometric functions and the exponential functions are closely related. See also Question **14(d)** in Exercise 7B.

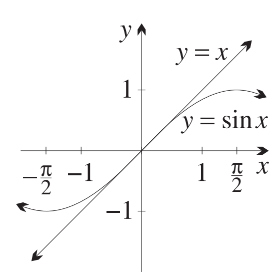
Some analogies between π and e



In the previous chapter, and in Chapter 11 of the Year 11 book, we discussed how choosing the special number e as the base of the exponential function makes the derivative of $y = e^x$ is exactly $y' = e^x$.

In particular, the tangent to $y = e^x$ at the y-intercept has gradient exactly 1.

The choice of radian measure, based on the special number π, was motivated in exactly the same way. As has just been explained, the derivative of $y = \sin x$ using radian measure is exactly $y' = \cos x$.



In particular, the tangent to $y = \sin x$ at the origin has gradient exactly 1.

Both numbers $\pi = 3.141\,592\ldots$ and $e = 2.718\,281\ldots$ are irrational. The number π is associated with the area of a circle and e is associated with areas under the rectangular hyperbola. These things are further hints of connections between trigonometric and exponential functions.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

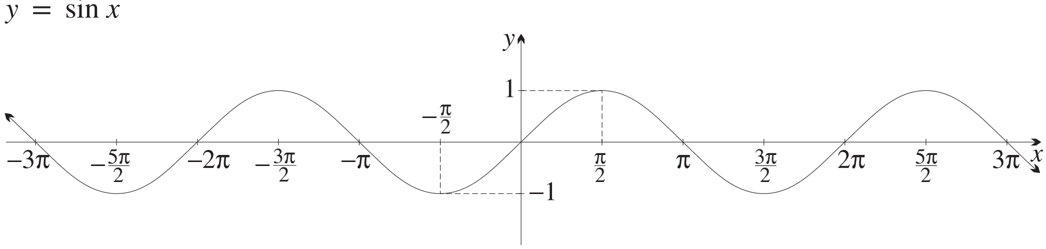
Sketches of the six trigonometric functions in radians

In Section 11J of the Year 11 book, we sketched all six trigonometric functions in radians. The six graphs are repeated on this page. Here are the key properties:

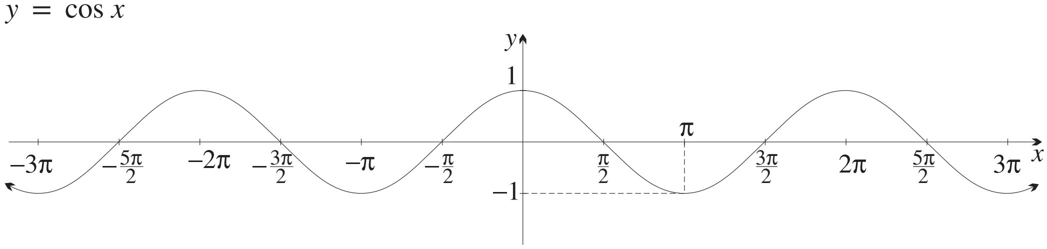
- $\sin x$ and $\cos x$ each have amplitude 1. The others do not have an amplitude.
- $\sin x$ and $\cos x$ (and their reciprocals $\sec x$ and $\operatorname{cosec} x$ each have period 2π, $\tan x$ (and its reciprocal $\cot x$) have period π.

Exercise 11J in the Year 11 book investigated in some detail the symmetry properties of these six trigonometric functions.

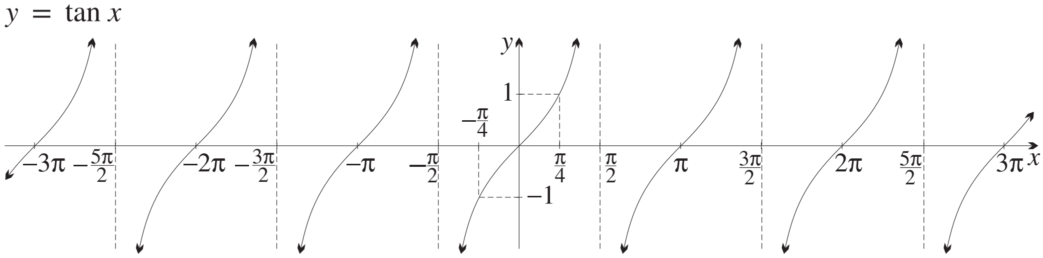
$y = \sin x$



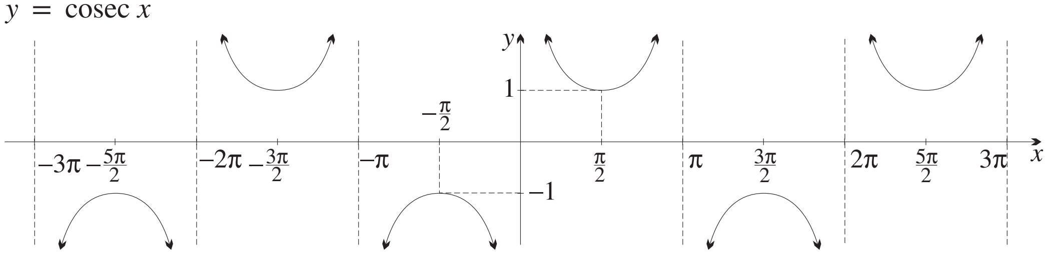
$y = \cos x$



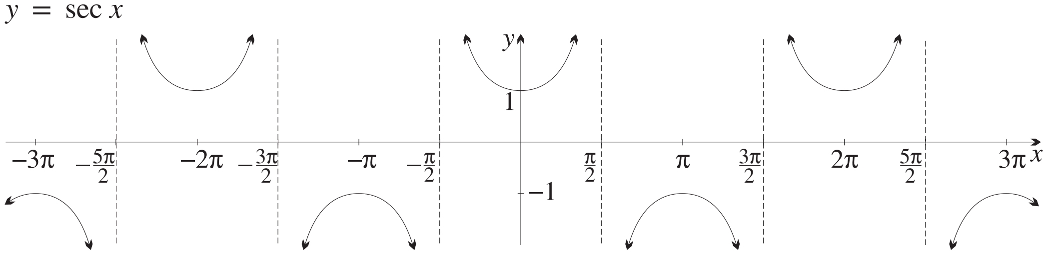
$y = \tan x$



$y = \operatorname{cosec} x$



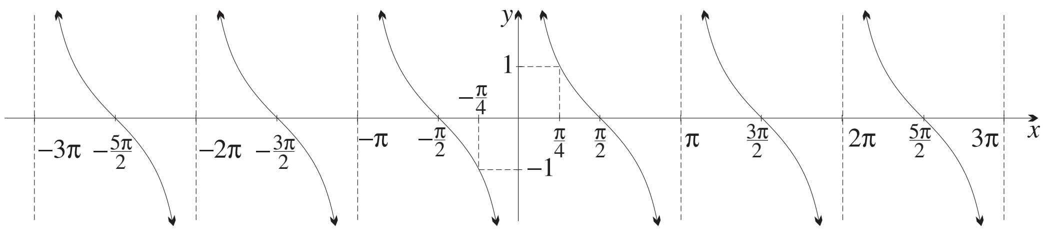
$y = \sec x$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$y = \cot x$



Exercise 7B

FOUNDATION

1 Use the standard forms to differentiate with respect to x.

a $y = \sin x$ **b** $y = \cos x$ **c** $y = \tan x$
d $y = 2\sin x$ **e** $y = \sin 2x$ **f** $y = 3\cos x$
g $y = \cos 3x$ **h** $y = \tan 4x$ **i** $y = 4\tan x$
j $y = 2\sin 3x$ **k** $y = 2\tan 2x$ **l** $y = 4\cos 2x$
m $y = -\sin 2x$ **n** $y = -\cos 2x$ **o** $y = -\tan 2x$
p $y = \tan \frac{1}{2}x$ **q** $y = \cos \frac{1}{2}x$ **r** $y = \sin \frac{x}{2}$
s $y = 5\tan \frac{1}{5}x$ **t** $y = 6\cos \frac{x}{3}$ **u** $y = 12\sin \frac{x}{4}$

2 Differentiate with respect to x.

a $\sin 2\pi x$ **b** $\tan \frac{\pi}{2}x$ **c** $3\sin x + \cos 5x$
d $4\sin \pi x + 3\cos \pi x$ **e** $\sin(2x - 1)$ **f** $\tan(1 + 3x)$
g $2\cos(1 - x)$ **h** $\cos(5x + 4)$ **i** $7\sin(2 - 3x)$
j $10\tan(10 - x)$ **k** $6\sin\left(\frac{x+1}{2}\right)$ **l** $15\cos\left(\frac{2x+1}{5}\right)$

3 Find the first, second, third and fourth derivatives of:

a $y = \sin 2x$ **b** $y = \cos 10x$ **c** $y = \sin \frac{1}{2}x$ **d** $y = \cos \frac{1}{3}x$

4 If $f(x) = \cos 2x$, find $f'(x)$ and then find:

a $f'(0)$ **b** $f'\left(\frac{\pi}{12}\right)$ **c** $f'\left(\frac{\pi}{6}\right)$ **d** $f'\left(\frac{\pi}{4}\right)$

5 If $f(x) = \sin\left(\frac{1}{4}x + \frac{\pi}{2}\right)$, find $f'(x)$ and then find:

a $f'(0)$ **b** $f'(2\pi)$ **c** $f'(-\pi)$ **d** $f'(\pi)$

DEVELOPMENT

6 Find $\dfrac{dy}{dx}$ using the product rule.

a $y = x\sin x$ **b** $y = 2x\tan 2x$ **c** $y = x^2\cos 2x$ **d** $y = x^3\sin 3x$

7 Find $\dfrac{dy}{dx}$ using the quotient rule.

a $y = \dfrac{\sin x}{x}$ **b** $y = \dfrac{\cos x}{x}$ **c** $y = \dfrac{x^2}{\cos x}$ **d** $y = \dfrac{x}{1 + \sin x}$

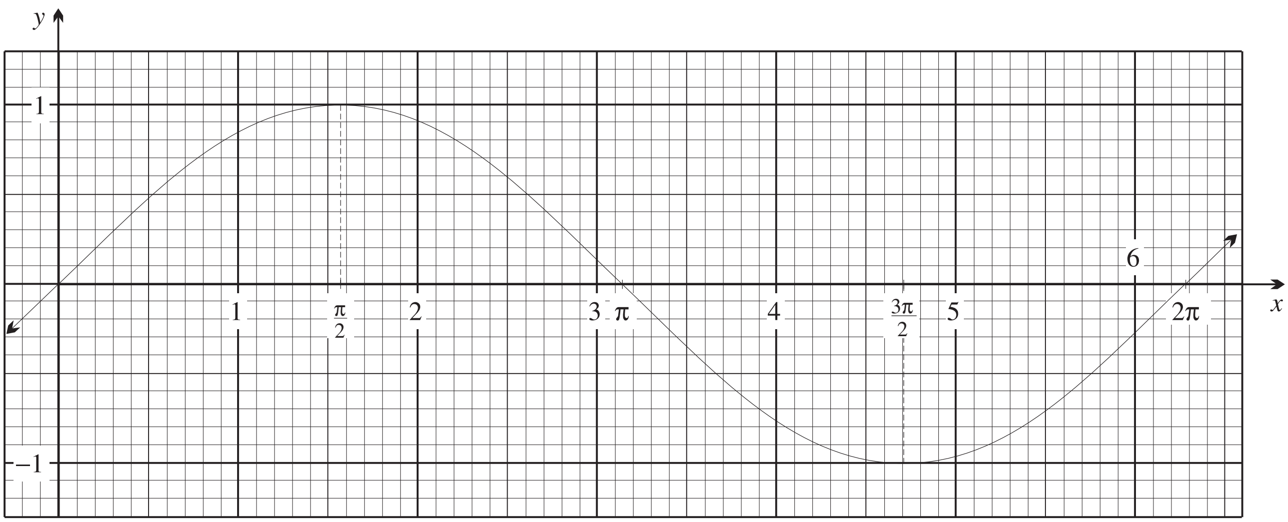
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Find $\dfrac{dy}{dx}$ using the chain rule. (Hint: Remember that $\cos^2 x$ means $(\cos x)^2$.)

a $y = \sin(x^2)$
b $y = \sin(1 - x^2)$
c $y = \cos(x^3 + 1)$
d $y = \sin\dfrac{1}{x}$
e $y = \cos^2 x$
f $y = \sin^3 x$
g $y = \tan^2 x$
h $y = \tan\sqrt{x}$

9



a Photocopy the sketch above of $f(x) = \sin x$. Carefully draw tangents at the points where $x = 0, 0.5, 1, 1.5, \ldots, 3$, and also at $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

b Measure the gradient of each tangent correct to two decimal places, and copy and complete the following table.

x	0	0.5	1	1.5	$\frac{\pi}{2}$	2	2.5	3	π	3.5	4	4.5	$\frac{3\pi}{2}$	5	5.5	6	2π
$f'(x)$																	

c Use these values to plot the graph of $y = f'(x)$.

d What is the equation of this graph?

10 [Technology]

Most graphing programs can graph the derivative of a function. Start with $y = \sin x$, as in the previous question, then graph y', y'', y''' and y'''', and compare your results with the graphs printed in the theory introducing this exercise.

11 Differentiate:

a $f(x) = e^{\tan x}$
b $f(x) = e^{\sin 2x}$
c $f(x) = \sin\left(e^{2x}\right)$
d $f(x) = \log_e(\cos x)$
e $f(x) = \log_e(\sin x)$
f $f(x) = \log_e(\cos 4x)$

12 Differentiate these functions.

a $y = \sin x \cos x$
b $y = \sin^2 7x$
c $y = \cos^5 3x$
d $y = (1 - \cos 3x)^3$
e $y = \sin 2x \sin 4x$
f $y = \tan^3(5x - 4)$

13 Find $f'(x)$, given that:

a $f(x) = \dfrac{1}{1 + \sin x}$
b $f(x) = \dfrac{\sin x}{1 + \cos x}$
c $f(x) = \dfrac{1 - \sin x}{\cos x}$
d $f(x) = \dfrac{\cos x}{\cos x + \sin x}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 a Sketch $y = \cos x$, for $-3\pi \le x \le 3\pi$.

b Find y', y'', y''' and y'''', and sketch them underneath the first graph.

c What geometrical interpretations can be given of the facts that:

i $y'' = -y$? **ii** $y'''' = y$?

15 [Technology]

The previous question is well suited to a graphing program, and the results should be compared with those of successive differentiation of $\sin x$.

16 a If $y = e^x \sin x$, find y' and y'', and show that $y'' - 2y' + 2y = 0$.

b If $y = e^{-x} \cos x$, find y' and y'', and show that $y'' + 2y' + 2y = 0$.

17 Consider the function $y = \frac{1}{3}\tan^3 x - \tan x + x$.

a Show that $\dfrac{dy}{dx} = \tan^2 x \sec^2 x - \sec^2 x + 1$.

b Hence use the identity $\sec^2 x = 1 + \tan^2 x$ to show that $\dfrac{dy}{dx} = \tan^4 x$.

18 a Copy and complete: $\log_b\left(\frac{P}{Q}\right) = \ldots$.

b If $f(x) = \log_e\left(\dfrac{1 + \sin x}{\cos x}\right)$, show that $f'(x) = \sec x$.

19 a By writing $\sec x$ as $(\cos x)^{-1}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

b Similarly, show that $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

c Similarly, show that $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

20 Show that $\dfrac{d}{dx}\left(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right) = \sin^4 x \cos^3 x$.

ENRICHMENT

21 a If $y = \sin x$, prove:

i $\dfrac{dy}{dx} = \sin\left(\frac{\pi}{2} + x\right)$ **ii** $\dfrac{d^2y}{dx^2} = \sin(\pi + x)$ **iii** $\dfrac{d^3y}{dx^3} = \sin\left(\frac{3\pi}{2} + x\right)$

b Deduce an expression for $\dfrac{d^ny}{dx^n}$.

22 a Show that $\frac{1}{2}(\sin(m + n)x + \sin(m - n)x) = \sin mx \cos nx$.

b Hence, without using the product rule, differentiate $\sin mx \cos nx$.

c Simplify $\frac{1}{2}(\cos(m + n)x + \cos(m - n)x)$, and hence differentiate $\cos mx \cos nx$.

23 Show that the function $y = e^{-x}(\cos 2x + \sin 2x)$ is a solution of the differential equation $y'' + 2y' + 5y = 0$.

24 a If $y = \ln(\tan 2x)$, show that $\dfrac{dy}{dx} = 2\sec 2x \operatorname{cosec} 2x$.

b If $y = \ln\left(\dfrac{\sqrt{2} - \cos x}{\sqrt{2} + \cos x}\right)$, show that $\dfrac{dy}{dx} = \dfrac{2\sqrt{2}\sin x}{1 + \sin^2 x}$.

25 At the start of this section, we differentiated $\sin x$ by first principles. Using that working as a guide, differentiate $\cos x$ by first principles. You will need this sums-to-products identity from Question **10** of Exercise 17G in the Year 11 book,

$$\cos P - \cos Q = -2\sin\tfrac{1}{2}(P + Q)\sin\tfrac{1}{2}(P - Q).$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

7C Applications of differentiation

Differentiation of the trigonometric functions can be applied in the usual way to the analysis of a number of functions that are very significant in the practical application of calculus. It can also be used to solve *optimisation problems* meaning problems about maxima and minima).

Tangents and normals

As always, the derivative is used to find the gradients of the relevant tangents, then point–gradient form is used to find their equations.

Example 11 — 7C

Find the equation of the tangent to $y = 2\sin x$ at the point P where $x = \frac{\pi}{6}$.

SOLUTION

When $x = \frac{\pi}{6}$, $\quad y = 2\sin\frac{\pi}{6}$

$$= 1 \quad \left(\text{because } \sin\tfrac{\pi}{6} = \tfrac{1}{2}\right),$$

so the point P has coordinates $\left(\frac{\pi}{6}, 1\right)$.

Differentiating, $\dfrac{dy}{dx} = 2\cos x$.

When $x = \frac{\pi}{6}$, $\quad \dfrac{dy}{dx} = 2\cos\frac{\pi}{6}$

$$= \sqrt{3} \quad \left(\text{because } \cos\tfrac{\pi}{6} = \tfrac{1}{2}\sqrt{3}\right),$$

so the tangent at $P\left(\frac{\pi}{6}, 1\right)$ has gradient $\sqrt{3}$.

Hence its equation is $y - y_1 = m(x - x_1)$ (point−gradient form)

$$y - 1 = \sqrt{3}\left(x - \tfrac{\pi}{6}\right)$$

$$y = x\sqrt{3} + 1 - \tfrac{\pi}{6}\sqrt{3}.$$

Example 12 — 7C

a Find the equations of the tangents and normals to the curve $y = \cos x$ at $A\left(-\frac{\pi}{2}, 0\right)$ and $B\left(\frac{\pi}{2}, 0\right)$.

b Show that the four lines form a square, sketch, and find the other two vertices.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

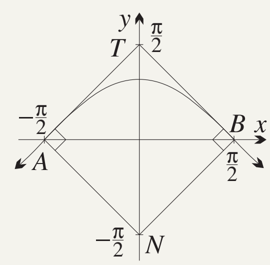
a The function is $y = \cos x,$

and the derivative is $y' = -\sin x.$

$$
\begin{aligned}
\text{Hence}\qquad \text{gradient of tangent at } A\left(-\tfrac{\pi}{2}, 0\right) &= -\sin\left(-\tfrac{\pi}{2}\right)\\
&= 1,\\
\text{and}\qquad \text{gradient of normal at } A\left(-\tfrac{\pi}{2}, 0\right) &= -1.\\
\text{Similarly,}\qquad \text{gradient of tangent at } B\left(\tfrac{\pi}{2}, 0\right) &= -\sin \tfrac{\pi}{2}\\
&= -1,\\
\text{and}\qquad \text{gradient of normal at } B\left(\tfrac{\pi}{2}, 0\right) &= 1.
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence the tangent at } A \text{ is}\qquad y - 0 &= 1 \times \left(x + \tfrac{\pi}{2}\right)\\
y &= x + \tfrac{\pi}{2},\\
\text{and the normal at } A \text{ is}\qquad y - 0 &= -1 \times \left(x + \tfrac{\pi}{2}\right)\\
y &= -x - \tfrac{\pi}{2}.\\
\text{Similarly, the tangent at } B \text{ is}\qquad y - 0 &= -1 \times \left(x - \tfrac{\pi}{2}\right)\\
y &= -x + \tfrac{\pi}{2},\\
\text{and the normal at } B \text{ is}\qquad y - 0 &= 1 \times \left(x - \tfrac{\pi}{2}\right)\\
y &= x - \tfrac{\pi}{2}.
\end{aligned}
$$

b Hence the two tangents meet on the y-axis at $T\left(0, \frac{\pi}{2}\right)$, and the two normals meet on the y-axis at $N\left(0, -\frac{\pi}{2}\right)$. Because adjacent sides are perpendicular, $ANBT$ is a rectangle, and because the diagonals are perpendicular, it is also a rhombus, so the quadrilateral $ANBT$ is a square.



Example 13 7C

a Find the equation of the tangent to $y = \tan 2x$ at the point on the curve where $x = \frac{\pi}{8}$.

b Find the x-intercept and y-intercept of this tangent.

c Sketch the situation.

d Find the area of the triangle formed by this tangent and the coordinate axes.

SOLUTION

a

$$
\begin{aligned}
\text{The function is}\qquad y &= \tan 2x,\\
\text{and differentiating,}\qquad y' &= 2\sec^2 2x.\\
\text{When } x = \tfrac{\pi}{8},\qquad y &= \tan \tfrac{\pi}{4}\\
&= 1\\
\text{and}\qquad y' &= 2\sec^2 \tfrac{\pi}{4}\\
&= 2 \times \left(\sqrt{2}\right)^2\\
&= 4,\\
\text{so the tangent is } y - 1 &= 4\left(x - \tfrac{\pi}{8}\right)\\
y &= 4x - \tfrac{\pi}{2} + 1.
\end{aligned}
$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

b When $x = 0$, $y = 1 - \frac{\pi}{2}$

$$= \frac{2 - \pi}{2},$$

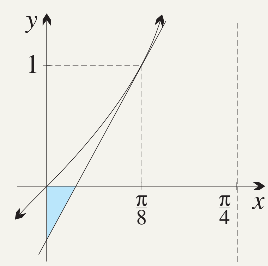
and when $y = 0$, $0 = 4x - \frac{\pi}{2} + 1$

$$4x = \frac{\pi}{2} - 1$$

$$4x = \frac{\pi - 2}{2}$$

$\boxed{\div 4}$ $$x = \frac{\pi - 2}{8}.$$

c The sketch is drawn opposite.



d Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times \frac{\pi - 2}{2} \times \frac{\pi - 2}{8}$$

$$= \frac{(\pi - 2)^2}{32} \text{ square units.}$$

Curve sketching

Curve-sketching problems involving trigonometric functions can be long, with difficult details. Nevertheless, the usual steps of the 'curve-sketching menu' still apply and the working of each step is done exactly the same as usual.

Sketching these curves using either a computer package or a graphics calculator would greatly aid understanding of the relationships between the equations of the curves and their graphs.

Note: With trigonometric functions, it is often easier to determine the nature of stationary points from an examination of the second derivative than from a table of values of the first derivative.

Example 14 7C

Consider the curve $y = \sin x + \cos x$ in the interval $0 \le x \le 2\pi$.

a Find the values of the function at the endpoints of the domain.
b Find the x-intercepts of the graph.
c Find any stationary points and determine their nature.
d Find any points of inflection and sketch the curve.

SOLUTION

a When $x = 0$, $y = \sin 0 + \cos 0 = 1$,
and when $x = 2\pi$, $y = \sin 2\pi + \cos 2\pi = 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b To find the x-intercepts, put $y = 0$.

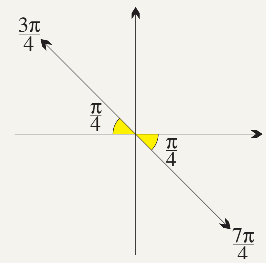
Then $\sin x + \cos x = 0$

$$\sin x = -\cos x$$

$$\tan x = -1 \quad \text{(dividing through by } \cos x\text{)}.$$

Hence x is in quadrant 2 or 4, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.



c Differentiating, $\quad y' = \cos x - \sin x$,

so y' has zeroes when $\sin x = \cos x$,

that is, $\quad \tan x = 1 \quad$ (dividing through by $\cos x$).

Hence x is in quadrant 1 or 3, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

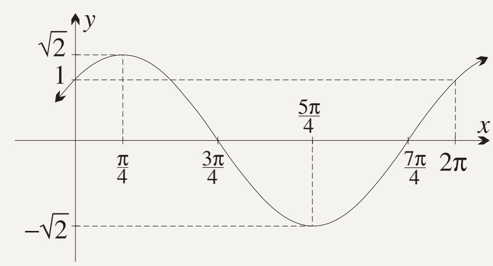
When $x = \frac{\pi}{4}$, $\quad y = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$

$$= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= \sqrt{2},$$

and when $x = \frac{5\pi}{4}$, $\quad y = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}$

$$= -\sqrt{2}.$$



Differentiating again, $\quad y'' = -\sin x - \cos x$,

so when $x = \frac{\pi}{4}$, $\quad y'' = -\sqrt{2}$,

and when $x = \frac{5\pi}{4}$, $\quad y'' = \sqrt{2}$.

Hence $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is a maximum turning point,

and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a minimum turning point.

d The second derivative y'' has zeroes when $-\sin x - \cos x = 0$,

that is, at the zeroes of y, which are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

x	0	$\frac{3\pi}{4}$	π	$\frac{7\pi}{4}$	2π
y''	-1	0	1	0	-1
	$\frown$	$\cdot$	$\smile$	$\cdot$	$\frown$

Hence the x-intercepts $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$ are also inflections.

Note: The final graph is simply a wave with the same period 2π as $\sin x$ and $\cos x$, but with amplitude $\sqrt{2}$. It is actually $y = \sqrt{2}\cos x$ shifted right by $\frac{\pi}{4}$. Any function of the form $y = a\sin x + b\cos x$ has a similar graph.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 15 7C

[A harder example]

Sketch the graph of $f(x) = x - \sin x$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$.

e Find any stationary points and examine their nature.

f Find any points of inflection.

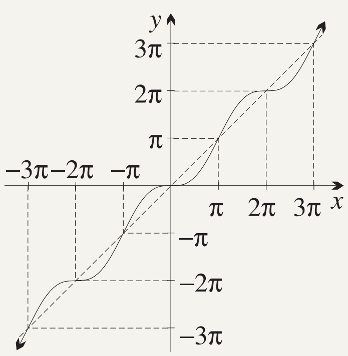
Note: This function is essentially the function describing the area of a segment, if the radius in the formula $A = \frac{1}{2}r^2(x - \sin x)$ is held constant while the angle x at the centre varies.

SOLUTION

a The domain of $f(x) = x - \sin x$ is the set of all real numbers.

b $f(x)$ is odd, because both $\sin x$ and x are odd.

c The function is zero at $x = 0$ and nowhere else,
because $\sin x < x$, for $x > 0$,
and $\sin x > x$, for $x < 0$.



d The value of $\sin x$ always remains between -1 and 1,
so for $f(x) = x - \sin x$, $\quad f(x) \to \infty$ as $x \to \infty$,
and $\quad f(x) \to -\infty$ as $x \to -\infty$.

e Differentiating, $f'(x) = 1 - \cos x$,
so $f'(x)$ has zeroes whenever $\cos x = 1$,
that is, for $x = \ldots, -2\pi, 0, 2\pi, 4\pi, \ldots$.
But $f'(x) = 1 - \cos x$ is never negative, because $\cos x$ is never greater than 1,
thus the curve $f(x)$ is always increasing except at its stationary points.
Hence each stationary point is a stationary inflection,
and these points are $\ldots, (-2\pi, -2\pi), (0, 0), (2\pi, 2\pi), (4\pi, 4\pi), \ldots$.

f Differentiating again, $f''(x) = \sin x$,
which is zero for $x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$.
We know that $\sin x$ changes sign around each of these points,
so $\ldots, (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi), \ldots$ are also inflections.
Because $f'(\pi) = 1 - (-1) = 2$, the gradient at these other inflections is 2.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 7C

FOUNDATION

Technology: The large number of sketches in this exercise should allow many of the graphs to be drawn first on a computer. Such sketching should be followed by an algebraic explanation of the features.

Many graphing packages allow tangents and normals to be drawn at specific points so that diagrams can be drawn of the earlier questions in the exercise.

1 Find the gradient of the tangent to each of the following curves at the point indicated.

a $y = \sin x$ at $x = 0$ **b** $y = \cos x$ at $x = \frac{\pi}{2}$ **c** $y = \sin x$ at $x = \frac{\pi}{3}$

d $y = \cos x$ at $x = \frac{\pi}{6}$ **e** $y = \sin x$ at $x = \frac{\pi}{4}$ **f** $y = \tan x$ at $x = 0$

g $y = \tan x$ at $x = \frac{\pi}{4}$ **h** $y = \cos 2x$ at $x = \frac{\pi}{4}$ **i** $y = -\cos \frac{1}{2}x$ at $x = \frac{2\pi}{3}$

j $y = \sin \frac{x}{2}$ at $x = \frac{2\pi}{3}$ **k** $y = \tan 2x$ at $x = \frac{\pi}{6}$ **l** $y = \sin 2x$ at $x = \frac{\pi}{12}$

2 **a** Show that the line $y = x$ is the tangent to the curve $y = \sin x$ at $(0, 0)$.

b Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.

c Show that the line $y = \frac{\pi}{2} - x$ is the tangent to the curve $y = \cos x$ at $\left(\frac{\pi}{2}, 0\right)$.

3 Find the equation of the tangent at the given point on each of the following curves.

a $y = \sin x$ at $(\pi, 0)$ **b** $y = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$

c $y = \cos x$ at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ **d** $y = \cos 2x$ at $\left(\frac{\pi}{4}, 0\right)$

e $y = \sin 2x$ at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **f** $y = x \sin x$ at $(\pi, 0)$

4 Find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on each of the following curves where the gradient of the tangent is zero.

a $y = 2 \sin x$ **b** $y = 2 \sin x - x$

c $y = 2 \cos x + x$ **d** $y = 2 \sin x + \sqrt{3}x$

5 The point $P\left(\frac{\pi}{6}, \frac{1}{2}\right)$ lies on the curve $y = 2 \sin x - \cos 2x$.

a Show that the tangent at P has equation $2\sqrt{3}x - y = \frac{1}{3}\pi\sqrt{3} - \frac{1}{2}$.

b Show that the normal at P has equation $x + 2\sqrt{3}y = \frac{\pi}{6} + \sqrt{3}$.

6 **a** Show that $y = \sin^2 x$ has derivative $y' = 2 \sin x \cos x$.

b Find the gradients of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

c Find the equations of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

d Suppose that the tangent meets the x-axis at P, the normal meets the y-axis at Q and O is the origin. Show that $\triangle OPQ$ has area $\frac{1}{32}(\pi^2 - 4)$ units^2.

7 **a** Differentiate $y = e^{\sin x}$.

b Hence find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on the curve $y = e^{\sin x}$ where the tangent is horizontal.

8 **a** Differentiate $y = e^{\cos x}$.

b Hence find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on the curve $y = e^{\cos x}$ where the tangent is horizontal.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

9 **a** Find the first and second derivatives of $y = \cos x + \sqrt{3} \sin x$.

b Find the stationary points in the domain $0 \le x \le 2\pi$, and use the second derivative to determine their nature.

c Find the points of inflection.

d Hence sketch the curve, for $0 \le x \le 2\pi$.

10 **a** Find the derivative of $y = x + \sin x$, and show that $y'' = -\sin x$.

b Find the stationary points in the domain $-2\pi < x < 2\pi$, and determine their nature.

c Find the points of inflection.

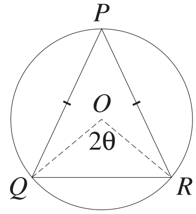
d Hence sketch the curve, for $-2\pi \le x \le 2\pi$.

11 Find any stationary points and inflections of the curve $y = 2 \sin x + x$ in the interval $0 \le x \le 2\pi$, then sketch the curve.

12 An isosceles triangle has equal sides of length 10 cm. The angle θ between these equal sides is increasing at the rate of 3° per minute. Show that the area of the triangle is increasing at $\frac{5\sqrt{3}\pi}{12}$ cm^2 per minute at the instant when $\theta = 30°$.

13 A rotating light L is situated at sea 180 metres from the nearest point P on a straight shoreline. The light rotates through one revolution every 10 seconds. Show that the rate at which a ray of light moves along the shore at a point 300 metres from P is 136π m/s.

14 An isosceles triangle PQR is inscribed in a circle with centre O of radius 1 unit, as shown in the diagram to the right. Let $\angle QOR = 2\theta$, where θ is acute.



a Join PO and extend it to meet QR at M. Then prove that $QM = \sin\theta$ and $OM = \cos\theta$.

b Show that the area A of $\triangle PQR$ is $A = \sin\theta(\cos\theta + 1)$.

c Hence show that, as θ varies, $\triangle PQR$ has its maximum possible area when it is equilateral.

15 **a** Show that $\dfrac{d}{d\theta}\left(\dfrac{2 - \sin\theta}{\cos\theta}\right) = \dfrac{2\sin\theta - 1}{\cos^2\theta}$.

b Hence find the maximum and minimum values of the expression $\dfrac{2 - \sin\theta}{\cos\theta}$ in the interval $0 \le \theta \le \frac{\pi}{4}$, and state the values of θ for which they occur.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 a Find the first and second derivatives of $y = 2\sin x + \cos 2x$.
b Show that $y' = 0$ when $\cos x = 0$ or $\sin x = \frac{1}{2}$. (You will need to use the formula $\sin 2x = 2\sin x\cos x$.)
c Hence find the stationary points in the interval $-\pi \le x \le \pi$ and determine their nature.
d Sketch the curve for $-\pi \le x \le \pi$ using this information.

17 a Find the first and second derivatives of $y = e^{-x}\cos x$. (Note that this function models damped oscillations.)
b Find the stationary points for $-\pi \le x \le \pi$ and determine their nature.
c Find the points of inflection for $-\pi \le x \le \pi$.
d Hence sketch the curve for $-\pi \le x \le \pi$.

ENRICHMENT

18 A straight line passes through the point $(2, 1)$ and has positive x- and y-intercepts at P and Q respectively. Suppose $\angle OPQ = \alpha$, where O is the origin.
a Explain why the line has gradient $-\tan\alpha$.
b Find the x- and y-intercepts in terms of α.
c Show that the area of $\triangle OPQ$ is given by $A = \dfrac{(2\tan\alpha + 1)^2}{2\tan\alpha}$.
d Hence show that this area is maximised when $\tan\alpha = \frac{1}{2}$.

19 a Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.
b Using a diagram, explain why $\tan x > x$ for $0 < x < \frac{\pi}{2}$.
c Let $f(x) = \dfrac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$. Find $f'(x)$ and show that $f'(x) < 0$ in the given domain.
d Sketch the graph of $f(x)$ over the given domain, and hence explain why $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$.

20 Find the stationary points and hence sketch, for $0 \le x \le 2\pi$:
a $y = \sin^2 x + \cos x$
b $y = \sin^3 x\cos x$
c $y = \tan^2 x - 2\tan x$

21 Let $f(x) = \dfrac{\sin x}{x}$. Remember that we proved in Section 7A that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$.
a Write down the domain of $f(x)$, show that $f(x)$ is even, find the zeroes of $f(x)$, and determine $\displaystyle\lim_{x\to\infty} f(x)$.
b Differentiate $f(x)$, and hence show that $f(x)$ has stationary points when $\tan x = x$.
c Sketch $y = \tan x$ and $y = x$ on one set of axes, and hence use your calculator to estimate the turning points for $0 \le x \le 4\pi$. Give the x-coordinates in the form $\lambda\pi$, with λ to no more than two decimal places.
d Using this information, sketch $y = f(x)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7D Integrating the trigonometric functions

As always, the standard forms for differentiation can be reversed to give standard forms for integration.

The standard forms for integrating the trigonometric functions

When the standard forms for differentiating $\sin x$, $\cos x$ and $\tan x$ are reversed, they give three new standard integrals.

First, $\dfrac{d}{dx}\sin x = \cos x$, and reversing this, $\displaystyle\int \cos x\,dx = \sin x$.

Secondly, $\dfrac{d}{dx}\cos x = -\sin x$, and reversing this, $\displaystyle\int (-\sin x)\,dx = \cos x$

$\boxed{\times\ (-1)}$ $\displaystyle\int \sin x\,dx = -\cos x$.

Thirdly, $\dfrac{d}{dx}\tan x = \sec^2 x$, and reversing this, $\displaystyle\int \sec^2 x\,dx = \tan x$.

This gives three new standard integrals. These three standard forms should be carefully memorised — pay attention to the signs in the first two standard forms.

8 STANDARD TRIGONOMETRIC INTEGRALS

- $\displaystyle\int \cos x\,dx = \sin x + C$, for some constant C
- $\displaystyle\int \sin x\,dx = -\cos x + C$, for some constant C
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$, for some constant C

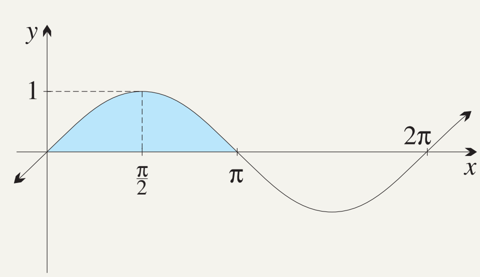
No calculation involving a primitive may cross an asymptote.

Example 16 7D

The curve $y = \sin x$ is sketched below. Show that the first arch of the curve, as shaded in the diagram, has area 2 square units.

SOLUTION

Because the region is entirely above the x-axis,



$$\begin{aligned}\text{area} &= \int_0^{\pi} \sin x\,dx\\ &= \Big[-\cos x\Big]_0^{\pi}\\ &= -\cos\pi + \cos 0\\ &= -(-1) + 1\end{aligned}$$

(the graph of $y = \cos x$ shows that $\cos\pi = -1$)

$= 2$ square units.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Note: This simple answer confirms again that radians are the right units to use for calculus with trigonometric functions. Similar simple results were obtained earlier when e was used as the base for powers. For example, Question 39c of the Chapter 6 Review gathered together three remarkably simple results:

$$\int_1^e \frac{1}{x}\,dx = \int_1^e \log_e x\,dx = \int_0^1 xe^x\,dx = 1$$

Example 17 — 7D

Evaluate these definite integrals.

a $\displaystyle\int_0^{\pi} \cos x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ **c** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\pi} \cos x\,dx &= \Big[\sin x\Big]_0^{\pi}\\ &= \sin\pi - \sin 0\\ &= 0\end{aligned}$$

(Use the graph of $y = \sin x$ to see that $\sin\pi = 0$ and $\sin 0 = 0$.)

b
$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec^2 x\,dx &= \Big[\tan x\Big]_0^{\frac{\pi}{3}}\\ &= \tan\tfrac{\pi}{3} - \tan 0\\ &= \sqrt{3}\end{aligned}$$

(Here $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan 0 = 0$.)

c This integral is meaningless because it crosses the asymptote at $x = \frac{\pi}{2}$.

Replacing x by $ax + b$

Reversing the standard forms for derivatives in Section 7B gives a further set of standard forms. Again, the constants of integration have been ignored until the boxed statement of the standard forms.

First, $$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b),$$

so $$\int a\cos(ax+b)\,dx = \sin(ax+b)$$

and dividing by a, $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b).$$

Secondly, $$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b),$$

so $$\int -a\sin(ax+b)\,dx = \cos(ax+b)$$

and dividing by $-a$, $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b).$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Thirdly, $$\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b),$$

so $$\int a\sec^2(ax+b)\,dx = \tan(ax+b).$$

and dividing by a, $$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b).$$

The result is extended forms of the three standard integrals. These extended standard forms should also be carefully memorised.

9 STANDARD INTEGRALS FOR FUNCTIONS OF $ax + b$

- $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C,$ for some constant C
- $\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C,$ for some constant C
- $\displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C,$ for some constant C

Example 18 — 7D

Evaluate these definite integrals.

a $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x\,dx$ **b** $\displaystyle\int_\pi^{2\pi} \sin \tfrac{1}{4}x\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos 3x\,dx &= \tfrac{1}{3}\Big[\sin 3x\Big]_0^{\frac{\pi}{6}}\\ &= \tfrac{1}{3}\left(\sin\tfrac{\pi}{2} - 3\sin 0\right)\\ &= \tfrac{1}{3}\end{aligned}$$

b
$$\begin{aligned}\int_\pi^{2\pi} \sin\tfrac{1}{4}x\,dx &= -4\Big[\cos\tfrac{1}{4}x\Big]_\pi^{2\pi} \qquad \text{(because the reciprocal of } \tfrac{1}{4} \text{ is 4)}\\ &= -4\cos\tfrac{\pi}{2} + 4\cos\tfrac{\pi}{4}\\ &= 0 + 4\times\frac{\sqrt{2}}{2}\\ &= 2\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx &= \tfrac{1}{2}\Big[\tan(2x+\pi)\Big]_0^{\frac{\pi}{8}}\\ &= \tfrac{1}{2}\left(\tan\tfrac{5\pi}{4} - \tan\pi\right)\\ &= \tfrac{1}{2}(1-0) \qquad \left(\tfrac{5\pi}{4} \text{ is in quadrant 3 with related angle } \tfrac{\pi}{4}\right)\\ &= \tfrac{1}{2}\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The primitives of tan *x* and cot *x*

The primitives of tan x and cot x can be found by using the ratio formulae $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$ and then applying the standard form from the previous chapter, in either one of its two versions,

$$\int \frac{u'}{u}\,dx = \log_e |u| + C \qquad \text{or} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C.$$

Example 19 7D

Find primitives of these functions.

a $\cot x$ **b** $\tan x$

SOLUTION

a
$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log_e |\sin x| + C.$$

Let $u = \sin x$. Then $u' = \cos x$. $\displaystyle\int \frac{u'}{u}\,dx = \log_e |u|$

OR

Let $f(x) = \sin x$. Then $f'(x) = \cos x$. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$

b
$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$$

Let $u = \cos x$. Then $u' = -\sin x$. $\displaystyle\int \frac{u'}{u}\,dx = \log_e |u|$

OR

Let $f(x) = \cos x$. Then $f'(x) = -\sin x$. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$

$= -\log_e |\cos x| + C.$ (This can also be written as $\log_e |\sec x|$.)

Note: Do not run across a zero of $\sin x$ when using part **a**, or a zero of $\cos x$ when using part **b**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Finding a function whose derivative is known

If the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be found.

Example 20 7D

The derivative of a certain function is $y' = \cos x$, and the graph of the function has y-intercept $(0, 3)$. Find the original function $f(x)$ and then find $f\left(\frac{\pi}{2}\right)$.

SOLUTION

Here $\quad y' = \cos x$,

and taking the primitive, $\quad y = \sin x + C$, for some constant C.

When $x = 0$, $y = 3$, so substituting $x = 0$,

$$3 = \sin 0 + C$$
$$C = 3.$$

Hence $\quad y = \sin x + 3$.

When $x = \frac{\pi}{2}$, $\quad y = \sin\frac{\pi}{2} + 3$

$= 4$, because $\sin\frac{\pi}{2} = 1$.

Example 21 7D

Given that $f'(x) = \sin 2x$ and $f(\pi) = 1$:

a find the function $f(x)$,

b find $f\left(\frac{\pi}{4}\right)$.

SOLUTION

a Here $\quad f'(x) = \sin 2x$,

and taking the primitive, $\quad f(x) = -\frac{1}{2}\cos 2x + C$, for some constant C.

It is known that $f(\pi) = 1$, so substituting $x = \pi$,

$$1 = -\tfrac{1}{2}\cos 2\pi + C$$
$$1 = -\tfrac{1}{2} \times 1 + C$$
$$C = 1\tfrac{1}{2}.$$

Hence $\quad f(x) = -\frac{1}{2}\cos 2x + 1\frac{1}{2}$.

b Substituting $x = \frac{\pi}{4}$, $\quad f\left(\frac{\pi}{4}\right) = -\frac{1}{2} \times \cos\frac{\pi}{2} + 1\frac{1}{2}$

$= 1\frac{1}{2}$, because $\cos\frac{\pi}{2} = 0$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Given a chain-rule derivative, find an integral

As always, the results of a chain-rule differentiation can be reversed to give a primitive.

Example 22 — 7D

a Use the chain rule to differentiate $\cos^5 x$.

b Hence find $\int_0^{\pi} \sin x \cos^4 x \, dx$.

SOLUTION

a Let $y = \cos^5 x$.

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -5 \sin x \cos^4 x$.

Let $u = \cos x$.
Then $y = u^5$.
Hence $\frac{du}{dx} = -\sin x$
and $\frac{dy}{du} = 5u^4$.

b From part **a**, $\frac{d}{dx}(\cos^5 x) = -5 \sin x \cos^4 x$.

Reversing this, $\int (-5 \sin x \cos^4 x) \, dx = \cos^5 x$.

$\boxed{\div (-5)}$ $\int \sin x \cos^4 x \, dx = -\frac{1}{5} \cos^5 x$.

Hence $\int_0^{\pi} \sin x \cos^4 x \, dx = -\frac{1}{5}\Big[\cos^5 x\Big]_0^{\pi}$

$= -\frac{1}{5}(-1 - 1)$

$= \frac{2}{5}$.

Using a formula for the reverse chain rule

The integral in the worked example above could have been done using the reverse chain rule for powers of u or $f(x)$.

Example 23 — 7D

Use the reverse chain rule to find $\int \sin x \cos^4 x \, dx$.

SOLUTION

$\int_0^{\pi} \sin x \cos^4 x \, dx$

$= -\int (-\sin x) \cos^4 x \, dx$

$= -\frac{1}{5} \cos^5 x + C$.

Let $u = \cos x$.
Then $u' = -\sin x$.

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1}$$

OR

Let $f(x) = \cos x$.
Then $f'(x) = -\sin x$.

$$\int \big(f(x)\big)^n \frac{du}{dx} dx = \frac{\big(f(x)\big)^{n+1}}{n+1}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 7D

FOUNDATION

1 Find the following indefinite integrals.

a $\int \sec^2 x\,dx$
b $\int \cos x\,dx$
c $\int \sin x\,dx$
d $\int -\sin x\,dx$
e $\int 2\cos x\,dx$
f $\int \cos 2x\,dx$
g $\int \frac{1}{2}\cos x\,dx$
h $\int \cos \frac{1}{2}x\,dx$
i $\int \sin 2x\,dx$
j $\int \sec^2 5x\,dx$
k $\int \cos 3x\,dx$
l $\int \sec^2 \frac{1}{3}x\,dx$
m $\int \sin \frac{x}{2}\,dx$
n $\int -\cos \frac{1}{5}x\,dx$
o $\int -4\sin 2x\,dx$
p $\int \frac{1}{4}\sin \frac{1}{4}x\,dx$
q $\int 12\sec^2 \frac{1}{3}x\,dx$
r $\int 2\cos \frac{x}{3}\,dx$

2 Find the value of:

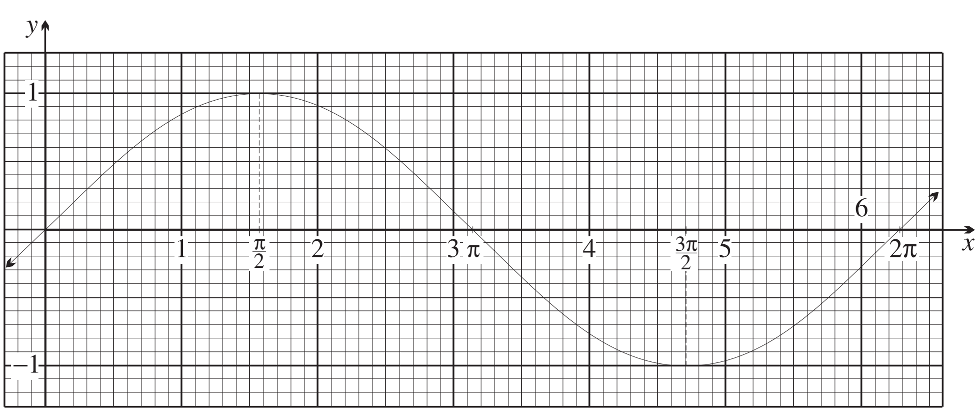
a $\int_0^{\frac{\pi}{2}} \cos x\,dx$
b $\int_0^{\frac{\pi}{6}} \cos x\,dx$
c $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$
d $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$
e $\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx$
f $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$
g $\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right)dx$
h $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{1}{2}x\right)dx$
i $\int_0^{\pi} (2\sin x - \sin 2x)\,dx$

3 a The gradient function of a certain curve is given by $\frac{dy}{dx} = \sin x$. If the curve passes through the origin, find its equation.

b Another curve passing through the origin has gradient function $y' = \cos x - 2\sin 2x$. Find its equation.

c If $\frac{dy}{dx} = \sin x + \cos x$, and $y = -2$ when $x = \pi$, find y as a function of x.

4



The graph of $y = \sin x$ is sketched above.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

a The first worked exercise in the notes for this section proved that $\int_0^{\pi} \sin x\,dx = 2$. Count squares on the graph of $y = \sin x$ above to confirm this result.

b On the same graph of $y = \sin x$, count squares and use symmetry to find:

i $\int_0^{\frac{\pi}{4}} \sin x\,dx$ **ii** $\int_0^{\frac{\pi}{2}} \sin x\,dx$ **iii** $\int_0^{\frac{3\pi}{4}} \sin x\,dx$

iv $\int_0^{\frac{5\pi}{4}} \sin x\,dx$ **v** $\int_0^{\frac{3\pi}{2}} \sin x\,dx$ **vi** $\int_0^{\frac{7\pi}{4}} \sin x\,dx$

c Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$, and confirm the results of part **b**.

5 [Technology]
Programs that sketch the graph and then approximate definite integrals would help reinforce the previous very important investigation. The investigation could then be continued past $x = \pi$, after which the definite integral decreases again.

Similar investigation with the graphs of $\cos x$ and $\sec^2 x$ would also be helpful, comparing the results of computer integration with the exact results obtained by integration using the standard primitives.

6 Find the following indefinite integrals.

a $\int \cos(x + 2)\,dx$ **b** $\int \cos(2x + 1)\,dx$ **c** $\int \sin(x + 2)\,dx$

d $\int \sin(2x + 1)\,dx$ **e** $\int \cos(3x - 2)\,dx$ **f** $\int \sin(7 - 5x)\,dx$

g $\int \sec^2(4 - x)\,dx$ **h** $\int \sec^2\left(\frac{1-x}{3}\right)dx$ **i** $\int \sin\left(\frac{1-x}{3}\right)dx$

7 **a** Find $\int \left(6\cos 3x - 4\sin \tfrac{1}{2}x\right)dx$.

b Find $\int \left(8\sec^2 2x - 10\cos \tfrac{1}{4}x + 12\sin \tfrac{1}{3}x\right)dx$.

8 **a** If $f'(x) = \pi\cos \pi x$ and $f(0) = 0$, find $f(x)$ and $f\left(\tfrac{1}{3}\right)$.

b If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(x)$ and $f\left(\tfrac{1}{6}\right)$.

c If $f''(x) = 18\cos 3x$ and $f'(0) = f\left(\tfrac{\pi}{2}\right) = 1$, find $f(x)$.

DEVELOPMENT

9 Find the following indefinite integrals, where a, b, u and v are constants.

a $\int a\sin(ax + b)\,dx$ **b** $\int \pi^2\cos \pi x\,dx$

c $\int \frac{1}{u}\sec^2(v + ux)\,dx$ **d** $\int \frac{a}{\cos^2 ax}\,dx$

ISBN 978-1-108-76630-2

10 **a** Copy and complete $1 + \tan^2 x = \dots$, and hence find $\int \tan^2 x \, dx$.

b Simplify $1 - \sin^2 x$, and hence find the value of $\int_0^{\frac{\pi}{3}} \frac{2}{1 - \sin^2 x} \, dx$.

11 **a** Copy and complete $\int \frac{f'(x)}{f(x)} \, dx = \dots$.

b Hence show that $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x} \, dx \doteqdot 0.4$.

12 **a** Use the fact that $\tan x = \frac{\sin x}{\cos x}$ to show that $\int_0^{\frac{\pi}{4}} \tan x \, dx = \frac{1}{2} \ln 2$.

b Use the fact that $\cot x = \frac{\cos x}{\sin x}$ to find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \, dx$.

13 **a** Find $\frac{d}{dx}(\sin^5 x)$, and hence find $\int \sin^4 x \cos x \, dx$.

b Find $\frac{d}{dx}(\tan^3 x)$, and hence find $\int \tan^2 x \sec^2 x \, dx$.

14 **a** Differentiate $e^{\sin x}$, and hence find the value of $\int_0^{\frac{\pi}{2}} \cos x e^{\sin x} \, dx$.

b Differentiate $e^{\tan x}$, and hence find the value of $\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x} \, dx$.

15 **a** Show that $\frac{d}{dx}(\sin x - x \cos x) = x \sin x$, and hence find $\int_0^{\frac{\pi}{2}} x \sin x \, dx$.

b Show that $\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right) = \sin^3 x$, and hence find $\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$.

16 Use the reverse chain rule $\int f'(x)\,\big(f(x)\big)^n dx = \frac{\big(f(x)\big)^{n+1}}{n + 1}$, to evaluate:

a $\int_0^{\pi} \sin x \cos^8 x \, dx$ **b** $\int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx$ **c** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin^7 x \, dx$

d $\int_0^{\frac{\pi}{6}} \cos x \sin^n x \, dx$ **e** $\int_0^{\frac{\pi}{3}} \sec^2 x \tan^7 x \, dx$ **f** $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^n x \, dx$

17 **a** Show, by finding the integral in two different ways, that for constants C and D,

$$\int \sin x \cos x \, dx = \tfrac{1}{2}\sin^2 x + C = -\tfrac{1}{4}\cos 2x + D.$$

b How may the two answers be reconciled?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 Find $\dfrac{d}{dx}\big(x\sin 2x\big)$, and hence find $\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$.

19 **a** Show that $\dfrac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$.

b Hence find $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$.

ENRICHMENT

20 **a** Show that $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.

b Hence find:

i $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin 3x\cos 2x\,dx$

ii $\displaystyle\int_0^{\pi} \sin 3x\cos 4x\,dx$

c Show that $\displaystyle\int_{-\pi}^{\pi} \sin mx\cos nx\,dx = 0$ for positive integers m and n:

i using the primitive,

ii using symmetry arguments.

21 **a** Find the values of A and B in the identity

$$A(2\sin x + \cos x) + B(2\cos x - \sin x) = 7\sin x + 11\cos x.$$

b Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\,dx = \frac{1}{2}(5\pi + 6\ln 2)$.

22 [The power series for $\sin x$ and $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .]$$

a We know that $\cos t \le 1$, for t positive. Integrate this inequality over the interval $0 \le t \le x$, where x is positive, and hence show that $\sin x \le x$.

b Change the variable to t, integrate the inequality $\sin t \le t$ over $0 \le t \le x$, and hence show that $\cos x \ge 1 - \dfrac{x^2}{2!}$.

c Do it twice more, and show that:

i $\sin x \ge x - \dfrac{x^3}{3!}$

ii $\cos x \le 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

d Now use induction (informally) to show that for all positive integers n,

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \le \sin x + \frac{x^{4n+3}}{(4n+3)!},$$

and use this inequality to conclude that $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$ converges, with limit $\sin x$.

e Proceeding similarly, prove that $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$ converges, with limit $\cos x$.

f Use evenness and oddness to extend the results of (d) and (e) to negative values of x.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7E Applications of integration

The trigonometric integrals can now be used to find areas in the usual way.

Finding areas by integration

As always, *a sketch is essential*, because areas below the x-axis are represented as a negative number by the definite integral.

It is best to evaluate the separate integrals first and then make a conclusion about areas.

Example 24 7E

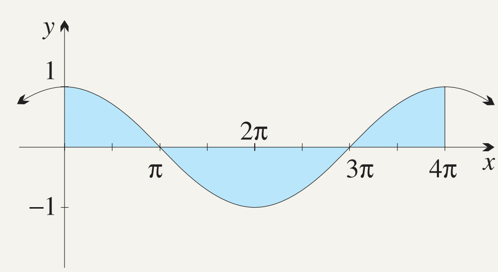
a Sketch $y = \cos \frac{1}{2}x$ in the interval $0 \le x \le 4\pi$, marking both x-intercepts.

b Hence find the area between the curve and the x-axis, for $0 \le x \le 4\pi$.

SOLUTION

a The curve $y = \cos \frac{1}{2}x$ has amplitude 1, and the period is $2\pi \div \frac{1}{2} = 4\pi$.
The two x-intercepts in the interval are $x = \pi$ and $x = 3\pi$.

b We must integrate separately over the three intervals $[0, \pi]$ and $[\pi, 3\pi]$ and $[3\pi, 4\pi]$.



First,
$$\int_0^{\pi} \cos \tfrac{1}{2}x\,dx = \Big[2\sin \tfrac{1}{2}x\Big]_0^{\pi}$$
$$= 2\sin\tfrac{\pi}{2} - 2\sin 0$$
$$= 2 - 0$$
$$= 2,$$
which is positive, because the curve is above the x-axis for $0 \le x \le \pi$.

Secondly,
$$\int_{\pi}^{3\pi} \cos \tfrac{1}{2}x\,dx = \Big[2\sin \tfrac{1}{2}x\Big]_{\pi}^{3\pi}$$
$$= 2\sin\tfrac{3\pi}{2} - 2\sin\tfrac{\pi}{2}$$
$$= -2 - 2$$
$$= -4,$$
which is negative, because the curve is below the x-axis for $\pi \le x \le 3\pi$.

Thirdly,
$$\int_{3\pi}^{4\pi} \cos \tfrac{1}{2}x\,dx = \Big[2\sin \tfrac{1}{2}x\Big]_{3\pi}^{4\pi}$$
$$= 2\sin 2\pi - 2\sin\tfrac{3\pi}{2}$$
$$= 0 - (-2)$$
$$= 2,$$
which is positive, because the curve is above the x-axis for $3\pi \le x \le 4\pi$.

Hence
$$\text{total area} = 2 + 4 + 2$$
$$= 8 \text{ square units.}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Finding areas between curves

The next worked example uses the principle that if $y = f(x)$ is above $y = g(x)$ throughout some interval $a \le x \le b$, then the area between the curves is given by the formula

$$\text{area between the curves} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 25 7E

a Show that the curves $y = \sin x$ and $y = \sin 2x$ intersect when $x = \frac{\pi}{3}$.

b Sketch these curves in the interval $[0, \pi]$.

c Find the area contained between the curves in the interval $\left[0, \frac{\pi}{3}\right]$.

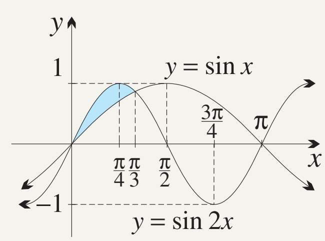
SOLUTION

a The curves intersect at $x = \frac{\pi}{3}$ because $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{1}{2}\sqrt{3}$.

b The curves are sketched to the right below.

c In the interval $0 \le x \le \frac{\pi}{3}$, the curve $y = \sin 2x$ is always above $y = \sin x$,

$$\begin{aligned}\text{so area between} &= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx \\ &= \Big[-\tfrac{1}{2}\cos 2x + \cos x\Big]_0^{\frac{\pi}{3}} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right).\end{aligned}$$



Because $\cos 0 = 1$ and $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{2\pi}{3} = -\frac{1}{2}$,

$$\begin{aligned}\text{area} &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right) - \left(-\tfrac{1}{2} + 1\right) \\ &= \tfrac{1}{4}\text{ square units.}\end{aligned}$$

Example 26 7E

a Show that in the interval $0 \le x \le 2\pi$, the curves $y = \sin x$ and $y = \cos x$ intersect when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$.

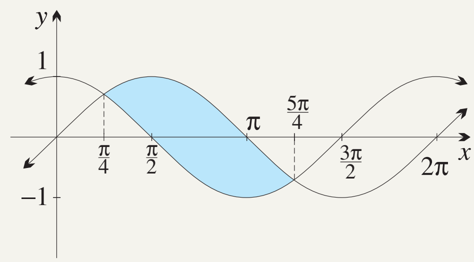
b Sketch the curves in this interval and find the area contained between them.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Put $\sin x = \cos x$.
Then $\tan x = 1$
$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$,
so the curves intersect at the points
$\left(\frac{\pi}{4}, \frac{1}{2}\sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{2}\sqrt{2}\right)$.



b
$$\begin{aligned}\text{Area between} &= \int_{\frac{\pi}{4}}^{-\frac{5\pi}{4}} (\sin x - \cos x)\,dx \\ &= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -\left(-\tfrac{1}{2}\sqrt{2}\right) - \left(-\tfrac{1}{2}\sqrt{2}\right) + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2} \\ &= 2\sqrt{2} \text{ square units.}\end{aligned}$$

Exercise 7E

FOUNDATION

Technology: Some graphing programs can perform numerical integration on specified regions. Such programs would help to confirm the integrals in this exercise and to investigate quickly further integrals associated with these curves.

1 Find the exact area between the curve $y = \cos x$ and the x-axis:

a from $x = 0$ to $x = \frac{\pi}{2}$,   **b** from $x = 0$ to $x = \frac{\pi}{6}$.

2 Find the exact area between the curve $y = \sec^2 x$ and the x-axis:

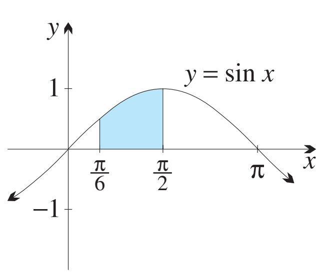
a from $x = 0$ to $x = \frac{\pi}{4}$,   **b** from $x = 0$ to $x = \frac{\pi}{3}$.

3 Find the exact area between the curve $y = \sin x$ and the x-axis:

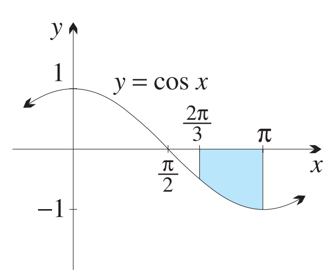
a from $x = 0$ to $x = \frac{\pi}{4}$,   **b** from $x = 0$ to $x = \frac{\pi}{6}$.

4 Calculate the area of the shaded region in each diagram below (and then observe that the two regions have equal area).

a



b



5 Find the area enclosed between each curve and the x-axis over the specified domain.

a $y = \sin x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$ **b** $y = \sin 2x$, from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$

c $y = \cos x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$ **d** $y = \cos 3x$, from $x = \frac{\pi}{12}$ to $x = \frac{\pi}{6}$

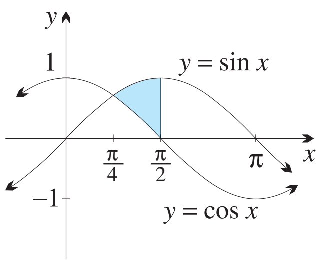
e $y = \sec^2 x$, from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$ **f** $y = \sec^2 \frac{1}{2}x$, from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

ISBN 978-1-108-76630-2

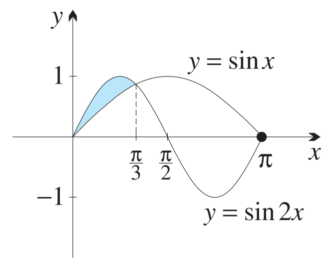
Cambridge University Press
Updated July 2022

6 Calculate the area of the shaded region in each diagram below.

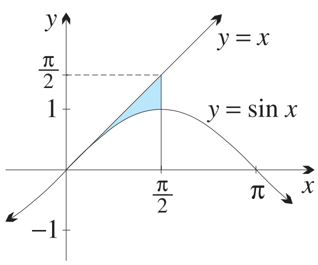
a



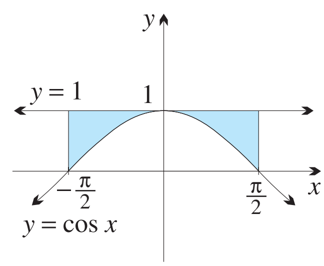
b



c

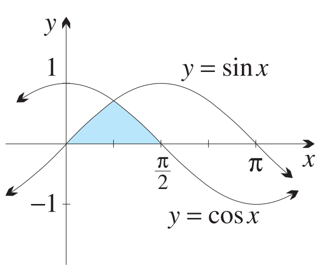


d

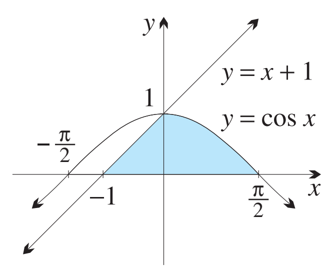


7 Calculate the area of the shaded region in each diagram below.

a



b



DEVELOPMENT

8 Find, using a diagram, the area bounded by one arch of each curve and the x-axis.

a $y = \sin x$ **b** $y = \cos 2x$

9 Sketch the area enclosed between each curve and the x-axis over the specified domain, and then find the exact value of the area. (Make use of symmetry wherever possible.)

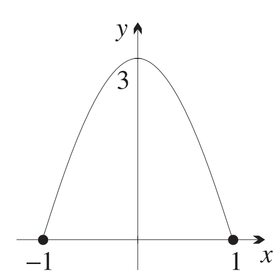
a $y = \cos x$, from $x = 0$ to $x = \pi$ **b** $y = \sin x$, from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$

c $y = \cos 2x$, from $x = 0$ to $x = \pi$ **d** $y = \sin 2x$, from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$

e $y = \sin x$, from $x = -\frac{5\pi}{6}$ to $x = \frac{7\pi}{6}$ **f** $y = \cos 3x$, from $x = \frac{\pi}{6}$ to $x = \frac{2\pi}{3}$

10 a Sketch the curve $y = 2\cos \pi x$, for $-1 \le x \le 1$, clearly marking the two x-intercepts.

b Find the exact area bounded by the curve $y = 2\cos \pi x$ and the x-axis, between the two x-intercepts.



11 An arch window 3 metres high and 2 metres wide is made in the shape of the curve $y = 3\cos\left(\frac{\pi}{2}x\right)$, as shown to the right. Find the area of the window in square metres, correct to one decimal place.

12 The graphs of $y = x - \sin x$ and $y = x$ are sketched together in a worked exercise in Section 7C. Find the total area enclosed between these graphs, from $x = 0$ to $x = 2\pi$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 The region R is bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$. Show that R has area $\ln 2$ square units.

14 a Sketch the region bounded by the graphs of $y = \sin x$ and $y = \cos x$, and by the vertical lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

b Find the area of the region in part **a**.

15 a Show by substitution that $y = \sin x$ and $y = \cos 2x$ meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

b On the same number plane, sketch $y = \sin x$ and $y = \cos 2x$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{6}$.

c Hence find the area of the region bounded by the two curves.

16 a Show that $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sin x + \cos x$.

b Hence, or otherwise, find the exact area under one arch of the curve $y = \sin x + \cos x$.

17 a Show that for all positive integers n:

i $\displaystyle\int_0^{2\pi} \sin nx\, dx = 0$ **ii** $\displaystyle\int_0^{2\pi} \cos nx\, dx = 0$

b Sketch each of the following graphs, and then find the area between the curve and the x-axis, from $x = 0$ to $x = 2\pi$.

i $y = \sin x$ **ii** $y = \sin 2x$ **iii** $y = \sin 3x$ **iv** $y = \sin nx$ **v** $y = \cos nx$

18 a Show that $\displaystyle\int_0^{n} (1 + \sin 2\pi x)\, dx = n$, for all positive integers n.

b Sketch $y = 1 + \sin 2\pi x$, and interpret the result geometrically.

19 Sketch $y = |\sin x|$ for $0 \le x \le 6\pi$, and hence evaluate $\displaystyle\int_0^{6\pi} |\sin x|\, dx$.

20 a Using the fact that $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$, explain why $x^2 \sin x < x^3 < x^2 \tan x$ for $0 < x < \frac{\pi}{2}$.

b Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} x^2 \sin x\, dx < \frac{\pi^4}{4^5} < \int_0^{\frac{\pi}{4}} x^2 \tan x\, dx$.

21 a Given that $y = \dfrac{1}{1 + \sin x}$, show that $y' = -\dfrac{\cos x}{(1 + \sin x)^2}$.

b Hence explain why the function $y = \dfrac{1}{1 + \sin x}$ is decreasing for $0 < x < \frac{\pi}{2}$.

c Sketch the curve for $0 \le x \le \frac{\pi}{2}$, and hence show that $\dfrac{\pi}{4} < \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x}\, dx < \frac{\pi}{2}$.

22 Use symmetry arguments to help evaluate:

a $\displaystyle\int_{-4\pi}^{4\pi} \sin 3x\, dx$ **b** $\displaystyle\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x\, dx$

c $\displaystyle\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}} \cos x\, dx$ **d** $\displaystyle\int_{-\pi}^{\pi} \sec^2 \tfrac{1}{3} x\, dx$

e $\displaystyle\int_{-\pi}^{\pi} (3 + 2x + \sin x)\, dx$ **f** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 2x + \cos 3x + 3x^2)\, dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

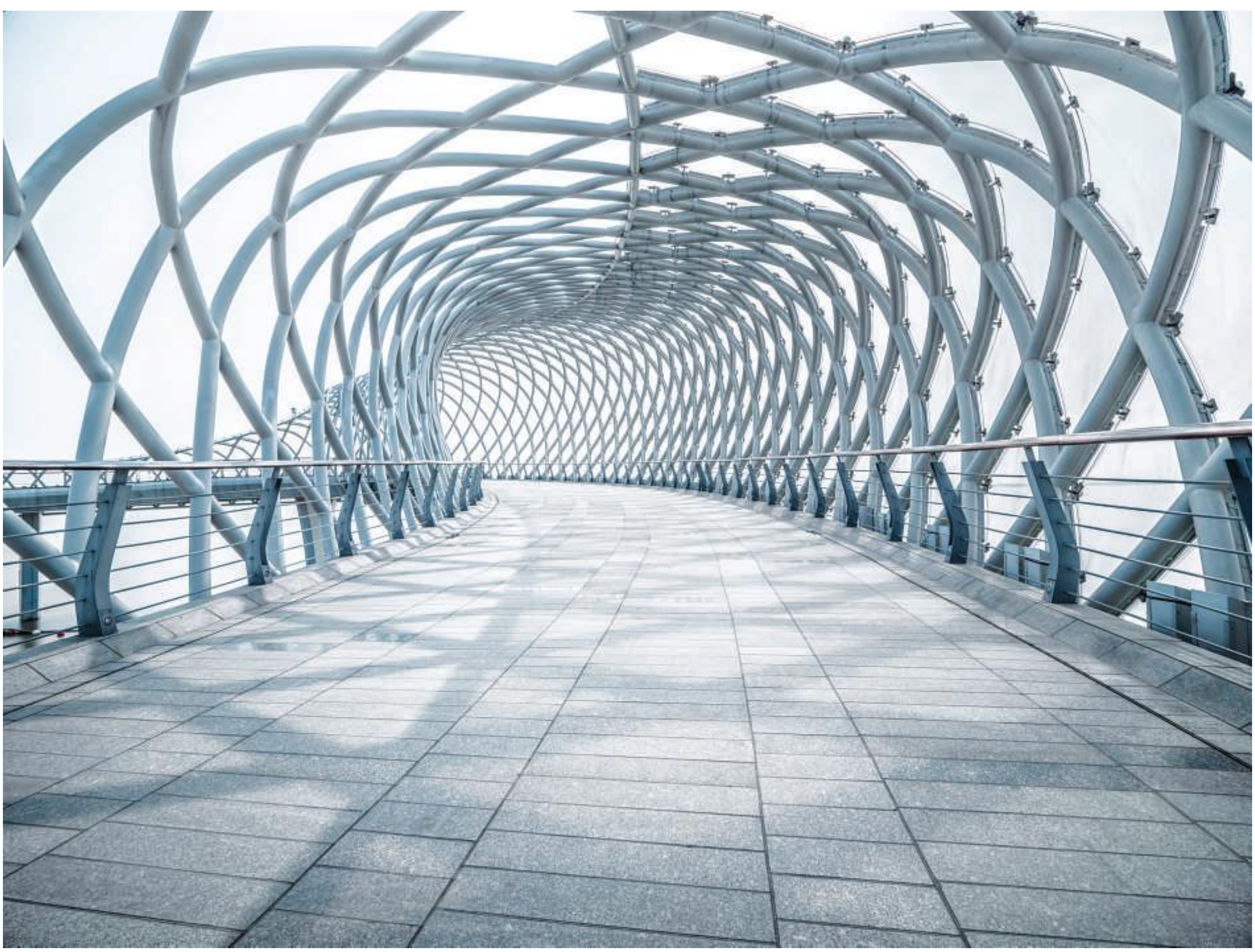
ENRICHMENT

23 a Show that $\dfrac{d}{dx}\Big(-\tfrac{1}{2}e^{-x}(\sin x + \cos x)\Big) = e^{-x}\sin x$.

b Find $\displaystyle\int_0^N e^{-x}\sin x\,dx$, and show that $\displaystyle\int_0^\infty e^{-x}\sin x\,dx$ converges to $\dfrac{1}{2}$.

c Find $\displaystyle\int_0^\pi e^{-x}\sin x\,dx$, $\displaystyle\int_{2\pi}^{3\pi} e^{-x}\sin x\,dx, \ldots$, and show that the areas of the arches above the x-axis form a GP with limiting sum $\dfrac{e^\pi}{2(e^\pi - 1)}$.

d Show that the areas of the arches below the x-axis also form a GP, and hence show that the total area contained between the curve and the x-axis, to the right of the y-axis, is $\dfrac{e^\pi + 1}{2(e^\pi - 1)}$. Also confirm by subtraction the result of part **b**.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 7 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 7 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Differentiate with respect to x.

a $y = 5\sin x$ **b** $y = \sin 5x$ **c** $y = 5\cos 5x$

d $y = \tan(5x - 4)$ **e** $y = x\sin 5x$ **f** $y = \dfrac{\cos 5x}{x}$

g $y = \sin^5 x$ **h** $y = \tan(x^5)$ **i** $y = e^{\cos 5x}$

j $y = \log_e(\sin 5x)$

2 Find the gradient of the tangent to $y = \cos 2x$ at the point on the curve where $x = \frac{\pi}{3}$.

3 a Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{3}$.

b Find the equation of the tangent to $y = x\cos x$ at the point where $x = \frac{\pi}{2}$.

4 Find the x-coordinates of the stationary points on each curve, for $0 \le x \le 2\pi$.

a $y = x + \cos x$ **b** $y = \sin x - \cos x$

5 Find:

a $\displaystyle\int 4\cos x\,dx$ **b** $\displaystyle\int \sin 4x\,dx$ **c** $\displaystyle\int \sec^2 \tfrac{1}{4}x\,dx$

6 Find the value of:

a $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x\,dx$ **c** $\displaystyle\int_0^{\frac{1}{3}} \pi\sin \pi x\,dx$

7 Find the value of $\displaystyle\int_0^{\frac{1}{4}} \sin 3x\,dx$, correct to three decimal places.

8 A curve has gradient function $y' = \cos \frac{1}{2}x$ and passes through the point $(\pi, 1)$. Find its equation.

9 a Sketch the curve $y = 2\sin 2x$, for $0 \le x \le \pi$, and then shade the area between the curve and the x-axis from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

b Calculate the shaded area in part **a**.

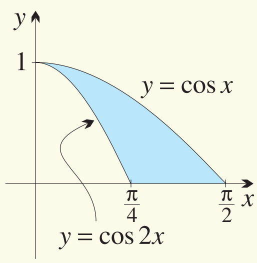
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

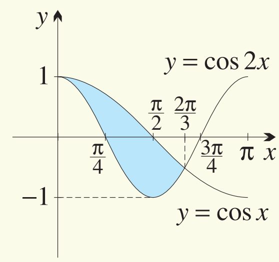
Review

10 Find the area of the shaded region in each diagram below.

a

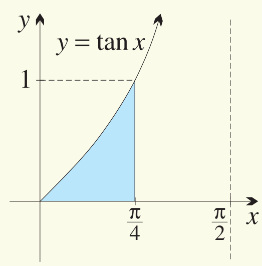


b



11 a Write $\tan x$ in terms of $\sin x$ and $\cos x$.

b Hence find the exact area of the shaded region in the diagram below.



The following questions are more difficult:

12 Consider the curve $y = 2\cos x + \sin 2x$ for $-\pi \leq x \leq \pi$.

a Find the x- and y-intercepts.

b Find the stationary points and determine their nature. (You will need to use the identity $\cos 2x = 1 - 2\sin^2 x$.)

c Hence sketch the curve over the given domain.

13 a Find the first and second derivatives of $y = e^x \sin x$. (Note that this function models oscillations such as feedback loops which grow exponentially.)

b Find the stationary points for $-\pi \leq x \leq \pi$ and determine their nature.

c Find the points of inflexion for $-\pi \leq x \leq \pi$.

d Hence sketch the curve for $-\pi \leq x \leq \pi$.

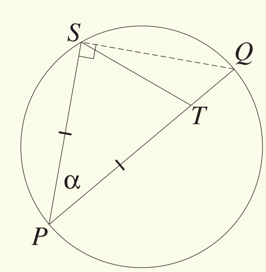
14 The angle θ between two radii OP and OQ of a circle of radius 6 cm is increasing at the rate of 0.1 radians per minute.

a Show that the area of sector OPQ is increasing at the rate of 1.8 cm^2/min.

b Find the rate at which the area of $\triangle OPQ$ is increasing at the instant when $\theta = \frac{\pi}{4}$.

c Find the value of θ for which the rate of increase of the area of the segment cut off by the chord PQ is at its maximum.

15 PQ is a diameter of the given circle and S is a point on the circumference. T is the point on PQ such that $PS = PT$. Let $\angle SPT = \alpha$.



a Show that the area A of $\triangle SPT$ is $A = \frac{1}{2}d^2\cos^2\alpha\sin\alpha$, where d is the diameter of the circle.

b Hence show that the maximum area of $\triangle SPT$ as S varies on the circle is $\frac{1}{9}d^2\sqrt{3}$ units^2.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

16 Find:

a $\displaystyle\int e^{2x}\cos e^{2x}\,dx$

b $\displaystyle\int \frac{\sin e^{-2x}}{e^{2x}}\,dx$

c $\displaystyle\int \frac{\sec^2 x}{3\tan x + 1}\,dx$

d $\displaystyle\int \frac{3\sin x}{4 + 5\cos x}\,dx$

e $\displaystyle\int \frac{1 - \cos^3 x}{1 - \sin^2 x}\,dx$

f $\displaystyle\int_0^{\pi} \sin x\cos^2 x\,dx$

17 a Show that $\tan^3 x = \tan x\sec^2 x - \tan x$.

b Hence find:

i $\displaystyle\int \tan^3 x\,dx$

ii $\displaystyle\int \tan^5 x\,dx$

18 a Sketch $y = 1 - \tan x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes.

b Find the area of R.

19 a Show that $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$.

b Hence find:

i $\displaystyle\int_0^{\frac{\pi}{2}} 2\cos 3x\cos 2x\,dx$

ii $\displaystyle\int_0^{\pi} \cos 3x\cos 4x\,dx$

c Find $\displaystyle\int \cos mx\cos nx\,dx$, where m and n are:

i distinct positive integers,

ii equal positive integers.

d Show that for positive integers m and n, $\displaystyle\int_{-\pi}^{\pi} \cos mx\cos nx\,dx = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Vectors

A *vector* in a Euclidean plane consists of a length and a direction.

The most obvious reason to introduce vectors is their usefulness in science. Many quantities in science are *scalars*, meaning that they have magnitude, but no direction is associated with them. Many other quantities, however, are vectors, meaning that they have a magnitude and a direction.

The contrast between scalars and vectors occurs throughout science. For example:

- Distance is a scalar that can be positive or zero, but displacement is a vector, because we move a certain distance in a certain direction.
- Speed is a scalar, but velocity is a vector because it is speed in a certain direction.
- Time is a scalar that can be positive or negative depending on our choice of ‘time zero’.
- Temperature is a scalar that cannot be less than absolute zero.
- The rotation of the Earth is a vector because its axis is tilted in a certain direction, but the Earth’s mass is a scalar.
- Force is a vector because you always push something in a certain direction, but pressure is a scalar — it acts in all directions.

A vector combines the magnitude and direction of a quantity into a single mathematical object.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8A Directed intervals and vectors

We can always specify a *vector* by giving its length and direction. For example,

'Walk 20 km east'.

The starting point is not part of the vector, so that the effect of this vector will be quite different depending on whether we start at Bathurst or at Bondi Beach.

We have already been working with vectors specified this way in trigonometry, such as when a ship sails at a certain speed in a certain direction. This section and the next develop two further ways of dealing with vectors:

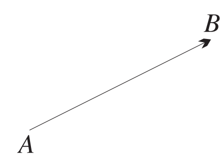
- Represent them by directed intervals that are free to wander around the plane.
- Represent them by *pairs of components*, which may be written in a column.

All three representations of a vector are equivalent, but the two ways introduced in this chapter allow a more general theory of vectors to be developed.

Vector notation

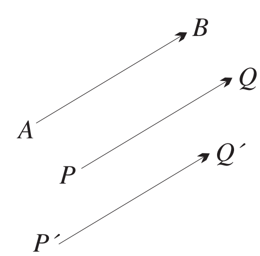
A vector can be handwritten as $\underset{\sim}{a}$ using a tilde underneath, or typeset in bold as $\mathbf{a}$. Boldface notation is clear in a printed text such as this book, but is unfortunately not appropriate for handwriting, so we have mostly avoided it. Remember that the one symbol $\underset{\sim}{a}$ or $\mathbf{a}$ holds both the length and the direction.

Directed intervals (or directed line segments)



Let A and B be distinct points in the Euclidean plane. The *directed interval* $\overrightarrow{AB}$, also called a *directed line segment*, is the interval AB together with the direction from A to B. The points A and B are called the *tail* and *head* of the directed interval, and the distance from A to B is called its *length*, written in this chapter as $|AB|$ to avoid ambiguity.

Representing a vector by directed intervals



Let $\underset{\sim}{a}$ be a vector, meaning that $\underset{\sim}{a}$ is a length and a direction. We can *represent* $\underset{\sim}{a}$ by any directed line interval $\overrightarrow{AB}$ that has the same length as $\underset{\sim}{a}$ and the same direction as $\underset{\sim}{a}$.

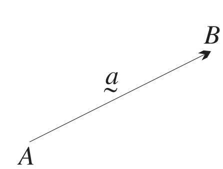
Now allow the directed interval $\overrightarrow{AB}$ to move around the plane using only translations. The resulting images $\overrightarrow{PQ}$, $\overrightarrow{P'Q'}$, ... of $\overrightarrow{AB}$ are again directed intervals, and each has the same direction and length as $\overrightarrow{AB}$, so they all represent the same vector $\underset{\sim}{a}$. Thus $\overrightarrow{AB}$, $\overrightarrow{PQ}$, $\overrightarrow{P'Q'}$, ... are distinct directed intervals, but they all represent the same vector.

This leaves the vector $\underset{\sim}{a}$ *free* to wander about all over the plane, the only restriction being that its length and direction are preserved as it wanders.

An alternative vector notation

We also write a vector as any directed interval that represents it, giving an alternative notation

$\underset{\sim}{a} = \overrightarrow{AB}$ (or using boldface notation, $\mathbf{a} = \overrightarrow{AB}$).



The notation $\overrightarrow{AB}$ for a vector is so common that from now on, the symbol $\overrightarrow{AB}$ will mean 'the vector $\overrightarrow{AB}$' unless it is explicitly referred to as 'the directed interval $\overrightarrow{AB}$'. In fact, the distinction between a vector and a directed interval is not strictly observed, either in language or in notation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The length of a vector

Denote by $|\underset{\sim}{a}|$ the length of a vector $\underset{\sim}{a}$. Thus $|\underset{\sim}{a}|$ is always positive or zero. The symbol $|\ldots|$ is generally used in mathematics for the magnitude of a quantity, for example, absolute value is written as $|-7| = 7$.

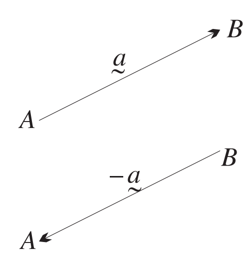
The word 'vector'

The Latin word *veho* means 'carry' or 'convey' from one place to another. In English, a 'vehicle' conveys passengers from where they are to their destination, and in medicine, a mosquito that carries a disease from one sick animal or person to another is called a 'vector'. Think of a geometric vector as moving things in the plane a certain distance in a certain direction.

1 VECTORS AND DIRECTED INTERVALS

- A *directed interval* $\overrightarrow{AB}$, or *directed line segment*, with *tail* A and *head* B, is the interval AB together with the *direction* from A to B. Its length is $|AB|$.
- A *vector* $\underset{\sim}{a}$ or $\mathbf{a}$ combines a length and a direction. It is *represented* by any directed interval $\overrightarrow{AB}$ with that length and direction, and we then write $\underset{\sim}{a} = \overrightarrow{AB}$.
- The length of a vector $\underset{\sim}{a}$ is written as $|\underset{\sim}{a}|$.

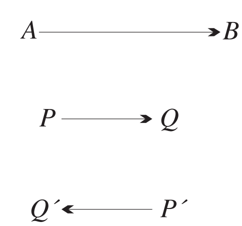
Opposite vectors and parallel vectors



The *opposite* of a directed interval $\overrightarrow{AB}$ is the directed interval $\overrightarrow{BA}$ with the same length and the opposite direction.

Correspondingly, the *opposite vector* $-\underset{\sim}{a}$ of a vector $\underset{\sim}{a} = \overrightarrow{AB}$ has the same length but opposite direction. It is thus represented by the opposite directed interval,

$$-\underset{\sim}{a} = \overrightarrow{BA}.$$



Two directed intervals $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are called *parallel* if the lines AB and PQ are parallel, whether in the same or the opposite direction. Similarly, two vectors $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{b} = \overrightarrow{PQ}$ are *parallel* if the lines AB and PQ are parallel, that is, if they are represented by parallel directed intervals.

Be careful: Parallel directed intervals or vectors may have the same direction or the opposite direction, because oppositely directed intervals or vectors are parallel.

The zero vector

When A and B are the same point, we have a *one-point directed interval* $\overrightarrow{AA}$, which is just the point A. It has no direction, and it has zero length.

The *zero vector* $\underset{\sim}{0}$ has length zero, and it is thus the only vector that does not have a direction. It can be represented by any one-point directed interval,

$$\underset{\sim}{0} = \overrightarrow{PP}, \quad \text{for all points } P.$$

The zero vector $\underset{\sim}{0}$ is the only vector that is its own opposite,

$$-\underset{\sim}{a} = \underset{\sim}{a} \quad \text{if and only if} \quad \underset{\sim}{a} = \underset{\sim}{0}.$$

Despite having no direction, the zero vector is defined to be *parallel to itself and to every other vector*.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 OPPOSITE VECTORS, PARALLEL VECTORS, AND THE ZERO VECTOR

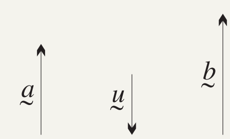
- The opposite of a vector $\underset{\sim}{a} = \overrightarrow{AB}$ is the vector $-\underset{\sim}{a} = \overrightarrow{BA}$ with the same length and the opposite direction.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are called *parallel* if the lines AB and PQ are parallel. Their directions may be the same or opposite.
- The zero vector $\underset{\sim}{0}$ has no length and no direction, and is the only vector that is its own opposite. It is parallel to every vector.

Example 1 — 8A

For non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{u}$ and $\underset{\sim}{b}$, we know from Euclidean geometry that:

$$\text{If } \underset{\sim}{a} \parallel \underset{\sim}{u} \text{ and } \underset{\sim}{u} \parallel \underset{\sim}{b}, \text{ then } \underset{\sim}{a} \parallel \underset{\sim}{b}.$$

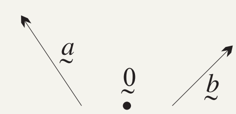
Is this statement true for all vectors?



SOLUTION

This statement is not true when the zero vector is allowed. Every vector is parallel to the zero vector, so if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are two vectors that are not parallel, then

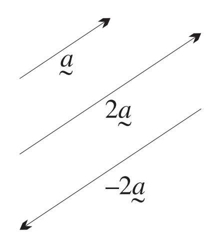
$$\underset{\sim}{a} \parallel \underset{\sim}{0} \text{ and } \underset{\sim}{0} \parallel \underset{\sim}{b}, \text{ but } \underset{\sim}{a} \nparallel \underset{\sim}{b}.$$



Multiplying a vector by a scalar

We can multiply a vector $\underset{\sim}{a}$ by a scalar λ.

- If either $\lambda = 0$ or $\underset{\sim}{a} = \underset{\sim}{0}$, then $\lambda\underset{\sim}{a} = \underset{\sim}{0}$ is the zero vector.
- Otherwise, $\lambda\underset{\sim}{a}$ has length $|\lambda| \times |\underset{\sim}{a}|$, and:
  — If $\lambda > 0$, then $\lambda\underset{\sim}{a}$ has the same direction as $\underset{\sim}{a}$.
  — If $\lambda < 0$, then $\lambda\underset{\sim}{a}$ has the opposite direction to $\underset{\sim}{a}$.



It is now easily checked that in all three cases,

$$|\lambda\underset{\sim}{a}| = |\lambda|\,|\underset{\sim}{a}| \qquad \text{and} \qquad \lambda\underset{\sim}{a} \parallel \underset{\sim}{a}.$$

It follows immediately that multiplication by 0, 1 and -1 behave as expected,

$$0 \times \underset{\sim}{a} = \underset{\sim}{0} \qquad \text{and} \qquad 1 \times \underset{\sim}{a} = \underset{\sim}{a} \qquad \text{and} \qquad (-1) \times \underset{\sim}{a} = -\underset{\sim}{a}.$$

and that multiplication by a scalar is associative, in the sense that if μ is a scalar,

$$\lambda(\mu\underset{\sim}{a}) = (\lambda\mu)\underset{\sim}{a}.$$

3 MULTIPLYING A VECTOR BY A SCALAR

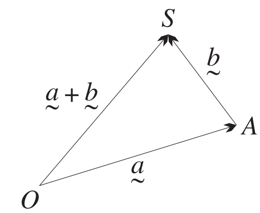
A vector can be multiplied by a scalar, and for all vectors $\underset{\sim}{a}$ and scalars λ and μ:

Multiplying by zero:	$0\underset{\sim}{a} = \underset{\sim}{0}$
Multiplying by 1:	$1\underset{\sim}{a} = \underset{\sim}{a}$
Multiplying by -1:	$(-1)\underset{\sim}{a} = -\underset{\sim}{a}$
Associative law:	$\lambda(\mu\underset{\sim}{a}) = (\lambda\mu)\underset{\sim}{a}$
Lengths:	$|\lambda\underset{\sim}{a}| = |\lambda|\,|\underset{\sim}{a}|$
Parallels:	$\lambda\underset{\sim}{a} \parallel \underset{\sim}{a}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Vector addition



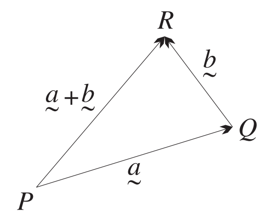
To add two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, represent them as

$$\underset{\sim}{a} = \overrightarrow{OA} \qquad \text{and} \qquad \underset{\sim}{b} = \overrightarrow{AS}$$

so that the head of the first is the tail of the second. The sum of the two vectors is then

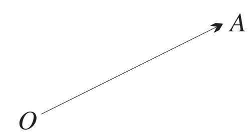
$$\overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}.$$

That is, the sum of a movement from O to A, followed by a movement from A to S, is a movement from O to S. Notice that the sum $\overrightarrow{OS}$ does not involve the point A. When I take a taxi from Orange to Sydney, all I care about is the result — I don't care if the driver goes through Armidale.



This definition is independent of the choice of the tail O used in the representation of the first vector. If we take another starting point P and construct $\underset{\sim}{a} = \overrightarrow{PQ}$ and $\underset{\sim}{b} = \overrightarrow{QR}$, then we can translate O to P, and the same translation will translate everything so that the directed interval $\overrightarrow{OB}$ is translated to the directed interval $\overrightarrow{PR}$.

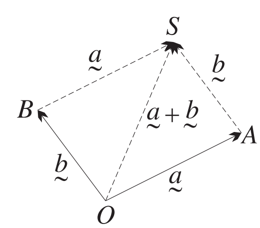
Zeroes and opposites



It is easily verified that addition of the zero vector changes nothing, and that adding a vector and its opposite gives the zero vector.

$$\overrightarrow{OA} + \overrightarrow{AA} = \overrightarrow{OA} \qquad \text{and} \qquad \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO}.$$

Addition by completing the parallelogram



For non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ that are not parallel, construct the sum $\underset{\sim}{a} + \underset{\sim}{b} = \overrightarrow{OS}$ as before. Now represent $\underset{\sim}{b}$ again as $\underset{\sim}{b} = \overrightarrow{OB}$ with tail O.

Then $OASB$ is a parallelogram because the opposite sides AS and OB are equal and parallel, and we have a second representation of $\underset{\sim}{a}$ as $\underset{\sim}{a} = \overrightarrow{BS}$.

This gives us a second method of adding vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

- Represent $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with a common tail O.
- Complete the parallelogram $AOBS$.
- Then $\underset{\sim}{a} + \underset{\sim}{b} = \overrightarrow{OS}$.

When $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel (including when one is zero), the resulting parallelogram is *degenerate* — meaning here that it is a one-dimensional (or zero-dimensional) object. In this case, we are just adding or subtracting lengths.

Addition is commutative

This follows immediately from the diagram above, where the parallelogram provides two representations of $\underset{\sim}{a}$ and two representations of $\underset{\sim}{b}$,

$$\begin{aligned} \underset{\sim}{a} + \underset{\sim}{b} &= \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}, \\ \text{and } \underset{\sim}{b} + \underset{\sim}{a} &= \overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BS} = \overrightarrow{OS}, \\ \text{so } \underset{\sim}{a} + \underset{\sim}{b} &= \underset{\sim}{b} + \underset{\sim}{a}, \quad \text{as required.} \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

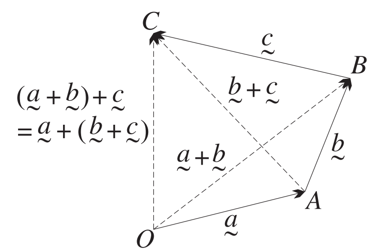
Addition is associative

Given any three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ we can represent them by directed intervals head-to-tail as

$$\underset{\sim}{a} = \overrightarrow{OA}, \quad \underset{\sim}{b} = \overrightarrow{AB}, \quad \underset{\sim}{c} = \overrightarrow{BC}.$$

Then $(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c}$
$$\begin{aligned} &= (\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BC} \\ &= \overrightarrow{OB} + \overrightarrow{BC} \\ &= \overrightarrow{OC}, \end{aligned}$$

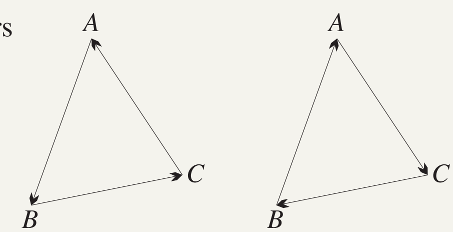
and $\underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$
$$\begin{aligned} &= \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC}) \\ &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OC}, \end{aligned}$$



which proves the associative law $(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$.

Example 2 — 8A

How can the sides of a triangle ABC be used to form three vectors whose sum is zero?



SOLUTION

Form the vectors so that they are head-to-tail.

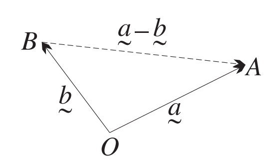
First way: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \underset{\sim}{0}$

Second way: $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BA} = \overrightarrow{AA} = \underset{\sim}{0}$

Subtraction of vectors

The *difference* $\underset{\sim}{a} - \underset{\sim}{b}$ of two vectors is defined in the usual way as the sum of $\underset{\sim}{a}$ and the opposite of $\underset{\sim}{b}$,

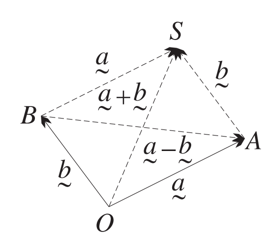
$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b}).$$



The best way to represent subtraction is to use a triangle. Represent the vectors as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with the same tail O. Then

$$\begin{aligned} \underset{\sim}{a} - \underset{\sim}{b} &= \overrightarrow{OA} - \overrightarrow{OB} \\ &= \overrightarrow{OA} + \overrightarrow{BO} \\ &= \overrightarrow{BO} + \overrightarrow{OA} \\ &= \overrightarrow{BA}. \end{aligned}$$

Thus $\underset{\sim}{a} - \underset{\sim}{b} = \overrightarrow{OA} - \overrightarrow{OB}$ is the vector $\overrightarrow{BA}$ with tail B and head A.



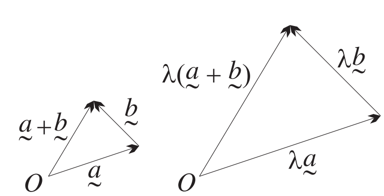
Now complete the parallelogram $AOBS$.

- The two directions of the diagonal OS represents the sums
  $\overrightarrow{OS} = \overrightarrow{OA} + \overrightarrow{OB}$ and $\overrightarrow{SO} = \overrightarrow{AO} + \overrightarrow{BO}$.
- The two directions of the diagonal AB represents the differences
  $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

Two distributive laws

First, multiplying a vector by a scalar is distributive over vector addition,

$$\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}, \quad \text{for all scalars } \lambda \text{ and vectors } \underset{\sim}{a} \text{ and } \underset{\sim}{b}.$$



This is easily proven by similarity, as shown in the diagram to the right.

Secondly, multiplying a vector by a scalar is distributive over scalar addition, as is easily seen if a diagram is drawn.

$$(\lambda + \mu)\underset{\sim}{a} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{a}, \quad \text{for all scalars } \lambda \text{ and } \mu \text{ and vectors } \underset{\sim}{a}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 VECTOR ADDITION AND SUBTRACTION

- To construct the sum $\underset{\sim}{a} + \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, place them head to tail as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = AB$. Then $\underset{\sim}{a} = \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}$.
- Alternatively, represent them as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with a common tail O, and complete the parallelogram $OASB$. Then $\underset{\sim}{a} + \underset{\sim}{b} = \overrightarrow{OS}$.
- To construct the difference $\underset{\sim}{a} - \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, represent them as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with a common tail O. Then the difference is the vector $\underset{\sim}{a} - \underset{\sim}{b} = \overrightarrow{BA}$ with tail B and head A.
- For all vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ and all scalars λ and μ:

Adding the zero vector:	$\underset{\sim}{a} + \underset{\sim}{0} = \underset{\sim}{a}$
Adding the opposite:	$\underset{\sim}{a} + (-\underset{\sim}{a}) = \underset{\sim}{0}$
Commutative law:	$\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$
Associative law:	$(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$
Two distributive laws:	$\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$
	$(\lambda + \mu)\underset{\sim}{a} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{a}$

The next worked example shows how vectors can be used to prove geometric theorems.

Example 3 8A

Let M be the midpoint of the side BC in $\triangle ABC$. Let $BA = \underset{\sim}{b}$ and $CA = \underset{\sim}{c}$.

a Explain why $\overrightarrow{BC} = \underset{\sim}{b} - \underset{\sim}{c}$, and hence express $\overrightarrow{BM}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

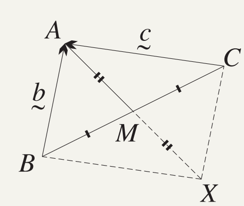
b Express $\overrightarrow{MA}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Why is the formula symmetric in $\underset{\sim}{b}$ and $\underset{\sim}{c}$?

c Extend AM to X so that $AM = MX$, and express $\overrightarrow{XA}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

d Express $\overrightarrow{XC}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

e What geometric theorem have you proven in part **d**?

SOLUTION



a Placing vectors head to tail,

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BA} - \overrightarrow{CA} = \underset{\sim}{b} - \underset{\sim}{c}.$$

Hence $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c})$.

b $\overrightarrow{MA} = \overrightarrow{MB} + \overrightarrow{BA} = -\overrightarrow{BM} + \overrightarrow{BA} = -\frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c}) + \underset{\sim}{b} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})$.
If we exchange B and C in the previous arguments, the result is the same.

c $\overrightarrow{XA} = 2 \times \overrightarrow{MA} = \underset{\sim}{b} + \underset{\sim}{c}$.

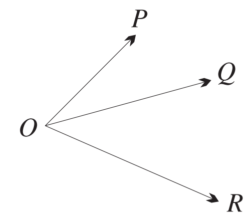
d $\overrightarrow{XC} = \overrightarrow{XA} + \overrightarrow{AC} = \overrightarrow{XA} - \overrightarrow{CA} = (\underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{c} = \underset{\sim}{b}$.

e Because $\overrightarrow{BA} = \underset{\sim}{b} = \overrightarrow{XC}$, the intervals BA and XC are equal and parallel, so $ABXC$ is a parallelogram.
We have proven that if the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Choosing a reference point or origin — position vectors



Points are not vectors. It is often useful, however, in a diagram or a physical situation, to refer everything back to some conveniently chosen reference point or *origin* O. Then there is a correspondence between each point P in the plane and its *position vector* $\overrightarrow{OP}$ drawn with tail O and head P.

For example, surveyors in a town may place a marker on top of a nearby hill and refer every point in the town back to that reference point.

In this situation, every point P has a unique position vector $\overrightarrow{OP}$, and every vector $\underset{\sim}{a}$ is the position vector of a unique point P — represent the vector with tail O, and then the head is P.

5 AN ORIGIN AND POSITION VECTORS

- Choose a convenient *origin* O as a reference point.
- Each point P in the plane then corresponds to the *position vector* $\overrightarrow{OP}$.
- Conversely, any vector $\underset{\sim}{a}$ can be drawn as a position vector $\overrightarrow{OP}$, and thus corresponds to the point P at the head.
- Subtraction of position vectors is particularly straightforward, because they have a common tail O.

The next worked example illustrates the final dotpoint in Box 5.

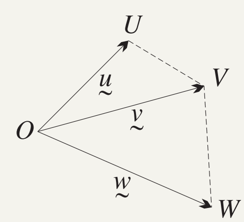
Example 4 8A

Three points U, V and W have position vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ with respect to a chosen origin O.

a Explain in two ways why $\overrightarrow{UV} = \underset{\sim}{v} - \underset{\sim}{u}$.

b Hence write down a condition on $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ for U, V and W to be collinear.

SOLUTION



a First, $\overrightarrow{UV} = \overrightarrow{UO} + \overrightarrow{OV} = -\overrightarrow{OU} + \overrightarrow{OV} = -\underset{\sim}{u} + \underset{\sim}{v}$.
Secondly, because $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are drawn as position vectors with a common tail O,
$\underset{\sim}{v} - \underset{\sim}{u}$ can be represented as $\overrightarrow{UV}$ with tail U and head V.

b The points are collinear when W lies on UV,
that is, when $\overrightarrow{VW}$ is a multiple of $\overrightarrow{UV}$,
that is, when $\underset{\sim}{w} - \underset{\sim}{v} = \lambda(\underset{\sim}{v} - \underset{\sim}{u})$, for some scalar λ,
better expressed as $\underset{\sim}{w} = \underset{\sim}{v} + \lambda(\underset{\sim}{v} - \underset{\sim}{u})$, for some scalar λ.
(There are many other possible forms.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 8A

FOUNDATION

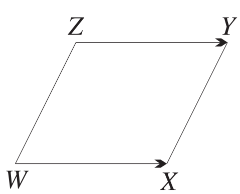
1 In each part, draw a diagram showing the displacement vector $\overrightarrow{AC}$. Then calculate the magnitude and direction of $\overrightarrow{AC}$. (Give magnitude correct to the nearest km where necessary, and direction as a true bearing correct to the nearest degree where necessary).

- **a** Sarah drives 100 km east from A to B, and then 40 km west from B to C.
- **b** William walks 6 km south from A to B, and then 4 km east from B to C.
- **c** A boat sails 25 km east from A to B, and then 15 km northeast from B to C.

2 Vikram cycled 28 km north from P to Q, then 19 km east from Q to R, and finally 12 km south from R to S.

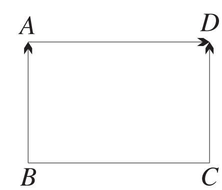
- **a** Draw a diagram representing his journey, and show the displacement vector $\overrightarrow{PS}$.
- **b** Determine the magnitude and direction of $\overrightarrow{PS}$ (in km correct to one decimal place, and as a true bearing correct to the nearest degree).

3



In the diagram above, $\overrightarrow{WX} = \overrightarrow{ZY}$.
Explain why $WXYZ$ is a parallelogram.

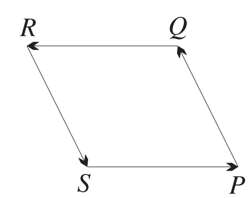
4



In the diagram above, $\overrightarrow{BA} = \overrightarrow{CD}$ and $\overrightarrow{BA} \perp \overrightarrow{AD}$.
What type of special quadrilateral is $ABCD$?
Give reasons for your answer.

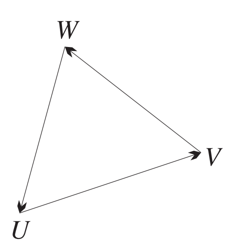
5 In the diagram to the right, $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$ and $\overrightarrow{SP}$ all have the same magnitude.

- **a** What type of special quadrilateral is $PQRS$? Justify your answer.
- **b** What can be said about the directions of $\overrightarrow{PQ}$ and $\overrightarrow{RS}$? Justify your answer.

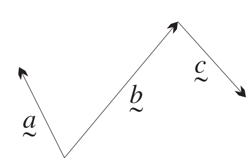


6 Suppose that u and v are non-zero vectors. By drawing a parallelogram, show that $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$.

7 In the diagram to the right, what is $\overrightarrow{UV} + \overrightarrow{VW} + \overrightarrow{WU}$?



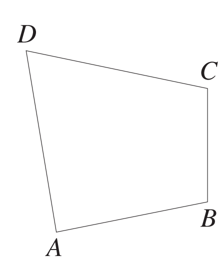
8



Copy the diagram above, and on it draw vectors representing $\underset{\sim}{b} - \underset{\sim}{a}$ and $\underset{\sim}{b} + \underset{\sim}{c}$.

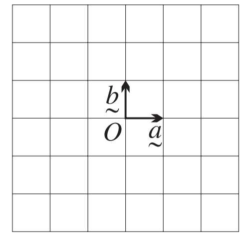
9 From the diagram to the right, write down a single vector equal to:

- **a** $\overrightarrow{AC} + \overrightarrow{CD}$
- **b** $\overrightarrow{BC} + \overrightarrow{CA}$
- **c** $\overrightarrow{AD} - \overrightarrow{AB}$
- **d** $\overrightarrow{AC} - \overrightarrow{BC}$



ISBN 978-1-108-76630-2

10 Copy the diagram to the right, and on it show *position vectors* representing:

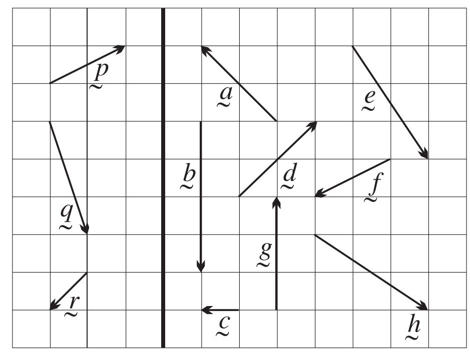


a $2\underset{\sim}{a} + 3\underset{\sim}{b}$

b $3\underset{\sim}{a} - \underset{\sim}{b}$

c $-2\underset{\sim}{a} - 2\underset{\sim}{b}$

11 Find the vector in the diagram to the right that represents:



a $-\underset{\sim}{p}$

b $-2\underset{\sim}{r}$

c $\underset{\sim}{p} + \underset{\sim}{q}$

d $\underset{\sim}{q} + \underset{\sim}{r}$

e $-\underset{\sim}{p} - \underset{\sim}{r}$

f $-\underset{\sim}{q} + \underset{\sim}{r}$

g $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r}$

h $\underset{\sim}{p} - \underset{\sim}{q} + \underset{\sim}{r}$

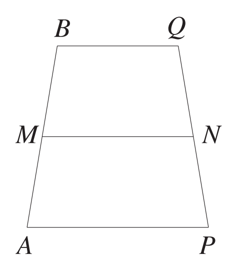
DEVELOPMENT

12 In $\triangle ABC$, the point M is the midpoint of BC. If $\overrightarrow{AC} = \underset{\sim}{u}$ and $\overrightarrow{BC} = \underset{\sim}{v}$, write in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$:

a $\overrightarrow{AB}$

b $\overrightarrow{AM}$

13 In the diagram, M and N are the midpoints of AB and PQ. Let $\underset{\sim}{u} = \overrightarrow{AM}$, $\underset{\sim}{v} = \overrightarrow{PN}$, $\underset{\sim}{a} = \overrightarrow{AP}$, $\underset{\sim}{b} = \overrightarrow{BQ}$ and $\underset{\sim}{p} = \overrightarrow{MN}$.



a Explain why $\overrightarrow{MB} = \underset{\sim}{u}$ and $\overrightarrow{NQ} = \underset{\sim}{v}$.

b Find $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

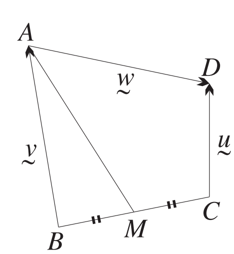
c Hence show that $\underset{\sim}{a} + \underset{\sim}{b} = 2\underset{\sim}{p}$.

14 In $\triangle ABC$, the point P lies on BC so that $BP:PC = 1:2$. Let $\overrightarrow{AB} = \underset{\sim}{p}$, $\overrightarrow{AC} = \underset{\sim}{q}$ and $\overrightarrow{AP} = \underset{\sim}{r}$.

a Draw a diagram, then write $\overrightarrow{BC}$, and then $\overrightarrow{BP}$, in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

b Prove that $\underset{\sim}{r} = \frac{2}{3}\underset{\sim}{p} + \frac{1}{3}\underset{\sim}{q}$.

15 In the diagram, $ABCD$ is a quadrilateral and M is the midpoint of BC. Let $\overrightarrow{CD} = \underset{\sim}{u}$, $\overrightarrow{BA} = \underset{\sim}{v}$ and $\overrightarrow{AD} = \underset{\sim}{w}$.



a Find $\overrightarrow{MB}$ in terms of $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

b Hence find $\overrightarrow{MA}$ in terms of $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

16 Suppose that $WXYZ$ is a quadrilateral, and that P, Q and R are the midpoints of WX, YZ and PQ respectively. Let $\overrightarrow{RW} = \underset{\sim}{w}$, $\overrightarrow{RX} = \underset{\sim}{x}$, $\overrightarrow{RY} = \underset{\sim}{y}$ and $\overrightarrow{RZ} = \underset{\sim}{z}$.

a Write $\overrightarrow{WX}$, and then $\overrightarrow{WP}$, in terms of $\underset{\sim}{w}$ and $\underset{\sim}{x}$.

b Find $\overrightarrow{RP}$ in terms of $\underset{\sim}{w}$ and $\underset{\sim}{x}$.

c Find $\overrightarrow{RQ}$ in terms of $\underset{\sim}{y}$ and $\underset{\sim}{z}$.

d Deduce that $\underset{\sim}{w} + \underset{\sim}{x} + \underset{\sim}{y} + \underset{\sim}{z} = \underset{\sim}{0}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 Suppose that P is a point on the line AB such that $\overrightarrow{AP} = k\,\overrightarrow{AB}$, where k is a constant.

a Show on a diagram the position of P relative to A and B if:

i $k > 1$ **ii** $0 < k < 1$ **iii** $k < 0$

b Find the value of k for which:

i $\overrightarrow{AP} = \frac{3}{2}\overrightarrow{PB}$ **ii** $\overrightarrow{AP} = -\frac{3}{2}\overrightarrow{PB}$ **iii** $\overrightarrow{AP} = -\frac{2}{3}\overrightarrow{PB}$

c Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{p}$ be the respective position vectors of the points A, B and P relative to an origin O. Prove that $\underset{\sim}{p} = (1 - k)\underset{\sim}{a} + k\underset{\sim}{b}$.

18 Just above Box 4 of the text, we claimed that the distributive law $\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$ is easily proven using the diagram. Explain this proof.

19 a Why is there only one zero vector?

b If you were the dictator of the world and you could put the origin anywhere on the surface of the Earth, where would you put it and why? What 'origins' have been chosen in the past, and which are still used?

ENRICHMENT

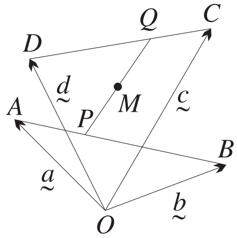
20 In $\triangle ABC$, X is the midpoint of BC, Y is the midpoint of AC and P is the point on AX such that $AP : PX = 2 : 1$. Let $\overrightarrow{AB} = \underset{\sim}{u}$ and $\overrightarrow{AC} = \underset{\sim}{v}$.

a Show that $\overrightarrow{PY} = \frac{1}{6}(\underset{\sim}{v} - 2\underset{\sim}{u})$.

b Hence show that the points B, P and Y are collinear.

c What geometric theorem have you proven?

21 In the diagram, the points A, B, C and D have respective position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ relative to an origin O. The point P divides AB in the ratio $1 : 3$, Q divides DC in the ratio $3 : 1$ and M is the midpoint of PQ.



a Show that P has position vector $\frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$.

b Express $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$.

c Hence show that M has position vector $\frac{1}{8}(3\underset{\sim}{a} + \underset{\sim}{b} + 3\underset{\sim}{c} + \underset{\sim}{d})$.

ISBN 978-1-108-76630-2

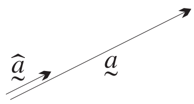
Cambridge University Press Updated July 2022

8B Components and column vectors

This section introduces *components*, which are another way to specify a vector. The components can be stacked in a *column vector*, which again allows a vector to be dealt with as a single object.

First, however, we need to define *unit vectors* and *perpendicular vectors*.

Unit vectors



A *unit vector* is a vector $\underset{\sim}{a}$ of length 1 , that is, $|\underset{\sim}{a}| = 1$. Every non-zero vector $\underset{\sim}{a}$ has a *corresponding unit vector* $\hat{\underset{\sim}{a}}$ with the same direction, namely

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \text{ the vector } \underset{\sim}{a} \text{ divided by its length } |\underset{\sim}{a}|.$$

There are two unit vectors parallel to $\underset{\sim}{a}$, namely $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ and $-\hat{\underset{\sim}{a}} = -\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

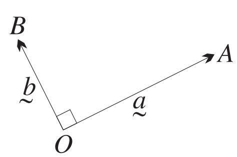
The first has the same direction as $\underset{\sim}{a}$, and the second has the opposite direction.

6 UNIT VECTORS

- A vector of length 1 is called a *unit vector*.
- If $\underset{\sim}{a} \neq \underset{\sim}{0}$, the vector $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ is a unit vector with the same direction as $\underset{\sim}{a}$,

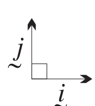
  and the vector $-\hat{\underset{\sim}{a}} = -\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ is a unit vector with the opposite direction.

Perpendicular vectors



Two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are called *perpendicular* if their directions are perpendicular. That is, when they are represented as $\underset{\sim}{a} = OA$ and $\underset{\sim}{b} = OB$ with common tail O, then the angle $\angle AOB$ is a right angle.

Choosing a basis

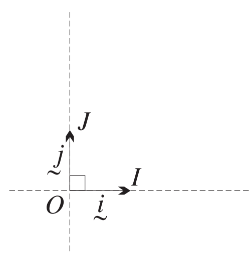


Choose a fixed *basis* consisting of two vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ such that:

- $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are both unit vectors,
- $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are perpendicular.

The correct name is *orthonormal basis*, but we will not be dealing with any other types of basis, so we shall omit the qualification.

Choosing an origin and forming axes



In Section 8A we described choosing a reference point or origin O in the plane. Then each point P in the plane has position vector $\overrightarrow{OP}$, and each vector $\underset{\sim}{u}$, when drawn with tail O as $\underset{\sim}{u} = \overrightarrow{OA}$, is the position vector of the head A.

In particular, we can represent $\underset{\sim}{i} = \overrightarrow{OI}$ and $\underset{\sim}{j} = \overrightarrow{OJ}$ as position vectors, thus forming a pair of *axes* at right angles. Each axis then becomes a number line by assigning 0 to the origin O and 1 to the point I and J.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

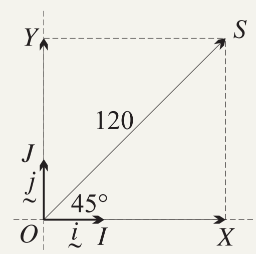
7 CHOOSING A BASIS AND AN ORIGIN, AND FORMING AXES

- Choose a *basis* consisting of perpendicular unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.
- Choose a point O as the *origin*.
- The basis vectors can now be drawn as position vectors $\underset{\sim}{i} = \overrightarrow{OI}$ and $\underset{\sim}{j} = \overrightarrow{OJ}$.
- The basis and the origin thus form a pair of *axes* OI and OJ.

The next worked example prepares the way for the central idea of components.

Example 5 8B

A ship S leaves a port O and sails north-east at 20 km/h. Choose a convenient origin and axes. Then describe its position after 6 hours by giving its position vector as a sum of multiples of the basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.



SOLUTION

The obvious choice of origin is the port O, and one obvious choice of basis vectors is to take $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the directions east and north. Then after 6 hours, the ship is 120 km from O.
Complete the rectangle $OXSY$, where X lies on OI and Y lies on OJ.

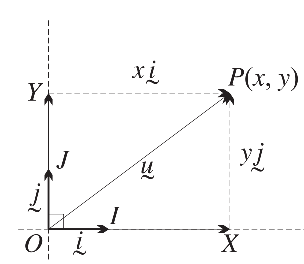
Then by simple trigonometry, $\overrightarrow{OX} = 60\sqrt{2}\underset{\sim}{i}$ and $\overrightarrow{OY} = 60\sqrt{2}\,\underset{\sim}{j}$.

The position vector $\overrightarrow{OS}$ is the vector sum of $\overrightarrow{OX}$ and $\overrightarrow{OY}$,

so $\qquad \overrightarrow{OS} = 60\sqrt{2}\underset{\sim}{i} + 60\sqrt{2}\,\underset{\sim}{j}$.

Note: There is no problem with exchanging the chosen basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ so that $\underset{\sim}{i}$ points north and $\underset{\sim}{j}$ points east — in fact this would probably conform better to the conventions of navigation. In mathematics, however, we tend to take a basis in which $\underset{\sim}{j}$ is the vector $\underset{\sim}{i}$ rotated 90° anti-clockwise.

The components of a vector



We can quickly generalise the construction in the previous worked example. Suppose that we have an origin and basis, then take any vector $\underset{\sim}{u}$, and represent it as a position vector $\underset{\sim}{u} = \overrightarrow{OP}$. Complete the rectangle $PXOY$, where X lies on OI and Y lies on OJ.

Let $\overrightarrow{OX} = x\underset{\sim}{i}$ and $\overrightarrow{OY} = y\underset{\sim}{j}$. Then x and y are called the *scalar components* of the vector $\underset{\sim}{u}$, and we can write

$$P = (x, y) \qquad \text{and} \qquad \underset{\sim}{u} = \overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}.$$

This last form $\underset{\sim}{u} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$ is called the *component form* of the vector $\underset{\sim}{u}$, and the terms $x\,\underset{\sim}{i}$ and $y\underset{\sim}{j}$ are called the *vector components* of $\underset{\sim}{u}$. The rectangle $PXOY$ represents taking the sum of the two vector components $x\,\underset{\sim}{i}$ and $y\underset{\sim}{j}$.

The unqualified term *component* may mean either scalar or vector component. It is usually clear from the context which is intended.

These components are independent of the choice of the origin O, because a translation from O to a different choice O' of origin moves one figure onto the other. A different choice of basis vectors, however, would change the components.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

8 THE COMPONENTS OF A VECTOR

Suppose that a basis and an origin have been chosen, and let $\underset{\sim}{u}$ be any vector, represented as a position vector $\underset{\sim}{u} = \overrightarrow{OP}$.

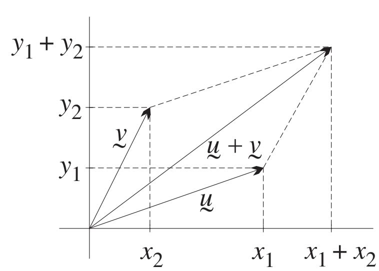
- Complete the rectangle $OXPY$, and let $\overrightarrow{OX} = x\underset{\sim}{i}$ and $\overrightarrow{OY} = y\underset{\sim}{j}$.
  Then x and y are called the *scalar components* of the vector $\underset{\sim}{u}$, and
  $$P = (x,\ y) \qquad \text{and} \qquad \underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j},$$
- The last form is called the *component form* of the vector $\underset{\sim}{u}$, and the vectors $\overrightarrow{OX} = x\underset{\sim}{i}$ and $\overrightarrow{OY} = y\underset{\sim}{j}$ are called the *vector components* of $\underset{\sim}{u}$.
- The rectangle $OXPY$ represents $\underset{\sim}{u}$ as the sum $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j}$.
- The components are independent of the choice of origin O, but depend very much on the choice of basis $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

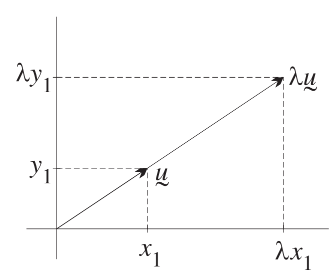
The point $P(x, y)$ naturally corresponds to the position vector $x\underset{\sim}{i} + y\underset{\sim}{j}$.

Two identities for components

Let $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ be vectors, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are a basis. Let λ be a scalar. Two identities are clear from the diagrams below them:

$$\underset{\sim}{u} + \underset{\sim}{v} = (x_1 + x_2)\,\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j} \qquad \text{and} \qquad \lambda\underset{\sim}{u} = (\lambda x_1)\,\underset{\sim}{i} + (\lambda y_1)\underset{\sim}{j}.$$





Alternatively, the identities can be proven using the distributive laws from Box 4 and the associative law from Box 3.

Column vector notation

Let $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ be a vector. We can now write $\underset{\sim}{u}$ as a *column vector*, placing its scalar components in a vertical stack,

$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad \text{or alternatively as} \qquad \underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

In column vector notation, the zero vector $\underset{\sim}{0}$ and the basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are

$$\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now let $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ be another vector and λ be a scalar.
The two identities for components above now become

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

and
$$\lambda\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda y_1 \end{bmatrix}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

We can therefore write the vector $\underset{\sim}{u}$ in terms of the basis vectors and components as

$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(which in turn equals } x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\text{).}$$

9 COLUMN VECTORS

- Each vector $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ can be written as a *column vector*
$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{or} \quad \underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$
- The basis vectors are $\underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Addition and multiplication by a scalar are done component-wise:
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \quad \text{and} \quad \lambda\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda y_1 \end{bmatrix}.$$
- The vector $\underset{\sim}{u}$ can be written as a sum of the basis vectors as
$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

There is a natural correspondence between a point $P(x, y)$ and its position vector $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$, that is, between $P(x, y)$ and $\begin{bmatrix} x \\ y \end{bmatrix}$.

Column vectors can be written with square brackets or with round brackets, that is, as $\begin{bmatrix} x \\ y \end{bmatrix}$ or as $\begin{pmatrix} x \\ y \end{pmatrix}$, but we have mostly avoided the round brackets notation because of possible confusion with ${}^{x}C_{y}$ in combinatorics, and because square brackets fit more neatly on a printed line.

Example 6 — 8B

a **i** Given $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$, find $2\underset{\sim}{u}$, $3\underset{\sim}{v}$, and $2\underset{\sim}{u} + 3\underset{\sim}{v}$.

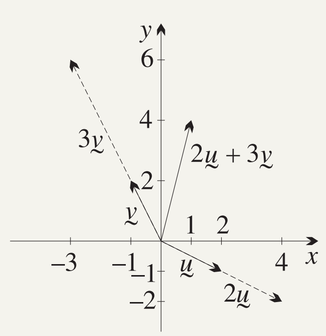
ii Draw all five vectors as position vectors. What figure do the origin and the heads of $2\underset{\sim}{u}$, $3\underset{\sim}{v}$, and $2\underset{\sim}{u} + 3\underset{\sim}{v}$ form?

b **i** Given $\underset{\sim}{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $\underset{\sim}{v} - \underset{\sim}{u}$.

ii Draw $\underset{\sim}{u} = \overrightarrow{OU}$ and $\underset{\sim}{v} = \overrightarrow{OV}$ as position vectors, and mark $\underset{\sim}{v} - \underset{\sim}{u}$.

SOLUTION

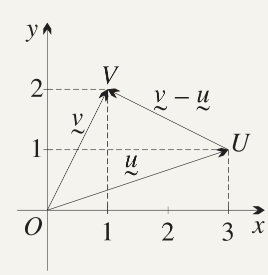
a



i $2\underset{\sim}{u} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$, $3\underset{\sim}{v} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$,
$2\underset{\sim}{u} + 3\underset{\sim}{v} = \underset{\sim}{i} + 4\underset{\sim}{j}$.

ii They form a parallelogram.

b



i $\underset{\sim}{v} - \underset{\sim}{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

ii $\underset{\sim}{v} - \underset{\sim}{u}$ has tail U and head V.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 7 8B

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -5\underset{\sim}{i} + 4\underset{\sim}{j}$. Find λ and μ so that

$$\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} = -4\underset{\sim}{i} - \underset{\sim}{j}.$$

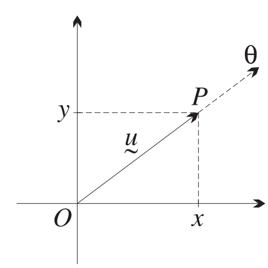
SOLUTION

We need simultaneous equations for the solution.

$$\begin{array}{lrcll}
\text{First,} & \lambda\underset{\sim}{a} + \mu\underset{\sim}{b} & = & (2\underset{\sim}{i} - 3\underset{\sim}{j}) + \mu(-5\underset{\sim}{i} + 4\underset{\sim}{j}) & \\
 & & = & (2\lambda - 5\mu)\underset{\sim}{i} + (-3\lambda + 4\mu)\underset{\sim}{j}). & \\
\text{Hence} & 2\lambda - 5\mu & = & -4 & (1) \\
\text{and} & -3\lambda + 4\mu & = & -1. & (2) \\
\text{Taking } (1) \times 4, & 8\lambda - 20\mu & = & -16 & (1A) \\
\text{and } (2) \times 5, & -15\lambda + 20\mu & = & -5. & (2A) \\
\text{Adding } (1A) + (2A), & -7\lambda & = & -21 & \\
 & \lambda & = & 3, & \\
\text{and substituting into } (2), & -9 + 4\mu & = & -1 & \\
 & \mu & = & 2. &
\end{array}$$

Length and angle

When there are an origin and axes, we can import all the trigonometry from Chapter 6 of the Year 11 book. Let $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a non-zero vector, represented in the diagram to the right as a position vector $\overrightarrow{OP}$, and let θ be the angle of the ray OP.



- The length and angle are given by Pythagoras' theorem and trigonometry,

$$|\underset{\sim}{u}|^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

  There will be two answers for θ, oriented in opposite directions, so the quadrant will need to be identified to distinguish between them.
- Conversely, the components of the vector can be recovered from θ and $|\underset{\sim}{u}|$,

$$x = |\underset{\sim}{u}|\cos\theta \quad \text{and} \quad y = |\underset{\sim}{u}|\sin\theta.$$

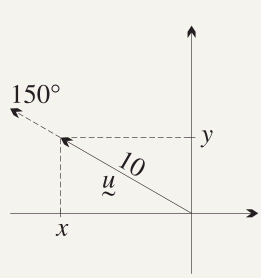
Example 8 8B

a Find the vector $\underset{\sim}{u}$ of length 10 whose position vector has angle 150°.

b Find the length and angle (nearest degree) of $\underset{\sim}{v} = -\underset{\sim}{i} - 2\underset{\sim}{j}$.

SOLUTION

a

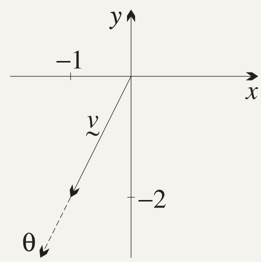


$$\begin{aligned}
\underset{\sim}{u} &= (10\cos 150°)\underset{\sim}{i} + (10\sin 150°)\underset{\sim}{j} \\
&= 10 \times \left(-\frac{\sqrt{3}}{2}\right)\underset{\sim}{i} + 10 \times \tfrac{1}{2}\underset{\sim}{j} \\
&= -5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}
\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b



$$|\underset{\sim}{v}|^2 = (-1)^2 + (-2)^2 = 5$$
$$|\underset{\sim}{v}| = \sqrt{5}$$
$$\tan\theta = \frac{-2}{-1} = 2$$
$$\theta \doteqdot 63° + 180° \quad \text{(third quadrant)}$$
$$\doteqdot 243°$$

10 LENGTH AND ANGLE OF POSITION VECTORS

Let $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a non-zero position vector relative to a chosen origin and basis, and let $\underset{\sim}{u}$ have trigonometric angle θ. Then

- $x = |\underset{\sim}{u}| \cos\theta$ and $y = |\underset{\sim}{u}| \sin\theta$,
- $|\underset{\sim}{u}|^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$ (the two answers for θ need to be distinguished).

Exercise 8B

FOUNDATION

1 If $\underset{\sim}{a} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$, find:

a $|\underset{\sim}{a}|$ **b** $2\underset{\sim}{a}$ **c** $|2\underset{\sim}{a}|$ **d** $-5\underset{\sim}{a}$ **e** $|-5\underset{\sim}{a}|$

2 Suppose that $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$. Find:

a $\underset{\sim}{a} + \underset{\sim}{b}$ **b** $|\underset{\sim}{a} + \underset{\sim}{b}|$ **c** $\underset{\sim}{a} - \underset{\sim}{b}$ **d** $|\underset{\sim}{a} - \underset{\sim}{b}|$ **e** $-3\underset{\sim}{a} - 2\underset{\sim}{b}$ **f** $|-3\underset{\sim}{a} - 2\underset{\sim}{b}|$

3 Given that $\underset{\sim}{a} = \begin{bmatrix} -17 \\ 3 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} -7 \\ -13 \end{bmatrix}$, find:

a $\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}$ **b** $|\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}|$ **c** $-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c}$ **d** $|-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c}|$

4 **a** For $\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$, write $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{u} + \underset{\sim}{v}$ as column vectors, and sketch them as position vectors.

b For $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, write $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ in component form and sketch them as position vectors. Also sketch $\underset{\sim}{a} - \underset{\sim}{b}$ as the vector subtraction of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

5 If $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{v} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v}$, find:

a $\hat{\underset{\sim}{u}}$ **b** $\hat{\underset{\sim}{v}}$ **c** $\hat{\underset{\sim}{w}}$

6 Given $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -12 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -15 \\ -8 \end{bmatrix}$, show that:

a $|2\underset{\sim}{a}| = 2|\underset{\sim}{a}|$ **b** $|-\underset{\sim}{b}| = |\underset{\sim}{b}|$ **c** $|\underset{\sim}{a} + \underset{\sim}{b}| < |\underset{\sim}{a}| + |\underset{\sim}{b}|$ **d** $|\underset{\sim}{a} + \underset{\sim}{b}| > \big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big|$

7 **a** Test whether the points P, Q and R are collinear if

$$\overrightarrow{OP} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \quad\text{and}\quad \overrightarrow{OQ} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad\text{and}\quad \overrightarrow{OR} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}.$$

b Test whether the points A, B and C are collinear if

$$\overrightarrow{OA} = 3\underset{\sim}{i} + 8\underset{\sim}{j} \quad\text{and}\quad \overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j} \quad\text{and}\quad \overrightarrow{OC} = -4\underset{\sim}{i} - \underset{\sim}{j}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{m}$ be the respective position vectors representing the points $A(4,-7)$, $B(6,3)$ and M, where M is the midpoint of AB. Write in component form:

a $\underset{\sim}{a}$ **b** $\underset{\sim}{b}$ **c** $\underset{\sim}{m}$

9 Suppose that P is the point $(4,-1)$, Q is the point $(-3,5)$ and O is the origin. Write as column vectors:

a $\overrightarrow{OP}$ **b** $2\,\overrightarrow{OP} - \overrightarrow{OQ}$ **c** $\overrightarrow{PQ}$ **d** $\overrightarrow{QP}$

10 If A and B are the points $(1,-1)$ and $(7,3)$ respectively, find:

a $\overrightarrow{AB}$ in component form,
b $|AB|$,
c the unit vector in the direction of $\overrightarrow{AB}$.

DEVELOPMENT

11 Find the magnitude and direction (as an angle of inclination) of each vector.

a $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$ **b** $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ **c** $\underset{\sim}{c} = -3\sqrt{3}\underset{\sim}{i} + 3\underset{\sim}{j}$ **d** $\underset{\sim}{d} = \begin{bmatrix} -\sqrt{6} \\ -\sqrt{6} \end{bmatrix}$

12 Write, in component form, a vector whose magnitude and direction are:

a 4 and $-\frac{\pi}{4}$, **b** $2\sqrt{6}$ and $\frac{2\pi}{3}$, **c** 2 and $-\frac{5\pi}{6}$, **d** $2\sqrt{2}$ and $\frac{5\pi}{12}$.

13 Given that $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$, find the values of λ_1 and λ_2 such that $\underset{\sim}{c} = \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b}$.

14 The triangle ABC has vertices $A(2\sqrt{3},3)$, $B(3\sqrt{3},4)$ and $C(2\sqrt{3},5)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{CB}$ as column vectors.
b Hence find $|\overrightarrow{AB}|$ and $|\overrightarrow{CB}|$.
c What type of special triangle is ABC? Why?

15 The quadrilateral $ABCD$ has vertices $A(-7,-5)$, $B(8,-2)$, $C(10,9)$ and $D(-5,6)$. Show that $\overrightarrow{AD} = \overrightarrow{BC}$, and hence explain why $ABCD$ is a parallelogram.

16 The quadrilateral $PQRS$ has vertices $P(-3,-4)$, $Q(2,-2)$, $R(4,3)$ and $S(-1,1)$.

a Show that $\overrightarrow{PQ} = \overrightarrow{SR}$.
b Show that $|\overrightarrow{PQ}| = |\overrightarrow{QR}|$.
c What type of special quadrilateral is $PQRS$? Give reasons for your answer.

17 The quadrilateral $OABC$ has vertices $O(0,0)$, $A(5,-3)$, $B(7,4)$ and $C(2,7)$.

a Show that $\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{OB}$.
b What type of special quadrilateral is $OABC$? Give a reason for your answer.

18 The parallelogram $WXYZ$ has vertices $W(-6,4)$, $X(6,2)$, $Y(4,9)$ and $Z(a,b)$. Use vectors to find the values of a and b.

ENRICHMENT

19 The points A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. Find the three possible position vectors representing the point D if the points A, B, C and D are the vertices of a parallelogram.

ISBN 978-1-108-76630-2

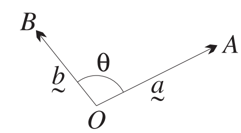
Cambridge University Press
Updated July 2022

8C The dot product (or scalar product)

This section introduces an important product associated with vectors. The *dot product* or *scalar product* $\underset{\sim}{a} \cdot \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is a scalar — the dot here is a raised dot. The dot product is closely associated with the cosine rule.

In Section 8E, we will introduce *projections*, and see that the dot product is also closely related to the projection of one vector onto another.

The angle between two non-zero vectors



To define the *angle between two non-zero vectors* $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$, represent the vectors as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with the same tail. The non-reflex angle $\angle AOB$ is taken as the angle between the vectors.

This definition is independent of the choice of O, because if P is any other point, then the translation from O to P maps the whole figure across.

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be non-zero vectors with angle θ between them. Here are some immediate consequences of the definition.

- If λ is a positive scalar, then the angle between $\lambda\underset{\sim}{a}$ and $\underset{\sim}{a}$ is $0°$.
- If λ is a negative scalar, then the angle between $\lambda\underset{\sim}{a}$ and $\underset{\sim}{a}$ is $180°$.
- $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction if and only if the angle between them is $0°$.
- $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have opposite directions if and only if the angle between them is $180°$.
- The angle between $-\underset{\sim}{a}$ and $-\underset{\sim}{b}$ is θ.
- The angle between $\underset{\sim}{a}$ and $-\underset{\sim}{b}$, and between $-\underset{\sim}{a}$ and $\underset{\sim}{b}$, is $180° - \theta$.
- The angle between the unit vectors $\hat{\underset{\sim}{a}}$ and $\hat{\underset{\sim}{b}}$ is θ.

There is no meaning to the angle between a vector and the zero vector.

11 THE ANGLE BETWEEN TWO NON-ZERO VECTORS

Represent two non-zero vectors $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with a common tail. Then the non-reflex angle $\angle AOB$ is called the *angle between the two vectors*.

The dot product or scalar product — geometric formula

There are two equivalent formulae for the dot product of two vectors. We will begin with the geometric formula using the cosine function, and then develop an alternative formula using the components of the vectors.

The *dot product* or *scalar product* of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined geometrically using the same diagram that was used above for the angle between two vectors.

- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are both non-zero vectors, then $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is the product of their lengths times the cosine of the angle between them,
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \times \cos\theta, \quad \text{where } \theta \text{ is the angle between the vectors.}$$
- If either a or b is the zero vector, then the dot product is zero,
$$\underset{\sim}{a} \cdot \underset{\sim}{0} = 0 \quad \text{and} \quad \underset{\sim}{0} \cdot \underset{\sim}{b} = 0.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Hence for non-zero vectors, $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is positive if and only if θ is acute or $0°$, and $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is negative if and only if θ is obtuse or $180°$.

Notice also that $\cos 90° = 0$, so the dot product of two perpendicular vectors is zero. Most importantly, the dot product of two basis vectors is zero.

12 THE DOT PRODUCT OR SCALAR PRODUCT OF TWO VECTORS — GEOMETRIC FORMULA

- Define the dot or scalar product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as follows.
  — If both vectors are non-zero and the angle between them is θ, define
  $$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \times \cos\theta\,.$$
  — If $\underset{\sim}{a}$ or $\underset{\sim}{b}$ is the zero vector, define $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.
- For all vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and all scalars λ,
  Perpendicular: $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ if and only if $\underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$ or $\underset{\sim}{a} \perp \underset{\sim}{b}$.
  Length: $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$, which means that $|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$.
  Commutative law: $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$
  Associative law: $\lambda\,(\underset{\sim}{a} \cdot \underset{\sim}{b}) = (\lambda \underset{\sim}{a}) \cdot \underset{\sim}{b}$
  Distributive law: $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

The second dotpoint contains five important laws for the dot product. The first four are all immediate consequences of the definition. The last is a distributive law, which will be easier to prove after we have developed the component formula.

The product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ has two names — *dot product* and *scalar product*.
- It is called the *dot product* because the notation is a raised dot.
- It is called *scalar product* because the answer is a scalar.

(Naming $\underset{\sim}{u} \cdot \underset{\sim}{v}$ the 'dot product' after its notation links it with another vector product called the 'cross product' $\underset{\sim}{u} \times \underset{\sim}{v}$, which you will meet in later years.)

Be careful: If you decide to use the term 'scalar product', do not confuse $\lambda\underset{\sim}{v}$, which is 'multiplication by a scalar', with the 'scalar product' $\underset{\sim}{u} \cdot \underset{\sim}{v}$.

Example 9 8C

An isosceles right-angled triangle $\triangle ABC$ has sides of length $|AB| = 3\sqrt{2}$ and $|BC| = |CA| = 3$. Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\underset{\sim}{c} = \overrightarrow{CA}$.

a Find:
i $\underset{\sim}{a} \cdot \underset{\sim}{a}$ **ii** $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **iii** $3\underset{\sim}{a} \cdot (-2\underset{\sim}{c})$ **iv** $\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

b Without expanding the brackets, find $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$.

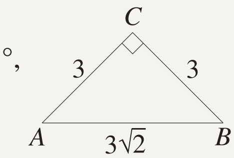
c Without calculating LHS and RHS, explain why $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

We use the formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}| \cos\theta$. Notice that the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $135°$, not $45°$, and $\cos 135° = -\dfrac{1}{\sqrt{2}}$. Similarly, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{c}$ is $135°$.



a **i** $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$
$\qquad = 18$

ii $\underset{\sim}{a} \cdot \underset{\sim}{b} = 3\sqrt{2} \times 3 \times \cos 135°$
$\qquad = -9$

iii $3\underset{\sim}{a} \cdot (-2\underset{\sim}{c}) = -6 \times 3\sqrt{2} \times 3 \times \cos 135°$
$\qquad = 54$

iv $\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{c} \cdot \underset{\sim}{a}$
$\qquad = 0 - (-9)$
$\qquad = 9$

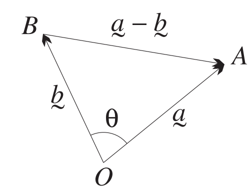
b Because $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, it follows that $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}).(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{0} \cdot \underset{\sim}{0} = 0$.

c $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c}$
$\qquad = \underset{\sim}{a} \cdot \underset{\sim}{c}$, because $\underset{\sim}{b} \perp \underset{\sim}{c}$.

The dot product, the cosine rule and Pythagoras' theorem

Let OAB be a triangle, and let $\theta = \angle AOB$.
Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$. Then the third side represents the vector $\underset{\sim}{a} - \underset{\sim}{b} = \overrightarrow{BA}$.
The cosine rule tells us that

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA|\,|OB| \cos\theta$$
$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$$



Replacing $|\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$ by $\underset{\sim}{a} \cdot \underset{\sim}{b}$,

$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}.$$

This can be solved for $\underset{\sim}{a} \cdot \underset{\sim}{b}$ to obtain another formula for the dot product,

$$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a} - \underset{\sim}{b}|^2.$$

We remarked in Section 6J of the Year 11 book that the cosine rule can be regarded as 'Pythagoras' theorem with an error term'. This last formula now characterises the dot product as half that error term.

13 THE DOT PRODUCT, THE COSINE RULE AND PYTHAGORAS' THEOREM

The cosine rule can be written in vector form as

$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}.$$

This can be solved for $\underset{\sim}{a} \cdot \underset{\sim}{b}$ to obtain another formula for the dot product,

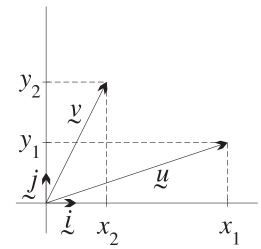
$$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a} - \underset{\sim}{b}|^2.$$

Thus the dot product is half the error term in Pythagoras' theorem.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The dot product — component formula



Now suppose that we have a basis $\underset{\sim}{i}$ and $\underset{\sim}{j}$ and an origin O, and that

$$\underset{\sim}{u} = x_1\,\underset{\sim}{i} + y_1\,\underset{\sim}{j} \qquad \text{and} \qquad \underset{\sim}{v} = x_2\,\underset{\sim}{i} + y_2\,\underset{\sim}{j}.$$

Using the formula in Box 13 above,

$$2\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u} - \underset{\sim}{v}|^2,$$

and applying Pythagoras' theorem in the form of the distance formula,

$$\begin{aligned} 2\underset{\sim}{u} \,.\, \underset{\sim}{v} &= \left(x_1{}^2 + y_1{}^2\right) + \left(x_2{}^2 + y_2{}^2\right) - \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right) \\ &= 2x_1x_2 + 2y_1y_2 \qquad \text{(after expanding)}, \end{aligned}$$

so $\quad \underset{\sim}{u} \,.\, \underset{\sim}{v} = x_1x_2 + y_1y_2. \qquad\qquad (*)$

In words, take the product of the scalar components in the $\underset{\sim}{i}$ direction, and add the product of the scalar components in the $\underset{\sim}{j}$ direction.

In column vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1x_2 + y_1y_2.$

Conversely, we could have taken the component formula as the definition of the dot product, and worked backwards through the working above to prove the original geometric formula.

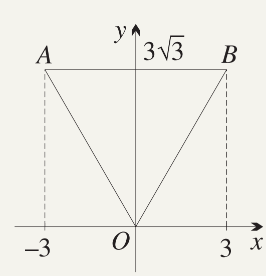
- If you are working with a basis $\underset{\sim}{i}$ and $\underset{\sim}{j}$, use the component formula.
- If you are working in a plane with no obvious basis, use the geometric form (or perhaps the error-in-Pythagoras'-theorem form).

14 THE DOT PRODUCT OR SCALAR PRODUCT — COMPONENT FORM

Let $\underset{\sim}{u} = x_1\,\underset{\sim}{i} + y_1\,\underset{\sim}{j}$ and $\underset{\sim}{v} = x_2\,\underset{\sim}{i} + y_2\,\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are a basis.

- The dot product can be written using components as
  $$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2\,.$$
- Alternatively, using column vector notation,
  $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1x_2 + y_1y_2\,.$$
- This component form of the dot product is equivalent to the geometric form, and either can be taken as the definition of the dot product.

Example 10 8C



The diagram to the right shows a triangle OAB. Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ be the position vectors of A and B.

a Find $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\overrightarrow{AB}$ as column vectors.

b Using the component formula for the dot product, find:

i $\underset{\sim}{a} \cdot \underset{\sim}{a}$ **ii** $\underset{\sim}{a} \cdot \underset{\sim}{b}$

iii $\underset{\sim}{a} \cdot \overrightarrow{AB}$ **iv** $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \overrightarrow{AB}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 3\sqrt{3} \end{bmatrix}$, $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

b i $\underset{\sim}{a} \cdot \underset{\sim}{a} = \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix}$
$= 9 + 27$
$= 36.$

ii $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3\sqrt{3} \end{bmatrix}$
$= -9 + 27$
$= 18.$

iii $\underset{\sim}{a} \cdot \overrightarrow{AB} = \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix}$
$= -18 + 0$
$= -18.$

iv $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \overrightarrow{AB} = \begin{bmatrix} 0 \\ 6\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix}$
$= 0 + 0$
$= 0.$

Note: We suggest that the reader now prove that $\triangle ABC$ is equilateral, and then recalculate each part using the geometric formula for the dot product.

Proving the distributive law for dot product

We can now use the components and column vector notation to prove the distributive law stated in Box 12.

Let $\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ and $\underset{\sim}{w} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$.

$$\begin{aligned} \text{Then } \underset{\sim}{u} \cdot (\underset{\sim}{v} + \underset{\sim}{w}) &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 + x_3 \\ y_2 + y_3 \end{bmatrix} \\ &= x_1(x_2 + x_3) + y_1(y_2 + y_3) \\ &= (x_1x_2 + x_1x_3) + (y_1y_2 + y_1y_3) \\ &= (x_1x_2 + y_1y_2) + (x_1x_3 + y_1y_3) \\ &= \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}. \end{aligned}$$

Writing the components of a vector using the dot product

First, the basis vectors each have length 1 and are perpendicular, so

$$\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = 1 \qquad \text{and} \qquad \underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{i} = 0.$$

Now we can write the scalar components of any vector u using the dot product.

Using the distributive law and the dot products of the basis vectors,

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{i} &= (x\underset{\sim}{i} + y\underset{\sim}{j}) \cdot \underset{\sim}{i} \\ &= x(\underset{\sim}{i} \cdot \underset{\sim}{i}) + y(\underset{\sim}{j} \cdot \underset{\sim}{i}) \\ &= x. \end{aligned} \qquad \text{and} \qquad \begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{j} &= (x\underset{\sim}{i} + y\underset{\sim}{j}) \cdot \underset{\sim}{j} \\ &= x(\underset{\sim}{i} \cdot \underset{\sim}{j}) + y(\underset{\sim}{j} \cdot \underset{\sim}{j}) \\ &= y. \end{aligned}$$

Thus the scalar components of $\underset{\sim}{u}$ are $x = \underset{\sim}{u} \cdot \underset{\sim}{i}$ and $y = \underset{\sim}{u} \cdot \underset{\sim}{j}$, and

$$\underset{\sim}{u} = (\underset{\sim}{u} \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{u} \cdot \underset{\sim}{j})\underset{\sim}{j}, \quad \text{that is,} \quad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u} \cdot \underset{\sim}{i} \\ \underset{\sim}{u} \cdot \underset{\sim}{j} \end{bmatrix}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 WRITING THE COMPONENTS OF A VECTOR USING THE DOT PRODUCT

- Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be a chosen basis. Then
$$\underset{\sim}{i}\cdot\underset{\sim}{i}=\underset{\sim}{j}\cdot\underset{\sim}{j}=1 \quad\text{and}\quad \underset{\sim}{i}\cdot\underset{\sim}{j}=\underset{\sim}{j}\cdot\underset{\sim}{i}=0\,.$$
- Let $\underset{\sim}{u}=x\underset{\sim}{i}+y\underset{\sim}{j}$ be any vector.
Then the scalar components of $\underset{\sim}{u}$ are $x=\underset{\sim}{u}\cdot\underset{\sim}{i}$ and $y=\underset{\sim}{u}\cdot\underset{\sim}{j}$, and
$$\underset{\sim}{u}=(\underset{\sim}{u}\cdot\underset{\sim}{i})\underset{\sim}{i}+(\underset{\sim}{u}\cdot\underset{\sim}{j})\underset{\sim}{j},\quad\text{that is,}\quad \underset{\sim}{u}=\begin{bmatrix}\underset{\sim}{u}\cdot\underset{\sim}{i}\\ \underset{\sim}{u}\cdot\underset{\sim}{j}\end{bmatrix}.$$

Exercise 8C

FOUNDATION

1 Find $\underset{\sim}{a}\cdot\underset{\sim}{b}$ given:

a $\underset{\sim}{a}=\begin{bmatrix}3\\1\end{bmatrix}, \underset{\sim}{b}=\begin{bmatrix}2\\4\end{bmatrix}$

b $\underset{\sim}{a}=\begin{bmatrix}-8\\-5\end{bmatrix}, \underset{\sim}{b}=\begin{bmatrix}6\\-14\end{bmatrix}$

c $\underset{\sim}{a}=\begin{bmatrix}6u\\-2v\end{bmatrix}, \underset{\sim}{b}=\begin{bmatrix}3v\\9u\end{bmatrix}$

d $\underset{\sim}{a}=\begin{bmatrix}x-1\\x-2\end{bmatrix}, \underset{\sim}{b}=\begin{bmatrix}x-1\\x+2\end{bmatrix}$

2 If θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, find $\underset{\sim}{a}\cdot\underset{\sim}{b}$ given:

a $|\underset{\sim}{a}|=6, |\underset{\sim}{b}|=5$ and $\theta=60°$,

b $|\underset{\sim}{a}|=4, |\underset{\sim}{b}|=3$ and $\theta=45°$.

3 Find the angle θ between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ (nearest degree if necessary) if

a $|\underset{\sim}{u}|=4, |\underset{\sim}{v}|=5$ and $\underset{\sim}{u}\cdot\underset{\sim}{v}=-10$,

b $|\underset{\sim}{u}|=3, |\underset{\sim}{v}|=5$ and $\underset{\sim}{u}\cdot\underset{\sim}{v}=12$.

4 Write down the value of:

a $4\underset{\sim}{i}\cdot 2\underset{\sim}{j}$

b $-5\underset{\sim}{i}\cdot 3\underset{\sim}{j}$

c $4\underset{\sim}{i}\cdot 2\underset{\sim}{i}$

d $-5\underset{\sim}{j}\cdot 3\underset{\sim}{j}$

5 Calculate the value of:

a $(4\underset{\sim}{i}+2\underset{\sim}{j})\cdot(4\underset{\sim}{i}+2\underset{\sim}{j})$

b $(-5\underset{\sim}{i}+3\underset{\sim}{j})\cdot(-5\underset{\sim}{i}+3\underset{\sim}{j})$

c $(4\underset{\sim}{i}+2\underset{\sim}{j})\cdot(-5\underset{\sim}{i}+3\underset{\sim}{j})$

6 Find $\underset{\sim}{u}\cdot\underset{\sim}{v}$ and hence determine in each part whether the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

a $\underset{\sim}{u}=\begin{bmatrix}-4\\5\end{bmatrix}, \underset{\sim}{v}=\begin{bmatrix}7\\6\end{bmatrix}$

b $\underset{\sim}{u}=\begin{bmatrix}-4\\-6\end{bmatrix}, \underset{\sim}{v}=\begin{bmatrix}18\\-12\end{bmatrix}$

c $\underset{\sim}{u}=\begin{bmatrix}-1\\a^{-2}\end{bmatrix}, \underset{\sim}{v}=\begin{bmatrix}a^{-1}\\a\end{bmatrix}$

DEVELOPMENT

7 Suppose that A, B and C are the points $(2, 5)$, $(5, 14)$ and $(-2, 13)$ respectively. It is known that the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $45°$.

a Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form.

b Find $\overrightarrow{AB}\cdot\overrightarrow{AC}$ using the result $\underset{\sim}{u}\cdot\underset{\sim}{v}=x_1x_2+y_1y_2$.

c Confirm your answer to part **b** using the result $\underset{\sim}{u}\cdot\underset{\sim}{v}=|\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$.

8 Suppose that P, Q and R are the points $(\sqrt{3}, 8)$, $(3\sqrt{3}, 14)$ and $(5\sqrt{3}, 12)$ respectively. It is known that the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $30°$.

a Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in component form.

b Find $\overrightarrow{PQ}\cdot\overrightarrow{PR}$ using the result $\underset{\sim}{u}\cdot\underset{\sim}{v}=x_1x_2+y_1y_2$.

c Confirm your answer to part **b** using the result $\underset{\sim}{u}\cdot\underset{\sim}{v}=|\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 If θ is the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, find the exact value of $\cos\theta$ given:

a $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}, \underset{\sim}{b} = 5\underset{\sim}{j}$ **b** $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **c** $\underset{\sim}{a} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$

10 Find the values of λ for which the vectors $\underset{\sim}{u} = \lambda^2\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} - (2 + 2\lambda)\underset{\sim}{j}$ are perpendicular.

11 a Given that $|\underset{\sim}{a}| = 6$, $|\underset{\sim}{b}| = 2$ and $\theta = \frac{\pi}{3}$, find:

i $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **ii** $2\underset{\sim}{a} \cdot (-5)\underset{\sim}{b}$ **iii** $4\underset{\sim}{a} \cdot 0\underset{\sim}{b}$

iv $\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ **v** $\underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ **vi** $(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

b Repeat part **a** if instead $\theta = \frac{2\pi}{3}$. **c** Repeat part **a** if instead $\theta = \frac{\pi}{2}$.

12 The quadrilateral $ABCD$ has vertices $A(-3, -6)$, $B(1, -4)$, $C(-2, 2)$ and $D(-6, 0)$.

a Show that $\overrightarrow{AB} = \overrightarrow{DC}$.

b Show that $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$.

c What type of special quadrilateral is $ABCD$? Give reasons for your answer.

13 The quadrilateral $PQRS$ has vertices $P(-8, 3)$, $Q(3, 7)$, $R(7, 18)$ and $S(-4, 14)$.

a Show that the diagonals PR and QS bisect each other by showing that $\frac{1}{2}\overrightarrow{PR} = \overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QS}$.

b Show that that the diagonals are perpendicular by showing that $\overrightarrow{PR} \cdot \overrightarrow{QS} = 0$.

c What type of special quadrilateral is $PQRS$? Give reasons for your answer.

14 Suppose that A, P and Q are the points $(-3, 3)$, $(2, 9)$ and $(10, 0)$ respectively.

a Write $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ as column vectors.

b Hence find $\angle PAQ$ correct to the nearest degree.

15 The quadrilateral $PQRS$ has vertices $P(1, 2)$, $Q(8, 3)$, $R(6, 13)$ and $S(4, 9)$. Use the scalar (that is, dot) product to find, correct to the nearest minute, the acute angle between the diagonals of the quadrilateral.

16 The point $P(r\cos\theta, r\sin\theta)$ varies on the circle $x^2 + y^2 = r^2$.
Let A and B be the points $(-r, 0)$ and $(r, 0)$ respectively.
Use the dot product to show that $\angle APB = 90°$, provided that $P \neq A$ or B.

17 Triangle ABC has vertices $A(2, 1)$, $B(10, 4)$ and $C(5, 13)$.

a Show that $\cos\angle ABC = \dfrac{13}{\sqrt{7738}}$.

b Find the exact value of $\sin\angle ABC$.

c Hence find the area of triangle ABC.

18 A triangle APB has area $10\ \text{u}^2$. Suppose that $\overrightarrow{PA} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\overrightarrow{PB} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $|\overrightarrow{PB}| = 4\sqrt{5}$.

a Show that $3a + b = 20$ or $3a + b = -20$.

b Hence find all the possibilities for $\overrightarrow{PB}$.

19 We claimed in Box 14 and in the text above it that the geometric formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$ for the dot product could be derived from the component formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2$, where $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are non-zero vectors with angle θ between them.

To prove this claim, let $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ be two vectors, drawn as position vectors $\underset{\sim}{u} = \overrightarrow{OA}$ and $\underset{\sim}{v} = \overrightarrow{OB}$, and let $\angle AOB = \theta$.

a Write down the cosine rule with AB^2 as the subject.

b Substitute expressions for the *squared lengths only* using the distance formula.

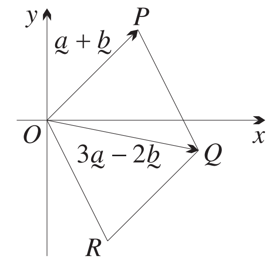
c Expand the brackets, and hence prove that $x_1x_2 + y_1y_2 = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

20 In the diagram to the right, the points P and Q have respective position vectors $\underset{\sim}{a} + \underset{\sim}{b}$ and $3\underset{\sim}{a} - 2\underset{\sim}{b}$, and $OPQR$ is a parallelogram.



a Express $\overrightarrow{PR}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Now suppose that $OPQR$ is a square. Use dot products to prove that $|\underset{\sim}{a}|^2 = 2|\underset{\sim}{b}|^2$.

21 The points A, B, C and D are the vertices of a quadrilateral $ABCD$, and have respective position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ relative to an origin O.

a State, in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$, a condition for the diagonals AC and BD of the quadrilateral to be:

i perpendicular, **ii** the same length.

b Suppose that $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, $\underset{\sim}{c} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ and $\underset{\sim}{d} = \begin{bmatrix} m \\ n \end{bmatrix}$. If the diagonals AC and BD are perpendicular and have the same length, find the possible values of m and n.

22 a Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be three vectors with sum $\underset{\sim}{0}$. Expand $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$ using the distributive law to prove that

$$|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 = -2(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} + c \cdot \underset{\sim}{a}).$$

Explain why the sum of three squared lengths appears to be a negative number.

b In $\triangle ABC$, let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\underset{\sim}{c} = \overrightarrow{CA}$. Use the form of the cosine rule given in Box 13 to prove the identity in part **a**.

c Calculate the LHS and RHS of the identity separately for:

i an equilateral triangle of side length 1,

ii a right-angled isosceles triangle whose equal sides have length 1,

iii a right-angled triangle with hypotenuse of length 2 and one side of length 1.

23 a In a quadrilateral $ABCD$, let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$, $\underset{\sim}{c} = \overrightarrow{CD}$ and $\underset{\sim}{d} = \overrightarrow{DA}$, so that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} = \underset{\sim}{0}$. Expand $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d})$ using the distributive law to prove that

$$\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{d} + \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{d} + \underset{\sim}{c} \cdot \underset{\sim}{d} = -\tfrac{1}{2}(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2).$$

b Evaluate all six terms on the LHS to confirm this result for:

i a rectangle with sides k and ℓ,

ii a parallelogram with sides k and ℓ and angle θ at A.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8D Geometric problems

In Years 7–10, you will have developed and proven a systematic sequence of theorems in Euclidean geometry. The development of those theorems was based on Pythagoras' theorem, parallel lines, congruence, and similarity, and moved systematically through triangles, quadrilaterals and circles.

It is possible to prove those same geometric theorems using vectors, again in a systematic sequence of theorems. This short course in vectors cannot do this, however, because some of those theorems have already been used when we were developing vectors, and we would therefore be in danger of circular arguments.

Instead, Exercise 8D is an unsystematic demonstration of the use of vectors to prove geometric theorems. It should be clear in each question which definition of the geometric object is being used, and which properties and tests are being assumed in the question.

The striking thing about this exercise is that the vector proofs are usually dramatically different from the traditional proofs of Euclidean geometry. Geometry is extremely subtle, and many contrasting approaches are possible, even to its fundamental objects.

Proving theorems using vectors

As in all mathematics, there are no rules that are guaranteed to lead to a proof, but the following principles are important.

- Always begin by drawing a diagram and labelling all the points.
- Then introduce vectors — it may be useful to choose one of the vertices, or a point outside the figure, as a reference point or origin.
- Use the sum and product of vectors to complete a triangle.
- Parallel vectors are multiples of each other.
- The dot product is zero when two non-zero vectors are at right angles.
- Otherwise the dot product may allow access to $\cos\theta$.

Try to make your diagram, and your choice of letters, as symmetric as possible.

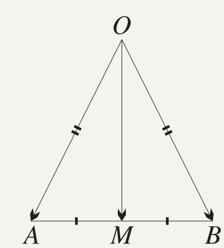
The three examples below, however, show that vector proofs of geometric theorems may be more elaborate than the Euclidean proof. It would be useful to compare the straightforward Euclidean proofs of these results, and perhaps of some others in Exercise 8D.

Example 11 8D

Prove that the line joining the apex of an isosceles triangle to the midpoint of the base is perpendicular to the base.

SOLUTION

Let $\triangle OAB$ be isosceles with $OA = OB$, and let M be the midpoint of AB.
Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, so that $|\underset{\sim}{a}| = |\underset{\sim}{b}|$.



$$\begin{aligned}
\text{Then}\quad \overrightarrow{OM} &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}\\
&= \underset{\sim}{a} + \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})\\
&= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}),\\
\text{so}\quad \overrightarrow{OM}\cdot\overrightarrow{AB} &= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})\cdot(\underset{\sim}{b} - \underset{\sim}{a})\\
&= \tfrac{1}{2}(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2)\\
&= 0, \quad \text{which proves that } OM \perp AB.
\end{aligned}$$

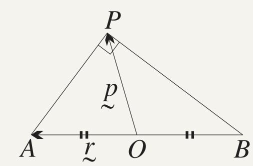
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 12 8D

An interval AB with centre O subtends a right angle at a point P. Prove that the circle with diameter AB passes through P.

SOLUTION



Let $\underset{\sim}{r} = \overrightarrow{OA}$, where O is the midpoint of AB, and let $\underset{\sim}{p} = \overrightarrow{OP}$.

Then $\overrightarrow{OB} = -\underset{\sim}{r}$, so $\quad \overrightarrow{BP} = \underset{\sim}{p} + \underset{\sim}{r}$,

and $\quad \overrightarrow{AP} = \underset{\sim}{p} - \underset{\sim}{r}$.

Because $AP \perp BP$, $\quad \overrightarrow{BP} \cdot \overrightarrow{AP} = 0$

$$(\underset{\sim}{p} + \underset{\sim}{r}) \cdot (\underset{\sim}{p} - \underset{\sim}{r}) = 0$$
$$\underset{\sim}{p} \cdot \underset{\sim}{p} - \underset{\sim}{p} \cdot \underset{\sim}{r} + \underset{\sim}{r} \cdot \underset{\sim}{p} - \underset{\sim}{r} \cdot \underset{\sim}{r} = 0$$
$$\underset{\sim}{p} \cdot \underset{\sim}{p} = \underset{\sim}{r} \cdot \underset{\sim}{r}$$
$$|\underset{\sim}{p}| = |\underset{\sim}{r}|$$

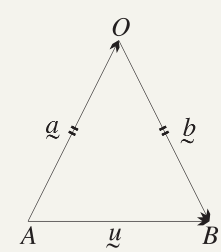
Hence $OP = OA = OB$, and the circle with diameter AB passes through P.

Note: This result is the converse of the well-known 'angle in a semi-circle' theorem proven in Question 7 of Exercise 8D.

Example 13 8D

Prove that the base angles of an isosceles triangle are equal.

SOLUTION



Let $\triangle OAB$ be isosceles with $OA = OB$.

Let $\underset{\sim}{a} = \overrightarrow{AO}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ and $\overrightarrow{AB} = \underset{\sim}{u}$.

Then $\underset{\sim}{u} = \underset{\sim}{a} + \underset{\sim}{b}$, so using the geometric dot product formula,

$$|\underset{\sim}{a}|\,|\underset{\sim}{u}| \cos A = \underset{\sim}{a} \cdot \underset{\sim}{u} = \underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$$
$$|\underset{\sim}{b}|\,|\underset{\sim}{u}| \cos B = \underset{\sim}{b} \cdot \underset{\sim}{u} = \underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}.$$

Subtracting, and using the fact that $|\underset{\sim}{b}| = |\underset{\sim}{a}|$, and so also that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$,

$$|\underset{\sim}{u}|\,|\underset{\sim}{a}|\,(\cos A - \cos B) = 0.$$

Hence $\cos A = \cos B$, so $A = B$ because cosine is one-to-one in the interval $[0, \pi]$.

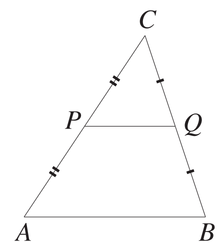
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 8D

FOUNDATION

1 In $\triangle ABC$ to the right, P is the midpoint of AC and Q is the midpoint of BC.
Let $\overrightarrow{AC} = \underset{\sim}{a}$ and $\overrightarrow{CB} = \underset{\sim}{b}$.

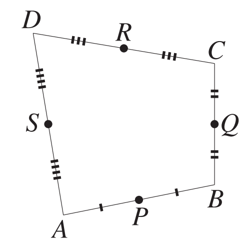


a Write $\overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Write $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

c Hence explain why $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$ and half its length.

2 In the diagram to the right, $ABCD$ is a quadrilateral. The points P, Q, R and S are the midpoints of AB, BC, CD and DA respectively.
Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$, $\overrightarrow{AD} = \underset{\sim}{d}$ and $\overrightarrow{DC} = \underset{\sim}{c}$.



a Explain why $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{d} + \underset{\sim}{c}$.

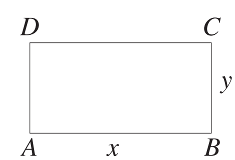
b Express $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

c Express $\overrightarrow{SR}$ in terms of $\underset{\sim}{d}$ and $\underset{\sim}{c}$.

d Hence show that $\overrightarrow{PQ} = \overrightarrow{SR}$.

e Deduce that the quadrilateral $PQRS$ is a parallelogram.

3 The rectangle $ABCD$ to the right has side lengths $|AB| = x$ and $|BC| = y$.
Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$ and $\overrightarrow{CD} = \underset{\sim}{c}$.



a Express $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and $\overrightarrow{BD}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

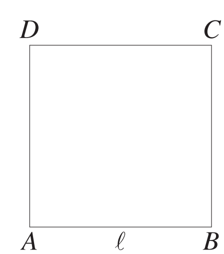
b What is the value of $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

c Write $\underset{\sim}{a} \cdot \underset{\sim}{a}$ in terms of x.

d Show that $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ and $(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} + \underset{\sim}{c})$ are both equal to $x^2 + y^2$.

e What have we proven in part **d** about the diagonals of a rectangle?

4 The square $ABCD$ to the right has side length ℓ.
Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$.



a Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

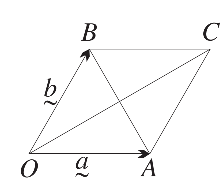
b What is the value of $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

c Write $\underset{\sim}{a} \cdot \underset{\sim}{a}$ in terms of ℓ.

d Find $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$.

e What have we proven in part **d** about the diagonals of a square?

5 In the diagram to the right, $OACB$ is a rhombus.
Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



a Why is $|\underset{\sim}{a}| = |\underset{\sim}{b}|$?

b By squaring the result in part **a**, show that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$.

c Express $\overrightarrow{OC}$ and $\overrightarrow{BA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

d Hence show that $\overrightarrow{OC} \cdot \overrightarrow{BA} = 0$.

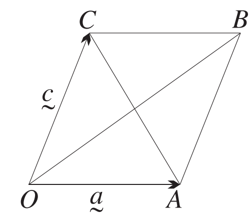
e What have we just proven about the diagonals of a rhombus?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

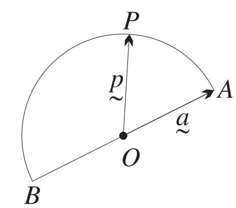
DEVELOPMENT

6 In the diagram to the right, $OABC$ is a parallelogram whose diagonals OB and AC are equal. The points A and C have respective position vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ relative to O.



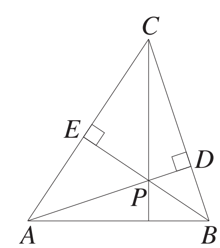
- **a** Explain why $\overrightarrow{CB} = \underset{\sim}{a}$.
- **b** Write $\overrightarrow{OB}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{a}$.
- **c** Write $\overrightarrow{AC}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{a}$.
- **d** Explain why $|\underset{\sim}{c} + \underset{\sim}{a}| = |\underset{\sim}{c} - \underset{\sim}{a}|$.
- **e** Use the result in part **d**, and the fact that $|\underset{\sim}{v}|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$, to show that $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$.
- **f** What can we now say about a parallelogram whose diagonals are equal?

7 In the diagram to the right, O is the centre of a semi-circle APB whose diameter is AB. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OP} = \underset{\sim}{p}$.



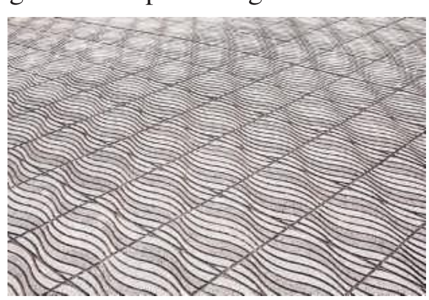
- **a** Write $\overrightarrow{OB}$ in terms of $\underset{\sim}{a}$.
- **b** Express $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{p}$.
- **c** Hence prove that $\angle APB = 90°$.
- **d** What circle geometry theorem have we just proven?

8 In the diagram to the right, AD and BE are altitudes of $\triangle ABC$ intersecting at P. Let $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{p}$ be the respective position vectors of A, B, C and P relative to an origin O.



- **a** Explain why $(\underset{\sim}{p} - \underset{\sim}{a}) \cdot (\underset{\sim}{c} - \underset{\sim}{b}) = 0$.
- **b** Explain why $(\underset{\sim}{p} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$.
- **c** Hence show that $(\underset{\sim}{p} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$.
- **d** Deduce that the three altitudes of a triangle are concurrent.

9 Prove, using vectors, that the diagonals of a parallelogram bisect each other.



10 Use vectors to prove that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides.

11 Suppose that $OABC$ is a parallelogram. Let M be the midpoint of OA and let P be the point of intersection of MC and OB. Prove, using vectors, that $OP = \frac{1}{3}OB$.

ENRICHMENT

12 Consider a triangle OAB, with P, Q and R the midpoints of OB, OA and AB respectively. Suppose that the medians AP and BQ intersect at C, and let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

- **a** Explain why $\overrightarrow{AC} = \lambda_1\left(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}\right)$ and $\overrightarrow{BC} = \lambda_2\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}\right)$, where λ_1 and λ_2 both lie between 0 and 1.
- **b** Hence prove that the three medians of a triangle are concurrent, and that C divides each median in the ratio $2:1$.

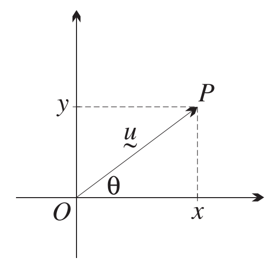
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8E Projections

In Section 8C on the dot product, we expressed a position vector $\underset{\sim}{u} = \overrightarrow{OP}$ as a sum of its vector components

$$\underset{\sim}{u} = (\underset{\sim}{u} \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{u} \cdot \underset{\sim}{j})\underset{\sim}{j}$$



where $x = \underset{\sim}{u} \cdot \underset{\sim}{i}$
and $y = \underset{\sim}{u} \cdot \underset{\sim}{j}$

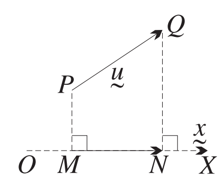
The necessary constructions were given in Section 8B and involved dropping perpendiculars to the horizontal and vertical axes. The constructions are examples of *projections* — in this case, projections onto the unit vector $\underset{\sim}{i}$ and the unit vector $\underset{\sim}{j}$.

This short section generalises the projection construction so that we can project any vector onto any other vector. Projection is a geometric idea, and its formulae can be expressed using the dot product.

Projections

Let $\underset{\sim}{x} = \overrightarrow{OX}$ be a non-zero reference vector.
We can *project* any vector $\underset{\sim}{u} = \overrightarrow{PQ}$ onto this vector $\underset{\sim}{x}$.

- Drop perpendiculars PM and QN to the line OX.
- The *projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$* is the vector $\overrightarrow{MN}$.
- We write the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ as $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN}$.



If $\underset{\sim}{u}$ is the zero vector, then N coincides with M, and the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ is $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN} = \underset{\sim}{0}$, the zero vector. Similarly, if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{OX}$ are perpendicular, then again M and N coincide and $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MM} = \underset{\sim}{0}$. In all other situations, the projection is non-zero.

16 PROJECTIONS

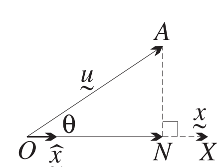
Let $\underset{\sim}{x} = \overrightarrow{OX}$ be a non-zero vector, and let $\underset{\sim}{u} = \overrightarrow{PQ}$ be any vector.

- To *project* $\underset{\sim}{u} = \overrightarrow{PQ}$ onto $\underset{\sim}{x}$, drop perpendiculars PM and QN to the line OX.
  The *projection* of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ is the vector $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN}$.

A better diagram for a geometric formula

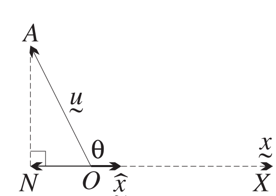
It is easier to introduce angles if we represent $\underset{\sim}{u}$ as $\underset{\sim}{u} = \overrightarrow{OA}$ with tail at O. Let θ be the angle between $\underset{\sim}{u}$ and $\underset{\sim}{x}$, then $ON = |\underset{\sim}{u}|\cos\theta$ using trigonometry. The unit vector in the direction OX is $\hat{\underset{\sim}{x}}$, so we can write

$$\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN} = \overrightarrow{ON} = \big(|\underset{\sim}{u}|\cos\theta\big)\,\hat{\underset{\sim}{x}}.$$



When θ is obtuse, then $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ points in the opposite direction from $\underset{\sim}{x}$. The formula takes this into account because then $\cos\theta$ is negative. See the diagram to the right.

The cases where $\theta = 0°$ or $\theta = 90°$ or $\theta = 180°$ or $\underset{\sim}{u} = \underset{\sim}{0}$ are left to the reader. In all cases, the projection vector $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ has length $|\underset{\sim}{u}|\ |\cos\theta|$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 PROJECTIONS — GEOMETRIC FORMULAE

Let $\underset{\sim}{u}$ and $\underset{\sim}{x} = \overrightarrow{OX}$ be non-zero vectors, and let θ be the angle between them. Represent $\underset{\sim}{u} = OA$ with tail at O, and drop the perpendicular AN to OX.

- The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ is the vector $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{ON}$.
- $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \left(|\underset{\sim}{u}|\cos\theta\right)\hat{\underset{\sim}{x}}$ and $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ has length $|\underset{\sim}{u}|\ |\cos\theta|$.

Projections and the dot product

Now we can combine projections with the dot product, using the definition of the dot product as $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\ |\underset{\sim}{v}|\cos\theta$.

First,
$$\begin{aligned}\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} &= (|\underset{\sim}{u}|\cos\theta)\,\hat{\underset{\sim}{x}}\\ &= (|u||\hat{\underset{\sim}{x}}|\cos\theta)\hat{\underset{\sim}{x}}, \text{ because } |\hat{\underset{\sim}{x}}| = 1,\\ &= (\underset{\sim}{u}\cdot\hat{\underset{\sim}{x}})\,\hat{\underset{\sim}{x}}. \qquad (1)\end{aligned}$$

Secondly,
$$\hat{\underset{\sim}{x}} = \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|},$$

so from (1),
$$\begin{aligned}\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} &= \frac{(\underset{\sim}{u}\cdot\underset{\sim}{x})\,\underset{\sim}{x}}{|\underset{\sim}{x}| \times |\underset{\sim}{x}|}\\ &= \frac{\underset{\sim}{u}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}} \times \underset{\sim}{x}. \qquad (2)\end{aligned}$$

Projections and the vector components

In particular, replacing the vector $\underset{\sim}{x}$ in equation (1) above by either of the two basis vectors $\underset{\sim}{i}$ or $\underset{\sim}{j}$ gives the previous formulae for the vector components of $\underset{\sim}{u}$, that is,

$$\text{proj}_{\underset{\sim}{i}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{i})\,\underset{\sim}{i} \text{ and } \text{proj}_{\underset{\sim}{j}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{j})\underset{\sim}{j}.$$

Hence the vector components of any vector $\underset{\sim}{u}$ are the projections onto $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

The 'completing the rectangle' construction that we used below Box 7 in Section 8B can now be interpreted as the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i}$ and the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{j}$.

18 PROJECTIONS AND THE DOT PRODUCT — VECTOR COMPONENTS

Let $\underset{\sim}{x} = \overrightarrow{OX}$ be a non-zero vector, and let $\underset{\sim}{u}$ be a vector.

- The projection $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ can be expressed using the dot product,
$$\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\hat{\underset{\sim}{x}})\,\hat{\underset{\sim}{x}} \qquad \text{and} \qquad \text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}} \times \underset{\sim}{x}$$
- Projection satisfies the two laws,
  — $\text{proj}_{\underset{\sim}{x}}(\lambda\underset{\sim}{u}) = \lambda(\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u})$, for any scalar λ.
  — $\text{proj}_{\underset{\sim}{x}}(\underset{\sim}{u} + \underset{\sim}{v}) = \text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} + \text{proj}_{\underset{\sim}{x}}\underset{\sim}{v}$, for any other vector $\underset{\sim}{v}$.
- The vector components of $\underset{\sim}{u}$ are the projections of $\underset{\sim}{u}$ onto $\underset{\sim}{i}$ and $\underset{\sim}{j}$,
$$\text{proj}_{\underset{\sim}{i}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{i})\underset{\sim}{i} \qquad \text{and} \qquad \text{proj}_{\underset{\sim}{j}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{j})\underset{\sim}{j}.$$

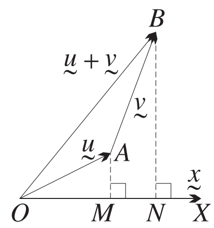
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The two identities of the second dotpoint follow easily from the formulae in the first dotpoint of Box 18.

In particular, the second of these two identities is demonstrated geometrically by the diagram to the right,

$$\text{proj}_{\underset{\sim}{x}}(\underset{\sim}{u}+\underset{\sim}{v}) = \text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} + \text{proj}_{\underset{\sim}{x}}\underset{\sim}{v}.$$



Example 14 8E

a Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, where $|OA| = 6$, $|OB| = 10$ and $\angle AOB = 60°$. Find $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$ as a multiple of $\underset{\sim}{b}$.

b Find the projection of $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ onto $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

SOLUTION

a
$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= (|\underset{\sim}{a}|\cos 60°)\,\hat{\underset{\sim}{b}}\\ &= 6 \times \tfrac{1}{2} \times \tfrac{1}{10}\underset{\sim}{b}\\ &= \tfrac{3}{10}\underset{\sim}{b}\end{aligned}$$

b
$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}\\ &= \frac{-3+12}{1+9} \times (-\underset{\sim}{i} + 3\underset{\sim}{j})\\ &= \tfrac{9}{10}(-\underset{\sim}{i} + 3\underset{\sim}{j})\\ &= -\tfrac{9}{10}\underset{\sim}{i} + \tfrac{27}{10}\underset{\sim}{j}\end{aligned}$$

Exercise 8E

FOUNDATION

1 Write down the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$ **b** $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{j}$ **c** $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$

2 Write down the length of the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$ **b** $\underset{\sim}{a} = -2\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{j}$ **c** $\underset{\sim}{a} = -6\sqrt{2}\underset{\sim}{i} + 8\sqrt{2}\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$

3 Write down the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \begin{bmatrix}2\\1\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}4\\0\end{bmatrix}$ **b** $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{j}$ **c** $\underset{\sim}{a} = \begin{bmatrix}5\\-3\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}-6\\0\end{bmatrix}$

4 Use trigonometry to find the length of the projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ if:

a $|\overrightarrow{OA}| = 6$ and $\angle AOB = 30°$, **b** $|\overrightarrow{OA}| = 6\sqrt{6}$ and $\angle AOB = 45°$.

DEVELOPMENT

5 Show that the projection of $\underset{\sim}{a} = \begin{bmatrix}10\\-2\end{bmatrix}$ onto $\underset{\sim}{b} = \begin{bmatrix}1\\-7\end{bmatrix}$ has length $\dfrac{12\sqrt{2}}{5}$.

6 Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}2\\2\end{bmatrix}$ **b** $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$ **c** $\underset{\sim}{a} = \begin{bmatrix}-5\\5\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}-6\\8\end{bmatrix}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 Find the component of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}, \underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j}$

b $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}, \underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

8 Find the magnitude of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = -2\underset{\sim}{i}, \underset{\sim}{b} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$

b $\underset{\sim}{a} = 6\underset{\sim}{i} - 4\underset{\sim}{j}, \underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

9 Find the projection of $\overrightarrow{AB}$ onto $-6\underset{\sim}{i} + 4\underset{\sim}{j}$ given $A(-3, -7)$ and $B(1, 5)$.

10 Find the length of the projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$ given $A(1, 3)$, $B(6, 18)$, $C(9, 4)$ and $D(19, 24)$.

11 Find the possible values of λ if the projection of $\lambda\underset{\sim}{i} + 4\underset{\sim}{j}$ onto $12\underset{\sim}{i} - 5\underset{\sim}{j}$ has length $\frac{140}{13}$.

12 Use the identities in the first dotpoint of Box 18 to prove the two identities in the second dotpoint.

ENRICHMENT

13 Let P be the point $(25, -5)$ and ℓ be the line $x + 3y + 10 = 0$.

a What is the gradient of ℓ?

b Hence write down a vector $\underset{\sim}{v}$ that is parallel to ℓ.

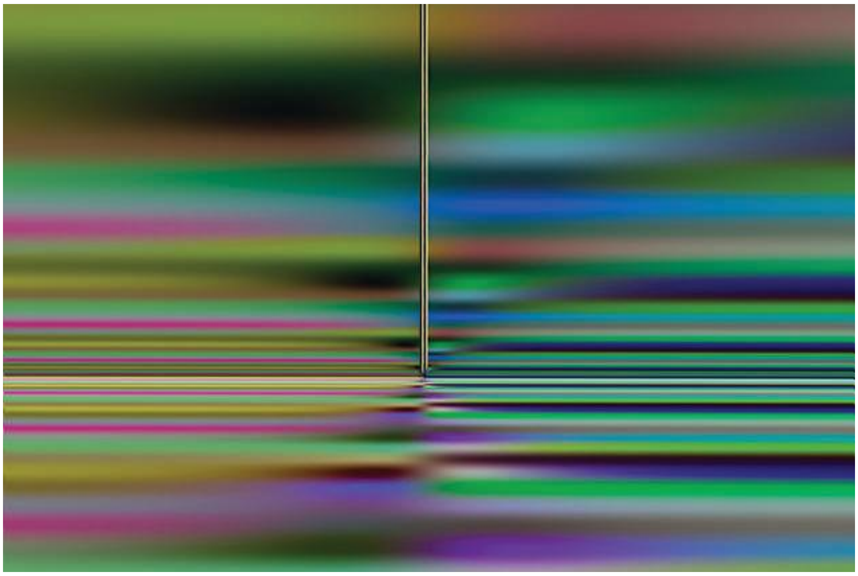
c Show that ℓ passes through the point $A(2, -4)$.

d Write down $\overrightarrow{AP}$ in component form.

e Find the length of the projection of $\overrightarrow{AP}$ onto $\underset{\sim}{v}$.

f Hence find the perpendicular distance from P to ℓ.

14 Use the approach in the previous question to prove that the perpendicular distance from the origin to the line $ax + by + c = 0$ is given by $\dfrac{|c|}{\sqrt{a^2 + b^2}}$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8F Applications to physical situations

Physics is full of problems in two- and three-dimensional space where vectors make the situation clearer. More advanced topics, such as electro-magnetic forces and waves, cannot be properly understood without vectors. At this stage, however, simple trigonometry will often solve problems quickly, and vectors may seem an unnecessary complication. The worked examples below use various methods, and each is done two ways. Readers should compare and contrast the methods used.

Without a great deal more physics, the only reasonable applications involve displacements, velocities, accelerations and forces. Some terminology is required, and some introduction to the relationship between force and acceleration.

Displacement and velocity

The first two worked examples involve only displacement and velocity.

Example 15 — 8F

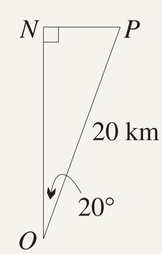
I walk 20 km in a direction N20°E. Find how far north I have gone:

a using a map of my journey,

b using projection vectors.

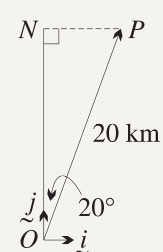
SOLUTION

Let the walk begin at O and end at P.
Let N be the point north of O and west of P.



a By trigonometry, $ON = 20\cos 20°$
$\qquad\qquad\doteqdot 18.8.$

b Using projections, let $\underset{\sim}{j}$ be a unit vector pointing north.



$$\text{Then}\qquad |\overrightarrow{ON}| = \text{proj}_{\underset{\sim}{j}}\overrightarrow{OP}$$
$$= (20\cos 20°)\underset{\sim}{j}$$
$$\doteqdot 18.8\underset{\sim}{j}\,.$$

Hence I have gone 18.8 km north.

Example 16 — 8F

A ship leaves port and sails north-east in a straight line at an angle to the straight north-south coastline. Its speed along the coast is 20 km/h, and its speed in the water is 25 km/h (there is no current). Find its direction of motion and its speed away from the coast:

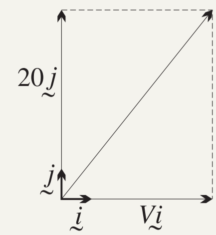
a using a velocity resolution diagram,

b using projection vectors.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION



Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors east and north, and let V be the speed away from the coast.

Then the ship's velocity vector is $\underset{\sim}{v} = V\underset{\sim}{i} + 20\underset{\sim}{j}$.

We know that $V^2 + 20^2 = 25^2$,

so $V = 15$,

and the ship's velocity vector is $15\underset{\sim}{i} + 20\underset{\sim}{j}$.

Let θ be the acute angle between the ship's direction and shoreline.

a Using trigonometry,
$$\cos\theta = \frac{20}{25}$$
$$\theta \doteqdot 37°.$$

b Using projections,
$$\begin{aligned}\text{velocity along the coast} &= \text{proj}_{\underset{\sim}{j}}\,\underset{\sim}{v}\\ 20\underset{\sim}{j} &= (|\underset{\sim}{v}|\cos\theta)\,\underset{\sim}{j}\\ 20\underset{\sim}{j} &= (25\cos\theta)\,\underset{\sim}{j}\\ \cos\theta &= \tfrac{20}{25}\\ \theta &\doteqdot 37°.\end{aligned}$$

Hence the ship is travelling about N37°E, leaving the coast at 15 km/h.

The resultant of two vectors

The sum $\underset{\sim}{u} + \underset{\sim}{v}$ of two vectors is often called the *resultant* of the two vectors. This terminology is used particularly for the sum of two forces, when the sum or resultant of two forces can be regarded as a single force acting on the object, as in the next worked example.

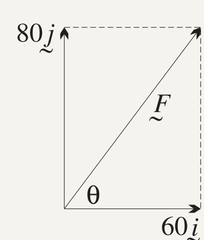
Example 17 8F

Peter and Paul are pulling a large box using ropes. They can never cooperate, and they end up pulling the box in different directions, Peter pulling east with a force of 60 newtons, and Paul pulling north with force of 80 newtons. Find the resultant force:

a using a forces diagram,

b using projection vectors.

SOLUTION



Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors east and north.

Then the resultant force is $\underset{\sim}{F} = 60\underset{\sim}{i} + 80\underset{\sim}{j}$,

so $|\underset{\sim}{F}| = 100$.

Let θ be the acute angle between $\underset{\sim}{i}$ and $\underset{\sim}{F}$.

a Using trigonometry,
$$\tan\theta = \tfrac{80}{60}$$
$$\theta \doteqdot 53°.$$

b Using projections,
$$\begin{aligned}\text{proj}_{\underset{\sim}{i}}\,\underset{\sim}{F} &= 60\underset{\sim}{i}\\ (100\cos\theta)\,\underset{\sim}{i} &= 60\\ \cos\theta &= \tfrac{60}{100}\\ \theta &\doteqdot 53°\end{aligned}$$

Hence the resultant force is about 100 newtons in a direction N37°E.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Forces and their units

Newton's second law of motion states that

$$F = ma, \qquad \text{meaning that} \qquad \text{force} = \text{mass} \times \text{acceleration}.$$

We will have a great deal more to say about acceleration in the next two chapters, but this law is needed now so that the units of force can be introduced — these units are called 'newtons', with symbol N, in honour of Sir Isaac Newton.

In words, Newton's second law says that if a body of mass m kg is accelerating at $a \text{ m/s}^2$, then the sum of all the forces acting on the body has magnitude $F = ma$ newtons, and acts in the same direction as the acceleration. Thus 1 newton is the force required to accelerate a body of mass 1 kg at 1 m/s^2.

There is a second unit of force called 'kilograms weight'. A mass of m kilograms that is free to move is pulled downwards by gravity with a force that accelerates it at about 10 m/s^2. The physical value has symbol g, and a better approximation is $g = 9.8 \text{ m/s}^2$ ($g = 9.832 \text{ m/s}^2$ at the poles and $g = 9.780 \text{ m/s}^2$ at the equator).

This means that the downwards gravitational force on a mass m kg is mg newtons.

In particular, because $g \doteqdot 10 \text{ m/s}^2$, a force of 1 newton is about the downwards force that you feel when you hold a 100 g apple in your open hand.

19 NEWTON'S SECOND LAW AND THE UNITS OF FORCE

- One newton, written in symbols as 1 N, is the force required to accelerate a body of mass 1 kg at 1 m/s^2.
- Newton's second law of motion says that
  $$F = ma.$$
  'If a body of mass m kg is accelerating at $a \text{ m/s}^2$, then the sum of all the forces acting on the body has magnitude $F = ma$ newtons, and acts in the same direction as the acceleration.'
- One kilogram weight is the downward force due to gravity on a mass of 1 kg at the Earth's surface.
- Acceleration due to gravity at the Earth's surface has the symbol g, whose approximate value is 9.8 m/s^2 (or 10 m/s^2 in round figures).
- One newton is therefore about $\frac{1}{10}$ kg weight — about the downward force due to gravity of a 100-gram apple on your open hand.

Example 18 8F

A parcel P of mass 8 kg is resting on a plane inclined at 30° to the horizontal.

a What component of the gravitation force is acting on the parcel in the direction down the plane? Answer the question:

i using a diagram of forces, **ii** using projection vectors.

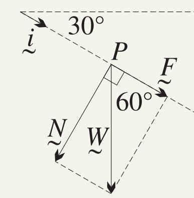
b What is the force in newtons, taking $g \doteqdot 9.8 \text{ m/s}^2$?

c What frictional force is acting on the parcel in the direction up the plane?

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

SOLUTION



Let $\underset{\sim}{W}$ be the weight of the parcel, represented as a vector down from P.

a i Regard the weight as the resultant of a force $\underset{\sim}{F}$ down the plane, and a force $\underset{\sim}{N}$ normal to the plane.

By simple trigonometry, $|\underset{\sim}{F}| = |\underset{\sim}{W}| \cos 60°$
$= 8 \times \frac{1}{2}$
$= 4.$

ii Let $\underset{\sim}{W}$ be the weight of the parcel, represented as a vector down from P.

Let $\underset{\sim}{i}$ be a unit vector down the plane.

Then the force down the plane is the projection of $\underset{\sim}{W}$ onto i,

$$\underset{\sim}{F} = \text{proj}_{\underset{\sim}{i}}\underset{\sim}{W}$$
$$= (\underset{\sim}{W} \cdot \underset{\sim}{i})\,\underset{\sim}{i} \quad \text{OR} \quad (|\underset{\sim}{W}| \cos 60°)\,\underset{\sim}{i}$$
$$= 4\underset{\sim}{i},$$

Hence the force down the plane is 4 kg weight.

b In newtons, this is about $4 \times 9.8 = 39.2$ N.

c The parcel is not accelerating (it is not even moving), so the frictional force is equal and opposite to the component of the weight acting down the plane. Hence the frictional force is 4 kg weight up the plane.

Exercise 8F

FOUNDATION

In this exercise take $g = 9.8\ \text{m/s}^2$.

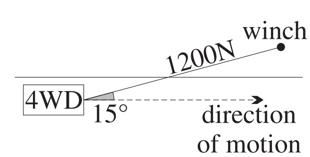
1 A ball is thrown at an angle of 30° to the horizontal with an initial speed of 20 m/s. Find the initial horizontal and vertical components of the velocity of the ball.

2 A particle has initial position vector $(4\underset{\sim}{i} + 5\underset{\sim}{j})$ metres. It moves with a constant velocity of $(3\underset{\sim}{i} - 2\underset{\sim}{j})$ m/s. Find its position vector after 7 seconds.

3 Find the magnitude of the resultant of the forces $(2\underset{\sim}{i} - 3\underset{\sim}{j})$ N, $(4\underset{\sim}{i} + \underset{\sim}{j})$ N and $(-3\underset{\sim}{i} + 3\underset{\sim}{j})$ N.

4 Two forces of magnitude 30 N and 16 N act away from a point P and are perpendicular. Find the magnitude and direction of the resultant force (measured from the 30 N force correct to the nearest degree).

5 In the diagram, a 4-wheel drive vehicle is bogged on a muddy road. A winch is pulling the vehicle with a force of 1200 N. The chain connecting the winch to the vehicle makes an angle of 15° with the direction of motion. Calculate, correct to the nearest newton, the magnitude of the component of the force:



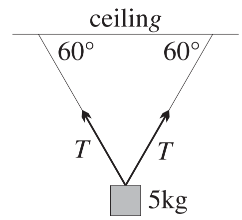
a in the direction of motion,

b perpendicular to the direction of motion.

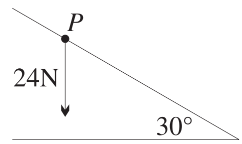
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 In the diagram, an object of mass 5 kg is suspended from a horizontal ceiling by two strings of equal length. Each string makes an angle of 60° with the ceiling. Calculate, correct to 3 significant figures, the equal tensions in the two strings.



7 The diagram shows an object of weight 24 N at rest at P on an inclined plane. Find the component of the weight:



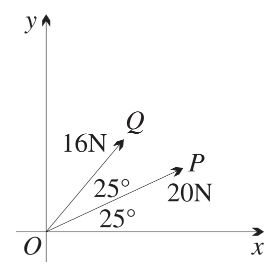
a down the plane,

b perpendicular to the plane.

8 In the diagram, the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent forces of magnitude 20 N and 16 N respectively.

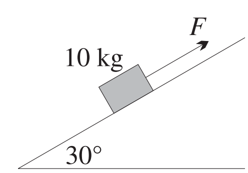
a Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in component form.

b Calculate, correct to 2 significant figures, the magnitude and direction of the resultant of the two forces.



DEVELOPMENT

9 In the diagram, an object of mass 10 kg is kept at rest on a smooth plane inclined at 30° to the horizontal by a force of F newtons acting parallel to the plane. Find the value of F.



10 A river is flowing at a speed of 1.5 m/s. Sam wants to row from point A on one bank to point B on the other bank directly opposite A. He intends to maintain a constant speed of 2.5 m/s. In what direction, correct to the nearest degree, should Sam row? Give your answer as an angle of inclination to the line AB.

11 Two dogs Brutus and Nitro are simultaneously tugging on a bone. Brutus is pulling with a force of 12 N in a direction 45° west of north, while Nitro is pulling with a force of 16 N in a direction 30° south of east. Calculate, correct to two significant figures, the magnitude and direction of the resultant force.

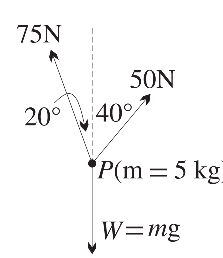
12 Three forces act on an object of mass 5 kg. These forces are represented by the vectors $9\underset{\sim}{i} - 2\underset{\sim}{j}$, $-3\underset{\sim}{i} + 10\underset{\sim}{j}$ and $18\underset{\sim}{i} - \underset{\sim}{j}$. Calculate the magnitude and direction of the acceleration of the object.

13 The position of a plane flying horizontally in a straight line at a constant speed is plotted on a radar screen. One unit on the screen represents 1 km in the air. At 12 noon the position vector of the plane is $40\underset{\sim}{i} + 16\underset{\sim}{j}$. Five minutes later its position vector is $33\underset{\sim}{i} + 40\underset{\sim}{j}$. Find:

a the position vector of the plane at 12:15 pm,

b the velocity of the plane as a vector in km/h.

14 The diagram shows an object of mass 5 kg being raised by forces of magnitude 75 N and 50 N.



a Find the weight of the object.

b Find, correct to the nearest newton, the magnitude of the resultant of the three forces acting on the object.

c Find, correct to the nearest degree, the angle this resultant makes with the upward vertical direction.

ISBN 978-1-108-76630-2

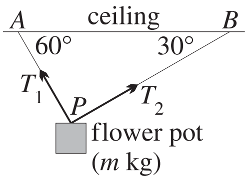
Cambridge University Press
Updated July 2022

15 David can row at 5 m/s in still water. He starts rowing from a point on the south bank of a river that is flowing due east at 3 m/s and steers the boat at 90° to the bank. He is also being blown by a wind from the north-east at 4 m/s.

a Express the velocity of the boat as a component vector.

b Hence find the speed of the boat, correct to 2 significant figures, and the bearing on which the boat is travelling correct to the nearest tenth of a degree.

16 In the diagram, a flowerpot of mass m kg is hung from a ceiling by two chains. Let the tensions in the chains AP and BP be T_1 and T_2 newtons respectively. The third force acting at P is the weight of the flowerpot.

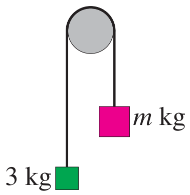


a By finding the horizontal component of the resultant of the three forces acting at P, show that $T_1 = \sqrt{3}\,T_2$.

b By finding the vertical component of the resultant of the three forces acting at P, show that $\sqrt{3}\,T_1 + T_2 = 19.6m$ newtons.

c Find the mass of the flowerpot, given that $T_2 = 98$ N.

17 The diagram shows objects of mass 3 kg and m kg attached to the ends of a light inextensible string that passes over a smooth pulley. The 3 kg object is accelerating at 4.9 m/s^2 upwards. Let the tension in the string be T newtons.

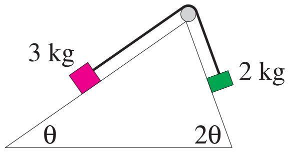


a Find the value of T.

b Find the value of m.

18 Two forces, of magnitude p newtons and q newtons, have a resultant of $2\sqrt{7}$ N when they act at 90° to each other. When they act at 30° to each other, however, the magnitude of the resultant is $2\sqrt{13}$ N. Find the values of p and q.

19 The diagram shows objects of mass 3 kg and 2 kg on connected smooth planes inclined at angles of θ and 2θ to the horizontal. The objects are attached to the ends of a light inextensible string that passes over a smooth pulley. Let T newtons be the tension in the string, and suppose that the 3 kg object is accelerating at a m/s^2 up its plane.



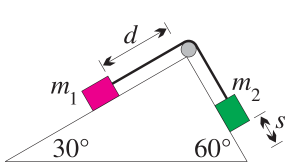
a Find, in terms of a, T, g and θ, an equation for the motion of the 3 kg object up its plane.

b Write down a similar equation for the motion of the 2 kg object down the other plane.

c Show that the system is in equilibrium when $\cos\ \theta = \frac{3}{4}$.

ENRICHMENT

20 In the diagram, objects of mass m_1 and m_2 are held at rest on adjoining smooth inclined planes. They are connected by a light inextensible string that passes over a smooth pulley.



a Show that when the objects are released, the object of mass m_1 will accelerate *towards the pulley* if $m_1 < \sqrt{3}\,m_2$.

b Assuming that the condition in part **a** is satisfied, show that the object of mass m_1 will hit the pulley with speed $\sqrt{\dfrac{dg(\sqrt{3}\,m_2 - m_1)}{m_1 + m_2}}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 8 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 8 Multiple-choice quiz

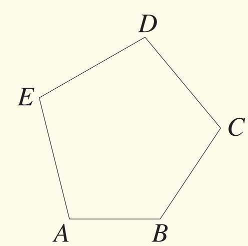
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

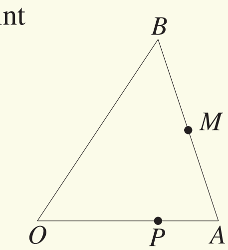
1 A ship sailed 133 km from port A to port B on a bearing of 068°T, then it sailed 98 km from port B to port C on a bearing of 116°T. Draw a diagram showing the displacement vector $\overrightarrow{AC}$, then calculate the magnitude and direction of $\overrightarrow{AC}$ (correct to the nearest tenth of a km, and as a true bearing correct to the nearest degree).

2 In the diagram to the right, write down a single vector equal to:



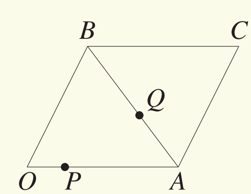
a $\overrightarrow{AE} + \overrightarrow{ED}$
b $\overrightarrow{AD} + \overrightarrow{DC}$
c $\overrightarrow{DA} + \overrightarrow{AC}$
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}$
e $\overrightarrow{AD} - \overrightarrow{AC}$
f $\overrightarrow{EB} - \overrightarrow{ED}$

3 In the triangle OAB shown in the diagram, M is the midpoint of AB and P is the point on OA such that $OP : PA = 2 : 1$. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Express, in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$:



a $\overrightarrow{AB}$
b $\overrightarrow{OM}$
c $\overrightarrow{PM}$

4 In the diagram, $OACB$ is a parallelogram. The point P divides OA in the ratio $1 : 3$ and Q divides AB in the ratio $3 : 4$. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



a Express $\overrightarrow{PA}$ in terms of $\underset{\sim}{a}$.
b Express $\overrightarrow{AQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
c Show that $\overrightarrow{PQ} = \frac{9}{28}\underset{\sim}{a} + \frac{3}{7}\underset{\sim}{b}$.
d Show that $\overrightarrow{QC} = \frac{3}{7}\underset{\sim}{a} + \frac{4}{7}\underset{\sim}{b}$.
e Hence show that the points P, Q and C are collinear.

5 If A and B are the points $(-4, 2)$ and $(2, 10)$ respectively, find:

a $\overrightarrow{AB}$ in component form,
b $|AB|$,
c a unit vector in the direction of $\overrightarrow{AB}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 Given that $\underset{\sim}{v} = \begin{bmatrix} 2a \\ a \end{bmatrix}$, where $a > 0$, find:

a $|\underset{\sim}{v}|$

b $\hat{\underset{\sim}{v}}$

c $\underset{\sim}{v} + \underset{\sim}{v}$

d $\underset{\sim}{v} \cdot \underset{\sim}{v}$

7 Determine in each part whether the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

a $\underset{\sim}{a} = \begin{bmatrix} x - 1 \\ 1 - x \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} x + 1 \\ 1 + x \end{bmatrix}$

b $\underset{\sim}{a} = \begin{bmatrix} 5x \\ 5x - 1 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 1 - 2x \\ 2x \end{bmatrix}$

8 Find, correct to the nearest minute, the angle between the vectors $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$.

9 A quadrilateral $PQRS$ has vertices $P(-4, -5)$, $Q(10, 5)$, $R(5, 12)$ and $S(-9, 2)$.

a Show that $\overrightarrow{PQ} = \overrightarrow{SR}$.

b Show that $\overrightarrow{PQ} \cdot \overrightarrow{PS} = 0$.

c What type of special quadrilateral is $PQRS$?

10 Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ given:

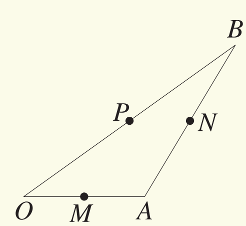
a $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$

b $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

11 Find $|\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a}|$ given $\underset{\sim}{a} = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$.

12 Suppose that A, B and C are the points $(-3, 1)$, $(4, 8)$ and $(2, -5)$ respectively. Use vector methods to find $\angle ABC$ correct to the nearest degree.

13 In $\triangle OAB$ to the right, M, N and P are the midpoints of OA, AB and OB respectively. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



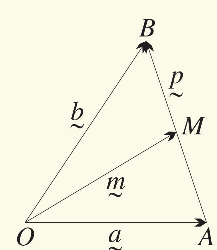
a Express $\overrightarrow{MA}$ in terms of $\underset{\sim}{a}$.

b Express $\overrightarrow{AN}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

c Hence show that $\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{b}$.

d Explain why $MNBP$ is a parallelogram.

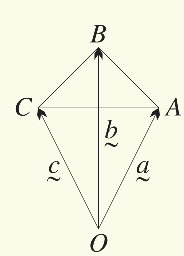
14 In $\triangle OAB$ to the right, M is the midpoint of AB. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{AB} = \underset{\sim}{p}$.



a Write $\underset{\sim}{p}$ and $\underset{\sim}{m}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Hence prove that $|\underset{\sim}{p}|^2 + 4|\underset{\sim}{m}|^2 = 2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{b}|^2$.

15 The diagram shows a kite $OABC$. Note that $OA = OC$ and $AB = CB$. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



a Explain why $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{c} \cdot \underset{\sim}{c}$.

b Explain why $(\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = (\underset{\sim}{b} - \underset{\sim}{c}) \cdot (\underset{\sim}{b} - \underset{\sim}{c})$.

c Hence prove that the diagonals of the kite are perpendicular.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

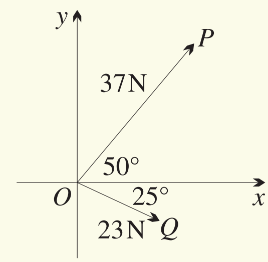
16 Draw a parallelogram $ABCD$ with $|AB| = |CD| = x$ and $|BC| = |DA| = y$. Let $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{b} = \overrightarrow{BC}$.

a Write $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Expand and simplify $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ and $(\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$.

c Hence prove that the diagonals of a parallelogram are equal if and only if the parallelogram is a rectangle.

17 In the diagram, the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent forces of magnitude 37 N and 23 N respectively.



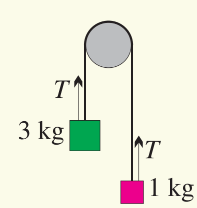
a Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in component form.

b Calculate, correct to 3 significant figures, the magnitude and direction of the resultant of the two forces.

18 A small fishing boat can travel at 8 km/h in still water. It is being steered due east, but there is a current running south at 2 km/h and a breeze blowing the boat south-west at 4 km/h . Find the resultant velocity of the boat, giving the speed correct to 2 decimal places and the direction correct to the nearest degree.

19 A light fitting of mass 2 kg is hung from a timber beam by two identical chains each inclined at 45° to the beam. Taking $g = 9.8\ \text{m}/\text{s}^2$, find, correct to the nearest newton, the tension in each of the chains.

20 The diagram shows two objects of mass 3 kg and 1 kg attached to the ends of a light inextensible string that passes over a smooth pulley. The tension in the string is T Newtons. Take $g = 9.8\ \text{m}/\text{s}^2$.



a Find the downwards acceleration of the 3 kg object.

b Find the value of T.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 Motion and rates

Anyone watching objects in motion can see that they often make patterns with a striking simplicity and predictability. These patterns are related to the simplest objects in geometry and arithmetic. A thrown ball traces out a parabolic path. A cork bobbing in flowing water traces out a sine wave. A rolling billiard ball moves in a straight line, rebounding symmetrically off the table edge. The stars and planets move in more complicated, but highly predictable, paths across the sky. The relationship between physics and mathematics, logically and historically, begins with these and many similar observations.

The first three sections of this chapter, however, only begin to introduce the relationship between calculus and motion. Because this is a mathematics course, not a physics course, our attention will not be on the nature of space and time, but on the striking alternative interpretations that the physical world brings to the first and second derivatives. The first derivative of displacement is velocity, which we can see. The second derivative is acceleration, which we can feel.

Motion is just one example of a rate. We have met rates briefly several times in Year 11 — Section 8G (Applications of these functions — exponential rates), Section 9J (Rates of change — using the derivative), Sections 11F and 16B–16C (Exponential growth and decay) and Section 16A (Related rates). The last three sections of this chapter unify and extend examples of rates in general, using now a much larger array of functions. Related rates are reviewed here, but not exponential growth because that will be revisited in Chapter 13: Differential equations. Rates also provide the context to clarify the ideas of increasing and concave up in an interval, rather than at a point as in Chapter 4.

The examples of motion and rates in this chapter also provide models of the linear, quadratic, exponential and trigonometric functions, because all these functions can be brought into play at once.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9A Average velocity and speed

This first section sets up the mathematical description of motion in one dimension, using a function to describe the relationship between time and the position of an object in motion. Average velocity is the gradient of the chord on this displacement–time graph. This will lead, in the next section, to the description of instantaneous velocity as the gradient of a tangent.

Motion in one dimension

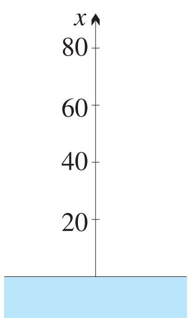
When a particle is moving in one dimension (meaning along a line) its position is varying over time. That position can be specified at any time t by a single number x, called the *displacement*, and the whole motion can be described by giving x as a function of the *time* t.

Suppose, for example, that a ball is hit vertically upwards from ground level and lands 8 seconds later in the same place. Its motion can be described approximately by the following quadratic equation and table of values,

$$x = 5t(8 - t)$$

t	0	2	4	6	8
x	0	60	80	60	0

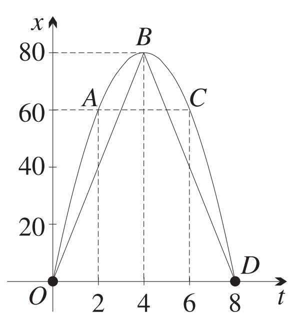
Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line.



This vertical line has been made into a number line, with the ground as the origin, upwards as the positive direction, and metres as the units of displacement.

Time has also become a number line. The origin of time is when the ball is thrown, and the units of time are seconds.

The graph to the right is the resulting graph of the equation of motion $x = 5t(8 - t)$. The horizontal axis is time and the vertical axis is displacement — the graph must not be mistaken as a picture of the ball's path.



The graph is a section of a parabola with vertex at $(4, 80)$, which means that the ball achieves a maximum height of 80 metres after 4 seconds. When $t = 8$, the height is zero, and the ball strikes the ground again. The equation of motion therefore has quite restricted domain and range,

$$0 \le t \le 8 \qquad \text{and} \qquad 0 \le x \le 80.$$

Most equations of motion have this sort of restriction on the domain of t. In particular, *it is a convention of this course that negative values of time are excluded unless the question specifically allows it.*

1 MOTION IN ONE DIMENSION

- Motion in one dimension is specified by giving the displacement x on the number line as a function of time t after time zero.
- Negative values of time are excluded unless otherwise stated.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 1 9A

Consider the example above, where $x = 5t(8 - t)$.

a Find the height of the ball after 1 second.

b At what other time is the ball at this same height above the ground?

SOLUTION

a When $t = 1$, $x = 5 \times 1 \times 7$
$= 35$.

Hence the ball is 35 metres above the ground after 1 second.

b To find when the height is 35 metres, solve the equation $x = 35$.
Substituting into $x = 5t(8 - t)$ gives

$$\begin{aligned} 5t(8 - t) &= 35 \\ \boxed{\div 5} \qquad t(8 - t) &= 7 \\ 8t - t^2 - 7 &= 0 \\ \boxed{\times (-1)} \qquad t^2 - 8t + 7 &= 0 \\ (t - 1)(t - 7) &= 0 \\ t &= 1 \text{ or } 7. \end{aligned}$$

Hence the ball is 35 metres high after 1 second and again after 7 seconds.

Average velocity

During its ascent, the ball in the worked example above moved 80 metres upwards. This is a change in displacement of +80 metres in 4 seconds, giving an average velocity of 20 metres per second.

Average velocity thus equals the gradient of the chord *OB* on the displacement–time graph (be careful, because there are different scales on the two axes). Hence the formula for average velocity is the familiar gradient formula.

2 AVERAGE VELOCITY

Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$. Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

That is, on the displacement–time graph,

$$\text{average velocity} = \text{gradient of the chord.}$$

During its descent, the ball moved 80 metres downwards in 4 seconds, which is a change in displacement of $0 - 80 = -80$ metres. The average velocity is therefore -20 metres per second, which is equal to the gradient of the chord *BD*.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 2 9A

Consider again the example $x = 5t(8 - t)$. Find the average velocities of the ball:

a during the first second,

b during the fifth second.

SOLUTION

The first second stretches from $t = 0$ to $t = 1$ and the fifth second stretches from $t = 4$ to $t = 5$. The displacements at these times are given in the table to the right.

t	0	1	4	5
x	0	35	80	75

a Average velocity during 1st second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{35 - 0}{1 - 0}$$
$$= 35 \text{ m/s.}$$

b Average velocity during 5th second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{75 - 80}{5 - 4}$$
$$= -5 \text{ m/s.}$$

Distance travelled

The change in displacement can be positive, negative or zero. *Distance*, however, is always positive or zero. In the previous example, the change in displacement during the 4 seconds from $t = 4$ to $t = 8$ is -80 metres, but the distance travelled is 80 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the total distance travelled by the ball is $80 + 80 = 160$ metres, even though the ball's change in displacement over the first 8 seconds is zero because the ball is back at its original position on the ground.

3 DISTANCE TRAVELLED

- The *distance travelled* takes into account any journey and return.
- Distance travelled can never be negative.

Average speed

The *average speed* is the distance travelled divided by the time taken. Average speed, unlike average velocity, can never be negative.

4 AVERAGE SPEED

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Average speed can never be negative.

During the 8 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 160 metres, so

$$\text{average velocity} = \frac{0 - 0}{8 - 0} = 0 \text{ m/s,}$$

$$\text{average speed} = \frac{160}{8} = 20 \text{ m/s.}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 3 9A

Find the average velocity and the average speed of the ball:

a during the eighth second, **b** during the last six seconds.

SOLUTION

The eighth second stretches from $t = 7$ to $t = 8$ and the last six seconds stretch from $t = 2$ to $t = 8$. The displacements at these times are given in the table to the right.

t	0	2	7	8
x	0	60	35	0

a During the eighth second, the ball moves 35 metres down from $x = 35$ to $x = 0$.

$$\begin{aligned} \text{Hence} \quad \text{average velocity} &= \frac{0 - 35}{8 - 7} \\ &= -35 \text{ m/s.} \\ \text{Also} \quad \text{distance travelled} &= 35 \text{ metres,} \\ \text{so} \quad \text{average speed} &= 35 \text{ m/s.} \end{aligned}$$

b During the last six seconds, the ball rises 20 metres from $x = 60$ to $x = 80$, and then falls 80 metres from $x = 80$ to $x = 0$.

$$\begin{aligned} \text{Hence} \quad \text{average velocity} &= \frac{0 - 60}{8 - 2} \\ &= -10 \text{ m/s.} \\ \text{Also} \quad \text{distance travelled} &= 20 + 80 \\ &= 100 \text{ metres,} \\ \text{so} \quad \text{average speed} &= \frac{100}{6} \\ &= 16\tfrac{2}{3} \text{ m/s.} \end{aligned}$$

Exercise 9A

FOUNDATION

1 A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

t	0	1	2	3
x				

- **a** Copy and complete the table of values to the right.
- **b** Hence find the average velocity:
  - **i** during the first second,
  - **ii** during the first two seconds,
  - **iii** during the first three seconds,
  - **iv** during the third second.
- **c** Use the table of values above to sketch the displacement–time graph. Then add the chords corresponding to the average velocities calculated in part **b**.

2 A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

t	0	1	2	3	4
x					

- **a** Copy and complete the table of values to the right.
- **b** Hence sketch the displacement–time graph.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

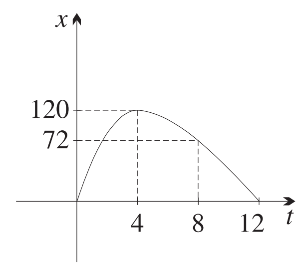
c Find the total distance travelled during the first 4 seconds. Then find the average speed during this time.

d Find the average velocity during the time:

i from $t = 0$ to $t = 2$, **ii** from $t = 2$ to $t = 4$, **iii** from $t = 0$ to $t = 4$.

e Add to your graph the chords corresponding to the average velocities in part **d**.

3 A piece of cardboard is shot 120 metres vertically into the air by an explosion and floats back to the ground, landing at the same place. The graph to the right gives its height x metres above the ground t seconds after the explosion.



a Copy and complete the following table of values.

t	0	4	8	12
x				

b What is the total distance travelled by the cardboard?

c Find the average speed of the cardboard during its travels, using the formula

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

d Find the average velocity during:

i the ascent, **ii** the descent, **iii** the full 12 seconds.

DEVELOPMENT

4 Michael the mailman rides his bicycle 1 km up a hill at a constant speed of 10 km/hr. He then turns around and rides back down the hill at a constant speed of 30 km/hr.

a How many minutes does he take to travel:

i the first kilometre, when he is riding up the hill,

ii the second kilometre, when he is riding back down again?

b Use these values to draw a displacement–time graph, with the time axis in minutes.

c What is his average speed over the total 2 km journey?

d What is the average of his speeds up and down the hill?

5 Sadie the snail is crawling up a 6-metre-high wall. She takes an hour to crawl up 3 metres, then falls asleep for an hour and slides down 2 metres, repeating the cycle until she reaches the top of the wall. Let x be Sadie's height in metres after t hours.

a Copy and complete the table of values of Sadie's height up the wall.

t	0	1	2	3	4	5	6	7
x								

b Hence sketch the displacement–time graph.

c How long does Sadie take to reach the top?

d What total distance does she travel, and what is her average speed?

e What is her average velocity over this whole time?

f Which places on the wall does she visit exactly three times?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance x is in centimetres and time t is in seconds.

a Copy and complete the table of values to the right, and sketch the curve.

t					
x	0	2	4	6	8

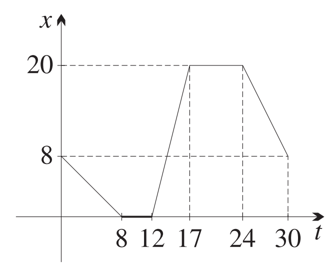
b Hence find the average velocity as the particle moves:

i from $x = 0$ to $x = 2$, **ii** from $x = 2$ to $x = 4$,

iii from $x = 4$ to $x = 6$, **iv** from $x = 0$ to $x = 6$.

c What does the equality of the answers to parts **ii** and **iv** of part **b** tell you about the corresponding chords in part **c**?

7 Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x in metres from the front gate after t seconds.



a What is her average velocity:

i during the first 8 seconds,

ii while she is driving forwards,

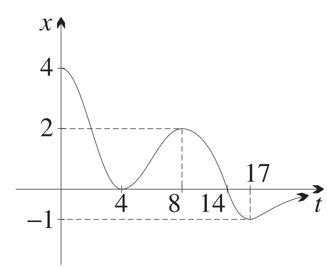
iii while she is reversing the second time?

b Find the total distance she travelled, and her average speed, over the 30 seconds.

c Find her change in displacement, and her average velocity, over the 30 seconds.

d What would her average speed have been if she had not paused at the gate and at the garage?

8 A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The graph shows the height x in metres of the weight above the water as a function of time t seconds after she first drops it.



a How many times is the weight:

i at $x = 3$, **ii** at $x = 1$, **iii** at $x = -\frac{1}{2}$?

b At what times is the weight:

i at the water surface, **ii** above the water surface?

c How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?

d What is the weight's greatest depth under the water and when does it occur?

e What happens to the weight eventually?

f What is its average velocity:

i during the first 4 seconds, **ii** from $t = 4$ to $t = 8$, **iii** from $t = 8$ to $t = 17$?

g What distance does it travel:

i over the first 4 seconds, **ii** over the first 8 seconds,

iii over the first 17 seconds, **iv** eventually?

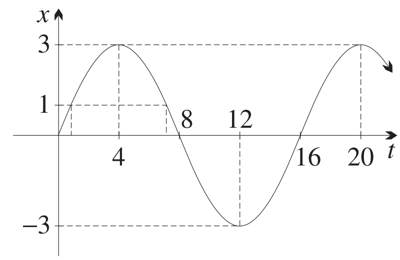
h What is its average speed over the first:

i 4, **ii** 8, **iii** 17 seconds?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 A particle is moving according to $x = 3\sin\frac{\pi}{8}t$, in units of centimetres and seconds. Its displacement–time graph is sketched to the right.



a Use $T = \dfrac{2\pi}{n}$ to confirm that the period is 16 seconds.

b Find the maximum and minimum values of the displacement.

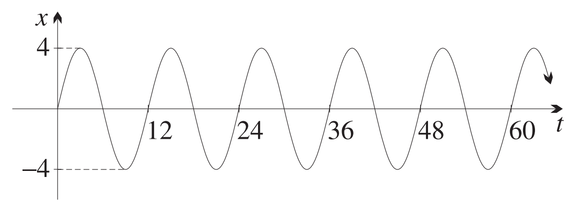
c Find the first two times when the displacement is maximum.

d Find the first two times when the particle returns to its initial position.

e When, during the first 20 seconds, is the particle on the negative side of the origin?

f Find the total distance travelled during the first 16 seconds, and the average speed.

10 A particle is moving according to the equation $x = 4\sin\frac{\pi}{6}t$, in units of metres and seconds. The graph of its displacement for the first minute is sketched to the right.



a Find the amplitude and period.

b How many times does the particle return to the origin by the end of the first minute?

c Find at what times it visits $x = 4$ during the first minute.

d Find how far it travels during the first 12 seconds, and its average speed in that time.

e Find the values of x when $t = 0$, $t = 1$ and $t = 3$. Hence show that the average speed during the first second is twice the average speed during the next 2 seconds.

11 A particle moves according to $x = 10\cos\frac{\pi}{12}t$, in units of metres and seconds.

a Find the amplitude, and use the formula $T = 2\pi/n$ to show that the period of the motion is 24 seconds.

b Sketch the displacement–time graph over the first 36 seconds.

c When, during the first 36 seconds, is the particle at the origin?

d Where is the particle initially? What is the maximum distance the particle reaches from its initial position, and when, during the 36 seconds, is it there?

e How far does the particle move during the first 36 seconds, and what is its average speed over this time?

f Use the fact that $\cos\frac{\pi}{3} = \frac{1}{2}$ to copy and complete the table of values to the right:

t	4	8	12	16	20	24
x						

g From the table, find the average velocity during the first 4 seconds, the second 4 seconds and the third 4 seconds.

h Use the graph and the table of values to find when, in the first 24 seconds, the particle is more than 15 metres from its initial position.

12 A balloon rises so that its height h in metres after t minutes is $h = 8000\left(1 - e^{-0.06t}\right)$.

a What height does it start from, and what happens to the height as $t \to \infty$?

b Copy and complete the table to the right, correct to the nearest metre.

t	0	10	20	30
x				

c Sketch the displacement–time graph of the motion.

d Find the balloon's average velocity during the first 10 minutes, the second 10 minutes and the third 10 minutes, correct to the nearest metre per minute.

e Use your calculator to show that the balloon has reached 99% of its final height after 77 minutes, but not after 76 minutes.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 Two engines, Thomas and Henry, move on close parallel tracks. They start at the origin, and are together again at time $t = e - 1$. Thomas' displacement–time equation, in units of metres and minutes, is $x = 300 \log(t + 1)$, and Henry's is $x = kt$, for some constant k.

a Sketch the two graphs.

b Show that $k = \dfrac{300}{e - 1}$.

c Use calculus to find the maximum distance between Henry and Thomas during the first $e - 1$ minutes, and the time when it occurs (in exact form, and then correct to the nearest metre or the nearest second).

ENRICHMENT

14 [Arithmetic mean, geometric mean and harmonic mean]
Let a and b be two positive numbers. The *arithmetic mean* and *(positive) geometric mean* of a and b are

$$\text{arithmetic mean} = \frac{a+b}{2} \qquad \text{and} \qquad \text{geometric mean} = \sqrt{ab}\,,$$

and their *harmonic mean* is the number h such that

$$\frac{1}{h} \text{ is the arithmetic mean of } \frac{1}{a} \text{ and } \frac{1}{b}.$$

Suppose that town B lies on the road between town A and town C, and that a cyclist rides from A to B at a constant speed U, and then rides from B to C at a constant speed V.

a Prove that if town B lies midway between towns A and C, then the cyclist's average speed W over the total distance AC is the harmonic mean of U and V.

b Now suppose that the distances AB and BC are not equal.

i Show that if W is the arithmetic mean of U and V, then

$$AB : BC = U : V.$$

ii Show that if W is the geometric mean of U and V, then

$$AB : BC = \sqrt{U} : \sqrt{V}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9B Velocity and acceleration as derivatives

If I drive the 160 km from Sydney to Newcastle in 2 hours, my average velocity is 80 km per hour. But my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. Just as an average velocity corresponds to the gradient of a chord on the displacement–time graph, so an instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous velocity and speed

From now on, the words *velocity* and *speed* alone will mean instantaneous velocity and instantaneous speed.

5 INSTANTANEOUS VELOCITY AND INSTANTANEOUS SPEED

- The *instantaneous velocity* v of a particle in motion is the gradient of the tangent on the displacement–time graph,

  $$v = \frac{dx}{dt} \qquad \text{which can also be written as} \qquad v = \dot{x}.$$

- The dot over any symbol means differentiation with respect to time.
- The *instantaneous speed* is the absolute value $|v|$ of the velocity.

The notation $\dot{x}$, originally introduced by Newton, is yet another way of writing the derivative. The dot over the x, or over any symbol, stands for differentiation with respect to time t. Thus the symbols v, $\dfrac{dx}{dt}$ and $\dot{x}$ are all symbols for the velocity.

Example 4 9B

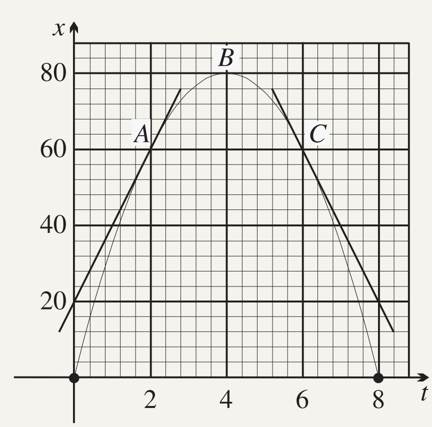
Here again is the displacement–time graph of the ball moving with equation $x = 5t(8 - t)$.

a Differentiate to find the equation for the velocity v, draw up a table of values at 2-second intervals and sketch the velocity–time graph.

b Measure the gradients of the tangents that have been drawn at A, B and C on the displacement–time graph and compare your answers with the table of values in part **a**.

c With what velocity was the ball originally hit?

d What is its impact speed when it hits the ground?



ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

SOLUTION

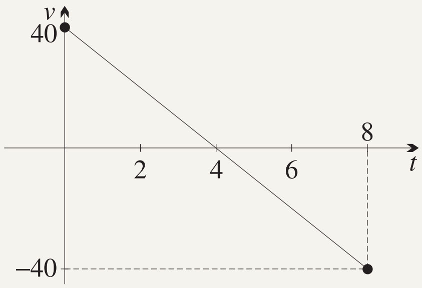
a The equation of motion is $x = 5t(8 - t)$

$x = 40t - 5t^2,$

and differentiating, $v = 40 - 10t.$

The graph of velocity is a straight line, with v-intercept 40 and gradient -10.

t	0	2	4	6	8
v	40	20	0	-20	-40



b These values agree with the measurements of the gradients of the tangents at A where $t = 2$, at B where $t = 4$, and at C where $t = 6$.
(Be careful to take account of the different scales on the two axes.)

c When $t = 0$, $v = 40$, so the ball was originally hit upwards at 40 m/s.

d When $t = 8$, $v = -40$, so the ball hits the ground again at 40 m/s.

Vector and scalar quantities

Displacement and velocity are *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards and a negative displacement would mean it was below ground level. Distance and speed, however, are *scalar quantities* — distance is the magnitude of the displacement, and speed is the magnitude of the velocity — and neither can be negative.

Finding when a particle is stationary

A particle is said to be *stationary* when its velocity v is zero, that is, when $\dfrac{dx}{dt} = 0$. This is the origin of the word 'stationary point', used in Chapter 4 and in Year 11 to describe a point on a graph where the derivative is zero. For example, the ball in the first example was stationary for an instant at the top of its flight when $t = 4$, because the velocity was zero at the instant when its motion changed from upwards to downwards.

6 FINDING WHEN A PARTICLE IS STATIONARY

- A particle is *stationary* when its velocity is zero.
- To find when a particle is stationary, put $v = 0$ and solve for t.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 5 — 9B

A particle moves so that its distance in metres from the origin at time t seconds is given by $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

a Find the velocity and speed when $t = 4$.

b Find the times when the particle is stationary.

c Find its distance from the origin at these times.

SOLUTION

The displacement function is $\quad x = \frac{1}{3}t^3 - 6t^2 + 27t - 18,$

and differentiating, $\quad v = t^2 - 12t + 27$

$\qquad = (t - 3)(t - 9).$

a When $t = 4$, $\quad v = -5$,

so the velocity is -5 m/s and the speed is 5 m/s.

b When $v = 0$, $\quad t = 3$ or 9,

so the particle is stationary after 3 seconds and again after 9 seconds.

c When $t = 3$, $\quad x = 9 - 54 + 81 - 18$

$\qquad = 18,$

and when $t = 9$, $\quad x = 243 - 486 + 243 - 18$

$\qquad = -18.$

Thus the particle is 18 metres from the origin on both occasions.

Acceleration as the second derivative

A particle is said to be *accelerating* if its velocity is changing. The *acceleration* of an object is defined to be the rate at which the velocity is changing. Thus acceleration a is the derivative $\dfrac{dv}{dt} = \dot{v}$ of the velocity with respect to time.

Velocity is already the derivative of displacement, so acceleration is the second derivative $\dfrac{d^2x}{dt^2} = \ddot{x}$ of displacement. The double-dot means the second derivative.

7 ACCELERATION AS A DERIVATIVE

- Acceleration is the first derivative of velocity with respect to time, and the second derivative of displacement with respect to time:

$$a = \frac{dv}{dt} = \dot{v} \qquad \text{and} \qquad a = \frac{d^2x}{dt^2} = \ddot{x}.$$

Note: Be very careful with the symbol a, because in this context a is the acceleration function, whereas elsewhere the letter a is usually used for a constant. Because of this issue, we tend to use $\ddot{x}$ for acceleration more often than a.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 6 — 9B

Consider again the ball moving with displacement function $x = 5t(8 - t)$.

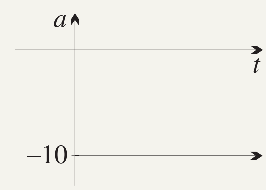
a Find the velocity function v and the acceleration function a, and sketch the acceleration function.

b Find and describe the displacement, velocity and acceleration when $t = 2$.

c State when the ball is speeding up and when it is slowing down, explaining why this can happen when the acceleration is constant.

SOLUTION

a The displacement is $x = 40t - 5t^2$.
Differentiating, $v = 40 - 10t$.
Differentiating again, $a = -10$, which is a constant.
Hence the acceleration is always $10\ \text{m/s}^2$ downwards.



b Substituting $t = 2$ into the functions x, v and a,

$$x = 60 \quad \text{and} \quad v = 20 \quad \text{and} \quad a = -10.$$

Thus when $t = 2$, the displacement is 60 metres above the ground, the velocity is 20 m/s upwards, and the acceleration is $10\ \text{m/s}^2$ downwards.

c During the first 4 seconds, the ball has positive velocity, meaning that it is rising, and the ball is slowing down by 10 m/s every second.

During the last 4 seconds, however, the ball has negative velocity, meaning that it is falling, and the ball is speeding up by 10 m/s every second.

Units of acceleration

In the previous example, the particle's velocity was decreasing by 10 m/s every second. The particle is said to be 'accelerating at −10 metres per second, per second', written in symbols as $-10\ \text{m/s}^2$ or as $-10\ \text{ms}^{-2}$.

The units of acceleration correspond with the indices of the second derivative $\dfrac{d^2x}{dt^2}$.

Acceleration would normally be regarded as a vector quantity, that is, with a direction built into it. This is why the particle's acceleration is written with a minus sign as $-10\ \text{m/s}^2$. Alternatively, one can omit the minus sign and specify the direction instead, writing '$10\ \text{m/s}^2$ in the downwards direction'.

Example 7 — 9B

In worked Example 5, we examined the function $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

a Find the acceleration function, and find when the acceleration is zero.

b Where is the particle at this time and what is its velocity?

SOLUTION

a The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.
Differentiating, $v = t^2 - 12t + 27$,
and differentiating again,
$$\ddot{x} = 2t - 12$$
$$= 2(t - 6).$$
Thus the acceleration is zero when $t = 6$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b When $t = 6$, $v = 36 - 72 + 27$
$= -9$,
and $x = 72 - 216 + 162 - 18$
$= 0$.
Thus when $t = 6$, the particle is at the origin, moving with velocity -9 m/s.

Trigonometric equations of motion

When a particle's motion is described by a sine or cosine function, it moves backwards and forwards and is therefore stationary over and over again. It is best to work with the graphs drawn.

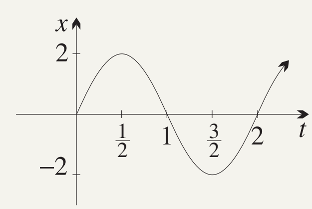
Example 8 9B

A particle's displacement function is $x = 2 \sin \pi t$.

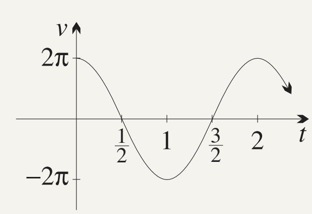
a Find its velocity and acceleration functions, and graph all three functions in the time interval $0 \le t \le 2$.
b Find the times within the time interval $0 \le t \le 2$ when the particle is at the origin, and find its speed and acceleration at those times.
c Find the times within the time interval $0 \le t \le 2$ when the particle is stationary, and find its displacement and acceleration at those times.
d Briefly describe the motion.

SOLUTION

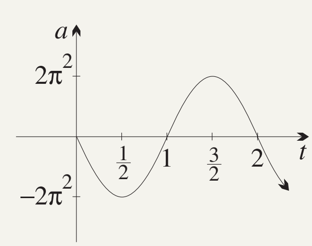
a The displacement function is $x = 2 \sin \pi t$,
which has amplitude 2 and period $\dfrac{2\pi}{\pi} = 2$.
Differentiating, $v = 2\pi \cos \pi t$,
which has amplitude 2π and period 2.
Differentiating again, $a = -2\pi^2 \sin \pi t$,
which has amplitude $2\pi^2$ and period 2.



b The particle is at the origin when $x = 0$,
and reading from the displacement graph,
this occurs when $t = 0, 1$ or 2.
Reading now from the velocity and acceleration graphs,
when $t = 0$ or 2, $v = 2\pi$ and $a = 0$,
and when $t = 1$, $v = -2\pi$ and $a = 0$,
so in all cases the speed is 2π and the acceleration is zero.



c The particle is stationary when $v = 0$,
and reading from the velocity graph above,
this occurs when $t = \frac{1}{2}$ or $1\frac{1}{2}$.
Reading from the displacement and acceleration graphs,
when $t = \frac{1}{2}$, $x = 2$ and $a = -2\pi^2$,
and when $t = 1\frac{1}{2}$, $x = -2$ and $a = 2\pi^2$.



d The particle oscillates forever between $x = -2$ and $x = 2$, with period 2, beginning at the origin and moving first to $x = 2$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Motion with exponential functions — limiting values of displacement and velocity

Sometimes a question will ask what happens to the particle 'eventually', or 'as time goes on'. This simply means taking the limit of the displacement and the velocity as $t \to \infty$. Particles whose motion is described by an exponential function are the most usual examples of this. Remember that $e^{-x} \to 0$ as $x \to \infty$.

Example 9 — 9B

A particle is moving so that its height x metres above the ground at time t seconds after time zero is $x = 2 - e^{-3t}$.

a Find the velocity and acceleration functions, and sketch the three graphs of displacement, velocity and acceleration.

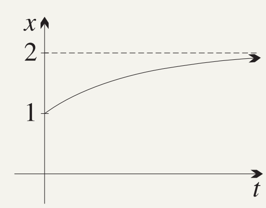
b Find the initial values of displacement, velocity and acceleration.

c What happens to the displacement, velocity and acceleration eventually?

d Briefly describe the motion.

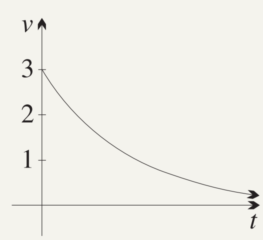
SOLUTION

a The displacement function is $x = 2 - e^{-3t}$.
Differentiating, $v = 3e^{-3t}$,
and differentiating again, $a = -9e^{-3t}$.



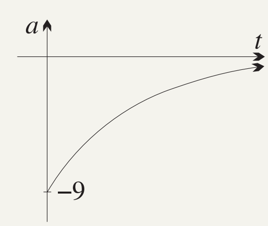
b Substituting $t = 0$ and using $e^0 = 1$,

$x = 1$ and $v = 3$ and $a = -9$.



c As t increases, that is, as $t \to \infty$, $e^{-3t} \to 0$.
Hence *eventually* (meaning as $t \to \infty$),

$x \to 2$ and $v \to 0$ and $a \to 0$.



d The particle starts 1 metre above the ground, with initial velocity of 3 m/s upwards.
It is constantly slowing down, and it moves towards a limiting position at height 2 metres.

Extension — Newton's second law of motion

Newton's second law of motion — a law of physics, not of mathematics — says that when a force is applied to a body free to move, the body accelerates with an acceleration proportional to the resultant force and inversely proportional to the mass of the body. Written symbolically,

$$F = ma,$$

where m is the mass, F is the resultant force, and a is the acceleration. (The units of force are chosen to make the constant of proportionality 1 — in units of kilograms, metres and seconds, the units of force are, appropriately, called *newtons*.)

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

This means that acceleration is felt in our bodies as a force, as we all know when a car that we are in accelerates away from the lights, or comes to a stop quickly. In this way, the second derivative becomes directly observable to our senses as a force, just as the first derivative, velocity, is observable to our sight.

There is no need to formalise this, but it is helpful to have an intuitive idea that force and acceleration are closely related.

Exercise 9B

FOUNDATION

Note: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The mathematics should be well known, but the physical interpretations can be confusing.

1 A particle is moving with displacement function $x = 20 - t^2$, in units of metres and seconds.
- **a** Differentiate to find the velocity v as a function of time t.
- **b** Differentiate again to find the acceleration a.
- **c** Find the displacement, velocity and acceleration when $t = 3$.
- **d** What are the distance from the origin and the speed when $t = 3$?

2 A particle's displacement function is $x = t^2 - 10t$, in units of centimetres and seconds.
- **a** Differentiate to find v as a function of t.
- **b** What are the displacement, the distance from the origin, the velocity and the speed after 3 seconds?
- **c** When is the particle stationary and where is it then?

3 A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2$.
- **a** Differentiate to find v as a function of t, and differentiate again to find a.
- **b** Where is the particle initially and what are its speed and acceleration then?
- **c** At time $t = 3$, is the particle to the left or to the right of the origin?
- **d** At time $t = 3$, is the particle travelling to the left or to the right?
- **e** At time $t = 3$, is the particle accelerating to the left or to the right?
- **f** Show that the particle is stationary when $t = 4$ and find where it is at this time.
- **g** Show that the particle is at the origin when $t = 6$ and find its velocity and speed at this time.

4 A cricket ball is thrown vertically upwards. Its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.
- **a** Find v and a as functions of t, and show that the ball is always accelerating downwards. Then sketch graphs of x, v and a against t.
- **b** Find the speed at which the ball was thrown.
- **c** Find when it returns to the ground (that is, when $x = 0$) and show that its speed then is equal to the initial speed.
- **d** Find its maximum height above the ground and the time taken to reach this height.
- **e** Find the acceleration at the top of the flight, and explain why the acceleration can be non-zero when the ball is stationary.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 If $x = e^{-4t}$, find the velocity function $\dot{x}$ and the acceleration function $\ddot{x}$.

a Explain why neither x, nor $\dot{x}$, nor $\ddot{x}$ can ever change sign, and state their signs.

b Using the displacement function, find where the particle is:

i initially (substitute $t = 0$), **ii** eventually (take the limit as $t \to \infty$).

c What are the particle's velocity and acceleration:

i initially, **ii** eventually?

6 Find the velocity function v and the acceleration function a for a particle P moving according to $x = 2\sin \pi t$.

a Show that P is at the origin when $t = 1$ and find its velocity and acceleration then.

b When $t = \frac{1}{3}$, in what direction is the particle:

i moving, **ii** accelerating?

DEVELOPMENT

7 A particle moves according to $x = t^2 - 8t + 7$, in units of metres and seconds.

a Find the velocity $\dot{x}$ and the acceleration $\ddot{x}$ as functions of time t.

b Sketch the graphs of the displacement x, velocity $\dot{x}$ and acceleration $\ddot{x}$.

c When is the particle:

i at the origin, **ii** stationary?

d What is the maximum distance from the origin, and when does it occur:

i during the first 2 seconds,

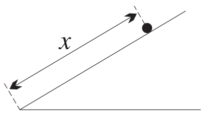
ii during the first 6 seconds,

iii during the first 10 seconds?

e What is the particle's average velocity during the first 7 seconds? When and where is its instantaneous velocity equal to this average?

f How far does it travel during the first 7 seconds, and what is its average speed?

8 A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.



a Find the functions for velocity v and acceleration a.

b Sketch the graphs of displacement x and velocity v.

c In which direction is the ice moving, and in which direction is it accelerating:

i when $t = 2$, **ii** when $t = 4$?

d When is the ice stationary, for how long is it stationary, where is it then, and is it accelerating then?

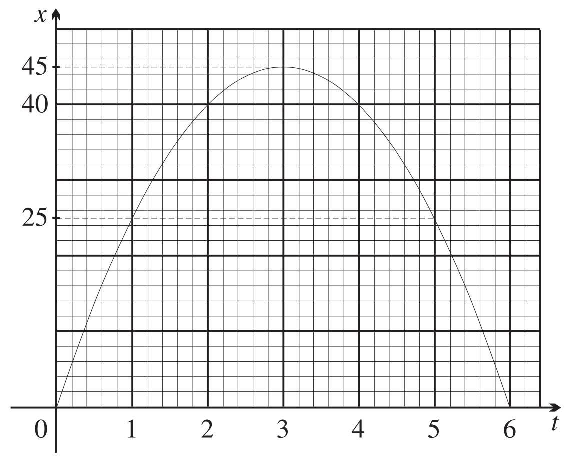
e Show that the average velocity over the first 2 seconds is 4 m/s. Then find the time and place at which the instantaneous velocity equals this average velocity.

f Show that the average speed during the first 3 seconds, the next 3 seconds and the first 6 seconds are all the same.

ISBN 978-1-108-76630-2

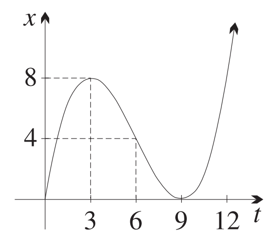
Cambridge University Press
Updated July 2022

9 A stone was thrown vertically upwards. The graph to the right shows its height x metres at time t seconds after it was thrown.



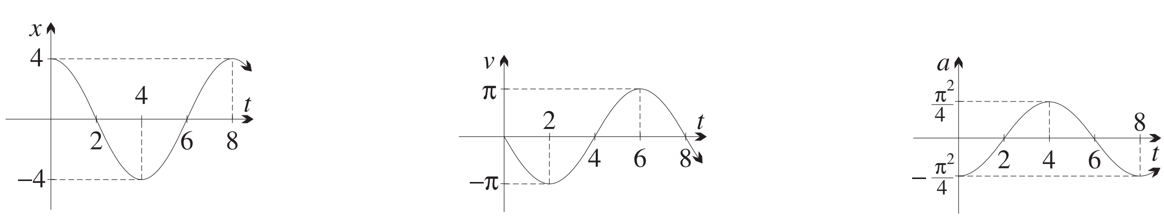
- **a** What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?
- **b** Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ and 6.
- **c** For what length of time was the stone stationary at the top of its flight?
- **d** The graph is concave down everywhere. How is this relevant to the motion?
- **e** Draw a graph of the instantaneous velocity of the stone from $t = 0$ to $t = 6$. What does this velocity–time graph tell you about what happened to the velocity during these 6 seconds?

10 A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.



- **a** In the first 10 seconds, what is its maximum distance from the origin and when does it occur?
- **b** By examining the gradient, find when the particle is:
  - **i** stationary,
  - **ii** moving to the right,
  - **iii** moving to the left.
- **c** When does it return to the origin, what is its velocity then, and in which direction is it accelerating?
- **d** When is its acceleration zero, where is it then, and in what direction is it moving?
- **e** By examining the concavity, find the time interval during which the particle's acceleration is negative.
- **f** At about what times are:
  - **i** the displacement,
  - **ii** the velocity, about the same as those at $t = 2$?
- **g** Sketch (roughly) the graphs of velocity v and acceleration a.

11 A particle is moving according to $x = 4\cos\frac{\pi}{4}t$, where the units are metres and seconds. The displacement, velocity and acceleration graphs are drawn below, for $0 \le t \le 8$.



- **a** Differentiate to find the functions for velocity v and the acceleration a.
- **b** What are the particle's maximum displacement, velocity and acceleration, and when, during the first 8 seconds, do they occur?
- **c** How far does it travel during the first 20 seconds, and what is its average speed?
- **d** Show by substitution that $x = 2$ when $t = 1\frac{1}{3}$ and when $t = 6\frac{2}{3}$. Hence use the graph to find when $x < 2$ during the first 8 seconds.
- **e** When, during the first 8 seconds, is:
  - **i** $v = 0$,
  - **ii** $v > 0$?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 A particle is oscillating on a spring so that its height is $x = 6 \sin 2t$ cm at time t seconds.

a Find v and $\ddot{x}$ as functions of t, and sketch graphs of x, v and $\ddot{x}$, for $0 \le t \le 2\pi$.

b Show that $\ddot{x} = -kx$, for some constant k, and find k.

c When, during the first π seconds, is the particle:

i at the origin,

ii stationary,

iii moving with zero acceleration?

d When, during the first π seconds, is the particle:

i below the origin,

ii moving downwards,

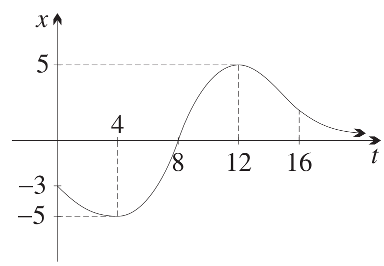
iii accelerating downwards?

e Find the first time the particle has:

i displacement $x = 3$,

ii speed $|v| = 6$.

13 A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



a At what times is this particle:

i below the origin,

ii moving downwards,

iii accelerating downwards?

b At about what time is its speed greatest?

c At about what times are:

i the distance from the origin,

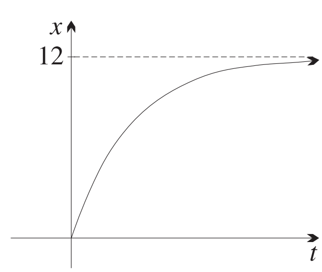
ii the velocity, about the same as those at $t = 3$?

d How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?

e How far will the particle eventually travel?

f Draw an approximate sketch of the graph of v as a function of time.

14 A large stone is falling through a layer of mud. Its depth x metres below ground level at time t minutes is given by $x = 12 - 12e^{-0.5t}$, and its displacement–time graph is drawn.



a Find the velocity and acceleration functions, and sketch them.

b In which direction is the stone always:

i travelling,

ii accelerating?

c What happens to the position, velocity and acceleration of the particle as $t \to \infty$?

d Find when the stone is halfway between the origin and its final position. Show that its speed is then half its initial speed, and its acceleration is half its initial acceleration.

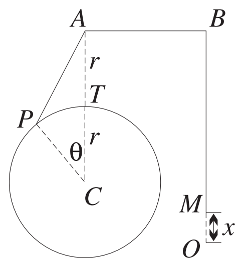
e How long, correct to the nearest minute, will it take for the stone to reach within 1 mm of its final position?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

15 The diagram to the right shows a point P that is rotating anticlockwise in a circle of radius r and centre C at a steady rate. A string passes over fixed pulleys at A and B, where A is distant r above the top T of the circle, and connects P to a mass M on the end of the string. At time zero, P is at T, and the mass M is at the point O. Let x be the height of the mass above the point O at time t seconds later, and θ be the angle $\angle TCP$ through which P has moved.



a Show that $x = -r + r\sqrt{5 - 4\cos\theta}$, and find the range of x.

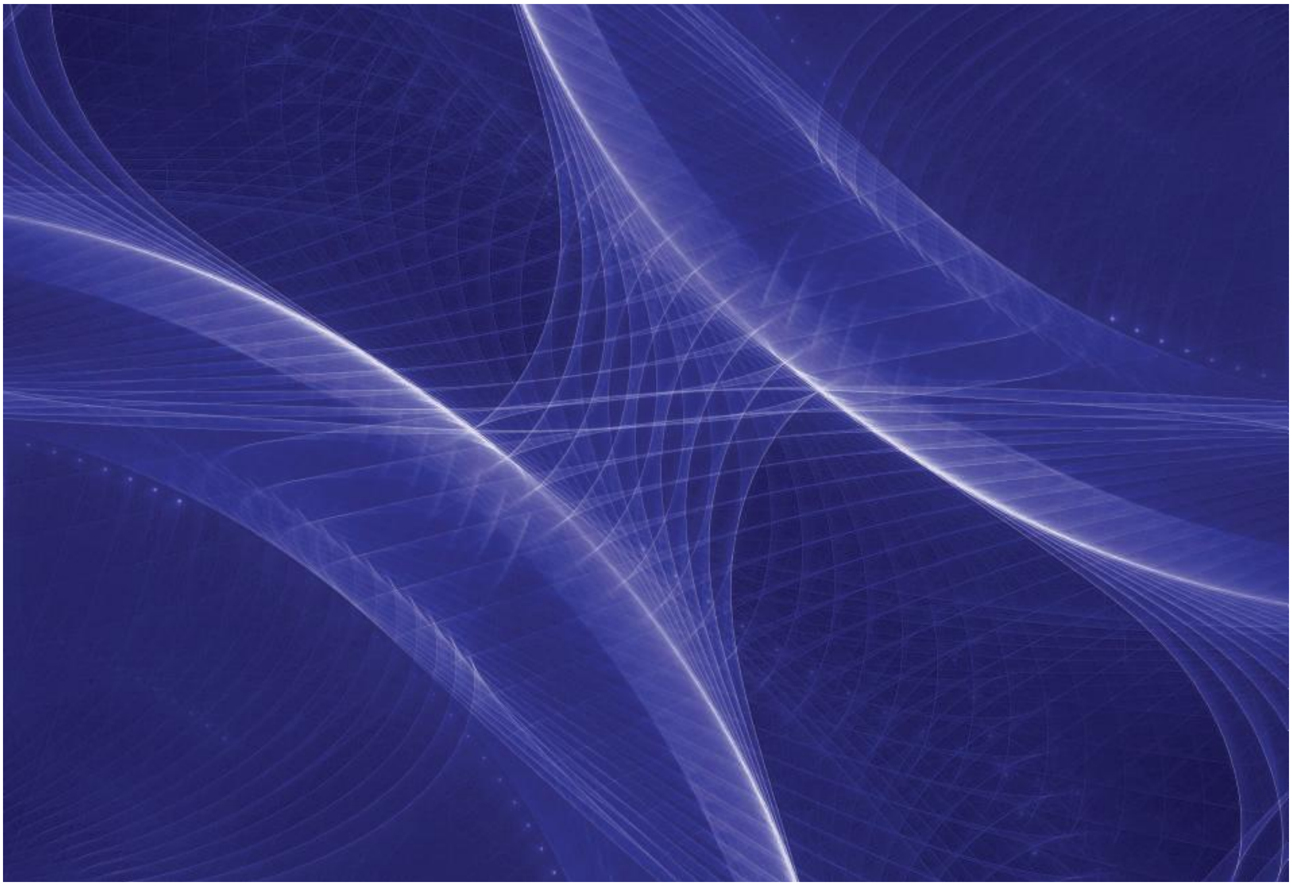
b Find $\dfrac{dx}{d\theta}$, and find for what values of θ the mass M is travelling:

i upwards, **ii** downwards.

c Show that $\dfrac{d^2x}{d\theta^2} = -\dfrac{2r(2\cos^2\theta - 5\cos\theta + 2)}{(5 - 4\cos\theta)^{\frac{3}{2}}}$. Find for what values of θ the speed of M is maximum, and find $\dfrac{dx}{d\theta}$ at these values of θ.

d Explain geometrically why these values of θ give the maximum speed, and why they give the values of $\dfrac{dx}{d\theta}$ they do.

16 [This question will require resolution of forces.]
At what angle α should the surface in Question **8** be inclined to the horizontal to produce these equations?



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9C Integrating with respect to time

The inverse process of differentiation is integration. Thus if the acceleration function is known, integration will generate the velocity function. In the same way, if the velocity function is known, integration will generate the displacement function.

Using initial conditions

Taking the primitive of a function always involves a constant of integration. Determining such a constant requires an *initial condition* (or *boundary condition*) to be known. For example, the problem may tell us the velocity when $t = 0$, or give us the displacement when $t = 3$.

In this chapter, the constants of integration cannot ever be omitted.

8 INTEGRATING WITH RESPECT TO TIME

- Given the acceleration function a, integrate to find the velocity function v.
- Given the velocity function v, integrate to find the displacement function x.
- Never omit the constants of integration.
- Use an initial or boundary condition to evaluate each constant of integration.

Example 10 **9C**

A particle's acceleration function is $\ddot{x} = 24t$. Initially it is at the origin, moving with velocity -12 cm/s.

a Integrate, substituting the initial condition, to find the velocity function.

b Integrate again to find the displacement function.

c Find when the particle is stationary and find the displacement then.

d Find when the particle returns to the origin and the acceleration then.

SOLUTION

a The given acceleration function is $\ddot{x} = 24t$. (1)

Integrating, $v = 12t^2 + C$, for some constant C.

When $t = 0$, $v = -12$, so $-12 = 0 + C$,

so $C = -12$, and $v = 12t^2 - 12$. (2)

b Integrating again, $x = 4t^3 - 12t + D$, for some constant D.

When $t = 0$, $x = 0$, so $0 = 0 - 0 + D$,

so $D = 0$, and $x = 4t^3 - 12t$. (3)

c Put $v = 0$. Then from (2),

$$\begin{aligned} 12t^2 - 12 &= 0 \\ t^2 &= 1 \\ t &= 1 \text{ (because } t \geq 0\text{).} \end{aligned}$$

Hence the particle is stationary after 1 second.

When $t = 1$, $x = -8$, so at this time its displacement is $x = -8$ cm.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d Put $x = 0$. Then using (3), $4t^3 - 12t = 0$

$$4t(t^2 - 3) = 0,$$

so $t = 0$ or $t = \sqrt{3}$ (because $t \geq 0$).

Hence the particle returns to the origin after $\sqrt{3}$ seconds,

and at this time, $\ddot{x} = 24\sqrt{3}\ \text{cm/s}^2$.

The acceleration due to gravity

Since the time of Galileo, it has been known that near the surface of the Earth, a body that is free to fall accelerates downwards at a constant rate, whatever its mass and whatever its velocity, provided that air resistance is ignored. This acceleration is called the *acceleration due to gravity* and is conventionally given the symbol g. The value of this acceleration is about $9.8\ \text{m/s}^2$, or in rounder figures, $10\ \text{m/s}^2$.

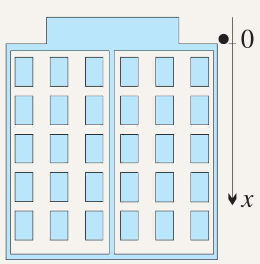
The acceleration is downwards. Thus if upwards is taken as positive, the acceleration is $-g$, but if downwards is taken as positive, the acceleration is g.

9 THE ACCELERATION DUE TO GRAVITY

- A body that is falling accelerates downwards at a constant rate $g \doteqdot 9.8\ \text{m/s}^2$, provided that air resistance is ignored.
- If upwards is taken as positive, start with the function $a = -g$ and integrate.
- If downwards is taken as positive, start with the function $a = g$ and integrate.

Example 11 9C

A stone is dropped from the top of a high building. How far has it travelled, and how fast is it going, after 5 seconds? Take $g = 9.8\ \text{m/s}^2$.



SOLUTION

Let x metres be the distance travelled t seconds after the stone is dropped.
This simple sentence puts the origin of space at the top of the building,
it puts the origin of time at the instant when the stone is dropped,
and it makes downwards the positive direction.

Then $a = 9.8$ (given). (1)

Integrating, $v = 9.8t + C$, for some constant C.
Because the stone was dropped, its initial speed was zero,
and substituting, $0 = 0 + C$,
so $C = 0$, and $v = 9.8t$. (2)

Integrating again, $x = 4.9t^2 + D$, for some constant D.
Because the initial displacement of the stone was zero,

$0 = 0 + D$,

so $D = 0$, and $x = 4.9t^2$. (3)

When $t = 5$, $v = 49$ (substituting into (2) above)
and $x = 122.5$ (substituting into (3) above).
Hence the stone has fallen 122.5 metres and is moving downwards at 49 m/s.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

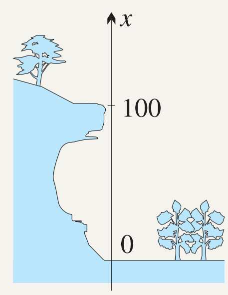
Making a convenient choice of the origin and the positive direction

Physical problems do not come with origins and directions attached. Thus it is up to us to choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible.

The previous worked example made reasonable choices, but the next worked example makes quite different choices. In all such problems, the physical interpretation of negatives and displacements is the mathematician's responsibility, and the final answer should be given in ordinary language.

Example 12 **9C**

A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of 40 m/s and it falls back past the lookout onto the valley floor below.



How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10\ \text{m/s}^2$.)

SOLUTION

Let x be the distance above the valley floor t seconds after the stone is thrown. Again, this simple sentence puts the origin of space at the valley floor, and puts the origin of time at the instant when the stone is thrown. It also makes upwards positive, so that $a = -10$ because the acceleration is downwards.

As discussed, $\ddot{x} = -10$. (1)

Integrating, $v = -10t + C$, for some constant C.

Because $v = 40$ when $t = 0$, $40 = 0 + C$,

so $C = 40$, and $v = -10t + 40$. (2)

Integrating again, $x = -5t^2 + 40t + D$, for some constant D.

Because $x = 100$ when $t = 0$, $100 = 0 + 0 + D$,

so $D = 100$, and $x = -5t^2 + 40t + 100$. (3)

The stone hits the ground when $x = 0$, so using (3) above,

$$\begin{aligned} -5t^2 + 40t + 100 &= 0 \\ t^2 - 8t - 20 &= 0 \\ (t - 10)(t + 2) &= 0 \\ t &= 10 \text{ or } -2. \end{aligned}$$

The ball was not in flight at $t = -2$, so the ball hits the ground after 10 seconds.

Substituting $t = 10$ into equation (2), $v = -100 + 40 = -60$,

so the ball hits the ground at 60 m/s.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Formulae from physics cannot be used

This course requires that even problems where the acceleration is constant, such as the two above, must be solved by integrating the acceleration function. Many readers will know of three very useful equations for motion with constant acceleration a,

$$v = u + at \qquad \text{and} \qquad s = ut + \tfrac{1}{2}at^2 \qquad \text{and} \qquad v^2 = u^2 + 2as.$$

These equations automate the integration process, and so cannot be used in this course. Question **18** in Exercise 9C develops a proper proof of these results.

Integrating trigonometric functions

The next worked example applies the same methods of integration to motion involving trigonometric functions.

Example 13 9C

The velocity of a particle initially at the origin is $v = \sin\frac{1}{4}t$, in units of metres and seconds.

a Find the displacement function.
b Find the acceleration function.
c Find the values of displacement, velocity and acceleration when $t = 4\pi$.
d Briefly describe the motion, and sketch the displacement–time graph.

SOLUTION

a The velocity is $v = \sin\frac{1}{4}t.$ (1)
Integrating (1), $x = -4\cos\frac{1}{4}t + C$, for some constant C.
Substituting $x = 0$ when $t = 0$,
$0 = -4 \times 1 + C,$
$C = 4.$
Thus $x = 4 - 4\cos\frac{1}{4}t.$ (2)

b Differentiating (1), $\ddot{x} = \frac{1}{4}\cos\frac{1}{4}t.$ (3)

c When $t = 4\pi$, $x = 4 - 4 \times \cos\pi$, using (2),
$= 8$ metres.
Also $v = \sin\pi$, using (1),
$= 0$ m/s,
and $\ddot{x} = \frac{1}{4}\cos\pi$, using (3),
$= -\frac{1}{4}$ m/s^2.

d The particle oscillates between $x = 0$ and $x = 8$ with period 8π seconds.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Integrating exponential functions

The next worked example involves exponential functions. The velocity function approaches a limit 'as time goes on'.

Example 14 9C

The acceleration of a particle is given by $a = e^{-2t}$ (in units of metres and seconds), and the particle is initially stationary at the origin.

a Find the velocity and displacement functions.

b Find the displacement when $t = 10$.

c Sketch the velocity–time graph and describe briefly what happens to the velocity of the particle as time goes on.

SOLUTION

a The acceleration is $a = e^{-2t}$. (1)

Integrating, $v = -\frac{1}{2}e^{-2t} + C$, for some constant C.

It is given that when $t = 0$, $v = 0$,

so $0 = -\frac{1}{2} + C$

$C = \frac{1}{2}$,

and $v = -\frac{1}{2}e^{-2t} + \frac{1}{2}$. (2)

Integrating again, $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t + D$, for some constant D.

It is given that when $t = 0$, $x = 0$,

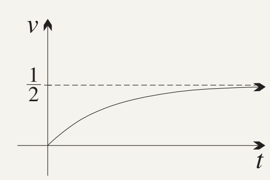
so $0 = \frac{1}{4} + D$

$D = -\frac{1}{4}$,

and $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$. (3)

b When $t = 10$, $x = \frac{1}{4}e^{-20} + 5 - \frac{1}{4}$

$= 4\frac{3}{4} + \frac{1}{4}e^{-20}$ metres.



c Using equation (2), the velocity is initially zero, and increases so that the limiting velocity as time goes on is $\frac{1}{2}$ m/s.

Note: A car moving off from the kerb on level ground obeys this sort of equation if the accelerator is pressed down not too far and kept in that position.

Exercise 9C FOUNDATION

1 A particle is moving with velocity function $v = 3t^2 - 6t$, in units of metres and seconds. At time $t = 0$, its displacement is $x = 4$.

a Integrate, substituting the initial condition, to find the displacement function.

b Show that the particle is at the origin when $t = 2$ and find its velocity then.

c Differentiate the given velocity function to find the acceleration function.

d Show that the acceleration is zero when $t = 1$, and find the displacement then.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 A stone is dropped from a lookout 80 metres high. Take $g = 10\ \text{m/s}^2$ and downwards as positive, so that the acceleration function is $a = 10$.

a Using the lookout as the origin, find the velocity and displacement as functions of t. (Hint: When $t = 0$, $v = 0$ and $x = 0$.)

b Find the time that the stone takes to fall, and its impact speed.

c Where is it, and what is its speed, halfway through its flight time?

d How long does it take to go halfway down, and what is its speed then?

3 A stone is thrown downwards from the top of a 120-metre building, with an initial speed of 25 m/s. Take $g = 10\ \text{m/s}^2$ and take upwards as positive, so that $a = -10$.

a Using the ground as the origin, find the acceleration, velocity and height x of the stone t seconds after it is thrown. (Hint: When $t = 0$, $v = -25$ and $x = 120$.)

b Find the time it takes to reach the ground, and the impact speed.

c What is the average speed of the stone during its descent?

4 a Find the velocity function $\dot{x}$ and the displacement function x of a particle whose initial velocity and displacement are zero if:

i $\ddot{x} = 6t$ **ii** $\ddot{x} = e^{-3t}$ **iii** $\ddot{x} = \cos \pi t$ **iv** $\ddot{x} = 12(t + 1)^{-2}$

b Find the acceleration function a and the displacement function x of a particle whose initial displacement is -2 if:

i $v = -4$ **ii** $v = e^{\frac{1}{2}t}$ **iii** $v = 8 \sin 2t$ **iv** $v = \sqrt{t}$

5 A particle is moving with acceleration $\ddot{x} = 12t$. Initially, it has velocity -24 m/s and is 20 metres on the positive side of the origin.

a Find the velocity function $\dot{x}$ and the displacement function x.

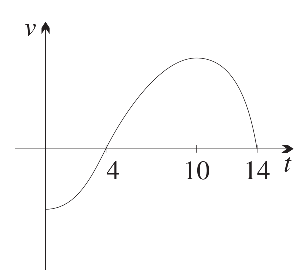
b When does the particle return to its initial position, and what is its speed then?

c What is the minimum displacement, and when does it occur?

d Find x when $t = 0, 1, 2, 3$ and 4, and sketch the displacement-time graph.

DEVELOPMENT

6 The graph to the right shows a particle's velocity–time graph.



a When is the particle moving forwards?

b When is the acceleration positive?

c When is it furthest from its starting point?

d When is it furthest in the negative direction?

e About when does it return to its starting point?

f Sketch the graphs of acceleration and displacement, assuming that the particle is initially at the origin.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

7 A car moves along a straight road from its front gate, where it is initially stationary. During the first 10 seconds, it has a constant acceleration of 2 m/s^2, it has zero acceleration during the next 30 seconds, and it decelerates at 1 m/s^2 for the final 20 seconds until it stops.

a What is the car's speed after 20 seconds?

b Show that the car travels:

i 100 metres during the first 10 seconds,

ii 600 metres during the next 30 seconds,

iii 200 metres during the last 20 seconds.

c Sketch the graphs of acceleration, velocity and distance from the gate.

8 A particle is moving with velocity $\dot{x} = 16 - 4t \text{ cm/s}$ on a horizontal number line.

a Find a and x. (The function x will have a constant of integration.)

b When does it return to its original position, and what is its speed then?

c When is the particle stationary? Find the maximum distances right and left of the initial position during the first 10 seconds, and the corresponding times and accelerations.

d How far does it travel in the first 10 seconds, and what is its average speed?

9 A mouse emerges from his hole and moves out and back along a line. His velocity at time t seconds is

$$v = 4t(t-3)(t-6) = 4t^3 - 36t^2 + 72t \text{ cm/s}.$$

a When does he return to his original position, and how fast is he then going?

b How far does he travel during this time, and what is his average speed?

c What is his maximum speed, and when does it occur?

d If a video of these 6 seconds were played backwards, could this be detected?

10 A body is moving with its acceleration proportional to the time elapsed, that is $\ddot{x} = kt$, for some constant of proportionality k. When $t = 1$, $v = -6$, and when $t = 2$, $v = 3$.

a Find the functions $\dot{x}$ and v. (Hint: Integrate, using the usual constant C of integration. Then find C and k by substituting the two given values of t.)

b When does the body return to its original position?

11 A particle moves from $x = -1$ with velocity $v = \dfrac{1}{t+1}$. How long does it take to get to the origin, and what are its speed and acceleration then? Describe its subsequent motion.

12 A body moving vertically through air experiences an acceleration $a = -40e^{-2t} \text{ m/s}^2$ (we are taking upwards as positive). Initially, it is thrown upwards with speed 15 m/s.

a Taking the origin at the point where it is thrown, find the velocity function $\dot{x}$ and the displacement function x, and find when the body is stationary.

b Find its maximum height and its acceleration then.

c Describe the velocity of the body as $t \to \infty$.

13 A moving particle is subject to an acceleration of $\ddot{x} = -2\cos t \text{ m/s}^2$. Initially it is at $x = 2$, moving with velocity 1 m/s, and it travels for 2π seconds.

a Find the functions v and x.

b When is the acceleration positive?

c When and where is the particle stationary, and when is it moving backwards?

d What are the maximum and minimum velocities, and when and where do they occur?

e Find the change in displacement and the average velocity.

f Sketch the displacement–time graph, and hence find the distance travelled and the average speed.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 Once again, the trains Thomas and Henry are on parallel tracks, level with each other at time zero.

Thomas is moving with velocity $v_T = \dfrac{20}{t+1}$ and Henry with velocity $v_H = 5$.

a Who is moving faster initially, and by how much?

b Find the displacements x_T and x_H of the two trains, if they start at the origin.

c Use your calculator to find during which second the trains are level, and find the speed at which the trains are drawing apart at the end of this second.

d When is Henry furthest behind Thomas, and by how much (nearest metre)?

15 A ball is dropped from a lookout 180 metres high. At the same time, a stone is fired vertically upwards from the valley floor with speed V m/s. Take $g = 10$ m/s^2.

a Find for what values of V a collision in the air will occur. Find, in terms of V, the time and the height when collision occurs, and prove that the collision speed is V m/s.

b Find the value of V for which they collide halfway up the cliff, and the time taken.

16 A falling body experiences both the gravitational acceleration g and air resistance that is proportional to its velocity. Thus a typical equation of motion is $\ddot{x} = -10 - 2v$ m/s^2. Suppose that the body is dropped from the origin.

a By writing $\ddot{x} = \dfrac{dv}{dt}$ and taking reciprocals, find t as a function of v, and hence find v as a function of t. Then find x as a function of t.

b Describe the motion of the particle.

ENRICHMENT

17 [A proof of three constant-acceleration formulae from physics — not to be used elsewhere.]

a A particle moves with constant acceleration a. Its initial velocity is u, and at time t it is moving with velocity v and its distance from its initial position is s. Show that:

i $v = u + at$ **ii** $s = ut + \frac{1}{2}at^2$ **iii** $v^2 = u^2 + 2as$

b Solve questions 2 and 3 using formulae **ii** and **i**, and again using **iii** and **i**.

18 Particles P_1 and P_2 move with velocities $v_1 = 6 + 2t$ and $v_2 = 4 - 2t$, in units of metres and seconds. Initially, P_1 is at $x = 2$ and P_2 is at $x = 1$.

a Find x_1, x_2 and the difference $D = x_1 - x_2$.

b Prove that the particles never meet, and find the minimum distance between them.

c Prove that the midpoint M between the two particles is moving with constant velocity, and find its distance from each particle after 3 seconds.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

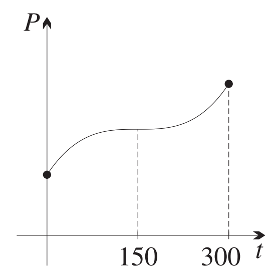
9D Rates and differentiation

When a quantity varies over time, we saw in Section 9J of the Year 11 book that we can differentiate to find the rate at which it is increasing or decreasing at each time t, and differentiate again to find the rate at which that rate is increasing or decreasing. This section reviews those methods in the larger context.

In Chapter 4, we carefully defined a function to be *increasing at* $x = a$ if $f'(a)$ is positive, and *decreasing at* $x = a$ if $f'(a)$ is negative. These were precise *pointwise* definitions. But at the same time we have continued, right from the beginning of Year 11, to use only vague language about a function being *increasing or decreasing over an interval*. These ideas will now be made precise as well.

Increasing and decreasing in an interval

Suppose that the price P of a share in newly-launched company t days after launching is given by the curve $P = P(t)$ drawn to the right.



The share price P is said to be *increasing in the interval* $0 \le t \le 300$ because every chord slopes upwards, that is

$$P(a) < P(b), \quad \text{for all } a < b \text{ in the interval } [0, 300].$$

Notice that except for the isolated point $t = 150$ where the tangent is horizontal, the function P is increasing at every point in the interval. Such a function is clearly increasing in the interval, because if every other tangent slopes upwards, then every chord without exception slopes upwards.

10 INCREASING AND DECREASING IN AN INTERVAL

Suppose that a function $f(x)$ is defined in an interval I. The interval may be bounded or unbounded, and may be open or closed or neither.

- The function $f(x)$ is called *increasing in the interval I* if every chord within the interval slopes upwards, that is,

$$f(a) < f(b), \quad \text{for all } a < b \text{ in the interval.}$$

- The function $f(x)$ is called *decreasing in the interval I* if every chord within the interval slopes downwards, that is,

$$f(a) > f(b), \quad \text{for all } a < b \text{ in the interval.}$$

If $f(x)$ is differentiable in the interval, and is increasing at every point in the interval (except perhaps for some isolated points where the tangent is horizontal), then clearly it is increasing in the interval.

Concave up and down in an interval

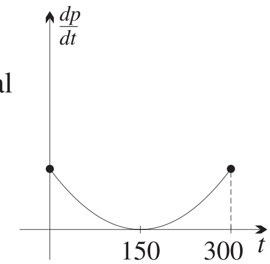
We have also been sloppy when talking about concavity over an interval rather than at a point. The graph above is said to be *concave down in the interval* $[0, 150]$ because every chord lies *under the curve*. The significance of this is that the share price, which is increasing in this interval, is *increasing at a decreasing rate*.

The graph is *concave up in the interval* $[150, 300]$ because every chord in this interval lies *above the curve*. The significance of this is that the share price, which is increasing in this interval as well, is *increasing at an increasing rate*.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Sketched to the right is the gradient function $\dfrac{dP}{dt}$ of the share price P. This gradient function is decreasing in the interval $[0, 150]$, corresponding to the fact that the original curve is concave down in the interval $[0, 150]$. The gradient function $\dfrac{dP}{dt}$ is increasing in the remaining part $[150, 300]$ of the interval, corresponding to the fact that the original function P is concave up in the interval $[150, 300]$.



11 CONCAVE UP AND CONCAVE DOWN IN AN INTERVAL

Suppose that a function $f(x)$ is defined and continuous in an interval I, which may be bounded or unbounded, and may be open or closed or neither.

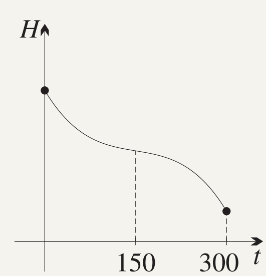
- The function $f(x)$ is called *concave up in the interval I* if every chord within the interval lies above the curve.
- The function $f(x)$ is called *concave down in the interval I* if every chord within the interval lies below the curve.

Concavity is usually a matter of common sense. These may or may not help.

- If $f(x)$ is differentiable in the interval, and $f'(x)$ is increasing in the interval, then $f(x)$ is concave up in the interval.
- If $f(x)$ is doubly differentiable in the interval and concave up at every point in the interval (except perhaps from some isolated points where $f''(x)$ is zero), then it is concave up in the interval.

Example 15 9D

Drought hit Kookaburra Valley last year, and the height of the Everyflow River dropped alarmingly, as shown in the graph to the right of the river height H at Emu Bridge t days after 1st January. Use the features of the graph to describe the behaviour of the river height. What happened in the period just before the 150th day?



SOLUTION

The river height was decreasing for the whole 300 days.
The graph is concave up in the interval $[0, 150]$,
and the height was decreasing at a decreasing rate.
The graph is concave down in the interval $[150, 300]$,
and the height was decreasing at an increasing rate.
It probably rained a little in the period just before $t = 150$.

Note: When we say that the height is 'decreasing at a decreasing rate', we are saying that $\dfrac{dH}{dt}$ is negative and that $\left|\dfrac{dH}{dt}\right|$ is decreasing. Similarly, when we say that the height is 'decreasing at an increasing rate', we are saying that $\dfrac{dH}{dt}$ is negative and that $\left|\dfrac{dH}{dt}\right|$ is increasing. People naturally use the right language here, but thinking about the situation may lead to confusion.

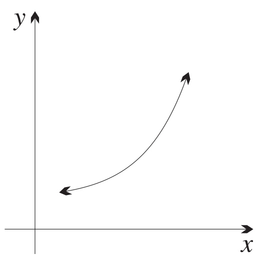
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

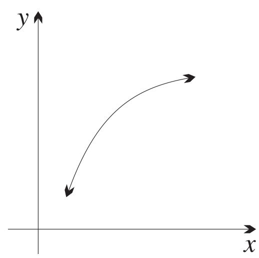
12 INCREASING OR DECREASING AT AN INCREASING OR DECREASING RATE

Suppose that a function $f(x)$ is defined and continuous in an interval I.

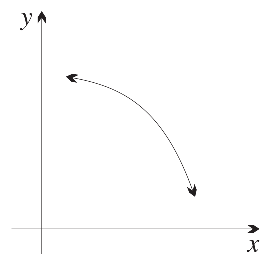
Increasing at an increasing rate: $f(x)$ is increasing and concave up in I.
Increasing at a decreasing rate: $f(x)$ is increasing and concave down in I.
Decreasing at an increasing rate: $f(x)$ is decreasing and concave down in I.
Decreasing at a decreasing rate: $f(x)$ is decreasing and concave up in I.



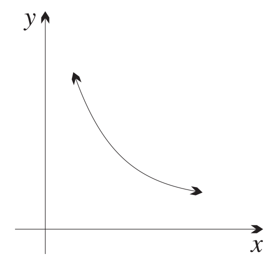
Increasing at an increasing rate



Increasing at a decreasing rate



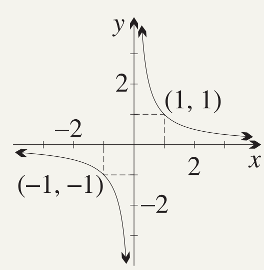
Decreasing at an increasing rate



Decreasing at a decreasing rate

Example 16 9D

Describe the branches of the function $y = \dfrac{1}{x}$ in the terms of this section.

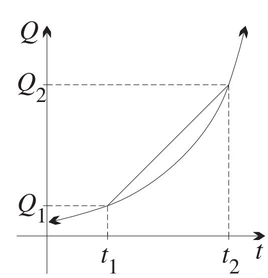


SOLUTION

In the interval $(-\infty, 0)$, the curve is decreasing at an increasing rate, and concave down. In the interval $(0, \infty)$, the curve is decreasing at a decreasing rate, and concave up.

Average rates and instantaneous rates

Worked Example 17 below is an example of a rates question that uses the new language of this section. It also uses differentiation to find a rate, and the second derivative to classify turning points and find inflections. First, however, here is a quick summary of average and instantaneous rates from Section 9J of the Year 11 book.



Suppose that a quantity Q is given as a function of time t, as in the diagram to the right. There are two types of rates.

- An *average rate of change* corresponds to a chord. From the usual gradient formula,

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- An *instantaneous rate of change* corresponds to a tangent. The instantaneous rate of change at time t_1 is the value of the derivative $\dfrac{dQ}{dt}$ at time $t = t_1$,

$$\text{instantaneous rate} = \frac{dQ}{dt}, \text{ evaluated at } t = t_1.$$

A rate of change always means the instantaneous rate of change unless otherwise stated, and is the gradient of the corresponding tangent.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 17 9D

For the first 8 months after its first listing, the share price P in cents of the new company Avocado Marketing followed the cubic function $P = (t - 4)^3 - 12x + 100$, where the time t is in months after listing.

a What were the initial share price and its final share price?

b What were the rate of change of the price, and the rate of change of the rate of change?

c When was the share price at a local maximum or minimum, and what were those values?

d Find any points of inflection, and sketch the curve.

e Describe the behaviour of the price in different intervals of time using the terms in Box 12.

f What was the average rate of increase of the share price over the whole 8 months?

SOLUTION

a When $t = 0$, $\quad P = (-4)^3 + 0 + 100$
$\qquad = 36$ cents.

When $t = 8$, $\quad P = 4^3 - 96 + 100$
$\qquad = 68$ cents.

b Differentiating, $\quad \dfrac{dP}{dt} = 3(t - 4)^2 - 12$

$$\frac{d^2P}{dt^2} = 6(t - 4).$$

c Put $\dfrac{dP}{dt} = 0$ to find the stationary points,

$$\begin{aligned} 3(t-4)^2 &= 12 \\ t - 4 &= 2 \text{ or } -2 \\ t &= 2 \text{ or } 6. \end{aligned}$$

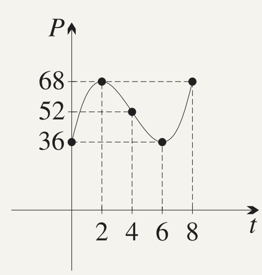
When $t = 2$, $\quad \dfrac{d^2P}{dt^2} = -12 < 0$ and $P = -8 - 24 + 100 = 68$,

and when $t = 6$, $\quad \dfrac{d^2P}{dt^2} = 12 > 0$ and $P = 8 - 72 + 100 = 36$,

so the share price had a local maximum of 68 cents after 2 months, and a local minimum of 36 cents after 6 months.

d There is a zero of $\dfrac{d^2P}{dt^2}$ when $t = 4$, and the value there is $P = 0 - 48 + 100 = 52$.

x	2	4	6
$\dfrac{d^2P}{dt^2}$	-12	0	12
	⌢	·	⌣



The table shows that $\dfrac{d^2P}{dt^2}$ changes sign around the point, so $(4, 52)$ is an inflection.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

e In the interval $[0, 2]$, the price is increasing at a decreasing rate.
In the interval $[2, 4]$, the price is decreasing at an increasing rate.
In the interval $[4, 6]$, the price is decreasing at a decreasing rate.
In the interval $[6, 8]$, the price is increasing at an increasing rate.

f The price at time $t = 0$ was 36 cents, and at time $t = 8$ it was 68 cents.

$$\begin{aligned}\text{Hence average rate of increase} &= \frac{68 - 36}{8}\\ &= 4 \text{ cents per month.}\end{aligned}$$

Exercise 9D

FOUNDATION

1 Grain is pouring into a storage silo. After t minutes there are V tonnes of grain in the silo, where $V = 20t$.

a How much grain is in the silo after 4 minutes?
b Show that the silo was empty to begin with.
c If the silo takes 18 minutes to fill, what is its capacity?
d At what rate is the silo being filled?

2 The amount F litres of fuel in a tank t minutes after it starts to empty is given by $F = 200(20 - t)^2$. Initially the tank is full.

a Find the initial amount of fuel in the tank.
b Find the quantity of fuel in the tank after 15 minutes.
c Find the time taken for the tank to empty, and hence write down the domain of F.
d Show that $\dfrac{dF}{dt} = -400(20 - t)$, and hence find the rate at which the tank is emptying after 5 minutes.
e The value of $\dfrac{dF}{dt}$ is negative for all values of t in the domain. Explain why this is expected in this situation.

3 Grape juice is being pumped into a vat at the rate of $\dfrac{dV}{dt} = 300$ litres per minute, where V litres is the volume of grape juice in the tank after t minutes. The tank already has 1500 litres in it when the pump starts. Rachael correctly guesses that $V = kt + C$, but she does not know the values of k and C.

a Use the initial value to determine C.
b Substitute the formula for V into the equation for $\dfrac{dV}{dt}$ to find the value of k.
c The tank can hold 6000 litres. How long does the pump need to run to fill the tank?

4 Using either the graph of $y = 2^x$ or a suitable reflection in one of the axes, draw a graph to represent a function that is:

a increasing at a decreasing rate,
b decreasing at an increasing rate,
c decreasing at a decreasing rate,
d increasing at an increasing rate.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 Consider the function $y = \sin x$ with domain $0 \le x \le 2\pi$. Sketch the function and then answer the following questions.

a For what values of x in the domain is the function:

i increasing at a decreasing rate, **ii** decreasing at an increasing rate,

iii decreasing at a decreasing rate, **iv** increasing at an increasing rate.

b Use the graph or your answers to part **a** to state where the function is:

i concave up, **ii** concave down.

6 An object is projected vertically, and its height h metres at time t seconds is given by

$h = 180\left(1 - e^{-\frac{1}{3}t}\right) - 30t$.

a Find the rate at which the height is changing.

b What is the initial speed?

c The object reaches its maximum height at time T. Find T and find and the maximum height, correct to the nearest centimetre.

d Find the height, correct to the nearest centimetre, and the speed at time $2T$.

e What is the eventual speed of the object?

DEVELOPMENT

7 When a certain jet engine starts operating, the rate of fuel burn, R kg per minute, t minutes after startup is given by $R = 10 + \dfrac{10}{1 + 2t}$.

a What is the rate of fuel burn after:

i 2 minutes, **ii** 7 minutes?

b What limiting value does R approach as t increases?

c Show that $\dfrac{dR}{dt} < 0$ and that $\dfrac{d^2R}{dt^2} > 0$ for $t \ge 0$.

d Describe the graph of R against t as either increasing or decreasing at either an increasing or decreasing rate.

e Draw a sketch of R as a function of t to confirm your answer to the previous part.

8 For a certain brand of medicine, the amount M present in the blood after t hours is given by $M = 9te^{-t}\mu\text{g}$, for $0 \le t \le 9$. (The symbol μg means 'micrograms'.)

a What are the values of M at $t = 0$ and $t = 9$? Evaluate the latter correct to one decimal place.

b Determine $\dfrac{dM}{dt}$ and hence find the turning point.

c Determine $\dfrac{d^2M}{dt^2}$ and hence find the inflection point.

d Sketch a graph of M against t, showing these features.

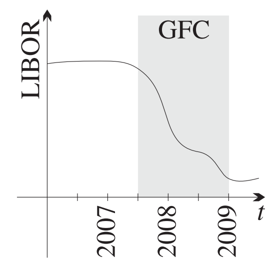
e When is the amount of medicine in the blood a maximum?

f When is the amount of medicine increasing most rapidly?

g When is the amount of medicine decreasing most rapidly?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



9 To the right is a simplified graph showing the effects of the Global Financial Crisis (GFC), from July in 2007 to the end of 2008, on the London Interbank Offered Rate (LIBOR). The LIBOR is sometimes used as a measure of the strength of the world economy.

a According to this graph, when was the crisis at its worst?

b What feature of the graph indicates the end of the crisis in January 2009.

c Why might an economist at the time have been optimistic in July of 2008?

d Sketch a possible graph of $\frac{dL}{dt}$, the derivative of the LIBOR, as a function of time t.

10 The number U of unemployed people at time t was studied over a period of time. At the start of this period, the number of unemployed was 600000.

a Throughout the study, $\frac{dU}{dt} > 0$. What can be deduced about U over this period?

b The study also found that $\frac{d^2U}{dt^2} < 0$. What does this indicate about the changing unemployment level?

c Sketch a graph of U against t, showing this information.

11 A scientist studying an insect colony estimates the number $N(t)$ of insects after t months to be $N(t) = \frac{A}{2 + e^{-t}}$.

a When the scientist begins measuring, the number of insects in the colony is estimated to be 3×10^5. Find A.

b What is the population of the colony one month later?

c How many insects would you expect to find in the nest after a long time?

d Find an expression for the rate at which the population increases with time.

e Hence show that $\frac{dN}{dt} = \frac{N(1 - 2N)}{9 \times 10^5}$.

12 The inflation rate I as a percentage can be modeled using the Consumer Price Index C according to the equation

$$I = \frac{100}{C} \times \frac{dC}{dt}\ \%.$$

The treasury department in the nation of Mercatura has predicted that

$$C(t) = -\tfrac{1}{5}t^3 + 3t^2 + 200$$

for the next eight years, where t is the number of years from now.

a Find an expression for $I(t)$.

b Hence evaluate $I(4)$ correct to two decimal places.

c According to this model, there are two years in which the inflation rate is 0. What are those years, and why must the later value be rejected?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

13 The standard normal distribution, which will be studied later in this course, has probability density function $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

a Show that $\phi(x)$ is an even function.

b Explain why $\phi(x) > 0$ for all values of x.

c Evaluate $\phi(0)$ and determine $\lim\limits_{x \to \infty} \phi(x)$.

d Show that $\phi'(x) = -x\phi(x)$, and hence find where the function is decreasing.

e Show that $\phi''(x) = (x^2 - 1)\phi(x)$, and hence locate the two inflection points.

f Sketch the graph of $y = \phi(x)$ showing all these details.

g Using the graph or the signs of $\phi'(x)$ and $\phi''(x)$, determine where in the domain $\phi(x)$ is decreasing at an increasing rate, and where it is decreasing at a decreasing rate.

h How is the latter evident in the graph?

14 In an ideal situation, a sound wave decays away with distance according to the function $y = 2e^{-ax}\cos x$ for $x \ge 0$, where a is a positive constant.

a Find the y-intercept and x-intercepts.

b Use the product rule to show that $y' = -2e^{-ax}(a\cos x + \sin x)$.

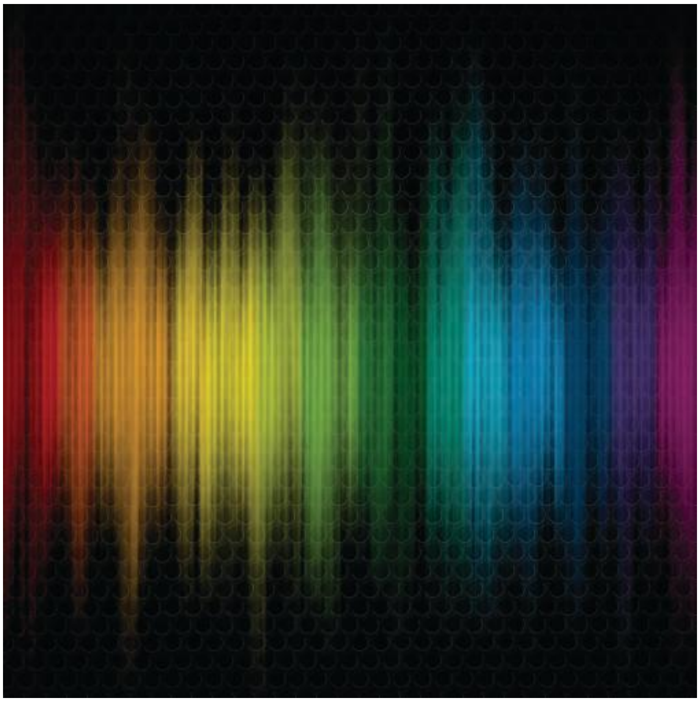
c Use the product rule again to show that $y'' = 2e^{-ax}\big((a^2 - 1)\cos x + 2a\sin x\big)$.

d Show that when $a = \tan\frac{\pi}{12}$, the stationary points are at $x = \frac{11\pi}{12}, \frac{23\pi}{12}, \frac{35\pi}{12}, \ldots$

e It is known that $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. Show that $y'' = 0$ at $x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots$ You may assume that these are the inflection points of the curve.

f Sketch the curve showing this information, approximating the y-coordinates of the stationary and inflection points correct to one decimal place.

g Show that in any interval of 2π, the ratio of where y increases at an increasing rate to where y increases at a decreasing rate is 5 : 7.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9E Review of related rates

In Section 16A of the Year 11 book, we applied the chain rule to rates so that we could deal easily with related rates. This section reviews the ideas and methods.

The example below is a bubble on water, growing over time. The radius and the volume are both increasing. Because volume is a function of radius, we can differentiate with respect to time using the chain rule to express the rate of change of the volume in terms of the rate of change of the radius.

Be very careful with units in all these calculations — this caution holds throughout the chapter, and anywhere where calculus is being applied.

Example 18 9E

A hemispherical bubble on the water surface is growing in size.

a Write down the formula for the volume V mm^3 in terms of the radius r mm, then differentiate with respect to time using the chain rule.

b Find the rate at which the volume is increasing when the radius is 6 mm, if the radius is increasing at 3 mm/s.

c Find the rate at which the radius is increasing when the volume is 18 mm^3, if the volume is increasing at 90 mm^3/s.

SOLUTION

a The volume is
$$V = \tfrac{1}{2} \times \tfrac{4\pi}{3} r^3 = \tfrac{2\pi}{3} r^3.$$

Differentiating,
$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$
$$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}.$$

b Substituting $r = 6$ and $\frac{dr}{dt} = 3$,
$$\frac{dV}{dt} = 2\pi \times 36 \times 3 = 216\pi \text{ mm}^3/\text{s}.$$

c When $V = 18$,
$$18 = \tfrac{2\pi}{3} r^3$$
$$r^3 = 27\pi^{-1}$$
$$r = 3\pi^{-\frac{1}{3}}.$$

Substituting $r = 3\pi^{-\frac{1}{3}}$ and $\frac{dV}{dt} = 90$,
$$90 = 2\pi \times 9\pi^{-\frac{2}{3}} \times \frac{dr}{dt}$$
$$\frac{dr}{dt} = 5\pi^{-\frac{1}{3}} \text{ mm/s}$$
$$\doteqdot 3.4 \text{ mm/s}.$$

13 RELATED RATES

- Express one quantity as a function of the other quantity.
- Then differentiate with respect to time t using the chain rule.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 9E

FOUNDATION

Note: This exercise reviews material covered in Exercise 16A of the Year 11 volume.

1 The sides of a square of side length x metres are increasing at a rate of $0.1\ \text{m/s}$.

a Show that the rate of increase of the area is given by $\dfrac{dA}{dt} = 0.2\,x\ \text{m}^2/\text{s}$.

b At what rate is the area of the square increasing when its sides are 5 metres long?

c What is the side length when the area is increasing at $1.4\ \text{m}^2/\text{s}$?

d What is the area when the area is increasing at $0.6\ \text{m}^2/\text{s}$?

2 The diagonal of a square is decreasing at a rate of $\frac{1}{2}\ \text{m/s}$.

a Find the area A of a square with a diagonal of length ℓ.

b Hence show that the rate of change of area is $\dfrac{dA}{dt} = -\frac{1}{2}\ell\ \text{m}^2/\text{s}$.

c Find the rate at which the area is decreasing when:

i the diagonal is 10 metres,

ii the area is $18\ \text{m}^2$.

d What is the length of the diagonal when the area is decreasing at $17\ \text{m}^2/\text{s}$?

3 The radius r of a sphere is increasing at a rate of $0.3\ \text{m/s}$. In both parts, approximate π using a calculator and give your answer correct to three significant figures.

a Show that the sphere's rate of change of volume is $\dfrac{dV}{dt} = 1.2\pi r^2$, and find the rate of increase of its volume when the radius is 2 metres.

b Show that the sphere's rate of change of surface area is $\dfrac{dS}{dt} = 2.4\pi r$, and find the rate of increase of its surface area when the radius is 4 metres.

4 Jules is blowing up a spherical balloon at a constant rate of $200\ \text{cm}^3/\text{s}$.

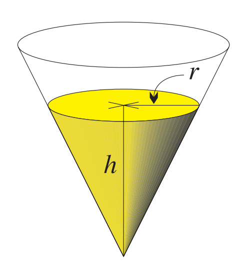
a Show that $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$.

b Hence find the rate at which the radius is growing when the radius is 15 cm.

c Find the radius and volume when the radius is growing at $0.5\ \text{cm/s}$.

DEVELOPMENT

5 An artist pours plaster of Paris into a conical mould at a rate of $5\ \text{cm}^3/\text{s}$. At time t, let the top of the plaster have radius r, and let the depth be h, where $h = 2r$.



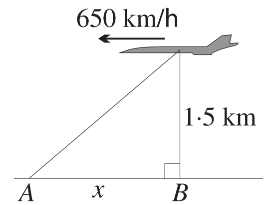
a Find the volume of plaster, and show that it is the same as that of a hemisphere with the same radius.

b Find the rate at which the radius is increasing when the depth of the plaster is 10 cm.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 An observer at A in the diagram is watching a plane fly overhead, and he tilts his head so that he is always looking directly at the plane. The aircraft is flying at $650\ \text{km}/\text{h}$ at an altitude of 1.5 km. Let θ be the angle of elevation of the plane from the observer, and suppose that the distance from A to B, directly below the aircraft, is x km.

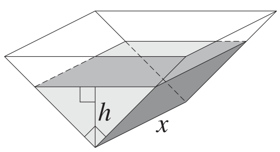


a By writing $x = \dfrac{3}{2\tan\theta}$, show that $\dfrac{dx}{d\theta} = -\dfrac{3}{2\sin^2\theta}$.

b Hence find the rate at which the observer's head is tilting when the angle of inclination to the plane is $\frac{\pi}{3}$. Convert your answer from radians per hour to degrees per second, correct to the nearest degree.

7 A boat is observed from the top of a 100-metre-high cliff. The boat is travelling towards the cliff at a speed of 50 m/min. How fast is the angle of depression changing when the angle of depression is 15°? Convert your answer from radians per minute to degrees per minute, correct to the nearest degree.

8 The water trough in the diagram is in the shape of an isosceles right triangular prism, x metres long. It is found that during summer in a drought, the amount of water lost to evaporation each day in cm^3 is equal to 10% of the surface area in cm^2.



a Find a formula for the volume $V\ \text{cm}^3$ of water in the trough when the depth is h cm.

b Show that the rate at which the depth of the water is changing is constant.

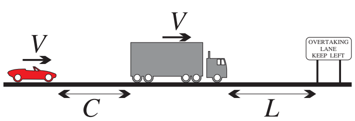
9 The volume of a sphere is increasing at a rate numerically equal to its surface area at that instant. Show that $\dfrac{dr}{dt} = 1$.

10 A point moves anticlockwise around the circle $x^2 + y^2 = 1$ at a uniform speed of $2\ \text{m}/\text{s}$.

a Find an expression for the rate of change of its x-coordinate in terms of x, when the point is above the x-axis. (The units on the axes are metres.)

b Use your answer to part **a** to find the rate of change of the x-coordinate as it crosses the y-axis at $P(0, 1)$. Why should this answer have been obvious without this formula?

11 A car is travelling C metres behind a truck, both travelling at a constant speed of $V\ \text{m}/\text{s}$. The road widens L metres ahead of the truck and there is an overtaking lane. The car accelerates at a uniform rate so that it is exactly alongside the truck at the beginning of the overtaking lane.



a What is the acceleration of the car?

b Show that the speed of the car as it passes the truck is $V\left(1 + \dfrac{2C}{L}\right)$.

c The objective of the driver of the car is to spend as little time alongside the truck as possible. What strategies could the driver employ?

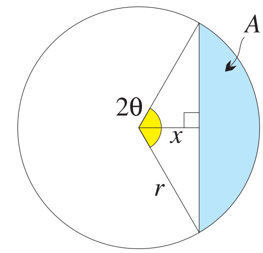
d The speed limit is 100 km/h, and the truck is travelling at 90 km/h and is 50 metres ahead of the car. How far before the overtaking lane should the car begin to accelerate if applying the objective in part **c**?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

12 The diagram shows a chord distant x from the centre of a circle. The radius of the circle is r, and the chord subtends an angle 2θ at the centre.



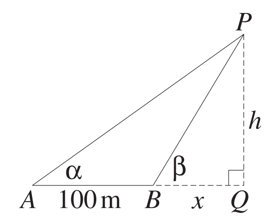
a Show that the area of the segment cut off by this chord is $A = r^2(\theta - \sin\theta\cos\theta)$.

b Explain why $\dfrac{dA}{dt} = \dfrac{dA}{d\theta} \times \dfrac{d\theta}{dx} \times \dfrac{dx}{dt}$.

c Show that $\dfrac{d\theta}{dx} = -\dfrac{1}{\sqrt{r^2 - x^2}}$.

d Given that $r = 2$, find the rate of increase in the area if $\dfrac{dx}{dt} = -\sqrt{3}$ when $x = 1$.

13 The diagram shows two radars at A and B 100 metres apart. An aircraft at P is approaching and the radars are tracking it, hence the angles α and β are changing with time.



a Show that $x\tan\beta = (x + 100)\tan\alpha$.

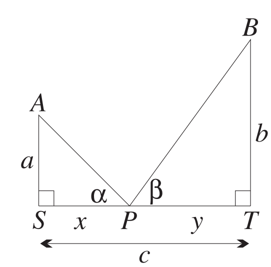
b Keeping in mind that x, α and β are all functions of time, use the chain and product rules to show that

$$\frac{dx}{dt} = \frac{\dot{\alpha}(x + 100)\sec^2\alpha - \dot{\beta}x\sec^2\beta}{\tan\beta - \tan\alpha}.$$

c Use part **a** to find the value of x and the height of the plane when $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$.

d At the angles given in part **c**, it is found that $\dfrac{d\alpha}{dt} = \frac{5}{36}\left(\sqrt{3} - 1\right)$ radians per second and $\dfrac{d\beta}{dt} = \frac{5}{18}\left(\sqrt{3} - 1\right)$ radians per second. Find the speed of the plane.

14 [A proof that for reflected light, the angle of incidence is equal to the angle of reflection.] Suppose that a light source is at A above a reflective surface ST, and the reflected light is observed at B. Further suppose that at the point of reflection P, the angle of incidence is $(90° - \alpha)$ and the angle of reflection is $(90° - \beta)$. Let $AS = a$, $BT = b$ and $ST = c$. Also let $SP = x$ and $PT = y$. We will assume that light travels between two points along a path that takes the shortest time and therefore AP and PB are straight lines. We will also assume that the speed of light is constant.



a Show that the distance APB is $s = \sqrt{a^2 + x^2} + \sqrt{b^2 + y^2}$.

b Let v be the speed of light, then $vt = s$. Also note that $y = c - x$. Use these results and the chain rule where necessary to show that

$$v\frac{dt}{dx} = \frac{\frac{x}{a}}{\sqrt{1 + \left(\frac{x}{a}\right)^2}} - \frac{\frac{y}{b}}{\sqrt{1 + \left(\frac{y}{b}\right)^2}}.$$

c Hence show that t has a stationary point when $\dfrac{x}{a} = \dfrac{y}{b}$.

d By considering the values of $\dfrac{dt}{dx}$ when P is at the points S and T, show that this stationary point is a minimum.

e Complete the proof by showing that $\alpha = \beta$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9F Rates and integration

In many situations, what is given is the rate $\frac{dQ}{dt}$ at which a quantity Q is changing. The original function Q can then be found by integration — and as with motion, never omit the constant of integration. To evaluate the constant of integration, use an *initial* or *boundary condition*, giving the value of Q at some particular time t.

14 FINDING THE QUANTITY FROM THE RATE

Suppose that the rate of change of a quantity Q is known as a function of time t.

- Integrate to find Q as a function of time.
- Never omit the constant of integration.
- Use an initial or boundary condition to evaluate the constant of integration.

Example 19 9F

A tank contains 40 000 litres of water. When the draining valve is opened, the volume V in litres of water in the tank decreases at a variable rate given by $\frac{dV}{dt} = -1500 + 30t$, where t is the time in seconds after opening the valve. Once the water stops flowing, the valve shuts off.

a When does the water stop flowing?

b Give a common-sense reason why the rate $\frac{dV}{dt}$ is negative up to this time.

c Integrate to find the volume of water in the tank at time t, and sketch the graph of volume V as a function of time t.

d How much water has flowed out of the tank and how much remains?

SOLUTION

a Put $\quad \frac{dV}{dt} = 0.$

Then $\quad -1500 + 30t = 0$

$\quad t = 50,$

so it takes 50 seconds for the flow to stop.

b During this 50 seconds, the water is flowing out of the tank.

Hence the volume V is decreasing, so the derivative $\frac{dV}{dt}$ is negative.

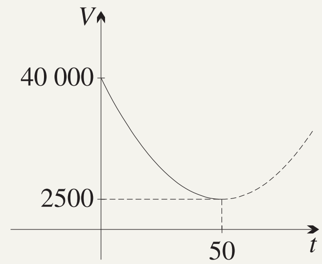
c Integrating, $\quad V = -1500t + 15t^2 + C$, for some constant C.

It is given that when $t = 0$, $V = 40\,000$,

and substituting, $\quad 40\,000 = 0 + 0 + C$

$\quad C = 40\,000.$

Hence $\quad V = 40\,000 - 1500t + 15t^2.$



d When $t = 50$, $\quad V = 40\,000 - (1500 \times 50) + (15 \times 2500)$

$\quad = 2500.$

Hence the tank still holds 2500 litres when the valve closes,

so $40\,000 - 2500 = 37\,500$ litres has flowed out during the 50 seconds.

ISBN 978-1-108-76630-2

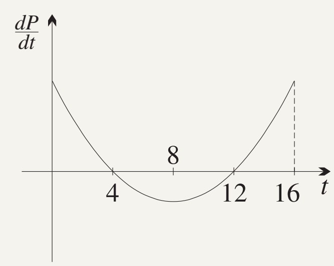
Cambridge University Press
Updated July 2022

Questions with a diagram or a graph instead of an equation

In some problems about rates, a graph of some function is known, but its equation is unknown. Such problems require careful attention to zeroes and turning points and points of inflection. An approximate sketch of another graph often needs to be drawn.

Example 20 9F

Frog numbers were increasing in the Ranavilla district, but during a long drought, the rate of increase fell and actually became negative for a few years. The rate $\dfrac{dP}{dt}$ of population growth of the frogs has been graphed to the right as a function of the time t years after careful observations began.



a When was the frog population neither increasing nor decreasing?
b When was the frog population decreasing and when was it increasing?
c When was the frog population decreasing most rapidly?
d When, during the first 12 years, was the frog population at a maximum?
e When, during the years $4 \le t \le 16$, was the frog population at a minimum?
f Draw a possible graph of the frog population P against time t.

SOLUTION

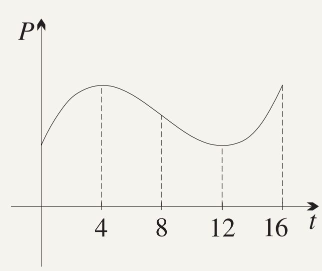
a The graph shows that $\dfrac{dP}{dt}$ is zero when $t = 4$ and again when $t = 12$.
These are the times when the frog population was neither increasing nor decreasing.

b The graph shows that $\dfrac{dP}{dt}$ is negative when $4 < t < 12$,
so the population was decreasing during the years $4 < t < 12$.
The graph shows that $\dfrac{dP}{dt}$ is positive when $0 < t < 4$ and when $12 < t < 16$,
so the population was increasing during the years $0 < t < 4$ and during $12 < t < 16$.

c The frog population was decreasing most rapidly when $t = 8$.

d The population was at a maximum when $x = 4$,
because from parts **a** and **b**, the population was rising before this and falling afterwards.

e Similarly, the population was minimum when $x = 12$.



f All that matters is to draw the possible graph of P so that its gradients are consistent with the graph of $\dfrac{dP}{dT}$. These things were discussed above in parts **a**–**d**. Also, the frog population must never fall below zero.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Rates involving the exponential function

Many natural events involve a quantity that dies away gradually, with an equation that involves the exponential function. The next worked example uses the standard form $\int e^{ax+b} = \frac{1}{a}e^{ax+b} + C$ to evaluate the primitive of $3\,e^{-0.02t}$. The full working is

$$\begin{aligned}\int 3\,e^{-0.02t}dt &= 3 \times \frac{1}{-0.02} \times e^{-0.02t} + C \\ &= -3 \times \frac{100}{2} \times e^{-0.02t} + C \\ &= -150\,e^{-0.02t} + C.\end{aligned}$$

Example 21 — 9F

During a drought, the flow rate $\frac{dV}{dt}$ of water from Welcome Well gradually diminishes according to the formula $\frac{dV}{dt} = 3\,e^{-0.02t}$, where V is the volume in megalitres of water that has flowed out during the first t days after time zero.

a Show that $\frac{dV}{dt}$ is always positive, and explain the physical significance of this.

b Find the volume V as a function of time t.

c How much water will flow from the well during the first 100 days?

d Describe the behaviour of V as $t \to \infty$, and find what percentage of the total flow comes in the first 100 days. Then sketch the function.

SOLUTION

a Because $e^x > 0$ for all x, $\frac{dV}{dt} = 3\,e^{-0.02t}$ is always positive.
The volume V is always increasing, because V is the volume that has flowed out of the well, and the water doesn't flow backwards into the well.

b The given rate is $\quad \frac{dV}{dt} = 3\,e^{-0.02t}$.

Integrating, $\quad V = -150\,e^{-0.02t} + C$ (using the calculation above).

When $t = 0$, no water has flowed out, so $V = 0$,

and substituting, $\quad 0 = -150 \times e^0 + C$

$C = 150.$

Hence $\quad V = -150\,e^{-0.02t} + 150$

$= 150\left(1 - e^{-0.02t}\right).$

c When $t = 100$, $\quad V = 150\,(1 - e^{-2})$

$\doteqdot 129.7$ megalitres.

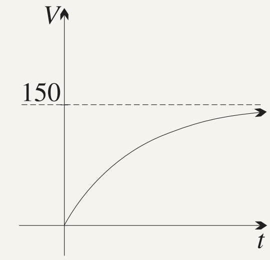
d As $t \to \infty$, $e^{-0.02t} \to 0$, so $V \to 150$.

Hence $\quad \frac{\text{flow in first 100 days}}{\text{total flow}} = \frac{150\,(1 - e^{-2})}{150}$

$= 1 - e^{-2}$

$= 0.864\,66\ldots$

$\doteqdot 86.5\%.$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 9F

FOUNDATION

1 Twenty-five wallabies are released on Wombat Island and the population is observed over the next six years. It is found that the rate of increase in the wallaby population is given by $\frac{dP}{dt} = 12t - 3t^2$, where time t is measured in years.

a Show that $P = 25 + 6t^2 - t^3$.

b After how many years does the population reach a maximum? (Hint: Let $\frac{dP}{dt} = 0$.)

c What is the maximum population?

d When does the population increase most rapidly? (Hint: Let $\frac{d^2P}{dt^2} = 0$.)

2 Water is flowing out of a tank at the rate of $\frac{dV}{dt} = 10t - 250$, where V is the volume in litres remaining in the tank at time t minutes after time zero.

a When does the water stop flowing?

b Given that the tank still has 20 litres left in it when the water flow stops, show that the volume V at any time is given by $V = 5t^2 - 250t + 3145$.

c How much water was initially in the tank?

3 The rate at which a perfume ball loses its scent over time is $\frac{dP}{dt} = -\frac{2}{t+1}$, where t is measured in days.

a Find P as a function of t if the initial perfume content is 6.8.

b How long will it be before the perfume in the ball has run out and it needs to be replaced? (Answer correct to the nearest day.)

4 A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by $\frac{dV}{dt} = -2 + \frac{1}{10}t \text{ m}^3/\text{s}$.

a What is the initial flow rate, when the tap is fully on?

b How long does it take to turn the tap off?

c Given that when the tap has been turned off there are still 500 m^3 of water left in the tank, find V as a function of t.

d Hence find how much water is released during the time it takes to turn the tap off.

e Suppose that it is necessary to let out a total of 300 m^3 from the tank. How long should the tap be left fully on before gradually turning it off?

5 The velocity of a particle is given by $\frac{dx}{dt} = e^{-0.4t}$.

a Does the particle ever stop moving?

b If the particle starts at the origin, show that its displacement x as a function of t is given by $x = \frac{5}{2}\left(1 - e^{-0.4t}\right)$.

c When does the particle reach $x = 1$? (Answer correct to two decimal places.)

d Where does the particle eventually move to? (That is, find its limiting position.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

6 A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time t, is given by $\dfrac{dx}{dt} = 250\left(e^{-0.2t} - 1\right)$, where x is the height above the ground.

a What is its initial speed?

b What is its eventual speed?

c Find x as a function of t, if it is initially 200 metres above the ground.

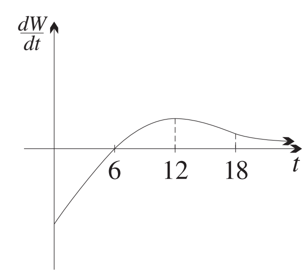
7 Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\dfrac{dI}{dt} = -5 + 4\cos \frac{\pi}{12}t$, where I is the tonnage of ice on the mountain at time t in hours since 2:00 am on 20th October.

a It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find I as a function of t.

b Explain, from the given rate, why the ice is always melting.

c The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.

8 The graph to the right shows the rate $\dfrac{dW}{dt}$ at which the average weight W of bullocks at St Vidgeon station was changing t months after a drought was officially proclaimed.



a When was the average weight decreasing and when was it increasing?

b When was the average weight at a minimum?

c When was the average weight increasing most rapidly?

d What appears to have happened to the average weight as time went on?

e Sketch a possible graph of the average weight W.

9 As a particle moves around a circle, its angular velocity is given by $\dfrac{d\theta}{dt} = \dfrac{1}{1 + t^2}$.

a Given that the particle starts at $\theta = \frac{\pi}{4}$, find θ as a function of t.

b Hence find t as a function of θ.

c Using the result of part **a**, show that $\frac{\pi}{4} \le \theta < \frac{3\pi}{4}$, and hence explain why the particle never moves through an angle of more than $\frac{\pi}{2}$.

10 The flow of water into a small dam over the course of a year varies with time and is approximated by $\dfrac{dW}{dt} = 1.2 - \cos^2 \frac{\pi}{12}t$, where W is the volume of water in the dam, measured in thousands of cubic metres, and t is the time measured in months from the beginning of January.

a What is the maximum flow rate into the dam, and when does this happen?

b Given that the dam is initially empty, find W.

c The capacity of the dam is 25 200 m^3. Show that it will be full in three years.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 A certain brand of medicine tablet is in the shape of a sphere with diameter 5 mm. The rate at which the pill dissolves is $\dfrac{dr}{dt} = -k$, where r is the radius of the sphere at time t hours, and k is a positive constant.

a Show that $r = \frac{5}{2} - kt$.

b The pill dissolves completely in 12 hours. Find k.

12 Sand is poured onto the top of a pile in the shape of a cone at a rate of $0.5\ \text{m}^3/\text{s}$. The apex angle of the cone remains constant at 90°. Let the base have radius r and let the height of the cone be h.

a Find the volume of the cone, and show that it is one quarter of the volume of a sphere with the same radius.

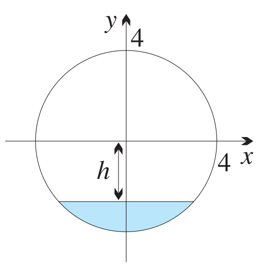
b Find the rate of change of the radius of the cone as a function of r.

c By taking reciprocals and integrating, find t as a function of r, given that the initial radius of the pile was 10 metres.

d Hence find how long it takes, correct to the nearest second, for the pile to grow another 2 metres in height.

ENRICHMENT

13 **a** The diagram shows the spherical cap formed when the region between the lower half of the circle $x^2 + y^2 = 16$ and the horizontal line $y = -h$ is rotated about the y-axis. Find the volume V so formed.



b The cap represents a shallow puddle of water left after some rain. When the sun comes out, the water evaporates at a rate proportional to its surface area (which is the circular area at the top of the cap).

i Find this surface area A.

ii We are told that $\dfrac{dV}{dt} = -kA$. Show that the rate at which the depth of the water changes is $-k$.

iii The puddle is initially 2 cm deep and the evaporation constant is known to be $k = 0.025\ \text{cm}/\text{min}$. Find how long it takes for the puddle to evaporate.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Chapter 9 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 9 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

t	2	4
x		

1 For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity from $t = 2$ to $t = 4$. The units in each part are centimetres and seconds.

a $x = 20 + t^2$ **b** $x = (t + 2)^2$ **c** $x = t^2 - 6t$ **d** $x = 3^t$

2 For each displacement function below, find the velocity function and the acceleration function. Then find the displacement, the velocity and the acceleration of the particle when $t = 5$. All units are metres and seconds.

a $x = 40t - t^2$ **b** $x = t^3 - 25t$ **c** $x = 4(t - 3)^2$
d $x = 50 - t^4$ **e** $x = 4\sin \pi t$ **f** $x = 7e^{3t-15}$

3 A ball rolls up an inclined plane and back down again. Its distance x metres up the plane after t seconds is given by $x = 16t - t^2$.

a Find the velocity function v and the acceleration function a.
b What are the ball's position, velocity, speed and acceleration after 10 seconds?
c When does the ball return to its starting point, and what is its velocity then?
d When is the ball farthest up the plane, and where is it then?
e Sketch the displacement–time graph, the velocity–time graph and the acceleration–time graph.

4 Differentiate each velocity function below to find the acceleration function a. Then integrate v to find the displacement function x, given that the particle is initially at $x = 4$.

a $v = 7$ **b** $v = 4 - 9t^2$ **c** $v = (t - 1)^2$
d $v = 0$ **e** $v = 12\cos 2t$ **f** $v = 12e^{-3t}$

5 For each acceleration function below, find the velocity function v and the displacement function x, given that the particle is initially stationary at $x = 2$.

a $a = 6t + 2$ **b** $a = -8$ **c** $a = 36t^2 - 4$
d $a = 0$ **e** $a = 5\cos t$ **f** $a = 7e^t$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

6 A particle is moving with acceleration function $\ddot{x} = 6t$, in units of centimetres and seconds. Initially it is at the origin and has velocity 12 cm/s in the negative direction.

a Find the velocity function $\dot{x}$ and the displacement function x.

b Show that the particle is stationary when $t = 2$.

c Hence find its maximum distance on the negative side of the origin.

d When does it return to the origin, and what are its velocity and acceleration then?

e What happens to the particle's position and velocity as time goes on?

7 A stone is thrown vertically upwards with velocity 40 m/s from a fixed point B situated 45 metres above the ground. Take $g = 10\ \text{m/s}^2$.

a Taking upwards as positive, explain why the acceleration function is $a = -10$.

b Using the ground as the origin, find the velocity function v and the displacement function x.

c Hence find how long the stone takes to reach its maximum height, and find that maximum height.

d Show that the time of flight of the stone until it strikes the ground is 9 seconds.

e With what speed does the stone strike the ground?

f Find the height of the stone after 1 second and after 2 seconds.

g Hence find the average velocity of the stone during the 2nd second.

8 The acceleration of a body moving along a line is given by $\ddot{x} = \sin t$, where x is the distance from the origin O at time t seconds.

a Sketch the acceleration–time graph.

b From your graph, state the first two times after $t = 0$ when the acceleration is zero.

c Integrate to find the velocity function, given that the initial velocity is -1 m/s.

d What is the first time when the body stops moving?

e The body is initially at $x = 5$.

i Find the displacement function x.

ii Find where the body is when $t = \frac{\pi}{2}$.

9 The velocity of a particle is given by $v = 20\,e^{-t}$, in units of metres and seconds.

a What is the velocity when $t = 0$?

b Why is the particle always moving in a positive direction?

c Find the acceleration function a.

d What is the acceleration at time $t = 0$?

e The particle is initially at the origin. Find the displacement function x.

f What happens to the acceleration, the velocity and the displacement as t increases?

10 The stud farm at Benromach sold a prize bull to a grazier at Dalmore, 300 kilometres west. The truck delivering the bull left Benromach at 9:00 am, driving over the dirt roads at a constant speed of 50 km/hr. At 10:00 am, the driver realised that he had left the sale documents behind, so he drove back to Benromach at the same speed. He then drove the bull and the documents straight to Dalmore at 60 km/hr.

a Draw the displacement–time graph of his displacement x kilometres west of Benromach at time t hours after 9:00 am.

b What total distance did he travel?

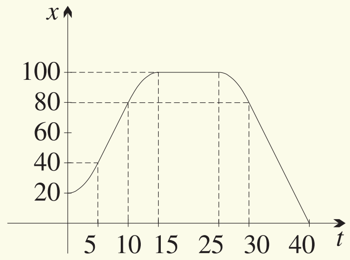
c What was his average road speed for the whole journey?

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Review

11 Crispin was trying out his bicycle in Abigail Street. The graph below shows his displacement in metres north of the oak tree after t seconds.



a Where did he start from, and what was his initial speed?

b What was his velocity:

i from $t = 5$ to $t = 10$,

ii from $t = 15$ to $t = 25$,

iii from $t = 30$ to $t = 40$?

c In what direction was he accelerating:

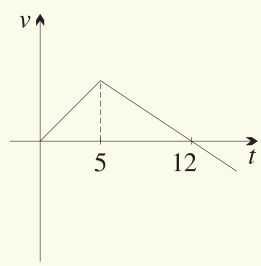
i from $t = 0$ to $t = 5$,

ii from $t = 10$ to $t = 15$,

iii from $t = 25$ to $t = 30$?

d Draw a possible sketch of the velocity–time graph.

12 A small rocket was launched vertically from the ground. The graph to the right shows its velocity–time graph. After a few seconds the motor cut out. A few seconds later the rocket reached its maximum height and then began to fall back towards the ground.



a When did the motor cut out?

b When was the rocket stationary, when was it moving upwards and when was it moving downwards?

c When was the rocket accelerating upwards, and when was it accelerating downwards?

d When was the rocket at its maximum height?

e Sketch the acceleration–time graph.

f Sketch the displacement–time graph.

13 **a** Draw rough sketches of these functions for $0 \le x \le \frac{\pi}{2}$.

i $y = \sin x$ **ii** $y = \cos x$ **iii** $y = \tan x$ **iv** $y = \cot x$

b For the graphs you drew, and over the given domain, which functions are:

i increasing at a decreasing rate **ii** decreasing at an increasing rate

iii decreasing at a decreasing rate **iv** increasing at an increasing rate

14 The kakapo is a critically endangered species of parrot in New Zealand. In 2017 the New Zealand Department of Conservation released a number of kakapo on Little Barrier Island. Biologists hoped that the population would grow slowly at first and later more quickly. It was also expected that after several more years the population would then grow at a slower and slower rate. Answer the following questions, assuming the biologists' predictions were correct.

a Explain why a graph of the kakapo population K over time t has an inflection point.

b Sketch a possible graph of K as a function of t, showing the inflection point.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

15 The volume V litres of water in a tank at time t minutes is given by $V = 3(50 - 2t)^2$, for $0 \le t \le 25$.

a What is the initial volume of liquid in the tank?

b Determine $\dfrac{dV}{dt}$.

c Use $\dfrac{dV}{dt}$ to explain why the tank must be emptying.

d Does the outflow increase or decrease with time?

16 The velocity of a particle is given by $\dfrac{dx}{dt} = 3 - 2e^{-\frac{1}{5}t}$, with x measures in metres and t in seconds.

a Draw a graph of the velocity versus time.

b Is the particle accelerating or decelerating? Confirm your answer by finding $\dfrac{d^2x}{dt^2}$.

c What is the eventual velocity of the particle?

d The particle starts at the origin. Find x as a function of t.

17 James had a full drink bottle containing 500 ml of Gatorade™. He drank from it so that the volume V ml of Gatorade™ in the bottle changed at a rate given by $\dfrac{dV}{dt} = \left(\frac{2}{5}t - 20\right)$ ml/s.

a Find a formula for V.

b Show that it took James 50 seconds to drink the contents of the bottle.

c How long, correct to the nearest second, did it take James to drink half the contents of the bottle?

18 A 5-metre ladder is leaning against a wall, and the base is sliding away from the wall at 5 cm/s. Find the rate at which:

a the height,

b the angle of inclination, is changing when the foot is 1.4 m from the wall.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

10 Projectile motion

This short chapter deals with just one case of motion in two dimensions — the motion of a projectile, such as a thrown ball or a shell fired from a gun.

A *projectile* is something that is thrown or fired into the air, and subsequently moves under the influence of gravity alone. We regard it as a point.

- Missiles and aeroplanes are not projectiles, because they have motors on them that keep pushing them forwards.
- We shall ignore air resistance, so we will not be dealing with objects such as sheets of metal or pieces of paper where air resistance cannot reasonably be ignored.
- Our projectiles are always moving close to the Earth's surface, where the acceleration due to gravity is a constant $g \doteqdot 9.8\ \text{m/s}^2$.

Everyone can see that a projectile moves in a path that looks like a parabola. Our task is to set up the equations that describe this motion so that we can solve problems.

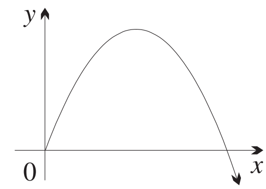
Vectors are useful in describing projectile motion, particularly when it comes to describing the velocity, which keeps changing direction. But apart from their components, they are not necessary in this topic. Their great advantage is that they provide a more concise description of the situation.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10A Projectile motion — the time equations



The diagram to the right shows the sort of path we would expect a projectile to move in. The two-dimensional space in which it moves has been made into a number plane by choosing an origin — in this case the point from which the projectile was fired — and measuring horizontal distance x and vertical distance y from this origin.

We could put time t on the graph, which would require a third dimension for the t-axis. But we will instead treat t as a parameter, because at each time t during the flight, the projectile is at some point $P(x, y)$ on its path.

These pronumerals x, y and t for horizontal distance, vertical distance and time, will often be used in this chapter without further introduction.

Using position vectors to specify a point

The choice of basis vectors should coincide with the chosen axes. Thus $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the positive directions of the x-axis and y-axis respectively.

Then a point such as $P(30, 15)$ can also be described by stating that its position vector is $\overrightarrow{OP} = 30\underset{\sim}{i} + 15\underset{\sim}{j}$.

Specifying the velocity

With two-dimensional motion, *velocity* is a vector. Its *speed* is its magnitude, and it is travelling in particular *direction*. When an object is moving through the air, we can describe its velocity at a particular time t in two different ways:

- We can give its speed and the angle at which it is moving. For example, at some instant of time a ball may be moving at 12 m/s with an angle of inclination of $60°$ or $-60°$. This *angle of inclination* is always measured from the horizontal, and is taken as negative if the object is travelling downwards.
- We can also specify the velocity at that instant by giving the rates $\dot{x}$ and $\dot{y}$ at which the horizontal displacement x and the vertical displacement y are changing. Using the basis vector $\underset{\sim}{i}$ and $\underset{\sim}{j}$, we can write the velocity as a vector
  $$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}.$$

The horizontal and vertical velocities are projections of $\underset{\sim}{v}$ onto the basis vectors,
$$\dot{x} = \underset{\sim}{v} \cdot \underset{\sim}{i} \text{ and } \dot{y} = \underset{\sim}{v} \cdot \underset{\sim}{j}.$$

1 TWO WAYS TO SPECIFY VELOCITY

The velocity of a projectile at some particular time t can be described in two ways:

- We can give the speed and angle of inclination. The *angle of inclination* is the acute angle between the path and the horizontal. It is positive if the object is travelling upwards, and negative if the object is travelling downwards.

OR

- We can give the horizontal and vertical components $\dot{x}$ and $\dot{y}$. This allows the velocity to be written as a vector
  $$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j} \text{ where } \dot{x} = \underset{\sim}{v} \cdot \underset{\sim}{i} \text{ and } \dot{y} = \underset{\sim}{v} \cdot \underset{\sim}{j}.$$

ISBN 978-1-108-76630-2

The resolution of velocity

The conversion from one way of describing velocity to the other requires a diagram to *resolve the velocity into its horizontal and vertical components*, or to convert back from components to speed and direction, as in the worked examples below. Vectors are the basis of the calculations, but usually the geometric diagram is sufficient for the working.

Example 1 — 10A

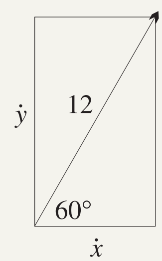
Use a diagram to resolve the velocity into its horizontal and vertical components, for a projectile moving with speed 12 m/s and angle of inclination:

a $60°$, **b** $-60°$.

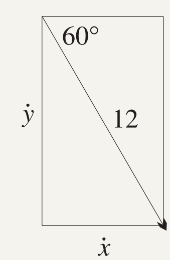
Give your answer as an expression for the vector velocity $\underset{\sim}{v}$ in vector components.

SOLUTION

a
$$\begin{aligned}\dot{x} &= 12\cos 60° \\ &= 6 \text{ m/s}, \\ \dot{y} &= 12\sin 60° \\ &= 6\sqrt{3} \text{ m/s}, \\ \text{so } \underset{\sim}{v} &= 6\underset{\sim}{i} + 6\sqrt{3}\,\underset{\sim}{j} \text{ m/s}.\end{aligned}$$



b
$$\begin{aligned}\dot{x} &= 12\cos 60° \\ &= 6 \text{ m/s}, \\ \dot{y} &= -12\sin 60° \\ &= -6\sqrt{3} \text{ m/s}, \\ \text{so } v &= 6\underset{\sim}{i} - 6\sqrt{3}\,\underset{\sim}{j} \text{ m/s}.\end{aligned}$$



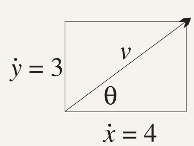
Example 2 — 10A

Find the speed v and angle of inclination θ (correct to the nearest degree) of a projectile whose velocity vector is:

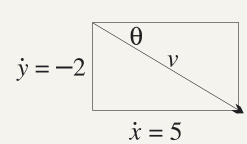
a $\underset{\sim}{v} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ m/s **b** $\underset{\sim}{v} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$ m/s

SOLUTION

a
$$\begin{aligned}v^2 &= 4^2 + 3^2 \\ v &= 5 \text{ m/s}, \\ \tan\theta &= \tfrac{3}{4} \\ \theta &\doteqdot 37°.\end{aligned}$$



b
$$\begin{aligned}v^2 &= 5^2 + 2^2 \\ v &= \sqrt{29} \text{ m/s}, \\ \tan\theta &= -\tfrac{2}{5} \\ \theta &\doteqdot -22°.\end{aligned}$$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 RESOLUTION OF VELOCITY

To convert between velocity given in terms of speed v and angle of inclination θ, and velocity given in terms of horizontal and vertical components $\dot{x}$ and $\dot{y}$:

- Use a diagram to *resolve the velocity into its horizontal and vertical components*.
- Alternatively, use the conversion equations

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} v^2 = \dot{x}^2 + \dot{y}^2 \\ \tan\theta = \dfrac{\dot{y}}{\dot{x}} \end{cases}$$

(where ambiguity between θ and $180^\circ - \theta$ may need to be clarified).

The independence of the vertical and horizontal motion

We have already seen that gravity affects every object free to move by accelerating it downwards with the same constant acceleration g, where $g \doteqdot 9.8 \text{ m/s}^2$ (or 10 m/s^2 in round figures).

In this course, a projectile is unaffected by air resistance, and has no motor. No force acts on a projectile except for the downwards force of gravity.

- Because this acceleration is downwards, it affects the vertical component $\dot{y}$ of the velocity according to $\ddot{y} = -g$.
- It has no effect, however, on the horizontal component $\dot{x}$, and thus $\ddot{x} = 0$.

Every projectile motion is therefore governed by this same pair of equations,

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g.$$

In vector form, the acceleration vector $\underset{\sim}{a}$ is $\underset{\sim}{a} = -g\underset{\sim}{j}$, because the horizontal component of acceleration is zero.

3 THE FUNDAMENTAL EQUATIONS OF PROJECTILE MOTION

- Projectile motion is governed by the acceleration vector $\underset{\sim}{a} = -g\underset{\sim}{j}$.
- In practice, however, we work with the equations for the horizontal and vertical components of acceleration,

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g.$$

- Unless otherwise indicated, every question on projectile motion should begin with these two equations.
- The working will usually involve four integrations, two for $\ddot{x}$, and two for $\ddot{y}$.
- There will be four corresponding substitutions of the boundary conditions.

The integrations will yield six equations — the original equations for vertical and horizontal acceleration, two equations for vertical and horizontal velocity, and two equations for vertical and horizontal displacement.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 3 10A

A ball is thrown with initial speed 40 m/s and angle of inclination 30° from the top of a stand 25 metres above the ground.

a Using the stand as the origin and $g = 10\text{ m/s}^2$, find the six equations of motion.

b Find how high it rises, how long it takes to get there, what its speed is then, and how far it is horizontally from the stand.

c Find the flight time, the horizontal range, and the impact speed and angle.

SOLUTION

Initially, $x = y = 0$, and $\dot{x} = 40\cos 30° = 20\sqrt{3}$, $\quad \dot{y} = 40\sin 30° = 20$.

a To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$.

When $t = 0$, $\dot{x} = 20\sqrt{3}$

$20\sqrt{3} = C_1$,

so $\dot{x} = 20\sqrt{3}$. (2)

Integrating, $x = 20t\sqrt{3} + C_2$.

When $t = 0$, $x = 0$

$0 = C_2$,

so $x = 20t\sqrt{3}$. (3)

To begin, $\ddot{y} = -10$. (4)

Integrating, $\dot{y} = -10t + C_3$.

When $t = 0$, $\dot{y} = 20$

$20 = C_3$,

so $\dot{y} = -10t + 20$. (5)

Integrating, $y = -5t^2 + 20t + C_4$.

When $t = 0$, $y = 0$

$0 = C_4$,

so $y = -5t^2 + 20t$. (6)

b At the top of its flight, the vertical component of velocity is zero,

so put $\dot{y} = 0$.

From (5), $-10t + 20 = 0$

$t = 2$ seconds (the time taken).

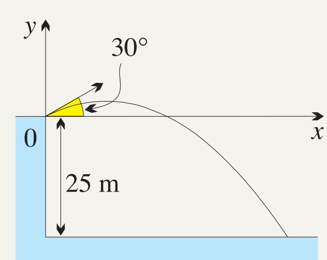
When $t = 2$, from (6), $y = -20 + 40$

$= 20$ metres (the maximum height above the stand).

When $t = 2$, from (3), $x = 40\sqrt{3}$ metres (the horizontal distance from the stand).

Because the vertical component of velocity is zero, the speed there is $\dot{x} = 20\sqrt{3}$ m/s.

c It hits the ground when it is 25 metres below the stand,



so put $y = -25$.

From (6), $-5t^2 + 20t = -25$

$t^2 - 4t - 5 = 0$

$(t - 5)(t + 1) = 0$,

so it hits the ground when $t = 5$ ($t = -1$ is inadmissible).

From (3), when $t = 5$, $x = 100\sqrt{3}$ metres,

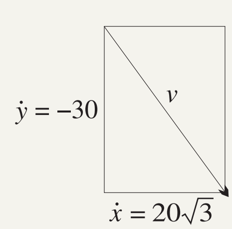
so the horizontal range is $100\sqrt{3}$ metres.

Using a diagram to resolve the velocity at the impact,

$\dot{x} = 20\sqrt{3}$ and $\dot{y} = -50 + 20 = -30$,

so $v^2 = 1200 + 900$

$v = 10\sqrt{21}$ m/s (the impact speed),



and $\tan\theta = -\dfrac{30}{20\sqrt{3}}$

$\theta \doteqdot -40°54'$, and the impact angle is about $40°54'$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using pronumerals for initial speed and angle of inclination

Many problems in projectile motion require the initial speed or angle of inclination to be found so that the projectile behaves in some particular fashion. Often the muzzle speed of a gun will be fixed, but the angle at which it is fired can easily be altered — in such situations there are usually two solutions, corresponding to a low-flying shot and a 'lobbed' shot that goes high into the air.

Example 4 — 10A

A gun at O fires shells with an initial speed of 200 m/s, but a variable angle of inclination α. Take $g = 10\ \text{m/s}^2$.

a Find the two possible angles at which the gun can be set so that it will hit a fortress F that is 2 km away on top of a mountain 1000 metres high.

b Show that the inclination of the lower angle to OF is the same as the inclination of the higher angle to the vertical.

c Find the corresponding flight times and the impact speeds and angles.

SOLUTION

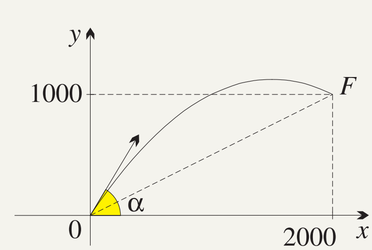
Place the origin at the gun, so that initially, $x = y = 0$.

Resolving the initial velocity, $\dot{x} = 200\cos\alpha \quad \dot{y} = 200\sin\alpha$

To begin, $\ddot{x} = 0. \qquad (1)$
Integrating, $\dot{x} = C_1$.
When $t = 0$, $\dot{x} = 200\cos\alpha$
$200\cos\alpha = C_1$,
so $\dot{x} = 200\cos\alpha. \qquad (2)$
Integrating, $x = 200t\cos\alpha + C_2$.
When $t = 0$, $x = 0$
$0 = C_2$,
so $x = 200t\cos\alpha. \qquad (3)$

To begin, $\ddot{y} = -10. \qquad (4)$
Integrating, $\dot{y} = -10t + C_3$.
When $t = 0$, $\dot{y} = 200\sin\alpha$
$200\sin\alpha = C_1$,
so $\dot{y} = -10t + 200\sin\alpha. \qquad (5)$
Integrating, $y = -5t^2 + 200t\sin\alpha + C_4$.
When $t = 0$, $y = 0$
$0 = C_4$,
so $y = -5t^2 + 200t\sin\alpha. \qquad (6)$

a Because the fortress is 2 km away, $x = 2000$,
so from (3),
$$200t\cos\alpha = 2000$$
$$t = \frac{10}{\cos\alpha}.$$
Because the mountain is 1000 metres high, $y = 1000$,
so from (6),
$$-5t^2 + 200t\sin\alpha = 1000.$$
Hence
$$-\frac{500}{\cos^2\alpha} + \frac{2000\sin\alpha}{\cos\alpha} - 1000 = 0$$
$$\sec^2\alpha - 4\tan\alpha + 2 = 0.$$



But $\sec^2\alpha = \tan^2\alpha + 1$,
so
$$\tan^2\alpha - 4\tan\alpha + 3 = 0$$
$$(\tan\alpha - 3)(\tan\alpha - 1) = 0$$
$$\tan\alpha = 1 \text{ or } 3$$
$$\alpha = 45^\circ \text{ or } \tan^{-1}3\ (\doteqdot 71^\circ 34').$$

ISBN 978-1-108-76630-2

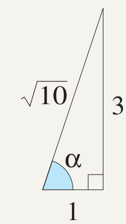
Cambridge University Press
Updated July 2022

b $\angle OFX = \tan^{-1}\frac{1}{2} \doteqdot 26°34'$, so the 45° shot is inclined at $18°26'$ to OF, and the $71°34'$ shot is inclined at $18°26'$ to the vertical.

c When $\alpha = 45°$, from (a), $\quad t = \dfrac{10}{\cos\alpha} = 10\sqrt{2}$ seconds,

and when $\quad t = 10\sqrt{2}$, from (5), $\dot{y} = -100\sqrt{2} + 200 \times \frac{1}{2}\sqrt{2} = 0$,

so from (2), the shell hits horizontally at $100\sqrt{2}$ m/s.



When $\alpha = \tan^{-1}3$, $\quad \cos\alpha = \dfrac{1}{\sqrt{10}}$, and $\sin\alpha = \dfrac{3}{\sqrt{10}}$,

so from (a), $\quad t = \dfrac{100}{\cos\alpha} = 10\sqrt{10}$ seconds,

and when $t = 10\sqrt{10}$, from (5), $\quad \dot{y} = -100\sqrt{10} + 60\sqrt{10} = -40\sqrt{10}$,

and from (2), $\quad \dot{x} = 20\sqrt{10}$,

so $\quad v^2 = 16\,000 + 4000 = 20\,000$

$\quad v = 100\sqrt{2}$ m/s,

and using resolution of velocity, $\quad \tan\theta = \dfrac{\dot{y}}{\dot{x}} = -2$

$\quad \theta = -\tan^{-1}2 \ (\doteqdot -63°26')$

so the shell hits at $100\sqrt{2}$ m/s, at about $63°26'$ to the horizontal.

Exercise 10A

FOUNDATION

Note: In this exercise take $g = 10\ \text{m/s}^2$ unless otherwise indicated.

1 A particle is projected from the origin with a speed of $30\sqrt{2}$ m/s at an angle of 45° to the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 30$ and $\dot{y} = -10t + 30$.
- **b** Hence find x and y in terms of t.
- **c** Put $y = 0$ to find when the particle returns to the x-axis.
- **d** Hence find the horizontal distance travelled by the particle.
- **e** Put $\dot{y} = 0$ to find when the particle reaches its greatest height above the x-axis.
- **f** Find the greatest height.

2 A particle is projected from horizontal ground with a speed of 40 m/s at an angle of 30° to the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 20\sqrt{3}$ and $\dot{y} = -10t + 20$.
- **b** Hence find x and y in terms of t.
- **c** Find:
  - **i** when the particle returns to the ground,
  - **ii** the horizontal distance travelled by the particle,
  - **iii** the greatest height reached above the ground.

3 A particle is projected from the origin with a speed of 20 m/s at an angle of 60° to the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 10$ and $\dot{y} = -10t + 10\sqrt{3}$.
- **b** Hence find x and y in terms of t.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Use Pythagoras' theorem to find, correct to one decimal place:

i the distance of the particle from the origin after one second,

ii the speed of the particle after one second.

4 A particle is projected from ground level with a speed of 60 m/s at an angle of $\tan^{-1}\frac{4}{3}$ to the horizontal.

a Show that $\dot{x} = 36$ and $\dot{y} = -10t + 48$.

b Find x and y as functions of t.

c Find, correct to one decimal place, the distance of the particle from the point of projection after 3 seconds.

d Show that the velocity of the particle after 3 seconds is $18\sqrt{5}$ m/s at an angle of $\tan^{-1}\frac{1}{2}$ above the horizontal. (Note that velocity is a vector quantity, so both the speed and the direction must be specified.)

5 A particle is projected from the origin. Its initial velocity vector is $8\underset{\sim}{i} + 6\underset{\sim}{j}$.

a Express its velocity at time t in the form $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$.

b Express its displacement at time t in the form $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$.

c Find:

i the initial speed of the particle,

ii the position vector of the particle after 2 seconds,

iii the position vector of the particle when it reaches its greatest height.

6 A particle is projected from the origin. Its horizontal and vertical components of displacement after t seconds are $x = 40t$ and $y = -5t^2 + 25t$ respectively.

a Find the initial values of $\dot{x}$ and $\dot{y}$.

b Hence show that the initial velocity is $5\sqrt{89}$ at an angle of approximately 32° to the horizontal.

DEVELOPMENT

7 A particle is projected from the origin with initial speed V at an angle of α to the horizontal. Two seconds later it passes through the point with position vector $8\underset{\sim}{i} - 12\underset{\sim}{j}$.

a Show that $V\cos\alpha = 4$ and $V\sin\alpha = 4$, and hence write down the initial velocity vector of the particle.

b Find the position vector of the particle after 0.5 seconds.

8 A particle is projected from a point O with initial speed V m/s at an angle of θ to the horizontal. After 2 seconds its horizontal and vertical displacements from O are both 30 m.

a Show that $V\cos\theta = 15$ and $V\sin\theta = 25$.

b Hence show that $V = 5\sqrt{34}$ m/s and $\theta = \arctan\frac{5}{3}$.

9 A stone is thrown from the top of an 11 m high vertical tower standing on level ground. The initial speed is 12 m/s and the initial direction is 30° *below* the horizontal.

a Starting with the horizontal and vertical components of acceleration, show that $\dot{x} = 6\sqrt{3}$ and $\dot{y} = -10t - 6$.

b If the origin is at the point of projection, find x and y in terms of t.

c How long will it take for the stone to hit the ground?

d How far from the base of the tower, in metres correct to one decimal place, will the stone hit the ground?

e Show that the stone will hit the ground at an angle of $\tan^{-1}\frac{8\sqrt{3}}{9}$ below the horizontal.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 A ball is tossed with initial speed 6 m/s at an angle of 45° to the horizontal. It hits the ground at a point that is 2 m below the point of projection. Find, correct to two decimal places:

a the time of flight of the ball,

b the horizontal distance travelled by the ball.

11 A projectile was fired from level ground and just cleared a wall that is 10 m high and 20 m from the point of projection. If the angle of projection was 36°, find the initial speed of the projectile in m/s correct to the nearest integer.

12 A ball was projected from ground level and, when at its highest point, it just cleared a 3-metre wall. Given that the initial velocity of the ball was 20 m/s at an angle of α to the horizontal, prove that $\alpha = \sin^{-1}\frac{\sqrt{15}}{10}$.

13 Two particles P_1 and P_2 are projected from the origin with initial velocity vectors $20\underset{\sim}{i} + 30\underset{\sim}{j}$ and $60\underset{\sim}{i} + 50\underset{\sim}{j}$ respectively. If P_2 is projected 2 seconds after P_1, determine whether the particles collide and, if so, when they collide.

14 A particle is projected from the origin with initial speed $20\sqrt{2}$ m/s. On its flight it passes through the point $(20, 15)$ at time t. Let θ be the angle of projection.

a Show that $\sqrt{2}\,t\cos\theta = 1$ and $4\sqrt{2}\,t\sin\theta - t^2 = 3$.

b Hence show that $\tan^2\theta - 8\tan\theta + 7 = 0$.

c Hence find, correct to the nearest minute where necessary, the two possible values of θ.

15 A particle is projected with initial speed 50 m/s. On its flight it passes through a point P that is at a horizontal distance of 100 m and a vertical distance of 25 m from the point of projection.

a Show that the angle of projection α is $\tan^{-1}\frac{1}{2}$ or $\tan^{-1}\frac{9}{2}$.

b For each of the possible angles of projection, find:

i the time it takes for the particle to reach P,

ii the velocity of the particle as it passes through P, giving answers correct to one decimal place where necessary.

16 A particle is projected from level ground with initial speed V m/s at an angle of θ to the horizontal.

a Prove that:

i the greatest height is $\dfrac{V^2\sin^2\theta}{2g}$ metres,

ii the horizontal range is $\dfrac{V^2\sin 2\theta}{g}$ metres.

b If the horizontal range is five times the greatest height, prove that $\theta = \arctan\frac{4}{5}$.

ENRICHMENT

17 A particle is projected from the floor of a horizontal tunnel that is 2 m high. If the particle just touches the ceiling of the tunnel, prove that the horizontal range inside the tunnel is $\sqrt{\dfrac{16}{g}(V^2 - 4g)}$ metres.

18 A projectile is fired from level ground with initial speed V at an angle of θ to the horizontal. Suppose that the greatest height of the projectile is h.

a Prove that $V^2\sin^2\theta = 2gh$.

b Prove that the particle is at height $\dfrac{h}{2}$ when $t = \dfrac{\left(\sqrt{2}+1\right)\sqrt{h}}{\sqrt{g}}$ or $t = \dfrac{\left(\sqrt{2}-1\right)\sqrt{h}}{\sqrt{g}}$.

c If the ratio of the speed of the particle at height $\dfrac{h}{2}$ to the speed at height h is $\sqrt{5}:\sqrt{2}$, find θ.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10B Projectile motion — the equation of path

The formulae for x and y in terms of t give a parametric equation of the physical path of the projectile through the x–y plane. Eliminating t will give the cartesian equation of the path, which is always an upside-down parabola. Many questions are solved more elegantly by consideration of the equation of path. Unless the question gives it, however, the equation of path must be derived each time.

The general case

The working below derives the equation of path in the general case of a projectile fired from the origin with initial speed V and angle of elevation α.

Resolving the initial velocity, $\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha$.

To begin, $\ddot{x} = 0$, (1)
Integrating, $\dot{x} = C_1$.
When $t = 0$, $\dot{x} = V\cos\alpha$
$V\cos\alpha = C_1$,
so $\dot{x} = V\cos\alpha$. (2)
Integrating, $x = Vt\cos\alpha + C_2$.
When $t = 0$, $x = 0$
$0 = C_2$,
so $x = Vt\cos\alpha$. (3)

To begin, $\ddot{y} = -g$. (4)
Integrating, $\dot{y} = -gt + C_3$.
When $t = 0$, $\dot{y} = V\sin\alpha$
$V\sin\alpha = C_3$,
so $\dot{y} = -gt + V\sin\alpha$. (5)
Integrating, $y = -\frac{1}{2}gt^2 + Vt\sin\alpha + C_4$.
When $t = 0$, $y = 0$
$0 = C_4$,
so $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$. (6)

From (3), $$t = \frac{x}{V\cos\alpha}.$$

Substituting into (6), $$y = -\frac{gx^2}{2V^2\cos^2\alpha} + \frac{Vx\sin\alpha}{V\cos\alpha}$$
$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha.$$

The Pythagorean identity $\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$ gives an alternative form
$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\alpha) + x\tan\alpha.$$

This working must always be shown unless the equation is given in the question.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 THE EQUATION OF PATH

- The path of a projectile fired from the origin with initial speed V and angle of elevation α is
$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \qquad \text{(not to be memorised)}.$$
- The Pythagorean identities give an alternative form
$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\alpha) + x\tan\alpha.$$
- This last equation is quadratic in x, $\tan\alpha$ and V, and is linear in g and y.
- Differentiation of the equation of path gives the gradient of the path for any value of x. This provides an alternative approach to finding the angle of inclination of a projectile in flight.

Example 5 10B

Use the equation of path above in these questions.

a Show that the range on level ground is $\dfrac{V^2}{g}\sin 2\alpha$, and hence find the maximum range.

b Arrange the equation of path as a quadratic in $\tan\alpha$, and hence show that with a given initial speed V and variable angle α of elevation, a projectile can be fired through the point $P(x, y)$ if and only if $2V^2gy \le V^4 - g^2x^2$.

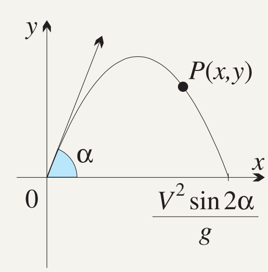
c What does this mean geometrically?

SOLUTION

a Put $y = 0$, then $\dfrac{gx^2\sec^2\alpha}{2V^2} = x\tan\alpha$,

so $x = 0$ or $\dfrac{gx}{2V^2\cos^2\alpha} = \dfrac{\sin\alpha}{\cos\alpha}$

$$x = \frac{2V^2}{g}\cos\alpha\sin\alpha$$
$$x = \frac{V^2}{g}\sin 2\alpha.$$



Hence the projectile lands $\dfrac{V^2}{g}\sin 2\alpha$ away from the origin.

Because $\sin 2\alpha$ has a maximum value of 1 when $\alpha = 45°$,

the maximum range is $\dfrac{V^2}{g}$ when $\alpha = 45°$.

b Multiplying the equation of path through by $2V^2$,
$$2V^2y = -gx^2(1 + \tan^2\alpha) + 2V^2x\tan\alpha$$
$$2V^2y + gx^2 + gx^2\tan^2\alpha - 2V^2x\tan\alpha = 0$$
$$gx^2\tan^2\alpha - 2V^2x\tan\alpha + (2V^2y + gx^2) = 0.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

This equation is now a quadratic in $\tan\alpha$. Using the standard theory of quadratics, it will have a solution for $\tan\alpha$ when $\triangle \ge 0$,

$$(2V^2x)^2 - 4 \times gx^2 \times (2V^2y + gx^2) \ge 0$$
$$4V^4x^2 - 8x^2V^2gy - 4g^2x^4 \ge 0$$
$$\boxed{\div\, 4x^2} \qquad 2V^2gy \le V^4 - g^2x^2.$$

c Rearranging again, $y \le \dfrac{V^4 - g^2x^2}{2V^2g}$.

This means that the target $P(x,\ y)$ must lie on or under the parabola $y = \dfrac{V^4 - g^2x^2}{2V^2g}$,

whose vertex $\left(0, \dfrac{V^2}{2g}\right)$ lies on the y-axis, and whose zeroes are $x = \dfrac{V^2}{g}$ and $x = -\dfrac{V^2}{g}$.

Exercise 10B

FOUNDATION

Note: In this exercise take $g = 10\ \text{m/s}^2$ unless otherwise indicated.

1 Suppose that a particle is projected from the origin and its parabolic path has equation
$y = -\frac{5}{324}x^2 + \frac{4}{3}x$.

a Find the height of the particle when it has travelled a horizontal distance of 12 m.

b Find how far the particle has travelled horizontally when its height is 19 m.

c Find $\dfrac{dy}{dx}$ and hence show that:

i the angle of projection is $\tan^{-1}\frac{4}{3}$,

ii when the horizontal distance travelled is 18 m, the direction of motion is $\tan^{-1}\frac{7}{9}$ above the horizontal,

iii when the horizontal distance travelled is 54 m, the direction of motion is $\tan^{-1}\frac{1}{3}$ below the horizontal.

2 A particle is projected from a point O. Its parabolic motion is governed by the parametric equations $x = 48t$ and $y = -5t^2 + 20t$, where x metres and y metres are the respective horizontal and vertical components of the displacement from O after t seconds.

a Show that the parabolic path has Cartesian equation $y = -\frac{5}{2304}x^2 + \frac{5}{12}x$.

b Use the Cartesian equation of the parabola to find:

i the horizontal range of the particle,

ii the greatest height of the particle,

iii the angle of projection (correct to one decimal place),

iv the direction in which the particle is moving when $x = 120$ (correct to the nearest degree).

3 A particle is projected from the origin O with initial velocity vector $3\underset{\sim}{i} + \underset{\sim}{j}$.

a What is the initial speed of the particle?

b Show that the parabolic path has parametric equations $x = 3t$ and $y = t - 5t^2$.

c Hence show that the Cartesian equation of the path is $9y = 3x - 5x^2$.

d Find, correct to one decimal place, the direction of motion when:

i $x = 0.15$ **ii** $t = 0.15$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

4 An object is tossed from the top of a 20 m tower with initial velocity vector $5\underset{\sim}{i}$.

a If the point of projection is the origin, find the Cartesian equation of the path of the object.

b Find how far the object lands from the base of the tower.

c Find, correct to the nearest degree, the direction in which the object is travelling when it hits the ground.

DEVELOPMENT

5 A ball is tossed from the origin O that is 6 m above ground level. The initial velocity is 24 m/s at an angle of 30° above the horizontal.

a Find $\dot{x}$, $\dot{y}$, x and y as functions of t.

b Show that the Cartesian equation of the path of the ball is $y = \frac{1}{\sqrt{3}}x - \frac{5}{432}x^2$.

c If D metres is the horizontal distance that the ball has travelled when it strikes the ground, show that $5D^2 - 144\sqrt{3}\,D - 2592 = 0$.

d Hence find D correct to one decimal place.

6 A stone is thrown from the top of a 60 m cliff at an angle of 27° below the horizontal. It lands in the ocean 35 m from the base of the cliff.

a Taking the origin at the point of projection, show that the path of the stone has Cartesian equation $y = -x\tan 27° - \dfrac{5x^2}{V^2\cos^2 27°}$, where V is the initial speed.

b Hence find:

i the initial speed of the stone correct to one decimal place,

ii the direction in which the stone is moving, correct to the nearest degree, when it lands in the ocean.

7 A particle is projected from the origin with initial speed V at an angle of θ to the horizontal. You may assume that the equation of its path is $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2V^2}$.

a Show that the horizontal range of the particle is $\dfrac{V^2\sin 2\theta}{g}$.

b If the initial speed is 30 m/s and the horizontal range is to be 75 m, find the two possible values of θ correct to one decimal place.

8 A particle is projected from the origin with initial speed V at an angle of θ to the horizontal. You may assume the equation of its parabolic path (as given, for example, in Question 7).

a Find $\dfrac{dy}{dx}$, and hence show that the vertex of the parabola is $\left(\dfrac{V^2\sin 2\theta}{2g}, \dfrac{V^2\sin^2\theta}{2g}\right)$.

b If the initial speed is 20 m/s and the greatest height is 15 m, find θ.

9 A gun can fire a shell with a constant initial speed V and a variable angle of elevation α. Assume that t seconds after being fired, the horizontal and vertical displacements x and y of the shell from the gun are given by

$$x = Vt\cos\alpha \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + Vt\sin\alpha.$$

a Show that the Cartesian equation of the shell's path may be written as

$$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Suppose that $V = 200\ \text{m/s}$, and that the shell hits a target positioned 3 km horizontally and 0.5 km vertically from the gun. Show that $\tan\alpha = \dfrac{4+\sqrt{3}}{3}$ or $\tan\alpha = \dfrac{4-\sqrt{3}}{3}$, and hence find the two possible values of α, correct to the nearest minute.

10 A particle is projected from a point O with speed 34 m/s at an angle of θ above the horizontal.

a Show that the parabolic path of the particle has equation $y = x\tan\theta - \dfrac{5x^2\sec^2\theta}{1156}$.

b During its flight the particle passes through a point that is 11 m above O and 30 m horizontally from O. Show that the two possible angles of projection are $\theta = \arctan\frac{8}{15}$ and $\theta = \arctan\frac{538}{75}$.

ENRICHMENT

11 A boy throws a ball with speed V m/s at an angle of 45° to the horizontal.

a Derive expressions for the horizontal and vertical components of the displacement of the ball from the point of projection.

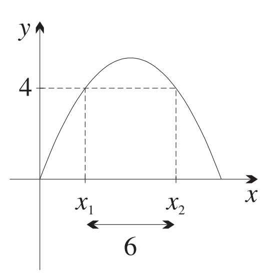
b Hence show that the Cartesian equation of the path of the ball is $y = x - \dfrac{gx^2}{V^2}$.

c The boy is now standing on a hill inclined at an angle θ to the horizontal. He throws the ball at the same angle of elevation of 45° and at the same speed of V m/s. If he can throw the ball 60 metres measured down the hill, but only 30 metres measured up the hill, use the result in part (b) to show that

$$\tan\theta = 1 - \frac{30g\cos\theta}{V^2} = \frac{60g\cos\theta}{V^2} - 1,$$

and hence that $\theta = \tan^{-1}\frac{1}{3}$.

12 A particle is projected from the origin with speed V m/s at an angle α to the horizontal.



a Assuming that the coordinates of the particle at time t are $\left(Vt\cos\alpha, Vt\sin\alpha - \frac{1}{2}gt^2\right)$, prove that the horizontal range R of the particle is $\dfrac{V^2\sin 2\alpha}{g}$.

b Hence prove that the path of the particle has equation $y = x\left(1 - \dfrac{x}{R}\right)\tan\alpha$.

c Suppose that $\alpha = 45°$ and that the particle passes through two points 6 metres apart and 4 metres above the point of projection, as shown in the diagram. Let x_1 and x_2 be the x-coordinates of the two points.

i Show that x_1 and x_2 are the roots of the equation $x^2 - Rx + 4R = 0$.

ii Use the identity $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_2x_1$ to find R.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 10 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 10 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Note: In this exercise take $g = 10\text{ m/s}^2$ unless otherwise indicated.

1 A particle is projected from horizontal ground with a speed of 60 m/s at an angle of 40° above the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 60\cos 40°$ and $\dot{y} = -10t + 60\sin 40°$.
- **b** Hence find x and y in terms of t.
- **c** Find, correct to one decimal place:
  - **i** when the particle returns to the ground,
  - **ii** the horizontal distance travelled by the particle,
  - **iii** the greatest height reached above the ground.

2 A rock is thrown from the top of a vertical cliff of height 40 metres. Its initial velocity is 30 m/s at an angle of 60° above the horizontal. Find, correct to two decimal places where necessary:

- **a** the greatest height reached by the rock above the point of projection,
- **b** the distance that the rock lands from the base of the cliff,
- **c** the speed at which the rock hits the ground.

3 A particle is projected from the origin O with a speed of 25 m/s at an angle of $\tan^{-1}\frac{4}{3}$ to the horizontal.

- **a** Show that $\dot{x} = 15$ and $\dot{y} = -10t + 20$.
- **b** Find x and y as functions of t.
- **c** Find the distance of the particle from the point of projection after one second.
- **d** Show that the velocity of the particle after one second is $5\sqrt{13}$ m/s at an angle of $\tan^{-1}\frac{2}{3}$ above the horizontal.
- **e** If R is the horizontal range of the particle and H is the greatest height, show that $R = 3H$.

4 An object is projected from level ground with initial speed 10 m/s at an angle of 45° to the horizontal.

- **a** Find $\dot{x}$, x, $\dot{y}$ and y by integration from $\ddot{x} = 0$ and $\ddot{y} = -10$. Then, by eliminating t, show that the equation of the parabolic path is $y = x - \frac{1}{10}x^2$.
- **b** Use the equation of the path to find the horizontal range and the maximum height.
- **c** Suppose first that the stone hits a wall 8 metres away.
  - **i** Find how far up the wall the stone hits.
  - **ii** Differentiate the equation of the path, and hence find the direction of the object when it hits the wall.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

d Suppose now that the stone hits a ceiling 2.1 metres high.

i Find the horizontal distance travelled before impact.

ii Find the angle at which the stone hits the ceiling.

5 A particle is projected from the origin. Its initial velocity vector is $24\underset{\sim}{i} + 18\underset{\sim}{j}$.

a Express its velocity at time t in component vector form.

b Express its displacement at time t in component vector form.

c **i** Find the initial speed of the particle,

ii Find the position vector of the particle after 4 seconds,

iii Find the position vector of the particle when it reaches its greatest height.

6 A ball is thrown with initial speed V at an angle of α to the horizontal. Two seconds later it passes through the point with position vector $24\sqrt{5}\underset{\sim}{i} + 28\underset{\sim}{j}$.

a Find the initial velocity vector of the ball.

b Find the position vector of the ball after 3 seconds.

c Is the ball rising or falling after 3 seconds? Justify your answer.

7 A particle is projected from a point O on level ground with initial speed V m/s at an angle of θ to the horizontal. Its horizontal range is 108 m, and its flight time is 3 seconds.

a Show that $V = 39$ m/s and $\theta = \arctan \frac{5}{12}$.

b Find the greatest height reached by the particle.

8 Steve tosses an apple to Adam, who is sitting near him. Adam catches the apple at exactly the same height that Steve released it. Suppose that the initial speed of the apple is $V = 5$ m/s, and the initial angle α of elevation is given by $\tan \alpha = 2$.

a Use a velocity resolution diagram to find the initial values of $\dot{x}$ and $\dot{y}$.

b Find $\dot{x}$, x, $\dot{y}$ and y by integrating $\ddot{x} = 0$ and $\ddot{y} = -10$, taking the origin at Steve's hands.

c Show by substitution into y that the apple is in the air for less than 1 second.

d Find the greatest height above the point of release reached by the apple.

e Show that the flight time is $\frac{2}{5}\sqrt{5}$ seconds, and hence find the horizontal distance travelled by the apple.

f Find $\dot{x}$ and $\dot{y}$ at the time Adam catches the apple. Then use a diagram to resolve the velocity and show that the final speed equals the initial speed, and the final angle of inclination is the opposite of the initial angle of elevation.

g The path of the apple is a parabolic arc. By eliminating t from the equations for x and y, find its equation in Cartesian form.

9 A golf ball was hit from level ground and just cleared a 12 m tall tree that was 30 m from the point of projection. If the angle of projection was 26°, find the initial speed of the ball in m/s correct to the nearest integer.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

10 A particle is projected from a point O with initial speed V at an angle of θ above the horizontal.

a Assuming that the displacement functions are $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$, prove that the Cartesian equation of the parabolic path is $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2V^2}$.

b During its flight the particle passes through a point that is 6 m above O and 30 m horizontally from O. If $V = 25$ m/s, show that the two possible angles of projection are $\theta = \arctan\frac{1}{2}$ and $\theta = \arctan\frac{11}{3}$.

11 A ball was thrown uphill from the base of a hill inclined at 30° to the horizontal. The initial velocity was 15 m/s at 60° to the horizontal.

a Show that the parabolic path of the ball has parametric equations

$$x = \tfrac{15}{2}t \qquad \text{and} \qquad y = -5t^2 + \tfrac{15\sqrt{3}}{2}t.$$

b Hence find the Cartesian equation of the path.

c **i** Find how far up the hill the ball landed (measuring along the slope),

ii Find the time of flight.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 Trigonometric equations

Trigonometric equations occur whenever trigonometric functions are being analysed, and careful study of them is essential. Various approaches to them have already been developed in Section 6H, Section 11H and part of Chapter 17 of the Year 11 book, and in parts of Chapter 7 of this book.

This chapter develops those methods further, with particular emphasis on the use of the compound-angle formulae.

Expressions of the form $a\cos\theta + b\sin\theta$ that are the sum of a sine wave and a cosine wave dominate Sections 11B and 11C. Such sums of waves turn out to be extremely important in physics and science generally. The methods of Section 11B lead eventually to the analysis of every periodic function into the sum of sine and cosine waves, and so are essential, for example, in the analysis of data sent by radio through the air, or by light waves through a fibre.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11A Equations involving compound angles

This section reviews the various compound-angle formulae discussed so far and uses them to solve trigonometric equations. Two further forms of the $\cos 2\theta$ formula are needed, and will be needed again in the next chapter when integrating.

A review of the compound-angle formulae

For reference, here are the compound-angle formulae that were proven in Chapter 17 of the Year 11 book.

The basic compound-angle formulae:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

The double-angle formulae and the $\cos 2\theta$ formulae:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

The t-formulae: Let $t = \tan\frac{1}{2}\theta$. Then

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

Products to sums:

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$$

The enrichment section of Exercise 17G in the Year 11 book also introduced the sums-to-products formulae that reverse the products-to-sums formulae given here.

Expressing $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$

Using the second and third $\cos 2\theta$ formulae above, we can write the squares $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$.

$$\begin{aligned}\text{From } \cos 2\theta &= 2\cos^2\theta - 1, \\ 2\cos^2\theta &= 1 + \cos 2\theta \\ \cos^2\theta &= \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

$$\begin{aligned}\text{From } \cos 2\theta &= 1 - 2\sin^2\theta, \\ 2\sin^2\theta &= 1 - \cos 2\theta \\ \sin^2\theta &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

1 EXPRESSING $\sin^2\theta$ AND $\cos^2\theta$ IN TERMS OF $\cos 2\theta$

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta \qquad \sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Adding these two identities gives

$$\cos^2\theta + \sin^2\theta = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) + \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) = 1,$$

as expected from the Pythagorean identities. This observation may help you to memorise the two new formulae.

Example 1 11A

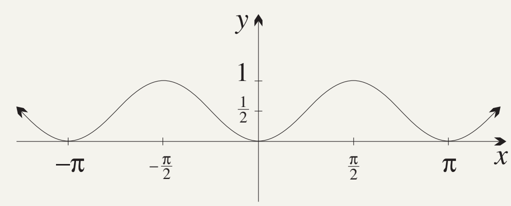
Sketch $y = \sin^2 x$, and state its amplitude, period and range.

SOLUTION

Using the identities above,

$$y = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x.$$

This is the graph of $y = \cos 2x$ turned upside down, then stretched vertically by the factor $\tfrac{1}{2}$, then shifted up $\tfrac{1}{2}$. Its period is π, and its amplitude is $\tfrac{1}{2}$. Because it oscillates around $\tfrac{1}{2}$ rather than 0, its range is $0 \le y \le 1$.



Equations with more than one trigonometric function, but the same angle

This is where trigonometric identities are essential.

2 EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION

- Trigonometric identities can usually be used to produce an equation in only one trigonometric function.
- Many trigonometric equations can be solved by more than one method.

Example 2 11A

Solve the equation $2\tan\theta = \sec\theta$, for $0° \le \theta \le 360°$:

a using the ratio identities,

b by squaring both sides.

SOLUTION

a
$$\begin{aligned} 2\tan\theta &= \sec\theta \\ \frac{2\sin\theta}{\cos\theta} &= \frac{1}{\cos\theta} \\ \sin\theta &= \tfrac{1}{2} \\ \theta &= 30° \text{ or } 150° \end{aligned}$$

b
$$\begin{aligned} \text{Squaring, } 4\tan^2\theta &= \sec^2\theta \\ 4\sec^2\theta - 4 &= \sec^2\theta \\ \sec^2\theta &= \tfrac{4}{3} \\ \cos\theta &= \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3} \\ \theta &= 30°,\ 150°,\ 210° \text{ or } 330°. \end{aligned}$$

Checking each solution, $\theta = 30°$ or $150°$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The dangers of squaring an equation

In part **b** of the previous worked example, we had to check each solution because the equation had been squared.

Squaring can introduce extra solutions. For example:

- $\sin x = 1$, for $0 \le x \le 2\pi$, has one solution $x = \frac{\pi}{2}$.
- $\sin^2 x = 1$, for $0 \le x \le 2\pi$, has two solutions $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Equations involving different angles

When different angles are involved in the one trigonometric equation, use compound-angle identities to change all the trigonometric functions to functions of the one angle.

3 EQUATIONS INVOLVING DIFFERENT ANGLES

- Use compound-angle identities to change all the trigonometric functions to functions of the one angle.

Example 3 — 11A

Solve $\tan 4x = -\tan 2x$, for $0 \le x \le \frac{\pi}{2}$, using the $\tan 2\theta$ formula.

SOLUTION

$$\begin{aligned}
\tan 4x &= -\tan 2x \\
\frac{2\tan 2x}{1 - \tan^2 2x} &= -\tan 2x \\
2\tan 2x &= -\tan 2x + \tan^3 2x \\
\tan^3 2x - 3\tan 2x &= 0 \\
\tan 2x\,(\tan^2 2x - 3) &= 0 \\
\tan 2x &= 0, \text{ or } \tan 2x = \sqrt{3}, \text{ or } \tan 2x = -\sqrt{3}.
\end{aligned}$$

The restriction on $2x$ is $0 \le 2x \le \pi$, so the solutions are

$$\begin{aligned}
2x &= 0 \text{ or } \pi, \text{ or } 2x = \tfrac{\pi}{3}, \text{ or } 2x = \tfrac{2\pi}{3} \\
x &= 0 \text{ or } \tfrac{\pi}{6} \text{ or } \tfrac{\pi}{3} \text{ or } \tfrac{\pi}{2}.
\end{aligned}$$

Alternatively, substitute $t = \tan 2x$ after the second line.

Equations involving different angles and functions

The six trigonometric functions are very closely related. The best approach is usually:

4 APPROACHING TRIGONOMETRIC EQUATIONS

- First, try to get all the angles the same.
- Then try to get all the trigonometric functions the same.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 4 11A

Solve $\cos 2x = 4\sin^2 x - 14\cos^2 x$, for $0 \le x \le 2\pi$:

a by changing all the angles to x,

b by changing all the angles to $2x$.

SOLUTION

a
$$\begin{aligned}\cos 2x &= 4\sin^2 x - 14\cos^2 x\\ \cos^2 x - \sin^2 x &= 4\sin^2 x - 14\cos^2 x\\ 15\cos^2 x &= 5\sin^2 x\\ \tan x &= \sqrt{3} \text{ or } -\sqrt{3}\\ x &= \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3}\end{aligned}$$

b
$$\begin{aligned}\cos 2x &= 4\sin^2 x - 14\cos^2 x\\ \cos 2x &= 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) - 14\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)\\ 10\cos 2x &= -5\\ \cos 2x &= -\tfrac{1}{2}\\ 2x &= \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \tfrac{8\pi}{3} \text{ or } \tfrac{10\pi}{3}\\ x &= \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3}\end{aligned}$$

Homogeneous equations

Expressions *homogeneous in* $\sin x$ *and* $\cos x$ were mentioned in Section 17F of the Year 11 book in the context of proving the t-formulae. The idea is also useful when solving trigonometric equations.

5 HOMOGENEOUS EQUATIONS

- An equation is called *homogeneous* in $\sin x$ and $\cos x$ if the sum of the indices of $\sin x$ and $\cos x$ in each term is the same.
- To solve an equation homogeneous in $\sin x$ and $\cos x$, divide through by a suitable power of $\cos x$ to produce an equation in $\tan x$ alone.

The expansions of $\sin 2x$ and $\cos 2x$ are homogeneous of degree 2 in $\sin x$ and $\cos x$.
Also, $1 = \sin^2 x + \cos^2 x$ can be regarded as being homogeneous of degree 2.

Example 5 11A

Solve $\sin 2x + \cos 2x = \sin^2 x + 1$, for $0 \le x \le 2\pi$.

SOLUTION

Expanding, $2\sin x\cos x + (\cos^2 x - \sin^2 x) = \sin^2 x + (\sin^2 x + \cos^2 x)$

$$\begin{aligned}3\sin^2 x - 2\sin x\cos x &= 0\\ \boxed{\div \cos^2 x} \qquad 3\tan^2 x - 2\tan x &= 0\\ \tan x(3\tan x - 2) &= 0\\ \tan x = 0 \text{ or } \tan x &= \tfrac{2}{3}.\end{aligned}$$

Hence $x = 0, \pi$ or 2π, or $x \doteqdot 0.588$ or 3.730.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 11A

FOUNDATION

1 Consider the equation $\sin 2x - \cos x = 0$.

a By using a double-angle formula and then factoring, show that $\cos x = 0$ or $\sin x = \frac{1}{2}$.

b Hence solve the equation for $0 \le x \le 2\pi$.

2 Consider the equation $\cos 2x - \cos x = 0$.

a By using a double-angle formula and then factoring, show that $\cos x = 1$ or $-\frac{1}{2}$.

b Hence solve the equation for $0 \le x \le 2\pi$.

3 Consider the equation $\sin\left(x + \frac{\pi}{4}\right) = 2\cos\left(x - \frac{\pi}{4}\right)$.

a Use compound-angle formulae to show that $\tan x = -1$.

b Hence solve the equation for $0 \le x \le 2\pi$.

DEVELOPMENT

4 Use compound-angle formulae to solve, for $0 \le \theta \le 2\pi$:

a $\sin\left(\theta + \frac{\pi}{6}\right) = 2\sin\left(\theta - \frac{\pi}{6}\right)$

b $\cos\left(\theta - \frac{\pi}{6}\right) = 2\cos\left(\theta + \frac{\pi}{6}\right)$

c $\cos 4\theta\cos\theta + \sin 4\theta\sin\theta = \frac{1}{2}$

d $\cos 3\theta = \cos 2\theta\cos\theta$

(Hint: In part **d**, write $\cos 3\theta$ as $\cos(2\theta + \theta)$.)

5 Use double-angle formulae to solve, for $0 \le x \le 2\pi$:

a $\sin 2x = \sin x$

b $\sin 2x + \sqrt{3}\cos x = 0$

c $3\sin x + \cos 2x = 2$

d $\cos 2x + 3\cos x + 2 = 0$

e $\tan 2x + \tan x = 0$

f $\sin 2x = \tan x$

6 Solve, for $0° \le \theta \le 360°$, giving solutions correct to the nearest minute where necessary:

a $2\sin 2\theta + \cos\theta = 0$

b $2\cos^2\theta + \cos 2\theta = 0$

c $2\cos 2\theta + 4\cos\theta = 1$

d $8\sin^2\theta\cos^2\theta = 1$

e $3\cos 2\theta + \sin\theta = 1$

f $\cos 2\theta = 3\cos^2\theta - 2\sin^2\theta$

g $10\cos\theta + 13\cos\frac{1}{2}\theta = 5$

h $\tan\theta = 3\tan\frac{1}{2}\theta$

i $\cos^2 2\theta = \sin^2\theta$
$\left(\text{Hint: Use } \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta.\right)$

j $\cos 2\theta + 3 = 3\sin 2\theta$
(Hint: Write 3 as $3\cos^2\theta + 3\sin^2\theta$.)

7 Consider the equation $\tan\left(\frac{\pi}{4} + \theta\right) = 3\tan\left(\frac{\pi}{4} - \theta\right)$.

a Show that $\tan^2\theta - 4\tan\theta + 1 = 0$.

b Hence use the quadratic formula to solve the equation for $0 \le \theta \le \pi$.

8 Given the equation $2\cos x - 1 = 2\cos 2x$:

a Show that $\cos x = \frac{1}{4}\left(1 + \sqrt{5}\right)$ or $\cos x = \frac{1}{4}\left(1 - \sqrt{5}\right)$.

b Hence solve the equation for $0 \le x \le 2\pi$.

9 a Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

b Hence solve the equation $\sin^2 3\theta - \sin^2\theta = \sin 2\theta$, for $0 \le \theta \le \pi$.

10 a Show that $\sin 3x = 3\sin x - 4\sin^3 x$.

b Hence solve the equation $\sin 3x + \sin 2x = \sin x$, for $0 \le x \le 2\pi$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 a Given the equation $\sin\left(\theta + \frac{\pi}{6}\right) = \cos\left(\theta - \frac{\pi}{4}\right)$, show that $\tan\theta = \sqrt{6} - \sqrt{3} - \sqrt{2} + 2$.

b Hence solve the equation for $0 \le \theta \le 2\pi$.

12 Use compound-angle formulae, other trigonometric identities, and factoring to solve for $0° \le \alpha \le 360°$, giving solutions correct to the nearest minute where necessary.

a $\sec^2\alpha = 2\sec\alpha$

b $\sec^2\alpha - \tan\alpha - 3 = 0$

c $\operatorname{cosec}^3 2\alpha = 4\operatorname{cosec} 2\alpha$

d $\sqrt{3}\operatorname{cosec}^2 \frac{1}{2}\alpha + \cot\frac{1}{2}\alpha = \sqrt{3}$

e $\sqrt{3}\operatorname{cosec}^2\alpha = 4\cot\alpha$

f $\cot\alpha + 3\tan\alpha = 5\operatorname{cosec}\alpha$

ENRICHMENT

13 a Use the product-to-sum identity $2\cos A\cos B = \cos(A + B) + \cos(A - B)$ to prove the sum-to-product identity $\cos P + \cos Q = 2\cos\left(\frac{P + Q}{2}\right)\cos\left(\frac{P - Q}{2}\right)$.

b Hence solve the equation $\cos 4x + \cos x = 0$, for $0 \le x \le \pi$.

14 Consider the equation $\sin\theta + \cos\theta = \sin 2\theta$, for $0° \le \theta \le 360°$.

a By squaring both sides, show that $\sin^2 2\theta - \sin 2\theta - 1 = 0$.

b Hence solve for θ over the given domain, giving solutions correct to the nearest minute. (Hint: Beware of the fact that squaring can create invalid solutions.)

15 a Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

b By substituting $x = 2\cos\theta$, show that the equation $x^3 - 3x - 1 = 0$ has roots $2\cos 20°$, $-2\sin 10°$ and $-2\cos 40°$.

c Use a similar approach to find, correct to three decimal places, the three real roots of the equation $x^3 - 12x = 8\sqrt{3}$.

16 a If $t = \tan x$, show that $\tan 4x = \dfrac{4t(1 - t^2)}{1 - 6t^2 + t^4}$.

b If $\tan 4x \tan x = 1$, show that $5t^4 - 10t^2 + 1 = 0$.

c Show that $\sin A\sin B = \frac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)$ and that $\cos A\cos B = \frac{1}{2}\big(\cos(A - B) + \cos(A + B)\big)$.

d Hence show that $\frac{\pi}{10}$ and $\frac{3\pi}{10}$ both satisfy $\tan 4x\tan x = 1$.

e Hence write down, in trigonometric form, the four real roots of the polynomial equation $5x^4 - 10x^2 + 1 = 0$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

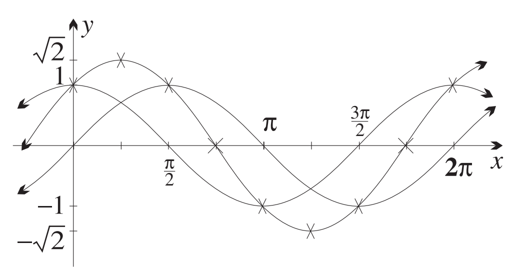
11B The sum of sine and cosine functions

We have seen that the sine and cosine curves are the same, except that the sine wave is the cosine wave shifted right by $\frac{\pi}{2}$. This section analyses what happens when the sine and cosine curves are added, and, more generally, when multiples of the two curves are added. The surprising result is that $y = a \sin x + b \cos x$ is still a sine or cosine wave, whatever the values of a and b are, but shifted sideways so that usually the zeroes no longer lie on multiples of $\frac{\pi}{2}$, and stretched vertically.

These forms for $a \sin x + b \cos x$ give a systematic method of solving any equation of the form $a \cos x + b \sin x = c$. In the next section, an alternative method of solution using the t-formulae is developed.

Sketching $y = \sin x + \cos x$ by graphical methods

The diagram to the right shows the two graphs of $y = \sin x$ and $y = \cos x$. From these two graphs, the sum function $y = \sin x + \cos x$ has been drawn on the same diagram — the crosses represent obvious points to mark on the graph of the sum.



- The new graph has the same period 2π as $y = \sin x$ and $y = \cos x$. It looks like a wave, and within $[0, 2\pi]$ there are zeroes at the two values $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ where $\sin x$ and $\cos x$ take opposite values.
- The new amplitude is bigger than 1. The value at $x = \frac{\pi}{4}$ is $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$, so if the maximum occurs there, as seems likely, the amplitude is $\sqrt{2}$.

This would indicate that the resulting sum function is $y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$, because it is the stretched sine curve shifted left $\frac{\pi}{4}$. We can check this by expansion,

$$\begin{aligned}\sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right) &= \sqrt{2}\left(\sin x \cos \tfrac{\pi}{4} + \cos x \sin \tfrac{\pi}{4}\right)\\ &= \sin x + \cos x,\ \text{ because } \cos \tfrac{\pi}{4} = \sin \tfrac{\pi}{4} = \frac{1}{\sqrt{2}}.\end{aligned}$$

That is exactly what we expected from the sketches of the graphs.

The general algebraic approach — the auxiliary angle

It is true in general that any function of the form $f(x) = a \sin x + b \cos x$ can be written as a single wave function. There are four possible forms in which the wave can be written, and the process is done by expanding the standard form and equating coefficients of $\sin x$ and $\cos x$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

6 AUXILIARY-ANGLE METHOD

- Any function of the form $f(x) = a\sin x + b\cos x$, where a and b are constants (not both zero), can be written in any one of the four forms:
$$\begin{array}{ll} y = R\sin(x - \alpha) & \qquad y = R\cos(x - \alpha) \\ y = R\sin(x + \alpha) & \qquad y = R\cos(x + \alpha) \end{array}$$
where $R > 0$ and $0° \le \alpha < 360°$. The constant $R = \sqrt{a^2 + b^2}$ is the same for all forms, but the *auxiliary angle* α depends on which form is chosen.
- To begin the process, expand the standard form and equate coefficients of $\sin x$ and $\cos x$. Be careful to identify the quadrant in which α lies.

The next worked example continues with the example given at the start of the section, and shows the systematic algorithm used to obtain the required form.

EXAMPLE 6 11B

Express $y = \sin x + \cos x$ in the two forms:

a $R\sin(x + \alpha)$,

b $R\cos(x + \alpha)$,

where, in each case, $R > 0$ and $0 \le \alpha < 2\pi$. Then sketch the curve, showing all intercepts and turning points in the interval $0 \le x \le 2\pi$.

SOLUTION

a Expanding, $\quad R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$,

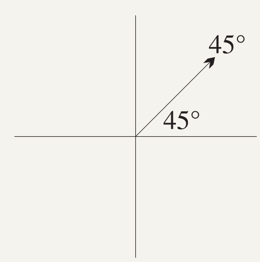
so for all x, $\quad \sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$.

Equating coefficients of $\sin x$, $R\cos\alpha = 1$, (1)

equating coefficients of $\cos x$, $R\sin\alpha = 1$. (2)

Squaring and adding, $R^2 = 2$,

and because $R > 0$, $R = \sqrt{2}$.



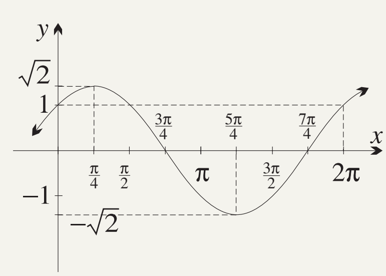
From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)

and from (2), $\sin\alpha = \dfrac{1}{\sqrt{2}}$, (2A)

so α is in the 1st quadrant, with related angle $\frac{\pi}{4}$.

Hence $\quad \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

The graph is $y = \sin x$ shifted left by $\frac{\pi}{4}$ and stretched vertically by a factor of $\sqrt{2}$.



Thus the x-intercepts are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$,

there is a maximum of $\sqrt{2}$ when $x = \frac{\pi}{4}$,

and a minimum of $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.

ISBN 978-1-108-76630-2

b Expanding, $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$,

so for all x, $\sin x + \cos x = R\cos x\cos\alpha - R\sin x\sin\alpha$.

Equating coefficients of $\cos x$, $R\cos\alpha = 1$, (1)

equating coefficients of $\sin x$, $R\sin\alpha = -1$. (2)

Squaring and adding, $R^2 = 2$,

and because $R > 0$, $R = \sqrt{2}$.

From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)

and from (2), $\sin\alpha = -\dfrac{1}{\sqrt{2}}$, (2A)

so α is in the 4th quadrant, with related angle $\frac{\pi}{4}$.

Hence $\sin x + \cos x = \sqrt{2}\cos\left(x + \frac{7\pi}{4}\right)$.

The graph above could equally well be obtained from this.

It is $y = \cos x$ shifted left by $\frac{7\pi}{4}$ and stretched vertically by a factor of $\sqrt{2}$.

Approximating the auxiliary angle

Unless special angles are involved, the auxiliary angle will need to be approximated on the calculator. Degrees or radian measure may be used, but the next worked example uses degrees to make the working a little more intuitive.

Example 7 11B

a Express $y = 3\sin x - 4\cos x$ in the form $y = R\cos(x-\alpha)$, where $R > 0$ and $0° \le \alpha < 360°$, giving α correct to the nearest degree.

b Sketch the curve, showing, correct to the nearest degree, all intercepts and turning points in the interval $-180° \le x \le 180°$.

SOLUTION

a Expanding, $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$,

so far all x, $3\sin x - 4\cos x = R\cos x\cos\alpha + R\sin x\sin\alpha$.

Equating coefficients of $\cos x$, $R\cos\alpha = -4$, (1)

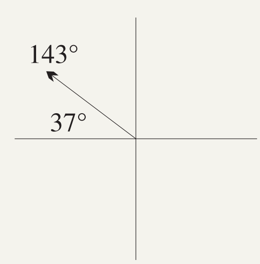
equating coefficients of $\sin x$, $R\sin\alpha = 3$. (2)

Squaring and adding, $R^2 = 25$,

and because $R > 0$, $R = 5$.

From (1), $\cos\alpha = -\frac{4}{5}$, (1A)

and from (2), $\sin\alpha = \frac{3}{5}$, (2A)



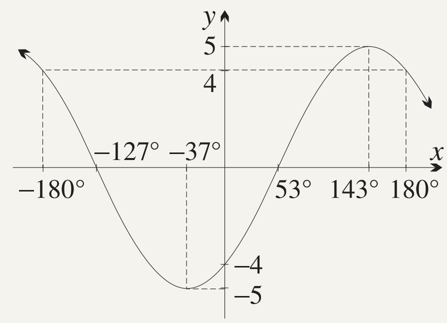
so α is in the 2nd quadrant, with related angle about 37°.

Hence $3\sin x - 4\cos x = 5\cos(x-\alpha)$, where $\alpha \doteqdot 143°$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b The graph is $y = \cos\ x$ shifted right by $\alpha \doteqdot 143°$ and stretched vertically by a factor of 5. Thus the x-intercepts are $x \doteqdot 53°$ and $x \doteqdot -127°$, there is a maximum of 5 when $x \doteqdot 143°$, and a minimum of -5 when $x \doteqdot -37°$.



A note on the calculator and approximations for the auxiliary angle

In the previous worked examples, the exact value of α is $\alpha = 180° - \sin^{-1}\frac{3}{5}$, because α is in the second quadrant. It is this value that is obtained on the calculator, and if there are subsequent calculations to do, as in the equation solved below, this value should be stored in memory and used whenever the auxiliary angle is required. Re-entry of the approximation may lead to rounding errors.

Solving equations of the form $a \sin x + b \cos x = c$, and inequations

Once the LHS has been put into one of the four standard forms, the solutions can easily be obtained. It is always important to keep track of restrictions on the compound angle. The worked example below continues with the previous example.

Example 8 — 11B

a Using the previous worked example, solve the equation $3 \sin x - 4 \cos x = -2$, for $-180° \le x \le 180°$, correct to the nearest degree.

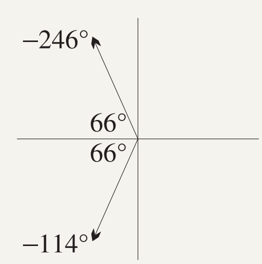
b Hence use the graph to solve $3 \sin x - 4 \cos x \le -2$, for $-180° \le x \le 180°$.

SOLUTION

a Using $3 \sin x - 4 \cos x = 5 \cos\,(x - \alpha)$, where $\alpha \doteqdot 143°$,

$$5 \cos\,(x - \alpha) = -2, \text{ where } -323° \le x - \alpha \le 37°$$

$$\cos\,(x - \alpha) = -\frac{2}{5}.$$



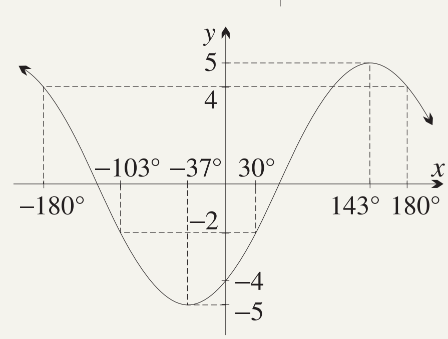
Hence $x - \alpha$ is in quadrant 2 or 3, with related angle about 66°,

so $x - \alpha \doteqdot -114°$ or $-246°$

$x \doteqdot 30°$ or $-103°$.

Be careful to use the calculator's memory here.
Never re-enter approximations of the angles.

b The graph to the right shows the previously drawn graph of $y = 3 \sin x - 4 \cos x$ with the horizontal line $y = -2$ added. This roughly verifies the two answers obtained in part **a**.
It also shows that the solution of the inequality $3 \sin x - 4 \cos x \le -2$ is $-103° \le x \le 30°$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 THE AUXILIARY-ANGLE METHOD FOR EQUATIONS OF THE FORM $a \sin x + b \cos x = c$

- Get the LHS into one of the four forms

  $R \sin (x + \alpha)$ or $R \sin (x - \alpha)$ or $R \cos (x + \alpha)$ or $R \cos (x - \alpha)$.

  Then solve the resulting equation.

Exercise 11B

FOUNDATION

1 Find R and α exactly, if $R > 0$ and $0 \le \alpha < 2\pi$, and:

a $R \sin \alpha = \sqrt{3}$ and $R \cos \alpha = 1$, **b** $R \sin \alpha = 3$ and $R \cos \alpha = 3$.

2 Find R (exactly) and α (correct to the nearest minute), if $R > 0$ and $0° \le \alpha < 360°$, and:

a $R \sin \alpha = 5$ and $R \cos \alpha = 12$, **b** $R \cos \alpha = 2$ and $R \sin \alpha = 4$.

3 a If $\cos x - \sin x = A \cos (x + \alpha)$, show that $A \cos \alpha = 1$ and $A \sin \alpha = 1$.

b Find the positive value of A by squaring and adding.

c Find α, if $0 \le \alpha < 2\pi$.

d State the maximum and minimum values of $\cos x - \sin x$, and the first positive values of x for which they occur.

e Solve the equation $\cos x - \sin x = -1$, for $0 \le x \le 2\pi$.

f Write down the amplitude and period of $\cos x - \sin x$. Hence sketch $y = \cos x - \sin x$, for $0 \le x \le 2\pi$. Indicate on your sketch the line $y = -1$ and the solutions of the equation in part **e**.

4 Sketch $y = \cos x$ and $y = \sin x$ on one set of axes. Then, by taking differences of heights, sketch $y = \cos x - \sin x$. Compare your sketch with that in the previous question.

5 a If $\sqrt{3} \cos x - \sin x = B \cos (x + \theta)$, show that $B \cos \theta = \sqrt{3}$ and $B \sin \theta = 1$.

b Find B, if $B > 0$, by squaring and adding.

c Find θ, if $0 \le \theta < 2\pi$.

d State the greatest and least possible values of $\sqrt{3} \cos x - \sin x$, and the values of x closest to $x = 0$ for which they occur.

e Solve the equation $\sqrt{3} \cos x - \sin x = 1$, for $0 \le x \le 2\pi$.

f Sketch $y = \sqrt{3} \cos x - \sin x$, for $0 \le x \le 2\pi$. On the same diagram, sketch the line $y = 1$. Indicate on your diagram the solutions of the equation in part **e**.

6 Let $4 \sin x - 3 \cos x = A \sin (x - \alpha)$, where $A > 0$ and $0° \le \alpha < 360°$.

a Show that $A \cos \alpha = 4$ and $A \sin \alpha = 3$.

b Show that $A = 5$ and $\alpha = \tan^{-1} \frac{3}{4}$.

c Hence solve the equation $4 \sin x - 3 \cos x = 5$, for $0° \le x \le 360°$. Give the solution(s) correct to the nearest minute.

7 Consider the equation $2 \cos x + \sin x = 1$.

a Let $2 \cos x + \sin x = B \cos (x - \theta)$, where $B > 0$ and $0° \le \theta < 360°$. Show that $B = \sqrt{5}$ and $\theta = \tan^{-1} \frac{1}{2}$.

b Hence find, correct to the nearest minute where necessary, the solutions of the equation, for $0° \le x \le 360°$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Let $\cos x - 3\sin x = D\cos(x + \phi)$, where $D > 0$ and $0° \le \phi < 360°$.

a Show that $D = \sqrt{10}$ and $\phi = \tan^{-1}3$.

b Hence solve $\cos x - 3\sin x = 3$, for $0° \le x \le 360°$. Give the solutions correct to the nearest minute where necessary.

9 Consider the equation $\sqrt{5}\sin x + 2\cos x = -2$.

a Transform the LHS into the form $C\sin(x + \alpha)$, where $C > 0$ and $0° \le \alpha < 360°$.

b Find, correct to the nearest minute where necessary, the solutions of the equation, for $0° \le x \le 360°$.

DEVELOPMENT

10 Solve each equation, for $0° \le x \le 360°$, by transforming the LHS into a single-term sine or cosine function. Give solutions correct to the nearest minute.

a $3\sin x + 5\cos x = 4$ **b** $6\sin x - 5\cos x = 7$

c $7\cos x - 2\sin x = 5$ **d** $9\cos x + 7\sin x = 3$

11 Find A and α exactly, if $A > 0$ and $0 \le \alpha < 2\pi$, and:

a $A\sin\alpha = 1$ and $A\cos\alpha = -\sqrt{3}$, **b** $A\cos\alpha = -5$ and $A\sin\alpha = -5$.

12 Find A (exactly) and α (correct to the nearest minute), if $A > 0$ and $0° \le \alpha < 360°$, and:

a $A\cos\alpha = 5$ and $A\sin\alpha = -4$, **b** $A\sin\alpha = -11$ and $A\cos\alpha = -2$.

13 a i Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and $0 < \theta < 2\pi$.

ii Hence solve $\sqrt{3}\cos x + \sin x = 1$, for $0 \le x < 2\pi$.

b i Express $\cos x - \sin x$ in the form $B\sin(x + \alpha)$, where $B > 0$ and $0 < \alpha < 2\pi$.

ii Hence solve $\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

c i Express $\sin x - \sqrt{3}\cos x$ in the form $C\sin(x + \beta)$, where $C > 0$ and $0 < \beta < 2\pi$.

ii Hence solve $\sin x - \sqrt{3}\cos x = -1$, for $0 \le x < 2\pi$.

d i Express $-\cos x - \sin x$ in the form $D\cos(x - \phi)$, where $D > 0$ and $0 < \phi < 2\pi$.

ii Hence solve $-\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

14 a i Express $2\cos x - \sin x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 360°$. (Write α correct to the nearest minute.)

ii Hence solve $2\cos x - \sin x = 1$, for $0° \le x < 360°$. Give the solutions correct to the nearest minute where necessary.

b i Express $-3\sin x - 4\cos x$ in the form $S\cos(x - \beta)$, where $S > 0$ and $0 < \beta < 2\pi$. (Write β correct to four decimal places.)

ii Hence solve $-3\sin x - 4\cos x = 2$, for $0 \le x < 2\pi$. Give the solutions correct to two decimal places.

15 Solve, for $0° \le x \le 360°$, giving solutions correct to the nearest minute:

a $2\sec x - 2\tan x = 5$ **b** $2\operatorname{cosec} x + 5\cot x = 3$

16 a Given the equation $\sin\theta + \cos\theta = \cos 2\theta$, show that $\tan\theta = -1$ or $\cos\theta - \sin\theta = 1$.

b Hence solve $\sin\theta + \cos\theta = \cos 2\theta$, for $0 \le \theta < 2\pi$.

17 Solve, for $0 \le x \le 2\pi$:

a $\sin x - \cos x = \sqrt{1.5}$ **b** $\sqrt{3}\sin 2x - \cos 2x = 2$ **c** $\sin 4x + \cos 4x = 1$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 a Show that $\left(\sqrt{3}+1\right)\cos 2x+\left(\sqrt{3}-1\right)\sin 2x=2\sqrt{2}\cos\left(2x-\frac{\pi}{12}\right)$.

b Hence solve $\left(\sqrt{3}+1\right)\cos 2x+\left(\sqrt{3}-1\right)\sin 2x=2$, for $-\pi\le x\le\pi$.

19 a i Show that $\sin x-\cos x=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$.

ii Hence sketch the graph of $y=\sin x-\cos x$, for $0\le x\le 2\pi$.

iii Use your sketch to determine the values of x in the domain $0\le x\le 2\pi$ for which $\sin x-\cos x>1$.

b Use a similar approach to part **a** to solve, for $0\le x\le 2\pi$:

i $\sin x+\sqrt{3}\cos x\le 1$

ii $\sin x-\sqrt{3}\cos x<-1$

iii $\left|\sqrt{3}\sin x+\cos x\right|<1$

iv $\cos x-\sin x\ge\frac{1}{2}\sqrt{2}$

ENRICHMENT

20 a Prove that:

i $\sin\theta=\cos\left(\theta-\frac{\pi}{2}\right)$

ii $\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$

b Using the identity $\sin x+\sqrt{3}\cos x=2\sin\left(x+\frac{\pi}{3}\right)$, and the identities in part **a**, express $\sin x+\sqrt{3}\cos x$ in each of the other three standard forms.

c Repeat part **b**, this time starting with the identity $\cos x-\sin x=\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$.

21 a Prove that $\sin(\theta+\pi)=-\sin\theta$.

b Given the identity $\sqrt{3}\sin x+\cos x=2\sin\left(x+\frac{\pi}{6}\right)$, use reflection in the y-axis, the fact that $\sin x$ is odd and $\cos x$ is even, and part **a**, to prove that:

i $-\sqrt{3}\sin x+\cos x=2\sin\left(x+\frac{5\pi}{6}\right)$

ii $-\sqrt{3}\sin x-\cos x=2\sin\left(x+\frac{7\pi}{6}\right)$

iii $\sqrt{3}\sin x-\cos x=2\sin\left(x-\frac{\pi}{6}\right)$

22 a Show that if $\cos(x-\alpha)=\cos\beta$, then $\tan x=\tan(\alpha+\beta)$ or $\tan x=\tan(\alpha-\beta)$.

b Show that $2\cos x+11\sin x=5\sqrt{5}\cos\left(x-\tan^{-1}\frac{11}{2}\right)$.

c Consider the equation $2\cos x+11\sin x=10$, for $0\le x<2\pi$.

i By writing the equation in the form $\cos(x-\alpha)=\cos\beta$ and using part **a**, show that $\tan x=\frac{4}{3}$ or $-\frac{24}{7}$.

ii Deduce that the equation has roots $\tan^{-1}\frac{4}{3}$ and $\pi-\tan^{-1}\frac{24}{7}$.

iii Prove that one of the roots is twice the other.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11C Using the *t*-formula to solve equations

The t-formulae provide a quite different method of solving an equation of the form $a \sin x + b \cos x = c$. Simply substitute $t = \tan \frac{1}{2}x$, and then everything will quickly be reduced to a quadratic equation.
The advantages of this method are that it is far more automatic, and that only a single approximation is involved. But there are two important disadvantages.

- The intuition about the LHS being a shifted wave function is lost.
- If $x = 180°$ happens to be a solution, it will not be found by this method, because $\tan \frac{1}{2}x$ is not defined at $x = 180°$.

The t-formulae are included in the formula review at the start of Section 11A.

Example 9 11C

Solve $3 \sin x - 4 \cos x = -2$, for $-180° \le x \le 180°$, correct to the nearest minute, using the substitution $t = \tan \frac{1}{2}x$.

SOLUTION

Using $\sin x = \dfrac{2t}{1 + t^2}$ and $\cos x = \dfrac{1 - t^2}{1 + t^2}$, the equation becomes

$$\frac{6t}{1 + t^2} - \frac{4 - 4t^2}{1 + t^2} = -2, \text{ provided that } x \ne 180°$$

$$6t - 4 + 4t^2 = -2 - 2t^2$$

$$6t^2 + 6t - 2 = 0$$

$$3t^2 + 3t - 1 = 0, \text{ which has discriminant } \triangle = 21,$$

$$\tan \tfrac{1}{2}x = -\tfrac{1}{2} + \tfrac{1}{6}\sqrt{21} \text{ or } -\tfrac{1}{2} - \tfrac{1}{6}\sqrt{21}\,.$$

Because $-180° \le x \le 180°$, the restriction on $\frac{1}{2}x$ is $-90° \le \frac{1}{2}x \le 90°$,

so $\frac{1}{2}x = 14.775\,961\ldots°$ or $-51.645\,859\ldots°$ (only round at the last step)
$x \doteqdot 29°33'$ or $-103°18'$.

The problem when $x = 180°$ is a solution

The substitution $t = \tan \frac{1}{2}x$ fails when $x = 180°$, because $\tan 90°$ is undefined. We must always be aware of this possibility, and be prepared to add this answer to the final solution. The situation can easily be recognised in either of the following ways:

- The terms in t^2 cancel out, leaving a linear equation in t.
- The coefficient of $\cos x$ is the opposite of the constant term.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 10 11C

Solve $7\sin x - 4\cos x = 4$, for $0° \le x \le 360°$, by using the substitution $t = \tan \frac{1}{2}x$.

SOLUTION

Substituting $t = \tan \frac{1}{2}x$ gives

$$\frac{14t}{1+t^2} - \frac{4-4t^2}{1+t^2} = 4\text{, provided that } x \ne 180°,$$

$$14t - 4 + 4t^2 = 4 + 4t^2$$

$$14t = 8.$$

Warning: The terms in t^2 have cancelled out — check $t = 180°$!

Hence $\tan \frac{1}{2}x = \frac{4}{7}$

$x \doteqdot 59°29'.$

But $x = 180°$ is also a solution, because then LHS $= 7 \times 0 - 4 \times (-1) =$ RHS,

so $x = 180°$ or $x \doteqdot 59°29'$.

A summary of methods of solving $a\sin x + b\cos x = c$

Here then is a summary of the two approaches to the solution.

8 SOLVING EQUATIONS OF THE FORM $a\sin x + b\cos x = c$:

- **The auxiliary-angle method:** Get the LHS into one of the forms
  $R\sin(x + \alpha)$ or $R\sin(x - \alpha)$ or $R\cos(x + \alpha)$ or $R\cos(x - \alpha)$,
  then solve the resulting equation.

- **Using the t-formulae:** Substitute $t = \tan \frac{1}{2}x$ and then solve the resulting quadratic in t.
  Be aware that $x = 180°$ may also be a solution if:
  — the terms in t^2 cancel out, leaving a linear equation in t, or equivalently,
  — the coefficient of $\cos x$ is the opposite of the constant term.

Exercise 11C

FOUNDATION

1 Consider the equation $\cos x - \sin x = 1$, where $0 \le x \le 2\pi$.

a Using the substitutions $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$, where $t = \tan \frac{1}{2}x$, show that the equation can be written as $t^2 + t = 0$.

b Hence show that $\tan \frac{1}{2}x = 0$ or -1, where $0 \le \frac{1}{2}x \le \pi$.

c Hence solve the given equation for x.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 Consider the equation $\sqrt{3}\sin x + \cos x = 1$.

a Show that the equation can be written as $t^2 = \sqrt{3}\,t$, where $t = \tan\frac{1}{2}x$.

b Hence solve the equation, for $0 \le x \le 2\pi$.

3 a Show that the equation $4\cos x + \sin x = 1$ can be written as $(5t + 3)(t - 1) = 0$, where $t = \tan\frac{1}{2}x$.

b Hence solve the equation, for $0^\circ \le x \le 360^\circ$. Give the solutions correct to the nearest minute where necessary.

4 a Show that the equation $3\sin x - 2\cos x = 2$ can be written as $3t - 2 = 0$, where $t = \tan\frac{1}{2}x$.

b Hence solve the equation for $0^\circ \le x \le 360^\circ$, giving solutions correct to the nearest minute where necessary. (Remember to check $x = 180^\circ$ as a possible solution, given that the resulting equation in t is linear.)

5 a Show that the equation $6\sin x - 4\cos x = 5$ can be written as $t^2 - 12t + 9 = 0$, where $t = \tan\frac{1}{2}x$.

b Show that $\tan\frac{1}{2}x = 6 + 3\sqrt{3}$ or $6 - 3\sqrt{3}$.

c Hence show that $77^\circ 35'$ and $169^\circ 48'$ are the solutions (correct to the nearest minute) of the given equation over the domain $0^\circ \le x \le 360^\circ$.

DEVELOPMENT

6 Solve each equation, for $0^\circ \le x \le 360^\circ$, using the t-formulae, where $= \tan\frac{1}{2}x$. Give solutions correct to the nearest minute where necessary.

a $5\sin x + 4\cos x = 5$

b $7\cos x - 6\sin x = 2$

c $3\sin x - 2\cos x = 1$

d $5\cos x + 6\sin x = -5$

7 Use t-formulae to prove that the equation $8\tan\theta - 4\sec\theta = 1$ has solutions $2\tan^{-1}\frac{1}{3}$ and $2\tan^{-1}5$ in the domain $0 \le \theta \le 2\pi$.

8 Solve the equation $2\sin 2x + \cos 2x = 2$, for $0^\circ \le x \le 360^\circ$, using the substitution $t = \tan x$. Give solutions correct to the nearest tenth of a degree where necessary.

9 Suppose that $a\cos x = 1 + \sin x$, where $0^\circ < x < 90^\circ$.

a Prove that $\dfrac{a - 1}{a + 1} = t$, where $t = \tan\frac{1}{2}x$.

b Hence find, correct to the nearest minute, the acute angle x that satisfies the equation $2\cos x - \sin x = 1$.

10 Let θ_1 and θ_2, where $\theta_1 > \theta_2$, be the solutions of the equation $6\cos\theta + 17\sin\theta = 18$ in the domain $0 \le \theta \le 2\pi$. Show that $\tan\left(\dfrac{\theta_1 - \theta_2}{2}\right) = \dfrac{1}{18}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

11 Consider the equation $a\cos x + b\sin x = c$, where a, b and c are constants.

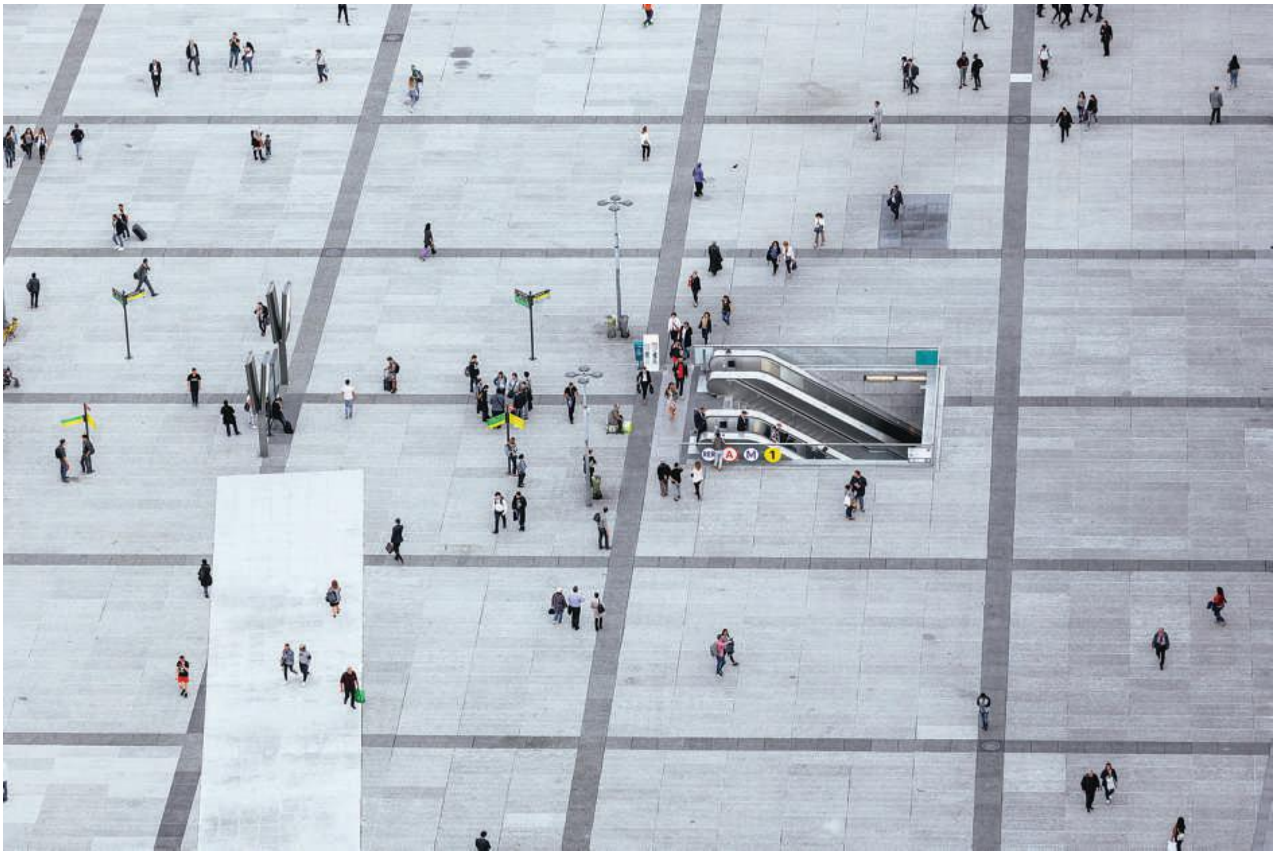
a Show that the equation can be written in the form $(a + c)t^2 - 2bt - (a - c) = 0$, where $t = \tan\frac{1}{2}x$.

b Show that the root(s) of the equation are real if $c^2 \le a^2 + b^2$.

c Suppose that $\tan\frac{1}{2}\alpha$ and $\tan\frac{1}{2}\beta$ are distinct real roots of the quadratic equation in part **a**. Prove that $\tan\frac{1}{2}(\alpha + \beta) = \dfrac{b}{a}$.

12 Given the equation $(2k - 1)\cos\theta + (k + 2)\sin\theta = 2k + 1$, where k is a constant, use t-formulae to prove that $\tan\theta = \dfrac{4}{3}$ or $\dfrac{2k}{k^2 - 1}$.

13 Let θ_1, θ_2, θ_3 and θ_4 be solutions of the equation $a\cos 4\theta + b\sin 4\theta = c$ such that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are distinct. Use the product of the roots of a quartic equation to prove that $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 = 1$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 11 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 11 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Solve, for $0 \le x \le 2\pi$:

a $\sin 2x + \sin x = 0$

b $\cos 2x + \cos x = 0$

c $\cos 2x + 5\sin x + 2 = 0$

d $2\sin\left(x - \frac{\pi}{6}\right) = \cos\left(x - \frac{\pi}{3}\right)$

2 **a** Express $\sin x - \cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Hence solve $\sin x - \cos x = \sqrt{2}$, for $0 \le x \le 2\pi$.

3 **a** Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x - \theta)$, where $A > 0$ and $0 < \theta < \frac{\pi}{2}$.

b Hence solve $\sqrt{3}\cos x + \sin x = -1$, for $0 \le x \le 2\pi$.

4 **a** Express $2\sin x + \sqrt{5}\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and α is acute.

b Hence solve $2\sin x + \sqrt{5}\cos x = 3$, for $0^\circ \le x \le 360^\circ$, writing the solution in degrees correct to one decimal place.

5 **a** Express $3\cos x - 2\sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and θ is acute.

b Hence solve $3\cos x - 2\sin x = 1$, for $0^\circ \le x \le 360^\circ$, writing the solutions correct to the nearest minute.

6 Use a suitable t-formula to solve $\sin x = \tan\frac{1}{2}x$, for $0 \le x \le 2\pi$.

7 Consider the equation $7\sin x + \cos x = 5$.

a Show that the equation can be written as $3t^2 - 7t + 2 = 0$, where $t = \tan\frac{x}{2}$.

b Hence show that the equation has solutions $x = 2\tan^{-1}\frac{1}{3}$ and $x = 2\tan^{-1}2$.

8 Use t-formulae to solve $4\sin x - 2\cos x = 3$, for $0 \le x \le 2\pi$. Write the solutions correct to 2 decimal places.

9 **a** Use compound- and double-angle formulae to prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

b Hence solve $\cos 3x + \sin 2x + \cos x = 0$, for $0 \le x \le 2\pi$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 Further calculus

This chapter is mostly concerned with three different ways in which integration can be extended to a greater range of functions. Much of the material concerns the use of trigonometry in integration, and readers should be aware of the structured progression of trigonometry in the two books.

- Trigonometric identities have been developing through Chapters 6, 11 and 17 of the Year 11 book.
- Trigonometric equations also began in those chapters, and were the main subject of the previous chapter.
- Trigonometric integrals were introduced in Chapter 7 of this book and are developed further in this chapter.

Trigonometry has many purposes and applications, but its principal significance in calculus is its role in integration, and more generally in the differential equations that are the subject of the next chapter.

Sections 12A–12B develop the calculus of the inverse trigonometric functions, with the surprising result that inverse trigonometric functions are required for the integration of two purely algebraic functions. The standard forms are:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C \qquad \text{and} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C.$$

Section 12C uses the $\cos 2\theta$ formula to integrate $\sin^2 x$ and $\cos^2 x$ and reviews the reverse chain rule for trigonometric integrals. Then Sections 12D–12E generalise the reverse chain rule to a more general method of integration by substitution, which applies to all integrals, whether trigonometric or not.

Section 12F moves in a different direction, using calculus to find the volumes of solids generated by rotating a curve about the x-axis or the y-axis.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12A Inverse trigonometric functions — differentiating

The inverse trigonometric functions were defined in Chapter 17 of the Year 11 book. We can now apply the normal processes of calculus to them. This section develops their derivatives and applies them to curve-sketching and maximisation.

Differentiating $\sin^{-1} x$ and $\cos^{-1} x$

To differentiate $y = \sin^{-1} x$ and $y = \cos^{-1} x$, we change to the inverse functions and use the known derivatives of the sine and cosine functions. We need to keep track of the restrictions to the domain so that the choice can be made later between positive and negative square roots.

A Let $y = \sin^{-1} x$.
Then $x = \sin y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,
so $\dfrac{dx}{dy} = \cos y$.
Because y is in the first or fourth quadrant, $\cos y$ is positive,

$$\text{so} \quad \cos y = +\sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

$$\text{Thus} \quad \frac{dx}{dy} = \sqrt{1 - x^2},$$

$$\text{so} \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

$$\text{Hence} \quad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.$$

B Let $y = \cos^{-1} x$.
Then $x = \cos y$, where $0 \le y \le \pi$,
so $\dfrac{dx}{dy} = -\sin y$.
Because y is in the first or second quadrant, $\sin y$ is positive,

$$\text{so} \quad \sin y = +\sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}.$$

$$\text{Thus} \quad \frac{dx}{dy} = -\sqrt{1 - x^2},$$

$$\text{so} \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

$$\text{Hence} \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

Differentiating $\tan^{-1} x$

The problem of which square root to choose does not arise when differentiating $y = \tan^{-1} x$.

Let $y = \tan^{-1} x$.
Then $x = \tan y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$\text{so} \quad \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y.$$

$$\text{Hence} \quad \frac{dx}{dy} = 1 + x^2$$

and $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, giving the standard form $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1 + x^2}$.

1 STANDARD FORMS FOR DIFFERENTIATION

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 1 12A

Differentiate these functions.

a $y = x\tan^{-1} x$

b $y = \sin^{-1}(ax + b)$

SOLUTION

a
$$\begin{aligned} y &= x\tan^{-1} x \\ y' &= vu' + uv' \\ &= \tan^{-1} x \times 1 + x \times \frac{1}{1 + x^2} \\ &= \tan^{-1} x + \frac{x}{1 + x^2} \end{aligned}$$

Let $u = x$
and $v = \tan^{-1} x$.
Then $u' = 1$
and $v' = \dfrac{1}{1 + x^2}$.

b
$$\begin{aligned} y &= \sin^{-1}(ax + b) \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{\sqrt{1 - (ax + b)^2}} \times a \\ &= \frac{a}{\sqrt{1 - (ax + b)^2}} \end{aligned}$$

Let $u = ax + b$,
then $y = \sin^{-1} u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1 - u^2}}$.

Linear extensions

The method used in part **b** above can be applied to all three inverse trigonometric functions, giving a further set of standard forms.

2 FURTHER STANDARD FORMS FOR DIFFERENTIATION

$$\frac{d}{dx}\sin^{-1}(ax + b) = \frac{a}{\sqrt{1 - (ax + b)^2}}$$

$$\frac{d}{dx}\cos^{-1}(ax + b) = -\frac{a}{\sqrt{1 - (ax + b)^2}}$$

$$\frac{d}{dx}\tan^{-1}(ax + b) = \frac{a}{1 + (ax + b)^2}$$

Example 2 12A

a Find the points A and B on the curve $y = \cos^{-1}(x - 1)$ where the tangent has gradient -2.

b Sketch the curve, showing these points.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Differentiating, $y' = -\dfrac{1}{\sqrt{1 - (x - 1)^2}}$.

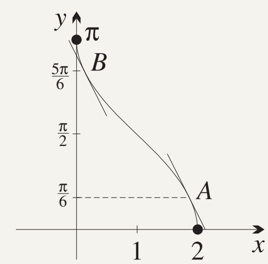
Put $y' = -2$.

Then $-\dfrac{1}{\sqrt{1 - (x - 1)^2}} = -2$

$$1 - (x - 1)^2 = \tfrac{1}{4}$$
$$(x - 1)^2 = \tfrac{3}{4}$$
$$x - 1 = \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3}$$
$$x = 1 + \tfrac{1}{2}\sqrt{3} \text{ or } 1 - \tfrac{1}{2}\sqrt{3},$$

so the points are $A\left(1 + \frac{1}{2}\sqrt{3}, \frac{\pi}{6}\right)$ and $B\left(1 - \frac{1}{2}\sqrt{3}, \frac{5\pi}{6}\right)$.

b



Functions whose derivatives are zero are constants

Several identities involving inverse trigonometric functions can be obtained by showing that some derivative is zero, and hence that the original function is a constant. The following identity is the clearest example — it has been proven already in Section 17C of the Year 11 book using symmetry arguments.

Example 3 12A

a Differentiate $\sin^{-1} x + \cos^{-1} x$.

b Hence prove the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

SOLUTION

a $$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1 - x^2}} + \frac{-1}{\sqrt{1 - x^2}}$$
$$= 0$$

b Hence $\sin^{-1} x + \cos^{-1} x = C$, for some constant C.

Substitute $x = 0$, then $0 + \frac{\pi}{2} = C$,

so $C = \frac{\pi}{2}$, and $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, as required.

Curve sketching using calculus

The usual methods of curve sketching can now be extended to curves whose equations involve the inverse trigonometric functions. The next worked example applies calculus to sketching the curve $y = \cos^{-1} \cos x$, which was sketched without calculus in Section 17C of the Year 11 book.

Example 4 12A

Use calculus to sketch $y = \cos^{-1} \cos x$.

SOLUTION

The function is periodic with the same period 2π as $\cos x$. A simple table of test values gives some key points. The shape of the curve joining these points can be obtained by calculus.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	...
y	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	...

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Differentiating using the chain rule,

$$\frac{dy}{dx} = \frac{\sin x}{\sqrt{1 - \cos^2 x}}$$

$$= \frac{\sin x}{\sqrt{\sin^2 x}}.$$

Let $u = \cos x$,
then $y = \cos^{-1} u$.
Hence $\dfrac{du}{dx} = -\sin x$
and $\dfrac{dy}{du} = -\dfrac{1}{\sqrt{1 - u^2}}$.

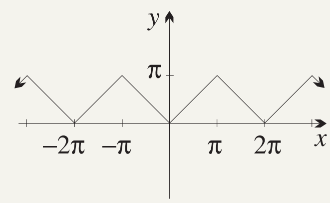
When $\sin x$ is positive, $\sqrt{\sin^2 x} = \sin x$,

so $\dfrac{dy}{dx} = 1$.

When $\sin x$ is negative, $\sqrt{\sin^2 x} = -\sin x$,

so $\dfrac{dy}{dx} = -1$.

Hence $\dfrac{dy}{dx} = \begin{cases} 1, & \text{for } x \text{ in quadrants 1 and 2,} \\ -1, & \text{for } x \text{ in quadrants 3 and 4,} \end{cases}$



This means that the graph consists of a series of intervals, each with gradient 1 or −1.

Chain-rule extensions to the standard forms

The usual chain-rule extensions to the standard forms can be used with the inverse trigonometric forms. They provide an alternative to the chain-rule setting out.

3 CHAIN-RULE EXTENSIONS TO THE STANDARD FORMS

$$\frac{d}{dx}\sin^{-1} u = \frac{u'}{\sqrt{1 - u^2}} \qquad \text{OR} \qquad \frac{d}{dx}\sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1 - \big(f(x)\big)^2}}$$

$$\frac{d}{dx}\cos^{-1} u = -\frac{u'}{\sqrt{1 - u^2}} \qquad \frac{d}{dx}\cos^{-1} f(x) = -\frac{f'(x)}{\sqrt{1 - \big(f(x)\big)^2}}$$

$$\frac{d}{dx}\tan^{-1} u = \frac{u'}{1 + u^2} \qquad \frac{d}{dx}\tan^{-1} f(x) = \frac{f'(x)}{1 + \big(f(x)\big)^2}$$

Example 5 — 12A

Use the chain-rule extension formulae to differentiate:

a $\tan^{-1} e^{-5x}$

b $\cos^{-1}(3x^2 + 2)$

SOLUTION

a Here $u = e^{-5x}$ and $u' = -5e^{-5x}$,

so $$\frac{d}{dx}(\tan^{-1} e^{-5x}) = \frac{u'}{1 + u^2} = \frac{-5e^{-5x}}{1 + e^{-10x}}.$$

OR

Here $f(x) = e^{-5x}$ and $f'(x) = -5e^{-5x}$,

so $$\frac{d}{dx}(\tan^{-1} e^{-5x}) = \frac{f'(x)}{1 + \big(f(x)\big)^2} = \frac{-5e^{-5x}}{1 + e^{-10x}}.$$

b Here $u = 3x^2 + 2$ and $u' = 6x$,

so $$\frac{d}{dx}\big(\cos^{-1}(3x^2 + 2)\big) = -\frac{u'}{\sqrt{1 - u^2}} = -\frac{6x}{\sqrt{1 - (3x^2 + 2)^2}}$$

OR

Here $f(x) = 3x^2 + 2$ and $f'(x) = 6x$,

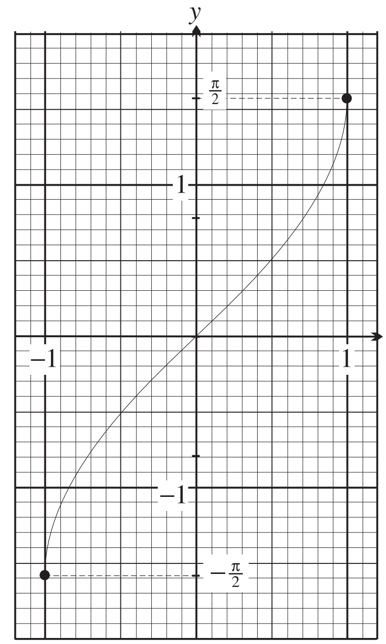
so $$\frac{d}{dx}\big(\cos^{-1}(3x^2 + 2)\big) = -\frac{f'(x)}{\sqrt{1 - \big(f(x)\big)^2}} = -\frac{6x}{\sqrt{1 - (3x^2 + 2)^2}}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 12A

FOUNDATION



1 **a** Photocopy the graph of $y = \sin^{-1} x$ shown to the right. Then carefully draw a tangent at each x value in the table. Then, by measurement and calculation of rise/run, find the gradient of each tangent correct to two decimal places and fill in the second row of the table.

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\dfrac{dy}{dx}$									

b Check your gradients using $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$.

2 Differentiate with respect to x:

a $\cos^{-1} x$ **b** $\tan^{-1} x$ **c** $\sin^{-1} 2x$
d $\tan^{-1} 3x$ **e** $\cos^{-1} 5x$ **f** $\sin^{-1}(-x)$
g $\sin^{-1} x^2$ **h** $\tan^{-1} x^3$ **i** $\tan^{-1}(x + 2)$
j $\cos^{-1}(1 - x)$ **k** $x \sin^{-1} x$ **l** $(1 + x^2)\tan^{-1} x$
m $\sin^{-1} \frac{1}{5}x$ **n** $\tan^{-1} \frac{1}{4}x$ **o** $\cos^{-1} \sqrt{x}$
p $\tan^{-1} \sqrt{x}$ **q** $\tan^{-1} \dfrac{1}{x}$

3 Find the gradient of the tangent to each curve at the point indicated:

a $y = 2\tan^{-1} x$, at $x = 0$ **b** $y = \sqrt{3}\sin^{-1} x$, at $x = \frac{1}{2}$
c $y = \tan^{-1} 2x$, at $x = -\frac{1}{2}$ **d** $y = \cos^{-1} \frac{x}{2}$, at $x = \sqrt{3}$

4 Find, in the form $y = mx + b$, the equation of the tangent and the normal to each curve at the point indicated:

a $y = 2\cos^{-1} 3x$, at $x = 0$ **b** $y = \sin^{-1} \frac{x}{2}$, at $x = \sqrt{2}$

5 **a** Show that $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$.

b Hence explain why $\sin^{-1} x + \cos^{-1} x$ is a constant function, and use any convenient value of x in its domain to find the value of the constant.

6 Use the method of the previous question to show that each of these functions is a constant function, and find the value of the constant.

a $\cos^{-1} x + \cos^{-1}(-x)$ **b** $2\sin^{-1} \sqrt{x} - \sin^{-1}(2x - 1)$

DEVELOPMENT

7 **a** If $f(x) = x\tan^{-1} x - \frac{1}{2}\ln(1 + x^2)$, show that $f''(x) = \dfrac{1}{1 + x^2}$.

b Is the graph of $y = f(x)$ concave up or concave down at $x = -1$?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Show that the gradient of the curve $y = \dfrac{\sin^{-1} x}{x}$ at the point where $x = \frac{1}{2}$ is $\frac{2}{3}(2\sqrt{3} - \pi)$.

9 Find the derivative of each function in simplest form:

a $x\cos^{-1} x - \sqrt{1 - x^2}$ **b** $\sin^{-1} e^{3x}$ **c** $\sin^{-1} \frac{1}{4}(2x - 3)$

d $\tan^{-1} \frac{1}{1 - x}$ **e** $\sin^{-1} e^x$ **f** $\log_e \sqrt{\sin^{-1} x}$

g $\sin^{-1} \sqrt{\log_e x}$ **h** $\sqrt{x}\sin^{-1} \sqrt{1 - x}$ **i** $\tan^{-1} \frac{x + 2}{1 - 2x}$

10 a i If $y = (\sin^{-1} x)^2$, show that $y'' = \dfrac{2 + \frac{2x\sin^{-1} x}{\sqrt{1 - x^2}}}{1 - x^2}$.

ii Hence show that $(1 - x^2)y'' - xy' - 2 = 0$.

b Show that $y = e^{\sin^{-1} x}$ satisfies the differential equation $(1 - x^2)y'' - xy' - y = 0$.

11 Consider the function $y = \sin^{-1} 2x$.

a Write down the range of the function.

b Make x the subject of the equation.

c Find $\dfrac{dx}{dy}$, and explain why it is never negative.

d Use the result that $\dfrac{dy}{dx}$ is the reciprocal of $\dfrac{dx}{dy}$ to find $\dfrac{dy}{dx}$.

12 Use the approach in the previous question to find $\dfrac{dy}{dx}$ given:

a $y = \sin^{-1} \frac{x}{2}$ **b** $y = \cos^{-1} (x - 1)$ **c** $y = \tan^{-1} \sqrt{x}$

13 Consider the function $f(x) = \cos^{-1} x^2$.

a What is the domain of $f(x)$?

b About which line is the graph of $y = f(x)$ symmetric?

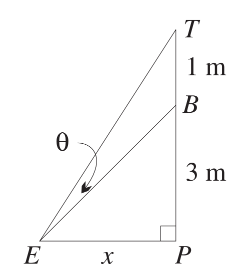
c Find $f'(x)$.

d Show that $y = f(x)$ has a maximum turning point at $x = 0$.

e Show that $f'(x)$ is undefined at the endpoints of the domain. What is the geometrical significance of this?

f Sketch the graph of $y = f(x)$.

14 A picture TB that is 1 metre tall is hung on a wall so that its bottom edge B is 3 metres above the eye E of a viewer. Let the distance EP be x metres, and let θ be the angle that the picture subtends at E.

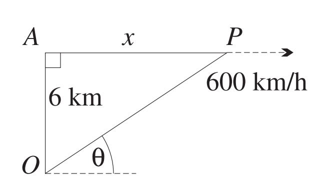


a Show that $\theta = \tan^{-1} \frac{4}{x} - \tan^{-1} \frac{3}{x}$.

b Show that θ is maximised when the viewer is $2\sqrt{3}$ metres from the wall.

c Show that the maximum angle subtended by the picture at E is $\tan^{-1} \frac{\sqrt{3}}{12}$.

15 A plane P at a constant altitude of 6 km and at a constant speed of 600 km/h is flying directly away from an observer at O on the ground. The point A on the path of the plane lies directly above O. Let the distance AP be x km, and let the angle of elevation of the plane from the observer be θ.



a Show that $\theta = \tan^{-1} \frac{6}{x}$.

b Show that $\dfrac{d\theta}{dt} = \dfrac{-3600}{x^2 + 36}$ radians per hour.

c Hence find, in radians per second, the rate at which θ is decreasing at the instant when the distance AP is 3 km.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 a State the domain of $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$, and its symmetry.

b Show that $f'(x) = 0$ for all values of x in the domain.

c Show that $f(x) = \begin{cases} \frac{\pi}{2}, & \text{for } x < 0, \\ -\frac{\pi}{2}, & \text{for } x > 0 \end{cases}$ and hence sketch the graph of $f(x)$.

17 Consider the function $f(x) = \cos^{-1} \frac{1}{x}$.

a State the domain of $f(x)$. (Hint: Think about it rather than relying on algebra.)

b Recalling that $\sqrt{x^2} = |x|$, show that $f'(x) = \dfrac{1}{|x|\sqrt{x^2 - 1}}$.

c Comment on $f'(1)$ and $f'(-1)$.

d Use the expression for $f'(x)$ in part **b** to write down separate expressions for $f'(x)$ when $x > 1$ and when $x < -1$.

e Explain why $f(x)$ is increasing for $x > 1$ and for $x < -1$.

f Find:

i $\lim\limits_{x \to \infty} f(x)$ **ii** $\lim\limits_{x \to -\infty} f(x)$

g Sketch the graph of $y = f(x)$.

18 Use the formulae for the chain-rule standard forms in Box 3 to differentiate:

a $\tan^{-1} e^{3x}$ **b** $\sin^{-1} x^3$ **c** $\cos^{-1} (\log_e x)$

ENRICHMENT

19 a What is the domain of $g(x) = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$?

b Show that $g'(x) = \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{x}{|x|\sqrt{1 - x^2}}$.

c Hence determine the interval over which $g(x)$ is constant, and find this constant.

20 In question **9 i**, you proved that $\dfrac{d}{dx} \tan^{-1} \dfrac{x + 2}{1 - 2x}$ was $\dfrac{1}{1 + x^2}$, which is also the derivative of $\tan^{-1} x$. What is going on?

21 The function $f(x)$ is defined by the rule $f(x) = \sin^{-1} (\sin x)$.

a State the domain and range of $f(x)$, and whether it is even, odd or neither.

b Show that $f'(x) = \dfrac{\cos x}{|\cos x|}$.

c Is $f'(x)$ defined when $\cos x = 0$?

d What are the only two values that $f'(x)$ takes if $\cos x \neq 0$, and when does each of these values occur?

e Sketch the graph of $f(x)$ using the above information and a table of values if necessary.

22 In Question 14, construct the circle that passes through T and B and is tangent to the horizontal line ℓ through P. Then the point E on ℓ at which TB subtends the greatest angle is the point where the circle touches the line ℓ.

a Prove this using Euclidean geometry.

b Explain how to construct this circle, and E, using straight edge and compasses.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12B Inverse trigonometric functions — integrating

This section deals with the integrals associated with the inverse trigonometric functions, and with the standard applications of those integrals.

The basic standard forms

Differentiation of the inverse trigonometric functions yields purely algebraic functions,

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

These are remarkable results, and indicate once again that trigonometric functions are very closely related to algebraic functions associated with squares and square roots. The relationship was already clear when the trigonometric functions were defined using the circle, whose equation $x^2 + y^2 = r^2$ is Pythagoras' theorem, which is purely algebraic.

This section concerns integration, and we begin by reversing the three standard forms for differentiation.

4 STANDARD FORMS FOR INTEGRATION

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C \qquad \text{OR} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

Thus some purely algebraic functions require the inverse trigonometric functions for their integration. We have seen this sort of phenomenon before with the standard form $\int \frac{1}{x}\,dx = \log_e|x|$, where the logarithmic function was required for the integration of another purely algebraic function.

The functions $y = \dfrac{1}{\sqrt{1-x^2}}$ and $y = \dfrac{1}{1+x^2}$

The primitives of both these functions have now been obtained, and they should therefore be regarded as reasonably standard functions whose graphs should be known. The sketch of each function and some important definite integrals associated with them are developed in Questions 15 and 16 in Exercise 12B.

Example 6 12B

Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ using both standard forms given in Box 4.

SOLUTION

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx &= \Big[\sin^{-1}x\Big]_0^{\frac{1}{2}} \\ &= \sin^{-1}\tfrac{1}{2} - \sin^{-1}0 \\ &= \tfrac{\pi}{6} - 0 \\ &= \tfrac{\pi}{6}\end{aligned}$$

OR

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx &= \Big[-\cos^{-1}x\Big]_0^{\frac{1}{2}} \\ &= -\cos^{-1}\tfrac{1}{2} + \cos^{-1}0 \\ &= -\tfrac{\pi}{3} + \tfrac{\pi}{2} \\ &= \tfrac{\pi}{6}\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 7 12B

Evaluate these definite integrals exactly or correct to four significant figures.

a $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

b $\displaystyle\int_0^4 \frac{1}{1+x^2}\,dx$

SOLUTION

a $$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1} x\Big]_0^1\\ &= \tan^{-1} 1 - \tan^{-1} 0\\ &= \tfrac{\pi}{4}\end{aligned}$$

b $$\begin{aligned}\int_0^4 \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1} x\Big]_0^4\\ &= \tan^{-1} 4 - \tan^{-1} 0\\ &\doteqdot 1.326\end{aligned}$$

More general standard forms

When constants are involved, the calculation of the primitive becomes fiddly. These standard integrals are commonly used.

5 STANDARD FORMS WITH ONE CONSTANT

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad \text{OR} \quad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = -\cos^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Proof

A $$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx\\ &= \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \times \frac{1}{a}\,dx\\ &= \sin^{-1}\frac{x}{a} + C\end{aligned}$$

Let $u = \dfrac{x}{a}$.

Then $\dfrac{du}{dx} = \dfrac{1}{a}$.

$$\int \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\,dx = \sin^{-1} u$$

B $$\begin{aligned}\int \frac{1}{a^2+x^2}\,dx &= \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\,dx\\ &= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a}\,dx\\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C\end{aligned}$$

Let $u = \dfrac{x}{a}$.

Then $\dfrac{du}{dx} = \dfrac{1}{a}$.

$$\int \frac{1}{1+u^2}\frac{du}{dx}\,dx = \tan^{-1} u$$

These forms also hold when a is negative. Can you prove this?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 8 12B

Evaluate these four indefinite integrals. In parts **a** and **b**, the formulae can be applied immediately, but in parts **c** and **d**, the coefficients of x^2 need to be taken out first.

a $\displaystyle\int \frac{2}{\sqrt{9 - x^2}}\,dx = 2\sin^{-1}\frac{x}{3} + C$

b $\displaystyle\int \frac{1}{8 + x^2}\,dx = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x}{2\sqrt{2}} + C$

c $\displaystyle\int \frac{6}{49 + 25x^2}\,dx$

$$= \frac{6}{25}\int \frac{1}{\frac{49}{25} + x^2}\,dx$$

$$= \frac{6}{25} \times \frac{5}{7}\tan^{-1}\frac{x}{7/5} + C$$

$$= \frac{6}{35}\tan^{-1}\frac{5x}{7} + C$$

d $\displaystyle\int \frac{1}{\sqrt{5 - 3x^2}}\,dx$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\frac{5}{3} - x^2}}\,dx$$

$$= \tfrac{1}{3}\sqrt{3}\,\sin^{-1} x\sqrt{\tfrac{3}{5}} + C$$

Because manipulating the constants in parts **c** and **d** is still difficult, some prefer to remember these fuller versions of the standard forms:

6 STANDARD FORMS WITH TWO CONSTANTS

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C \quad \text{OR} \quad -\frac{1}{b}\cos^{-1}\frac{bx}{a} + C$$

$$\int \frac{1}{a^2 + b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + C$$

These forms can be proven in the same manner as the forms with a single constant, or they can be developed from those forms in the same way as was done in parts **c** and **d** above. With these more general forms, parts **c** and **d** can be written down without any intermediate working.

Reverse chain rule

In the usual way, the standard forms can be extended to give forms appropriate for the reverse chain rule. These forms are the reversals of the standard forms at the end of Section 12A.

7 THE REVERSE CHAIN RULE

$$\int \frac{u'}{\sqrt{1 - u^2}}\,dx = \sin^{-1}u + C \quad \text{OR} \quad \int \frac{f'(x)}{\sqrt{1 - \big(f'(x)\big)^2}}\,dx = \sin^{-1}f'(x) + C$$

$$\int \frac{u'}{\sqrt{1 - u^2}}\,dx = -\cos^{-1}u + C \qquad \int \frac{f'(x)}{\sqrt{1 - \big(f'(x)\big)^2}}\,dx = -\cos^{-1}f'(x) + C$$

$$\int \frac{u'}{1 + u^2}\,dx = \tan^{-1}u + C \qquad \int \frac{f'(x)}{1 + \big(f'(x)\big)^2}\,dx = \tan^{-1}f'(x) + C$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 9 12B

Find primitives of:

a $\dfrac{e^x}{1 + e^{2x}}$ **b** $\dfrac{x}{\sqrt{1 - x^4}}$

SOLUTION

a Let $u = e^x$, then $u' = e^x$. OR Let $f(x) = e^x$, then $f'(x) = e^x$.

With both notations, $\displaystyle\int \frac{e^x}{1 + e^{2x}}\,dx = \int \frac{e^x}{1 + (e^x)^2}\,dx$

$= \tan^{-1} e^x + C$, for some constant C

b Let $u = x^2$, then $u' = 2x$. OR Let $f(x) = x^2$, then $f'(x) = 2x$.

With both notations, $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx = \tfrac{1}{2}\int \frac{2x}{\sqrt{1 - (x^2)^2}}\,dx$

$= \tfrac{1}{2}\sin^{-1} x^2 + C$, for some constant C.

Given a derivative, find an integral

As always, the result of a product-rule differentiation can be used to obtain an integral. In particular, this allows the primitives of the inverse trigonometric functions to be obtained.

Example 10 12B

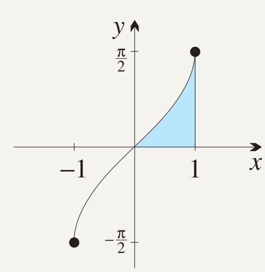
a Differentiate $x \sin^{-1} x$, and hence find a primitive of $\sin^{-1} x$.

b Find the shaded area under the curve $y = \sin^{-1} x$ from $x = 0$ to $x = 1$.

SOLUTION

a Let $y = x \sin^{-1} x$.

Using the product rule with $u = x$ and $v = \sin^{-1} x$,



$$\frac{dy}{dx} = \sin^{-1} x + \frac{x}{\sqrt{1 - x^2}}.$$

Hence $\displaystyle\int \sin^{-1} x\,dx + \int \frac{x}{\sqrt{1 - x^2}}\,dx = x \sin^{-1} x$

$$\int \sin^{-1} x\,dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}}\,dx.$$

Using the reverse chain rule,

$$-\int \frac{x}{\sqrt{1 - x^2}}\,dx = \tfrac{1}{2}\int (1 - x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$= \tfrac{1}{2} \times (1 - x^2)^{\frac{1}{2}} \times \frac{2}{1}$$

$$= \sqrt{1 - x^2},$$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$.

$\displaystyle\int u^{-\frac{1}{2}}\frac{du}{dx}\,dx = u^{\frac{1}{2}} \times 2.$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

so $\displaystyle\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1 - x^2} + C,$

and $\displaystyle\int_0^1 \sin^{-1} x\,dx = \Big[x\sin^{-1} x + \sqrt{1 - x^2} \Big]_0^1$

$= (1 \times \frac{\pi}{2} + 0) - (0 + 1)$

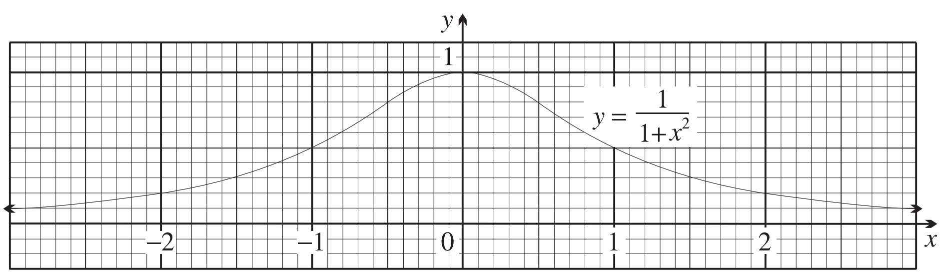
$= \frac{\pi}{2} - 1$ square units.

Note: We have already established in Section 7D that the area under $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ is exactly 1 square unit. This means that the area between $y = \sin^{-1} x$ and the y-axis is 1, and subtracting this area from the rectangle of area $\dfrac{\pi}{2}$ in the diagram above gives the same value $\frac{\pi}{2} - 1$ for the shaded area.

Exercise 12B

FOUNDATION

1 a



Find each definite integral correct to two decimal places from the graph by counting the number of little squares in the region under the curve:

i $\displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx$ **ii** $\displaystyle\int_0^2 \frac{1}{1 + x^2}\,dx$

iii $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1 + x^2}\,dx$ **iv** $\displaystyle\int_{-3}^{-1} \frac{1}{1 + x^2}\,dx$

b Check your answers to **a** by using the fact that $\tan^{-1} x$ is a primitive of $\dfrac{1}{1 + x^2}$.

2 Find:

a $\displaystyle\int \frac{-1}{\sqrt{1 - x^2}}\,dx$ **b** $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\,dx$ **c** $\displaystyle\int \frac{1}{9 + x^2}\,dx$

d $\displaystyle\int \frac{1}{\sqrt{\frac{4}{9} - x^2}}\,dx$ **e** $\displaystyle\int \frac{1}{2 + x^2}\,dx$ **f** $\displaystyle\int \frac{-1}{\sqrt{5 - x^2}}\,dx$

3 Find the exact value of:

a $\displaystyle\int_0^3 \frac{1}{\sqrt{9 - x^2}}\,dx$ **b** $\displaystyle\int_0^2 \frac{1}{4 + x^2}\,dx$ **c** $\displaystyle\int_0^1 \frac{1}{\sqrt{2 - x^2}}\,dx$

d $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\frac{1}{2}}{\frac{1}{4} + x^2}\,dx$ **e** $\displaystyle\int_{\frac{1}{6}\sqrt{3}}^{\frac{1}{6}} \frac{-1}{\sqrt{\frac{1}{9} - x^2}}\,dx$ **f** $\displaystyle\int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\sqrt{\frac{9}{4} - x^2}}\,dx$

4 Find the equation of the curve, given that:

a $y' = (1 - x^2)^{-\frac{1}{2}}$ and the curve passes through the point $(0, \pi)$.

b $y' = 4(16 + x^2)^{-1}$ and the curve passes through the point $(-4, 0)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 **a** If $y' = \dfrac{1}{\sqrt{36 - x^2}}$ and $y = \frac{\pi}{6}$ when $x = 3$, find the value of y when $x = 3\sqrt{3}$.

b Given that $y' = \dfrac{2}{4 + x^2}$ and that $y = \frac{\pi}{3}$ when $x = 2$, find y when $x = \frac{2}{\sqrt{3}}$.

DEVELOPMENT

6 Find:

a $\displaystyle\int \frac{1}{\sqrt{1 - 4x^2}}\,dx$ **b** $\displaystyle\int \frac{1}{1 + 16x^2}\,dx$ **c** $\displaystyle\int \frac{-1}{\sqrt{1 - 2x^2}}\,dx$

d $\displaystyle\int \frac{1}{\sqrt{4 - 9x^2}}\,dx$ **e** $\displaystyle\int \frac{1}{25 + 9x^2}\,dx$ **f** $\displaystyle\int \frac{-1}{\sqrt{3 - 4x^2}}\,dx$

7 Find the exact value of:

a $\displaystyle\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1 - 9x^2}}\,dx$ **b** $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1 + 4x^2}\,dx$ **c** $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1 - 3x^2}}\,dx$

d $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9 - 4x^2}}\,dx$ **e** $\displaystyle\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3 + 4x^2}\,dx$ **f** $\displaystyle\int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5 + 2x^2}\,dx$

8 **a** Shade the region bounded by $y = \sin^{-1} x$, the x-axis and the vertical line $x = \frac{1}{2}$.

b Show that $\dfrac{d}{dx}\left(x \sin^{-1} x + \sqrt{1 - x^2}\right) = \sin^{-1} x$.

c Hence find the exact area of the region.

9 **a** Shade the region bounded by the curve $y = \sin^{-1} x$, the y-axis and the line $y = \frac{\pi}{6}$.

b Find the exact area of this region.

c Hence use an alternative approach to confirm the area in the previous question.

10 **a** Show that $\dfrac{d}{dx}\left(\cos^{-1}(2 - x)\right) = \dfrac{1}{\sqrt{4x - x^2 - 3}}$.

b Hence find $\displaystyle\int_1^2 \frac{1}{\sqrt{4x - x^2 - 3}}\,dx$.

11 **a** Differentiate $\tan^{-1} \frac{1}{2} x^3$.

b Hence find $\displaystyle\int \frac{x^2}{4 + x^6}\,dx$.

12 **a** Differentiate $x \tan^{-1} x$.

b Hence find $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

13 Without finding any primitives, use arguments from symmetry or geometry to evaluate:

a $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \sin^{-1} x\,dx$ **b** $\displaystyle\int_{-5}^{5} \tan^{-1} x\,dx$ **c** $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x\,dx$

d $\displaystyle\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{x}{\sqrt{1 - x^2}}\,dx$ **e** $\displaystyle\int_{-3}^{3} \frac{x}{1 + x^2}\,dx$ **f** $\displaystyle\int_{-6}^{6} \sqrt{36 - x^2}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 a Given that $f(x) = \dfrac{x}{1+x^2} - \tan^{-1}x$:

i find $f(0)$,

ii show that $f'(x) = \dfrac{-2x^2}{(1+x^2)^2}$.

b Hence:

i explain why $f(x) < 0$ for all $x > 0$,

ii find $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$.

15 Consider the function $f(x) = \dfrac{1}{\sqrt{4-x^2}}$.

a Sketch the graph of $y = \sqrt{4-x^2}$.

b Hence sketch the graph of $y = f(x)$.

c Write down the domain and range of $f(x)$, and describe its symmetry.

d Find the area between the curve and the x-axis from $x = -1$ to $x = 1$.

e Find the total area between the curve and the x-axis. [**Note:** This is an example of an unbounded region having a finite area.]

16 Consider the function $f(x) = \dfrac{4}{x^2+4}$.

a What is the axis of symmetry of $y = f(x)$?

b What are the domain and range?

c Show that the graph of $f(x)$ has a maximum turning point at $(0, 1)$.

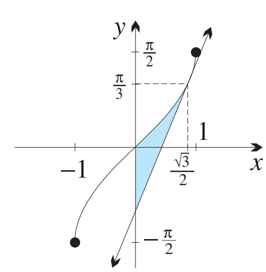
d Find $\lim\limits_{x\to\infty} f(x)$, and hence sketch $y = f(x)$. On the same axis, sketch $y = \frac{1}{4}(x^2+4)$.

e Calculate the area bounded by the curve and the x-axis from $x = -2\sqrt{3}$ to $x = \frac{2}{3}\sqrt{3}$.

f Find the exact area between the curve and the x-axis from $x = -a$ to $x = a$, where a is a positive constant.

g By letting a tend to infinity, find the total area between the curve and the x-axis. [**Note:** This is another example of an unbounded region having a finite area.]

17 The diagram to the right shows the region bounded by the curve $y = \sin^{-1}x$, the y-axis and the tangent to the curve at the point $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$. Show that the region has area $\frac{1}{4}$ unit^2.



18 a Use the trapezoidal rule with five points to approximate $I = \displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$, expressing your answer in simplest fraction form.

b Find the exact value of I, and hence show that $\pi \doteqdot \frac{5323}{1700}$.

19 Show that $\displaystyle\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$.

20 Find, using the reverse chain rule:

a $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

b $\displaystyle\int_0^1 \frac{1}{e^{-x}+e^x}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

21 **a** Show that $\dfrac{d}{dx}\left(\tan^{-1}\left(\frac{3}{2}\tan x\right)\right) = \dfrac{6}{5\sin^2 x + 4}$.

b Hence find the area under the curve $y = \dfrac{1}{5\sin^2 x + 4}$ $x = 0$ to $x = 7$.

c Why is the calculation asked for in part **b** invalid and completely wrong?

22 [The power series for $\tan^{-1} x$]

Let x be a positive real number.

a Find the sum of the geometric series $1 - t^2 + t^4 - t^6 + \cdots + t^{4n}$, and hence show that for $0 < t < x$,

$$\frac{1}{1 + t^2} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n}.$$

b Find $1 - t^2 + t^4 - t^6 + \cdots + t^{4n} - t^{4n+2}$, and hence show that for $0 < t < x$,

$$1 - t^2 + t^4 - t^6 + \cdots + t^{4n} < \frac{1}{1 + t^2} + t^{4n+2}.$$

c By integrating the inequalities of parts **a** and **b** from $t = 0$ to $t = x$, show that

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n + 1} < \tan^{-1} x + \frac{x^{4n+3}}{4n + 3}.$$

d By taking limits as $n \to \infty$, show that for $0 \le x \le 1$,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

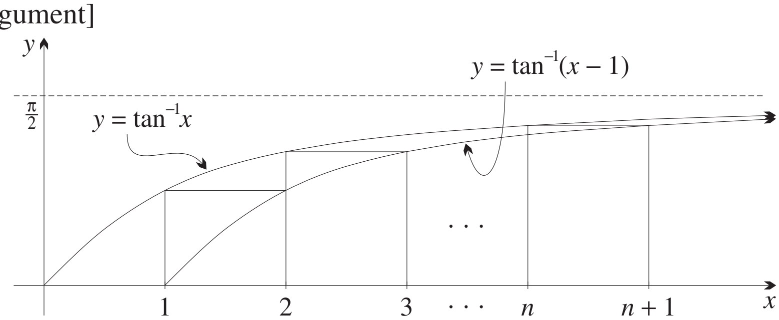
e Use the fact that $\tan^{-1} x$ is an odd function to prove this identity for $-1 \le x < 0$.

f [Gregory's series] Use a suitable substitution to prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

g By combining the terms in pairs, show that $\dfrac{\pi}{8} = \dfrac{1}{1 \times 3} + \dfrac{1}{5 \times 7} + \dfrac{1}{9 \times 11} + \cdots$, and use the calculator to find how close an approximation to π can be obtained by taking 10 terms.

23 [A sandwiching argument]



In the diagram, n rectangles are constructed between the two curves $y = \tan^{-1} x$ and $y = \tan^{-1}(x - 1)$ in the interval $1 \le x \le n + 1$.

a Write down an expression for S_n, the sum of the areas of the n rectangles.

b Differentiate $x\tan^{-1} x$ and hence find a primitive of $\tan^{-1} x$.

c Show that for all $n \ge 1$,

$$n\tan^{-1} n - \tfrac{1}{2}\ln(n^2 + 1) < S_n < (n + 1)\tan^{-1}(n + 1) - \tfrac{1}{2}\ln\left(\tfrac{n^2}{2} + n + 1\right) - \tfrac{\pi}{4}$$

d Deduce that $1562 < \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 + \cdots + \tan^{-1} 1000 < 1565$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12C Further trigonometric integrals

The principal purpose of this section is the integration of $\sin^2 x$ and $\cos^2 x$. Trigonometric integrals in general are quickly reviewed, particularly reverse-chain-rule integrations in preparation for the next two sections.

Six standard forms for integration

Reversing the derivatives of the six standard forms gives six standard integrals:

8 SIX STANDARD INTEGRALS

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad {}^*\int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$${}^*\int \sec x \tan x\,dx = \sec x + C \qquad {}^*\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$$

The full list is there only for completeness, and so that the patterns can be seen. The three integrals marked * need not be memorised — they are the reversals of derivatives in Question 19 of Exercise 7B.

The primitives of $\sin^2 x$ and $\cos^2 x$

These two integrals are very important. The keys to finding them are the two further forms of the cos 2θ formulae that we established in Section 11A,

$$\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta \qquad \text{and} \qquad \cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta,$$

and it is better to remember these identities rather than the actual integrals.

Example 11 **12C**

Evaluate:

a $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

b $\displaystyle\int_0^{\frac{5\pi}{3}} \cos^2 \tfrac{1}{2}x\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx &= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx \\ &= \left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} \\ &= \left(\tfrac{\pi}{4} - 0\right) - (0 - 0) \\ &= \tfrac{\pi}{4}\end{aligned}$$

b
$$\begin{aligned}\int_0^{\frac{5\pi}{3}} \cos^2 \tfrac{1}{2}x\,dx &= \int_0^{\frac{5\pi}{3}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos\ x\right) dx \\ &= \left[\tfrac{1}{2}x - \tfrac{1}{2}\sin\ x\right]_0^{\frac{5\pi}{3}} \\ &= \left(\tfrac{5\pi}{6} - \tfrac{1}{4}\right) - (0 - 0) \\ &= \tfrac{5\pi}{6} - \tfrac{1}{4}\end{aligned}$$

Here are the primitives of the squares of all six trigonometric functions. Two are given as standard forms. The other four are given only in terms of the identities that are needed to obtain them. The forms marked * are there for completeness.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 INTEGRATING THE SQUARES OF THE TRIGONOMETRIC FUNCTIONS

$$\int \sin^2 x\,dx = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)dx \qquad \int \cos^2 x\,dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)dx$$

$$\int \sec^2 x\,dx = \tan x + C \qquad {}^*\int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx \qquad {}^*\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)\,dx$$

Trigonometric integrals using the reverse chain rule

It remains to demonstrate the use of the reverse chain rule in trigonometric integrals, in preparation for the next two sections on substitution.

Example 12 12C

Integrate $\cot x$ using the reverse chain rule.

SOLUTION

$$\begin{aligned}\int \cot x\,dx &= \int \frac{\cos x}{\sin x}\,dx\\ &= \log_e|\sin x| + C\end{aligned}$$

Let $u = \sin x$.
Then $u' = \cos x$,
and $\displaystyle\int \frac{u'}{u}\,dx = \log_e|u|$.

Example 13 12C

Use the reverse chain rule to find primitives of:

a $y = \sin x\cos^4 x$

b $y = \cos x\sin^n x$

SOLUTION

a
$$\begin{aligned}\int \sin x\cos^4 x\,dx &= -\int (-\sin x)\cos^4 x\,dx\\ &= -\tfrac{1}{5}\cos^5 x + C\end{aligned}$$

Let $u = \cos x$.
Then $\dfrac{du}{dx} = -\sin x$,
and $\displaystyle\int u^4\frac{du}{dx}\,dx = \tfrac{1}{5}u^5$.

b
$$\int \cos x\sin^n x\,dx = \frac{\sin^{n+1} x}{n+1} + C$$

Let $u = \sin x$.
Then $\dfrac{du}{dx} = \cos x$,
and $\displaystyle\int u^n\frac{du}{dx}\,dx = \frac{u^n}{n+1}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 12C

FOUNDATION

1 Use double-angle results to show that:

a $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

b $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

c $\sin 3x \cos 3x = \frac{1}{2}\sin 6x$

d $2\sin^2 \frac{x}{2} = 1 - \cos x$

2 Without a calculator, find the value of:

a $\cos^2 15°$

b $\sin^2 \frac{5\pi}{12}$

c $\sin 105° \cos 105°$

d $\sin^2 \frac{7\pi}{8}$

3 Express $\sin^2 \theta$ in terms of $\cos 2\theta$, and hence find:

a $\int \sin^2 x\,dx$

b $\int \sin^2 2x\,dx$

c $\int \sin^2 \frac{1}{4}x\,dx$

d $\int \sin^2 3x\,dx$

4 Express $\cos^2 \theta$ in terms of $\cos 2\theta$, and hence find:

a $\int \cos^2 x\,dx$

b $\int \cos^2 6x\,dx$

c $\int \cos^2 \frac{1}{2}x\,dx$

d $\int \cos^2 10x\,dx$

DEVELOPMENT

5 Use the results $\sin^2 \theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $\cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ to evaluate:

a $\int_0^{\pi} \sin^2 x\,dx$

b $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$

c $\int_0^{\frac{\pi}{6}} \sin^2 \frac{1}{2}x\,dx$

d $\int_0^{\frac{\pi}{16}} \cos^2 2x\,dx$

e $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 \left(x + \frac{\pi}{12}\right)\,dx$

f $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 \left(x - \frac{\pi}{6}\right)\,dx$

6 Use the products-to-sums formulae reviewed in Section 11A to find:

a $\int \sin 3x \cos 2x\,dx$

b $\int \cos 3x \sin x\,dx$

c $\int_0^{\frac{\pi}{4}} 2\cos 2x \cos x\,dx$

d $\int_0^{\frac{\pi}{3}} \sin 5x \sin 2x\,dx$

7 a Sketch the graph of $y = \cos 2x$, for $0 \le x \le 2\pi$.

b Hence sketch, on the same diagram, $y = \frac{1}{2}(1 + \cos 2x)$ and $y = \frac{1}{2}(1 - \cos 2x)$.

c Deduce graphically that $\cos^2 x + \sin^2 x = 1$.

8 Use the reverse chain rule to find:

a $\int \sin^3 x \cos x\,dx$ (Let $u = \sin x$.)

b $\int \sin^6 x \cos x\,dx$

c $\int \cos^5 x \sin x\,dx$ (Let $u = \cos x$.)

d $\int \cos^8 x \sin x\,dx$

e $\int e^x \sin e^x\,dx$ (Let $u = e^x$.)

f $\int e^x \cos 5e^x\,dx$

g $\int \tan x\,dx$ (Let $u = \cos x$)

h $\int \cot 7x\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 **a** Find the range of the function $y = \cos x \sin x$.

b Integrate $y = \cos x \sin x$ in three ways:

i using the $\sin 2x$ formula,

ii using the reverse chain rule in two different ways.

c Reconcile the three results.

10 **a** By writing $\sin^4 x$ as $(\sin^2 x)^2$, show that $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$.

b Find a similar result for $\cos^4 x$.

c Hence find:

i $\displaystyle\int_0^{\pi} \sin^4 x\,dx$ **ii** $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4 x\,dx$

d Use the Pythagorean identity to find $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\,dx$.

11 Evaluate using the standard forms $\displaystyle\int \sec^2 x\,dx = \tan x$ and $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x$:

a $\displaystyle\int \tan^2 2x\,dx$ **b** $\displaystyle\int \cot^2 \tfrac{1}{2}x\,dx$ **c** $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3\tan^2 3x\,dx$ **d** $\displaystyle\int_{\frac{\pi}{24}}^{\frac{\pi}{8}} \cot^2 4x\,dx$

12 Integrate these functions, using the reverse chain rule or otherwise.

a $\tan x \sec^2 x$ **b** $\dfrac{\sin^2 x}{1 + \cos x}$ **c** $\dfrac{1 + \cos^3 x}{\cos^2 x}$

ENRICHMENT

13 Define $F(x) = \displaystyle\int_0^x \sin^2 t\,dt$, where $0 \le x \le 2\pi$.

a Show that $F(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$.

b Explain why $F'(x) = \sin^2 x$. Hence state the values of x in the given domain for which $F(x)$ is:

i stationary, **ii** increasing, **iii** decreasing.

c Explain why $F(x)$ never differs from $\frac{1}{2}x$ by more than $\frac{1}{4}$.

d Find any points of inflection of $F(x)$ in the given domain.

e Sketch, on the same diagram, the graphs of $y = F(x)$ and $y = F'(x)$ over the given domain, and observe how they are related.

f **i** For what value of k is $\displaystyle\int_0^k \sin^2 x\,dx = \frac{3\pi}{2}$?

ii For what values of k is $\displaystyle\int_0^k \sin^2 x\,dx = \frac{n\pi}{2}$, where n is an integer?

14 Find the value of $\displaystyle\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\,dt\right)$, explaining your reasoning carefully.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12D Integration by substitution

The reverse chain rule as we have been using it so far does not cover all the situations where the chain rule can be used in integration. This section and the next develop a more general method called *integration by substitution*.

The first stage, covered in this section, begins by translating the reverse chain rule into a slightly more flexible notation. It involves substitutions of the form

'Let $u =$ some function of x.'

The reverse chain rule — an example

Here is an example of the reverse chain rule as we have been using it. The working is set out in full on the right.

Example 14 — 12D

Find $\int x(1-x^2)^4\,dx$.

SOLUTION

$$\begin{aligned}\int x(1-x^2)^4\,dx &= -\tfrac{1}{2}\int(-2x)\,(1-x^2)^4\,dx\\ &= -\tfrac{1}{2}\times\tfrac{1}{5}(1-x^2)^5 + C\\ &= -\tfrac{1}{10}(1-x^2)^5 + C\end{aligned}$$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$,

and $\displaystyle\int u^4\frac{du}{dx}\,dx = \tfrac{1}{5}u^5$.

Rewriting this example as integration by substitution

We shall now rewrite this using a new notation. The key to this new notation is that the derivative $\dfrac{du}{dx}$ is treated as a fraction — the du and the dx are split apart, so that the statement

$$\frac{du}{dx} = -2x \qquad \text{is written instead as} \qquad du = -2x\,dx.$$

The new variable u no longer remains in the working column on the right, but is brought over into the main sequence of the solution on the left.

Example 15 — 12D

Find $\int x(1-x^2)^4\,dx$, using the substitution $u = 1 - x^2$.

SOLUTION

$$\begin{aligned}\int x(1-x^2)^4\,dx &= \int u^4\left(-\tfrac{1}{2}\right)du\\ &= -\tfrac{1}{2}\times\tfrac{1}{5}u^5 + C\\ &= -\tfrac{1}{10}(1-x^2)^5 + C\end{aligned}$$

Let $u = 1 - x^2$.

Then $du = -2x\,dx$,

and $x\,dx = -\tfrac{1}{2}\,du$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 16 **12D**

Find $\int \sin x \sqrt{1 - \cos x}\, dx$, using the substitution $u = 1 - \cos x$.

SOLUTION

$$\int \sin x \sqrt{1 - \cos x}\, dx = \int u^{\frac{1}{2}}\, du$$
$$= \tfrac{2}{3} u^{\frac{3}{2}} + C$$
$$= \tfrac{2}{3}(1 - \cos x)^{\frac{3}{2}} + C$$

Let $u = 1 - \cos x$.
Then $du = \sin x\, dx$.

An advance on the reverse chain rule

Some integrals that can be done in this way could only be done by the reverse chain rule in a rather clumsy manner.

Example 17 **12D**

Find $\int x\sqrt{1 - x}\, dx$, using the substitution $u = 1 - x$.

SOLUTION

$$\int x\sqrt{1 - x}\, dx = \int (1 - u)\sqrt{u}\,(-du)$$
$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$
$$= \tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}} + C$$
$$= \tfrac{2}{5}(1 - x)^{\frac{5}{2}} - \tfrac{2}{3}(1 - x)^{\frac{3}{2}} + C$$

Let $u = 1 - x$.
Then $du = -\,dx$,
and $x = 1 - u$.

Substituting the limits of integration in a definite integral

A great advantage of this new method is that the limits of integration can be changed from values of x to values of u. There is then no need ever to go back to x. The first worked example below repeats the previous integrand, but this time within a definite integral.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 18 — 12D

Find $\int_0^1 x\sqrt{1-x}\,dx$, using the substitution $u = 1 - x$.

SOLUTION

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= -\int_1^0 (1-u)\sqrt{u}\,du\\ &= -\int_1^0 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du\\ &= -\left[\tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right]_1^0\\ &= -0 + \left(\tfrac{2}{3} - \tfrac{2}{5}\right)\\ &= \tfrac{4}{15}\end{aligned}$$

Let $u = 1 - x$.
Then $du = -dx$,
and $x = 1 - u$.
When $x = 0, u = 1$,
when $x = 1, u = 0$.

Example 19 — 12D

Find $\int_0^{\pi} \sin x \cos^6 x\,dx$, using the substitution $u = \cos x$.

SOLUTION

$$\begin{aligned}\int_0^{\pi} \sin x\cos^6 x\,dx &= -\int_1^{-1} u^6\,du\\ &= -\tfrac{1}{7}\Big[u^7\Big]_1^{-1}\\ &= -\tfrac{1}{7}\times(-1) + \tfrac{1}{7}\times 1\\ &= \tfrac{2}{7}\end{aligned}$$

Let $u = \cos x$.
Then $du = -\sin x\,dx$.
When $x = 0, u = 1$,
when $x = \pi, u = -1$.

Exercise 12D

FOUNDATION

1 Consider the integral $\int 2x(1+x^2)^3\,dx$, and the substitution $u = 1 + x^2$.

- **a** Show that $du = 2x\,dx$.
- **b** Show that the integral can be written as $\int u^3\,du$.
- **c** Hence find the primitive of $2x(1+x^2)^3$.
- **d** Check your answer by differentiating it.

2 Repeat the previous question for each indefinite integral and substitution.

- **a** $\int 2(2x+3)^3\,dx$ (Let $u = 2x + 3$.)
- **b** $\int 3x^2(1+x^3)^4\,dx$ (Let $u = 1 + x^3$.)
- **c** $\int \dfrac{2x}{(1+x^2)^2}\,dx$ (Let $u = 1 + x^2$.)
- **d** $\int \dfrac{3}{\sqrt{3x-5}}\,dx$ (Let $u = 3x - 5$.)
- **e** $\int \sin^3 x\cos x\,dx$ (Let $u = \sin x$.)
- **f** $\int \dfrac{4x^3}{1+x^4}\,dx$ (Let $u = 1 + x^4$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Consider the integral $\int \frac{x}{\sqrt{1-x^2}}\,dx$, and the substitution $u = 1 - x^2$.

a Show that $x\,dx = -\frac{1}{2}\,du$.

b Show that the integral can be written as $-\frac{1}{2}\int u^{-\frac{1}{2}}\,du$.

c Hence find the primitive of $\frac{x}{\sqrt{1-x^2}}$.

4 Repeat the previous question for each indefinite integral and substitution.

a $\int x^3(x^4+1)^5\,dx$ (Let $u = x^4 + 1$.)

b $\int x^2\sqrt{x^3-1}\,dx$ (Let $u = x^3 - 1$.)

c $\int x^2 e^{x^3}\,dx$ (Let $u = x^3$.)

d $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^3}\,dx$ (Let $u = 1 + \sqrt{x}$.)

e $\int \tan^2 2x \sec^2 2x\,dx$ (Let $u = \tan 2x$.)

f $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$ (Let $u = \frac{1}{x}$.)

5 Find the exact value of each definite integral, using the given substitution.

a $\int_0^1 x^2(2+x^3)^3\,dx$ (Let $u = 2 + x^3$.)

b $\int_0^1 \frac{2x^3}{\sqrt{1+x^4}}\,dx$ (Let $u = 1 + x^4$.)

c $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\,dx$ (Let $u = \cos x$.)

d $\int_{\frac{1}{2}\sqrt{3}}^{1} x\sqrt{1-x^2}\,dx$ (Let $u = 1 - x^2$.)

e $\int_1^{e^2} \frac{\ln x}{x}\,dx$ (Let $u = \ln x$.)

f $\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}}\,dx$ (Let $u = \sqrt{x}$.)

g $\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x\,dx$ (Let $u = \sin 2x$.)

h $\int_0^1 \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx$ (Let $u = \sin^{-1}x$.)

i $\int_0^2 \frac{x+1}{\sqrt[3]{x^2+2x}}\,dx$ (Let $u = x^2 + 2x$.)

j $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$ (Let $u = \tan x$.)

DEVELOPMENT

6 Use the substitution $u = x^3$ to find the exact area bounded by the curve $y = \frac{x^2}{1+x^6}$, the x-axis and the line $x = 1$.

7 Evaluate each definite integral, using the substitution $u = \sin x$.

a $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx$

b $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$

c $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

d $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x}\,dx$

8 Find each indefinite integral, using the given substitution.

a $\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$ (Let $u = e^{2x}$.)

b $\int \frac{1}{x\ln x}\,dx$ (Let $u = \ln x$.)

c $\int \frac{\tan x}{\ln\cos x}\,dx$ (Let $u = \ln\cos x$.)

d $\int \tan^3 x \sec^4 x\,dx$ (Let $u = \tan x$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 **a** A curve has gradient function $\dfrac{e^{2x}}{1 + e^{4x}}$ and passes through the point $(0, \frac{\pi}{8})$. Use the substitution $u = e^{2x}$ to find its equation.

b If $y'' = \dfrac{x}{(4 - x^2)^{\frac{3}{2}}}$, and when $x = 0$, $y' = 1$ and $y = \frac{1}{2}$, use the substitution $u = 4 - x^2$ to find y' and then find y as a function of x.

10 **a** Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

b Use the substitution $u = \sec x$, and the standard form $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$, to find:

i $\displaystyle\int_0^{\frac{\pi}{3}} 2^{\sec x} \sec x \tan x\,dx$

ii $\displaystyle\int_0^{\frac{\pi}{4}} \sec^5 x \tan x\,dx$

11 Evaluate each integral, using the given substitution.

a $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \sin^2 x}\,dx$ (Let $u = \sin^2 x$.)

b $\displaystyle\int_1^{e} \frac{\ln x + 1}{(x \ln x + 1)^2}\,dx$ (Let $u = x \ln x$.)

12 Use the substitution $u = \sqrt{x - 1}$ to find $\displaystyle\int \frac{1}{2x\sqrt{x - 1}}\,dx$.

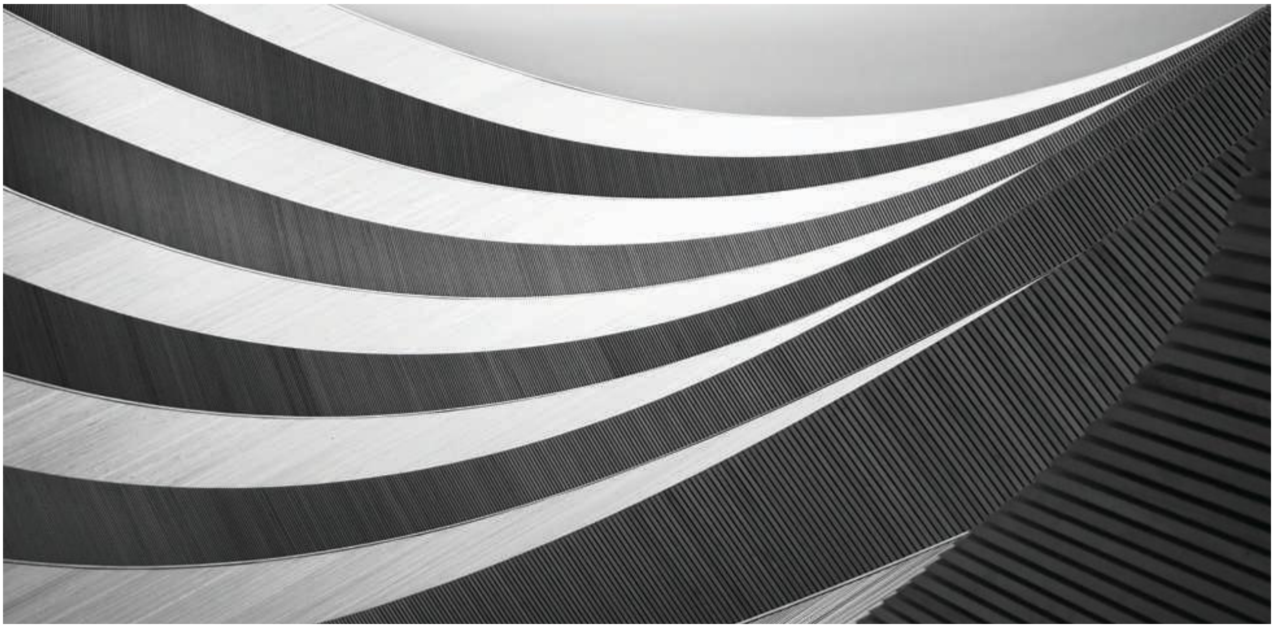
13 **a** Use the substitution $u = \sqrt{x}$ to find $\displaystyle\int \frac{1}{\sqrt{x(1 - x)}}\,dx$.

b Evaluate the integral in part **a** again, using the substitution $u = x - \frac{1}{2}$.

c Hence show that $\sin^{-1}(2x - 1) = 2\sin^{-1}\sqrt{x} - \frac{\pi}{2}$, for $0 < x < 1$.

ENRICHMENT

14 Use the substitution $u = x - \dfrac{1}{x}$ to show that $\displaystyle\int_1^{\frac{1}{2}(\sqrt{6}+\sqrt{2})} \frac{1 + x^2}{1 + x^4}\,dx = \frac{\pi}{4\sqrt{2}}$.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12E Further integration by substitution

The second stage of integration by substitution reverses the previous procedure and replaces x by a function of u. The substitutions are therefore of the form

'Let $x =$ some function of u.'

Substituting x by a function of u

As a first example, here is a quite different substitution which solves the integral given in a worked example of the last section.

Example 20 **12E**

Find $\int_0^1 x\sqrt{1-x}\,dx$, using the substitution $x = 1 - u^2$.

SOLUTION

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= \int_1^0 (1-u^2)u(-2u)\,du\\ &= -2\int_1^0 (u^2-u^4)\,du\\ &= -2\Big[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\Big]_1^0\\ &= -0 + 2\left(\tfrac{1}{3}-\tfrac{1}{5}\right)\\ &= \tfrac{4}{15}\end{aligned}$$

Let	x	$= 1 - u^2$.
Then	dx	$= -2u\,du$,
and	$\sqrt{1-x}$	$= u$.
When	x	$= 0, u = 1$,
when	x	$= 1, u = 0$.

This question is a good example of how an integral may be evaluated in contrasting ways. The next integral uses a trigonometric substitution, but can also be done using areas of segments.

Example 21 **12E**

Find $\int_{3\sqrt{2}}^{6} \sqrt{36 - x^2}\,dx$:

a using the substitution $x = 6\sin u$,

b using the formula for the area of a segment.

SOLUTION

a
$$\begin{aligned}\int_{3\sqrt{2}}^{6}\sqrt{36-x^2}\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 6\cos u \times 6\cos u\,du\\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 36\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2u\right)du\\ &= \Big[\,18u + 9\sin 2u\,\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\\ &= (9\pi + 0) - \left(\tfrac{9\pi}{2} + 9\right)\\ &= \tfrac{9}{2}(\pi - 2)\end{aligned}$$

Let	x	$= 6\sin u$.
Then	dx	$= 6\cos u\,du$,
and	$\sqrt{36-x^2}$	$= 6\cos u$.
When	$x = 3\sqrt{2}$,	$u = \frac{\pi}{4}$,
when	$x = 6$,	$u = \frac{\pi}{2}$.

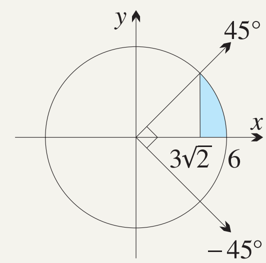
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b The integral is sketched opposite. The shaded area is half the segment subtending an angle of 90°.

Hence $\displaystyle\int_{3\sqrt{2}}^{6} \sqrt{36 - x^2}\, dx = \tfrac{1}{2} \times \tfrac{1}{2} \times 6^2\left(\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right)$

$= 9\left(\tfrac{\pi}{2} - 1\right).$



Note: Careful readers may notice a problem here, in that given the value $x = 3\sqrt{2}$, u is determined by $\sin u = \frac{1}{2}\sqrt{2}$, so there are infinitely many possible values of u. A similar problem occurred in the previous worked example, where $0 = 1 - u^2$ had two solutions. These problems arise because the functions involved in the substitutions were $x = 1 - u^2$ and $x = 6\sin u$, whose inverses were not functions. A full account of all this would require substitutions by restrictions of the functions given above so that they had inverse functions. In practice, however, this is rarely necessary, and it is certainly not a concern of this course.

As a rule of thumb, work with positive square roots, and with trigonometric functions, work in the same quadrants as were involved in the definitions of the inverse trigonometric functions in Chapter 17 of the Year 11 book.

Here is a summary of Sections 12D and 12E.

10 INTEGRATION USING THE REVERSE CHAIN RULE AND SUBSTITUTION

The chain rule can be used in reverse in various ways to find integrals.

- The reverse chain rule straightforwardly reverses the steps of differentiation by the chain rule.
- A reverse-chain-rule formula can be developed for each standard form.
- Substitution can be done by, 'Let u be a function of x.'
- Substitution can be done by, 'Let x be a function of u.'
- When substituting into a definite integral, the limits of integration can be substituted as well, so that there is no need ever to return from u to x.

Exercise 12E

FOUNDATION

1 Consider the integral $I = \displaystyle\int x(x - 1)^5\, dx$, and let $x = u + 1$.

a Show that $dx = du$.

b Show that $I = \displaystyle\int u^5(u + 1)\, du$.

c Hence find I.

d Check your answer by differentiating it.

2 Using the same substitution as in the previous question, find:

a $\displaystyle\int \frac{x}{\sqrt{x - 1}}\, dx$

b $\displaystyle\int \frac{x}{(x - 1)^2}\, dx$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Consider the integral $J = \int x\sqrt{x+1}\,dx$, and let $x = u^2 - 1$.

a Show that $dx = 2u\,du$.

b Show that $J = 2\int (u^4 - u^2)\,du$.

c Hence find J.

d Check your answer by differentiating it.

4 Using the same substitution as in the previous question, find:

a $\int x^2\sqrt{x+1}\,dx$

b $\int \dfrac{2x+3}{\sqrt{x+1}}\,dx$

5 Find each indefinite integral using the given substitution.

a $\int \dfrac{x-2}{x+2}\,dx$ (Let $x = u - 2$.)

b $\int \dfrac{2x+1}{\sqrt{2x-1}}\,dx$ $\left(\text{Let } x = \tfrac{1}{2}(u+1).\right)$

c $\int 3x\sqrt{4x-5}\,dx$ $\left(\text{Let } x = \tfrac{1}{4}(u^2+5).\right)$

d $\int \dfrac{1}{1+\sqrt{x}}\,dx$ $\left(\text{Let } x = (u-1)^2.\right)$

6 Evaluate, using the given substitution:

a $\int_0^1 x(x+1)^3\,dx$ (Let $x = u - 1$.)

b $\int_0^{\frac{1}{2}} \dfrac{1+x}{1-x}\,dx$ (Let $x = 1 - u$.)

c $\int_0^1 \dfrac{3x}{\sqrt{3x+1}}\,dx$ $\left(\text{Let } x = \tfrac{1}{3}(u-1).\right)$

d $\int_0^1 \dfrac{2-x}{(2+x)^3}\,dx$ (Let $x = u - 2$.)

e $\int_0^4 x\sqrt{4-x}\,dx$ $\left(\text{Let } x = 4 - u^2.\right)$

f $\int_1^5 \dfrac{x}{(2x-1)^{\frac{3}{2}}}\,dx$ $\left(\text{Let } x = \tfrac{1}{2}(u^2+1).\right)$

g $\int_0^4 \dfrac{1}{3+\sqrt{x}}\,dx$ $\left(\text{Let } x = (u-3)^2.\right)$

h $\int_0^7 \dfrac{x^2}{\sqrt[3]{x+1}}\,dx$ $\left(\text{Let } x = u^3 - 1.\right)$

DEVELOPMENT

7 **a** Consider the integral $I = \int \dfrac{1}{\sqrt{5-4x-x^2}}\,dx$, and let $x = u - 2$.

Show that $I = \int \dfrac{1}{\sqrt{9-u^2}}\,du$, and hence find I.

b Use a similar approach to find:

i $\int \dfrac{1}{x^2+2x+4}\,dx$ (Let $x = u - 1$.)

ii $\int \dfrac{1}{\sqrt{4-2x-x^2}}\,dx$ (Let $x = u - 1$.)

iii $\int_1^2 \dfrac{1}{\sqrt{3+2x-x^2}}\,dx$ (Let $x = u + 1$.)

iv $\int_3^7 \dfrac{1}{x^2-6x+25}\,dx$ (Let $x = u + 3$.)

8 **a** Consider the integral $J = \int \dfrac{1}{\sqrt{4-x^2}}\,dx$, and let $x = 2\sin\theta$.

Show that $J = \int 1\,d\theta$, and hence show that $J = \sin^{-1}\tfrac{x}{2} + C$.

b Using a similar approach, find:

i $\int \dfrac{1}{9+x^2}\,dx$ (Let $x = 3\tan\theta$.)

ii $\int \dfrac{-1}{\sqrt{3-x^2}}\,dx$ (Let $x = \sqrt{3}\cos\theta$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iii $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$ (Let $x = \frac{1}{2}\sin\theta$.)

iv $\displaystyle\int \frac{1}{1+16x^2}\,dx$ (Let $x = \frac{1}{4}\tan\theta$.)

v $\displaystyle\int_0^3 \frac{1}{\sqrt{36-x^2}}\,dx$ (Let $x = 6\sin\theta$.)

vi $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}\,dx$ (Let $x = \frac{2}{3}\tan\theta$.)

9 **a** Consider the integral $I = \displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$, and let $x = \sin\theta$.

Show that $I = \displaystyle\int \sec^2\theta\,d\theta$, and hence show that $I = \dfrac{x}{\sqrt{1-x^2}} + C$.

b Similarly, use the given substitution to find:

i $\displaystyle\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$ (Let $x = 2\tan\theta$.)

ii $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$ (Let $x = \sin\theta$.)

iii $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ (Let $x = 2\sin\theta$.)

iv $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ (Let $x = 5\cos\theta$.)

v $\displaystyle\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$ (Let $x = 3\tan\theta$.)

vi $\displaystyle\int_2^4 \frac{1}{x^2\sqrt{x^2-4}}\,dx$ (Let $x = 2\sec\theta$.)

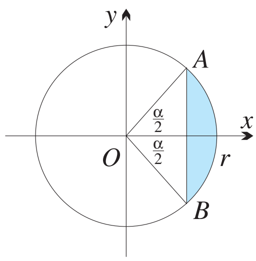
10 Find the equation of the curve $y = f(x)$ if $f'(x) = \dfrac{\sqrt{x^2-9}}{x}$ and $f(3) = 0$.
(Hint: Use the substitution $x = 3\sec\theta$.)

11 Find the exact area of the region bounded by $y = \dfrac{x^3}{\sqrt{3-x^2}}$, the x-axis and the line $x = 1$.
(Hint: Use the substitution $x = \sqrt{3}\sin\theta$, followed by the substitution $u = \cos\theta$.)

12 [These are confirmations rather than proofs, because the calculus of trigonometric functions was developed on the basis of the formulae in parts **a** and **b**.]

a Use integration to confirm that the area of a circle is πr^2.
(Hint: Find the area bounded by the semi-circle $y = \sqrt{r^2-x^2}$ and the x-axis and double it. Use the substitution $x = r\sin\theta$.)

b The shaded area in the diagram to the right is the segment of a circle of radius r cut off by the chord AB subtending an angle α at the centre O.



i Show that the area is $I = 2\displaystyle\int_{r\cos\frac{1}{2}\alpha}^{r} \sqrt{r^2-x^2}\,dx$.

ii Let $x = r\cos\theta$, and show that $I = -2r^2\displaystyle\int_{\frac{1}{2}\alpha}^{0} \sin^2\theta\,d\theta$.

iii Hence confirm that $I = \frac{1}{2}r^2(\alpha - \sin\alpha)$.

c Use a similar approach to confirm that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. Then justify the formula by regarding the ellipse as the unit circle stretched horizontally by a factor of a and vertically by a factor of b.

ENRICHMENT

13 **a** Use the substitution $x = -u$ to show that $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx = \int_{-2}^{2} \frac{x^2e^x}{e^x+1}\,dx$.

b Hence find $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

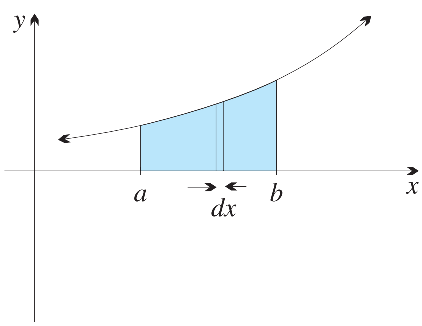
12F Volumes of rotation

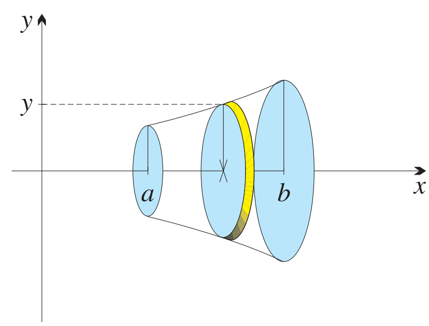
When a region in the coordinate plane is rotated about either the x-axis or the y-axis, a solid region in three dimensions is generated, called a *solid of revolution*. The process is similar to shaping wood on a lathe, or making pottery on a wheel, because such shapes have rotational symmetry and circular cross-sections.

The volumes of such solids can be found using a simple integration formula. The well-known formulae for the volumes of cones and spheres can finally be proven by this method.

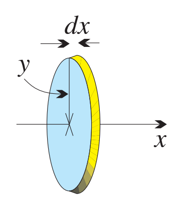
Rotating a region about the x-axis

The first diagram below shows the region under the curve $y = f(x)$ in the interval $a \le x \le b$, and the second shows the solid generated when this region is rotated about the x-axis.





Imagine the solid sliced like salami perpendicular to the x-axis into infinitely many circular slices, each of width dx. One of the slices is shown in the right-hand diagram, and again in more detail below. The vertical strip in the left-hand diagram is what generates this slice when it is rotated about the x-axis.



Now radius of circular slice $= y$, the height of the strip,

so area of circular slice $= \pi y^2$, because it is a circle.

The slice is a thin cylinder of infinitesimal thickness dx,

so volume of circular slice $= \pi y^2\, dx$ (area $\times$ thickness).

To get the total volume, we add all the slices from $x = a$ to $x = b$,

so volume of solid $= \displaystyle\int_a^b \pi y^2\, dx$.

11 VOLUMES OF REVOLUTION ABOUT THE x-AXIS

The volume of the solid generated when the region between a curve and the x-axis from $x = a$ to $x = b$ is rotated about the x-axis is

$$\text{volume} = \int_a^b \pi y^2\, dx \text{ cubic units.}$$

If the curve is below the x-axis, so that y is negative, then the volume calculated is still positive, because y^2 rather than y occurs in the formula.

Unless other units are specified, ‘cubic units’ (u^3), should be used, by analogy with the areas of regions discussed in Section 3F.

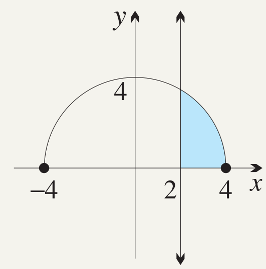
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 22 12F

The shaded region cut off the semi-circle $y = \sqrt{16 - x^2}$ by the line $x = 2$ is rotated about the x-axis. Find the volume generated.

SOLUTION



Squaring, $y^2 = 16 - x^2$,

$$\begin{aligned}\text{so} \quad \text{volume} &= \int_2^4 \pi y^2\, dx \\ &= \pi \int_2^4 (16 - x^2)\, dx \\ &= \pi \Big[16x - \tfrac{1}{3}x^3 \Big]_2^4 \\ &= \pi\left(64 - \tfrac{64}{3} - 32 + \tfrac{8}{3}\right) \\ &= \tfrac{40\pi}{3} \text{ cubic units.}\end{aligned}$$

Volumes of revolution about the y-axis

To calculate the volume when a region is rotated about the y-axis, we exchange x and y.

12 VOLUMES OF REVOLUTION ABOUT THE y-AXIS

The volume of the solid generated when the region between a curve and the y-axis from $y = a$ to $y = b$ is rotated about the y-axis is

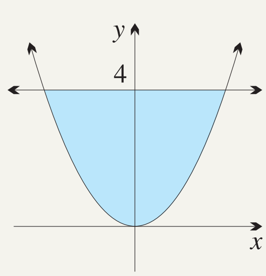
$$\text{volume} = \int_a^b \pi x^2\, dy \text{ cubic units.}$$

When y is given as a function of x, the equation will need to be written with x^2 as the subject.

Example 23 12F

Find the volume of the solid formed by rotating the region between $y = x^2$ and the line $y = 4$ about the y-axis.

SOLUTION



$$\begin{aligned}\text{Because } x^2 = y, \text{ volume} &= \int_0^4 \pi x^2\, dy \\ &= \pi \int_0^4 y\, dy \\ &= \pi \Big[\tfrac{1}{2}y^2 \Big]_0^4 \\ &= 8\pi \text{ cubic units.}\end{aligned}$$

Volume by subtraction

When rotating the region between two curves lying above the x-axis, the two integrals need to be subtracted, as if the outer volume has been formed first, and the inner volume then cut away from it. The two volumes

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

can always be calculated separately and subtracted, but if the two integrals have the same limits of integration, it may be more convenient to combine them.

13 ROTATING THE REGION BETWEEN CURVES

- The volume of the solid generated when the region between two curves from $x = a$ to $x = b$ is rotated about the x-axis is
$$\text{volume} = \int_a^b \pi(y_2{}^2 - y_1{}^2)\,dx \quad (\text{where } y_2 > y_1 > 0).$$
- Similarly, the volume of the solid generated when the region between two curves from $y = a$ to $y = b$ is rotated about the y-axis is
$$\text{volume} = \int_a^b \pi(x_2{}^2 - x_1{}^2)\,dy \quad (\text{where } x_2 > x_1 > 0).$$

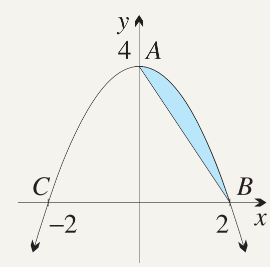
Example 24 12F

The curve $y = 4 - x^2$ meets the y-axis at $A(0, 4)$ and the x-axis at $B(2, 0)$ and $C(-2, 0)$. Find the volume generated when the region between the curve and the line AB is rotated:

a about the x-axis, **b** about the y-axis.

SOLUTION

a The parabola is $y_2 = 4 - x^2$, and the line AB is $y_1 = 4 - 2x$,



$$\begin{aligned}
\text{so volume} &= \int_0^2 \pi(y_2{}^2 - y_1{}^2)\,dx \\
&= \pi\int_0^2 \big((16 - 8x^2 + x^4) - (16 - 16x + 4x^2)\big)\,dx \\
&= \pi\int_0^2 (x^4 - 12x^2 + 16x)\,dx \\
&= \pi\Big[\tfrac{1}{5}x^5 - 4x^3 + 8x^2\Big]_0^2 \\
&= \pi\big(\tfrac{32}{5} - 32 + 32\big) \\
&= \tfrac{32\pi}{5} \text{ cubic units.}
\end{aligned}$$

b The parabola is $x_2{}^2 = 4 - y$, and the line is $x_1 = 2 - \frac{1}{2}y$,

$$\begin{aligned}
\text{so volume} &= \int_0^4 \pi(x_2{}^2 - x_1{}^2)\,dy \\
&= \pi\int_0^4 \big((4 - y) - (4 - 2y + \tfrac{1}{4}y^2)\big)\,dy \\
&= \pi\int_0^4 (-\tfrac{1}{4}y^2 + y)\,dy \\
&= \pi\Big[-\tfrac{1}{12}y^3 + \tfrac{1}{2}y^2\Big]_0^4 \\
&= \pi\big(-\tfrac{16}{3} + 8\big) \\
&= \tfrac{8\pi}{3} \text{ cubic units.}
\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Note: One would not normally expect the volumes of revolution about the two different axes to be equal. This is because an element of area will generate a larger element of volume if it is moved further away from the axis of rotation.

Cones and spheres

The formulae for the volumes of cones and spheres may have been learnt earlier, but they cannot be proven without arguments involving integration. The proofs of both results are developed in Question 12 of Exercise 12F, and these questions should be carefully worked through.

14 VOLUME OF CONES AND SPHERES

- For a cone, $V = \frac{1}{3}\pi r^2 h$.
- For a sphere, $V = \frac{4}{3}\pi r^3$.

Exercise 12F

FOUNDATION

1 **a** Sketch the region bounded by the line $y = 3x$ and the x-axis between $x = 0$ and $x = 3$.

b When this region is rotated about the x-axis, a right circular cone is formed. Find the radius and height of the cone and hence find its volume.

c Evaluate $\pi \int_0^3 y^2\,dx = \pi \int_0^3 9x^2\,dx$ in order to check your answer.

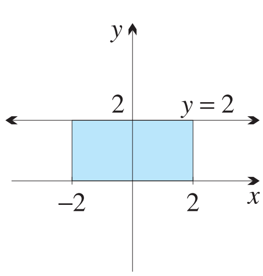
2 **a** Sketch the region bounded by the curve $y = \sqrt{9 - x^2}$ and the x-axis between $x = -3$ and $x = 3$.

b When this region is rotated about the x-axis, a sphere is formed. Find the radius of the sphere and hence find its volume.

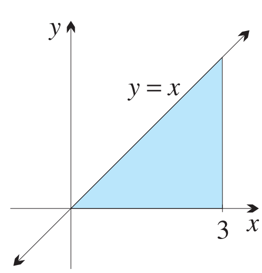
c Evaluate $\pi \int_{-3}^3 y^2\,dx = \pi \int_{-3}^3 (9 - x^2)\,dx$ in order to check your answer.

3 Calculate the volume generated when each shaded region is rotated about the x-axis.

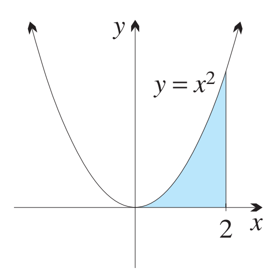
a



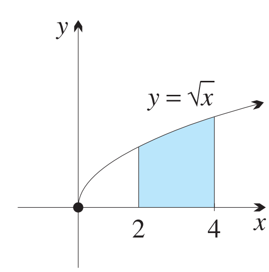
b



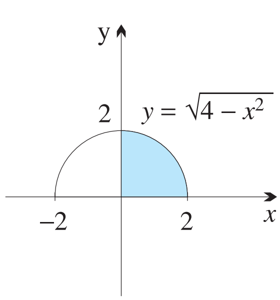
c



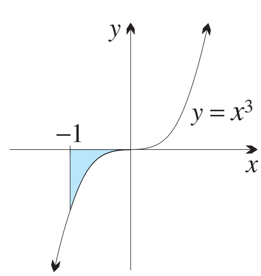
d



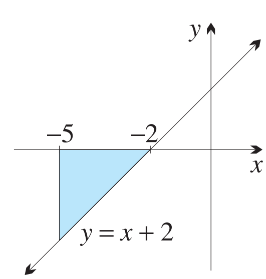
e



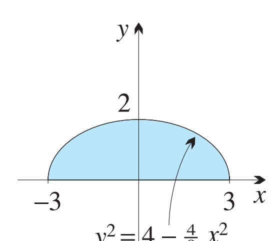
f



g



h

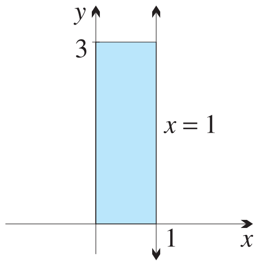


ISBN 978-1-108-76630-2

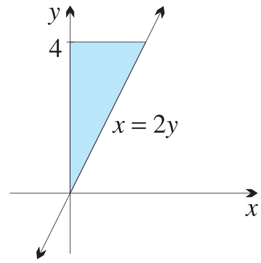
Cambridge University Press
Updated July 2022

4 Calculate the volume generated when each shaded region is rotated about the y-axis.

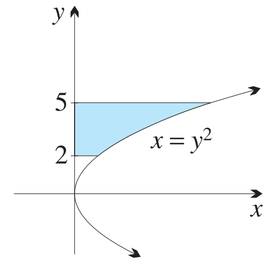
a



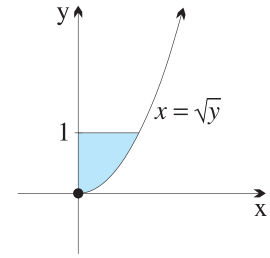
b



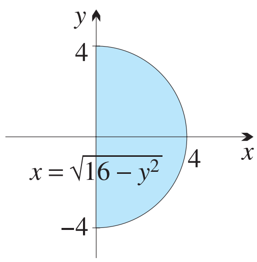
c



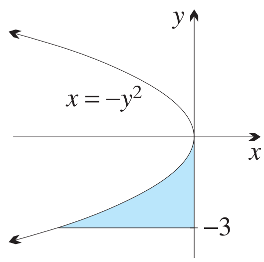
d



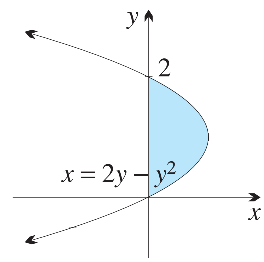
e



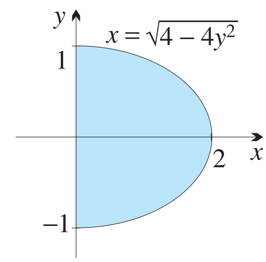
f



g



h



DEVELOPMENT

5 The region between the curve $y = e^x$ and the x-axis from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the solid generated.

6 Find the volume generated when the region between the curve $y = \dfrac{1}{\sqrt{x}}$ and the x-axis, from $x = 2$ to $x = 4$, is rotated about the x-axis.

7 The region between the curve $y = \dfrac{1}{x^2}$ (called a *truncus*) and the y-axis from $y = 1$ to $y = 6$ is rotated about the y-axis. Calculate the exact volume of the solid formed.

8 **a** Write $\tan^2 x$ in terms of $\sec^2 x$.
b The region bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$ is rotated about the x-axis. Find the volume of the solid generated.

9 **a** Write $\sin^2 x$ in terms of $\cos 2x$.
b The region bounded by the curve $y = \sin x$, the x-axis and the vertical line $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the volume of the solid generated.

10 Find the volume of the solid formed by rotating the region with the given boundaries about the x-axis. (A sketch of each region will be needed.)
a $y = x + 3$, $x = 3$, $x = 5$ and $y = 0$
b $y = 1 + \sqrt{x}$, $x = 1$, $x = 4$ and $y = 0$
c $y = 5x - x^2$ and $y = 0$
d $y = x^3 - x$ and $y = 0$

11 Find the volume of the solid formed by rotating the region with the given boundaries about the y-axis. (A sketch of each region will be needed.)
a $x = y - 2$, $y = 1$ and $x = 0$
b $x = y^2 + 1$, $y = 0$, $y = 1$ and $x = 0$
c $x = y(y - 3)$ and $x = 0$
d $y = 1 - x^2$ and $y = 0$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

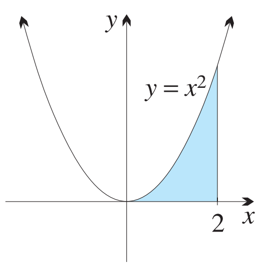
12 A vat is designed by rotating about the x-axis the region between the curve $y = 1 + \frac{1}{x}$ and the x-axis from $x = \frac{1}{2}$ to $x = 3$. Show that its volume is $\frac{\pi}{6}(25 + 12\ln 6)$ u^3.

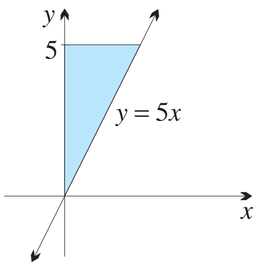
13 A metal stud is created by rotating about the x-axis the region contained between the curve $y = e^x - e^{-x}$ and the x-axis from $x = 0$ to $x = \frac{1}{2}$. Show that the volume of the stud is $\frac{\pi}{2}(e - 2 - e^{-1})$ u^3.

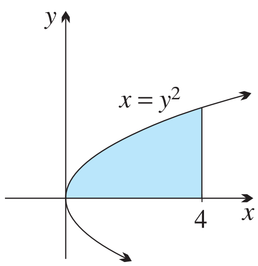
14 A champagne flute is designed by rotating about the x-axis the region between the curve $y = 4 + 4\sin\frac{x}{4}$ and the x-axis from $x = 4\pi$ to $x = 6\pi$. Find, correct to 4 significant figures, the capacity of the flute, if 1 unit = 1 cm.

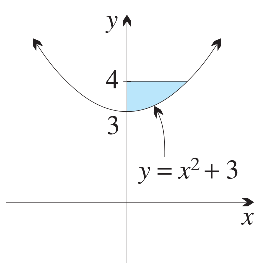
15 The region R is bounded by the parabola $y = x^2$ and the x-axis from $x = 0$ to $x = 4$.
- **a** Find the volume of the cylinder formed when the region between the line $x = 4$ and the y-axis from $y = 0$ to $y = 16$ is rotated about the y-axis.
- **b** Find the volume of the solid formed when the region between the parabola $y = x^2$ and the y-axis from $y = 0$ to $y = 16$ is rotated about the y-axis.
- **c** Hence find the volume of the solid formed when R is rotated about the y-axis.

16 Find the volume of the solid generated by rotating each region about:
- **i** the x-axis,
- **ii** the y-axis. (Hint: In some cases a subtraction of volumes will be necessary.)

a


b


c


d


17 a Sketch the region bounded by the curves $y = x^2$ and $y = x^3$.
- **b** Find the volume of the solid generated when this region is rotated about:
  - **i** the x-axis,
  - **ii** the y-axis.

18 a On the same number plane sketch the graphs of $xy = 5$ and $x + y = 6$, clearly indicating their points of intersection.
- **b** Find the volume of the solid generated when the region bounded by the two curves is rotated about the x-axis.

19 The region bounded by the hyperbola $y = 2 - \frac{2}{x}$, the vertical line $x = 1$ and the horizontal line $y = 1$ is rotated about the x-axis. Find the volume of the solid formed.

20 a Find the equation of the tangent to the curve $y = x^3 + 2$ at the point where $x = 1$.
- **b** Draw a diagram showing the region bounded by the curve, the tangent and the y-axis.
- **c** Calculate the volume of the solid formed when this region is rotated about:
  - **i** the x-axis,
  - **ii** the y-axis.

21 A rubber washer is generated by rotating the region between the curve $y = \log_e x$, the x-axis and the line $x = 2$ about the y-axis. Find the exact volume of the washer.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

22 In this question some standard volume formulae will be proven.

a A right circular cone of height h and radius r is generated by rotating about the x-axis the region bounded by the line $y = \dfrac{rx}{h}$, the x-axis and the line $x = h$. Show that the volume of the cone is $\frac{1}{3}\pi r^2 h$.

b A cylinder of height h and radius r is generated by rotating about the x-axis the region bounded by the line $y = r$, the x-axis, the y-axis and the line $x = h$. Show that the volume of the cylinder is $\pi r^2 h$.

c i A sphere of radius r is generated by rotating about the x-axis the region between the semi-circle $y = \sqrt{r^2 - x^2}$ and the x-axis. Show that the volume of the sphere is $\frac{4}{3}\pi r^3$.

ii A spherical cap of height h is formed by rotating about the x-axis the region between the semi-circle $y = \sqrt{r^2 - x^2}$ and the x-axis from $x = r - h$ to $x = r$. Show that the volume of the cap is $\frac{1}{3}\pi h^2(3r - h)$.

ENRICHMENT

23 a Multiply $\sec\theta$ by $\dfrac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$, and hence find $\displaystyle\int \sec\theta\, d\theta$.

b The region R is bounded by $y = \dfrac{x}{\sqrt{x^2 + 16}}$, the x-axis and the line $x = 4$. Show that the volume generated by rotating R about the y-axis is $16\pi\left(\sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right)$ units³. (Hint: Use the substitution $y = \sin\theta$ and the result in part **a**.)

24 Consider the curves $f(x) = x^n$ and $f(x) = x^{n+1}$, where n is a positive integer.

a Find the points of intersection of the curves.

b Show that the volume V_n of the solid generated when the region bounded by the two curves is rotated around the x-axis is given by $\pi\left(\dfrac{1}{2n + 1} - \dfrac{1}{2n + 3}\right)$ cubic units.

c Describe the solid whose volume is given by $\displaystyle\lim_{n\to\infty}(V_1 + V_2 + V_3 + \cdots + V_n)$.

d Find the volume of the solid in part **c**.

e Deduce that the series $\dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \dfrac{1}{7 \times 9} + \cdots$ has a limiting sum of $\dfrac{1}{6}$.

25 a Sketch the region bounded by the parabola $y = -x^2 + 6x - 8$ and the x-axis.

b By completing the square, or otherwise, show that the equation of the curve is $x = 3 + \sqrt{1 - y}$ for $3 \le x \le 4$, and $x = 3 - \sqrt{1 - y}$ for $2 \le x \le 3$.

c The region in part **a** is rotated about the y-axis. Show that the volume of the solid formed is given by $V = \displaystyle\int_0^1 12\pi\sqrt{1 - y}\, dy$, and hence calculate the exact volume.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 12 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 12 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Find the derivative of:

a $y = \sin^{-1} 3x$ **b** $y = \tan^{-1} \frac{x}{3}$ **c** $y = \cos^{-1}(1 - x)$

d $y = x^2 \tan^{-1} x$ **e** $y = \tan^{-1}\left(\frac{1}{2}x + 1\right)$ **f** $y = \sin^{-1} \frac{1}{x}$

2 Show that $y = \tan^{-1} x$ satisfies the differential equation $(1 + x^2)y'' + 2xy' = 0$.

3 a Show that the functions $y = \cos^{-1} x$ and $y = \sin^{-1} \sqrt{1 - x^2}$ have the same derivative for $0 \le x \le 1$.

b Explain the significance of the result in **a**.

4 Find:

a $\displaystyle\int \frac{1}{3 + x^2}\,dx$ **b** $\displaystyle\int \frac{1}{\sqrt{3 - x^2}}\,dx$

c $\displaystyle\int \frac{1}{9 + 4x^2}\,dx$ **d** $\displaystyle\int \frac{-1}{\sqrt{16 - 9x^2}}\,dx$

5 Evaluate:

a $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + 9x^2}\,dx$ **b** $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}}\,dx$

6 Find:

a $\displaystyle\int \cos^2 x\,dx$ **b** $\displaystyle\int \sin^2 x\,dx$

c $\displaystyle\int \cos^2 2x\,dx$ **d** $\displaystyle\int \sin^2 4x\,dx$

7 Find the exact value of:

a $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 3x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{6}} \cos^2 \tfrac{1}{2}x\,dx$

8 Show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 2x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 4x\,dx = \frac{\pi}{4}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

9 Find each indefinite integral using the given substitution.

a $\int 5(5x - 1)^5\,dx$ [Let $u = 5x - 1$.]

b $\int 2x(x^2 + 2)^2\,dx$ [Let $u = x^2 + 2$.]

c $\int \frac{4x^3}{(x^4 + 1)^2}\,dx$ [Let $u = x^4 + 1$.]

d $\int \frac{1}{\sqrt{4x + 3}}\,dx$ [Let $u = 4x + 3$.]

e $\int \sin^2 x \cos x\,dx$ [Let $u = \sin x$.]

f $\int \tan^3 x \sec^2 x\,dx$ [Let $u = \tan x$.]

10 Evaluate each definite integral using the given substitution.

a $\int_{-1}^{0} x^2(1 + x^3)^4\,dx$ [Let $u = 1 + x^3$.]

b $\int_{0}^{\frac{\pi}{2}} \cos^3 x \sin x\,dx$ [Let $u = \cos x$.]

c $\int_{1}^{\sqrt{2}} x\sqrt{x^2 - 1}\,dx$ [Let $u = x^2 - 1$.]

d $\int_{1}^{e} \frac{(\ln x)^2}{x}\,dx$ [Let $u = \ln x$.]

e $\int_{\frac{1}{2}}^{1} \frac{e^{\frac{1}{x}}}{x^2}\,dx$ [Let $u = \frac{1}{x}$.]

f $\int_{0}^{\frac{\pi}{8}} \frac{\sec^2 2x}{1 + \tan 2x}\,dx$ [Let $u = \tan 2x$.]

11 Find each indefinite integral using the given substitution.

a $\int \frac{x}{x - 1}\,dx$ [Let $x = u + 1$.]

b $\int \frac{x - 1}{\sqrt{x + 2}}\,dx$ [Let $x = u - 2$.]

c $\int x\sqrt{2x + 1}\,dx$ [Let $x = \frac{1}{2}u^2 - \frac{1}{2}$.]

d $\int \frac{1}{4 + \sqrt{x}}\,dx$ [Let $x = (u - 4)^2$.]

12 Evaluate, using the given substitution:

a $\int_{1}^{2} x(x - 1)^4\,dx$ [Let $x = u + 1$.]

b $\int_{1}^{5} \frac{x}{x + 3}\,dx$ [Let $x = u - 3$.]

c $\int_{0}^{15} \frac{x}{\sqrt{x + 1}}\,dx$ [Let $x = u^2 - 1$.]

d $\int_{2}^{3} \frac{1}{2}x\sqrt{x - 2}\,dx$ [Let $x = u^2 + 2$.]

13 a State the domain and range of the function $y = \sqrt{9 - x}$.

b Sketch the graph of the function.

c Calculate the area of the region bounded by the curve and the coordinate axes.

d Calculate the volume of the solid formed when this region is rotated about:

i the x-axis, **ii** the y-axis.

14 A horn is created by rotating about the x-axis the region between the curve $y = \frac{1}{\sqrt{4 - x}}$ and the x-axis from $x = 0$ to $x = 3\frac{3}{4}$. Find the volume of the horn.

15 A vase is designed by rotating the parabola $y^2 = 18(x - 6)$ from $y = -6$ to $y = 6$ about the y-axis. Find the exact volume of the vase.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 a Write $\cos^2 2x$ in terms of $\cos 4x$.

b The region bounded by the curve $y = \cos 2x$ and the x-axis from $x = -\frac{\pi}{6}$ to $x = \frac{\pi}{6}$ is rotated about the x-axis. Show that the solid generated has volume $\frac{\pi}{24}(4\pi + 3\sqrt{3})\ u^3$.

17 A tank is created by rotating about the x-axis the region between the curve $y = 1 + e^{-x}$ and the x-axis from $x = 1$ to $x = 3$. Find its volume correct to two decimal places.

18 a Sketch $y = 1 - \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes.

b Find the volume generated when R is rotated about the x-axis.

19 a Find the stationary points of the curve $y = x + \dfrac{1}{x}$ and sketch its graph.

b Show that the line $y = \frac{5}{2}$ intersects the curve when $x = \frac{1}{2}$ or $x = 2$.

c Find the volume of the solid generated when the region between the curve and the line $y = \frac{5}{2}$ is rotated about the x-axis.

20 a On the same set of axes sketch the curves $y = \sqrt{9 - x^2}$ and $y = 18 - 2x^2$.

b Find the area bounded by the two curves.

c Find the volume of the solid formed when this region is rotated about the x-axis

21 The region under the graph $y = 2^{x+1}$ between $x = 1$ and $x = 3$ is rotated about the x-axis.

a Write down the definite integral representing the volume of the solid that is formed.

b Use the trapezoidal rule with five function values to approximate the volume of the solid, giving your answer as multiple of π.

c Find the exact value of the volume, then approximate is as an integer multiple of π. Why is the exact answer smaller than the trapezoidal-rule approximation?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 Differential equations

A *differential equation* is an equation that involves a derivative.

For example, we saw in Question 4 of the Chapter 16 Review in the Year 11 book that the radioactive decay of a mass M of caesium-137, with a half-life of about 30.2 years, obeys the differential equation

$$\frac{dM}{dt} = -kM, \text{ where } k \doteqdot 0.023,$$

which says that the rate at which caesium-137 is decreasing is proportional to the mass M present at that time t.

When calculus is used in science or engineering or elsewhere, it is very common that the behaviour being studied is modelled by a differential equation, as in the example with caesium-137. This chapter is a first introduction to an extremely important branch of calculus.

Changes of pronumeral can be confusing when learning methods. The first four sections are all explained using only the standard pronumerals x and y. Examples from science and elsewhere require other pronumerals such as t for time and M for mass, and will not be discussed until Section 13E.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13A Differential equations

We begin by explaining what a differential equation is and what a solution of a differential equation is, and by introducing some basic terminology.

In this chapter, all logarithms have base e, and we will usually write $\log x$ or $\ln x$ instead of $\log_e x$.

Differential equations and their order

A *differential equation*, or DE for short, is an equation that involves a derivative. Here are some examples of DEs.

$$y' = -7y \qquad y' = 1 + y^2 \qquad y'' + 49y = 0$$
$$yy' + x = 0 \qquad y' = y(y-1) \qquad x^2y'' - xy' + y = 0$$

The *order of a differential equation* is the order of the highest derivative that occurs within it. For example, the DEs in the last column above have order 2 because they involve the second derivative, whereas the other four have order 1.

The chapter is mostly concerned with first-order DEs, with a few simple higher-order examples.

Example 1 13A

State whether these equations are differential equations, and if they are, state their order.

a $x^2y'' - xy' + y = 0$ **b** $x^2y^2 - xy + 1 = 0$

c $y' = x^2 + 1$ **d** $y = y''''$

SOLUTION

Parts **a**, **c** and **d** are DEs of orders 2, 1 and 4 respectively.
Part **b** is not a DE because it does not involve a derivative.

What is a solution of a differential equation?

DEs are equations, and they have solutions. The solutions are not numbers, but functions of x, or more generally relations in x and y. We can test whether a function is a solution by substituting it into the DE, as in the next worked example.

Example 2 13A

Test whether each function is a solution of the differential equation $y' = -7y$.

a $y = 20e^{7x}$ **b** $y = 20e^{-7x}$

SOLUTION

a Differentiating, $y' = 140e^{7x}$, so substituting into the DE $y' = -7y$,
$\text{LHS} = 140e^{7x}$ and $\text{RHS} = -7 \times 20e^{7x}$.
Because LHS $\neq$ RHS, the function is not a solution.

b Differentiating, $y' = -140e^{-7x}$, so substituting into the DE $y' = -7y$,
$\text{LHS} = -140e^{-7x}$ and $\text{RHS} = -7 \times 20e^{-7x}$.
Because LHS $=$ RHS, the function is a solution.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Note: The differential equation $y' = -7y$ in part **b** above only involves algebraic operations, yet the solution $y = 20e^{-7x}$ involves an exponential function. The next worked example below is also algebraic, but its solution involves a trigonometric function. This behaviour is typical for DEs, just as we have already seen with integrals such as $\displaystyle\int \frac{dx}{1 + x^2} = \tan^{-1} x + C$.

Example 3 13A

Show that $y = \tan x$ is a solution of the differential equation $y' = 1 + y^2$.

SOLUTION

Substituting $y = \tan x$ into the DE,

$$\text{LHS} = \frac{d}{dx}\tan x \qquad\qquad \text{RHS} = 1 + \tan^2 x$$
$$= \sec^2 x, \qquad\qquad = \sec^2 x.$$

Hence $y = \tan x$ is a solution of the DE.

Indefinite integrals are equivalent to DEs

An indefinite integral such as $\displaystyle\int 2x\,dx$ says, 'Find a function whose derivative is $2x$'. It is thus equivalent to the DE

$$y' = 2x,$$

and the primitive $y = x^2$ is a solution of this DE because on substitution,

$$\text{LHS} = y' = \frac{d}{dx}(x^2) = 2x = \text{RHS}.$$

In general, the indefinite integral $\displaystyle\int f(x)\,dx$ is equivalent to the DE $y' = f(x)$.

1 DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

- A *differential equation* or *DE* is an equation that involves a derivative. Its solutions are functions, or in general, relations.
- The *order of a differential equation* is the order of the highest derivative that occurs within it.
- To test if a function is a solution of a DE, substitute it into the DE.
- An indefinite integral is equivalent to a differential equation because finding $\displaystyle\int f(x)\,dx$ means finding a solution of the DE $y' = f(x)$.

Indefinite integrals and the family of solutions

We saw in Section 4*I* that the solutions of the DE

$$y' = 2x$$

involve a single arbitrary constant,

$$y = x^2 + C, \text{ for some constant } C,$$

producing the family of solution curves sketched on the next page.

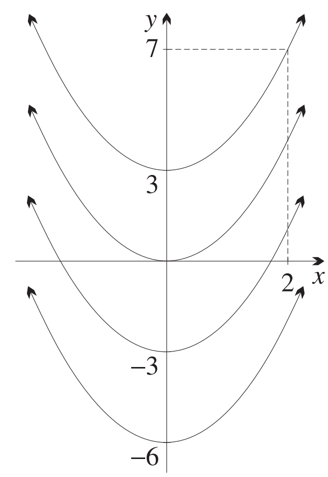
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

When working with motion in Chapter 9, we often started with acceleration and integrated twice, which needed two arbitrary constants. Our most common such differential equation was the second-order DE

$$\ddot{x} = -g,$$

whose solution is $x = -\frac{1}{2}gt^2 + At + B$, for constants A and B. This is a two-parameter family of solutions, not nearly so easy to represent graphically.



General solution of a differential equation

The general solution of any differential equation normally involves arbitrary constants, and so forms a family of curves. The next two examples demonstrate this. They show a solution family with one arbitrary constant, and a solution family with two arbitrary constants.

Example 4 — 13A

Show that $y = Ae^x - x - 1$ is a solution of $y' = x + y$, for all values of A.

SOLUTION

Substituting $y = Ae^x - x - 1$ into the DE $y' = x + y$,

$$\begin{aligned} \text{LHS} &= y' \\ &= Ae^x - 1, \end{aligned} \quad \text{and} \quad \begin{aligned} \text{RHS} &= x + (Ae^x - x - 1) \\ &= Ae^x - 1. \end{aligned}$$

Hence $y = Ae^x - x - 1$ is a solution for all values of A.

Note: Three curves in the family of solutions are drawn in worked Example 7.

Example 5 — 13A

a Show that $y = \sin 7x$ and $y = \cos 7x$ satisfy $y'' = -49y$.

b Show that $y = A\sin 7x + B\cos 7x$ satisfies $y'' = -49y$, for all A and B.

c Find the value of the second derivative y'' when $y = 5$.

d What transformation maps each solution of the DE to its second derivative?

SOLUTION

a
$$\begin{aligned} &\frac{d^2}{dx^2}(\sin 7x) \\ &= \frac{d}{dx}(7\cos 7x) \\ &= -49\sin 7x \\ &= -49y \end{aligned}$$

$$\begin{aligned} &\frac{d^2}{dx^2}(\cos 7x) \\ &= \frac{d}{dx}(-7\sin 7x) \\ &= -49\cos 7x \\ &= -49y \end{aligned}$$

b
$$\begin{aligned} &\frac{d^2}{dx^2}(A\sin 7x + B\cos 7x) \\ &= \frac{d}{dx}(7A\cos 7x - 7B\sin 7x) \\ &= -49A\sin 7x - 49B\cos 7x \\ &= -49(A\sin 7x - B\cos 7x) \\ &= -49y \end{aligned}$$

c Because $y'' = -49y$, if $y = 5$, then $y'' = -49 \times 5 = -245$.

d Because $y'' = -49y$, every solution is mapped to its second derivative by a reflection in the x-axis followed by a vertical dilation with factor 49.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

How many arbitrary constants are there in the general solution?

Normally, the most general solution of an nth-order differential equation has n arbitrary constants, but we cannot prove this in the present course.

We can only sometimes prove that even a solution with the expected number of arbitrary constants is the most general solution of the DE, in the sense that every solution can be obtained from it by substituting suitable arbitrary constants.

Initial value problems and particular solutions

We saw, especially in motion and rates, that arbitrary constants in the general solution can be evaluated provided that we have suitable *initial conditions* (or *boundary conditions*). The resulting curve is a *particular solution*, and the problem of finding a particular solution given a DE and suitable initial conditions is called an *initial value problem* or *IVP*. There may also be restrictions on the variables.

A particular solution may or may not be a connected curve. Many familiar functions such as $y = \frac{1}{x}$ and $y = \tan x$ have two or more branches, and a solution curve (or *integral curve*) may be just one branch of such a function.

The next two worked examples pick out particular family members that satisfy the initial conditions. The first is equivalent to the familiar indefinite integral $\int 2x\, dx$, whose solution family is sketched above.

Example 6 — 13A

Solve the differential equation $y' = 2x$, given that $P(2, 7)$ lies on the curve.

SOLUTION

Integrating, $y = x^2 + C$, for some constant C.
When $x = 2,\ y = 7$, so $7 = 4 + C$,
so $C = 3$ and $y = x^2 + 3$, as on the diagram drawn below Box 1.

Now we can use the same substitution method to pick out three particular solutions of the family in worked Example 4, given three different initial conditions.

Example 7 — 13A

We showed in worked Example **4** that $y = Ae^x - x - 1$, where A is a constant, is the general solution of $y' = x + y$.

a Find the particular solution passing through the origin.
b Find A if $y = -2$ when $x = 0$, and write down this particular solution.
c Find the particular solution through $(0, -1)$.
d Analyse the gradient and concavities of the solutions in parts **a**–**c**, identifying any stationary points and inflections.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

e By substituting $y' = 0$ into the DE, identify the set of points where the solution curves have gradient zero.

f Sketch the three solutions of the DE, and draw dashed the answer to part **e**.

g Do all solution curves to this DE have a stationary point?

SOLUTION

a Substituting $(0, 0)$, $\quad 0 = A - 0 - 1,$
so $A = 1$, giving $\quad y = e^x - x - 1.$

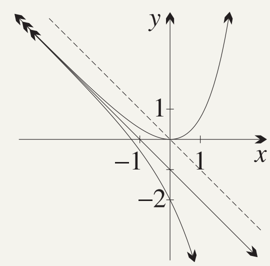
b Substituting $(0, -2)$, $\quad -2 = A - 0 - 1,$
so $A = -1$, giving $\quad y = -e^x - x - 1.$

c Substituting $(0, -1)$, $\quad -1 = A - 0 - 1,$
so $A = 0$, giving $\quad y = -x - 1.$

d For part **a**, $\quad y' = e^x - 1 \quad$ and $\quad y'' = e^x,$
so there is a minimum turning point at $(0, 1)$, and the curve is always concave up.
For part **b**, $\quad y' = -e^x - 1 \quad$ and $\quad y'' = -e^x,$
so the curve is always decreasing and always concave down.
Part **c** is the line $y = -x - 1$.

e Substituting $y' = 0$ into the DE $y' = x + y$ gives the line $x + y = 0$.

f



g Clearly, no. The solution curve in part **a** crosses $x + y = 0$, and has a stationary point there. But the other two solution curves do not cross $x + y = 0$, so they do not have any stationary points.

Note: Further detail about this DE is given in worked Example 13 in Section 13B.

Initial values with higher-order differential equations

An initial condition is usually a point on the curve. For higher-order DEs, however, the initial conditions may involve also a point on the graph of the derivative, that is, a value of y' for some particular value of x. For two arbitrary constants, we need two initial conditions, which usually form two simultaneous equations.

Example 8 — 13A

We showed in worked Example **5b** that $y = A\sin 7x + B\cos 7x$ is a solution of $y'' = -49y$ for all A and B. Find the solution for which $y(0) = 1$ and $y'(0) = 14$.

SOLUTION

The solution is $\quad y = A\sin 7x + B\cos 7x,$
and the derivative is $\quad y' = 7A\cos 7x - 7B\sin 7x.$
When $x = 0,\ y = 1,$ so $\quad 1 = 0 + B, \qquad (1)$
and when $x = 0,\ y' = 14,$ so $\quad 14 = 7A + 0. \qquad (2)$
Hence $A = 2$ and $B = 1$, and $\quad y = 2\sin 7x + \cos 7x.$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 A FAMILY OF SOLUTIONS, AND PARTICULAR SOLUTIONS

- The general solution of a DE normally involves one or more arbitrary constants, giving a *family* of solutions.
- The number of arbitrary constants is normally equal to the degree of the DE.
- These constants can be evaluated to give *particular solutions* if one or more *initial conditions* (or *boundary conditions*) are known.
- These conditions are usually points on the curve, but if higher derivatives occur, they may also involve the values of derivatives for particular values of x.
- A *solution curve* (or *integral curve*) may be just one of the connected branches of a function.
- The problem of finding a particular solution, given the DE and suitable initial conditions, is called an *initial value problem* or *IVP*.

A relation can be a solution of a differential equation

A relation that is not a function can also be a solution of theº differential equation. We will not be dealing with such situations in any detail, and some simple examples will be sufficient. Such examples will normally need the chain rule (and possibly alsothe product and quotient rules) to differentiate expressions involving y. For example, the next worked example requires differentiating y^2 and xy with respect to x,

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2yy',$$

$$\frac{d}{dx}(xy) = y\frac{d}{dx}(x) + x\frac{d}{dx}(y) = y + xy'.$$

Example 9 13A

a Differentiate the equation $x^2 + y^2 = a^2$ of the circle with respect to x, where $a > 0$, and show that each equation in the family of curves satisfies the differential equation $y' = -\frac{x}{y}$. What geometrical significance does this have?

b Differentiate the equation $xy = a^2$ of the rectangular hyperbola with respect to x, where $a > 0$, and show that the equation satisfies the differential equation $y' = -\frac{y}{x}$. What geometrical significance does this have?

SOLUTION

a We differentiate $x^2 + y^2 = a^2$.

Using the working above, $2x + \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 0$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}, \text{ as required.}$$

The radius OP has gradient $\frac{y}{x}$, and from the DE, the tangent has gradient $y' = -\frac{x}{y}$.

Thus the radius and tangent are perpendicular.

(If $y = 0$, then the the division by y above was invalid — this corresponds to the vertical tangents to the circle at the x-intercepts, when y' is undefined.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b We differentiate $\quad xy = a^2$.

Using the working above, $y + xy' = 0$

$$y' = -\frac{y}{x}, \quad \text{as required.}$$

(There is no division by 0 because x is a solution of $xy = a^2$, so cannot be zero.)

The interval OP has gradient $\frac{y}{x}$, and from the DE, the tangent has gradient $-\frac{y}{x}$, which is the opposite of this. Thus the interval OP and the tangent form an isosceles triangle with the x-axis.

These two examples — one a function and the other a relation that is not a function — show that relations form families of solutions of a DE, with arbitrary constants, in the same way that functions do.

3 RELATIONS CAN BE SOLUTIONS OF DIFFERENTIAL EQUATIONS

- Relations that are not functions can be solutions of differential equations, with the usual arbitrary constants and families of curves.
- Forming a differential equation from a relation that is not a function will involve the chain rule.

First-order linear differential equations

A first-order differential equation is called *linear* if it can be put into the form

$y' + f(x)y = g(x)$, where $f(x)$ and $g(x)$ are functions of x.

For example, here are some first-order linear differential equations:

$$y' = 5x^3 \qquad y' - 6y = 0 \qquad y' - 4xy = 3x^2$$
$$x^2y' + xy = 0 \qquad y - e^x y' = 1 \qquad y'\sin x - y\cos x = 1$$

The first row are all in the given form already. Either of the terms $f(x)\,y$ or $g(x)$ can be missing, but y' must be present for it to be a differential equation.

The first DE is equivalent to an indefinite integral. The second is the exponential growth DE — we will discuss these equations later in Sections 13D–13E.

The second row can all be put into the standard form. Careful readers will notice that division by x^2 or $\sin x$ involves division by zero, but in this introductory course, we will mostly not bother very much with such difficulties during the solution, but instead sort out any difficulties afterwards. Compare the remarks about division by zero in worked Example 9 above.

An alternative definition of a first-order linear differential equation uses the form

$f_1(x)y' + f(x)y = g(x)$, $\quad$ where $f_1(x)$, $f(x)$ and $g(x)$ are functions of x.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 10 13A

Classify these DEs in terms of their order and whether they are linear.

a $y' = \dfrac{y}{x}$ **b** $y' = \dfrac{x}{y}$

SOLUTION

a $y' = \dfrac{y}{x}$ is first-order linear because it can be put into the form $y' - \dfrac{y}{x} = 0$.

b $y' = \dfrac{x}{y}$ is first-order, but it is not linear, because it becomes $yy' = x$.

4 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

- A *first-order linear DE* is a DE that can be put into the form
$$y' + f(x)y = g(x).$$
- An indefinite integral is equivalent to a first-order linear DE with $f(x) = 0$,
$$y' = g(x).$$
- An exponential growth DE is a first-order linear DE with $g(x) = 0$ and $f(x)$ a constant, that is
$$y' - ky = 0, \quad \text{where } k \text{ is a constant.}$$

Exercise 13A FOUNDATION

Note: In this chapter, all logarithms have base e, and we will usually write $\log x$ or $\ln x$ instead of $\log_e x$.

1 In each case, state the order of the differential equation

a $y' - y = x$ **b** $y'y = 3x$ **c** $y' + 4y' - y = \sin x$

d $y' + y\cos x = e^x$ **e** $y'' - \frac{1}{2}(y')^2 = 0$ **f** $y' + y^2 = 1$

g $y' + xy = 0$ **h** $xy'' + y' = x^2$ **i** $y'' - xy' + e^x y = 0$

2 For each of the first-order differential equations in Question **1**, state whether it is linear or non-linear.

3 For each of the differential equations in Question **1**, state how many arbitrary constants will appear in the general solution.

4 Show by substitution that the given function is a particular solution of the differential equation.

a $y = 5x^3$; $xy' - 3y = 0$ **b** $y = x^2 - 1$; $xy' - 2y = 2$

c $y = 3e^{-x}$; $y' + y = 0$ **d** $y = \sqrt{x^2 + 4}$; $y'y = x$

5 Find the general solution of the differential equation by integration.

a $y' = 2x - 3$ **b** $y' = 12e^{-2x} + 4$ **c** $y' = \sec^2 x$

d $y' = 6\cos 2x + 9\sin 3x$ **e** $y' = \sqrt{1 - 5x}$ **f** $y' = 4x\cos x^2$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

6 Show by substitution that the given function with arbitrary constant C is the general solution of the differential equation.

a $y = Ce^{x} - x - 1;\quad y' = x + y$

b $y = Cxe^{-x};\quad xy' = y(1 - x)$

c $y = \sin(x + C);\quad (y')^2 = 1 - y^2$

d $y = \dfrac{C}{x} + 2;\quad \dfrac{dy}{dx} = \dfrac{2 - y}{x}$

7 Verify that the given function is a particular solution of the differential equation.

a $x^2y'' - 2xy' + 2y = 6;\quad y = x^2 - 2x + 3$

b $y'' - 6y' + 5y = 0;\quad y = 2e^{x} + e^{5x}$

c $y'' + \pi^2 y = 0;\quad y = \cos \pi x - 3\sin \pi x$

d $y'' + 4y' + 5y = 0;\quad y = e^{-2x}\sin x$

e $x^2y'' + xy' + y = 0;\quad y = \cos(\log x)$

8 Solve these second-order differential equations by integrating twice.

a $y'' = 2$

b $y'' = \cos 2x$

c $y'' = e^{\frac{1}{2}x}$

d $y'' = \sec^2 x$

9 **a** Show that $y = e^{-x}$ and $y = e^{3x}$ are each solutions of the equation $y'' - 2y' - 3y = 0$.

b Now show that $y = Ae^{-x} + Be^{3x}$ is also a solution of this equation for any values of the constants A and B.

10 Verify by substitution that the given function is the general solution of the differential equation for all values of the constants A, B and C.

a $y''' = 6;\quad y = x^3 + Ax^2 + Bx + C$

b $y'' + 3y' + 2y = 4x;\quad y = Ae^{-x} + Be^{-2x} + 2x - 3$

c $y'' + 4y = 0;\quad y = A\cos 2x + B\sin 2x$

d $y'' + 2y' + 2y = 0;\quad y = Ae^{-x}\cos x$

e $y'' = y(x^2 - 1);\quad y = Ae^{-\frac{1}{2}x^2}$

11 Solve these initial value problems. In each case, use integration to find the general solution, then use the initial condition to evaluate the constant.

a $y' = 1;\quad y(2) = 1$

b $y' = 2x - 3;\quad y(0) = 2$

c $y' = 3x^2 + 6x - 9;\quad y(1) = 2$

d $y' = \sin x;\quad y(\pi) = 3$

e $y' = 6e^{2x};\quad y(0) = 0$

f $y' = 3\sqrt{x} - 2;\quad y(4) = 7$

12 When an indefinite integral involves the reciprocal of a quadratic, the expression needs to be separated into the sum of two fractions, as in the following examples.

a Consider the initial value problem $y' = \dfrac{1}{x(1 - x)}$, where $y\left(\frac{1}{2}\right) = 0$.

i Show that $\dfrac{1}{x(1 - x)} = \dfrac{1}{x} + \dfrac{1}{1 - x}$.

ii Find the general solution of the DE.

iii Hence solve the IVP.

b Consider the initial value problem $y' = \dfrac{4}{(2 - x)(2 + x)}$, where $y(0) = 1$.

i Show that $\dfrac{4}{(2 - x)(2 + x)} = \dfrac{1}{2 - x} + \dfrac{1}{2 + x}$.

ii Find the general solution of the DE.

iii Hence solve the IVP.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 **a** **i** Differentiate both sides of the equation $x^2 + y^2 = 9$ with respect to x, using the chain rule where necessary.

ii Hence show that $\dfrac{dy}{dx} = -\dfrac{x}{y}$ at each point on the circle.

iii Part **ii** needs qualification. Where is the differential equation undefined and why is this expected?

b Likewise, find $\dfrac{dy}{dx}$ for these curves:

i the parabola $y^2 = x + 4$

ii the hyperbola $xy = c^2$

iii the ellipse $9x^2 + 16y^2 = 144$

iv the hyperbola $x^2 - 4y^2 = 4$

v the hyperbola $xy - y^2 = 1$

vi the folium of Descartes $x^3 + y^3 = 3xy$

14 **a** Show that $y = \sin x$ is a solution of $y'' + y = 0$.

b Find the value of the second derivative y'' when $y = 12$.

c Suppose that $y = f(x)$ is a solution of this differential equation. What does the equation say about $f''(x)$? Answer in terms of a transformation of $f(x)$.

15 Use integration and the initial conditions to find the solution of $y'' = \sec^2 x$ with $y(0) = 1$ and $y'(0) = 1$.

16 Show that the function $y = \tan x$ is a solution of the initial value problem $y'' = 2yy'$, with $y\left(\frac{\pi}{4}\right) = 1$ and $y'\left(\frac{\pi}{4}\right) = 2$.

ENRICHMENT

17 For each DE, use substitution to find the values of λ, if any, that make the function $y = Ae^{\lambda x}$ a solution for all values of A.

a $y'' - 4y' + 3y = 0$

b $y'' + 2y' + y = 0$

c $y'' - 3y' + 4y = 0$

18 Consider the function $y = \sec x$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

a Find a linear first-order initial value problem that has this solution.

b Find a non-linear first-order IVP that has this solution.

19 Consider the initial value problem $y' = -2xy$ with $y(0) = 1$. Suppose that the function $y = f(x)$ is a solution.

a Write down the y-intercept of the graph of $y = f(x)$.

b Calculate the gradient of this graph at the y-intercept.

c **i** Differentiate the differential equation to show that $y'' = (4x^2 - 2)y$.

ii Hence determine the concavity of $y = f(x)$ at the y-intercept.

d Determine the value of $f'''(0)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

20 Consider the differential equation $(y')^2 - xy' + y = 0$

a Show that $y = cx - c^2$ is a general solution of this equation.

b Draw the particular solutions corresponding to $c = -2,\ -1\frac{1}{2},\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2},\ 2,$ for the domain $-4 \le x \le 4$. What do you notice?

c Find the coordinates of the point where the line corresponding to $c = p$ intersects the line corresponding to $c = p + h$.

d Show that in the limit as $h \to 0$, the coordinates of this point are $x = 2p$ and $y = p^2$.

e Eliminate p from these two equations and show that the resulting curve is a solution of the original differential equation. Try to explain what has happened.

21 Find the general solution of $y^{(n)} = 1$. How many arbitrary constants did you use?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13B Slope fields

Before we embark on any more algebra, this section takes a visual approach to solving a DE. A *slope field* is a visual display that provides an impression of possible solution curves. The method only applies to a first-order differential equation of a special type.

Slope fields (or gradient fields or direction fields)

A slope field can only be drawn for a first-order differential equation that can be written with y' as the subject,

$$y' = G(x, y), \quad \text{where } G(x, y) \text{ is an expression in } x \text{ and } y. \qquad (*)$$

At each point P on a solution curve, the tangent to the curve at P has gradient given by equation $(*)$. This allows us to draw a *line element* at P, of a fixed short length, inclined at the same angle as the tangent at that point.

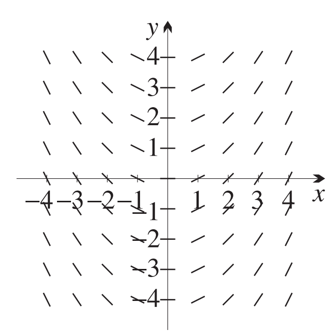
- Choose a suitable grid of points $P(x, y)$ in the plane.
- Use the equation $(*)$ to construct a table of values of the gradients at these grid points.
- At each point, draw a short line element of fixed length with that gradient.
- Centre each line element on the grid point whose tangent it represents.

Our first example will be equivalent to an indefinite integral, where we have already seen how to construct families of curves. Consider the differential equation

$$y' = \tfrac{1}{2}x.$$

All the rows in the table of slopes are the same because y is not involved.

y \ x	-4	-3	-2	-1	0	1	2	3	4
4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
0	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-1	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2



The slope field suggests a family of parabolas, all with the x-axis as axis of symmetry, and all vertical translations of each other. We know that this is true because by integration, the general solution is

$$y = \tfrac{1}{4}x^2 + C.$$

Constructing the table of gradients and the slope field is a time-consuming procedure. Refer to Question 4 in Exercise 13B for instructions how to use the free WolframAlpha.com website to draw a slope field of a differential equation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Terminology — slope fields or direction fields or gradient fields

Each short line element displays the gradient of the solution curve at that point. It does not indicate direction, and its length does not have any significance, so it must not be confused with a vector. The terms *slope field* or *gradient field* therefore seem preferable to another commonly used term *direction field*. Some authors distinguish all three terms.

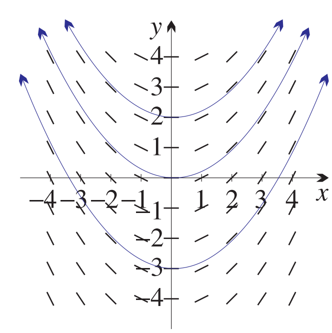
5 SLOPE FIELDS

Suppose that a DE can be written with y' as the subject, that is, as

$$y' = G(x, y)\text{ , where }G(x, y)\text{ is an expression in }x\text{ and }y.$$

- At each point in the coordinate plane where $G(x, y)$ is defined, the solution curve has the gradient given by this equation.
- Choose a suitable grid of points in the plane.
- Draw up a table showing the gradients at these grid points.
- Display these gradients by short tilted *line elements* of a fixed length, centred on these grid points in the coordinate plane.
- The resulting diagram is called a *slope field* (or *gradient field* or *direction field*).

Sketching a solution curve (or integral curve)



Place a pencil on the diagram and follow the gradients. Keep in mind that the line elements represent gradients. On the right, three solution curves have been drawn on the slope field we drew.

This is not an accurate procedure, and it it is quite unlike joining up the dots when sketching a curve from plotted points. For example, the curves do not pass through the centres of neighbouring line elements, and drawing a solution curve means threading the curve through the slope field. The purpose is to get a global view of what is happening, usually before any detailed calculations.

- Apart from some strange singularities, two solution curves can never cross, because the gradient at each point can only have one value.
- An *isocline* is a curve passing through points where the tangents have equal gradient. In the slope field above, every vertical line is an isocline — this is because the DE is equivalent to an indefinite integral, so all the solution curves are vertical translations of each other, and the gradient is therefore independent of y.
- In the slope field above, the y-axis is a special isocline because it joins all the points where the gradient is zero. Every solution curve has a stationary point where it crosses the y-axis.
- If you are given an initial condition, such as the origin, start there and draw the approximate curve, in both directions, as far as any restrictions allow. If there is no initial condition, draw at least three representative curves in different places on the plane.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 11 13B

a Construct a slope table for the differential equation $y'y + x = 0$.
b From the table, draw the slope field.
c What family of curves look as if they are solution curves?
d Check your conjecture by differentiation.

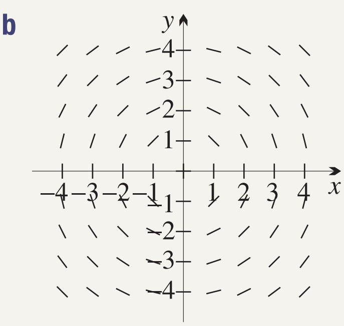
SOLUTION

a Solving $y'y + x = 0$ for y' gives $y' = -\frac{x}{y}$, provided that $y \neq 0$.

Construct a table of grid points for y' as follows.

$y \backslash x$	-4	-3	-2	-1	0	1	2	3	4
4	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$	-1
3	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1	$-\frac{4}{3}$
2	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
1	4	3	2	1	0	-1	-2	-3	-4
0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
-1	-4	-3	-2	-1	0	1	2	3	4
-2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-3	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$
-4	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

b



c It looks as if the solution curves are circles with centre the origin.

d Test whether $x^2 + y^2 = r^2$ is a solution, for any constant r.
Differentiating using the chain rule, $2x + 2yy' = 0$, as required.

Vertical line element

The slope field above has vertical line elements on the x-axis. Why was this done when vertical lines do not have gradient?

- The line elements on the x-axis are drawn vertical because they arise from division of a non-zero number x by y, which is zero. (You could even enter ∞ instead of $*$ into the table of grid points.)
- This does not include the origin, where the central element $\frac{0}{0}$ in the table has no meaning at all.

Some things to look for in a slope field

The diagram looks as if it is self-interpreting, but some advice about what to look for is useful. Here are four points to think about when looking at the diagram:

- Look at where the slope is zero, that is, where the line elements are horizontal. These are the places where a solution curve has a stationary point.
- Look at where the line elements slope upwards (positive gradient), and where they slope downwards (negative gradient). This tells you where a solution curve is increasing and where it is decreasing.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

For example, looking at the slope field drawn in the previous worked Example:

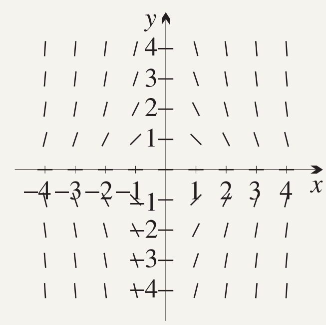
- The points with gradient zero are the y-axis (excluding the origin).
- The gradients are negative in quadrants 1 and 3, and positive in 2 and 4.
- The isoclines — curves joining points where the gradients are equal — are all the lines through the origin. That is, they are the radii of the circles.

Example 12 13B

Which DEs are possibly graphed in the diagram to the right?

A $y' = -xy$ **B** $y' = -x^2y$

C $y' = -xy^2$ **D** $y' = -x - y$



SOLUTION

- The slopes are zero on the y-axis. This excludes option D.
- The slopes are positive in the second quadrant. This excludes option B.
- The slopes are negative in the third quadrant. This excludes option C.
- Option A has the right sign in each quadrant. Notice also that the gradients become steeper away from the axes, as they should in option A.

6 SLOPE FIELDS AND THE SOLUTION CURVES

- Two solution curves can never cross (apart from some strange singularities).
- This is not an accurate procedure. In particular, asymptotic behaviour may not be clear.
- Precede the sketch by looking for *isoclines*, which are curves through points of equal gradient in the slope field.
- If there is an initial condition, construct the *solution curve* (or *integral curve*) beginning at a given initial point. Otherwise draw about three solution curves.

An isocline that is an asymptote

The next worked example extends worked Example 7 in the previous section by looking at the slope field of the differential equation. It also gives some further insight into the importance of isoclines.

Example 13 13B

a Sketch the slope field of the differential equation $y' = x + y$.

b Sketch representative solution curves, including the solution curve through the origin.

c Identify all the isoclines of the slope field.

d Explain the significance of the isocline $y = -x$.

e Explain the significance of the isocline $y = -x - 1$.

f Show that $y = Ae^x - x - 1$ is a solution, for all constants A.

g Explain these equations in terms of the slope field, including isoclines.

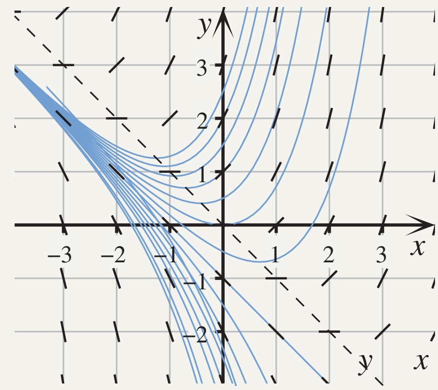
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a

$y \backslash x$	-4	-3	-2	-1	0	1	2	3	4
4	0	1	2	3	4	5	6	7	8
3	-1	0	1	2	3	4	5	6	7
2	-2	-1	0	1	2	3	4	5	6
1	-3	-2	-1	0	1	2	3	4	5
0	-4	-3	-2	-1	0	1	2	3	4
-1	-5	-4	-3	-2	-1	0	1	2	3
-2	-6	-5	-4	-3	-2	-1	0	1	2
-3	-7	-6	-5	-4	-3	-2	-1	0	1
-4	-8	-7	-6	-5	-4	-3	-2	-1	0



The most obvious solution curve is the line $y = -x - 1$.

b The solution through the origin is drawn here. As we saw in worked Example 7, the line $y = -x - 1$ is also a solution, and there are also solutions below $y = -x - 1$. Turn back now to the diagram for worked Example 7 to see how the three solution curves drawn there fit onto this slope field.

c The table of values above makes it clear that every line with gradient -1 is an isocline.

d Every solution curve that crosses the isocline $y = -x$ has a stationary point at the intersection, and given the slopes on both sides, it will be a minimum turning point. The curves below $y = -x - 1$, however, do not cross this line.

e The isocline $y = -x - 1$ is exceptional in many ways.

— It is a solution of the DE, because it satisfies $y' = x + y$. When the line is substituted into the DE, both sides equal -1.

— This isocline is a line, and its gradient equals the gradient of the slope field at each point on it.

— No other solution curve crosses this curve, and the other solution curves fall into two groups on each side of this line.

— The other solution curves are asymptotic to this line in the second quadrant. Look at the diagram, and look back again to worked Example 7.

f Substituting into $y' = x + y$,

$$\begin{aligned} \text{LHS} &= \frac{d}{dx}(Ae^x - x - 1) \\ &= Ae^x - 1 \\ \text{RHS} &= x + (Ae^x - x - 1) \\ &= \text{LHS}, \end{aligned}$$

so $y = Ae^x - x - 1$ is a solution, for all constants A.

g The isocline $y = -x - 1$ is a solution curve — it corresponds to $A = 0$.

The solution curves above $y = -x - 1$ curl upwards, corresponding to $A > 0$.

The solution curves below $y = -x - 1$ curl downwards, corresponding to $A < 0$.

The fact that every solution curve is asymptotic to the isocline $y = -x - 1$ corresponds to the limit $\lim\limits_{x \to -\infty} (Ae^{kx} - 1) = -1$.

ISBN 978-1-108-76630-2

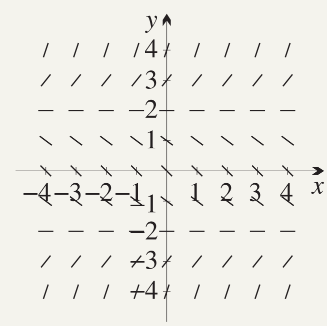
Cambridge University Press
Updated July 2022

Constant solutions are horizontal asymptotes

The derivative of a constant function is zero. Hence any constant solutions $y = k$ of a DE stands out on the slope field. Look at the diagrams below — they are the two lines consisting of horizontal line elements. A constant solution is an isocline, and it divides the other solutions into those above and those below. Usually many of the solution curves in the family have this line as a horizontal asymptote on the left or right.

Constant solutions are particular examples of the more general phenomenon of *equilibrium solutions*, for reasons that the next worked example will make clear.

Example 14 13B



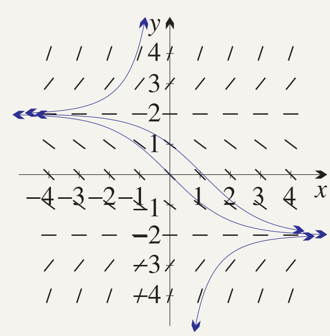
a Sketch possible solution curves to the slope field in the upper diagram to the right. It is the slope field of the differential equation $y' = \frac{1}{4}(y - 2)(y + 2)$.

b From the slope field, identity the constant solutions, that is, the equilibrium solutions.

c Substitute into the DE to show that they are solutions.

d If the horizontal axis is time, describe the behaviour of the solution curves near those constant solutions, and distinguish between them.

SOLUTION



a Four different connected solution curves are drawn in the lower diagram to the right. (Whether or not they have vertical asymptotes will become clear when the logistic equation is solved in Section 13D.)

b The horizontal isoclines $y = 2$ and $y = -2$ stand out because both consist of points where the gradient is zero.

c They are solution curves because substituting $y = 2$ or $y = -2$ into the DE gives LHS $= y' = 0$ and RHS $= 0$.

d The time x moves forwards, so the two constant solutions have different meanings, because $y = -2$ is an asymptote on the right, and $y = 2$ is an asymptote on the left.

— For $y = -2$: As time increases, the middle and bottom solution curves have the limit $y = -2$. This means that over time, the system moves towards *equilibrium* at $y = -2$.

— For $y = 2$: If the curve starts at the point $P(-5, 2.001)$, it will go up without bound. If the curve starts at the point $P(-5, 1.999)$, it will go down, then move towards the horizontal asymptote $y = -2$. Thus a minutely small change in initial situation — even a quantum fluctuation — may produce a huge change in the final situation.

Thus $y = 2$ is an equilibrium solution in the sense that with this solution, y does not change over time. But the equilibrium is an *unstable equilibrium* because the slightest fluctuation will send it permanently away from $y = 2$.

The equilibrium at $y = -2$, however, is a *stable equilibrium* because any slight change will see the system return to where it was.

Note: The differential equation in this example is a *logistic equation*. Such equations will be discussed further in Sections 13D–13E.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Isoclines and the slopefield

The worked examples in the previous discussion should have made it clear how useful isoclines can be in examining the solution curves of a DE. Here is a summary of some uses that we have made of them.

- Isoclines consisting of horizontal line elements show where solution curves that cross them have stationary points.
- Some isoclines have a particular property that makes them stand out — they are lines consisting of line elements with the same gradient as the line.
  - Such an isocline is a solution curve, so no other solution curve crosses it.
  - It therefore divides the other solutions into two groups on either side.
  - Normally many of the solution curves have this line as an asymptote.
- Constant solutions have this property, and are the most important isoclines of all. They are horizontal lines consisting of horizontal line elements, and are seen immediately on the slope field.
- The presence of an isocline can often be used to help identify a correct DE from a list of options. Any option that does not have the requisite isocline can immediately be eliminated from consideration.

Using technology to deal with slope fields

Everyone needs to plot by hand a few slope fields after first preparing the tables and doing the calculations. But this is laborious, and technology is a great assistance. There is software available, there are online resources, and there are some calculators with screens that will do the job. Here are some suggestions:

- The WolframAlpha.com website is free for the tasks required in this chapter. Question 4 gives some initial instructions about commands, but use the website's help functions.
- A link to a Desmos calculator set up to plot slope fields has been provided with Question 4 in the interactive textbook.
- If your calculator has a screen, check if it can handle slope fields.

Exercise 13B

FOUNDATION

1 In each case, find the value of y' at the given point.

a $y' = 2x - 3$ at $(1, 1)$ **b** $y' = 2\cos x - 1$ at $(0, 0)$ **c** $y' = 4 - y^2$ at $(0, 1)$

d $y' = \dfrac{1}{1 + y}$ at $(3, 1)$ **e** $y' = \dfrac{y}{x} + 1$ at $(-2, 1)$ **f** $y' = xy - x$ at $(1, -2)$

2 Answer these questions for the differential equation $y' = \frac{1}{2}x - 1$.

a Copy and complete the table of values for the slope field.

b Draw a number plane with a scale of 1 cm = 1 unit with domain $[-1, 5]$ and range $[-1, 5]$.

c Through each grid point in the table, draw a line element $\frac{1}{2}$ cm long, centred on the point and with gradient as given in the table.

d Notice that the vertical line $x = 1$ is an isocline. Why is this expected from the table?

e Check for any other isoclines evident in the table or graph.

y \ x	-1	0	1	2	3	4	5
5	$-\frac{3}{2}$	-1	$-\frac{1}{2}$				
4	$-\frac{3}{2}$						
3							
2							
1							
0							
-1							

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

f The slope field indicates a positive gradient to the right of $x = 2$ and a negative gradient to the left of $x = 2$. What will be the concavity of a solution curve?

g Starting at the origin, draw an integral curve (solution curve) to the right and to the left. What type of curve might this be?

h Draw two more integral curves, starting at $(0, 2)$ and $(0, 3)$, making sure that none of the curves cross.

3 Consider the differential equation $y' + y = x$.

a Make y' the subject.

b Copy and complete the table of values for the slope field.

c Draw a number plane with a scale of 2 cm = 1 unit with domain $[-1, 2]$ and range $[-2, 1]$.

d Through each grid point in the table, draw a line element $\frac{1}{2}$ cm long, centred on the point and with gradient as given in the table.

e Look carefully at the table for matching entries. Check that these agree with the isoclines in your graph.

$y \backslash x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-2	$-\frac{3}{2}$	-1				
$\frac{1}{2}$	$-\frac{3}{2}$	-1					
0							
$-\frac{1}{2}$							
-1							
$-\frac{3}{2}$							
-2							

f Look carefully at your graph. What concavity would you expect for the solution curve passing through the origin?

g Draw this solution curve.

h Now add solution curves that pass through $(-1, -1)$ and $(1, -1)$, making sure that none of the curves cross.

i Which line do all your solution curves appear to have as an asymptote?

j Is this line a solution of the differential equation?

4 [Technology]

Various mathematical applications can be used to save time plotting slope fields. A Desmos slope field plotter is provided in the interactive textbook. In addition, the slope field for $y' = x + y$ can be plotted in the free internet application WolframAlpha.com by using the command

```
slope field x + y, {x, −4, 4}, {y, −4, 4}.
```

The two terms in braces are optional, and are used to indicate the domain $-4 \le x \le 4$ and range $-4 \le y \le 4$.

Use WolframAlpha or Desmos to plot the following slope fields. In each case:

i identify any points or isoclines where $y' = 0$,

ii identify any other obvious isoclines,

iii state how the gradients of the line elements change along the line $x = 1$, from bottom to top, and

iv state how the gradients of the line elements change along the line $y = 2$, from left to right.

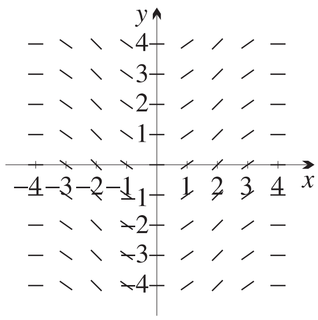
a $y' = -y^2$ **b** $y' = -\dfrac{1}{x^2}$ **c** $y' = \cos\left(\frac{\pi}{4}x\right)$

d $y' = 1 - x + y$ **e** $y' = \dfrac{2y}{x} - y$ **f** $y' = \dfrac{2x}{y+1} - x$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

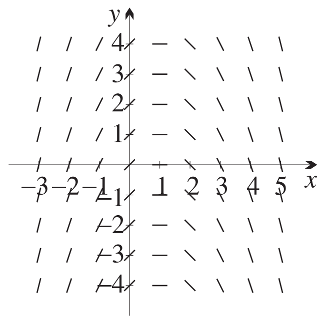
5 For each slope field below, draw the solution curves that pass through the two given points. Ensure that at each point on each curve, the gradient is roughly the average of the slopes indicated at nearby points.

a



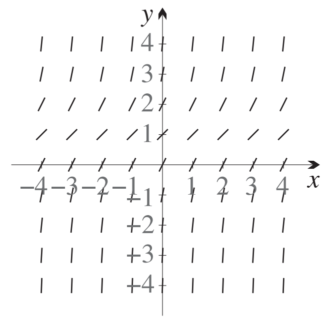
i $(0, 0)$ **ii** $(2, 0)$

b



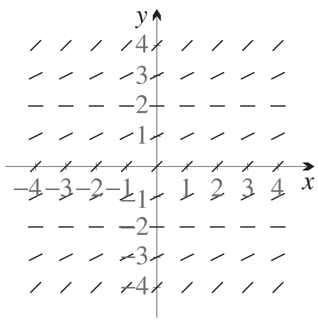
i $(0, 0)$ **ii** $(1, 3)$

c



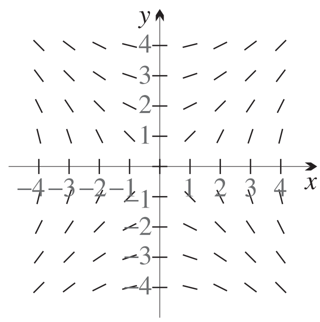
i $(0, 1)$ **ii** $(1, 0)$

d



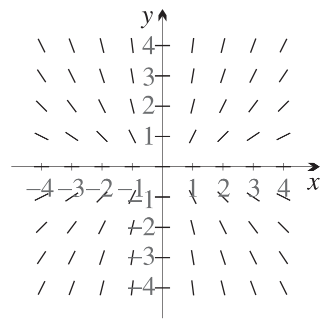
i $(-2, 0)$ **ii** $(2, 0)$

e



i $(0, 2)$ **ii** $(-2, 0)$

f

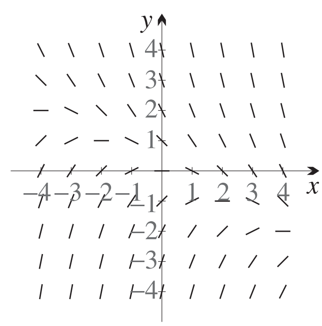


i $(1, 1)$ **ii** $(-2, 1)$

6 For each slope field in the previous question, use the isoclines to determine whether y' is a function of x alone, a function of y alone, or a combination of both.

DEVELOPMENT

7 The slope field for the differential equation $y' = -\frac{1}{2}x - y$ is drawn to the right. Make a copy of the slope field and answer the following questions.



a On your copy of the slope field, draw the solution curves through the points $(-2, 0)$ and $(2, 0)$.

b Look carefully at the slope of the line elements on the vertical line $x = 1$, from bottom to top.

i Do the gradients increase or decrease?

ii Do your solution curves converge (get closer) or diverge (further apart) as they cross $x = 1$, from left to right?

c Explain why the line $y = \frac{1}{2} - \frac{1}{2}x$ is an isocline.

d Show that the equation of this isocline is also a solution of the DE, then add the solution to your copy.

e What do you notice about the isocline and the two solution curves you have drawn?

8 Consider the differential equation $y' = \dfrac{9 - y^2}{9}$.

a Draw the slope field for this DE.

b What are the constant solutions for this DE?

c Are these constant solutions isoclines?

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

d Consider the slope of the line elements on the vertical line $x = 1$.

 i If $y > 0$, will the solution curves converge or diverge as they cross $x = 1$ from left to right?

 ii What happens to the solution curves if $y < 0$?

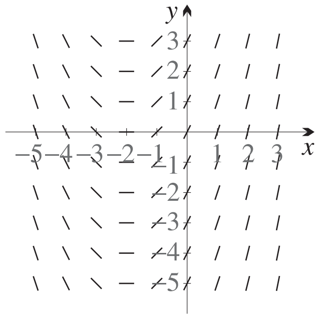
 iii Is the same true as the solution curves cross other vertical lines from left to right?

 iv What do you conclude about the constant solutions?

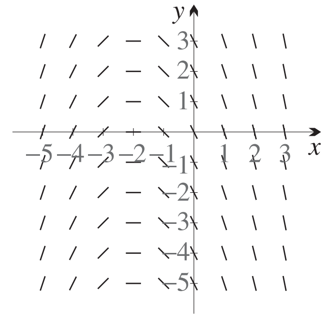
e Confirm all your answers by drawing the solution curve through $(0, 0)$.

9 By considering isoclines and constant solutions, determine which of the slope fields shown below corresponds to the differential equation $y' = -1 - y$.

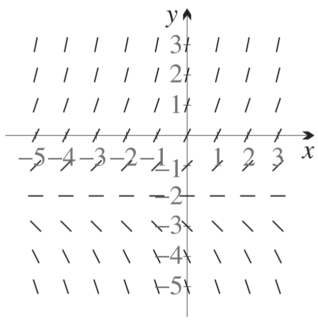
A



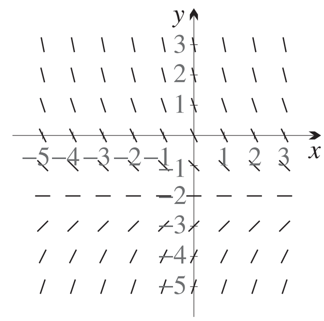
B



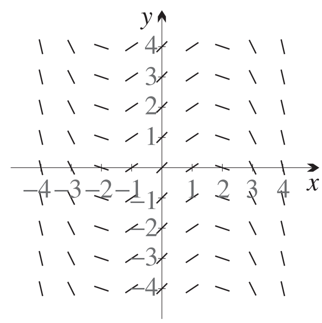
C



D



10 Consider the slope field shown to the right. Look for any of the important features of the slope field: constant solutions, points where $y' = 0$, isoclines, converging or diverging solution curves. Hence determine which of the following DEs corresponds to the slope field.

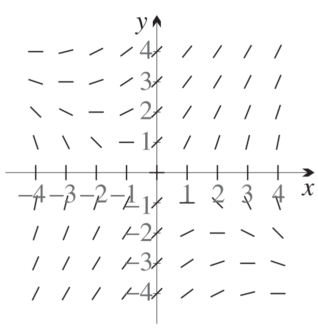


A $y' = \frac{1}{3}(x^2 - 3)$ **B** $y' = \frac{1}{3}(y^2 - 3)$

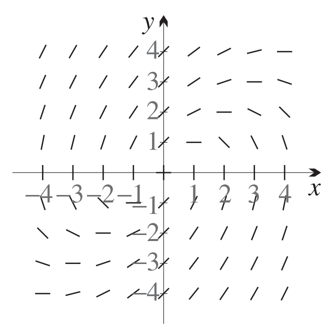
C $y' = \frac{1}{3}(3 - x^2)$ **D** $y' = \frac{1}{3}(3 - y^2)$

11 Which slope field below corresponds to the DE $y' = 1 - \dfrac{x}{y}$?

A



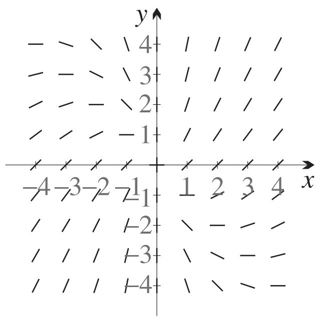
B



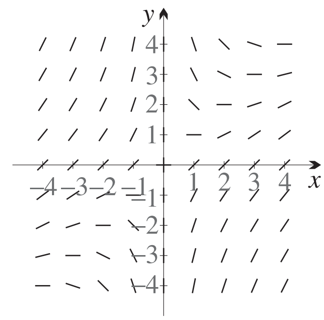
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

C

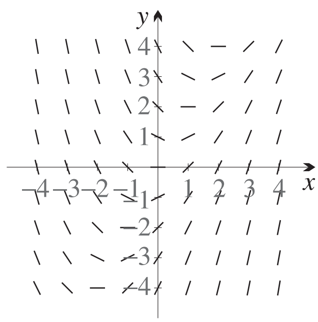


D

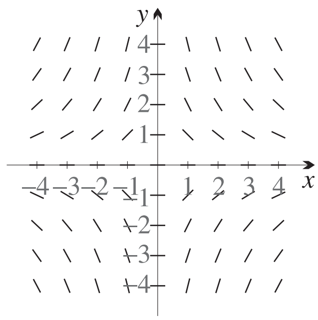


12 In each part, determine which DE corresponds to the slope field shown.

a **A** $y' = x + \frac{1}{2}y$
B $y' = x - \frac{1}{2}y$
C $y' = \frac{1}{2}x + y$
D $y' = \frac{1}{2}x - y$



b **A** $y' = \dfrac{2xy}{1 + y^2}$
B $y' = \dfrac{2xy}{1 + x^2}$
C $y' = \dfrac{-2xy}{1 + y^2}$
D $y' = \dfrac{-2xy}{1 + x^2}$



13 **a** **i** Draw the slope field for the differential equation $y' = -\dfrac{y}{x}$.
ii Add the solution curves through $(2, 2)$ and $(-2, -2)$ to your graph.
b **i** On a separate number plane draw the slope field for $y' = \dfrac{x}{y}$.
ii Add the solution curves through $(2, 0)$ and $(-2, 0)$.
c **i** Show that the hyperbola $xy = 4$ is the solution of the IVP in part **a**.
ii Show that the hyperbola $x^2 - y^2 = 4$ is the solution of the IVP in part **b**.
iii Graph these two hyperbolas (without their slope fields) on a third number plane.
d If you have drawn your graph in part **c** carefully enough, then the hyperbolas will be perpendicular where they intersect. Why?

14 **a** **i** Draw the slope field for the differential equation $y' = \dfrac{2y}{x}$.
ii Add the solution curve through $(2, 2)$ to your graph.
b **i** On a separate number plane draw the slope field for $y' = -\dfrac{x}{2y}$.
ii Add the solution curve through $(2, 0)$ $\big($it also passes through $(-2, 0)\big)$.
c **i** Show that the parabola $y = \frac{1}{2}x^2$ is the solution of the IVP in part **a**.
ii Show that the ellipse $x^2 + 2y^2 = 4$ is the solution of the IVP in part **b**.
iii Graph these two (without their slope fields) on a third number plane.
d If you have drawn your graph in part **c** carefully enough, then the parabola and ellipse will be perpendicular where they intersect. Why?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 [Shifting]

a Show that the equation of the circle with radius 4 and centre the origin satisfies the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

b What is the equation of this circle if it is shifted 3 units right and 1 units up?

c Show that this new circle satisfies the differential equation $\frac{dy}{dx} = -\frac{x-3}{y-1}$.

d It should be clear from this that if a curve is translated h units right and k units up, then the new DE is obtained by replacing x by $(x-h)$ and y by $(y-k)$.

The hyperbola $x^2 - y^2 = 1$ satisfies the DE $y' = \frac{x}{y}$, and the slope field is graphed in Question **5e**.

i Write down the DE for the shifted hyperbola $(x-1)^2 - (y+2)^2 = 1$.

ii Sketch its slope field by shifting the one in Question **5e**.

16 Slope fields can be used to draw the solution curves for differential equations that cannot be solved algebraically. For example, the function $\Phi(x)$ used in statistics is defined by the integral formula

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2}\, dt\,.$$

Differentiating both sides gives the differential equation:

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}.$$

a Plot the slope field for $y = \Phi(x)$, that is, for $y' = \frac{1}{\sqrt{2\pi}} e^{-x^2}$.

b It is known that $y = \Phi(x)$ has two asymptotes, $y = 1$ and $y = 0$. Using these asymptotes and the slope field, sketch the integral curve through $\left(0, \frac{1}{2}\right)$.

c Use your graph to estimate $\Phi(1)$ correct to one decimal place.

17 [An alternative method]

Here is an alternative method, based on isoclines, for drawing slope fields and solution curves. Consider the differential equation $y' = xy$.

a What two isoclines correspond to $y' = 0$? Add horizontal line elements to these.

b Write down the equation of the isoclines for the constant gradient $y' = C$, where $C \neq 0$, and name the type of curve this is.

c Graph the isoclines for each value $C = 1, -1, 2, -2, 3, -3,$ for $-4 \le x \le 4$ and $-4 \le y \le 4$.

d On the isocline for $C = 1$, add line elements at 45° to the horizontal, roughly equally spaced along the isocline.

e Likewise add the line elements for $C = 2$ at angle 63° (note that $\tan 63° \doteqdot 2$).

f Add the line elements for $C = 3$ at angle 72°.

g Similarly add line elements for the isoclines where $C = -1, -2, -3$.

h Draw the integral curve (solution curve) through $(0, 1)$. Ensure that your curve crosses each isocline at the correct angle.

i Likewise draw the integral curves through $\left(0, \frac{1}{4}\right)$ and $\left(0, -\frac{1}{8}\right)$.

j **i** What is an advantage of this technique?

ii What is a disadvantage of this technique?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 [A comparison of methods]
The alternative method for drawing the slope field presented in the previous question is impractical when the expression for y' is a complicated or an unknown curve. In some instances, however, it is much preferable. Consider the differential equation $y' = -\frac{x}{y}$.

a i What can be said about the line elements of the slope field for points on the y-axis other than the origin?

ii The value of y' at points on the x-axis is undefined. By considering $\lim_{y \to 0^+} y'$ and $\lim_{y \to 0^-} y'$ when $x \neq 0$, what can be said about the line elements of the slope field for points on the x-axis?

iii Draw the slope field using a grid of points as in Questions **2** and **3** or by using software such as the free internet application WolframAlpha.

iv What shape seems to be suggested?

b i What is the equation of the isocline corresponding to $y' = C$?

ii On a new number plane, draw by hand the isoclines for $C = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \sqrt{3}$ and $\sqrt{3}$. (Hint: What angles of inclination correspond to these gradients?)

iii Explain why each line element is perpendicular to its isocline.

iv Add line elements to each isocline at equally spaced intervals.

v Is the shape of part **c** clearer now?

ENRICHMENT

19 Prove the following three statements about isoclines.

a If $y' = f(x)$, then the isoclines $y' = c$ are the vertical lines.

b If $y' = g(y)$, then the isoclines $y' = c$ are the horizontal lines.

c If $y = cx + b$ is a solution of a first-order DE for a specific value of the constant c, then this line is also an isocline. (Compare this result with Question **7c**.)

20 None of the integral curves in previous questions crosses itself. In this question you will investigate what happens when a curve crosses itself.

The folium of Descartes has equation $x^3 + y^3 = 3axy$ for different values of the constant a. In this question put $a = 1$, so that $x^3 + y^3 = 3xy$.

a Show that $y' = \dfrac{y - x^2}{y^2 - x}$.

b Draw the slope field, with $-2 \le x \le 2$ and $-2 \le y \le 2$.

c The integral curve is horizontal where $y' = 0$. What curve do these points lie on?

d The integral curve is vertical where $\dfrac{1}{y'} = 0$. What curve do these points lie on?

e Use the slope field to plot the integral curve that passes through $(\sqrt[3]{2}, \sqrt[3]{4})$. You may assume that this curve also passes through $(\sqrt[3]{4}, \sqrt[3]{2})$ and the origin.

f What happens at the origin?

g The integral curve appears to have an asymptote that is also an isocline. Write down its equation, show that it is indeed an isocline, and show that this is also a solution of the DE in part **a**.

h In which line does the integral curve have symmetry? Explain this in terms of:

i the equation of the folium $x^3 + y^3 = 3axy$,

ii the differential equation $y' = \dfrac{y - x^2}{y^2 - x}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13C Separable differentiable equations

Many particular types of differential equations can be solved by systematic approaches. *Separable DEs*, after a suitable rearrangement, can be solved just by integration.

Separable differential equations

A first-order differential equation is called *separable* if y' can be written as the product of a function of x and a function of y,

$$y' = f(x)\, g(y).$$

There are three types of separable DEs.

- The general case is $y' = f(x)\, g(y)$, where neither function is a constant.
- The case $y' = f(x)$, where $g(y) = 1$, is equivalent to an indefinite integral.
- The case $y' = g(y)$, where $f(x) = 1$, will be discussed in Section 13D.

Solving a separable differential equation

The key step is to separate the dx and dy in the derivative, exactly as we were doing in the last chapter when integrating by substitution.

We write the DE as $\quad \dfrac{1}{g(y)}\, dy = f(x)\, dx,$

then integrate both sides, $\quad \displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$

Note: The chain rule is the justification of this. First, it allows us to separate the dx and the dy, as is the first line above. Secondly, it allows us to cancel dx.

The more complete argument is $\quad \dfrac{dy}{dx} = f(x)\, g(y)$

$\boxed{\div g(y)}$ $\quad \dfrac{1}{g(y)} \dfrac{dy}{dx} = f(x),$

integrating with respect to x, $\quad \displaystyle\int \frac{1}{g(y)} \frac{dy}{dx}\, dx = \int f(x)\, dx$

and cancelling dx, $\quad \displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$

Example 15 — 13C

a Solve $y' = -2xe^{y}$. **b** Find the solution curve through $(0, 0)$.

SOLUTION

a Arrange the equation as $\quad -e^{-y}\, dy = 2x\, dx \quad$ (Why was this convenient?)

and integrate, $\quad \displaystyle\int -e^{-y} dy = \int 2x\, dx$

$$\begin{aligned} e^{-y} &= x^2 + C, \text{ for some constant } C \\ -y &= \log(x^2 + C) \\ y &= -\log(x^2 + C). \end{aligned}$$

b Substituting $(0, 0)$ gives $\quad 0 = -\log C,$

so $C = 1$ and $\quad y = -\log(x^2 + 1).$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 16 13C

a Solve the DE $y' = \dfrac{x}{y}$.

b Find the solution passing through $P(4, 5)$.

SOLUTION

a The DE is $\dfrac{dy}{dx} = \dfrac{x}{y}$,

and separating dy and dx, $y\,dy = x\,dx$.

Now we can integrate, $\displaystyle\int y\,dy = \int x\,dx$,

$\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C$, for some constant C,

and putting $D = 2C$, $y^2 - x^2 = D$, for some constant D.

b Substituting the point $P(4, 5)$, $25 - 16 = D$,

so $D = 9$, and the solution is $y^2 - x^2 = 9$.

7 SEPARABLE DIFFERENTIAL EQUATIONS

- A first-order DE is called *separable* if it can be put into the form
  $$y' = f(x)\,g(y).$$
- The general case is $y' = f(x)\,g(y)$, where neither function is a constant.
  - The case $y' = f(x)$, where $g(y) = 1$, is equivalent to an indefinite integral.
  - The case $y' = g(y)$, where $f(x) = 1$, will be discussed in Section 13D.
- To solve the general case, integrate after putting it into the form
  $$\frac{1}{g(y)}\,dy = f(x)\,dx.$$

Look for constant solutions before solving

It is easy to check whether a constant function $y = k$ is a solution of a DE because the derivative y' is zero. This step is important because when the DE is solved by the methods above, constant solutions are often missing because of a division by zero or other issue.

Constant solutions were discussed in Section 13B — they correspond on the slope field to horizontal isoclines consisting of horizontal line elements, and they are usually horizontal asymptotes of nearby solution curves.

The first step in solving any DE is therefore, ‘Look for constant solutions.’

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 17 13C

a Solve $y' = -xy^2$.

b Find the particular solution given that:

i $y(1) = \frac{1}{2}$, **ii** $y(2) = 0$.

SOLUTION

a First, the constant function $y = 0$ is trivially a solution of the DE.

Now we can write the DE as $-y^{-2}\,dy = x\,dx$, because $y \neq 0$,

and integrating, $$\int -y^{-2}\,dy = \int x\,dx,$$

so for some constant C, $$y^{-1} = \tfrac{1}{2}x^2 + C. \qquad (*)$$

b i Substituting $y(1) = \frac{1}{2}$ into $(*)$, $2 = \frac{1}{2} + C$,

so $C = \frac{3}{2}$ and $$y^{-1} = \tfrac{1}{2}x^2 + \tfrac{3}{2}$$

$$y = \frac{2}{x^2 + 3}.$$

ii Substituting $y(2) = 0$ into $(*)$ is impossible because of division by zero, but the first solution $y = 0$ satisfies $y(2) = 0$, so it is the required solution.

8 CONSTANT SOLUTIONS

- Always check first whether any constant functions are solutions of the DE.
- These are horizontal isoclines consisting of horizontal line elements, and they are usually horizontal asymptotes of nearby solution curves.

We remarked in worked Example 14 that a constant solution corresponds to *stable equilibrium* when it is an asymptote on the right to nearby solution curves, and corresponds to *unstable equilibrium* when it is an asymptote on the left.

Dealing with absolute values in the solution

Many solutions obtained by the methods used with separable DEs result in absolute values because $\dfrac{1}{x}$ has primitive $\log|x|$. This requires some care because although an arbitrary constant C may take any value, the power e^C is never negative and never zero.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 18 13C

a Solve the DE $y' = x(1 - y)$.

b Find the particular solution passing through:

i the origin, **ii** $(1, 1)$.

SOLUTION

a First, the constant function $y = 1$ is trivially a solution.

Now we can write the DE as $\dfrac{dy}{1 - y} = x\,dx$, because $y \neq 1$,

and integrating, $\displaystyle\int \frac{dy}{y - 1} = -\int x\,dx$

$$\log|y - 1| = -\tfrac{1}{2}x^2 + C, \text{ for some constant } C$$

$$|y - 1| = e^{-\frac{1}{2}x^2 + C},$$

and putting $A = e^C$, $|y - 1| = Ae^{-\frac{1}{2}x^2}$, where A is positive.

Hence $y - 1 = Ae^{-\frac{1}{2}x^2}$, where A can be positive or negative.

We have already remarked that $y - 1 = 0$ is a solution of the original DE,

so the general solution is $y - 1 = Ae^{-\frac{1}{2}x^2}$, for any constant A,

$$y = 1 + Ae^{-\frac{1}{2}x^2}. \qquad (*)$$

b i Substituting $(0, 0)$ into $(*)$ gives $0 = 1 + A \times 1$,

so $A = -1$, and the particular solution is $y = 1 - e^{-\frac{1}{2}x^2}$.

ii Substituting $(1, 1)$ into $(*)$ gives $1 = 1 + Ae^{-\frac{1}{2}}$

so $A = 0$, giving the constant function $y = 1$ identified on the first line.

Example 19 13C

a Solve the DE $y' = xy$.

b Find the solution for which:

i $y(2) = 3$, **ii** $y(3) = 0$.

SOLUTION

a First, the constant function $y = 0$ is trivially a solution.

Now we can write the DE as $\dfrac{dy}{y} = x\,dx$, because $y \neq 0$,

and integrating, $\displaystyle\int \frac{dy}{y} = \int x\,dx$

$$\log|y| = \tfrac{1}{2}x^2 + C, \text{ for some constant } C,$$

so $|y| = e^C\, e^{\frac{1}{2}x^2}$

$$|y| = A\,e^{\frac{1}{2}x^2}, \text{ where } A > 0.$$

Hence $y = A\,e^{\frac{1}{2}x^2}$, where A can be positive or negative.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

We have already remarked that $y = 0$ is a solution of the original DE,

so the general solution is $y = A\,e^{\frac{1}{2}x^2}$, for any constant A. (∗)

b **i** Substituting $y(2) = 3$ gives $3 = Ae^2$,

so $A = 3e^{-2}$ and $y = 3e^{\frac{1}{2}x^2-2}$.

ii This solution is the constant solution $y = 0$.

9 DEALING WITH ABSOLUTE VALUE IN SOLUTIONS

When interpreting absolute value signs in a solution

- Acknowledge that e^C is always positive.
- Modify the solution when removing the absolute value signs.
- Modify the solution again if any constant function is a solution.

Exercise 13C

FOUNDATION

1 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x-1}{y+1}$.

a Multiply through by $y + 1$ and then by dx, so that the variables are separated.

b Use the result $\displaystyle\int (x+a)^n\,dx = \frac{1}{n+1}(x+a)^{n+1} + C$ to find the general solution of this differential equation. Write your answer without using fractions.

2 Likewise find the general solutions of these separable equations. Make y the subject of the solution in each case.

a $\dfrac{dy}{dx} = xe^{-y}$

b $\dfrac{dy}{dx} = 4x^3(1+y^2)$

3 **a** Explain why $y = 0$, where $x \neq 0$, is a solution of $\dfrac{dy}{dx} = -\dfrac{y^2}{x}$. (Always look first for constant solutions.)

b Use the method of separable DEs to find the other solutions.

4 Suppose that the solution curve for $\dfrac{dy}{dx} = -\dfrac{x}{y}$ passes through $(1, \sqrt{3})$.

a Separate the variables and hence write down the corresponding equation of integrals.

b The general solution of this DE is a relation, not a function. Find the general solution, writing your answer without fractions.

c Hence determine the equation of the solution curve through the given point.

5 Likewise, for each DE, find the solution curve passing through the given point.

a $\dfrac{dy}{dx} = \dfrac{x}{y}$, through $(0, 1)$

b $\dfrac{dy}{dx} = (1+x)(1+y^2)$, with $y(-1) = 0$

c $\dfrac{dy}{dx} = -2y^2x$, with $y(1) = \frac{1}{2}$

d $\dfrac{dy}{dx} = e^{-y}\sec^2 x$, through $(\frac{\pi}{4}, \log 2)$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

6 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2y + 4}{x}$.

a Find the constant solution, substituting to show that it is a solution of the DE.
b Use separation of variables to find the other solutions of the DE.
c How can the solutions in parts **a** and **b** be combined?

7 Consider the differential equation $y' = -xy$.

a Find the constant solution, substituting to show it is a solution of the DE.
b Use separation of variables to find the other solutions of the DE.
c How can the solutions in parts **a** and **b** be combined?

8 Use a similar approach to Questions **5** and **6** to solve these DEs.

a $\dfrac{dy}{dx} = \dfrac{2 - y}{x}$ **b** $\dfrac{dy}{dx} = \dfrac{xy}{1 + x^2}$ **c** $\dfrac{dy}{dx} = \dfrac{-2y}{x}$

d $\dfrac{dy}{dx} = y \sin x$ **e** $\dfrac{dy}{dx} = \dfrac{3y}{x^2}$ **f** $\dfrac{dy}{dx} = \dfrac{y(1 - x)}{x}$

9 **a** Find all the constant solutions of $\dfrac{dy}{dx} = 3x^2 \cos^2 y$ in the interval $-2\pi \le y \le 2\pi$.

b Find all the non-constant solutions.

10 Consider the initial value problem $\dfrac{dy}{dx} = \dfrac{2y}{x - 1}$ with $y(2) = 1$.

a Show that the constant solution of the DE is not a solution of the IVP.
b Use separation of variables to find the general solution of the DE.
c Hence solve the IVP.

11 Consider the initial value problem $\dfrac{dy}{dx} = (y - 1)\tan x$ with $y(\frac{\pi}{4}) = 3$.

a Show that the constant solution of the DE is not a solution of the IVP.
b Use separation of variables to find the general solution of the DE.
c Hence solve the IVP.

12 Use a similar approach to Questions **8** and **9** to solve these IVPs.

a $\dfrac{dy}{dx} = \dfrac{y}{x}$ with $y(2) = 1$ **b** $\dfrac{dy}{dx} = \dfrac{y}{2x}$ with $y(1) = 2$

c $\dfrac{dy}{dx} = \dfrac{-2xy}{1 + x^2}$ with $y(1) = 2$ **d** $\dfrac{dy}{dx} = -\dfrac{y}{x}$ with $y(2) = 1$

e $\dfrac{dy}{dx} = y \cos x$ with $y(\frac{\pi}{2}) = 1$ **f** $\dfrac{dy}{dx} = \dfrac{y(2 - x)}{x^2}$ with $y(2) = \frac{1}{2}$

13 **a** Differentiate $\log(\log x)$.

b Hence find the general solution of $(x \log x)y' = y$.

14 **a** Show that $\dfrac{x}{x + 2} = 1 - \dfrac{2}{x + 2}$.

b Hence solve the initial value problem $(x + 2)y' - xy = 0$ with $y(0) = 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 **a** Use the double-angle formulae to rewrite $2\cos^2 x$ in terms of $\cos 2x$

b The solution of the DE $\dfrac{dy}{dx} = \dfrac{2\cos^2 x}{y}$ is a relation and not a function. Use part **a** to find its equation, given that it passes through $(0, \sqrt{2})$.

16 **a** Let $y = x \times u$, where u is an unknown function of x. Use the product rule to find an expression for y'.

b Consider the differential equation $xy' = 2x + 2y$.

i Use the result of part **a** to write a corresponding differential equation for u that is separable.

ii Solve this DE for u.

iii Hence write down the general solution of $xy' = 2x + 2y$.

ENRICHMENT

17 Suppose that $(x^2 + 1)y' + (y^2 + 1) = 0$ with $y(0) = 1$.

a Find a general solution of this DE.

b Show that the general solution is equivalent to $\dfrac{y + x}{1 - xy} = D$.

c Hence find the solution of the IVP. Make y the subject of your answer.

18 For the unwary mathematician, the initial value problem

$$\frac{dy}{dx} = xy^{\frac{1}{2}} \text{ with } y(2) = 1$$

appears to have two solutions:

$$y_1 = \tfrac{1}{16}x^4 \text{ and } y_2 = \tfrac{1}{16}(x^2 - 8)^2.$$

a Show that both y_1 and y_2 satisfy the initial condition $y(2) = 1$.

b Show that both y_1 and y_2 satisfy the modified DE $(y')^2 = x^2 y$.

c Draw the slope field for the original DE, $y' = xy^{\frac{1}{2}}$. Then add both y_1 and y_2 to the graph and observe that y_2 clearly does not follow the slope field.

d Explain algebraically why y_2 is not a solution, and then correctly derive y_1.

19 [Picard Iterations]

Suppose that $y' = f(x, y)$ with $y(x_0) = y_0$ has solution $y(x)$.

a Explain why $y(x) = y_0 + \displaystyle\int_{x_0}^{x} f\big(t, y(t)\big)\, dt$.

b Now consider the sequence $y_0, y_1(x), y_2(x)\ldots$, where y_0 is constant, and where

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$\text{with } y_{n+1}(x) = y_0 + \int_{x_0}^{x} f\big(t, y_n(t)\big)\, dt.$$

Suppose that $y' = -xy$ with $y(0) = 1$. That is, $f(x, y) = -xy$ and $y_0 = 1$.

i Find the solution of this IVP using separation of variables.

ii Use the formulae above to find $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$.

iii It can be shown that the function $y_n(x)$ is a series approximation that converges to the solution of the IVP as $n \to \infty$. Use $y_4(x)$ with the value $x = \frac{1}{2}$ to approximate e correct to 4 decimal places.

iv Investigate better approximations by using higher values of n.

ISBN 978-1-108-76630-2

13D $y' = g(y)$ and the logistic equation

We now consider equations of the form $y' = g(y)$. The exponential growth DE has this form, and so does the logistic equation, which develops the natural growth DE by modelling amongst other things populations restricted by predators or lack of food.

This section, however, is still mostly about the equations rather than the situations that they are modelling, so we continue to use the pronumerals x and y. Section 13E will explain some models and use a variety of pronumerals.

Solving $y' = g(y)$

There are two equivalent approaches to solving a DE of the form $\dfrac{dy}{dx} = g(y)$.

- Regard it as a separable DE, write it as $\dfrac{1}{g(y)}\,dy = dx$, and integrate.
- Take reciprocals, write it as $\dfrac{dx}{dy} = \dfrac{1}{g(y)}$, and integrate.

Example 20 — 13D

Solve $y' = e^y$ using both approaches.

SOLUTION

EITHER regard it as a separable DE, $\quad e^{-y}\,dy = dx$

and integrate, $$-\int e^{-y}\,dy = -\int dx$$
$$e^{-y} = -x + C, \quad \text{for some constant } C$$
$$x = -e^{-y} + C.$$

OR take reciprocals, $$\frac{dx}{dy} = e^{-y}$$

and integrate, $$x = -e^{-y} + C, \quad \text{for some constant } C.$$

After either approach, it may be appropriate to solve for y, giving
$$y = -\log(-x + C).$$

10 SOLVING $y' = g(y)$

- Solve $y' = g(y)$ by integration using either of the forms
$$\frac{1}{g(y)}\,dy = dx \qquad \text{OR} \qquad \frac{dx}{dy} = \frac{1}{g(y)}.$$

The exponential growth DE

Exponential growth has a very simple DE
$$y' = ky, \quad \text{where} \quad k \neq 0,$$
which says in words that the rate of change of the quantity (such as population or mass of a radioactive isotope) is proportional to the quantity. Growth occurs when k is positive, and decay when k is negative.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapters 11 and 16 of the Year 11 book presented exponential growth without discussing this DE in any detail. We will now examine this DE — think of x as time.

These DEs are a special case of a linear DE, because they can be put into the form $y' + f(x)y = g(x)$. In this case $f(x)$ is a constant and $g(x) = 0$.

Example 21 13D

Solve $y' = ky$, where $k \neq 0$, using both approaches.

SOLUTION

First, the constant function $y = 0$ is trivially a solution. Otherwise divide by y.

Rearranging, $\quad \frac{1}{y}\,dy = k\,dx,$

and integrating, $\quad \int \frac{1}{y}\,dy = \int k\,dx,$

so for some constant C, $\quad \log|y| = kx + C,$

Hence $\quad |y| = e^{kx+C},$

and putting $A = e^C$, $\quad |y| = Ae^{kx}.$

OR

Taking reciprocals, $\quad \frac{dx}{dy} = \frac{1}{ky},$

and integrating, $\quad x = \int \frac{1}{ky}\,dx,$

so for some constant C, $\quad x = \frac{1}{k}\log|y| + C.$

Hence $\quad kx - kC = \log|y|$

$\quad |y| = e^{kx-kC},$

and putting $A = e^{-kC}$, $\quad |y| = Ae^{kx}.$

With either working, $|y| = Ae^{kx}$ where A is positive.

Hence $\quad y = Ae^{kx}$ where A can be positive or negative,

and because the constant function $y = 0$ is trivially a solution,

$\quad y = Ae^{kx}$, where A can be any real number.

Example 22 13D

Solve the differential equation $y' = -3y$ given the initial value $y(2) = 50$.

SOLUTION

First, the constant function $y = 0$ is trivially a solution.

Rearranging, $\quad \frac{1}{y}\,dy = -3\,dx,$

and integrating, $\quad \log|y| = -3x + C,$ for some constant C,

so $\quad |y| = e^{-3x+C}$

$\quad |y| = Ae^{-3x},$ where $A = e^C$.

Hence $\quad y = Ae^{-3x},$ where A can be positive or negative,

and because the constant function $y = 0$ is trivially a solution,

the general solution is $\quad y = Ae^{-3x},$ where A can be any real number.

Substituting $y(2) = 50$ gives $50 = Ae^{-6}$,

so $A = 50e^6$, and $\quad y = 50e^{-3(x-2)}.$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Autonomous DEs

A differential equation that does not involve the independent variable x is called *autonomous*. All the DEs in this section are autonomous, and the title of this could have been, '*First-order autonomous differential equations*'.

Think of x as time. The exponential growth DE above is independent of time, which means that the differential equation describing such phenomena — populations, radioactive decay, the cooling of a kettle of hot water taken off the stove — are true for all times. These things are laws of physics, and laws of physics are usually independent of time because the laws hold at all times. The most general physical laws also hold in all places, provided that we factor into the DE any gravitational forces, so these general laws are independent of both space and time — think of Newton's second law of motion $F = m\ddot{x}$, and of the motion of a mass oscillating on a spring in accordance with $\ddot{x} = -n^2x$. Thus the absence here of a variable in an equation has amazing significance in the physical world.

The logistic equation — solving the differential equation

A *logistic differential equation* has a very general definition, but in this course it is a DE of the form

$y' = ky(P - y)$, where P and k are non-zero constants.

The example below is the simplest logistic equation — both constants P and k are set equal to 1.

Solving the logistic equation directly involves converting a single fraction into the sum of two fractions using a procedure known as *partial fractions*. Partial fractions are not in the course, so such a decomposition will be given in each question where it is needed.

Example 23 13D

a Show that $\dfrac{1}{y(1-y)} = \dfrac{1}{y} + \dfrac{1}{1-y}$. (This is called a *partial fractions decomposition*.)

b Hence solve $y' = y(1-y)$, writing the solution with y as the subject.

SOLUTION

a $\text{RHS} = \dfrac{(1-y)+y}{y(1-y)} = \text{LHS}.$

b First, the constant functions $y = 0$ and $y = 1$ are trivially solutions.

Otherwise, rearranging, $\dfrac{1}{y(1-y)}\,dy = dx,$

and using part **a**, $\left(\dfrac{1}{y} + \dfrac{1}{1-y}\right)dy = dx.$

$$\begin{aligned}
\text{Integrating,}\quad \log|y| - \log|1-y| &= x + C, \text{ for some constant } C\\
\log\left|\frac{y}{1-y}\right| &= x + C\\
\left|\frac{y}{1-y}\right| &= e^{x+C}\\
\left|\frac{y}{1-y}\right| &= Ae^{x}, \text{ where } A = e^{C} \text{ is positive.}
\end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Hence $\dfrac{y}{1-y} = Ae^x$, where A can be positive or negative.

Making y the subject,
$$y = Ae^x - Ae^x y$$
$$Ae^x y + y = Ae^x$$
$$y = \frac{Ae^x}{Ae^x + 1}.$$

This is not a good form because the arbitrary constant occurs twice.

Dividing top and bottom by Ae^x, $y = \dfrac{1}{1 + Be^{-x}}$, where $B = A^{-1}$.

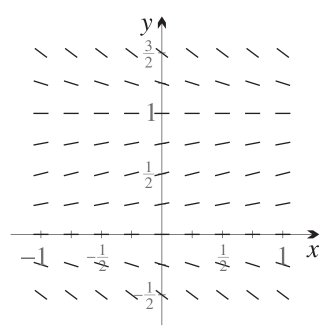
Hence $y = 0$, or $y = 1$, or $y = \dfrac{1}{1 + Be^{-x}}$, for some non-zero constant B.

Note: The solution $y = 1$ corresponds to $B = 0$, and the solution $y = 0$, corresponds to $B \to \infty$. It is probably better to consider these two solutions as special cases, and add to the general solution the condition, 'where B is non-zero'.

The logistic equation — the slope field and three types of solution

To the right is the slope field of this DE $y' = y(1-y)$. Go back now to worked Example 14 in Section 13B to see some solution curves drawn on a very similar slope field. The slope field makes clear some things that are difficult to make out from the algebra above.

- The two constant solutions $y = 0$ and $y = 1$ are clear from the slope field — in fact they are seen first.
- No solution curve ever crosses the two horizontal lines $y = 0$ and $y = 1$. This means that in practice the other connected solution curves fall into three distinct groups:
  — those with range $y > 1$,
  — those with range $0 < y < 1$,
  — those with range $y < 0$.



The next worked example picks out a solution curve in each of the three regions.

Example 24 13D

In worked Example 23, we obtained the general solution $y = \dfrac{1}{1 + Be^{-x}}$ of the differential equation $y' = y(1-y)$.

a Use this to find connected solution curves passing through:

i $(0, \frac{1}{2})$, **ii** $(0, 2)$, **iii** $(0, -1)$.

In each case, identify any asymptotes and any symmetries, state the domain and range of the connected curve, and briefly describe the situation if x is time and y is population.

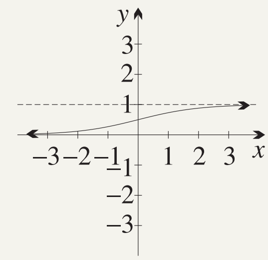
b How are the solution curves in parts **a ii** and **a iii** related?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a i Substituting $(0, \frac{1}{2})$, $\quad \dfrac{1}{2} = \dfrac{1}{1 + B}$,



so $B = 1$, and the solution curve is $\quad y = \dfrac{1}{1 + e^{-x}}$.

The domain is all real x, and the range is $0 < y < 1$.

Taking limits, $\lim\limits_{x \to \infty} y = 1$ and $\lim\limits_{x \to -\infty} y = 0$,

so $y = 0$ and $y = 1$ are horizontal asymptotes.

The solution curve has point symmetry in $(0, \frac{1}{2})$, because

$$\begin{aligned} y(-x) &= \frac{1}{1 + e^{x}} \times \frac{e^{-x}}{e^{-x}} \\ &= \frac{e^{-x}}{e^{-x} + 1} \\ &= \frac{e^{-x} + 1}{e^{-x} + 1} - \frac{1}{e^{-x} + 1} \\ &= 1 - \frac{1}{1 + e^{-x}} \\ &= 1 - y(x). \end{aligned}$$

On the left of the x-axis, the population begins very small and increases, first at an increasing rate. Then on the right of the x-axis, the population increases at a decreasing rate, approaching a limiting value, which we take as the stable population.

ii Substituting $(0, 2)$ $\quad 2 = \dfrac{1}{1 + B}$,

so $B = -\frac{1}{2}$, and the solution curve is $\quad y = \dfrac{1}{1 - \frac{1}{2}e^{-x}}$.

There is a vertical asymptote when $\quad e^{-x} = 2$,

that is, $\quad x = -\log 2$,

so that $y \to \infty$ as $x \to (-\log 2)^{+}$.

Our connected solution curve does not cross this asymptote,

so because the initial value is at $x = 0$, the domain is $x > -\log 2$.

Taking limits, $\quad \lim\limits_{x \to \infty} y = 1$,

giving a horizontal asymptote at $y = 1$, and range $y > 1$.

The population is originally greater than the stable population, and then decreases at a decreasing rate, approaching the same stable population as in part **i**.

iii Substituting $(0, -1)$, $\quad -1 = \dfrac{1}{1 + B}$,

so $B = -2$, and the solution curve is $\quad y = \dfrac{1}{1 - 2e^{-x}}$.

There is a vertical asymptote when $\quad e^{-x} = \frac{1}{2}$

that is, $\quad x = \log 2$

so that $y \to -\infty$ as $x \to (\log 2)^{-}$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Our connected solution curve does not cross this asymptote,
so because the initial value is at $x = 0$, the domain is $x < \log 2$.
Taking limits, $\lim_{x \to -\infty} y = 0$,
giving a horizontal asymptote at $y = 0$, and range $y < 0$.

The negative values of y mean that this solution curve has no meaning for populations.

b The function $y = \dfrac{1}{1 - \frac{1}{2}e^{-x}}$ that we found in part **a ii** has another disconnected branch to the left of $y = -\log 2$, and this branch is below $y = 0$. Similarly, the function $y = \dfrac{1}{1 - 2e^{-x}}$ that we found in part **a iii** has another disconnected branch to the right of $y = -\log 2$, and this branch is above $y = 1$. In general, each solution curve of the DE above $y = 1$ is paired with a solution curve below $y = 0$, and vice versa.

In applications, however, it is rare for more than one of these branches to have any meaning, or for there to be any physical relationship between them.

Using the differential equation to find the second derivative

The middle solution curve that we found in part **a i** looks as if it has an inflection at $\left(0, \frac{1}{2}\right)$, and the symmetry that we established proves this. But our usual methods of examining the first and second derivatives look complicated because of the need to differentiate $y = \dfrac{1}{1 + e^{-x}}$ twice. The next worked example show how we can use the differential equation itself to make this process far quicker.

Example 25 13D

In worked Example 24, we were examining solution curves of $y' = y(1 - y)$.

a Prove that $y'' = y'(1 - 2y) = y(1 - y)(1 - 2y)$.

b Use this result and the DE itself to analyse the gradient and concavity of the solution curves in parts **a i–a iii** of worked Example 24.

SOLUTION

a Expanding, $y' = y - y^2$.

Using the chain rule,

$$y'' = \frac{d}{dy}(y - y^2) \times \frac{dy}{dx}$$

$$y'' = (1 - 2y)y'$$

$$= y(1 - y)(1 - 2y).$$

b First, the DE $y' = y(1 - y)$ tells us that y' is positive for $0 < y < 1$ and negative for $y < 0$ or $y > 1$. Hence the solution curves in the middle region are always increasing, and the solution curves in the top and bottom region are always decreasing.

Secondly, $y'' = y(1 - y)(1 - 2y)$ tells us that y'' is positive for $0 < y < \frac{1}{2}$ or $y > 1$, and negative for $\frac{1}{2} < y < 1$ or $y < 0$. Hence the solution curves in the top region are always concave up, and the solution curves in the bottom region are always concave down, The solution curves in the middle region change concavity from up to down at $y = \frac{1}{2}$, giving a point of inflection there.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The summary below uses a more general form of the logistic equation.

11 THE LOGISTIC DIFFERENTIAL EQUATION

- The logistic equation is a first-order DE of the form
  $$y' = ky(P - y), \text{ where } k \text{ and } P \text{ are constants,}$$
  where the following dotpoints assume that P and k are positive.
- The constant functions $y = 0$ and $y = P$ are solutions of the DE.
- The general solution consists otherwise of three groups of solution curves:
  — Solution curves between $y = 0$ and $y = P$, with domain all real numbers.
  — Solution curves above $y = P$.
  — Solution curves below $y = 0$.
- The solution curves in the second group are the upper branches of two-branch functions whose lower branch is a solution in the third group, and vice versa.
- In applications, one group, or even two groups, may have no significance.

But look at Question 15 in Exercise 13E, where the third group does have significance — it describes the way in which a population moves to extinction if its numbers ever fall below a certain threshold.

Note: Go back to the slope field above worked Example 24 (and see also worked Example 14 in Section 13B). The slope field displays the solution curves in the middle group very nicely. But it fails to identify that every solution curve in the upper or lower groups has a vertical asymptote. As is often the case with slope fields, some aspects of a problem are very clearly displayed, but other aspects may be deceptive.

Exercise 13D

FOUNDATION

1 Consider the differential equation $\dfrac{dy}{dx} = -y$.

a What is the constant solution of this equation?
b Use separation of variables to solve the DE.
c Make y the subject of this solution, simplifying the constant part of the expression.
d Check that the constant solution is included in your answer to part **c**.
e Find the solution for the initial condition $y(0) = 2$.

2 Consider the differential equation $\dfrac{dy}{dx} = 3y$.

a What is the constant solution of this equation?
b Write down the DE obtained by taking the reciprocal of both sides.
c Use direct integration to obtain x as a function of y.
d Make y the subject of this solution, simplifying the constant part of the expression.
e Check that the constant solution is included in your answer to part **d**.
f Find the solution for the initial condition $y(0) = -1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Find the solutions of these autonomous IVPs $\big($*autonomous* means that ithas the form $y' = g(y)\big)$. Use either of the methods given in Questions **1** and **2**.

a $y' - y = 0$, with $y(0) = -3$
b $y' + 2y = 0$, with $y(0) = 1$
c $y' = -3y$, with $y(0) = 2$
d $y' = 2y$, with $y(0) = -1$

4 Consider the differential equation $\dfrac{dy}{dx} = 2 - y$.

a What is the constant solution of this equation?
b Use separation of variables or take reciprocals to solve this DE.
c Make y the subject, simplifying the constant part of the expression.
d Check that the constant solution is included in your answer to part **c**.
e Find the solution for the initial condition $y(0) = 3$.

5 Follow the procedures in Question **4** to solve these IVPs.

a $y' = 1 - y$, with $y(0) = 3$
b $y' = y - 1$, with $y(0) = 0$
c $y' = \frac{1}{2}(y + 1)$, with $y(0) = 1$
d $y' = 2(3 - y)$, with $y(0) = 4$

6 Solve these initial value problems.

a $y' = 2y^2$, with $y(0) = 3$
b $y' = -y^2$, with $y(0) = 1$
c $y' = 1 + y^2$, with $y(\frac{\pi}{4}) = 1$
d $y' = -e^y$, with $y(0) = 0$
e $y' = e^{-y}$, with $y(3) = 0$
f $y' = y^{\frac{2}{3}}$, with $y(0) = 1$

DEVELOPMENT

7 Consider the differential equation $y' = ky$, where k is an unknown constant.

a Find the general solution of this DE.
b Evaluate the arbitrary constant, given the initial condition $y(0) = 20$.
c Finally, evaluate k given the condition $y(2) = 5$.
d Simplify your solution, and hence evaluate $y(3)$.

8 Once again consider the differential equation $y' = ky$, where k is an unknown constant.

a Find the general solution of this DE.
b Evaluate the arbitrary constant, given the initial condition $y(0) = 8$.
c Finally, evaluate k given the condition $y(2) = 18$.
d Simplify your solution, and hence evaluate $y(4)$.

9 In solving certain problems involving support beams in engineering, the fourth-order differential equation $y'''' = \lambda^4 y$ is encountered, which is sometimes written as $y^{(4)} = \lambda^4 y$.

a Show that $y = Ae^{\lambda x} + Be^{-\lambda x} + C\cos \lambda x + D\sin \lambda x$ is a solution of the DE.
b A beam rests on a support at a point O. At a horizontal distance x along the beam, the downwards deflection is y. Thus $y(0) = 0$, and we may also assume that $y''(0) = 0$. Find the value of C.
c If the beam is also resting on a support at $x = 10$, then both $y(10) = 0$ and $y''(10) = 0$. From this it can be shown that $\lambda = \frac{n\pi}{10}$.
  i Use these results to show that $A = B = 0$.
  ii Write down the solution of the beam IVP.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 **a** Find the general solution of the DE $y' = e^{-y}$

b Describe the family of curves that you have found.

c Draw the slope field for this DE.

i Draw the solution curve that passes through the origin.

ii Draw two other solution curves.

iii Describe how your two solution curves can be obtained by simple transformations.

iv What feature of the slope field makes this possible?

d Evaluate the arbitrary constant, given that the curve passes through $(0, 1)$.

11 The solution of $y'y = 2$ is a relation that is not a function.

a Find the general solution of the DE.

b Describe the family of curves you have found.

c Draw the slope field for this DE.

i Draw the solution curve that passes through the origin.

ii Draw two other solution curves.

iii Describe how your two solution curves can be obtained by simple transformations of the curve in part **i**.

iv What feature of the slope field makes this possible.

d Evaluate the arbitrary constant, given that the curve passes through $(0, 1)$.

12 Let $L(x) = \dfrac{1}{1 + e^{-x}}$ and consider the curve $y = L(x)$

a What is the y-intercept?

b Explain why the function is always positive.

c Determine $\lim\limits_{x \to \infty} L(x)$ and $\lim\limits_{x \to -\infty} L(x)$.

d Find L' and hence show that the curve has no stationary points.

e **i** Show that $L' = \dfrac{1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$.

ii Use the result in part **i** to find L''.

iii Hence find the point of inflection of $y = L(x)$.

f Sketch $y = L(x)$.

g **i** Show by substitution that $y = L(x)$ is a solution of the logistic DE $y' = y(1 - y)$.

ii Use this DE to prove the formula for $L'(x)$ given in part **e i**.

iii Use this DE again to prove the formula for $L''(x)$ that you found in part **e ii**.

13 **a** Show that $\dfrac{1}{y(1 - y)} = \dfrac{1}{y} + \dfrac{1}{(1 - y)}$.

b Consider the logistic differential equation $y' = y(1 - y)$.

i What are the constant solutions of this equation?

ii Use part **a** to find the general solution of the logistic DE.

c Show that the solution is the result of shifting the function in Question **12**, and determine the shift.

d The constant solutions cannot be obtained in the usual way from the general solution. Use the general solution $y = \dfrac{1}{1 + Be^{-x}}$ to answer the following.

i Find $\lim\limits_{B \to \infty} y$. Is this one of the constant solutions?

ii Find $\lim\limits_{B \to 0^+} y$. Is this one of the constant solutions?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 A more generalised version of the logistic equation is $y' = ry(1 - y)$, for some constant r.

a Find the constant solutions.

b Use part **a** of Question **13** to find the general solution.

c Suppose that the initial condition is $y(0) = y_0$. Determine the value of the arbitrary constant B given in the answer to part **a**.

d Show that the solution given in the answer to part **a** is the result of shifting the function $y = \dfrac{1}{1 + e^{-rx}}$ right by $\frac{1}{r}\log B$.

e By using the answers to Question **13d**, or otherwise, what happens as:

i $y_0 \to 0^+$, **ii** $y_0 \to 1^-$.

15 Draw the slope field for the logistic differential equation $y' = y(1 - y)$.

a Add the two constant solutions $y = 0$ and $y = 1$ to the graph.

b Add the solution curve $y = \dfrac{1}{1 + e^{-x}}$ to the slope field. Notice that this curve occupies the part of the number plane between the two constant solutions.

c Show by substitution that $y = \dfrac{1}{1 - e^{-x}}$ is also a solution of the DE.

d Follow the curve-sketching menu to add this second function to the graph. Notice that this curve has two branches that both lie outside the two constant solutions.

16 **a** Show that $\dfrac{1}{(y - 1)(y - 3)} = \dfrac{1}{2}\left(\dfrac{1}{y - 3} - \dfrac{1}{y - 1}\right)$.

b Consider the modified logistic equation $y' = -(1 - y)(3 - y)$.

i What are the constant solutions?

ii Use part **a** to find the general solution of the given DE.

c Using the general solution given in the answers:

i Which of the constant solutions is found by taking $B \to 0^+$?

ii Which of the constant solutions is found by taking $B \to \infty$?

d Draw the slope field, the two constant solutions, and the solutions with $B = -1$ and $B = 1$.

e **i** Find an expression for y'' in terms of y alone.

ii Hence determine the location of the inflection point for the solution curve with $B = 1$.

17 Some first-order differential equations can be made much simpler by using a substitution. Here is a very important example.

Once again, consider the logistic equation, $y' = ry(1 - y)$.

a Make the substitution $v = \dfrac{1}{y}$ and show that $v' = r(1 - v)$.

b Solve the differential equation for v by any appropriate means.

c Hence find the general solution of the logistic equation.

18 Some second-order differential equations can be turned into first-order equations with a substitution. Here is a simple example.

Consider the second-order initial value problem $y'' = 2(1 - y')$, with $y(0) = 1$ and $y'(0) = 0$.

a Put $v = y'$. Write down the corresponding differential equation for v.

b Write down the initial condition for v.

c Solve the initial value problem for v.

d Hence write down y' as a function of x.

e Finally integrate to find y as a function of x.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

19 **a** Prove that if $y = f(x)$ is a solution of the autonomous differential equation $y' = g(y)$, then $y = f(x - C)$ is also a solution.

b Describe the graph of $y = f(x - C)$ as a transformation of $y = f(x)$.

c Describe the isoclines of $y' = g(y)$.

d How are parts **b** and **c** related?

e Review your solutions of any of the first-order differential equations in this exercise in light of this result.

20 This question combines several techniques from earlier in the exercise to solve the simple harmonic motion differential equation. That equation is $\dfrac{d^2y}{dx^2} + y = 0$, which is autonomous.

a Begin by putting $v = \dfrac{dy}{dx}$.

i Use the chain rule to show that $\dfrac{d^2y}{dx^2} = v\dfrac{dv}{dy}$.

ii Hence write down a separable differential equation in terms of v and y alone.

b The general solution of the DE in part **a ii** is a relation, not a function. Find it.

c Explain the situation geometrically for different cases of arbitrary constant.

d Assuming that $C = r^2$, what are the standard parametric equations?

e Confirm that using these parametric equations gives $y' = v$.

f Use the results of Question **19** to write down the general solution of the original DE.

g Hence show that $y = A\cos x + B\sin x$ is the general solution of the simple harmonic motion differential equation, $y'' + y = 0$:

i first by expanding the result of the previous part,

ii then by direct substitution into the DE.

21 Care must be taken when the solution of a DE involves inverse functions. Solutions may be inadvertently lost or added, as in the following example. It demonstrates that any solution of a DE should be thoroughly checked before it is accepted as correct.

Consider the initial value problem $\dfrac{dy}{dx} = -\sqrt{1 - y^2}$, with $y(0) = 1$.

a Find the implicit general solution of the DE.

b Evaluate the unknown constant by applying the initial condition.

c Making y the subject without taking care, it would seem that the solution of the IVP is $y = \cos x$. Explain why this solution is not valid for all values of x. It may help to substitute this solution into each side of the DE.

d What is the correct solution of the IVP in which y is the subject?

22 It is possible for an IVP to have multiple solutions. Consider the initial value problem

$$\frac{dy}{dx} = 3y^{\frac{2}{3}}, \text{ with } y(0) = 0.$$

Show that each of the following functions is a solution of this IVP.

a $y = x^3$

b $y = 0$

c $y = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } x \geq 0 \end{cases}$

d $y = \begin{cases} x^3 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13E Applications of differential equations

In this final section, we apply differential equations to some situations in science where they occur naturally. In particular, we shall examine the use of the exponential growth DE and the logistic equation to model physical situations such as population growth. Mostly our variables will no longer be x and y, and in particular, our independent variable will typically be t for time.

First, however, we discuss another approach to solving DEs when we know the form of the solution, but not the values of the constants in the formula.

Evaluating unknown constants when the form of the solution is known

People who work with DEs learn to recognise these equations. They may not know the solution, but often they do know what the solution looks like. The next worked example demonstrates how to find the actual solution in two such situations.

Example 26 13E

a Find the values of b and c so that $y = e^{3x} + bx + c$ is a solution of $y' = 3y - 18x$.

b Find n so that $y = 10\cos nx$ is a solution of the second-order DE $y'' = -49y$.

SOLUTION

a Substituting $y = e^{3x} + bx + c$ into the DE,

$$\text{LHS} = 3e^{3x} + b, \qquad \begin{aligned}\text{RHS} &= 3(e^{3x} + bx + c) - 18x\\ &= 3e^{3x} + (3b - 18)x + 3c.\end{aligned}$$

Equating coefficients of x, $\quad 3b - 18 = 0 \quad (1)$

and equating constants, $\quad 3c = b, \quad (2)$

so $b = 6$ and $c = 2$, giving the solution $\quad y = 3e^{3x} + 6x + 2.$

b Substituting $y = 10\cos nx$ into the DE,

$$\begin{aligned}\text{LHS} &= \frac{d}{dx}(-10n\sin nx)\\ &= -10n^2\cos nx,\end{aligned} \qquad \begin{aligned}\text{RHS} &= -49 \times 10\cos nx\\ &= -490\cos nx.\end{aligned}$$

Hence $-10n^2 = -490$

$n = 7$ or $n = -7$,

giving the solutions $y = 10\cos 7x$ and $y = 10\cos(-7x)$,

which are the same function because cosine is an even function.

Solving the natural growth DE in a practical situation

This example demonstrates the formation of the differential equation and the procedure to solve it.

Example 27 13E

The rabbits on Bandicoot Island are increasing. Fifty years ago, 100 rabbits were released, and now there are 10 000 rabbits. Assume in this example that the rate of increase of rabbits is proportional to the number of rabbits.

a Find the equation for the population N at time t years after they were introduced.

b Find how many rabbits there will be in another:

i 25 years, **ii** 50 years.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

SOLUTION

a Writing the assumption about the rate of increase of rabbits as a DE,

$$\frac{dN}{dt} = kN, \text{ for some constant of proportionality } k.$$

First, the constant function $N = 0$ is trivially a solution of the DE.

$$\begin{aligned}
&\text{Otherwise, dividing by } N, & \frac{1}{N}\,dN &= k\,dt,\\
&\text{and integrating,} & \log|N| &= kt + C, \text{ for some constant } C,\\
&& |N| &= Ae^{kt}, \text{ where } A = e^{C} > 0,\\
&& N &= Ae^{kt}, \text{ where } A \text{ can be positive or negative.}\\
&\text{Because } N = 0 \text{ is a solution,} & N &= Ae^{kt}, \text{ for any constant } A.\\
&\text{When } t = 0,\ N = 100, \text{ so} & 100 &= A \times 1\\
&\text{so} & N &= 100e^{kt}.\\
&\text{When } t = 50,\ N = 10\,000, \text{ so} & 10\,000 &= 100e^{50k}\\
&& k &= \tfrac{1}{50}\log 100.\\
&\text{Hence} & N &= 100e^{kt}, \text{ where } k = \tfrac{1}{50}\log 100.
\end{aligned}$$

b

$$\begin{aligned}
&\text{Substituting } t = 75, & kt &= \frac{1}{50} \times \log 100 \times 75\\
&& &= 1.5\log 100\\
&\text{so population after another 25 years} & &= 100 \times e^{1.5\log 100}\\
&& &= 1\,000\,000 \text{ rabbits.}\\
&\text{Substituting } t = 100, & \text{kt} &= 2\log 100\\
&\text{so population after another 50 years} & &= 100 \times \text{e}^{2\log 100}\\
&& &= 1\,000\,000 \text{ rabbits.}
\end{aligned}$$

The logistic equation can model a limit to exponential growth

The rabbits on Bandicoot Island can't increase forever because there is a limit to the amount of food on the island, and there may be predators. The most straightforward model to limit that growth is the logistic equation discussed in the previous section.

Example 28 13E

It is estimated that the maximum carrying capacity of the Bandicoot Island is 20000 rabbits. Use the population values in the previous worked example, but model the population growth by the logistic equation

$$\frac{dN}{dt} = kN(P - N), \text{ where } P = 20000 \text{ is the maximum population.}$$

a Prove the partial fractions decomposition $\dfrac{1}{N(P - N)} = \dfrac{1}{P}\left(\dfrac{1}{N} + \dfrac{1}{P - N}\right)$.

b Find the equation for the population N at time t years after introduction.

c Find how many rabbits there will be in another:

i 25 years, **ii** 50 years.

d Comment on these results in comparison with the previous worked example.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

SOLUTION

a RHS $= \dfrac{1}{P} \times \dfrac{(P-N)+N}{N(P-N)} =$ LHS.

b First, the constant functions $N = 0$ and $N = P$ are trivially solutions.

Rearranging the DE, $\dfrac{1}{N(P-N)}\,dN = k,dt$

and by part **a**, $\left(\dfrac{1}{N} + \dfrac{1}{P-N}\right)dN = Pk\,dt.$

Integrating, $\log|N| - \log|P-N| = Pkt + C$, for some constant C,

$$\log\left|\frac{N}{P-N}\right| = Pkt + C$$

$$\left|\frac{N}{P-N}\right| = Ae^{Pkt},\ \text{where } A = e^C > 0,$$

$$\frac{N}{P-N} = Ae^{Pkt},\ \text{where } A \text{ can be positive or negative.}$$

Making N the subject,

$$N = APe^{Pkt} - ANe^{Pkt}$$

$$ANe^{Pkt} + N = APe^{Pkt}$$

$$N = \frac{APe^{Pkt}}{Ae^{Pkt}+1}.$$

Dividing through by Ae^{Pkt} and replacing A^{-1} by B,

$$N = \frac{P}{1+Be^{-Pkt}}.$$

Now we substitute $P = 20000$ and apply the initial conditions.

When $t = 0$, $N = 100$, so

$$100 = \frac{20\,000}{1+B}$$

$$1 + B = 200$$

so $B = 199$, and

$$N = \frac{P}{1+199e^{-Pkt}}$$

When $t = 50$, $N = 10000$, so

$$10000 = \frac{20000}{1+199e^{-20000\times k\times 50}}$$

$$1 + 199e^{-1\,000\,000k} = 2$$

$$e^{-1\,000\,000k} = \frac{1}{199}$$

$$1000000k = \log 199$$

$$k = \frac{\log 199}{1000000}.$$

Hence $N = \dfrac{P}{1+Be^{-Pkt}}$, where $P = 20\,000$, $B = 199$ and $k = \dfrac{\log 199}{1\,000000}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Substituting $t = 75$, $$-Pkt = -20000 \times \frac{\log 199}{1000000} \times 75$$
$$= -1.5 \log 199$$
so population after another 25 years $= \dfrac{20\,000}{1 + 199e^{-1.5 \log 199}}$
$\doteqdot 18676$ rabbits.

Substituting $t = 100$, $$-Pkt = -20000 \times \frac{\log 199}{1000000} \times 100$$
$$= -2 \log 199$$
so population after another 50 years $= \dfrac{20000}{1 + 199e^{-2 \log 199}}$
$\doteqdot 19900$ rabbits.

d The reduction in the predicted population using the logistic model is dramatic. In the previous worked example, the population increased as a GP, whereas in this worked example the population rises rapidly to approach the limit of 20000.

Notice that in this worked example, the value 50 years ago was 100, and the value in 50 years time is $19000 = 20000 - 100$. The function $N = \dfrac{P}{1 + Be^{-Pkt}}$ has point symmetry in the point $(50, 10\,000)$. See worked Example 24 in Section 13D.

Exercise 13E

FOUNDATION

1 **a** In each case find the values of a and b given that:
- **i** $y = ax^2 + bx$ is a solution of $y' = 1 - 4x$,
- **ii** $y = e^{-x}(a\cos x + b\sin x)$ is a solution of $y' = 2e^{-x}\sin x$,
- **iii** $y = ax + b + 3e^{-x}$ is a solution of $y' = x - y$.

b In each case find the values of a, b and c given that:
- **i** $y = ax^2 + bx + c + 4e^{-2x}$ is a solution of $y' + 2y = x^2 - 3x - 4$,
- **ii** $y = ax^2 + bx + c - \sin 2x$ is a solution of $y'' + 4y = x^2 + 5x$.

c Find the possible values of λ given that $y = 5e^{\lambda x}$ is a solution of $y'' + 5y' + 6y = 0$.

2 In a laboratory, a scientist has a sample of radioactive material. The material decays at a rate proportional to the amount present. That is,
$$\frac{dR}{dt} = kR,$$
where R is the amount present at time t days, and k is an unknown constant.

a Find the general solution of this DE.

b Initially there is 100 grams of the material. Determine the arbitrary constant in your solution.

c After 4 days only 20 grams of the substance remains radioactive. Determine the value of k.

d Hence determine the amount present after 12 days.

e Show that $R = 100 \times \left(\frac{1}{5}\right)^{\frac{1}{4}t}$, then use this formula to check part **d** mentally.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 A metal ingot is put in a fridge until its temperature is 5°C. The ingot is then taken out of the fridge and left in a room maintained at 25°C. Let H be the temperature of the ingot after t minutes. Experiments show that the rate of change of temperature over time is proportional to the difference in temperature between the ingot and the room. That is,

$$\frac{dH}{dt} = k(H - 25), \quad \text{for some constant } k.$$

a Find the general solution of this DE.
b Use the initial condition to determine the arbitrary constant.
c After 10 minutes the ingot is at 15°C. Determine the value of k.
d Find how long it takes, correct to the nearest minute, for the temperature to reach 24°C.

4 A conical tank with height 12 m and radius 4 m is filled with water. The water in the tank evaporates at a rate proportional to the circular surface area exposed to the air. Let r metres be the radius of the surface at time t hours. (Recall that the volume of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)
a Write down a differential equation for the evaporation in terms of the radius.
b Use part a and the chain rule to find a DE for the rate of change of the radius.
c Use the given information to solve this DE.
d After 6 hours the depth of the water in the tank is $10\frac{1}{2}$ m. Determine the value of k.
e Hence give a formula for the volume of water at time t. Note any restrictions on t.

5 A certain tank is in the shape of a cylinder. It is filled with water to a height of 400 cm. A tap at the bottom of the tank is opened and, as the water empties, the rate of change of height of water in the tank is proportional to the square root of the height. That is:

$$\frac{dh}{dt} = k\sqrt{h}.$$

a State the initial condition, and explain why the constant k must be negative.
b Solve the IVP. You may assume that the arbitrary constant of integration is positive.
c After 20 minutes the height of water in the tank is 100 cm. Find the value of k.
d How long does the tank take to drain?
e Is the function you found in part b valid for larger values of t?

6 A certain curve has the property that the tangent at any point passes through the origin.
a Write down the gradient of the line from $(0, 0)$ to the point (x, y).
b Now suppose that (x, y) is on this curve. Write down a differential equation for this curve.
c Hence determine the general equation of this curve.
d Which special case is a solution of the problem, but not of the DE?

7 A certain curve has the property that the normal at any point passes through the origin.
a Write down the gradient of the line from $(0, 0)$ to the point (x, y).
b Now suppose that (x, y) is on this curve. Write down a differential equation for this curve.
c Hence determine the general equation of this curve, which is a relation and not a function.

8 A tangent is drawn to a curve, and it is found that its x-intercept is 1 less than the x-coordinate of the point of contact.
a Write down the gradient of the line from $(x - 1, 0)$ to the point (x, y).
b Now suppose that (x, y) is on this curve. Write down a differential equation for this curve.
c Hence determine the general equation of this curve.
d Find such a curve passing through $(0, 1)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

9 The atmospheric pressure P on the planet Nebula changes with altitude h at a rate proportional to P. That is,

$$\frac{dP}{dh} = kP, \quad \text{for some constant } k.$$

Let the atmospheric pressure at ground level be $P = P_0$.

a Find the general solution of this DE.

b Measurements from satellites orbiting the planet show that the pressure at 10 000 m is 40 kPa, and the pressure at 6000 m is 80 kPa. Find the value of k.

c Determine the pressure at ground level correct to the nearest kPa.

10 A tangent to a curve intersects the coordinate axes at A and B. It is found that the point of contact of the tangent is also the mid-point of AB.

a Let the point of contact with the curve be (x, y). Find the coordinates of A and B in terms of x and y.

b Use the gradient of AB to determine a differential equation for this curve.

c Hence determine the general equation of this curve.

11 According to Fick's law, diffusion across a cell membrane is governed by a differential equation. If $C(t)$ is the concentration of a solute in a cell, and S is the concentration of the solute in the surrounding medium, then

$$\frac{dC}{dt} = k(S - C), \quad \text{for some constant } k.$$

a Find the general solution of this DE.

b Suppose that initially $C(0) = C_0$, where $C_0 < S$. Solve the IVP.

12 In order to save an endangered species, it has been decided to release 40 animals on an island where there are no predators. The maximum number that can survive on the island is called the *carrying capacity*, which is 1000. It is assumed that the population N of these animals at time t years after release fits the logistic growth equation

$$\frac{dN}{dt} = kN(1000 - N), \quad \text{for some constant } k.$$

a Show that $\dfrac{1000}{N(1000 - N)} = \dfrac{1}{N} + \dfrac{1}{1000 - N}$.

b Use the result of part a to find the general solution of the logistic growth equation.

c Determine the arbitrary constant by applying the initial condition $N(0) = 40$, then simplify the function.

d Given that the population of animals after 1 year was 80, find the value of k correct to four significant figures.

e What will the population be after 5 years, correct to the nearest whole number?

13 In the mid-1800s, Verhulst estimated the population growth of the United States of America using the logistic differential equation

$$\dot{N} = kN(P - N), \quad \text{for some positive constant } k,$$

where N is the population in millions, and t is the number of years after 1850.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

a Show that $\dfrac{P}{N(P-N)} = \dfrac{1}{N} + \dfrac{1}{P-N}$.

b Use the result of part **a** to find the general solution of the logistic growth equation.

c It was estimated at the time that the carrying capacity of the USA was $P = 187.5$, and the population in 1850 was recorded as $N(0) = 23.2$, both in units of millions. Determine the arbitrary constant and simplify the function.

d The estimate used for k was $k = 1.6 \times 10^{-4}$, which predicted a population of 59.8 million in 1890. The actual population of the USA in 1890 was 63.0 million. Calculate a new value for k.

e Using the revised value of k, compare the predicted population for 1930 using this model with the actual population, which was 123.2 million.

f The population of the USA in 2018 was approximately 327 million. Comment on this value.

14 Once again consider the logistic initial value problem,

$$\frac{dN}{dt} = kN(P - N), \text{ with } N(0) = N_0.$$

a Use the result of Question **12a** to find the general solution of the logistic DE.

b Apply the initial condition, and hence show that $N = \dfrac{N_0 P}{N_0 + (P - N_0)e^{-kPt}}$.

c Suppose that the population is N_1 when $t = t_1$. Find a formula for k.

d Now suppose that $t_1 = 1$ and that $N(2) = N_2$. Find a quadratic equation for P with coefficients that only involve the values N_0, N_1 and N_2.

15 Biologists are modelling the population of an endangered species of fish in a river system. The indigenous population of the area are permitted to harvest 200 fish once a year in January as part of their culture. Data collected on a recent field trip in December indicate that there are 500 fish in the river system. The data also suggest that the following mathematical model should be used,

$$\frac{dy}{dt} = -2 + \tfrac{1}{24}y(16 - y),$$

where y is the population of fish in the river measured in hundreds at time t years after the next harvest.

a The term $\frac{1}{24}y(16 - y)$ on the right-hand side represents the familiar logistic growth model. What is the significance of the -2 in this equation?

b Show that the DE can be re-written as $\dfrac{dy}{dt} = -\frac{1}{24}(y - 4)(y - 12)$, and write down the initial condition assuming that the harvest goes ahead.

c Show that $\dfrac{24}{(y-4)(y-12)} = \dfrac{3}{y-12} - \dfrac{3}{y-4}$.

d Hence solve the IVP in part **b**.

e According to this model, the fish in the river will die out. When will that be?

f **i** If the most recent harvest had not occurred, what would the initial condition change to?

ii It can be shown that the solution of the DE for this initial condition is

$$y = \frac{4\left(3 + 7e^{-\frac{1}{3}t}\right)}{1 + 7e^{-\frac{1}{3}t}}.$$

Investigate what happens for this solution over time. Comment on the result.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 **a** Consider the integral $I = \displaystyle\int \frac{dx}{x \log x}$.

i Use the substitution $u = \log x$ to simplify this integral.

ii Determine the integral for u and then back-substitute to find the original integral.

b In studying the survival of a population after an epidemic, Gompertz proposed the following alternative differential equation for population growth

$$\frac{dN}{dt} = kN \log N,$$

where N is the population at time t. Use the results of part **a** to solve this differential equation.

ENRICHMENT

17 A tank initially holds 100 litres of a solution of water and 200 g of a radioactive substance. Water flows into one end of the tank through a pipe at a rate of 5 litres per minute and mixes with the solution. A pipe at the other end of the tank allows the mixed solution to flow out at the same rate. Each minute the radioactive substance decays at a rate of 0.1 times the amount present. Form and solve a differential equation for the situation and hence find a formula for M, the amount of radioactive material in the tank at time t.

18 An object is heated to 100°C and then placed in a room at 20°C. Let h be the temperature of the object after t minutes. The rate of change of temperature is proportional to its difference from the room temperature. That is,

$$\frac{dh}{dt} = k(h - 20), \text{ for some constant } k.$$

After 10 minutes the temperature of the object is 80°C.

a Solve the initial value problem and find the value of the constant k.

b At the 10-minute mark, refrigeration equipment is turned on, which lowers the temperature in the room by 1°C/min. You may assume that the rate of change of temperature continues to be proportional to the difference, and that the value of k is unchanged. Let $H(t)$ be the temperature of the object t minutes after the refrigeration is turned on.

i Write down the initial value problem for $H(t)$ in terms of k.

ii Let $y = H - 20 + t$. What is the corresponding IVP for y?

iii Find y, and hence determine $H(t)$.

19 Often mathematicians try to simplify a problem by removing constants and parameters from a differential equation. Here the logistic equation in Question **12** will be simplified to one such as those investigated in Section 13D. Let the IVP be

$$\frac{dN}{dt} = kN(P - N), \text{ with } N(0) = N_0.$$

a Put $N = P_y$, where y is an unknown function of t. Also put $r = kP$, and hence determine the corresponding IVP in terms of y, t and r.

b Now put $x = rt$ to obtain a differential equation in terms of y and x alone. What is the new initial condition?

c Next make the substitution $v = \dfrac{1}{y}$ to get a differential equation in v.

d Find the general solution for v, and hence write down the corresponding solution for y.

e Apply the initial condition to evaluate the arbitrary constant, then simplify y.

f Hence determine the solution of the original IVP.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

20 You have shown several times that the DE $y' = ry(1 - y)$, with initial condition $y(0) = \frac{1}{2}$, has solution $y = \dfrac{1}{1 + e^{-rx}}$.

a Draw the graphs for $r = 1,\ 2,\ 4$ and 8.

b For each fixed value of x, find $\lim\limits_{r \to \infty} y$ for:

i $x > 0$, **ii** $x = 0$, **iii** $x < 0$.

c Sketch the resulting function $u(x)$. This is called the *Heaviside step function*, and it has many applications. Electrical engineers use it as an ideal switch, and it models the quantum steps that occur in quantum mechanics.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 13 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 13 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 In each part, state the order of the DE, whether or not it is linear, and how many arbitrary constants will appear in the general solution.

a $y' + xy = \cos x$ **b** $y'' + x^2y' - 3y = e^x \sin x$ **c** $y''' + y''y' = x - 2$

2 Consider the DE $xy' = y(1 - x^2)$.

a Make y' the subject of the equation.

b Copy and complete the table of values for the slope field.

c Draw a number plane with a scale of 2 cm = 1 unit, with domain $[-2, 2]$ and range $[-2, 2]$.

d Through each grid point corresponding to the table, draw a line element $\frac{1}{2}$ cm long, centred on the point and with gradient as given in the table.

e Look carefully at the table for matching entries. Check that these agree with the isoclines in your graph.

y \ x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	3	$\frac{4}{3}$	0	-3	$*$				
$\frac{3}{2}$	$\frac{9}{4}$	$\frac{5}{4}$	0						
1									
$\frac{1}{2}$									
0									
$-\frac{1}{2}$									
-1									
$-\frac{3}{2}$									
-2									

f Look carefully at your graph. What asymptote seems to be suggested? Does this agree with any constant solutions?

g What symmetry seems to be suggested by the slope field? How is this evident in the equation for y' and in the table?

h In this instance, every solution curve passes through the origin. Add the integral curves (solution curves) that pass through $\left(1, -\frac{1}{2}\right)$, $(1, 1)$ and $(1, 2)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

3 Show by substitution that the given function with arbitrary constant C is a general solution of the accompanying differential equation.

a $y = Cx^2e^x$,
$xy' = y(2 + x)$

b $y = \sqrt{x^2 + C}$,
$y'y = x$

c $y = \dfrac{1}{x^2 + C}$,
$y' = -2xy^2$

4 Consider the differential equation $\dfrac{dy}{dx} = -\frac{1}{2}y$.

a What is the constant solution of this equation?

b Write down the DE obtained by taking the reciprocal of both sides.

c Use direct integration to obtain x as a function of y.

d Make y the subject of this solution, simplifying the constant part of the expression.

e Check that the constant solution is included in your answer to part **d**.

f Find the solution for the initial condition $y(0) = 3$.

5 In each case draw the slope field for the given DE using appropriate technology. Then **i** identify any points or isoclines where $y' = 0$, **ii** state how the gradients of the line elements change along the line $x = 1$, from top to bottom, and **iii** state how the gradients of the line elements change along the line $y = 1$, from left to right. **iv** Finally draw three appropriate solution curves.

a $y' = \frac{1}{4}(x^2 - 4)$ **b** $y' = \frac{1}{8}(y^2 - 4)$ **c** $y' = \frac{1}{2}(x + y)$

6 Verify that the given function is a general solution of the differential equation for all values of the constants A, B, C and D.

a $y = Ae^{-x} + Be^{-2x}$,
$y'' + 2y' + y = 0$

b $y = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$,
$y'' + 2y' + 5y = 0$

c $y = A\cos x + B\sin x + Ce^{2x}$,
$y''' - 2y'' + y' - 2y = 0$

d $y = Ae^{2x} + Be^{-2x} + C\cos 2x + D\cos 2x$
$y''' - 16y = 0$

DEVELOPMENT

7 Use separation of variables to solve each differential equation. Note that the solution of part **b** is a relation that is not a function.

a $\dfrac{dy}{dx} = \dfrac{-2xy}{1 + x^2}$ **b** $\dfrac{dy}{dx} = \dfrac{1 - x}{2 + y}$ **c** $\dfrac{dy}{dx} = \dfrac{y(x - 1)}{x}$

8 Solve each IVP by taking the reciprocal and using direct integration.

a $y' = \frac{1}{2}(1 - y)$, with $y(0) = 2$ **b** $y' = \frac{1}{5}(5 - y)$, with $y(0) = 2$

9 **a** Show that $\dfrac{1}{1 - x^2} = \dfrac{1}{2}\left(\dfrac{1}{1 + x} + \dfrac{1}{1 - x}\right)$.

b Hence solve $\dfrac{dy}{dx} = \dfrac{y}{1 - x^2}$ by separation of variables.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

10 Let $L(x) = \dfrac{1}{1 - e^{-x}}$ and consider the curve $y = L(x)$.

a Determine the domain and any intercepts.

b Determine $\lim\limits_{x \to \infty} L(x)$ and $\lim\limits_{x \to -\infty} L(x)$.

c Explain why there is a vertical asymptote at $x = 0$ and investigate the behaviour of function on either side.

d Evaluate $y = L(x)$ at $x = \log 2$ and at $x = -\log 2$.

e **i** Show that $L'(x) = \dfrac{-1}{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2}$.

ii Use part **i** to determine $L''(x)$.

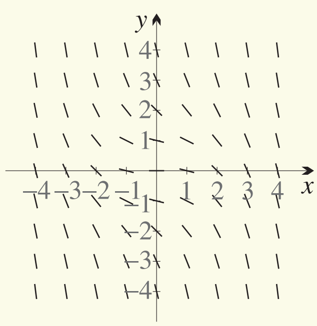
f Hence determine the concavity of the graph of $y = L(x)$.

g Sketch $y = L(x)$, showing all these features.

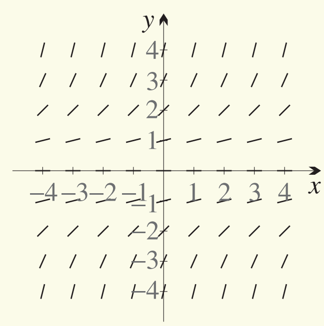
h Show by substitution that $y = L(x)$ is a solution of the logistic DE $y' = y(1 - y)$.

11 Which of the slope fields shown below corresponds to the DE $y' = \frac{1}{4}(x^2 + y^2)$?

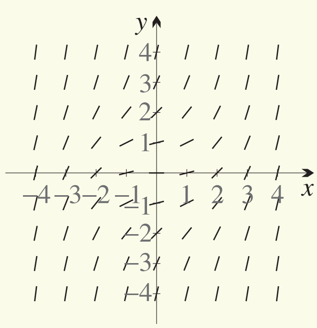
A



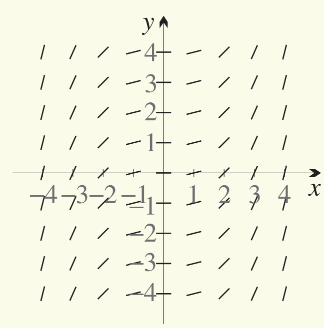
B



C



D



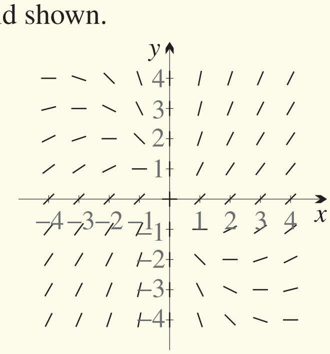
12 Determine which of the DEs corresponds to the slope field shown.

A $y' = 1 + \dfrac{x}{y}$

B $y' = 1 + \dfrac{y}{x}$

C $y' = 1 - \dfrac{x}{y}$

D $y' = 1 - \dfrac{y}{x}$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

13 Use separation of variables to solve initial value problem.

a $\dfrac{dy}{dx} = -\dfrac{y}{x}$, with $y(2) = 1$

b $\dfrac{dy}{dx} = (1 + 2x)e^{-y}$, with $y(1) = 0$

c $\dfrac{dy}{dx} = \dfrac{y^2}{\sqrt{x}}$, with $y(0) = -1$

14 a Show that $\dfrac{1}{y(1 - y)} = \dfrac{1}{y} + \dfrac{1}{(1 - y)}$.

b Consider the logistic differential equation $y' = y(1 - y)$.

i What are the constant solutions of this equation?

ii Use part **a** to find the general solution of the logistic DE.

iii Hence find the solution that passes through $\left(0, \frac{1}{4}\right)$.

15 a Show that $\dfrac{1}{(2 - y)(3 - y)} = \dfrac{1}{y - 3} - \dfrac{1}{y - 2}$.

b Consider the harvest logistic differential equation $y' = -\frac{1}{5}(3 - y)(2 - y)$.

i What are the constant solutions of this equation?

ii Use part **a** to find the general solution of the harvest logistic DE with initial condition $y(0) = 1$.

iii For what value of x does $y = 0$?

16 A company selling mobile phones has been using the logistic differential equation to model sales over the last two years. That is

$$\frac{dN}{dt} = kN(5 - N)$$

where N is the number of people in millions who bought a mobile phone t years after the company began tracking sales with this mathematical model. The value of the constant k is unknown.

a Show that $\dfrac{5}{N(5 - N)} = \dfrac{1}{N} + \dfrac{1}{5 - N}$.

b Use the result of part **a** to find the general solution of the logistic growth equation.

c When the company started using this model, they had already sold 1 million phones. Determine the arbitrary constant, and simplify your solution.

d It is now 2 years since the company started tracking phone sales in this way, and the company has sold 400 000 phones in that time. Find the value of k correct to four significant figures.

e According to this model, what will the projected sales in the next year be, correct to the nearest thousand?

17 Let the function $y = f(x)$ be a solution of IVP $y' = x(1 - 2y)$, with $y(0) = 1$.

a Differentiate the given DE, and hence find a formula for y''.

b Hence show that $y = f(x)$ has a maximum turning point at $x = 0$.

c Now solve the IVP by separation of variables to confirm your answers.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

18 a A differential equation of the form $y' = f(x)$ is solved. Explain how the solutions are related by a simple transformation.

b Likewise, a differential equation of the form $y' = g(y)$ is solved. You may assume that the general solution of the DE is a function and not a relation. Explain how the solutions are related by a simple transformation.

ENRICHMENT

19 The DE $(y')^2 = 1 - y^2$ appears to have two distinct general solutions, $y_1 = \cos(x + A)$ and $y_2 = \sin(x + B)$, as well as constant solutions.

a Determine the constant solutions of this DE.

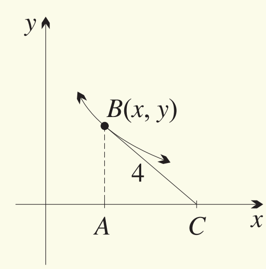
b Show that both y_1 and y_2 satisfy the DE.

c Expand the trigonometric functions in y_1 and y_2, and hence show that these two solutions are in fact identical apart from the values of the constants A and B.

d Determine the relationship between A and B.

20 [Tractrix]

A child pulls a box of toys over a smooth floor using a string of length 4 m. When the obvious coordinate system is applied, the child is initially at the origin, and the box is at $(0, 4)$. It is noted that as the child moves along the x-axis, the string is always tangent to the unknown path followed by the box of toys. Thus when the child is at C and the box is at $B(x, y)$, the line BC is tangent to the curve and $|BC| = 4$. The situation is shown in the diagram to the right.



a Use $\triangle ABC$ to show that $\dfrac{dy}{dx} = \dfrac{-y}{\sqrt{16 - y^2}}$.

b Note any restrictions on y and state the initial condition.

c State the constant solution of the DE. Is it a solution of the initial value problem?

d This IVP cannot be solved using the techniques in this course. Nevertheless a solution curve, called a *tractrix*, can be drawn as follows.

i Draw the slope field for this DE.

ii Add the solution curve that corresponds to the initial condition given.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 Series and finance

Chapter 1 introduced sequences and series, principally arithmetic sequences (or APs) and geometric sequences (or GPs). The treatment there was mostly theoretical, because the intention was to give a wider mathematical context for linear and exponential functions, the derivative, and the definite integral.

The first two sections of this chapter review sequences and series, with particular attention to the use of logarithms, and apply them to many more practical problems. The remaining three sections deal entirely with the role of sequences and series in financial situations — simple and compound interest, depreciation and inflation, superannuation, and paying off a loan.

Readers may or may not need the review of the theory in the first two sections, but the applications are new and need attention. The large number of questions in the financial sections are a result of the variety of ways in which questions can be asked — there are too many for a first encounter, and many of them could be left for later revision.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14A Applications of APs and GPs

This section and the next will review the main results about APs and GPs from Chapter 1 and apply them to a variety of problems, in preparation for the later sections on finance. Section 14B is particularly concerned with the use of logarithms in solving the exponential equations that arise when working with GPs.

Formulae for arithmetic sequences

Here are the essential definitions and formulae that are needed for problems involving APs.

1 ARITHMETIC SEQUENCES

- A sequence T_n is called an *arithmetic sequence* or *AP* if the difference between successive terms is constant. That is,

  $$T_n - T_{n-1} = d, \text{ for all } n \geq 2,$$

  where d is a constant, called the *common difference*.
- The nth term of an AP with first term a and common difference d is

  $$T_n = a + (n - 1)d.$$
- Three numbers a, x and b are in AP if

  $$b - x = x - a, \qquad \text{that is,} \qquad x = \frac{a + b}{2}.$$
- The sum S_n of the first n terms of an AP is

  $$S_n = \tfrac{1}{2}n(a + \ell) \qquad \text{(use when the last term } \ell = T_n \text{ is known),}$$

  or $$S_n = \tfrac{1}{2}n\big(2a + (n - 1)d\big) \qquad \text{(use when the difference } d \text{ is known).}$$

The word *series* is usually used when we are adding up the terms of a sequence. Typically, we will refer to

the sequence 1, 3, 5, . . . and the series $1 + 3 + 5 + \cdots$.

Example 1 [Salaries and APs] 14A

Georgia earned \$50 000 in her first year at Information Holdings, and her salary then increased every year by \$6000. She worked at the company for 12 years.

a What was her annual salary in her final year?

b What were her total earnings over the 12 years?

SOLUTION

Her annual salaries form a series, $50000 + 56000 + 62000 + \cdots$ with 12 terms.
This is an AP with $a = 50000$, $d = 6000$ and $n = 12$.

a Her final salary is the twelfth term T_{12} of the series.

$$\begin{aligned} \text{Final salary} &= a + 11d \qquad \text{(using the formula for } T_{12}\text{)} \\ &= 50\,000 + 66\,000 \\ &= \$116\,000. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

b Her total earnings are the sum S_{12} of the first twelve terms of the series.

Using the first formula for S_n,

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2}n(a + \ell)\\ &= \tfrac{1}{2} \times 12 \times (a + \ell)\\ &= 6 \times (50000 + 116000)\\ &= \$996000.\end{aligned}$$

Using the second formula for S_n,

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2}n\big(2a + (n - 1)d\big)\\ &\tfrac{1}{2} \times 12 \times (2a + 11d)\\ &= 6 \times (100000 + 66000)\\ &= \$996000.\end{aligned}$$

Counting when the years are named

Problems in which events happen in particular named years are notoriously tricky. The following problem becomes clearer when the years are stated in terms of 'years after 2005'.

Example 2 — 14A

Gulgarindi Council is very happy. It had 2870 complaints in 2006, but only 2170 in 2016. The number of complaints decreased by the same amount each year.

a What was the total number of complaints during these years?

b By how much did the number of complaints decrease each year?

c If the trend continued, in what year would there be no complaints at all?

SOLUTION

The first year is 2006, the second year is 2007, and the 11th year is 2016.
In general, the nth year of the problem is the nth year after 2005.
The successive numbers of complaints form an AP with $a = 2870$, $\ell = 2170$ and $n = 11$.

a $$\begin{aligned}\text{Total number of complaints} &= \tfrac{1}{2}n(a + \ell) \quad \text{(using the first formula for } S_n\text{)}\\ &= \tfrac{1}{2} \times 11 \times (2870 + 2170)\\ &= 27720.\end{aligned}$$

b Put
$$\begin{aligned}T_{11} &= 2170\\ a + 10d &= 2170 \qquad \text{(using the formula for } T_{11}\text{)}\\ 2870 + 10d &= 2170\\ 10d &= -700\\ d &= -70.\end{aligned}$$

Hence the number of complaints decreased by 70 each year.

c $$\begin{aligned}\text{The number of complaints is } T_n &= a + (n - 1)d \qquad \text{(using the formula for } T_n\text{)}\\ &= 2870 - 70(n - 1)\\ &= 2940 - 70n.\end{aligned}$$

Put
$$\begin{aligned}T_n &= 0 \qquad \text{(to find the year in which there are no complaints)}\\ 2940 - 70n &= 0\\ 70n &= 2940\\ n &= 42.\end{aligned}$$

Thus there would be no complaints at all in the year $2005 + 42 = 2047$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Formulae for geometric sequences

The formulae for GPs correspond roughly to the formulae for APs, except that the formula for the limiting sum of a GP has no analogy for arithmetic sequences.

2 GEOMETRIC SEQUENCES

- A sequence T_n is called a *geometric sequence* if the ratio of successive terms is constant. That is,
$$\frac{T_n}{T_{n-1}} = r, \text{ for all } n \geq 2,$$
where r is a constant, called the *common ratio*.
- The nth term of a GP with first term a and common ratio r is
$$T_n = ar^{n-1}.$$
- Neither the ratio nor any term of a GP can be zero.
- Three numbers a, x and b are in GP if
$$\frac{b}{x} = \frac{x}{a}, \qquad \text{that is,} \qquad x^2 = ab.$$
- The sum S_n of the first n terms of a GP is
$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \text{(easier when } r > 1\text{)},$$
$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \text{(easier when } r < 1\text{)}.$$
- The limiting sum S_∞ exists if and only if $-1 < r < 1$, that is, $|r| < 1$, and in this case,
$$S_\infty = \frac{a}{1 - r}, \qquad \text{that is,} \qquad \lim_{n\to\infty} S_n = \frac{a}{1 - r}.$$

Example 3 [An example with $r < 1$] 14A

Sales from the Gumnut Softdrinks Factory in the mountain town of Wadelbri are declining by 6% every year. In 2016, 50000 bottles were sold.

a How many bottles will be sold in 2025?

b How many bottles will be sold in total in the years 2016–2025?

SOLUTION

Here 2016 is the first year, 2017 is the second year, . . . , and 2025 is the 10th year.
The annual sales form a GP with $a = 50000$ and $r = 0.94$.

a The sales in 2025 are the 10th term T_{10}, because 2016–2025 consists of 10 years.

$$\begin{aligned} \text{Sales in 2025} &= ar^9 \\ &= 50\,000 \times 0.94^9 && \text{(using the formula for } T_{10}\text{)} \\ &\doteqdot 28\,650 && \text{(correct to the nearest bottle).} \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Total sales $= \dfrac{a(1-r^{10})}{1-r}$ (using the second formula for S_{10})

$= \dfrac{50\,000 \times (1-0.94^{10})}{0.06}$

$\doteqdot 384\,487$ (correct to the nearest bottle).

Limiting sums

If the ratio of a GP is between -1 and 1, that is, $0 < |r| < 1$, then the sum S_n of the first n terms of the GP converges to the limit $S_\infty = \dfrac{a}{1-r}$ as $n \to \infty$. In applications, this allows us to speak about the sum of the terms 'eventually', or 'as time goes on'.

Example 4 — 14A

Consider again the Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year and 50000 bottles were sold in 2016. Suppose now that the company continues in business indefinitely.

a What would the total sales from 2016 onwards be eventually?

b What proportion of those sales would occur by the end of 2025?

SOLUTION

The sales form a GP with $a = 50000$ and $r = 0.94$.
Because $-1 < r < 1$, the limiting sum exists.

a Eventual sales $= S_\infty$

$= \dfrac{a}{1-r}$

$= \dfrac{50\,000}{0.06}$

$\doteqdot 833\,333$ (correct to the nearest bottle).

b Using the results from part **a**, and from the previous worked example,

$\dfrac{\text{sales in 2016–2025}}{\text{eventual sales}} = \dfrac{50\,000 \times (1-0.94^{10})}{0.06} \times \dfrac{0.06}{50\,000}$ (using the exact values)

$= 1 - 0.94^{10}$

$\doteqdot 46.14\%$ (correct to the nearest 0.01%).

Exercise 14A — FOUNDATION

Note: The theory for this exercise was discussed in Chapter 1 and is reviewed in this section. The exercise is a medley of problems on APs and GPs, with two introductory questions to revise the formulae for APs and GPs.

1 **a** Five hundred terms of the series $102 + 104 + 106 + \cdots$ are added. What is the total?

b In a particular arithmetic series, there are 48 terms between the first term 15 and the last term -10. What is the sum of all the terms in the series?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c **i** Show that the series $100 + 97 + 94 + \cdots$ is an AP, and find the common difference.

ii Show that the nth term is $T_n = 103 - 3n$, and find the first negative term.

iii Find an expression for the sum S_n of the first n terms, and show that 68 is the minimum number of terms for which S_n is negative.

2 **a** The first few terms of a particular series are $2000 + 3000 + 4500 + \cdots$.

i Show that it is a geometric series, and find the common ratio.

ii What is the sum of the first five terms?

iii Explain why the series does not converge.

b Consider the series $18 + 6 + 2 + \cdots$.

i Show that it is a geometric series, and find the common ratio.

ii Explain why this geometric series has a limiting sum, and find its value.

iii Show that the limiting sum and the sum of the first ten terms are equal, correct to the first three decimal places.

3 A secretary starts on an annual salary of \$60 000, with annual increments of \$4000.

a Use the AP formulae to find his annual salary, and his total earnings, at the end of 10 years.

b In which year will his salary be \$84 000?

4 An accountant receives an annual salary of \$80 000, with 5% increments each year.

a Show that her annual salary forms a GP and find the common ratio.

b Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.

5 **a** What can be said about the terms of an AP in which:

i the common difference is zero,

ii the common difference is negative?

b Why can't the common ratio of a GP be zero?

c What can be said about the terms of a GP with common ratio r in which:

i $r < 0$, **ii** $r = 1$, **iii** $r = -1$, **iv** $0 < |r| < 1$?

6 Lawrence and Julian start their first jobs on low wages. Lawrence starts at \$50 000 per annum, with annual increases of \$5000. Julian starts at the lower wage of \$40 000 per annum, with annual increases of 15%.

a Find Lawrence's annual wages in each of the first three years and explain why they form an arithmetic sequence.

b Find Julian's annual wages in each of the first three years and explain why they form a geometric sequence.

c Show that the first year in which Julian's annual wage is the greater of the two will be the sixth year, and find the difference, correct to the nearest dollar.

7 **a** An initial salary of \$50 000 increases by \$3000 each year.

i Find a formula for T_n, the salary in the nth year.

ii In which year will the salary first be at least twice the original salary?

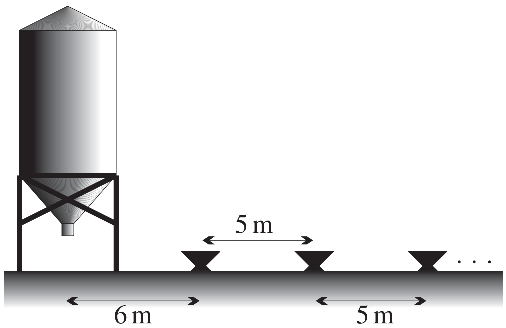
b An initial salary of \$50 000 increases by 4% each year. What will the salary be in the tenth year, correct to the nearest dollar?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

8 A farmhand is filling a row of feed troughs with grain. The distance between adjacent troughs is 5 metres, and the silo that stores the grain is 6 metres from the closest trough. He decides that he will fill the closest trough first and work his way to the far end. (He can only carry enough grain to fill one trough with each trip.)



a How far will the farmhand walk to fill the 1st trough and return to the silo? How far for the 2nd trough? How far for the 3rd trough?

b How far will the farmhand walk to fill the nth trough and return to the silo?

c If he walks a total of 62 metres to fill the furthest trough:

i how many feed troughs are there,

ii what is the total distance he will walk to fill all the troughs?

9 One Sunday, 120 days before Christmas, Aldsworth store publishes an advertisement saying ‘120 shopping days until Christmas’. Aldsworth subsequently publishes similar advertisements every Sunday until Christmas.

a How many times does Aldsworth advertise?

b Find the sum of the numbers of days published in all the advertisements.

c On which day of the week is Christmas?

10 The number of infections in an epidemic rose from 10000 on 1st July to 160000 on 1st of September.

a If the number of infections increased by a constant difference each month, what was the number of infections on 1st August?

b If the number of infections increased by a constant ratio each month, what was the number of infections on 1st August?

11 Theodor earns \$60000 in his first year of work, and his salary increases each year by a fixed amount \$$D$.

a Find D if his salary in his tenth year is \$117600.

b Find D if his total earnings in the first ten years are \$942000.

c If $D = 4400$, in which year will his salary first exceed \$120000?

d If $D = 4000$, show that his total earnings first exceed \$1200000 during his 14th year.

12 Margaret opens a hardware store. Sales in successive years form a GP, and sales in the fifth year are half the sales in the first year. Find the total sales of the company as time goes on, as a multiple of the first year’s sales \$$F$, correct to two decimal places.

13 [Limiting sums of trigonometric series]

a Consider the series $1 - \tan^2 x + \tan^4 x - \cdots$, where $0 < |x| < \frac{\pi}{2}$.

i For what values of x does the series converge?

ii What is the limit when it does converge?

iii What happens when $x = 0$?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Consider the series $1 + \cos^2 x + \cos^4 x + \cdots$.

i Show that the series is a GP, and find its common ratio.

ii For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?

iii Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \operatorname{cosec}^2 x$.

iv We omitted a qualification. What happens when $\cos x = 0$?

c Consider the series $1 + \sin^2 x + \sin^4 x + \cdots$.

i Show that the series is a GP, and find its common ratio.

ii For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?

iii Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \sec^2 x$.

iv We omitted a qualification. What happens when $\sin x = 0$?

14 Two bulldozers are sitting in a construction site facing each other. Bulldozer A is at $x = 0$ and bulldozer B is 36 metres away at $x = 36$. There is a bee sitting on the scoop at the very front of bulldozer A.

At 7:00 am the workers start up both bulldozers and start them moving towards each other at the same speed V m/s. The bee is disturbed by the commotion and flies at twice the speed of the bulldozers to land on the scoop of bulldozer B.

a Show that the bee reaches bulldozer B when it is at $x = 24$.

b Immediately the bee lands, it takes off again and flies back to bulldozer A. Where is bulldozer A when the two meet?

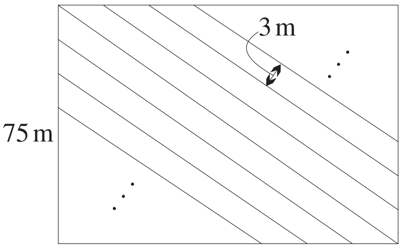
c Assume that the bulldozers keep moving towards each other and the bee keeps flying between the two, so that the bee will eventually have three feet on each bulldozer.

i Where will this happen?

ii How far will the bee have flown?

ENRICHMENT

15 The area available for planting in a particular paddock of a vineyard measures 100 metres by 75 metres. In order to make best use of the sun, the grape vines are planted in rows diagonally across the paddock, as shown in the diagram, with a 3-metre gap between adjacent rows.



a What is the length of the diagonal of the field?

b What is the length of each row on either side of the diagonal?

c Confirm that each row two away from the diagonal is 112.5 metres long.

d Show that the lengths of these rows form an arithmetic sequence.

e Hence find the total length of all the rows of vines in the paddock.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14B The use of logarithms with GPs

None of the exercises in the previous section asked about the number of terms in a given GP. Such questions require either trial-and-error or logarithms.

Trial-and-error may be easier to understand, but it is a clumsy method when the numbers are larger. Logarithms provide a better approach, but require understanding of the relationship between logarithms and indices.

Solving exponential inequations using trial-and-error

The next worked example shows how to solve such an equation using trial-and-error. Notice that:

- The powers of 3 get bigger because 3 is greater than 1,

$$3^0 = 1,\ 3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ \ldots$$

- The powers of 0.95 get smaller because 0.95 is smaller than 1,

$$0.95^0 = 1,\ 0.95^1 = 0.95,\ 0.95^2 = 0.903,\ 0.95^3 = 0.857,\ 0.95^4 = 0.815,\ \ldots$$

Example 5 **14B**

Use trial-and-error on your calculator to find the smallest integer n such that:

a $3^n > 400\,000$

b $0.95^n < 0.01$

SOLUTION

a Using the function labelled $\boxed{x^y}$

$$3^{11} = 177\,147$$
and $3^{12} = 531\,441$,

so the smallest such integer is 12.

b Using the function labelled $\boxed{x^y}$

$$0.95^{89} = 0.010\,408\ldots$$
and $0.95^{90} = 0.009\,888\ldots$,

so the smallest such integer is 90.

Note: In practice, quite a few more trial calculations are usually needed in order to trap the given number between two integer powers.

Solving exponential inequations using logarithms

To solve an exponential inequation using logarithms, the corresponding equation must first be solved. To do this:

- Convert the exponential equation to a logarithmic equation.
- Convert to logarithms base e or base 10 using the change-of-base formula

$$\log_b x = \frac{\log_e x}{\log_e b}$$ 'The log of the number over log of the base.'

Logarithms base e or base 10 can be used.

An alternative approach is to take logarithms base e or base 10 of both sides of the exponential equation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 6 14B

Use logarithms to find the smallest integer n such that:

a $3^n > 400000$ **b** $0.95^n < 0.01$

SOLUTION

a Put $3^n = 400\,000$ (the corresponding equation)

Then $n = \log_3 400\,000$.

$$n = \frac{\log_e 400\,000}{\log_e 3} \quad \text{(change-of-base formula)}$$

$$= 11.741\ldots.$$

Thus the smallest such integer is 12, because $3^{11} < 400\,000$ and $3^{12} > 400\,000$.

b Put $0.95^n = 0.01$.

Then $n = \log_{0.95} 0.01$

$$= \frac{\log_e 0.01}{\log_e 0.95} \quad \text{(change-of-base formula)}$$

$$= 89.781\ldots.$$

Thus the smallest such integer is 90, because $0.95^{89} > 0.01$ and $0.95^{90} < 0.01$.

3 SOLVING EXPONENTIAL INEQUATIONS

To solve an exponential inequation such as $3^n > 400000$ or $0.95^n < 0.01$:

- The first approach is to use trial-and-error with the calculator.
- The second approach is to use logarithms base e or base 10.
  - Write down the corresponding equation $3^n = 400000$ or $0.95^n = 0.01$.
  - Solve for n, giving $n = \log_3 400000$ or $n = \log_{0.95} 0.01$.
  - Convert this to logarithms base e or base 10, and approximate.
  - Then write down the solution of the corresponding inequation.
- Be aware that powers of 3 get bigger as the index increases because 3 is greater than 1, and powers of 0.95 get smaller because 0.95 is smaller than 1.

A third approach is to take logarithms base e or base 10 of both sides.

Example 7 14B

The profits of the Extreme Sports Adventure Company have been increasing by 15% every year since its formation, when its profit was \$60000 in the first year.

a During which year did its profit first exceed \$1200000?

b During which year did its total profit since foundation first exceed \$4000000?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

The successive profits form a GP with $a = 60000$ and $r = 1.15$.

a Put

$$\begin{aligned}
T_n &= 1\,200\,000 \quad \text{(the corresponding equation)}\\
ar^{n-1} &= 1\,200\,000\\
60\,000 \times 1.15^{n-1} &= 1\,200\,000\\
\boxed{\div 60\,000} \quad 1.15^{n-1} &= 20\\
n - 1 &= \log_{1.15} 20\\
n &= \frac{\log_e 20}{\log_e 1.15} + 1\\
n &\doteqdot 22.43,
\end{aligned}$$

so the profit first exceeds $1200000 during the 23rd year.

b Here

$$\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \quad \text{(using this form because } r > 1\text{)}\\
&= \frac{60\,000 \times (1.15^n - 1)}{0.15}\\
&= 400\,000 \times (1.15^n - 1).
\end{aligned}$$

Put

$$\begin{aligned}
S_n &= 4\,000\,000 \quad \text{(the corresponding equation)}\\
400\,000 \times (1.15^n - 1) &= 4\,000\,000\\
\boxed{\div 400\,000} \quad 1.15^n - 1 &= 10\\
1.15^n &= 11\\
n &= \log_{1.15} 11\\
&= \frac{\log_e 11}{\log_e 1.15}\\
&\doteqdot 17.16,
\end{aligned}$$

so the total profit since foundation first exceeds $4000000 during the 18th year.

Example 8 14B

Consider again the slowly failing Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year, with 50000 bottles sold in 2016.

During which year will sales first fall below 20000?

SOLUTION

The sales form a GP with $a = 50000$ and $r = 0.94$.

Put

$$\begin{aligned}
T_n &= 20\,000 \quad \text{(the corresponding equation)}\\
a\,r^{n-1} &= 20\,000\\
50\,000 \times 0.94^{n-1} &= 20\,000\\
\boxed{\div 50\,000} \quad 0.94^{n-1} &= 0.4\\
n - 1 &= \log_{0.94} 0.4\\
n &= \frac{\log_e 0.4}{\log_e 0.94} + 1\\
n &\doteqdot 15.81.
\end{aligned}$$

Hence sales will first fall below 20000 when $n = 16$, that is, in 2031.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 14B

FOUNDATION

1 Use trial-and-error (and your calculator) to find the smallest integer n such that:

a $2^n > 30$ b $2^n > 15000$ c $3^n > 10$ d $3^n > 5000000$

e $\left(\frac{1}{2}\right)^n < 0.1$ f $\left(\frac{1}{2}\right)^n < 0.005$ g $\left(\frac{1}{2}\right)^n < 0.0001$ h $\left(\frac{1}{3}\right)^n < 0.00001$

2 Use logarithms to find the smallest integer n such that:

a $2^n > 7000$ b $3^n > 20000$ c $\left(\frac{1}{2}\right)^n < 0.004$ d $\left(\frac{1}{3}\right)^n < 0.0002$

3 a Show that 10, 11, 12.1, . . . is a geometric sequence.
b State the first term and the common ratio.
c Use the formula $T_n = ar^{n-1}$ to write down the fifteenth term.
d Find the number of terms less than 60 using trial-and-error on your calculator.
e Repeat part **d** using logarithms.

4 An accountant receives an annual salary of \$40000, with 5% increments each year.
a Show that her annual salary forms a GP, and find the common ratio.
b Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.
c In which year will her salary first exceed \$70000?

5 An initial salary of \$50000 increases by 4% each year. In which year will the salary first be at least twice the original salary?

DEVELOPMENT

6 A certain company manufactures three types of shade cloth. The product with code SC50 cuts out 50% of harmful UV rays, SC75 cuts out 75% and SC90 cuts out 90% of UV rays. In the following questions, you will need to consider the amount of UV light let through.
a What percentage of UV light does each cloth let through?
b Show that two layers of SC50 would be equivalent to one layer of SC75 shade cloth.
c Use logarithms to find the minimum number of layers of SC50 that would be required to cut out at least as much UV light as one layer of SC90.
d Similarly find how many layers of SC50 would be required to cut out 99% of UV rays.

7 Yesterday, a tennis ball used in a game of cricket in the playground was hit onto the science block roof. Luckily it rolled off the roof. After bouncing on the playground it reached a height of 3 metres. After the next bounce it reached 2 metres, then $1\frac{1}{3}$ metres, and so on.
a What was T_n, the height in metres reached after the nth bounce?
b What was the height of the roof that the ball fell from?
c The last time the ball bounced, its height was below 1 cm for the first time. After that it rolled away across the playground.
i Show that if $T_n < 0.01$, then $\left(\frac{3}{2}\right)^{n-1} > 300$.
ii How many times did the ball bounce?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Olim, Pixi, Thi (pronounced 'tea'), Sid and Nee work in the sales division of a calculator company. Together they find that sales of scientific calculators are dropping by 150 per month, while sales of graphics calculators are increasing by 150 per month.

a Current sales of all calculators total 20000 per month, and graphics calculators account for 10% of sales. How many graphics calculators are sold per month?

b How many more graphics calculators will be sold per month by the sales team six months from now?

c Assuming that current trends continue, how long will it be before all calculators sold by the company are graphics calculators?

9 Madeleine opens a business selling computer stationery. In its first year, the business has sales of \$200000, and each year sales are 20% more than the previous year's sales.

a In which year do annual sales first exceed \$1000000?

b In which year do total sales since foundation first exceed \$2000000?

10 **a** Explain why 'increasing a quantity by 300%' means 'multiplying the quantity by 4'.

b A population is increasing by 25% every year. How many full years will it take the population to increase by over 300%?

11 Consider the geometric series $3, 2, \frac{4}{3}, \ldots$.

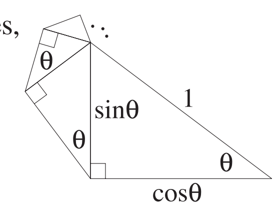
a Write down a formula for the sum S_n of the first n terms of the series.

b Explain why the geometric series has a limiting sum, and determine its value S.

c Find the smallest value of n for which $S - S_n < 0.01$.

ENRICHMENT

12 The diagram shows the first few triangles in a spiral of similar right-angled triangles, each successive one built with its hypotenuse on a side of the previous one.

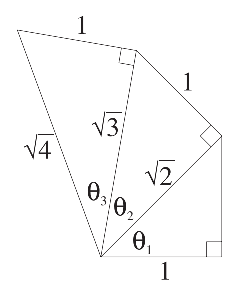


a What is the area of the largest triangle?

b Use the result for the ratio of areas of similar figures to show that the areas of successive triangles form a geometric sequence. What is the common ratio?

c Hence show that the limiting sum of the areas of the triangles is $\frac{1}{2}\tan\theta$.

13 The diagram shows the beginning of a spiral created when each successive right-angled triangle is constructed on the hypotenuse of the previous triangle. The altitude of each triangle is 1, and it is easy to show by Pythagoras' theorem that the sequence of hypotenuse lengths is $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$. Let the base angle of the nth triangle be θ_n. Clearly θ_n gets smaller, but does this mean that the spiral eventually stops turning? Answer the following questions to find out.



a Write down the value of $\tan\theta_n$.

b Show that $\displaystyle\sum_{n=1}^{k}\theta_n \geq \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$. (Hint: $\theta \geq \frac{1}{2}\tan\theta$, for $0 \leq \theta \leq \frac{\pi}{4}$.)

c By sketching $y = \dfrac{1}{x}$ and constructing the upper rectangle on each of the intervals

$1 \leq x \leq 2,\ 2 \leq x \leq 3,\ 3 \leq x \leq 4,\ \ldots$, show that $\displaystyle\sum_{n=1}^{k}\frac{1}{n} \geq \int_{1}^{k}\frac{1}{n}\,dn$.

d Does the total angle through which the spiral turns approach a limit?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14C Simple and compound interest

This section reviews the formulae for simple and compound interest. Simple interest is both an arithmetic sequence and a linear function. Compound interest is both a geometric sequence and an exponential function.

Simple interest, arithmetic sequences and linear functions

The formula for simple interest I should be well known from earlier years,

$$I = PRn,$$

where P is the principal invested, n is the number of units of time (such as days, weeks, months or years), and R is the interest rate per unit time.

- Regard n as a real number. Then the interest I is a linear function of n.
- Substitute $n = 1, 2, 3, \ldots$. Then the interest forms an arithmetic sequence

  $$T_1 = PR, \quad T_2 = 2PR, \quad T_3 = 3PR, \quad \ldots$$

  with first term PR and difference PR. Substituting $n = 0$ gives $T_0 = 0$, the 0th term of the sequence, when the interest due is still zero.

Thus APs and linear functions are closely related, as discussed in Chapter 1.

The simple interest formula gives the interest alone. To find the total amount at the end of n units of time, add the original principal P to the interest.

4 SIMPLE INTEREST

Suppose that a principal $\$P$ earns simple interest at a rate R per unit time.

- The simple interest $\$I$ earned over n units of time is

  $$I = PRn.$$

- Thus the interest is a linear function of n.
- Substituting $n = 1, 2, 3 \ldots$ gives an AP with first term PR and difference PR.
- To find the total amount at the end of n units of time, add the principal P.

Note: The interest rate here is a number. If the rate is given as a percentage, such as 7% p.a., then substitute $R = 0.07$. (The abbreviation 'p.a.' stands for *per annum*, which is Latin for 'per year' — always be careful of the units of time.)

Example 9 14C

A principal $\$P$ is invested at 6% p.a. simple interest.

a If the principal $\$P$ is \$3000, how much money will there be after seven years?

b Find the principal $\$P$, if the total at the end of five years is \$6500.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a Using the formula, interest $= PRn$

$= 3000 \times 0.06 \times 7$

$= \$1260.$

Hence final amount $= 3000 + 1260$ (principal + interest)

$= \$4260.$

b
$$\begin{aligned}
\text{Final amount after 5 years} &= 6500 \\
P + PRn &= 6500 \quad \text{(principal + interest)} \\
P + P \times 0.06 \times 5 &= 6500 \\
P + P \times 0.3 &= 6500 \\
P \times 1.3 &= 6500 \\
\boxed{\div\, 1.3} \qquad P &= \$5000.
\end{aligned}$$

Compound interest, geometric sequences and exponential functions

The formula for compound interest should also be well known from earlier years,

$$A_n = P(1 + R)^n,$$

where A_n is the final amount after n units of time (such as days, weeks, months or years), P is the principal, and R is the interest rate per unit time.

- Regard n as a real number. Then A_n is a multiple of an exponential function of n with base $1 + R$.
- Substitute the values $n = 1, 2, 3, \ldots$. Then the final amounts $A_1, A_2, A_3, \ldots$ after $1, 2, 3, \ldots$ units of time form a geometric sequence

  $$A_1 = P(1 + R)^1, \;\; A_2 = P(1 + R)^2, \;\; A_3 = P(1 + R)^3, \ldots$$

  with first term $P(1 + R)$ and common ratio $1 + R$. Substituting $n = 0$ gives $A_0 = P$, the 0th term of the sequence, when the amount due is still equal to the principal.

Thus GPs and exponential functions are closely related, as discussed in Chapter 1.

The compound interest formula gives the final amount. To find the interest, subtract the principal from the final amount.

5 COMPOUND INTEREST

Suppose that a principal $\$P$ earns compound interest at a rate R per unit time for n units of time, compounded every unit of time.

- The total amount A_n after n units of time is

  $$A_n = P(1 + R)^n.$$

- Thus the final amount is an exponential function with base $1 + R$.
- Substituting $n = 1, 2, 3, \ldots$ gives a GP with first term $P(1 + R)$ and ratio $1 + R$.
- To find the interest, subtract the principal from the final amount.

Note: The formula only works when compounding occurs after every unit of time. For example, if the interest rate is given as 24% per year with interest compounded monthly, then the units of time must be months, and the interest rate must be the rate per month, which is $R = 0.24 \div 12 = 0.02$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Proof Although the formula was developed in earlier years, it is important to understand how it arises and how the process of compounding generates a GP.

The initial principal is P, and the interest rate is R per unit time.
Hence the amount A_1 at the end of one unit of time is

$$A_1 = \text{principal} + \text{interest} = P + PR = P(1 + R).$$

This means that adding the interest is effected by multiplying by $1 + R$.
Thus the amount A_2 is obtained by multiplying A_1 by $1 + R$:

$$A_2 = A_1(1 + R) = P(1 + R)^2.$$

Continuing the process for the amounts $A_3, A_4, \ldots,$

$$A_3 = A_2(1 + R) = P(1 + R)^3,$$
$$A_4 = A_3(1 + R) = P(1 + R)^4,$$

so that when the money has been invested for n units of time,

$$A_n = A_{n-1}(1 + R) = P(1 + R)^n.$$

Example 10 14C

Amelda takes out a loan of \$5000 at a rate of 12% p.a., compounded monthly. She makes no repayments.

a Find the total amount owing after five years, and the interest alone.

b Find when the amount owing doubles, correct to the nearest month.

SOLUTION

Because the interest is compounded every month, the units of time must be months. The interest rate is therefore 1% per month, so $R = 0.01$.

a
$$\begin{aligned} A_{60} &= P(1 + R)^{60} \qquad \text{(converting 5 years to 60 months)} \\ &= 5000 \times 1.01^{60} \\ &\doteqdot \$9083. \end{aligned}$$

After five years, interest $= \$9083 - \$5000$ (subtracting the principal)
$= \$4083.$

b
$$\begin{aligned} \text{Put} \qquad A_n &= 2P. \\ \text{Then} \qquad P(1 + R)^n &= 2P \\ P \times 1.01^n &= 2P \\ \boxed{\div P} \qquad 1.01^n &= 2 \qquad \text{(so the value } P = 5000 \text{ is irrelevant)} \\ n &= \log_{1.01} 2 \\ &= \frac{\log_e 2}{\log_e 1.01} \\ &\doteqdot 70 \text{ months.} \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Depreciation

Depreciation is important when a business buys equipment because the equipment becomes worn or obsolete over time and loses its value. The company is required to record this loss of value as an expense in its accounts. Depreciation reduces the company's profit, which in turn reduces also the income tax payable. Depreciation is usually expressed as the loss per unit time of a percentage of the value of an item. The formula for depreciation is therefore the same as the formula for compound interest, except that the rate is negative.

6 DEPRECIATION

Suppose that goods originally costing $\$P$ depreciate at a rate R per unit time.

- The value A_n of the goods after n units of time is

$$A_n = P(1 - R)^n.$$

- Thus the final value is an exponential function of n with base $1 - R$.
- Substituting $n = 1, 2, 3, \ldots$ gives a GP with first term $P(1 - R)$ and ratio $1 - R$.
- To find the loss of value, subtract the final value from the original value.

Substituting $n = 0$ gives $A_0 = P$, the original value (0th term of the sequence).

Example 11 14C

An espresso machine bought for \$15000 on 1st January 2016 depreciates at $12\frac{1}{2}\%$ p.a.

a Find the depreciated value on 1st January 2025, and the loss of value over those nine years, correct to the nearest dollar.

b During which year will the value drop below 10% of the original cost?

SOLUTION

This is depreciation with $R = 0.125$, so $1 - R = 0.875$.

a
$$\begin{aligned}\text{Depreciated value} &= A_9 \qquad \text{(from 01/01/2016 to 01/01/2025 is 9 years)}\\ &= P(1-R)^n,\\ &= 15000 \times 0.875^9\\ &\doteqdot \$4510.\end{aligned}$$
$$\begin{aligned}\text{Loss of value} &\doteqdot 15000 - 4510 \qquad \text{(subtracting the depreciated value)}\\ &\doteqdot \$10490.\end{aligned}$$

b
$$\begin{aligned}\text{Put} \qquad A_n &= 0.1P\\ P \times 0.875^n &= 0.1P\\ \boxed{\div P} \qquad 0.875^n &= 0.1 \qquad \text{(so the value } P = 15000 \text{ is irrelevant here)}\\ n &= \log_{0.875} 0.1\\ &= \frac{\log_e 0.1}{\log_e 0.875}\\ &\doteqdot 17.24.\end{aligned}$$

Hence the depreciated value will drop below 10% during 2033.
(There are 17 years from 01/01/2016 to 01/01/2033, so the drop occurs during 2033.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 14C

FOUNDATION

1 Use the formula $I = PRn$ to find: i the simple interest, ii the total amount, when:
 a \$5000 is invested at 6% per annum for three years,
 b \$12 000 is invested at 6.15% per annum for seven years.

2 Use the formula $A = P(1 + R)^n$ to find, correct to the nearest cent: i the total value,
 ii the interest alone, of:
 a \$5000 invested at 6% per annum, compounded annually, for three years,
 b \$12 000 invested at 6.15% per annum, compounded annually, for seven years.

3 Use the formula $A = P(1 - R)^n$ to find, correct to the nearest cent: i the final value,
 ii the loss of value, of:
 a \$5000 depreciating at 6% per annum for three years,
 b \$12 000 depreciating at 6.15% per annum for seven years.

4 First convert the interest rate to the appropriate unit of time, then find the final value, correct to the nearest cent, when:
 a \$400 is invested at 12% per annum, compounded monthly, for two years,
 b \$10 000 is invested at 7.28% per annum, compounded weekly, for one year.

5 A man invested \$10 000 at 6.5% per annum simple interest.
 a Write down a formula for A_n, the total value of the investment at the end of the nth year.
 b Show that the investment exceeds \$20 000 at the end of 16 years, but not at the end of 15 years.

6 A company has just bought several cars for a total of \$229 000. The depreciation rate on these cars is 15% per annum.
 a What will be the net worth of the fleet of cars five years from now?
 b What will be the loss in value then?

7 Howard is arguing with Juno over who has the better investment. Each invested \$20 000 for one year. Howard has his invested at 6.75% per annum simple interest, while Juno has hers invested at 6.6% per annum compound interest.
 a If Juno's investment is compounded annually, who has the better investment, and what are the final values of the two investments?
 b Juno then points out that her interest is compounded monthly, not yearly. Now who has the better investment, and by how much?

DEVELOPMENT

8 a Find the total value of an investment of \$5000 earning 7% per annum simple interest for three years.
 b A woman invested an amount for nine years at a rate of 6% per annum. She earned a total of \$13 824 in simple interest. What was the initial amount she invested?
 c A man invested \$23 000 at 3.25% per annum simple interest. At the end of the investment period he withdrew all the funds from the bank, a total of \$31 222.50. How many years did the investment last?
 d The total value of an investment earning simple interest after six years is \$22 610. If the original investment was \$17 000, what was the interest rate?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 **a** The final value of an investment, after ten years earning 15% per annum, compounded yearly, was \$32 364. Find the amount invested, correct to the nearest dollar.

b The final value of an investment that earned 7% compound interest per annum for 18 years was \$40 559.20. What was the original amount, correct to the nearest dollar?

c A sum of money is invested at 4.5% interest per annum, compounded monthly. At the end of three years the value is \$22 884.96. Find the amount of the original investment, correct to the nearest dollar.

10 An insurance company recently valued my car at \$14 235. The car is three years old and the depreciation rate used by the insurance company was 10.7% per annum. What was the cost of the car, correct to the nearest dollar, when I bought it?

11 **a** What does \$6000 grow to at 8.25% per annum for three years, compounded monthly?

b How much interest is earned over the three years?

c What rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

12 An amount of \$10 000 is invested for five years at 4% p.a. interest, compounded monthly.

a Find the final value of the investment.

b What rate of simple interest, correct to two significant figures, would be needed to yield the same final balance?

c How many full months will it take for the money to exceed \$15 000?

13 The present value of a company asset is \$350 000. If it has been depreciating at $17\frac{1}{2}\%$ per annum for the last six years, what was the original value of the asset, correct to the nearest \$1000?

14 After six years compound interest, the final value of a \$30 000 investment was \$45 108.91. Find the rate of interest, correct to two significant figures, if it was compounded annually.

15 **a** Write down the formula for the total value A_n when a principal of \$6000 is invested at 12% p.a. compound interest for n years. Hence find the smallest number of complete years required for the investment to increase by a factor of 10.

b Xiao and Mai win a prize in the lottery and decide to put \$100 000 into a retirement fund offering 8.25% per annum interest, compounded monthly. How long will it be before their money has doubled? Round your answer up to the next month.

c My brother bought a new car on 1st August 2020, and it depreciates at 15% per annum. Every year afterwards, on the 1st of August, he calculates its depreciated value. How many times will he calculate that the car's value is more than 10% of its cost?

16 A bank customer earned \$7824.73 in interest on a \$40 000 investment at 6% per annum, compounded quarterly. Find the period of the investment, correct to the nearest quarter.

17 Thirwin, Neri, Sid and Nee each inherit \$10 000. Each invests the money for one year. Thirwin invests his money at 7.2% per annum simple interest. Neri invests hers at 7.2% per annum, compounded annually. Sid invests his at 7% per annum, compounded monthly. Nee invests in certain shares with a return of 8.1% per annum, but must pay stockbrokers' fees of \$50 to buy the shares initially and again to sell them at the end of the year. Who is furthest ahead at the end of the year?

18 **a** Find the interest on \$15 000 invested at 7% per annum simple interest for five years.

b Hence write down the total value of the investment.

c What rate of compound interest would yield the same amount if compounded annually? Give your answer correct to three significant figures.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

19 A student was asked to find the original value of an investment earning 9% per annum, compounded annually for three years, given its current value of \$54 391.22.

a She incorrectly thought that because she was working in reverse, she should use the depreciation formula. What value did she get?

b What is the correct answer, correct to the nearest dollar?

ENRICHMENT

20 **a** A bank lent \$1000 for one year at 12% p.a. compound interest. Find the amount owing at the end of the year, correct to the nearest cent, if the compounding occurred:

i annually, **ii** quarterly, **iii** monthly, **iv** daily.

b Question 17 of Exercise 6G outlined a proof that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$. Use this limit to find the amount after one year if the compounding had been continuous.

c What would the difference have been between annual compounding and continuous compounding if the period of the loan had been 10 years?

21 **a** Find the total value A_n if P is invested at a simple interest rate R for n periods.

b Show, by means of the binomial expansion, that the total value of the investment when compound interest is applied may be written as $A_n = P + PRn + P\sum_{k=2}^{n} {}^nC_k\, R^k$.

c Explain what each of the three terms of the formula in part **b** represents.

A possible project

Interest rates change, sometimes only every couple of years, sometimes every month. The Reserve Bank of Australia sets a benchmark interest rate called the *cash rate*. The historical cash rates are available, and a spreadsheet can be set up that will calculate the value of an amount that has earned this benchmark rate of interest over a number of years. Alternatively, the historical term deposit interest rates from a major bank could be used.

Inflation keeps varying also. A good question to ask is whether the amount has kept up with inflation, which also changes from month to month and needs a spreadsheet to calculate. There is also the problem that an investor has to pay tax on the interest, and the rate of tax varies over time and varies with the investor's income.

Thus even though the dollar value of a monetary asset may have increased over the years, its purchasing power may have gone backwards and taxation will have been lost. All this can be set up in a spreadsheet over the last say 30–40 years using data gathered from the web. A significant comparison is the price of housing over the same period, but there are many other interesting comparisons to be made.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

14D Investing money by regular instalments

Investment schemes such as superannuation — often called annuities — require money to be invested at regular intervals, for example every month or every year. This complicates things because each individual instalment earns compound interest for a different length of time. Calculating the value of these investments at some future time requires adding the terms of a GP.

This section and the next are applications of GPs. Learning new formulae is not recommended, because they will all need to be derived within each question.

Developing the GP and summing it

The most straightforward way to solve these problems is to find what each instalment grows to as it accrues compound interest. These final amounts form a GP, which can then be summed.

7 FINDING THE FUTURE VALUE OF AN INVESTMENT SCHEME

- Find what each instalment will amount to as it earns compound interest.
- Add up all these amounts using the formula for the sum of a GP.

Example 12 14D

Rawen's parents invested \$1000 in his name on the day that he was born. They continued to invest \$1000 for him on each birthday until his 20th birthday. On his 21st birthday they gave him the value of the investment.

If all the money earned interest of 7% compounded annually, what was the final value of the scheme, correct to the nearest dollar?

SOLUTION

The 1st instalment is invested for 21 years, and so amounts to 1000×1.07^{21}.
The 2nd instalment is invested for 20 years, and so amounts to 1000×1.07^{20}.

. . . .

The 20th instalment is invested for 2 years, and so amounts to 1000×1.07^{2}.
The 21st and last instalment is invested for 1 year, and so amounts to 1000×1.07^{1}.
Thus the total amount A_{21} at the end of 21 years is the sum

$$\begin{aligned} A_{21} &= \text{instalments plus interest} \\ &= (1000 \times 1.07^1) + (1000 \times 1.07^2) + \cdots + (1000 \times 1.07^{21}). \end{aligned}$$

This is a GP with first term $a = 1000 \times 1.07$, ratio $r = 1.07$, and 21 terms.

$$\begin{aligned} \text{Hence } A_{21} &= \frac{a(r^{21} - 1)}{r - 1} \qquad \text{(using the formula for } S_n \text{ for a GP with } r > 1) \\ &= \frac{1000 \times 1.07 \times (1.07^{21} - 1)}{0.07} \\ &\doteqdot \$48\,006. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 13 14D

Robin and Robyn are investing \$10 000 in a superannuation scheme on 1st July each year, beginning in the year 2010. The money earns compound interest at 8% p.a., compounded annually.

a How much will the fund amount to by 30th June 2030?

b How much will the fund amount to by the end of n years?

c Show that 2031 is the year when the fund first exceeds \$500 000 on 30th June.

d What annual instalment would have produced \$1 000 000 by 2030?

SOLUTION

a The 1st instalment is invested for 20 years, and so amounts to 10000×1.08^{20}.
The 2nd instalment is invested for 19 years, and so amounts to 10000×1.08^{19}.
The 19th instalment is invested for 2 years, and so amounts to 10000×1.08^{2}.
The 20th and last is invested for 1 year, and so amounts to 10000×1.08^{1}.
Thus the total amount A_{20} at the end of 20 years is the sum

$$\begin{aligned} A_{20} &= \text{instalments plus interest} \\ &= (10000 \times 1.08^1) + (10000 \times 1.08^2) + \cdots + (10000 \times 1.08^{20}). \end{aligned}$$

This is a GP with first term $a = 10000 \times 1.08$, ratio $r = 1.08$, and 20 terms.

$$\begin{aligned} \text{Hence } A_{20} &= \frac{a(r^{20} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{10000 \times 1.08 \times (1.08^{20} - 1)}{0.08} \\ &\doteqdot \$494\,229 \quad \text{(correct to the nearest dollar)}. \end{aligned}$$

b The 1st instalment is invested for n years, and so amounts to 10000×1.08^{n}.
The 2nd instalment is invested for $n - 1$ years, and so amounts to $10000 \times 1.08^{n-1}$.
The nth and last is invested for 1 year, and so amounts to 10000×1.08^{1}.
Thus the total amount A_n at the end of n years is the sum

$$\begin{aligned} A_{n} &= \text{instalments plus interest} \\ &= (10000 \times 1.08^1) + (10000 \times 1.08^2) + \cdots + (10000 \times 1.08^{n}). \end{aligned}$$

This is a GP with first term $a = 10000 \times 1.08$, ratio $r = 1.08$, and n terms.

$$\begin{aligned} \text{Hence } A_{n} &= \frac{a(r^{n} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{10000 \times 1.08 \times (1.08^{n} - 1)}{0.08} \\ &= 135000 \times (1.08^n - 1). \end{aligned}$$

c From part **a**, the total after 20 years is just under \$500 000.
Substituting $n = 21$ into the formula in part **b**,

$$\begin{aligned} A_{21} &= 135000 \times (1.08^{21} - 1) \\ &\doteqdot \$544\,568. \end{aligned}$$

Hence 2031 is the year when the fund first exceeds \$500 000 on 30th June.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

d Reworking part **b** with an instalment $\$M$ instead of \$10 000 gives the formula

$$A_n = 13.5 \times M \times (1.08^n - 1).$$

Substituting $n = 20$ and $A_{20} = 1\,000\,000$ into this formula,

$$1\,000\,000 = 13.5 \times M \times (1.08^{20} - 1)$$

$$M = \frac{1\,000\,000}{13.5 \times (1.08^{20} - 1)} \qquad \text{(making } M \text{ the subject)}$$

$$\doteqdot \$20\,234 \qquad \text{(correct to the nearest dollar)}.$$

Example 14 [Using logarithms to find n] 14D

Continuing with the previous example, use logarithms to find the year in which the fund first exceeds \$700 000 on 30th June.

SOLUTION

Substituting $M = 10\,000$ and $A_n = 700\,000$ into the formula found in part **b**,

$$700\,000 = 135\,000 \times (1.08^n - 1)$$

$$\boxed{\div 135\,000} \quad 1.08^n - 1 = \tfrac{700}{135}$$

$$\boxed{+1} \quad 1.08^n = \tfrac{835}{135}$$

$$n = \log_{1.08} \tfrac{835}{135} \qquad \text{(converting to a logarithmic equation)}$$

$$= \frac{\log_{10} \tfrac{835}{135}}{\log_{10} 1.08} \qquad \text{(using the change-of-base formula)}$$

$$\doteqdot 23.68.$$

Hence the fund first exceeds \$700 000 on 30th June when $n = 24$, that is, in 2034.

Example 15 [Monthly and weekly compounding] 14D

a Charmaine has a superannuation scheme with monthly instalments of \$600 for 10 years and an interest rate of 7.8% p.a., compounded monthly. What will the final value of her investment be?

b Charmaine was offered an alternative scheme with interest of 7.8% p.a., compounded weekly, and weekly instalments. What weekly instalments would have yielded the same final value as the scheme in part **a**?

c Which scheme would have cost her more per year?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a The monthly interest rate is $0.078 \div 12 = 0.0065$.
There are 120 months in 10 years.
The 1st instalment is invested for 120 months and so amounts to 600×1.0065^{120}.
The 2nd instalment is invested for 119 months and so amounts to 600×1.0065^{119}.
The 120th and last is invested for 1 month and so amounts to 600×1.0065^{1}.
Thus the total amount A_{120} at the end of 120 months is the sum

$$\begin{aligned} A_{120} &= \text{instalments plus interest} \\ &= (600 \times 1.0065^1) + (600 \times 1.0065^2) + \cdots + (600 \times 1.0065^{120}). \end{aligned}$$

This is a GP with first term $a = 600 \times 1.0065$, ratio $r = 1.0065$, and 120 terms.

$$\begin{aligned} \text{Hence } A_{120} &= \frac{a(r^{120} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{600 \times 1.0065 \times (1.0065^{10} - 1)}{0.0065} \\ &\doteqdot \$109\,257 \qquad \text{(retained in the memory for part } \mathbf{b}). \end{aligned}$$

b The weekly interest rate is $0.078 \div 52 = 0.0015$.
Let $\$M$ be the weekly instalment. There are 520 weeks in 10 years.
The 1st instalment is invested for 520 weeks and so amounts to $M \times 1.0015^{520}$.
The 2nd instalment is invested for 519 weeks and so amounts to $M \times 1.0015^{519}$.
The 520th and last is invested for 1 week and so amounts to $M \times 1.0015^{1}$.
Thus the total amount A_{520} at the end of 520 weeks is the sum

$$\begin{aligned} A_{520} &= \text{instalments plus interest} \\ &= M \times 1.0015 + M \times 1.0015^2 + \cdots + M \times 1.0015^{520}. \end{aligned}$$

This is a GP with first term $a = M \times 1.0015$, ratio $r = 1.0015$ and 520 terms.

$$\text{Hence } A_{520} = \frac{a(r^{520} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1)$$

$$A_{520} = \frac{M \times 1.0015 \times (1.0015^{520} - 1)}{0.0015}$$

$$A_{520} = \frac{M \times 10015 \times (1.0015^{520} - 1)}{15}.$$

Writing this formula with M as the subject,

$$M = \frac{15 \times A_{520}}{10015 \times (1.0015^{520} - 1)}.$$

But the final value A_{520} is to be the same as the final value in part **a**,
so substituting the answer to part **a** for A_{520} gives

$$M \doteqdot \$138.65 \qquad \text{(retain in the memory for part } \mathbf{c}).$$

c The weekly scheme in part **b** therefore costs about \$7210.04 per year, compared with \$7200 per year for the monthly scheme in part **a**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

An alternative approach using recursion

There is an alternative approach, using recursion, to developing the GPs involved in these calculations. Because the working is slightly longer, we have chosen not to display this method in the notes. It has, however, the great advantage that its steps follow the progress of a banking statement.

For those interested in the recursive method, it is developed in two structured questions, Questions 17 and 18, at the end of Exercise 14D. Most of the other questions can also be done by recursion if that is preferred (provided, of course, that some structuring within the question is ignored).

Exercise 14D

FOUNDATION

Note: Questions 1–5 of this exercise have been heavily structured to follow the approach given in the worked examples of this section. There are several other satisfactory approaches, including the recursive method outlined in Questions 17 and 18. If a different approach is chosen, the structuring in the first five questions below can be ignored.

1 Suppose that an instalment of \$500 is invested in a superannuation scheme on 1st January each year for four years, beginning in 2020. The money earns interest at 10% p.a., compounded annually.

a **i** What is the value of the first instalment on 31st December 2023?

ii What is the value of the second instalment on 31st December 2023?

iii What is the value of the third instalment on this date?

iv What is the value of the fourth instalment?

v What is the total value of the superannuation on 31st December 2023?

b **i** Write down the four answers to parts **i** to **iv** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, and hence check your answer to part **v** of part **a**.

2 Suppose that an instalment of \$1200 is invested in a superannuation scheme on 1st April each year for five years, beginning in 2015. The money earns interest at 5% p.a., compounded annually.

a In each part, round your answer correct to the nearest cent.

i What is the value of the first instalment on 31st March 2020?

ii What is the value of the second instalment on this date?

iii Do the same for the third, fourth and fifth instalments.

iv What is the total value of the superannuation on 31st March 2020?

b **i** Write down the answers to parts **i** to **iii** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part **iv** of part **a**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Joshua makes 15 contributions of \$1500 to his superannuation scheme on 1st April each year. The money earns compound interest at 7% per annum. He calculates what the scheme will be worth at a target date 15 years later.

a Let A_{15} be the total value of the fund at the target date.

i How much does the first instalment amount to at the target date?

ii How much does the second instalment amount to at the target date?

iii How much does the last contribution amount to invested for just one year?

iv Hence write down a series for A_{15}.

b Hence show that the final value of the fund is $A_{15} = \dfrac{1500 \times 1.07 \times (1.07^{15} - 1)}{0.07}$, and evaluate this correct to the nearest dollar.

4 Laura makes 24 contributions of \$250 to her superannuation scheme on the first day of each month. The money earns interest at 6% per annum, compounded monthly (that is, at 0.5% per month). She calculates the scheme's value at a target date 24 months later.

a Let A_{24} be the total value of the fund at the target date.

i How much does the first instalment amount to at the target date?

ii How much does the second instalment amount to at the target date?

iii What is the value of the last contribution, invested for just one month?

iv Hence write down a series for A_{24}.

b Hence show that the total value of the fund after contributions have been made for two years is $A_{24} = \dfrac{250 \times 1.005 \times (1.005^{24} - 1)}{0.005}$, and evaluate this correct to the nearest dollar.

5 A company makes contributions of \$3000 to the superannuation fund of one of its employees on 1st July each year. The money earns compound interest at 6.5% per annum. In this question, round all currency amounts correct to the nearest dollar.

a Let A_{25} be the value of the fund at the end of 25 years.

i How much does the first instalment amount to at the end of 25 years?

ii How much does the second instalment amount to at the end of 24 years?

iii How much does the last instalment amount to at the end of just one year?

iv Hence write down a series for A_{25}.

b Hence show that $A_{25} = \dfrac{3000 \times 1.065 \times (1.065^{25} - 1)}{0.065}$.

c What will be the value of the fund after 25 years, and what will be the total amount of the contributions?

DEVELOPMENT

6 Finster and Finster Superannuation offer a superannuation scheme with annual contributions of \$12000 invested at an interest rate of 9% p.a., compounded annually. Contributions are paid on 1st of January each year.

a Zoya decides to invest in the fund for the next 20 years. Show that the final value of her investment is given by $A_{20} = \dfrac{12000 \times 1.09 \times (1.09^{20} - 1)}{0.09}$.

b Evaluate A_{20}.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c By how much does this exceed the total contributions Zoya made?

d The company agrees to let Zoya make a higher contribution to the scheme. Let this instalment be M. Show that in this case $A_{20} = \dfrac{M \times 1.09 \times (1.09^{20} - 1)}{0.09}$.

e What would Zoya's annual contribution have to be in order for her superannuation to have a total value of \$1 000 000 at the end of the 20 years?

7 The company that Itsushi works for makes contributions to his superannuation scheme on 1st January each year. Any amount invested in this scheme earns interest at the rate of 7.5% p.a.

a Let M be the annual contribution. Show that the value of the investment at the end of the nth year is $A_n = \dfrac{M \times 1.075 \times (1.075^n - 1)}{0.075}$.

b Itsushi plans to have \$1 500 000 in superannuation when he retires in 25 years' time. Show that the company must contribute \$20 526.52 each year, correct to the nearest cent.

c The first year that Itsushi's superannuation is worth more than \$750 000, he decides to change jobs. Let this year be n.

i Show that n is the smallest integer solution of $(1.075)^n > \dfrac{750000 \times 0.075}{20526.52 \times 1.075} + 1$.

ii Evaluate the right-hand side and hence show that $(1.075)^n > 3.5492$.

iii Use logarithms or trial-and-error to find the value of n.

8 A person invests \$10 000 each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first payment is made on 1st January 2021 and the last payment is made on 1st January 2040.

a How much did the person invest over the life of the fund?

b Calculate, correct to the nearest dollar, the amount to which the 2021 payment has grown by the beginning of 2041.

c Find the total value of the fund when it is paid out on 1st January 2041.

d The person wants to reach a total value of \$1 000 000 in superannuation.

i Find a formula for A_n, the value of the investment after n years.

ii Show that the target is reached when $1.1^n > \dfrac{10}{1.1} + 1$.

iii At the end of which year will the superannuation be worth \$1 000 000?

e Suppose instead that the person wanted to achieve the same total investment of \$1 000 000 after only 20 years. What annual contribution would produce this amount? (Hint: Let M be the amount of each contribution.)

9 Each year on her birthday, Jane's parents put \$20 into an investment account earning $9\frac{1}{2}\%$ per annum compound interest. The first deposit took place on the day of her birth. On her 18th birthday, Jane's parents gave her the account and \$20 cash in hand.

a How much money had Jane's parents deposited in the account?

b How much money did she receive from her parents on her 18th birthday?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 A man about to turn 25 is getting married. He has decided to pay \$5000 each year on his birthday into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. If he dies before age 65, his wife will inherit the value of the insurance to that point. If he lives to age 65, the insurance company will pay out the value of the policy in full. Answer these questions correct to the nearest dollar.

- **a** The man is in a dangerous job. What will be the payout if he dies just before he turns 30?
- **b** The man's father died of a heart attack just before age 50. Suppose that the man also dies of a heart attack just before age 50. How much will his wife inherit?
- **c** What will the insurance company pay the man if he survives to his 65th birthday?

11 In 2021, the school fees at a private girls' school are \$20 000 per year. Each year the fees rise by $4\frac{1}{2}\%$ due to inflation.

- **a** Susan is sent to the school, starting in Year 7 in 2021. If she continues through to her HSC year, how much will her parents have paid the school over the six years?
- **b** Susan's younger sister is starting in Year 1 in 2021. How much will they spend on her school fees over the next 12 years if she goes through to her HSC?

ENRICHMENT

12 A woman has just retired with a payment of \$500 000, having contributed for 25 years to a superannuation fund that pays compound interest at the rate of $12\frac{1}{2}\%$ per annum. What was the size of her annual premium, correct to the nearest dollar?

13 At age 20, a woman takes out a life insurance policy under which she agrees to pay premiums of \$500 per year until she turns 65, when she is to be paid a lump sum. The insurance company invests the money and gives a return of 9% per annum, compounded annually. If she dies before age 65, the company pays out the current value of the fund plus 25% of the difference between the current value and what the value would have been had she lived until 65.

- **a** What is the value of the payout, correct to the nearest dollar, at age 65?
- **b** Unfortunately she dies at age 53, just before her 35th premium is due.
  - **i** What is the current value of the life insurance?
  - **ii** How much does the life insurance company pay her family?

14 A person pays \$2000 into an investment fund every year, and it earns compound interest at a rate of 6% p.a.

- **a** How much is the fund worth at the end of 10 years?
- **b** In which year will the fund reach \$70 000?

15 [Technology]
In the first column of a spreadsheet, enter the numbers from 1 to 30 on separate rows. In the first 30 rows of the second column, enter the formula

$$\frac{20256.52 \times 1.075 \times (1.075^n - 1)}{0.075}$$

for the value of a superannuation investment, where n is the value given in the first column.

- **a** Which value of n is the first to give a superannuation amount greater than \$750 000?
- **b** Compare this answer with your answer to question 7(c).
- **c** Try to do question 8(d) in the same way.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 [Technology]
Try checking your answers to questions 3 to 11 using a spreadsheet and its built-in financial functions. In particular, the built-in Excel™ function `FV(rate, nper, pmt, pv, type)`, which calculates the future value of an investment, seems to produce an answer different from what might be expected. Investigate this and explain the difference.

Note: The last two questions illustrate an alternative approach to superannuation questions, using a recursive method to generate the appropriate GP. The advantage of the method is that its steps follow the progress of a banking statement.

17 [The recursive method]
At the start of each month, Cecilia deposits $\$M$ into a savings scheme paying 1% per month, compounded monthly. Let A_n be the amount in her account at the end of the nth month.

a Explain why $A_1 = 1.01M$.
b Explain why $A_2 = 1.01(M + A_1)$, and why $A_{n+1} = 1.01(M + A_n)$, for $n \geq 2$.
c Use the recursive formulae in part **b**, together with the value of A_1 in part **a**, to obtain expressions for $A_2, A_3, \ldots, A_n$.
d Using the formula for the sum of n terms of a GP, show that $A_n = 101M(1.01^n - 1)$.
e If each deposit is \$100, how much will be in the fund after three years?
f Hence find, correct to the nearest cent, how much each deposit M must be if Cecilia wants the fund to amount to \$30 000 at the end of five years.

18 [The recursive method]
A couple saves \$100 at the start of each week in an account paying 10.4% p.a. interest, compounded weekly. Let A_n be the amount in the account at the end of the nth week.

a Explain why $A_1 = 1.002 \times 100$, and why $A_{n+1} = 1.002 \times (100 + A_n)$, for $n \geq 2$.
b Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.
c Using GP formulae, show that $A_n = 50\,100 \times (1.002^n - 1)$.
d Hence find how many weeks it will be before the couple has \$100 000.

A possible project

As discussed at the end of Exercise 14C, interest rates vary over time, and inflation means that the purchasing power of a matured superannuation fund is less than its dollar-value may have suggested some years ago. Taking all this into account requires a spreadsheet. But with superannuation there are many other considerations, and building these things into a spreadsheet as well would involve an extended project.

- Most superannuation funds have an insurance component, insuring against early death or disability. The cost of this insurance is built into the policy, but the details are not straightforward.
- All superannuation funds charge fees, which are calculated in various ways, perhaps depending on the balance, perhaps depending on the future value, perhaps depending on the number of transactions. This could also be investigated and built into the spreadsheet.
- The contributions to the fund are almost certainly a proportion of the salary. Thus some estimates must be made of future salary.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14E Paying off a loan

Long-term loans such as housing loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time. The calculations associated with paying off a loan are therefore similar to the investment calculations of the previous section.

Developing the GP and summing it

As with superannuation, the most straightforward method is to calculate the final value of each instalment as it earns compound interest, and then add these final values up as before, using the theory of GPs. But there is an extra complication — these instalments must be balanced against the initial loan, which is growing with compound interest. The loan is finally paid off when the amount owing is zero.

8 CALCULATIONS ASSOCIATED WITH PAYING OFF A LOAN

To find the amount A_n still owing after n units of time:

- Find what the principal, earning compound interest, would amount to if no instalments were paid.
- Find what each instalment will amount to as it earns compound interest, then add up all these amounts, using the formula for the sum of a GP.
- The amount A_n still owing at the end of n units of time is

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest}).$$

The loan is paid off when the amount A_n still owing is zero.

Note: When paying off a loan, the first payment is usually made one unit of time after the loan is taken out. But always read the question carefully!

Example 16 14E

Yianni and Eleni borrow \$20000 from the Town and Country Bank to go on a trip to Constantinople. Interest is charged at 12% per annum, compounded monthly. They start repaying the loan one month after taking it out, and their monthly instalments are \$300.

a How much will they still owe the bank at the end of six years?

b How much interest will they have paid in these six years?

SOLUTION

a The monthly interest rate is 1%, so $1 + R = 1.01$.

The initial loan of \$20000, after 72 months, amounts to 20000×1.01^{72}.

The 1st instalment is invested for 71 months, and so amounts to 300×1.01^{71}.

The 2nd instalment is invested for 70 months, and so amounts to 300×1.01^{70}.

The 71st instalment is invested for 1 month, and so amounts to 300×1.01^{1}.

The 72nd and last instalment is invested for no time at all, and so amounts to 300.

Hence the amount A_{72} still owing at the end of 72 months is

$$\begin{aligned} A_{72} &= (\text{principal plus interest}) - (\text{instalments plus interest}) \\ &= 20000 \times 1.01^{72} - (300 + 300 \times 1.01 + \cdots + 300 \times 1.01^{71}). \end{aligned}$$

The bit in brackets is a GP with first term $a = 300$, ratio $r = 1.01$, and 72 terms.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$$\text{Hence } A_{72} = 20000 \times 1.01^{72} - \frac{a(r^{72} - 1)}{r - 1} \qquad \text{(finding the sum of the GP)}$$

$$= 20000 \times 1.01^{72} - \frac{300 \times (1.01^{72} - 1)}{0.01}$$

$$= 20000 \times 1.01^{72} - 30000 \times (1.01^{72} - 1)$$

$$\doteqdot \$9529 \qquad \text{(correct to the nearest dollar).}$$

b Total instalments over six years $= 300 \times 72$

$= \$21\,600.$

Reduction in loan over six years $= 20000 - 9529$

$= \$10\,471.$

Hence interest charged $= 21\,600 - 10\,471$

$= \$11\,129$ (more than half the original loan).

Example 17 [Finding what instalments should be paid] 14E

Ali takes out a loan of \$10000 to buy a car. He will repay the loan in five years, paying 60 equal monthly instalments, beginning one month after he takes out the loan. Interest is charged at 6% p.a., compounded monthly.

Find how much the monthly instalment should be, correct to the nearest cent.

SOLUTION

The monthly interest rate is 0.5%, so $1 + R = 1.005$. Let each instalment be \$$M$.
First calculate the amount A_{60} still owing at the end of 60 months.
Then find M by setting A_{60} equal to zero.

The initial loan of \$10000, after 60 months, amounts to 10000×1.005^{60}.
The 1st instalment is invested for 59 months, and so amounts to $M \times 1.005^{59}$.
The 2nd instalment is invested for 58 months, and so amounts to $M \times 1.005^{58}$.
The 59th instalment is invested for 1 month, and so amounts to $M \times 1.005^{1}$.
The 60th and last instalment is invested for no time at all, and so amounts to M.
Hence the amount A_{60} still owing at the end of 60 months is

$$A_{60} = \text{(principal plus interest)} - \text{(instalments plus interest)}$$

$$= 10000 \times 1.005^{60} - (M + M \times 1.005 + \cdots + M \times 1.005^{59}).$$

The bit in brackets is a GP with first term $a = M$, ratio $r = 1.005$, and 60 terms.

$$\text{Hence } A_{60} = 10000 \times 1.005^{60} - \frac{a(r^{60} - 1)}{r - 1}$$

$$= 10000 \times 1.005^{60} - \frac{M(1.005^{60} - 1)}{0.005}$$

$$= 10000 \times 1.005^{60} - 200M(1.005^{60} - 1) \qquad \left(\text{because } 0.005 = \tfrac{1}{200}\right).$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

But the loan is exactly paid off in these 5 years, so $A_{60} = 0$.

Hence $\quad 10000 \times 1.005^{60} - 200M(1.005^{60} - 1) = 0$

$$200M(1.005^{60} - 1) = 10000 \times 1.005^{60}$$

$\boxed{\div 200}$ $$M(1.005^{60} - 1) = 50 \times 1.005^{60}$$

$$M = \frac{50 \times 1.005^{60}}{1.005^{60} - 1}$$

$$\doteqdot \$193.33.$$

Finding the length of the loan

A loan is fully repaid when the amount A_n still owing is zero. Thus finding the length of a loan means solving an equation for the index n, a process that requires logarithms.

Example 18 — 14E

Natasha and Richard take out a loan of \$200000 on 1st January 2002 to buy a house. They will repay the loan in monthly instalments of \$2200. Interest is charged at 12% p.a., compounded monthly.

a Find a formula for the amount owing at the end of n months.

b How much is owing after five years?

c How long does it takes to repay:

  i the full loan? **ii** half the loan?

d Why would instalments of \$1900 per month never repay the loan?

SOLUTION

a The monthly interest rate is 1%, so $1 + R = 1.01$.

The initial loan, after n months, amounts to 200000×1.01^n.

The 1st instalment is invested for $n - 1$ months and so amounts to $2200 \times 1.01^{n-1}$.

The 2nd instalment is invested for $n - 2$ months and so amounts to $2200 \times 1.01^{n-2}$.

The nth and last instalment is invested for no time at all and so amounts to 2200.

Hence the amount A_n still owing at the end of n months is

$$\begin{aligned} A_n &= (\text{principal plus interest}) - (\text{instalments plus interest}) \\ &= 200000 \times 1.01^n - (2200 + 2200 \times 1.01 + \cdots + 2200 \times 1.01^{n-1}). \end{aligned}$$

The bit in brackets is a GP with first term $a = 2200$, ratio $r = 1.01$, and n terms.

$$\begin{aligned} \text{Hence } A_n &= 200000 \times 1.01^n - \frac{a(r^n - 1)}{r - 1} \\ &= 200000 \times 1.01^n - \frac{2200 \times (1.01^n - 1)}{0.01} \\ &= 200000 \times 1.01^n - 220000 \times (1.01^n - 1) \\ &= 220000 - 20000 \times 1.01^n. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

b To find the amount owing after 5 years, substitute $n = 60$,

$$A_{60} = 220000 - 20000 \times 1.01^{60}$$
$$\doteqdot \$183666 \qquad \text{(This is still almost as much as the original loan!)}$$

c i To find when the loan is repaid, put $A_n = 0$,

$$\begin{aligned} 220000 - 20000 \times 1.01^n &= 0 \\ 20000 \times 1.01^n &= 220000 \\ \boxed{\div 20000} \qquad 1.01^n &= 11 \\ n &= \log_{1.01} 11 && \text{(converting to a logarithmic equation)} \\ n &= \frac{\log_{10} 11}{\log_{10} 1.01} && \text{(using the change-of-base formula)} \\ &\doteqdot 20 \text{ years and 1 month.} \end{aligned}$$

ii To find when the loan is half repaid, put $A_n = 100000$,

$$\begin{aligned} 220000 - 20000 \times 1.01^n &= 100000 \\ 20000 \times 1.01^n &= 220000 \\ \boxed{\div 20000} \qquad 1.01^n &= 6 \\ n &= \log_{1.01} 6 && \text{(converting to a logarithmic equation)} \\ n &= \frac{\log_{10} 6}{\log_{10} 1.01} && \text{(using the change-of-base formula)} \\ &\doteqdot 15 \text{ years.} \end{aligned}$$

Notice that this is about three-quarters, not half, the total time of the loan.

d With a loan of \$200000 at an interest rate of 1% per month,

$$\begin{aligned} \text{initial interest per month} &= 200000 \times 0.01 \\ &= \$2000. \end{aligned}$$

This means that at the start of the loan, \$2000 of the instalment is required just to pay the interest. Hence with repayments of only \$1900, the debt would increase rather than decrease.

The alternative approach using recursion

As with superannuation, the GP involved in a loan-repayment calculation can also be developed using a recursive method, whose steps follow the progress of a banking statement.

Again, this method is developed in two structured questions, Questions 18 and 19 at the end of Exercise 14E, and recursion can easily be applied to the other questions in the exercise provided that some internal structuring is ignored.

ISBN 978-1-108-76630-2

Exercise 14E

FOUNDATION

Note: As in the previous exercise, Questions 1 and 2 have been heavily structured to follow the approach given in the worked examples of this section. There are several other satisfactory approaches, including the recursive method outlined in Questions 18 and 19. If a different approach is chosen, the structuring in the first three questions below can be ignored.

1 On 1st January 2020, Lizbet borrows \$501 from a bank for four years at an interest rate of 10% p.a. She repays the loan with four equal instalments of \$158.05 at the end of each year.

a Use the compound interest formula to show that the initial loan amounts to \$733.51 at the end of four years.

b **i** What is the value of the first instalment on 31st December 2023, having been invested for three years?

ii What is the value of the second instalment on this date?

iii What is the value of the third instalment?

iv What is the value of the fourth (and last) instalment?

v Find the total value of all the instalments on 31st December 2023, and hence show that Lizbet has now repaid the loan.

c **i** Write down the four answers to parts **i**–**iv** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part **v** of part **b**.

2 Suppose that on 1st April 2020 a loan of \$5600 is made, which is repaid with equal instalments of \$1293.46 made on 31st March each year for five years, beginning in 2021. The loan attracts interest at 5% p.a., compounded annually.

a Use the compound interest formula to show that the initial loan amounts to \$7147.18 by 31st March 2025.

b In each part, round your answer correct to the nearest cent.

i What is the value of the first instalment on 31st March 2025?

ii What is the value of the second instalment on this date?

iii Do the same for the third, fourth and fifth instalments.

iv Find the total value of the instalments on 31st March 2025, and hence show that the loan has been repaid.

c **i** Write down your answers to parts **i**–**iii** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part **iv** of part **b**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Lome took out a loan with Tornado Credit Union for \$15000, to be repaid in 15 equal annual instalments of \$1646.92 on 1st April each year. Compound interest is charged at 7% per annum.

a Let A_{15} be the amount owed at the end of 15 years.

i Use the compound interest formula to show that $15000 \times (1.07)^{15}$ is owed on the initial loan after 15 years.

ii How much does the first instalment amount to at the end of the loan, having been invested for 14 years?

iii How much does the second instalment amount to at the end of 13 years?

iv What is the value of the second-last instalment?

v What is the worth of the last contribution, invested for no time at all?

vi Hence write down an expression involving a series for A_{15}.

b Show that the final amount owed is

$$A_{15} = 15000 \times (1.07)^{15} - \frac{1646.92 \times (1.07^{15} - 1)}{0.07}.$$

c Evaluate A_{15} and hence show that the loan has been repaid.

4 Matts signed a mortgage agreement for \$100000 with a bank for 20 years at an interest rate of 6% per annum, compounded monthly (that is, at 0.5% per month).

a Let M be the size of each repayment to the bank, and let A_{240} be the amount owing on the loan after 20 years.

i What does the initial loan amount to after 20 years?

ii Write down the amount that the first repayment grows to by the end of the 240th month.

iii Do the same for the second repayment and for the last repayment.

iv Hence write down a series expression for A_{240}.

b Hence show that $A_{240} = 100000 \times 1.005^{240} - 200 \times M(1.005^{240} - 1)$.

c Explain why the bank puts $A_{240} = 0$.

d Hence find M, correct to the nearest cent.

e How much will Matts have paid the bank over the period of the loan?

5 I took out a personal loan of \$10000 with a bank for five years at an interest rate of 18% per annum, compounded monthly (that is, at 1.5% per month).

a Let M be the size of each instalment to the bank, and let A_{60} be the amount owing on the loan after 60 months.

i What does the initial loan amount to after 60 months?

ii Write down the amount that the first instalment grows to by the end of the 60th month.

iii Do the same for the second instalment and for the last instalment.

iv Hence write down a series expression for A_{60}.

b Hence show that $0 = 10000 \times 1.015^{60} - \dfrac{M(1.015^{60} - 1)}{0.015}$.

c Hence find M, correct to the nearest dollar.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

6 A couple take out a \$165000 mortgage on a house, and they agree to pay the bank \$1700 per month. The interest rate on the loan is 9% per annum, compounded monthly, and the contract requires that the loan be paid off within 15 years.

a Let A_{180} be the balance on the loan after 15 years. Find a series expression for A_{180}.

b Show that $A_{180} = 165000 \times 1.0075^{180} - \dfrac{1700(1.0075^{180} - 1)}{0.0075}$.

c Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

7 A couple take out a \$250000 mortgage on a house, and they agree to pay the bank \$2000 per month. The interest rate on the loan is 7.2% per annum, compounded monthly, and the contract requires that the loan be paid off within 20 years.

a Let A_n be the balance on the loan after n months. Find a series expression for A_n.

b Hence show that $A_n = 250000 \times 1.006^n - \dfrac{2000(1.006^n - 1)}{0.006}$.

c Find the amount owing on the loan at the end of the tenth year, and state whether this is more or less than half the amount borrowed.

d Find A_{240}, and hence show that the loan is actually paid out in less than twenty years.

e If it is paid out after n months, show that $1.006^n = 4$, and hence that $n = \dfrac{\log 4}{\log 1.006}$.

f Find how many months early the loan is paid off.

8 A company borrows \$500000 from the bank at an interest rate of 5.25% per annum, compounded monthly, to be repaid in monthly instalments. The company repays the loan at the rate of \$10000 per month.

a Let A_n be the amount owing at the end of the nth month. Show that

$$A_n = 500000 \times 1.004375^n - \frac{10000(1.004375^n - 1)}{0.004375}.$$

b Given that the loan is paid off, use the result in part **a** to show that $1.004375^n = 1.28$.

c Use logarithms or trial-and-error to find how long it will take to pay off the loan. Give your answer in whole months.

9 As can be seen from these questions, the calculations involved with reducible loans are reasonably complex. For that reason, it is sometimes convenient to convert the reducible interest rate into a simple interest rate. Suppose that a mortgage is taken out on a \$180000 house at 6.6% reducible interest per annum for a period of 25 years, with payments of amount M made monthly.

a Using the usual pronumerals, explain why $A_{300} = 0$.

b Show that $A_{300} = 180000 \times 1.0055^{300} - \dfrac{M(1.0055^{300} - 1)}{0.0055}$.

c Find the size of each repayment to the bank.

d Hence find the total paid to the bank, correct to the nearest dollar, over the life of the loan.

e What amount is therefore paid in interest? Use this amount and the simple interest formula to calculate the simple interest rate per annum over the life of the loan, correct to two significant figures.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 A personal loan of \$15 000 is borrowed from the Min Hua Finance Company at a rate of $13\frac{1}{2}\%$ per annum over five years, compounded monthly. Let M be the amount of each monthly instalment.

a Show that $15000(1.01125)^{60} - \dfrac{M(1.01125^{60} - 1)}{0.01125} = 0$.

b What is the monthly instalment necessary to pay back the loan? Give your answer correct to the nearest dollar.

11 [Problems with rounding]
Most questions so far have asked you to round monetary amounts correct to the nearest dollar. This is not always wise, as this question demonstrates. A personal loan for \$30 000 is approved with the following conditions. The reducible interest rate is 13.3% per annum, with payments to be made at six-monthly intervals over five years.

a Find the size of each instalment, correct to the nearest dollar.

b Using this amount, show that $A_{10} \neq 0$, that is, the loan is not paid off in five years.

c Explain why this has happened.

12 A couple have worked out that they can afford to pay \$19 200 each year in mortgage payments. The current home loan rate is 7.5% per annum, with equal payments made monthly over a period of 25 years.

a Let P be the principal borrowed and A_{300} the amount owing after 25 years. Show that

$$A_{300} = P \times 1.00625^{300} - \frac{1600(1.00625^{300} - 1)}{0.00625}.$$

b Hence determine the maximum amount that the couple can borrow and still pay off the loan. Round your answer down to the nearest dollar.

13 The current credit card rate of interest on Bankerscard is 23% per annum, compounded monthly.

a If a cardholder can afford to repay \$1500 per month on the card, what is the maximum value of purchases that can be made in one day if the debt is to be paid off in two months?

b How much would be saved in interest payments if the cardholder instead saved up the money for two months before making the purchase?

ENRICHMENT

14 Some banks offer a ‘honeymoon’ period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed \$170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay \$1650 per month to the bank. The standard rate of interest is $8\frac{1}{2}\%$ p.a., but the bank also offers a special rate of 6% p.a. for one year to people buying their first home. (All interest rates are compounded monthly.)

a Calculate the amount the couple would owe at the end of the first year, using the special rate of interest.

b Use this value as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the loan contract?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 Over the course of years, a couple have saved \$300 000 in a superannuation fund. Now that they have retired, they are going to draw on that fund in equal monthly pension payments for the next 20 years. The first payment is at the beginning of the first month. At the same time, any balance will be earning interest at $5\frac{1}{2}\%$ per annum, compounded monthly. Let B_n be the balance left immediately after the nth payment, and let M be the amount of the pension instalment. Also, let $P = 300\,000$ and R be the monthly interest rate.

a Show that $B_n = P \times (1 + R)^{n-1} - \dfrac{M\big((1 + R)^n - 1\big)}{R}$.

b Why is $B_{240} = 0$?

c What is the value of M?

16 A company buys machinery for \$500 000 and pays it off by 20 equal six-monthly instalments, the first payment being made six months after the loan is taken out. If the interest rate is 12% p.a., compounded monthly, how much will each instalment be?

---

17 [Technology]
In the first column of a spreadsheet, enter the numbers from 1 to 60 on separate rows. In the first 60 rows of the second column, enter the formula

$$500\,000 \times 1.004375^n - \frac{10\,000 \times (1.004375^n - 1)}{0.004375}$$

for the balance of a loan repayment, where n is the value given in the first column.

a Observe the pattern of figures in the second column. Notice that the balance decreases more slowly at first and more quickly towards the end of the loan.

b Which value of n is the first to give a balance less than or equal to zero?

c Compare this answer with your answer to question 8.

d Try to do question 7 **f** in the same way.

Note: The next two questions illustrate the alternative approach to loan repayment questions, using a recursive method to generate the appropriate GP.

18 [The recursive method]
A couple buying a house borrow $\$P = \$150\,000$ at an interest rate of 6% p.a., compounded monthly. They borrow the money at the beginning of January, and at the end of every month, they pay an instalment of $\$M$. Let A_n be the amount owing at the end of n months.

a Explain why $A_1 = 1.005P - M$.

b Explain why $A_2 = 1.005A_1 - M$, and why $A_{n+1} = 1.005A_n - M$, for $n \geq 2$.

c Use the recursive formulae in part **b**, together with the value of A_1 in part **a**, to obtain expressions for $A_2, A_3, \ldots, A_n$.

d Using GP formulae, show that $A_n = 1.005^n P - 200M(1.005^n - 1)$.

e Hence find, correct to the nearest cent, what each instalment should be if the loan is to be paid off in 20 years?

f If each instalment is \$1000, how much is still owing after 20 years?

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

19 [The recursive method]
Eric and Enid borrow $\$P$ to buy a house at an interest rate of 9.6% p.a., compounded monthly. They borrow the money on 15th September, and on the 14th day of every subsequent month, they pay an instalment of $\$M$. Let A_n be the amount owing after n months have passed.

a Explain why $A_1 = 1.008P - M$, and why $A_{n+1} = 1.008A_n - M$, for $n \geq 2$.

b Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.

c Using GP formulae, show that $A_n = 1.008^n P - 125M(1.008^n - 1)$.

d If the maximum instalment they can afford is \$1200, what is the maximum they can borrow, if the loan is to be paid off in 25 years? (Answer correct to the nearest dollar.)

e Put $A_n = 0$ in part **c**, and solve for n. Hence find how long will it take to pay off the loan of \$100 000 if each instalment is \$1000. (Round up to the next month.)

A possible project

The remarks at the end of Exercises 14C and 14D about varying interest rates and inflation hold also for housing loans. Many loans also contain insurance against death or disability or loss of employment, and again there are fees, which are not easily found.

All this historical data can be found on the web and built into a spreadsheet. The spreadsheet could also take into account the increasing value of housing over past years. An interesting comparison could be made between the relative wealth of a couple who rented for a long period and invested their savings elsewhere, and a couple on the same (increasing) salary who purchased a home with a large mortgage.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 14 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 14 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Consider the series $31 + 44 + 57 + \cdots + 226$.

a Show that it is an AP and write down the first term and the common difference.

b How many terms are there in this series?

c Find the sum.

2 Consider the series $24 + 12 + 6 + \cdots$.

a Show that it is a geometric series and find the common ratio.

b Explain why this geometric series has a limiting sum.

c Find the limiting sum and the sum of the first 10 terms, and show that they are approximately equal, correct to the first three significant figures.

3 Use trial-and-error, and probably a calculator, to find the smallest integer such that:

a $2^n > 2000$ **b** $(1.08)^n > 2000$ **c** $(0.98)^n < 0.01$ **d** $\left(\frac{1}{2}\right)^n < 0.0001$

Then repeat parts **a**–**d** using logarithms.

4 On a certain day at the start of a drought, 900 litres of water flowed from the Neverfail Well. The next day, only 870 litres flowed from the well, and each day, the volume of water flowing from the well was $\frac{29}{30}$ of the previous day's volume. Find the total volume of water that would have flowed from the well if the drought had continued indefinitely.

5 The profits of a company are growing at 14% per year. If this trend continues, how many full years will it be before the profit has increased by over 2000%?

6 A chef receives an annual salary of \$35 000, with 4% increments each year.

a Show that her annual salaries form a GP and find the common ratio.

b Find her annual salary, and her total earnings, at the end of 10 years, each correct to the nearest dollar.

7 Darko's salary is \$47 000 at the beginning of 2004, and it will increase by \$4000 each year.

a Find a formula for T_n, his salary in the nth year.

b In which year will Darko's salary first be at least twice what it was in 2004?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Miss Yamada begins her new job in 2005 on a salary of \$53 000, and it is increased by 3% each year. In which year will her salary be at least twice her original salary?

9 **a** Find the value of a \$12 000 investment that has earned 5.25% per annum, compounded monthly, for five years.

b How much interest was earned over the five years?

c What annual rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

10 A Wolfsrudel car depreciates at 12% per annum. Jake has just bought one that is four years old at its depreciated value of \$25 000.

a What will the car's depreciated value be in another four years?

b Find the average loss in value over the next four years.

c What was the new price of the car?

d Find the average loss in value over the four years from when it was new.

11 Katarina has entered a superannuation scheme into which she makes annual contributions of \$8000. The investment earns interest of 7.5% per annum, compounded annually, with contributions made on 1st October each year.

a Show that after 15 years of contributions, the value of Katarina's investment is given by

$$A_{15} = \frac{8000 \times 1.075 \times (1.075^{15} - 1)}{0.075}.$$

b Evaluate A_{15}.

c By how much does A_{15} exceed the total contributions Katarina made over these years?

d Show that after 17 years of contributions, the value A_{17} of the superannuation is more than double Katarina's contributions over the 17 years.

12 Ahmed wishes to retire with superannuation worth half a million dollars in 25 years' time. On 1st August each year he pays a contribution to a scheme that gives interest of 6.6% per annum, compounded annually.

a Let M be the annual contribution. Show that the value of the investment at the end of the nth year is

$$A_n = \frac{M \times 1.066 \times (1.066^n - 1)}{0.066}.$$

b Hence show that the amount of each contribution is \$7852.46.

13 Alonso takes out a mortgage on a flat for \$159 000, at an interest rate of 6.75% per annum, compounded monthly. He agrees to pay the bank \$1415 each month for 15 years.

a Let A_{180} be the balance of the loan after 15 years. Find a series expression for A_{180}.

b Show that $A_{180} = 159000 \times 1.005625^{180} - \dfrac{1415(1.005625^{180} - 1)}{0.005625}$.

c Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

d What monthly payment, correct to the nearest cent, is needed in order to pay off the loan in 15 years?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

14 May-Eliane borrowed \$1.7 million from the bank to buy some machinery for her farm. She agreed to pay the bank \$18 000 per month. The interest rate is 4.5% per annum, compounded monthly, and the loan is to be repaid in 10 years.

a Let A_n be the balance of the loan after n months. Find a series expression for n.

b Hence show that $A_n = 1\,700\,000 \times 1.00375^n - \dfrac{18\,000\,(1.00375^n - 1)}{0.00375}$.

c Find the amount owing on the loan at the end of the fifth year, and state whether this is more or less than half the amount borrowed.

d Find A_{120}, and hence show that the loan is actually paid out in less than 10 years.

e If it is paid out after n months (that is, put $A_n = 0$), show that $1.00375^{\,n} = 1.5484$, and hence that

$$n = \frac{\log_{10} 1.5484}{\log_{10} 1.00375}.$$

f Find how many months early the loan is paid off.

15 **a** A woman saves \$3000 at the end of each month at 3% p.a. interest compounded monthly. What does she have at the end of $2\frac{1}{2}$ years?

b If she had wanted to have \$100 000 after these $2\frac{1}{2}$ years, what should her instalment have been?

16 **a** A retiree buys a \$600 000 annuity that will pay him an annual pension for 25 years. Interest is 4% p.a. compounded annually. What is his annual pension?

b If instead he wanted an annual pension of \$60 000, how much would he have to deposit?

17 Eduardo and Eleni borrow money for a trip to Europe, and repay it at 12% p.a. compounded monthly with monthly instalments of \$2000 over 3 years.

a What did they borrow for their trip?

b What do they still owe after 2 years?

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

15 Displaying and interpreting data

Data bombard us constantly from every direction — prices of cars, daily temperatures and rainfall, birth weights of babies, marks in different subjects, how much milk people have for breakfast — and we struggle to make sense of it all. The subject of statistics begins as *descriptive statistics*, which develops various systematic approaches, using tables, graphs and summary statistics, so that we can see the big picture. Very quickly, however, statistics is combined with *probability theory*, inviting the language of *prediction* to be used, and leading perhaps to a discussion of *causation*, which is so fundamental in science.

In the Year 11 book, the foundations of probability theory and discrete probability distributions were developed, ending with some limited discussion of sampling and the way in which probability theory is related to statistical observations. This chapter and the next work the other way around, beginning in Chapter 15 with the raw data and ways of organising raw data. Then Chapter 16 moves to the relationship of that data to probability theory and continuous distributions.

Sections 15A–15C deal with *univariate* data, meaning that there is just one variable involved, such as the prices of cars or the birth weights of babies. Data may also be *bivariate*, such as when we measure the heights and weights of people to investigate how height and weight are related. Sections 15D–15F introduce the possible *correlation* between two statistical variables and the associated *line of best fit*.

Many opportunities are provided for investigations, for serious use of technology, and for possible projects, particularly in Exercise 15F.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15A Displaying data

Raw data come in small, large and huge unsorted lists, mostly of numbers, but also of categories. It is usually unrewarding to make much sense of such a list just by scanning through it. The first task of *statistics* is to provide tools for the analysis of data — Chapters 12 and 13 of the Year 11 book began this task.

There are three successive stages to analysing raw data.

- Display the raw data in various *tables* and *graphs* (or *charts*) to gain some overview of it, and perhaps some initial insight into what is happening.
- Carry out calculations of *summary statistics*. For univariate data, the most important are the *measures of location*, such as the mean and median, and the *measures of spread*, such as the variance, standard deviation and interquartile range. For bivariate data, we also use the *correlation* and the *line of best fit*.
- Speculate about patterns, predictions, and possible causal factors of the data, using *probability theory* to calculate theoretical probability distributions, followed by tests as to how well the data fit any suggested distribution.

All this may then be followed by suggestions for further experiments, in which the statistician may well be heavily involved in *designing the experiments* that will yield the next sets of raw data.

This chapter is about data — the table, the graphs, and the summary statistics. The next chapter deals with continuous probability distributions, and in particular with the normal distribution. Discrete probability distributions were discussed in Chapter 13 of the Year 11 book.

The Desmos graphing calculator which is embedded in the interactive textbook has a number of features to help visualise data and calculate summary statistics. Refer to the interactive textbook and teacher resources for Desmos guides and statistics activities.

A review of random variables

Here is a quick review of random variables from Section 13A of the Year 11 book.

1 EXPERIMENTS AND RANDOM VARIABLES

Random variables

- A *deterministic experiment* is an experiment with one possible outcome.
- A *random experiment* is an experiment with more than one possible outcome.
- A *random variable*, usually denoted by an upper-case letter such as X, is the outcome when a random experiment is run, and the various possible outcomes of the experiment are called the *values* of the random variable.

Scores and frequency

- When an experiment is run many times, the outcomes are called *scores*, and the (finite) list of all the scores is called a *sample*.
- The *frequency* of an outcome or value is the number of times it occurs.

Types of random variables

- A random variable may be *numeric* (if its values are numbers) or *categorical*.
- A random variable is called *discrete* if it is numeric and its values can be *listed*, meaning that it is possible to write them down in a sequence $x_1, x_2, x_3, \ldots$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Recording the country of birth of a person chosen at random in Australia is a categorical variable. Recording the number of overseas countries visited by that person is a numeric variable, which is discrete because its possible values 0, 1, 2, . . . can be listed.

Recording the height of a person is a numeric variable, but if we regard peoples' heights as real numbers rather than rounded measurements, then the variable is not discrete because we cannot list the set of possible values. This is a *continuous variable*, whose precise definition will be given in Chapter 16.

Frequency tables and cumulative frequency tables

The most basic object for organising and inspecting raw data is a *frequency table*, as introduced in Chapter 13 (Year 11). This table of frequencies can be produced digitally using a spreadsheet or database, or by hand using tallies.

When the data are numeric, we can also produce a *cumulative frequency table*, which tells us the number of scores less than or equal to a given score. For example, at the start of Year 7, Cedar Heights High School gave 40 students a spelling test marked out of 10. Here are the raw results, presented as univariate data:

4, 7, 2, 8, 7, 6, 3, 2, 8, 2, 9, 5, 8, 5, 8, 3, 6, 7, 5, 2,
10, 6, 7, 5, 6, 6, 9, 1, 5, 7, 8, 1, 6, 5, 7, 10, 6, 7, 8, 6,

and here are the tallies, the frequencies and the cumulative frequencies.

Mark	0	1	2	3	4	5	6	7	8	9	10	Sum
Tally		||	||||	||	|	~~||||~~ |	~~||||~~ |||	~~||||~~ ||	~~||||~~ |	||	||	—
Frequency	0	2	4	2	1	6	8	7	6	2	2	40
Cumulative frequency	0	2	6	8	9	15	23	30	36	38	40	—

A quick glance at the cumulative frequencies suggests that 8–9 students have poor spelling, or perhaps they had little experience in earlier years doing tests.

2 CUMULATIVE FREQUENCY

- For numeric data, the *cumulative frequency* is the number of scores that are less than or equal to a given score.
- A frequency distribution table can be extended to a cumulative frequency distribution table by taking the accumulating sums of the frequencies.

If the values of a categorical dataset have been sorted into some sort of meaningful order, then a cumulative frequency table can also be produced — see the Pareto charts later in this section.

Finding the median from the cumulative frequencies

Two questions dominate the discussion of univariate data, as we saw in Chapter 13 (Year 11).

- Measures of location: Where is the centre of the distribution?
- Measures of spread: How spread out are the data?

We begin with the median, which is a measure of location.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 THE MEDIAN — A MEASURE OF LOCATION

The *median* of numeric data — with symbol Q_2 meaning second quartile — is the middle score, when the scores are written out in ascending order. More specifically:

- For an odd number of scores, the median is the middle score.
- For an even number of scores, the median is the average of the two middle scores.

A cumulative frequency table makes it easy to pick out the median.

It may be helpful to review how to find the median by writing the scores out in order. Suppose first that there are an odd number of scores:

$$\begin{array}{ccccccccc} 4 & 7 & 8 & 10 & 10 & 11 & 13 & 15 & 21 \\ & & & & \uparrow & & & & \end{array} \qquad \text{(9 scores)}$$

The median of the 9 scores is the middle score. This is the 5th score in the row, which is 10. Now suppose that there are an even number of scores:

$$\begin{array}{cccccccccc} 4 & 7 & 8 & 10 & 10 & & 11 & 11 & 13 & 13 & 21 \\ & & & & & \uparrow & & & & & \end{array} \qquad \text{(10 scores)}$$

There are now two middle scores — the 5th, which is 10, and the 6th, which is 11. The median is their average, which is $10\frac{1}{2}$.

Example 1 15A

Explain how to use the cumulative frequency table above of the 40 spelling test marks to calculate the median of the 40 scores.

SOLUTION

There are 40 scores, so the median is the average of the 20th and 21st scores.

From the table of cumulative frequencies, the 15th score is 5, and all scores from the 16th up to the 23rd score are 6.

Hence the 20th and 21st scores are both 6, so the median is 6.

Thus someone with a score of 6 is in the middle of the class.

Displaying categorical data

A glance at some newspapers shows the wide variety of tables and graphs that are used to display data. Such tables and graphs should be easy to read because one of their purposes is to display data to people who are interested in the subject matter of the table or graph, but may or may not know any statistics.

Tables and graphs may be intended as the first step of an open-minded analysis of an unknown situation. They may be a quick visual illustration of ideas expounded in accompanying text. They may be intended to argue a contentious point. Whatever their purpose, the first job of the reader is to look out for intended or unintended distortions created by the display. Statistics wrongly used routinely lead people astray.

Two examples displaying categorical data are given here — Pareto charts and two-way graphs. The exercises have examples of other displays, such as pie charts and bar charts. In every example, interpretation is vital.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Pareto charts

Any set of categorical data, or even discrete data, can be represented on a *Pareto chart*. Its main purpose, however, is to identify which problems in a business are most urgent, and it is classified as one of 'seven basic tools of quality'.

For example, Secure Roofs often arranges a repair, but for various reasons that repair does not take place on the scheduled day, causing loss of income for the company while salary and other expenses still have to be paid. The manager organised the last 200 such failures into six categories, as in the table to the right.

Problem	Frequency
Blackout	4
Employee not arriving	6
Illness of employee	16
Owner not home	64
Rain	88
Truck breakdown	22
Total	200

To construct a *Pareto chart*, first arrange the categories into descending order of frequency — this places the most serious issues first, because they are the first problems that need to be addressed. Then add a cumulative frequency column.

The Pareto chart consists of two graphs drawn together on the same chart:
- a frequency histogram with columns arranged in this descending order,
- a cumulative frequency polygon.

The chart usually has two vertical axes, one on the left and one on the right. On the left are the frequencies, on the right are the percentage frequencies.

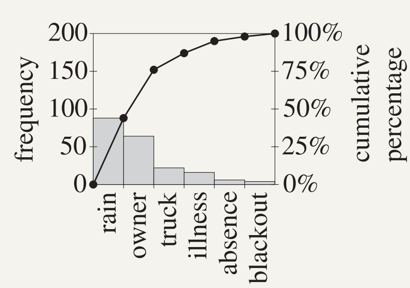
Example 2 15A

a Draw a Pareto chart of the data gathered by Secure Roofs.
b Describe what actions the manager may decide to take using the chart.

SOLUTION

a Here is the cumulative frequency table, with the categories arranged in descending order of frequency, and the cumulative frequencies calculated corresponding to that order. The Pareto chart is on the right.

Problem	Frequency	Cumulative
Rain	88	88
Owner	64	152
Truck	22	174
Illness	16	190
Absence	6	196
Blackout	4	200
Total	200	



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b With this chart, the manager can go through the issues from left to right and attempt to deal with what is causing problems for the business, from the most serious to the least serious.

- First, perhaps, he will first decide only to schedule external roof repairs three days ahead, when forecasts are more reliable.
- Then perhaps he will personally ring each owner two days ahead with a friendly reminder, following up with an SMS the evening before.
- Perhaps he has budgeted for a new truck next year.
- Perhaps he knows that he has little control over the other three issues.

There are no firmly established conventions for drawing the details of a Pareto chart, and the conventions we have chosen are certainly not universal. Here are some details about the chart as we have drawn it.

- The rectangles of the histogram join up with each other.
- The cumulative frequency polygon starts at the left-hand bottom corner of the left-hand rectangle because the initial sum is zero.
  — The next plot is at the right-hand top corner of the first rectangle.
  — Each subsequent plot is above the right-hand side of each rectangle.

We will have more to say about histograms and polygons in Section 15B.

The cumulative frequency polygon in a Pareto chart is always concave down because the issues have been arranged in descending order of frequency. (The term 'concave down' is being used more loosely here than in Chapter 9 — here every chord lies 'below or on the curve' rather than 'below the curve'.)

Two-way tables (contingency tables)

A *two-way table* or *contingency table* consists of two or more related frequency tables put together. In its simplest form, it has only four numbers in the table, yet it is surprisingly complicated to interpret. The topic anticipates the discussion of bivariate data in Section 15D, and it involves also conditional probability from Section 12G of Year 11.

A survey asked 200 adults what colour phone cover they preferred. Responses were recorded as black–brown (Dark) or as coloured (Colour), and the gender of the person was also recorded. The resulting *joint frequencies* are tabulated to the right.

	Dark	Colour
Men	38	12
Women	56	94

Let us ask the question, 'Do men prefer dark colours more than women do?' If we glanced just at the joint frequencies 38 and 56 under 'Dark', we might conclude that women prefer dark colours more than men. The data are deceptive, however, because far more women were questioned than men — for reasons that we have not been told. We need to take this into account.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 3 15A

Explain how to analyse the two-way table above to answer the question, 'Do men prefer dark colours more than women do?'

SOLUTION

We can find the sums 50 and 150 of the two rows, and the sums 94 and 106 of the two columns. The grand total is 200, which checks the additions. These five numbers are called *marginal frequencies*. The bias of the sample towards women is now clear.

	Dark	Colour	Sum
Men	38	12	50
Women	56	94	150
Sum	94	106	200

Each of the three rows, and each of the three columns, is a frequency table. The last row and the last column are called *marginal distributions*, and the inner two rows and columns are called *conditional distributions* (for reasons explained below).

Now we can answer the question. The proportion of men preferring dark covers is $\frac{38}{50} = 76\%$, and the proportion of women preferring dark covers is $\frac{56}{150} \doteqdot 37\%$, so the survey definitely says that men prefer dark covers more than women do. (Because the survey was biased towards women, the proportion of people surveyed preferring dark covers is $\frac{94}{200} = 47\%$, which is not the mean of 76% and 37%).

Similarly, the proportion of men preferring coloured covers is $\frac{12}{50} = 24\%$, and the proportion of women preferring coloured colours is $\frac{94}{150} \doteqdot 63\%$ (and the proportion of people surveyed preferring coloured covers is $\frac{106}{200} = 53\%$).

Conditional probability in two-way tables

The percentages in the worked example above are probabilities. If we choose a person from the survey at random,

$$P\,(\text{person prefers dark covers}) = \tfrac{94}{200} = 0.47.$$

Section 12G in the Year 11 book introduced *conditional probability*. To find the probability that the person prefers dark covers given that the person is a man (or a woman), we use the *reduced sample space* of 50 men (or 150 women),

$$\begin{aligned} P\,(\text{prefers dark} \mid \text{man}) &= \tfrac{38}{50} = 0.76 \\ P\,(\text{prefers dark} \mid \text{woman}) &= \tfrac{56}{150} \doteqdot 0.37. \end{aligned}$$

We can also find conditional probabilities the other way around. For a person chosen at random from the survey,

$$\begin{aligned} P\,(\text{person is a man}) &= \tfrac{50}{200} = 0.25 \\ P\,(\text{man} \mid \text{prefers dark}) &= \tfrac{38}{94} \doteqdot 0.40 \\ P\,(\text{man} \mid \text{prefers coloured}) &= \tfrac{12}{106} \doteqdot 0.11. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 TWO PARTICULAR DATA DISPLAYS

Histograms and polygons, and cumulative histograms and polygons, will be discussed further in the next section. In this section we have particularly looked at:

- *Pareto charts*, which consist of a frequency histogram and a cumulative frequency polygon drawn together, after the categories have been arranged in decreasing order of frequency. They are normally used with categorical data, for the purpose of displaying issues in descending order of importance.
- *Two-way tables* (or *contingency tables)*, by which we can investigate whether two variables are related, and make estimates of conditional probability.

The mode and the range

The *mode* is the most popular score, meaning the score with the greatest frequency ('mode' means 'fashion'). It is the simplest measure of location to identify because it is immediately obvious from the frequency table. It is even more obvious from the resulting histogram.

For example, in the frequency table of problems experienced by Secure Roofs, the mode is the problem 'Rain', with a frequency of 88. In the earlier table of spelling test scores, the mode is 6, which happens to coincide with the median, but this is not always the case.

Some frequency tables have two or more scores with the same maximum frequency, and are called *bimodal* or *trimodal* or *multimodal*.

The *range* is only defined for numeric data. It is the difference between the minimum and maximum scores. For example, with the 40 spelling test scores,

$$\text{minimum} = 1, \qquad \text{maximum} = 10, \qquad \text{range} = 10 - 1 = 9.$$

The range is the simplest measure of spread of a dataset.

This meaning of the word 'range' in statistics is quite different from its meaning in the language of functions, where it means the set of output values of a function.

5 MODE (A MEASURE OF LOCATION) AND RANGE (A MEASURE OF SPREAD)

- The *mode* of a dataset is the most popular score, that is, the score with the greatest frequency. A dataset may be *bimodal*, *trimodal* or multimodal.
- The *range* of a dataset is the difference between the minimum and the maximum scores.
- The mode is a measure of location, and the range is a measure of spread.

Exercise 15A

FOUNDATION

1 State whether each random variable is numeric or categorical. If it is numeric, state whether it is discrete or continuous. Comments may be appropriate.

a The favourite day of the week for a person chosen at random in Australia
b Height of Australian professional basketball players
c Age
d Political affiliation

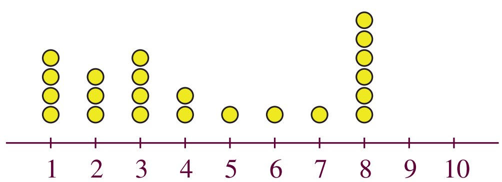
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

e Colour of a counter drawn from a cup containing 5 red and 6 blue counters.
f Sex (male or female) of a child attending a particular primary school
g Sum of the numbers when two dice are thrown
h Shoe size
i Examination scores

2 Find the median, mode and range of each dataset.
a 10 13 14 14 15 17 18
b 5 7 8 9 10 12 13 15 17
c 3 3 4 5 7 9 10 12 13 15
d 4 4 4 6 6 6 7 7 8 8 9 10
e 4 2 6 4 7 3 4 6 3 4 3 5 2 1 5 7 8
f 2 9 7 6 4 3 2 7 8 9 10 5 4 2 3 6 9 3

3 A shop sells individual cupcakes and keeps a record of how many cupcakes each customer purchases. The results are shown in the dot plot to the right.



a Construct a cumulative frequency table from the data.
b Find the median sales of cupcakes.
c Find the mode of the data.
d The shop intends to pre-package cupcakes to streamline its sales for many customers. Discuss the advantage of selling the cupcakes in packages of i 3, ii 4, iii 8.

4 A basketball coach begins each training session by challenging his star player to shoot as many hoops as he can in two minutes. Over twenty-one sessions he records these results:

4 3 5 4 5 4 6 7 5 6 8 6 6 8 9 10 7 8 9 7 9

a Construct a dot plot of the data. (This can be used as an alternative to a tally.)
b Copy and fill in the following frequency table.

score x	3	4	5	6	7	8	9	10
frequency f								
cumulative								

c What is the median number of hoops shot by the star player?
d In his twenty-second session he shoots 11 hoops in the 2 minutes allowed. What is his new median score?
e Is this frequency table a helpful way of displaying the scores?

5 An operator tracks the number of customers who pay the $4 fee to take his amusement ride on each day of the week. His data is displayed in the following table.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
13	32	35	38	57	75	65

a Draw a bar chart showing the data, with days of the week on the horizontal axis. Use a scale of 1 cm per 10 rides.
b Construct a table of cumulative frequencies, and draw a cumulative bar chart showing the number of rides sold up to and including that day of the week.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

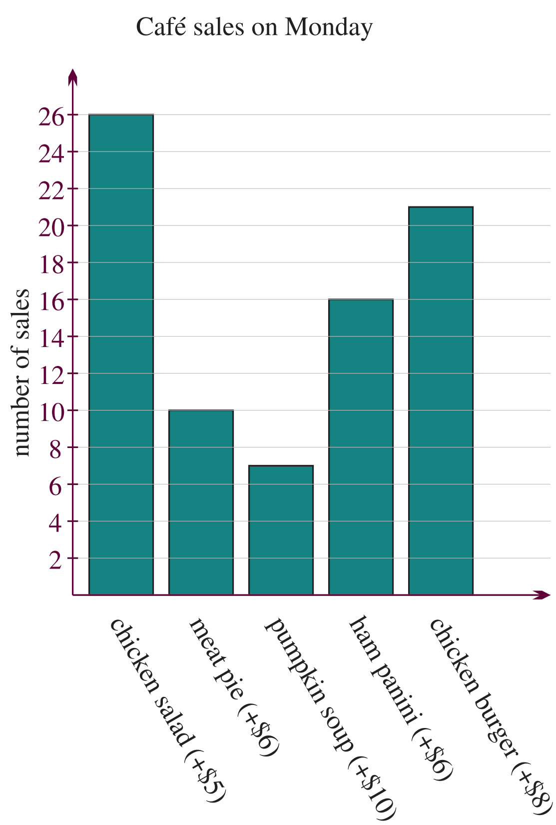
6 A survey of 1000 people in the Netherlands generated the following data shown in a contingency table relating hair and eye colour.

Colour	Blond hair	Red hair	Brown hair	Black hair	Total
Brown eyes	78	4	65	25	172
Blue eyes	324	10	46	9	389
Grey eyes	252	8	47	10	317
Green eyes	74	3	35	10	122
Total	728	25	193	54	1000

- **a** What is the most common hair and eye colour combination in the study?
- **b** What was the least common combination?
- **c** What is the probability that a blond-haired person also has blue eyes?
- **d** What is the probability that a black-haired person also has blue eyes?
- **e** What percentage of people with black hair had brown eyes?
- **f** What percentage of people with dark hair (brown or black) had brown eyes?
- **g** What percentage of people with light hair (blond or red) had lighter coloured eyes (blue, grey or green)?
- **h** Does there appear to be a link between hair colour and eye colour?
- **i** This study was carried out from a particular genetic population. Is it likely that similar results hold everywhere?

7 A café tracks its sales on a certain Monday to find what menu items are selling. It has a limited menu: chicken salad, meat pie, pumpkin soup, ham panini, and chicken burger. The café's results are shown on the graph to the right. The graph also records the markup (profit) on each choice, shown as (+$ markup).

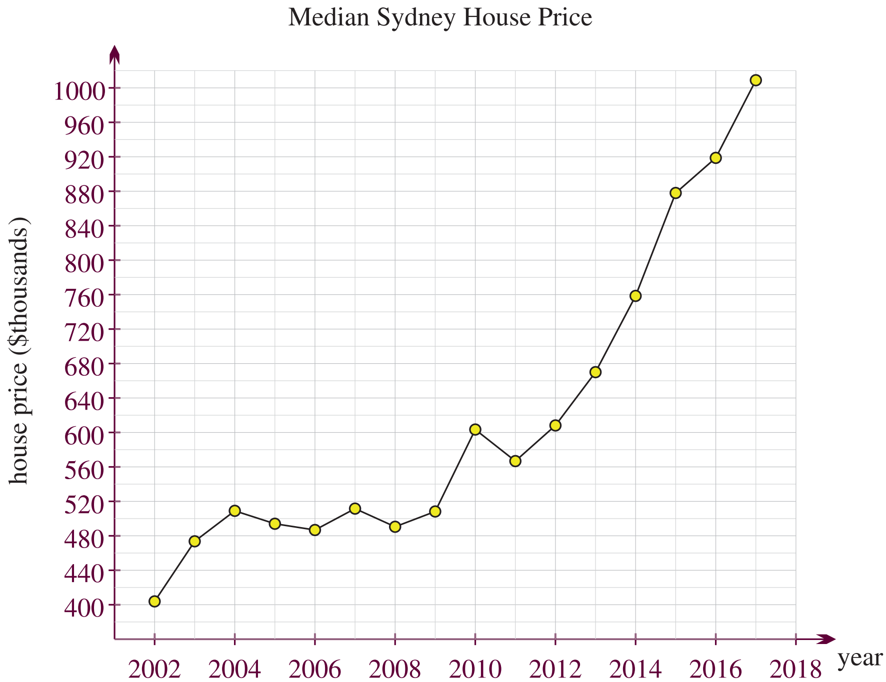
- **a** What is the total number of menu orders for the café on Monday?
- **b** Determine what percentage the sale of each menu option is of the total.
- **c** What is the profit, in dollars, obtained from each of the choices on the menu on the Monday?
- **d** What is the total profit, in dollars, for the café on the day?
- **e** The café has a policy to drop from the menu any choice with sales below 10%. Give two reasons why they should not drop the pumpkin soup from the menu.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 The median house price in Sydney from 2002 to 2017 is recorded in the line graph.



a Write down the median price of a Sydney house in 2002 and 2017, correct to the nearest ten thousand dollars.

b What is the percentage increase in house prices in Sydney from 2002 to 2017?

c What was the average increase in house price per year over this time?

d If this trend continues, what do you predict the median house price will be in 2030?

e What year saw the greatest increase in house prices?

f What year saw the greatest decrease in house prices? How much did prices change?

9 Owners of *The Happy Pet* boarding house for pets whose owners are out of town are looking to expand their business. A business analyst has asked them to keep track of the last 400 pets staying at their boarding house to determine what kind of pets they will need to accommodate in their planned expansion. This information is displayed in the Pareto chart to the right.

a What percentage of the last 400 pets at the boarding house were dogs? How many dogs was this?

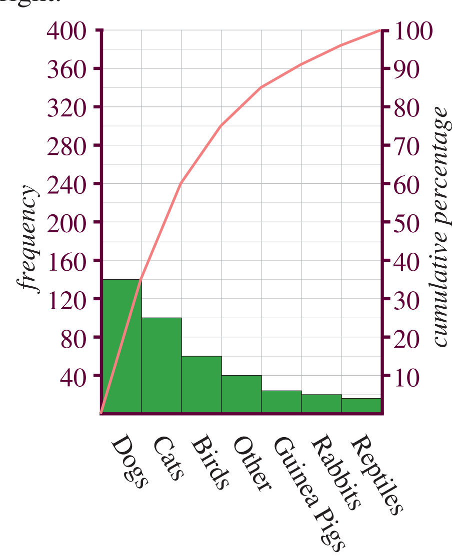
b Rabbits and guinea pigs can stay in the same type of cage. What percentage of the last 400 pets staying were rabbits or guinea pigs?

c To maximise business profits, the owners decide to concentrate on the three most common pets. What percentage of the last 400 pets were one of these three?

d What percentage of pets fell into one of the three least common categories?

e Comment on the size of the 'others' category.

f What other matters should the owners take into account, besides the numbers of pets looking for boarding?



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

10 A school is investigating reasons why students arrive late to class. Students who are late are asked to state a reason. The reasons given by the last 100 students are recorded in the table to the right.

a Construct a table with the categories ordered by decreasing frequency. Add a cumulative frequency column, which will also be the cumulative frequency percentage because there were 100 students in the survey.

b Construct a Pareto chart of the data. Use a scale of 1 cm per 10 units on each vertical scale.

c Explain why the cumulative frequency polygon of a Pareto chart will always be concave down.

d What percentage of the reasons are included in the first three categories?

e Comment on how the school could work to reduce the first three causes of tardiness.

Reasons late to class	Frequency
Didn't hear the bell	20
Held back in last class	27
Cancelled music lesson	10
Lost bag	5
Late back from lunch	3
Summons by senior teacher	2
Medical	3
Distance from last class	20
Other	10

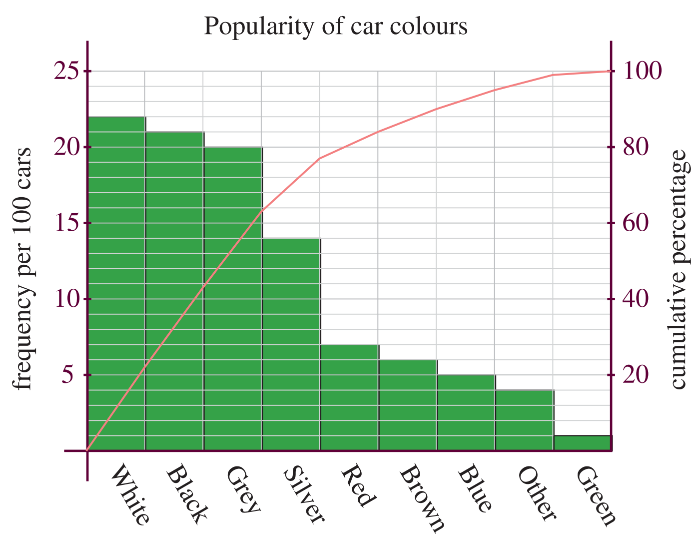
11 The colours of cars on the road are recorded in the following Pareto chart.

a What percentage of cars are brown?

b What percentage of cars are of one of the three most common colours, white, black or grey?

c How many cars are not one of the seven most popular colours?

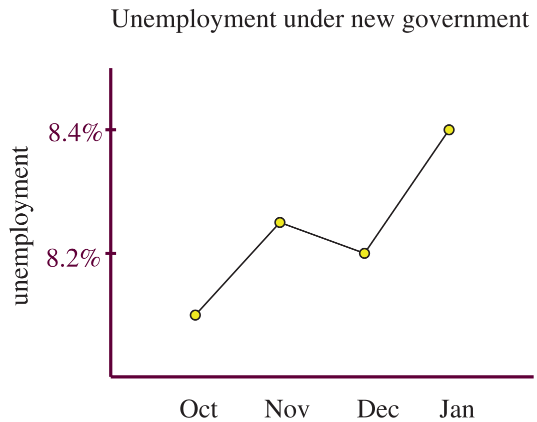
d This Pareto chart uses different scales on the two axes — is this confusing?



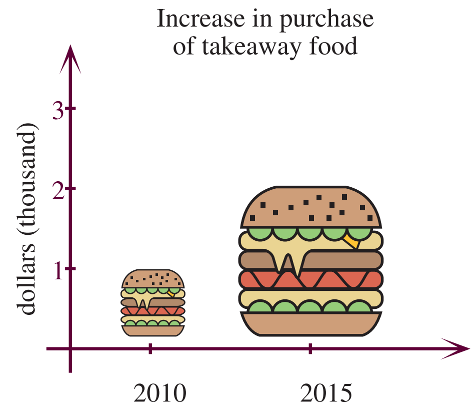
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 Statistics can easily be misinterpreted or deliberately used to mislead.

a 

The graph copied above was published in a newspaper. Can you suggest why it might be a misleading use of statistics and graphing?

b 

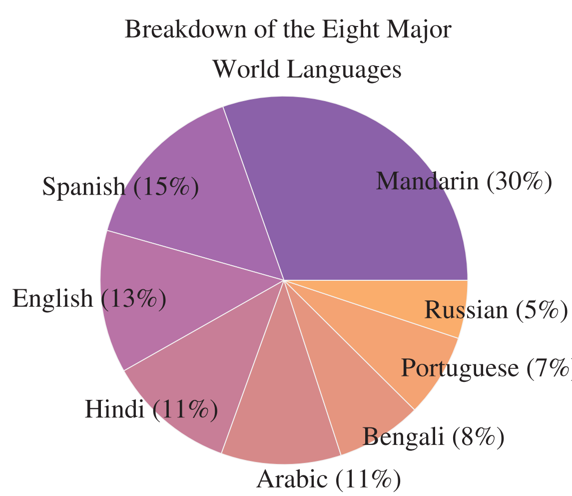
A study has been commissioned into the consumption of fast food. The graph to the right shows one of the results of this study. Discuss why this graph may be misleading if used in a local newspaper or television advertisement.

c A survey was designed to collect data to investigate the following scenarios. Comment on problems with the design of the experiment.

i A study is carried out to determine people's musical tastes. People are asked to fill in an online survey, and the results are then collated.

ii To investigate the growth in medical costs, a community group accesses data from their local hospital. The growth in total expenses at the hospital over time is displayed in a line graph.

13 In 2019, around 40% of the world's 7.7 billion population were first-language speakers of one of eight different languages. The sector chart to the right shows the breakdown of first-language speakers as a percentage of the top eight.



a What percentage of the 40% speak one of the three most common languages as a first language?

b How many people speak one of these eight as their first language?

c How many people in the world speak Mandarin as their first language?

d Around what percentage of the world's population speak English as their first language?

e Is this chart useful to a school deciding what languages to offer or a student deciding what language to learn?

14 The mean temperature for each month in Sydney (Observatory Hill) and Dubbo (airport) at 3pm is recorded in degrees Celsius on a radial chart.

For example, to read the mean temperatures for February, look at the radius marked 'Feb' — the mean temperature in Dubbo is about 30°C, and in Sydney it is about 25°C. The mean temperatures in January are on the adjacent radius, and each pair of values are joined by an unbroken interval for Sydney, and a broken interval for Dubbo.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the 3pm temperature as a measure of the temperature in Sydney and Dubbo:

a What is the 3pm temperature in Dubbo in
  i July, **ii** December?

b What is the 3pm temperature in Sydney in
  i August, **ii** March?

c What is the maximum 3pm mean temperature difference between the two locations, and in what month does it occur?

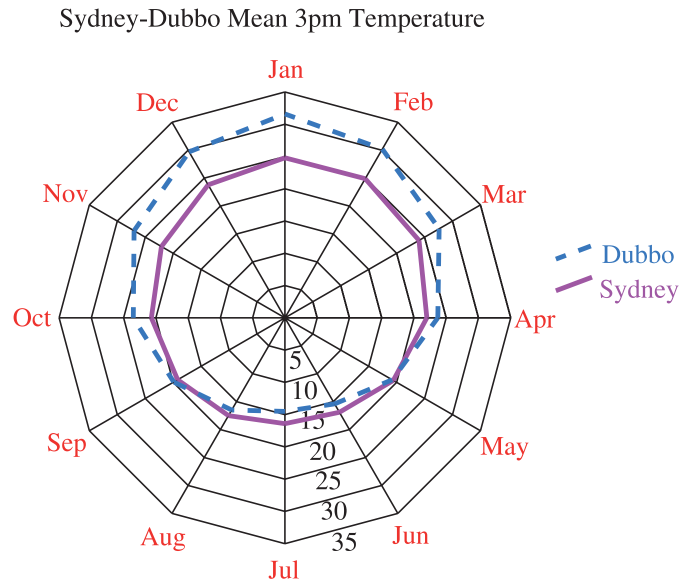
d In which months is the mean 3pm temperature in Sydney and Dubbo the same?

e In which months is Dubbo at least 5 degrees hotter than Sydney?

f Are there any months where Dubbo is colder than Sydney?

g Is this a good style of chart to display the data? Would there be a better type of chart to use?

h Why do you think that the designer of this chart chose to use dotted and solid lines, rather than just the cyan and magenta colours, to distinguish the two temperature lines?



15 A small class is working with pupils who have difficulty with English. After students had been in the class for some time, their examination results were recorded in the stacked bar chart below. The examination consisted of three sections: an interpretative exercise, an essay on a novel studied in class, and a creative writing exercise. Each section was awarded a mark out of 10.

a What was the examination out of in total?

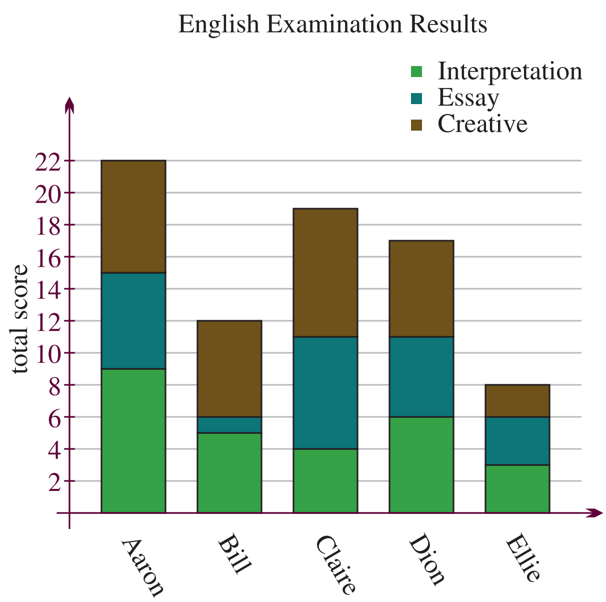
b What were the highest and lowest scores, as a percentage?

c Identify any of the three sections for which a pupil may need additional help.

d What percentage did Claire receive in Interpretation?

e What percentage did Dion receive in Essay?

f Students who receive 55% overall and at least 50% in each section leave this class. Who will be leaving the class following this examination?



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

16 The following two-way table shows the highest non-school qualification received by Australians aged between 15–64, with a break-down by age. The entries are percentages.

	15–24	25–34	35–44	45–54	55–64	Total 15–64
Postgraduate	0.4	6.4	6.2	5.3	4.7	4.6
Graduate diploma/cert	0.1	1.9	2.5	3.2	3.1	2.1
Bachelor degree	7.4	26.8	21.2	15.1	13.3	17.0
Adv diploma/diploma	4.2	9.6	11.1	11.7	9.1	9.1
Certificate I-IV	14.0	21.7	24.1	23.3	22.9	21.1
Other	2.1	3.0	2.7	2.5	2.8	2.6
None	71.8	30.6	32.2	38.9	44.1	43.5
Total (percent)	100%	100%	100%	100%	100%	100%
Total (thousands)	3122.5	3215.9	3118.1	2969.3	2422.3	14848.1

a *There is a greater percentage of people in the 25–34 age group with postgraduate degrees, compared with the 55–64 aged group. This suggests that more people are gaining postgraduate degrees in more recent times.* Comment on whether this is a reasonable interpretation of the data.

b What it the probability that an Australian chosen at random from the 45–54 age group has a post-school qualification?

c What it the probability that an Australian aged 15–64 chosen at random from the group with a post-school qualification lies in the age group 45–54?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15B Grouped data and histograms

The main purpose of organising data into tables and graphs is to see the data as a whole. When a table has too many rows or columns, or a graph has too much detail, such an overview is much more difficult. The usual approach in such situations is to *group* the data. This reduces the number of rows or columns on the tables, and reduces the amount of clutter on the graphs.

Grouping data

Here are the heights of 100 people in centimetres, from a file detailing individuals of all ages from the !Kung people of the Kalahari desert. The underlying random variable here is continuous (assuming that heights are real numbers), but height cannot be measured correct to more than a few significant figures.

151.765	139.7	136.525	156.845	145.415	163.83	149.225	168.91	147.955	165.1
154.305	151.13	144.78	149.9	150.495	163.195	157.48	121.92	105.41	86.36
161.29	156.21	129.54	109.22	146.4	148.59	147.32	137.16	125.73	114.3
147.955	161.925	146.05	146.05	142.875	142.875	147.955	160.655	151.765	171.45
147.32	147.955	144.78	121.92	128.905	97.79	154.305	143.51	146.7	157.48
127	110.49	97.79	165.735	152.4	141.605	158.8	155.575	164.465	151.765
161.29	154.305	145.415	145.415	152.4	163.83	144.145	129.54	129.54	153.67
142.875	146.05	167.005	91.44	165.735	149.86	147.955	137.795	154.94	161.925
147.955	113.665	159.385	148.59	136.525	158.115	144.78	156.845	179.07	118.745
170.18	146.05	147.32	113.03	162.56	133.985	152.4	160.02	149.86	142.875

The data seem to be given correct to 0.005 cm, which seems less than one can reliably measure, and the trailing zeroes that we normally insert are missing — always question the credibility of raw data. Perhaps heights were recorded in inches, then converted to centimetres. We have grouped the data in 10 cm intervals because that results in 10 classes, which is a good number for seeing the big picture. Here is the table of frequencies and cumulative frequencies.

interval	class centre	frequency	cumulative frequency
80–90	85	1	1
90–100	95	3	4
100–110	105	2	6
110–120	115	5	11
120–130	125	8	19
130–140	135	6	25
140–150	145	34	59
150–160	155	22	81
160–170	165	16	97
170–180	175	3	100

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The *class centre* on each row is the midpoint of the interval used in the grouping. The table could just as well have been written with rows instead of columns, and rows have been used later in the calculation of the mean and variance.

This makes the distribution of heights reasonably clear. A frequency distribution table based on the raw data, however, would be practically useless, because the frequency of almost every score is just 1 (and the histograms drawn below would also be useless).

Grouping data is a form of rounding. It is useful because it allows us to see the big picture, but it always involves ignoring information, and the summary statistics for the grouped data will only be an approximation of the summary statistics of the raw data. Never discard the original data.

For example, the median height is the average of the 50th and 51st heights. For the grouped data, both these heights are 145 cm (taking the class centre as the measurement), whereas if we work with the original data, the median is 147.955. Similarly, the range of the grouped data is $175 - 85 = 90$, but the range of the raw data is $179.07 - 86.36 = 92.71$.

When the underlying random variable is continuous, any data are already grouped by the rounding that all measurement involves. When those measurements involve several significant figures, as they do here, further grouping is usually required when displaying the data.

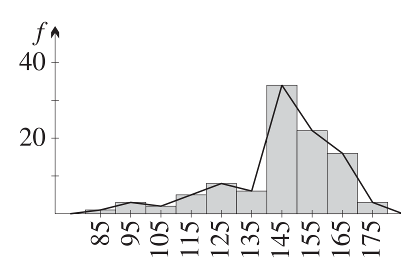
6 GROUPING DATA

- Numeric data, whether discrete or continuous, may be *grouped* so that the resulting tables and graphs give a clearer overview of the data.
- The grouping involves *intervals of equal width* and *class centres*.
- Grouping involves ignoring information. This may or may not be an issue.
- Data on a boundary should be treated consistently, and the treatment noted.

With a continuous variable, there may be data on the boundary, because data are always rounded. You can place these scores in the lower interval — this is consistent with the cumulative frequency convention in Section 15A. Or you can place them in the upper interval — this is also standard practice. But be consistent, and make a note about it if any boundary data actually occurred. This didn't happen with the data above.

Frequency histograms and frequency polygons

Whether or not the data have been grouped, a *frequency histogram* is the most basic way of displaying data in a graph. The diagram to the right shows the histogram of the grouped heights in the previous frequency table.



The *frequency polygon* has been added to the display. The two graphs can be drawn separately or together, and only one may be needed.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Some guidelines when drawing a frequency histogram

- For ungrouped data, each rectangle is centred on the value. For grouped data, each rectangle is centred on the class centre.
- The rectangles join up with no gaps.
- As a practical concern, too many columns in a histogram can make it difficult to interpret. Coarser grouping is the best solution here.
- The subintervals on the horizontal axis are often called *bins*.

Some guidelines when drawing a frequency polygon

- The plotted points are at the centre of the top of each rectangle.
- Join the plotted points with intervals.
- On the left, start the polygon on the horizontal axis, at the previous value or class centre.
- On the right, end the polygon on the horizontal axis, at the next value or class centre.

A question from the graph: Always ask questions about the data display.

- Why is the data so *skewed to the left*, with such low frequencies? Were children's heights included?

Histograms with discrete data

Histograms are designed for data from a continuous variable, which is the main reason why the rectangles should join up. When they are used for discrete data, be aware that the rectangles still have width, that they still join up (according to most conventions), and that they are centred on the values (or on the class centres for grouped data). These conventions will routinely involve numbers such as half-integers that are not possible values of the random variable.

Example 4 — 15B

Section 15A prepared a frequency table for 40 spelling test marks in Year 7.

Mark x	1	2	3	4	5	6	7	8	9	10
Frequency f	2	4	2	1	6	8	7	6	2	2

a Group the data by pairing the marks 1–2, 3–4, ...
b Draw a histogram and frequency polygon for the original data.
c Draw a histogram and frequency polygon for the grouped data.
d Comment on what the various displays have shown.

SOLUTION

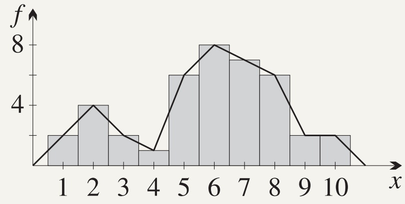
a

Interval	1–2	3–4	5–6	7–8	9–10
Class centre x	$1\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	$7\frac{1}{2}$	$9\frac{1}{2}$
Frequency f	6	3	14	13	4

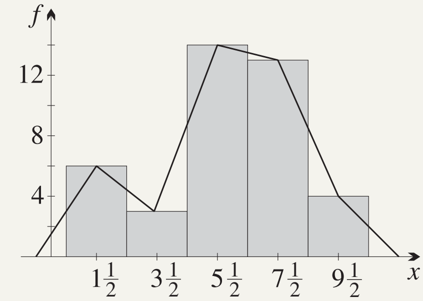
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b



c



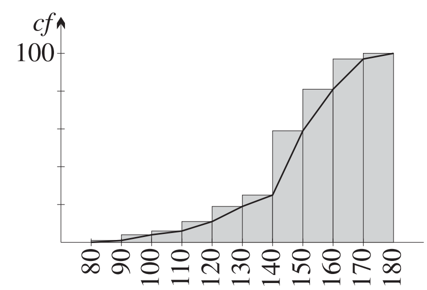
d The histogram of the grouped data perhaps makes it a little clearer that a significant group have difficulties either with spelling or with tests.

Note: The frequency polygon starts and finishes on the horizontal axis at the previous or next value or class centre. For the original data in part **b**, it starts at 0 and finishes at 11. For the grouped data, it starts at $-\frac{1}{2}$ and finishes at $11\frac{1}{2}$.

Cumulative frequency histograms and polygons (ogives)

A *cumulative frequency histogram* is drawn using the same procedures as for the earlier frequency histogram. The *cumulative frequency polygon*, also called an *ogive*, is drawn slightly differently corresponding to its cumulative nature.

We have drawn the two graphs together for the grouped table of heights at the start of this section, but again, each can be drawn separately.



Some guidelines when drawing a cumulative frequency histogram

- The rectangles of the frequency histogram are piled on top of each other to form the cumulative frequency histogram.
- The height of the last rectangle is the total size of the sample.

Some guidelines when drawing a cumulative frequency polygon

- The polygon starts at zero at the bottom left-hand corner of the first rectangle, when no scores have yet been accumulated.
- It passes through the top right-hand corner of each rectangle because it plots the scores less than or equal to the upper bound of the class interval.
- It finishes at the top right-hand corner of the last rectangle, and its height there equals the total size of the sample.

Example 5 15B

Section 15A also prepared a cumulative frequency table for the spelling test marks.

Mark x	1	2	3	4	5	6	7	8	9	10
Frequency f	2	4	2	1	6	8	7	6	2	2
Cumulative	2	6	8	9	15	23	30	36	38	40

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

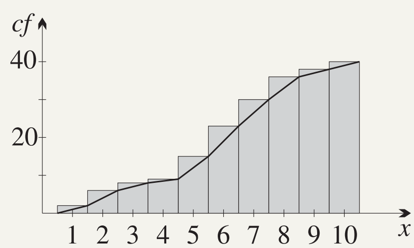
a Group the data by pairing the marks, adding the cumulative frequency.
b Draw a cumulative frequency histogram and ogive for the data.
c Draw a cumulative frequency histogram and ogive for the grouped data.
d Use the cumulative frequency tables to calculate the median using the original data and the grouped data, and compare them.

SOLUTION

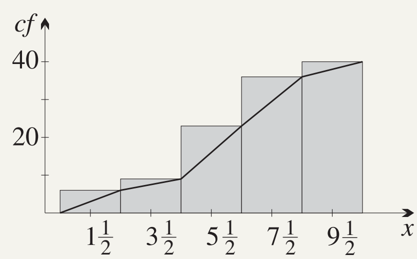
a

Interval	1–2	3–4	5–6	7–8	9–10
Class centre x	$1\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	$7\frac{1}{2}$	$9\frac{1}{2}$
Frequency f	6	3	14	13	4
Cumulative	6	9	23	36	40

b



c



d We calculated before that the median was the average of the 20th and 21st scores, which the cumulative frequencies tell us are both 6, so the median is 6.

From the grouped data, the class centres of the 20th and 21st scores are both $5\frac{1}{2}$, so the median is $5\frac{1}{2}$. Such discrepancies are normal after grouping.

7 HISTOGRAMS AND POLYGONS

- The rectangles of the *frequency histogram* and the *cumulative frequency histogram* join up. For ungrouped data, they are centred on the value, and for grouped data, they are centred on the class centre.
- The *frequency polygon* passes through the centres of the rectangles.
- On the left and right, the frequency polygon starts and finishes on the horizontal axis, centred on the previous or next value or class interval.
- The *cumulative frequency polygon* or *ogive* starts at the bottom left corner of the first rectangle.
- The ogive passes through the right-hand top corner of each rectangle.

The mean

In Year 11, we calculated the mean and variance of a sample in Section 13D, and we used relative frequencies in those calculations because our attention was on estimating probabilities from relative frequencies. This chapter, however, is about data, so in reviewing mean and variance, we will instead use formulae based only on the frequencies. The review also takes grouped data into account.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Recall that we use the symbols $\bar{x}$ and s for the mean and standard deviation when we are dealing with a sample, that is, with data, as we are in this chapter. When dealing with a population or a theoretical distribution, we use the symbol μ or $E(X)$ for the mean, and σ for the standard deviation.

Here again is the grouped frequency table of heights, arranged this time in rows, but it could as well be in columns.

class centre x	85	95	105	115	125	135	145	155	165	175	Sum
frequency f	1	3	2	5	8	6	34	22	16	3	100
$x \times f$	85	285	210	575	1000	810	4930	3410	2640	525	14470

The sum of the scores is the sum of the products $x \times f =$ score $\times$ frequency, except that with grouped data we use the class centres. This sum is 14470. The number of scores is the sum of the frequencies, which is 100. Hence

$$\text{mean} = \frac{\sum xf}{n} = \frac{14470}{100} = 144.70 \text{ cm.} \qquad \left(\sum xf \text{ means add all the products } xf.\right)$$

The authors also calculated the mean of the heights without grouping and obtained 145.352 cm. Thus the grouping reduced the mean by about $6\frac{1}{2}$ mm.

In Section 13D (Year 11) we wrote the mean as a weighted mean of the scores, weighting each score by its relative frequency using the formula $\bar{x} = \sum xf_r$, where f_r is the relative frequency. The relative frequencies are obtained by dividing each frequency by the number n of scores, that is, $f_r = \dfrac{f}{n}$, so the formula used above and the earlier formula are the same. To prove this, start with the earlier formula,

$$\bar{x} = \sum xf_r = \sum\left(x \times \frac{f}{n}\right) = \frac{\sum xf}{n}.$$

The variance and standard deviation

As explained in the Year 11 book, the variance and standard deviation are measures of spread, meaning that they measure how spread out the data are away from the centre. The standard deviation is the square root of the variance. Thus the variance has symbol s^2 for a sample and σ^2 or $\text{Var}(X)$ for a population or a theoretical distribution. In Section 13D of the Year 11 book, we developed two formulae for the variance of data,

$$s^2 = \sum (x - \bar{x})^2 f_r \qquad \text{and} \qquad s^2 = \sum x^2 f_r - \bar{x}^2.$$

Substituting $f_r = \dfrac{f}{n}$ for the relative frequency, these two formulae become

$$\begin{aligned} s^2 &= \sum (x - \bar{x})^2 f_r \\ &= \frac{\sum (x - \bar{x})^2 f}{n} \end{aligned} \qquad \text{and} \qquad \begin{aligned} s^2 &= \sum x^2 f_r - \bar{x}^2 \\ &= \frac{\sum x^2 f}{n} - \bar{x}^2. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

It is rare with data that the mean $\bar{x}$ is a round number, so the second form is the recommended form for calculation. The first form, however, makes it clear that the variance is a measure of spread — we are looking at deviations from the mean, squaring them so that they are all positive, then taking their weighted mean, weighted according to the frequencies.

8 MEAN AND STANDARD DEVIATION OF A SAMPLE

Suppose that data have been organised into a frequency table with scores x and frequencies f.

- The mean is $\bar{x} = \dfrac{\sum xf}{n}$ (where $\sum$ says take the sum over the distribution).
- The variance is $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2$.
- The standard deviation is the square root of the variance, and has the same units as the scores.
- With grouped data, use the class centres rather than the scores.

The actual definition of the variance is $s^2 = \dfrac{\sum (x - \bar{x})^2 f}{n}$. This formula is usually less suitable for calculation, but it makes it clear that we are taking the average of the squares of the deviations from the mean.

Here are the calculations for the variance and standard deviations of the heights.

x	85	95	105	115	125	135	145	155	165	175	Sum
f	1	3	2	5	8	6	34	22	16	3	100
xf	85	285	210	575	1000	810	4930	3410	2640	525	14 470
x^2f	7225	27 075	22 050	66 125	125 000	109 350	714 850	528 550	435 600	91 875	2 127 700

$$\begin{aligned} \text{Thus } \bar{x} &= \frac{\sum xf}{n} \\ &= \frac{14\,470}{100} \\ &= 144.7 \text{ cm,} \end{aligned}$$

$$\begin{aligned} \text{and } s^2 &= \frac{\sum x^2 f}{n} - \bar{x}^2 \\ &= \frac{2\,127\,700}{100} - 144.7^2 \\ &= 338.91, \\ s &\doteqdot 18.41 \text{ cm.} \end{aligned}$$

The authors used technology with the raw scores, and obtained $s^2 = 325.5407$ and $s \doteqdot 18.04$ cm. Grouping has produced results that are slightly different.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 6 15B

a Find the mean, variance and standard deviation of the Year 7 spelling test marks in worked Example 4.
b Calculate them again using the grouped data.

SOLUTION

a

x	1	2	3	4	5	6	7	8	9	10	Sum
f	2	4	2	1	6	8	7	6	2	2	40
$x \times f$	2	8	6	4	30	48	49	48	18	20	233
$x^2 \times f$	2	16	18	16	150	288	343	384	162	200	1579

Hence $\bar{x} = \dfrac{\sum xf}{n}$

$= \dfrac{233}{40}$

$= 5.825,$

and $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2$

$= \dfrac{1579}{40} - 5.825^2$

$= 5.544375,$

$s \doteqdot 2.355.$

b

Interval	1–2	3–4	5–6	7–8	9–10	Sum
Class centre x	1.5	3.5	5.5	7.5	9.5	—
Frequency f	6	3	14	13	4	40
$x \times f$	9	10.5	77	97.5	38	232
$x^2 \times f$	13.5	36.75	423.5	731.25	361	1566

Hence $\bar{x} = \dfrac{\sum xf}{n}$

$= \dfrac{232}{40}$

$= 5.8,$

and $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2$

$= \dfrac{1566}{40} - 5.8^2$

$= 5.51,$

$s \doteqdot 2.347.$

Again, the differences between the results of parts **a** and **b** arise from grouping.

A correction factor for the sample variance

These qualifications may not be required. We mentioned at the end of Section 13D in the Year 11 book that when we know the theoretical or the population mean μ, and we are sampling to find the variance, there is no problem with the formulae for the sample variance. When, however, we are sampling both to find the mean and to find the variance, then the sample mean will drift very slightly towards the sample results, with the effect that the sample variance will tend to be slightly smaller than it should be.

The standard solution is to multiply the sample variance by a correction factor $\dfrac{n}{n-1}$, where n is the size of the sample. Thus in the example using 100 heights, we were using a sample mean rather than a theoretical or population mean, so the correction factor is $\frac{100}{99}$. Using the correction factor would yield

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$\bar{x} = 144.7$ cm (as before) $\qquad s^2 = 338.91 \times \frac{100}{99} \doteqdot 342.3333 \qquad s = \sqrt{s^2} \doteqdot 18.50$ cm.

The larger the size n of the sample, the less difference the correction makes.

On the calculator, a button $\boxed{\sigma_n}$ or something equivalent does not apply this correction, and seems to be all that is required in this course. A button labelled $\boxed{\sigma_{n-1}}$ or equivalent applies the correction factor.

Currently in Excel 365, the function `STDEV.S` (S for 'sample') applies the correction, and the function `STDEV.P` (P for 'population') does not apply the correction, but earlier versions did things differently.

Like so many things in statistics, the distinction between a sample and a population is not straightforward. The Year 7 spelling test marks may be regarded as the record of spelling on that day from every member of the cohort — then the marks are a population. They may also be regarded as one set of estimates of an underlying random variable, 'spelling ability of each child', to be augmented by next week's spelling test and more in later weeks — then the marks are a sample. Scaling software, which massages results into aggregates and positions and ranks, tends to regard marks as a population. A classroom teacher, who is watching students learn and develop and is very aware that the choice and design of test questions are arbitrary, tends to regard marks as a sample.

A possible project

Systematic testing of the validity of this correction factor by taking a large number of samples from a known population could be developed into a project.

Perhaps theoretical probability distributions could also be considered, perhaps both discrete and continuous (as developed in Chapter 16).

Exercise 15B

FOUNDATION

1 a Copy and complete the following table to determine the mean and standard deviation for the data. (The mean is a whole number, so calculations using this formula will be straightforward.)

x	3	5	6	7	8	9	10	Sum
f	1	1	1	3	2	1	1	
$x \times f$								
$(x - \bar{x})^2$								
$(x - \bar{x})^2 f$								

$$\begin{aligned} \text{Mean} &= \bar{x} \\ &= \frac{\sum xf}{n} \\ &= \ldots \\ \text{Variance} &= s^2 \\ &= \frac{\sum (x - \bar{x})^2 f}{n} \\ &= \ldots \end{aligned}$$

b Repeat the calculation using the alternative formula for the variance.

x	3	5	6	7	8	9	10	Sum
f	1	1	1	3	2	1	1	
$x \times f$								
$x^2 \times f$								

$$\begin{aligned} \text{Mean} &= \bar{x} \\ &= \frac{\sum xf}{n} \\ &= \ldots \\ \text{Variance} &= s^2 \\ &= \frac{\sum x^2 f}{n} - \bar{x}^2 \\ &= \ldots \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 Use a table as in Question 1 to calculate manually the mean and standard deviation of the following datasets. Use the two forms for the variance in different parts — the means here are all whole numbers. Give your answers correct to two decimal places.

a 12, 14, 16, 17, 19, 21, 22, 23

b 2, 3, 3, 3, 6, 6, 7, 8, 8, 8, 9, 9, 10, 10, 13

c 40, 49, 50, 50, 51, 54, 57, 57, 57, 60, 65, 70

d 7, 8, 9, 9, 10, 10, 10, 10, 11, 11, 11, 11, 11, 12, 12, 12, 12, 13, 13, 14, 15

3 Use your calculator to find the mean and standard deviation of each dataset.

a 3, 7, 9, 10, 3, 4, 6, 8, 13, 6, 5, 12

b 4, 4, 4, 4, 5, 5, 7, 8, 8, 8

c 3.2, 3.6, 1.3, 2.4, 1.9, 4.1, 3.5, 4.1, 3.9, 2.3

d 34, 45, 23, 56, 34, 53, 23, 43, 37, 55, 52, 41, 43, 51, 57, 39

DEVELOPMENT

4 A census was carried out on the houses in Short Street to determine the number x of people in each household. The results are recorded in the frequency table below.

The population in this question is all the houses in the street. Because the data here are determined by a census of the whole population, statisticians use the symbol μ for the mean of the population (called the *population mean*) and the symbol σ for the standard deviation of the population (called the *population standard deviation*).

x	0	1	2	3	4	5	6	7	8
f	1	5	6	7	8	3	3	0	1

a How many houses are there in Short Street?

b Calculate the mean μ and standard deviation σ of the data.

c Group the data into the classes 0–2, 3–5 and 6–8 and construct a grouped frequency table.

d Calculate the mean and standard deviation of this grouped data.

e Why do your results from part **b** and **d** differ?

5 Xiomi recorded her time to get to work each day. Her results in minutes were:

22 30 23.5 27 25 21.5 39 30 32.5 33 35.5 37 42 22 23.5 27 29.5 23 34

a Write the data out in order, and determine the median.

b Group the data into classes by completing the following table.

class	20–24	24–28	28–32	32–36	36–40	40–44
class centre						
frequency						
cumulative						

c Find the median of the grouped data. Does it agree with your answer to part **a**?

d Draw a frequency histogram and polygon on the same chart.

e Draw a cumulative frequency histogram and polygon on the same chart.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

6 In a class experiment, students measured their heights. The results in centimetres were:

155 152 165 162 170 168 165 162 166 154 158 159
163 166 164 164 159 157 163 154 166 158 159 163

a Display the data in a frequency table.

b Calculate the median of the dataset.

c Why would it not be helpful to graph the data without first grouping it into classes?

d Group the data into the intervals 150–154, 154–158, 158–162, 162–166, 166–170 and display your results in a grouped frequency table. Include any scores on a boundaries in the lower group, thus x is in the group 150–154 if $150 < x \leq 154$.

e Calculate the median of the dataset from this grouped frequency table.

f Display your grouped data on a histogram with a frequency polygon joining the centres. Construct a cumulative frequency histogram and ogive — remember that the ogive starts at the bottom left-corner of the first rectangle and passes through the right-hand top corner of each rectangle.

g Trace the line at frequency 12 (50%) until it meets the ogive, and check whether this agrees with your answer for the median of the grouped data in part **e**.

h Construct a cumulative frequency histogram and ogive of the *ungrouped* data.

i Compare your grouped and ungrouped cumulative histograms in parts **f** and **h**. How similar are the graphs? Contrast the differences between the histogram of the grouped data in part **f** and what you would expect the histogram of the ungrouped data to look like (not drawn).

j Confirm that the line at frequency 12 meets your ungrouped ogive to give the same median as in part **g**.

ENRICHMENT

7 The Australian Bureau of Statistics (ABS) surveys important medical and physical information for the Australian population. According to their 1995 survey, the mean weight of a male over 18 was 82 kg, with a standard deviation of 13.6 kg.

The data were gathered from a sample of the whole population, but the quoted standard deviation has been calculated using the population standard deviation formula. Thus the *population variance* should be multiplied by the *correction factor* $\dfrac{n}{n-1}$, as discussed in the notes at the end of this section, to give the sample variance. This corrects for the drift of the calculated variance towards the sample results and away from the true population standard deviation.

a What would be the corrected sample standard deviation, assuming that the sample only surveyed:

i 10 people, **ii** 100 people, **iii** 1000 people?

b Actually, the ABS survey involved 10 000 people. What percentage change would the correction factor make to the standard deviation? Give your answer correct to three decimal places.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15C Quartiles and interquartile range

For numeric data, the spread of the data around the median can also be identified by the quartiles and the interquartile range.

Upper and lower quartiles

Write the scores in increasing order. The lower quartile, the median and the upper quartile attempt to divide this list into four equal parts.

An odd number of scores: Reliable Appliances sell toasters. Here are the numbers of toasters that they sold in each of 15 successive weeks.

$$19 \quad 16 \quad 18 \quad 15 \quad 16 \quad 19 \quad 17 \quad 21 \quad 16 \quad 16 \quad 20 \quad 18 \quad 30 \quad 19 \quad 21$$

First we write them out in a list in increasing order,

$$15 \quad 16 \quad 16 \quad 16 \quad 16 \quad 17 \quad 18 \quad \underset{\uparrow}{18} \quad 19 \quad 19 \quad 19 \quad 20 \quad 21 \quad 21 \quad 30$$

The number of scores in the list is 15, which is odd. The median Q_2 is the 8th score 18. Now divide the list into two sublists of 7, with the median in the middle,

$$15\ 16\ 16\ \underset{\uparrow}{16}\ 16\ 17\ 18 \qquad 18 \qquad 19\ 19\ 19\ \underset{\uparrow}{20}\ 21\ 21\ 30$$

The lower quartile Q_1 is the median of the left-hand list, which is 16, and the upper quartile Q_3 is the median of the right-hand list, which is 20. In summary:

$$Q_1 = 16 \qquad \text{and} \qquad Q_2 = 18 \qquad \text{and} \qquad Q_3 = 20.$$

An even number of scores: On the 16th week they sold 21 toasters, making 16 scores, which is even. The list can now be written out in two equal sublists,

$$15\ 16\ 16\ 16\ 16\ 17\ 18\ 18 \qquad 19\ 19\ 19\ 20\ 21\ 21\ 21\ 30$$

The median Q_2 is the average of the 8th and 9th scores, which is $18\frac{1}{2}$. The lower quartile Q_1 is the median of the left-hand list, which is 16, and the upper quartile Q_3 is the median of the right-hand list, which is $20\frac{1}{2}$. In summary:

$$Q_1 = 16 \qquad \text{and} \qquad Q_2 = 18\tfrac{1}{2} \qquad \text{and} \qquad Q_3 = 20\tfrac{1}{2}.$$

Interquartile range: The difference $Q_3 - Q_1$ between the upper and lower quartiles is called the *interquartile range* or IQR. It is the range of the middle 50% of the marks. Thus in the two examples above, the interquartile ranges are

$$\text{IQR} = Q_3 - Q_1 = 20 - 16 = 4 \qquad \text{and} \qquad \text{IQR} = Q_3 - Q_1 = 20\tfrac{1}{2} - 16 = 4\tfrac{1}{2}.$$

9 QUARTILES AND INTERQUARTILE RANGE

Suppose that a set of scores is arranged in increasing order.

An odd number of scores:

- Omit the middle score, thus separating the list into two sublists of equal size.
- The *lower* or *first quartile* Q_1 is the median of the left-hand list,
- The *upper* or *third quartile* Q_3 is the median of the right-hand list.

An even number of scores:

- Separate the list into two sublists of equal size.
- The *lower* or *first quartile* Q_1 is the median of the left-hand list,
- The *upper* or *third quartile* Q_3 is the median of the right-hand list.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The interquartile range — a measure of spread:
- The *interquartile range* or IQR is the difference $Q_3 - Q_1$.
- The interquartile range is a measure of spread.

Quartiles, like medians, are easily calculated from the cumulative frequency table. They can also be calculated for theoretical distributions, as in worked Example 7.

The five-number summary

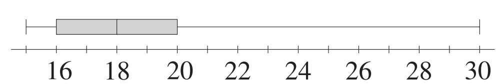
A data distribution can be usefully summarised in a *five-number summary*, which can then be displayed in a box-and-whisker plot. This summary presents the median, the quartiles, and the two extreme scores, that is,
- the minimum score (sometimes written as Q_0)
- the lower quartile Q_1
- the median Q_2
- the upper quartile Q_3
- the maximum score (sometimes written as Q_4).

Thus the five-number summaries of the two sets of weekly toaster-sale scores are:
15 weekly toaster-sale scores: 15, 16, 18, 20, 30
16 weekly toaster-sale scores: 15, 16, $18\frac{1}{2}$, $20\frac{1}{2}$, 30

Notice that the range is the difference between the first and last numbers, and the interquartile range is the difference between the second and second-last numbers. The symbols Q_0 and Q_4 are convenient, but are not standard notation.

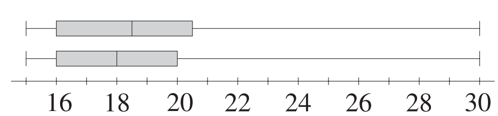
Box-and-whisker plots (box plots)

To the right is the *box-and-whisker plot* (also called *box plot*) of the 15 weekly toaster-sales scores. It displays the five-number summary in a clear diagram.



- The *box* extends from the lower quartile $Q_1 = 16$ to the upper quartile $Q_3 = 20$. Its length is the IQR.
- The vertical line within the box is the median $Q_2 = 18$.
- The *whiskers* extend left to the least score 15, and right to the greatest score 30, showing the range.

The second diagram shows the box plot of the 15 toaster-sale scores underneath, and above it the box plot of the subsequent 16 toaster-sale scores. It is a *parallel box plot*.



The addition of the one extra score 17 has increased the median and the upper quartile. The point here is to see immediately any significant differences in the overall picture.

Outliers

When scientists do experiments, they often end up with data that they secretly wish that they had not collected — perhaps they were not expecting these results, or the data do not fit their theories, or the data are ‘clearly’ the result of an experimental error, or they were ‘wrongly recorded’ by a research assistant, or they just look strange and the scientist doesn’t know what to do about them. These pieces of data are scores that lie a long way from most of the data collected and they consequently muck up the patterns that the other data create.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Such pieces of data are called *outliers*, and the inclusion or exclusion of these outliers from datasets causes serious arguments wherever statistics is used. And time and time again, outliers have been an indication of an inadequate theory that needed to be reformulated.

There are no generally accepted criteria for outliers, just a few contradictory rules that people argue about. In this course, we shall usually take a criterion based on quartiles and the interquartile range IQR. We usually take an outlier to be a score that lies

$$\text{more than } 1.5 \times \text{IQR below } Q_1 \qquad \text{or} \qquad \text{more than } 1.5 \times \text{IQR above } Q_3.$$

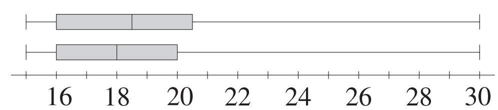
This criterion is very simple and can usually be calculated mentally.

- One problem with this test is that the gap between the suspect outlier and the next score or scores is also an important criterion, and *analysis of such gaps is missing from this test*.
- A second problem is that when there are very large datasets, we expect from the normal laws of probability to have scores many IQRs from the quartiles, but *the size of the sample is missing from this test*.

In the end, nothing can replace careful attention to the scores themselves. In most circumstances, outliers should be left in the dataset, but probably should be displayed differently and commented on.

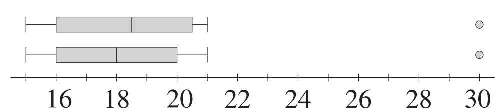
Outliers in box-and-whisker plots

Box-and-whisker plots are easily adapted to show outliers. You will have seen in both the sets of toaster-sales data that one week 30 toasters were sold, whereas the next highest score is 21.



The score of 30 is an outlier by two criteria — it is well separated from the other scores, and (in the case of the 15 scores) it is 10 above the upper quartile, which is 2.5 times the interquartile range.

We have therefore redrawn the parallel box plots to the right, with the right-hand whisker stopping at 20, and a circle placed at 30 as a code for the outlier. Outliers are often indicated on a box plot in this or a similar way.



Explaining outliers is most important — there was a toaster sale that week.

Example 7 — 15C

a Throw a die until a six occurs, and record the number n of throws needed. Do this 50 times, or use the results of class members, or simulate it using random numbers or a spreadsheet.

b Construct a frequency table and cumulative frequency table.

c Find the median, the quartiles, and the interquartile range.

d Draw a box plot and discuss any outliers.

e Let X be the number of tosses required to get a six. Explain why

$$P(X = 1) = \tfrac{1}{6}, \quad P(X = 2) = \tfrac{5}{6} \times \tfrac{1}{6}, \quad P(X = 3) = \left(\tfrac{5}{6}\right)^2 \times \tfrac{1}{6}, \quad \ldots,$$

and use GP theory to prove that the limiting sum of these probabilities is 1.

f Copy and complete the following cumulative discrete probability table, giving each value correct to three decimal places.

n	1	2	3	4	5	6	7	8	9	10	...
$P(X = n)$	0.167										
$P(X \le n)$	0.167										

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

g Hence find the theoretical mean, quartiles and interquartile ranges. What values of n are classified as outliers according to the IQR criterion? Then sketch a box plot of the theoretical results.

h Explain why both the box plot of the data and the box plot of the theoretical distribution are unsymmetric.

SOLUTION

a These results were obtained by simulation using random numbers based on part **e**. Note the gaps in the last two results.

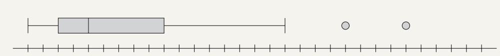
b

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	22	26
f	4	5	9	4	5	2	3	1	2	3	1	2	1	1	1	2	1	1	1	1
Cml	4	9	18	22	27	29	32	33	35	38	39	41	42	43	44	46	47	48	49	50

c The median Q_2 is the average of the 25th and 26th scores, which is 5. The lower quartile Q_1 is the 13th score, which is 3. The upper quartile Q_3 is the 38th score, which is 10.

d The IQR is $10 - 3 = 7$, so

$$Q_3 + 1.5 \times \text{IQR} = 10 + 10\tfrac{1}{2} = 20\tfrac{1}{2}.$$



The IQR criterion for outliers classifies the last two scores 22 and 26 as outliers. This is also a common-sense classification, because these last two scores are well separated from the other scores.

e Using standard probability techniques from Chapter 12 of the Year 11 book and using N to represent 'not rolling a six' and S to represent 'rolling a six',

$$P(X = 1) = \tfrac{1}{6} \qquad = \left(\tfrac{5}{6}\right)^0 \times \tfrac{1}{6},$$
$$P(X = 2) = P(\text{NS}) = \tfrac{5}{6} \times \tfrac{1}{6} \qquad = \left(\tfrac{5}{6}\right)^1 \times \tfrac{1}{6},$$
$$P(X = 3) = P(\text{NNS}) = \tfrac{5}{6} \times \tfrac{5}{6} \times \tfrac{1}{6} \qquad = \left(\tfrac{5}{6}\right)^2 \times \tfrac{1}{6},$$
$$\ldots$$

This is a GP with first term $a = \tfrac{1}{6}$ and ratio $r = \tfrac{5}{6}$,

so $$S_\infty = \frac{a}{1 - r} = \tfrac{1}{6} \div \left(1 - \tfrac{5}{6}\right) = 1.$$

f

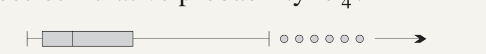
n	1	2	3	4	5	6	7	8	9	10
$P(X = n)$	0.167	0.139	0.116	0.096	0.080	0.067	0.056	0.047	0.039	0.032
$P(X \le n)$	0.167	0.306	0.421	0.518	0.598	0.665	0.721	0.767	0.806	0.838

g The median is $Q_2 = 4$, because it is the first score whose cumulative probability is at least $\tfrac{1}{2}$.

The lower quartile is $Q_1 = 2$ because it is the first score whose cumulative probability is $\tfrac{1}{4}$.

The upper quartile is $Q_3 = 8$ because it is the first score whose cumulative probability is $\tfrac{3}{4}$.

Hence the IQR is $8 - 2 = 6$. Thus the IQR criterion classifies as an outlier every score greater than $Q_3 + 1.5 \times \text{IQR} = 8 + 9 = 17$.



h The frequencies or probabilities are bunched up on the left and spread out or *skewed* on the right.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Median and quartiles vs mean and standard deviation

We now have two families of summary statistics:

- Mean, variance and standard deviation.
- Median, quartiles and interquartile range.

Earlier in this section, we used 15 weeks of toaster sales and found that

$$Q_1 = 16, \qquad Q_2 = 18, \qquad Q_3 = 20, \qquad \text{IQR} = 4.$$

After further calculation, the mean is 18.73 and the standard deviation is 3.53.

Now suppose that we replace the outlier 30 by 21, the highest of the other scores. The mean, quartiles and interquartile range do not change, but the mean changes to 18.13, and the standard deviation changes dramatically to 2.00. That is, the standard deviation may be very sensitive to outliers, the mean less so.

If Reliable Appliances is tallying up its cash flow and profits, they would use mean and standard deviation. If they were looking at their marketing and want to study toaster purchases outside exceptional situations such as sales, they would use the median and the quartiles, which are *robust* to outliers.

House prices are another much-discussed example. The prices of homes are stretched out or *skewed* on the right by very expensive homes, so that the median is a more useful measure of prices of ordinary homes than the mean. The upper quartile is also not affected by those very expensive homes, so that interquartile range may be a better measure of the spread than the standard deviation.

Summary statistics review

The summary statistics discussed in Sections 15A–15C were:

10 SUMMARY STATISTICS

Measures of location

Mode, median, mean

Measures of spread

Range, interquartile range, variance, standard deviation,

The five-number summary

minimum, first quartile Q_1, median Q_2, third quartile Q_3, maximum

A *box-and-whisker plot* is constructed from the five-number summary.

We have described data several times as being *skewed*. Recall from earlier years that data are skewed in the direction of the tail, not the peak.

- *Skewed to the right*, or *positively skewed*, means that there is bigger tail on the right-hand side.
- *Skewed to the left*, or *negatively skewed*, means that there is bigger tail on the left-hand side.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 15C

FOUNDATION

1 For each dataset, calculate the three measures of central tendency (mean, median and mode), and calculate the range.

a 4, 8, 5, 2, 9, 12, 8

b 12, 23, 18, 30, 24, 29, 19, 22, 25, 12

c 7, 6, 2, 5, 7, 3, 4, 5, 7, 6

d 54, 62, 73, 57, 61, 61, 54, 66, 73

2 Find the middle quartiles Q_1, Q_2, Q_3 and the interquartile range $\text{IQR} = Q_3 - Q_1$ for each dataset.

a 3, 7, 9, 13, 14, 17, 20

b 8, 12, 13, 17, 20, 24, 27, 31

c 4, 7, 8, 9, 11, 14, 17, 19, 20

d 2, 5, 7, 10, 13, 17

e 2, 4, 5, 7, 12, 13, 14

f 8, 10, 12, 15, 17, 21, 22

g 3, 4, 6, 7, 9, 11, 13, 14, 18

h 9, 12, 13, 15, 18, 21

3 Find the IQR of each unordered dataset.

a 15, 10, 12, 19, 1, 17, 13, 6, 2

b 1, 11, 14, 9, 0, 4

c 12, 7, 9, 11, 13, 2, 9

d 6, 3, 2, 12, 0, 6, 8, 4

e 7, 11, 7, 5, 10, 7

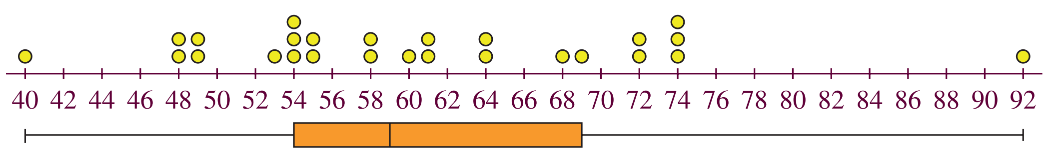
f 2, 9, 5, 4, 5, 9, 12

g 8, 3, 4, 1, 12, 2, 4, 11

h 10, 4, 5, 18, 11, 13, 2, 9, 7

DEVELOPMENT

4



The combined box plot and dot diagram above shows the exam scores for a small cohort of 26 students.

a Use your intuition to identify any *outliers*, thinking here of outliers as scores that are a long way from the rest of the data.

b Write down the five-number summary statistics: the minimum value, the lower quartiles Q_1, the median Q_2, the upper quartile Q_3, and the maximum value.

c The IQR criterion identifies *outliers* as those values less than $Q_1 - 1.5 \times \text{IQR}$ or more than $Q_3 + 1.5 \times \text{IQR}$. Use this criterion to identify any outliers.

d Do your answers to part **a** and **c** agree?

e Recalculate the three quartiles Q_1, Q_2, Q_3 if we:

i omit 40 only,

ii omit 92 only,

iii omit both 40 and 92.

f Do the quartiles and the IQR change much when outliers are removed?

g Calculate the mean and standard deviation of the dataset correct to one decimal place:

i with all values,

ii without 40,

iii without 92,

iv without 40 and 92.

h What is the change in the standard deviation in part **g iv** as a result of removing the outlying values, as a percentage of the standard deviation in part **g i**?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 **a** For each dataset, calculate the interquartile range and identify any outliers using the IQR criterion. Then draw a combined dot plot and box-and-whisker plot.

i 3, 5, 6, 7, 7, 9, 18
ii 9, 10, 10, 12, 12, 13, 18
iii 9, 10, 10, 12, 12, 13, 18, 19
iv 1, 3, 7, 9, 9, 9, 10, 10, 11, 12
v 1, 3, 8, 9, 9, 9, 10, 10, 11, 12
vi 5, 7, 7, 8, 8, 8, 8, 9
vii 5, 7, 7, 8, 8, 8, 8, 9, 9
viii 1, 1, 2, 2, 3, 10, 12, 12, 12, 12, 13, 13, 13, 13, 13, 14, 14, 15, 16, 22, 22, 23, 24, 25

b By reference to your answers in part **a**, comment on how well this definition of outliers seems to agree with your intuitive notion of an outlier as an extreme value.

c The dataset in part **a i** contained an outlier. Redraw the box-and-whisker plot for part **a i** with the whiskers excluding the outlier, and with the outlier shown as a single small circle on your diagram.

6 **a** Draw up a frequency table and hence determine the mean and standard deviation for the dataset

16, 12, 13, 14, 14, 14, 12, 16, 15, 24

b Construct a dot plot. Do any of the values seem to be *outliers*, that is isolated points, a long way from the rest of the data?

c Confirm that the IQR criterion of an outlier agrees with your instinct in part **b**.

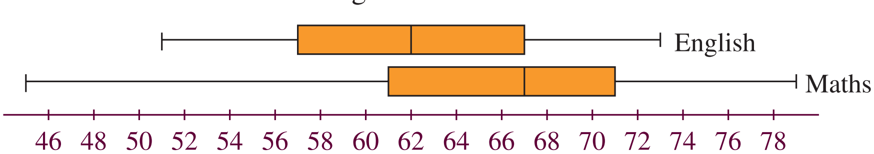
d Remove the outlier you should have found in parts **b**–**c**, and recalculate the mean and standard deviation.

e Has the outlier had a big effect on the mean and standard deviation? Use your tabular calculations to explain why this might be so.

f Has the outlier had a big influence on the median or IQR?

g The interquartile range and the standard deviation are both measures of spread. When might the interquartile range be a better measure than the standard deviation?

7 The results of a class in an English task and a mathematics task are shown in the parallel box plot below.



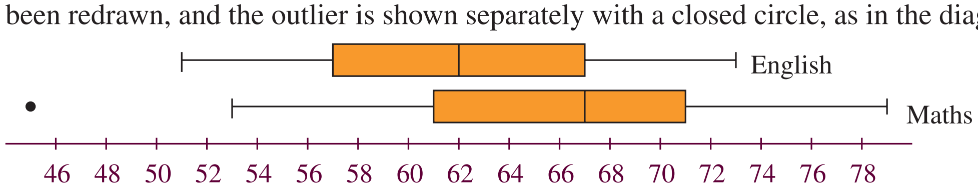
a Emily's result was in the bottom half of the English cohort. What can be said about her mark?

b What percentage of the students fall below 67 in mathematics?

c Comment on the spread of the two distributions by comparing the medians of the two sets of scores, and then by comparing the entire distributions.

d Xavier gets 66 in both English and mathematics. Is this a more impressive result in English or in mathematics, and why?

e Closer analysis shows that there is actually an outlier in the mathematics results. The box plots have been redrawn, and the outlier is shown separately with a closed circle, as in the diagram below:



i What was the outlier score in mathematics?

ii Compare again the spread of the bottom 25% of the students in mathematics and English, this time with the outlier excluded.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 An English class completes a writing and a speaking task. The results are displayed in the back-to-back stem-and-leaf plot below.

Writing task		Speaking task
5	3	7
	4	
6 6 5 2	5	1 3 4 7 8 8
9 9 8 8 7 7 4 4	6	3 5 5 6 6 7 8
5 5 4 4 2 1	7	1 1 3 5 7
	8	
1	9	3

a Genjo got 35 in the writing task. What was his score in the speaking task?
b For the writing task:
 i calculate the mean, median and range,
 ii calculate the interquartile range and determine any outliers.
c For the speaking task:
 i calculate the mean, median and range,
 ii calculate the interquartile range and determine any outliers.
d Which set of results was more impressive?

ENRICHMENT

9 [An investigation that could become a project] In this section we have given two tests for outliers — the IQR criterion, and graphing the data and applying common sense with particular attention to any gaps. The discussion of outliers is an important subject and books have been written on the subject. Interested students may like to investigate this further. Questions may include the following:
a Some statisticians label scores that are below $Q1 - 3 \times \text{IQR}$ or above $Q3 + 3 \times \text{IQR}$ as *extreme outliers*. Scores below $Q1 - 1.5 \times \text{IQR}$ or above $Q3 + 1.5 \times \text{IQR}$ that are not extreme outliers are called *mild outliers*. What questions in this exercise have included extreme outliers? Generate some data with both and note the difference on a dot plot and a box-and-whisker plot.
b What is the effect of having two or more outliers in a distribution — can they mask each other's existence from an IQR criterion test? Generate datasets to explore this. Include the possibility of multiple outliers at both ends of the dataset, or only on one end.
c Outliers can also be defined by measuring the number of standard deviations from the mean. Calculate the number of standard deviations that the outliers in this exercise are from the mean, and decide if this could be developed into a reasonable criterion. Start with the datasets in Question 5.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

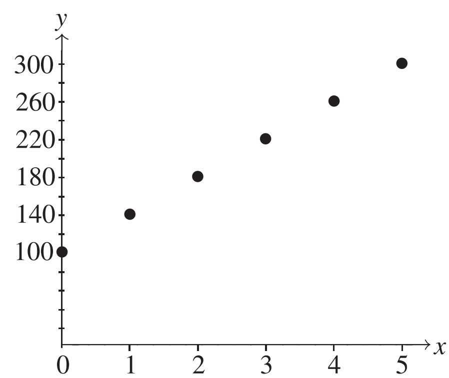
15D Bivariate data

For bivariate data, two further summary statistics are used — correlation, and the line of best fit. This section takes an intuitive approach to correlation and line of best fit, using displays and drawing the line of best fit by eye.

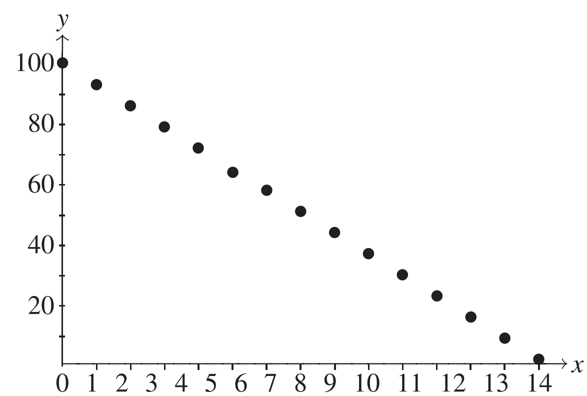
Section 15E will introduce formulae for correlation and the least squares version of the line of best fit. The last section, Section 15F, will use technology to analyse and display bivariate data.

Variables can be correlated without being related by a function

This course mostly concerns functions, where a variable y is completely determined by a variable x. Here are two linear functions, one with positive gradient, and one with negative gradient.

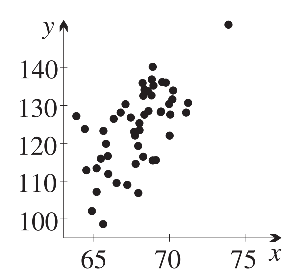


An electrician charges \$100 to visit a home, then \$40 for each power point. His total fee y for installing x power points in a home is $y = 100 + 40x$.



One hundred old cars were dumped in a park, and the council is removing 7 per day. The number y of cars remaining after x days is $y = 100 - 7x$.

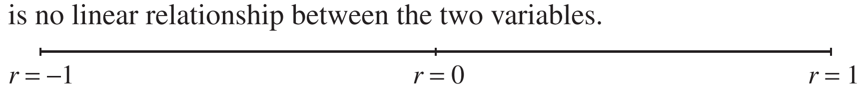
Now think about the relationship between the heights and weights of people. The *scatterplot* to the right plots the height x inches and weights y pounds of a group of people, with x as the *independent variable* and y as the *dependent variable*.



The cluster of dots is spread out because people of the same height don't all have the same weight, that is, y is not completely determined by x. And yet there seems to be more than a random relationship between the heights and weight of the people in the group, because it looks as if taller people tend to be heavier.

Correlation is the word for such a statistical relationship. In this section we will judge correlation by eye after the points are plotted on a scatterplot, then in Sections 15E–15F we will introduce a summary statistic called *Pearson's correlation coefficient* r. When y is determined by x, as in the two graphs above,

- $r = 1$, if they lie on a straight line and increase together,
- $r = -1$, if they lie on a straight line with one increasing, the other decreasing.
- $r = 0$, if there is no linear relationship between the two variables.



Correlations of 1 and -1 are called *perfect correlation*. We will perform some calculations of Pearson's correlation coefficient by hand in Section 15E, and by technology in Section 15F.

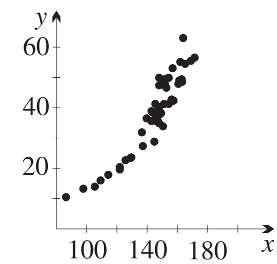
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The choice of which variable is independent and which is dependent is sometimes obvious, but is sometimes rather arbitrary, and in either case the choice may well be a matter for a scientist rather than a mathematician. A choice must be made, however, for the methods of these sections to work.

Heights and weights — a positive correlation

The raw data of the heights of people in Section 15B came also with the weights of those people.

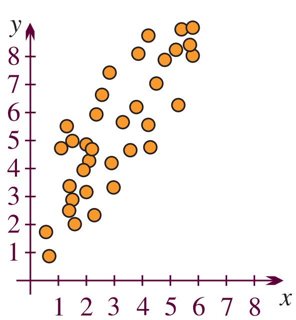


Here is the *scatterplot* of the first 50 pairs of measurements, with the height x cm taken as the independent variable on the horizontal axis, and the weight y kg taken as the dependent variable on the vertical axis.

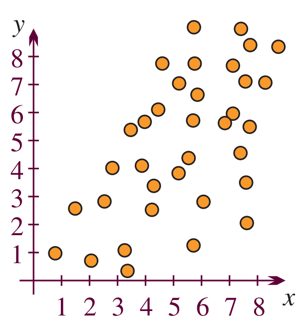
Each dot in the chart represents the height and weight of one person. Notice that the horizontal scale starts at 80 cm, not at zero.

This gives a visual demonstration that the weight in that group is very closely related to the height. We can classify this visually as a *strong correlation* — that the actual value of the correlation here is about 0.928, which is regarded as very strong (the correlation of the previous scatterplot is 0.64 — still strong, according to many published guides). The correlation is also *positive*, meaning that both variables increase together and that the slope of the cluster is positive.

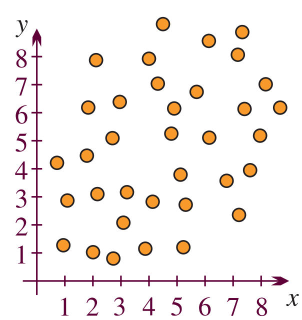
Here is a rough guide to positive correlations — it can only be a rough guide:



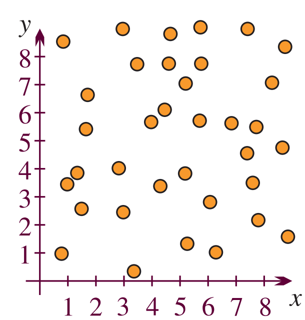
strong



moderate

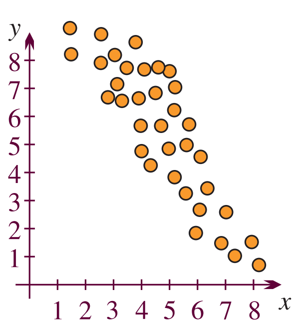


weak

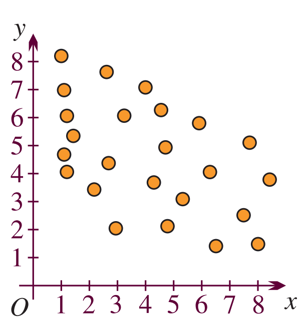


none

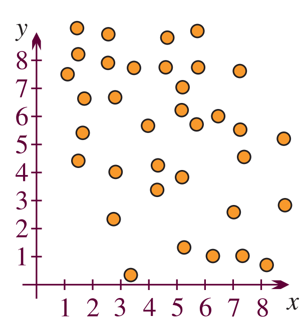
And here is a rough guide to negative correlations:



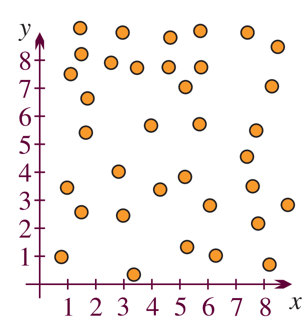
strong



moderate



weak



none

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Non-linear correlation

Data are always more complicated than whatever is said about them. The cluster of dots in the heights–weights scatterplot above has a definite curve in it. Perhaps it should be tested not against a straight line, which is all that we will do in this chapter, but against a curve. Should that curve be quadratic, or exponential? Perhaps it should be cubic, because the volume of similar figures is proportional to the cube of the height. Such reasoning shows how prediction and causation are always involved in any discussion about data.

11 CORRELATION

Bivariate data means data in the form of ordered pairs. Suppose that we have identified an *independent variable* x and a *dependent variable* y in bivariate data.

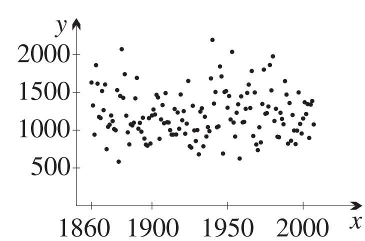
- A *scatterplot* graphs all these pairs on the one coordinate plane.
- *Linear correlation* occurs when the dots in the scatterplot tend to cluster in a shape vaguely like a line.
- Linear correlation is usually measured by *Pearson's correlation coefficient*, or simply the *correlation*, which is a real number r in the interval $-1 \le r \le 1$.

$r = -1$ $\qquad\qquad$ $r = 0$ $\qquad\qquad$ $r = 1$

- *Non-linear correlation* occurs when the dots in the scatterplot tend to cluster in a shape vaguely like some other curve.

Rainfall over the years — no correlation

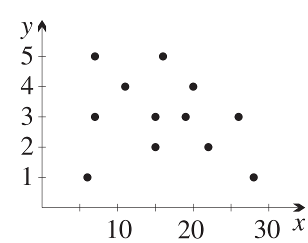
The Bureau of Meteorology has data for rainfall in Sydney for every day of every year since about 1858. We can download and massage that data to find the annual rainfall y in millimetres each year x from 1860–2007 — the scatterplot is drawn to the right.



There is no (linear) correlation. Pearson's correlation coefficient is -0.014, which is virtually zero. But look at the dots! You can see the drought years and the flood years.

Callers put on hold — a negative correlation

Customers of a large technology company often ring about problems, and are put on hold for short or long periods. At the end of their call they are asked to rank the company 1 (very dissatisfied) to 5 (very satisfied). Some preliminary test data on the waiting time x minutes and the rank y were taken, with the following results.



x	7	15	22	11	20	15	7	28	6	16	26	19
y	5	2	2	4	4	3	3	1	1	5	3	3

The correlation here is negative because the cluster slopes backwards, and it can be roughly characterised as weak (the calculated correlation is about -0.26).

The point $(6, 1)$ could be identified as an *outlier*. It has a big effect. Place your finger over this single dot. The correlation now looks moderate (the calculated value is about -0.57). Correlation is very sensitive to outliers when there are few values — both the visual impression and the calculated value.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

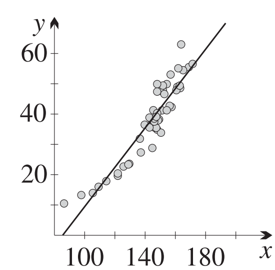
12 POSITIVE, NEGATIVE AND ZERO CORRELATION

- A positive slope in the cluster corresponds to $0 < r \le 1$, and is called *positive correlation*. A correlation of $r = 1$ means *perfect positive correlation*, that is, y is a linear function of x with some positive gradient.
- A negative slope in the cluster corresponds to $-1 \le r < 0$, and is called *negative correlation*. A correlation of $r = -1$ means *perfect negative correlation*, that is, y is a linear function of x with some negative gradient.
- $r = 0$ means that there is no linear correlation between the variables.

One qualification about zero correlation is needed. A horizontal line of points, or a cluster in the vague shape of a horizontal line, both have zero correlation.

The line of best fit

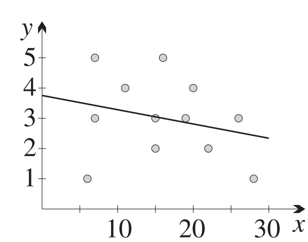
The purpose of correlation is to test whether the cluster of results are suggesting a line, sloping forwards or backwards. The line that they best cluster around is called the *line of best fit* or the *regression line*. The most common way of calculating it is to find the line that minimises the squares of the vertical distances from the points to the line, and the resulting line is called the *least squares regression line*. In Section 15E (a Challenge section) we will use formulae to calculate the gradient and y-intercept of this line of best fit, and in Section 15F we will use technology, but for now, we will estimate it by eye.



Here again is the scatterplot of the heights and weights of 50 individuals. We have drawn on it by eye a line of best fit. It has gradient about 0.65, and x-intercept about 85. Its equation is therefore about

$$y - 0 = 0.65(x - 85)$$
$$y = 0.65x - 55.25$$

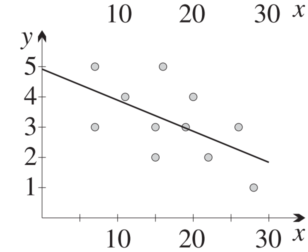
Using the formulae in the next section, the gradient and y-intercept of the least-squares regression line are about 0.649 and -55.52. Again, be careful because the horizontal scale starts at 80 cm, not zero.



The scatterplot of the callers on hold shows only weak correlation, so it is difficult to draw a line of best fit by eye. By the formulae, the gradient is about -0.047, and the y-intercept is about 3.75, giving the line

$$y = -0.037x + 3.75$$

that we have drawn on the graph.



In the second scatterplot, the outlier has been omitted (this is not the recommended procedure). The correlation is now moderate, and this makes it easier to draw the line by eye. The formulae in the next section tell us that the gradient is about -0.10, and the y-intercept is about 4.92. The resulting line

$$y = -0.1x + 4.92$$

has been drawn on the graph.

Why was the outlier there? Perhaps someone rang simply to cancel the service because of all sorts of other problems, and the short waiting time had no effect on his rank of 1. Should it be ignored? That is up to management and what questions they are asking.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Double, triple and multiple points

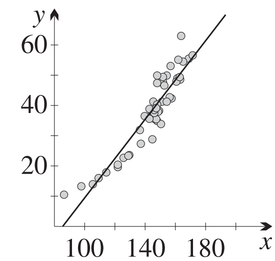
None of our diagrams or datasets so far have repeated points, but in practice repeated points often occur. There are different conventions — use ever larger circles, use a code of circles, squares and crosses, place numbers inside the circle, ... Use whatever is convenient.

It is important to be aware of repeated points when judging correlation and line of best fit by eye, otherwise one's judgement will be right out. See Question 4 of Exercise 15D.

13 REGRESSION — THE LINE OF BEST FIT

- Given bivariate data, we can calculate the *line of best fit* or *regression line*.
- When correlation can be seen clearly, we can draw the line of best fit by eye. It is important then to identify any multiple points in the scatter graph.
- Outliers may have a significant effect on correlation and the line of best fit.

Interpolation, extrapolation, prediction and causation



Interpolation mean *predicting further results within the range of the variables in the data*. That is reasonably straight forward once the line of best fit has been drawn. Interpolation is justified, provided that we are convinced that the sample we are working from is not biased.

Extrapolation mean *predicting further results outside the range of the variables in the data*. That can present a real problem, because the situation is often quite different outside the range of sample values. For example, given the scatterplot and line of best fit of 50 heights and weights, a person of height 85 cm would be predicted to have zero weight, and a baby of height 40 cm would float away!

Even when data have extremely high correlation — high enough for the relationship to be regarded as a function — extrapolation is dangerous. Newton's laws of motion cannot be extrapolated to speeds approaching the speed of light because of relativity theory, and cannot be extrapolated to very tiny particles because of quantum mechanics.

Causation is best left to scientists. If events A and B are correlated, there are four possibilities — A causes B, B causes A, A and B have a common cause C, and the correlation is a fluke. Events can have multiple causes, particularly in medicine and weather. Many phenomena are chaotic, such as the weather and eddies in flowing water, making prediction impossible and rendering the idea of causation extremely complicated.

14 INTERPOLATION, EXTRAPOLATION, PREDICTION AND CAUSATION

- Provided that the data are reliable, the line of best fit can reasonably be used for *interpolation*, meaning prediction of results within the range of the variables.
- It can also be used for *extrapolation*, meaning prediction of results outside the range of the variables in the data, but this requires caution and common sense because the results of extrapolation are often very misleading.
- Questions of causation are probably best left to scientists.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 15D

FOUNDATION

1 In the following relationships, identify the most reasonable choice for: **i** the independent variable, **ii** the dependent variable. Is there any uncertainty in your answer?

a height and weight,

b the area of a circle and its radius,

c the weight of meat purchased and the price,

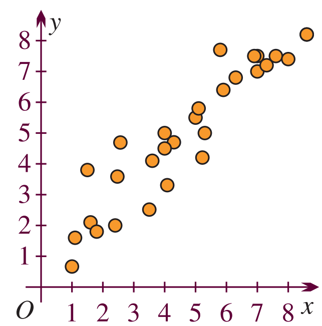
d the outcome for a player at Wimbledon Tennis championships and the player's previous world rank,

e outside temperature and household power consumption,

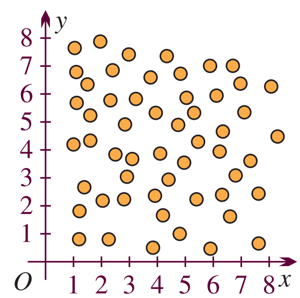
f pronumerals x and y for the many-to-one function $y = f(x)$.

2 By eye, decide if each of these graphs is an example of strong, moderate, weak or no linear correlation. Where there is a correlation, note whether it is a positive or negative correlation.

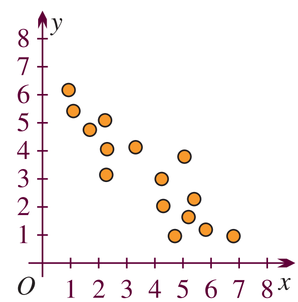
a



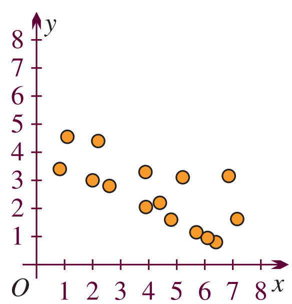
b



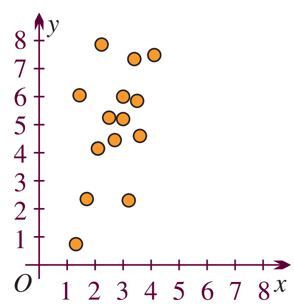
c



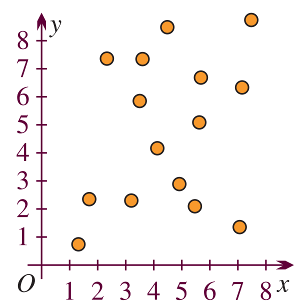
d



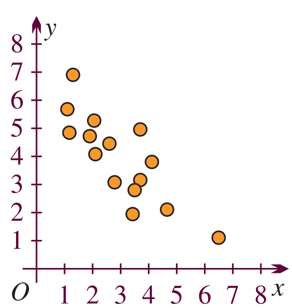
e



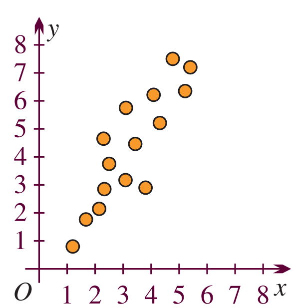
f



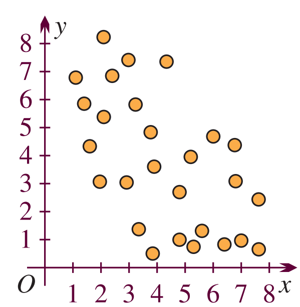
g



h



i



3 Plot each dataset on a separate diagram and draw a line of best fit by eye. Write down the equation of the line of best fit that you have drawn.

a $(0,3)\ (1,2)\ (2,4)\ (3,3)\ (4,5)\ (5,5)\ (6,7)\ (9,7)$

b $(0,30)\ (1,35)\ (2,45)\ (3,40)\ (4,55)\ (5,55)\ (6,70)\ (7,65)$

c $(0,16)\ (2,14)\ (3,8)\ (4,10)\ (6,7)\ (7,4)\ (9,4)\ (12,0)$

d $(0,6)\ (1,8)\ (2,7)\ (3,5)\ (4,4)\ (5,5)\ (6,4)\ (7,1)$

ISBN 978-1-108-76630-2

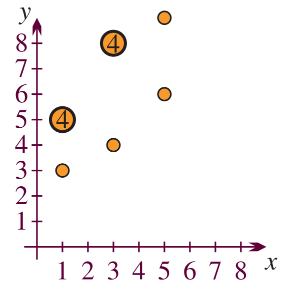
Cambridge University Press
Updated July 2022

4 The following datasets include repeated points, with the frequency indicated by a number on the plot. For each question, allowing for the extra weighting of the repeated points:

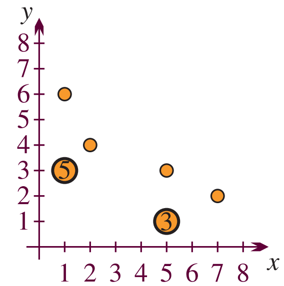
i Estimate the strength of the correlation

ii Copy the diagram and draw in a line of best fit by eye,.

a

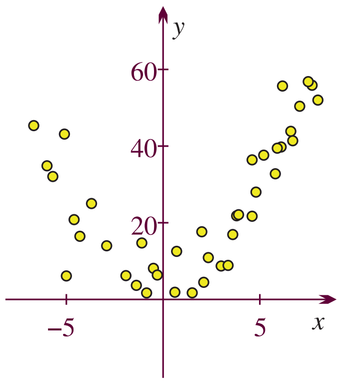


b

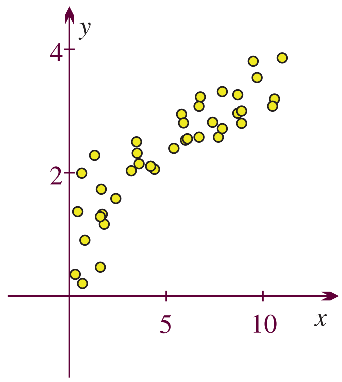


5 Not every dataset shows a linear relationship — some data are best modelled by a quadratic curve (a parabola), a circle (or semi-circle), a hyperbola, a square root, an exponential, or some other such curve. By eye, suggest what type of curve might model each dataset.

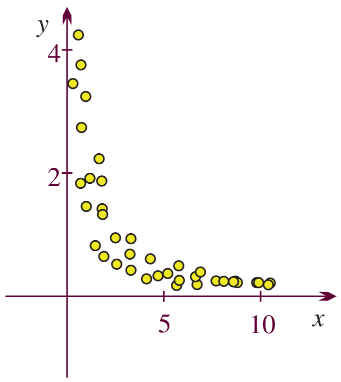
a



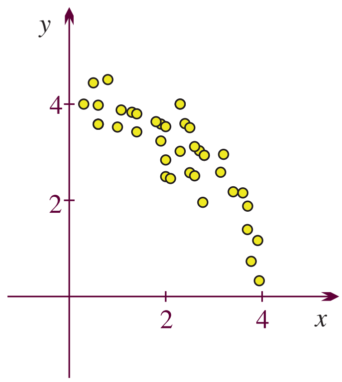
b



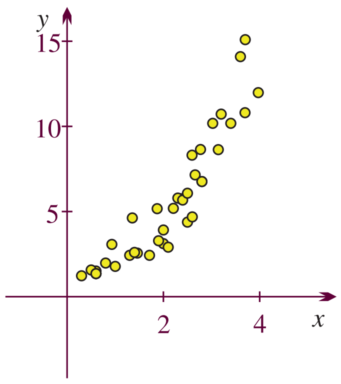
c



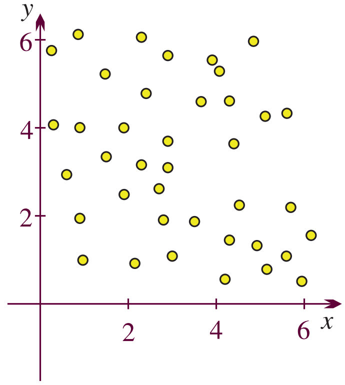
d



e



f

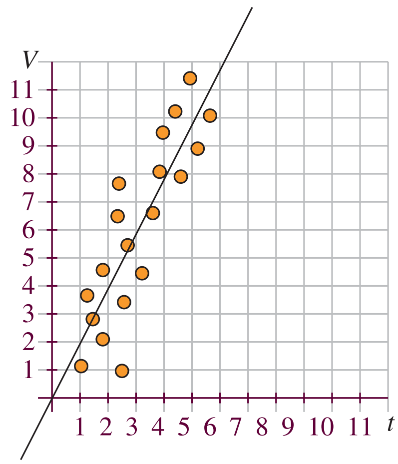


ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

6 Yasuf has conducted an experiment. He recorded the volume in litres of water that flowed through a pipe in a given time of between 1 and 6 minutes. Then he repeated this procedure many times over a period of several hours in the middle of the day. His results are shown in the scatterplot to the right. Yasuf has also drawn a line of best fit through the data.



a Use interpolation, by reading off the graph, to estimate the volume of water flowing through the pipe in: **i** 3 minutes, **ii** 5 minutes.

b Estimate the equation of the line of best fit.

c What is the value of the V-intercept, and why would you expect that value?

d What is the physical meaning of the gradient?

e Should Yasuf have drawn the line down through the origin into the third quadrant?

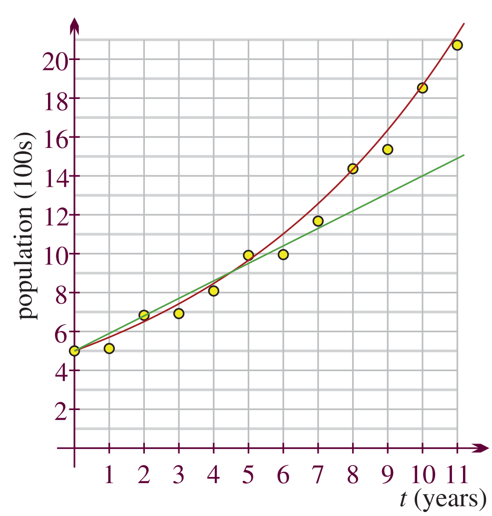
f Why do you think the correlation in this experiment is not perfect, that is, why don't all the points lie on a straight line?

g Use the equation of the line of best fit determined by Yasuf to extrapolate the amount of water flowing through the pipe in half an hour. Is this extrapolation reasonable?

h How long would be required for the pipe to disgorge 45 litres of water?

i Yasuf wishes to estimate how much water will flow through the pipe in one 24-hour day. Explain why extrapolating from these results may not be valid.

7 Population in the town of Hammonsville has been growing strongly over the last few years. When town planners first took a census in 2010, the population of people living in the town was 500. The population over the next 11 years is recorded in the scatterplot to the right. Planners have also attempted to fit the data with various curves to model future population growth.



a What was the population five years later in 2015?

b After seven years the planners began to draw a scatterplot, and they added a line of best fit in order to estimate population trends.

Place your fingers over the last four points on the scatterplot so that you see only the 8-point scatter graph that the planners had after 7 years.

i Find the equation of the line (population P in 100s as a function of time t years since 2010). Use the fact that the line passes through $(0, 5)$ and $(10, 14)$.

ii Does this line look a good fit for the seven-year period?

iii In 2017 predictors used this model to extrapolate the population after a further two years (that is, when $t = 9$). What was the error in their prediction, compared with the plotted population in that year?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

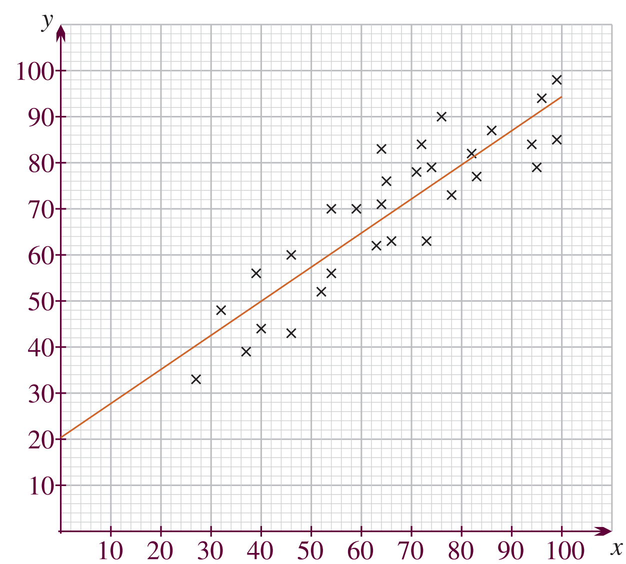
c Experts noticed that their model became an increasingly poor predictor as time went on, and instead attempted to fit the data with an exponential curve $P = 5 \times 2^{0.19t}$, where P is the population t years after 2010.

i Check the accuracy of this model by calculating its population prediction in 2019.

ii Would you expect this model to be accurate for the next 10 years?

d What does this question suggest about the general viability of using a line to fit data (that is, a line of best fit)?

8 Student percentage marks for assessment 1 (x) and assessment 2 (y) have been compared on a scatterplot, and a line of best fit has been drawn. There are no repeated points.



a What was the top mark in each assessment? Were they obtained by the same student?

b What was the bottom mark in each assessment? Were they obtained by the same student?

c Which assessment was more difficult? Give reasons for your claim.

d Some students were absent for one or the other of the assessment tasks, and their missing scores must be estimated. All scores are to be recorded as a whole number. Use interpolation on the line of best fit to estimate the score in the other assessment of a student who received:

i 40 in assessment 1, **ii** 60 in assessment 1, **iii** 80 in assessment 2,

iv 40 in assessment 2, **v** 15 in assessment 2.

e Read off the coordinates of the points on the line at $x = 0$ and $x = 100$. Hence find the equation of the line of best fit.

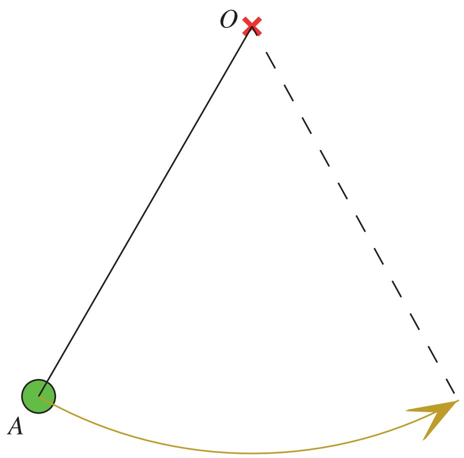
f Is this an accurate method of estimating a student's missing assessment score?

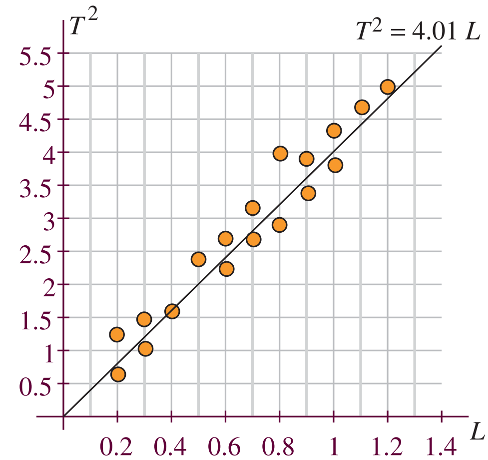
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

9 A class is carrying out an experiment. A weight is attached by a string to a fixed point O. It is drawn aside and released, allowing it to swing and then return to its original position. The time taken to return is called the *period* of the pendulum. The experiment requires students to measure the periods T seconds for different string lengths L metres. Theory suggests that the square of the period is related to the length of the string. The class's results are shown in the scatterplot below, graphing T^2 on the vertical axis and L on the horizontal axis.





a For a given length of string, what is the maximum difference between the square of the period predicted by the linear model and the square of the measured period?

b This particular maximum difference is significantly larger than for other points, and it appears to be an outlier in the data. Comment on possible causes.

c The class has claimed great accuracy in measuring the times for a period, as can be seen from the strong correlation. What methods may they have used to achieve this accuracy?

d Scientist have developed a theoretical model relating T and L. The model predicts that $T = 2\pi\sqrt{\dfrac{L}{g}}$, where $g \doteqdot 9.8\text{ m/s}^2$. Does this agree with these experimental results?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15E Formulae for correlation and regression

You must be able to use technology to calculate Pearson's correlation coefficient and the line of best fit from given data. The next section goes into some detail about those procedures.

This section presents the actual formulae for the line of best fit. They are rather elaborate, and calculations using them take more time and paper than the earlier calculations for mean and standard deviation. Nevertheless, calculating at least a few examples by hand can prevent statistics becoming a 'black-box' where the user of the results has no real idea what is happening. In a mathematics course, understanding is key.

These formulae could be regarded as Enrichment. The very short exercise has just one purpose — familiarity with the formulae.

The formula for Pearson's correlation coefficient

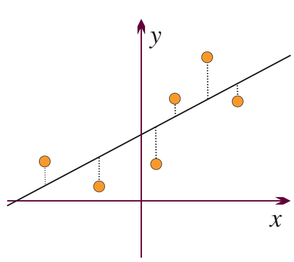
The standard measure of correlation is *Pearson's correlation coefficient* r. It tests only for linear correlation, that is, it gives a measure of how close the data are to being on a line of non-zero gradient,

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}.$$

We will not develop this formula, but these remarks should help to understand its significance and how to go about calculating it.

- We must first calculate the means $\bar{x}$ and $\bar{y}$ of the x-values and of the y-values. The point $(\bar{x}, \bar{y})$ will lie in the middle of the cluster on the scatterplot.
- We need all the *deviations from the mean*. These are the deviations $x - \bar{x}$ of the x-values from their mean $\bar{x}$, and the deviations $y - \bar{y}$ of the y-values from their mean $\bar{y}$.
- For the numerator, we take each product $(x - \bar{x})(y - \bar{y})$. This is the key object, because if x and y both lie on the same side of their means, the product is positive, and if they lie on opposite sides, the product is negative. Adding them up gives some sense of whether the variables are working together, or working contrary to each other and cancelling out.
- The denominator is necessary to normalise the quantity and make it a ratio. In particular, notice that the units of x cancel out and the units of y cancel out, so that the resulting quotient r is a pure number.
- The denominator is closely related to the formulae for the standard deviation of x and of y. In fact, the formula for r can be rewritten using the standard deviation, and standard deviation calculations can be re-used here.
- Pearson's correlation coefficient is unaffected by units and gradient (apart from the sign of the gradient). If we change metres to centimetres, or multiply all the y-values by $+7$, there is no change in r. That is, only the clustering and the sign of the gradient are relevant.

Formulae for the regression line



The standard method of finding the regression line is to find the line that minimises the sum of the squares of the vertical distances from each plot to the line — this line is called the *least squares regression line*. Again, we omit any derivation, and simply state that the line is

$$y = mx + b, \text{ where } m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} \quad \text{and} \quad b = \bar{y} - m\bar{x}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

- The numerator of the gradient m is the same as the numerator of r.
- The denominator of m has already been calculated when calculating r.
- The value of the y-intercept b ensures that the line passes through $(\bar{x}, \bar{y})$.

Thus once the calculations required for r have been done, the calculation of the line of best fit is very quick.

Calculations using the callers on hold

The example in Section 15D of the waiting times of callers on hold was deliberately engineered so that the means were whole numbers. Otherwise an alternative form of the formula would need to be developed, which is not appropriate in this course, or machine calculation would be necessary.

The sums of the first and second lines of the table allow the means $\bar{x}$ and $\bar{y}$ of the x-values and y-values to be calculated. These means are needed in the third and fourth lines to calculate the deviations.

													Sum
x	7	15	22	11	20	15	7	28	6	16	26	19	192
y	5	2	2	4	4	3	3	1	1	5	3	3	36
$x - \bar{x}$	−9	−1	6	−5	4	−1	−9	12	−10	0	10	3	0
$y - \bar{y}$	2	−1	−1	1	1	0	0	−2	−2	2	0	0	0
$(x - \bar{x})^2$	81	1	36	25	16	1	81	144	100	0	100	9	594
$(y - \bar{y})^2$	4	1	1	1	1	0	0	4	4	4	0	0	20
$(x - \bar{x})(y - \bar{y})$	−18	1	−6	−5	4	0	0	−24	20	0	0	0	−28

First, $\bar{x} = \dfrac{\sum x}{n} = \dfrac{192}{12} = 16,$ and $\bar{y} = \dfrac{\sum y}{n} = \dfrac{36}{12} = 3.$

Hence
$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}} = \frac{-28}{\sqrt{594 \times 20}} \doteqdot -0.25689.$$

For regression,
$$m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} = \frac{-28}{594} \doteqdot -0.04714,$$

and
$$b = \bar{y} - m\bar{x} = 3 + \frac{28 \times 16}{594} \doteqdot 3.75421.$$

Thus the correlation is -0.26, and the line of best fit is $y = -0.047x + 3.754$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 FORMULAE FOR CORRELATION AND REGRESSION

Let $\bar{x}$ and $\bar{y}$ be the means of the x-values and y-values of a set of bivariate data.

- Pearson's correlation coefficient r is given by

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}.$$

- The least-squares regression line is $y = mx + b$, where

$$m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} \qquad \text{and} \qquad b = \bar{y} - m\bar{x}.$$

Other forms of these formulae combine them with the formula for variance.

Classifying correlations

The verbal descriptions used in Section 15D for the strength of correlation are visual impressions of the scatterplot. There is no agreed relationship between those verbal descriptions and the values of the correlation coefficient calculated in this section and the next. The same criteria may not be appropriate for different disciplines or for different experiments within a discipline.

The authors regard the following as reasonably helpful suggestions. For positive correlations (and similarly for negative correlations),

Correlation	0.6–1.0	0.4–0.6	0.1–0.4	0.0–0.1
Description	strong	moderate	weak	virtually none

There are no rules — have the scatterplot at hand, think about the experiment, and be aware of any outliers.

According to this classification, the caller waiting times and ranks, with a correlation of about -0.26, show weak negative correlation. With the outlier (6, 1) removed, the correlation is about -0.57, which is moderate negative correlation.

The correlations of the set of eight scatterplots in Section 15D are:

Top row: 0.82, 0.59, 0.35, 0.07 Bottom row: -0.87, -0.5, -0.37, -0.09

Exercise 15E

Note: If this exercise is attempted, the best approach is perhaps to do the calculations once or twice to gain some familiarity, and then to combine calculations by hand with calculations by technology. Further opportunities to use technology for such calculations are given in Questions **1**–**2** of Exercise 15F.

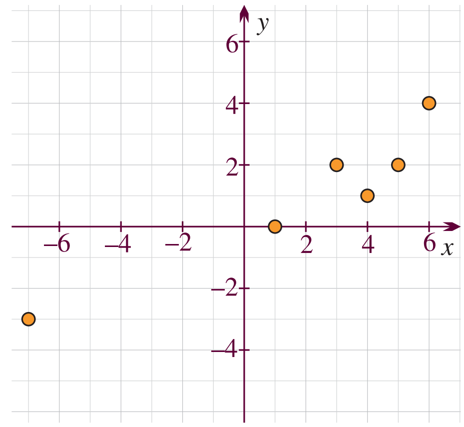
ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

1 A student is given the task of calculating the line of best fit for the small dataset:

$(-7, -3) \quad (1, 0) \quad (3, 2) \quad (4, 1) \quad (5, 2) \quad (6, 4)$

The data are shown on the scatterplot to the right.



a Does the correlation of the data appear linear, and if so, does the correlation appear strong, weak or moderate?

b Copy the scatterplot into your book. By eye, estimate and draw the line of best fit for the data.

c Copy the following table into your book. Complete the sum for the first two rows, and hence calculate the means $\bar{x}$ and $\bar{y}$.

							Sum
x	-7	1	3	4	5	6	
y	-3	0	2	1	2	4	
$x - \bar{x}$							
$y - \bar{y}$							
$(x - \bar{x})^2$							
$(y - \bar{y})^2$							
$(x - \bar{x})(y - \bar{y})$							

d Mark the point $(\bar{x}, \bar{y})$. Does it fall on your estimated line of best fit?

e Complete the last five rows of the table.

f Use the formula

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}$$

to find Pearson's correlation coefficient for the data, correct to two decimal places.

g Is r close to 1 or to -1? This would indicate that the line is a good fit for the data.

h Use the formula to calculate the gradient m of the least squares line of best fit,

$$m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}.$$

i Use the formula $b = \bar{y} - m\bar{x}$ to calculate the y-intercept of the line of best fit,

j Write down the line of best fit calculated from these formula, rounding m and b each correct to one decimal place. If this line differs from your estimate of the line of best fit in part **b**, add this line to your diagram.

2 Repeat the previous question for the following datasets.

a $(-2, 0) \quad (0, 0) \quad (1, 1) \quad (3, 1) \quad (4, 2) \quad (6, 2)$

b $(-3, -4) \quad (-2, -3) \quad (0, 1) \quad (2, 3) \quad (3, 4) \quad (6, 8)$

c $(-4, 7) \quad (-2, 6) \quad (-1, 1) \quad (0, -1) \quad (1, -3) \quad (2, 1) \quad (4, -4)$

d $(-2, 6) \quad (-1, 3) \quad (0, 4) \quad (1, 2) \quad (2, 0) \quad (6, -3)$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

15F Using technology with bivariate data

It is clear from Section 15E that calculations by hand are laborious. The right tools are statistics calculators, spreadsheets and statistical packages, all of which may be online. This section has several purposes, loosely grouped around technology.

- Small datasets suited to a statistics calculator to find r and line of best fit.
- Larger datasets for investigations using spreadsheets and other software.
- Investigation activities such as surveys to generate data for analysis.
- Investigations allowing the reader to search out raw data from the internet.

Waiting times of callers on hold — correlation and regression

The previous section calculated Pearson's correlation coefficient for the waiting times of callers and their ranking of the company (data in Section 15D). If you have a *calculator* that is capable of performing the calculations, then work out now how to use it to get the correlation and the line of best as calculated in Section 15E. The various calculators differ from one another, and we have not chosen not to give detailed instructions for any particular model. Find the instructions and read them.

For any extended work, however, a *spreadsheet* is the best tool to use (in the absence of specialised statistical software). Excel is widely used, and we have worked the example here using Excel. This example, and the questions in Exercise 15F, should be easily adapted to other technology, including online versions.

Excel has many different versions, and its functions are complicated. Use the help file or search online for guidance. In particular:

- The functions in Excel for Windows keep changing over time. All the functions used here were different some years ago.
- The Mac versions of Excel have their own peculiarities. For example, 'Fill down' and 'Fill right' are well-known difficulties on a Mac.

In particular, if you cannot find the search box in the fourth dotpoint below, search for 'Formula Builder' in the help file.

Here are the steps in finding the correlation and regression line using the most recent version of Excel 365. The dataset is drawn from the 'Waiting time of callers' just above Box 12 in Section 15D.

The data: Type the data into Excel from the table in Section 15D.

- Type 'Time x' into cell `A1` and 'Rank y' into cell `B1` — these are the headers.
- Then type the 12 data pairs into the cells `A2:A13` and `B2:B13`.

The means: Type 'Mean x' and 'Mean y' into cells `D1:E1`.

- Place the cursor into cell `D2` and type $=$ into it. This initial character $=$ is the code for Excel to interpret what follows in a cell as a function.
- You will notice that in the top left below the word `File` there is now a text box with a down arrow. Click on the arrow, then click on 'More functions'.
- In the resulting search box, enter 'Average'. Select `AVERAGE` and read its description. Perhaps also click the link to the help file, and perhaps compare this function with the function `AVERAGEA`.
- Double-clicking will bring up a dialogue. You only need Number1, and you can either enter the cells with the x-values as `A2:A13`, or select them in the spreadsheet so that Excel enters the cell labels. The result should be the mean 16.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

- With cell D2 selected, look at the text box above the row-and-column array. It should give the formula in the cell, which is =AVERAGE(A2:A13).
- Do the same for the mean of the y-values in cell E2 — the mean is 3. But Excel is cleverer than this! Instead, select cells D2:E2, and press Ctrl+R to Fill right. You can check that the formula in cell D2 has been copied to cell E2, except that column A in the formula has been changed automatically to column B. You can also see immediately what cells have been referenced by selecting cell E2 and pressing the F2 key.

The standard deviations: Type 'SD x' and 'SD y' into cells D4:E4.
- Then repeat the steps to insert the means into cells D5:E5, except search for 'standard deviation', and select the function STDEV.P.
- (As we remarked in an extension note at the end of Section 15B, it may be more correct, because this is a sample, to use the sample standard deviation STDEV.S rather than the population standard deviation STDEV.P.)

The correlation: Type 'Correlation' into cell D7.
- Insert the correlation into cell D8 as before — type = and click on the down arrow, select 'More functions', but search for 'correlation' and select PEARSON.
- Enter the x-values into the top box and the y-values into the second box.

The line of best fit: Type 'Regression' into cell D10, 'Gradient' into cell D11 and 'Intercept' into cell E11.
- To insert the gradient of the regression line into cell D12, search for 'Regression' and select SLOPE. Be careful here because things are reversed! Enter the y-values into the top box, and the x-values into the second box.
- To insert the y-intercept of the regression line into the cell E12, search for 'Regression' and select INTERCEPT. As before, enter the y-values into the top box, and the x-values into the second box.

	A	B	C	D	E
1	Time x	Rank y		Mean x	Mean y
2	7	5		16	3
3	15	2			
4	22	2		SD x	SD y
5	11	4		7.035624	1.290994
6	20	4			
7	15	3		Correlation	
8	7	3		-0.25689	
9	28	1			
10	6	1		Regression	
11	16	5		Gradient	Intercept
12	26	3		-0.04714	3.754209
13	19	3			

Waiting times of callers on hold — the scatterplot and regression line

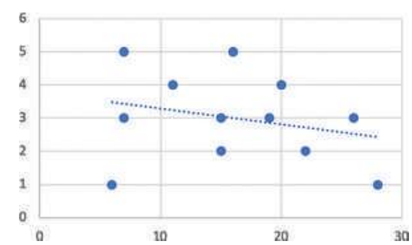
Excel can draw a scatterplot of the data with the regression line inserted.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The scatterplot: First select the 24 cells `A2:B2` down to `A13:B13` that contain all the data.

- Click on the 'Insert' tab at the top of Excel. Find the 'Charts' group, and click 'Recommended charts'. Then click on the 'All charts' tab.
- You will see all sorts of charts there (including box-and-whisker that we discussed in Section 15C), but the one we want is 'X Y Scatter'. Click on it, place the cursor on the second of the two charts that just shows the 12 dots all in one colour, and double-click.
- You now have a scatterplot placed onto your spreadsheet. You can move it and resize it in the usual ways. When you double-click on the chart, a menu appears on the far right allowing other changes to be made.



The line of best fit: Click on the chart again, and a new tab 'Design' appears on the very top of the Excel window.

- Click on the 'Design' tab, find the group 'Chart layouts' (on the far left), and select 'Add chart element'.
- Go to 'Trendline' and select 'Linear'. Now the least squares regression line appears, calculated as in Section 15E.

Saving the chart: You can export the chart to other software such as Word or image-processing software.

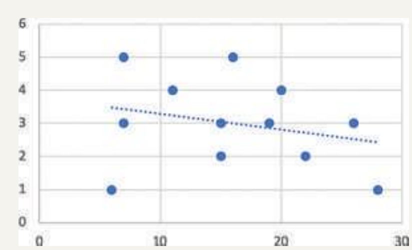
- Click on the chart, then right-click on the top border, then click 'Copy'.
- Paste the file where you want it.

Example 8 — 15F

Repeat all the steps above for the call waiting times with the outlier removed. Alternatively, your spreadsheet should allow you to remove one data point.

SOLUTION

- The means are about 16.91 for the x-values and 3.18 for the y-values.
- The standard deviations are about 6.64 for the x-values and 1.19 for the y-values.
- The correlation is about -0.57.
- The regression line is about $y = -0.1x + 4.92$.
- Excel's scatterplot with the regression line is copied to the right.



Internet data — investigations and possible projects

Exercise 15F is mostly concerned with processing data from the internet using technology. The methods are the same, whether there are 12 or 12 000 pairs of data.

Internet data, however, can be very rough. Always look carefully through the data for obvious anomalies. For example, in the heights and weights data introduced in Section 15D, many entries had the height field or the weight field missing, or both, and there were a few entries that seemed to have become corrupted — these are two of the few good reasons for omitting outliers.

The first few questions use technology for correlation and regression. Then the exercise is intended to provide investigations of various types. Most questions can easily be extended to projects by asking further questions and using and comparing several sources of data.

ISBN 978-1-108-76630-2

Exercise 15F

INVESTIGATION

Calculation of the gradient and the y-intercept of the line of best fit, and the calculation of Pearson's correlation coefficient, are best done using technology. These calculations may be done using a calculator with a regression analysis mode (also sometimes labelled '2 Var Stats' or '$A + Bx$'), with a spreadsheet program (such as Excel), with statistical software (such as the software package R), or with an online program designed to analyse such data. A number of such free online programs are available if you type *online line of regression* into a search engine.

The datasets in the first four questions are small, to enable practice with the technology before attempting to analyse some more realistic data. Further practice with technology is provided by the datasets in Exercise 15E.

1 Use technology to calculate Pearson's correlation coefficient and the equation of the line of best fit for each dataset.

- **a** $(1, 1.7)$ $(2, 1.9)$ $(3, 3.7)$ $(4, 4)$ $(5, 4.5)$ $(6, 6.7)$
- **b** $(-2, 3)$ $(0, 2)$ $(2, 2.3)$ $(3, 3.1)$ $(4, 4)$ $(5, 5.7)$ $(6, 6.1)$
- **c** $(3.6, 3.2)$ $(5.9, 3.9)$ $(7, 1)$ $(4.4, 5.2)$ $(3.4, 6.2)$ $(2.2, 5.7)$ $(5, 2.3)$ $(1.2, 8.2)$
- **d** $(6.3, 0.7)$ $(3.6, 4.9)$ $(4, 2.6)$ $(5.4, 1.2)$ $(9, 2.3)$ $(1.9, 1.8)$ $(1.4, 7.4)$ $(0.4, 3.6)$
- **e** $(2.1, 3)$ $(4.7, 6.9)$ $(2.8, 4)$ $(3.3, 5.5)$ $(1.3, 3.3)$ $(2.7, 4.8)$ $(4.9, 7.8)$ $(1.7, 2)$ $(3.8, 6.2)$ $(0.5, 1.8)$ $(3.3, 6.1)$ $(4.4, 6.6)$

2 The following datasets each include an *outlier point*. Recall that for our purposes, outliers are points that are a large vertical distance from the line of best fit in relation to the other points. For each dataset below:

- **i** Repeat the previous question and also look carefully at a scatterplot of the data.
- **ii** Note any points that appear to be outliers.
- **iii** Remove the outliers and calculate the correlation and line of best fit again.

- **a** $(0.9, 5.2)$ $(6.7, 8.8)$ $(3.9, 1.1)$ $(1.8, 6.7)$ $(4.6, 8.7)$ $(0.8, 3.9)$
- **b** $(2.9, 3.7)$ $(1.4, 5.6)$ $(4.4, 2.6)$ $(4.3, 5.6)$ $(2.5, 5.1)$ $(6.4, 1.3)$
- **c** $(4.7, 8.3)$ $(2.4, 2)$ $(5.3, 7.1)$ $(1.3, 2.9)$ $(3.6, 6.4)$ $(5.5, 9.5)$ $(1.8, 0.5)$ $(1.3, 2.9)$ $(6.5, 10.5)$ $(3.6, 2.2)$ $(6.1, 8.9)$ $(2.6, 4.9)$

3 Explain why in Question **2**, Pearson's correlation coefficient and the line of best fit seemed less affected by the outlier in part **c** than in the other datasets.

4 This question presents two datasets, each with a repeated point, which can have a significant and possibly overlooked effect on the line of best fit.

Dataset 1: $(1, 3)$ $(3, 3)$ $(3, 5)$ $(5, 5)$ and repeated point $(5, 7)$ with frequency 5

Dataset 2: $(5, 7)$ $(6, 8)$ $(4, 7)$ $(1, 4)$ $(3, 4)$ $(4, 4)$ $(1, 2)$ $(8, 7)$ $(0, 2)$ $(5, 5)$ and repeated point $(3, 6)$ with frequency 5

- **a** For each dataset, calculate the equation of the line of best fit, and Pearson's correlation coefficient:
  - **i** with the dataset as given,
  - **ii** if the repeated point is only included once in the dataset.
- **b** Comment on the strength of correlation in these examples.
- **c** Comment on the effect of the repeated point on the equations of the line of best fit.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 [Pareto charts]

Each of these suggested investigations involves a survey or collection of data, which is then collated into a Pareto chart.

- A preliminary class discussion or sample survey with a small number of respondents may be necessary to decide on categories for the data and chart.
- You will need to consider how to design your survey or data collection so that it is random — that you are, for example, collecting data in multiple locations, or conducting your school survey with respondents from a range of year groups.
- An online survey (search for *online questionnaire*) may be a good way to get a range of data, if the target group all have access to the online survey.
- How many respondents do you think you need to get accurate results?

a In Exercise 15A, a question explored the most common car colours and showed this information on a Pareto chart.

i Design your own experiment where you record the colour of the first 100 cars that pass the school. Let several groups do the experiment simultaneously and check agreement. You may need to discuss the colour categories carefully — it can be a matter of opinion if a car is silver or grey.

ii Do your results agree with the results shown on the chart in Exercise 15A?

b Investigate causes of customer dissatisfaction with the school canteen.

i First discuss in class suitable categories, such as prices, variety and opening hours. A limited number of categories is best.

ii Survey people to find their major cause of dissatisfaction — only one category can be chosen. In a good survey, it is important to survey a range of interest groups, such as all year groups and the teachers.

iii Alternatively, do a survey on causes of customer *satisfaction*.

c Investigate the reasons students are late to school.

i Decide on a number of categories in a class discussion, such as, ‘slept in’, ‘traffic’, ‘unwell’.

ii Design your questionnaire and show your results on a Pareto chart.

d Do the people at your school use recycling? Investigate the reasons why they don’t. This could be a question about recycling at school, at home, or across the spectrum of their lives.

i Decide on categories. Some examples are: unaware of environmental value of recycling, too inconvenient, no recycling bins at school, not aware how to recycle (because of issues such as bin labelling or lack of information on how to recycle old electronic items).

ii Complete your results and draw up a Pareto chart.

e Pareto charts are regarded as one of the *seven basic tools of quality control*, because they are frequently used to investigate questions about quality control, customer satisfaction, and so forth. Investigate what the other six tools are, and see if you can apply a number of them to a quality issue in your school or environment.

6 [Contingency tables]

Contingency tables investigate the interrelationship between different variables in a complex dataset. They work best where there are a limited number of categories (such as male/female and blond/red/brown/black hair).

a Survey a number of students to find their favourite style of music, recording also whether they are male or female.

i Display your results on a contingency table.

ii Investigate whether the favourite style of music appears to differ between males and females in your sample.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b Consider other surveys that might be explored using contingency tables. Some examples: favourite music and year group, gender and time spent playing video games (group these in categories such 1 hour per week and 2–3 hours per week), gender and time spent on homework, favourite winter sport and favourite summer sport, favourite social media and age.

7 [Scatterplots]
Many of these investigations generate large amounts of data. Excel or another spreadsheet program could be used to generate a scatterplot and to draw the line of best fit.

a Investigate the ages of students at your school and their heights.

i Plot their age in months on the horizontal axis and their height in centimetres on the vertical axis. Do they appear to be correlated? Can you draw a reasonable line of best fit through the data?

ii If your program does not allow you to shift the intersection of the axes, it may be clearer to plot their age in months above 11 years and their height in centimetres above 130 cm.

iii Find the equation of the line of best fit and Pearson's correlation coefficient using technology.

b Measure the lengths of students' forearms and their heights.

i Draw a scatterplot of height x and forearm length y.

ii Construct the line of best fit, and use Pearson's coefficient to decide if there is a good linear correlation between the quantities.

iii Vary this experiment to: leg length and height, stride length and height, foot length and height, arm span and height, hand and forearm length, length of thumb and middle finger, height of student and their father (or mother), circumference of head and height.

Large datasets: The following questions involve the analysis of large sets of data. There are many such datasets available on the internet. For your convenience, some of datasets for these questions may be downloaded from the Cambridge-GO website for this textbook.

No solutions are provided, and it is recommended that the solutions to this exercise should be discussed in class.

XLS

8 There is frequent mention in the media of rising sea levels over recent years, because of melting ice sheets and glaciers, and because of the expansion of seawater as it warms. This question uses satellite data provided by NASA to investigate the rise in sea levels since 1995. Further records are also available on the website if you wish to pursue historical records from coastal tide gauge records.

Source: https://climate.nasa.gov/vital-signs/sea-level/

a Download the data files provided on the Cambridge-GO website. Alternatively, press the button marked 'DOWNLOAD DATA' on the NASA webpage given above, and massage the data into a spreadsheet.

The text file on the Cambridge-GO website with extension .txt is provided should you wish to investigate more closely what the various columns in the provided spread-sheet mean and how the data are collected. The file GMSL.csv is provided as a comma-separated spreadsheet that can be read by any spreadsheet program. The file GMSL.xlsx is provided for those running Excel.

b Open GMSL.xlsx. Copy column `C` and column `F` to a new sheet at columns `A` and `B`. Column `C` is the date since 1993, in year and fraction of a year, and column `F` is the sea level measure known as Global Mean Sea Level (GMSL) with a 20-year reference mean taken as 0. Because our concern is the change in sea level, the zero reference point is not important.

c Construct a scatterplot of the data, with date on the horizontal axis and sea level on the vertical level.

d Find the line of best fit.

e Get Excel to display the R^2 value (which is the square of Pearson's correlation). How good is the fit?

f Get Excel to display the equation of the line of best fit. What is the meaning of the gradient here? Is the y-intercept significant?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

g It is probably more meaningful at a glance to adjust the vertical axis to be *change in sea level since the mean height in 1993* (when our data starts).

i Calculate the mean of the sea level values in 1993, storing this in cell `E1`.

ii Add a new column `C`, defined by `=B1-$E$1`, and fill this value down to the rest of the cells in column `C`.

iii Construct a new scatterplot from columns `A` and `C`.

h For further investigation and calculation:

i Eighty percent of the Maldives is less than one metre above sea level. How long will it take if this trend continues for that eighty percent to be under water?

ii Find the height above sea level of your current location, and estimate when it will be under water, if the trend continues.

9 An interesting investigation would be to repeat this question using instead the data from Fort Denison in Sydney, http://www.bom.gov.au/oceanography/projects/ntc/monthly/. The data at this URL go back to 1914, and is regarded worldwide as providing one of the most reliable set of measurements of past sea level. The data are presented quite differently, so you will need to adapt your methods and your questions.

10 Economists make use of linear correlation and regression to forecast a number of economic indicators. This question examines data from the Australian Bureau of Statistics (http://stat.data.abs.gov.au/) on the gross operating profits of the Australian mining industry, collected from 1994–2018.

a Download the spreadsheet CompanyProfits.xlsx from the Cambridge-GO website. Open the tab labelled Data1 and copy Columns `A` and `B` to a new spreadsheet. Delete rows `1–12`, leaving the data from 1995–2018. (The data from 1994 are incomplete, so we shall begin in 1995).

b In cell `C1` enter the formula `=(A1-DATE(1995,1,1))/365+1995`. This converts the date in cell `A1` to a year-and-decimal-fraction-of-a-year format, so that 1/Mar/1995 should convert to 1995.162 because it is $1/6 \doteqdot 0.162$ of the way through 1995.

c Fill the formula in `C1` down to the rest of the column, to convert all the dates to this more useful format.

d Create a scatterplot with column `C` on the horizontal axis, and column `B` on the vertical axis. It may be easier in your version of Excel if you first copy column `B` to column `D`, so that the data are in the expected order — first x-values, then y-values. Create the scatterplot of columns `C` and `D`.

e Determine the R^2 value and the formula for the line of best fit. How well are the data correlated to the straight line model?

f What are the correct units for the company profits on the vertical axis?

g The data do not fit perfectly on a straight line, but is it still a useful model for economic prediction?

11 Astronomers have discovered that for a certain large class of stars, brightness is well correlated with colour. These stars are on the so-called *main sequence*, which omits the very massive red giants and the relatively light white dwarves. Astronomers measure brightness by both apparent magnitude and absolute magnitude, the second of which is adjusted so that the measure of brightness does not depend on a star's distance from Earth. Colour is measured on the BV colour scale, which gives each colour a number. (The initials BV comes from the way the colour is determined by the use of Blue and Visible light filters.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

–0.33	–0.30	–0.02	0.30	0.58	0.81	1.40

BV Colour Indices

a Download the spreadsheet starseq.xlsx from the Cambridge-GO website. This dataset is also provided in a .csv format, which may be useful for those not using Excel.

b Copy column `C` to column `J`. In cell `K2` enter the formula `=B2+5*LOG(D2*10)`. This formula converts the star's apparent magnitude to its absolute magnitude. Fill cell `K2` down to the rest of the cells in column `K`, down to `K6221`. Add the heading *Absolute Magnitude* in `K1`. (You can find out more about this conversion formula if you search for *Convert Apparent Magnitude absolute magnitude* in a web browser).

c Create a scatterplot of the data in columns `J` and `K`.

d Because of the large number of points, the default size that Excel uses to display a point is too large. Click on a point and change the pointer option to a smaller size.

e Adjust the scale on your scatterplot so that the bulk of the data is visible on the plot, say horizontally from -0.5 to 2 and vertically -5 to 20. Excel will allow you to double-click on each axis and set the range of the axis.

f Traditionally, the vertical axis should show the axis flipped, with the negative numbers above the positive. Excel includes an option on the axis to display *Values in reverse order*.

g This dataset will NOT give a correlation coefficient close to -1, because the data include stars off the main sequence.

i Copy columns `J–K` to `O–P`, Replace `P2` by the formula

`=IF(ABS(B2$+5*LOG($D2*10)-7.5*O2)>3,NA(),B2$+5*LOG($D2*10)).`

This code is designed to remove all points (stars) vertically well separated from the apparent line of best fit $y = 7.5x$, which seems to model the main sequence. The values that have been eliminated are marked #NA, meaning the value is ignored.

ii Draw a scatterplot of these new data and check if the result gives a good correlation. Remember to adjust the point size and the default axes range, and to flip the direction of the vertical axis.

iii Is it valid to eliminate much of our data in this way?

h Compare your resulting plots with those found online under a search for Hertzsprung–Russell Diagram.

i [Extension] Investigate further the formula relating apparent and absolute magnitude, the BV colour scale, the parsec and arcsec scale for measuring distance, and the 'correct' way to choose stars on the main sequence, by mass.

j [Extension] The magnitude data may be separated into different columns on the basis of the colour index. When Excel constructs the scatterplot, data from separate columns may be coloured independently, illustrating the differing star colours on the main sequence.

12 Weather is notoriously difficult to model, but it is such an important phenomenon that much effort has been applied to modelling its behaviour. The Bureau of Meterology keeps historical and recent data on Australian weather in the data section of its website

http://www.bom.gov.au/climate/data/

Investigate correlation between data on temperature and rainfall for May (or some other month) over a number of years. Choose other variables on the BOM website for similar investigations of correlation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 15 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 15 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Write down the mean, median, mode and range of each dataset.

a 4 7 9 2 4 5 8 6 1 4 **b** 16 17 14 13 18 15 16 15 11

2 A market gardener is testing a new fertiliser on his coriander. Plant beds are set aside to test the new fertiliser. One set of beds have the new fertiliser, while others continue to use the old fertiliser. The total average yield in kilograms in each set of beds is recorded.

Old fertiliser: 1.8 1.4 1.6 2.1 1.9 2.3 1.8 2.1 1.9 1.8

New fertiliser: 2.1 1.6 1.7 2.2 1.9 2.2 1.9 2.2 2.0 1.8

a Find the median and the two quartiles Q_1 and Q_3 for the two datasets.
b Construct a parallel box-and-whisker diagram for the data.
c Do you have any recommendations for the market gardener?

3 **a** Draw up a dot plot of the data in the following list:

4 9 2 6 7 8 2 2 13 5 7 8 6 4 3 6 9 7 4 15 4 6 7 8 9 7 6

b Examine your plot and decide if would you call any of the data points outliers.
c Calculate the interquartile range and determine if the IQR criterion agrees with your assessment in part **b**.
d Draw up a box-and-whisker plot, showing any outliers by a separate dot outside the whiskers.

4 An athlete records his times in seconds running 100 m over a number of events and trials.

11.22 11.43 11.17 11.58 12.10 12.53 11.45 12.04
11.29 13.04 11.31 11.67 12.45 12.14 12.24 11.46

a Calculate the mean and standard deviation for the data using your calculator.
b Construct a frequency table for the data grouped into classes
10.8–11.2, 11.2–11.6, 11.6–12.0, 12.0–12.4, 12.4–12.8, 12.8–13.2.
c Calculate the mean and standard deviation for the grouped data. How does your answer in part **a** compare?
d Draw a frequency histogram for the data.
e Explain why you probably wouldn't want to use a wider class width, such as 0.5.
f Construct a cumulative frequency histogram and ogive for the grouped data.
g Use your cumulative frequency polygon (ogive) to estimate the median for the dataset.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

5 A restaurant wishes to streamline the rate at which it gets food out to its customers at their tables. It divides the evening service into two sittings — the first sitting (arrive 6:00 pm–leave 8:00 pm) and the second sitting (after 8:00 pm). Data are gathered over the course of one night on whether a customer orders an entrée or not.

	first sitting	second sitting	Total
order entrée	45	42	
no entrée	38	28	
Total			

a Copy and complete the table.
b How many people attended the restaurant that night?
c What is the probability that a customer ordered an entrée?
d What percentage of the customers attended the first sitting?
e The manager claims that someone attending the first session is more likely to order an entrée than someone attending the second sitting. Is this correct? Explain.
f It is discovered at the end of the evening that one customer absconded without paying her bill. Given that she ordered an entrée, what is the probability that she was in the first session?
g The next day the restaurant is expecting an exceptionally busy evening for the second sitting, with 90 customers placing a booking. Estimate how many entrées will be ordered.
h Can you suggest why there might be a difference in the ordering habits of those attending the restaurant?
i Why might this survey contain useful information for a restaurant?

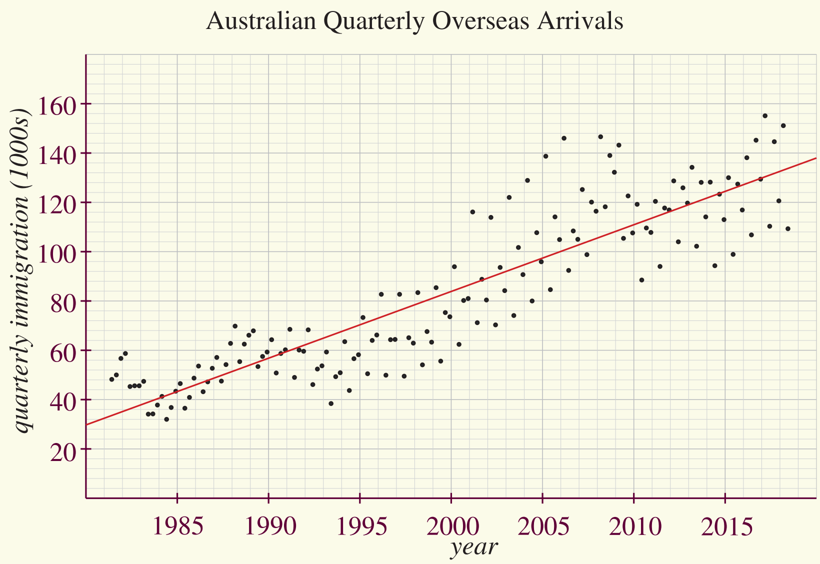
6 These small datasets are suitable for calculation by hand or by technology.
i Plot the points and estimate a line of best fit;
ii Calculate Pearson's correlation coefficient and the gradient and y-intercept of the line of best fit, correct to two decimal places.

a (1, 5) (2, 4) (6, 2) (5, 3) (3, 4) (7, 1) (4, 2)
b (7, 5) (0, 1) (2, 2) (6, 4) (3, 3) (5, 3) (1, 2) (4, 4)

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

7 The scatterplot below shows the immigration of people coming to live in Australia, measured in 1000 s. Measurements are made quarterly (March, June, September, December).



a How many people came to live in Australia in the first quarter of 2011?

b Estimate the arrivals in the four quarters of 2000.

c What was the total number of annual arrivals and the average quarterly arrival in 2000?

d Why can it be misleading to examine only the arrivals in one quarter?

e Read the estimate predicted by the line of best fit in the first quarter of 2000.

f What information is missing to make the data useful to someone investigating immigration into Australia?

g The line of best fit shown on the diagram is $y = 2.7x - 5328.8$, correct to one decimal place, where x is the year and y is the quarterly immigration in 1000 s.

  i Estimate the immigration rate for the first quarter of 2000 using this formula with $x = 2000.16$ (March 2000).

  ii Compare your estimate in part **i** with your estimate from part **e** and explain any discrepancy between these results.

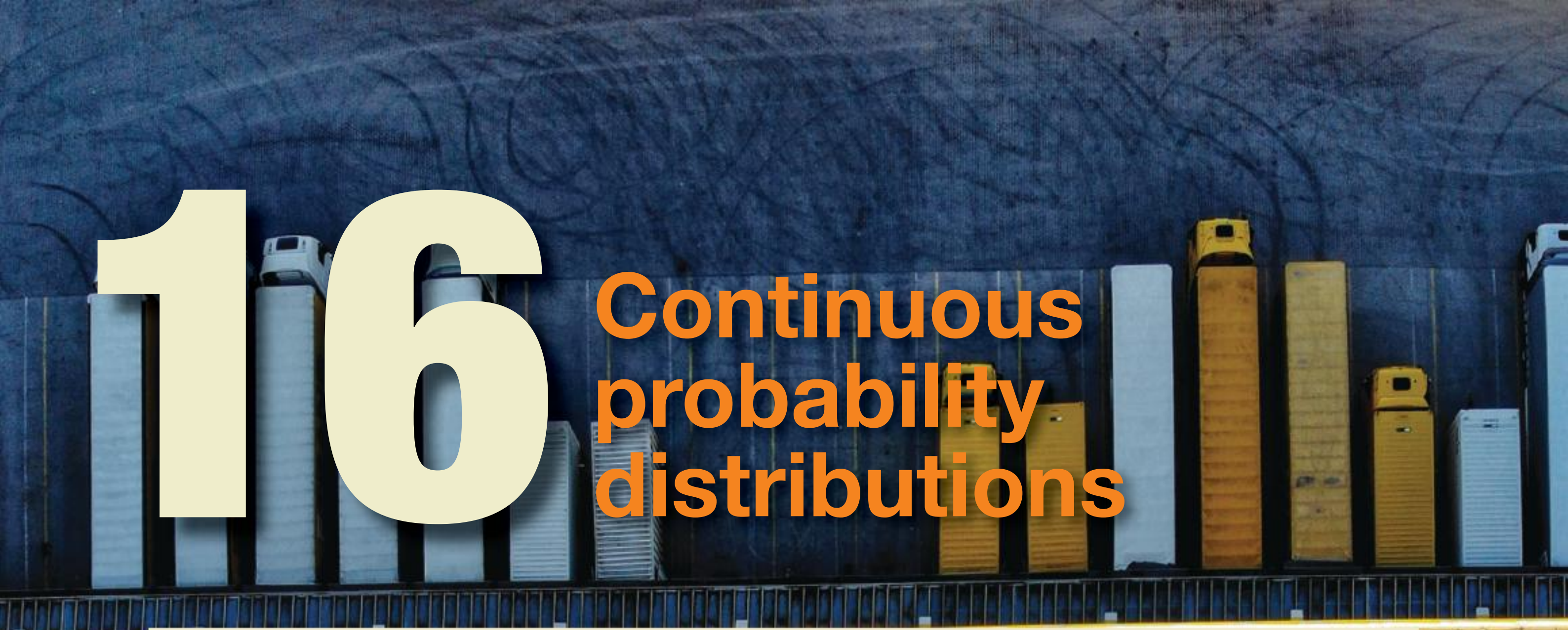
  iii Repeat your calculation from part **i** using the formula $y = 2.70633x - 5328.8$.

  iv Estimate the immigration rate for the year 2030, by using this formula with $x = 2030$ and then multiplying by 4.

  v Use your estimate from part **c** to estimate further the percentage increase in immigration over the 31 years from 2000 to 2030 inclusive.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



16 Continuous probability distributions

Last year we moved from calculating individual probabilities in Chapter 12 to calculating in Chapter 13 a whole set of closely related probabilities grouped together as a *discrete probability distribution*. *Random variables* became part of our machinery, and we calculated expected values (or means) and standard deviations of these random variables in theoretical probability distributions.

The final section of that chapter, Section 13D, only just began to combine the theoretical probabilities in a discrete probability distribution with the data collected from experiments. This present chapter now builds on the techniques of organising and displaying data in Chapter 15 to bring probabilities and data into a closer relationship. The key to this is *relative frequency*, because relative frequencies obtained from data are estimates of theoretical probabilities.

These methods, combined with grouping, allow us finally to give a coherent account of a *continuous random variable*, defining it in terms of a *probability density function*. Integration is needed to understand this material, because a probability is now interpreted as an area under a curve — a dramatic and unexpected idea.

The *normal distribution* is the most important of all continuous distributions, and indeed of all distributions. Sections 16D–16G develop the basic theory of normal distributions, and explain how to apply them to practical problems.

There are many calculations in this chapter, as in Chapter 15. Your pen-and-paper work can be assisted in several ways, all of which can be found in the interactive textbook or online:
- a scientific calculator and a table of values of the standard normal,
- the Desmos graphing calculator or a statistics calculator,
- a spreadsheet,
- specialised statistics software.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16A Relative frequency

Relative frequencies were introduced in Chapter 13 of the Year 11 book because they are estimates of the probabilities of the outcomes of an experiment. They are needed again in this chapter, where probability theory is central once more.

A brief review of the mean and variance of a discrete distribution

In Year 11, Sections 13B and 13C introduced the expected value and the variance of a discrete random variable X. Let $p(x) = P(X = x)$. Then the expected value $\mu = \mathrm{E}(X)$ is

$$\mu = \mathrm{E}(X) = \sum x\,p(x), \quad \text{summing over all values of the distribution.}$$

This is the *weighted mean of the values, weighted according to their probabilities*.

The variance $\mathrm{Var}(X) = \sigma^2$ is the square of the standard deviation σ. It is the expected value of the squared deviation from the mean,

$$\mathrm{Var}(X) = \mathrm{E}\Big((X - \mu)^2\Big) = \sum (x - \mu)^2\,p(x).$$

We proved also that the variance has an alternative form that is preferable when the mean is not an integer, and is therefore particularly suited to data,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = \sum x^2\,p(x) - \mu^2.$$

Setting out the calculations

Our model experiment in the Year 11 book was, ‘throw four coins and record the number of heads’. The second rows in the tables below show the theoretical probabilities of obtaining 0, 1, 2, 3 or 4 heads.

A Here is the way we set out the calculations of mean and variance when we use the definition of variance as $\mathrm{E}(X - \mu)^2$.

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$x\,p(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$(x - \mu)$	-2	-1	0	1	2	—	
$(x - \mu)^2$	4	1	0	1	4	—	
$(x - \mu)^2\,p(x)$	$\frac{4}{16}$	$\frac{4}{16}$	0	$\frac{4}{16}$	$\frac{4}{16}$	1	(the variance)

The mean is $\mu = 2$, the variance is $\sigma^2 = 1$, and the standard deviation is $\sigma = 1$.

B Here is our setting-out using the other variance formula $E(X^2) - \mu^2$. The calculations are more straightforward, whether or not μ is a whole number.

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$x\,p(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$x^2\,p(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$	5	(this is $\mathrm{E}(X^2)$)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$$\begin{aligned} \text{From the third row,} \quad \mathrm{E}(X) &= 2, \text{ which is the mean } \mu. \\ \text{From the last row,} \quad \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mu^2 \\ &= 5 - 2^2 \\ &= 1, \text{ which is } \sigma^2. \end{aligned}$$

Cumulative distribution function

In Year 11 we did not discuss the cumulative distribution function, but it is easily defined. With the four coins, it is the function $F(x)$ obtained by adding all the probabilities up to a certain point:

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

For example, $F(3) = p(0) + p(1) + p(2) + p(3)$,
and in general, $F(x) = p(0) + p(1) + \cdots + p(x)$, for $x = 0, 1, 2, 3, 4$.

Relative frequencies and the histogram and polygon

Relative frequency is the key connection between the probabilities in the theoretical distributions studied in Year 11 and the analysis of the datasets in Chapter 15 this year. In Year 11 we performed the experiment of tossing four coins 100 times, and obtained these results.

x	0	1	2	3	4	Sum
f	7	29	34	21	9	100
f_r	0.07	0.29	0.34	0.21	0.09	1

- The first line lists the *values* — the number x of heads can be $0, 1, 2, 3$ or 4.
- The second line lists the *frequencies* f — the experiment was run 100 times.
- The third line lists the *relative frequencies* $f_r = \dfrac{f}{100}$ — divide by 100 trials.

These relative frequencies are *estimates of the probabilities* of tossing $0, 1, 2, 3$ or 4 heads — they are often referred to as 'experimental probabilities'. Unless the experiments are biased in some way, these estimates will almost certainly be closer and closer to the theoretical probabilities as the number of trials increases.

Setting out the calculations using relative frequencies

Calculating the sample mean and variance of this dataset was explained in Section 13D of the Year 11 book.

x	0	1	2	3	4	Sum
f_r	0.07	0.29	0.34	0.21	0.09	1
xf_r	0	0.29	0.68	0.63	0.36	1.96
x^2f_r	0	0.29	1.36	1.89	1.44	4.98

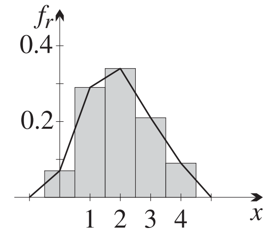
The sample mean $\bar{x}$, variance s^2 and standard deviation s can then be calculated from the table as follows. These are *estimates* of the expected value $\mu = E(X)$, variance $\sigma^2 = \mathrm{Var}(X)$ and standard deviation σ of the probability distribution.

$$\begin{aligned} \bar{x} &= \sum x f_r \\ &= 1.96 \end{aligned} \qquad \begin{aligned} s^2 &= \sum x^2 f_r - \bar{x}^2 \\ &= 4.98 - (1.96)^2 \\ &\doteqdot 1.14 \quad (\text{compare with } \sigma^2 = 1) \end{aligned} \qquad \begin{aligned} s &= \sqrt{s^2} \\ &\doteqdot 1.07 \text{ heads} \quad (\text{compare with } \sigma = 1) \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Histograms and polygons using relative frequencies



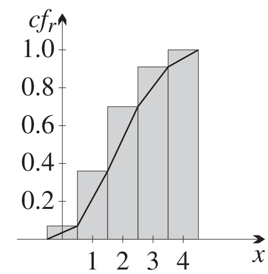
We can graph the relative frequencies in a *relative frequency histogram* and a *relative frequency polygon*. Look at the total area of the histogram rectangles, and the area under the polygon — the two areas are equal because each interval of the polygon cuts a triangle off one rectangle, and adds a triangle of the same area to an adjacent rectangle.

This particular histogram has a significant property — *the sum of the areas of the rectangles is 1*. This will always happen when the rectangles have width 1, because the sum of probabilities is 1. The Enrichment Question 12 in Exercise 16A deals with the situation when the rectangles have width different from 1.

These remarks about areas are made in preparation for using integration in Section 16B.

Cumulative relative frequencies

In the fourth line of the table below, we have calculated the *cumulative relative frequencies*.



x	0	1	2	3	4	Sum
f	7	29	34	21	9	100
f_r	0.07	0.29	0.34	0.21	0.09	1
$c\,f_r$	0.07	0.36	0.70	0.91	1	

The cumulative relative frequencies have been graphed in a *cumulative relative frequency histogram*. A *cumulative relative frequency polygon* or *ogive* can also be drawn, starting with accumulation zero at $x = -\frac{1}{2}$, and finishing with accumulation 1 at $x = 4\frac{1}{2}$.

Each cumulative frequency estimates the probabilities of obtaining a particular number of heads or fewer. For example,

$$\text{estimated probability of tossing 3 or fewer heads} = 0.91.$$

Discrete probability distributions and estimates from data

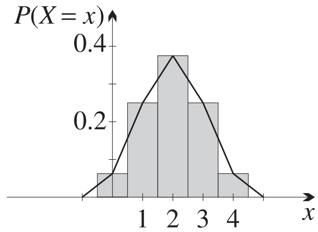
Relative frequencies obtained from data are estimates of probabilities. If we also know the theoretical probabilities, as we do for the four tossed coins, we can draw the same histograms and polygons as we have just done for data, but using probabilities and cumulative probabilities instead.

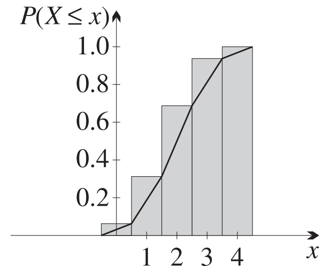
Here again is the theoretical probability distribution for the four tossed coins, and the two histograms and polygons — using decimals for easy comparison with the 100-trial data above. Compare these two diagrams with the diagrams above.

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625
$P(X \le x)$	0.0625	0.3125	0.6875	0.9375	1

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022





Probability estimates obtained from data are rarely exactly the same as the theoretical probabilities.

1 RELATIVE FREQUENCIES AND CUMULATIVE RELATIVE FREQUENCIES

- For a dataset, the relative frequencies and cumulative relative frequencies are obtained by dividing through by the total frequency.
- The relative frequencies of a dataset are estimates of probabilities. For this reason, they are often referred to as ‘experimental probabilities’.
- The cumulative distribution function $F(x)$ of any numeric probability distribution is the probability that the score is less than or equal to x,

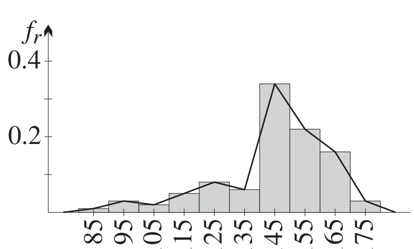
  $F(x) = P(X \le x)$, for all x in the domain.

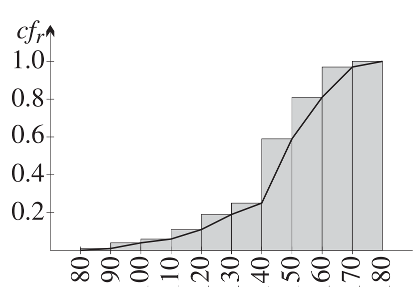
- The cumulative relative frequencies are estimates of the cumulative distribution function.
- The histograms and polygons of these relative frequencies and cumulative relative frequencies are drawn in the usual way.

Grouping data from a continuous random variable

When the underlying random variable is continuous, grouping is usually required. Here are the histograms and polygons for the heights x of 100 people from Section 15B, drawn this time using relative frequencies. The values of x are the class centres of the 10 cm intervals that were used in the grouping.

Class	80–90	90–100	100–110	110–120	120–130	130–140	140–150	150–160	160–170	170–180
x	85	95	105	115	125	135	145	155	165	175
f	1	3	2	5	8	6	34	22	16	3
f_r	0.01	0.03	0.02	0.05	0.08	0.06	0.34	0.22	0.16	0.03
F_r	0.01	0.04	0.06	0.11	0.19	0.25	0.59	0.81	0.97	1.00





ISBN 978-1-108-76630-2

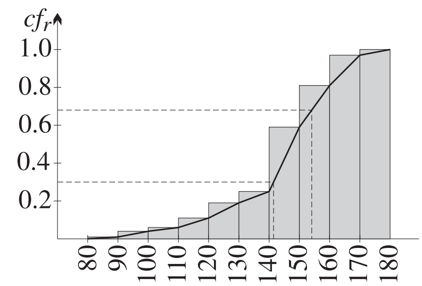
Cambridge University Press
Updated July 2022

Deciles and percentiles

We have seen that quartiles divide the scores into four equal lists. They can be read approximately from the cumulative relative frequency polygon by drawing horizontal lines at height 0.25 for the lower quartile Q_1, height 0.5 for the median Q_2, and height 0.75 for the upper quartile Q_3.

The graph to the right shows how *deciles* and *percentiles* can be found similarly.

- To find the 3rd decile, draw a horizontal line of height 0.3.
- To find the 68th percentile, draw a horizontal line of height 0.68.

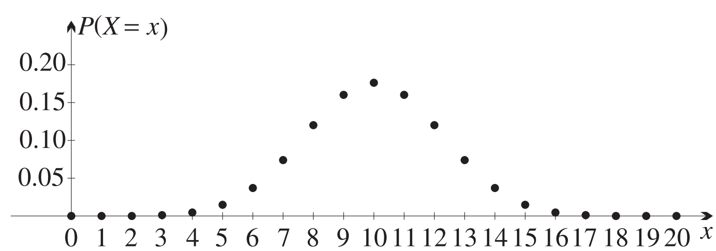


From the graph to the right, the 3rd decile is about 142 and the 68th percentile is about 154.

Note: This graphical method of finding medians and quartiles will give slightly different results from the method used in Sections 15A and 15C. Once the cumulative histogram is drawn, it may seem more natural to use the graph, and graphical methods fit better with the integrals that are needed in the continuous distributions of this chapter.

When does a discrete distribution begin to look continuous?

As the number of values of a discrete distribution increases, the graph of the distribution may suggest a curve. For example, when 20 coins are thrown and the number of heads recorded, the diagram below shows the graph of the resulting probability distribution (the calculations are omitted). There definitely seems to be a curve involved here.



This chapter is about continuous distributions. The more coins there are, the more difficult the calculations become, and the more attractive it is to work out some way to approximate the discrete distribution by a continuous distribution. This is one of many ways in which continuous distributions are useful.

Probability and area

Here is a rather simple probability problem that requires area and cannot possibly be reduced to a discrete probability distribution.

Example 1 — 16A

A point-chook is wandering randomly around a 20 m × 20 m square enclosure. It is just as likely to be at any one place in the enclosure as any other. A circle 10 metres in radius has been inscribed in the square. If Farmer Brown looks out at the enclosure, what is the probability that the chook is inside the circle?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

Here area of enclosure $= 20^2$

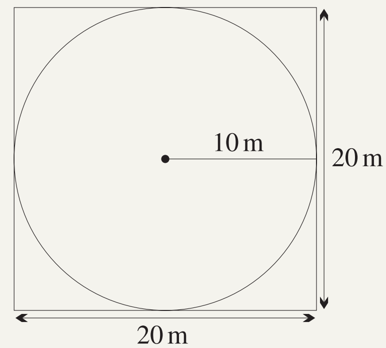
$= 400 \text{ m}^2$,

and area of circle $= \pi r^2$

$= 100\pi \text{ m}^2$,

so $P(\text{chook is inside the circle}) = \dfrac{\text{area of circle}}{\text{area of square}}$

$= \dfrac{\pi}{4}$.



In this problem it is completely obvious that we take the ratio of areas. Yet the calculations have nothing to do with the discrete sample spaces that we have spent so much time analysing. The answer $\frac{\pi}{4}$ is not even a rational number. The association of probability with area is fundamental to the way we shall deal with continuous probability distributions.

Exercise 16A

FOUNDATION

Questions 1–5 are a short review of ideas from Chapter 13 of the Year 11 book

1 The probabilities in a discrete probability distribution must all be non-negative and add to 1. Which of the following are valid discrete probability distributions?

A

x	1	2	3	4
$P(X = x)$	0.2	0.3	0.3	0.2

B

x	1	2	3	4
$P(X = x)$	1.4	0	-0.5	0.1

C

x	1	2	3	4
$P(X = x)$	0.15	0.2	0.4	0.25

D

x	1	2	3	4
$P(X = x)$	0.35	0.2	0.3	0.1

2 Two four-sided tetrahedral dice are thrown. The apex number on each die is read, and the sum of these two numbers is recorded. Let the random variable X be the outcome of this experiment.

a Record the possible outcomes and their probabilities in a probability table.

b Find:

i $P(X < 5)$ **ii** $P(X > 7)$

iii $P(X < 2)$ **iv** $P(X \le 10)$

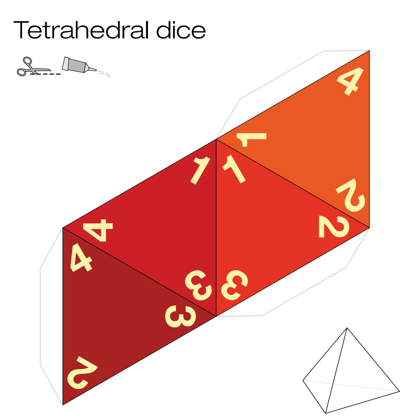
c Find the probability:

i that the sum is less than 4,

ii that the sum is odd,

iii that the sum is not 2,

iv that the sum is at least 6.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 A certain weighted spinner has five outcomes 1, 2, 3, 4 and 5. The spinner is known to be biased, and after a large number of trials, the relative frequencies of each outcome x were:

Score x	1	2	3	4	5	Total
f_r	0.1	0.2	0.45	0.15	0.1	
$x f_r$						
$x^2 f_r$						

a Complete the table.

b What is the significance of your total of row 2?

c Sum row 3, then use the formula $\bar{x} = \sum x f_r$ to calculate the sample mean $\bar{x}$.

d In your own words, what is the sample mean $\bar{x}$ measuring?

e Sum row 4, then use the formula $s^2 = \sum x^2 f_r - \bar{x}^2$ to calculate the sample variance.

f Hence calculate the sample standard deviation s, correct to two decimal places.

g In your own words, what is the sample standard deviation s measuring?

h What are the sample mean $\bar{x}$, and the sample standard deviation s, estimates of in this experiment?

i The spinner is thrown 100 times, and the outcome is recorded for each throw. State a reasonable estimate for the sum of these outcomes.

4 Data are recorded in the following table.

score x	3	4	5	6	7	Total
relative frequency f_r	0.04	0.21	0.35	0.25	0.15	

a Complete the table as in Question 3 to calculate the sample mean and sample standard deviation, correct to two decimal places.

b There are 5 values in this dataset and in the dataset in Question 3. Comment on the centre and spread of the data in this set, compared with Question 3.

5 These are the results from a class quiz where the maximum possible score is six marks.

score x	1	2	3	4	5	6	Total
frequency f	2	4	4	8	2	0	
$P(X = x)$							
$x \times P(X = x)$							
$x^2 \times P(X = x)$							

a Find the median and mode.

b Use the relative frequencies as probabilities to fill in the row for $P(X = x)$. Then complete the table.

c Use the formula $\mathrm{E}(X) = \sum x\, P(X = x)$ to estimate the expected value (also called the mean).

d Use the formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \sum x^2 P(X = x) - \mu^2$ to estimate the variance.

e Find the standard deviation.

f Comment on the class's performance in this quiz by reference to the distribution of these quiz scores.

g The teacher likes to record all quiz results out of 30, so he multiplies all these results by 5 before recording them in his markbook. What will be the mean and standard deviation of this new set of marks? You may find it helpful to remember the formulae

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \text{and} \qquad \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X).$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Start of Foundation for Section 16A of this chapter

6 A simple experiment has generated the following table of discrete data:

score x	1	2	3
frequency f	2	5	3

a i Construct a frequency histogram for the data. Add the frequency polygon to your diagram by joining the centres of the data points. Remember to join the ends of the polygon back to the horizontal axis.

ii Calculate the total area of the histogram rectangles.

iii Calculate the area under the frequency polygon, bounded by the horizontal axis.

iv What do you notice?

b i Copy the table, and add a row showing the relative frequency, obtained by dividing the frequencies by the total number of scores, which is 10.

ii Construct a relative frequency histogram for the data, including the relative frequency polygon.

iii Calculate the total area of the histogram rectangles.

iv Calculate the area under the relative frequency polygon, bounded by the horizontal axis.

v What do you notice?

vi What is the relationship between the relative frequencies and the probabilities $P(X = x)$ of the experiment's probability distribution?

7 a Copy and complete the following table by filling in the relative frequencies, cumulative frequencies and cumulative relative frequencies.

x	1	2	3	4	5	6	7	Total
f	3	1	4	3	1	3	1	
f_r								
cf								—
cf_r								—

b Construct a cumulative relative frequency histogram and polygon (ogive). Mark your vertical axis in divisions of $\frac{1}{8} = 0.125$.

c Use the ogive to read off the three quartiles Q_1, Q_2 and Q_3.

8 Repeat question 7 for the following dataset. Mark your vertical axis in divisions of 0.1.

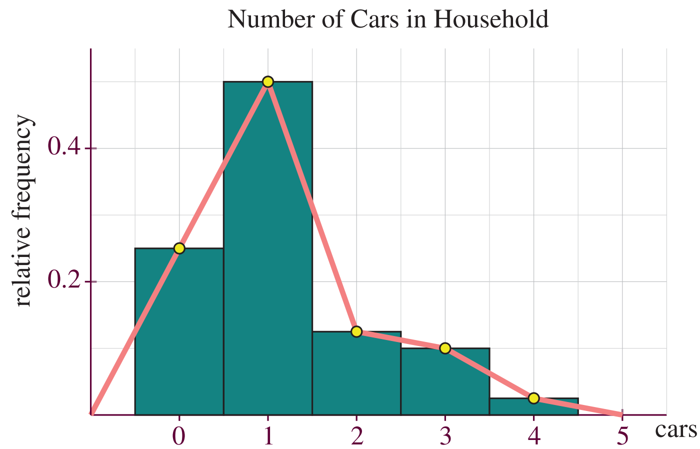
x	5	6	7	8	9	10	11
f	5	4	1	1	1	6	2

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

9 For town-planning purposes, the number of cars owned by each household in a suburb was recorded from census data. The results are displayed in the relative frequency histogram and polygon below. (This is a population, so the relative frequencies are probabilities.)



a What fraction of the households have no cars?

b What fraction of the households have fewer than 2 cars?

c What is the probability that a household chosen at random has three cars?

d Town planners will advise that additional on-street parking be provided if more than 40% of the households have 3 or more cars. Will they be advising that additional parking be provided? Explain your answer.

e Copy and complete the following table for this probability distribution.

x	0	1	2	3	4
$P(X = x)$					

f Show that the sum of the probabilities is 1. How is this related to the area of the rectangles of the histogram?

g Explain in your own words, and with reference to the graph above, why the area bounded by the relative frequency polygon and the horizontal axis will be the same as the area of the relative frequency histogram.

h Use your table to show that the mean number of cars per household is 1.15. What do you understand by this answer — how can a household have a fraction of a car?

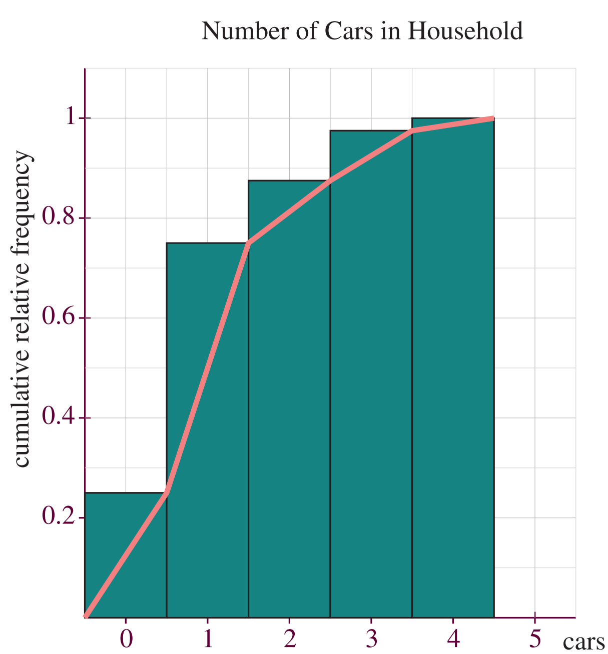
i A street in the suburb is selected at random. If there are 100 households in the street, how many cars would you expect to belong to the households in the street in total? Are your assumptions for this estimate reasonable?

j Copy and complete the following table for the cumulative relative frequencies of this probability distribution.

x	0	1	2	3	4
$P(X \le x)$					

k A town planner constructs a cumulative relative frequency polygon and histogram from these data. His graph is shown to the right. Confirm that your data agree with this graph.

l By drawing horizontal lines at heights 0.25, 0.5 and 0.75, find the three quartiles Q_1, Q_2 and Q_3.



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 **a** Construct a cumulative relative frequency histogram and polygon (ogive) for the following data.

x	0	1	2	3	4
f_r	0.1	0.3	0.2	0.1	0.3

b Estimate the 70th percentile (also called the 7th decile) by intersecting the horizontal line at height 0.7 with the ogive.

c Similarly estimate the first quartile Q_1 and the median Q_2 using horizontal lines and the ogive.

d Similarly estimate the third quartile Q_3 using the ogive.

e [Challenge] Use ratios on the last segment of the ogive to calculate the quartile Q_3 using the formula $3.5 + 0.05 \times \dfrac{4.5 - 3.5}{1 - 0.7}$. Compare your answer with part **d**.

11 To raise funds, a school running a musical performance also runs a set of stalls selling cheap items. The total amount spent at the stalls by each person attending was recorded.

Amount spent ($)	0–1	1–2	2–3	3–4	4–5	Total
class centre x	0.50	1.50	2.50	3.50	4.50	—
frequency f	20	5	15	40	20	

(Any value in the range $0 \le$ price < 1 is recorded in the class 0–1 etc.)

a Find the median and mode.

b Copy the table and add a row for the relative frequency.

c Calculate the expected value $\mathrm{E}(X)$ and the variance $\mathrm{Var}(X)$.

d Construct a relative frequency histogram, including the relative frequency polygon.

e Find the probability that an attendee spends between:

i \$0–\$1 **ii** \$1–\$2 **iii** \$2–\$3 **iv** \$3–\$4 **v** \$4–\$5.

f Find the sum of the probabilities in part **e**. What area does this represent?

g An attendee is chosen at random and asked how much they spent.

i Is the amount spent more likely to be \$0–\$3, or \$3–\$4?

ii Is the amount spent more likely to be \$0–\$1, or \$3–\$4?

h If the school also charged an entry fee of \$2, find the expected value and variance of this new distribution $Y = X + 2$. What does the value $E(Y)$ represent?

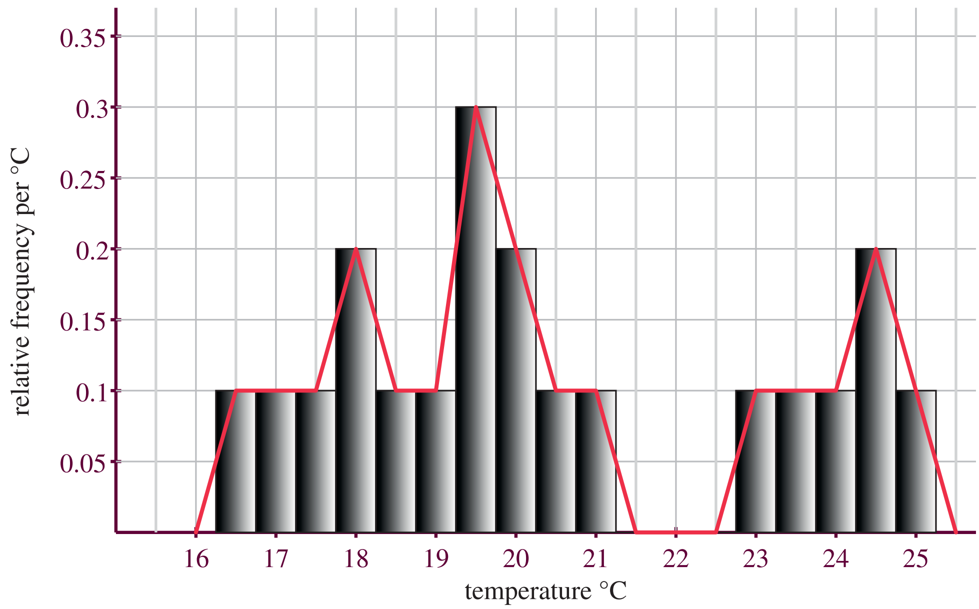
ENRICHMENT

12 We have seen several times that when the width of the rectangles is 1, the rectangles of the relative frequency histogram have total area 1. When the rectangles have a width w different from 1, however, then the total area is w. We can restore the area to 1 by using instead a scale of 'relative frequency per unit' on the vertical axis, as in this question.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The maximum temperature for the day over a period of twenty days at a local weather station is recorded. These temperatures are displayed in the relative frequency histogram and polygon below, where the rectangles each have width 0.5°C.



In this histogram, temperature (more accurately, the temperature class) is shown on the horizontal axis, and the *relative frequency per unit of temperature* is shown on the vertical axis. Temperature has been grouped in classes of 0.5°.

a Show that the total area under the histogram is 1. (Hint: Each box on the grid has an area of $0.025 = \frac{1}{40}$.)

b With this adjustment, the probability that the temperature will lie in a given class (or classes) is the area of the corresponding rectangle (or rectangles).

- **i** Find the probability that the maximum temperature is between 19.25°C and 19.75°C.
- **ii** Find the probability that the maximum temperature is between 16.25°C and 17.25°C.
- **iii** Find the probability that a day chosen at random is *warm*, if a warm day is defined to be one with a maximum of more than 22°C.

c The probability of a given temperature is proportional to the height of the frequency polygon at that point.

- **i** Estimate the relative likelihood of the maximum temperature being 17°C as compared with 20°C.
- **ii** What is the mode, that is, the most likely maximum temperature?

d The frequency polygon gives an estimate of the shape of the continuous probability distribution that would be obtained by successively grouping the data in narrower and narrower classes of temperatures. Use the area under the frequency polygon to estimate the probability that the maximum temperature on a given day is between:

- **i** 16.5°C and 17.5°C,
- **ii** 19°C and 20.5°C.

e Comment on the validity of using this histogram to decide on the probability of a given temperature at any time of the year.

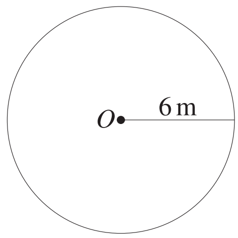
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16B Continuous distributions

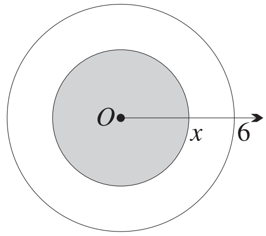
In a *continuous probability distribution*, the domain of the values is typically from a closed interval on the number line, such as [0, 6]. There are thus infinitely many values, which cannot even be listed. The probability of any one particular value is zero, and we want to talk instead about a probability such as $P(2 \le X \le 5)$, which is the probability that the value lies in the subinterval [2, 5] of [0, 6].

A cumulative distribution function



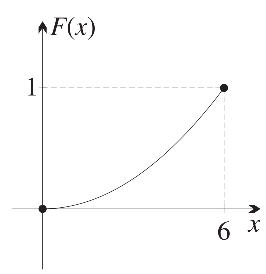
A point-chook is wandering randomly around a circular enclosure of radius 6 metres. It is just as likely to be in any one place in the enclosure as any other. Farmer Brown wants to know how far the chook is from the water at the centre O of the circle.

There are infinitely many distances from the centre within the enclosure. The probability that the chook is say exactly 2 metres from the centre is zero. Thus the tabular methods used with discrete probability distributions are useless here.



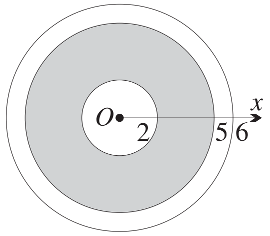
We can, however, approach the situation using cumulative frequency. Let $F(x)$ be the probability that when Farmer Brown looks out, the chook is no more than x metres from the centre.

$$\begin{aligned} F(x) &= \frac{\text{area of inner circle}}{\text{area of whole circle}} \\ &= \frac{\pi x^2}{\pi \times 6^2} \\ &= \tfrac{1}{36}x^2, \quad \text{where } 0 \le x \le 6. \end{aligned}$$



This function is a *cumulative distribution function* or *CDF*. It is continuous, and increases from $F(0) = 0$ on the left to $F(6) = 1$ on the right. A cumulative function is always non-decreasing. It can also be used to solve many more problems. For example, we can find the probability that the chook is between 2 metres and 5 metres from the centre by subtraction,

$$\begin{aligned} P(\text{chook is 2–5 metres from the centre}) &= F(5) - F(2) \\ &= \tfrac{1}{36}(25 - 4) \\ &= \tfrac{21}{25}. \end{aligned}$$



We can also calculate the median and the quartiles of the probability distribution in the obvious way.

For the first quartile,

put $F(x) = \frac{1}{4}$

$$\begin{aligned} \tfrac{1}{36}x^2 &= \tfrac{1}{4} \\ x^2 &= 9 \\ x &= 3. \end{aligned}$$

For the median,

put $F(x) = \frac{1}{2}$

$$\begin{aligned} \tfrac{1}{36}x^2 &= \tfrac{1}{2} \\ x^2 &= 18 \\ x &\doteqdot 4.24. \end{aligned}$$

For the third quartile,

put $F(x) = \frac{3}{4}$

$$\begin{aligned} \tfrac{1}{36}x^2 &= \tfrac{3}{4} \\ x^2 &= 27 \\ x &\doteqdot 5.20. \end{aligned}$$

We still have not precisely defined continuous probability distributions, but let us nevertheless summarise the discussion above.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 THE CUMULATIVE DISTRIBUTION FUNCTION

Let a continuous random variable X have values from a closed interval $[a, b]$.

- The *cumulative distribution function* or *CDF* for X is the function

$$F(x) = P(a \le X \le x), \text{ for all } x \text{ in the interval } [a, b].$$

- The cumulative distribution function is continuous and non-decreasing, with

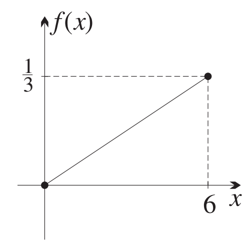
$$F(a) = 0 \qquad \text{and} \qquad F(b) = 1.$$

- It can be used to calculate medians, quartiles and percentiles.

A probability density function

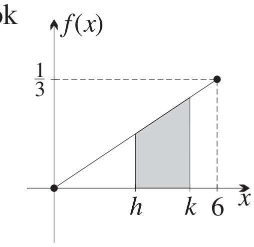
With a discrete distribution, the cumulative frequencies were obtained by adding all the probabilities up to a certain point — the same process produces the cumulative frequencies of a dataset. The continuous analogue of addition is integration, so we should expect the cumulative distribution function $F(x) = \frac{1}{36}x^2$ to be an *integral* over the values up to a certain point.

The fundamental theorem of calculus tells us that $F(x)$ is the integral of its derivative $F'(x)$. So we differentiate $F(x)$ to obtain what is called the *probability density function* $f(x)$,



$$\begin{aligned} f(x) &= \frac{d}{dx}\left(\tfrac{1}{36}x^2\right) \\ &= \tfrac{1}{18}x, \text{ where } 0 \le x \le 6. \end{aligned}$$

This linear graph of $f(x)$ is sketched above. It does not tell us the probability that the chook is x metres from the centre, because that probability is zero. Instead, it allows us to find by integration the probability that the chook is in some range of distances from centre.



The probability of the chook being in some range of positions is the area under the curve, which is found by integration.

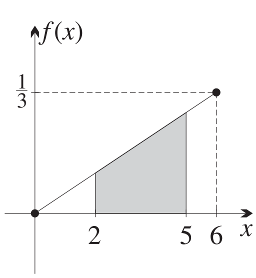
$$P(h \le x \le k) = \int_a^b f(x)\,dx.$$

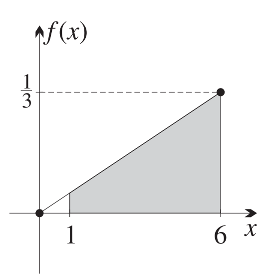
For example,

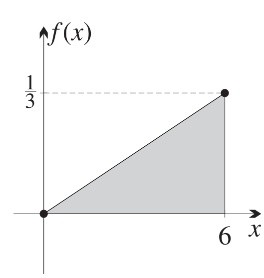
$$\begin{aligned} P(2 \le x \le 5) &= \int_2^5 \tfrac{1}{18}x\,dx \\ &= \left[\tfrac{1}{36}x^2\right]_2^5 \\ &= \tfrac{1}{36}(25 - 4) \\ &= \tfrac{21}{25}, \end{aligned}$$

$$\begin{aligned} P(x \ge 1) &= \int_1^6 \tfrac{1}{18}x\,dx \\ &= \left[\tfrac{1}{36}x^2\right]_1^6 \\ &= \tfrac{1}{36}(36 - 1) \\ &= \tfrac{35}{36}, \end{aligned}$$

$$\begin{aligned} P(0 \le X \le 6) &= \int_0^6 \tfrac{1}{18}x\,dx \\ &= \left[\tfrac{1}{36}x^2\right]_0^6 \\ &= \tfrac{1}{36}(36 - 0) \\ &= 1. \end{aligned}$$







ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

This probability density function has two important properties:

1 $f(x)$ is never negative, because $F(x)$ is cumulative and never decreases.

2 $\int_0^6 f(x)\,dx = 1$, because the chook is somewhere in the enclosure.

It turns out that the probability density function is more important than the cumulative distribution function when characterising a continuous distribution and working with it. It is also the best way to give a formal definition of a continuous probability distribution.

3 PROBABILITY DENSITY FUNCTIONS

- A *probability density function*, or *PDF*, is a function defined on a closed interval $[a, b]$ and satisfying two properties:
  1 $f(x) \geq 0$, for $a \leq x \leq b$.
  2 $\int_a^b f(x)\,dx = 1$.
- A *continuous probability distribution* is defined to be a probability distribution described by a probability density function.
- The x-coordinate of any global maximum of the PDF is called a *mode*.
- Probability is area under the curve. That is, for all closed subintervals $[h, k]$,

$$P(h \leq X \leq k) = \int_h^k f(x)\,dx.$$

- Later, we will allow a to be replaced by $-\infty$, and b to be replaced by ∞.

The probability of any particular value h, however, is always zero. That is,

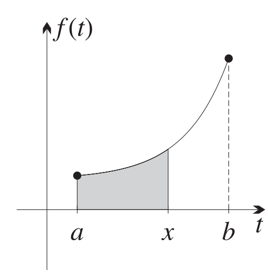
$$P(X = h) = \int_h^h f(x)\,dx = 0.$$

We remarked on such integrals in Box 8 of Section 5C. Because of this, it doesn't really matter whether $\leq$ or $<$ is used for intervals, or whether two adjacent intervals, say $[2, 4]$ and $[4, 5]$, overlap at the endpoints.

The CDF is the signed area function of the PDF

Suppose again that $f(x)$ is a probability density function defined on the closed domain $[a, b]$. Using the language of Section 5D, we can now use integration to define the cumulative distribution function $F(x)$ of $f(x)$ as the signed area function of the PDF,

$$F(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b.$$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 THE CUMULATIVE DISTRIBUTION FUNCTION AS THE SIGNED AREA FUNCTION

The *cumulative distribution function* $F(x)$, or *CDF* for short, of a probability density function, or PDF, is the signed area function

$$F(x) = P(X \le x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b,$$

and conversely $f(x) = F'(x)$ (apart possibly from isolated sharp corners).

Be pedantic and say 'probability density function' and 'cumulative distribution function'. 'Density' means at a point, and 'distribution' means over a range.

Uniform continuous distributions

An important special case of continuous probability distributions is a *uniform continuous distribution*. This is a distribution whose PDF is a constant function.

Example 2 16B

Tran does not know the times when trains leave Lakeside Station, but he does know that they leave precisely every fifteen minutes.

a He wants to know about the probability distribution of his waiting time if he arrives at the station at a random time, and what the PDF and CDF are.

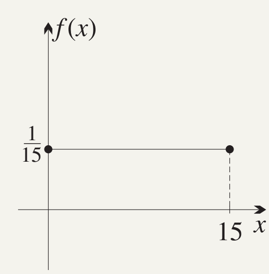
b He also wants to know the median and the 45th percentile, and the probability that he will wait between 5 and 10 minutes.

SOLUTION

a The waiting time x is anything from 0 to 15 minutes, and we have no reason to prefer any waiting time from any other waiting time. This means that the probability density function $f(x)$ is a constant function in the interval $[0, 15]$, and the probability distribution is therefore a *uniform continuous distribution* with values from the closed interval $[0, 15]$.

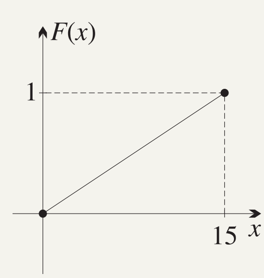
Because the area under the PDF is exactly 1 (the total probability),

$$f(x) = \tfrac{1}{15}, \quad \text{for } 0 \le x \le 15.$$



The CDF $F(x)$ is then found by integrating,

$$\begin{aligned} F(x) &= \int_0^x \tfrac{1}{15}\,dt \\ &= \Big[\tfrac{1}{15}t\Big]_0^x \\ &= \tfrac{1}{15}x. \end{aligned}$$



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b To find the probability that he waits between 5 and 10 minutes, either integrate the PDF or use the CDF.

$$\begin{aligned} P(5 \le X \le 10) &= \int_5^{10} \tfrac{1}{15}\,dx \\ &= \Big[\tfrac{1}{15}x\Big]_5^{10} \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3}, \end{aligned}$$

OR

$$\begin{aligned} P(5 \le X \le 10) &= F(10) - F(5) \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3}. \end{aligned}$$

For the median, put $F(x) = \tfrac{1}{2}$

$$\begin{aligned} \tfrac{1}{15}x &= \tfrac{1}{2} \\ x &= 7\tfrac{1}{2}. \end{aligned}$$

For the 45th percentile, put $F(x) = \tfrac{45}{100}$

$$\begin{aligned} \tfrac{1}{15}x &= \tfrac{9}{20} \\ x &= 6\tfrac{3}{4}. \end{aligned}$$

5 UNIFORM CONTINUOUS DISTRIBUTIONS

- A continuous distribution is called *uniform* if its density function is constant.
- Because the area under the graph is 1, a uniform continuous distribution defined on an interval $[a, b]$ has probability density function $y = \dfrac{1}{b - a}$.

Example 3 16B

Find the value of k that makes each function a probability density function. Then find the corresponding CDF $F(x)$. Hence find the median and quartiles.

a $f(x) = k$, where $0 \le x \le 10$,

b $f(x) = kx$, where $0 \le x \le 10$.

SOLUTION

a Put

$$\begin{aligned} \int_0^{10} k\,dx &= 1 \\ \Big[kx\Big]_0^{10} &= 1 \\ 10k - 0 &= 1 \\ k &= \tfrac{1}{10}, \end{aligned}$$

so $f(x) = \tfrac{1}{10}$.

Hence

$$\begin{aligned} F(x) &= \int_0^x \tfrac{1}{10}\,dt \\ &= \Big[\tfrac{1}{10}t\Big]_0^x \\ &= \tfrac{1}{10}x. \end{aligned}$$

When $F(x) = \tfrac{1}{2}, x = 5$,
when $F(x) = \tfrac{1}{4}, x = 2\tfrac{1}{2}$,
when $F(x) = \tfrac{3}{4}, x = 7\tfrac{1}{2}$,
so $Q_1 = 2\tfrac{1}{2}$, $Q_2 = 5$ and $Q_3 = 7\tfrac{1}{2}$.

b Put

$$\begin{aligned} \int_0^{10} kx\,dx &= 1 \\ \Big[\tfrac{1}{2}kx^2\Big]_0^{10} &= 1 \\ 50k - 0 &= 1 \\ k &= \tfrac{1}{50}, \end{aligned}$$

so $f(x) = \tfrac{1}{50}x$.

Hence

$$\begin{aligned} F(x) &= \int_0^x \tfrac{1}{50}t\,dt \\ &= \Big[\tfrac{1}{100}t^2\Big]_0^x \\ &= \tfrac{1}{100}x^2. \end{aligned}$$

When $F(x) = \tfrac{1}{2}$, $x = 5\sqrt{2}$,
when $F(x) = \tfrac{1}{4}$, $x = 5$,
when $F(x) = \tfrac{3}{4}$, $x = 5\sqrt{3}$,
so $Q_1 = 5$, $Q_2 = 5\sqrt{2}$ and $Q_3 = 5\sqrt{3}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Piecewise-defined probability density functions

The next worked example shows how to deal with a probability density function that is piecewise defined.

Example 4 16B

A probability density function is defined piecewise by

$$f(x) = \begin{cases} k(4 + x), & \text{for } -4 \le x \le 0, \\ k(4 - x), & \text{for } 0 \le x \le 4. \end{cases}$$

a Find the value of the constant k. Hence write the equation of $f(x)$ and sketch it.

b What is the probability that $0 \le X \le 2$?

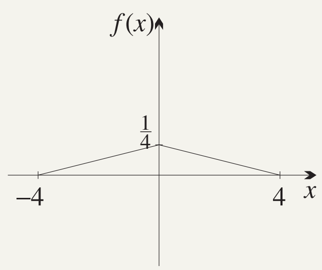
c Why is the median zero, and what is the mode?

d Find the CDF for $-4 \le x \le 0$, and the CDF for $0 \le x \le 4$. Then sketch the whole CDF.

SOLUTION

a The integral over the domain $[-4, 4]$ must be 1. Using areas of triangles is easier, but here is how to integrate piecewise over the domain by dissection.

$$\begin{aligned}\int_{-4}^{4} f(x)\,dx &= \int_{-4}^{0} k(4 + x)\,dx + \int_{0}^{4} k(4 - x)\,dx \\ &= k\Big[4x + \tfrac{1}{2}x^2\Big]_{-4}^{0} + k\Big[4x - \tfrac{1}{2}x^2\Big]_{0}^{4} \\ &= k\Big((0 + 0) - (-16 + 8) + (16 - 8) - (0 - 0)\Big) \\ &= 16k,\end{aligned}$$



so for the integral to be 1, the value of k must be $k = \frac{1}{16}$.

The function is therefore $f(x) = \begin{cases} \frac{1}{16}(4 + x), & \text{for } -4 \le x \le 0, \\ \frac{1}{16}(4 - x), & \text{for } 0 \le x \le 4. \end{cases}$

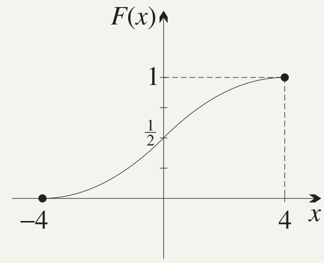
b
$$\begin{aligned}P(0 \le X \le 2) &= \int_{0}^{2} f(x)\,dx \\ &= \tfrac{1}{16}\int_{0}^{2} (4 - x)\,dx \qquad \text{(only the right-hand branch is relevant)} \\ &= \tfrac{1}{16}\Big[4x - \tfrac{1}{2}x^2\Big]_{0}^{2} \\ &= \tfrac{1}{16}\Big((8 - 2) - (0 - 0)\Big) \\ &= \tfrac{3}{8} \qquad \text{(or use the area of a trapezium).}\end{aligned}$$

c The areas to the left and right of $x = 0$ are equal, so the median is 0.
The mode is also $x = 0$, because there is a global maximum there.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d For $-4 \le x \le 0$,



$$\begin{aligned} F(x) &= \tfrac{1}{16}\int_{-4}^{x} (4 + t)\,dt \\ &= \tfrac{1}{32}\Big[(4 + t)^2\Big]_{-4}^{x} \\ &= \tfrac{1}{32}\Big((4 + x)^2 - 0\Big) \\ &= \tfrac{1}{32}(4 + x)^2. \end{aligned}$$

Hence $F(0) = \frac{1}{2}$, so for $0 \le x \le 4$,

$$\begin{aligned} F(x) &= \tfrac{1}{2} + \tfrac{1}{16}\int_{0}^{x} (4 - t)\,dt \\ &= \tfrac{1}{2} - \tfrac{1}{32}\Big[(4 - t)^2\Big]_{0}^{x} \\ &= \tfrac{1}{2} - \tfrac{1}{32}\Big((4 - x)^2 - 16\Big) \\ &= 1 - \tfrac{1}{32}(4 - x)^2. \end{aligned}$$

Distributions with unbounded domains

We have been using integrals (and sometimes area formulae) to find areas. In many important situations, the probability density function has a horizontal asymptote, however, which means that the possible values extend to infinity. For example, the diagram in Section 16A (page 717) involving 20 tossed coins suggested approximating that discrete distribution by a continuous curve with asymptotes on the left and right.

The radioactive isotope iodine-131 is often used in medicine for the treatment of thyroid cancer. It has a half-life of about 8 days. Suppose that we isolate a single nucleus of iodine-131, observe it constantly, and record the time X in days before it decays. Then using the fact that the isotope has a half-life of 8 days,

$$P(X > 8) = \tfrac{1}{2}, \qquad P(X > 16) = \tfrac{1}{4}, \qquad P(X > 24) = \tfrac{1}{8}, \qquad \dots$$

and taking the complementary events,

$$P(X \le 8) = \tfrac{1}{2}, \qquad P(X \le 16) = \tfrac{3}{4}, \qquad P(X \le 24) = \tfrac{7}{8}, \qquad \dots$$

In general, $\qquad P(X \le 8n) = 1 - 2^{-n}.$

This formula holds for all real values of $n \ge 0$, not just for whole numbers,
and to find $P(X \le x)$, put $\qquad x = 8n$,

then $n = \frac{1}{8}x$, giving $\quad P(X \le x) = 1 - 2^{-\frac{1}{8}x}$.

This last formula is the cumulative distribution function $F(x)$ for the experiment. The next worked example continues the story. We first change to base e, and write

$$F(x) = 1 - e^{-kx}, \qquad \text{where } k = \tfrac{1}{8}\ln 2 = 0.08664\dots \qquad \text{(store in memory).}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 5 16B

Let $f(x)$ and $F(x) = 1 - e^{-kx}$, where $k = \frac{1}{8}\ln 2$, be the PDF and CDF respectively for the experiment described above, observing the time x days that an iodine-131 nucleus survives before decaying.

a Explain why the domain of possible values is the unbounded interval $[0, \infty)$

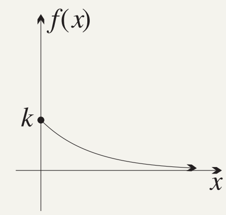
b Find the formula for the PDF, and sketch the CDF and PDF.

c Find the median, and show that it is the half-life.

d Find the probabilities that it decays on the first day and after the first day.

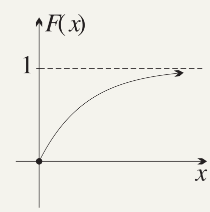
SOLUTION

a The experiment is extremely unlikely to last beyond a month or two, but it is minutely possible that it will continue for 10 years or even more. Thus we use the unbounded interval $[0, \infty)$ for the domain of possible values.



b The CDF is $\quad F(x) = 1 - e^{-kx}$.

Differentiating, $\quad f(x) = ke^{-kx}$, which is the PDF.



c To find the median, put

$$\begin{aligned} F(x) &= 0.5 \\ 1 - e^{-kx} &= \tfrac{1}{2} \\ e^{-kx} &= \tfrac{1}{2} \\ kx &= \ln 2, \end{aligned}$$

and using calculator or logs, $x = 8$ days, which is the half-life.

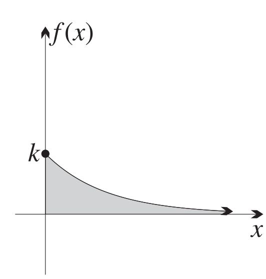
d

$$\begin{aligned} P(X \le 1) &= F(1) \\ &= 1 - e^{-k} \\ &\doteqdot 0.083 \end{aligned} \qquad \begin{aligned} P(X > 1) &= 1 - P(X \le 1) \\ &= e^{-k} \\ &\doteqdot 0.917 \end{aligned}$$

Improper integrals

Worked Example 5 above is quite sufficient preparation for the normal distribution later in the chapter. Readers may ask, however, how we can reasonably say that the area under the curve in the unbounded interval $[0, \infty)$ is 1 square unit, when it runs off to infinity! The integral involved here is called an *improper integral* — here is how to deal with it (regard this as Enrichment). The PDF is $y = ke^{-kx}$.

$$\begin{aligned} \text{area under curve over the interval } [0, \infty) &= \int_0^\infty ke^{-kx}\,dx \\ &= \Big[-e^{-kx}\Big]_0^\infty. \end{aligned}$$



Substituting $x = 0$ gives a value -1 for the primitive.
We cannot substitute $x = \infty$ because ∞ is not a number,
but we can take the limit of $-e^{-kx}$ as $x \to \infty$, which is 0,

$$\text{so} \qquad \begin{aligned} \int_0^\infty ke^{-kx}\,dx &= 0 - (-1) \\ &= 1. \end{aligned}$$

Thus we can reasonably say that the unbounded shaded area is 1 square unit.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 6 16B

a Find the shaded area under the curve $y = \frac{1}{x^2}$ over the interval $[1, \infty)$.

b Hence show that $y = \frac{1}{x^2}$, for $x \geq 1$, is a PDF, and find the CDF.

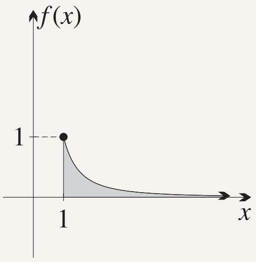
SOLUTION

a The improper integral over the closed interval $[1, \infty)$ is

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty$$

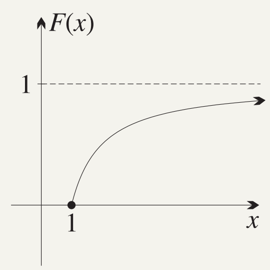
When $x = 1$, the primitive is -1.
We cannot substitute ∞ because ∞ is not a number, but we can take the limit as $x \to \infty$, which is 0.

so
$$\int_1^\infty \frac{1}{x^2}\,dx = 0 - (-1) = 1.$$



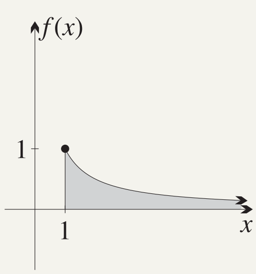
b Thus the area is 1 square unit, and the function $y = \frac{1}{x^2}$ is always positive in the interval $[1, \infty)$, so it is a PDF.

For the CDF,
$$F(x) = \int_1^x \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_1^x = 1 - \frac{1}{x}.$$



Example 7 16B

Show that the improper integral $\int_1^\infty \frac{1}{x}\,dx$ does not converge to a limit.



SOLUTION

Using the same procedure as before,

$$\int_1^\infty \frac{1}{x}\,dx = \left[\log_e x\right]_1^\infty.$$

Substituting $x = 1$ gives a value $\log_e 1 = 0$ for the primitive.
We cannot substitute $x = \infty$ because ∞ is not a number, but neither can we take the limit, because as $x \to \infty$, $\log_e x \to \infty$.
The conclusion is that the region has infinite area, and that the improper integral does not converge.

This example is rather striking because if we look only at the graph, the curves $y = e^{-x}$, $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$ all look so similar in their asymptotic behaviour.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 16B

FOUNDATION

1 **a** Sketch $f(x) = \frac{1}{2}$, where $0 \le x \le 2$. Then show that it satisfies the two conditions for a probability density function:

i Check from the graph that $f(x) \ge 0$, for all x in the domain.

ii Check that the area under the curve is 1, that is, that $\int_a^b f(x)\,dx = 1$.

b Repeat part **a** for $f(x) = \frac{1}{2}x$, where $0 \le x \le 2$.

c Repeat part **a** for $f(x) = \frac{1}{42}x$, where $4 \le x \le 10$.

2 Recall that a function $f(x)$ with domain the closed interval $[a, b]$ is called *a probability density function*, or *PDF* for short, if

$$f(x) \ge 0,\ \text{for all } x \text{ in the domain} \qquad \text{and} \qquad \int_a^b f(x)\,dx = 1.$$

Determine whether or not each function is a probability density function. If it is a PDF, find its mode (look for global maxima).

a $f(x) = 3x^2$, where $0 \le x \le 1$

b $f(x) = \frac{1}{4}x$, where $1 \le x \le 5$

c $f(x) = \dfrac{4 - 2x}{3}$, where $0 \le x \le 3$

d $f(x) = (n + 1)x^n$, where $0 \le x \le 1$

e $f(x) = \frac{1}{2}\sin x$, where $0 \le x \le \pi$

f $f(x) = \frac{1}{12}(3x^2 + 2x)$, where $0 \le x \le 2$

3 Let $f(x) = \frac{3}{4}(x^2 - 4x + 3)$ be a function defined on the closed interval $[0, 4]$.

a Show that $\int_0^4 f(x)\,dx = 1$.

b Show nevertheless that $f(x)$ is not a valid probability density function. (Hint: Sketch the graph of $y = f(x)$.)

4 For a distribution defined by a probability density function $f(x)$, the probability that x lies in the interval $[h, k]$ is the area given by $P(h \le X \le k) = \int_h^k f(x)\,dx$.

a Sketch the uniform probability density function $f(x) = \frac{1}{4}$, where $0 \le x \le 4$.

b Confirm that it satisfies the two requirements for a probability density function.

c By calculating areas, find:

i $P(0 \le X \le 1)$ **ii** $P(1 \le X \le 3)$ **iii** $P(X \le 2)$

iv $P(X = 2)$ **v** $P(X \le 3)$ **vi** $P(X \ge 1)$

d Confirm that $P(2 \le X \le 3) = P(x \le 3) - P(x \le 2)$.

5 Recall that for a probability density function, or PDF, defined on the interval $[a, b]$, the cumulative distribution function, or CDF, is $F(x) = \int_a^x f(t)\,dt$, for $a \le x \le b$.

For each PDF, calculate the CDF $F(x)$ and confirm that $F(b) = 1$.

a $f(x) = \frac{1}{32}x$, where $0 \le x \le 8$. (In this case $a = 0$ and $b = 8$.)

b $f(x) = \frac{3}{16}x^2$, where $-2 \le x \le 2$.

c $f(x) = \frac{3}{2}(1 - x^2)$, where $0 \le x \le 1$.

d $f(x) = \frac{1}{e}(e^x + 1)$, where $0 \le x \le 1$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

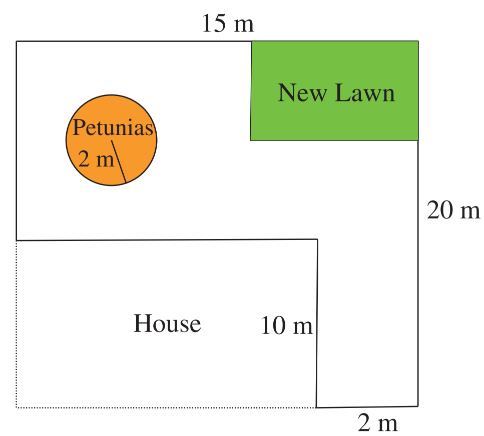
6 For the PDFs in parts **a** and **b** of Question **5**, use the CDF $F(x)$ to calculate:

a the median Q_2, by finding the value x such that $F(x) = 0.5$,

b the quartiles Q_1, by solving $F(x) = 0.25$, and Q_3, by solving $F(x) = 0.75$.

DEVELOPMENT

7 When Jack is at work, he shuts his dog Bud in the L-shaped backyard of his house. This is shown in the diagram to the right. Bud wanders around at random during the day, waiting for Jack to come home. Bud is the only cause of stress in Jack's quiet neighbourhood.



a When Bud is in the area directly to the right of the house, he will be anxious and howl for Jack to come home, to the distress of the neighbours, who shout at him. What is the probability that the neighbours will be stressed?

b If Bud is in the petunia patch, it will stress Jack's mother. What is the probability that Jack's mother will be stressed?

c If the neighbours or Jack's mother are shouting at Bud, Sally the cat cannot have a quiet sleep. What is the probability that Sally will be stressed?

d If the neighbours and Jack's mother aren't complaining, Jack's father is worried that Bud might be digging up his piece of new lawn. What is the probability that Jack's father will be stressed?

8 The function $y = 2x$, for $0 \le x \le 1$, is a probability density function.

a Sketch the graph and check that $f(x) \ge 0$.

b Use your diagram to show that the area bounded by the function and the x-axis is 1. Then check your result by integration.

c **i** Mark a point x between 0 and 1 on your diagram, and use area formulae to show that the cumulative distribution function is $P(X \le x) = x^2$.

ii Confirm your result by calculating the integral $P(X \le x) = \int_0^x 2t\,dt$.

d Use your expression for the cumulative distribution function to calculate the three quartiles Q_1, Q_2 and Q_3.

9 Find the value of the unknown constant c, given that each function $f(x)$ is a probability density function on the given domain.

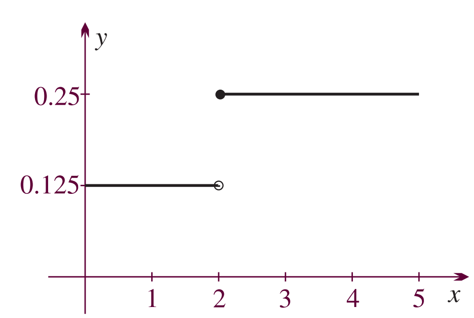
a $y = cx^4$, with domain $[0, 3]$

b $y = c$, with domain $[0, 6]$

c $y = c$, with domain $[-5, 5]$

d $f(x) = \frac{8}{3}(1 - x)$, with domain $[0, c]$

10 A function is graphed to the right.



a Verify that the function forms a valid PDF.

b Fill in the following table of values for the cumulative probabilities $P(X \le x)$.

x	0	1	2	3	4	5
$P(X \le x)$						

c Use your table to plot these points. Hence graph the cumulative probability function $P(X \le x)$ for $0 \le x \le 5$.

d Write down a formula for the CDF, writing your answer in piecewise notation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 A probability density function is defined by:

$$f(x) = \begin{cases} c, & \text{for } 0 \le x \le 5, \\ 2c, & \text{for } 5 < x \le 10. \end{cases}$$

a Sketch the probability density function.
b Find the value of c.
c Find an expression for the cumulative distribution function.
d Use your cumulative distribution function to find $P(1 < X < 7)$.

12 Define a probability density function by $f(x) = \frac{3}{32}x(4 - x), 0 \le x \le 4$.
a Sketch the probability density function, and state its mode.
b Confirm that $\int_0^4 f(x)\,dx = 1$.
c Write down $P(X \le 2)$. What other property of the curve enables us to do this without calculating an integral?
d Evaluate $P(X \le 1)$ and $P(X > 1)$. Explain why your two results add to 1.
e Evaluate $P(X \le 0.5)$ and $P(X \ge 3.5)$. What do you notice about your answers?
f Determine the cumulative distribution function, defined by $F(x) = P(X \le x)$, using the formula $F(x) = \int_0^x f(t)\,dt$.
g Use your cumulative distribution function (CDF) to evaluate:
i $P(X < 1.5)$
ii $P(1 < X < 1.5) = P(X < 1.5) - P(X < 1)$
iii $P(3 < X < 3.5)$
iv $P(2 < X < 2.5)$
h Graph the CDF in your book.
i By evaluating $P(X < 2)$ using your CDF, confirm that 50% of the data lie to the left of the line $x = 2$.
j [Technology] Plot the cumulative distribution function and determine the upper and lower quartiles, defined by $P(X < \mathrm{Q}_1) = 0.25$ and $P(X < \mathrm{Q}_3) = 0.75$.

13 Define $f(x) = ce^{-x}$, where $0 \le x \le 1$.
a Sketch the curve $y = f(x)$.
b Find c, given that $f(x)$ is a probability density function.
c Find the cumulative distribution function.
d Find the quartiles Q_1, Q_2 and Q_3.

14 Grouping approximates a continuous distribution by a discrete distribution. A few trials of an experiment generated data in the interval 1.5– 4.5, and the data were grouped in class intervals of width 1.

Class	1.5– 2.5	2.5– 3.5	3.5– 4.5
Class centre	2	3	4
Relative frequency	0.3	0.4	0.3

a Use this dataset to draw a relative frequency histogram and a relative frequency polygon.
b Find the total area of the histogram and the area under the polygon.
c On a new set of axes, draw the cumulative relative frequency histogram and polygon.
d Estimate the three quartiles Q_1, Q_2 and Q_3 by reading off the corresponding values on the horizontal axis for the relative frequencies 0.25, 0.5 and 0.75.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

e After running more trials and taking finer intervals, the experimenter decides that the data can best be modelled by the curve

$f(x) = \frac{3}{32}(x - 1)(5 - x)$, where $1 \le x \le 5$.

i Check that this curve is a probability density function.

ii Tabulate $f(x)$ for $x = 2$, 3 and 4, then graph it on top of the relative frequency polygon and compare the two (this would be easier with suitable technology).

iii Find the cumulative distribution function $F(x) = \int_1^x f(t)\,dt$, for $1 \le x \le 5$.

iv Substitute your three estimates for the quartiles into the cumulative distribution function. How close are your answers to 25%, 50% and 75%?

v [Technology] Graph the cumulative distribution function found by integration and read off the resulting estimates for the quartiles.

ENRICHMENT

15 This question and the next both involve improper integrals where the upper limit is ∞. Infinity is not a number, so you cannot substitute ∞. Instead, take the limit of the primitive as $x \to \infty$.

Let $f(x) = \dfrac{1}{x^2}$, where $x \ge 1$. Notice that this function is defined on an unbounded domain.

a Show that $f(x) > 0$ and $\int_1^\infty f(x)\,dx = 1$.

b Evaluate the cumulative distribution function $F(x)$.

c Confirm that $F(x) \to 1$ as $x \to \infty$. Why is this significant?

d Evaluate the three quartiles Q_1, Q_2 and Q_3.

16 Repeat the previous question for the function $f(x) = e^{-x}$, where $x \ge 0$, changing the limits 1 and ∞ of the integral to 0 and ∞.

17 Monte draws a unit square centred at $(0.5, 0.5)$ and inscribes a circle of radius 0.5 unit. He reasons that the probability of a random point in the square falling within the circle is proportional to the ratio of the areas of the shapes.

a Show that the ratio of the areas is $\frac{\pi}{4}$.

b Monte enters the code `=IF((RAND()-0.5)^2+(RAND()-0.5)^2<0.25,1,0)` in cell `A1` of a spreadsheet.

i In this code, the first `RAND()` selects the x-coordinate of a random point in the square, and the second `RAND()` selects the y-coordinate. Explain what value the code `(RAND()-0.5)^2+(RAND()-0.5)^2` will return for a random point.

ii What value does the code in `A1` return if the point is *inside* the circle?

iii What value does the code in `A1` return if the point is *outside* the circle?

c Monte fills the code down to the first 2000 cells of column `A`, and in cell `C1` enters `= 4*AVERAGE(A:A)`. What value is the code `AVERAGE(A:A)` measuring, and what value should C1 be approaching?

d Use more cells in column `A` and use the `RECALCULATE` feature on your spreadsheet to investigate the accuracy of this method of determining π.

e This method could be used on your calculator. On calculators that provide the command `RAN#` to generate a random number in the interval $[0, 1)$, type the code `(Ran#-0.5)`^{2}`+(Ran#-0.5)`^{2}. Every time you enter this command it should use a new random point. If it is less than 0.25, count the point as in the circle. To make this procedure more accurate, do it as a class exercise and combine your results.

f Look up further details of the *Monte Carlo method* on the web.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

16C Mean and variance of a distribution

The expected value (or mean) and the variance of a continuous probability distribution are obtained in almost the same ways as with discrete probability distributions. The main difference is that we replace the sum with its sigma notation $\sum$ by the integral with its integral notation $\int$. The Greek sigma and this early form of the letter S both correspond to the Latin letter S for 'sum'.

The expected value or mean of a continuous distribution

The *mean* or *expected value* of a discrete distribution is

$$\mathrm{E}(X) = \sum x\,p(x), \text{ summing over the whole distribution.}$$

The continuous analogue of addition is integration, so the continuous version is

$$\mathrm{E}(X) = \int_a^b x f(x)\,dx, \text{ integrating over the whole interval } [a, b].$$

The variance and standard deviation of a continuous distribution

The *variance* of a discrete distribution has two equivalent forms:

$$\mathrm{Var}(X) = E\big((X - \mu)^2\big) = \sum (x - \mu)^2 p(x),$$
$$\mathrm{Var}(X) = E(X^2) - \mu^2 = \sum x^2 p(x) - \mu^2.$$

The continuous analogues of these two forms are

$$\mathrm{Var}(X) = E\big((X - \mu)^2\big) = \int_a^b (x - \mu)^2 f(x)\,dx,$$
$$\mathrm{Var}(X) = E(X^2) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2.$$

The equality of these two expressions is proven in Question 7 of Exercise 16C.

Example 8 — 16C

Apply all this to the chook at the start of Section 16B, where $f(x) = \frac{1}{18}x$.

SOLUTION

$$\begin{aligned}\mu &= \int_0^6 x \times \tfrac{1}{18}x\,dx\\ &= \tfrac{1}{54}\Big[x^3\Big]_0^6\\ &= \tfrac{216}{54} - 0\\ &= 4,\end{aligned}$$

so the chook's mean or expected distance from the centre is 4 metres.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$$\begin{aligned}\sigma^2 &= \int_0^6 (x-4)^2 \times \tfrac{1}{18}x\,dx \\ &= \tfrac{1}{18}\int_0^6 (x^3 - 8x^2 + 16x)\,dx \\ &= \tfrac{1}{18}\Big[\tfrac{1}{4}x^4 - \tfrac{8}{3}x^3 + 8x^2\Big]_0^6 \\ &= \tfrac{1}{18}(324 - 576 + 288) \\ &= 2,\end{aligned}$$

OR

$$\begin{aligned}\sigma^2 &= \int_0^6 x^2 \times \tfrac{1}{18}x\,dx - 4^2 \\ &= \tfrac{1}{72}\Big[x^4\Big]_0^6 - 16 \\ &= (18 - 0) - 16 \\ &= 2.\end{aligned}$$

As with discrete distributions, the second form is usually easier for calculations.

The *standard deviation* is the square root of the variance, and has the same units as the values, so here $\sigma = \sqrt{2}$ metres.

6 MEAN OR EXPECTED VALUE, AND VARIANCE

Let $f(x)$ be a probability density function on a closed interval $[a, b]$.

- The *mean* or *expected value* $\mu = \mathrm{E}(X)$ is
$$\mathrm{E}(X) = \int_a^b x\,f(x)\,dx.$$
- The *variance* $\sigma^2 = \mathrm{Var}(X)$ is the expected value of the squared deviation from the mean,
$$\mathrm{Var}(X) = E\big((X-\mu)^2\big) = \int_a^b (x-\mu)^2 f(x)\,dx$$
- Alternatively, and usually easier in calculations, the variance is the expected value of the square, minus the square of the mean,
$$\mathrm{Var}(X) = E(X^2) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2.$$
- The standard deviation σ is the square root of the variance.

Example 9 16C

Find the mean and standard deviation of each PDF.

a $y = \frac{1}{8}$, for $0 \le x \le 8$

b $y = \frac{1}{50}x$, for $0 \le x \le 10$

SOLUTION

a $$\begin{aligned}\mu &= \int_0^8 \tfrac{1}{8}x\,dx \\ &= \Big[\tfrac{1}{16}x^2\Big]_0^8 \\ &= 4 - 0 \\ &= 4,\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \int_0^8 \tfrac{1}{8}x^2\,dx - 4^2 \\ &= \Big[\tfrac{1}{24}x^3\Big]_0^8 - 16 \\ &= \frac{512}{24} - 0 - 16 \\ &= \frac{16}{3},\end{aligned}$$

$$\sigma = \tfrac{4}{3}\sqrt{3}.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b $y = \frac{1}{50}x$, for $0 \le x \le 10$

$$\mu = \int_0^{10} \tfrac{1}{50}x^2\,dx$$
$$= \left[\tfrac{1}{150}x^3\right]_0^{10}$$
$$= \tfrac{1000}{150} - 0$$
$$= 6\tfrac{2}{3},$$

$$\sigma^2 = \int_0^{10} \tfrac{1}{50}x^3\,dx - \left(6\tfrac{2}{3}\right)^2$$
$$= \left[\tfrac{1}{200}x^4\right]_0^{10} - \tfrac{400}{9}$$
$$= \frac{10\,000}{200} - 0 - \tfrac{400}{9}$$
$$= \frac{50}{9},$$
$$\sigma = \tfrac{5}{3}\sqrt{2}.$$

Exercise 16C

FOUNDATION

1 A function is defined by $f(x) = \frac{1}{10}$, where $0 \le x \le 10$.
- **a** Show that $f(x)$ is a valid PDF (probability density function).
- **b** Calculate the expected value using the formula $E(X) = \int_a^b x\,f(x)\,dx$.
- **c** Does your answer for the expected value agree with your understanding of expected value as an average value?
- **d** Calculate the variance using the formula $\text{Var}(X) = \int_a^b (x - \mu)^2 f(x)\,dx$, then find the standard deviation σ.
- **e** Use the alternative formula for variance $\text{Var}(X) = E(X^2) - E(X)^2$ and confirm that your answer agrees with the previous result.

2 The previous question provides a mathematical model for selecting a random real number in the interval $[0, 10]$.

Use your calculator (or a spreadsheet) to generate a random number between 0 and 10 to as many decimal places as possible. Many calculators return a random number between 0 and 1, and you will need to multiply this answer by 10.
- **a** Generate 20 such numbers, recording them in a table.
- **b** Calculate the mean and standard deviation using your calculator.
- **c** Do your results agree with the theoretical probabilities in the previous question?
- **d** Our model includes the possibility of selecting a 10, but it is virtually certain that 10 will not be returned by the calculator's random number function. Does this affect the validity of our model and your results?

3 Define the function $f(x)$ by $f(x) = \frac{3}{2}x^2$, with domain $[-1, 1]$.
- **a** Confirm that it is a valid PDF. Confirm that it is a valid PDF, and find any modes.
- **b** Find the expected value $\mu = E(X)$.
- **c** Find the variance $\text{Var}(X)$ and the standard deviation σ.
- **d** Calculate $\int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$ to determine what percentage of the population defined by this distribution lies within one standard deviation of the mean.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 Repeat Question 3 for:

a $f(x) = 2x$, with domain $[0, 1]$

b $f(x) = |x|$, with domain $[-1, 1]$

c $f(x) = \frac{3}{64}x^2$, with domain $[0, 4]$ (final answer correct to three decimal places)

DEVELOPMENT

5 Consider the function defined by $f(x) = \dfrac{1}{c}$, for $0 \le x \le c$, where $c > 0$.

a Is this function a valid PDF?

b Calculate $\mathrm{E}(X)$. Is your answer as expected?

c Calculate $\mathrm{Var}(X)$.

d Compare your answer with the special case in Question 1.

e Use the results $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$ to find the expected value for the translated uniform probability distribution with density function $g(x) = \dfrac{1}{c}$, for $h \le x \le h + c$.

f Find the expected value and variance of the uniform probability distribution defined on the interval $h \le x \le k$.

6 **a** Show that the function $f(x)$ is a valid PDF,

$$f(x) = \begin{cases} \frac{1}{8}, & \text{for } 0 \le x < 2, \\ \frac{1}{4}, & \text{for } 2 \le x \le 5, \end{cases}$$

b Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

7 At the start of this chapter, we claimed that two expressions for the variance of a continuous distribution are equal,

$$\int_a^b (x - \mu)^2 f(x)\, dx = \int_a^b x^2 f(x)\, dx - \mu^2.$$

Prove this identity, starting with the LHS and expanding the integrand.

ENRICHMENT

Note: Except for Question 8, these questions involve improper integrals where the upper limit is ∞. Infinity is not a number, so you cannot substitute ∞. Instead, take the limit of the primitive as $x \to \infty$.

8 In Question 6 of Exercise 16A, we demonstrated that the area under a relative frequency polygon equals the area under a relative frequency histogram, and that both are equal to the total probability 1.

a Confirm that the relative frequency polygon in that question may be written piecewise as:

$$f(x) = \begin{cases} \frac{2}{10}x, & \text{for } 0 \le x \le 1, \\ \frac{1}{10}(3x - 1), & \text{for } 1 \le x \le 2, \\ \frac{1}{10}(-2x + 9), & \text{for } 2 \le x \le 3, \\ \frac{1}{10}(-3x + 12), & \text{for } 3 \le x \le 4. \end{cases}$$

b By integration, calculate $\mathrm{E}(X) = \displaystyle\int_0^4 x f(x)\, dx$ for the probability distribution using the PDF $f(x)$ defined above.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Compare the answer obtained by calculating the expected value for the discrete distribution using the table of values in Question 6 of Exercise 16A.

d What is your conclusion about the PDF as a generalisation of the frequency polygon?

9 Define $f(x) = \dfrac{3}{x^4}$, where $x \ge 1$.

a Show that $f(x)$ is a valid PDF.

b Evaluate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

c Calculate each probability.

i $P(X \le 4)$ **ii** $P(X \ge 2)$ **iii** $P(2 \le X \le 5)$

d Find the cumulative distribution function $F(x) = P(X \le x)$.

10 Consider the function $f(x) = e^{-x}$, where $x \ge 0$. In Question 16 of Exercise 16B, we showed that $f(x)$ is a valid PDF.

a Differentiate xe^{-x}, and hence integrate xe^{-x}.

b Evaluate $\mathrm{E}(X) = \displaystyle\int_0^\infty xe^{-x}\,dx$.

c Differentiate $x^2e^{-x} + 2xe^{-x} + 2e^{-x}$, and hence integrate x^2e^{-x}.

d Evaluate $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$ for the distribution with PDF $f(x)$.

11 Consider the function $f(x) = \dfrac{1}{\ln 2} \times \dfrac{1}{x(x+1)}$, where $x \ge 1$.

a Show that $f(x) = \dfrac{1}{\ln 2}\left(\dfrac{1}{x} - \dfrac{1}{x+1}\right)$.

b Hence show that $\displaystyle\int f(x)\,dx = \frac{1}{\ln 2} \times \ln\left(\frac{x}{x+1}\right)$, for $x \ge 1$.

c Hence show that $f(x)$ is a valid PDF in the domain $[1, \infty)$.

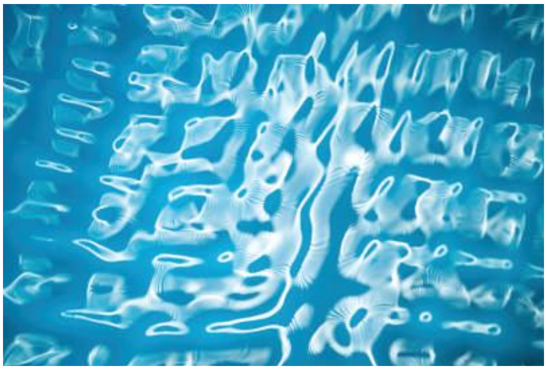
d Show that $\mathrm{E}(X)$ does not exist for this PDF.

12 a Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$.

b Explain why the function $f(x) = \dfrac{1}{\pi(1+x^2)}$, defined over the whole real line $(-\infty, \infty)$, is a valid PDF.

c What could a symmetry argument alone lead you to conclude about the value of $\mathrm{E}(X)$? Explain why $\mathrm{E}(X)$ is nevertheless undefined according to the definition in Box 6.

d Use the identity $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$ to show that $\mathrm{Var}(X)$ is not defined.



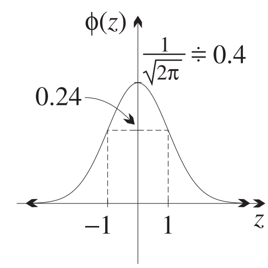
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16D The standard normal distribution

Gauss and the early statisticians realised that one particular group of continuous distributions — the *Gaussian* or *normal distributions* — are particularly important. They occur in a wide variety of situations, and for reasons that we shall explain later, are involved in the study of every distribution, continuous or discrete.

Their graphs are generally referred to as *bell-shaped curves*, and every normal distribution can be obtained from every other normal distribution by shifting and stretching. We saw the shape of such a curve emerging when 20 coins were tossed at the end of Section 16A.



The graph of the *standard normal distribution* is sketched to the right. The equation of its probability density function is

$$\phi(z) = \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \quad \text{(the Greek letter } \phi \text{ is phi, corresponding to Latin f).}$$

It is standard practice to use Z rather than X for the standard normal random variable, and z rather than x for its values, so that the PDF is $\phi(z)$.

Sketching the curve

We already have all the tools for sketching this function from its equation. The first thing is to stop looking at the complicated denominator $\sqrt{2\pi}$, which is just a constant. Use your calculator to find that $\dfrac{1}{\sqrt{2\pi}} \doteqdot 0.4$, and start thinking of the formula as $\phi(z) \doteqdot \frac{2}{5}e^{-\frac{1}{2}z^2}$, which looks much more friendly.

- The y-intercept is $\phi(0) = \dfrac{1}{\sqrt{2\pi}} \doteqdot 0.4$, because $e^0 = 1$.
- When z is non-zero, the index $-\frac{1}{2}z^2$ is negative, so $e^{-\frac{1}{2}z^2} < e^0 = 1$. Hence the value at $z = 0$ is a global maximum, and the mode is therefore $z = 0$.
- The function is defined for all values of z, and is positive for all values of z.
- The function is even, with line symmetry in the y-axis, because replacing z by $-z$ leaves the equation unchanged.
- As $z \to \infty$, and as $z \to -\infty$, the index $-\frac{1}{2}z^2$ quickly becomes a large negative number, so $e^{-\frac{1}{2}z^2}$ quickly becomes an extremely small positive number. Thus the z-axis is a horizontal asymptote in both directions.
- There are points of inflection at $z = -1$ and $z = 1$ (both have y-coordinate $\dfrac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \doteqdot 0.24$). We have left the proof of this to the end of this section.

Why is $\phi(z)$ a probability density function?

We have now established that $\phi(z)$ has the graph shown above, but why is it a probability density function? Certainly we can see that it is always positive. But we also need to establish that

$$\int_{-\infty}^{\infty} \phi(z)\, dz = 1.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Unfortunately, this integral cannot be established using the techniques in this course. This fact is one of a small number of things that readers will have to accept for now, and perhaps prove in later years.

Making the total area under the curve have the value 1 is the reason why the denominator $\sqrt{2\pi}$ has been put there, and proving the result requires a proof that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi} \doteqdot 2.5$. The best that can be done is to confirm this result by trapezoidal rule approximations along the lines of Question 19 in Exercise 16D.

The mean and variance of the standard normal distribution

- The mean of the standard normal distribution is $\mu = 0$.
- Its variance is $\sigma^2 = 1$, and its standard deviation is therefore 1.

The fact that the mean is 0 is clear from the graph, because $y = \phi(z)$ is even, with line symmetry about the y-axis.

Establishing that the variance is 1, however, is a little more difficult because of some fancy integration, and has also been left to the end of this section.

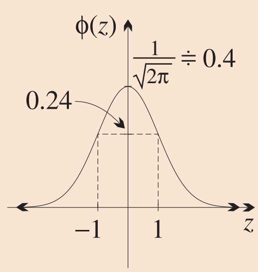
These results for the mean and standard deviation are closely tied to the turning point and inflections, so that μ and σ can be seen clearly on the graph.

- The mean $\mu = 0$ coincides with the maximum turning point at $z = 0$, which is the mode.
- The two inflections at $z = -1$ and $z = 1$ are each one standard deviation from the mean in opposite directions.

7 THE STANDARD NORMAL DISTRIBUTION

Let Z be the *standard normal random variable*.

- The probability density function of Z is $\phi(z) = \dfrac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$.
- The graph of the PDF is a *bell-shaped curve*, with global maximum at $z = 0$ (the mode) and points of inflection at $z = 1$ and $z = -1$.
- The mean is $\mu = 0$ and the standard deviation is $\sigma = 1$.
- The points of inflection are each one standard deviation from the mean.



In Section 16E we will be shifting and stretching the standard normal distribution. Whenever you see a curve that looks even vaguely normal, always look first at the turning point, then look at the two inflections and quickly estimate the standard deviation by eye.

Integrating to find probabilities

The probability that a standard normal random variable Z lies within one standard deviation of the mean is

$$P(-1 \le Z \le 1) = \int_{-1}^{1} \phi(z)\,dz.$$

Now we have a major inconvenience — the primitive of the function $\phi(z)$ cannot be written in terms of our usual range of exponential, trigonometric and algebraic functions. You have several options, all of which can be found online:

- Use a table of values for this integral — a short version is at the bottom of the next page.
- Use the Desmos graphing calculator or a statistics calculator that has the values of these integrals built in.

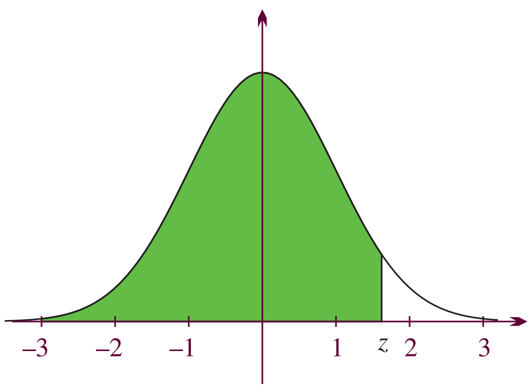
ISBN 978-1-108-76630-2

- Use a spreadsheet that has these integrals amongst its functions.
- Use specialised statistics software.

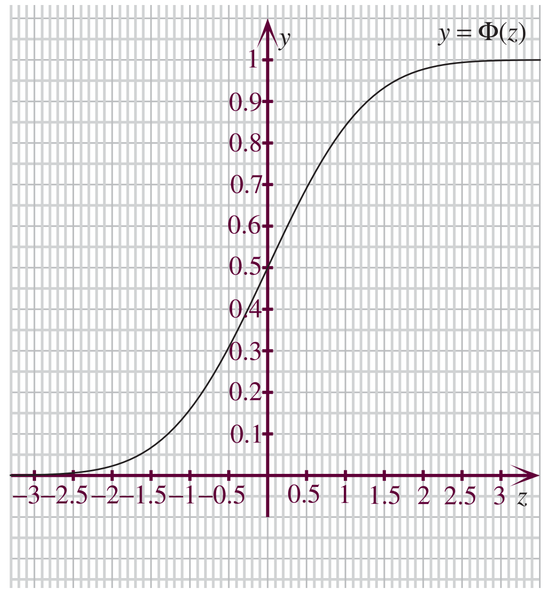
The cumulative distribution function

All these approaches will normally use the cumulative distribution function of the standard normal distribution. This CDF is usually denoted by $\Phi(z)$, using the uppercase version Φ of the Greek letter ϕ.

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt.$$



This is the graph of the standard normal probability density function $\phi(z)$.



This is the graph of the standard normal cumulative distribution function $\Phi(z)$.

The new curve $y = \Phi(z)$ has two horizontal asymptotes, $y = 0$ on the left, and $y = 1$ on the right. Because also $\phi(z)$ is even, $\Phi(z)$ has point symmetry in $(0, 0.5)$.

Here then are some further details about finding values of $\Phi(z)$.

- Below is a short table of values of $\Phi(z)$ for $0 \le z < 4$, in steps of 0.1.
- Statistical calculators should have this function built in.
- In Excel, the function is `NORM.S.DIST`. The function has two arguments.
  - The first argument is the value of z (or the cell containing that value).
  - Set the second argument to true to obtain the value of the CDF $\Phi(z)$, and set it to `false` for the value of the PDF $\phi(z)$.
- See the interactive textbook for using Desmos.

The following short table of values of $\Phi(z)$ will be quite sufficient for most purposes in this chapter. Because of the even symmetry of the PDF $\phi(z)$, there is no need to give values of the CDF $\Phi(z)$ for negative values of z.

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

A more detailed table giving the values of z to three decimal places is to be found in the Appendix to Chapter 17.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Calculating probabilities of a normally distributed random variable

Calculating other probabilities for Z requires juggling integrals, preferably while looking at a graph of the PDF $y = \phi(z)$. Always keep two things in mind.

- The total area under the curve $y = \phi(z)$ is 1.
- The curve $y = \phi(z)$ is even — it has line symmetry in the y-axis.

The next worked example demonstrates all the methods required.

Example 10 — 16D

a Look up $P(Z \le 2)$ and illustrate it as an area under $y = \phi(z)$.

b Illustrate each probability as an area under $y = \phi(z)$. Then calculate it using the value of $\Phi(2)$ found in part **a**. Keep looking back to the graph in part **a** while you juggle the intervals.

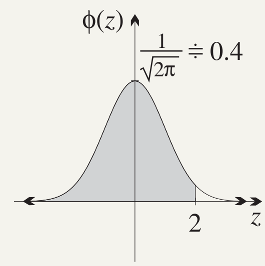
i $P(Z \ge 2)$ **ii** $P(Z \le -2)$ **iii** $P(0 \le Z \le 2)$ **iv** $P(-2 \le Z \le 2)$

SOLUTION

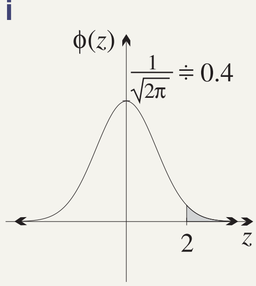
a From the table,

$$P(Z \le 2) = \Phi(2) = 0.9772.$$

The area under $y = \phi(z)$ corresponding to $\Phi(2)$ is shaded in the diagram to the right.

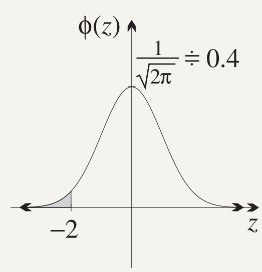


b **i**



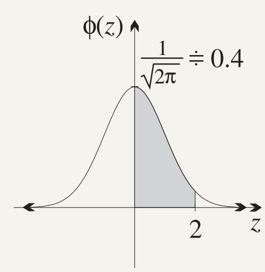
$P(Z \ge 2)$

ii



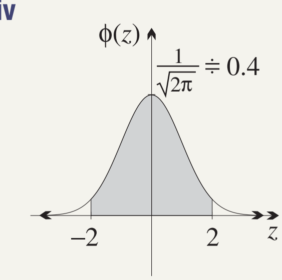
$P(Z \le -2)$

iii



$P(0 \le Z \le 2)$

iv



$P(-2 \le Z \le 2)$

i $P(Z \ge 2) = 1 - P(Z \le 2)$, because the total area is 1,
$\doteqdot 1 - 0.9772$
$\doteqdot 0.0228$.

The probability that $Z = 2$ exactly is zero,
so there is no need to distinguish between $\le$ and $<$ or between $\ge$ and $>$.

ii $P(Z \le -2) = P(Z \ge 2)$, because $\phi(z)$ is even,
$\doteqdot 0.0228$, from part **a**.

iii $P(0 \le Z \le 2)$
$= \Phi(2) - \Phi(0)$, using subtraction of areas,
$\doteqdot 0.9772 - 0.5$, because exactly half the scores are below the mean,
$\doteqdot 0.4772$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iv $P(-2 \le Z \le 2)$
$= \Phi(2) - \Phi(-2)$
$\doteqdot 0.9772 - 0.0228$, by part **b**,
$\doteqdot 0.9544$,

OR

$P(-2 \le Z \le 2)$
$= 2 \times P(0 \le Z \le 2)$, by symmetry,
$\doteqdot 2 \times 0.4772$, by part **c**,
$\doteqdot 0.9544$.

Example 11 — 16D

a Explain how to find $\Phi(0.7)$ from the table, and illustrate it.

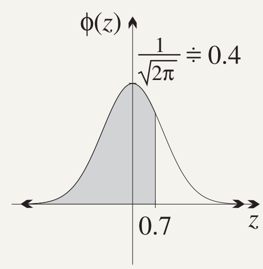
b Use symmetry and the table of values of $\Phi(z)$ to find:

i $P(-2.5 \le Z \le -0.3)$ **ii** $P(-1.7 \le Z \le 0.6)$

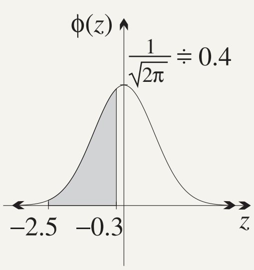
SOLUTION

a To find $\Phi(0.7)$ from the table (as illustrated to the right):
- Look at the first row because 0.7 starts with '0.'.
- Then go to the column headed '.7'.

$$P(Z \le 0.7) = \Phi(0.7) \doteqdot 0.7580.$$

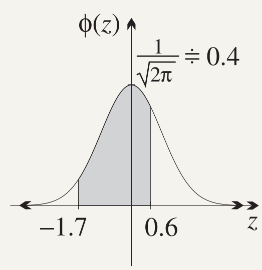


b i



$P(-2.5 \le Z \le -0.3)$
$= P(0.3 \le Z \le 2.5)$,
because $\phi(z)$ is even,
$= \Phi(2.5) - \Phi(0.3)$
$\doteqdot 0.9938 - 0.6179$
$\doteqdot 0.3759$.

ii



$P(-1.7 \le Z \le 0.6)$
$= P(-1.7 \le Z \le 0) + P(0 \le Z \le 0.6)$
$= P(0 \le Z \le 1.7) + P(0 \le Z \le 0.6)$
$= \Phi(1.7) - \Phi(0) + \Phi(0.6) - \Phi(0)$
$= \Phi(1.7) + \Phi(0.6) - 1$
$\doteqdot 0.6811$.

The empirical rule or the 68–95–99.7 rule

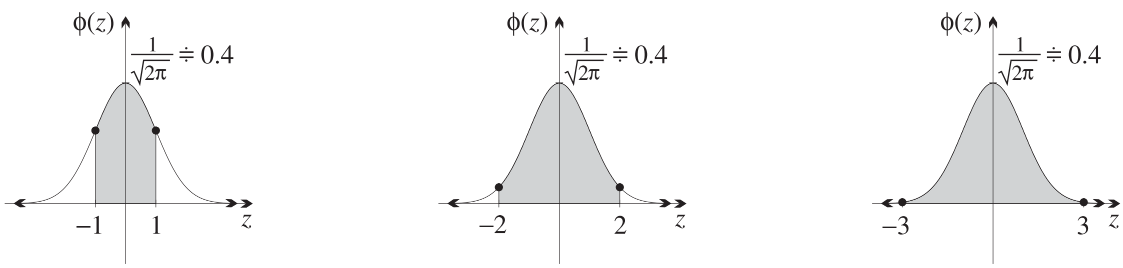
Sometimes statistics requires accurate results, and sometimes it uses very approximate methods. It turns out that in practical use, we constantly need to know the probabilities that a normally distributed variable is within 1, 2 or 3 standard deviations of the mean. That is intuitively straightforward, because
- Z has standard deviation $\sigma = 1$, and
- the two inflections make the region within one standard deviation of the mean stand out on the graph.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Here are those three results, derived from the table of values of $\Phi(z)$ and converted to the rounded percentages conventionally used in the empirical rule.

$$P(-1 \le Z \le 1) \doteqdot 0.6827 \doteqdot 68\%,$$
$$P(-2 \le Z \le 2) \doteqdot 0.9545 \doteqdot 95\%,$$
$$P(-3 \le Z \le 3) \doteqdot 0.9973 \doteqdot 99.7\%.$$



The percentages here are probabilities, but they are also predictions of what percentages of a normally distributed sample lie within 1, 2 or 3 standard deviations of the mean. These three results are so important that memorising them is part of learning to use the normal distribution, and together they are called the *empirical rule* or the *68–95–99.7 rule*.

8 THE EMPIRICAL RULE OR THE 68–95–99.7 RULE

In a normal distribution, the proportion of scores lying:

- within 1 standard deviation of the mean is 68%,
- within 2 standard deviations of the mean is 95%,
- within 3 standard deviations of the mean is 99.7%.

Example 12 **16D**

An experiment is run 1000 times. Its random variable is the standard normal variable Z. Answer these questions using the empirical rule only.

a How many scores greater than 2 would you expect?

b Find b if we would expect about 840 scores greater than b.

SOLUTION

a By the empirical rule, we expect $1000 \times 95\% = 950$ scores within $[-2, 2]$.
Because $\phi(z)$ is even, we expect $950 \div 2 = 475$ scores within $[0, 2]$.
Because 500 scores should be positive, we expect 25 scores greater than 2.

b We therefore expect 160 scores less than b, so in particular, b is negative.
Because $\phi(z)$ is even, we also expect 160 scores greater than $-b$.
Hence we expect $1000 - 160 - 160 = 680$ scores between $-b$ and b, so by the empirical rule, $b \doteqdot -1$.

Quartiles and percentiles

The graph of $y = \Phi(z)$ was drawn above on page 549. Approximation of the quartiles and percentiles can be found from this graph by drawing the appropriate horizontal lines. (We can also use interpolation on the table of values for $\Phi(z)$.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 13 16D

a Find the 9th decile of the standard normal distribution.

b Find the third quartile Q_3 and the first quartile Q_1 of $\Phi(z)$.

c Using the IQR criterion, what proportion of the scores of a standard normal random variable would be expected to be outliers?

SOLUTION

a For the 9th decile, we need to solve $\Phi(z) \doteqdot 0.9$.
From the table, $\Phi(1.2) \doteqdot 0.8849$ and $\Phi(1.3) \doteqdot 0.9032$, with difference 0.0183 making the 9th decile about 1.28.
This agrees with the horizontal line with height 0.9 on the graph of $\Phi(z)$.

b For the upper quartile, we need to solve $\Phi(z) \doteqdot 0.75$.
From the table, $\Phi(0.6) \doteqdot 0.7257$ and $\Phi(0.7) \doteqdot 0.7580$, with difference 0.0323.
Thus $Q_3 \doteqdot 0.675$, and because $\phi(z)$ is even, $Q_1 \doteqdot -0.675$.
This agrees with the horizontal line with height 0.75 on the graph of $\Phi(z)$.

c From part **b**, the IQR is about 1.35, so $Q_3 + 1.5 \times \text{IQR} \doteqdot 2.70$.
The IQR criterion is that an outlier lies outside $-2.70 \le z \le 2.70$.

$$\begin{aligned} \text{Hence } P(Z \text{ is an outlier}) &= P(Z > 2.70) + P(Z < -2.70) \\ &= 2 \times P(Z > 2.70) \\ &= 2\big(1 - \Phi(2.70)\big) \\ &\doteqdot 0.007, \end{aligned}$$

so roughly 7 in 1000 scores would be expected to be outliers.

Note: Correct to five significant figures, $Q_3 \doteqdot 0.67449$. It is standard practice to round percentiles and quartiles of the normal to two decimal places, giving

$$Q_1 \doteqdot -0.67 \qquad \text{and} \qquad Q_3 \doteqdot 0.67 \qquad \text{and} \qquad \text{IQR} \doteqdot 1.35,$$

and using the interquartile range criterion, an outlier is a score outside the interval

$$-2.70 \le z \le 2.70.$$

In a normal distribution, the IQR criterion makes just under 1% of scores outliers.

The points of inflection

Showing that $y = \phi(z)$ has inflections at $z = -1$ and $z = 1$ requires the second derivative of $\phi(z)$, and is reasonably straightforward.

Differentiation of $y = e^{-\frac{1}{2}z^2}$ requires the chain rule,

$$\begin{aligned} \frac{dy}{dz} &= \frac{dy}{du} \times \frac{du}{dz} \\ &= e^{-\frac{1}{2}z^2} \times (-z) \\ &= -ze^{-\frac{1}{2}z^2}. \end{aligned}$$

Let $u = -\frac{1}{2}z^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dz} = -z$,
and $\dfrac{dy}{du} = e^u$.

The function $\phi(z) = \dfrac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$ is a multiple of $e^{-\frac{1}{2}z^2}$,

so $\phi'(z) = -z\phi(z)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Hence $\phi(z)$ has a stationary point at $z = 0$, which is a maximum turning point, because $\phi(z)$ is increasing for $z < 0$ and decreasing for $z > 0$.

z	-1	0	1
$\phi'(z)$	$\phi(-1)$	0	$-\phi(1)$
sign	$+$	0	$-$
	/	—	\

Notice in both the tables to the right that $\phi(z)$ is always positive.

For the second derivative, $\phi''(z) = \dfrac{d}{dz}\big(\phi'(z)\big)$

$$= \frac{d}{dz}\big(-z\phi(z)\big),$$

and applying the product rule with $u = -z$ and $v = \phi(z)$,

$$\begin{aligned}\phi''(z) &= -\phi(z) - z\phi'(z)\\ &= -\phi(z) + z^2\phi(z)\\ &= \phi(z)(z^2 - 1).\end{aligned}$$

So there are points of inflection at $z = 1$ and at $z = -1$.

z	-2	-1	0	1	2
$\phi''(z)$	$3\phi(-2)$	0	$-\phi(0)$	0	$3\phi(2)$
sign	$+$	0	$-$	0	$+$
	$\smile$	$\cdot$	$\frown$	$\cdot$	$\smile$

The mean and standard deviation

The integrals involved in the calculation of mean and standard deviation require some rather sophisticated techniques. The mean is given by the integral

$$\text{E}(Z) = \int_{-\infty}^{\infty} z\phi(z)\,dz.$$

The integrand $z\phi(z)$ is an odd function, because it is the product of an odd function z and an even function $\phi(z)$. Hence the integral is zero.

This argument assumes that the integral converges. To avoid this assumption,

$$\begin{aligned}\text{E}(Z) &= \int_{-\infty}^{\infty} z\phi(z)\,dz\\ &= \Big[-\phi(z)\Big]_{-\infty}^{\infty} && \text{because we showed above that } \phi'(z) = -z\phi(z)\\ &= 0 - 0 && \text{because } \phi(z) \to 0 \text{ as } z \to \infty \text{ and as } z \to -\infty.\end{aligned}$$

Because the mean is zero, the variance is given by the integral

$$\text{Var}(Z) = \int_{-\infty}^{\infty} z^2\phi(z)\,dz.$$

We showed above while finding the second derivative of $\phi(z)$ that

$$\phi''(z) = \phi(z)(z^2 - 1),$$

and rearranging, $z^2\phi(z) = \phi''(z) + \phi(z)$.

Hence

$$\begin{aligned}\text{Var}(Z) &= \int_{-\infty}^{\infty} z^2\phi(z)dz\\ &= \int_{-\infty}^{\infty} \phi''(z)dz + \int_{-\infty}^{\infty} \phi(z)dz.\\ &= \Big[\phi'(z)\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(z)dz.\\ &= 0 + 1\\ &= 1.\end{aligned}$$

The first integral above is zero, because $\phi'(z) \to 0$ as $z \to \infty$ and as $z \to -\infty$. You can see this easily by looking at the graph of $y = \phi(z)$. On the far left, and also on the far right, the curve becomes flatter and flatter; that is, its gradient $\phi(z)$ converges to zero. (A more formal proof of these limits is not possible here.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 16D

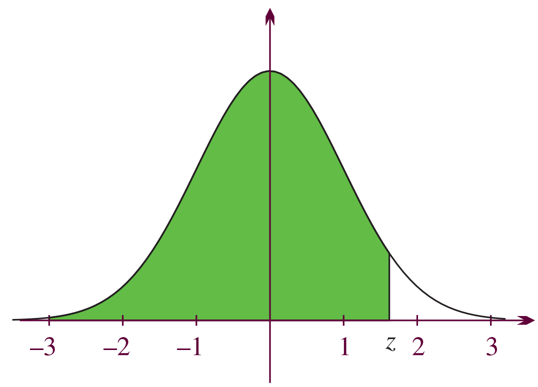
FOUNDATION

The purpose of this exercise is to build familiarity with the symmetry of the standard normal curve. It is not intended that any technology be used for the values of the standard normal distribution in the early questions, because it is important to maximise interaction with the curve and its shape.

The summary below is repeated as an appendix at the end of this chapter.

A brief summary of the standard normal probability distribution

The graph to the right is the *standard normal probability density function* $y = \phi(z)$.



The shaded area represents the value of the corresponding *cumulative distribution function*

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(t)dt.$$

The table below gives some values of the probabilities $\Phi(z) = P(Z \le z)$. For example,

$$P(Z \le 1.6) = \Phi(1.6) = \int_{-\infty}^{1.6} \phi(z)\,dz \doteqdot 0.9452.$$

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

A more detailed table giving the values of z to three decimal places is to be found in the Appendix to Chapter 17.

For many purposes, all that is required is the *empirical rule*, or *68–95–99.7 rule*,

$P(-1 \le Z \le 1) \doteqdot 68\%$
$P(-2 \le Z \le 2) \doteqdot 95\%$
$P(-3 \le Z \le 3) \doteqdot 99.7\%$

1 Use the table above to look up the following probabilities for the standard normal distribution. Record your answers correct to four decimal places.

a $P(Z \le 0)$ **b** $P(Z \le 1)$ **c** $P(Z \le 2)$ **d** $P(Z \le 1.5)$
e $P(Z < 0.4)$ **f** $P(Z \le 2.3)$ **g** $P(Z < 1.2)$ **h** $P(Z \le 5)$

2 Explain from the graph above why $P(Z > a) = 1 - P(Z \le a)$. Then use the standard normal table with this complementary result to find:

a $P(Z > 0)$ **b** $P(Z > 1)$ **c** $P(Z > 2)$ **d** $P(Z \ge 2.4)$
e $P(Z > 1.3)$ **f** $P(Z > 0.7)$ **g** $P(Z \ge 1.6)$ **h** $P(Z > 8)$

3 a Use the symmetry of the standard normal graph above to explain why if $a > 0$, then $P(Z < -a) = 1 - P(Z \le a)$. (You need not memorise this result).

b Use this result to find:

i $P(Z < -1.2)$ **ii** $P(Z \le -2.3)$ **iii** $P(Z < -0.2)$
iv $P(Z < -3.2)$ **v** $P(Z < -5)$ **vi** $P(Z \le -0.7)$
vii $P(Z < -1.6)$ **viii** $P(Z \le -1.4)$ **ix** $P(Z < -0)$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 a Use symmetry to explain why $P(Z \le 0) = 0.5$.
b Hence use symmetry and the standard normal table to find:
i $P(0 < Z \le 1.3)$ ii $P(0 < Z \le 2.4)$ iii $P(0 < Z \le 0.7)$
iv $P(-2.4 \le Z < 0)$ v $P(-1.1 \le Z < 0)$ vi $P(-0.7 \le Z < 0)$
vii $P(0 < Z \le 1.6)$ viii $P(-1.3 \le Z \le 0)$ ix $P(0 < Z \le 5)$
c Find:
i $P(-1.3 \le Z < 1.3)$ ii $P(-2.4 < Z \le 2.4)$ iii $P(-0.8 < Z < 0.8)$
iv $P(-2.9 < Z \le 2.9)$ v $P(-0.4 \le Z < 0.4)$ vi $P(-1.5 < Z \le 1.5)$

5 Match these eight probabilities into four pairs with equal values.
a $P(Z \le 2)$ b $P(Z \le -1)$ c $P(Z \le 1.2)$ d $P(Z = 4)$
e $P(Z < 2)$ f $P(Z = 2.3)$ g $P(Z \ge 1)$ h $P(Z > -1.2)$

6 Repeat the previous question for these eight values:
a $P(Z \le 5)$ b $P(Z > -1.7)$ c $P(Z < 5)$ d $P(Z \ge 2)$
e $P(Z = 3)$ f $P(Z \le -2)$ g $P(Z \le 1.7)$ h $P(Z = 1.2)$

DEVELOPMENT

7 a Explain why $P(a \le Z \le b) = P(Z \le b) - P(Z < a)$.
b Use this result to find:
i $P(1.2 \le Z < 1.5)$ ii $P(0.2 \le Z < 2.3)$ iii $P(0.6 \le Z < 1.7)$
iv $P(-2 \le Z < -1.2)$ v $P(-4 \le Z < -0.2)$ vi $P(-2.7 \le Z < -1)$
c Similarly find:
i $P(-1.5 \le Z < 2.2)$ ii $P(-0.9 \le Z < 1.2)$ iii $P(-2.9 \le Z < 1.3)$

8 Use just the two values $\Phi(1.2) = 0.8849$ and $\Phi(1.8) = 0.9641$, and the symmetry of $\phi(z)$, and your knowledge of its properties as a PDF, to find:
a $P(Z \le 0)$ b $P(Z = 4)$ c $P(Z > 1.8)$ d $P(Z \le 1.2)$
e $P(Z \ge 1.2)$ f $P(0 \le Z \le 1.2)$ g $P(Z \le -1.8)$ h $P(Z \ge -1.2)$
i $P(1.2 \le Z \le 1.8)$ j $P(-1.8 \le Z \le 1.2)$

9 Use the standard normal table to find:
a $P(Z \le 1.3)$ b $P(Z = 2.4)$ c $P(Z > 0.4)$ d $P(Z \le 1.7)$
e $P(Z \ge -1.3)$ f $P(0 \le Z \le 1.5)$ g $P(Z \le -0.8)$ h $P(Z \ge 0.2)$
i $P(1.1 \le Z \le 1.5)$ j $P(-1.3 \le Z \le 2.2)$

10 Use the standard normal table to find these probabilities. Recall from the probability chapter of the Year 11 book that ‘and’ and ‘or’ correspond to intersection and union.
a $P(Z \le 1.2 \text{ or } Z \ge 1.8)$ b $P(Z \le 1.8 \text{ and } Z \ge 1.2)$
c $P(Z \le 0.2 \text{ or } Z \ge 1.6)$ d $P(Z \le 2.4 \text{ and } Z \ge 1.7)$

11 Repeat any of the previous questions using a calculator or other technology in place of the standard normal tables.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 Use the empirical rule (also called the 68–95–99.7 rule) to find:

a $P(Z \le 0)$ **b** $P(Z \le 1)$ **c** $P(Z \le 2)$

d $P(Z < -1)$ **e** $P(0 \le Z \le 3)$ **f** $P(0 \le Z < 1)$

g $P(-2 \le Z \le 0)$ **h** $P(-3 < Z \le -2)$ **i** $P(-1 \le Z \le 1)$

j $P(-3 < Z \le 1)$ **k** $P(-2 \le Z < 1)$ **l** $P(-2 \le Z \le 7)$

13 Use the empirical rule to find the value of b in each case.

a $P(-b \le Z \le b) = 0.68$ **b** $P(0 \le Z \le b) = 0.475$

c $P(Z \ge b) = 84\%$ **d** $P(-2b \le Z \le b) = 0.815$

e $P(-3b \le Z \le 3b) = 0.997$ **f** $P(Z^2 \le b) = 0.95$

14 Use the standard normal table in reverse to find the value of a, given that:

a $P(Z < a) = 0.7257$ **b** $P(Z \le a) = 0.9893$

c $P(Z < -a) = 0.1151$ **d** $P(Z < a) = 0.2119$

e $P(-a \le Z < a) = 0.7286$ **f** $P(-a < Z \le a) = 0.9906$

15 A professional bowler discovers that when he bowls at a central target, his results form a standard normal distribution, where Z is the distance in centimetres from the target to where the ball hits on each bowl.

a Use the empirical rule to find the probability that his result lies:

i within 1 centimetre of the central target,

ii further to the left than 2 centimetres to the right of the target,

iii more than 3 centimetres from the target.

b In how many centimetres either side of the target do 50% of the bowls strike? You will need to use the standard normal table in reverse for this question.

16 Give a mathematical explanation, and also a practical explanation and example, for the result $P(Z = a) = 0$ for any a.

17 Consider the standard normal curve, $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.

a Test your knowledge of this curve:

i What is the domain?

ii Is it odd, even or neither?

iii Write down the equation of any axis of symmetry.

iv What is the area under the curve and above the horizontal axis?

v What are the z-coordinates of the points of inflection?

vi What are the coordinates of the maximum turning point?

vii What are the z-intercepts?

b Test your knowledge of the associated standard normal distribution:

i What is its mean? **ii** What is its mode?

iii What is its median? **iv** What is its standard deviation?

c Without looking, write down its probability density function.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 [Graphing the standard normal distribution]

The purpose of this question is to use our calculus and curve-sketching skills to draw a graph of $y = f(x)$, where

$$f(x) = e^{-\frac{1}{2}x^2},$$

and then use this graph to sketch the standard normal density function $y = \phi(x)$.

a Show that $f(x)$ is an even function.

b Show that $f'(x) = -xe^{-\frac{1}{2}x^2}$ and $f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$.

c Show that there is a unique stationary point. Find its coordinates and determine its nature.

d Show that there are two points of inflection, and that they occur one standard deviation either side of the mean. Find their coordinates.

e Explain what happens to $f(x)$ as $x \to \infty$ and $x \to -\infty$.

f Graph $y = f(x)$.

g Now use stretching to draw the graph of $y = \phi(x)$.

19 **a** **i** Use the trapezoidal rule with five function values (that is, four intervals) to estimate the integral $\int_0^1 \phi(z)dz$.

ii Double this value to estimate the probability that a value will lie within one standard deviation of the mean on the standard normal curve.

iii Why do you know that this will be an underestimate of the true result?

iv Is this in good agreement with the empirical rule and the standard normal table?

b Use the trapezoidal rule with five function values to determine the probability that:

i a value will lie within two standard deviations of the mean,

ii a value will lie within three standard deviations of the mean.

c Use a spreadsheet to increase your number of intervals to say 10, 20, 50, and 100, and observe the convergence.

ENRICHMENT

20 In this question, you may assume the result $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$, where $\phi(z) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ is the PDF of the standard normal distribution.

a Write down the integral for $\mathrm{E}(Z)$ and use the symmetry of the integrand to explain why $\mathrm{E}(Z) = 0$.

b Differentiate $ze^{-\frac{1}{2}z^2}$ and hence integrate $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} z^2e^{-\frac{1}{2}z^2}\,dz$.

c Evaluate $\mathrm{Var}(Z)$.

Calculators

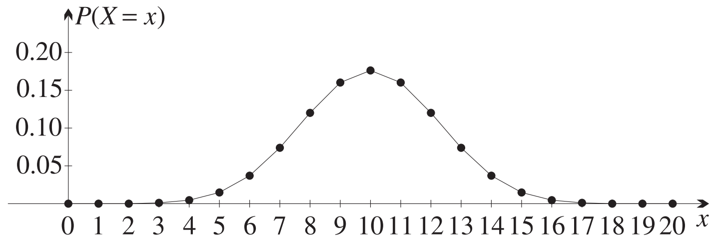
Once you are fluent with the diagrams for the standard normal and the associated calculations in this exercise, you should check on your calculator to see whether it has the function $\Phi(z)$. If it does, then as suggested in Question 11, practise until you can use the calculator confidently. The same questions will be quite adequate.

It is possible that the calculator also has the inverse function, denoted by $\Phi^{-1}(z)$ or something similar. This would allow you, for example, to find the value of z for which $\Phi(z) = 0.3$, so that there would be no need to use interpolation.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16E General normal distributions



In Section 16A we graphed the probabilities of obtaining x heads when 20 coins are thrown, and joined the 21 points to form a polygon. The polygon suggests very much that we should be approximating it with a bell-shaped normal curve. But the curve suggested by the graph is certainly not the standard normal curve, for two reasons:

- The mean is not zero.
- A glance at the inflections shows that the standard deviation is not 1.

This section extends the normal distribution to bell-shaped curves in general.

Shifting and stretching the standard normal distribution

We can estimate the means and the standard deviation from the graph and from the experiment.

- We know that the polygon above is symmetric about $x = 10$, because, for example, the probabilities of obtaining 7 heads from 20 throws, and 13 heads from 20 throws, are equal. Thus the mean is exactly 10.
- We can roughly estimate the standard deviation by looking at where the points of inflection would be if the points were joined up by a curve. The steepest intervals are the interval from $x = 7$ to $x = 8$, and the interval from $x = 12$ to $x = 13$. Let us estimate the points of inflection to be at $x = 7.5$ and $x = 12.5$. That would give a standard deviation of about 2.5.

Some further theory in the Extension 1 course (the *binomial distribution*) tells us that the true standard deviation is $\sigma = \sqrt{5}$, which is approximately 2.236. We now need to stretch and then shift the standard normal distribution to get a curve that may help understand the graph above. That is, we need to produce a normal distribution with $\mu = 10$ and $\sigma = \sqrt{5} \doteqdot 2.236$.

For the rest of this chapter, we will drop the approximately equals sign $\doteqdot$ because nearly all our numbers are estimates or approximations.

Stretching to accommodate the standard deviation

First, stretch the standard normal distribution horizontally by a factor of σ. This is done by replacing x by $\dfrac{x}{\sigma}$, as discussed in Section 3H. The standard normal is $y = \phi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, so the result is

$$y = \phi\left(\frac{x}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad \left(\text{always look at } \frac{1}{\sqrt{2\pi}} \text{ and think } \tfrac{2}{5} \text{ or } 0.4\right).$$

When this stretching is done, the inflections at $x = 1$ and $x = -1$ become inflections at $x = \sigma$ and $x = -\sigma$. This is because stretching transforms concave-up pieces of curve to concave-up pieces, and concave-down pieces of curve to concave-down pieces.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

This function, however, is not a probability density function, because the stretching has increased the area under the curve by a factor of σ, so that the area is now σ and not 1. To correct this, we have to stretch vertically by a factor of $\frac{1}{\sigma}$, giving what is once again a probability density function,

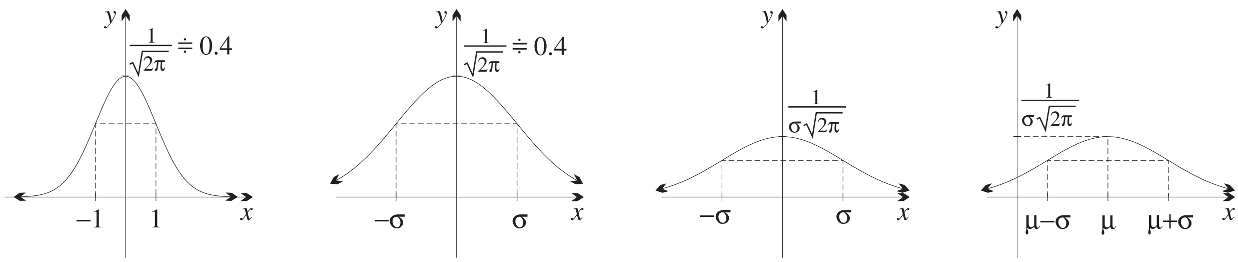
$$y = \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}.$$

Shifting to accommodate the mean

Once the standard deviation has been sorted out, shift the curve μ to the right to make the mean μ instead of 0. This is done by replacing x by $x - \mu$, giving

$$y = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

This time the area does not need to be adjusted, and the inflections continue to be one standard deviation from the mean, that is, at $x = \mu - \sigma$ and $x = \mu + \sigma$.



$y = \phi(x)$ $\qquad$ $y = \phi\left(\frac{x}{\sigma}\right)$ $\qquad$ $y = \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right)$ $\qquad$ $y = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$

Summarising the transformation

Let $f(x)$ be the new stretched and shifted probability density function. Taking account of the horizontal and the vertical stretches, and then the horizontal shift, we can write $f(x)$ in terms of $\phi(x)$,

$$f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right).$$

The last diagram above shows this transformed normal distribution. The continuous probability distribution described by the new function $f(x)$ is called the *normal distribution with mean μ and standard deviation σ*.

These are the important things to notice about the curves above.

- The first, third and fourth graphs are all normal distribution functions. In particular, all have area 1 under the curve.
- The third and fourth graphs both have standard deviation σ — look at the two points of inflection.
- The fourth graph has mean μ — look at the symmetry about $x = \mu$.
- The fourth graph has standard deviation σ — look at the two points of inflection σ to the right of the mean μ, and σ to the left of μ.

The four successive sketches above were drawn using the numerical values $\mu = 3$ and $\sigma = 2$. You can see in the fourth graph that if you take $\mu = 3$, then the inflections are at $x = 3 - 2 = 1$ and at $x = 3 + 2 = 5$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

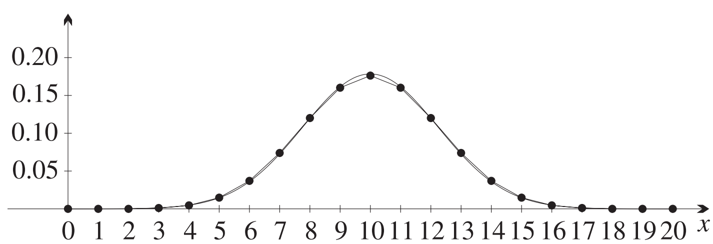
9 THE GENERAL NORMAL DISTRIBUTION

Let $f(x)$ be the probability density function describing a normal distribution with mean μ and standard deviation σ.

- The PDF $f(x)$ is obtained from the standard normal PDF by:
  - stretching horizontally with factor σ and vertically with factor $\dfrac{1}{\sigma}$,
  - then shifting right by μ units.
- The transformed PDF is therefore $f(x) = \dfrac{1}{\sigma}\phi\left(\dfrac{x-\mu}{\sigma}\right)$.
- Thus the transformed PDF has equation $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.
- The points of inflection of $f(x)$ are each one standard deviation from the mean, that is, at $x = \mu - \sigma$ and at $x = \mu + \sigma$.
- In any normal distribution, the mean, the median and the mode coincide.

Comparison with the 20 coin tosses

Graphed below is the polygon of the 20 coin tosses, together with the normal PDF with $\mu = 10$ and $\sigma = \sqrt{5}$. The vertical scale is the same for both graphs, but there is no name on the vertical axis. This is because for the discrete distribution the name is $P(X = x)$, and for the continuous distribution the name is $f(x)$ or y.



The fit is a very good approximation, but it is not exact. It looks very much as if the fit would get better with more and more coin tosses. The example clearly shows how useful the normal is in approximating complicated probability distributions — in this case a discrete distribution. Historically, this coin-tossing experiment was the first use of the normal to approximate another distribution.

Working with the general normal distribution — *z*-scores

We have seen that every normal distribution is obtained from the standard normal distribution by transformations. In order to work with any normal distribution, we need to convert back to the standard normal. The key to this is *z-scores*.

In any distribution, normal or not, the z-score of a score x is the number of standard deviations above the mean. This is easily calculated by the formula

$$z\text{-score} = \frac{x-\mu}{\sigma}.$$

We need to be able to convert from values of x to z-scores, and back from z-scores to values of x.

The two equations are

$$z = \frac{x-\mu}{\sigma} \quad \text{and} \quad x = \mu + \sigma z.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

For example, check conversions both ways for this table of scores and corresponding z-scores for a distribution with mean $\mu = 10$ and standard deviation $\sigma = 2$.

x	4	5	6	7	8	9	10	11	12	13	14	15	16
z-score	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3

10 THE z-SCORES OF A RANDOM VARIABLE

Suppose that X is a random variable, normal or not, with mean μ and standard deviation σ.

- The *z-score* of a score x is the number of standard deviations that x lies above the mean. If the z-score is negative, then x lies below the mean.
- Thus the conversions between z-scores and values of x are given by
$$z = \frac{x - \mu}{\sigma} \quad \text{and} \quad x = \mu + \sigma z.$$
- If the distribution is normal, the z-scores allow the values and features of the standard normal distribution to be applied.
- For sample data (not population data), use $\bar{x}$ for the mean and s for the standard deviation.

Example 14 16E

A dataset has mean $\bar{x} = 12$ and standard deviation $s = 3.60$. Answer these questions correct to two decimal places.

a What scores would be 1, 2 and 3 standard deviations from the mean?

b How many standard deviations from the mean are scores of 24, 11 and 7.7?

SOLUTION

a We can do part **a** either using the formula for conversion from z scores to x-values, or working verbally with 'the number of standard deviations from the mean'.

For $z = 1$,
$$\begin{aligned} x &= \bar{x} + sz \\ &= 12 + 3.60 \\ &= 15.60, \end{aligned}$$
and for $z = -1$,
$$\begin{aligned} x &= 12 - 3.60 \\ &= 8.40. \end{aligned}$$

For $z = 2$,
$$\begin{aligned} x &= \bar{x} + 2sz \\ &= 12 + 7.20 \\ &= 19.20, \end{aligned}$$
and for $z = -1$,
$$\begin{aligned} x &= 12 - 7.20 \\ &= 4.80. \end{aligned}$$

For $z = 3$,
$$\begin{aligned} x &= \bar{x} + 3sz \\ &= 12 + 10.80 \\ &= 22.80, \end{aligned}$$
and for $z = -1$,
$$\begin{aligned} x &= 12 - 10.80 \\ &= 1.20. \end{aligned}$$

OR

The scores one SD from $\bar{x}$ are $\bar{x} + s = 12 + 3.60 = 15.60,$ and $\bar{x} - s = 12 - 3.60 = 8.40.$

The scores two SDs from $\bar{x}$ are $\bar{x} + 2s = 12 + 7.20 = 19.20,$ and $\bar{x} - 2s = 12 - 7.20 = 4.80.$

The scores three SDs from $\bar{x}$ are $\bar{x} + 3s = 12 + 10.80 = 22.80,$ and $\bar{x} - 3s = 12 - 10.80 = 1.20.$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b For $x = 24$, $z = \dfrac{x - \bar{x}}{s}$
$= \dfrac{24 - 12}{3.6}$
$= 3.33$,

3.33 SDs above the mean.

For $x = 11$, $z = \dfrac{x - \bar{x}}{s}$
$= \dfrac{11 - 12}{3.6}$
$= -0.28$,

0.28 SDs below the mean.

For $x = 7.7$, $z = \dfrac{x - \bar{x}}{s}$
$= \dfrac{7.7 - 12}{3.6}$
$= -1.19$,

1.19 SDs below the mean.

Example 15 16E

A normally distributed random variable X has mean 100 and standard deviation 20.

a Write down the two conversion formulae between z-scores and values of x.

b Find: **i** $P(X \le 110)$ **ii** $P(X \ge 90)$

c Find (nearest whole number) the value of a such that $P(X \le a) = 0.98$.

SOLUTION

a $z = \dfrac{x - 100}{20}$ and $x = 100 + 20z$.

b **i** $P(X \le 110)$
$= P(Z \le 0.5)$
$= 0.69$

ii $P(X \ge 90) = P(Z \ge -0.5)$
$= P(Z \le 0.5)$ ($\phi(x)$ is even)
$= 0.69$

c From the table, $\Phi(2.0) = 0.9772$ and $\Phi(2.1) = 0.9821$,
so by interpolation, $\Phi(2.06) = 0.98$.
Converting back to x-values, $a = 100 + 20 \times 2.06 = 141$.

Quartiles, the empirical rule, and the IQR criterion for outliers

If a distribution is normal, we can use z-scores to apply results already calculated about the standard normal distribution. Suppose then that we have a normally distributed random variable X with mean μ and standard deviation σ.

The empirical rule, or 68–95–99.7 rule:

When the experiment is run a large number of times, these are the expectations.

- 68% lie within one standard deviation of the mean,
  — that is, 68% lie within the interval $\mu - \sigma \le x \le \mu + \sigma$.
- 95% lie within two standard deviations of the mean,
  — that is, 95% lie within the interval $\mu - 2\sigma \le x \le \mu + 2\sigma$.
- 99.7% lie within three standard deviations of the mean,
  — that is, 99.7% lie within the interval $\mu - 3\sigma \le x \le \mu + 3\sigma$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The first and third quartile:

- We saw in Section 16C that the third quartile of the standard normal is $z = 0.67$.
  This is 0.67 standard deviations above the mean,
  Hence the third quartile of the transformed distribution is $Q_3 = \mu + 0.67\sigma$.
  Alternatively, using the formula, $x = \mu + z\sigma = \mu + 0.67\sigma$.
- We saw in Section 16C that the first quartile of the standard normal is $z = -0.67$.
  This is 0.67 standard deviations below the mean,
  Hence the first quartile of the transformed distribution is $Q_1 = \mu - 0.67\sigma$.
  Alternatively, using the formula, $x = \mu + z\sigma = \mu - 0.67\sigma$.
- The standard normal has interquartile range 1.35,
  so for the transformed distribution, IQR $= 1.35\sigma$.

The IQR criterion for outliers:

- We showed below worked Example 13 that the IQR criterion characterises as outliers any scores lying outside the interval $-2.70 \le x \le 2.70$.
  Hence for the transformed distribution, we characterise as outliers any scores lying outside the interval $\mu - 2.70\sigma \le x \le \mu + 2.70\sigma$.

Example 16 — 16E

A dataset with 1000 scores is known to be a sample from a normally distributed variable X with mean $\mu = -32.6$ and standard deviation $\sigma = 5.7$.

a Describe what the empirical rule predicts about the data.

b Using z-scores, and finding $\Phi(x)$ from a table or using technology, predict roughly how many scores will:

i lie in $[-30, \infty)$, **ii** lie in $(-\infty, -40]$, **iii** lie in $[-40, -30]$.

c Using the IQR criterion $\mu - 2.70\sigma \le x \le \mu + 2.70\sigma$ for scores that are not outliers, roughly how many outliers would you expect?

SOLUTION

a About 680 scores will lie within one SD from the mean, that is, in $[-38.3, -26.9]$.
About 950 scores will lie within two SDs from the mean, that is, in $[-44.0, -21.2]$.
About 997 scores will lie within three SDs from the mean, that is, in $[-49.7, -15.5]$.

b **i** For -30,
$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{-30+32.6}{5.7} \\ &= 0.456, \\ \text{so } P(X \ge -30) &= P(Z \ge 0.456) \\ &= 0.324, \end{aligned}$$
predicting roughly 324 such scores.

ii For -40,
$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{-40+32.6}{5.7} \\ &= -1.298, \\ \text{so } P(X \le -40) &= P(Z \le -1.298) \\ &= P(Z \ge 1.298) \\ &= 1 - P(Z \le 1.298) \\ &= 1 - 0.903 \\ &= 0.097, \end{aligned}$$
predicting roughly 97 such scores.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iii $P(-40 \le X \le -30)$
$= 1 - (P(X \le -40) + P(X \ge -30))$ (no need for z-scores)
$= 1 - (0.324 + 0.097)$ (using parts **i** and **ii**)
$= 0.579$, predicting roughly 579 such scores.

c This does not need to be recalculated. Worked Example 13**c** showed that roughly 7 in 1000 scores are outliers by the IQR criterion for the standard normal, and because the calculation depended only on z-scores, this is valid for any normally distributed variable.

Exercise 16E

FOUNDATION

Note: There is a brief summary of the normal distribution, including a graph, a table and the empirical rule, in the Appendix at the end of this chapter.

1 In each part, calculate the z-scores corresponding to the given value of x, and state how many standard deviations each value of x lies above or below the mean.

a $\mu = 4, \sigma = 1, x = 5$
b $\mu = 13, \sigma = 3, x = 7$
c $\mu = 0.5, \sigma = 0.25, x = 0.75$
d $\mu = 1, \sigma = 3, x = -5$
e $\mu = 114, \sigma = 1.2, x = 120$
f $\mu = 2.35, \sigma = 0.05, x = 2.20$

2 **a** Use the formula $z = \dfrac{x - \mu}{\sigma}$ to find the z-score when:

i $\mu = 50, \sigma = 4$ and $x = 60$,
ii $\mu = 450, \sigma = 25$ and $x = 375$,
iii $\mu = 3.19, \sigma = 0.12$ and $x = 3.85$,
iv $\mu = 23, \sigma = 8$ and $x = 25$.

b Which of the results in part **a** are:

i furthest from the mean,
ii above the mean,
iii below the mean,
iv within 2 standard deviations from the mean,
v not within the middle 68% of the data?

3 Use z-scores to convert these probability statements for the normal random variable X with mean 4 and standard deviation 2 into probability statements on the standard normal random variable Z. For example, $P(X \le 7) = P(Z \le 1.5)$.

a $P(X \le 5)$
b $P(X > 4.5)$
c $P(X \le 2)$
d $P(X \ge 1)$
e $P(0 \le X \le 3)$
f $P(0.5 \le X \le 4.5)$

4 A certain quantity is normally distributed with mean 5 and standard deviation 2. Convert the following probabilities to probabilities involving the standard normal distribution, and then use the empirical rule to find them.

a $P(X \ge 5)$
b $P(3 \le X \le 7)$
c $P(X \le 9)$
d $P(X \ge 1)$
e $P(-1 \le X \le 7)$
f $P(1 \le X \le 3)$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 Use the empirical rule to find the following probabilities for a normally distributed random variable with the given parameters.
  a $P(10 \le X \le 18)$, given mean $\mu = 12$ and standard deviation $\sigma = 2$.
  b $P(X \ge 42)$, given mean $\mu = 37$ and standard deviation $\sigma = 5$.
  c $P(X \ge 4.5)$, given mean $\mu = 4$ and standard deviation $\sigma = 0.25$.

6 Find each probability for a normally distributed random variable X with the given parameters. You will need to use the table of values for the standard normal distribution, or a statistics calculator, or other technology such as a spreadsheet, or online resources.
  a $P(3 \le X \le 7)$, given mean $\mu = 5$ and standard deviation $\sigma = 0.8$.
  b $P(X \ge 20)$, where $\mu = 4$ and $\sigma = 10$.
  c $P(X \le 8)$, where $\mu = 12$ and $\sigma = 5$.
  d $P(X \ge -39)$, where $\mu = 0$ and $\sigma = 30$.
  e $P(X < 36)$, where $\mu = 20$ and $\sigma = 10$.
  f $P(3 < X \le 5)$, where $\mu = 8$ and $\sigma = 2$.

7 Explain what it means for a score x if the corresponding z-score is:
  a positive, b negative, c zero.

DEVELOPMENT

8 A distribution is known to be normal with mean $\mu = 73$ and $\sigma = 8$. A researcher records the following data values from this distribution:

  69, 80, 95, 50, 43, 90, 52, 98, 45

  a Write down the data values that lie within one standard deviation of the mean.
  b Write down the data values that lie within three standard deviations of the mean.
  c Write down the data values that lie more than two standard deviations below the mean.
  d Write down the data values that are more than two and a half standard deviations above the mean.
  e The researcher believes that these data values were obtained randomly. Do they seem to fit the expected distribution for a normal random variable? Construct a stem-and-leaf plot for the data and comment on the shape of the data.

9 The results of an English examination and a mathematics examination are approximately normally distributed with these parameters:

English:	$\mu = 65\%$	$\sigma = 10$
Mathematics:	$\mu = 62\%$	$\sigma = 15$

  a For each student below, determine the z-scores for the two results and state which is more impressive:
    i Student A's result in English (90%), or their result in mathematics (92%),
    ii Student B's result in English (57%), or their result in mathematics (53%),
    iii Student C's result in English (80%), or their result in mathematics (77%).
  b What is the probability that a mathematics student obtains over 95%?
  c What is the probability that a student's English mark is greater than the mean of the mathematics marks?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 The results of an experiment are known to be normal, with mean 50 and standard deviation 10. The experiment is run 600 times.

a Describe what the empirical rule predicts about the data.

b Using z-scores, and using a table of the standard normal, or a calculator, or statistics software, predict roughly how many scores will:

i lie in $[-\infty, 55]$, **ii** lie in $[35, 50]$, **iii** lie in $[38, 62]$.

c Using the IQR criterion $\mu - 2.70\sigma \le x \le \mu + 2.70\sigma$ for scores that are not outliers, roughly how many outliers would you expect?

ENRICHMENT

11 At a certain school, Biology has four assessments. The mean and standard deviation for these assessments are recorded in the table below.

Assessment	Mean	SD
1	60	10
2	65	8
3	75	4
4	63	12

a Jack obtained 50, 53 and 67 for the first three assessments, but was absent for the fourth assessment due to a fall.

i Find the z-score for each of Jack's results.

ii Use these z-scores to find Jack's average deviation from the mean.

iii Hence estimate a mark for Jack in the fourth assessment.

iv What are the advantages of this method over simply giving Jack the average for his scores in the first three assessments?

v Are there any disadvantages to the method?

b Jill obtains 64, 70 and 79 for the first three assessments, but due to a tumble could not attend the final assessment. Use the same method to estimate Jill's missing result.

Calculators

Once you are confident with z-scores and their use with probability calculations, you may want to check whether your calculator can handle z-scores, and if so, practise until you can do things quickly. Be careful, however, because automating these transformations gets in the way of understanding 'the number of standard deviations from the mean'.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16F Applications of the normal distribution

A great number of common situations follow a normal distribution, or follow it approximately enough for practical purposes. The questions in Exercise 16F are self-explanatory, given the previous theory, and one worked example should be sufficient introduction.

Example 17 — 16F

The Happytime Chocolate Company manufactures a 100 g chocolate–nougat bar. As with any manufacturing process, these chocolate bars do not all have precisely the same weight, and these bars are known to be normally distributed with standard deviation 2 g. To reduce the number of complaints, the company has adjusted its machinery so that the mean is 102 g (such an adjustment does not affect the standard deviation).

a Using the empirical rule where possible, and tables or technology otherwise, find the percentage of chocolate bars:

- **i** of weight less than the stated weight of 100 g,
- **ii** of weight greater than 105 g.

b What would the mean weight need to be for there to be 1 chocolate bar in 1000 under 100 g?

SOLUTION

a i A weight of 100 g is 1 standard deviation below the mean of 102 g.

By the empirical rule. $P(-1 \le Z \le 1) = 68\%$,

and using the complement, $P(Z < -1 \text{ or } Z > 1) = 32\%$,

so by the even symmetry, $P(Z < -1) = 32\% \div 2,$

$= 16\%.$

ii A weight of 105 g is 1.5 standard deviations above the mean of 102 g.

From the table. $P(Z \le 1.5) = 93\%$,

so using complements, $P(Z > 1.5) = 7\%$.

b The population of underweight chocolate bars is to be $\dfrac{1}{1000} = 0.001$,

so we search for 0.999 in the body of the table.

Reading the table backwards, $P(Z \le 3.1) = 0.999$,

so by symmetry and complements, $P(Z \le -3.1) = 0.001$.

Hence we need to make the mean 3.1σ above 100 g,

meaning that we set the controls so that $\mu = 100 + 3.1 \times 2 = 106.2$ g.

Exercise 16F — FOUNDATION

The first four questions of this exercise should be completed using the empirical rule (or the 68–95–97.7 rule) rather than the standard normal probability table or technology.

There is a brief summary of the normal distribution, including a graph, a table and the empirical rule, in the Appendix at the end of this chapter.

1 The results of a school's English examination are found to be normally distributed with mean 70 and standard deviation 10.

- **a** What percentage of the pupils score over 50?
- **b** What percentage of the pupils score under 80?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 The results in an examination are approximately normally distributed with mean 68 and standard deviation 9. In a cohort of 2000, how many students will be expected to score:

a more than 95, **b** less than 50, **c** between 59 and 86?

3 A machine produces screws that are an average of 2 cm long, with a standard deviation of 0.1 cm. The screw lengths are approximately normally distributed.

a What is the probability that a screw will be undersized, if this is taken to mean more than 2 standard deviations below the mean?

b In a batch of 2400, use z-scores to find how many screws are longer than 2.3 cm?

4 Apples of a certain variety are to be sold in packages in a supermarket. Their diameters are normally distributed with mean 68 mm and standard deviation 2 mm. Apples are discarded if their diameter is more than 72 mm or less than 64 mm. What percentage are discarded?

5 The IQ (*Intelligence Quotient*) test is designed to give a qualitative measure of a person's intelligence. In Australia, IQ is approximately normally distributed with mean 98 and standard deviation 15.

a According to one definition, a genius is defined to be someone with an IQ over 140. What percentage of the Australian population would this be?

b In a population of 25 million, how many geniuses would you expect?

DEVELOPMENT

6 A very famous and early experiment into cholesterol levels, called the Framingham study, found that the average cholesterol level in the population of adult males who did not go on to develop heart disease was 219 mg/mL, with standard deviation 41 mg/mL. Assuming that doctors call a reading of above 240 mg/mL *high*, what percentage of this population could be said to have high cholesterol?

7 In Australian adult males, height is found to be normally distributed with mean 176 cm and standard deviation 7.5 cm. A doorway is designed so that 90% of this population can enter without ducking.

a Read the supplied standard normal distribution table backwards to find the z-score such that $P(Z < z) = 90\%$, correct to 2 decimal places.

b Hence find the minimum height of the doorway.

c In the Dinaric alps, the mean and standard deviation of the heights of adult males are respectively 185 and 7.5 centimetres. A customer orders a special design for the doorway so that 95% of adult males can enter without ducking.

i Explain why a reasonable estimate from the table such that $P(Z < z) = 0.95$ is 1.65.

ii Find the minimum design height of the door.

8 A company has a machine designed to fill cereal boxes. It dispenses cereal according to a normal distribution with mean 500 g and standard deviation 2 g. To ensure that boxes are above the advertised weight at least 95% of the time, what weight should be recorded as the weight on each box?

9 The length of gestation (pregnancy) in human females is approximately normally distributed with mean 266 days and standard deviation 16 days.

a Nine months is about $0.75 \times 365 \doteqdot 274$ days. What percentage of females give birth before 274 days?

b If 266 days is considered 'on time', what percentage of females give birth more than:

i 1 week early, **ii** one week late?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 A certain study indicates that the pulse rate of an adult male aged 20–39 is about 71 with standard deviation about 9. The data are approximately normally distributed.

a What percentage of this population would be expected to have *bradycardia*, which is defined to be a slow pulse rate below 60 beats/minute?

b *Tachycardia* is defined to be a pulse rate greater than 100 beats/minute. What percentage of the population might be expected to fall in this category?

c Repeat part a–b for females aged 20–39, whose mean is about 76 and standard deviation is about 9.5.

ENRICHMENT

11 The apples in Question 4 must also fit within regulation weight guidelines. Suppose that the weights are normally distributed, with 97.7% of the apples weighing more than 100 g and 69.1% weighing less than 115 g. Find the mean and standard deviation of the weights of the apples. Use the supplied normal distribution table.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16G Investigations using the normal distribution

These questions are intended to be investigations using technology — statistical calculators, spreadsheets, statistics software, or online resources. Many of the investigations can be broadened or extended into projects. The exercise is long, and it is certainly not intended that all questions be attempted.

Many of the investigations use *sampling of the mean*. This concept is the reason why the normal distribution plays such a central role in all statistics. The underlying theorem is the *central limit theorem*.

The following Challenge paragraphs explain a particular case of the theorem very briefly. It would perhaps be better read after the idea has been encountered in one or more investigations.

Sampling of the mean

Suppose that we have a random variable X. The distribution may be discrete, or continuous and normal, or continuous and not normal. Suppose that this distribution has mean μ and standard deviation σ, neither of which we know.

We want to find the mean of this distribution, so we do the obvious thing — we *sample* the variable X. That is, we run n independent trials of the experiment and take the average of these results as our estimate for the mean μ. What this procedure has actually done is generate a new random variable Y. The procedure described is:

- Take n independent samples of the random variable X, thus generating n values of X.
- Find the mean of the n samples, and assign this mean to a random variable Y.

The central limit theorem says that in most situations, this new random variable Y:

- has the same mean μ as X (which is obvious),
- has variance $\dfrac{\sigma^2}{n}$, that is, its standard deviation is $\dfrac{\sigma}{\sqrt{n}}$ (nearly obvious),
- *tends towards a normal distribution as the number n of samples increases.*

The significance of this theorem is that sampling any random variable to find its mean generates approximately a normal distribution as more and more samples are taken. Thus the normal distribution is involved in the study of every distribution, continuous or discrete.

The normal approximation of the 'toss 20 coins' polygon in Section 16E is historically one of the first examples of the theorem. It will be studied in some detail in Sections 17C–17D of the final chapter on binomial distributions.

Exercise 16G

INVESTIGATION

Some of the questions below involve the use of a spreadsheet such as Excel, LibreOffice Calc or GoogleDocs. The instructions below are directly relevant for a recent version of Excel on Windows, but may be adapted depending on available software. More serious investigations could use a general programming language such as Python, or a statistical programming language such as R.

1 [Sampling of the mean] The data from many common experiments, with any distribution, can be displayed as normally distributed data using the following technique called *sampling from the mean*. This is the reason why the normal distribution is so important.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A student generated three real-valued random numbers between 0 and 15. The mean of these three numbers was recorded and the original three numbers discarded. This was repeated 1000 times, and the results were recorded as grouped data in the table below.

class	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8
class centre x	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
frequency f	2	5	24	46	88	116	138	158
relative frequency f_r								

class	8–9	9–10	10–11	11–12	12–13	13–14	14–15
class centre x	8.5	9.5	10.5	11.5	12.5	13.5	14.5
frequency f	144	113	78	47	27	10	4
relative frequency f_r							

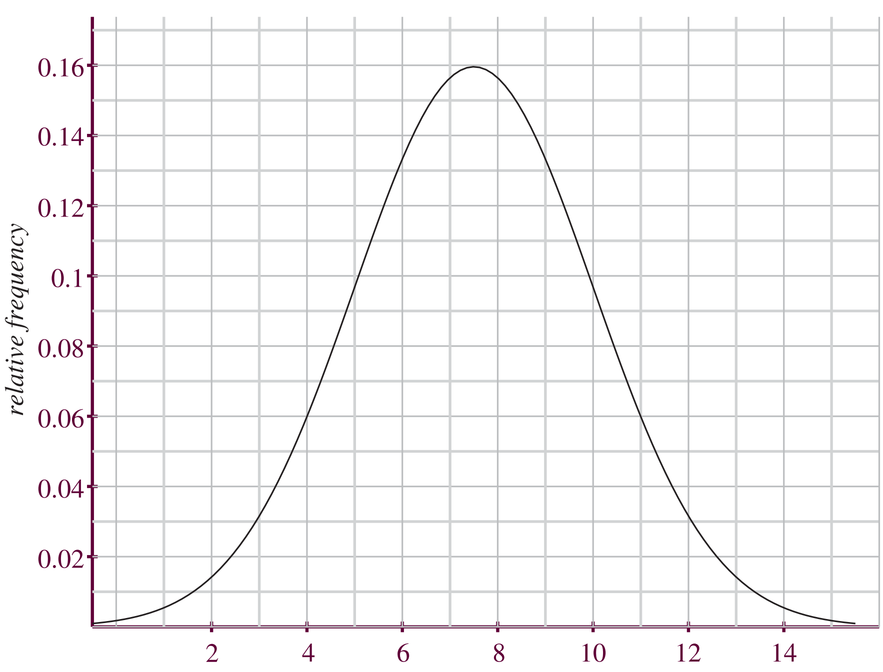
a Use your calculator or a spreadsheet to evaluate the mean and standard deviation of this data, using the given class centres and frequencies. Round your answers correct to 1 decimal place.

b Complete the table by filling in the relative frequencies. Now take the class centres x as the values, and take the relative frequencies as estimates of the probabilities $P(x)$, and evaluate the mean and standard deviation using the formulae

$$\begin{aligned} \mathrm{E}(X) &= \sum xP(x), \quad \text{and} \\ \mathrm{Var}(X) &= E(X^2) - \mathrm{E}(X)^2, \quad \text{where } \mathrm{E}(X^2) = \sum x^2P(x). \end{aligned}$$

Check that your answers agree with the previous results.

c



In the graph above, the normal probability density curve with the mean and standard deviation you just found has been plotted. Photocopy the graph and construct a relative frequency histogram for your grouped data on the same diagram. Add the relative frequency polygon by joining the top centres of the histogram rectangles. What do you notice?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d How do you think that you might improve the match between the normal curve and the relative frequency polygon?

e Using integration, calculate the mean $\mathrm{E}(Y)$ and variance $\mathrm{Var}(Y)$ for a uniform continuous random variable with $P(Y) = \frac{1}{15}$ on the interval $0 \le Y \le 15$. Confirm that the mean and variance obtained in part **a** above are related by the formulae:

$$\mathrm{E}(X) = \mathrm{E}(Y) \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{\mathrm{Var}(Y)}{n},$$

where $n = 3$ (because three random numbers were averaged at each stage).

2 In this question we will replicate the experiment discussed in the previous question using a spreadsheet. In this experiment we will generate 3 real-valued random numbers between 0 and 12 and calculate their mean. This step will be carried out 100 times, giving a set of 100 means that form a distribution called *sampling of the mean*.

a A fragment of a spreadsheet is shown below. In each cell `A2:C2` we have entered `=RAND()*12`. In cell `E2` we have entered a formula to calculate the mean of the three numbers. Enter all this in your spreadsheet.

	A	B	C	D	E	F
1					Average	
2	1.234	4.578	6.914		`=AVERAGE(A2:C2)`	

b Fill down row 2 one hundred times, finishing on row 101.

c Type the formula `=AVERAGE(E2:E101)` in cell `I2` and `=STDEV.P(E2:E101)` in cell `J2` to calculate the mean and standard deviation of this sample of means.

d We will now group the data in intervals of 1 unit. In cells `K3:V3` enter the starting value of each class, that is 0, 1, . . . , 11. Also record a final 12 in cell `W3` to record the end of the data. Cells `K4:V4` will calculate the class centre and cells `K5:V5` will calculate the frequency for each class. Finally cells `K6:V6` will divide the frequency by 100 (the number of times we ran the experiment) to calculate the relative frequency. The formulae are shown in the spreadsheet fragment below for cells `K4:K6`. You should fill the formulae from cells `K4:K6` across to cells `V4:V6`.

	…	I	J	K	L	M	N	O
1	…	Mean	SD					
2	…	5.901	1.897					
3	…			0	1	2	3	4
4	…			`=(L3+K3)/2`				
5	…			`=countif($E:$E, "<"&L3)−countif($E:$E,"<"&K3)`				
6	…			`=K5/100`				

(When using the code `$E:$E`, make sure that there are no other entries in column `E`.)

e Construct a histogram using the data from cells `K6:V6`. If your program allows, you should label the horizontal axis using your class centres and ensure that there are no gaps between the rectangles (Excel: Click on bars and select `FORMAT DATA SERIES`), because there should not be gaps between the rectangles of a histogram.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

f If your data do not generate a distribution that looks normal, you might like to recalculate with a fresh set of random numbers. In Excel this option is available under the 'Formulas' tab, option `Calculate Now` (shortcut `F9`).

g If random numbers are generated from the interval $0 \leq x \leq c$, then theory claims that

$$\mu \doteqdot \frac{c}{2} \qquad \text{and} \qquad \sigma^2 \doteqdot \frac{c^2}{12n}.$$

Test these two formulae agree with your results obtained in part **c**.

h If your distribution is normal, then approximately 68% of the data should be within 1 standard deviation of the mean. Theory predicts that the mean was 6 and the standard deviation was 2. Check that:

i approximately 68 of the 100 numbers fell in the interval $[4, 8]$,

ii approximately 95 of the 100 numbers fell in the interval $[2, 10]$.

i A more correct experimental approach to improve the normality of the distribution would be to run the experiment with more trials. Adapt the spreadsheet for 1000 trials. Remember to adjust cells `K6:V6` for the new experiment. Test your improved experiment by repeating part **h**.

3 In the experiment in Questions 1 and 2 we took means of a continuous distribution, but you can also take the mean of discrete data and approximate it by a normal distribution.

a In Question 2, replace cells `A2:C101` by `RANDBETWEEN(1,6)+RANDBETWEEN(1,6)`, simulating the result of throwing two dice and recording their sum. Note that this is not a uniform distribution.

b This could also be done as a practical experiment using a pair of dice.

4 [Normal approximation to the binomial] If we throw 10 coins, what is the probability of obtaining exactly 5 heads? This is called a *binomial probability*, because each coin produces one of only two possible outcomes, 0 and 1 (the prefix 'bi-' means 'two'). It is another discrete probability distribution that may be modelled by a continuous normal probability distribution.

a Throw 10 coins and record the number of heads (if you are short of cash, you could throw 10 dice and record the number of dice showing an even number).

b Repeat this 100 times, recording your results in a frequency table as you go:

Heads	0	1	2	3	4	5	6	7	8	9	10
Tally											
Freq											

c Calculate the mean and standard deviation of your distribution.

d Draw a histogram of your experiment and add the frequency polygon. Does it look normal?

e Assuming a mean of 5 and a standard deviation of about 1.5, one standard deviation either side of the mean should represents an outcome of 4, 5 or 6 heads. (Why?) Do you find 68% of the numbers fall within one standard deviation of the mean?

f To improve your results, repeat the experiment more times. If this is done as a class exercise in groups, groups could collate their data into one frequency table.

XLS

g [Technology] A spreadsheet is a good tool to record your data and construct a histogram.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

h [Technology] You may wish to try simulating the whole experiment in a spreadsheet. Here is a fragment of a spreadsheet to generate 10 coin flips and record the number of heads. The formula in cell `A1` is duplicated in cells `B1:J1` and returns a 1 for a head and a zero for a tail. The formula in cell `K1` counts the number of heads.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	`=randbetween(0,1)`				1	0	0	1	1	0	`=sum(A1:J1)`		

5 [For further investigation of Question 4]

a Theory indicates that if I count the number of heads on n throws and if p is the probability of a head, then

$$\bar{x} = \frac{n}{2} \qquad \text{and} \qquad s^2 = np(1 - p).$$

Check this for the experiment above.

b Try varying n.

c Try varying p — throw 15 dice and count the number of dice showing more than 4 $\left(\text{so that } p = \frac{1}{3}\right)$. Or, on your spreadsheet use the code `=if(randbetween(1,6)>4,1,0)`.

d Statisticians have a rule of thumb — for a fairly good approximation to a normal distribution, np and $n(1 - p)$ should be at least 5. Test this out. Another common rule of thumb replaces 5 by 10 - try this out as well.

6 [For further investigation] In Question 1 and 2 we generated a new set of data by averaging the means of three random numbers. If real numbers are generated from the interval $[0, c]$, the *central limit theorem* predicts that the mean and standard deviation of these data will be

$$\mu \doteqdot \frac{c}{2} \qquad \text{and} \qquad \sigma^2 \doteqdot \frac{c^2}{12n},$$

and that the approximation to normal should be increasingly good as $n \to \infty$. Investigate what happens as you use larger and larger values of n. Does your distribution look increasingly normal?

Note that if you wish to use an interval length not equal to 1 when grouping your data, then you will need to graph relative frequency per unit of width on the vertical axis.

7 It is reasonable to suppose that height follows a bell-shaped distribution, because most of a fairly homogeneous population cluster around the mean height and rapidly tails off further from the mean.

Collect the height of students in your year group. This could be done in classes and results shared, or results could be entered in an online survey.

a Using the techniques of this section, group the results, then graph the histogram and frequency polygon. You will need to choose your interval width so that enough students lie in the central classes to generate a good histogram.

b Does the curve look normal?

c Calculate the mean and standard deviation.

d Assuming that the results are approximately normal, test whether the expected number of students lie within one and two standard deviations of the mean.

e Can you improve the normality of your results by restricting your population to a certain age group or ethnic group?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

8 Using DESMOS, or other graph sketching program, we can investigate the normal distribution curve with mean μ and standard deviation σ,

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Your curve sketching program may recognise entering *pi* for 3.14159265. . . and *e* for 2.718281828. . ., but many programs will not recognise *sigma* or *mu* and you will need to replace them with *s* and *m* (or other pronumerals):

$$y = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}}.$$

- **a** Create sliders for m and s, if possible restricting $-6 \le s \le 6$ and $-3 \le m \le 3$.
- **b** Adjust the vertical scale (say $-0.1 \le y \le 1.1$) so that the graph with its 'bell shape' is clearly displayed.
- **c** Set $s = 1$ and verify that adjusting m shifts the central mean and median of the symmetric curve.
- **d** With $s = 1$ and $m = 0$, determine the highest point of the curve. Using the equation above, what are the exact coordinates of this maximum point on the curve?
- **e** What is the effect of adjusting s to the 'fatness' and height of the curve?
- **f** What are the heights of the curve when $s = 1$, $s = 2$ and $s = 4$? Comment.
- **g** In DESMOS, define the function

$$f(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}} \qquad \text{and then enter} \qquad \int_{-s}^{s} f(x)\,dx$$

  Check the claim that 68% of the data lie within one standard deviation of the mean.
- **h** Adjust the limits of the integral and check that 95.4% of the data lie within two standard deviations of the mean, and 99.7% of the data lie within three standard deviations of the mean.
- **i** Graph-sketching programs are not always good at handling ∞. Use the table for the standard normal curve and check the value obtained for $P(X \le 1)$ obtained by DESMOS with

$$\int_{-3}^{1} f(x)\,dx.$$

  How small a value is needed for the lower limit to get a value accurate to 9 decimal places, assuming that the answer should be $P(X \le 1) \doteqdot 0.841344746$?

9 [Investigation] The *Galton board* is a machine designed to generate a bell-shaped curve by means of a set of steel balls falling through a triangular array of pegs. It is possible to buy Galton boards, or there are animations of this device on the web. The original Galton board was designed by Sir Francis Galton (1822–1911).

- **a** Test this device or view animations of the device in action. How good a normal approximation does it produce?
- **b** How does it work?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 16 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 16 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

Note: There is a brief summary of the normal distribution, including a graph, a table and the empirical rule, in the Appendix at the end of this chapter.

1 A simple experiment measures the length of time in hours that a certain drug is retained in a patient's system. The following preliminary data were recorded:

0.9	1.4	2.1	2.3	2.6	2.2	2.4	2.7	3.6	3.7
4.1	4.3	4.4	4.4	4.7	5.1	5.2	6.1	6.3	7.1

a Complete the following table for these data.

x	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	Sum
cc	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	—
Tally									
f									—
cf									—
f_r									
cf_r									—

b Draw a relative frequency histogram and polygon for the dataset.

c Draw a cumulative relative frequency histogram and polygon (ogive) for the dataset.

d By adding appropriate horizontal lines to your graph, find:

i the median Q_2,

ii the quartiles Q_1 and Q_3,

iii the ninth decile,

iv the eighty-fifth percentile.

e The dataset appears (almost) bimodal, with many data points falling in two specific intervals. Advise the medical researcher how to proceed next.

2 State whether each of these sentences is true or false.

a The ogive is joined to the top centre of each rectangle of the cumulative frequency histogram.

b The area under the frequency polygon is 1.

c The probability density function of a continuous probability distribution is the analogue of the relative frequency polygon of a discrete distribution.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d A probability density function $f(x)$ defined on the interval $a \le x \le b$ satisfies the two conditions $f(x) \ge 0$, for all x in the interval, and $\int_a^b f(x)\,dx = 1$.

e Every normal distribution is related to the standard normal distribution by stretches and a horizontal shift.

f Approximately 99% of all data lie within three standard deviations of the mean.

3 Let $f(x) = \frac{1}{20}$, where $-10 \le x \le 10$.

a Show that $f(x)$ is a probability density function.

b What special name is given to this type of distribution, where the density function takes the same value across its domain?

c Calculate its expected value.

d Calculate its variance and standard deviation.

4 Let $f(x) = \frac{3}{16}(4 - x^2)$, $0 \le x \le 2$.

a Show that $f(x)$ is a probability density function (PDF).

b Find its cumulative density function (CDF).

c The CDF is graphed to the right. Use this graph to estimate:

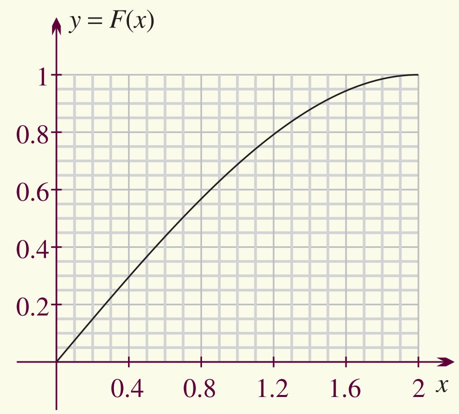
i the three quartiles Q_1, Q_2 and Q_3,

ii the sixth decile,

iii $P(X \le 1.2)$,

iv $P(X \ge 0.3)$,

v $P(0.2 \le X \le 0.4)$.



5 Use your standard normal table and a knowledge of the symmetry of the curve to find:

a $P(Z < 0)$ **b** $P(Z < 1.3)$ **c** $P(-1.8 < Z < 1.8)$

d $P(Z > 0.5)$ **e** $P(Z < -0.2)$ **f** $P(-0.1 < Z < 1.2)$

6 Find the given probability for the normal distribution with given mean and standard deviation. Use the empirical rule (the 68–95–99.7 rule) to estimate:

a $P(X \le 16)$ if $\mu = 10, \sigma = 3$ **b** $P(X \ge 3.5)$ if $\mu = 5, \sigma = 1.5$

c $P(1.85 \le X \le 2.3)$ if $\mu = 2, \sigma = 0.15$ **d** $P(13.65 \le X \le 14.1)$ if $\mu = 15, \sigma = 0.45$

7 Repeat the previous question, but this time use your standard normal distribution tables.

a $P(X \le 22.5)$ if $\mu = 20, \sigma = 5$ **b** $P(X \ge 62)$ if $\mu = 50, \sigma = 10$

c $P(3.96 \le X \le 4.3)$ if $\mu = 4, \sigma = 0.2$ **d** $P(6.79 \le X \le 8.09)$ if $\mu = 5.75, \sigma = 1.3$

8 A washing machine manufacturer has tested the design of its machines and found them to have an expected life of 6 years 4 months with a standard deviation of 15 months.

a If a family buys one of their machines, what is the probability that it will last more than eight years?

b The manufacturer is deciding on whether to launch a promotion and advertise a five-year warranty on its machines. How many machines could they expect to come to the end of their life within the five-year period?

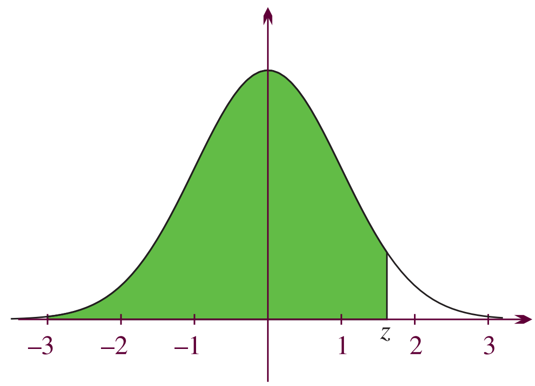
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Appendix: The standard normal distribution

A brief summary of the standard normal probability distribution

The graph to the right is the *standard normal probability density function* $y = \phi(z)$.



The shaded area represents the value of the corresponding *cumulative distribution function*

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(t)\, dt.$$

The table below gives some values of the probabilities $\phi(z) = P(Z \le z)$. For example,

$$P(Z \le 1.6) = \Phi(1.6) = \int_{-\infty}^{1.6} \phi(z)\, dz \doteqdot 0.9452.$$

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

A more detailed table giving the values of z to three decimal places is to be found in the Appendix to Chapter 17.

For many purposes, all that is required is the *empirical rule*, or *68–95–99.7 rule*,

$$P(-1 \le Z \le 1) \doteqdot 68\%$$
$$P(-2 \le Z \le 2) \doteqdot 95\%$$
$$P(-3 \le Z \le 3) \doteqdot 99.7\%$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 Binomial distributions

This chapter deals with binomial probability. It moves quickly through calculations of individual probabilities in Section 17A, to binomial distributions in Section 17B. These two sections combine ideas from all of Chapters 12–15 of the Year 11 book, and this may be an ideal time to revise that material.

The last two sections introduce normal approximation to a binomial distribution. Section 17C does this in the obvious way, expressing a question about a binomial distribution in terms of the normal approximation.

The final section introduces the sample proportions that binomial distributions produce, and uses the normal distribution to approximate the probabilities of the sample proportions rather than of the original binomial distribution. This approach may seem conceptually rather demanding at first, but the calculations involved turn out to be simple variations of those introduced in Section 17C.

Plenty of examples of binomial data are presented in the text and in the exercises, together with simulations of binomial experiments using technology.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17A Binomial probability

Section 17A is about individual binomial probabilities, in preparation for the later sections on binomial probability distributions. The discussion combines probability with the expansion of the binomial $(x + y)^n$, showing their close relationship.

We will be concerned with multi-stage experiments of a rather special form. The stages are independent, each stage has just two outcomes, ‘success’ and ‘failure’, with the same probabilities at each stage, and the random variable for the whole experiment records only the number of successes, not their order.

Bernoulli trials

It is convenient to have a name for the independent stages and describe them carefully. A *Bernoulli trial* or *Bernoulli experiment* is a single-stage random experiment with just two outcomes, conventionally called ‘success’ and ‘failure’, to which we usually assign the probabilities p and $q = 1 - p$.

The classic example of a Bernoulli trial is tossing a coin, where ‘success’ is heads and ‘failure’ is tails. The probability of success is then $p = \frac{1}{2}$, and the probability of failure is $q = 1 - p = \frac{1}{2}$.

The other classic example is throwing a die, provided that we define ‘success’ — if we define ‘success’ to be ‘throwing a six’, and record only that, then we have a Bernoulli trial with $p = \frac{1}{6}$.

A Bernoulli trial is completely determined by just one parameter — the probability p of ‘success’.

Binomial experiments

A *binomial experiment* is an n-stage experiment in which:

- each stage is a Bernoulli trial with the same probability p of success,
- the stages are independent — no stage affects any other stage,
- the random variable X is the number of successes — order is irrelevant.

Think now about tossing a coin 12 times and counting the number of heads. Or think about throwing four dice and counting the number of sixes.

A binomial experiment is thus completely determined by just two parameters — the number n of trials, and the probability p of success at each stage.

Each stage of a binomial experiment is trivially a binomial experiment with just one stage. Thus a Bernoulli trial is a special case of a binomial experiment.

Example 1 17A

Identify some further examples of Bernoulli trials.

SOLUTION

- Choose an adult Australian — did they vote in the last election?
- Choose a shopper in the street — have they visited Wooldi today?
- Ask a whale-spotting boat returning to port — did you spot a whale?
- Visit your letter-box — is there mail in it or not?
- Is there intelligent life elsewhere in the universe — yes or no?
- Choose a person at random — were they offered the last job they applied for?

ISBN 978-1-108-76630-2

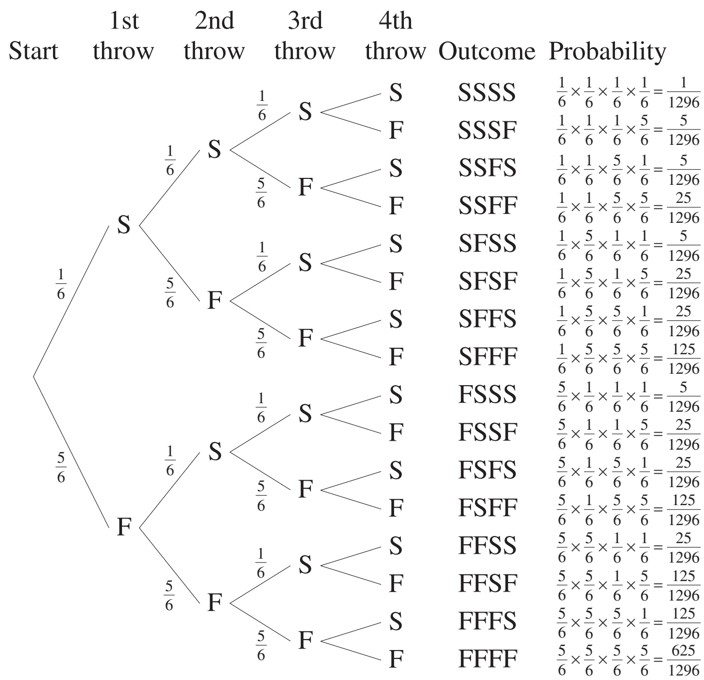
Example — repeatedly attempting to throw a six on a die

We will develop Bernoulli trials further in the next section. Let us now turn to the problem of calculating individual probabilities in a binomial distribution. Here is a classic example.

A die is thrown four times. Find the probabilities of getting $0,\ 1,\ 2,\ 3$ or 4 sixes.

Let S (success) be 'throwing a six' and F (failure) be 'not throwing a six', so that $p = \frac{1}{6}$ and $q = 1 - p = \frac{5}{6}$.

Here is the probability tree diagram showing the sixteen possible outcomes taking account of order, and their respective probabilities.



When we ignore order, these 16 outcomes collapse to five, and the resulting binomial random variable X has five possible values, $0,\ 1,\ 2,\ 3$ and 4.

The outcome 'two sixes', for example, can be obtained in ${}^4C_2 = 6$ different ways: SSFF, SFSF, SFFS, FSSF, FSFS, FFSS, because there are $\dfrac{4!}{2! \times 2!} = {}^4C_2$ ways of arranging two Ss and two Fs.

Each of these six outcomes has the same probability $\left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$, so

$$P(X = 2) \qquad = {}^4C_2 \times \left(\tfrac{1}{6}\right)^2 \times \left(\tfrac{5}{6}\right)^2.$$

Similar arguments apply to the probabilities of getting $0,\ 1,\ 3$ and 4 sixes:

Result	Probability	Approximation
0 sixes	${}^4C_0 \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^4$	0.482 25
1 six	${}^4C_1 \times \left(\frac{1}{6}\right)^1 \times \left(\frac{5}{6}\right)^3$	0.385 80
2 sixes	${}^4C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$	0.115 74
3 sixes	${}^4C_3 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^1$	0.015 43
4 sixes	${}^4C_4 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right)^0$	0.000 77
—	—	Sum $= 1$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The five probabilities of course add up to 1, because no other outcomes are possible. This is also clear because the five probabilities are the successive terms in the binomial expansion of $\left(\frac{1}{6}+\frac{5}{6}\right)^4 = 1^4 = 1$,

$$\left(\tfrac{1}{6}+\tfrac{5}{6}\right)^4 = {}^4C_0 \times \left(\tfrac{1}{6}\right)^0 \times \left(\tfrac{5}{6}\right)^4 + {}^4C_1 \times \left(\tfrac{1}{6}\right)^1 \times \left(\tfrac{5}{6}\right)^3 + {}^4C_2 \times \left(\tfrac{1}{6}\right)^2 \times \left(\tfrac{5}{6}\right)^2 + {}^4C_3 \times \left(\tfrac{1}{6}\right)^3 \times \left(\tfrac{5}{6}\right)^1 + {}^4C_4 \times \left(\tfrac{1}{6}\right)^4 \times \left(\tfrac{5}{6}\right)^0.$$

Remember that

$$(x+y)^4 = {}^4C_0\,x^0y^4 + {}^4C_1\,x^1y^3 + {}^4C_2\,x^2y^2 + {}^4C_3\,x^3y^1 + {}^4C_4\,x^4y^0.$$

Binomial probability — the general case

Suppose that a multi-stage experiment consists of n identical stages, and at each stage the probability of ‘success’ is p and of ‘failure’ is q, where $p + q = 1$. Then

$$P(x \text{ successes and } n-x \text{ failures in that order}) = p^x q^{n-x}.$$

But there are ${}^nC_x$ ways of ordering x successes and $(n-x)$ failures, so

$$P(x \text{ successes and } n-x \text{ failures in any order}) = {}^nC_x\, p^x q^{n-x}.$$

This is the term in p^xq^{n-x} in the expansion of the binomial $(p+q)^n$.

1 BERNOULLI TRIALS AND BINOMIAL PROBABILITY

- A *Bernoulli trial* or *Bernoulli experiment* is an experiment with two outcomes, ‘success’ and ‘failure’, usually assigned the probabilities p and $q = 1 - p$.
- A *binomial random variable* is the outcomes of an n-stage experiment in which:
  - — each stage is a Bernoulli trial with the same probability p of success,
  - — the stages are independent,
  - — the random variable X is the number of successes, not their order.
- Suppose that the probabilities of ‘success’ and ‘failure’ in any stage of an n-stage binomial experiment are p and $q = 1 - p$ respectively. Then

  $$P(x \text{ successes}) = {}^nC_x\, p^x q^{n-x}.$$
- This probability is the term in p^xq^{n-x} in the expansion of $(p+q)^n = 1^n = 1$, and the sum of the $n + 1$ binomial probabilities is 1.
- In particular, if $p = q = \frac{1}{2}$, then the formula simplifies to

  $$P\left(x \text{ successes}\right) = {}^nC_x \left(\tfrac{1}{2}\right)^n.$$

The pronumeral x is usually used for the values of a discrete random variable. In this chapter we will therefore mostly use x in place of the r that was used in Chapters 14–15 of the Year 11 book.

The next worked Example shows how to use complementary events and cases to answer questions.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 2 17A

Six cards are drawn at random from a pack of 52 playing cards. Each card is replaced and the pack is shuffled before the next card is drawn. Find, as fractions with denominator 4^6, the probability that:

a two are clubs,

b one is a club,

c at least one is a club,

d at least four are clubs.

SOLUTION

There are 13 clubs in the pack, so at each stage the probability of drawing a club is $\frac{1}{4}$. Applying the formula with $p = \frac{1}{4}$ and $q = \frac{3}{4}$:

a $$\begin{aligned} P(\text{two are clubs}) &= {}^6C_2 \times \left(\tfrac{1}{4}\right)^2 \times \left(\tfrac{3}{4}\right)^4 \\ &= \frac{15 \times 3^4}{4^6} \\ &= \frac{1215}{4^6} \end{aligned}$$

b $$\begin{aligned} P(\text{one is a club}) &= {}^6C_1 \times \left(\tfrac{1}{4}\right)^1 \times \left(\tfrac{3}{4}\right)^5 \\ &= \frac{6 \times 3^5}{4^6} \\ &= \frac{1458}{4^6} \end{aligned}$$

c $$\begin{aligned} P(\text{at least one is a club}) &= 1 - P(\text{all are non-clubs}) \\ &= 1 - \left(\tfrac{3}{4}\right)^6 \quad \left(\text{or } 1 - {}^6C_0 \times \left(\tfrac{1}{4}\right)^0 \times \left(\tfrac{3}{4}\right)^6\right) \\ &= \frac{3367}{4^6} \end{aligned}$$

d $P(\text{at least four are clubs})$

$$\begin{aligned} &= P(\text{four are clubs}) + P(\text{five are clubs}) + P(\text{six are clubs}) \\ &= {}^6C_4 \times \left(\tfrac{1}{4}\right)^4 \times \left(\tfrac{3}{4}\right)^2 + {}^6C_5 \times \left(\tfrac{1}{4}\right)^5 \times \left(\tfrac{3}{4}\right)^1 + {}^6C_6 \times \left(\tfrac{1}{4}\right)^6 \times \left(\tfrac{3}{4}\right)^0 \\ &= \frac{15 \times 3^2 + 6 \times 3 + 1}{4^6} \\ &= \frac{154}{4^6} \end{aligned}$$

An example where $p = q = \frac{1}{2}$

A particular case of binomial probability is when the probabilities p and q of ‘success’ and ‘failure’ are both $\frac{1}{2}$.

Example 3 17A

If a coin is tossed 100 times, what is the probability that it comes up heads exactly 50 times (correct to four significant figures)?

SOLUTION

Taking $p = q = \frac{1}{2}$, $$\begin{aligned} P(50\text{ heads}) &= {}^{100}C_{50} \times \left(\tfrac{1}{2}\right)^{50} \times \left(\tfrac{1}{2}\right)^{50} \\ &= {}^{100}C_{50} \times \left(\tfrac{1}{2}\right)^{100} \\ &\doteqdot 0.0796. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Note: This is a fairly low probability. Should we have expected a higher probability than this? Hardly, because any result from about 45 to 55 heads would be unlikely to surprise us. In general, as already stated in Box 1 above,

$$P\left(x \text{ heads in } n \text{ tosses of a coin}\right) = {}^nC_x \times \left(\tfrac{1}{2}\right)^n.$$

Experimental probabilities and binomial probability

Some of the most straightforward and important applications of binomial theory arise in situations where the probabilities of 'success' and 'failure' are determined experimentally.

Example 4 — 17A

A light bulb is classed as 'defective' if it burns out in under 1000 hours. A company making light bulbs finds, after careful testing, that 1% of its bulbs are defective. If it packs its bulbs in boxes of 50, find, correct to three significant figures:

a the probability that a box contains no defective bulbs,

b the probability that at least two bulbs in a box are defective.

SOLUTION

In this case, $p = 0.01$ and $q = 0.99$. Let X be the number of defective bulbs in the box. Then:

a $P(X = 0) = 0.99^{50} \doteqdot 0.605$

b
$$\begin{aligned} P(X \ge 2) &= 1 - \big(P(X = 0) + P(X = 1)\big) \\ &= 1 - \big(0.99^{50} + {}^{50}C_1 \times 0.01^1 \times 0.99^{49}\big) \\ &\doteqdot 0.0894 \end{aligned}$$

An example where each stage is a compound event

Sometimes, when each stage of the experiment is itself a compound event, it may take some work to find the probability of success at each stage.

Example 5 — 17A

Joe King and his sister Fay make shirts for a living. Joe works more slowly, but more accurately, making 20 shirts a day, of which 2% are defective. Fay works faster, making 30 shirts a day, of which 4% are defective. If they send out their shirts in randomly mixed parcels of 30 shirts, what is the probability (correct to three significant figures) that no more than two shirts in a box are defective?

SOLUTION

If a shirt is chosen at random from one parcel, then using the product rule and the addition rule, the probability p that the shirt is defective is

$$\begin{aligned} p &= P(\text{Joe made it, and it is defective}) + P(\text{Fay made it, and it is defective}) \\ &= \frac{20}{50} \times \frac{2}{100} + \frac{30}{50} \times \frac{4}{100} \\ &= \frac{4}{125}, \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

so $p = \frac{4}{125}$ and $q = \frac{121}{125}$.
Let X be the number of defective shirts.
Then $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \left(\tfrac{121}{125}\right)^{30} + {}^{30}C_1 \times \left(\tfrac{121}{125}\right)^{29} \times \tfrac{4}{125} + {}^{30}C_2 \times \left(\tfrac{121}{125}\right)^{28} \times \left(\tfrac{4}{125}\right)^2$$

$$\doteqdot 0.930.$$

Exercise 17A

FOUNDATION

Unless otherwise specified, leave your answers in unsimplified form.

1 Assume that the probability that a child is female is $\frac{1}{2}$, that sex is independent from child to child, and that there are only two sexes. Giving your answers as fractions in simplest form, find the probability that in a family of five children:

a all are boys,
b there are two girls and three boys,
c there are four boys and one girl,
d at least one will be a girl.

2 In a one-day cricket game, a batsman has a chance of $\frac{1}{5}$ of hitting a boundary every time he faces a ball. If he faces all six balls in an over, what is the probability that he will hit exactly two boundaries, assuming that successive strikes are independent?

3 A jury roll contains 2000 names, 700 of females and 1300 of males. Twelve jurors are randomly selected.

a Explain why it is reasonable to make the approximation that the probability of selecting a male does not change with each selection.
b What is the probability of ending up with an all-male jury?

4 A die is rolled twelve times. Find the probability that 5 appears on the uppermost face:

a exactly three times,
b exactly eight times,
c ten or more times (that is ten, eleven or twelve times).

5 A die is rolled six times. Let N denote the number of times that the number 3 is shown on the uppermost face. Find, correct to four decimal places:

a $P(N = 2)$
b $P(N < 2)$
c $P(N \ge 2)$

6 An archer finds that on average he hits the bulls-eye nine times out of ten. Assuming that successive attempts are independent, find the probability that in twenty attempts:

a he scores at least eighteen hits,
b he misses at least once.

7 A torch manufacturer finds that on average 9% of the bulbs are defective. What is the probability that in a randomly selected batch of ten one-bulb torches:

a there will be no more than two with defective bulbs,
b there will be at least two with defective bulbs.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

8 A coin is tossed four times and the result is recorded. Janice wins if there are exactly two heads.

- **a** List all the ways that this could occur, that is, in any order.
- **b** By counting, verify that there are exactly 6 cases where Janice wins and find the probability of this outcome.
- **c** Explain why this is the same as the number of ways of ordering the word HHTT, and use combinatorics to count how many such arrangements exist. Does your answer agree with part **b**?
- **d** Show that this is equivalent to choosing two of the coins and placing them heads up, while placing the other two coins tails up.

9 A poll indicates that 55% of people support the policies of the Working Together Party. If five people are selected at random, what is the probability that a majority of them will support WTP party policies? Give your answer correct to three decimal places.

10 The probability that a small earthquake occurs somewhere in the world on any one day is 0.95. Assuming that earthquake frequencies on successive days are independent (this assumption is probably false), what is the probability that a small earthquake occurs somewhere in the world on exactly 28 of January's 31 days? Leave your answer in index form.

11 The probability that a jackpot prize will be won in a given lottery is 0.012.

- **a** Find, correct to five decimal places, the probability that the jackpot prize will be won:
  - **i** exactly once in ten independent lottery draws,
  - **ii** at least once in ten independent lottery draws.
- **b** The jackpot prize is initially \$10 000 and increases by \$10 000 each time the prize is not won. Find, correct to five decimal places, the probability that the jackpot prize will exceed \$200 000 when it is finally won.

12 **a** How many times must a die be rolled so that the probability of rolling at least one six is greater than 95%?

- **b** How many times must a coin be tossed so that the probability of tossing at least one tail is greater than 99%?

13 Five families have three children each.

- **a** Find, correct to three decimal places, the probability that:
  - **i** at least one of these families has three boys,
  - **ii** each family has more boys than girls.
- **b** What assumptions have been made in arriving at your answer?

14 Comment on the validity of the following arguments:

- **a** 'In the McLaughlin Library, 10% of the books are mathematics books. Hence if I go to a shelf and choose five books from that shelf, then the probability that all five books are mathematics books is 10^{-5}.'
- **b** 'During an election, 45% of voters voted for party A. Hence if I select a street at random, and then select a voter from each of four houses in the street, the probability that exactly two of those voters voted for party A is ${}^4C_2 \times (0.45)^2 \times (0.55)^2$.'

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 During winter it rains on average 18 out of 30 days. Five winter days are selected at random. Find, correct to four decimal places, the probability that:

a the first two days chosen will be fine and the remainder wet,

b more rainy days than fine days have been chosen.

16 A tennis player finds that, on average, he gets his serve in in eight out of every ten attempts, and that he serves an ace (a serve which is in the boundaries and not touched by his opponent) once every fifteen serves. He serves four times. Assuming that successive serves are independent events, find, correct to six decimal places, the probability that:

a all four serves are in,

b he hits at least three aces,

c he hits exactly three aces and the other serve lands in.

17 A man is restoring ten old cars, six of them manufactured in 1955 and four of them manufactured in 1962. When he tries to start them, on average the 1955 models will start 65% of the time and the 1962 models will start 80% of the time. Find, correct to four decimal places, the probability that at any time:

a exactly three of the 1955 models and one of the 1962 models will start,

b exactly four of the cars will start. (Hint: You will need to consider five cases.)

18 An apple exporter deals in two types of apples, Red Delicious and Golden Delicious. The ratio of Red Delicious to Golden Delicious is 4:1. The apples are randomly mixed together before they are boxed. One in every fifty Golden Delicious and one in every one hundred Red Delicious apples will need to be discarded because they are undersized.

a What is the probability that an apple selected from a box will need to discarded?

b If ten apples are randomly selected from a box, find the probability that:

i all the apples will have to be discarded,

ii half of the apples will have to be discarded,

iii less than two apples will be discarded.

19 One bag contains three red and five white balls, and another bag contains four red and four white balls.

a One bag is chosen at random, a ball is selected from that bag, its colour is noted, and then it is replaced. Find the probability that the ball chosen is red.

b If the operation in part **a** is carried out eight times, find the probability that:

i exactly three red balls are drawn,

ii at least three red balls are drawn.

20 **a** If six dice are rolled one hundred times, how many times would you expect the number of even numbers showing to exceed the number of odd numbers showing?

b If eight coins are tossed sixty times, how many times would you expect the number of heads to exceed the number of tails?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

21 A game is played using a barrel containing twenty similar balls numbered 1 to 20. The game consists of drawing four balls, without replacement, from the twenty balls in the barrel. Thus the probability that any particular number is drawn in any game is 0.2.

a Find as a decimal the probability that the number 19 is drawn in exactly two of the next five games played.

b Find as a decimal the probability that the number 19 is drawn in at least two of the next five games played.

c Let n be an integer, where $4 \le n \le 20$.

i What is the probability that, in any one game, all four selected numbers are less than or equal to n?

ii Show that the probability that, in any one game, n is the largest of the four numbers drawn is $\dfrac{{}^{n-1}C_3}{{}^{20}C_4}$.

22 a Expand $(a + b + c)^3$.

b In a survey of football supporters, 65% supported Hawthorn, 24% followed Collingwood and 11% followed Sydney. Use the expansion in part **a** to find, correct to five decimal places, the probability that if three people are randomly selected:

i one supports Hawthorn, one supports Collingwood and one supports Sydney,

ii exactly two of them support Collingwood,

iii at least two of them support the same team.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17B Binomial distributions

Now that we have dealt with individual binomial probabilities, we can look at the whole distribution. The binomial distribution is a discrete distribution, and Chapter 13 of the Year 11 book was devoted to discrete distributions. As we have done before, we will draw its graph and work with its mean and variance.

Notation for binomial distributions

Some notation is convenient. The binomial distribution consisting of n independent trials, each with probability p of success, is denoted by $B(n,p)$ or $\text{Bin}(n,p)$. In particular, a Bernoulli experiment with probability p is a special case of a binomial experiment, so is denoted by $B(1,p)$.

To say that X is a binomial random variable for the distribution $B(n,p)$, we can use the symbolic notation

$$X \sim B(n,p) \qquad \text{OR} \qquad X \sim \text{Bin}(n,p).$$

In particular, $X \sim B(1,p)$ means that X is a Bernoulli random variable (there is only one stage) with probability p of success.

2 BINOMIAL AND BERNOULLI DISTRIBUTIONS

- The binomial distribution with n independent Bernoulli stages, each with probability p of success, is denoted by $B(n,p)$ or $\text{Bin}(n,p)$.
- The Bernoulli distribution with probability p is therefore $B(1,p)$.
- $X \sim B(n,p)$ means that X is a random variable for the distribution $B(n,p)$.

The mean and variance of a Bernoulli distribution

A Bernoulli trial with probabilities p of 'success' and $q = 1 - p$ of 'failure' has mean p and variance pq,

$$\mu = p \qquad \text{and} \qquad \sigma^2 = pq,$$

so that its standard deviation is $\sigma = \sqrt{pq}$.

These formulae are easy to prove because there are only two outcomes.

We use the formula $\mu = \sum xP(X = x)$,

where the sum is taken over the distribution, meaning for $x = 0$ and $x = 1$,

so
$$\begin{aligned} \mu &= 0 \times P(X = 0) + 1 \times P(X = 1) \\ &= 0 \times q + 1 \times p \\ &= p. \end{aligned}$$

Similarly,
$$\begin{aligned} \sigma^2 &= \sum x^2 P(X = x) - \mu^2, \text{ summing over the distribution,} \\ &= (0^2 \times q + 1^2 \times p) - p^2 \\ &= p - p^2 \\ &= p(1 - p) \\ &= pq \quad \text{(either form is an appropriate answer)}. \end{aligned}$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The mean and variance of a binomial distribution

The mean and variance of the binomial random variable $X \sim B(n, p)$ with n trials and probabilities p and q of 'success' and 'failure' are

$$\mu = np \qquad \text{and} \qquad \sigma^2 = npq.$$

so that the standard deviation is $\sigma = \sqrt{npq}$.

These two results $\mu = np$ and $\sigma^2 = npq$ seem to follow immediately from the results $\mu = p$ and $\sigma^2 = pq$ for Bernoulli trials just by multiplying by n. And they do — it is true in general that if we have a number of independent random variables, then the mean of the sum is the sum of the means, and the variance of the sum is the sum of the variances.

Unfortunately, however, those two very general theorems are too difficult to prove, and instead we must prove the result for binomial distributions directly. The proofs below are not conceptually difficult, but they do require a sequence of computations. We begin with a lemma about binomial coefficients.

Lemma: Let n and x be whole numbers with $x \le n$.

a If $x \ge 1$, then $\qquad x \times {}^nC_x = n \times {}^{n-1}C_{x-1}$.

b If $x \ge 2$, then $\quad x(x-1) \times {}^nC_x = n(n-2) \times {}^{n-2}C_{x-2}$.

Proof:

a

$$\begin{aligned} x \times {}^nC_x &= x \times \frac{n!}{x! \times (n-x)!} \\ &= \frac{x \times n \times (n-1)!}{x \times (x-1)! \times (n-x)!} \\ &= n \times \frac{(n-1)!}{(x-1)! \times (n-x)!} \\ &= \text{RHS} \end{aligned}$$

b

$$\begin{aligned} & x(x-1) \times {}^nC_x \\ &= x(x-1) \times \frac{n!}{x! \times (n-x)!} \\ &= \frac{x(x-1) \times n(n-1) \times (n-2)!}{x(x-1) \times (x-2)! \times (n-x)!} \\ &= n(n-1) \times \frac{(n-2)!}{(x-2)! \times (n-x)!} \\ &= \text{RHS} \end{aligned}$$

Proving that the mean is *np*

This proof requires binomial expansions, together with part **a** of the lemma above.

$$\begin{aligned} E(X) &= \sum xP(X = x), \text{ summed over the distribution,} \\ &= 0 \times {}^nC_0 \times p^0q^n + 1 \times {}^nC_1 \times p^1q^{n-1} + 2 \times {}^nC_2 \times p^2q^{n-2} + \cdots \\ &\qquad + n \times {}^nC_n \times p^nq^0. \end{aligned}$$

The first term is 0, then we apply part **a** of the lemma to the remaining terms,

$$\begin{aligned} E(X) &= n \times {}^{n-1}C_0\, p^1q^{n-1} + n \times {}^{n-1}C_1\, p^2q^{n-2} + \cdots + n \times {}^{n-1}C_{n-1}\, p^nq^0 \\ &= np\left({}^{n-1}C_0\, p^0q^{n-1} + {}^{n-1}C_1\, p^1q^{n-2} + \cdots + {}^{n-1}C_{n-1}\, p^{n-1}q^0\right). \end{aligned}$$

The bit in brackets is the binomial expansion of $(p+q)^{n-1}$, where $p + q = 1$, so

$$\begin{aligned} E(X) &= np(p+q)^{n-1} \\ &= np \times 1 \\ &= np. \end{aligned}$$

Proving that the variance is *npq*

This proof again requires binomial expansions. First, however, we need to develop yet another formula for the variance.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

$$
\begin{aligned}
\operatorname{Var}(X) &= \mathrm{E}(X^2) - \big(\mathrm{E}(X)\big)^2.\\
&= \mathrm{E}\big(X(X-1)\big) + \mathrm{E}(X) - \big(\mathrm{E}(X)\big)^2.\\
&= \mathrm{E}\big(X(X-1)\big) + np - n^2p^2, \text{ because } \mathrm{E}(X) = np.
\end{aligned}
$$

We can find $\mathrm{E}\big(X(X-1)\big)$ using binomial expansions and part **b** of the lemma,

$$
\begin{aligned}
\mathrm{E}\big(X(X-1)\big) &= \sum x(x-1)P(X=x), \text{ summed over the distribution,}\\
&= 0 \times (-1) \times {}^n\mathrm{C}_0 \times p^0q^n + 1 \times 0 \times {}^n\mathrm{C}_1 \times p^1q^{n-1} + 2 \times 1 \times {}^n\mathrm{C}_2 \times p^2q^{n-2}\\
&\quad + 3 \times 2 \times {}^n\mathrm{C}_3 \times p^3q^{n-3} + \cdots + n(n-1) \times {}^n\mathrm{C}_n \times p^nq^0.
\end{aligned}
$$

The first two terms are 0, then applying part **b** of the lemma,

$$
\begin{aligned}
&= n(n-1) \times {}^{n-2}\mathrm{C}_0 \times p^2q^{n-2} + n(n-1) \times {}^{n-2}\mathrm{C}_1 \times p^3q^{n-3}\\
&\quad + n(n-1) \times {}^{n-2}\mathrm{C}_2 \times p^4q^{n-4} + \cdots + n(n-1) \times {}^{n-2}\mathrm{C}_{n-2} \times p^nq^0.\\
&= n(n-1)p^2({}^{n-2}\mathrm{C}_0\,p^0q^{n-2} + {}^{n-2}\mathrm{C}_1\,p^1q^{n-3} + + \cdots + {}^{n-2}\mathrm{C}_{n-2}\,p^{n-2}q^0).
\end{aligned}
$$

The bit in brackets is the binomial expansion of $(p+q)^{n-2}$, where $p + q = 1$, so

$$
\begin{aligned}
&= n(n-1)p^2(p+q)^{n-2}\\
&= n^2p^2 - np^2.
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } \operatorname{Var}(X) &= \mathrm{E}\big(X(X-1)\big) + np - n^2p^2\\
&= n^2p^2 - np^2 + np - n^2p^2\\
&= np(1-p)\\
&= npq.
\end{aligned}
$$

3 MEAN AND VARIANCE OF A BINOMIAL DISTRIBUTION

- For a binomial random variable $X \sim B(n, p)$, where $q = 1 - p$,

  $\mu = np$ and $\sigma^2 = npq$ and $\sigma = \sqrt{npq}$.

- In particular, for a Bernoulli random variable $X \sim B(1, p)$,

  $\mu = p$ and $\sigma^2 = pq$ and $\sigma = \sqrt{pq}$.

The symbols $\mathrm{E}(X)$ and μ are interchangeable — μ is more concise, but the notation $\mathrm{E}(X)$ indicates that when we run the experiment, there is a sense in which we are 'expecting' to get $\mathrm{E}(X)$. Similarly, σ^2 and $\operatorname{Var}(X)$ are interchangeable.

Example 6 17B

A binomial random variable has parameters $n = 20$ and $p = 0.1$.

a Find the mean, variance and standard deviation.

b What is the probability of getting the mean when the experiment is run?

c What is the probability that the result is within one standard deviation of the mean?

d Give an example that this distribution could model.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a $\mu = np$
$= 20 \times 0.1$
$= 2$

$\sigma^2 = npq$ (where $q = 1 - p = 0.9$)
$= 20 \times 0.1 \times 0.9$
$= 1.8$

$\sigma = \sqrt{1.8}$
$\doteqdot 1.342$

b
$$\begin{aligned} P(X = 2) &= {}^{20}C_2\, p^2 q^{18} \\ &= {}^{20}C_2 \times (0.1)^2 \times (0.9)^{18} \\ &\doteqdot 0.285. \end{aligned}$$

c
$$\begin{aligned} P(\mu - \sigma \le X \le \mu + \sigma) &= P(0.658 \le X \le 3.342) \\ &= P(X = 1 \text{ or } X = 2 \text{ or } X = 3) \\ &= {}^{20}C_1\, p^1 q^{19} + {}^{20}C_2\, p^2 q^{18} + {}^{20}C_3\, p^3 q^{17} \\ &= 20 \times (0.1) \times (0.9)^{19} + 190 \times (0.1)^2 \times (0.9)^{18} \\ &\qquad + 1140 \times (0.1)^3 \times (0.9)^{17} \\ &\doteqdot 0.745. \end{aligned}$$

d Choose a busy intersection with traffic lights on the way to work. On 20 mornings, when crossing at the lights, look at the number plate of the left-most front vehicle stopped at the lights, and record whether the last digit-character is a 7. In a small country town, it is just possible that the events may not be independent, but in a city, independence is virtually certain.

Example 7 17B

A binomial random variable has parameters $n = 100$ and $p = \frac{1}{5}$.

a Find the mean and standard deviation.

b Keeping $p = \frac{1}{5}$, what would n need to be increased to so that the standard deviation is less than 1% of the number n of trials?

c Keeping $n = 100$, what must p be decreased to so that the standard deviation is less than 1? Can p be increased to give a standard deviation less than 1?

SOLUTION

a $\mu = np$
$= 100 \times \frac{1}{5}$
$= 20,$

$\sigma^2 = npq$ (where $q = 1 - p = \frac{4}{5}$)
$= 100 \times \frac{1}{5} \times \frac{4}{5}$
$= 16,$

$\sigma = \sqrt{16}$
$= 4.$

b Put $\sigma < \dfrac{n}{100}.$

Squaring, $npq < \dfrac{n^2}{10\,000}$

$n^2 > 10\,000 \times \frac{4}{25} \times n,$ because $pq = \frac{1}{5} \times \frac{4}{5} = \frac{4}{25}.$

Hence $n > 1600,$ because n is positive.

c Put $\sigma < 1.$

Squaring, $npq < 1$

$100p(1 - p) < 1$

$100p^2 - 100p + 1 > 0.$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

The quadratic $100p^2 - 100p + 1 = 0$, with $a = 100$, $b = -100$ and $c = 1$, has axis of symmetry $p = \frac{1}{2}$ and discriminant $\Delta = 9600 = 40^2 \times 6$,

$$\text{so its roots are } p = \frac{100 - 40\sqrt{6}}{200} \quad \text{and} \quad p = \frac{100 + 40\sqrt{6}}{200}$$

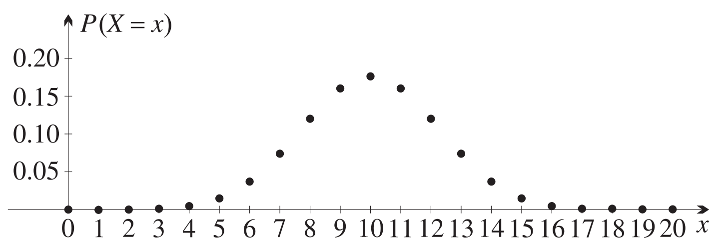
$$= \tfrac{1}{2} - \tfrac{1}{5}\sqrt{6} \qquad\qquad = \tfrac{1}{2} + \tfrac{1}{5}\sqrt{6}$$

$$\doteqdot 0.0101 \qquad\qquad \doteqdot 0.9899,$$

so p should be decreased to less than $\frac{1}{2} - \frac{1}{5}\sqrt{6}$ (roughly, less than 0.01), or increased to more than $\frac{1}{2} + \frac{1}{5}\sqrt{6}$ (roughly, greater than 9.99).

Skewed and non-skewed binomial distributions

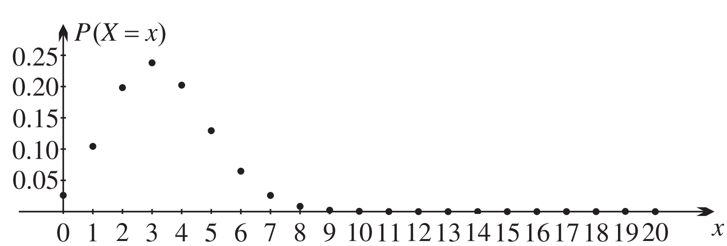
If $p = q = \frac{1}{2}$, such as when we toss a coin a number of times, the distribution is symmetric. For example, in Section 15E we graphed the distribution when a coin is tossed 20 times and the number of heads is recorded.



The mean is $\mu = np = 10$, and the distribution is symmetric about $x = 10$, with standard deviation $\sigma = \sqrt{npq} = \sqrt{5} \doteqdot 2.236$. This symmetry is easily seen from the symmetry of the binomial coefficient. For example, ${}^{20}C_7 = {}^{20}C_{13}$, so

$$P(X = 7) = {}^{20}C_7 \times \left(\tfrac{1}{2}\right)^7 \times \left(\tfrac{1}{2}\right)^{13} = {}^{20}C_{13} \times \left(\tfrac{1}{2}\right)^{13} \times \left(\tfrac{1}{2}\right)^7 = P(X = 13).$$

When, however, we throw a die 20 times, recording 'success' when the result is 6, the distribution is decidedly skewed to the right (positive skewness). Here $p = \frac{1}{6}$ and $q = \frac{5}{6}$, with mean $\mu = 3\frac{1}{3} \doteqdot 3.333$ and standard deviation $\sigma = \frac{5}{3} \doteqdot 1.667$.

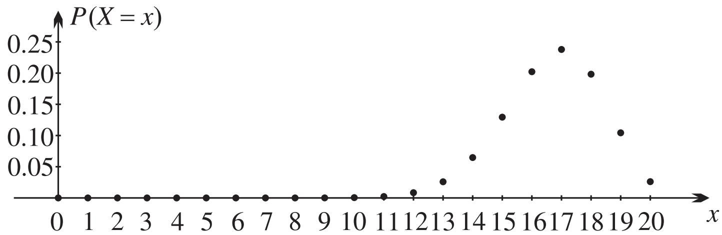


Remember that *skewed to the right* means that there is a *longer tail on the right-hand side*, and the peak is on the left-hand side.

On the other hand, when we throw the die 20 times but record 'success' as 'not getting 6', then the distribution is skewed to the left (negative skewness). Here $p = \frac{5}{6}$ and $q = \frac{1}{6}$, with mean $\mu = 16\frac{2}{3} \doteqdot 16.667$ and the same standard deviation $\sigma = \frac{5}{3} \doteqdot 1.667$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



Again, remember that *skewed to the left* means that there is a *longer tail on the left-hand side*, and the peak is on the right-hand side.

The graphs are reflections of each other in $x = 10$. For example,

$P(7\text{ heads}) = {}^{20}C_7 \times \left(\frac{1}{6}\right)^7 \times \left(\frac{5}{6}\right)^{13} = {}^{20}C_{13} \times \left(\frac{5}{6}\right)^{13} \times \left(\frac{1}{6}\right)^7 = P(13\text{ non-heads}).$

Question 11 in Exercise 17B shows how, for a fixed number of trials, the standard deviation decreases as the distribution becomes more skewed. Here is a simple example.

Example 8 17B

Two binomial random variables X and Y both consist of 100 Bernoulli trials, For X, $p = q = \frac{1}{2}$, and for Y, $p = \frac{1}{10}$. Find the ratio of their standard deviations.

SOLUTION

For X,
$$\begin{aligned} \sigma_X{}^2 &= npq \\ &= 100 \times \tfrac{1}{2} \times \tfrac{1}{2} \\ &= 25, \end{aligned}$$
so $\sigma_X = 5$.

For Y,
$$\begin{aligned} \sigma_Y{}^2 &= npq \\ &= 100 \times \tfrac{1}{10} \times \tfrac{9}{10} \\ &= 9, \end{aligned}$$
so $\sigma_X = 3$.

Hence the ratio of the two standard deviations is $\sigma_X : \sigma_Y = 5:3$.

4 SKEWED BINOMIAL DISTRIBUTIONS

Suppose that a binomial distribution consists of n independent Bernoulli trials, each with probability p of success and probability $q = 1 - p$ of failure.

- Reversing ‘success’ and ‘failure’ reverses the probabilities p and q, and reflects the graph in $x = \frac{1}{2}n$.
- If $p = q = \frac{1}{2}$, then the distribution is symmetric about $x = \frac{1}{2}$.
- If $p < \frac{1}{2}$, the distribution is skewed to the right (positive skewness).
  If $p > \frac{1}{2}$, the distribution is skewed to the left (negative skewness).
- For distributions with the same number of trials, the standard deviation gets closer to 0 or 1 as the skewness becomes more pronounced.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Simulating the experiment and graphing the data

The probabilities in a binomial distribution are estimates of what will happen when the experiment is run. Let us see what happens in simulations of two binomial experiments. The first experiment is symmetric, the second is skewed to the right.

Calculation and simulation in both worked examples should be done in a spreadsheet (the binomial function is currently `BINOM.DIST` in Excel).

Example 9 — 17B

The experiment is, 'toss 10 coins and count the number of heads'.

a Find the mean and standard deviation, and complete a table of the probability distribution.

b Simulate the experiment 100 times, complete the table of relative frequencies, and find the mean and standard deviation of the data.

c Draw the theoretical frequency histograms and polygon from part **a**. Then draw the frequency histograms and polygon of the simulation in part **b**.

SOLUTION

a Here $p = q = \frac{1}{2}$ and $n = 10$, so

$$\mu = np = 5, \qquad \sigma^2 = npq = 2.5, \qquad \sigma = \sqrt{2.5} \doteqdot 1.58.$$

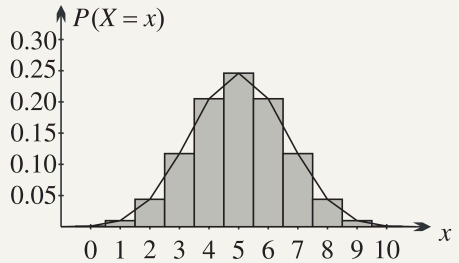
x	0	1	2	3	4	5	6	7	8	9	10
$P(X = x)$	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001

b After running the experiment 100 times,

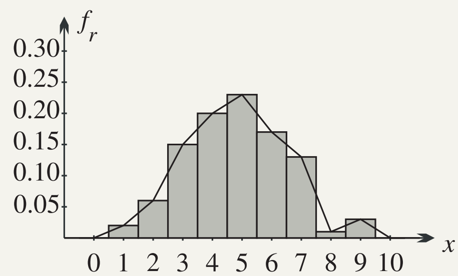
x	0	1	2	3	4	5	6	7	8	9	10
f	0	2	6	15	20	23	17	13	1	3	0
f_r	0	0.02	0.06	0.15	0.2	0.23	0.17	0.13	0.01	0.03	0

Calculation gives $\bar{x} = 4.82$ and $s \doteqdot 1.700$.

c



The theoretical distribution



The simulation

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 10 17B

The experiment is, 'roll 10 dice and count the number of sixes'.

a Find the mean and standard deviation, and complete a table of the probability distribution.

b Simulate the experiment 100 times, complete the table of relative frequencies, and find the mean and standard deviation of the data.

c Draw the theoretical frequency histograms and polygon from part **a**. Then draw the frequency histograms and polygon of the simulation in part **b**.

SOLUTION

a Here $p = \frac{1}{6}$ and $q = \frac{5}{6}$ and $n = 10$, so

$\mu = np = 1.667$, $\sigma^2 = npq = 1.389$, $\sigma \doteqdot 1.179$.

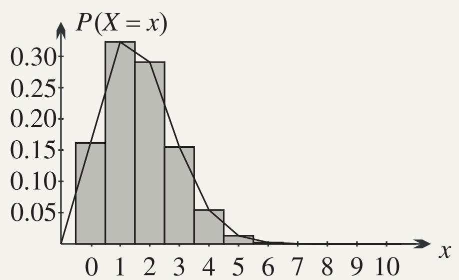
x	0	1	2	3	4	5	6	7	8	9	10
$P(X = x)$	0.162	0.323	0.291	0.155	0.054	0.013	0.002	0.0002	0.00002	0	0

b After running the experiment 100 times,

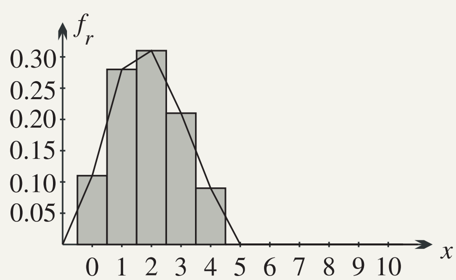
x	0	1	2	3	4	5	6	7	8	9	10
f	11	28	31	21	9	0	0	0	0	0	0
f_r	0.11	0.28	0.31	0.21	0.09	0.00	0.00	0.00	0.00	0.00	0.00

Calculation gives $\bar{x} = 1.89$ and $s \doteqdot 1.13$.

c



The theoretical distribution



The simulation

Exercise 17B FOUNDATION

1 Identify which of the following experiments may be modelled using a binomial random variable. If so, identify the variable.

- **a** It rains four days in a row. The number of rainy days is recorded.
- **b** A die is thrown ten times, and after each throw it is recorded whether the result is less than five.
- **c** Maddy is playing a simple game of cards. She turns up a card from the top of the pack. If it is an ace of spades she wins. Otherwise the card is returned to the pack, which is shuffled. The number of plays until she wins is recorded.
- **d** The probability of a head on tossing an unfair coin is 0.4. A coin is tossed twenty times and the number of heads is recorded.
- **e** Quality control testers have been given a random sample of 20 pens from a batch. It is known that 3% of the pens in the batch are defective. The testers record the number of faulty pens in the sample.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

f A pupil records the time for his journey to school over a thirty-day period.

g At stage n an experimenter selects one number from 1–100 and records a success if the number matches n. This experiment is repeated 50 times (that is, up to $n = 50$) and the experimenter wishes to know how many successes occur.

2 In a simple experiment, 6 fair coins are tossed and the number of heads is recorded. A student wishes to compare the results with theoretical predictions.

a Copy and complete the table below using the formula $P(X = x) = {}^nC_x p^x q^{n-x}$ for binomial probability.

Number of heads	0	1	2	3	4	5	6	Total
Number of ways can occur probability								

b Read off the mode (the most common result).

c Use this table to determine the expected value and variance of this discrete probability distribution.

d Compare your results to those obtained using the two formulae $E(X) = np$ and $\text{Var}(X) = npq$, where $q = 1 - p$.

e Explain, in your own words, why the result for the expected value is not a surprise.

3 For each situation, construct a table for the theoretical distribution associated with the given random variable and calculate the mode, mean and standard deviation.

a A coin is tossed five times and X is the number of heads that are face up.

b A die is thrown five times and X is the number of times 5 or 6 occurs.

c Five cards are drawn in turn with replacement from a standard pack of 52 cards and X is the number of court cards (jack, queen, king) that are drawn. Give your result correct to 3 decimal places.

4 At the School fête, few people have entered any of the 24 meat raffles, so to support the school, Larry buys 8 of the 20 tickets in each raffle.

a Calculate the expected value and standard deviation for the random variable of the number of Larry's wins.

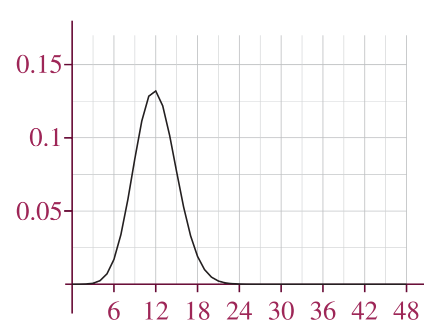
b If Larry wins 6 raffles, calculate how many standard deviations Larry's result is below the mean, as a measure of his poor luck.

5 **a** If a player throws 6 dice, what is the probability of getting at least two sixes? Let the random variable X record the number of sixes. What is the expected value and standard deviation for X?

b Repeat the question for:

i 12 dice, **ii** 24 dice.

6 The graph to the right shows the relative frequency polygon for the binomial distribution with $n = 48$ and $p = 0.25$.



a Is the distribution skewed to the left (negatively) or to the right (positively) or not at all?

b Find the mean and standard deviation for the distribution.

c What is the mode of the distribution? Estimate the probability of this most likely outcome.

d Copy the graph and shade the region no more than two standard deviations from the mean.

e Use the given sketch for $p = 0.25$ to assist in sketching the graph of the distribution for $p = 0.75$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 [Technology]
Large binomial calculations can be done in a spreadsheet, as in the following question. This question extends over two exercises — it continues in Exercise 17C — so be careful to save your spreadsheet for later use.

A student wishes to use Excel to explore the theoretical probabilities that occur when a coin is tossed 100 times.

- **a** In cell `F1` enter 100 (for the number of trials), and in cell `F2` enter 0.5 (for the probability p of obtaining a head). Add suitable labels in `E1` and `E2`.
- **b** Enter the numbers 0–100 in cells `A0:A101`.
- **c** In cell `B1` enter the formula = `BINOM.DIST($A1,$F$1,$F$2,FALSE)` to calculate the binomial probability of `A1` heads in 100 tosses for $p = 0.5$.
- **d** Use `Fill Down` to fill `B1` down to the first 101 rows in the second column.
- **e** What is the probability of obtaining at least 60 heads? (Select `B61:B101` and your spreadsheet program may show the sum on a bottom status line. Otherwise use a `SUM` command).
- **f** What is the probability of obtaining between 30 and 55 heads inclusive?
- **g** What is the mode?
- **h** Find the smallest integer i such that more than 50% of the data lies in the range $[50 - i, 50 + i]$.
- **i** Draw a histogram of the data in column `B`, with column `A` as the x-axis labels (this may well be the default). What famous shape does it remind you of?
- **j** Save all your work for future use in Exercise 17C.

DEVELOPMENT

8 Ms Taylor sets her class a test of 48 multiple-choice questions, each with four options A–D. Let the random variable X be the number of questions a person gets correct.

- **a** Calculate the expected value $E(X)$. How can this value be understood in this context?
- **b** What is the standard deviation σ of the random variable?
- **c** Ms Taylor is annoyed to discover that Fayola, one of her students, claims to have got 24 just by guessing. How many standard deviations is Fayola's score above the mean?

Ms Taylor writes a new test, with 100 questions, each with five options A–E.

- **d** What is the expected value and standard deviation of this new distribution with random variable Y?
- **e** Fayola gets 40 questions right this time, and again she claims to have achieved this result just by guessing. If true, would this be more or less unusual than her previous result?
- **f** Idette gets 75% in the first test and 60% in the second. Which is the more unusual of her results, if she is guessing?

9 A certain drug is found to be helpful for 70% of patients who take it. Two research teams are attempting to improve its effectiveness by adjusting the delivery system for the drug. Both conduct random trials to test the effectiveness of their changes. Team A runs a trial with 50 patients, of whom 45 show improvement using the drug. Team B runs a trial with 90 patients, of whom 74 show improvement. Use the mean and standard deviation of the two trials to decide which adjustment shows stronger evidence of an improvement.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 It is known that 15% of a large constituency voted for the Working Together Party (WTP) at the last election. A poll of 100 people is taken and they are asked whether they voted WTP.

a What is the mean and standard deviation for this poll?

b What is the probability that in the sample, the number of WTP voters lies within half a standard deviation of the mean?

11 In this question, take the number n of stages of a binomial variable $X \sim B(n, p)$ to be fixed, and allow p to vary.

a Find σ^2 as a quadratic in p, and graph it.

b Explain the symmetry of the graph.

c Show that $\sigma^2 < np$ and $\sigma^2 < nq$.

d Show that the maximum value of σ^2 is $\dfrac{n}{4}$.

e Let n be fixed. Show that $\sigma \to 0$ as $p \to 0^+$ and as $p \to 1^-$.

12 A fair coin is tossed n times. The histogram for the resulting binomial distribution is labelled $0, 1, 2, \ldots, n$ on the horizontal axis, and each column is 1 unit wide. How many columns are entirely contained in the interval one standard deviation or less from the mean, when n is:

a 16, **b** 36, **c** 64?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

13 Continuing with Question 12 above, as p moves away from 0.5, the standard deviation, and hence the number of columns contained in 1 standard deviation, decreases. Find the limit of the ratio of the number of columns entirely contained in 1 standard deviation for $p = 0.5$ to the number contained for $p = 0.25$, as $n \to \infty$.

14 In experimental trials, it may be useful to ensure that there is at least one positive result. For example, the trial may be difficult or expensive to run, and at least one successful result may be required. Suppose that in a certain set of n independent Bernoulli trials, the probability of success at each stage is p.

a Show that the probability of obtaining at least one success is $P(X \geq 1) = 1 - (1 - p)^n$.

b Show that the number of trials required to ensure that the probability of obtaining a success is at least 95% is $n = \dfrac{\log(0.05)}{\log(1 - p)}$.

c Copy and complete the table below to show the number of trials required to ensure at least 95% probability of success for the given value of p.

p	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.05
n									

d A certain journal refuses to publish experimental studies that do not yield a positive result, meaning a result agreeing with the experimental hypothesis. Explain why this is a dangerous practice.

15 If a Bernoulli trial occurs a fixed number n of times, and if the stages are independent, then the resulting distribution is a *binomial distribution*. If instead the experiment continues until a success is obtained and the number of trials is recorded, the resulting distribution is called a *geometric distribution* (and is also called a *discrete waiting-time distribution*).

Suppose that at each stage the probability of success is p and the probability of failure is $q = 1 - p$, and that X is the first trial producing a success. Thus the possible values of X are $1, 2, 3, \ldots$.

a For this geometric distribution:

i Show that $P(X = x) = pq^{x-1}$.

ii Show that $\mu = p + 2pq + 3pq^2 + \cdots$.

iii Show that $\mu - q\mu = p + pq + pq^2 + \cdots$, and hence calculate μ.

b During the day at a certain medical practice, the waiting room is constantly at capacity, with 20 patients waiting to be seen by a doctor. Every 5 minutes, a new patient will be chosen at random to see a doctor and a new patient will arrive at the practice. What is the mean waiting time to see a doctor?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17C Normal approximations to a binomial

A pollster asks a sample of 10 000 people whether they intend to vote for the Working Together Party in the next election. The WTP won $\frac{1}{3}$ of the vote in the last election, and using the assumption that this will be the case in the next election, he wants to know the probability that his survey will give him a result within [3300, 3400].

Using binomial probability, the calculation is

$$\begin{aligned} P(3300 \le X \le 3400) &= {}^{10\,000}C_{3300}\left(\tfrac{1}{3}\right)^{3300}\left(\tfrac{2}{3}\right)^{6700} + {}^{10\,000}C_{3301}\left(\tfrac{1}{3}\right)^{3301}\left(\tfrac{2}{3}\right)^{6699} \\ &\quad + \cdots + {}^{10\,000}C_{3399}\left(\tfrac{1}{3}\right)^{3399}\left(\tfrac{2}{3}\right)^{6601} + {}^{10\,000}C_{3400}\left(\tfrac{1}{3}\right)^{3400}\left(\tfrac{2}{3}\right)^{6600}. \end{aligned}$$

This is a dreadful calculation because of all the huge factorials and massive sizes of the numbers. Even with a computer, there are serious problems about controlling all the errors in the calculation. And if the calculation involves the whole population of Australia — over 25 000 000 — the situation is even more difficult.

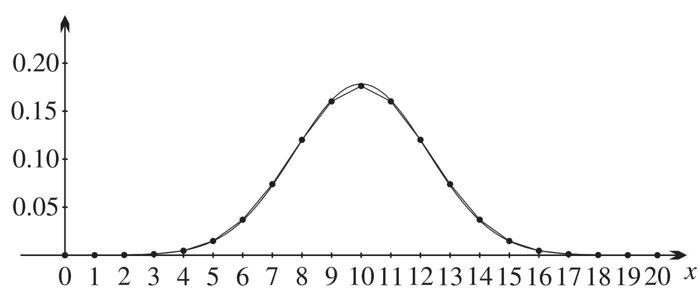
Fortunately, binomial distributions can be approximated very well by the normal distribution, and the larger the numbers are, the better the approximation. This section is about approximating a binomial distribution by a normal distribution.

An example of approximations

In Section 16E, we took the binomial distribution of the number of heads in 20 coin tosses, where $n = 20$ and $p = q = \frac{1}{2}$, so that

$$\mu = np = 20 \times \tfrac{1}{2} = 10 \quad \text{and} \quad \sigma^2 = npq = 20 \times \tfrac{1}{2} \times \tfrac{1}{2} = 5,$$

and $\sigma = \sqrt{5}$. We graphed the frequency polygon, then we superposed a normal curve with the same mean $\mu = 10$ and the same standard deviation $\sigma = \sqrt{5}$. The result looks reasonably good,



Thus the normal PDF approximates the frequency polygon of the binomial, and taking cumulative frequencies, it follows that the normal CDF approximates the ogive of the binomial. We cannot in this course give theoretical arguments why such approximations work — we can only look at the pictures and confirm the results. Then we can use these results in problems.

For example, the standard deviation is $\sqrt{5} \doteqdot 2.236$. Suppose that we want to find the probability that when we toss 20 coins, the number of heads will be within one standard deviation of the mean. These numbers are small enough for us to calculate it two ways, as in the next worked example.

Notation for normal distributions

The notation $N(\mu, \sigma^2)$ is used for the normal distribution with mean μ and variance σ^2. Thus in the diagram above, we are claiming that the binomial distribution $B\left(20, \frac{1}{2}\right)$ is approximated by the normal distribution $N(10, 5)$. In general, $B(n, p)$ is approximated by $N(np, npq)$, where $q = 1 - p$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 11 17C

A coin is tossed 20 times, and X is the number of heads. Find the probability that the number of heads is greater than 12:

a using a binomial distribution,

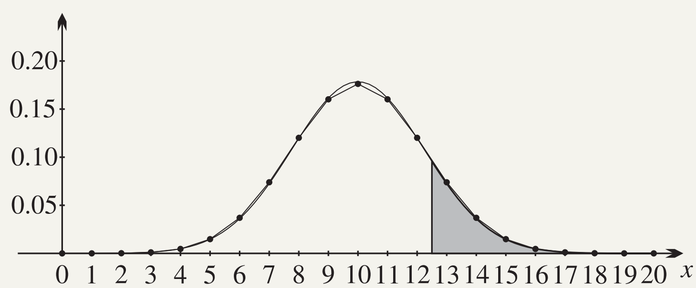
b using the normal approximation.

SOLUTION

a
$$\begin{aligned} P(\text{at least 13 heads}) &= {}^{20}C_{13}\left(\tfrac{1}{2}\right)^{20} + {}^{20}C_{14}\left(\tfrac{1}{2}\right)^{20} + \cdots + {}^{20}C_{20}\left(\tfrac{1}{2}\right)^{20} \\ &= \left(\tfrac{1}{2}\right)^{20}\left({}^{20}C_{13} + {}^{20}C_{14} + \cdots + {}^{20}C_{20}\right) \\ &= \left(\tfrac{1}{2}\right)^{20} \times 137\,980 \\ &\doteqdot 0.132. \end{aligned}$$

b To approximate the binomial by the normal distribution $N(10, 5)$, we treat the discrete random variable *X as if it were a continuous normal variable*. Because we are using a continuous density function, we need to follow the boundaries of the histograms and find the probability, not that $X \ge 13$, but that $X \ge 12.5$, which is the left-hand boundary of the first histogram.

$$P(\text{at least 13 heads}) \doteqdot P(X \ge 12.5).$$



This adjustment from 13 to 12.5 is called the *continuity correction* — it is the correction needed when a discrete distribution is treated as if it were continuous.

Using z-scores,
$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &\doteqdot 1.118, \end{aligned}$$

so
$$\begin{aligned} P(\text{at least 13 heads}) &\doteqdot P(Z > 1.118) \\ &\doteqdot 1 - \Phi(1.118) \\ &\doteqdot 1 - 0.868 \\ &\doteqdot 0.132. \end{aligned}$$

Continuity corrections

We are approximating a discrete binomial variable by a continuous normal variable. We thus integrate from halfway between the values of the discrete distribution. In the example above, that means from 12.5.

In the next worked example, we are adding the five values at $x = 8, 9, 10, 11$ and 12, so we integrate over the interval $[7.5, 12.5]$.

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Example 12 17C

A coin is tossed 20 times, and X is the number of heads. Find the probability that the number of heads is within one standard deviation of the mean:

a using a binomial distribution, **b** using the normal approximation.

Comment on the results of the calculations.

SOLUTION

a $P(X \text{ is within one standard deviation of the mean}) = P(7.764 \le X \le 12.236)$

$$= P(X = 8,\ 9,\ 10,\ 11 \text{ or } 12)$$

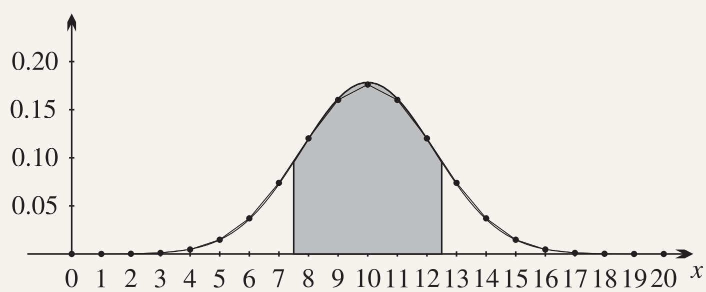
$$= {}^{20}C_8\left(\tfrac{1}{2}\right)^{20} + {}^{20}C_9\left(\tfrac{1}{2}\right)^{20} + {}^{20}C_{10}\left(\tfrac{1}{2}\right)^{20} + {}^{20}C_{11}\left(\tfrac{1}{2}\right)^{20} + {}^{20}C_{12}\left(\tfrac{1}{2}\right)^{20}$$

$$= \frac{{}^{20}C_8 + {}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11} + {}^{20}C_{12}}{2^{20}}$$

$$= \frac{772\,616}{1024 \times 1024}$$

$$\doteqdot 0.737.$$

b We are approximating the binomial distribution by the normal distribution $N(10, 5)$, so we treat the discrete random X as if it were a continuous normal variable, and calculate $P(7.5 \le X \le 12.5)$.



Using z-scores, the two limits of integration are

$$z = \frac{x - \mu}{\sigma} = \frac{12.5 - 10}{\sqrt{5}} = 1.118, \qquad z = \frac{x - \mu}{\sigma} = \frac{7.5 - 10}{\sqrt{5}} = -1.118,$$

so $P(X \text{ is within one standard deviation of the mean}) \doteqdot P(-1.118 \le Z \le 1.118)$

$$\doteqdot 0.736.$$

The results in parts **a** and **b** are in excellent agreement.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

5 NORMAL APPROXIMATION TO A BINOMIAL DISTRIBUTION

- A binomial distribution can be approximated by the normal distribution with the same mean and standard deviation.
- Thus the binomial distribution $B(n, p)$ or $\text{Bin}(n, p)$ can be approximated by the normal distribution $N(np, npq)$, where $q = 1 - p$.
- $N(\mu, \sigma^2)$ means the normal distribution with mean μ and variance σ^2.
- When approximating, we treat the discrete binomial variable X as if it were a continuous normal variable.
- For small values of n, apply the *continuity correction*. This means integrating between half-intervals, corresponding to the boundaries of the cumulative frequency histogram. For example, to approximate $P(X = 8, 9, 10, 11 \text{ or } 12)$, we treat X as a continuous normal variable and find $P(7.5 \le X \le 12.5)$.

The continuity correction for large numbers

In the example in the introduction to this chapter, the numbers are large enough so that the continuity correction is negligible, as the next worked example shows.

Example 13 17C

Compute $P(3300 \le X \le 3400)$ in the example in the introduction:

a ignoring the continuity correction,

b using the continuity correction.

Then comment on the results of the calculations.

SOLUTION

Here $n = 10\,000$ and $p = \frac{1}{3}$, so $\mu = 3333\frac{1}{3}$ and $\sigma^2 = 2222\frac{2}{9}$ and $\sigma \doteqdot 47.1416$.
We will treat the discrete random variable X as if it were a continuous normal variable with the same mean $3333\frac{1}{3}$ and standard deviation 47.1416.

a Without the continuity correction, we are finding $P(3300 \le X \le 3400)$.
The z scores are -0.7071 and 1.4142,
so
$$P(3300 \le X \le 3400) \doteqdot P(-0.7071 \le Z \le 1.4142)$$
$$\doteqdot 0.6816.$$

b With the continuity correction, we are finding $P(3299.5 \le X \le 3400.5)$.
The z scores are -0.7177 and 1.4248,
so
$$P(3299.5 \le X \le 3400.5) \doteqdot P(-0.7177 \le Z \le 1.4248)$$
$$\doteqdot 0.6859.$$

No pollster would ever need the third decimal place of the probabilities, so there is no problem whatsoever ignoring the continuity correction.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the normal approximation to calculate a binomial coefficient

One of the consequences of the normal approximation to the binomial is that for large parameters, an individual binomial coefficient can be approximated. These calculations demonstrate very well how vital the continuous correction can be.

Example 14 17C

Approximate ${}^{100}C_{53}$ using the normal approximation to the binomial distribution $B\left(100, \frac{1}{2}\right)$.
Check how good the approximation is.

SOLUTION

We know that ${}^{100}C_{53} \times \left(\frac{1}{2}\right)^{100} = P(X = 53)$.
We treat the discrete binomial variable X as if it were a continuous normal variable,

$$ {}^{100}C_{53} \times \left(\tfrac{1}{2}\right)^{100} = P(52.5 \le X \le 53.5). $$

The binomial distribution has mean $\mu = np = 50$ and SD $\sigma = \sqrt{npq} = \sqrt{25} = 5$.
So also does the normal approximation, and the limits have z-scores 0.5 and 0.7.

$$ \begin{aligned} \text{Hence} \quad {}^{100}C_{53} \times \left(\tfrac{1}{2}\right)^{100} &\doteqdot P(0.5 \le Z \le 0.7) \\ &\doteqdot 0.066574 \\ {}^{100}C_{53} &\doteqdot 2^{100} \times 0.066574 \\ &\doteqdot 8.438 \times 10^{28}. \end{aligned} $$

This compares with ${}^{100}C_{53} \doteqdot 8.441 \times 10^{28}$ (all according to Excel).

Sampling with replacement — when is a survey a binomial experiment?

Suppose that I choose ten cards in succession *without replacement* from a pack of 52 cards and count the number of aces. This is not a binomial experiment because the probabilities of getting an ace change as each card is removed from the pack. If I *replace the card each time*, however, then I have a binomial experiment.

More generally, a survey that questions n people chosen at random from N people without replacement is not a binomial experiment. A survey that chooses each person independently of all previous choices, however, thus allowing the same person possibly to be questioned more than once, is a binomial experiment because the stages are identical and independent.

Now suppose that the pool of N people from which the sample is chosen is very large. This allows us to ignore these distinctions, because whether replacement is used or not, the probabilities will be virtually the same in all stages. This is certainly the case with a sample chosen from all Australians, but in a college of say 3000 people and a sample that is small relative to 3000, the effect is also almost negligible.

How good an approximation is the normal?

This depends on how much accuracy is needed. The diagrams below show that n should be larger for a skewed distribution, meaning that p is near 0 or 1, than when the distribution is reasonably symmetric, meaning that p is near 0.5. A common rule of thumb therefore is to require

$$ np > 5 \quad \text{and} \quad nq > 5, $$

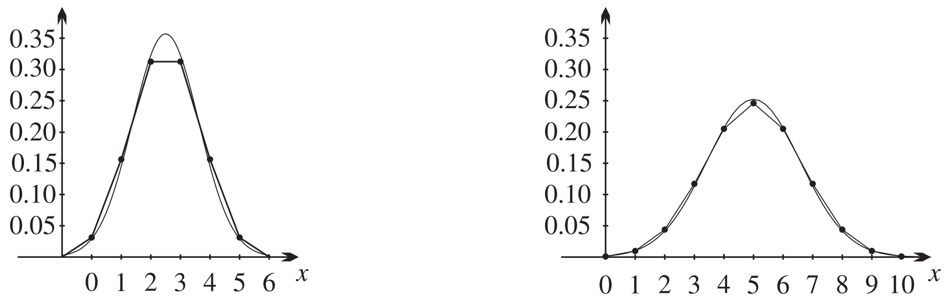
but numbers other than 5 are often stated, and there are other tests.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

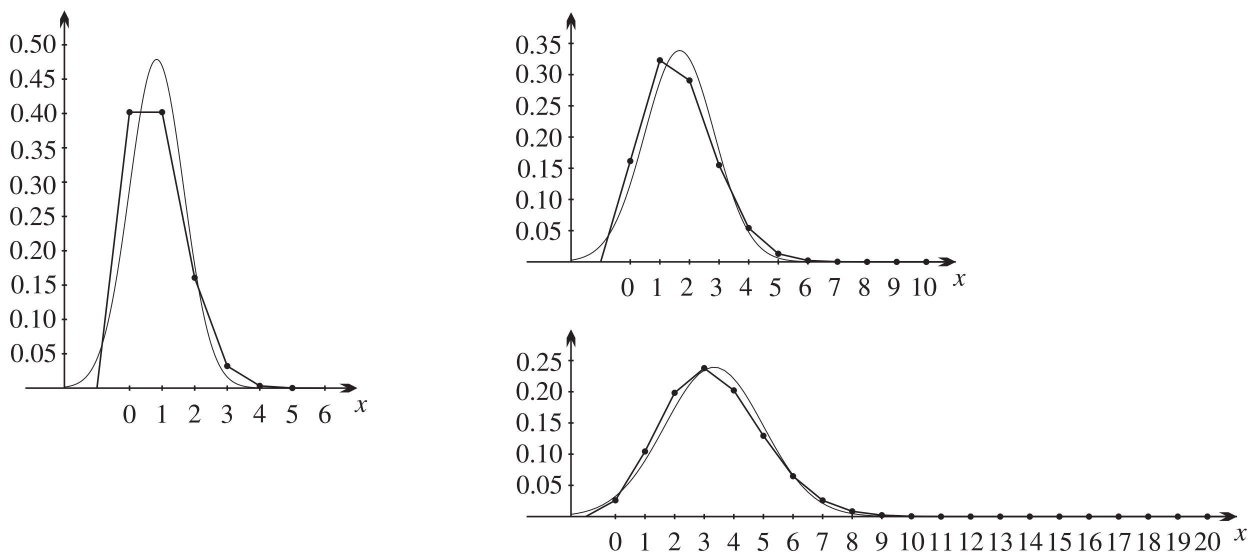
When $p = \frac{1}{2}$, we have already drawn the approximation when $n = 20$. Here are the ogive and the normal approximation, still for $p = \frac{1}{2}$, when $n = 5$ and $n = 10$.

a



When $p = \frac{1}{6}$, which arises when throwing a die and recording if it is a six, here are the graphs for $n = 5$, $n = 10$ and $n = 20$

b



6 A RULE OF THUMB FOR USING THE NORMAL APPROXIMATION

- For a fixed number n of Bernoulli trials, the normal approximation to a binomial distribution is less satisfactory when the distribution is skewed.
- For a fixed probability p, the normal approximation to a binomial distribution is more satisfactory when the number of trials is large.
- A common rule of thumb — one amongst many — for using the normal approximation to a binomial distribution is to require

$$np > 5 \quad \text{and} \quad nq > 5,$$

The normal approximation to a binomial distribution is a special case

We remarked at the start of Section 16G in the last chapter that the normal approximation to a binomial distribution was one of the first examples of a very general theorem in statistics called the *central limit theorem*. This would be a good occasion to read again the short account of the central limit theorem given there.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

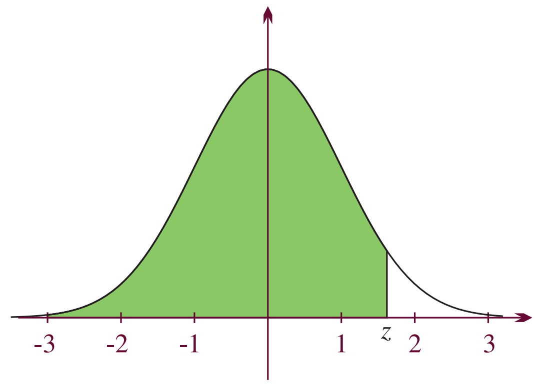
Exercise 17C

FOUNDATION

The standard normal probability distribution

The shaded area in the graph to the right represents a value of the *cumulative standard normal distribution function*

$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \Phi(t)\,dt.$$



The table below gives some further values of the probabilities $P(Z \le z) = \Phi(z)$, allowing two decimal places for z — after that, use interpolation. For example,

$$P(Z \le 1.627) = \Phi(1.627) = \int_{-\infty}^{1.627} \Phi(z)\,dz \doteqdot 0.9474 + 0.7(0.9484 - 0.9474) \doteqdot 0.9481.$$

	second decimal place									
z	+ .00	+ .01	+ .02	+ .03	+ .04	+ .05	+ .06	+ .07	+ .08	+ .09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

	second decimal place									
z	+ .00	+ .01	+ .02	+ .03	+ .04	+ .05	+ .06	+ .07	+ .08	+ .09
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Note: You may find in this exercise and the next that your answers differ slightly from the text, depending whether or not you interpolate, and whether you use the supplied tables or alternatives such as statistical calculators and spreadsheets that provide more accurate values.

1 A certain binomial distribution for the random variable X consists of 20 independent trials, each with probability of success $p = 0.3$.

a Write this binomial distribution in symbolic form.

b Calculate the probability of obtaining 9, 10 or 11 successes.

c Confirm that $np > 5$ and $nq > 5$, suggesting that a normal approximation to the binomial could be used to estimate this probability.

d Calculate the normal approximation by determining $P(8.5 \le X \le 11.5)$, treating X as approximately $N(\mu, \sigma^2)$, and using the normal tables at the start of this exercise.

e Find the percentage error in the normal approximation. Does the approximation seems fairly accurate for this value of n and p?

2 Repeat the steps of Question **1** for these cases.

a $n = 50$, $p = 0.5$, find $P(18 \le X \le 20)$,

b $n = 20$, $p = 0.4$, find $P(8 \le X \le 9)$,

c $n = 30$, $p = 0.3$, find $P(5 \le X \le 7)$,

d $n = 40$, $p = 0.2$, find $P(9 \le X \le 12)$,

e $n = 22$, $p = 0.6$, find $P(13 \le X \le 15)$,

f $n = 80$, $p = 0.1$, find $P(10 \le X \le 13)$,

g $n = 500$, $p = 0.25$, find $P(100 \le X \le 103)$,

h $n = 200$, $p = 0.9$, find $P(170 \le X \le 172)$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 A barrel contains 600 pink and 400 blue counters. At each stage of an experiment, the barrel is stirred well, then a counter is removed, its colour is noted, and it is returned to the barrel. This experiment is repeated 20 times.

a Explain why each stage of the experiment is a Bernoulli trial.

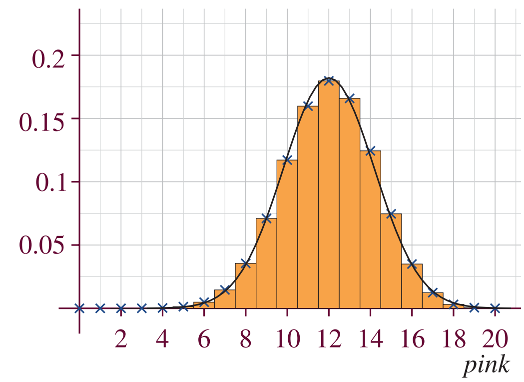
b Explain why the full experiment is binomial.

c Is it necessary to return the counter after each draw?

d Write down the probability of drawing a pink counter, and find the mean and standard deviation for this binomial distribution.

e Find the probability of drawing exactly 14 pink counters, correct to 3 decimal places.

f A student constructs a histogram for this distribution and overlays the normal distribution with the same mean and standard deviation. He notes the strong agreement, and to test this he uses his standard normal tables to calculate the area corresponding to drawing 14 pink counters.



i Explain why the required area for the normal distribution with random variable X is $P(13.5 < X < 14.5)$.

ii Hence find this area. Is it in strong agreement with your answer to part **e**?

4 In a college of 3000 pupils, 1320 are girls. A teacher selects a group of 15 pupils at random from the college rolls, without revealing the result, and asks her students to determine the probability that the group has more than 8 girls in it.

a Write down the probability p of selecting a girl from the population of 3000.

b Write down the mean and standard deviation of the binomial distribution obtained by selecting 15 pupils from the population.

c Use the exact binomial distribution to determine the probability of obtaining 9 or 10 girls.

d Is the sample size big enough to use the normal approximation to the binomial? Use the criterion $np > 5$ and $n(1 - p) > 5$.

e Use the normal approximation to the binomial to estimate the probability in part **c**.

f Find the percentage error in the estimation.

5 A commonly used rule of thumb states that the normal approximation to a binomial distribution will be reasonable if $np > 5$ and $n(1 - p) > 5$. This means that if p is further away from 0.5, the sample size needs to be bigger to get a reasonable approximation.

According to this rule, how big does the sample need to be if:

a $p = 0.5$, **b** $p = 0.25$, **c** $p = 0.125$, **d** $p = 0.01$,
e $p = 0.75$, **f** $p = 0.875$, **g** $p = 0.9$, **h** $p = 0.55$?

6 According to some estimates, eight per cent of males in the world are colour-blind. A representative random sample includes 854 people, and a statistician wishes to determine the probability that between 7% and 9% of the people in the sample are colour-blind.

a Give a reason why the researcher might want to use a normal approximation to the binomial to calculate this probability.

b Comment on the assumption that the sample is representative and random.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Calculate the required probability, using a normal approximation without a continuity correction.

d As a measure of the accuracy of ignoring any continuity correction, calculate the probability $P(76 < X < 76.5)$ associated with the boundary of 76 successes.

DEVELOPMENT

7 A horticulturalist is attempting to cross two species of flowers to strengthen certain characteristics. One of the plants has red flowers and the other's flowers are white, but the horticulturalist wishes to retain the strong red colour in the offspring. According to Mendel's theory of inheritance, there is a 25% chance that the flowers of the offspring will be red. The horticulturalist crosses 15 pairs of parent plants and notes the colour of their offspring plant. Find the probability that of the offspring:

a none will have red flowers,

b there will be at least one with red flowers,

c at least twenty per cent will have red flowers (use a normal approximation here).

8 Suppose that it is known that 45% of eighteen-year-olds in a particular city do not have a driver's licence. If a random sample of 20 eighteen-year-olds is taken in the city, what is the probability that more than half of them will not have a driver's licence?

9 Long-term studies show that 60% of the residents and visitors to Nashville Tennessee prefer Country music to Western. The local council provides Country music for those eating their lunch in the park to listen to. How confident can they be that in a group of thirty, more than twenty of them prefer Country? Comment on the assumption of independence in this question.

10 [Technology]

This question continues from Question **7** of Exercise 17B, and you will need to retrieve your saved spreadsheet. Do not overwrite this spreadsheet — make a copy and work only with the copy.

The student is continuing to explore the theoretical probabilities that occur if a coin is tossed 100 times, but this time he wonders what happens if $p \neq 0.5$. He previously drew a histogram of the data in column `B`, with column `A` as the x-axis labels (this may well be the default) and noted the visual similarity with a bell-shaped curve, like a normal distribution.

a Change the probability to $p = 0.1$ in cell `F2` and let the spreadsheet recalculate.

b What binomial situation could your new spreadsheet be modelling?

c Comment on the shape of the curve for different values of p. Try $p = 0.1, 0.4, 0.5, 0.7, 0.95$. In particular, comment on the centre and spread of the distribution for the different values of p.

11 [Technology simulation]

In this question we simulate the tossing of twenty coins. This binomial experiment is repeated 100 times to generate data approximating the distribution $B(20, 0.5)$. The authors have used a recent version of Excel for this exercise, and your spreadsheet program may use different commands for some of the required functions.

a Open a new spreadsheet.

b In cell `B2` enter the code `=RANDBETWEEN(0, 1)`. This will generate 1 if the toss is a success (head) and a 0 otherwise (failure).

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c Select the 20 cells `B2:U2`, and `Fill Right` using `Ctrl+R` or some other method. This represents the twenty tosses of the coin.

d In cell `W1` add the heading 'Heads', and in cell `W2` add the code `=SUM(B2:U2)`. This counts the number of heads in the 20 tosses.

e Select the 22 cells `B2:W2`, then extend this selection to a block from row `2` to row `101`, and `Fill Down` to cells `B101:W101` using `Ctrl+D` or some other method. This represents the 100 repetitions of the experiment, 'toss the coin 20 times'.

f In cell `AB32` enter the formula `=AVERAGE(W2:W101)`, and in cell `AB33` enter the formula `=STDEV.P(W2:W101)`. Add suitable headings in the preceding cells. Confirm that the results for the mean and standard deviation agree with the values predicted by the formulae $\mu = np$ and $\sigma = \sqrt{npq}$.

g Press the `Calculate now` option (Excel Formulas tab, or try function key `F9` or `CTRL+ALT+SHIFT+F9`), and see the data updating as the spreadsheet tosses another 100 lots of twenty coins.

h Enter the following values and formula in cells `Z1:AB3`.

	···	Z	AA	AB
1	···	Num.heads	f_r	cf_r
2	···	0	=COUNTIF($W$2:$W$101,$Z2/100)	=AA2
3	···	=1+Z2	=COUNTIF($W$2:$W$101,$Z3/100)	=AA3+AB2

Select cells `Z3:AB3`, then extend this selection to a block from row `3` to row `22`, and `Fill Down` to cells `Z22:AB22`. We now have the results of the simulation experiment. Graph the results in cells `AA2-AA22`, using the entries in cells `Z2-Z22` as horizontal axis (category) labels in a bar graph. Be careful to adjust the gap between the bars to zero so that it is a histogram.

i Add a second graph on separate axes using the cumulative frequency entries in column `AB`.

j We want to add the theoretical normal distribution to these results as a comparison. Add the following entries:

	···	AC	AD
1	···	theoretical f_r	theoretical cf_r
2	···	=BINOM.DIST(Z2,20,0.5,FALSE)	AC2
3	···	=BINOM.DIST(Z3,20,0.5,FALSE)	=AC3+AD2

k Again, select cells `AC3:AD3`, extend the selection to a block from row `3` to row `22`, and `Fill Down` to cells `AC22:AD22`. This gives us the theoretical results. Add the series in column `AC` to the first graph and the series in column `AD` to the second graph (the cumulative graph) — this is done by the `Select Data` option when you right-click on the graph area. Click each graph and format them as line graphs (polygons).

l Recalculate a few times and notice how good the agreement of the experiment and theoretical graphs is in each case. This is particularly true for the cumulative graph, which has a smoothing effect on the data.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

ENRICHMENT

12 Berry punnets distributed by a certain grower are exported in large batches. The receiver selects 10 punnets at random from each batch and tests whether any must be rejected because of rotten berries. The fraction p of punnets that are not of acceptable standard in the entire batch is unknown. The receiver needs a reasonable method of determining the quality of the batch without testing the entire batch, because the berries are destroyed in the process.

a One possible method is to reject the entire batch if any punnets in the sample are rejected.

i Suppose that $p = 0.05$, and determine the probability that the batch will be rejected.

ii Similarly determine and draw up a table showing the probability of rejection for $p = 0, 0.1, 0.2, 0.3, 0.4$. Include the case $p = 0.05$ in your table.

iii Plot a graph of p (on the horizontal axis) and the probability of rejecting the batch (on the vertical axis). This is called the *operating characteristic curve* (OC).

b Repeat part **a** if samples of 15 punnets are taken, but the batch is only rejected if two or more punnets are rejected. Include both plots on the same axes.

c Comment on whether part **a** or part **b** is the better method of ensuring the standard of the whole batch.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17D Sample proportions

So far, we have recorded the result of a binomial experiment as the number of successes. Often, however, it is more interesting to record instead the *proportion of the Bernoulli trials that were successes* — this is called the *sample proportion*.

Sample proportions have many purposes, but one important purpose is to estimate the probability p of each Bernoulli trial in the binomial experiment. For example, before every election, surveys estimate the proportion of Australians voting for the Working Together Party. If a survey of 2000 finds 700 WTP voters (the number of successes), then the the sample proportion is 0.35 (the proportion of successes), so the survey has estimated the WTP vote to be 35% of the population.

Population proportions

The probability p that a voter chosen at random will vote for the Working Together Party is a *population proportion*,

$$p = \frac{\text{number of Australians voting WTP}}{\text{number of Australian voters}}.$$

Many probabilities arise in this way, as we have often seen. Sample proportions apply to all binomial situations, but may be intuitively easier to understand if we think of the probability p arising as a population proportion, as in this voting example.

Sample proportions

Let $X \sim B(n, p)$ be a binomial variable consisting of n independent Bernoulli trials, each with probability p of success. The *sample proportion* $\hat{p}$ is the proportion of successes when the experiment is run. Thus *the sample proportion is a new random variable denoted by* $\hat{p}$ *and defined by*

$$\hat{p} = \frac{X}{n}.$$

For example, I survey 20 voters, and 7 say that they will vote WTP. This is a binomial experiment with 20 independent Bernoulli stages, and random variable X with 21 possible values $0, 1, 2, 3, \dots, 20$. I can record my result as

$$X = 7 \qquad \text{or as} \qquad \hat{p} = \frac{7}{20}.$$

The binomial random variable X has 21 possible values $0, 1, 2, 3, \dots, 20$, and the new random variable $\hat{p}$ has 21 possible values $0 = \frac{0}{20}, \frac{1}{20}, \frac{2}{20}, \dots, 1 = \frac{20}{20}$.

The sample proportion distribution — mean and variance

A sample proportion random variable $\hat{p}$ is not a binomial random variable X because its values are fractions between 0 and 1 rather than whole numbers from 0 to n. But it is closely related to the corresponding binomial variable — the probability of each value of $\hat{p}$ is the same as the probability of the corresponding value of X,

$$P\left(\hat{p} = \frac{x}{n}\right) = P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x},$$

so the various graphs of a sample proportion distribution can be obtained from the graphs of the corresponding binomial distributions by relabelling the axes and changing the numbers on the horizontal axis.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The mean and variance of $\hat{p}$ come directly from the mean and variance of X,

$$\begin{aligned} \mathrm{E}(\hat{p}) &= \mathrm{E}\left(\frac{X}{n}\right) \\ &= \frac{E(X)}{n} \\ &= \frac{np}{n} \\ &= p, \end{aligned} \qquad \begin{aligned} \mathrm{Var}(\hat{p}) &= \mathrm{Var}\left(\frac{X}{n}\right) \\ &= \frac{\mathrm{Var}(X)}{n^2} \\ &= \frac{npq}{n^2} \\ &= \frac{pq}{n}, \end{aligned} \qquad \begin{aligned} \mathrm{SD} &= \sqrt{\frac{npq}{n^2}} \\ &= \frac{\sqrt{npq}}{n}. \end{aligned}$$

The first formula $\mathrm{E}(\hat{p}) = p$ proves what we said in the introduction to this section — running a binomial experiment and recording the sample proportion $\hat{p}$ gives an estimate of the Bernoulli probability p.

7 THE SAMPLE PROPORTION $\hat{p}$

Suppose that a binomial experiment with random variable X consists of n independent Bernoulli trials, each with probability p.

- The *sample proportion* is the random variable $\hat{p} = \dfrac{X}{n}$.
- The values of $\hat{p}$ are the $n + 1$ values $0, 1, 2, \ldots, n$ of X divided by n. Thus the values of $\hat{p}$ are $0 = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, 1 = \frac{n}{n}$.
- The probability of each value of $\hat{p}$ equals the probability of the corresponding value of X,
$$P\left(\hat{p} = \frac{x}{n}\right) = P(X = x) = {}^{n}\mathrm{C}_x\, p^x q^{n-x}.$$
- The mean, variance and standard deviation of $\hat{p}$ are
$$\mathrm{E}(\hat{p}) = p, \qquad \mathrm{Var}(\hat{p}) = \frac{pq}{n}, \qquad \text{standard deviation} = \frac{\sqrt{npq}}{n}.$$
- The sample proportion $\hat{p}$ gives an estimate of the probability p.

Here is a simple example comparing a binomial distribution with the corresponding sample proportion distribution.

Example 15 — 17D

A coin is tossed six times. Let X be the binomial variable and $\hat{p}$ the corresponding sample proportion random variable.

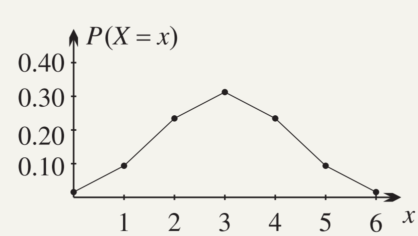
a Tabulate the probabilities of X, find the mean and variance, and draw the frequency polygon.

b Tabulate the probabilities of $\hat{p}$, find the mean and variance, and draw the frequency polygon.

SOLUTION

a Using the formula $P\,(x \text{ heads}) = {}^{6}\mathrm{C}_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{6-x}$,

x	0	1	2	3	4	5	6
prob	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$



For the binomial variable X,

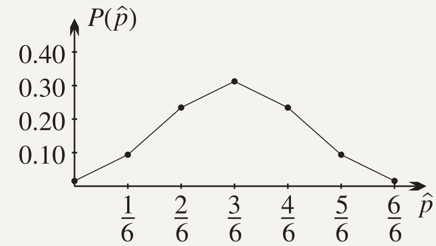
$$\mathrm{E}(X) = np = 3 \quad \text{and} \quad \mathrm{Var}(X) = npq = 1\tfrac{1}{2}.$$

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

b The sample proportion is $\hat{p} = \dfrac{X}{n}$, and the corresponding probabilities are the same, so

$\hat{p}$	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
prob	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$



For the sample proportion $\hat{p}$,

$$\mathrm{E}(\hat{p}) = p = \tfrac{1}{2} \quad \text{and} \quad \mathrm{Var}(\hat{p}) = \frac{pq}{n} = \tfrac{1}{24}.$$

Why all this machinery of sample proportions?

Marketers and public opinion pollsters are very keen to find sample proportions for the Australian population. But surveys are expensive, and pollsters need to know how effective they can be with the small samples that they can afford or have time to carry out.

These things are also crucial for health research and public policy. Estimating a quantity such as the monthly unemployment statistics, or the number of Australians who require disability assistance services, is dependent on sample survey methods.

Working with the sample proportion not only provides an immediate estimate of p, but it also provides a straightforward approach to assessing how accurate the estimate is likely to be.

For example, the next worked example calculates the probability that a pollster will get within 5% and 10% of the correct result when using a sample of 20 voters to estimate the proportion voting for the WTP.

Example 16 — 17D

A binomial experiment with 20 Bernoulli trials is being used to estimate the probability p of success in the Bernoulli experiment. If the actual value of p is $\frac{1}{3}$, find the probability that the experiment will yield an estimate within:

a 0.05 of the actual value,

b 0.1 of the actual value.

SOLUTION

Notice first that the estimate cannot be exactly $p = \frac{1}{3} \doteqdot 0.333$ because the size of the sample is 20, which is not a multiple of 3.

a To obtain an estimate within 0.05 of the actual value $p \doteqdot 0.333$, the sample proportion would need to be $\hat{p} = \frac{6}{20} = 0.30$ or $\hat{p} = \frac{7}{20} = 0.35$.

$$\begin{aligned} P\left(\hat{p} = \tfrac{6}{20} \text{ or } \hat{p} = \tfrac{7}{20}\right) &= {}^{20}C_6 \left(\tfrac{1}{3}\right)^6 \left(\tfrac{2}{3}\right)^{14} + {}^{20}C_7 \left(\tfrac{1}{3}\right)^7 \left(\tfrac{2}{3}\right)^{13} \\ &= 0.36425 \ldots \ \big(\text{save this value in the memory}\big). \end{aligned}$$

Hence the probability that the estimate is within 0.05 of p is about 0.364.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b To obtain an estimate within 0.1 of the actual value $p \doteqdot 0.333$, the sample proportion would need to be $\hat{p} = \frac{5}{20} = 0.25$ or $\frac{6}{20}$ or $\frac{7}{20}$ or $\frac{8}{20} = 0.40$.

$$P\left(\hat{p} = \tfrac{5}{20} \text{ or } \hat{p} = \tfrac{8}{20}\right) = {}^{20}C_5 \left(\tfrac{1}{3}\right)^5 \left(\tfrac{2}{3}\right)^{15} + {}^{20}C_8 \left(\tfrac{1}{3}\right)^8 \left(\tfrac{2}{3}\right)^{12}$$
$$= 0.29368\ldots,$$

Adding the answer to part **a**, the probability that the estimate is within 0.1 of the actual value is about 0.658.

8 USING THE SAMPLE PROPORTION

The sample proportion $\hat{p}$ can often be used to estimate the probability of obtaining an acceptable estimate to the probability p of a Bernoulli experiment.

Using the normal approximation to the sample proportion

We discussed in Section 17C how to use normal approximations to a binomial distribution to obtain reasonable approximations to the binomial distribution far more quickly. The probabilities of the sample proportions are just the probabilities of the corresponding values of the binomial variable, so the same approximation methods can be used here. We cannot prove in this course that the normal distribution approximates $\hat{p}$ — it is another example of the central limit theorem discussed at the start of Section 16G in the last chapter.

The normal approximation to the sample proportion thus has the same mean p as the sample proportion distribution, and the same variance $\dfrac{pq}{n}$, so we can approximate it using the normal distribution $N\left(p, \dfrac{pq}{n}\right)$.

As an example, let us increase the numbers in the previous worked example back up to realistic levels, and use the normal approximation to obtain the result. Political surveys before elections attempt to reduce the margin of error in their estimates to 1%–2%, that is, they attempt to estimate p correct to within 0.01–0.02.

Example 17 17D

A survey of 2000 Australian voters is being used to estimate the probability p that a random voter will vote for the Working Together Party. Assuming that the actual value of p is $\frac{1}{3}$, use the normal approximation to the sample proportion distribution to find the probability that the experiment will yield an estimate:

a within 0.01 of the actual value,

b within 0.02 of the actual value.

SOLUTION

The numbers in the sample are large enough for us to ignore continuity corrections.
The normal approximation to $\hat{p}$ has

$$\text{mean } = \tfrac{1}{3} \doteqdot 0.3333 \qquad \text{and} \qquad \text{variance } = \frac{pq}{n} = \frac{2}{9} \times \frac{1}{2000} = \frac{1}{9000},$$

so taking the square root, the standard deviation is about 0.01054.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

a We need to find $P(0.3233 \le \hat{p} \le 0.3433)$, so we find z-scores.

For 0.3233, $z = \dfrac{-0.01}{0.01054}$ $\doteqdot -0.9487$. For 0.3433, $z = \dfrac{0.01}{0.01054}$ $\doteqdot 0.9487$.

Hence the probability that the estimate is within 0.01 of the actual value is about

$$P(-0.9487 \le Z \le 0.9487) \doteqdot 0.66$$

b We need to find $P(0.3133 \le \hat{p} \le 0.3533)$. The z-scores are now;

For 0.3233, $z = \dfrac{-0.02}{0.01054}$ $\doteqdot -1.8974$. For 0.3433, $z = \dfrac{0.02}{1.01054}$ $\doteqdot 1.8974$.

Hence the probability that the estimate is within 0.02 of the actual value is about

$$P(-1.8974 \le Z \le 1.8974) \doteqdot 0.94$$

Example 18 17D

Election surveys usually claim that their margin of error is about 2%, and the last worked example has addressed this. What are some other issues that pollsters need to take into account?

SOLUTION

- Can I be sure that the sample is unbiased?
- Are people answering honestly?
- Are opinions changing as the election approaches?
- What is to be done with people who refuse to answer or 'don't know'?
- Were there language or cultural problems that interfered with the interview?

The data, the sample proportion, and the normal approximation

The situation has now become quite complicated because we have:

1 values of the sample proportion distribution $\hat{p}$ obtained from surveys,
2 the sample proportion distribution, if we assume a value for the mean p,
3 the normal approximation to the sample proportion distribution, also if we assume a value for the mean p.

We conclude this chapter by drawing one more picture, using the dataset that we gained by simulation of a binomial experiment in worked Example 9 of Section 16B. We are interested in the distribution of the sample proportion $\hat{p}$, that is, the distribution of the estimate of the probability, and we will use the cumulative distributions in our comparisons.

Notice that the density function of the normal approximation to $\hat{p}$ is obtained from the density function of the approximation to X by stretching horizontally by a factor of $\dfrac{1}{n}$, and then stretching vertically by a factor of n so that the area under the curve remains 1. Therefore we can't superpose the graph of that normal density function on the graph of the frequency polygon because the heights don't match. Instead, we will compare the cumulative graphs to see the agreement.

ISBN 978-1-108-76630-2

Example 19 17D

In worked Example 9 of Section 17B, we ran 100 simulations of the binomial experiment of tossing 10 coins and recording the number of heads. Now imagine that the purpose of each simulation was to produce an estimate of the probability of tossing heads, that is, a value of the random variable $\hat{p}$. Note that this random variable $\hat{p}$ has the 11 values $0 = \frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \ldots, 1 = \frac{10}{10}$.

a From the data, produce a table showing the cumulative relative frequency of obtaining each value of $\hat{p}$, and find the mean and standard deviation.

b Use binomial probability to calculate the cumulative probabilities of obtaining the values of $\hat{p}$, and find the mean and standard deviation.

c Use the normal approximation to the sample proportion distribution to calculate the cumulative probabilities of values of $\hat{p}$, and find the mean and standard deviation. Apply the continuous correction, because $n = 10$ is small.

d On one graph, draw the cumulative relative frequency polygon of the data, the cumulative probability polygon of the distribution, and the CDF of the normal approximation.

SOLUTION

The calculations in parts **a** and **b** were done in worked Example 9, and we only need to divide the values through by 10.

a After running the experiment 100 times, each relative frequency is the estimated probability of obtaining each estimate of $\hat{p}$.

$\hat{p}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_r	0.00	0.02	0.06	0.15	0.2	0.23	0.17	0.13	0.01	0.03	0.00
cf_r	0.00	0.02	0.08	0.23	0.43	0.66	0.83	0.96	0.97	1.00	1.00

After calculation, $\hat{p}$ has mean 0.482 and standard deviation about 0.170.

b From previous calculations, $\mu = 0.5$ and $\sigma = \frac{1}{10}\sqrt{2.5} \doteqdot 0.158$.

$\hat{p}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$P(\hat{p})$	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001
Cmlve	0.001	0.011	0.055	0.172	0.377	0.623	0.828	0.945	0.989	0.999	1.000

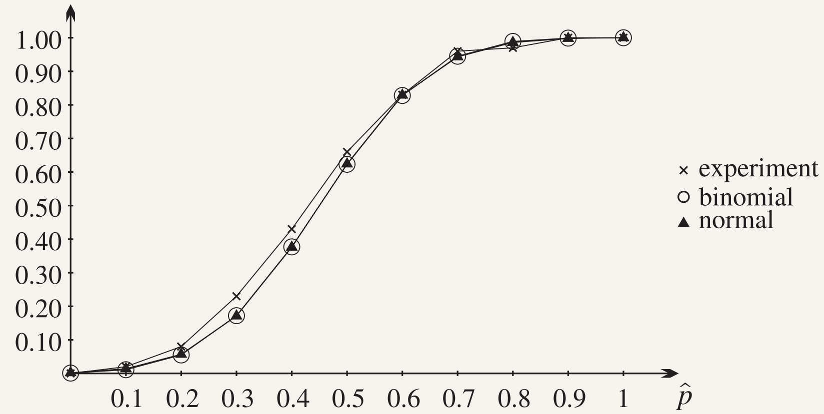
c The mean and standard deviation of the normal approximation are the same as for the sample proportion distribution. We need z-scores first, and we apply the continuity correction with 0.1 as the gaps in the values for $\hat{p}$. For example, for $\hat{p} = 0.7$ we find the z-score corresponding to $\hat{p} = 0.75$.

$\hat{p}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
z	−2.85	−2.21	−1.581	−0.949	−0.316	0.316	0.949	1.581	2.21	2.85	3.48
$\Phi(z)$	0.002	0.014	0.057	0.171	0.376	0.624	0.829	0.943	0.986	0.998	1.000

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

d



At the resolution of this graph, the theoretical probability distribution curve and the normal approximation to it are not separated. Look at the tables.

Exercise 17D

FOUNDATION

Note: In the previous exercise we used continuity corrections when we approximated a discrete binomial distribution by a continuous normal distribution. For large values of n this correction is less necessary. In this section, however, using the continuity correction when using sample proportions leads to much messier calculations, and the numbers are mostly large, so it will seldom be applied.

1 **a** A student tosses five coins and the results are: H T H T T. What is the sample proportion $\hat{p}$ of heads in this experiment?

b A student selects ten cards from a standard pack with replacement, and records the suit: ♡♣♠◇◇♠♡♠♠◇. What is the sample proportion $\hat{p}$ of spades in this sample?

c A manufacturer takes a sample of 12 items from their recent batch of gizmos, testing each item to see if it passes quality control (P) or not (F). The results were: P P P F F P P P P F P P. What is the proportion $\hat{p}$ of items that pass?

2 A single fair coin is tossed five times, and the number x of heads is recorded.

a Copy and complete the upper table to the right, using the binomial probability formula.

x	0	1	2	3	4	5
$P(X = x)$						

b Copy and complete the lower table to the right to convert part **a** to a table of probabilities of the sample proportions.

$\hat{p}$	
$P(\hat{p})$	

You will need to divide each score x by 5 to obtain the corresponding sample proportion.

c Calculate the mean of the second table.

d How would you interpret this mean?

3 Every Saturday for 20 weeks, a marketer surveyed five people chosen at random in a suburban shopping centre. The first question is, ‘Do you live in the suburb?’, and the weekly frequency of the number x of ‘Yes’ answers is given in the upper table.

x	0	1	2	3	4	5
f	1	1	3	2	6	7

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

a Copy the table to the right, and complete it to show a table in which the two rows are the sample proportions $\hat{p}$ and the relative frequencies f_r.

$\hat{p}$	
f_r	

b Calculate the mean of this second table.

c What is this mean an estimate of?

4 A coin is tossed 10 times, and a student is asked to calculate the probability that more than 75% of the coins will show heads.

a If more than 75% of the coins show heads, how many coins would this be?

b Use the binomial probability formula to determine the probability of obtaining more than 75% heads. (On this occasion the calculation is easily done without a normal approximation.)

5 A die is thrown 50 times. What is the probability that less than 9% of the time the result will be a six? Do this:

a using an exact binomial calculation, finding the probability of $0,\ 1,\ 2,\ 3$ or 4 sixes

b using a normal approximation to the sample proportion, and finding the probability $P(\hat{p} \le 0.09)$.

6 Information has been recorded about whether the 32 members of a class buy their lunch regularly at the school canteen:

1.	James	N	2.	Kate	N	3.	Xavier	N	4.	Jimmy	N
5.	Clyde	Y	6.	Bob	N	7.	Liam	N	8.	Agata	N
9.	Irene	N	10.	Aqila	Y	11.	Sonny	Y	12.	Andrea	N
13.	Magarida	N	14.	Terry	N	15.	Iman	N	16.	Lucie	Y
17.	Ping	N	18.	Maddy	Y	19.	Kamal	Y	20.	Xue	Y
21.	Odette	N	22.	Billy	N	23.	Chang	N	24.	Nahla	N
25.	Craig	Y	26.	Jerry	N	27.	Zahra	Y	28.	Jun	N
29.	Nara	Y	30.	Dakarai	Y	31.	Lerato	N	32.	Sahar	Y

In order to generate random samples, each student has been given a unique identifying number.

a Calculate the fraction of students who buy their lunch regularly at the canteen. This is called the *population proportion*.

b Kamal generates the five random numbers 12 15 3 30 17, thus generating the sample of 5 students Andrea, Iman, Xavier, Dakarai, Ping. What proportion of these five students buy their lunch regularly at the canteen? This is called a *sample proportion*.

c Copy the table below. Then use the following sets of five random numbers to generate 10 sample proportions for $n = 5$. Enter your results in the table — the first sample was dealt with in part **b** and is already included in the tally. (Notice that repetition is allowed — this is sampling with replacement.)

12	15	3	30	17	3	25	17	17	20	27	7	24	26	2	20	9	21	10	16
26	6	11	5	25	29	24	23	27	3	22	11	25	9	8	27	14	22	11	20
9	28	1	17	1	10	32	24	30	13										

$\hat{p}$	0	0.2	0.4	0.6	0.8	1.0
Tally		I				
Frequency						

d Draw a dot plot for the distribution obtained in part **c**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

DEVELOPMENT

7 In a local election, 20% of the people voted independent. What is the probability that if 500 people are chosen for a random survey, more than 22% of them voted independent? Use a normal approximation to the sample proportion.

8 A card is selected from a standard pack and it is then returned. This experiment is repeated 80 times. What is the probability that a hearts card turns up between 20% and 30% of the time, inclusive? Do this:

a By interpreting it as between 16 to 24 hearts inclusive and using a normal approximation with a continuity correction;

b Using sample proportion without any correction.

9 A farmer knows that a certain type of seed is 70% likely to germinate when planted. He plants 300 seeds at the start of the season. Use the normal approximation for the sample proportion to find the probability that:

a at least 65% will germinate,

b between 65% and 75% will germinate.

10 A medication causes a painful reaction in 5% of users.

a In a group of 100 people, find the probability that:

i no one reacts,

ii less than two per cent of the people react.

b In a larger study into patients' reactions to this medication, 1000 patients are given the medication (to which 5% are known to have a painful reaction). Researchers find that less than 3% of the patients in this study have a reaction.

i Use the normal approximation to the sample proportion to determine the probability of this happening by chance.

ii What should the researchers conclude?

11 A student is asked to redo Question 3 from Exercise 17C, but using sample proportions. A barrel contains 600 pink and 400 blue counters. At each stage of an experiment, the barrel is stirred well, then a counter is removed, its colour is noted, and it is returned to the barrel. This experiment is repeated 20 times.

The random variable for this sample proportion is $\hat{p} = \dfrac{X}{n}$, where X is the corresponding binomial random variable with $n = 20$ trials.

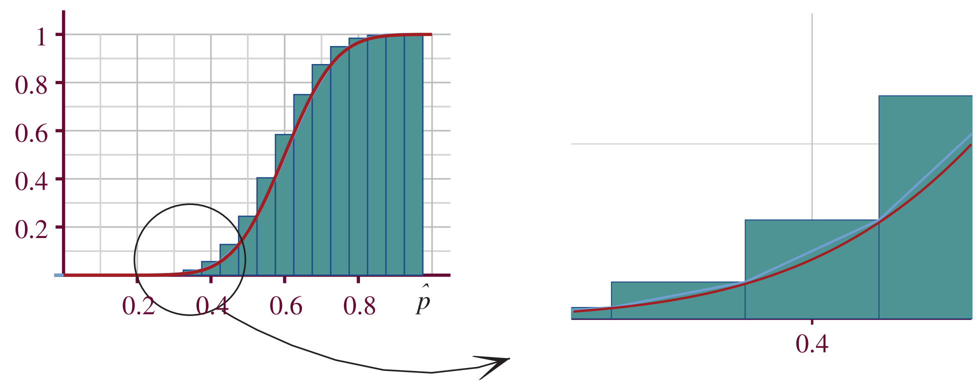
The student redraws his cumulative histogram to explore the idea of *sample proportion*. On the horizontal axis, he divides his results by 20 to record the fraction of pink counters occurring in the draw.

a What is the mean and standard deviation of the sample proportion random variable $\hat{p}$? Recall the formulae $\mu = p$ and $\sigma^2 = \dfrac{pq}{n}$, where $q = 1 - p$.

He overlays the cumulative normal curve with the same mean and standard deviation on top, and finds a very good agreement.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022



b Explain why the distribution of $\hat{p}$ with values between 0 and 1 is not a continuous distribution.

c To confirm his previous result, he decides to calculate the probability that $\hat{p} \le 0.4$, meaning that no more than 40% of the counters are pink. Assuming that the distribution is approximately that of a normal random variable $N(\mu, \sigma^2)$, calculate the required probability $P(\hat{p} \le 0.4)$. Do *not* attempt any continuity correction.

d Calculate the percentage difference in the exact answer 5.7% obtained by a direct binomial calculation. Is there good agreement?

e Calculate the probability that no more than 75% of the counters are pink, that is, $P(\hat{p} \le 0.75)$.

f Compare the percentage error in this second answer with the exact result 95% and explain why the approximation is better than parts **c**–**d**.

12 In this question we investigate the accuracy of a normal approximation to the sample proportion for various sample sizes n.

A coin is tossed repeatedly and the proportion of heads is recorded. The exact theoretical probability of obtaining at most 52% heads is calculated using the binomial distribution, and recorded in the second row of the table below for differing sample sizes.

n	1000	500	100	50	25
exact	0.9026	0.8262	0.6914	0.6641	0.6550
approx					
% error					

a Use a normal approximation for the sample proportion to fill in the second row of the table.

b In each case calculate the percentage error in the approximation for each of these samples. Record your results in the third row of the table.

c Comment on the accuracy of your approximations for various sample sizes n.

13 A manufacturer distributes tins of pineapple under a recognised brand name, and also as a generic supermarket no-name product. In a trial, 50 customers are given a tin of each and later asked to express a preference. The customers in the trial must choose one product as their favourite. Assuming that there is no difference between the two products, the probability of choosing the branded pineapple should be 0.5. It is found that more than 60% of customers prefer the branded pineapple. Calculate the probability of this using the normal approximation for the sample proportion, then comment.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14 Long-term trials have showed that 30% of patients with a certain disease respond to treatment by a company's drug. In further trials, 100 patients chosen at random from those with the disease are given a higher than usual dosage of the drug, and 40% respond positively. What is the probability that 40% or more could respond positively purely by chance?

15 The variance of the sample proportion $\hat{p}$ for a binomial distribution is $\sigma^2 = \dfrac{pq}{n}$. The variance measures the spread of the distribution of $\hat{p}$ around the population proportion p, thus it is customary to take n sufficiently large to ensure that σ is small.

a Assume that 70% of residents on a college campus are living at home, and that researchers want to choose a sufficiently large sample to mirror this statistic. How big will a sample need to be to ensure that the standard deviation of $\hat{p}$ is less than:

i 4% **ii** 3% **iii** 2% **iv** 1% **v** k%

b Repeat this question if a new survey finds that the number of residents living at home is 80%.

c Repeat for $p = 50\%$.

16 [Simulation]

a Ten coins are tossed.

i What is the probability of getting exactly six heads?

ii If this experiment is carried out 40 times, use your answer to part **i** to predict how many times you expect to get exactly 6 heads.

b Toss 10 coins and record the number of heads.

i Repeat this experiment 40 times, and copy and complete the table below, filling in the tally and the frequency.

Number of heads	0	1	2	3	4	5	6	7	8	9	10
Proportion of heads $\hat{p}$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.1
Expected frequency	0.0	0.4	1.8	4.7	8.2	9.8	8.2	4.7	1.8	0.4	0.0
Tally											
Frequency											

ii Does your answer to **a ii** agree with the expected frequency of six heads?

iii Draw a dot plot for $\hat{p}$ and note the shape of the curve. For a sufficiently large number of throws, we would expect to get a bell-shaped curve.

ENRICHMENT

17 Two dice are thrown and success is recorded if the sum is at least 9.

a Find the probability of success.

b Use the exact binomial distribution to find the probability of at most four successes on 20 throws.

c Use a normal approximation to the binomial to estimate the probability of at most four successes without continuity correction.

d Repeat part **c** with continuity correction.

e Use sample proportion to estimate the probability of at most 20% successes. Do not use continuity correction. Explain why your result agrees with part **c**.

f Use sample proportion, but this time use an estimate with continuity correction, $P\left(0 - \frac{1}{40} \le \hat{p} \le 0.2 + \frac{1}{40}\right)$. Explain why your result agrees with part **d**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 [Confidence intervals]

a Let $X \sim N(\mu, \sigma^2)$.

i Show that 95% of the data lie within 1.96σ of the mean μ, that is,

$$P(\mu - 1.96\sigma < X < \mu + 1.96\sigma) \doteqdot 95\%.$$

ii We interpret this to mean that a randomly chosen value X will lie within the interval $[\mu - 1.96\sigma, \mu + 1.96\sigma]$ ninety-five per cent of the time. Draw a diagram to show that for a given value x of the random variable, μ will lie within the interval $[x - 1.96\sigma, x + 1.96\sigma]$ ninety-five per cent of the time.

b In a sample of 100 school pupils, 67 surf the internet more than 7 hours a week. Thus we might estimate that 67% of all school pupils do the same, but how confident can we be of this estimate?

i Estimate the population standard deviation using this single sample, from the formula $\sigma^2 = p(1 - p)/n$, where we use the estimate $p \doteqdot \hat{p} = 67\%$.

ii Find the margin of error 1.96σ and find the 95% confidence interval, which is $[\hat{p} - 1.96\sigma, \hat{p} + 1.96\sigma]$, indicating that 95% of the time we take such a sample, the population proportion will fall within this interval.

iii Using the estimate $p \doteqdot 67\%$, how large does the sample size n need to be to reduce the margin of error to 1%?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 17 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 17 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 A marksman finds that on average he hits the target five times out of six. Assuming that successive shots are independent events, find the probability that in four shots:

a he has exactly three hits,

b he has exactly two misses.

2 Five out of six people surveyed think that Tasmania is the most beautiful state in Australia. What is the probability that in a group of 15 randomly selected people, at least 13 of them think that Tasmania is the most beautiful state in Australia?

3 There are ten questions in a multiple-choice test, and each question has five possible answers, only one of which is correct. What is the probability of answering exactly seven questions correctly by chance alone? Give your answer correct to three significant figures.

4 A card is selected from a pack, its suit is noted, and it is returned. How many times must this be done so that the probability of a heart is more than 95%?

5 An eight-sided die is inscribed with the digits 1–8.

a What is the probability of obtaining an 8 when the die is thrown?

b Six eight-sided dice are thrown. Construct a table for the distribution of the random variable X that counts the number of eights that occur. Record your results correct to 4 decimal places.

c A player needs to get exactly three eights in order to win. How often would you predict this to occur in 1000 throws of the six dice?

d Repeat part **c** if he needs a throw of three or more eights.

6 Are the following experiments Bernoulli trials? If so, state the probability of success p and failure q.

a A coin is tossed, and it is noted if the result is heads or tails.

b Two dice are thrown, and the player wins if the sum is more than 10.

c Tests show that 4 out of every 1000 items pass quality control. Consider the random variable 'number of passes' where an item is selected at random from the manufacturing process.

d A card is drawn from a pack, and its suit is noted.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Review

7 The binomial distribution $B(n, p)$ consists of n independent Bernoulli trials. Find the mean, variance and standard deviation for each distribution.

a $B(20, 0.2)$ **b** $B(70, 0.5)$ **c** $B(6, 0.8)$

d $B(120, 0.4)$ **e** $B(300, 0.1)$ **f** $B(5, 0.25)$

8 A company manufactures mobile phone cases using a mixture of machinery and traditional techniques. Data shows that the probability that a random case will fail quality control is 5%. An inspector selects a random batch of 60 cases from the warehouse. Let X be the binomial random variable of the number of cases that do not pass inspection.

a What is the mean, variance and standard deviation for this distribution?

b Find the probability that the number of cases that fail to pass lies within one standard deviation of the mean.

c New company standards insist that the number of failures in the batch must be no more that one standard deviation above the mean. Batches that fail to meet this standard are rejected. What is the probability of this?

d Due to the new regulations and the number of rejected batches, the company improves its manufacturing process so that the new experimental probability of failure is reduced to 2%. Repeat part **c** to find the new probability that a batch will be rejected.

9 A coin is tossed 80 times.

a Use the exact binomial formula to find the probability of 38, 39 or 40 heads.

b What is the mean and standard deviation for this binomial distribution?

c Calculate np and nq, and state whether this is a situation where a normal approximation may be used.

d In your own words, explain why we calculate $P(37.5 \le X \le 40.5)$ rather than $P(38 \le X \le 40)$.

e Find the probability of 38, 39 or 40 heads using a normal approximation.

f What is the percentage error in this normal approximation?

g Clearly in the example above, there was no need to use an approximation, because the probability could be calculated directly. Calculate now the probability of at least 50 heads using a normal approximation (but do not estimate the percentage error).

10 The sum of two dice is recorded.

a Find the probability that the sum is at least 10.

b Use a normal approximation to find the probability that the sum is at least 10 in more than 14 out of 80 throws.

11 A sample of 100 voters are asked whether they intend to vote for the Working Together Party at the next election. Thirty-five respond in the affirmative.

a What is the corresponding sample proportion?

b If a further sample is taken, would you expect the same number of respondents to indicate yes?

c What type of distribution will result if further such samples are taken and the sample proportions are recorded?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 A bag contains 3 red balls and 2 white balls. Five balls are selected in turn, with replacement, and the number of red balls is recorded.

a Construct a table showing the theoretical sample proportions of red balls that are selected.

b Find the probability that the proportion of red balls is less than:

i 40%, **ii** 50%.

c Find the mean, variance and standard deviation for the random variable $\hat{p}$ tabulated in part **a**.

13 A fair die is to be thrown 500 times.

a Find the mean and standard deviation for the sample proportion of sixes in the theoretical distribution for this experiment.

b In one sample of 500 throws, the number of sixes was 70. How many standard deviations is this result below the mean?

14 Long-term records show that the percentage of male babies born in a large hospital is 53%. A study is carried out on the effect of a high potassium diet (white beans, salmon, avocados, almonds, apples and mushrooms) on increasing the probability that the baby will be male. In the group of 653 births under this study, with the mother following this diet, more than 54% were male. What is the probability of this happening by chance? Use a normal approximation for the sample proportion with no continuity correction.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Answers

Answers are not provided for certain questions of the type 'show that' or 'prove that'. Please see worked solutions in these cases for a model.

Chapter 1

Exercise 1A

1 a 850, 1000, 1150, 1300, 1450, 1600, 1750, 1900, 2050, 2200, 2350, 2500, . . .
b 9 months

2 a 36, 46, 56, 66 **b** 24, 48, 96, 192
c 26, 22, 18, 14 **d** 3, $1\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{8}$
e 1, −1, 1, −1 **f** 16, 25, 36, 49
g $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$ **h** −2, 1 $-\frac{1}{2}$, $\frac{1}{4}$

3 a 3, 8, 13, 18 **b** 5, 25, 125, 625
c 4, 2, 0, −2 **d** 70, 700, 7000, 70 000
e 1, 8, 27, 64 **f** 2, 6, 12, 20
g −1, 1, 1, −1 **h** −3, 9, −27, 81

4 a 11, 61, 111, 161 **b** 15, 12, 9, 6
c 5, 10, 20, 40 **d** −100, −20, −4, $-\frac{4}{5}$

5 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62
a 5 **b** 4 **c** 52 **d** 7th term
e Yes; 17th term.
f No; they all end in 2 or 7.

6 $\frac{3}{4}$, $1\frac{1}{2}$, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536
a 10 **b** 3
c 384 **d** 9th term
e Yes; 8th term. **f** No

7 a 13, 14, 15, 16, 17. Add 1.
b 9, 14, 19, 24, 29. Add 5.
c 10, 5, 0, −5, −10. Subtract 5.
d 6, 12, 24, 48, 96. Multiply by 2.
e −7, 7, −7, 7, −7. Multiply by −1.
f 40, 20, 10, 5, $2\frac{1}{2}$. Divide by 2.

8 a $40 = T_{13}$
c $100 = T_{33}$, 200 is not a term, $1000 = T_{333}$.

9 a $44 = T_5$, 200 and 306 are not terms.
b 40 is not a term, $72 = T_6$, $200 = T_{10}$.
c $8 = T_3$, 96 is not a term, $128 = T_7$.

10 a The 9 terms T_1 to T_9 are less than 100.
b $T_6 = 64$

11 a 52 **b** $T_{21} = 103$

12 a 5, 17, 29, 41 **b** 12, 2, −8, −18
c 20, 10, 5, $2\frac{1}{2}$ **d** −1, −1, 1, −1

13 a $T_n = T_{n-1} + 5$ **b** $T_n = 2T_{n-1}$
c $T_n = T_{n-1} - 7$ **d** $T_n = -T_{n-1}$

14 a 1, 0, −1, 0, T_n where n is even.
b 0, −1, 0, 1, T_n where n is odd.
c −1, 1, −1, 1. No terms are zero.
d 0, 0, 0, 0. All terms are zero.

15 a $28 = T_7$, $70 = T_{10}$ **b** 5 terms

16 a $1\dfrac{1}{2} = T_4$, $96 = T_{10}$ **b** $T_7 = 12$

17 a $y = 10x - 4$ **b** $y = 2^{x-1} \times 3$
c $y = 42 - 4x$ **d** $y = 48 \times 2^{-x}$
e Here $T_n = (-1)^n$, but there is no curve and no real-valued function.
f $y = x^2$ **g** $y = \dfrac{x}{x + 1}$
h Here $T_n = (-2)^{5-n}$, but there is no curve and no real-valued function.

18 a $\frac{4}{5}$, $\frac{n}{n + 1}$ **b** $\frac{1}{30} = T_5$

19 a $0.9 = T_{10}$, $0.99 = T_{100}$
b $n^2 : (n^2 - 1)$ **c** $\dfrac{1}{n}$

20 a 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .
b 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, . . .
c The sum of two odd integers is even, and the sum of an even and an odd integer is odd.
d The first is 2, 4, 6, 10, 16, . . . , which is $2F_{n+1}$. The second is 0, 2, 2, 4, 6, . . . , which is $2F_{n-1}$.

22 a The 20th number is 10, and −20 is the 41st number on the list.
b Start by writing down the successive diagonals 1, 2, $\frac{1}{2}$, 3, $\frac{2}{2}$, $\frac{1}{3}$, 4, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, . . . Then remove every fraction that can be cancelled because it has previously been listed.
c The number x is not on the list because it differs from the nth number on the list at the nth decimal place.

Exercise 1B

1 a 18, 23, 28 **b** 5, −5, −15 **c** 9, $10\frac{1}{2}$, 12

2 a 3, 5, 7, 9 **b** 7, 3, −1, −5
c 30, 19, 8, −3 **d** −9, −5, −1, 3
e $3\frac{1}{2}$, $1\frac{1}{2}$, $-\frac{1}{2}$, $-2\frac{1}{2}$ **f** 0.9, 1.6, 2.3, 3

3 a AP: $a = 3, d = 4$ **b** AP: $a = 11, d = -4$
c AP: $a = 23, d = 11$ **d** AP: $a = -12, d = 5$
e not an AP **f** not an AP
g AP: $a = 8, d = -10$ **h** AP: $a = -17, d = 17$
i AP: $a = 10, d = -2\frac{1}{2}$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 a 67 **b** -55 **c** $50\frac{1}{2}$

5 a $T_n = 4n - 3$

b $T_n = 107 - 7n$

c $T_n = -19 + 6n$

6 a $a = 6 \; d = 10$

b 86, 206, 996

c $T_n = 10n - 4$

7 a $d = 3, T_n = 5 + 3n$

b $d = -6, T_n = 27 - 6n$

c not an AP

d $d = 4, T_n = 4n - 7$

e $d = 1\frac{1}{4}, T_n = \frac{1}{4}(2 + 5n)$

f $d = -17, T_n = 29 - 17n$

g $d = \sqrt{2}, T_n = n\sqrt{2}$

h not an AP

i $d = 3\frac{1}{2}, T_n = \frac{1}{2}(7n - 12) = \frac{7}{2}n - 6$

8 a $T_n = 170 - 5n$ **b** 26 terms

c $T_{35} = -5$

9 a $T_n = 23 - 3n, T_8 = -1$

b $T_n = 85 - 3n, T_{29} = -2$

c $T_n = 25 - \frac{1}{2}n, T_{51} = -\frac{1}{2}$

10 a 11 terms **b** 34 terms **c** 16 terms

d 13 terms **e** 9 terms **f** 667 terms

11 a 11, 15, 19, 23, $a = 11, d = 4$

b $T_{50} + T_{25} = 314, T_{50} - T_{25} = 100$

d $815 = T_{202}$

e $T_{248} = 999, T_{249} = 1003$

f $T_{49} = 203, \ldots, T_{73} = 299$ lie between 200 and 300, making 25 terms.

12 a i $T_n = 8n$

ii $T_{63} = 504, T_{106} = 848$

iii 44 terms

b $T_{91} = 1001, T_{181} = 1991$, 91 terms

c $T_{115} = 805, T_{285} = 1995$, 171 terms

13 a $d = 3$; 7, 10, 13, 16

b $d = 8; T_{20} = 180$

c $d = -2; T_{100} = -166$

14 a \$500, \$800, \$1100, \$1400, . . .

b \$4700

c cost $= 200 + 300n$

d 32

15 a 180, 200, 220, . . . **b** 400 km

c length $= 160 + 20n$ **d** 19 months

16 a 2120, 2240, 2360, 2480

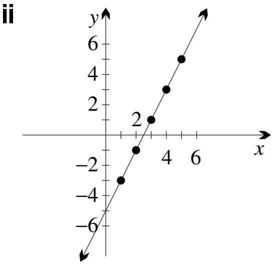
b $A_n = 2000 + 120n, A_{12} = 3440$

c 34 years

17 a 9, 6, 3, 0, -3, . . . and $T_n = 12 - 3n$

b i $T_n = 2n - 5, f(x) = 2x - 5$

ii



18 a $d = 4, x = 1$ **b** $d = 6x, x = \frac{1}{3}$

19 a $d = \log_3 2, T_n = n\log_3 2$

b $d = -\log_a 3, T_n = \log_a 2 + (4 - n)\log_a 3$

c $d = x + 4y, T_n = nx + (4n - 7)y$

d $d = -4 + 7\sqrt{5}, T_n = 9 - 4n + (7n - 13)\sqrt{5}$

e $d = -1.88, T_n = 3.24 - 1.88n$

f $d = -\log_a x, T_n = \log_a 3 + (3 - n)\log_a x$

20 The 13 terms $T_{28} = 19, \ldots, T_{40} = -17$ have squares less than 400.

21 a $a = m + b, d = m$

b gradient $= d$, y-intercept $= a - d$

22 a $a = \lambda a_1 + \mu a_2, d = \lambda d_1 + \mu d_2$

b $A(1, 0)$ is 1, 1, 1 , . . ., $A(0, 1)$ is 0, 1, 2 . . ., $A(a, d) = aA(1, 0) + dA(0, 1)$.

Exercise 1C

1 a 8, 16, 32 **b** 3, 1, $\frac{1}{3}$

c $-56, -112, -224$ **d** $-20, -4, -\frac{4}{5}$

e $-24, 48, -96$ **f** $200, -400, 800$

g $-5, 5, -5$ **h** $1, -\frac{1}{10}, \frac{1}{100}$

i 40, 400, 4000

2 a 12, 24, 48, 96 **b** $5, -10, 20, -40$

c 18, 6, 2, $\frac{2}{3}$ **d** $18, -6, 2, -\frac{2}{3}$

e $6, -3, 1\frac{1}{2}, -\frac{3}{4}$ **f** $-7, 7, -7, 7$

3 a GP: $a = 4, r = 2$

b GP: $a = 16, r = \frac{1}{2}$

c not a GP

d GP: $a = -1000, r = \frac{1}{10}$

e GP: $a = -80, r = -\frac{1}{2}$

f GP: $a = 29, r = 1$

g not a GP

h GP: $a = -14, r = -1$

i GP: $a = 6, r = \frac{1}{6}$

4 a 40 **b** $\frac{3}{10}$ **c** -56

d -8 **e** -88 **f** 120

5 a 3^{n-1} **b** $5 \times 7^{n-1}$ **c** $8 \times \left(-\frac{1}{3}\right)^{n-1}$

6 a $a = 7, r = 2$

b $T_6 = 224, T_{50} = 7 \times 2^{49}$

c $T_n = 7 \times 2^{n-1}$

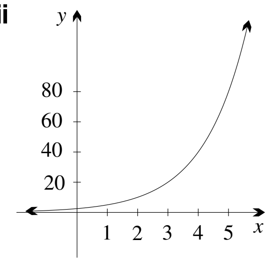
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 a $a = 10, r = -3$
b $T_6 = -2430, T_{25} = 10 \times (-3)^{24} = 10 \times 3^{24}$
c $T_n = 10 \times (-3)^{n-1}$
8 a $T_n = 10 \times 2^{n-1}, T_6 = 320$
b $T_n = 180 \times \left(\frac{1}{3}\right)^{n-1}, T_6 = \frac{20}{27}$
c not a GP
d not a GP
e $T_n = \frac{3}{4} \times 4^{n-1}, T_6 = 768$
f $T_n = -48 \times \left(\frac{1}{2}\right)^{n-1}, T_6 = -1\frac{1}{2}$
9 a $r = -1, T_n = (-1)^{n-1}, T_6 = -1$
b $r = -2, T_n = -2 \times (-2)^{n-1} = (-2)^n, T_6 = 64$
c $r = -3, T_n = -8 \times (-3)^{n-1}, T_6 = 1944$
d $r = -\frac{1}{2}, T_n = 60 \times \left(-\frac{1}{2}\right)^{n-1}, T_6 = -\frac{15}{8}$
e $r = -\frac{1}{2}, T_n = -1024 \times \left(-\frac{1}{2}\right)^{n-1}, T_6 = 32$
f $r = -6, T_n = \frac{1}{16} \times (-6)^{n-1}, T_6 = -486$
10 a $T_n = 2^{n-1}$, 7 terms
b $T_n = -3^{n-1}$, 5 terms
c $T_n = 8 \times 5^{n-1}$, 7 terms
d $T_n = 7 \times 2^{n-1}$, 6 terms
e $T_n = 2 \times 7^{n-1}$, 5 terms
f $T_n = 5^{n-3}$, 7 terms
11 a $r = 2$; 25, 50, 100, 200, 400
b i $r = 2$ **ii** $r = 0.1$ or -0.1
iii $r = -\frac{3}{2}$ **iv** $r = \sqrt{2}$ or $-\sqrt{2}$
12 a 50, 100, 200, 400, 800, 1600, $a = 50, r = 2$
b $6400 = T_8$
c $T_{50} \times T_{25} = 5^4 \times 2^{75}, T_{50} \div T_{25} = 2^{25}$
e The six terms $T_6 = 1600, \ldots, T_{11} = 51\,200$ lie between 1000 and 100 000.
13 The successive thicknesses form a GP with 101 terms, and with $a = 0.1$ mm and $r = 2$. Hence thickness $= T_{101} = \frac{2^{100}}{10}$ mm $\doteqdot 1.27 \times 10^{23}$ km $\doteqdot 1.34 \times 10^{10}$ light years, which is close to the present estimate of the distance to the Big Bang.

14 a $P \times 1.07, P \times (1.07)^2, P \times (1.07)^3$
b $A_n = P \times (1.07)^n$
c 11 full years to double, 35 full years to increase tenfold.
15 a $W_1 = 20\,000 \times 0.8, W_2 = 20\,000 \times (0.8)^2, W_3 = 20\,000 \times (0.8)^3, W_n = 20\,000 \times (0.8)^n$
b 11 years
16 a $r = \sqrt{2}, T_n = \sqrt{6} \times \left(\sqrt{2}\right)^{n-1} = \sqrt{3} \times \left(\sqrt{2}\right)^n$
b $r = ax^2, T_n = a^n x^{2n-1}$
c $r = \frac{y}{x}, T_n = -x^{2-n} y^{n-2}$
17 a $T_n = 2x^n, x = 1$ or -1
b $T_n = x^{6-2n}, x = \frac{1}{3}$ or $-\frac{1}{3}$
c $T_n = 2^{-16} \times 2^{4n-4} x = 2^{4n-20} x, x = 6$

18 a $T_n = 2^{8-3n}$
19 a $\frac{4}{5}$, 4, 20, 100, 500, ... and $T_n = \frac{4}{25} \times 5^n$
b i $T_n = \frac{5}{2} \times 2^n, f(x) = \frac{5}{2} \times 2^x$
ii



20 a $a = kb, r = b$ **b** $f(x) = ar^{x-1}$
21 a $a = cb, r = b$ **b** $f(x) = \frac{a}{r} \times r^x$
22 a first term $= aA$, ratio $= rR$
b $W_n = (A + a) r^{n-1}$

Exercise 1D

1 a 11 **b** 23 **c** -31
d -8 **e** 12 **f** 10
2 a 6 or -6 **b** 12 or -12 **c** 30 or -30
d 14 or -14 **e** 5 **f** -16
3 a 10; 8 or -8 **b** 25; 7 or -7
c $20\frac{1}{2}$; 20 or -20 **d** $-12\frac{1}{2}$; 10 or -10
e -30; 2 **f** 0; 6
g -3; 1 **h** 24; -3
i 40; 45 **j** 84; -16
k $-5\frac{3}{4}$; -36 **l** -21; 7
4 a 7, 14, 21, 28, 35, 42
b 27, 18, 12, 8
c 48, $36\frac{3}{4}$, $25\frac{1}{2}$, $14\frac{1}{4}$, 3
d 48, 24, 12, 6, 3 or 48, -24, 12, -6, 3
5 a $d = 3, a = -9$
b $d = -9, a = 60$
c $d = 3\frac{1}{2}, a = -4\frac{1}{2}$
6 a $r = 2, a = 4$ $r = 4, a = \frac{1}{16}$
b $r = 3$ and $a = \frac{1}{9}$, or $r = -3$ and $a = -\frac{1}{9}$
c $r = \sqrt{2}$ and $a = \frac{3}{2}$, or $r = -\sqrt{2}$ and $a = \frac{3}{2}$
7 a $T_8 = 37$ **b** $T_2 = 59$ **c** $T_2 = \frac{3}{8}$
8 a $n = 13$ **b** $n = 8$
c $n = 11$ **d** $n = 8$
9 b $n = 19$ **c** $n = 29$
d $n = 66$ **e** 10 terms
f 37 terms
10 a $T_n = 98 \times \left(\frac{1}{7}\right)^{n-1}$, 10 terms
b $T_n = 25 \times \left(\frac{1}{5}\right)^{n-1} = \left(\frac{1}{5}\right)^{n-3}$, 11 terms
c $T_n = (0.9)^{n-1}$, 132 terms
11 a about 78% **b** 152 sheets

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

12 a $a = 28, d = -1$
b $a = \frac{1}{3}$ and $r = 3$, or $a = \frac{2}{3}$ and $r = -3$
c $T_6 = -2$
13 a $x = 10; 9, 17, 25$
b $x = -2; -2, -6, -10$
c $x = 2; -1, 5, 11$
d $x = -4; -14, -4, 6$
14 a $x = -\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$
b $x = 1; 1, 2, 4$ or $x = 6; -4, 2, -1$
15 a i $x = -48$ **ii** $x = 6$
b i $x = 0.100\,01$
ii $x = 0.002$ or $x = -0.002$
c i $x = 0.398$ **ii** $x = 20$
d i They can't form an AP. **ii** $x = 9$
e i $x = 2$ **ii** $x = 4$ or $x = 0$
f i $x = \sqrt{5}$ **ii** $x = 2$ or $x = -2$
g i $x = \frac{3}{2}\sqrt{2}$ **ii** $x = 2$ or $x = -2$
h i $x = 40$ **ii** 2^5 or -2^5
i i $x = 0$ **ii** They can't form a GP.
16 a $a = 6\frac{1}{4}$ and $b = 2\frac{1}{2}$, or $a = 4$ and $b = -2$
b $a = 1, b = 0$
18 c $r = 1, \frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1}{2} - \frac{1}{2}\sqrt{5}$
d 1, 2, 4, 8, . . .
19 a Squares can't be negative.
c 1, 1, 1 is an AP and a GP. 1, 5, 9 is an AP and 1, 3, 9 is a GP.
20 b $\frac{T_8}{T_1} = \left(\frac{1}{2}\right)^{\frac{7}{12}} \doteqdot 0.6674 \doteqdot \frac{2}{3}$
c $\frac{T_5}{T_1} = \left(\frac{1}{2}\right)^{\frac{4}{12}} \doteqdot 0.7937 \doteqdot \frac{4}{5}$
d $\frac{T_6}{T_1} = \left(\frac{1}{2}\right)^{\frac{5}{12}} \doteqdot 0.7491 \doteqdot \frac{3}{4}$,
$\frac{T_4}{T_1} = \left(\frac{1}{2}\right)^{\frac{3}{12}} \doteqdot 0.8409 \doteqdot \frac{5}{6}$
e $\frac{T_3}{T_1} = \left(\frac{1}{2}\right)^{\frac{2}{12}} \doteqdot 0.8908 \doteqdot \frac{8}{9}$,
$\frac{T_2}{T_1} = \left(\frac{1}{2}\right)^{\frac{1}{12}} \doteqdot 0.9439 \doteqdot \frac{17}{18}$
21 e $\angle OTM = 90°$ because it is an angle in a semi-circle, so OT is a tangent. Now use similar triangles to prove that $OT^2 = OA \times OB$.

Exercise 1E

1 a 24 **b** 80 **c** $3\frac{3}{4}$ **d** 20
2 a $-2, 3, -3$ **b** $120, 121, 121\frac{1}{3}$ **c** 60, 50, 30
d 0.1111, 0.111 11, 0.111 111
3 a S_n: 2, 7, 15, 26, 40, 57, 77
b S_n: 40, 78, 114, 148, 180, 210, 238
c S_n: $2, -2, 4, -4, 6, -6, 8$
d S_n: 7, 0, 7, 0, 7, 0, 7
4 a 42 **b** 75 **c** 15 **d** 174
e 100 **f** 63 **g** 117 **h** -1
i 0 **j** 404 **k** 7 **l** -7
5 c 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, . . .
6 a T_n: 1, 3, 5, 7, 9, 11, 13
b T_n: 2, 4, 8, 16, 32, 64, 128
c T_n: $-3, -5, -7, -9, -11, -13, -15$
d T_n: $8, -8, 8, -8, 8, -8, 8$
7 a T_n: 1, 1, 1, 2, 3, 5, 8, 13
b T_n: 3, 1, 3, 4, 7, 11, 18, 29
8 a 2, 8, 26, 80, 242
b 2, 6, 18, 54, 162
c $T_n = 2 \times 3^{n-1}$
9 a $T_n = 5 \times 2^n$
b $T_n = 16 \times 5^{n-1}$
c $T_n = 3 \times 4^{n-2}$
10 a $T_n = 6n$, 6, 12, 18
b $T_n = 6 - 2n$, 4, 2, 0
c $T_n = 4$, 4, 4, 4
d $T_n = 3n^2 - 3n + 1$, 1, 7, 19
e $T_n = 2 \times 3^{-n}, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}$,
f $T_n = -6 \times 7^{-n}, -\frac{6}{7}, -\frac{6}{49}, -\frac{6}{343}$
11 a $\sum_{n=1}^{40} n^3$ **b** $\sum_{n=1}^{40} \frac{1}{n}$ **c** $\sum_{n=1}^{20} (n+2)$ **d** $\sum_{n=1}^{12} 2^n$
e $\sum_{n=1}^{10} (-1)^n n$ **f** $\sum_{n=1}^{10} (-1)^{n+1} n$ or $\sum_{n=1}^{10} (-1)^{n-1} n$
12 a $T_1 = 8, T_n = 2n + 3$ for $n \geq 2$
b $T_1 = -7, T_n = 14 \times 3^{n-1}$ for $n \geq 2$
c $T_1 = 1, T_n = \dfrac{-1}{n(n-1)}$ for $n \geq 2$
d $T_n = 3n^2 - n + 1$ for $n \geq 1$
The formula holds for $n = 1$ when $S_0 = 0$.
13 a $T_1 = 2, T_n = 2^{n-1}$ for $n \geq 2$
b 2, 2, 4, 8, 16, . . .
c The derivative of e^x is the original function e^x. Remove the initial term 2 from the sequence in part **b**, and the successive differences are the original sequence.
14 b $T_1 = 1$ and $T_n = 3n^2 - 3n + 1$ for $n \geq 2$
c $U_1 = 1$ and $U_n = 6n$ for $n \geq 2$
d 1, 7, 19, 37, 61, 91, . . . and 1, 6, 12, 18, 24, 30, . . .
e The derivative of x^3 is the quadratic $3x^2$, and its derivative is the linear function $6x$. Taking successive differences once gives a quadratic, and taking them twice gives a linear function.
15 b 3 **c** 1000

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 1F

1 77

2 a $n = 100, 5050$ **b** $n = 50, 2500$ **c** $n = 50, 2550$ **d** $n = 100, 15\,150$ **e** $n = 50, 7500$ **f** $n = 9000, 49\,504\,500$

3 a 180 **b** 78 **c** -153 **d** -222

4 a 222 **b** -630 **c** 78 400 **d** 0 **e** 65 **f** 30

5 a 101 terms, 10 100 **b** 13 terms, 650 **c** 11 terms, 275 **d** 100 terms, 15 250 **e** 11 terms, 319 **f** 10 terms, $61\frac{2}{3}$

6 a 500 terms, 250 500 **b** 2001 terms, 4 002 000 **c** 3160 **d** 1440

7 a $S_n = \frac{1}{2}n(5 + 5n)$ **b** $S_n = \frac{1}{2}n(17 + 3n)$ **c** $S_n = n(1 + 2n)$ **d** $\frac{1}{2}n(5n - 23)$ **e** $S_n = \frac{1}{4}n(21 - n)$ **f** $\frac{1}{2}n\left(2 + n\sqrt{2} - 3\sqrt{2}\right)$

8 a $\frac{1}{2}n(n + 1)$ **b** n^2 **c** $\frac{3}{2}n(n + 1)$ **d** $100n^2$

9 a 450 legs. No creatures have the mean number of 5 legs. **b** 16 860 years **c** \$352 000

10 a $a = 598, \ell = 200, 79\,800$ **b** $a = 90, \ell = -90, 0$ **c** $a = -47, \ell = 70, 460$ **d** $a = 53, \ell = 153, 2163$

11 a 10 terms, $55 \log_a 2$ **b** 11 terms, 0 **c** 6 terms, $3(4 \log_b 3 - \log_b 2)$ **d** $15(\log_x 2 - \log_x 3)$

12 a $\ell = 22$ **b** $a = -7.1$ **c** $d = 11$ **d** $a = -3$

13 b i 16 terms **ii** more than 16 terms
c 5 terms or 11 terms
d $n = 18$ or $n = -2$, but n must be a positive integer.
e $n = 4, 5, 6, \ldots, 12$
f Solving $S_n > 256$ gives $(n - 8)^2 < 0$, which has no solutions.

14 a $S_n = n(43 - n)$, 43 terms **b** $S_n = \frac{3}{2}n(41 - n)$, 41 term **c** $S_n = 3n(n + 14)$, 3 terms **d** $\frac{1}{4}n(n + 9)$, 6 terms

15 a $n = 17, a = -32$ **b** $n = 11, a = 20$

16 a 20 rows, 29 logs on bottom row **b** $S_n = 5n^2$, 7 seconds **c** 11 trips, deposits are 1 km apart.

17 a $d = -2, a = 11, S_{10} = 20$ **b** $a = 9, d = -2, T_2 = 7$ **c** $d = -3, a = 28\frac{1}{2}, T_4 = 19\frac{1}{2}$

18 a 300 **c** 162

19 a $S_n = \frac{1}{2}n(n + 1)$
b i n ends in 4, 5, 9 or 0.
ii n has remainder 3 or 0 after division by 4.
c i $n = 28$ **ii** $n = 14$ **iii** $n = 12$ **iv** $n = 19$ **v** $n = 3$ **vi** $n = 11$ **vii** $n = 20$

Exercise 1G

1 728

2 2801 kits, cats, sacks, wives and man

3 a 1093 **b** 547

4 a $1023, 2^n - 1$ **b** $242, 3^n - 1$ **c** $-11\,111, -\frac{1}{9}\left(10^n - 1\right)$ **d** $-781, -\frac{1}{4}\left(5^n - 1\right)$ **e** $-341, \frac{1}{3}\left(1 - (-2)^n\right)$ **f** $122, \frac{1}{2}\left(1 - (-3)^n\right)$ **g** $-9091, -\frac{1}{11}\left(1 - (-10)^n\right)$ **h** $-521, -\frac{1}{6}\left(1 - (-5)^n\right)$

5 a $\frac{1023}{64}, 16\left(1 - \left(\frac{1}{2}\right)^n\right)$ **b** $\frac{364}{27}, \frac{27}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$ **c** $\frac{605}{9}, \frac{135}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$ **d** $\frac{211}{24}, \frac{4}{3}\left(\left(\frac{3}{2}\right)^n - 1\right)$ **e** $\frac{341}{64}, \frac{16}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$ **f** $\frac{182}{27}, \frac{27}{4}\left(1 - \left(-\frac{1}{3}\right)^n\right)$ **g** $-\frac{305}{9}, -\frac{135}{4}\left(1 - \left(-\frac{1}{3}\right)^n\right)$ **h** $\frac{55}{24}, \frac{4}{15}\left(1 - \left(-\frac{3}{2}\right)^n\right)$

6 a $5\left((1.2)^n - 1\right), 25.96$
b $20\left(1 - (0.95)^n\right), 8.025$
c $100\left((1.01)^n - 1\right), 10.46$
d $100\left(1 - (0.99)^n\right), 9.562$

7 a i 2^{63} **ii** $2^{64} - 1$
b 615 km^3

8 a $S_n = \left(\left(\sqrt{2}\right)^n - 1\right)\left(\sqrt{2} + 1\right)$, $S_{10} = 31\left(\sqrt{2} + 1\right)$
b $S_n = \frac{1}{2}\left(1 - \left(-\sqrt{5}\right)^n\right)\left(\sqrt{5} - 1\right)$, $S_{10} = -1562\left(\sqrt{5} - 1\right)$

9 a $a = 6, r = 2, 762$ **b** $a = 9, r = 3, 3276$ **c** $a = 12, r = \frac{1}{2}, \frac{765}{32}$

10 a $\frac{1}{8} + \frac{3}{4} + \frac{9}{2} + 27 + 162 = 194\frac{3}{8}$ or $\frac{1}{8} - \frac{3}{4} + \frac{9}{2} - 27 + 162 = 138\frac{7}{8}$
b $15\frac{3}{4}$ **c** 1562.496 **d** 7 **e** 640

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

11 a **i** 0.01172 tonnes **ii** 11.99 tonnes
b 4.9×10^{-3}g
c **i** $S_n = 10P(1.1^{10} - 1)$
ii \$56.47
12 a 34 010 and 26 491 **c** 3.30
13 b $n = 8$ **c** 14 terms
d $S_{14} = 114\,681$
14 a 41 powers of 3 **b** 42 terms
15 a 6 terms **b** 8 terms **c** 5 terms **d** 7 terms
16 a $\dfrac{n+1}{n}$ **b** $\dfrac{3}{2^n + 1}$
17 a $r = 2$ or $r = -2$ **c** $r = 3^{-\frac{1}{2}}$ or $-3^{-\frac{1}{2}}$
18 112
19 a **i** 2 097 151 **ii** $\frac{6560}{4374}$
b $r = 4$ and $n = 4$
c $n = 6$ and $\ell = -1215$
20 a $3 \times 3^n + 6 \times 2^n - 9$ **b** $2 \times 2^n + n^2 + 4n - 2$
c $a = 1, d = 3, b = 3,$
$S_n = \frac{3}{2}n^2 + \frac{5}{2}n - 6 + 6 \times 2^n$
21 c iii $2^2 - 1 = 3$ and 6, $2^3 - 1 = 7$ and 28,
$2^5 - 1 = 31$ and 496, $2^7 - 1 = 127$ and 8128,
$2^{13} - 1 = 8191$ and $33\,500\,336$
(Care: $2^{11} - 1 = 23 \times 89$ is not prime.)

Exercise 1H

1 a 18, 24, 26, $26\frac{2}{3}$, $26\frac{8}{9}$, $26\frac{26}{27}$
b $S_\infty = 27$
c $S_\infty - S_6 = 27 - 26\frac{26}{27} = \frac{1}{27}$
2 a 24, 12, 18, 15, $16\frac{1}{2}$, $15\frac{3}{4}$
b $S_\infty = 16$
c $S_\infty - S_6 = 16 - 15\frac{3}{4} = \frac{1}{4}$
3 a $a = 8, r = \frac{1}{2}, S_\infty = 16$
b $a = -4, r = \frac{1}{2}, S_\infty = -8$
c $a = 1, r = -\frac{1}{3}, S_\infty = \frac{3}{4}$
d $a = 36, r = -\frac{1}{3}, S_\infty = 27$
e $r = -\frac{1}{2}, r = -\frac{1}{2}, S_\infty = 40$
f $r = -\frac{1}{5}, r = -\frac{1}{5}, S_\infty = 50$
4 a $r = -\frac{1}{2}, S_\infty = \frac{2}{3}$
b $r = -\frac{3}{2}$, no limiting sum
c $r = \frac{1}{3}, S_\infty = 18$ **d** $r = \frac{1}{10}, S_\infty = 1111\frac{1}{9}$
e $r = -\frac{1}{5}, S_\infty = -\frac{5}{3}$ **f** $r = \frac{1}{5}, S_\infty = -\frac{5}{6}$
5 a The successive down-and-up distances form a GP with $a = 12$ and $r = \frac{1}{2}$.
b $S_\infty = 24$ metres
6 a T_n: 10, 10, 10, 10, 10, 10. S_n: 10, 20, 30, 40, 50, 60. $S_n \to \infty$ as $n \to \infty$.
b T_n: 10, −10, 10, −10, 10, −10.
S_n: 10, 0, 10, 0, 10, 0. S_n oscillates between 10 and 0 as $n \to \infty$.
c T_n: 10, 20, 40, 80, 160, 320.
S_n: 10, 30, 70, 150, 310, 630. $S_n \to \infty$ as $n \to \infty$.
d T_n: 10, −20, 40, −80, 160, −320.
S_n: 10, −10, 30, −50, 110, −210. S_n oscillates between larger and larger positive and negative numbers as $n \to \infty$.
7 a $S_\infty - S_4 = 160 - 150 = 10$
b $S_\infty - S_4 = 111\frac{1}{9} - 111\frac{1}{10} = \frac{1}{90}$
c $S_\infty - S_4 = 55\frac{5}{9} - 32\frac{4}{5} = 22\frac{34}{45}$
8 a $a = 2000$ and $r = \frac{1}{5}$
b $S_\infty = 2500$
c $S_\infty - S_4 = 4$
9 a $S_\infty = 10\,000$ **b** $S_\infty - S_{10} \doteqdot 3487$
10 a $r = 1.01$, no limiting sum
b $r = (1.01)^{-1}, S_\infty = 101$
c $r = \frac{1}{4}, S_\infty = \frac{64}{3}\sqrt{5}$
d $\frac{7}{6}\left(7 + \sqrt{7}\right)$
e $4\left(2 - \sqrt{2}\right)$
f $5\left(5 - 2\sqrt{5}\right)$
g $r = \frac{1}{3}\sqrt{10} > 1$, so there is no limiting sum.
h $\frac{1}{3}\sqrt{3}$
11 a $a = \frac{1}{3}, r = \frac{1}{3}, S_\infty = \frac{1}{2}$ **b** $a = \frac{7}{2}, r = \frac{1}{2}, S_\infty = 7$
c $a = -24, r = -\frac{3}{5}, S_\infty = -15$
12 a $S_\infty = \frac{5}{1-x}, x = \frac{1}{2}$ **b** $S_\infty = \frac{5}{1+x}, x = -\frac{2}{3}$
c $S_\infty = \frac{3x}{2}, x = \frac{4}{3}$ **d** $S_\infty = \frac{3x}{4}, x = \frac{8}{3}$
13 b **i** 96 **ii** 32 **iii** 64 **iv** 32
14 a $-1 < x < 1, \frac{7}{1-x}$ **b** $-\frac{1}{3} < x < \frac{1}{3}, \frac{2x}{1-3x}$
c $0 < x < 2, \frac{1}{2-x}$ **d** $-2 < x < 0, -\frac{1}{x}$
15 a $-\sqrt{2} < x < \sqrt{2}$ and $x \neq 0, S_\infty = \frac{1}{2-x^2}$
b $x \neq 0, S_\infty = \frac{1+x^2}{x^2}$
16 b $r = -3$, which is impossible.
d **i** $S_\infty > 3$ **ii** $S_\infty < -4$
iii $S_\infty > \frac{1}{2}a$ **iv** $S_\infty < \frac{1}{2}a$
17 a $w = \frac{v}{1-v}$ **b** $v = \frac{w}{1+w}$ **c** v
18 a $r = \frac{4}{5}$
b $18 + 6 + 2 + \cdots$ or $9 + 6 + 4 + \cdots$
c $r = \frac{5}{6}$
d **i** $r = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ ($r = -\frac{1}{2} - \sqrt{5} < -1$, so it is not a possible solution.)
ii $r = \frac{1}{2}$ **iii** $r = \frac{1}{2}\sqrt{2}$ or $-\frac{1}{2}\sqrt{2}$

Exercise 1I

1 a $0.3 + 0.03 + 0.003 + \cdots = \frac{1}{3}$
b $0.1 + 0.01 + 0.001 + \cdots = \frac{1}{9}$
c $0.7 + 0.07 + 0.007 + \cdots = \frac{7}{9}$
d $0.6 + 0.06 + 0.006 + \cdots = \frac{2}{3}$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 a $0.27 + 0.0027 + 0.000027 + \cdots = \frac{3}{11}$
b $\frac{81}{99} = \frac{9}{11}$ **c** $\frac{1}{11}$ **d** $\frac{4}{33}$ **e** $\frac{26}{33}$
f $\frac{1}{37}$ **g** $\frac{5}{37}$ **h** $\frac{5}{27}$

3 a $12 + (0.4 + 0.04 + \cdots) = 12\frac{4}{9}$
b $7 + 0.81 + 0.0081 + \cdots = 7\frac{9}{11}$
c $8.4 + (0.06 + 0.006 + \cdots) = 8\frac{7}{15}$
d $0.2 + (0.036 + 0.00036 + \cdots) = \frac{13}{55}$

4 a $0.\dot{9} = 0.9 + 0.09 + 0.009 + \cdots = \frac{0.9}{1 - 0.1} = 1$
b $2.7\dot{9} = 2.7 + (0.09 + 0.009 + 0.0009 + \cdots)$
$= 2.7 + \frac{0.09}{1 - 0.1} = 2.7 + 0.1 = 2.8$

5 a $\frac{29}{303}$ **b** $\frac{25}{101}$ **c** $\frac{3}{13}$ **d** $\frac{3}{7}$
e $0.25 + (0.0057 + 0.000057 + \cdots) = \frac{211}{825}$
f $1\frac{14}{135}$ **g** $\frac{1}{3690}$ **h** $7\frac{27}{35}$

6 If $\sqrt{2}$ were a recurring decimal, then we could use the methods of this section to write it as a fraction.

7 a Notice that $\frac{1}{9} = 0.\dot{1}$, $\frac{1}{99} = 0.\dot{0}\dot{1}$, $\frac{1}{999} = 0.\dot{0}0\dot{1}$, and so on. If the denominator of a fraction can be made a string of nines, then the fraction will be a multiple of one of these recurring decimals.
b Periods: 1, 6, 1, 2, 6, 3, 3, 5, 4, 5

8 d The fourth sentence should be changed to, 'Imagine that each real number T_n in the sequence is written as an infinite decimal string of digits 0.dddddd . . . , where each d represents a digit. Add an infinite string of zeroes to every terminating decimal, and if there is an infinite string of 9s, rewrite the decimal as a terminating decimal.'

Chapter 1 review exercise

1 14, 5, −4, −13, −22, −31, −40, −49
a 6 **b** 4 **c** −31
d T_8 **e** No **f** $T_{11} = -40$

2 a 52, −62, −542, −5 999 942
b 20 no, $10 = T_8$, $-56 = T_{19}$, −100 no
c $T_{44} = -206$
d $T_{109} = -596$

3 a 4, 7, 7, 7, 7, 7, . . .
b 0, 1, 2, 3, 4, 5, 6, . . .
c $T_1 = 5$, $T_n = 2n - 1$ for $n > 1$
d $T_1 = 3$, $T_n = 2 \times 3^{n-1}$ for $n > 1$

4 a 82 **b** −15 **c** 1 **d** $\frac{63}{64}$

5 a −5, 5, −5, 5, −5, 5, −5, 5 **b** −5, 0
c Take the opposite. **d** 5, −5, −5

6 a AP, $d = 7$ **b** AP, $d = -121$
c neither **d** GP, $r = 3$
e neither **f** GP, $r = -\frac{1}{2}$

7 a $a = 23$, $d = 12$
b $T_{20} = 251$, $T_{600} = 7211$
d $143 = T_{11}$, 173 is not a term.
e $T_{83} = 1007$, $T_{165} = 1991$
f 83 (Count both T_{83} and T_{165}.)

8 a $a = 20$, $d = 16$ **b** $T_n = 4 + 16n$
c 12 cases, \$4 change **d** 18

9 a $a = 50$, $r = 2$
b $T_n = 50 \times 2^{n-1}$ (or 25×2^n)
c $T_8 = 6400$, $T_{12} = 102\,400$
d $1600 = T_6$, 4800 is not a term.
e 320 000 **f** 18 terms

10 a $a = 486$, $r = \frac{1}{3}$
b 486, 162, 54, 18, 6, 2 (no fractions)
c 4 **d** $S_6 = 728$ **e** 729

11 a 75 **b** 45 or −45

12 a 11 111 **b** −16 400 **c** 1025

13 a $n = 45$, $S_{45} = 4995$ **b** $n = 101$, $S_{101} = 5050$
c $n = 77$, $S_{77} = 2387$

14 a 189 **b** −1092 **c** $-157\frac{1}{2}$

15 a 300
b $r = -\frac{3}{2} < -1$, so there is no limiting sum.
c $-303\frac{3}{4}$

16 a $-3 < x < -1$ **b** $S_\infty = -\frac{2 + x}{1 + x}$

17 a $\frac{13}{33}$ **b** $\frac{52}{111}$ **c** $12\frac{335}{1100} = 12\frac{67}{220}$

18 a $d = 5$, 511 **b** −1450 **c** $r = -2$, −24
d $d = -5$ **e** $n = 2$ or $n = 8$
f $r = -\frac{1}{3}$ **g** 16

Chapter 2

Exercise 2A

1 A When $n = 1$, RHS $= 1^2$
$= \text{LHS}$,
so the statement is true for $n = 1$.
B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose
$$1 + 3 + 5 + \cdots + (2k - 1) = k^2. \qquad (**)$$
We prove the statement for $n = k + 1$.
That is, we prove
$$1 + 3 + 5 + \cdots + (2k + 1) = (k + 1)^2.$$
$$\begin{aligned} \text{LHS} &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1), \\ &\qquad \text{by the induction hypothesis } (**), \\ &= (k + 1)^2 \\ &= \text{RHS}. \end{aligned}$$

Mathematics Extension 1 Year 12 ISBN 978-1-108-76630-2 Cambridge University Press
Cambridge Maths Stage 6 Updated July 2022

C It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

3 1 and $\frac{1}{2}$

6 b It is not true for $n = 1$.

7 b If it is true for $n = k$, it does not follow that it is true for $n = k + 1$.

8 a $P(n) = (n + 1)(4n^2 + 14n + 9)$

Exercise 2B

1 A When $n = 1$, $7^n - 1 = 6$, which is divisible by 6, so the statement is true for $n = 1$.

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose $7^k - 1 = 6m$, for some integer m. $\quad (**)$
We prove the statement for $n = k + 1$.
That is, we prove $7^{k+1} - 1$ is divisible by 6.

$$\begin{aligned} 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 7 \times (6m + 1) - 1, \\ &\quad \text{by the induction hypothesis } (**), \\ &= 42m + 6 \\ &= 6(7m + 1), \\ &\quad \text{which is divisible by 6, as required.} \end{aligned}$$

C It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

3 a 0, 10, 120, 1330, 14 640, ...
The expression is always divisible by 10.

Chapter 2 review exercise

3 a 0, 5, 55, 485. . . .
The expression is always divisible by 5.

4 b The limiting sum is 1

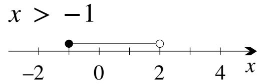
Chapter 3

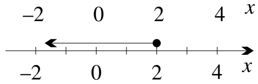
Exercise 3A

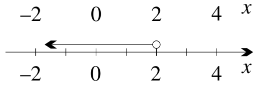
1 a i $-1 \leq x \leq 2$ **ii** $[-1, 2]$

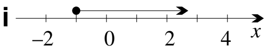
b i $-1 < x \leq 2$ **ii** $(-1, 2]$

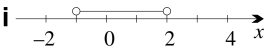
c i $x > -1$ **ii** $(-1, \infty)$

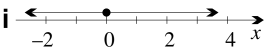
2 a i  **ii** $[-1, 2)$

b i  **ii** $(-\infty, 2]$

c i  **ii** $(-\infty, 2)$

3 a i  **ii** $x \geq -1$

b i  **ii** $-1 < x < 2$

c i 

ii **R** (There is no way of writing the interval using inequalities.)

4 a i $2^4 = 16$ **ii** $8 + 1 = 9$
iii $2^8 = 256$ **iv** $4 + 1 = 5$

b i 2^{x+1} **ii** $2^x + 1$
iii 2^{2^x} **iv** $x + 2$

5 a $(-\infty, 1)$ **b** $(0, 2)$ **c** $(0, 1)$ **d** $(4, \infty)$

6 a $-1 \leq x \leq 0$ or $x \geq 1$

b $-5 \leq x \leq -2$ or $x \geq 1$

c $x < -2$ or $x > 4$

d $-2 \leq x \leq 2$

e $x < -2$ or $0 < x < 2$

f $-1 \leq x < 0$ or $2 < x \leq 3$

7 a $x \neq -\dfrac{3}{2}$ **b** $x \leq 2$ **c** all real x

d $x > -1$ **e** $x > -3$ **f** all real x

8 a i $-1 < x < 1$ or $2 \leq x \leq 3$

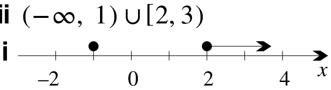
ii $(-1, 1) \cup [2, 3]$

b i $x < 1$ or $x \geq 2$

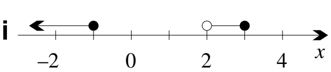
ii $(-\infty, 1) \cup [2, \infty)$

c i $x < 1$ or $2 \leq x < 3$

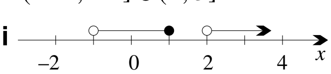
ii $(-\infty, 1) \cup [2, 3)$

9 a i 

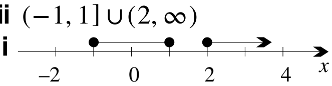
ii $[-1, -1] \cup [2, \infty)$

b i 

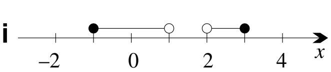
ii $(-\infty, -1] \cup (2, 3]$

c i 

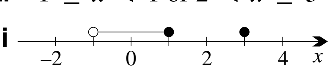
ii $(-1, 1] \cup (2, \infty)$

10 a i 

ii $-1 \leq x \leq 1$ or $x \geq 2$

b i 

ii $-1 \leq x < 1$ or $2 < x \leq 3$

c i 

ii $-1 < x \leq 1$ or $x = 3$

11 a $[-1, 0] \cup [1, \infty)$ **b** $[-5, -2] \cup [1, \infty)$

c $(-\infty, -2) \cup (4, \infty)$ **d** $[-2, 2]$

e $(-\infty, -2) \cup (0, 2)$ **f** $[-1, 0) \cup (2, 3]$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 a $x < -1$ or $x \geq 2$

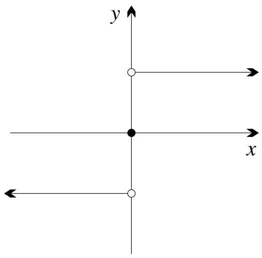
b $-1 < x \leq 1$ or $x > 3$

c $x < -1$ or $-1 < x < 2$

13 a $[0,\ 1) \cup (1,\ \infty)$ **b** $(-\infty, -1) \cup (6,\ \infty)$

c $(-1,\ 3)$ **d** $(-\infty,\ \infty)$

14 a



b $x \geq 0$

15 a $x \neq 0$

b $h'(x) = \dfrac{-4}{(e^x - e^{-x})^2}$

The denominator is a square so is positive.

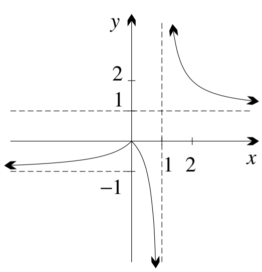
Thus $h'(x)$ is negative for $x \neq 0$.

16 a i $x \neq 1$

ii $(0, 0)$

iii $[0, 0] \cup (1, \infty)$

iv

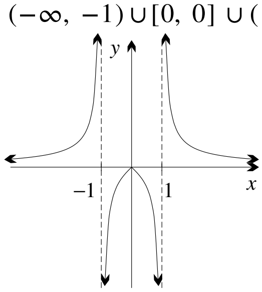


b i $x \neq -1$ or 1

ii $(0, 0)$

iii $(-\infty,\ -1) \cup [0,\ 0] \cup (1, \infty)$

iv



17 a Both sides equal $\sin\left(e^{1-x^2} + \dfrac{\pi}{3}\right)$

b LHS $= (f \circ g)h(x) = f\Big(g\big(h(x)\big)\Big)$,

RHS $= f\big(g \circ h(x)\big) = f\Big(g\big(h(x)\big)\Big) =$ LHS.

18 a It has one endpoint at 5 which it contains.

b It does not contain any end points.

c It contains all its endpoints. (There are none!)

Exercise 3B

1 a $f(x) \to 0 \quad x \to \infty \qquad x \to -\infty$

b $f(x) \to 1$ as $x \to \infty$ and $x \to -\infty$

c $f(x) \to -2$ as $x \to \infty$ and $x \to -\infty$

d $f(x) \to \dfrac{1}{2}$ as $x \to \infty$ and $x \to -\infty$

e $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

f $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

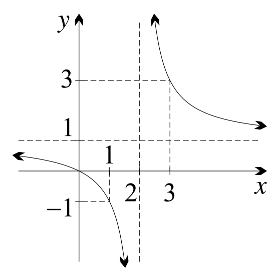
2 a $x \neq 2$

b $x = 0$ and $y = 0$

c $y \to 1$ as $x \to \infty$ and as $x \to -\infty$.

d $x = 2$ is a vertical asymptote, $y \to \infty$ as $x \to 2^+$,

$y \to -\infty$ as $x \to 2^-$.

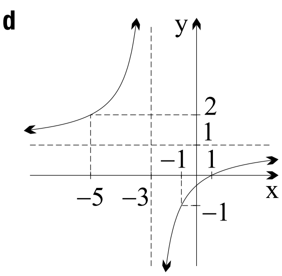


3 a $x = -3$

b $x = 1$ and $y = -\dfrac{1}{3}$

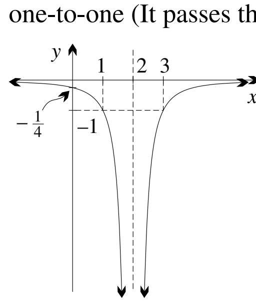
c $y \to 1$ as $x \to \infty$ and as $x \to -\infty$, $y \to -\infty$ as $x \to -3^+$, $y \to \infty$ as $x \to -3^-$.

d



e one-to-one (It passes the horizontal line test.)

4



domain: $x \neq 2$,

$y \to 0$ as $x \to \infty$ and as $x \to -\infty$

vert'l asymptote $x = 2$,

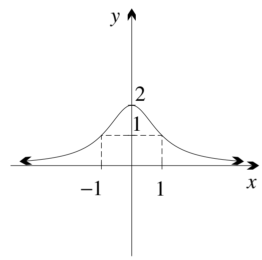
as $x \to 2^+$, $y < 0$ so $y \to -\infty$,

as $x \to 2^-$, $y < 0$ so $y \to -\infty$

ISBN 978-1-108-76630-2

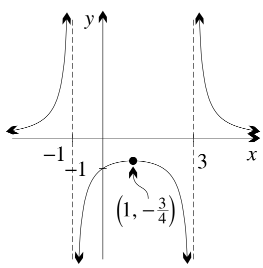
Cambridge University Press
Updated July 2022

5



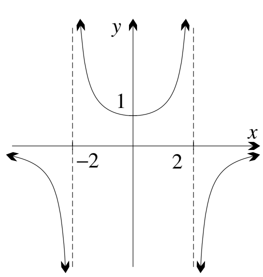
a $y \to 0$ as $x \to \infty$ and $x \to -\infty$.
b $x^2 + 1$ is never 0.
c $y' = -4x(x^2 + 1)^{-2}$
e $0 < y \leq 2$
f many-to-one (It fails the horizontal line test.)

6



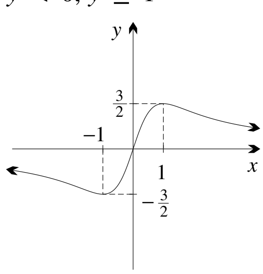
a $x \neq -1, 3$
b $(0, -1)$
c $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.
e as $x \to 3^+$, $y > 0$ so $y \to \infty$, as $x \to 3^-$, $y < 0$ so $y \to -\infty$, as $x \to 1^+$, $y < 0$ so $y \to -\infty$, and as $x \to 1^-$, $y > 0$ so $y \to \infty$
f $y \leq -\dfrac{3}{4}, y > 0$

7 a

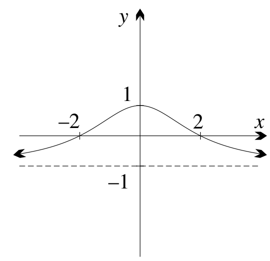


b $y < 0, y \geq 1$

8

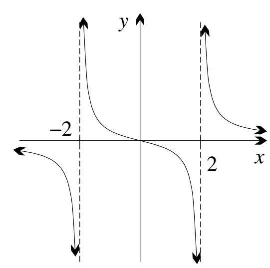


9



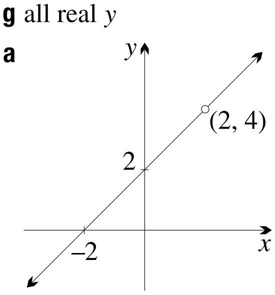
10 a $y = \dfrac{(x + 2)(x + 3)}{(x - 1)(x - 3)}$, $x = 1$, $x = 3$ and $y = 1$
b $y = \dfrac{(x - 1)^2}{(x + 1)(x + 4)}$, $x = -1$, $x = -4$ and $y = 1$
c $y = \dfrac{x - 5}{(x - 2)(x + 5)}$, $x = -5$, $x = 2$ and $y = 0$
d $y = \dfrac{(1 - 2x)(1 + 2x)}{(1 - 3x)(1 + 3x)}$, $x = \dfrac{1}{3}$, $x = -\dfrac{1}{3}$ and $y = \dfrac{4}{9}$

11

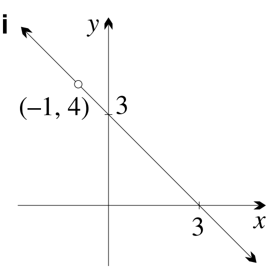


a odd
b domain: $x \neq 2$ and $x \neq -2$,
asymptotes: $x = 2$ and $x = -2$
d $y = 0$
e $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$ $f'(x) < 0$ for $x \neq 2$ & $x \neq -2$
g all real y

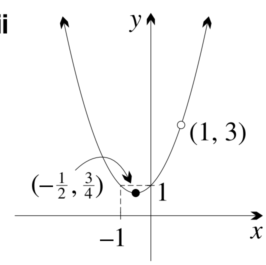
12 a



b i



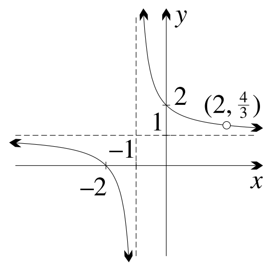
ii



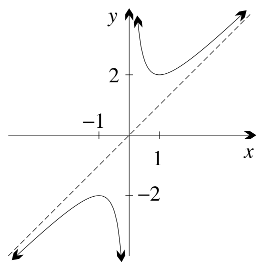
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iii



13



a point symmetry in the origin

b domain: $x \neq 0$, asymptote: $x = 0$

e $(-1, -2)$ and $(1, 2)$

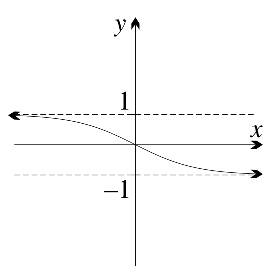
h $y \geq 2$ or $y \leq -2$

14 a 1

b -1

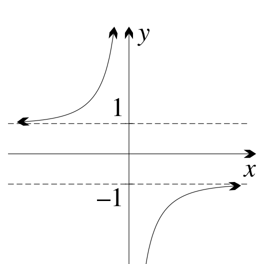
c $(0, 0)$

d

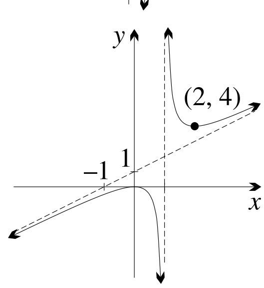


e odd

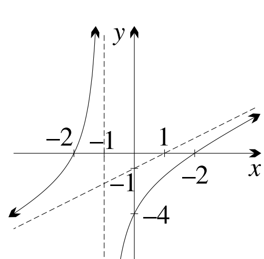
f



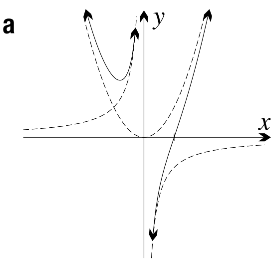
15 a



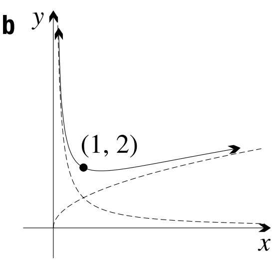
b



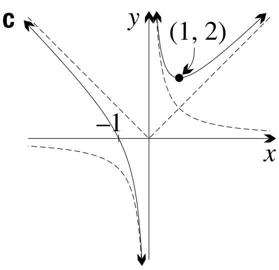
16 a



b



c



Exercise 3C

1 a $y = \dfrac{9}{(x - 3)(x + 3)}$

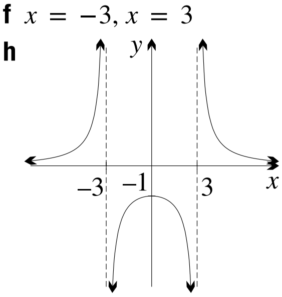
b $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

c symmetry in the y-axis

d $(0, -1)$ **e** $-3 < x < 3$

f $x = -3, x = 3$ **g** $y = 0$

h



i $y' = \dfrac{-18x}{(x^2 - 9)^2}$ so $y'(0) = 0$

2 a $y = \dfrac{x}{(2 - x)(2 + x)}$

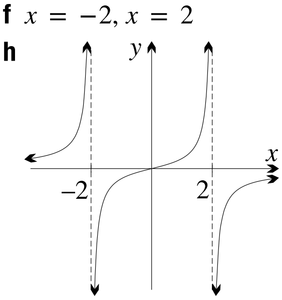
b $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

c point symmetry in the origin

d $(0, 0)$ **e** $x < -2$ or $0 \leq x < 2$

f $x = -2, x = 2$ **g** $y = 0$

h



i $y' > 0$ in the domain.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 a $\dfrac{2\left(x - 2\frac{1}{2}\right)}{(x-1)(x-4)}$

b $x \neq 1$ and $x \neq 4$

c The domain is not symmetric about $x = 0$.

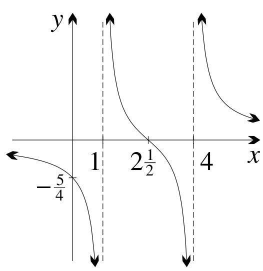
d $\left(0, -\frac{5}{4}\right)$ and $\left(2\frac{1}{2}, 0\right)$

e $1 < x < 2\frac{1}{2}$ or $x > 4$

f $x = 1$ and $x = 4$

g $y = 0$

h



4 a $y = x(x-2)(x+2)$

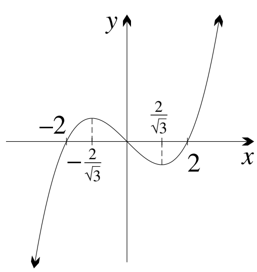
b $(-\infty, \infty)$

c $(-2, 0)$, $(0, 0)$, $(2, 0)$

d point symmetry in the origin

e no

g



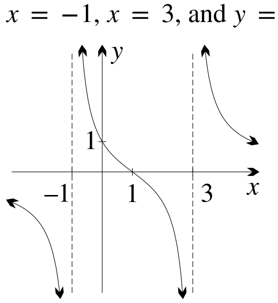
5 $y = \dfrac{3(x-1)}{(x-3)(x+1)}$

a domain: $x \neq -1$ and $x \neq 3$ intercepts: $(1, 0)$ and $(0, 1)$

b The domain is not symmetric about $x = 0$.

c $x = -1$, $x = 3$, and $y = 0$

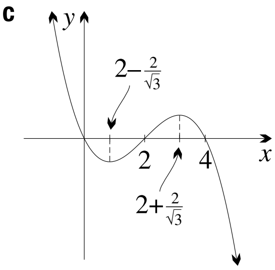
d



6 $y = -x(x-2)(x-4)$

a $-\infty < x < \infty$ $(0, 0)$, $(2, 0)$, $(4, 0)$

c



7 $y = \dfrac{(x+1)^2}{(x-1)(x+3)}$

a domain:

$x \neq -3$ and $x \neq 1$

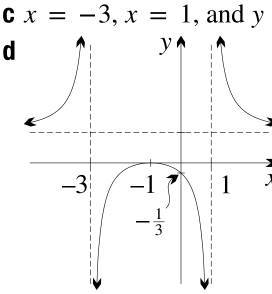
intercepts:

$(-1, 0)$ and $\left(0, -\frac{1}{3}\right)$

b The domain is not symmetric about $x = 0$.

c $x = -3$, $x = 1$, and $y = 1$

d



e $y \leq 0$ or $y > 1$

8 $f(x) = \dfrac{(x-2)(x+2)}{x(x-4)}$

a domain:

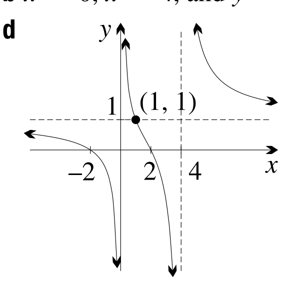
$x \neq 0$ and $x \neq 4$

intercepts:

$(-2, 0)$ and $(2, 0)$

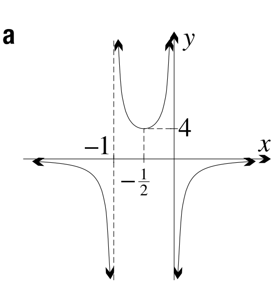
b $x = 0$, $x = 4$, and $y = 1$

d

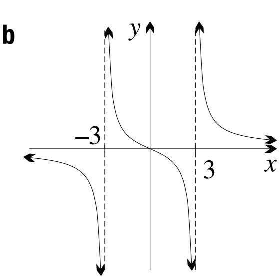


e all real y

9 a



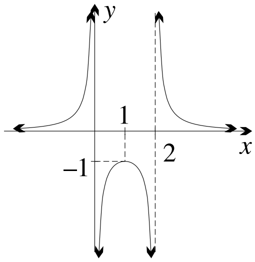
b



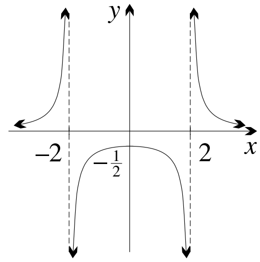
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

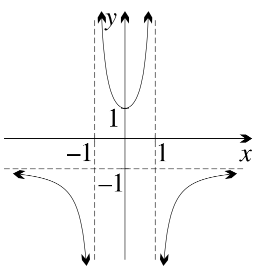
10 a



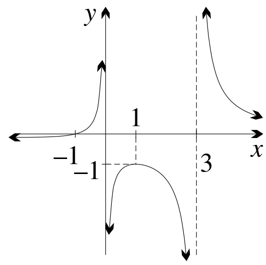
b



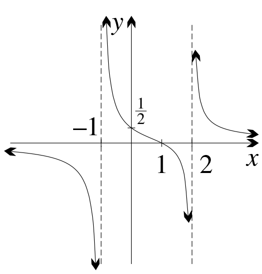
11 a



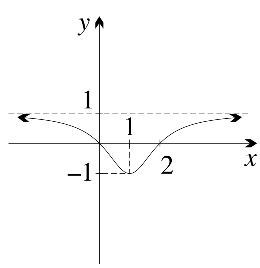
b



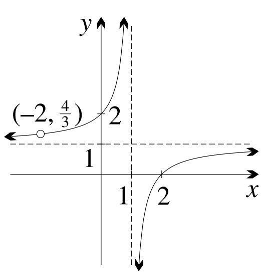
c



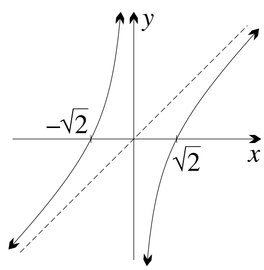
d



e



f



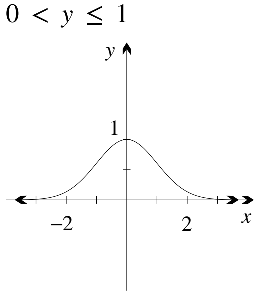
12 a domain: all real x intercept: $(0, 1)$

b It is an even function with asymptote $y = 0$.

c $(0, 1)$

d $y' = -xe^{-x^2/2}$, so $y' = 0$ at $x = 0$

e $0 < y \le 1$



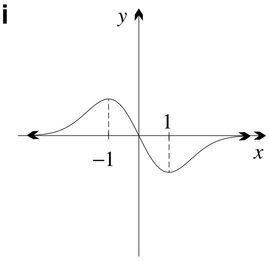
f $e^{-\frac{1}{2}} < 2^{-\frac{1}{2}}$ so $y = 2^{-\frac{1}{2}x^2}$ is higher, except at $x = 0$ where they are equal.

13 a **i** $e^x > x$ for $x \ge 0$

ii Replace x with $\frac{1}{2}x^2$ in part **i** and take reciprocals.

iii 0

b **i**

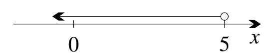


ii $f' = e^{-\frac{1}{2}x^2}(x^2 - 1)$ so $f'(x) = 0$ at $x = -1$ or 1.

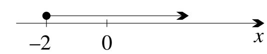
The graph shows that $f(x)$ is greatest at $x = -1$ and least at $x = 1$.

Exercise 3D

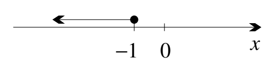
1 a $x < 5$



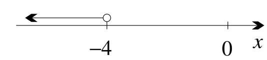
b $x \ge -2$



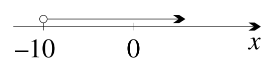
c $x \le -1$



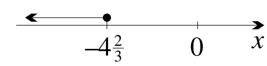
d $x < -4$



e $x > -10$



f $x \le -4\frac{2}{3}$



2 a $x < -2$, $(-\infty, -2)$

b $x \ge -4$, $[-4, \infty)$

c $x < 6$, $(-\infty, 6)$

3 $5x - 4 < 7 - \frac{1}{2}x$, with solution $x < 2$

4 a $-\frac{1}{2} \le x \le 1\frac{1}{2}$, $\left[-\frac{1}{2}, 1\frac{1}{2}\right]$

b $-4 < x < 2$, $(-4, 2)$

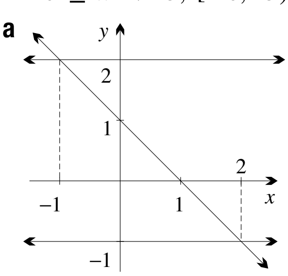
c $\frac{1}{3} < x \le 4$, $\left(\frac{1}{3}, 4\right]$

d $-2 < x \le 7$, $(-2, 7]$

e $-2 \le x < 0$, $[-2, 0)$

f $-6 \le x < 15$, $[-6, 15)$

5 a

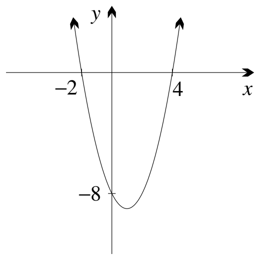


b $-1 \le x < 2$. The solution of the inequation is where the diagonal line lies between the horizontal lines.

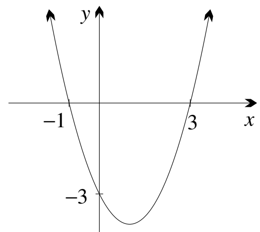
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

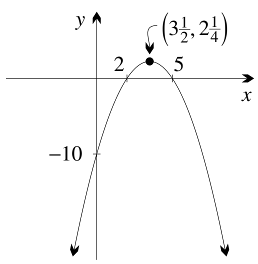
6 a $-2 < x < 4$



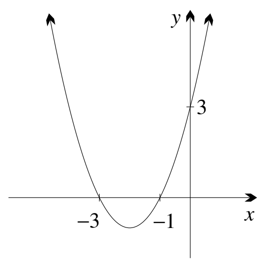
b $x < -1$ or $x > 3$



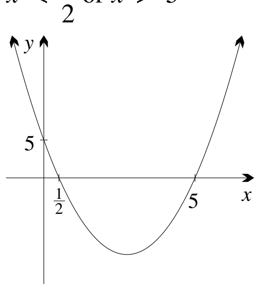
c $2 \le x \le 5$



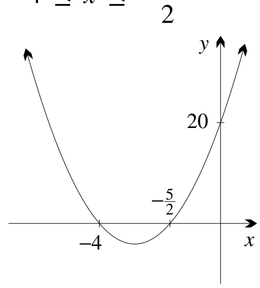
d $x \le -3$ or $x \ge -1$



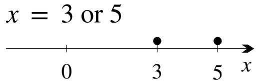
e $x < \dfrac{1}{2}$ or $x > 5$



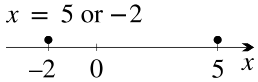
f $-4 \le x \le -\dfrac{5}{2}$



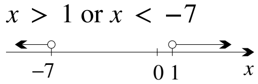
7 a $x = 3$ or 5



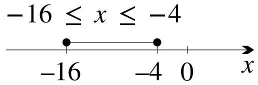
b $x = 5$ or -2



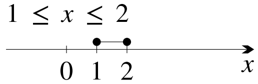
c $x > 1$ or $x < -7$



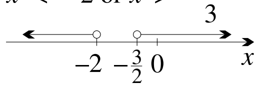
d $-16 \le x \le -4$



e $1 \le x \le 2$



f $x < -2$ or $x > -\dfrac{2}{3}$



8 a $x < 0$ or $x \ge 2$
b $-1 < x < 2$
c $x > \frac{3}{2}$ or $x < -\frac{1}{2}$
d $x \le \frac{1}{8}$ or $x > \frac{3}{4}$

9 a $y = \begin{cases} 2x - 1 & \text{for} \quad x \ge 2 \\ 3 & \text{for} \quad x < 2 \end{cases}$

b $y = \begin{cases} x + 9 & \text{for} \quad x \ge -2 \\ -3x + 1 & \text{for} \quad x < -2 \end{cases}$

c $y = \begin{cases} 4x - 4 & \text{for} \quad x \ge -1 \\ -2x + 2 & \text{for} \quad x < -1 \end{cases}$

10 a $x \ge 3$ **b** $0 < x \le 3$
c $-4 \le x \le 4$ **d** $x < -4$
e $0 < x < 8$ **f** $\dfrac{1}{25} \le x \le 625$

11 a $0 < x < 3$ **b** $x < -3$ or $x > 2$
c $x < 0$ **d** $x < 0$ or $1 < x < 5$
e $-4 < x < 1$

12 a i $y = \begin{cases} 3x & \text{for } x \ge 0 \\ -x & \text{for } x < 0 \end{cases}$ **ii** $x < -1$ or $x > \frac{1}{3}$

b i $y = \begin{cases} 4x - 8 & \text{for } x \ge 2 \\ 4 - 2x & \text{for } x < 2 \end{cases}$ so $1 \le x \le 2\frac{1}{2}$

ii $y = \begin{cases} \frac{1}{2}x + 1 & \text{for } x \ge -1 \\ -\frac{3}{2}x - 1 & \text{for } x < -1 \end{cases}$ so $-\frac{8}{3} < x < 4$

13 a $x < -4$ or $x > 3$
b $x \le -3$ or $x > -1$
c $\frac{1}{2} < x \le \frac{4}{5}$

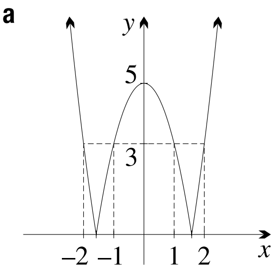
14 a $0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$
b $-\frac{\pi}{2} < x \le 0$ or $\frac{\pi}{4} \le x < \frac{\pi}{2}$

15 a $-5 \le x < -3$ or $-1 < x \le 1$
b $-\frac{1}{2} < x \le 1$ or $2 \le x < 3\frac{1}{2}$

16 a false: $x = 2$ and $y = -2$
b true
c false: $x = 2$ and $y = -2$
d true
e true
f false: $x = -2$

17 a It assumes $x + 1 = |x + 1|$. **b** $x < 1$

18 a



Intercepts at $x = -\sqrt{\frac{5}{2}}$ and $x = \sqrt{\frac{5}{2}}$

b $x \le -2$ or $-1 \le x \le 1$ or $x \ge 2$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

19 $x^2 + xy + y^2 = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x + y)^2$ or otherwise.

20 a Start with $(x - y)^2 \geq 0$.

b Replace x with x^2 and y with y^2, and divide.

21 a $2(a^2 + b^2 + c^2 - ab - bc - ac)$

b $2(a^3 + b^3 + c^3 - 3abc)$

Exercise 3E

1 a 1 **b** 2 **c** 3 **d** 2 **e** 2 **f** 3

2 a $x = \frac{1}{2}$

b $x = -\frac{3\pi}{4}$ or $\frac{\pi}{4}$

c $x \doteqdot -2.1,\ 0.3,\ 1.9$

d $x = 1$ or $x \doteqdot 3.5$

e $x = 1$ or $x \doteqdot -1.9$

f $x = 0$, $x \doteqdot -1.9$ or 1.9

3 a $x \leq -3$ **b** $0 \leq x \leq 2$ **c** $x = 1$

4 a $x < -2$ or $x > 1$

b $0 \leq x \leq 1$

c $-1 < x < 0$ or $x > 1$

5 a $\sqrt{2} \doteqdot 1.4$, $\sqrt{3} \doteqdot 1.7$

b $x = -1$ or $x = 2$

c $x < -1$ or $x > 2$

d $x = -2$ or $x = 1$, $-2 \leq x \leq 1$

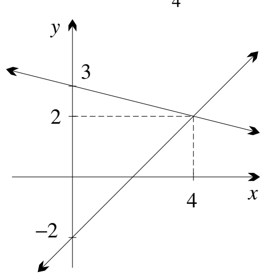
e $x \doteqdot 1.62$ or $x \doteqdot -0.62$

f i Draw $y = -x$; $x = 0$ or $x = -1$.

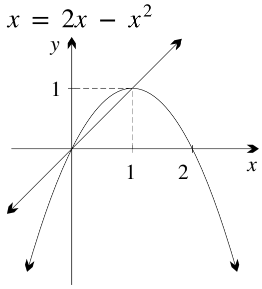
ii Draw $y = x + \frac{1}{2}$; $x \doteqdot 1.37$ or $x \doteqdot -0.37$.

iii Draw $y = \frac{1}{2}x + \frac{1}{2}$; $x = 1$ or $x = -\frac{1}{2}$.

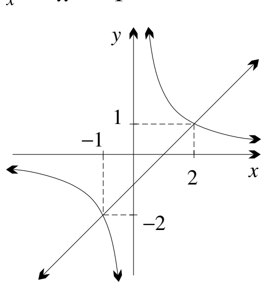
6 a $(4, 2)$, $x - 2 = 3 - \frac{1}{4}x$



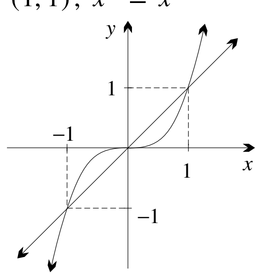
b $(0, 0)$ and $(1, 1)$, $x = 2x - x^2$



c $(-1, -2)$ and $(2, 1)$, $\frac{2}{x} = x - 1$



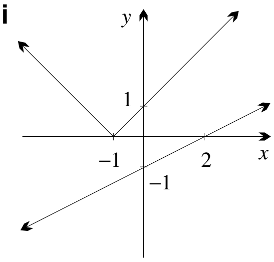
d $(-1, -1)$, $(0, 0)$ and $(1, 1)$, $x^3 = x$



7 a $x \geq 4$ **b** $0 < x < 1$

c $x < -1$ or $0 < x < 2$ **d** $-1 < x < 0$ or $x > 1$

8 a i

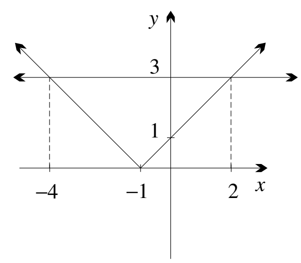


ii The graph of $y = |x + 1|$ is always above the graph of $y = \frac{1}{2}x - 1$.

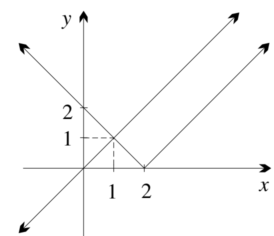
b The curve is always above the line.

c The two lines are parallel and thus the first is always below the second.

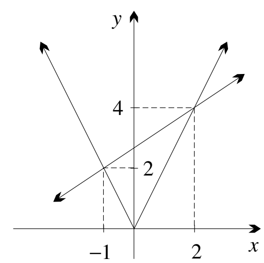
9 a $(-4, 3)$, $(2, 3)$



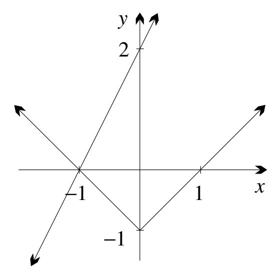
b $(1, 1)$



c $(-1, 2)$, $(2, 4)$



d $(-1, 0)$



10 a $-4 \leq x \leq 2$ **b** $x < 1$

c $x \leq -1$ or $x \geq 2$ **d** $x < -1$

11 a Divide by e^x to get $e^x = e^{1-x}$

b Multiply by $\cos x$ to get $\sin x = \cos x$

c Subtract 1 then divide by x to get $x^2 - 4 = -\frac{1}{x}$

12 a The table below traps the solution between $x = -1.690$ and $x = -1.6905$, so it is $x = -1.690$, correct to three decimal places.

x	-2	-1.7	-1.6	-1.68
2^x	0.25	0.3078	0.3299	0.3121
$x + 2$	0	0.3	0.4	0.32

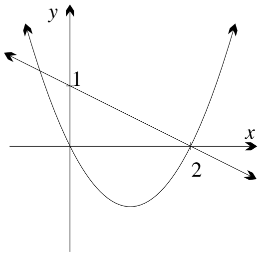
x	-1.69	-1.691	-1.6905
2^x	0.3099	0.3097	0.3098
$x + 2$	0.31	0.309	0.3095

b Part (c): $x \doteqdot -2.115$. Part (e): $x \doteqdot -1.872$.

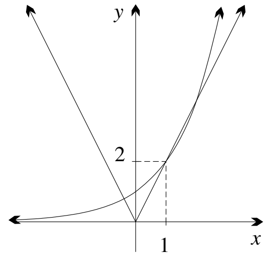
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

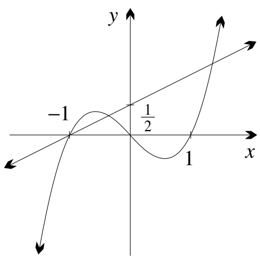
13 a 2 solutions



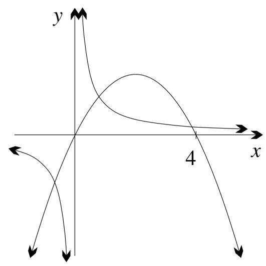
b 3 solutions



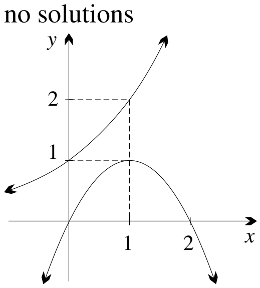
c 3 solutions



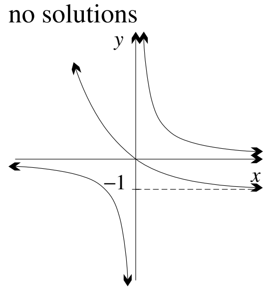
d 3 solutions



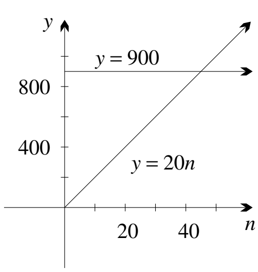
e no solutions



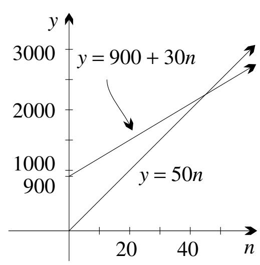
f no solutions



14 a



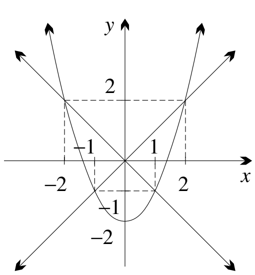
b



In both cases the break-even point is $n = 45$.

Total sales are \$2250 at that point.

15 a

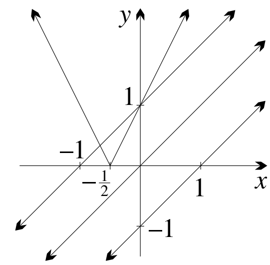


b $x = 2$ or -2

c $x < -2$ or $x > 2$

d $-1 \le x \le 1$

16



c $c > \frac{1}{2}$

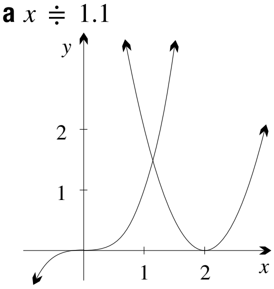
17 b $-\dfrac{3\sqrt{2}}{2} < b < \dfrac{3\sqrt{2}}{2}$

18 a 2

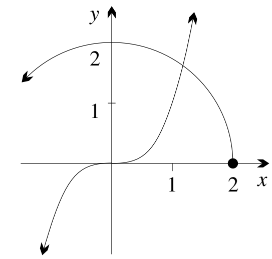
b The solutions are not integers.

c $x = \frac{1}{11}$ or $\frac{7}{3}$

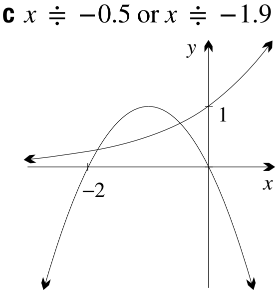
19 a $x \doteqdot 1.1$



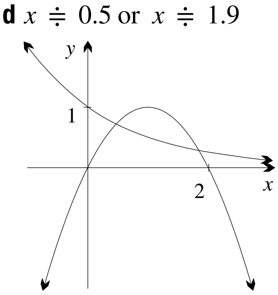
b $x \doteqdot 1.2$



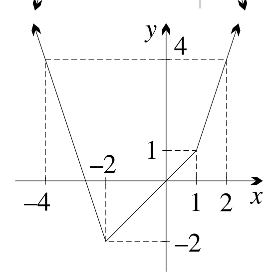
c $x \doteqdot -0.5$ or $x \doteqdot -1.9$



d $x \doteqdot 0.5$ or $x \doteqdot 1.9$



20



a $y = \begin{cases} -3x - 8, & \text{for } x < -2, \\ x, & \text{for } -2 \le x < 1, \\ 3x - 2, & \text{for } x \ge 1. \end{cases}$

b $-3\frac{1}{3} \le x \le -2\frac{1}{3}$ or $-1 \le x \le 1\frac{1}{3}$

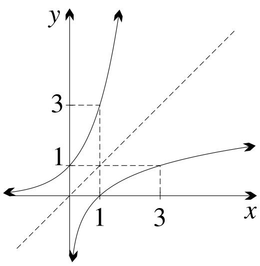
21 b $b < m$

c $-p \le m \le p$ and $b < -\frac{qm}{p}$

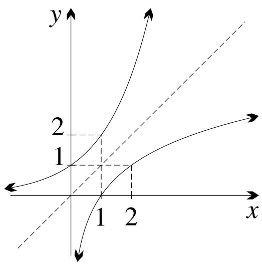
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

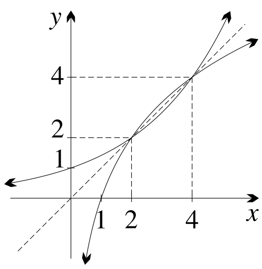
22 a i



ii



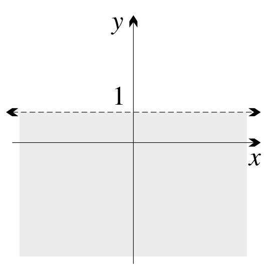
iii



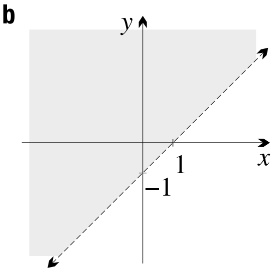
b 0, 1 or 2

Exercise 3F

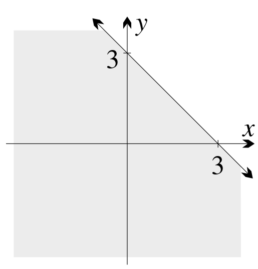
1 a



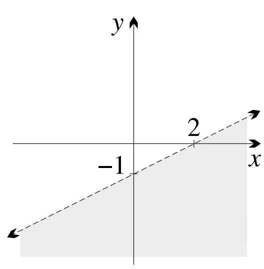
b



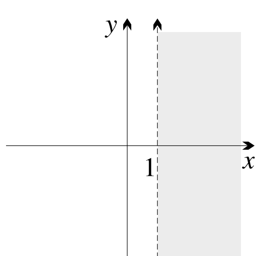
c



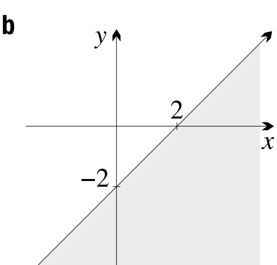
d



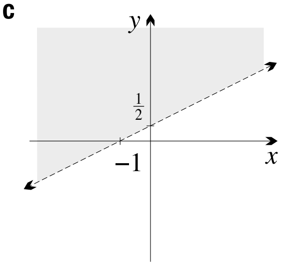
2 a



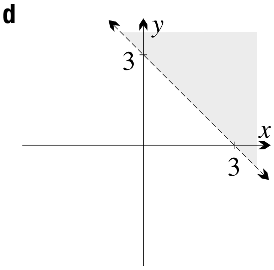
b



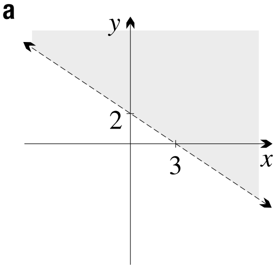
c



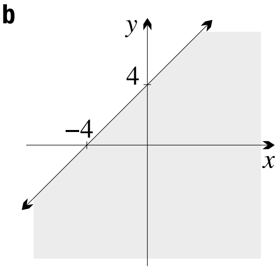
d



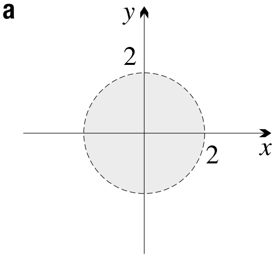
3 a



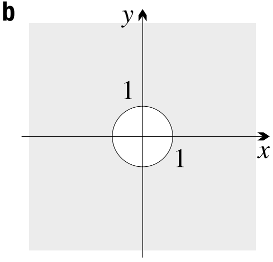
b



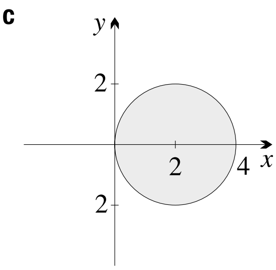
4 a



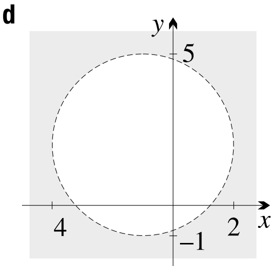
b



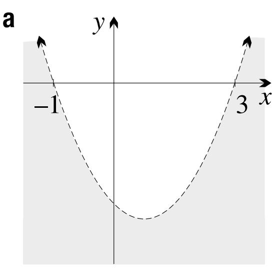
c



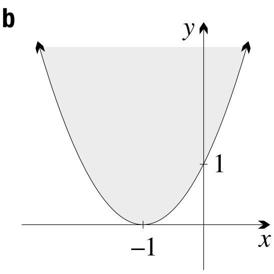
d



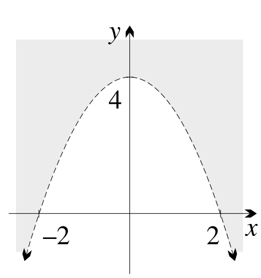
5 a



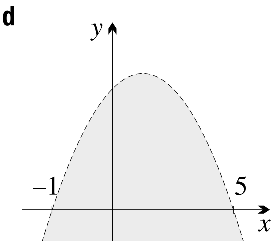
b



c



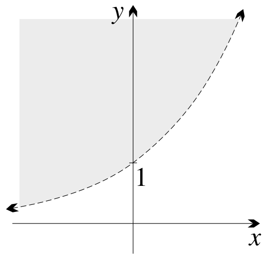
d



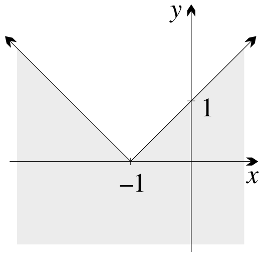
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

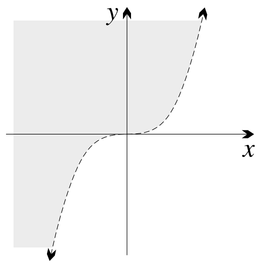
6 a



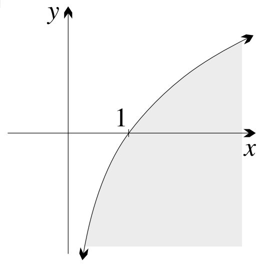
b



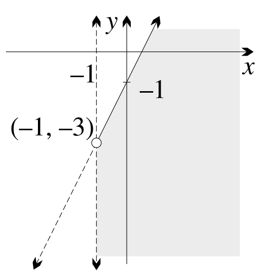
c



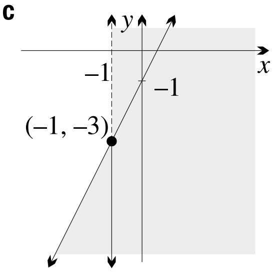
d



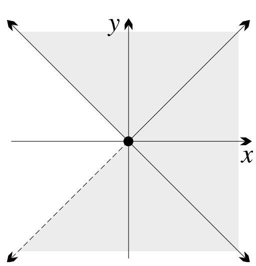
7 b



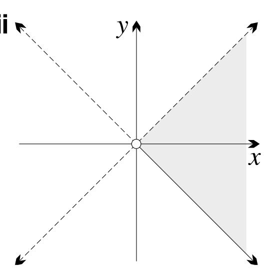
c



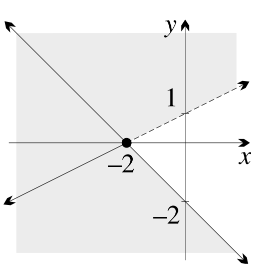
8 a i



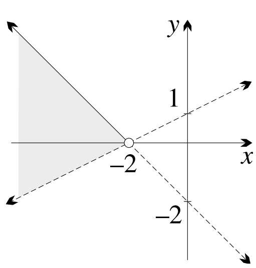
ii



b i



ii



9 a $x \geq 0$ and $y \geq 0$ **b** $x \leq 0$ and $y \geq 0$

c $x \leq 0$ and $y \leq 0$ **d** $x \geq 0$ and $y \leq 0$

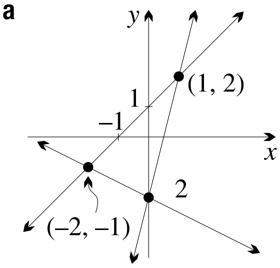
e $x \geq 0$ or $y \geq 0$ **f** $x \geq 0$ or $y \leq 0$

10 a $y < x$ and $y \leq 2 - x$

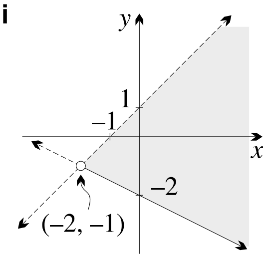
b $y \leq -\frac{1}{2}x - 1$ or $y \geq 2 - 2x$

c $y < x + 2$ or $y > 4x - 1$

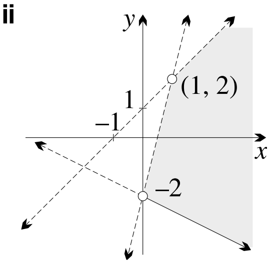
11 a



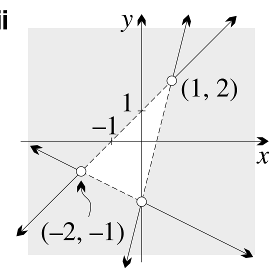
b i



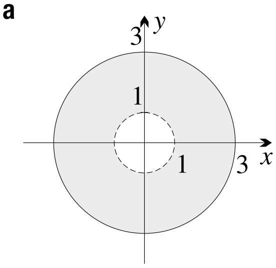
ii



iii

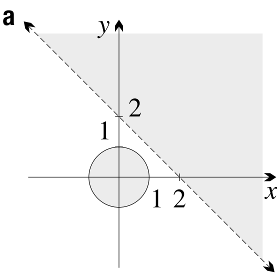


12 a



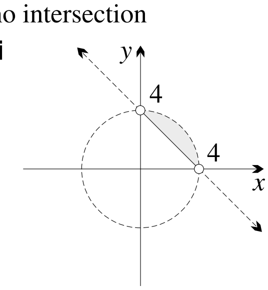
b whole plane

13 a

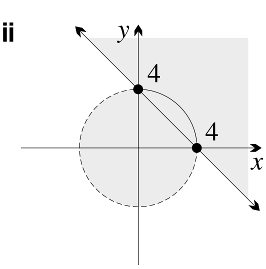


b no intersection

14 b i



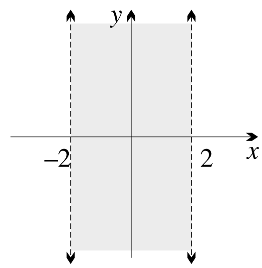
ii



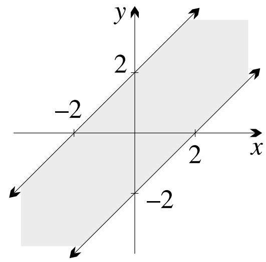
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

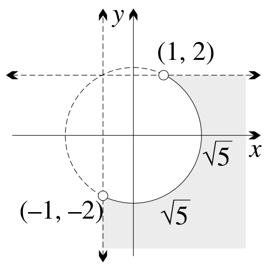
15 a $x < 2$ and $x > -2$



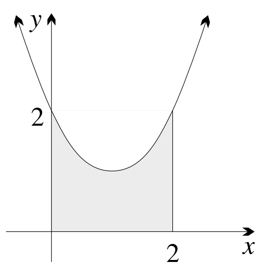
b $x - y \leq 2$ and $x - y \geq -2$



16

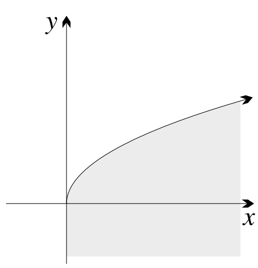


17



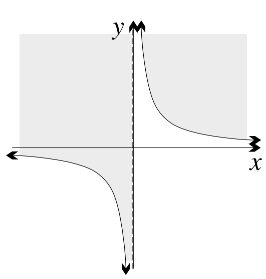
18 b The curve is undefined for $x < 0$.

c

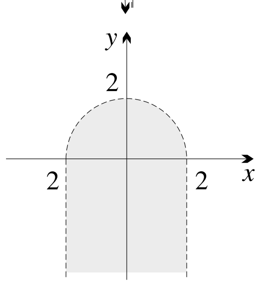


19 a The curve is undefined when $x = 0$.

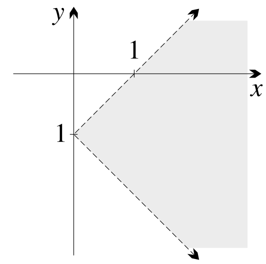
b



20

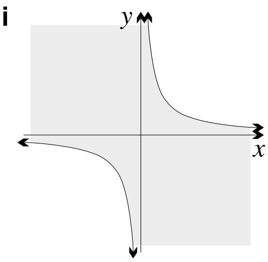


21

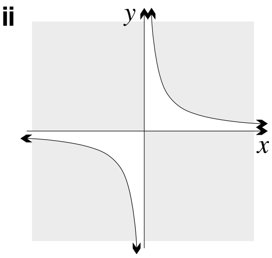


22 a 6

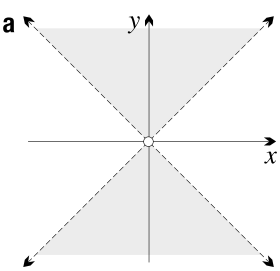
b i



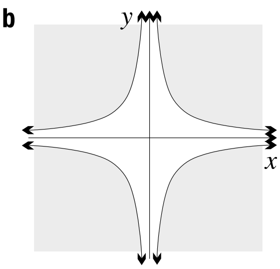
ii



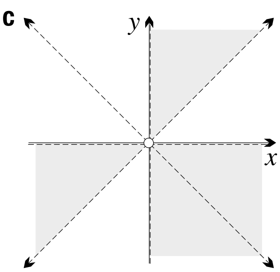
23 a



b

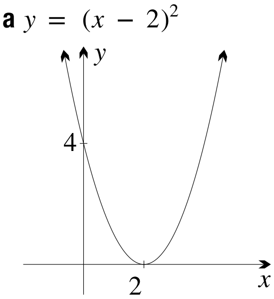


c

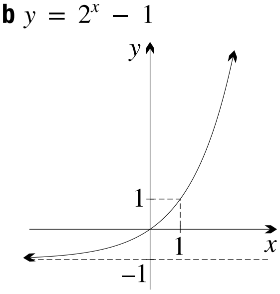


Exercise 3G

1 a $y = (x - 2)^2$



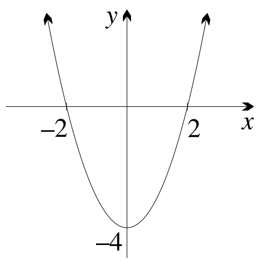
b $y = 2^x - 1$



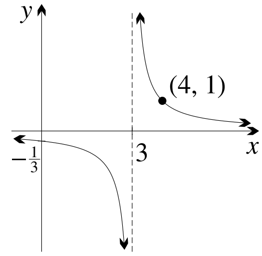
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

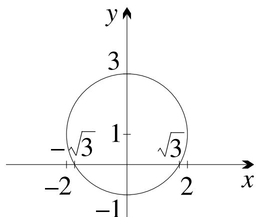
c $y = x^2 - 4$



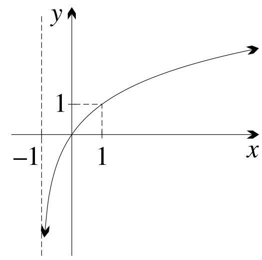
d $y = \frac{1}{x-3}$



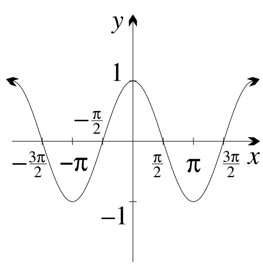
e $x^2 + (y-1)^2 = 4$



f $y = \log_2(x+1)$

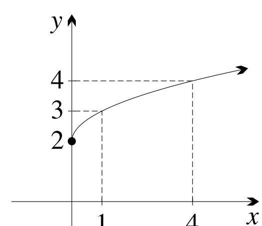


g $y = \sin(x + \frac{\pi}{2})$

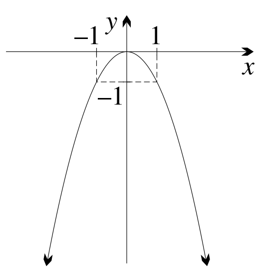


This is also $y = \cos x$.

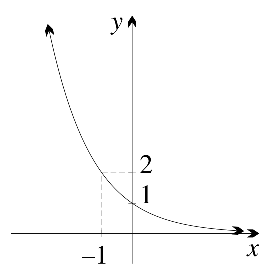
h $y = \sqrt{x} + 2$



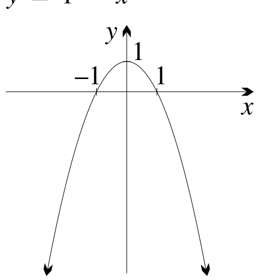
2 a $y = -x^2$



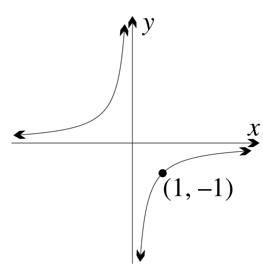
b $y = 2^{-x}$



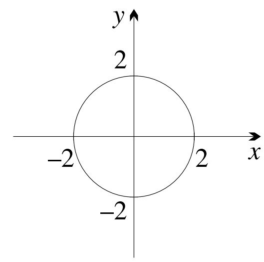
c $y = 1 - x^2$



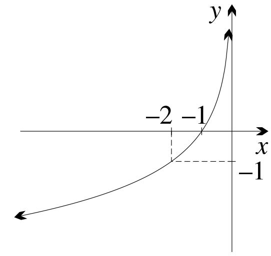
d $y = -\frac{1}{x}$



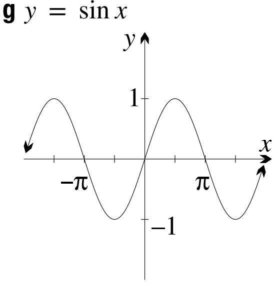
e $x^2 + y^2 = 4$



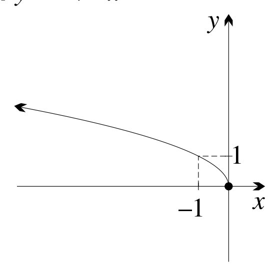
f $y = -\log_2(-x)$



g $y = \sin x$



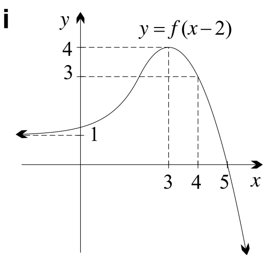
h $y = \sqrt{-x}$



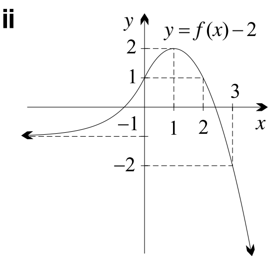
3 In part **e** the circle is symmetric in the y-axis.
In part **g** $y = \sin x$ is an odd function, and so is unchanged by a rotation of 180°.

4 a $r = 2,\ (-1, 0)$ **b** $r = 1,\ (1, 2)$
c $r = 2,\ (2, 0)$ **d** $r = 5,\ (0, 3)$

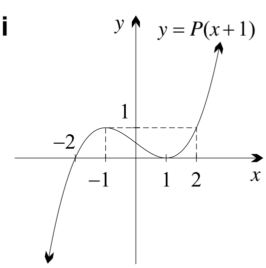
5 a i



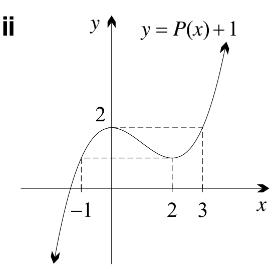
ii



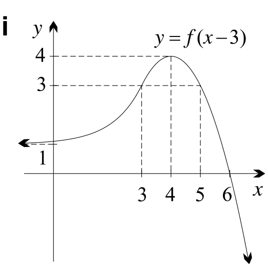
b i



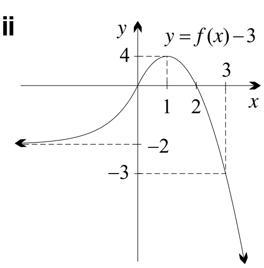
ii



6 a i



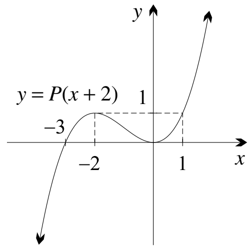
ii



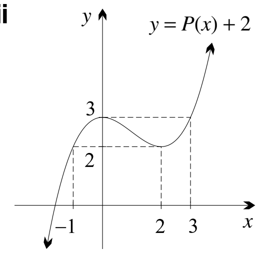
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b **i**



ii



7 **a** $y = (x + 1)^2 + 2$ **b** $y = \frac{1}{x - 2} + 3$

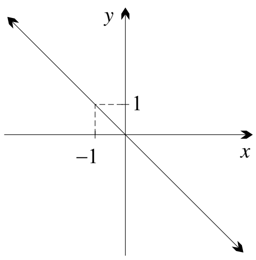
c $y = \cos\left(x - \frac{\pi}{3}\right) - 2$ **d** $y = e^{x+2} - 1$

8 **a** From $y = -x$:

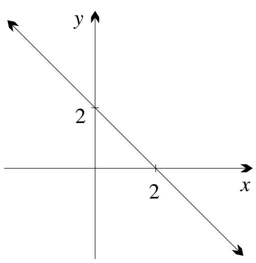
i shift up 2 (or right 2)

ii shift down 2 (or left 2)

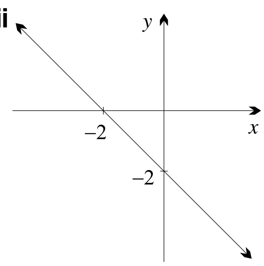
iii reflect in x-axis (or y-axis) and shift up 4 (or left 4)



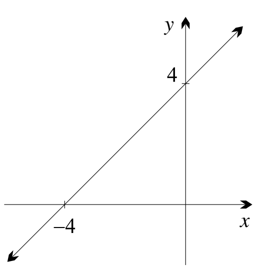
i



ii



iii

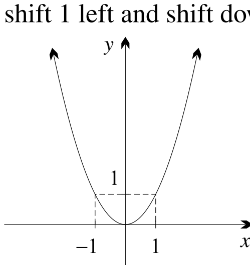


b From $y = x^2$:

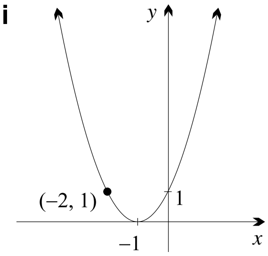
i shift 1 left

ii shift 1 left and reflect in x-axis

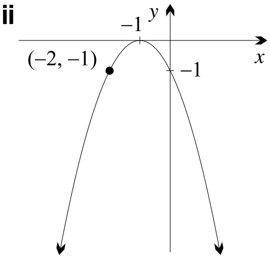
iii shift 1 left and shift down 1



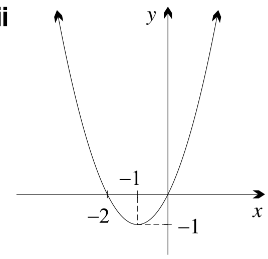
i



ii



iii

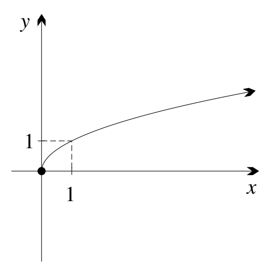


c From $y = \sqrt{x}$:

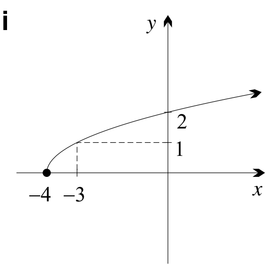
i shift 4 left

ii reflect in x-axis

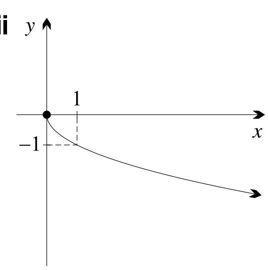
iii shift 4 left and reflect in x-axis



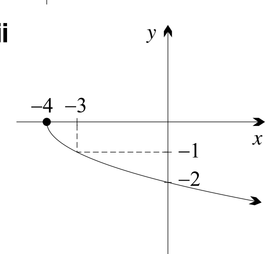
i



ii



iii



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

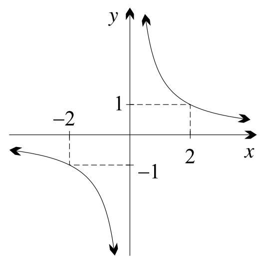
Cambridge University Press Updated July 2022

d From $y = \dfrac{2}{x}$:

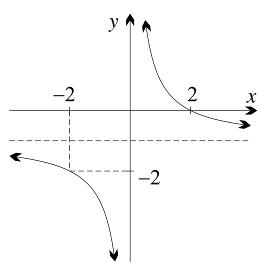
i shift down 1

ii shift down 1, left 2

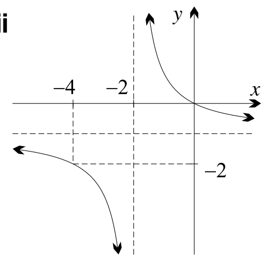
iii reflect in the x-axis or in the y-axis



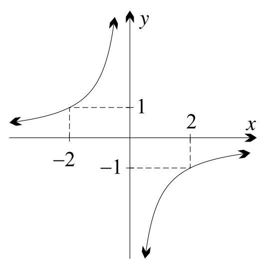
i



ii



iii

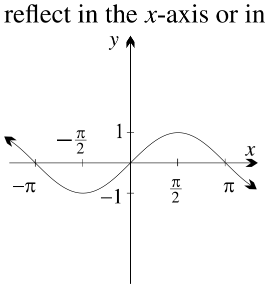


e From $y = \sin x$:

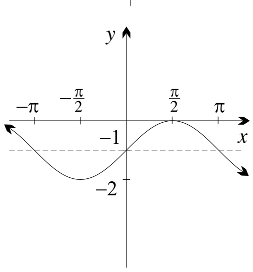
i shift down 1

ii shift down 1, left 2

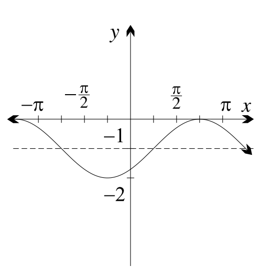
iii reflect in the x-axis or in the y-axis



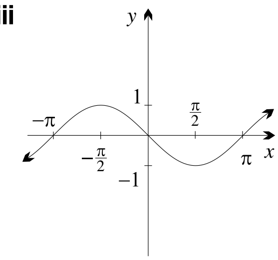
i



ii



iii



9 a $(1, -2)$ and $(-1, 2)$

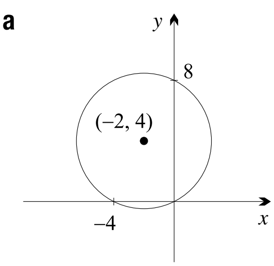
b i $y = x^3 - 3x + 1$

ii $(1, -1)$ and $(-1, 3)$

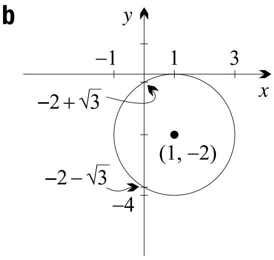
c i $y = x^3 + 3x^2 - 2$

ii $(0, -2)$ and $(-2, 2)$

10 a



b



11 a The parabola $y = x^2$ shifted left 2, down 1. $y + 1 = (x + 2)^2$

b The hyperbola $xy = 1$ shifted right 2, down 1. $y + 1 = \frac{1}{x - 2}$

c The exponential $y = 2^x$ reflected in the x-axis, shifted 1 up. $y = 1 - 2^x$

d The curve $y = \cos x$ reflected in the x-axis and shifted 1 up. $y = 1 - \cos x$

12 a The parabola $y = x^2$ reflected in the x-axis, then shifted 3 right and 1 up. $y - 1 = -(x - 3)^2$

b The curve $y = \log_2 x$ reflected in the y-axis, then shifted right 2, down 1. $y + 1 = -\log_2(x - 2)$

c The half parabola $y = \sqrt{x}$ reflected in the x-axis , then shifted left 4 and 2 up. $y - 2 = -\sqrt{x + 4}$

13 a i The result is a rotation of 180°, so odd functions are unchanged.

ii $\mathcal{I}$ and $\mathcal{H}$ do not commute.

b i $y = f(2a - x)$

ii $x = a$

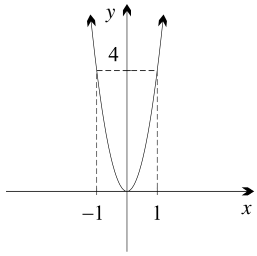
iii $g(a + t) = g(a - t)$ so $g(x)$ is symmetric in $x = a$.

ISBN 978-1-108-76630-2

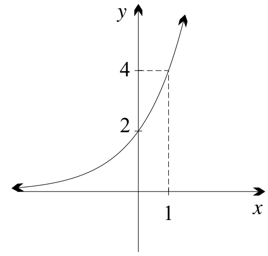
Cambridge University Press
Updated July 2022

Exercise 3H

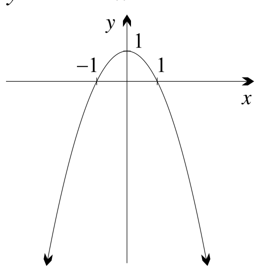
1 a $y = 4x^2$



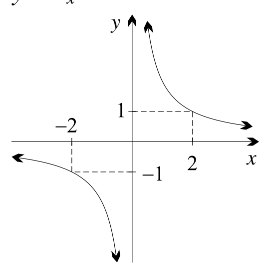
b $y = 2 \times 2^x = 2^{x+1}$



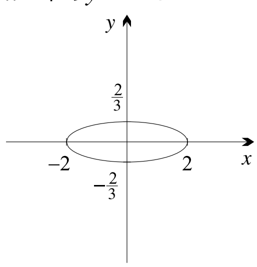
c $y = 1 - x^2$



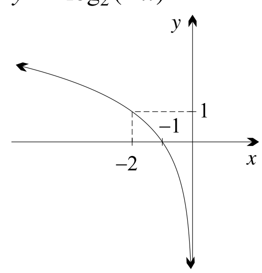
d $y = \frac{2}{x}$



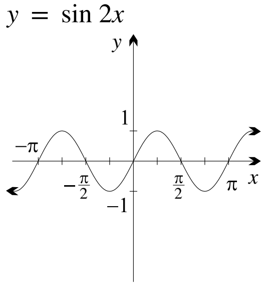
e $x^2 + 9y^2 = 4$



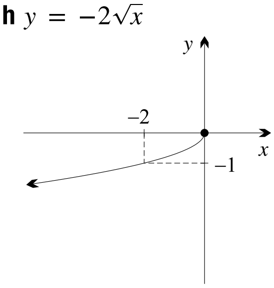
f $y = \log_2(-x)$



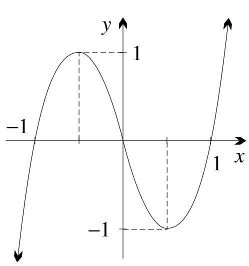
g $y = \sin 2x$



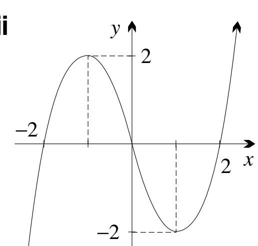
h $y = -2\sqrt{x}$



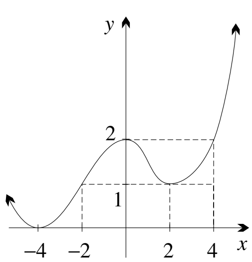
2 a i



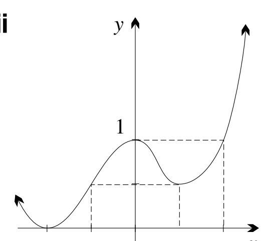
ii



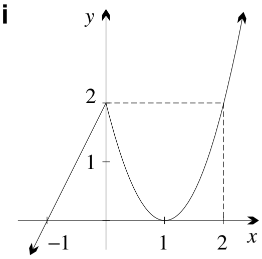
b i



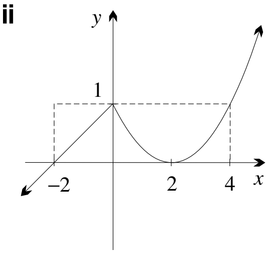
ii



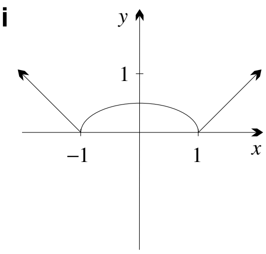
c i



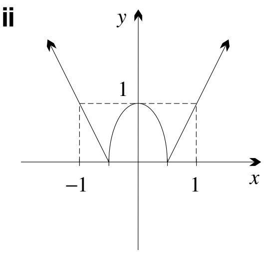
ii



d i



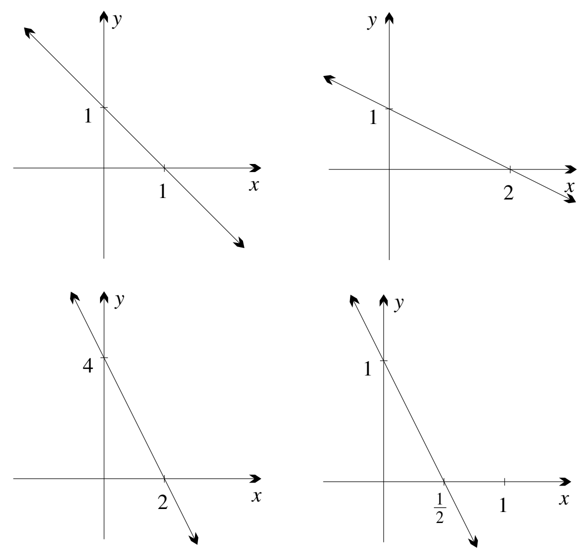
ii



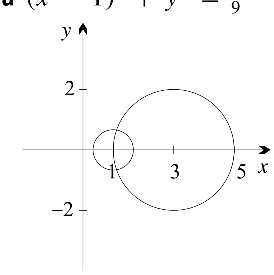
3 a stretch horizontally by factor 2

b stretch horizontally by factor 2, vertically by factor 4

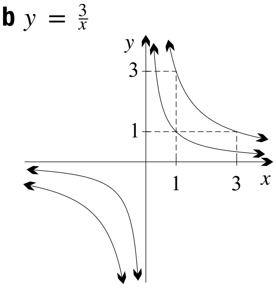
c stretch horizontally by factor $\frac{1}{2}$



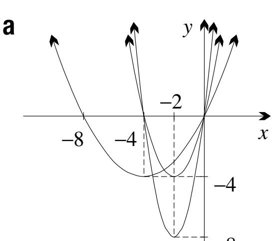
4 a $(x - 1)^2 + y^2 = \frac{4}{9}$



b $y = \frac{3}{x}$



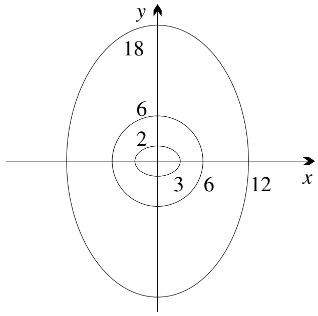
5 a



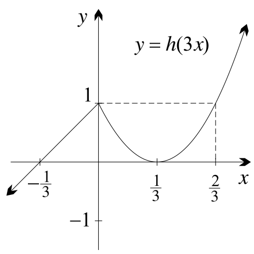
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

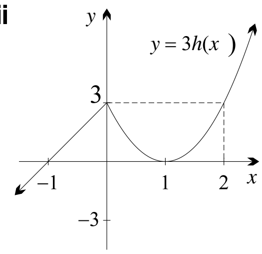
b



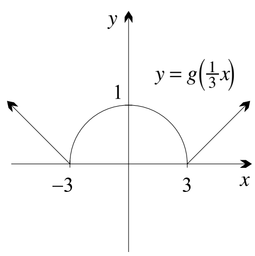
6 a i



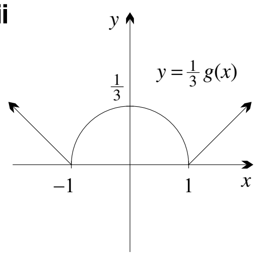
ii



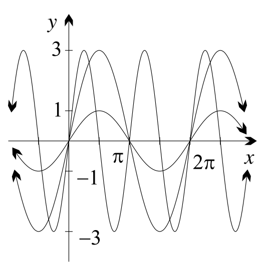
b i



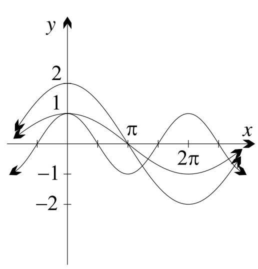
ii



7 a



b



8 a $(1, -2)$ and $(-1, 2)$

b i $y = 2x^3 - 6x$

ii $(1, -4)$ and $(-1, 4)$

c i $y = \frac{1}{27}x^3 - x$

ii $(3, -2)$ and $(-3, 2)$

9 a vertical factor 3 **b** horizontal factor $\frac{1}{2}$

c horizontal factor 4 **d** vertical factor 2

10 a $y = \frac{2}{x}$ **b** $y = \frac{2}{x}$

c Both dilations give the same graph.

d yes: by factor $\sqrt{2}$

11 a $y = 4x^2$ **b** $y = 4x^2$

c Both dilations give the same graph.

d no

12 a $M(0) = 3$ **b** 53 years

c i The mass has been dilated by factor 2, so $M = 6 \times 2^{-\frac{1}{53}t}$

14 a The unit circle $x^2 + y^2 = 1$, horizontally by 3, vertically by 2. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

b The exponential $y = 3^x$, vertically by -2. $y = -2 \times 3^x$

c The curve $y = \tan x$, horizontally by 3, vertically by 2. $y = 2\tan\frac{x}{3}$

15 a i stretch vertically by factor 2, $\frac{y}{2} = 2^x$, or translate left by 1, $y = 2^{(x+1)}$

ii stretch along both axes by k, $\dfrac{y}{k} = \dfrac{1}{\frac{x}{k}}$, or stretch horizontally by k^2, $y = \dfrac{1}{\frac{x}{k^2}}$

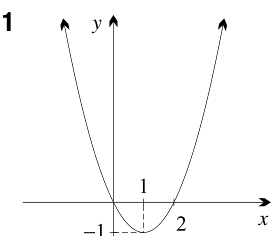
iii reciprocal, $y = \frac{1}{3^x}$, or reflect in the y-axis , $y = 3^{-x}$

16 vertically by factor a^2

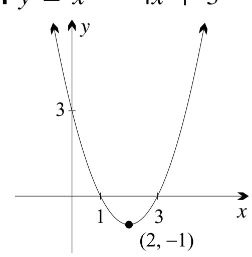
17 stretch horizontally by factor $\sqrt{3}$ and vertically by factor $3\sqrt{3}$

Exercise 3I

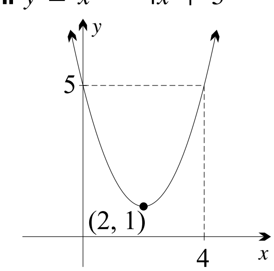
1



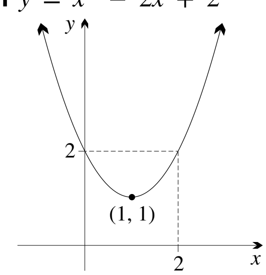
a i $y = x^2 - 4x + 3$



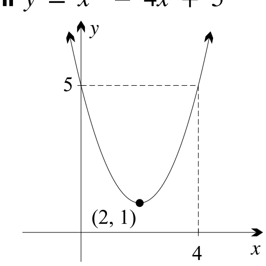
ii $y = x^2 - 4x + 5$



b i $y = x^2 - 2x + 2$



ii $y = x^2 - 4x + 5$

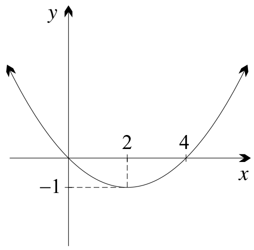


c yes

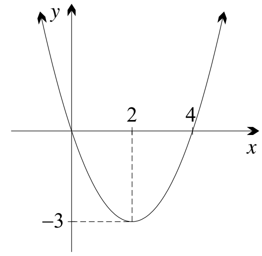
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

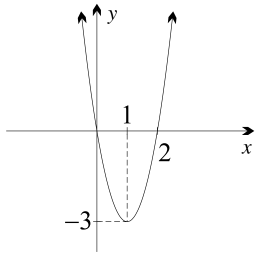
2 a i $y = \frac{1}{4}x^2 - x$



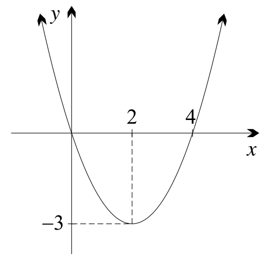
ii $y = \frac{3}{4}x^2 - 3x$



b i $y = 3x^2 - 6x$

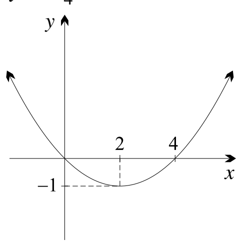


ii $y = \frac{3}{4}x^2 - 3x$

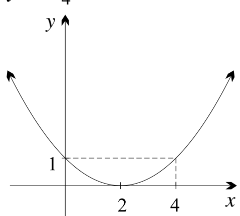


c yes

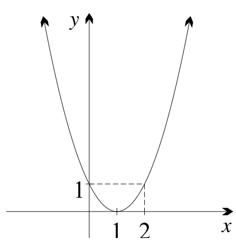
3 a i $y = \frac{1}{4}x^2 - x$



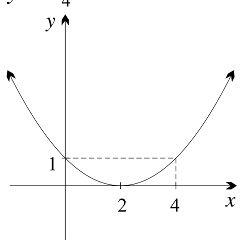
ii $y = \frac{1}{4}x^2 - x + 1$



b i $y = x^2 - 2x + 1$

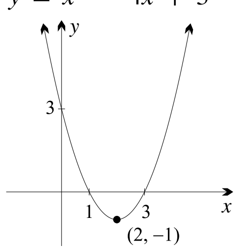


ii $y = \frac{1}{4}x^2 - x + 1$

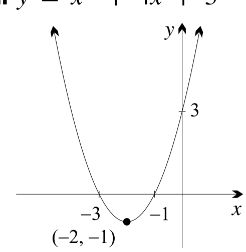


c yes

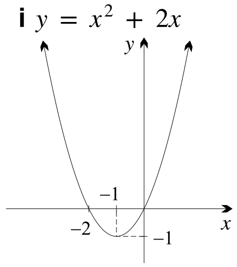
4 a i $y = x^2 - 4x + 3$



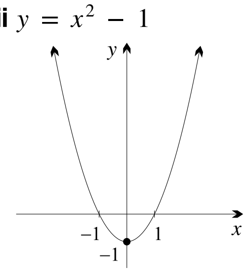
ii $y = x^2 + 4x + 3$



b i $y = x^2 + 2x$



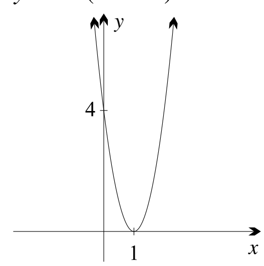
ii $y = x^2 - 1$



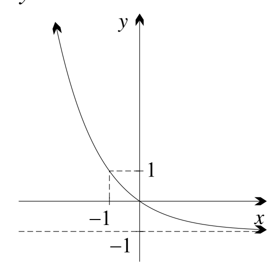
c no: order matters

5 a no **b** no **c** yes **d** yes **e** no **f** yes

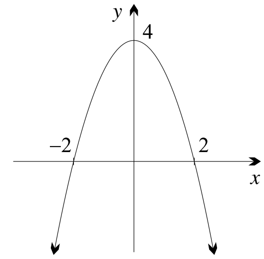
6 a $y = 4(x - 1)^2$



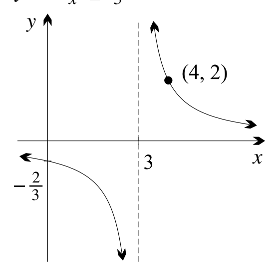
b $y = 2^{-x} - 1$



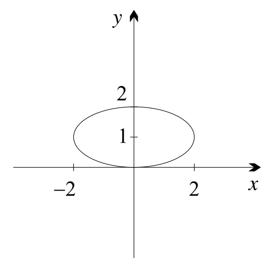
c $y = 4 - x^2$



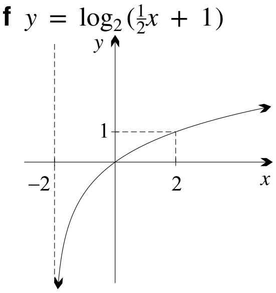
d $y = \frac{2}{x - 3}$



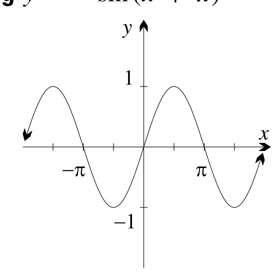
e $x^2 + 4(y - 1)^2 = 4$



f $y = \log_2(\frac{1}{2}x + 1)$

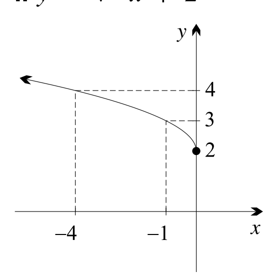


g $y = -\sin(x + \pi)$

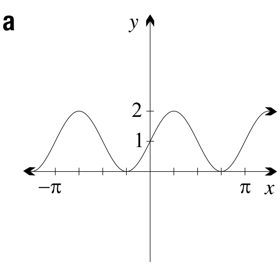


This is also $y = \sin x$.

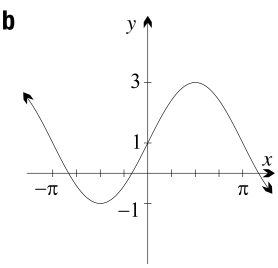
h $y = \sqrt{-x} + 2$



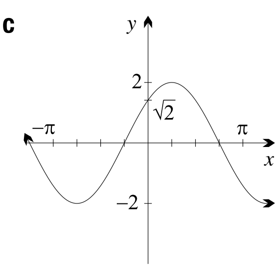
7 a



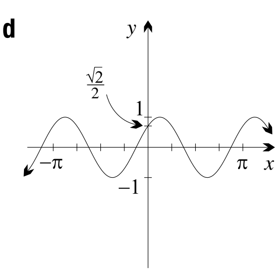
b



c



d



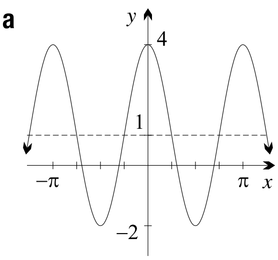
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

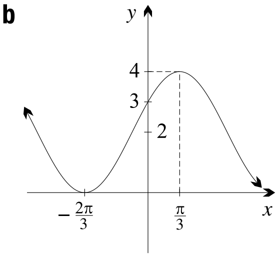
8 a $y = \frac{1}{4}(x + 2)^2 - 4$ **b** $y = \frac{1}{4}(x + 1)^2 - 4$

c $y = 2 - 2^x$ **d** $y = \frac{2}{x - 2} + 1$

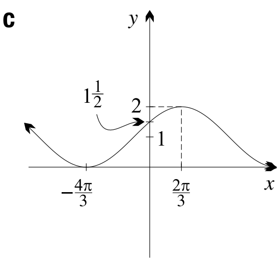
10 a



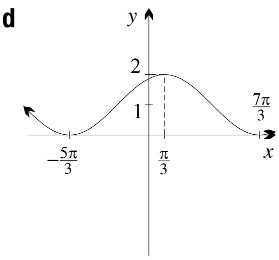
b



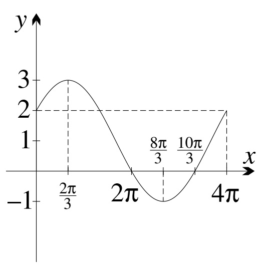
c



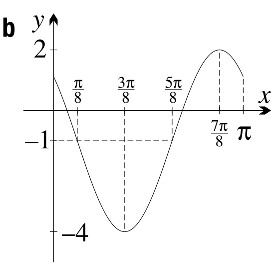
d



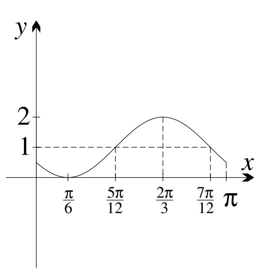
11 a



b



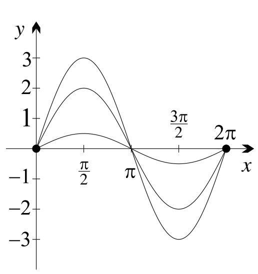
c



12 d If the transformations are dilations or reflections then they commute. A reflection is a special type of dilation, and any pair of dilations commute. If one of the transformations is a translation in a different direction to the other transformation, for example $\mathcal{H}$ and $\mathcal{V}$, then they commute.

Exercise 3J

1 a

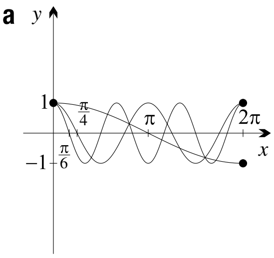


i $\frac{1}{2}$ **ii** 2 **iii** 3

b The graph $y = \sin x$ is stretched vertically by a factor of k.

c The amplitude increases. The bigger the amplitude, the steeper the wave.

2 a

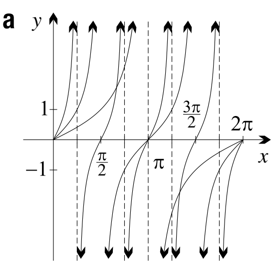


i 4π **ii** π **iii** $\frac{2\pi}{3}$

b The graph $y = \cos x$ is stretched horizontally by a factor of $\dfrac{1}{n}$.

c The period decreases.

3 a

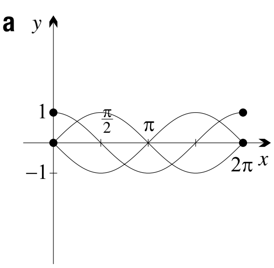


i π **ii** 2π **iii** $\frac{\pi}{2}$

b The graph $y = \tan x$ is stretched horizontally by a factor of $\frac{1}{a}$.

c The period decreases.

4 a

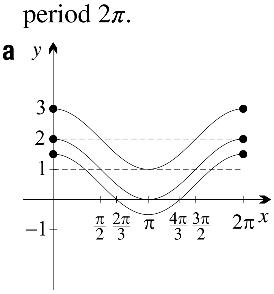


i $\frac{\pi}{2}$ **ii** π **iii** 2π or 0

b The graph $y = \sin x$ is shifted α units to the left.

c The graph stays the same, because $y = \sin x$ has period 2π.

5 a



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

i Range: $0 \le y \le 2$ or $[0, 2]$, mean value: 1

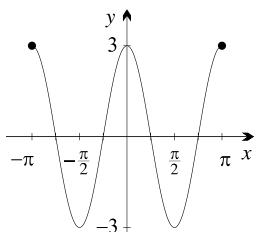
ii Range: $[1, 3]$, mean value: 2

iii Range: $\left[\frac{1}{2}, \frac{3}{2}\right]$, mean value: $\frac{1}{2}$

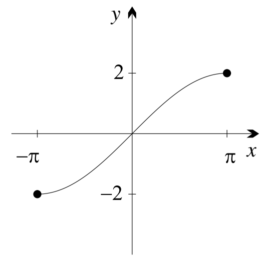
b The graph $y = \cos x$ is shifted c units up, and the mean value is c.

c It moves up.

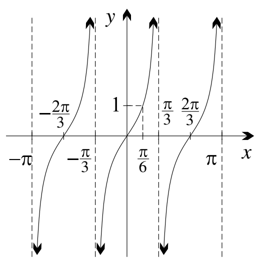
6 a period $= \pi$, amplitude $= 3$



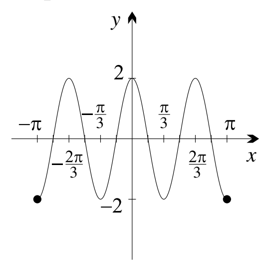
b period $= 4\pi$, amplitude $= 2$



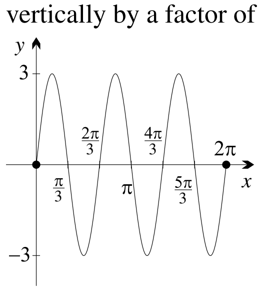
c period $= \frac{2\pi}{3}$, no amplitude



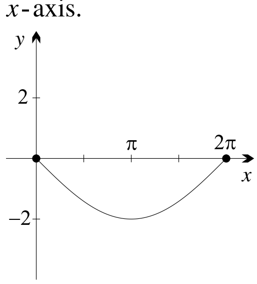
d period $= \frac{2\pi}{3}$, amplitude $= 2$



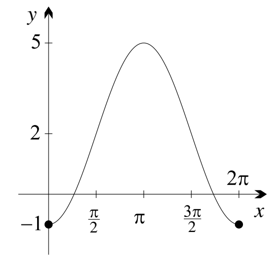
7 a Stretch horizontally by a factor of $\frac{1}{3}$, then stretch vertically by a factor of 3.



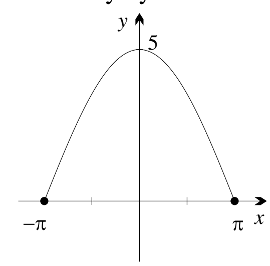
b Stretch horizontally with factor 2, then stretch vertically with factor 2, then reflect in the x-axis.



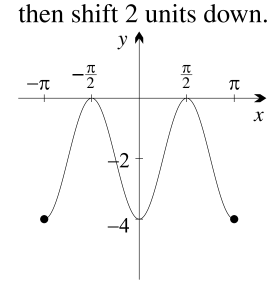
c Shift $\frac{\pi}{2}$ units right, then stretch vertically by a factor of 3, then shift 2 units up.



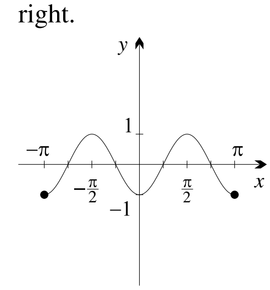
8 a Stretch horizontally by a factor of 2, then stretch vertically by a factor of 5.



b Stretch horizontally by a factor of $\frac{1}{2}$, then stretch vertically by a factor of 2, then reflect in the x-axis, then shift 2 units down.



c Stretch horizontally by a factor of $\frac{1}{2}$, then shift $\frac{\pi}{2}$ units right.



9 a i Dilate horizontally with factor $\frac{1}{5}$, then shift left $\frac{\pi}{3}$.

ii Period $= \frac{2\pi}{5}$, phase $= \frac{\pi}{3}$.

b i Dilate horizontally with factor 3, then shift left π.

ii Period $= 6\pi$, phase $= \pi$.

c i Dilate horizontally with factor $\frac{5}{3}$, then shift left $-\frac{2\pi}{5}$.

ii Period $= \frac{5\pi}{3}$, phase $= -\frac{2\pi}{5}$.

10 a i period $= \pi$, phase $= -\frac{\pi}{3}$, amplitude $= 3$, mean value $= 0$.

ii period $= 6\pi$, phase $= \pi$, amplitude $= \frac{5}{2}$, mean value $= 5$.

iii period $= \frac{\pi}{3}$, phase $= \frac{\pi}{8}$, no amplitude, no mean value.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 a i $y = \sin 6\left(x - \frac{\pi}{12}\right)$. Stretch horizontally with factor $\frac{1}{6}$, then shift left $-\frac{\pi}{12}$.

ii period $= \frac{\pi}{3}$, phase $= -\frac{\pi}{12}$.

iii Shift left $-\frac{\pi}{2}$, then stretch horizontally with factor $\frac{1}{6}$.

b i $y = \cos \frac{1}{4}(x + 2\pi)$. Stretch horizontally with factor 4, then shift left 2π.

ii period $= 8\pi$, phase $= 2\pi$.

iii Shift left $\frac{\pi}{2}$, then stretch horizontally with factor 4.

c i $y = \tan \frac{3}{5}\left(x - \frac{4\pi}{3}\right)$. Stretch horizontally with factor $\frac{5}{3}$, then shift left $-\frac{4\pi}{3}$.

ii period $= \frac{5\pi}{3}$, phase $= -\frac{4\pi}{3}$.

iii Shift left $-\frac{4\pi}{5}$, then stretch horizontally with factor $\frac{5}{3}$.

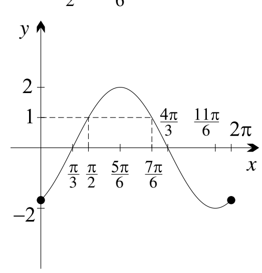
12 a i $y = \sin 3\left(x + \frac{\pi}{6}\right)$. Stretch horizontally with factor $\frac{1}{3}$, then shift $\frac{\pi}{6}$ units left.

ii $y = \frac{1}{4}\sin 4\left(x - \frac{\pi}{4}\right) - 4$. Stretch horizontally with factor $\frac{1}{4}$, then shift $-\frac{\pi}{4}$ units left, then stretch vertically by a factor of $\frac{1}{4}$, then shift 4 units down.

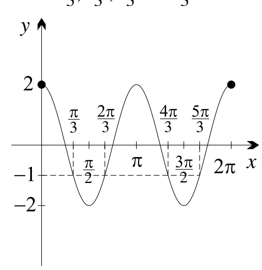
iii $y = 6\sin \frac{1}{2}\left(x + \frac{\pi}{2}\right) + 5$. Stretch horizontally with factor 2, then shift $\frac{\pi}{2}$ units left, then stretch vertically with factor 6, then shift up 5.

b Part(i): period $= \frac{2\pi}{3}$, phase $= \frac{\pi}{6}$, amplitude $= 1$, mean value $= 0$.
Part(ii): period $= \frac{2\pi}{4} = \frac{\pi}{2}$, phase $= -\frac{\pi}{4}$, amplitude $= \frac{1}{4}$, mean value $= -4$.
Part(iii): period $= 4\pi$, phase $= \frac{\pi}{2}$, amplitude $= 6$, mean value $= 5$.

13 a $x = \frac{\pi}{2}$ or $\frac{7\pi}{6}$



b $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

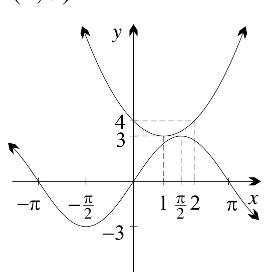


14 a $x \doteqdot 1.675$

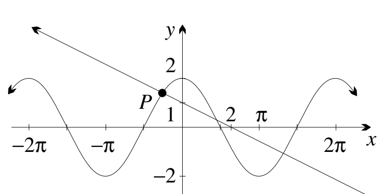
b $x \doteqdot 0.232$ or 1.803

15 a $(1, 3)$

b



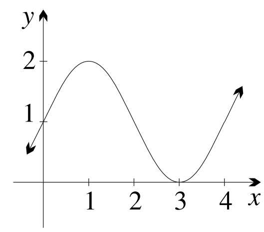
16



c 3

d P is in the second quadrant.

17



a 4

c the origin

d $m > \frac{1}{4}$ or $m = 0$

18 a 10 metres **b** 6 metres

c 2 pm **d** 9.20 am

19 c amplitude $= 5$

20 a ii 1 **iii** $0 < k < 1$

b ii 1.3

iii $\angle AOB = 2\theta \doteqdot 2.6$ radians

c ii $\ell > 300$

Chapter 3 review exercise

1 a i $-1 < x < 2$ **ii** $(-1, 2)$

b i $-1 \le x < 2$ **ii** $[-1, 2)$

c i $x \le 2$ **ii** $(-\infty, 2]$

2 a i 0 **ii** 4 **iii** 8 **iv** 0

b i $x^2 + 2x$ **ii** x^2

iii $x^4 - 2x^2$ **iv** $x + 2$

3 a $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

b $f(x) \to \frac{1}{2}$ as $x \to \infty$ and $x \to -\infty$

c $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

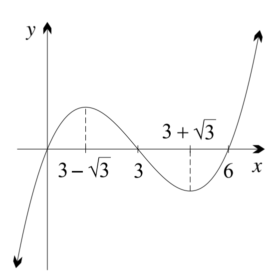
4 $y = x(x - 3)(x - 6)$

a $-\infty < x < \infty$

b $(0, 0)$, $(3, 0)$, $(6, 0)$

c no asymptotes

e $y' = 3(x^2 - 6x + 6)$ so $y' = 0$ at $x = 3 - \sqrt{3}$ or $x = 3 + \sqrt{3}$



5 a $-4 \le x < 2$, $[-4, 2)$

b $-\frac{3}{2} < x < 0$, $\left(-\frac{3}{2}, 0\right)$

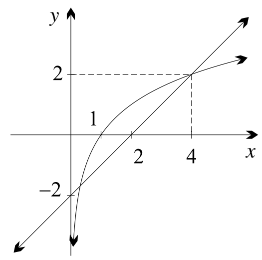
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

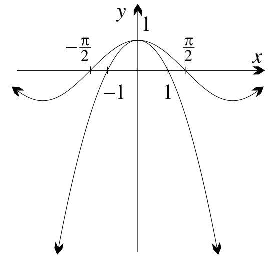
c $-3 \le x < \frac{1}{2}$, $[-3, \frac{1}{2})$

d $-4 \le x \le 10$, $[-6, 10]$

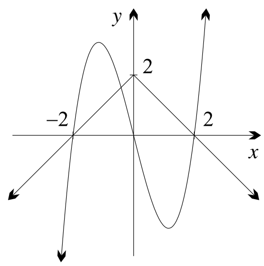
6 a 2 solutions



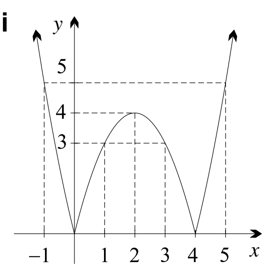
b 1 solution



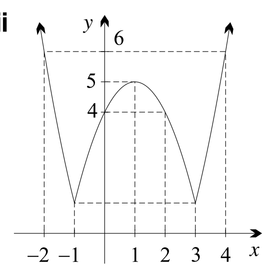
c 3 solutions



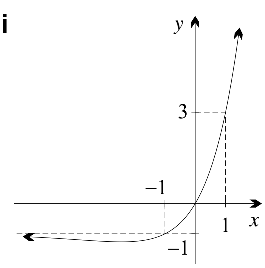
7 a i



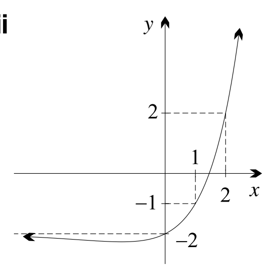
ii



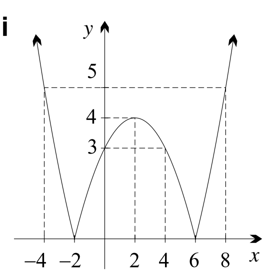
b i



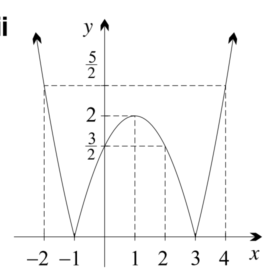
ii



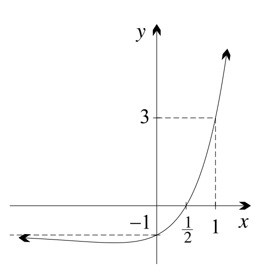
8 a i



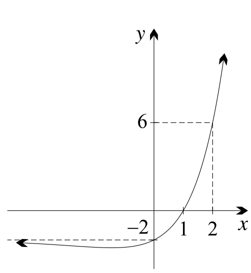
ii



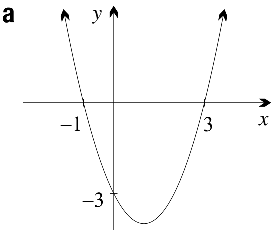
b i



ii

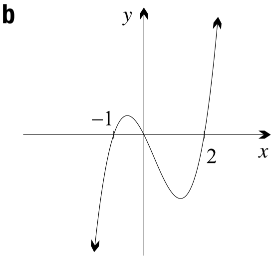


9 a



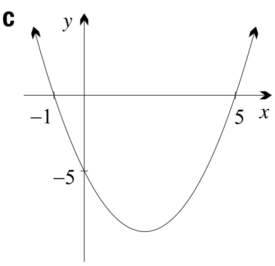
$-1 \le x \le 3$

b



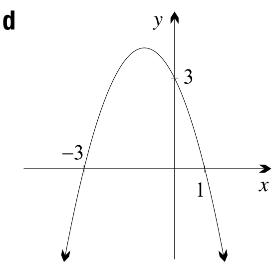
$x \le -1$ or $0 \le x \le 2$

c



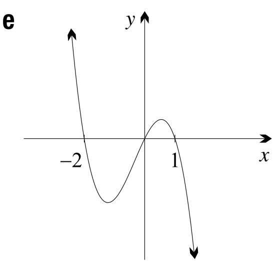
$-1 \le x \le 5$

d



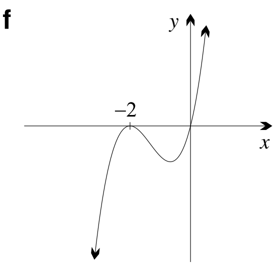
$x \le -3$ or $x \ge 1$

e



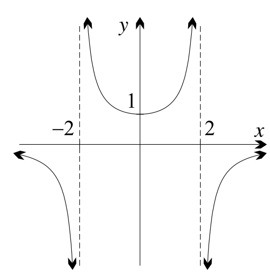
$x \le -2$ or $0 \le x \le 1$

f



$x \le 0$

10



a $x \neq -2, 2$

b $(0, 1)$

c $y \to 0$ as $x \to \infty$ and as $x \to -\infty$

e as $x \to 2^+$, $y < 0$ so $y \to -\infty$, as $x \to 2^-$, $y > 0$ so $y \to \infty$, as $x \to -2^+$, $y > 0$ so $y \to \infty$, and as $x \to -2^-$, $y < 0$ so $y \to -\infty$

f $(-\infty, 0) \cup [1, \infty)$

11 a $y = \frac{3(x+1)}{(x+3)(x-1)}$

b domain: $x \neq 1$ and $x \neq -3$ intercepts: $(-1, 0)$ and $(0, -1)$

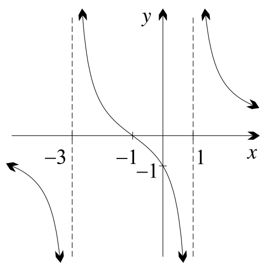
c The domain is not symmetric about $x = 0$.

d $x = -3$, $x = 1$, and $y = 0$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

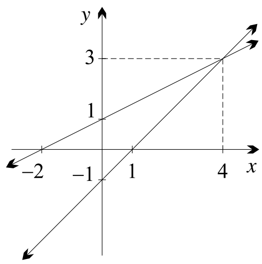
e



12 a $x = -3\frac{1}{2}$ or $3\frac{1}{2}$ **b** $x = \frac{1}{3}$ or 1

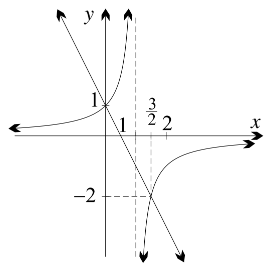
c $-3 \le x \le \frac{1}{3}$ **d** $x < -2$ or $x > -\frac{1}{3}$

13 a



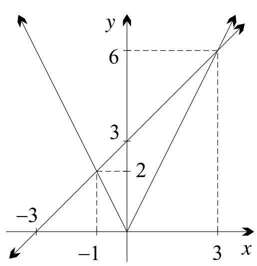
$x \ge 4$

b



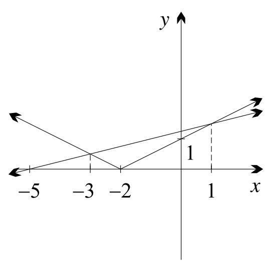
$0 < x < 1$ or $x > 1\frac{1}{2}$

c



$-1 \le x \le 3$

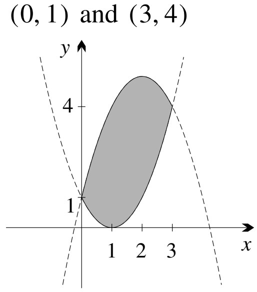
d



$x < -3$ or $x > 1$

14 a $(0, 1)$ and $(3, 4)$

b



15 a $y = (x - 2)^2 + 1$ **b** $y = \frac{1}{x + 2} - 3$

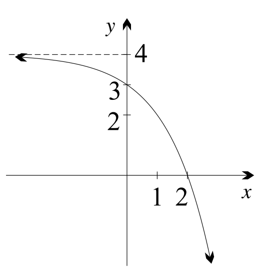
c $y = \sin(x + \frac{\pi}{6}) - 1$ **d** $y = e^{x-2} + 1$

16 a horizontal factor 2 **b** vertical factor $\frac{1}{2}$

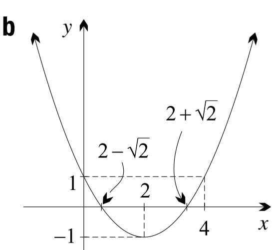
c vertical factor $\frac{1}{3}$ **d** horizontal factor 2

17 a yes **b** no **c** no **d** yes

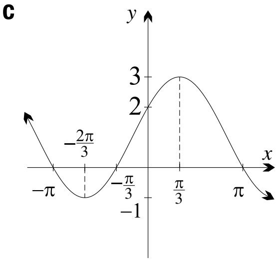
18 a



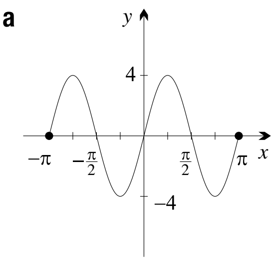
b



c

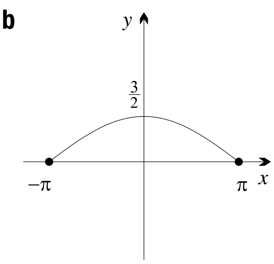


19 a



amplitude is 4, period is π

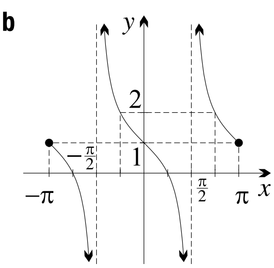
b



amplitude is $\frac{3}{2}$, period is 4π

20 a Reflect in the x-axis , then shift up 1 unit.

b



21 a i Stretch horizontally with factor $\frac{1}{5}$, then stretch vertically with factor 3, then shift down 2.

ii Stretch horizontally with factor $\frac{1}{4}$, then shift left $\frac{\pi}{6}$, then stretch vertically with factor 4.

iii $y = \cos 2(x - \frac{\pi}{6}) + 1$. Stretch horizontally with factor $\frac{1}{2}$, then shift left $-\frac{\pi}{6}$ units, then shift up 1.

b i period $= \frac{2\pi}{5}$, phase $= 0$, amplitude $= 3$, mean value $= -2$.

ii period $= \frac{\pi}{2}$, phase $= \frac{\pi}{6}$, amplitude $= 4$, mean value $= 0$.

iii period $= \pi$, phase $= -\frac{\pi}{6}$, amplitude $= 1$, mean value $= 1$.

22 a 3

b 3 solutions, 1 positive solution

c Outside this domain the line is beyond the range of the sine curve.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 4

Exercise 4A

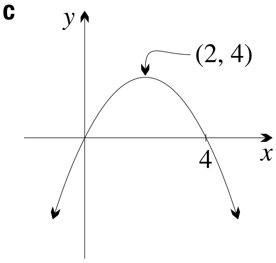
1 a A, G and I **b** C and E

c B, D, F and H

3 a $4 - 2x$

b i $x < 2$ **ii** $x > 2$ **iii** $x = 2$

c

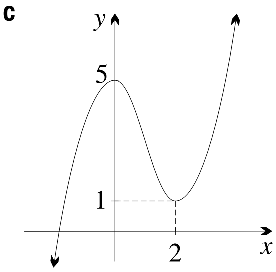


4 a $3x^2 - 6x$

b i $x < 0$ or $x > 2$ **ii** $0 < x < 2$

iii $x = 0$ or $x = 2$

c



5 a $\dfrac{3}{x^2}$

b The function is not continuous at $x = 0$.

6 a $x > 2$ **b** $x < -3$

c $x > 1$ or $x < -1$ **d** $x < 0$ or $x > 2$

7 a $-\frac{1}{3} < x < 1$ **b** $x < -2$ or $x > 4$

8 a $f'(x) = x^2 + 2x + 5$

b $f'(x) = (x + 1)^2 + 4 > 0$ for all x.

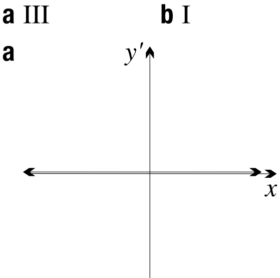
c The graph of $f(x)$ is increasing for all x, so it has exactly one x-intercept.

9 a $f'(x) = -\dfrac{6}{(x - 3)^2} < 0$ for $x \neq 3$.

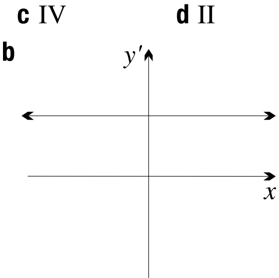
b $f'(x) = \dfrac{x^2(x^2 + 3)}{(x^2 + 1)^2} > 0$ for $x \neq 0$.

11 a III **b** I **c** IV **d** II

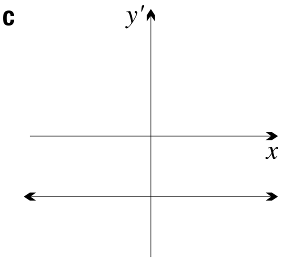
12 a



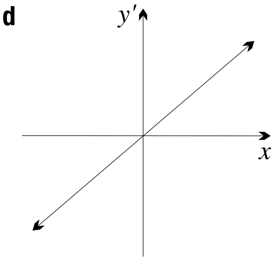
b



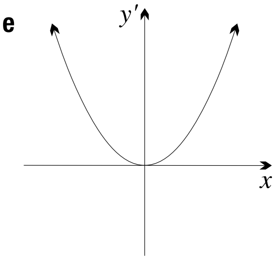
c



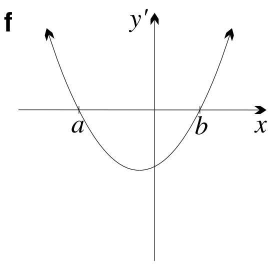
d



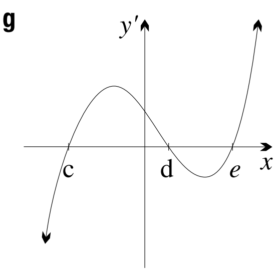
e



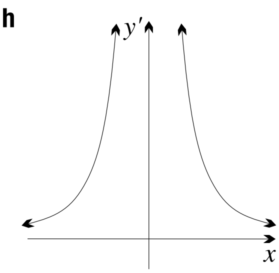
f



g



h

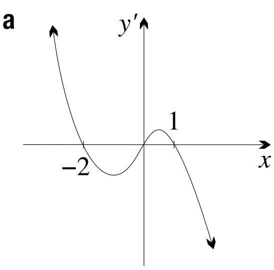


13 $-2 < x < 0$

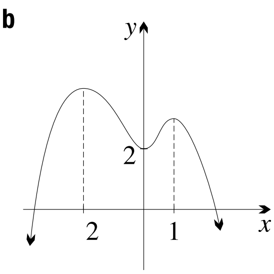
14 a i $f'(x) = \dfrac{-4x}{(x^2 + 1)^2}$ **ii** $f(0) = 1$

b $f(x)$ is a continuous function that is increasing for $x < 0$, decreasing for $x > 0$ and stationary at $x = 0$. So it reaches its maximum value at $x = 0$.

15 a



b



Exercise 4B

1 a $x = 3$ **b** $x = -2$ **c** $x = 1$ or -1

2 a $(2, 3)$ **b** $(4, 0)$

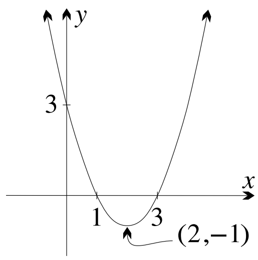
c $(1, -2)$ **d** $(1, 0)$

e $(0, 0)$ and $(2, -4)$ **f** $(1, -2)$

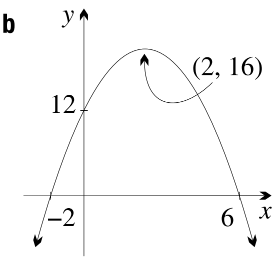
Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

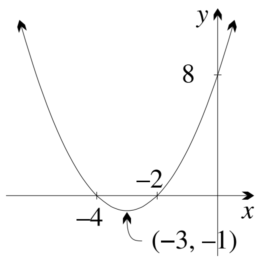
3 a



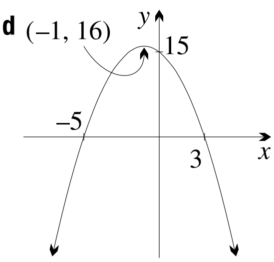
b



c

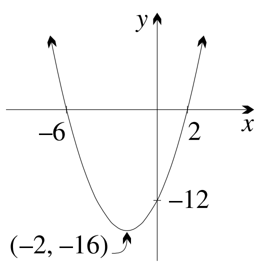


d

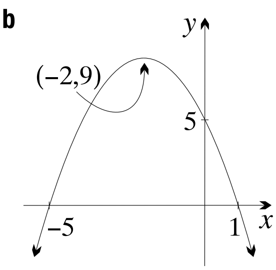


4 a minimum **b** maximum **c** minimum
d horizontal (or stationary) point of inflection

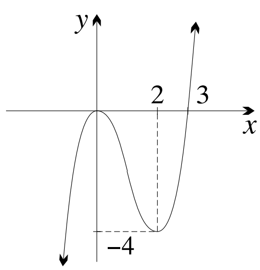
5 a



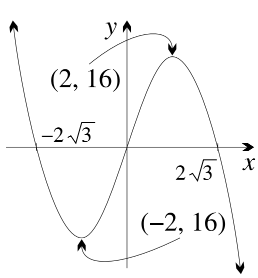
b



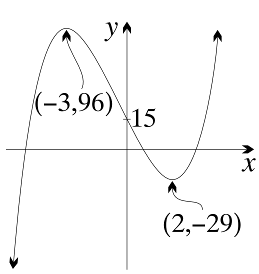
6 c



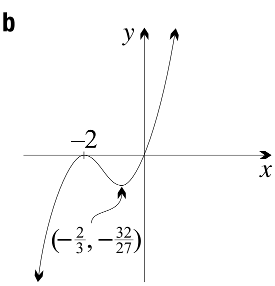
7



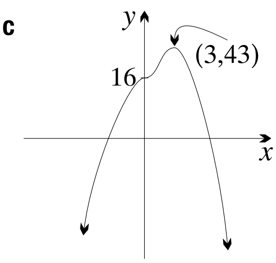
8 a



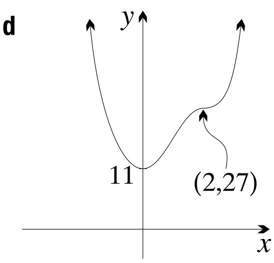
b



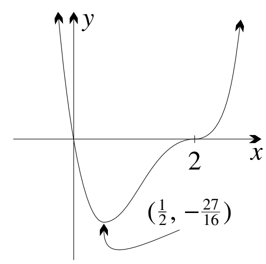
c



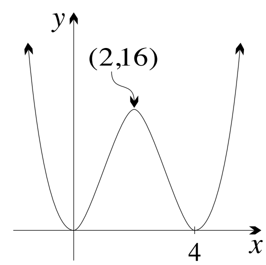
d



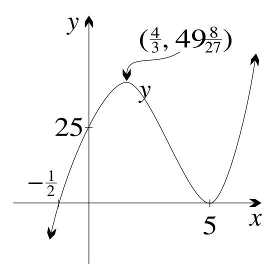
9



10



11



12 a $a = -8$ **b** $a = 2$

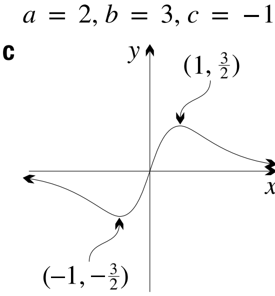
13 a $a = 2$ and $c = 3$ **b** $b = -3$ and $c = -24$

14 b $a = b = -1, c = 6$

15 a The curve passes through the origin.
c $a = -1$

16 $a = 2, b = 3, c = -12, d = 7$

17 c



d **i** no roots **ii** 1 root
iii 2 roots **iv** 1 root

19 c Hint: Consider the equation $P'(x) = 0$.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

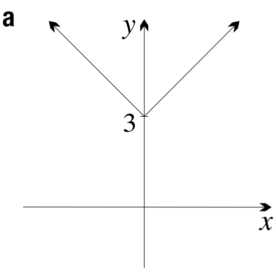
Cambridge University Press
Updated July 2022

Exercise 4C

1 a A maximum turning point, B minimum turning point
b C minimum
c D horizontal point of inflection, E maximum turning point
d F minimum turning point, G maximum, H minimum turning point
e I minimum
f J horizontal point of inflection, K minimum turning point, L maximum turning point

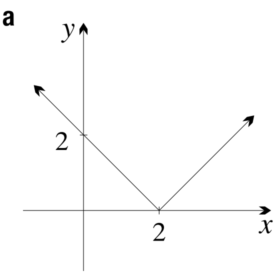
2 a $x = 0$ turning point, $x = 3$ horizontal point of inflection
b $x = -2$ turning point, $x = 4$ turning point
c $x = 0$ turning point, $x = 1$ discontinuity of y'
d $x = 0$ horizontal point of inflection, $x = 1$ discontinuity of y'
e $x = 0$ turning point, $x = 1$ discontinuity of y'
f $x = 0$ horizontal point of inflection, $x = 1$ discontinuity of y'
g $x = -1$ turning point, $x = 1$ turning point, $x = 0$ discontinuity of y'
h $x = 1$ turning point, $x = 0$ discontinuity of y'
i $x = 2$ turning point, $x = -2$ discontinuity of y', $x = 1$ discontinuity of y'

3 a



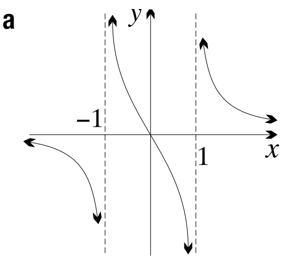
b When $x > 0$, $y' = 1$. When $x < 0$, $y = -1$.
c y' is not defined at $(0, 3)$.

4 a

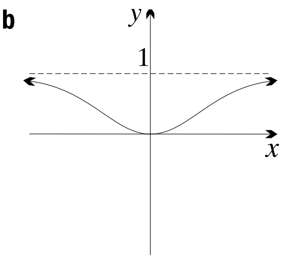


b When $x > 2$, $y' = 1$. When $x < 2$, $y' = -1$
c y' is not defined at $(2, 0)$.

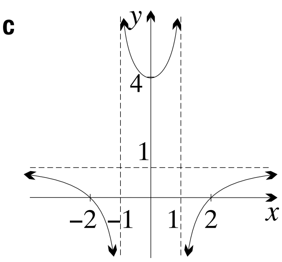
5 a



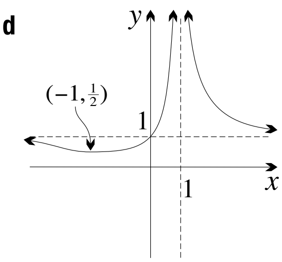
b



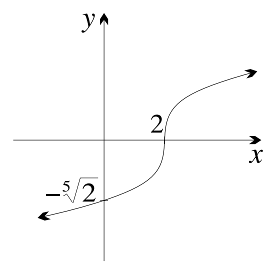
c



d

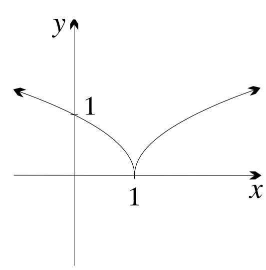


6



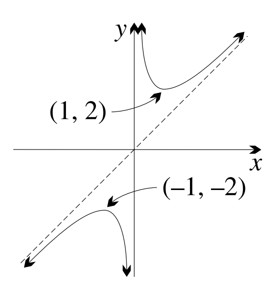
a $f'(x) = \dfrac{1}{5}(x - 2)^{-\frac{4}{5}}$
b There is a vertical tangent at $(2, 0)$.

7



b There is a cusp at $(1, 0)$.
c $f(x) = (x - 1)^{\frac{2}{3}}$ in Question 7 has a global minimum at $(1, 0)$.

8



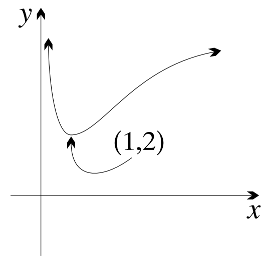
a $x \neq 0$
b Zeroes at $x = 1$ and $x = -1$, discontinuity at $x = 0$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

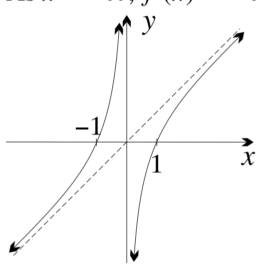
d The oblique asymptote is $y = x$ because $y - x \to 0$ as $|x| \to \infty$. The y-axis is a vertical asymptote.

9

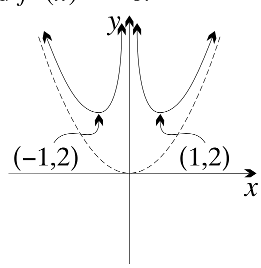


a $x > 0$
b $x = 1$
c $(1, 2)$ is a minimum turning point.
d As $x \to \infty$, $f(x) \to \infty$ and $f'(x) \to 0$.

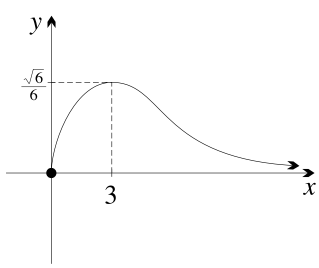
10 a



b

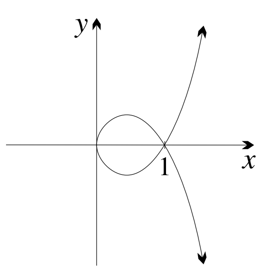


11

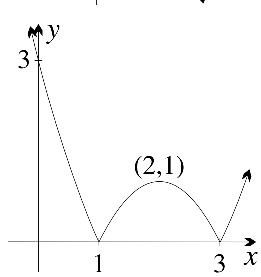


a domain: $x \geq 0$. horizontal asymptotes: $x = 0$
c $\left(3, \frac{1}{6}\sqrt{6}\right)$ is a maximum turning point.
d As $x \to 0^+$, $y \to 0$ and $y' \to \infty$, so the curve emerges vertically from the origin. (Notice that $y(0) = 0$, so the origin lies on the curve.)

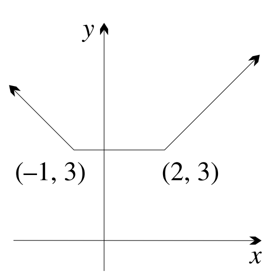
12 b



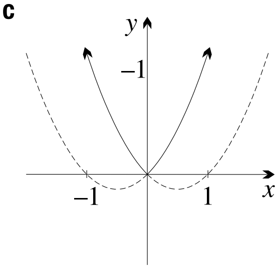
13 a



b



c



Exercise 4D

1 a $3x^2, 6x, 6$ **b** $10x^9, 90x^8, 720x^7$
c $7x^6, 42x^5, 210x^4$ **d** $2x, 2, 0$
e $8x^3, 24x^2, 48x$ **f** $15x^4, 60x^3, 180x^2$
g $-3, 0, 0$ **h** $2x - 3,\ 2,\ 0$
i $12x^2 - 2x, 24x - 2, 24$
j $20x^4 + 6x^2, 80x^3 + 12x, 240x^2 + 12$

2 a $2x + 3, 2$ **b** $3x^2 - 8x, 6x - 8$
c $2x - 1, 2$ **d** $6x - 13, 6$
e $30x^4 - 36x^3, 120x^3 - 108x^2$
f $32x^7 + 40x^4, 224x^6 + 160x^3$

3 a $0.3x^{-0.7}, -0.21x^{-1.7}, 0.357x^{-2.7}$
b $-\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}$ **c** $-\frac{2}{x^3}, \frac{6}{x^4}, -\frac{24}{x^5}$
d $-\frac{15}{x^4}, \frac{60}{x^5}, -\frac{300}{x^6}$ **e** $2x - \frac{1}{x^2}, 2 + \frac{2}{x^3}, -\frac{6}{x^4}$

4 a $-\frac{3}{x^4}, \frac{12}{x^5}$ **b** $-\frac{4}{x^5}, \frac{20}{x^6}$
c $-\frac{6}{x^3}, \frac{18}{x^4}$ **d** $-\frac{6}{x^4}, \frac{24}{x^5}$

5 a $2(x + 1), 2$
b $9(3x - 5)^2, 54(3x - 5)$
c $8(4x - 1), 32$
d $-11(8 - x)^{10}, 110(8 - x)^9$

6 a $\frac{-1}{(x + 2)^2}, \frac{2}{(x + 2)^3}$ **b** $\frac{2}{(3 - x)^3}, \frac{6}{(3 - x)^4}$
c $\frac{-15}{(5x + 4)^4}, \frac{300}{(5x + 4)^5}$ **d** $\frac{12}{(4 - 3x)^3}, \frac{108}{(4 - 3x)^4}$

7 a $\frac{1}{2\sqrt{x}}, \frac{-1}{4x\sqrt{x}}$ **b** $\frac{1}{3}x^{-\frac{2}{3}}, -\frac{2}{9}x^{-\frac{5}{3}}$
c $\frac{3}{2}\sqrt{x}, \frac{3}{4\sqrt{x}}$ **d** $-\frac{1}{2}x^{-\frac{3}{2}}, \frac{3}{4}x^{-\frac{5}{2}}$
e $\frac{1}{2\sqrt{x + 2}}, \frac{-1}{4(x + 2)^{\frac{3}{2}}}$ **f** $\frac{-2}{\sqrt{1 - 4x}}, \frac{-4}{(1 - 4x)^{\frac{3}{2}}}$

8 a $f'(x) = 3x^2 + 6x + 5, f''(x) = 6x + 6$
b i 5 **ii** 14 **iii** 6 **iv** 12

9 a i 15 **ii** 12 **iii** 6 **iv** 0
b i -8 **ii** 48 **iii** -192 **iv** 384

ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

10 a $\dfrac{1}{(x+1)^2}, \dfrac{-2}{(x+1)^3}$ **b** $\dfrac{7}{(2x+5)^2}, \dfrac{-28}{(2x+5)^3}$

11 $(x-1)^3(5x-1), 4(x-1)^2(5x-2)$

12 a $1, -1$ **b** $-\frac{1}{3}$

13 a $nx^{n-1}, n(n-1)x^{n-2}, n(n-1)(n-2)x^{n-3}$

b $n(n-1)(n-2)\ldots 1, 0$

15 $a = 3, b = 4$

Exercise 4E

1

Point	A	B	C	D	E	F	G	H	I
y'	0	+	0	−	0	−	0	+	0
y''	+	0	−	0	0	0	+	0	0

2 a concave down **b** concave up

c concave up **d** concave down

3 a minimum **b** maximum

c minimum **d** minimum

4 a $y'' = 2$, so $y'' > 0$ for all values of x.

b $y'' = -6$, so $y'' < 0$ for all values of x.

5 a $y'' = 6x - 6$

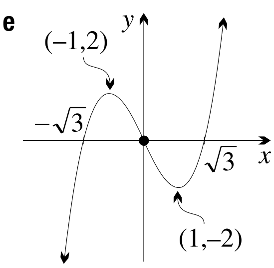
b i $x > 1$ **ii** $x < 1$

6 a $y'' = 6x - 2$

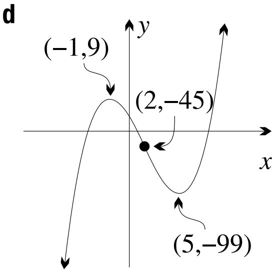
b i $x > \frac{1}{3}$ **ii** $x < \frac{1}{3}$

7 $x = 0$ and $x = 2$, but not $x = -3$.

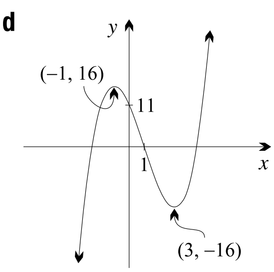
8 e



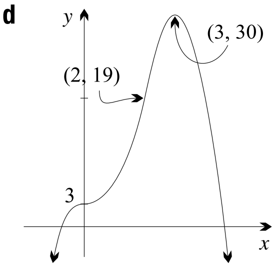
9 d



10 d



11 d



12 a $x > 2$ or $x < -1$ **b** $-1 < x < 2$

c $x > \frac{1}{2}$ **d** $x < \frac{1}{2}$

13 a $y' = 3x^2 + 6x - 72, y'' = 6x + 6$

d $75x + y - 13 = 0$

14 b $f''(x) = g''(x) = 0$, no

c $f(x)$ has a horizontal (or stationary) point of inflection, $g(x)$ has a minimum turning point.

15 a $y'' = 6x - 2a, a = 6$

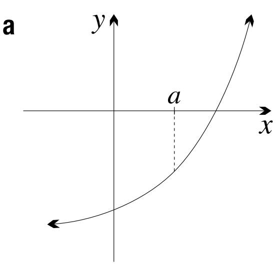
b $y'' = 6x + 4a, a > 1\frac{1}{2}$

c $y'' = 12x^2 + 6ax + 2b, a = -5, b = 6$

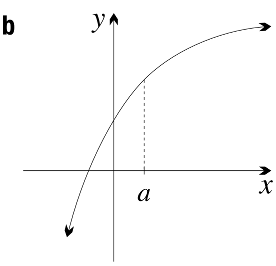
d $a > -\frac{2}{3}$

16 a Increasing. **b** Concave down.

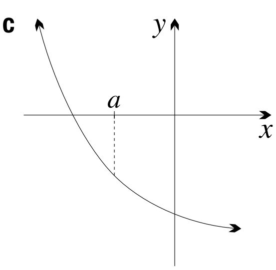
17 a



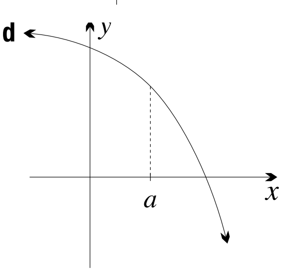
b



c



d



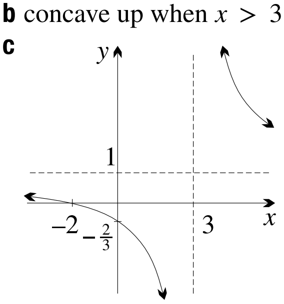
18 a $y' = (x-3)^2 + 2 \geq 2$ for all real x.

b There is a point of inflection at $x = 3$.

c One, because the function is continuous and increasing for all real x.

19 b concave up when $x > 3$, concave down when $x < 3$

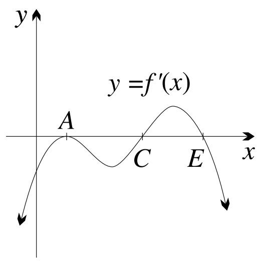
c



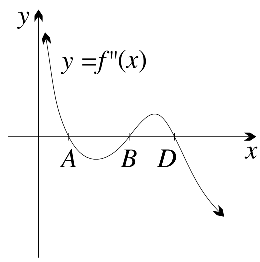
Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

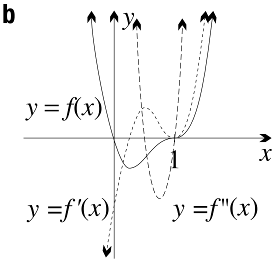
20 a



b

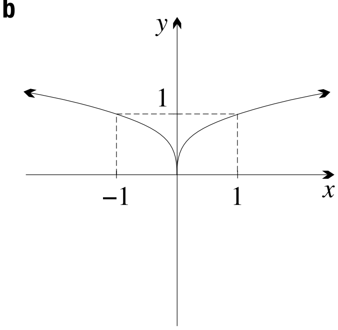


21 b



22 $a = 2, b = -3, c = 0$ and $d = 5$

23 b



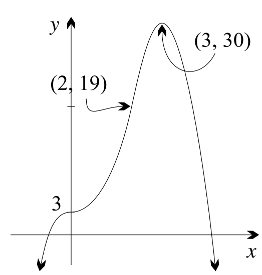
Exercise 4F

1 a $(6, 0)$ **b** $(4, 32)$ **c** $(2, 16)$

2 a $x = -1$ or $x = 2$ **b** $x = 0$

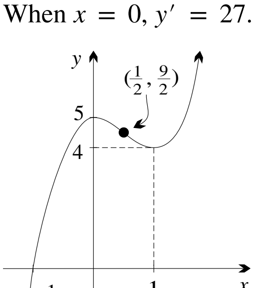
c $-1 < x < 2$ **d** $x < 0$

3 a Show that $f(-x) = -f(x)$. Point symmetry in the origin.



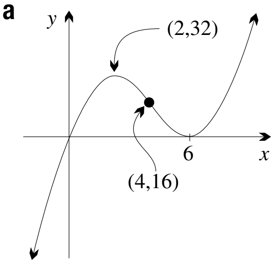
e When $x = 0$, $y' = 27$.

4

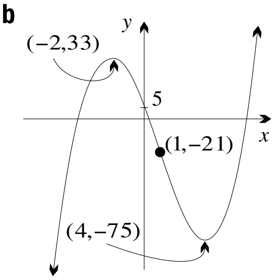


When $x = \frac{1}{2}$, $y' = 1\frac{1}{2}$.

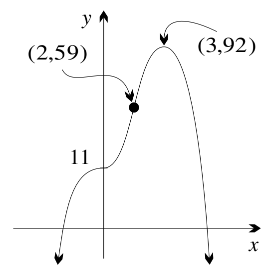
5 a



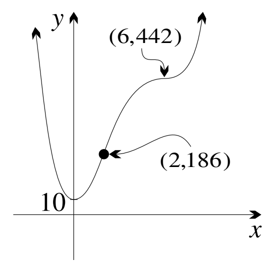
b



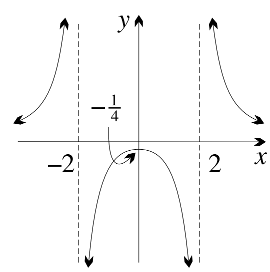
6



7



8



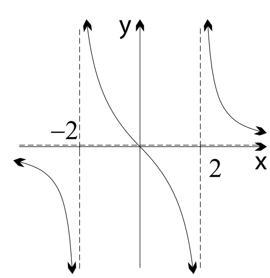
c line symmetry in the y-axis

d domain: $x \neq 2$ and $x \neq -2$, asymptotes: $x = 2$ and $x = -2$

e $y = 0$

g $y > 0$ or $y \leq -\frac{1}{4}$

9



c gradient $= -\frac{1}{4}$

d domain: $x \neq 2$ and $x \neq -2$, asymptotes: $x = 2$ and $x = -2$

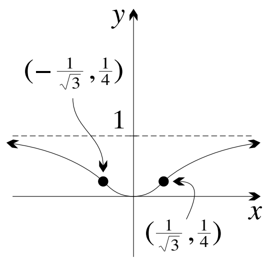
e $y = 0$

f point symmetry in the origin

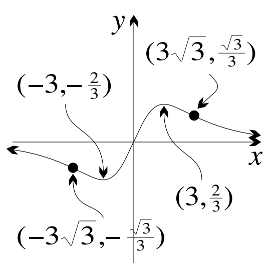
i all real y

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

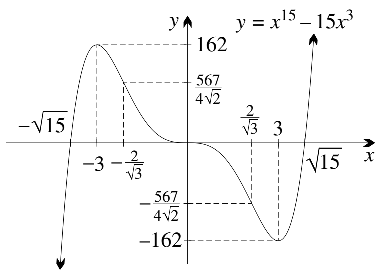
10 e



11 e

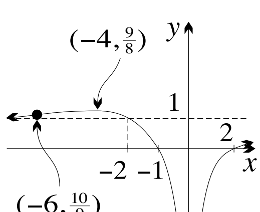


12

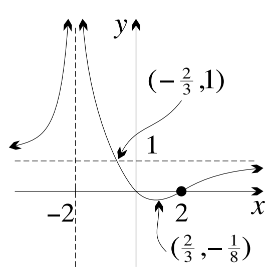


x-intercepts at 0, $\sqrt{15}$ and $-\sqrt{5}$, maximum turning point at $(-3, 162)$, horizontal point of inflection at $(0, 0)$, minimum turning point at $(3, -162)$, points of inflection at $(-\frac{3}{\sqrt{2}}, \frac{567}{4\sqrt{2}})$ and $(\frac{3}{\sqrt{2}}, -\frac{567}{4\sqrt{2}})$.

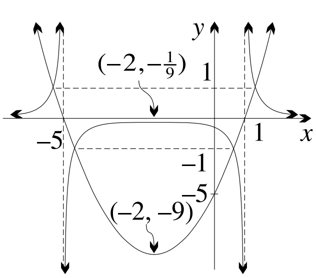
13 a



b



14 a



b $(-2 + 2\sqrt{2}, -1)$, $(-2 - 2\sqrt{2}, -1)$, $(-2 + \sqrt{10}, 1)$, $(-2 - \sqrt{10}, 1)$

Exercise 4G

1 a A local maximum, B local minimum
b C global maximum, D local minimum, E local maximum, F global minimum
c G global maximum, H horizontal point of inflection
d I horizontal point of inflection, J global minimum

2 a $0, 4$ **b** $2, 5$ **c** $0, 4$ **d** $0, 5$
e $0, 2\sqrt{2}$ **f** $-1, -\frac{1}{4}$ **g** $-1, 2$

3 a $-1, 8$ **b** $-49, 5$ **c** $0, 4$ **d** $0, 9$

4 a global minimum -5, global maximum 20
b global minimum -5, local maximum 11, global maximum 139
c global minimum 4, global maximum 11

Exercise 4H

1 a $P = 12x - 2x^2$ **b** 3 **c** 18
2 a $Q = 2x^2 - 16x + 64$
b 4 **c** 32
3 After 2 hours and 40 minutes.
4 c 10 **d** 200 m^2
5 d 24 cm
6 b $x = 30$ m and $y = 20$ m
7 c $h = 2$, $w = \frac{3}{2}$
8 a $\frac{x}{4}, \frac{10 - x}{4}$ **c** 5 **d** $\frac{25}{8}\text{ cm}^2$
9 a $R = x(47 - \frac{1}{3}x)$ **b** $-\frac{8}{15}x^2 + 32x - 10$ **c** 30
10 c 4 cm by 4 cm by 2 cm
11 c $\frac{10}{3}$
12 b Width $16\sqrt{3}$ cm and depth $16\sqrt{6}$ cm.
13 b 15 cm by 5 cm by 3.75 cm
14 c 48 cm^2
15 a $c = \pi x^2 a + 2\pi xhb$
16 d $2(\sqrt{10} + 1)$ cm by $4(\sqrt{10} + 1)$ cm
17 b 80 km/h **c** \$400
18 a $I_c = \frac{W}{x^2} + \frac{2w}{(30 - x)^2}$ **b** 13.27 cm

Exercise 4I

1 c $\frac{10}{3\pi}$
d $\frac{1000}{27\pi}\text{ m}^3$
2 c $20\sqrt{10}\,\pi\text{ cm}^3$
3 a $S = \pi r_1^2 + \pi(k - r_1)^2$
5 d $r = 8$
8 b $V = \frac{S}{2}r - \frac{1}{2}\pi r^3$
9 $2:\sqrt{3}$
11 c $2\pi R^2$
12 $r:h = 1:2$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 4J

1 a $\frac{1}{7}x^7 + C$ **b** $\frac{1}{4}x^4 + C$
c $\frac{1}{11}x^{11} + C$ **d** $\frac{3}{2}x^2 + C$
e $5x + C$ **f** $\frac{1}{2}x^{10} + C$
g $3x^7 + C$ **h** C

2 a $\frac{1}{3}x^3 + \frac{1}{5}x^5 + C$ **b** $x^4 - x^5 + C$
c $\frac{2}{3}x^3 + \frac{5}{8}x^8 + C$ **d** $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + c$
e $3x - 2x^2 + 2x^8 + C$ **f** $x^3 - x^4 - x^5 + C$

3 a $\frac{1}{3}x^3 - \frac{3}{2}x^2 + C$ **b** $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C$
c $x^3 + \frac{11}{2}x^2 - 4x + C$ **d** $\frac{5}{6}x^6 - x^4 + C$
e $x^8 + \frac{1}{2}x^4 + C$
f $\frac{1}{2}x^2 + \frac{1}{4}x^4 - 3x - x^3 + C$

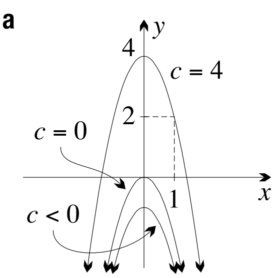
4 a i $y = x^2 + 3x + 3$ **ii** $y = x^2 + 3x + 4$
b i $y = 3x^3 + 4x + 1$ **ii** $y = 3x^3 + 4x - 2$
c i $y = x^3 - 2x^2 + 7x$
ii $y = x^3 - 2x^2 + 7x - 7$

5 a $-\dfrac{1}{x} + C$ **b** $-\dfrac{1}{2x^2} + C$
c $\dfrac{1}{x^2} + C$ **d** $\dfrac{1}{x^3} + C$
e $-\dfrac{1}{x} + \dfrac{1}{2x^2} + C$

6 a $\frac{2}{3}x^{\frac{3}{2}} + C$ **b** $2\sqrt{x} + C$ **c** $\frac{3}{4}x^{\frac{4}{3}} + C$
d $4\sqrt{x} + C$ **e** $\frac{5}{8}x^{\frac{8}{5}} + C$

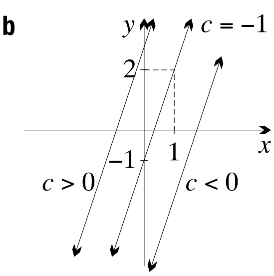
7 a $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ **b** $y = \frac{2}{3}x^{\frac{3}{2}} - 16$

8 a



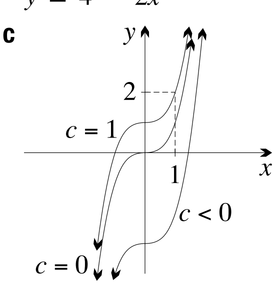
$y = -2x^2 + C$,
$y = 4 - 2x^2$

b



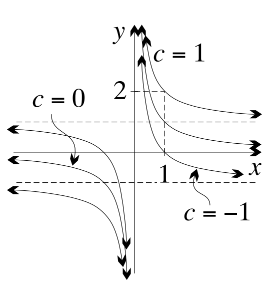
$y = 3x + C$,
$y = 3x - 1$

c



$y = x^3 + C$,
$y = x^3 + 1$

d



$y = \frac{1}{x} + C$,
$y = \frac{1}{x} + 1$

9 a $\frac{1}{4}(x + 1)^4 + C$ **b** $\frac{1}{6}(x - 2)^6 + C$
c $\frac{1}{3}(x + 5)^3 + C$ **d** $\frac{1}{10}(2x + 3)^5 + C$
e $\frac{1}{21}(3x - 4)^7 + C$ **f** $\frac{1}{20}(5x - 1)^4 + C$
g $-\frac{1}{4}(1 - x)^4 + C$ **h** $-\frac{1}{28}(1 - 7x)^4 + C$
i $\dfrac{-1}{3(x - 2)^3} + C$ **j** $\dfrac{1}{9(1 - x)^9} + C$

10 a $\frac{2}{3}(x + 1)^{\frac{3}{2}} + C$ **b** $\frac{2}{3}(x - 5)^{\frac{3}{2}} + C$
c $-\frac{2}{3}(1 - x)^{\frac{3}{2}} + C$ **d** $\frac{1}{3}(2x - 7)^{\frac{3}{2}} + C$
e $\frac{2}{9}(3x - 4)^{\frac{3}{2}} + C$

11 a $y = \frac{1}{5}(x - 1)^5$ **b** $\frac{1}{8}(2x + 1)^4 - \frac{9}{8}$
c $y = \frac{1}{3}(2x + 1)^{\frac{3}{2}}$

12 a $y = \frac{3}{5}x^5 - \frac{1}{4}x^4 + x$
b $y = -\frac{1}{4}x^4 + x^3 + 2x - 2$
c $y = -\frac{1}{20}(2 - 5x)^4 + \frac{21}{20}$

13 30

14 The rule gives the primitive of x^{-1} as $\dfrac{x^0}{0}$, which is undefined. This problem will be addressed in Chapter 6.

15 $y = x^3 + 2x^2 - 5x + 6$

17 $y = -x^3 + 4x^2 + 3$

18 $f(x) = \dfrac{1}{x} + 1$ for $x > 0$, and $f(x) = \dfrac{1}{x} + 3$ for $x < 0$

Chapter 4 review exercise

1 a C and H **b** A and F
c B, D, E and G **d** A, B, G and H
e D **f** C, E and F

2 a $f'(x) = 3x^2 - 2x - 1$
b i decreasing **ii** stationary
iii increasing **iv** increasing

3 a $2x - 4$
b i $x > 2$ **ii** $x < 2$ **iii** $x = 2$

4 a $f'(x) = 3x^2$, increasing
b $f'(x) = 2x - 1$, increasing
c $f'(x) = 5(x - 1)^4$, stationary
d $f'(x) = -\dfrac{4}{(x - 3)^2}$, decreasing

5 a $7x^6$, $42x^5$
b $3x^2 - 8x$, $6x - 8$
c $5(x - 2)^4$, $20(x - 2)^3$
d $-\dfrac{1}{x^2}$, $\dfrac{2}{x^3}$

6 a concave up **b** concave down

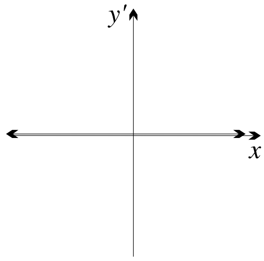
7 a $12x - 6$
b i $x > \frac{1}{2}$ **ii** $x < \frac{1}{2}$

8 a $x < 1$ or $x > 3$ **c** $x > 2$
b $1 < x < 3$ **d** $x < 2$

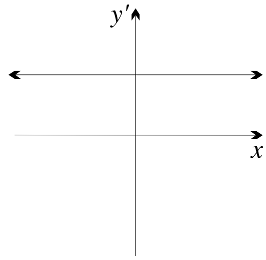
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

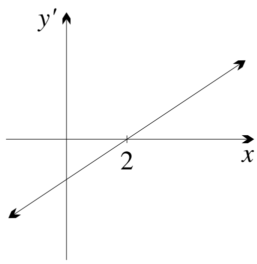
9 a



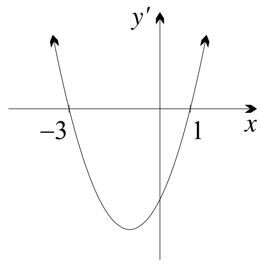
b



c

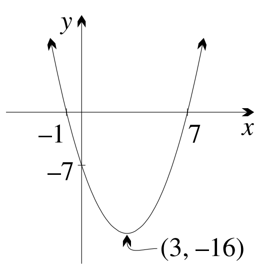


d

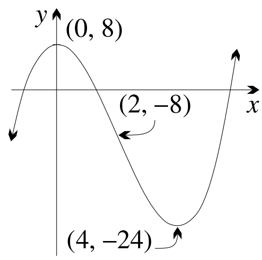


10 a $P(-1, 3)$, $Q(\frac{1}{3}, \frac{49}{27})$ **b** $x > -\frac{1}{3}$
c $\frac{49}{27} < k < 3$

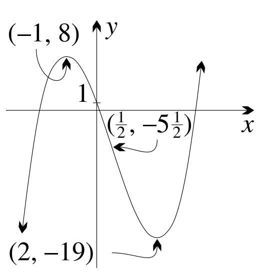
11 a



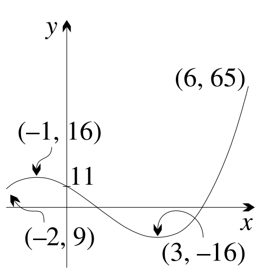
b



c

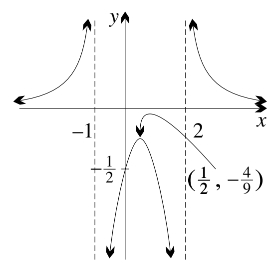


12 a

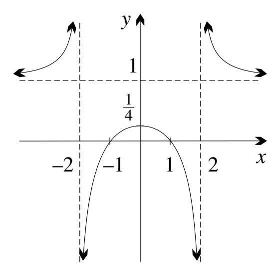


b 65 and -16

13 a $a = -2$ **b** $a = 3$ and $b = 6$
14 b -16
15 a 175 **b** 0 **c** 256
16 b $\frac{1600}{27}$ cm^3
17 b 30 cm by 40 cm
18 b $r = 8$ m
19 a $\frac{1}{8}x^8 + C$ **b** $x^2 + C$
c $4x + C$ **d** $2x^5 + C$
e $4x^2 + x^3 - x^4 + C$
20 a $x^3 - 3x^2 + C$ **b** $\frac{1}{3}x^3 - 2x^2 - 5x + C$
c $\frac{4}{3}x^3 - 6x^2 + 9x + C$
21 a $\frac{1}{6}(x + 1)^6 + C$ **b** $\frac{1}{8}(x - 4)^8 + C$
c $\frac{1}{8}(2x - 1)^4 + C$
22 a $-\frac{1}{x} + C$ **b** $\frac{2}{3}x^{\frac{3}{2}} + C$
23 $f(x) = x^3 - 2x^2 + x + 3$
24 25
25 c Maximum turning point
d $x = -1$ and $x = 2$
e As $x \longrightarrow \infty$, $f(x) \longrightarrow 0$.
f



26 a $(1, 0)$, $(-1, 0)$ and $(0, \frac{1}{4})$
f



27 a $S = 16x + 4h$ **c** 27 m by 9 m by 18 m
28 b $\dfrac{\pi R^2 h(r - R)}{r}$ **b** $\frac{4}{27}\pi r^2 h$
29 a $4\sqrt{3}$ cm^2

Chapter 5

Exercise 5A

1 a $\frac{1}{2}$ u^2
b The area under the curve is less than the area of the triangle.
2 a $\frac{1}{16}$ u^2 **b** $\frac{5}{16}$ u^2
c The area under the curve is less than the combined area of the triangle and trapezium.
3 b The gaps between the upper line segments and the curve are getting smaller.
4 a 6 **b** 12 **c** 8 **d** 9
e 2 **f** $\frac{25}{2}$ **g** 6 **h** 20
5 a 8 **b** 25 **c** 9 **d** 24
e 36 **f** 24 **g** 9 **h** 8

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

6 a 15 **b** 15 **c** 25 **d** 40
e $\frac{25}{2}$ **f** 12 **g** 16 **h** 24
i 8 **j** 18 **k** 4 **l** 16
m 4 **n** 16 **o** $\frac{25}{2}$ **p** $\frac{25}{2}$

7 a 8π **b** $\frac{25}{4}\pi$

8 a $\frac{7}{32}u^2$ **b** $\frac{15}{32}u^2$

c The sum of the areas of the lower rectangles is less than the exact area under the curve which is less than the sum of the areas of the upper rectangles. Note that $\int_0^1 x^2\,dx = \frac{1}{3}$.

9 d As the number of rectangles increases, the interval within which the exact area lies becomes smaller. Note that $\int_0^1 2^x\,dx = \frac{1}{ln\,2} \doteqdot 1.44$.

10 d As the number of rectangles increases, the interval within which the exact area lies becomes smaller. Note that $\int_2^4 \ln x\,dx = 6\ln 2 - 2 \doteqdot 2.16$.

11 e The interval is getting smaller.

f Yes, they appear to be getting closer and closer to the exact value.

13 a You should count approximately 133 squares. $\frac{133}{400} \doteqdot 0.33$. We shall see later that $\int_0^1 x^2\,dx = \frac{1}{3}$.

b The exact values are:

i $\frac{1}{24}$ **ii** $\frac{7}{24}$

14 b 0.79 **c** 3.16

17 a $\frac{1}{3} + \dfrac{1}{6n^2}$

b The lines P_0P_1, P_1P_2 … lie above the curve. Therefore the combined area of the trapezia is greater than the area under the curve.

Exercise 5B

1 a 1 **b** 15 **c** 16
d 84 **e** 19 **f** 243
g 62 **h** 2 **i** 1

2 a i 4 **ii** 25

iii 1 (Note that $\int_4^5 dx$ means $\int_4^5 1\,dx$.)

b Each function is a horizontal line, so each integral is a rectangle.

3 a 30 **b** 6 **c** 33
d 18 **e** 132 **f** 2
g 23 **h** 44 **i** 60

4 a 2 **b** 2 **c** 9
d 30 **e** 96 **f** 10

5 a $13\frac{1}{2}$ **b** $4\frac{2}{3}$ **c** $29\frac{1}{4}$
d 2 **e** $20\frac{5}{6}$ **f** 98

6 a 24 **b** 18 **c** $2\frac{2}{3}$
d 21 **e** $\frac{1}{4}$ **f** $\frac{8}{15}$

7 a 42 **b** 14 **c** 62
d $8\frac{1}{3}$ **e** $6\frac{2}{3}$ **f** 6

8 a $\frac{1}{24}$ **b** $\frac{20}{27}$ **c** $\frac{7}{8}$

9 a i $\frac{1}{10}$ **ii** $\frac{5}{36}$ **iii** 15
b i $\frac{1}{2}$ **ii** $\frac{15}{32}$ **iii** 7

10 a ii 8 **b ii** 6

11 a $k = 1$ **b** $k = 4$ **c** $k = 8$
d $k = 3$ **e** $k = 3$ **f** $k = 2$

12 a $1 + \frac{\pi}{2}$ **b** $2\frac{1}{2}$

13 a $\frac{3}{2}$ **b** $\frac{5}{8}$ **c** $42\frac{1}{3}$

14 a $13\frac{1}{3}$ **b** $8\frac{59}{120}$ **c** $\frac{1}{24}$

15 a x^2 is never negative.

b The function has an asymptote $x = 0$, which lies in the given interval. Hence the integral is meaningless and the use of the fundamental theorem is invalid.

c Part **ii** is meaningless because it crosses the asymptote at $x = 3$.

16 a i x^2 **ii** $x^3 + 3x$ **iii** $\frac{1}{x}$ **iv** $(x^3 - 3)^4$

17 a $(a - x)u(x)$

Exercise 5C

1 The values are 6 and -6, which differ by a factor of -1.

2 a $LHS = RHS = 2$ **b** $LHS = RHS = 6\frac{3}{4}$

c $LHS = RHS = 0$

3 a The interval has width zero.

b $y = x$ is an odd function.

4 a The area is below the x-axis.

b The area is above the x-axis.

c The areas above and below the x-axis are equal.

d The area below the x-axis is greater than the area above.

5 a The area is above the x-axis.

b The area is below the x-axis.

c $y = 1 - x^2$ is an even function and so is symmetrical about the y-axis.

d The area under the parabola from 0 to $\frac{1}{2}$ is greater than the area from $\frac{1}{2}$ to 1.

6 a -7 **b** 5

7 The area under the line $y = 2x$ from $x = 0$ to $x = 1$ is greater than the area under $y = x$.

8 The area below the x-axis is greater than the area above.

9 a i 6 **ii** -6.

The integrals are opposites because the limits have been reversed.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

b **i** 5 **ii** 5.
The factor 20 can be taken out of the integral.
c **i** 45 **ii** 30 **iii** 15.
An integral of a sum is the sum of the integrals.
d **i** 48 **ii** 3 **iii** 45.
The interval $0 \le x \le 2$ can be dissected into the intervals $0 \le x \le 1$ and $1 \le x \le 2$.
e **i** 0 **ii** 0.
An integral over an interval of zero width is zero.

10 a 0. The interval has zero width.
b 0. The interval has zero width.
c 0. The integrand is odd.
d 0. The integrand is odd.
e 0. The integrand is odd.
f 0. The integrand is odd.

11 a The curves meet at $(0, 0)$ and at $(1, 1)$.
b In the interval $0 \le x \le 1$, the curve $y = x^3$ is below the curve $y = x^2$.
c $\frac{1}{4}$ and $\frac{1}{3}$

12 a $\frac{\pi}{2} - \frac{1}{2}$ **b** $1 - \frac{\pi}{2}$

13 a **i** 5 **ii** $\frac{25}{2}$
b **i** 25 **ii** $17\frac{1}{2}$ **iii** $27\frac{1}{2}$

14 a **i** $\frac{1}{2}$ **ii** 18 **iii** 8
b **i** $-\frac{1}{2}$ **ii** -18 **iii** -8

15 a The function is odd, so the integral is zero.
b The function is even, so its graph is symmetrical about the y-axis.

16 a false **b** true **c** false **d** false

17 b **i** 3 **ii** 4 **iii** $-\frac{7}{3}$
iv $\frac{10}{3}$ **v** $60\frac{2}{3}$ **vi** 33

18 a True, as the function is odd.
b True, as $\sin 4x°$ is odd and $\cos 2x°$ is even.
c False, as $2^{-x^2} > 0$ for all x.
d True, as $2^x < 3^x$ for $0 < x < 1$.
e False, as $2^x > 3^x$ for $-1 < x < 0$.
f True, as $t^n > t^{n+1}$ for $0 \le t \le 1$ and hence
$$\frac{1}{1 + t^n} \le \frac{1}{1 + t^{n+1}}.$$

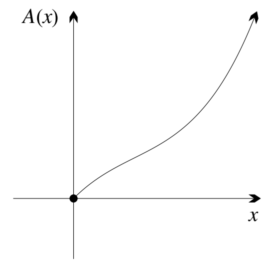
19 a The integral is $-\frac{1}{N} + 1$, which converges to 1 as $N \to \infty$.
b The integral is $-1 + \frac{1}{\varepsilon}$, which diverges to ∞ as $\varepsilon \to 0^+$.
c The integral is $2\sqrt{N} - 2$, which diverges to ∞ as $N \to \infty$.
d The integral is $2 - 2\sqrt{\varepsilon}$, which converges to 2 as $\varepsilon \to 0^+$.

Exercise 5D

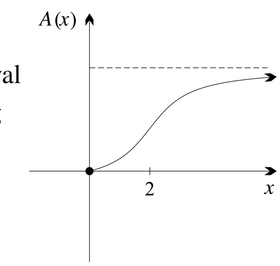
1 a $A(x) = \frac{3}{2}x^2$ **b** $A'(x) = 3x$

2 a $y = 3, A(x) = 3x$
b $y = 2t, A(x) = x^2, A'(x) = 2x$
c $y = 2 + t, A(x) = 2x + \frac{1}{2}x^2, A'(x) = 2 + x$
d $y = 5 - t, A(x) = 5x - \frac{1}{2}x^2, A'(x) = 5 - x$

3 a $A(x)$ is increasing at a decreasing rate in the interval $0 \le x < 2$, and increasing at an increasing rate for $x > 2$. It has an inflection at $x = 2$.



b $A(x)$ is increasing at an increasing rate in the interval $0 \le x \le 2$, and increasing at a decreasing rate for $x \ge 2$. It has a point of inflection at $x = 2$.



4 a $A'(x) = \dfrac{1}{x}$ **b** $A'(x) = \dfrac{1}{1 + x^3}$
c $A'(x) = e^{-\frac{1}{2}x^2}$

5 a $A'(x) = 3x^2 - 12, A(x) = x^3 - 12x + 11$
b $A'(x) = x^3 + 4x, A(x) = \frac{1}{4}x^4 + 2x^2 - 12$
c $A'(x) = \dfrac{1}{x^2}, A(x) = \frac{1}{2} - \frac{1}{x}$

6 a **i** x^2 **ii** $x^3 + 3x$ **iii** $\frac{1}{x}$ **iv** $(x^3 - 3)^4$

7 a The function $y = A(x)$ is zero at $x = 0$, and is increasing at an increasing rate. (Your freehand curve $y = A(x)$ should look like $y = e^x - 1$, for $x \ge 0$, but you won't be able to calculate this formula until the next chapter.).
The curve is increasing at an increasing rate.
b The function $y = A(x)$ is zero at $x = 1$, and is increasing at an increasing rate.
c The function $y = A(x)$ is zero at $x = 1$, and is increasing at a decreasing rate. (Your freehand curve $y = A(x)$ should look like $y = \log_e x$, but you won't be able to calculate this formula until the next chapter.)

8 a The values are 0, 1, 0, -1, 0. The curve looks like $y = \sin x$, and this suggests that $\sin x$ has derivative $\cos x$.
b The values are 0, 1, 2, 1, 0. The graph looks like $y = 1 - \cos x$, which suggest that the derivative of $\cos x$ is $-\sin x$.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

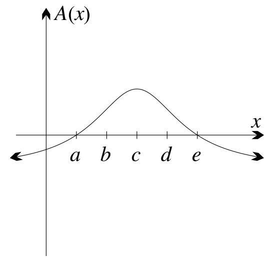
9 a $A(x)$ is increasing when $f(t)$ is positive, that is, for $t < c$, and is decreasing for $t > c$.

b $A(x)$ has a maximum turning point at $x = c$, and no minimum turning points.

c $A(x)$ has inflections when $f'(t)$ changes sign, that is, at $x = b$ and $x = d$.

d Because of the point symmetry of $f(t)$, there are two zeroes of $A(x)$ are $x = a$ and $x = e$.

e $A(x)$ is positive for $a < x < e$ and negative for $x < a$ and for $x > e$.



10 a The function is continuous at every real number, so it is a continuous function.

b The domain is $x \neq 2$, and y is continuous at every value in its domain, so it is a continuous function.

c Zero now lies in the domain, and y is not continuous at $x = 0$, so it is not a continuous function.

d The domain is $x \geq 0$, and y is continuous at every value in its domain, so it is a continuous function.

e The domain is $x > 0$, and y is continuous at every value in its domain, so it is a continuous function.

f The domain is $x \geq 0$, and y is not continuous at $x = 0$, so it is not a continuous function.

Exercise 5E

1 a $4x + C$ **b** $x + C$ **c** C **d** $-2x + C$
e $\frac{x^2}{2} + C$ **f** $\frac{x^3}{3} + C$ **g** $\frac{x^4}{4} + C$ **h** $\frac{x^8}{8} + C$

2 a $x^2 + C$ **b** $2x^2 + C$ **c** $x^3 + C$ **d** $x^4 + C$
e $x^{10} + C$ **f** $\frac{x^4}{2} + C$ **g** $\frac{2x^6}{3} + C$ **h** $\frac{x^9}{3} + C$

3 a $\frac{x^2}{2} + \frac{x^3}{3} + C$ **b** $\frac{x^5}{5} - \frac{x^4}{4} + C$
c $\frac{x^8}{8} + \frac{x^{11}}{11} + C$ **d** $x^2 + x^5 + C$
e $x^9 - 11x + C$ **f** $\frac{x^{14}}{2} + \frac{x^9}{3} + C$
g $4x - \frac{3x^2}{2} + C$ **h** $x - \frac{x^3}{3} + \frac{x^5}{5} + C$
i $x^3 - 2x^4 + \frac{7x^5}{5} + C$

4 a $-x^{-1} + C$ **b** $-\frac{1}{2}x^{-2} + C$ **c** $-\frac{1}{7}x^{-7} + C$
d $-x^{-3} + C$ **e** $-x^{-9} + C$ **f** $-2x^{-5} + C$

5 a $\frac{2}{3}x^{\frac{3}{2}} + C$ **b** $\frac{3}{4}x^{\frac{4}{3}} + C$ **c** $\frac{4}{5}x^{\frac{5}{4}} + C$
d $\frac{3}{5}x^{\frac{5}{3}} + C$ **e** $2x^{\frac{1}{2}} + C$ **f** $\frac{8}{3}x^{\frac{3}{2}} + C$

6 a $\frac{1}{3}x^3 + x^2 + C$ **b** $2x^2 - \frac{1}{4}x^4 + C$
c $\frac{5}{3}x^3 - \frac{3}{4}x^4 + C$ **d** $\frac{1}{5}x^5 - \frac{5}{4}x^4 + C$
e $\frac{1}{3}x^3 - 3x^2 + 9x + C$ **f** $\frac{4}{3}x^3 + 2x^2 + x + C$
g $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C$ **h** $4x - 3x^3 + C$
i $\frac{1}{3}x^3 - \frac{1}{2}x^4 - 3x + 3x^2 + C$

7 a $\frac{1}{2}x^2 + 2x + C$ **b** $\frac{1}{2}x^2 + \frac{1}{3}x^3 + C$
c $\frac{1}{6}x^3 - \frac{1}{16}x^4 + C$

8 a $-\frac{1}{x} + C$ **b** $-\frac{1}{2x^2} + C$
c $-\frac{1}{4x^4} + C$ **d** $-\frac{1}{9x^9} + C$
e $-\frac{1}{x^3} + C$ **f** $-\frac{1}{x^5} + C$
g $-\frac{1}{x^7} + C$ **h** $-\frac{1}{3x} + C$
i $-\frac{1}{28x^4} + C$ **j** $\frac{1}{10x^2} + C$
k $\frac{1}{4x^4} - \frac{1}{x} + C$ **l** $-\frac{1}{2x^2} - \frac{1}{3x^3} + C$

9 a $\frac{2}{3}x^{\frac{3}{2}} + C$ **b** $\frac{3}{4}x^{\frac{4}{3}} + C$
c $2\sqrt{x} + C$ **d** $\frac{3}{5}x^{\frac{5}{3}} + C$

10 a 18 **b** 12 **c** 4 **d** $\frac{3}{5}$

11 a $\frac{1}{6}(x + 1)^6 + C$ **b** $\frac{1}{4}(x + 2)^4 + C$
c $-\frac{1}{5}(4 - x)^5 + C$ **d** $-\frac{1}{3}(3 - x)^3 + C$
e $\frac{1}{15}(3x + 1)^5 + C$ **f** $\frac{1}{32}(4x - 3)^8 + C$
g $-\frac{1}{14}(5 - 2x)^7 + C$ **h** $-\frac{1}{40}(1 - 5x)^8 + C$
i $\frac{1}{24}(2x + 9)^{12} + C$ **j** $\frac{3}{22}(2x - 1)^{11} + C$
k $\frac{4}{35}(5x - 4)^7 + C$ **l** $-\frac{7}{8}(3 - 2x)^4 + C$

12 a $\frac{3}{5}\left(\frac{1}{3}x - 7\right)^5 + C$ **b** $\frac{4}{7}\left(\frac{1}{4}x - 7\right)^7 + C$
c $-\frac{5}{4}\left(1 - \frac{1}{4}x\right)^4 + C$

13 a $-\frac{1}{2(x + 1)^2} + C$ **b** $-\frac{1}{3(x - 5)^3} + C$
c $-\frac{1}{3(3x - 4)} + C$ **d** $\frac{1}{4(2 - x)^4} + C$
e $-\frac{3}{5(x - 7)^5} + C$ **f** $-\frac{1}{2(4x + 1)^4} + C$
g $\frac{2}{15(3 - 5x)^3} + C$ **h** $\frac{1}{5 - 20x} + C$
i $-\frac{7}{96(3x + 2)^4} + C$

14 a $\frac{3}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + C$ **b** $\frac{1}{2}x^2 - 4x + C$
c $2x^2 - \frac{8}{3}x^{\frac{3}{2}} + x + C$

15 a **i** $\frac{2}{3}$ **ii** 2 **iii** 12
b **i** $5\frac{1}{3}$ **ii** $96\frac{4}{5}$ **iii** 4

16 a 2 **b** $-\frac{13}{6}$ **c** $12\frac{1}{6}$

17 $\int x^{-1}\,dx = \frac{x^0}{0} + C$ is meaningless. Chapter 6 deals with the resolution of this problem.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 a $\frac{1}{3}(2x - 1)^{\frac{3}{2}} + C$ **b** $-\frac{1}{6}(7 - 4x)^{\frac{3}{2}} + C$
c $\frac{3}{16}(4x - 1)^{\frac{4}{3}} + C$ **d** $\frac{2}{3}\sqrt{3x + 5} + C$

19 a $\frac{242}{5}$ **b** 0 **c** $121\frac{1}{3}$
d 1 **e** $\frac{13}{6}$ **f** 2
g 0 **h** $\frac{112}{9}$ **i** $8\frac{2}{5}$

20 b i $\frac{1}{5}x(x - 1)^5 - \frac{1}{30}(x - 1)^6 + C$
ii $\frac{2}{3}x(1 + x)^{\frac{3}{2}} - \frac{4}{15}(1 + x)^{\frac{5}{2}} + C$

Exercise 5F

1 a 4 u^2 **b** 26 u^2 **c** 81 u^2
d 12 u^2 **e** 9 u^2 **f** $6\frac{2}{3}\text{ u}^2$
g $\frac{128}{3}\text{ u}^2$ **h** 6u^2 **i** $\frac{1}{4}\text{ u}^2$
j $57\frac{1}{6}\text{ u}^2$ **k** 36u^2 **l** 60u^2

2 a 25u^2 **b** 8u^2 **c** 4u^2
d 108u^2 **e** $\frac{9}{2}\text{u}^2$ **f** $34\frac{2}{3}\text{ u}^2$
g 18u^2 **h** 2u^2

3 a $\frac{4}{3}\text{u}^2$ **b** $\frac{27}{2}\text{u}^2$ **c** $\frac{81}{4}\text{u}^2$ **d** $46\frac{2}{5}\text{ u}^2$

4 a $\frac{9}{2}\text{u}^2$ **b** $\frac{4}{3}\text{u}^2$ **c** $\frac{45}{4}\text{u}^2$ **d** 9 u^2

5 b $4\frac{1}{2}\text{u}^2$ **c** 2 u^2 **d** $6\frac{1}{2}\text{u}^2$
e $2\frac{1}{2}$. This is the area above the x-axis minus the area below it.

6 b $10\frac{2}{3}\text{u}^2$ **c** $2\frac{1}{3}\text{u}^2$ **d** 13 u^2
e $-8\frac{1}{3}$. This is the area above the x-axis minus the area below it.

7 b $2\frac{2}{3}\text{u}^2$ **c** $\frac{5}{12}\text{u}^2$ **d** $3\frac{1}{12}\text{u}^2$
e $-2\frac{1}{4}$. This is the area above the x-axis minus the area below it.

8 a $11\frac{2}{3}\text{u}^2$ **b** $128\frac{1}{2}\text{u}^2$ **c** 4 u^2
d $8\frac{1}{2}\text{u}^2$ **e** $32\frac{3}{4}\text{u}^2$ **f** $11\frac{1}{3}\text{u}^2$

9 a 13u^2 **b** $2\frac{1}{2}\text{u}^2$ **c** $9\frac{1}{3}\text{u}^2$ **d** $7\frac{1}{3}\text{u}^2$

10 a i 64 u^2 **ii** 128 u^2 **iii** $64\frac{4}{5}\text{u}^2$
b i 50 u^2 **ii** 18 u^2 **iii** $\frac{32}{3}\text{u}^2$

11 8 u^2

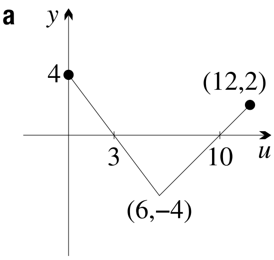
12 a $(2, 0)$, $\left(0, 4\sqrt{2}\right)$, $\left(0, -4\sqrt{2}\right)$
b $\frac{16\sqrt{2}}{3}\text{u}^2$
ii $x = 2 - \frac{y^2}{16}$

13 a $y = \frac{1}{3}x^3 - 2x^2 + 3x$
b The curve passes through the origin, $(1, 1\frac{1}{3})$ is a maximum turning point and $(3, 0)$ is a minimum turning point.
c $\frac{4}{3}\text{u}^2$

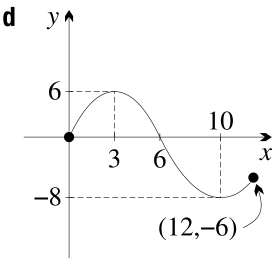
15 a $2 : n + 1$ **b** $1 : n + 1$

16 b $a^2 = \frac{1}{2}\left(3 + \sqrt{5}\right)$, $a^4 = \frac{1}{2}\left(7 + 3\sqrt{5}\right)$,
$a^5 = \frac{1}{2}\left(11 + 5\sqrt{5}\right)$
c Areas are $\frac{1}{5}\text{u}^2$, $\frac{1}{10}\text{u}^2$ and $\frac{1}{10}\text{u}^2$.

17 a



b maximum at $(3, 6)$, minimum at $(10, -8)$
c 0, 6
d



e 24 u^2

18 a $-\dfrac{1}{n + 1}$ **b** $\dfrac{1}{n + 1}$

Exercise 5G

1 a $\frac{1}{6}\text{u}^2$ **b** $\frac{1}{4}\text{u}^2$ **c** $\frac{3}{10}\text{u}^2$ **d** $\frac{1}{12}\text{u}^2$
e $\frac{2}{35}\text{u}^2$ **f** $20\frac{5}{6}\text{u}^2$ **g** 36 u^2 **h** $20\frac{5}{6}\text{u}^2$

2 a $\frac{4}{3}\text{u}^2$ **b** $\frac{1}{6}\text{u}^2$ **c** $\frac{4}{3}\text{u}^2$ **d** $4\frac{1}{2}\text{u}^2$

3 a $37\frac{1}{3}\text{u}^2$ **b** $\frac{9}{4}\text{u}^2$

4 a $16\frac{2}{3}\text{u}^2$ **b** $9\frac{1}{3}\text{u}^2$

5 a $4\frac{1}{2}\text{ u}^2$

6 a $\frac{4}{3}\text{u}^2$

7 a 36 u^2

8 a $4\frac{1}{2}\text{u}^2$ **b** $20\frac{5}{6}\text{u}^2$ **c** $2\frac{2}{3}\text{u}^2$

9 c 36 u^2

10 c $\frac{4}{3}\text{u}^2$

11 a $4\frac{1}{2}\text{u}^2$ **b** $20\frac{5}{6}\text{u}^2$ **c** $21\frac{1}{3}\text{u}^2$

12 c $\frac{1}{3}\text{u}^2$

13 b $y = x - 2$ **c** $5\frac{1}{3}\text{u}^2$

14 c 108 u^2

15 a The points are $(-4, -67)$, $(1, -2)$, and $(2, 5)$.
c $73\frac{5}{6}\text{u}^2$

16 a $((0, 0)$, $(\frac{1}{2}, \frac{1}{8})$, $(1, 0)$ **b** $\frac{1}{16}\text{u}^2$

17 a $-1 < x < 1$ or $x > 4$ **b** $21\frac{1}{12}\text{u}^2$

18 b $y = 2x - 7$ **c** $\frac{7}{12}\text{u}^2$

19 $1 - \dfrac{1}{\sqrt[3]{2}}$

Exercise 5H

1 a 40 **b** 22 **c** -26

2 a 164

3 30

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

4 a The curve is concave up, so the chord is above the curve, and the area under the chord will be greater than the area under the curve.

b The curve is concave down, so the chord is underneath the curve, and the area under the chord will be less than the area under the curve.

5 b 10

c $10\frac{2}{3}$, the curve is concave down.

d $6\frac{1}{4}\%$

6 b $10\frac{1}{10}$

c y'' is positive in the interval $1 \le x \le 5$, so the curve is concave up.

7 b 24.7

c $24\frac{2}{3}$. y'' is negative in the interval $9 \le x \le 16$, so the curve is concave down.

8 a 0.73 **b** 4.5 **c** 3.4 **d** 37

9 a 1.12 **b** 0.705 **c** 22.9 **d** 0.167

10 9.2 metres

11 $550\,\text{m}^2$

12 $2950\,\text{m}^2$

13 a 0.7489

b $\pi \doteqdot 3.0$, the approximation is less than the integral, because the curve is concave down.

15 d 876 400

Exercise 5I

1 a $8(2x+3)^3$

b i $(2x+3)^4 + C$ **ii** $2(2x+3)^4 + C$

2 a $9(3x-5)^2$

b i $(3x-5)^3 + C$ **ii** $3(3x-5)^3 + C$

3 a $20(1+4x)^4$

b i $(1+4x)^5 + C$ **ii** $\frac{1}{2}(1+4x)^5 + C$

4 a $-8(1-2x)^3$

b i $(1-2x)^4 + C$ **ii** $\frac{1}{4}(1-2x)^4 + C$

5 a $-4(4x+3)^{-2}$

b i $(4x+3)^{-1} + C$ **ii** $-\frac{1}{4}(4x+3)^{-1} + C$

6 a $(2x-5)^{-\frac{1}{2}}$

b i $(2x-5)^{\frac{1}{2}} + C$ **ii** $\frac{1}{3}(2x-5)^{\frac{1}{2}} + C$

7 a $8x(x^2+3)^3$

b i $(x^2+3)^4 + C$ **ii** $5(x^2+3)^4 + C$

8 a $15x^2(x^3-1)^4$

b i $(x^3-1)^5 + C$ **ii** $\frac{1}{5}(x^3-1)^5 + C$

9 a $\dfrac{2x}{\sqrt{2x^2+3}}$

b i $\sqrt{2x^2+3} + C$

ii $\frac{1}{2}\sqrt{2x^2+3} + C$

10 a $\dfrac{3(\sqrt{x}+1)^2}{2\sqrt{x}}$

b i $(\sqrt{x}+1)^3 + C$ **ii** $\frac{2}{3}(\sqrt{x}+1)^3 + C$

11 a $12(x^2+2x)(x^3+3x^2+5)^3$

b i $(x^3+3x^2+5)^4 + C$

ii $\frac{1}{12}(x^3+3x^2+5)^4 + C$

12 a $-7(2x+1)(5-x^2-x)^6$

b i $(5-x^2-x)^7 + C$

ii $-\frac{1}{7}(5-x^2-x)^7 + C$

13 a $\frac{1}{4}(5x+4)^4 + C$ **b** $\frac{1}{6}(1-3x)^6 + C$

c $\frac{1}{8}(x^2-5)^8 + C$ **d** $\frac{1}{5}(x^3+7)^5 + C$

e $\frac{-1}{3x^2+2} + C$ **f** $2\sqrt{9-2x^3} + C$

14 a $\frac{1}{3}(5x^2+3)^3 + C$ **b** $\frac{1}{4}(x^2+1)^4 + C$

c $\frac{1}{6}(1+4x^3)^6 + C$ **d** $\frac{1}{30}(1+3x^2)^5 + C$

e $-\frac{1}{32}(1-x^4)^8 + C$ **f** $\frac{2}{3}(x^3-1)^{\frac{3}{2}} + C$

g $\dfrac{1}{15}(5x^2+1)^{\frac{3}{2}} + C$ **h** $2\sqrt{x^2+3} + C$

i $\frac{1}{4}\sqrt{4x^2+8x+1} + C$ **j** $-\dfrac{1}{4(x^2+5)^2} + C$

15 a $\frac{32}{15}$ **b** $\frac{7}{144}$ **c** $\frac{1}{12}$ **d** 936

16 a $\frac{1}{6}(1-\frac{1}{x})^6 + C$ **b** $\dfrac{1}{3}$

17 a $x \ge 1$ or $x \le -1$

b $\dfrac{2x^2-1}{\sqrt{x^2-1}}$ **c** $\dfrac{16}{3}\sqrt{2}\,\text{u}^2$

18 a horizontal points of inflection at $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$, maximum at $(1, 216)$, minimum at $(-1, -216)$

b $600\frac{1}{4}\,\text{u}^2$

Chapter 5 review exercise

1 a 1 **b** $\frac{3}{2}$ **c** 609

d $\dfrac{2}{5}$ **e** -12 **f** $8\frac{2}{3}$

g 8 **h** -10 **i** $21\frac{1}{3}$

2 a $4\frac{2}{3}$ **b** $-1\frac{2}{3}$ **c** $\dfrac{1}{3}$

3 a $-1\frac{1}{2}$ **b** 15 **c** $-6\frac{1}{6}$

4 a ii $k = 6$ **b ii** $k = 3$

5 a 0. The integral has zero width.

b 0. The integrand is odd.

c 0. The integrand is odd.

6 a 8 **b** $\frac{3}{2}$

7 a i $4x - \frac{1}{2}x^2 + 10$ **ii** $\frac{1}{2} - x^{-1}$

b i $4 - x$ **ii** x^{-2}

c i $x^5 - 5x^3 + 1$

ii $\dfrac{x^2+4}{x^2-1}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

8 a $\frac{x^2}{2} + 2x + C$

b $\frac{x^4}{4} + x^3 - \frac{5x^2}{2} + x + C$

c $\frac{x^3}{3} - \frac{x^2}{2} + C$

d $-\frac{x^3}{3} + \frac{5x^2}{2} - 6x + C$

e $-x^{-1} + C$

f $-\frac{1}{6x^6} + C$

g $\frac{2x^{\frac{3}{2}}}{3} + c$

h $\frac{1}{5}(x + 1)^5 + C$

i $\frac{1}{12}(2x - 3)^6 + C$

9 a $9\frac{1}{3}\text{u}^2$ **b** 4u^2 **c** $\frac{4}{3}\text{u}^2$ **d** 1u^2

e $\frac{1}{6}\text{u}^2$ **f** $\frac{4}{15}\text{u}^2$ **g** $\frac{1}{6}\text{u}^2$ **h** $4\frac{1}{2}\text{u}^2$

10 b $\frac{4}{3}\text{u}^2$

11 a 9 **b** 0.56

12 a $18(3x + 4)^5$

b i $(3x + 4)^6 + C$ **ii** $\frac{1}{2}(3x + 4)^6 + C$

13 a $6x(x^2 - 1)^2$

b i $(x^2 - 1)^3 + C$ **ii** $\frac{1}{6}(x^2 - 1)^3 + C$

14 a $\frac{1}{5}(x^3 + 1)^5 + C$ **b** $-\frac{1}{2(x^2 - 5)^2} + C$

Chapter 6

Exercise 6A

1 a 2^{10} **b** e^7 **c** 2^4

d e^3 **e** 2^{12} **f** e^{30}

2 a e^{7x} **b** e^{2x} **c** e^{10x}

d e^{-5x} **e** e^{5x} **f** e^{-12x}

3 a 7.389 **b** 0.04979 **c** $e^1 \doteqdot 2.718$

d $e^{-1} \doteqdot 0.3679$ **e** $e^{\frac{1}{2}} \doteqdot 1.649$ **f** $e^{-\frac{1}{2}} \doteqdot 0.6065$

4 a $y' = e^x$ and $y'' = e^x$

b 'The curve $y = e^x$ is always concave up, and is always increasing at an increasing rate.'

5 a gradient $= e$, $y = ex$.

b $y = x + 1$

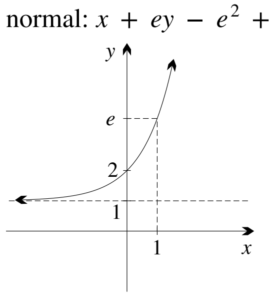
c $y = \frac{1}{e}(x + 2)$

6 a $P = (1, e - 1)$

b $\frac{dy}{dx} = e^x$. When $x = 1$, $\frac{dy}{dx} = e$.

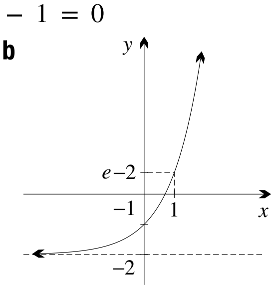
c tangent: $ex - y - 1 = 0$,
normal: $x + ey - e^2 + e - 1 = 0$

7 a



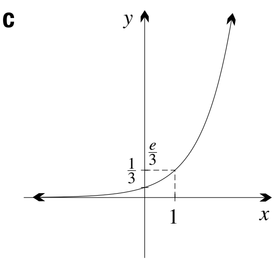
Shift e^x up 1

b



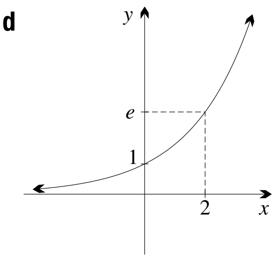
Shift e^x down 2

c



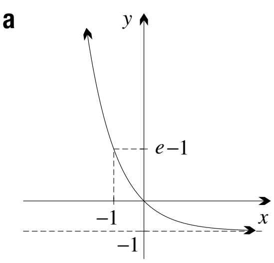
Stretch e^x vertically with factor $\frac{1}{3}$

d



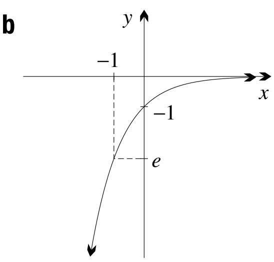
Stretch e^x horizontally with factor 2

8 a



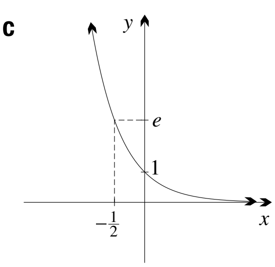
Shift e^{-x} down 1

b



Reflect e^{-x} in x-axis

c



Stretch e^{-x} horizontally with factor $\frac{1}{2}$

9 It is a vertical dilation of $y = e^x$ with factor $-\frac{1}{3}$. Its equation is $y = -\frac{1}{3}e^x$.

10 a $e^{2x} - 1$ **b** $e^{6x} + 3e^{4x} + 3e^{2x} + 9$

c $1 - 2e^{3x}$ **d** $e^{-4x} + 2 + e^{4x}$

11 a $e^{2x} + e^x$ **b** $e^{-2x} - e^{-x}$

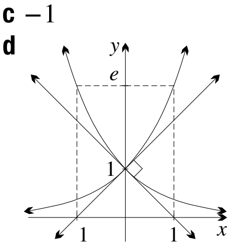
c $e^{20x} + 5e^{30x}$ **d** $2e^{-4x} + 3e^{-5x}$

12 a 1

b Reflection in y-axis

c -1

d



e Horizontal dilation with factor -1

13 a e^x, e^x, e^x, e^x

b $e^x + 3x^2$, $e^x + 6x$, $e^x + 6$, e^x

c $4e^x$, $4e^x$, $4e^x$, $4e^x$

d $5e^x + 10x$, $5e^x + 10$, $5e^x$, $5e^x$. In part **c**, the gradient equals the height.

14 a 1, 45° **b** e, 69°48′

c e^{-2}, 7°42′ **d** e^5, 89°37′

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

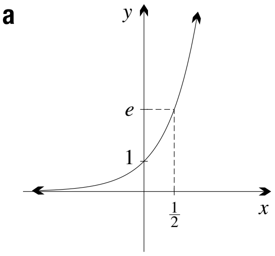
Cambridge University Press
Updated July 2022

15 a $e - 1$

b $\dfrac{dy}{dx} = e^x$. When $x = 1$, $\dfrac{dy}{dx} = e$.

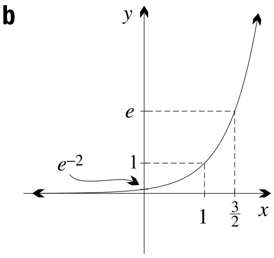
c $y = ex - 1$

16 a



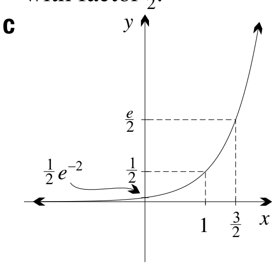
Stretch horizontally with factor $\frac{1}{2}$.

b



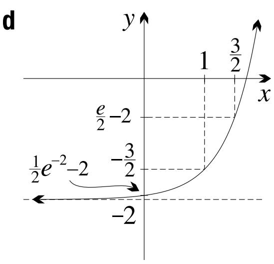
Shift right 1.

c



Stretch vertically with factor $\frac{1}{2}$.

d



Shift down 2.

17 a Shift left 2. Alternatively, $y = e^2 e^x$, so it is a vertical dilation with factor e^2.

b Stretch vertically with factor 2. Alternatively, $y = e^{\log_e 2}\, e^x = e^{x + \log_e 2}$, so it is a shift left $\log_e 2$.

Exercise 6B

1 a $7e^{7x}$ **b** $12e^{3x}$

c $2e^{\frac{1}{3}x}$ **d** e^{-2x}

e $y' = 3e^{3x+4}$ **f** $y' = 4e^{4x-3}$

g $y' = -3e^{-3x+4}$ **h** $y' = -2e^{-2x-7}$

2 a $e^x - e^{-x}$ **b** $2e^{2x} + 3e^{-3x}$

c $\frac{e^x + e^{-x}}{2}$ **d** $\frac{e^x - e^{-x}}{3}$

e $e^{2x} + e^{3x}$ **f** $e^{4x} + e^{5x}$

3 a $y' = 3e^{3x}$ **b** $y' = 2e^{2x}$

c $y' = 2e^{2x}$ **d** $y' = 6e^{6x}$

e $y' = 3e^{3x}$ **f** $y' = -e^{-x}$

g $y' = -3e^{-3x}$ **h** $y' = -5e^{-5x}$

4 a i $-e^{-x}, e^{-x}, -e^{-x}, e^{-x}$

ii Successive derivatives alternate in sign. More precisely, $f^{(n)}(x) = \begin{cases} e^{-x}, & \text{if } n \text{ is even,} \\ -e^{-x}, & \text{if } n \text{ is odd.} \end{cases}$

b i $2e^{2x}, 4e^{2x}, 8e^{2x}, 16e^{2x}$

ii Each derivative is twice the previous one. More precisely, $f^{(n)}(x) = 2^n e^{2x}$.

5 a $2e^{2x} + e^x$ **b** $e^{-x} - 4e^{-2x}$

c $2e^{2x} + 2e^x$ **d** $2e^{2x} + 6e^x$

e $2e^{2x} - 2e^x$ **f** $2e^{2x} - 4e^x$

g $2(e^{2x} + e^{-2x})$ **h** $10(e^{10x} + e^{-10x})$

6 a $a\,e^{ax+b}$ **b** $2xe^{x^2}$

c $-xe^{-\frac{1}{2}x^2}$ **d** $2xe^{x^2+1}$

e $-2xe^{1-x^2}$ **f** $2(x + 1)\,e^{x^2+2x}$

g $(1 - 2x)\,e^{6+x-x^2}$ **h** $(3x - 1)\,e^{3x^2-2x+1}$

7 a $(x + 1)\,e^x$ **b** $(1 - x)\,e^{-x}$

c xe^x **d** $(3x + 4)\,e^{3x-4}$

e $(2x - x^2)\,e^{-x}$ **f** $4xe^{2x}$

g $(x^2 + 2x - 5)\,e^x$ **h** $x^2e^{2x}(3 + 2x)$

8 a $y' = \frac{x - 1}{x^2}e^x$ **b** $y' = (1 - x)\,e^{-x}$

c $y' = \frac{(x - 2)\,e^x}{x^3}$ **d** $y' = (2x - x^2)\,e^{-x}$

e $y' = \frac{x}{(x + 1)^2}e^x$ **f** $y' = -xe^{-x}$

g $y' = (7 - 2x)\,e^{-2x}$ **h** $y' = (x^2 - 2x - 1)\,e^{-x}$

9 a $2e^{2x} + 3e^x$ **b** $4e^{4x} + 2e^{2x}$

c $-2e^{-2x} - 6e^{-x}$ **d** $-6e^{-6x} + 18e^{-3x}$

e $3e^{3x} + 2e^{2x} + e^x$ **f** $12e^{3x} + 2e^{2x} + e^{-x}$

10 a $-5e^x(1 - e^x)^4$ **b** $16e^{4x}(e^{4x} - 9)^3$

c $-\frac{e^x}{(e^x - 1)^2}$ **d** $-\frac{6e^{3x}}{(e^{3x} + 4)^3}$

12 a $f'(x) = 2e^{2x+1}$, $f'(0) = 2e$, $f''(x) = 4e^{2x+1}$, $f''(0) = 4e$

b $f'(x) = -3e^{-3x}$, $f'(1) = -3e^{-3}$, $f''(x) = 9e^{-3x}$, $f''(1) = 9e^{-3}$

c $f'(x) = (1 - x)\,e^{-x}$, $f'(2) = -e^{-2}$, $f''(x) = (x - 2)\,e^{-x}$, $f''(2) = 0$

d $f'(x) = -2xe^{-x^2}$, $f'(0) = 0$, $f''(x) = (4x^2 - 2)\,e^{-x^2}$, $f''(0) = -2$

13 a $y' = ae^{ax}$ **b** $y' = -ke^{-kx}$

c $y' = Ake^{kx}$ **d** $y' = -B\ell e^{-\ell x}$

e $y' = pe^{px+q}$ **f** $y' = pCe^{px+q}$

g $y' = \dfrac{pe^{px} - qe^{-qx}}{r}$ **h** $e^{ax} - e^{-px}$

14 a $3e^x(e^x + 1)^2$

b $4(e^x - e^{-x})(e^x + e^{-x})^3$

c $(1 + 2x + x^2)\,e^{1+x} = (1 + x)^2e^{1+x}$

d $(2x^2 - 1)\,e^{2x-1}$

e $\frac{e^x}{(e^x + 1)^2}$

f $-\frac{2e^x}{(e^x - 1)^2}$

15 a $y' = -e^{-x}$

b $y' = e^x$

c $y' = e^{-x} - 4e^{-2x}$

d $y' = -12e^{-4x} - 3e^{-3x}$

e $y' = e^x - 9e^{3x}$

f $y' = -2e^{-x} - 2e^{-2x}$

17 a $y' = \frac{1}{2}\sqrt{e^x}$ **b** $y' = \frac{1}{3}\sqrt[3]{e^x}$

c $y' = -\frac{1}{2\sqrt{e^x}}$ **d** $y' = -\frac{1}{3\sqrt[3]{e^x}}$

e $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **f** $-\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

g $-\frac{1}{x^2}e^{\frac{1}{x}}$ **h** $\frac{1}{x^2}e^{-\frac{1}{x}}$

i $\left(1 + \frac{1}{x^2}\right)e^{x-\frac{1}{x}}$ **j** $e^x e^{e^x} = e^{x+e^x}$

20 a -5 or 2

b $-\frac{1}{2}\left(1 + \sqrt{5}\right)$ or $-\frac{1}{2}\left(1 - \sqrt{5}\right)$

Exercise 6C

1 a $A = \left(\frac{1}{2}, 1\right)$ **b** $y' = 2e^{2x-1}$ **c** $y = 2x$

2 a $R = \left(-\frac{1}{3}, 1\right)$ **b** $y' = 3e^{3x+1}$

c $-\frac{1}{3}$ **d** $3x + 9y - 8 = 0.$

3 a $x - ey + e^2 + 1 = 0$

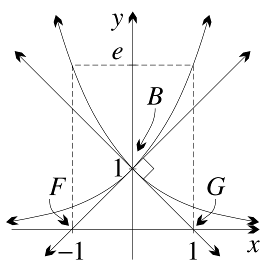
b $x = -e^2 - 1, y = e + e^{-1}$

c $\frac{1}{2}(e^3 + 2e + e^{-1})$

4 a $y = x + 1$ **b** $y = -x + 1$

c $F(-1, 0), G(1, 0)$

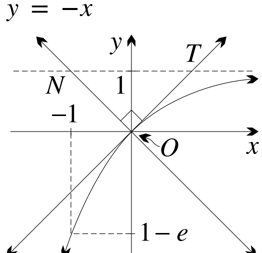
d



e isosceles right triangle, 1 square unit

5 b $y = -x$ **c** $y = 1$

d

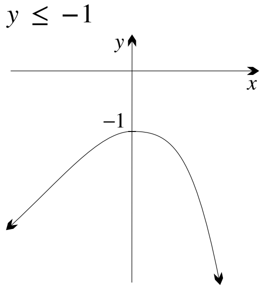


e 1 square unit

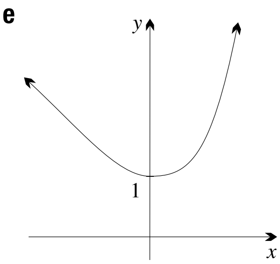
6 a $y' = 1 - e^x, y'' = -e^x$

c maximum turning point at $(0, -1)$

d $y \leq -1$



e



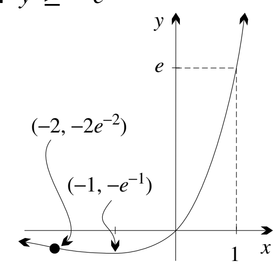
7 b $y = e^t(x - t + 1)$

c The x-intercept of each tangent to $y = e^x$ is 1 unit left of the x-value of the point of contact.

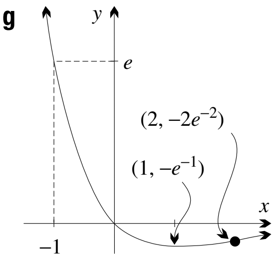
8 a There is a zero at $x = 0$, it is positive for $x > 0$ and negative for $x < 0$. It is neither even nor odd.

e They all tend towards ∞.

f $y \geq -e^{-1}$

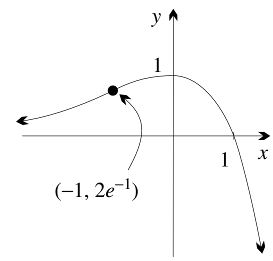


g



9 a

x	0	1	2
y	1	0	$-e^2$
sign	+	0	−



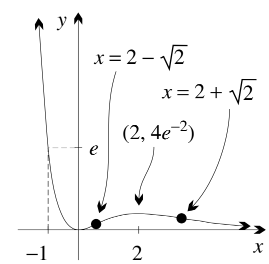
b $y' = -xe^x$,

$y'' = -(x + 1)e^x$

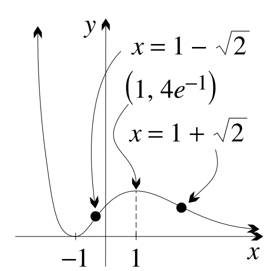
d They all tend to $-\infty$.

e $y \leq 1$

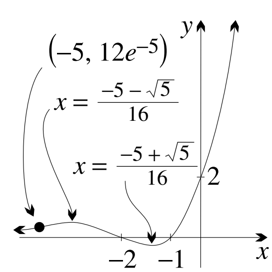
10 d $y \geq 0$



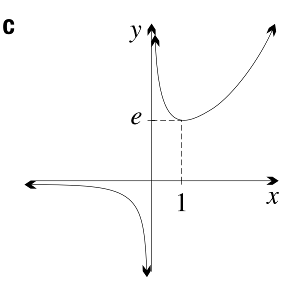
11 d $y \geq 0$



12

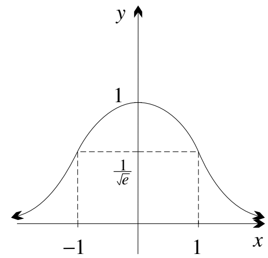


13 a $x \neq 0$, $y < 0$ or $y \geq e$ **c**

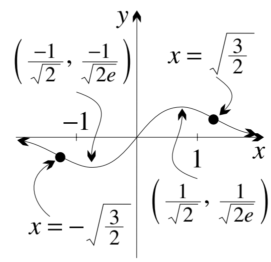


Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

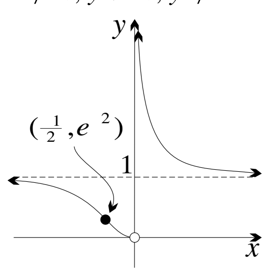
14 a $y' = -xe^{-\frac{1}{2}x^2}$,
$y'' = (x^2 - 1)e^{-\frac{1}{2}x^2}$
d $0 < y \le 1$

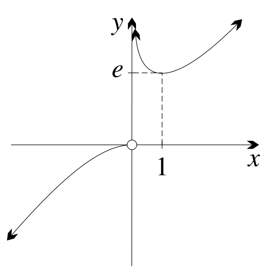


15 d $-\frac{1}{\sqrt{2e}} \le y \le \frac{1}{\sqrt{2e}}$



16 a **i** $y \to 0$ **ii** $y \to \infty$
b **i** $y \to -\infty$ **ii** $y \to 0$
17 $x = 1$ or $x = -1$
18 d $x \ne 0, y > 0, y \ne 1$ **e**





Exercise 6D

1 a $\frac{1}{2}e^{2x} + C$ **b** $\frac{1}{3}e^{3x} + C$
c $3e^{3x} + C$ **d** $2e^{2x} + C$
e $5e^{2x} + C$ **f** $4e^{3x} + C$
g $\frac{1}{4}e^{4x+5} + C$ **h** $\frac{1}{4}e^{4x-2} + C$
i $2e^{3x+2} + C$ **j** $e^{4x+3} + C$
k $-\frac{1}{2}e^{7-2x} + C$ **l** $-\frac{1}{6}e^{1-3x} + C$
2 a $e - 1$ **b** $e^2 - e$
c $e - e^{-3}$ **d** $e^2 - 1$
e $\frac{1}{2}(e^4 - 1)$ **f** $4(e^5 - e^{-10})$
g $2(e^{12} - e^{-4})$ **h** $\frac{3}{2}(e^{18} - e^{-6})$
i $\frac{1}{2}(e^3 - e^{-1})$ **j** $\frac{1}{4}(e^{-3} - e^{-11})$
k $\frac{1}{3}(e^{-1} - e^{-4})$ **l** $\frac{e^2}{2}(e^2 - 1)$
m $\frac{e}{3}(e^2 - 1)$ **n** $2e^4(e^3 - 1)$
o $3e^3(e^4 - 1)$ **p** $4e^2(e^3 - 1)$
3 a $-e^{-x} + C$ **b** $-\frac{1}{2}e^{-2x} + C$
c $-\frac{1}{3}e^{-3x} + C$ **d** $e^{-3x} + C$
e $-3e^{-2x} + C$ **f** $4e^{2x} + C$

4 a $f(x) = \frac{1}{2}e^{2x} + C$, for some constant C
b $C = -2\frac{1}{2}$, so $f(x) = \frac{1}{2}e^{2x} - 2\frac{1}{2}$
c $f(1) = \frac{1}{2}e^2 - 2\frac{1}{2}$, $f(2) = \frac{1}{2}e^4 - 2\frac{1}{2}$
5 a $f(x) = x + 2e^x - 1$, $f(1) = 2e$
b $f(x) = 2 + x - 3e^x$, $f(1) = 3 - 3e$
c $f(x) = 1 + 2x - e^{-x}$, $f(1) = 3 - e^{-1}$
d $f(x) = 1 + 4x + e^{-x}$, $f(1) = 5 + e^{-1}$
e $f(x) = \frac{1}{2}e^{2x-1} + \frac{5}{2}$, $f(1) = \frac{1}{2}(e + 5)$
f $f(x) = 1 - \frac{1}{3}e^{1-3x}$, $f(1) = 1 - \frac{1}{3}e^{-2}$
g $f(x) = 2e^{\frac{1}{2}x+1} - 6$, $f(1) = 2e^{\frac{3}{2}} - 6$
h $f(x) = 3e^{\frac{1}{3}x+2} - 1$, $f(1) = 3e^{\frac{7}{3}} - 1$
6 a $\frac{1}{2}e^{2x} + e^x + C$ **b** $\frac{1}{2}e^{2x} - e^x + C$
c $e^{-x} - e^{-2x} + C$ **d** $\frac{1}{2}e^{2x} + 2e^x + x + C$
e $\frac{1}{2}e^{2x} - 2e^x + x + C$ **f** $\frac{1}{2}e^{2x} - 4e^x + 4x + C$
g $\frac{1}{2}(e^{2x} + e^{-2x}) + C$ **h** $\frac{1}{10}(e^{10x} + e^{-10x}) + C$
7 a $\frac{1}{7}e^{7x+q} + C$ **b** $\frac{1}{3}e^{3x-k} + C$
c $\frac{1}{s}e^{sx+1} + C$ **d** $\frac{1}{k}e^{kx-1} + C$
e $e^{px+q} + C$ **f** $e^{mx+k} + C$
g $\frac{A}{s}e^{sx-t} + C$ **h** $\frac{B}{k}e^{kx-\ell} + C$
8 a $-e^{1-x} + C$ **b** $-\frac{1}{3}e^{1-3x} + C$
c $-\frac{1}{2}e^{-2x-5} + C$ **d** $-2e^{1-2x} + C$
e $2e^{5x-2} + C$ **f** $-4e^{5-3x} + C$
9 a $x - e^{-x} + C$ **b** $e^x - e^{-x} + C$
c $\frac{1}{2}e^{-2x} - e^{-x} + C$ **d** $e^{-3x} - \frac{1}{2}e^{-2x} + C$
e $e^{-3x} - e^{-2x} + C$ **f** $e^{-x} - e^{-2x} + C$
10 a $y = e^{x-1}$, $y = e^{-1}$
b $y = e^2 + 1 - e^{2-x}$, $y = e^2 + 1$
c $f(x) = e^x + \frac{x}{e} - 1$, $f(0) = 0$
d $f(x) = e^x - e^{-x} - 2x$
11 a $e^2 - e$
b $\frac{1}{2}(e^2 - e^{-2}) + 4(e - e^{-1}) + 8$
c $e + e^{-1} - 2$
d $\frac{1}{4}(e^4 - e^{-4}) + \frac{1}{2}(e^{-2} - e^2)$
e $e - e^{-1}$
f $e - e^{-1} + \frac{1}{2}(e^{-2} - e^2)$
12 a **i** $2xe^{x^2+3}$ **ii** $e^{x^2+3} + C$
b **i** $2(x - 1)e^{x^2-2x+3}$ **ii** $\frac{1}{2}e^{x^2-2x+3} + C$
c **i** $(6x + 4)e^{3x^2+4x+1}$ **ii** $\frac{1}{2}e^{3x^2+4x+1} + C$
d **i** $3x^2e^{x^3}$ **ii** $\frac{1}{3}(1 - e^{-1})$
13 a $-\frac{1}{2}e^{-2x} + C$ **b** $-\frac{1}{3}e^{-3x} + C$
c $2e^{\frac{1}{2}x} + C$ **d** $3e^{\frac{1}{3}x} + C$
e $-2e^{-\frac{1}{2}x} + C$ **f** $-3e^{-\frac{1}{3}x} + C$
14 a $y' = xe^x + e^x$, $e^2 + 1$
b $y' = -xe^{-x} + e^{-x}$, $-1 - e^2$
15 a $2e^{\frac{1}{2}x} + \frac{2}{3}e^{-\frac{3}{2}x} + C$ **b** $\frac{3}{2}e^{\frac{2}{3}x} - \frac{3}{4}e^{-\frac{4}{3}x} + C$
16 b 0

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

17 a $\frac{1}{2}e^{x^2} + C$ **b** $2e^{x^2-7} + C$
c $\frac{1}{2}e^{3x^2+4x+1} + C$ **d** $\frac{1}{3}e^{x^3-3x^2} + C$
e $-e^{x^{-1}} + C$ **f** $\frac{2}{3}e^{-x\sqrt{x}} + C$

20 b 1.1276
d $e^{0.5} = \alpha + \sqrt{\alpha^2 - 1}, e^{-0.5} = \alpha - \sqrt{\alpha^2 - 1}$

Exercise 6E

1 a i $e - 1 \doteqdot 1.72$
ii $1 - e^{-1} \doteqdot 0.63$
iii $1 - e^{-2} \doteqdot 0.86$
iv $1 - e^{-3} \doteqdot 0.95$
b The total area is exactly 1.

2 a $\frac{1}{2}(e^6 - 1) \doteqdot 201.2$ square units
b $1 - e^{-1} \doteqdot 0.6321$ square units
c $3(1 - e^{-1}) \doteqdot 1.896$ square units

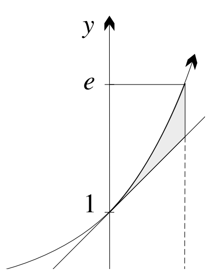
3 a $e(e^2 - 1)\,\text{u}^2$ **b** $\frac{1}{2}(e - e^{-1})\,\text{u}^2$
c $\frac{1}{2}e(e^2 - 1)\,\text{u}^2$ **d** $3e^2(e - 1)\,\text{u}^2$

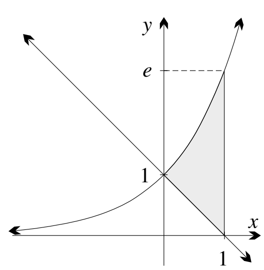
4 a $2(e - e^{-\frac{1}{2}})\,\text{u}^2$ **b** $(1 - e^{-1})\,\text{u}^2$

5 a $2(e^2 - e^{-2}) \doteqdot 14.51\,\text{u}^2$
b $18 + e^3 - e^{-3} \doteqdot 38.04\,\text{u}^2$

6 a $(1 + e^{-2})\,\text{u}^2$ **b** $1\,\text{u}^2$
c $e^{-1}\,\text{u}^2$ **d** $(3 + e^{-2})\,\text{u}^2$
e $1\,\text{u}^2$ **f** $(9 + e^{-2} - e)\,\text{u}^2$

7 a $\int_0^1 (e^x - 1 - x)\,dx = \left(e - 2\frac{1}{2}\right)\text{u}^2$ **b** $\int_0^1 (e^x - 1 + x)\,dx = \left(e - 1\frac{1}{2}\right)\text{u}^2$



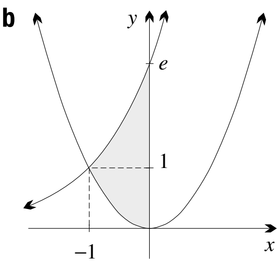


8 a The region is symmetric, so the area is twice the area in the first quadrant.
b $2 - \frac{2}{e}$ square units

9 a The region is symmetric, so the area is twice the area in the first quadrant.
b 2 square units

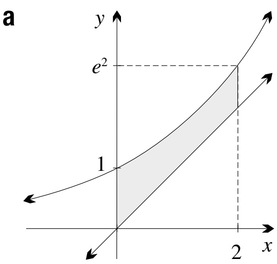
10 b 0
c The region is symmetric, so the area is twice the area in the first quadrant.
d $2(e^3 + e^{-3} - 2)$ square units

11 b



c $\left(e - 1\frac{1}{3}\right)\text{u}^2$

12 a



$(e^2 - 3)$ square units

b intercepts $(0, 7)$ and $(3, 0)$ and area $24 - \frac{7}{\log_e 2}$ square units

13 a $e - 1 \doteqdot 1.7183$
b 1.7539
c The trapezoidal-rule approximation is greater. The curve is concave up, so all the chords are above the curve.

14 a $-\frac{1}{2}e^{-x^2}$
b From $x = 0$ to $x = 2$, area $= \frac{1}{2} - \frac{1}{2}e^{-4}$ square units. The function is odd, so the area (not signed) from $x = -2$ to $x = 2$ is $1 - e^{-4}$ square units.

15 a i $1 - e^N$ **ii** 1
b i $1 - e^{-N}$ **ii** 1
c $\int_0^N 2xe^{-x^2}\,dx = 1 - e^{-N^2}$, thus in the limit as $N \to \infty$ this is just 1.

16 a $2(e - e^{\sqrt{\delta}})$
b It approaches $2(e - 1)$.

17 a $1 - (1 + N)e^{-N}$ **b** 1 **c** 2

Exercise 6F

1 a 2.303 **b** -2.303 **c** 11.72
d -12.02 **e** 3.912 **f** -3.912

2 a ln 20 **b** ln 5 **c** ln 80

3 a 3 **b** -1 **c** -2 **d** $\frac{1}{2}$
e 5 **f** 0.05 **g** 1 **h** e

4 b $1 = e^0$, so $\log_e 1 = \log_e e^0 = 0$.
d $e = e^1$, so $\log_e e = \log_e e^1 = 1$.

5 a $\log_e x = 6$ **b** $x = e^{-2}$ or $x = 1/e^2$
c $e^x = 24$ **d** $x = \log_e \frac{1}{3}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

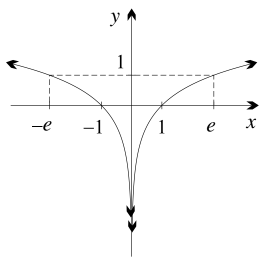
6 a $\dfrac{\log_e 7}{\log_e 2} \doteqdot 2.807$ **b** $\dfrac{\log_e 25}{\log_e 10} \doteqdot 1.398$

c $\dfrac{\log_e 0.04}{\log_e 3} \doteqdot -2.930$

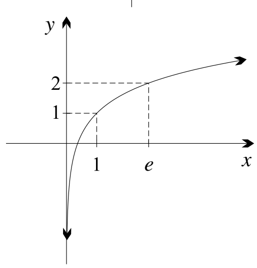
7 a Reflection in $y = x$, which reflects lines with gradient 1 to lines of gradient 1. The tangent to $y = e^x$ at its y-intercept has gradient 1, so its reflection also has gradient 1.

b Reflection in the y-axis, which is also a horizontal dilation with factor -1.

c

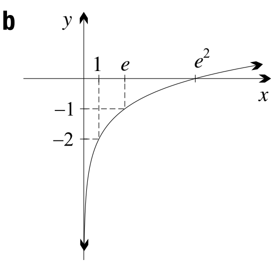


8 a



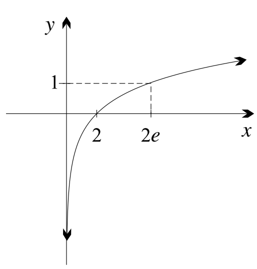
Shift $y = \log_e x$ up 1.

b



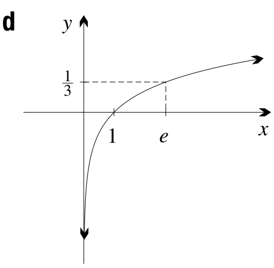
Shift $y = \log_e x$ down 2.

c



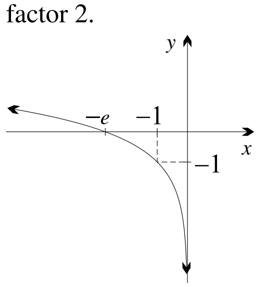
Stretch $y = \log_e x$ horizontally with factor 2.

d



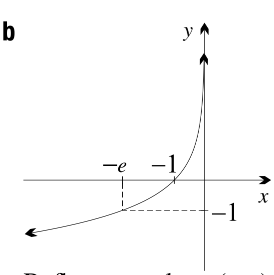
Stretch $y = \log_e x$ vertically with factor $\frac{1}{3}$.

9 a



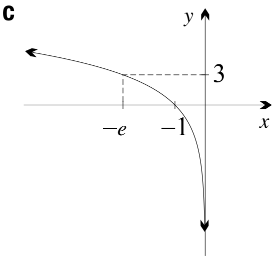
Shift $y = \log_e(-x)$ down 1.

b



Reflect $y = \log_e(-x)$ in the x-axis.

c



Stretch $y = \log_e(-x)$ vertically with factor 3.

10 a e **b** $-\frac{1}{e}$ **c** 6

d $\frac{1}{2}$ **e** $2e$ **f** 0

g e **h** 1 **i** 0

11 It is a horizontal dilation of $y = \log_e(-x)$ with factor $\frac{1}{2}$. Its equation is $y = \log_e(-2x)$.

12 a $x = 1$ or $x = \log_2 7$

b $x = 2$ ($3^x = -1$ has no solutions.)

c i $x = 2$ or $x = 0$

ii $x = 0$ or $x = \log_3 4$

iii $x = \log_3 5$ ($3^x = -4$ has no solutions.)

iv The quadratic has no solutions because $\Delta < 0$

v $x = 2$

vi $x = 1$ or 2

13 a $x = 0$ **b** $x = \log_e 2$

c $x = 0$ or $x = \log_e 3$ **d** $x = 0$

14 a $x = 1$ or $x = \log_4 3 \doteqdot 0.792$

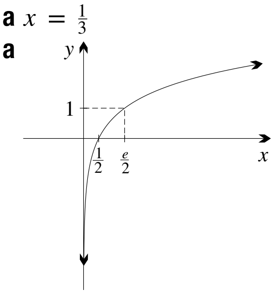
b $x = \log_{10} \frac{1+\sqrt{5}}{2} \doteqdot 0.209$. $\log_{10} \frac{1-\sqrt{5}}{2}$ does not exist because $\frac{1-\sqrt{5}}{2}$ is negative.

c $x = -1$ or $x = \log_{\frac{1}{5}} 2 \doteqdot -0.431$

15 a $x = e$ or $x = e^4$ **b** $x = 1$ or $x = e^3$

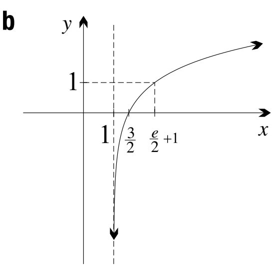
16 a $x = \frac{1}{3}$ **b** $x = 3$ or 4

17 a



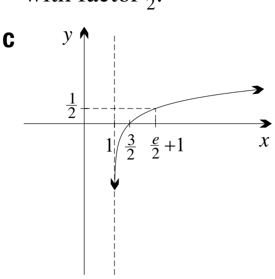
Stretch horizontally with factor $\frac{1}{2}$.

b



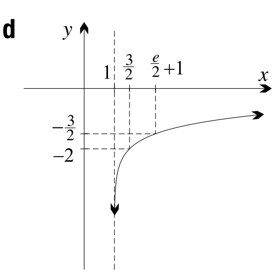
Shift right 1.

c



Stretch vertically with factor $\frac{1}{2}$.

d



Shift down 2.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

18 First, the base must be positive because powers of negative numbers are not well defined when the index is a real number, so a negative number can't be used as a base for logarithms. Secondly, the base cannot be 1 because all powers of 1 are 1, and in any case, $\log_e 1 = 0$ and you can't divide by zero.

19 a Stretch horizontally with factor $\frac{1}{5}$. Alternatively, $y = \log_e x + \log_e 5$, so it is a shift up $\log_e 5$.

b Shift up 2.
Alternatively, $y = \log_e x + \log_e e^2 = \log_e e^2 x$, so it is a dilation horizontally with factor e^{-2}.

20 $2\frac{28}{39}$

Exercise 6G

1 a $y' = \frac{1}{x+2}$ **b** $y' = \frac{1}{x-3}$
c $y' = \frac{3}{3x+4}$ **d** $y' = \frac{2}{2x-1}$
e $y' = \frac{-4}{-4x+1}$ **f** $y' = \frac{-3}{-3x+4}$
g $y' = \frac{-2}{-2x-7} = \frac{2}{2x+7}$ **h** $y' = \frac{6}{2x+4} = \frac{3}{x+2}$
i $y' = \frac{15}{3x-2}$

2 a $y = \log_e 2 + \log_e x,\ y' = \frac{1}{x}$
b $y = \log_e 5 + \log_e x,\ y' = \frac{1}{x}$
c $\frac{1}{x}$ **d** $\frac{1}{x}$ **e** $\frac{4}{x}$ **f** $\frac{3}{x}$ **g** $\frac{4}{x}$ **h** $\frac{3}{x}$

3 a $y' = \frac{1}{x+1},\ y'(3) = \frac{1}{4}$
b $y' = \frac{2}{2x-1},\ y'(3) = \frac{2}{5}$
c $y' = \frac{2}{2x-5},\ y'(3) = 2$
d $y' = \frac{4}{4x+3},\ y'(3) = \frac{4}{15}$
e $y' = \frac{5}{x+1},\ y'(3) = \frac{5}{4}$
f $y' = \frac{12}{2x+9},\ y'(3) = \frac{4}{5}$

4 a $\frac{1}{x}$ **b** $\frac{-1}{x+1}$
c $1 + \frac{4}{x}$ **d** $8x^3 + \frac{3}{x}$
e $\frac{2}{2x-1} + 6x$ **f** $3x^2 - 3 + \frac{5}{5x-7}$

5 a $y = 3\ln x,\ y' = \frac{3}{x}$
b $y = 2\ln x,\ y' = \frac{2}{x}$
c $y = -3\ln x,\ y' = -\frac{3}{x}$
d $y = -2\ln x,\ y' = -\frac{2}{x}$
e $y = \frac{1}{2}\ln x,\ y' = \frac{1}{2x}$
f $y = \frac{1}{2}\ln(x+1),\ y' = \frac{1}{2(x+1)}$

6 a $\frac{1}{x}$ **b** $\frac{1}{x}$ **c** $\frac{3}{x}$
d $-\frac{6}{x}$ **e** $1 + \frac{1}{x}$ **f** $12x^2 - \frac{1}{x}$

7 a $\frac{2x}{x^2+1}$ **b** $\frac{-2x}{2-x^2}$ **c** $\frac{e^x}{1+e^x}$

8 a $\frac{2x+3}{x^2+3x+2}$ **b** $\frac{6x^2}{1+2x^3}$ **c** $\frac{e^x}{e^x-2}$
d $1 - \frac{2x+1}{x^2+x}$ **e** $2x + \frac{3x^2-1}{x^3-x}$
f $12x^2 - 10x + \frac{4x-3}{2x^2-3x+1}$

9 a $1, 45°$ **b** $\frac{1}{3}, 18°26'$
c $2, 63°26'$ **d** $\frac{1}{4}, 14°2'$

10 a $1 + \log_e x$
b $\frac{2x}{2x+1} + \log_e(2x+1)$
c $\frac{2x+1}{x} + 2\log_e x$
d $x^3(1 + 4\log_e x)$
e $\log_e(x+3) + 1$
f $\frac{2(x-1)}{2x+7} + \log_e(2x+7)$
g $e^x\left(\frac{1}{x} + \log_e x\right)$
h $e^{-x}\left(\frac{1}{x} - \log_e x\right)$

11 a $\frac{1-\log_e x}{x^2}$ **b** $\frac{1-2\log_e x}{x^3}$
c $\frac{\log_e x - 1}{(\log_e x)^2}$ **d** $\frac{x(2\log_e x - 1)}{(\log_e x)^2}$
e $\frac{1 - x\log_e x}{xe^x}$ **f** $\frac{e^x(x\log_e x - 1)}{x(\log_e x)^2}$

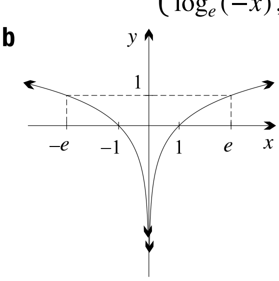
12 a $\frac{3}{x}$ **b** $\frac{1}{3x}$ **c** $-\frac{1}{x}$ **d** $\frac{1}{2x-4}$
e $\frac{1}{1+x} + \frac{1}{1-x}$
f $\frac{1}{x} + \frac{1}{2(x+1)}$
g $y = x\log_e 2,\ y' = \log_e 2$
h $y = x,\ y' = 1$
i $y = x\log_e x,\ y' = 1 + \log_e x$

13 a $f'(x) = \frac{1}{x-1},\ f'(3) = \frac{1}{2},\ f''(x) = -\frac{1}{(x-1)^2},$
$f''(3) = -\frac{1}{4}$
b $f'(x) = \frac{2}{2x+1},\ f'(0) = 2,\ f''(x) = -\frac{4}{(2x+1)^2},$
$f''(0) = -4$
c $f'(x) = \frac{2}{x},\ f'(2) = 1,\ f''(x) = -\frac{2}{x^2},$
$f''(2) = -\frac{1}{2}$
d $f'(x) = 1 + \log_e x,\ f'(e) = 2,\ f''(x) = \frac{1}{x},$
$f''(e) = \frac{1}{e}$

14 a $\log_e x,\ x = 1$
b $x(1 + 2\log_e x),\ x = e^{-\frac{1}{2}}$
c $\frac{1-\log_e x}{x^2},\ x = e$
d $\frac{4(\log_e x)^3}{x},\ x = 1$
e $\frac{8}{x}(2\log_e x - 3)^3,\ x = e^{\frac{3}{2}}$
f $\frac{-1}{x(\log_e x)^2}$ is never zero.
g $\frac{1}{x\log_e x}$ is never zero.
h $1 + \ln x,\ \left(\frac{1}{e}, -\frac{1}{e}\right)$
i $\frac{x-1}{x^2},\ (1, 1)$

16 a $\log_e|x| = \begin{cases} \log_e x, & \text{for } x > 0, \\ \log_e(-x), & \text{for } x < 0. \end{cases}$

b



c For $x > 0$, $\log_e|x| = \log_e x$, so $\frac{d}{dx}\log_e x = \frac{1}{x}$.
For $x < 0$, $\log_e|x| = \log_e(-x)$, and using the standard form, $\frac{d}{dx}\log_e(-x) = -\frac{1}{-x} = \frac{1}{x}$.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

d $\log_e 0$ is undefined. In fact, $\log_e x \to -\infty$ as $x \to 0$, so $x = 0$ is an asymptote.

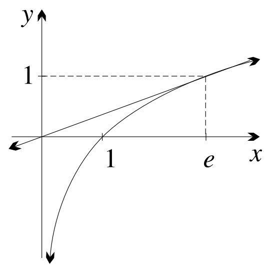
17 d **i** 2 **ii** 2.5937 **iii** 2.7048
iv 2.7169 **v** 2.7181

Exercise 6H

1 a $y = \frac{1}{e}x$ **b** $y = x - 1$ **c** $y = ex - 2$

d $y = -x + 1$. When $x = 0$, $y = 1$.

2 a As P moves to the left along the curve, the tangent becomes steeper, so it does not pass through the origin. As P moves right, the angle of the tangent becomes less steep, hence it does not pass through the origin.



b There are no tangents through each point below the curve. There are two tangents through each point above the curve and to the right of the y-axis. There is one tangent through each point on the curve, and through each point on and to the left of the y-axis.

3 a $y = 4x - 4$, $y = -\frac{1}{4}x + \frac{1}{4}$

b $y = x + 2$, $y = -x + 4$

c $y = 2x - 4$, $y = -\frac{1}{2}x - 1\frac{1}{2}$

d $y = -3x + 4$, $y = \frac{1}{3}x + \frac{2}{3}$

4 b $y = 3x - 3, -3$, $y = -\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}$

c $\frac{5}{3}$ square units

5 a $(2, \log_e 2)$, $y = \frac{1}{2}x - 1 + \log_e 2$,
$y = -2x + 4 + \log_e 2$

b $\left(\frac{1}{2}, -\log_e 2\right)$, $y = 2x - 1 - \log_e 2$,
$y = -\frac{1}{2}x + \frac{1}{4} - \log_e 2$

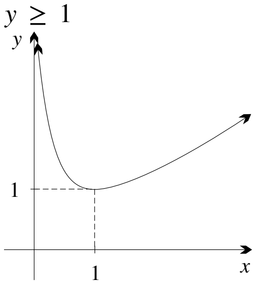
6 a $x > 0$. The domain is not symmetric about the origin, so the function is certainly not even or odd.

b $y' = 1 - \frac{1}{x}$, $y'' = \frac{1}{x^2}$

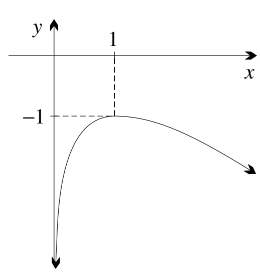
c $y'' > 0$, for all x

d $(1, 1)$

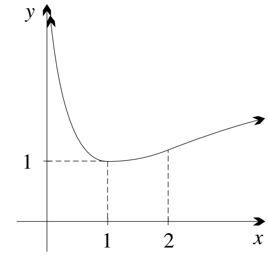
e $y \geq 1$



f



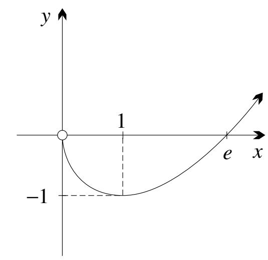
7 a $x > 0$ **d** $y \geq 1$



8 a $x > 0$, $(e, 0)$

b

x	1	e	e^2
y	-1	0	e^2
sign	$-$	0	$+$



c $y'' = \frac{1}{x}$

d $(1, -1)$ is a minimum turning point.

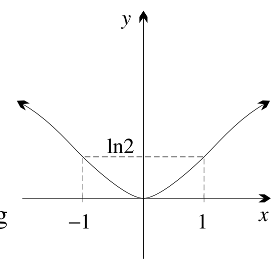
e It is concave up throughout its domain.

f $y \geq -1$

9 a all real x

b Even

c It is zero at $x = 0$, and is positive otherwise because the logs of numbers greater than 1 are positive.



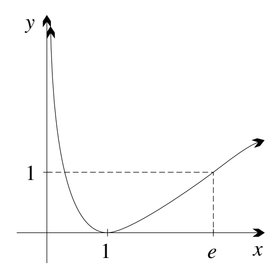
e $(0, 0)$ is a minimum turning point.

f $(1, \log_e 2)$ and $(-1, \log_e 2)$

g $y \geq 0$

10 a $x > 0$

b It is zero at $x = 1$, and is positive otherwise because squares cannot be negative.

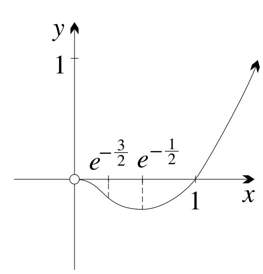


c $y' = \frac{2}{x}\ln x$

e $y \geq 0$

11 a $x > 0$. Minimum at $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$

c $y \to 0$ as $x \to 0^+$,
$y' \to 0$ as $x \to 0^+$,
hence the graph becomes horizontal approaching the origin.



d $y \geq -\frac{1}{2e}$

12 a $x > 0$. $y \to 0^+$ as $x \to \infty$ so the x-axis is a horizontal asymptote.
$y \to -\infty$ as $x \to 0^+$ so the y-axis is a vertical asymptote.

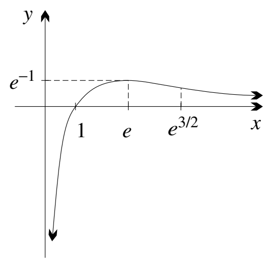
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

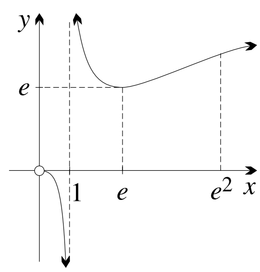
b $y' = \frac{1}{x^2}(1 - \log x)$, $y'' = \frac{1}{x^3}(2\log_e x - 3)$

d $\left(e^{\frac{3}{2}}, \frac{3}{2}e^{-\frac{3}{2}}\right)$

e $y \le e^{-1}$



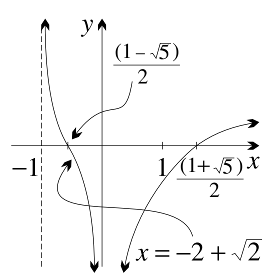
13 $x > 0, x \ne 1, y < 0$ or $y \ge e$. $x = 1$ is a vertical asymptote and the curve becomes horizontal approaching the origin.



14 a $x > -1$ or $x \ne 0$

c $x = -2$ is outside the domain.

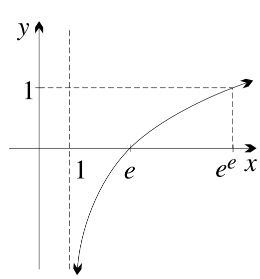
d one at $x = -2 + \sqrt{2}$



15 a $x > 1$

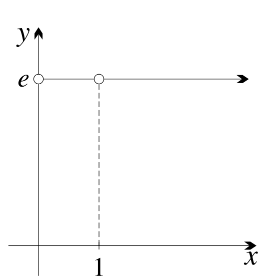
c $y' = \frac{1}{x\ln x}$, which can never be zero, $y'' = -\frac{1 + \ln x}{(x\ln x)^2}$

d The value $x = e^{-1}$ is outside the domain.



16 $\displaystyle\lim_{x\to\infty}\frac{\log_e x}{x} = 0$ and $\displaystyle\lim_{x\to 0^+} x\log_e x = 0$

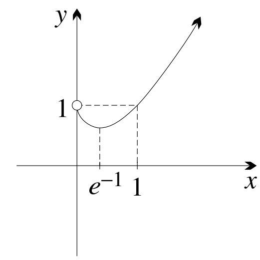
17 $y = e$ for all x in the domain, which is $x > 0$, $x \ne 1$.



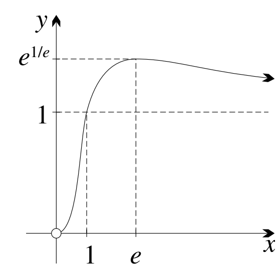
18 a $y' = x^x(1 + \log_e x)$

b Stationary point at $(e^{-1}, e^{-1/e})$, and gradient 1 at $x = 1$.

c domain: $x > 0$ (note that 0^0 is undefined), range: $y \ge e^{-1/e}$



19 b $y' = x^{-2}x^{\frac{1}{x}}(1 - \log_e x)$



Exercise 6I

1 a $2\log_e|x| + C$ **b** $\frac{1}{3}\log_e|x| + C$
c $\frac{4}{5}\log_e|x| + C$ **d** $\frac{3}{2}\log_e|x| + C$

2 a $\frac{1}{4}\log_e|4x + 1| + C$ **b** $\frac{1}{5}\log_e|5x - 3| + C$
c $2\log_e|3x + 2| + C$ **d** $3\log_e|5x + 1| + C$
e $\log_e|4x + 3| + C$ **f** $-\log_e|3 - x| + C$
g $-\frac{1}{2}\log_e|7 - 2x| + C$ **h** $\frac{4}{5}\log_e|5x - 1| + C$
i $-4\log_e|1 - 3x| + C$

3 a $\log_e 5$
b $\log_e 3$
c $\log_e|-2| - \log_e|-8| = -2\log_e 2$
d The integral is meaningless because it runs across an asymptote at $x = 0$.
e $\frac{1}{2}(\log_e 8 - \log_e 2) = \log_e 2$
f $\frac{1}{5}(\log_e|-75| - \log_e|-25|) = -\frac{1}{5}\log_e 3$

4 a $\log_e 2 \doteqdot 0.6931$
b $\log_e 3 - \log_e 5 \doteqdot -0.5108$
c $-\frac{1}{2}\log_e 7 \doteqdot -0.9730$
d $\frac{3}{2}\log_e 3 \doteqdot 1.648$
e $\log_e\frac{5}{2} \doteqdot 0.9163$
f The integral is meaningless because it runs across an asymptote at $x = 5\frac{1}{2}$.

5 a 1 **b** 2 **c** 3 **d** $\frac{1}{2}$

6 a $x + \log_e|x| + C$ **b** $\frac{1}{5}x + \frac{3}{5}\log_e|x| + C$
c $\frac{1}{9}\log_e|x| - \frac{8}{9}x + C$ **d** $3x - 2\log_e|x| + C$
e $x^2 + x - 4\log_e|x| + C$
f $\frac{1}{3}x^3 - \log_e|x| - \frac{2}{x} + C$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

7 a $\log_e|x^2 - 9| + C$
b $\log_e|3x^2 + x| + C$
c $\log_e|x^2 + x - 3| + C$
d $\log_e|2 + 5x - 3x^2| + C$
e $\frac{1}{2}\log_e|x^2 + 6x - 1| + C$
f $\frac{1}{4}\log_e|12x - 3 - 2x^2| + C$
g $\log_e(1 + e^x) + C$
h $-\log_e(1 + e^{-x}) + C$
i $\log_e(e^x + e^{-x}) + C$
The denominators in parts **g–i** are never negative, so the absolute value sign is unnecessary.

8 a $\frac{1}{3}\log_e|3x - k| + C$ **b** $\frac{1}{m}\log_e|mx - 2| + C$
c $\log_e|px + q| + C$ **d** $\frac{4}{s}\log_e|sx - t| + C$

9 a $f(x) = x + 2\ln|x|, f(2) = 2 + 2\ln 2$
b $f(x) = x^2 + \frac{1}{3}\ln|x| + 1, f(2) = 5 + \frac{1}{3}\ln 2$
c $f(x) = 3x + \frac{5}{2}\ln|2x - 1| - 3,$
$f(2) = 3 + \frac{5}{2}\ln 3$
d $f(x) = 2x^3 + 5\ln|3x + 2| - 2,$
$f(2) = 14 + 5\ln 8$

10 a $f(x) = x + \ln|x| + \frac{1}{2}x^2$
b $g(x) = x^2 - 3\ln|x| + \frac{4}{x} - 6$

11 a $y = \frac{1}{4}(\log_e|x| + 2), x = e^{-2}$
b $y = 2\log_e|x + 1| + 1$
c $y = \log_e\left|\frac{x^2 + 5x + 4}{10}\right| + 1, y(0) = \log_e\frac{4}{10} + 1$
d $y = 2\log_e|x| + x + C, y = 2\log_e|x| + x,$
$y(2) = \log_e 4 + 2$
e $f(x) = 2 + x - \log_e|x|, f(e) = e + 1$

12 a $\log_e|x^3 - 5| + C$
b $\log_e|x^4 + x - 5| + C$
c $\frac{1}{4}\log_e|x^4 - 6x^2| + C$
d $\frac{1}{2}\log_e|5x^4 - 7x^2 + 8| + C$
e $2\log_e 2$
f $\log_e\frac{4(e + 1)}{e + 2}$

13 a **i** $y' = \log_e x$
ii $x\log_e x - x + C$ and $\frac{\sqrt{e}}{2}$
b **i** $y' = 4x\log_e x$
ii $\frac{1}{2}x^2\log_e x - \frac{1}{4}x^2$ and $2\log 2 - 1 - \frac{e^2}{4}$
c $\frac{2\log_e x}{x}$ and $\frac{3}{8}$
d $\ln(\ln x) + C$

14 a $a = e^5$ **b** $a = e^{-4}$
c $a = -e^2$ **d** $a = -e^{-1}$

15 a $\log_e(e^x + 1) + \mathrm{C}$ **b** $\frac{1}{3}(e^3 - e^{-3}) + 2$
c $(x + 1)e^x$

16 The key to all this is that
$\log_e|5x| = \log_e 5 + \log_e|x|,$
so that $\log_e|x|$ and $\log_e|5x|$ differ only by a constant $\log_e 5$. Thus $C_2 = C_1 - \frac{1}{5}\log_e 5$, and because C_1 and C_2 are arbitrary constants, it does not matter at all. In particular, in a definite integral, adding a constant doesn't change the answer, because it cancels out when we take $F(b) - F(a)$.

17 $y = \begin{cases} \log_e x + 1, & \text{for } x < 0, \\ \log(-x) + 2, & \text{for } x > 0. \end{cases}$

18 d **i** $\log_e\frac{3}{2} \doteqdot 0.41$
ii $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$
e $\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots,$
$\log_e\frac{1}{2} \doteqdot -0.69$
f Using $x = \frac{1}{2}$, $\log_e 3 \doteqdot 1.0986$.

Exercise 6J

1 b $e \doteqdot 2.7$

2 **i** $\log_e 5 \doteqdot 1.609\,\mathrm{u}^2$
ii $1\mathrm{u}^2$
iii $2\log_e 2 \doteqdot 1.386\,\mathrm{u}^2$

3 a $(\log_e 3 - \log_e 2)$ square units
b $\log_e 2 - \log_e\frac{1}{2} = 2\log_e 2$ square units

4 a $\frac{1}{3}(\log_e 5 - \log_e 2)\,\mathrm{u}^2$ **b** $9\mathrm{u}^2$
c $2\log_e 2 + \frac{15}{8}\mathrm{u}^2$ **d** $\log_e 3 + 8\frac{2}{3}\mathrm{u}^2$

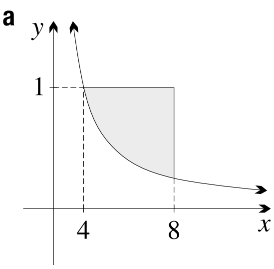
5 a $(6 - 3\log_e 3)\,\mathrm{u}^2$ **b** $(4 - \log_e 3)\,\mathrm{u}^2$

6 a $\left(3\frac{3}{4} - 2\log_e 4\right)\mathrm{u}^2$ **b** $\left(3\frac{3}{4} - 2\log_e 4\right)\mathrm{u}^2$

7 a $2\log_e 2\mathrm{u}^2$ **b** $(1 - \log_e 2)\,\mathrm{u}^2$

8 a $(\log_e 4)\,\mathrm{u}^2$ **b** $(6 - 3\log_e 3)\,\mathrm{u}^2$

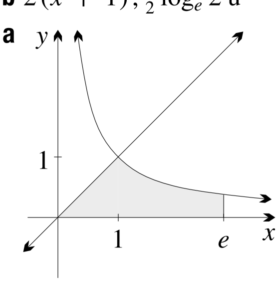
9 a $\frac{1}{2}\mathrm{u}^2$ **b** $2\log_e\frac{4}{3}\mathrm{u}^2$

10 a


b $4(1 - \log_e 2)$

11 a $(\frac{1}{3}, 3)$ and $(1, 1)$ **b** $(\frac{4}{3} - \log_e 3)\,\mathrm{u}^2$

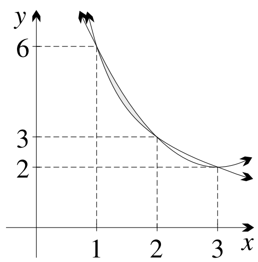
12 a $2x, \frac{1}{2}\log_e 5 \doteqdot 0.805\mathrm{u}^2$
b $2(x + 1), \frac{1}{2}\log_e 2\,\mathrm{u}^2$

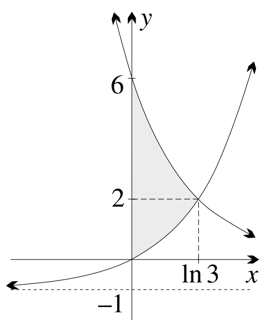
13 a


b $\frac{3}{2}\mathrm{u}^2$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

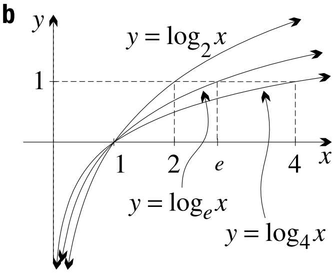
Cambridge University Press
Updated July 2022

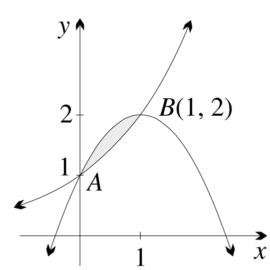
14 a 3.9828 square units
b $5\log_e 5 - 4 \doteqdot 4.0472$ square units
c The estimate is less. The curve is concave down, so the chords are below the curve.
15 b $(2 - \log_e 3)\ \text{u}^2$
16 b


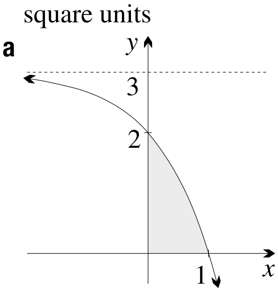
c $(2 - 6\log_e \frac{4}{3})\ \text{u}^2$
17 a The upper rectangle has height 2^{-n}, the lower rectangle has height 2^{-n-1}, both rectangles have width $2^{n+1} - 2^n = 2^n$.
18 b $(\ln 3, 2)$
c


d $(2 + \ln 3)\ \text{u}^2$
19 a $e - 2$ square units **b** e^{-1} square units
c $e - 2 + e^{-1}$ square units
20 $2\log_e(\sqrt{x} + 1) + C$
21 b i $\log_e(1 + \sqrt{2})$ **ii** $\log_e(2 + \sqrt{3})$
23 a i The region is above the x-axis, and contained within the rectangle $ABCO$.
ii
$$0 < \Big[\log_e t\Big]_1^{\sqrt{x}} < \sqrt{x}$$
$$0 < \log_e \sqrt{x} - \log_e 1 < \sqrt{x}$$
$$0 < \tfrac{1}{2}\log_e x < \sqrt{x}$$
$$\boxed{\times \tfrac{2}{x}} \quad 0 < \frac{\log_e x}{x} < \frac{2}{\sqrt{x}}.$$
iii The third term has limit 0 as $x \to \infty$, so the result follows by the sandwiching principle.

Exercise 6K

1 a 1.58 **b** 3.32 **c** 2.02 **d** −4.88
2 a $y' = \frac{1}{x\log_e 2}$ **b** $y' = \frac{1}{x\log_e 10}$ **c** $y' = \frac{3}{x\log_e 5}$
3 a $y' = \frac{1}{x\log_e 3}$ **b** $y' = \frac{1}{x\log_e 7}$ **c** $y' = \frac{5}{x\log_e 6}$
4 a $3^x\log_e 3$ **b** $4^x\log_e 4$ **c** $2^x\log_e 2$
5 a $y' = 10^x\log_e 10$ **b** $y' = 8^x\log_e 8$
c $y' = 3 \times 5^x\log_e 5$
6 a $\frac{2^x}{\log_e 2} + C$ **b** $\frac{6^x}{\log_e 6} + C$
c $\frac{7^x}{\log_e 7} + C$ **d** $\frac{3^x}{\log_e 3} + C$
7 a $\frac{1}{\log_e 2} \doteqdot 1.443$ **b** $\frac{2}{\log_e 3} \doteqdot 1.820$
c $\frac{24}{5\log_e 5} \doteqdot 2.982$ **d** $\frac{15}{\log_e 4} \doteqdot 10.82$
8 b


9 a $\frac{1}{\log_e 2}$ **b** $y = \frac{1}{\log_e 2}(x - 1)$
c i $y = \frac{1}{\log_e 3}(x - 1)$ **ii** $y = \frac{1}{\log_e 5}(x - 1)$
10 a $\frac{6}{\log_e 2} \doteqdot 8.6562$ **b** $2 + \frac{8}{3\log_e 3} \doteqdot 4.4273$
c $\frac{99}{\log_e 10} - 20 \doteqdot 22.9952$
11 $y = \frac{\log_e x}{\log_e 10},\ y' = \frac{1}{x\log_e 10}$
a $\frac{1}{10\log_e 10}$
b $x - 10y\log_e 10 + 10(\log_e 10 - 1) = 0$
c $x = \frac{1}{\log_e 10}$
12 a $y = \frac{1}{\log_e 2}\left(\frac{x}{3} - 1 + \log_e 3\right),\ y = \frac{x}{3} - 1 + \log_e 3,$
$y = \frac{1}{\log_e 4}\left(\frac{x}{3} - 1 + \log_e 3\right)$
b They all meet the x-axis at $(3 - 3\log_e 3, 0)$
13 b $\left(\frac{5}{3} - \frac{1}{\log_e 2}\right)\ \text{u}^2$


14 intercepts $(0, 7)$ and $(3, 0)$, area $\left(24 - \frac{7}{\log_e 2}\right)$ square units
15 a


b $\left(3 - \frac{2}{\log_e 3}\right)\text{u}^2$
16 b $\displaystyle\int_{-\frac{1}{2}}^{0} x + 1 - 4^x\,dx$ **c** $\frac{3}{8} - \frac{1}{2\log_e 4}$
18 a $x\log_e x - x + C$ **b** $10 - \frac{9}{\log_e 10}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

19 a i $y' = \frac{1}{x \log_e 3}$

ii $y' = \frac{2}{(2x + 3) \log_e 7}$

iii $y' = -\frac{45}{(4 - 9x) \log_e 6}$

b i $y' = 10^x \log_e 10$

ii $y' = 4 \times 8^{4x-3} \log_e 8$

iii $y' = -21 \times 5^{2-7x} \log_e 5$

c i $\frac{3^{5x}}{5 \log_e 3} + C$ **ii** $\frac{6^{2x+7}}{2 \log_e 6} + C$

iii $-\frac{5 \times 7^{4-9x}}{9 \log_e 7} + C$

20 a $y = e^{kx}$ and $y = \frac{1}{k} \log_e x$

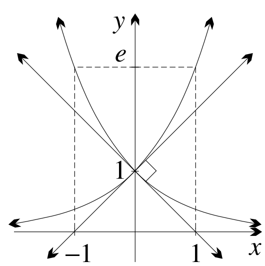
b The functions $y = a^x$ and $y = \log_e x$ are inverse, so they are symmetric in the line $y = x$. The common tangent is therefore the line $y = x$, which has gradient 1. (This argument would be invalid if there were more than one intersection point.)

c $k\,e^{kx} = 1$ and $\frac{1}{kx} = 1$

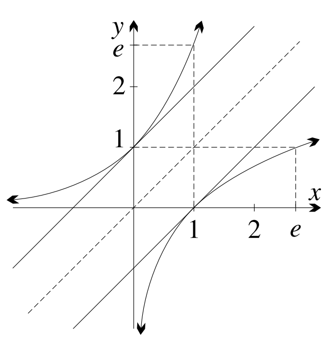
d $k = \frac{1}{e}$, $a = e^{\frac{1}{e}}$

Chapter 6 review exercise

1 a Each graph is reflected onto the other in the line $x = 0$. The tangents have gradients 1 and -1, and are at right angles.



b Each graph is reflected onto the other in the line $y = x$. The tangents both have gradients 1, and are thus parallel.



2 a 54.60 **b** 2.718 **c** 0.2231 **d** 0.6931

e -0.3010 **f** -5.059 **g** 130.6 **h** 0.5925

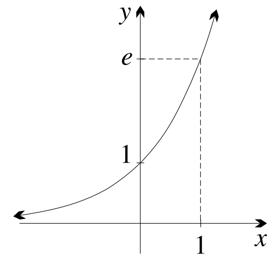
3 a 2.402 **b** 5.672 **c** 5.197 **d** 3.034

4 a e^{5x} **b** e^{6x} **c** e^{-4x} **d** e^{9x}

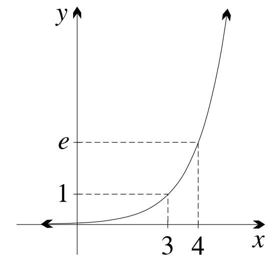
5 a $x = 2$

b $x = \log_e 4\,(= 2\log_e 2)$ or $\log_e 7$

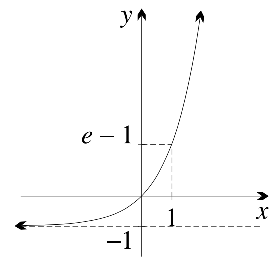
6 a $y > 0$



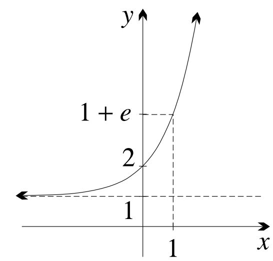
b $y > 0$



c $y > 1$

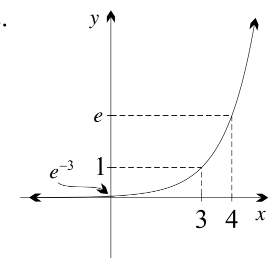


d $y > -1$



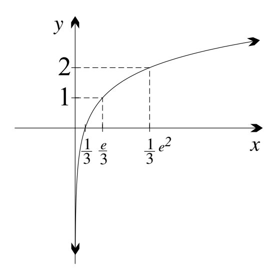
7 a i Shift $y = e^x$ right 3 units.

ii $y = e^{-3}e^x$ or $\frac{y}{e^{-3}} = e^x$, so dilate vertically with factor e^{-3}.



b i $y = \log_e \frac{x}{1/3}$, so dilate $y = \log_e x$ horizontally with factor $\frac{1}{3}$.

ii $y = \log_e x + \log_e 3$, or $y - \log_e 3 = \log_e x$, so shift $y = \log_e x$ up $\log_e 3$.



8 a e^x **b** $3e^{3x}$ **c** $2e^{2x+3}$ **d** $-e^{-x}$

e $-3e^{-3x}$ **f** $6e^{2x+5}$ **g** $2e^{\frac{1}{2}x}$ **h** $4e^{6x-5}$

9 a $5e^{5x}$ **b** $4e^{4x}$ **c** $-3e^{-3x}$ **d** $-6e^{-6x}$

10 a $3x^2e^{x^3}$ **b** $(2x - 3)\,e^{x^2-3x}$

c $e^{2x} + 2xe^{2x} = e^{2x}(1 + 2x)$

d $6e^{2x}(e^{2x} + 1)^2$

e $\frac{e^{3x}(3x - 1)}{x^2}$

f $2xe^{x^2}(1 + x^2)$

g $5(e^x + e^{-x})(e^x - e^{-x})^4$

h $\frac{4xe^{2x}}{(2x + 1)^2}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 a $y' = 2e^{2x+1},\ y'' = 4e^{2x+1}$

b $y' = 2xe^{x^2+1},\ y'' = 2e^{x^2+1}(2x^2+1)$

12 $y = e^2x - e^2$, x-intercept 1, y-intercept $-e^2$.

13 a $\frac{1}{3}$

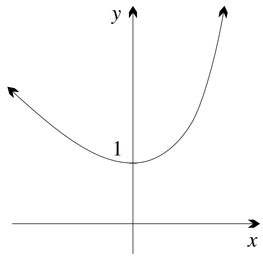
b When $x = 0,\ y'' = 9$, so the curve is concave up there.

14 a $y' = e^x - 1,\ y'' = e^x$

b $(0, 1)$ is a minimum turning point.

c $y'' = e^x$, which is positive for all x.

d Range: $y \geq 1$



15 $\left(\frac{1}{2}, \frac{1}{2e}\right)$ is a maximum turning point.

16 a $\frac{1}{5}e^{5x} + C$ **b** $-2e^{2-5x} + C$

c $5e^{\frac{1}{5}x} + C$ **d** $\frac{3}{5}e^{5x-4} + C$

17 a $e^2 - 1$ **b** $\frac{1}{2}(e^2 - 1)$

c $e - 1$ **d** $\frac{1}{3}(e^2 - 1)$

e $\frac{1}{2}e^2(e - 1)$ **f** $4(e - 1)$

18 a $-\frac{1}{5}e^{-5x} + C$ **b** $\frac{1}{4}e^{4x} + C$

c $-2e^{-3x} + C$ **d** $\frac{1}{6}e^{6x} + C$

e $-\frac{1}{2}e^{-2x} + C$ **f** $e^x - \frac{1}{2}e^{-2x} + C$

g $\frac{1}{3}e^{3x} + e^x + C$ **h** $x - 2e^{-x} - \frac{1}{2}e^{-2x} + C$

19 a $2 - e^{-1}$ **b** $\frac{1}{2}(e^4 + 3)$

c $2(1 - e^{-1})$ **d** $\frac{1}{3}(e - 2)$

e $e - e^{-1}$ **f** $\frac{1}{2}(e^2 + 4e - 3)$

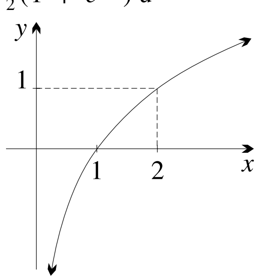
20 $f(x) = e^x + e^{-x} - x + 1,\ f(1) = e + e^{-1}$

21 a $3x^2e^{x^3}$ **b** $\frac{1}{3}(e - 1)$

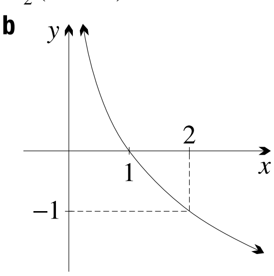
22 a $3.19\ \text{u}^2$ **b** $0.368\ \text{u}^2$

23 a $\frac{1}{2}(1 + e^{-2})\ \text{u}^2$ **b** $\frac{1}{2}(3 - e)\ \text{u}^2$

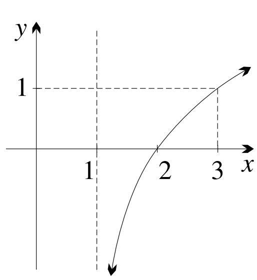
24 a



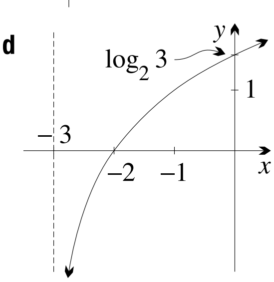
b



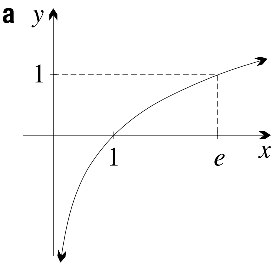
c



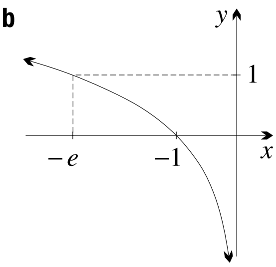
d



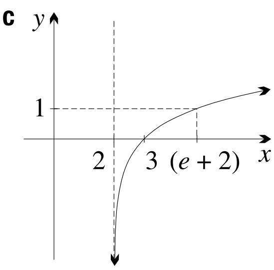
25 a



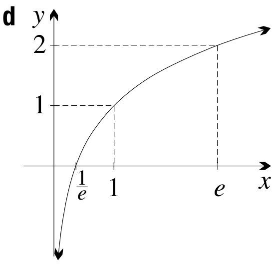
b



c



d



26 a e **b** 3 **c** -1 **d** e

27 a $\dfrac{1}{x}$ **b** $\dfrac{1}{x}$ **c** $\dfrac{1}{x+4}$

d $\dfrac{2}{2x-5}$ **e** $\dfrac{10}{5x-1}$ **f** $1 + \dfrac{1}{x}$

g $\dfrac{2x-5}{x^2-5x+2}$ **h** $\dfrac{15x^4}{1+3x^5}$

i $8x - 24x^2 + \dfrac{2x}{x^2-2}$

28 a $\dfrac{3}{x}$ **b** $\dfrac{1}{2x}$

c $\dfrac{1}{x} + \dfrac{1}{x+2}$ **d** $\dfrac{1}{x} - \dfrac{1}{x-1}$

29 a $1 + \log_e x$ **b** $\dfrac{e^x}{x} + e^x \log_e x$

c $\dfrac{\ln x - 1}{(\ln x)^2}$ **d** $\dfrac{1 - 2\ln x}{x^3}$

30 $y = 3x + 1$

32 a $\log_e |x| + C$ **b** $3\log_e |x| + C$

c $\frac{1}{5}\log_e |x| + C$ **d** $\log_e |x + 7| + C$

e $\frac{1}{2}\log_e |2x - 1| + C$ **f** $-\frac{1}{3}\log_e |2 - 3x| + C$

g $\log_e |2x + 9| + C$ **h** $-2\log_e |1 - 4x| + C$

33 a $\log_e \frac{3}{2}$ **b** $\frac{1}{4}\log_e 13$

c 1 **d** 1

34 a $\log_e(x^2 + 4) + C$

b $\log_e |x^3 - 5x + 7| + C$

c $\frac{1}{2}\log_e |x^2 - 3| + C$

d $\frac{1}{4}\log_e |x^4 - 4x| + C$

35 $\log_e 2\ \text{u}^2$

36 b $12 - 5\log_e 5\ \text{u}^2$

37 a e^x **b** $2^x \log_e 2$

c $3^x \log_e 3$ **d** $5^x \log_e 5$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

38 a $e^x + C$ **b** $\dfrac{2^x}{\log_e 2} + C$

c $\dfrac{3^x}{\log_e 3} + C$ **d** $\dfrac{5^x}{\log_e 5} + C$

39 a $x \log_e x - x + C$ **b** $xe^x - e^x + C$

40 a $8 \log_e 2$ **b** $\dfrac{1}{8 \log_e 2}$

c The curves $y = 2^x$ and $y = \log_2 x$ are reflections of each other in $y = x$. This reflection exchanges A and B, and exchanges their tangents. Because it also exchanges rise and run, the gradients are reciprocals of each other.

41 a $\dfrac{7}{\ln 2}$ and $\dfrac{7}{8 \ln 2}$

b When $y = 2^x$ is transformed successively by a vertical dilation with factor 8 and a shift right 3 units, the result is the same graph $y = 2^x$. The region in the second integral is transformed to the region in the first integral by this compound transformation.

Chapter 7

Exercise 7A

1 a The entries under 0.2 are 0.198 669, 0.993 347, 0.202 710, 1.013 550, 0.980 067.

b 1 and 1

3 a $\frac{\pi}{90}$ **b** $\sin 2^\circ = \sin \frac{\pi}{90} \doteqdot \frac{\pi}{90}$ **c** 0.0349

4 a The entries under 5° are 0.087 27, 0.087 16, 0.9987, 0.087 49, 1.003, 0.9962.

b $\sin x < x < \tan x$

c i 1 **ii** 1

d $x \le 0.0774$ (correct to four decimal places), that is, $x \le 4°26'$.

6 a 1 **b** 2 **c** $\frac{1}{2}$ **d** $\frac{3}{2}$ **e** $\frac{5}{3}$ **f** 8

7 87 metres

8 $26'$

13 a $AB^2 = 2r^2(1 - \cos x)$, arc $AB = rx$

b The arc is longer than the chord, so $\cos x$ is larger than the approximation.

15 a $\sin(A - B) = \sin A \cos B - \cos A \sin B$

b 6

Exercise 7B

1 a $\cos x$ **b** $-\sin x$ **c** $\sec^2 x$ **d** $2 \cos x$
e $2 \cos 2x$ **f** $-3 \sin x$ **g** $-3 \sin 3x$ **h** $4 \sec^2 4x$
i $4 \sec^2 x$ **j** $6 \cos 3x$ **k** $4 \sec^2 2x$ **l** $-8 \sin 2x$
m $-2 \cos 2x$ **n** $2 \sin 2x$ **o** $-2 \sec^2 2x$ **p** $\frac{1}{2} \sec^2 \frac{1}{2}x$
q $-\frac{1}{2} \sin \frac{1}{2}x$ **r** $\frac{1}{2} \cos \frac{x}{2}$ **s** $\sec^2 \frac{1}{5}x$ **t** $-2 \sin \frac{x}{3}$
u $3 \cos \frac{\pi}{4}$

2 a $2\pi \cos 2\pi x$ **b** $\frac{\pi}{2} \sec^2 \frac{\pi}{2}x$
c $3 \cos x - 5 \sin 5x$ **d** $4\pi \cos \pi x - 3\pi \sin \pi x$
e $2 \cos(2x - 1)$ **f** $3 \sec^2(1 + 3x)$
g $2 \sin(1 - x)$ **h** $-5 \sin(5x + 4)$
i $-21 \cos(2 - 3x)$ **j** $-10 \sec^2(10 - x)$
k $3 \cos\left(\frac{x+1}{2}\right)$ **l** $-6 \sin\left(\frac{2x+1}{5}\right)$

3 a $2 \cos 2x, -4 \sin 2x, -8 \cos 2x, 16 \sin 2x$

b $-10 \sin 10x, -100 \cos 10x, 1000 \sin 10x, 10\,000 \cos 10x$

c $\frac{1}{2} \cos \frac{1}{2}x, -\frac{1}{4} \sin \frac{1}{2}x, -\frac{1}{8} \cos \frac{1}{2}x, \frac{1}{16} \sin \frac{1}{2}x$

d $-\frac{1}{3} \sin \frac{1}{3}x, -\frac{1}{9} \cos \frac{1}{3}x, \frac{1}{27} \sin \frac{1}{3}x, \frac{1}{81} \cos \frac{1}{3}x$

4 $-2 \sin 2x$

a 0 **b** -1 **c** $-\sqrt{3}$ **d** -2

5 $\frac{1}{4} \cos\left(\frac{1}{4}x + \frac{\pi}{2}\right)$

a 0 **b** $-\frac{1}{4}$ **c** $\frac{1}{8}\sqrt{2}$ **d** $-\frac{1}{8}\sqrt{2}$

6 a $x \cos x + \sin x$ **b** $2(\tan 2x + 2x \sec^2 2x)$
c $2x(\cos 2x - x \sin 2x)$ **d** $3x^2(\sin 3x + x \cos 3x)$

7 a $\frac{x \cos x - \sin x}{x^2}$ **b** $\frac{-x \sin x - \cos x}{x^2}$
c $\frac{x(2 \cos x + x \sin x)}{\cos^2 x}$ **d** $\frac{1 + \sin x - x \cos x}{(1 + \sin x)^2}$

8 a $2x \cos(x^2)$ **b** $-2x \cos(1 - x^2)$
c $-3x^2 \sin(x^3 + 1)$ **d** $-\frac{1}{x^2} \cos\left(\frac{1}{x}\right)$
e $-2 \cos x \sin x$ **f** $3 \sin^2 x \cos x$
g $2 \tan x \sec^2 x$ **h** $\frac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$

9 d $y = \cos x$

11 a $e^{\tan x} \sec^2 x$ **b** $2e^{\sin 2x} \cos 2x$
c $2e^{2x} \cos(e^{2x})$ **d** $-\tan x$
e $\cot x$ **f** $-4 \tan 4x$

12 a $\cos^2 x - \sin^2 x$

b $14 \sin 7x \cos 7x$

c $-15 \cos^4 3x \sin 3x$

d $9 \sin 3x(1 - \cos 3x)^2$

e $2(\cos 2x \sin 4x + 2 \sin 2x \cos 4x)$

f $15 \tan^2(5x - 4) \sec^2(5x - 4)$

13 a $\frac{-\cos x}{(1 + \sin x)^2}$ **b** $\frac{1}{1 + \cos x}$
c $\frac{-1}{1 + \sin x}$ **d** $\frac{-1}{(\cos x + \sin x)^2}$

14 c i The graphs are reflections of each other in the x-axis.

ii The graphs are identical.

16 a $y' = e^x \sin x + e^x \cos x$, $y'' = 2e^x \cos x$

b $y' = -e^{-x} \cos x - e^{-x} \sin x$, $y'' = 2e^{-x} \sin x$

18 a $\log_b P - \log_b Q$

 ISBN 978-1-108-76630-2 Cambridge University Press
Updated July 2022

21 b $\sin\left(\frac{n\pi}{2} + x\right)$

22 b $\frac{1}{2}\Big((m + n)\cos(m + n)x + (m - n)\cos(m - n)x\Big)$

b $\frac{1}{2}\Big(\cos(m + n)x + \cos(m - n)x\Big)$
$= \cos mx \cos nx,$
$-\frac{1}{2}\Big((m + n)\sin(m + n)x + (m - n)\sin(m - n)x\Big)$

Exercise 7C

1 a 1 **b** -1 **c** $\frac{1}{2}$ **d** $-\frac{1}{2}$
e $\frac{1}{\sqrt{2}}$ **f** 1 **g** 2 **h** -2
i $\frac{\sqrt{3}}{4}$ **j** $\frac{1}{4}$ **k** 8 **l** $\sqrt{3}$

3 a $y = -x + \pi$ **b** $2x - y = \frac{\pi}{2} - 1$
c $x + 2y = \frac{\pi}{6} + \sqrt{3}$ **d** $y = -2x + \frac{\pi}{2}$
e $x + y = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ **f** $y = -\pi x + \pi^2$

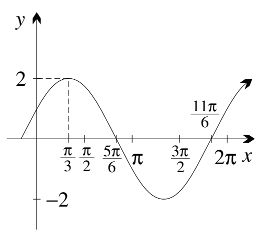
4 a $\frac{\pi}{2}, \frac{3\pi}{2}$ **b** $\frac{\pi}{3}, \frac{5\pi}{3}$ **c** $\frac{\pi}{6}, \frac{5\pi}{6}$ **d** $\frac{5\pi}{6}, \frac{7\pi}{6}$

6 b 1 and -1
c $x - y = \frac{\pi}{4} - \frac{1}{2}, x + y = \frac{\pi}{4} + \frac{1}{2}$

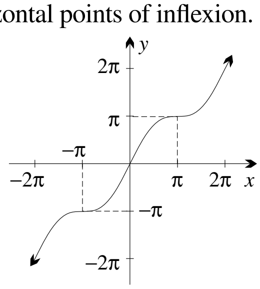
7 a $y' = \cos x e^{\sin x}$ **b** $\frac{\pi}{2}, \frac{3\pi}{2}$

8 a $y' = -\sin x e^{\cos x}$ **b** $0, \pi, 2\pi$

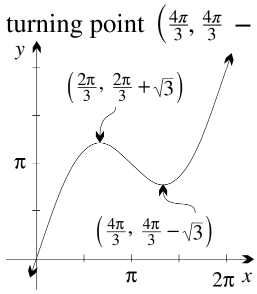
9 a $y' = -\sin x + \sqrt{3}\cos x, y'' = -\cos x - \sqrt{3}\sin x$
b maximum turning point $\left(\frac{\pi}{3}, 2\right)$, minimum turning point $\left(\frac{4\pi}{3}, -2\right)$
c $\left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$
d



10 a $y' = 1 + \cos x$
b $(-\pi, -\pi)$ and (π, π) are horizontal points of inflexion.
c $(0, 0)$
d

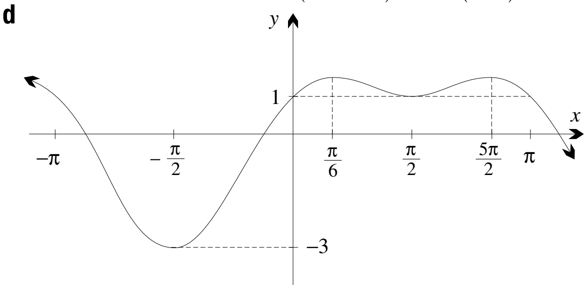


11 maximum turning point $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right)$, minimum turning point $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$, inflexion (π, π)

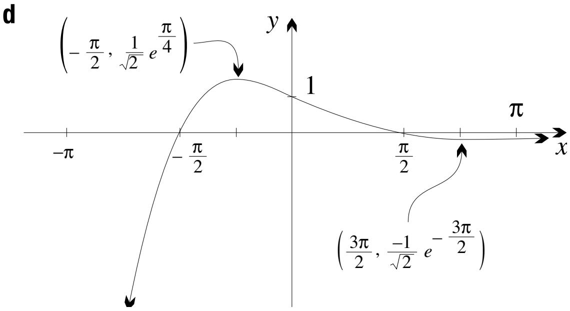


15 b minimum $\sqrt{3}$ when $\theta = \frac{\pi}{6}$, maximum 2 when $\theta = 0$

16 a $y' = 2\cos x - 2\sin 2x, y'' = -2\sin x - 4\cos 2x$
c maximum turning points $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{3}{2}\right)$, minimum turning points $\left(-\frac{\pi}{2}, -3\right)$ and $\left(\frac{\pi}{2}, 1\right)$
d



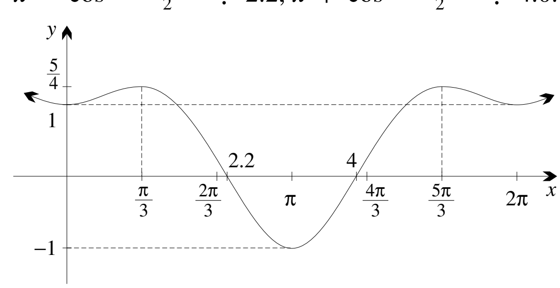
17 a $y' = -e^{-x}(\cos x + \sin x), y'' = 2e^{-x}\sin x$
b minimum turning point $\left(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}\right)$, maximum turning point $\left(-\frac{\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}\right)$
c $(-\pi, -e^{\pi}), (0, 1), (\pi, -e^{-\pi})$
d



18 a The angle of inclination is $\pi - \alpha$ and so $m = \tan(\pi - \alpha) = -\tan\alpha$.
b $P = \left(\frac{1}{\tan\alpha} + 2, 0\right), Q = (0, 2\tan\alpha + 1)$

19 b Over the given domain, the graph of $y = \tan x$ is above the graph of $y = x$.
c $f'(x) = \frac{x\cos x - \sin x}{x^2}$
d From the sketch, we see that $f(x) > \frac{2}{\pi}$ over the given domain.

20 a maximum turning points $\left(\frac{\pi}{3}, \frac{5}{4}\right), \left(\frac{5\pi}{3}, \frac{5}{4}\right)$, minimum turning points $(0, 1), (\pi, -1), (2\pi, 1)$, x-intercepts $\pi - \cos^{-1}\frac{\sqrt{5} - 1}{2} \doteqdot 2.2, \pi + \cos^{-1}\frac{\sqrt{5} - 1}{2} \doteqdot 4.0.$

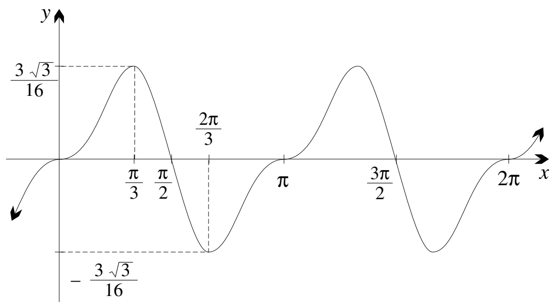


Mathematics Extension 1 Year 12 ISBN 978-1-108-76630-2

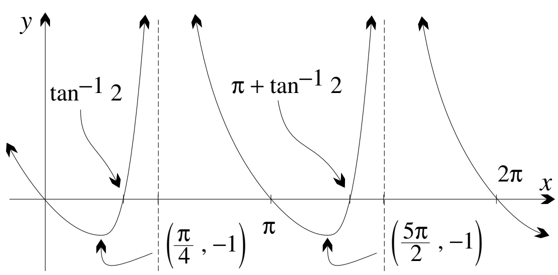
Cambridge University Press
Cambridge Maths Stage 6

Updated July 2022

b maximum turning points $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{16}\right)$, $\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{16}\right)$,
minimum turning points $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{16}\right)$, $\left(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{16}\right)$
horizontal points of inflexion $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$



c minimum turning points $\left(\frac{\pi}{4}, -1\right)$, $\left(\frac{5\pi}{4}, -1\right)$, vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, x-intercepts $0, \pi, 2\pi$, $\tan^{-1} 2 \doteqdot 1.1$, $\pi + \tan^{-1} 2 \doteqdot 4.25$

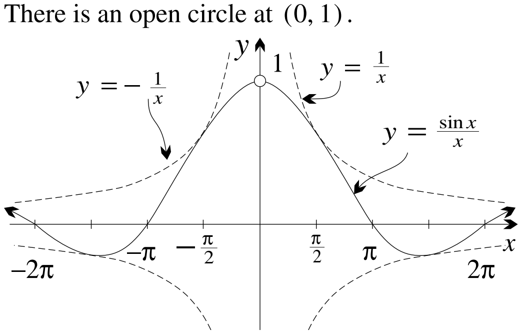


21 a Domain: $x \neq 0$, $f(x)$ is even because it is the ratio of two odd functions, the zeroes are $x = n\pi$ where n is an integer, $\lim\limits_{x \to \infty} f(x) = 0$.

b $f'(x) = \frac{x\cos x - \sin x}{x^2}$, which is zero when $\tan x = x$.

c The graph of $y = x$ crosses the graph of $y = \tan x$ just to the left of $x = \frac{3\pi}{2}$, of $x = \frac{5\pi}{2}$ and of $x = \frac{7\pi}{2}$. Using the calculator, the three turning points of $y = f(x)$ are approximately $(1.43\pi, -0.217)$, $(2.46\pi, 0.128)$ and $(3.47\pi, -0.091)$.

d There is an open circle at $(0, 1)$.



Exercise 7D

1 a $\tan x + C$ **b** $\sin x + C$
c $-\cos x + C$ **d** $\cos x + C$
e $2\sin x + C$ **f** $\frac{1}{2}\sin 2x + C$
g $\frac{1}{2}\sin x + C$ **h** $2\sin\frac{1}{2}x + C$
i $-\frac{1}{2}\cos 2x + C$ **j** $\frac{1}{5}\tan 5x + C$
k $\frac{1}{3}\sin 3x + C$ **l** $3\tan\frac{1}{3}x + C$
m $-2\cos\frac{x}{2} + C$ **n** $-5\sin\frac{1}{5}x + C$
o $2\cos 2x + C$ **p** $-\cos\frac{1}{4}x + C$
q $36\tan\frac{1}{3}x + C$ **r** $6\sin\frac{x}{3} + C$

2 a 1 **b** $\frac{1}{2}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\sqrt{3}$ **e** 1
f $\frac{3}{4}$ **g** 2 **h** 1 **i** 4

3 a $y = 1 - \cos x$
b $y = \sin x + \cos 2x - 1$
c $y = -\cos x + \sin x - 3$

6 a $\sin(x + 2) + C$ **b** $\frac{1}{2}\sin(2x + 1) + C$
c $-\cos(x + 2) + C$ **d** $-\frac{1}{2}\cos(2x + 1) + C$
e $\frac{1}{3}\sin(3x - 2) + C$ **f** $\frac{1}{5}\cos(7 - 5x) + C$
g $-\tan(4 - x) + C$ **h** $-3\tan\left(\frac{1 - x}{3}\right) + C$
i $3\cos\left(\frac{1 - x}{3}\right) + C$

7 a $2\sin 3x + 8\cos\frac{1}{2}x + C$
b $4\tan 2x - 40\sin\frac{1}{4}x - 36\cos\frac{1}{3}x + C$

8 a $f(x) = \sin \pi x$, $f\left(\frac{1}{3}\right) = \frac{1}{2}\sqrt{3}$
b $f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sin \pi x$, $f\left(\frac{1}{6}\right) = \frac{1}{\pi}$
c $f(x) = -2\cos 3x + x + \left(1 - \frac{\pi}{2}\right)$

9 a $-\cos(ax + b) + C$ **b** $\pi \sin \pi x + C$
c $\frac{1}{u^2}\tan(v + ux) + C$ **d** $\tan ax + C$

10 a $1 + \tan^2 x = \sec^2 x$, $\tan x - x + C$
b $1 - \sin^2 x = \cos^2 x$, $2\sqrt{3}$

11 a $\log_e f(x) + C$

12 a $\int \tan x = -\ln\cos x + C$
b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx = \Big[\log\sin x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \log 2$

13 a $5\sin^4 x\cos x$, $\frac{1}{5}\sin^5 x + C$
b $3\tan^2 x\sec^2 x$, $\frac{1}{3}\tan^3 x + C$

14 a $\cos x e^{\sin x}$, $e - 1$ **b** $\sec^2 x e^{\tan x}$, $e - 1$

15 a 1 **b** $\frac{5}{24}$

16 a $\frac{2}{9}$ **b** $\frac{1}{n + 1}$ **c** 0
d $\frac{1}{2^{n+1}(n + 1)}$ **e** $10\frac{1}{8}$ **f** $\frac{1}{n + 1}$

17 b $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, so $\frac{1}{2}\sin^2 x + C$
$= \frac{1}{4} - \frac{1}{4}\cos 2x + C = -\frac{1}{4}\cos 2x + \left(C + \frac{1}{4}\right)$
$= -\frac{1}{4}\cos 2x + D$, where $D = C + \frac{1}{4}$.

18 $\sin 2x + 2x\cos 2x$, $\frac{\pi - 2}{8}$

19 b $\frac{4}{3}$

20 b i $\frac{6}{5}$ **ii** $-\frac{6}{7}$

21 a $A = 5, B = 3$

Exercise 7E

1 a 1 square unit **b** $\frac{1}{2}$ square unit
2 a 1 square unit **b** $\sqrt{3}$ square units

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 a $1 - \frac{1}{\sqrt{2}}$ square units **b** $1 - \frac{\sqrt{3}}{2}$ square units

4 a $\frac{1}{2}\sqrt{3}$ u^2 **b** $\frac{1}{2}\sqrt{3}$ u^2

5 a $\frac{1}{2}$ u^2 **b** $\frac{1}{2}$ u^2

c $1 - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(2 - \sqrt{3}\right)$ u^2

d $\frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{1}{6}\left(2 - \sqrt{2}\right)$ u^2

e $\frac{2}{3}\sqrt{3}$ u^2

f 4 u^2

6 a $\left(\sqrt{2} - 1\right)$ u^2 **b** $\frac{1}{4}$ u^2

c $\left(\frac{\pi^2}{8} - 1\right)$ u^2 **d** $(\pi - 2)$ u^2

7 a $\left(2 - \sqrt{2}\right)$ u^2 **b** $1\frac{1}{2}$ u^2

8 a 2 u^2 **b** 1 u^2

9 a 2 u^2 **b** $\sqrt{2}$ u^2 **c** 2 u^2

d $\frac{1}{2}$ u^2 **e** 4 u^2 **f** 1 u^2

10 b $\frac{4}{\pi}$ u^2

11 3.8 m^2

12 4 u^2

14 b $\frac{1}{2}(3 + \sqrt{3})$ u^2

15 c $\frac{3}{4}\sqrt{3}$ u^2

16 b $2\sqrt{2}$ u^2

17 b They are all 4 u^2.

18 b The curve is below $y = 1$ just as much as it is above $y = 1$, so the area is equal to the area of a rectangle n units long and 1 unit high.

19 12

20 a Since $x^2 > 0$ for $0 < x < \frac{\pi}{2}$, $x^2 \sin x < x^2 \times x < x^2 \tan x$.

21 b $\cos x$ and $(1 + \sin x)^2$ are both positive in the given domain, so y' is negative there.

22 a 0, since the integrand is odd.

b 0, since the integrand is odd.

c 2, since the integrand is even.

d $6\sqrt{3}$, since the integrand is even.

e 6π. The first term is even, the other two are odd.

f $-\frac{2}{3} + \frac{\pi^2}{4}$. The first term is odd, the other two are even.

Chapter 7 review exercise

1 a $5 \cos x$ **b** $5 \cos 5x$

c $-25 \sin 5x$ **d** $5 \sec^2 (5x - 4)$

e $\sin 5x + 5x \cos 5x$ **f** $\frac{-5x \sin 5x \; - \; \cos 5x}{x^2}$

g $5 \sin^4 x \cos x$ **h** $5x^4 \sec^2 (x^5)$

i $-5 \sin 5x e^{\cos 5x}$ **j** $\frac{5 \cos 5x}{\sin 5x} = 5 \cot 5x$

2 $-\sqrt{3}$

3 a $y = 4x + \sqrt{3} - \frac{4\pi}{3}$ **b** $y = -\frac{\pi}{2}x + \frac{\pi^2}{4}$

4 a $\frac{\pi}{2}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$

5 a $4 \sin x + C$ **b** $-\frac{1}{4} \cos 4x + C$

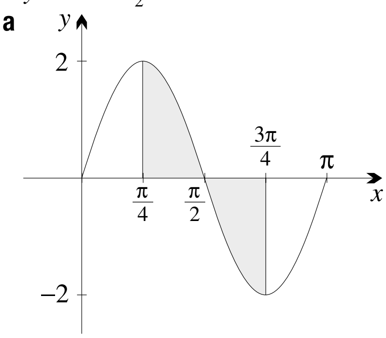
c $4 \tan \frac{1}{4}x + C$

6 a $\sqrt{3} - 1$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$

7 0.089

8 $y = 2 \sin \frac{1}{2}x - 1$

9 a



b 2 u^2

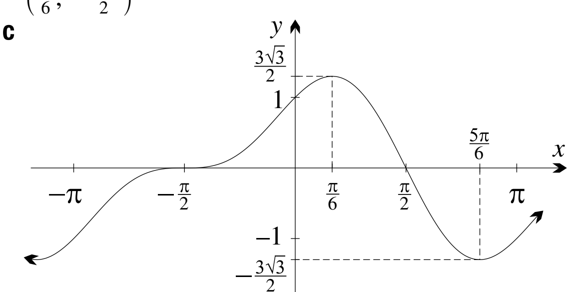
10 a $\frac{1}{2}$u^2 **b** $\frac{3\sqrt{3}}{4}$u^2

11 a $\tan x = \frac{\sin x}{\cos x}$ **b** $\frac{1}{2} \ln 2$ u^2

12 a $\left(\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), (0, 2)$

b Horizontal point of inflexion $\left(-\frac{\pi}{2}, 0\right)$, maximum turning point $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$, minimum turning point $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$

c

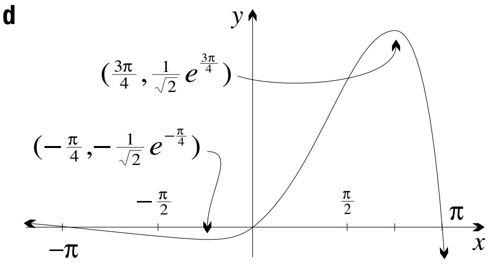


13 a $y' = e^x(\cos x + \sin x)$, $y'' = 2e^x \cos x$

b Minimum turning point $\left(-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}\right)$, maximum turning point $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$.

c $\left(-\frac{\pi}{2}, -e^{-\frac{\pi}{2}}\right), \left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$

d



14 b $\frac{9\sqrt{2}}{10}$ cm^2/ min **c** $\theta = \pi$

16 a $\frac{1}{2} \sin e^{2x} + C$ **b** $\frac{1}{2} \cos e^{-2x} + C$

c $\frac{1}{3} \log_e (3 \tan x + 1) + C$

d $-\frac{3}{5} \log_e (4 + 5 \cos x) + C$

e $\tan x - \sin x + C$

f $\frac{2}{3}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 b **i** $\frac{1}{2}\tan^2 x + \log_e(\cos x) + C$

ii $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \log_e(\cos x) + C$

18 b $\left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)\mathrm{u}^2$

19 b **i** $1\frac{1}{5}$ **ii** 0

c **i** $\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C$

ii $\frac{\sin(m+n)x}{2(m+n)} + x + C$

Chapter 8

Exercise 8A

1 a 60 km, 090°T **b** 7 km, 146°T

c 37 km, 073°T

2 b 24.8 km, 050°T

3 The opposite sides WX and ZY are parallel and equal, so $WXYZ$ is a parallelogram.

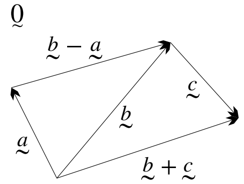
4 The opposite sides BA and CD are parallel and equal, and $\angle BAD = 90°$. So $ABCD$ is a parallelogram with an interior angle of 90°, so it is a rectangle.

5 a The 4 sides are equal, so $PQRS$ is a rhombus.

b The opposite sides of a rhombus are parallel, so $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have opposite directions.

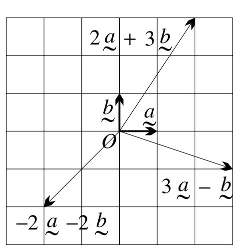
7 $\underset{\sim}{0}$

8



9 a $\overrightarrow{AD}$ **b** $\overrightarrow{BA}$ **c** $\overrightarrow{BD}$ **d** $\overrightarrow{AB}$

10



11 a $\underset{\sim}{f}$ **b** $\underset{\sim}{d}$ **c** $\underset{\sim}{h}$ **d** $\underset{\sim}{b}$

e $\underset{\sim}{c}$ **f** $\underset{\sim}{a}$ **g** $\underset{\sim}{e}$ **h** $\underset{\sim}{g}$

12 a $\underset{\sim}{u} - \underset{\sim}{v}$ **b** $\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v}$

13 a $\overrightarrow{AM}$ and $\overrightarrow{MB}$ have the same length and direction. Similarly $\overrightarrow{PN}$ and $\overrightarrow{NQ}$ have the same length and direction.

b $\underset{\sim}{a} = \underset{\sim}{p} + \underset{\sim}{u} - \underset{\sim}{v}, \underset{\sim}{b} = \underset{\sim}{p} - \underset{\sim}{u} + \underset{\sim}{v}$

14 a $\overrightarrow{BC} = \underset{\sim}{q} - \underset{\sim}{p}, \overrightarrow{BP} = \frac{1}{3}(\underset{\sim}{q} - \underset{\sim}{p})$

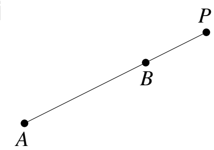
15 a $\overrightarrow{MB} = \frac{1}{2}(\underset{\sim}{u} - \underset{\sim}{v} - \underset{\sim}{w})$

b $\overrightarrow{MA} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v} - \underset{\sim}{w})$

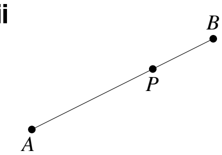
16 a $\overrightarrow{WX} = \underset{\sim}{x} - \underset{\sim}{w}, \overrightarrow{WP} = \frac{1}{2}(\underset{\sim}{x} - \underset{\sim}{w})$

b $\overrightarrow{RP} = \frac{1}{2}(\underset{\sim}{w} + \underset{\sim}{x})$ **c** $\overrightarrow{RQ} = \frac{1}{2}(\underset{\sim}{y} + \underset{\sim}{z})$

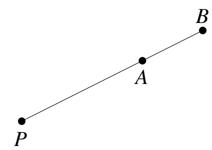
17 a **i**



ii



iii



b **i** $k = \frac{3}{5}$

ii $k = 3$

iii $k = -2$

18 The triangles are similar by the SAS similarity test — the angles between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and between $\lambda\underset{\sim}{a}$ and $\lambda\underset{\sim}{b}$ are equal, and the matching sides are in ratio $1:\lambda$. It now follows that the head of the vector $\lambda\underset{\sim}{b}$ is the head of the vector $\lambda(\underset{\sim}{a} + \underset{\sim}{b})$.

19 a Two zero vectors each have zero length and no direction, and so are equal.

b Rome for administration (in the distant past), Greenwich UK for longitude, Jerusalem and Mecca for religious ceremonies, the North and South Poles for maps. The obelisk in Macquarie Place, Sydney, remains the origin for road distances in NSW. It is inscribed on the front,

'THIS OBELISK WAS ERECTED IN MACQUARIE PLACE A.D. 1818, TO RECORD THAT ALL THE PUBLIC ROADS LEADING TO THE INTERIOR OF THE COLONY ARE MEASURED FROM IT. L. MACQUARIE ESQ GOVERNOR'

20 c The three medians of a triangle are concurrent, and their point of intersection trisects each median. (A *median* of a triangle is the line joining a vertex to the midpoint of the opposite side.)

21 b $\overrightarrow{PQ} = \frac{1}{4}(3\underset{\sim}{c} + \underset{\sim}{d} - 3\underset{\sim}{a} - \underset{\sim}{b})$

Exercise 8B

1 a 10 **b** $16\underset{\sim}{i} + 12\underset{\sim}{j}$ **c** 20

d $-40\underset{\sim}{i} - 30\underset{\sim}{j}$ **e** 50

2 a $3\underset{\sim}{i} - \underset{\sim}{j}$ **b** $\sqrt{10}$ **c** $\underset{\sim}{i} + 7\underset{\sim}{j}$

d $5\sqrt{2}$ **e** $-8\underset{\sim}{i} - \underset{\sim}{j}$ **f** $\sqrt{65}$

3 a $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ **b** $5\sqrt{2}$ **c** $\begin{bmatrix} 12 \\ 20 \end{bmatrix}$ **d** $4\sqrt{34}$

4 a $\underset{\sim}{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \underset{\sim}{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\ \underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

b $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}, \underset{\sim}{b} = 4\underset{\sim}{i} + \underset{\sim}{j}, \underset{\sim}{a} - \underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j}$

5 a $\frac{1}{\sqrt{5}}\underset{\sim}{i} + \frac{2}{\sqrt{5}}\underset{\sim}{j}$ **b** $-\frac{4}{5}\underset{\sim}{i} + \frac{3}{5}\underset{\sim}{j}$ **c** $-\frac{3}{\sqrt{34}}\underset{\sim}{i} + \frac{5}{\sqrt{34}}\underset{\sim}{j}$

7 a Yes. The intervals PQ and QR both have gradient -1.

b No. The intervals AB and BC have different gradients.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

8 a $4\underset{\sim}{i} - 7\underset{\sim}{j}$ **b** $6\underset{\sim}{i} + 3\underset{\sim}{j}$ **c** $5\underset{\sim}{i} - 2\underset{\sim}{j}$

9 a $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} 11 \\ -7 \end{bmatrix}$ **c** $\begin{bmatrix} -7 \\ 6 \end{bmatrix}$ **d** $\begin{bmatrix} 7 \\ -6 \end{bmatrix}$

10 a $6\underset{\sim}{i} + 4\underset{\sim}{j}$ **b** $2\sqrt{13}$ **c** $\frac{3}{\sqrt{13}}\underset{\sim}{i} + \frac{2}{\sqrt{13}}\underset{\sim}{j}$

11 a $2\sqrt{2}, \dfrac{\pi}{4}$ **b** $2, -\dfrac{\pi}{3}$

c $6, \dfrac{5\pi}{6}$ **d** $2\sqrt{3}, -\dfrac{3\pi}{4}$

12 a $2\sqrt{2}\underset{\sim}{i} - 2\sqrt{2}\underset{\sim}{j}$ **b** $-\sqrt{6}\underset{\sim}{i} + 3\sqrt{2}\underset{\sim}{j}$

c $-\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j}$

d $\left(\sqrt{3} - 1\right)\underset{\sim}{i} + \left(\sqrt{3} + 1\right)\underset{\sim}{j}$

13 $\lambda_1 = 6, \lambda_2 = -4$

14 a $\overrightarrow{AB} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $\overrightarrow{CB} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$

b $|\overrightarrow{AB}| = |\overrightarrow{CB}| = 2$

c Equilateral, because each side has length 2.

15 The opposite sides AD and BC are parallel and equal.

16 c It is rhombus, because it is a parallelogram with adjacent sides equal.

17 b It is a parallelogram, because its diagonals bisect each other.

18 $a = -8$ and $b = 11$

19 $\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}, \underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{a}, \underset{\sim}{c} + \underset{\sim}{a} - \underset{\sim}{b}$

Exercise 8C

1 a 10 **b** 22 **c** 0 **d** $2x^2 - 2x - 3$

2 a 15 **b** $6\sqrt{2}$

3 a $\cos\theta = -\frac{1}{2}, \theta = 120°$

b $\cos\theta = 0.8, \theta \doteqdot 37°$

4 a 0 **b** 0 **c** 8 **d** -15

5 a 20 **b** 34 **c** -14

6 a no **b** yes **c** yes

7 a $\overrightarrow{AB} = 3\underset{\sim}{i} + 9\underset{\sim}{j}, \overrightarrow{AC} = -4\underset{\sim}{i} + 8\underset{\sim}{j}$

b 60

8 a $\overrightarrow{PQ} = 2\sqrt{3}\underset{\sim}{i} + 6\underset{\sim}{j}, \overrightarrow{PR} = 4\sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j}$

b 48

9 a $\frac{3}{5}$ **b** $\frac{1}{\sqrt{5}}$ **c** $\frac{10}{\sqrt{221}}$

10 $\lambda = -\frac{2}{3}$ or 2

11 a i 6 **ii** -60 **iii** 0 **iv** 42 **v** 10 **vi** 32

b i -6 **ii** 60 **iii** 0 **iv** 30 **v** -2 **vi** 32

c i 0 **ii** 0 **iii** 0 **iv** 36 **v** 4 **vi** 32

12 c It is a rectangle, because it is a parallelogram with a right-angle.

13 c It is a rhombus, because its diagonals bisect each other at 90°.

14 a $\overrightarrow{AP} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \overrightarrow{AQ} = \begin{bmatrix} 13 \\ -3 \end{bmatrix}$

b 63°

15 58°8′

17 b $\dfrac{87}{\sqrt{7738}}$ **c** 43.5 u^2

18 b $\overrightarrow{PB} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} -8 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

20 a $\overrightarrow{PR} = \underset{\sim}{a} - 4\underset{\sim}{b}$

21 a i $(\underset{\sim}{c} - \underset{\sim}{a}) \cdot (\underset{\sim}{d} - \underset{\sim}{b}) = 0$

ii $|\underset{\sim}{c} - \underset{\sim}{a}| = |\underset{\sim}{d} - \underset{\sim}{b}|$ (other answers are possible)

b $m = 4$ and $n = 1$ or $m = 6$ and $n = 15$

22 a A dot product is negative when the angle is obtuse.

c i They are both 3. When calculating the RHS, be careful to take the exterior angles as the angles between the vectors

ii They are both 4. **iii** They are both 8.

Exercise 8D

1 a $\overrightarrow{AB} = \underset{\sim}{a} + \underset{\sim}{b}$ **b** $\overrightarrow{PQ} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

c $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$ so $PQ \parallel AB$ and $PQ = \frac{1}{2}AB$.

2 a $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{d} + \underset{\sim}{c}$ **b** $\overrightarrow{PQ} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

c $\overrightarrow{SR} = \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{c})$

3 a $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}, \overrightarrow{BD} = \underset{\sim}{b} + \underset{\sim}{c}$

b $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

c $\underset{\sim}{a} \cdot \underset{\sim}{a} = x^2$

d $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = x^2 + y^2$,

$(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = y^2 + x^2$

e The diagonals of a rectangle have equal length.

4 a $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}, \overrightarrow{BD} = \underset{\sim}{b} - \underset{\sim}{a}$

b $a \cdot b = 0,$

c $a \cdot a = \ell^2,$

d 0

e The diagonals of a square meet at right angles.

5 a The sides of a rhombus are equal.

c $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$ and $\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b}$

e They are perpendicular.

6 a The opposite sides of a parallelogram are parallel and equal.

b $\overrightarrow{OB} = \underset{\sim}{c} + \underset{\sim}{a}$ **c** $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

d The diagonals OB and AC are equal.

f It is a rectangle.

7 a $\overrightarrow{OB} = -\underset{\sim}{a}$

b $\overrightarrow{AP} = \underset{\sim}{p} - \underset{\sim}{a}$ and $\overrightarrow{BP} = \underset{\sim}{p} + \underset{\sim}{a}$

d An angle in a semi-circle is a right-angle.

8 a P lies on the altitude from A to BC.

b P lies on the altitude from B to CA.

d From **iii**, $\overrightarrow{CP} \cdot \overrightarrow{BA} = 0$, so $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{BA}$, so P lies on the altitude from C to BA.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 8E

1 a $\underset{\sim}{i}$ **b** $2\underset{\sim}{j}$ **c** $-3\underset{\sim}{i}$

2 a 2 **b** 4 **c** $6\sqrt{2}$

3 a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **b** $3\underset{\sim}{j}$ **c** $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

4 a $3\sqrt{3}$ **b** $6\sqrt{3}$

6 a $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ **b** $\frac{3}{5}\underset{\sim}{i} - \frac{1}{5}\underset{\sim}{j}$ **c** $\begin{bmatrix} -\frac{21}{5} \\ \frac{28}{5} \end{bmatrix}$

7 a $\frac{6}{5}\underset{\sim}{i} + \frac{2}{5}\underset{\sim}{j}$ **b** $\frac{27}{10}\underset{\sim}{i} + \frac{9}{10}\underset{\sim}{j}$

8 a $\frac{6}{\sqrt{13}}$ **b** $\frac{14}{\sqrt{5}}$

9 $-\frac{36}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}$

10 $7\sqrt{5}$

11 $\lambda = \frac{40}{3}$ or -10

13 a $-\frac{1}{3}$

b $-3\underset{\sim}{i} + \underset{\sim}{j}$ is one such vector.

d $23\underset{\sim}{i} - \underset{\sim}{j}$ **e** $7\sqrt{10}$ **f** $2\sqrt{10}$

Exercise 8F

1 $10\sqrt{3}$ m/s, 10 m/s

2 $25\underset{\sim}{i} - 9\underset{\sim}{j}$

3 $\sqrt{10}$ N

4 34 N at 28° to the 30 N force

5 a $1200\cos 15° \doteqdot 1159$ N

b $1200\sin 15° \doteqdot 311$ N

6 28.3 N

7 a 12 N **b** $12\sqrt{3}$ N

8 a $\overrightarrow{OP} = (20\cos\ 25°)\,\underset{\sim}{i} + (20\sin\ 25°)\underset{\sim}{j}$

$\overrightarrow{OQ} = (16\cos\ 50°)\,\underset{\sim}{i} + (16\sin\ 50°)\underset{\sim}{j}$

b 35 N, 36° above the horizontal

9 $F = 49$

10 37°

11 5.4 N, 5.2° north of east

12 5 m/s^2 at an angle of $\tan^{-1}\frac{7}{24}$ above the horizontal

13 a $19\underset{\sim}{i} + 88\underset{\sim}{j}$ **b** $-84\underset{\sim}{i} + 288\underset{\sim}{j}$

14 a 49 N **b** 60 N **c** 6°

15 a $\left(3 - 2\sqrt{2}\right)\underset{\sim}{i} + \left(5 - 2\sqrt{2}\right)\underset{\sim}{j}$

b 2.2 m/s, 4.5°T

16 c 20 kg

17 a $T = 44.1$ **b** $m = 9$

18 $2\sqrt{3}$ and 4

19 a $3a = T - 3g\sin\theta$ **b** $2a = 2g\sin\ 2\theta\ - T$

Chapter 8 review exercise

1 211.5 km, 088°T

2 a $\overrightarrow{AD}$ **b** $\overrightarrow{AC}$ **c** $\overrightarrow{DC}$ **d** 0 **e** $\overrightarrow{CD}$ **f** $\overrightarrow{DB}$

3 a $\underset{\sim}{b} - \underset{\sim}{a}$ **b** $\frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$ **c** $\frac{1}{6}(3\underset{\sim}{b} - \underset{\sim}{a})$

4 a $\overrightarrow{PA} = \frac{3}{4}\underset{\sim}{a}$ **b** $\overrightarrow{AQ} = \frac{3}{7}(\underset{\sim}{b} - \underset{\sim}{a})$

5 a $6\underset{\sim}{i} + 8\underset{\sim}{j}$ **b** 10 **c** $\frac{3}{5}\,\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$

6 a $\sqrt{5}\ a$ **b** $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ **c** $\begin{bmatrix} 4a \\ 2a \end{bmatrix}$ **d** $5a^2$

7 a yes **b** no, unless $x = 0$

8 137°44′

9 c a rectangle

10 a $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ **b** $\frac{33}{10}\,\underset{\sim}{i} + \frac{11}{10}\underset{\sim}{j}$

11 $\frac{84}{\sqrt{153}}$

12 36°

13 a $\overrightarrow{MA} = \frac{1}{2}\underset{\sim}{a}$ **b** $\overrightarrow{AN} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

d $\overrightarrow{MN} = \overrightarrow{PB} = \frac{1}{2}\underset{\sim}{b}$, so a pair of opposite sides are parallel and equal.

14 a $\underset{\sim}{p} = \underset{\sim}{b} - \underset{\sim}{a}, m = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

15 a $|a|^2 = |c|^2$

b $|\overrightarrow{AB}|^2 = |\overrightarrow{CB}|^2$

16 a $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}, \overrightarrow{BD} = \underset{\sim}{b} - \underset{\sim}{a}$

b $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = x^2 + y^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b}$ and $(b - a) \cdot (b - a) = x^2 + y^2 - 2a \cdot b$

c If the parallelogram is a rectangle, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, so the diagonals are equal. Conversely, if the diagonals are equal, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, so the parallelogram is a rectangle.

17 a $\overrightarrow{OP} = (37\cos 50°)\,\underset{\sim}{i} + (37\sin 50°)\underset{\sim}{j}$,

$\overrightarrow{OQ} = (23\cos 25°)\,\underset{\sim}{i} - (23\sin 25°)\underset{\sim}{j}$

b 48.4 N, 22.7° above the horizontal

18 7.08 km/h, 133°T

19 14 N

20 a 4.9 m/s^2 **b** $T = 14.7$

Chapter 9

Exercise 9A

1a $x = -4, -3, 0, 5$

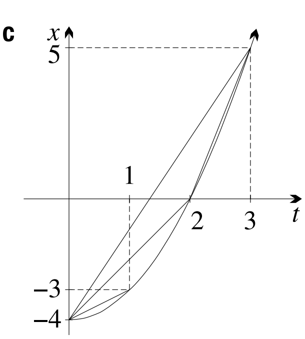
b **i** 1 m/s

ii 2 m/s

iii 3 m/s

iv 5 m/s

c



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

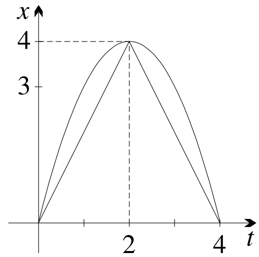
Cambridge University Press Updated July 2022

2 a $x = 0, 3, 4, 3, 0$

c The total distance travelled is 8 metres. The average speed is 2 m/s.

d **i** 2 m/s **ii** -2 m/s **iii** 0 m/s

e



3 a $x = 0, 120, 72, 0$ **b** 240 metres

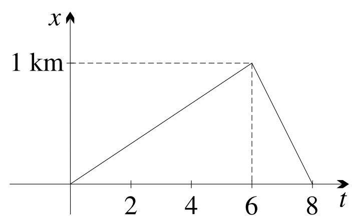
c 20 m/s

d **i** 30 m/s **ii** -15 m/s **iii** 0 m/s

4 a **i** 6 minutes

ii 2 minutes

b

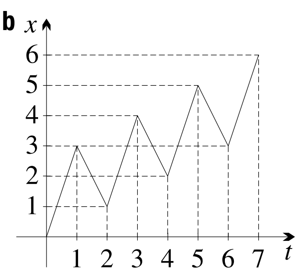


c 15 km/hr

d 20 km/hr

5 a $x = 0, 3, 1, 4, 2, 5, 3, 6$

b



c 7 hours

d 18 metres, $2\frac{4}{7}$ m/hr

e $\frac{6}{7}$ m/hr

f Those between 1 and 2 metres high or between 4 and 5 metres high

6 a $t = 0, 1, 4, 9, 16$

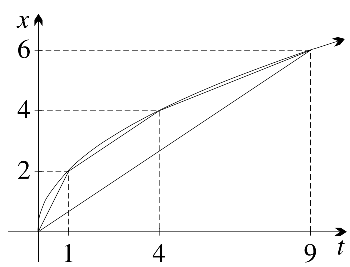
b **i** 2 cm/s

ii $\frac{2}{3}$ cm/s

iii $\frac{2}{5}$ cm/s

iv $\frac{2}{3}$ cm/s

c They are parallel.



7 a **i** -1 m/s

ii 4 m/s

iii -2 m/s

b 40 metres, $1\frac{1}{3}$ m/s

c 0 metres, 0 m/s

d $2\frac{2}{19}$ m/s

8 a **i** once

ii three times

iii twice

b **i** when $t = 4$ and when $t = 14$

ii when $0 \le t < 4$ and when $4 < t < 14$

c It rises 2 metres, at $t = 8$.

d It sinks 1 metre, at $t = 17$.

e As $t \to \infty$, $x \to 0$, meaning that eventually it ends up at the surface.

f **i** -1 m/s **ii** $\frac{1}{2}$ m/s **iii** $-\frac{1}{3}$ m/s

g **i** 4 metres **ii** 6 metres

iii 9 metres **iv** 10 metres

h **i** 1 m/s **ii** $\frac{3}{4}$ m/s **iii** $\frac{9}{17}$ m/s

9 b $x = 3$ and $x = -3$ **c** $t = 4, t = 20$

d $t = 8, t = 16$ **e** $8 < t < 16$

f 12 cm, $\frac{3}{4}$ cm/s

10 a amplitude: 4 metres, period: 12 seconds

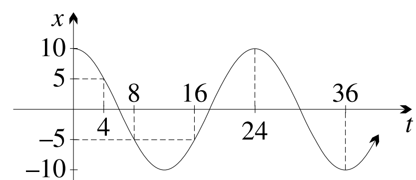
b 10 times

c $t = 3, 15, 27, 39, 51$

d It travels 16 metres with average speed $1\frac{1}{3}$ m/s.

e $x = 0, x = 2$ and $x = 4$, 2 m/s and 1 m/s

11 amplitude: 10 metres, period: 24 seconds



c It is at $x = 0$ when $t = 6$, 18 and 30.

d When $t = 0$, $x = 10$. The maximum distance is 20 metres, when $t = 12$ and 36.

e 60 metres, $1\frac{2}{3}$ m/s

f 10, 5, -5, -10, -5, 5, 10

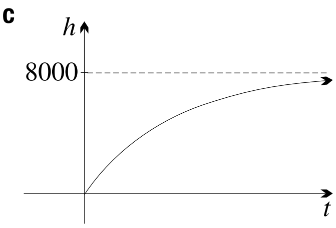
g $-1\dfrac{1}{4}$ m/s, $-2\dfrac{1}{2}$ m/s, $-1\dfrac{1}{4}$ m/s

h $x = -5$ when $t = 8$ or $t = 16$, $x < -5$ when $8 < t < 16$.

12 a When $t = 0$, $h = 0$. As $t \to \infty$, $h \to 8000$.

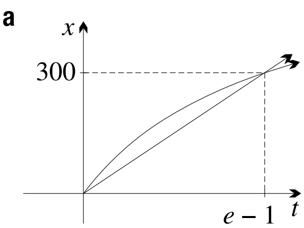
b 0, 3610, 5590, 6678

c



d 361 m/ min , 198 m/ min , 109 m/ min

13 a



c The maximum distance is $300\log(e - 1) - \dfrac{300(e - 2)}{e - 1} \doteqdot 37$ metres when $t = e - 2 \doteqdot 43''$.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 9B

1 a $v = -2t$

b $a = -2$

c $x = 11$ metres, $v = -6$ m/s, $a = -2$ m/s^2

d distance from origin: 11 metres, speed: 6 m/s

2 a $v = 2t - 10$

b displacement: -21 cm, distance from origin: 21 cm, velocity: $v = -4$ cm/s, speed: $|v| = 4$ cm/s

c When $v = 0$, $t = 5$ and $x = -25$.

3 a $v = 3t^2 - 12t$, $a = 6t - 12$

b When $t = 0$, $x = 0$ cm, $|v| = 0$ cm/s and $a = -12$ cm/s^2.

c left $(x = -27 \text{ cm})$

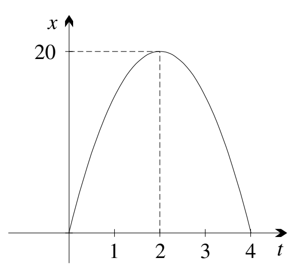
d left $(v = -9 \text{ cm/s})$

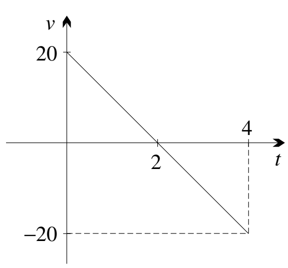
e right $(a = 6 \text{ cm/s}^2)$

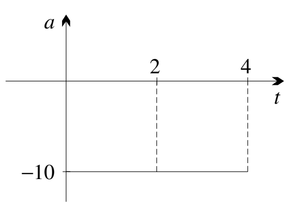
f When $t = 4$, $v = 0$ cm/s and $x = -32$ cm.

g When $t = 6$, $x = 0$, $v = 36$ cm/s and $|v| = 36$ cm/s.

4 a $x = 5t(4 - t)$
$v = 20 - 10t$
$a = -10$







b 20 m/s

c It returns at $t = 4$; both speeds are 20 m/s.

d 20 metres after 2 seconds

e -10 m/s^2. Although the ball is stationary, its velocity is changing, meaning that its acceleration is non-zero.

5 $\dot{x} = -4e^{-4t}$, $\ddot{x} = 16e^{-4t}$

a e^{-4t} is positive, for all t, so $\dot{x}$ is always negative and $\ddot{x}$ is always positive.

b i $x = 1$ **ii** $x = 0$

c i $\dot{x} = -4$, $\ddot{x} = 16$ **ii** $\dot{x} = 0$, $\ddot{x} = 0$

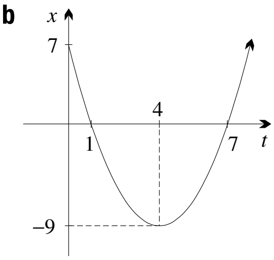
6 $v = 2\pi \cos \pi t$, $a = -2\pi^2 \sin \pi t$

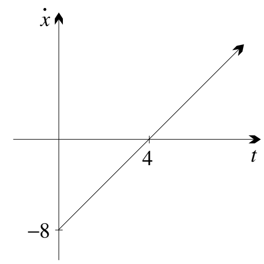
a When $t = 1$, $x = 0$, $v = -2\pi$ and $a = 0$.

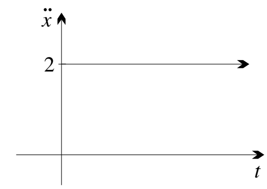
b i right $(v = \pi)$ **ii** left $\left(a = -\pi^2\sqrt{3}\right)$

7 a $x = (t - 7)(t - 1)$
$\dot{x} = 2(t - 4)$
$\ddot{x} = 2$

b







c i $t = 1$ and $t = 7$ **ii** $t = 4$

d i 7 metres when $t = 0$

ii 9 metres when $t = 4$

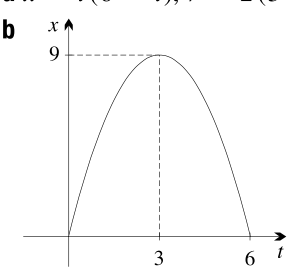
iii 27 metres when $t = 10$

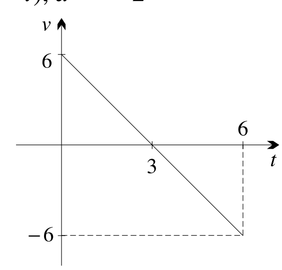
e -1 m/s, $t = 3\frac{1}{2}$, $x = -8\frac{3}{4}$

f 25 metres, $3\frac{4}{7}$ m/s

8 a $x = t(6 - t)$, $v = 2(3 - t)$, $a = -2$

b





c i When $t = 2$, it is moving upwards and accelerating downwards.

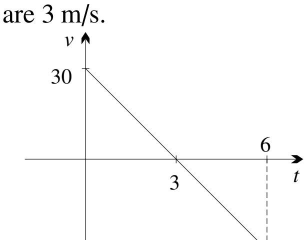
ii When $t = 4$, it is moving downwards and accelerating downwards.

d $v = 0$ when $t = 3$. It is stationary for zero time, 9 metres up the plane, and is accelerating downwards at 2 m/s^2.

e 4 m/s. When $v = 4$, $t = 1$ and $x = 5$.

f All three average speeds are 3 m/s.

9 a 45 metres, 3 seconds, 15 m/s



b 30 m/s, 20, 10, 0, -10, -20, -30

c 0 seconds

d The acceleration was always negative.

e The velocity was decreasing at a constant rate of 10 m/s every second.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

10 a 8 metres, when $t = 3$

b **i** when $t = 3$ and $t = 9$ (because the gradient is zero)

ii when $0 \le t < 3$ and when $t > 9$ (because the gradient is positive)

iii when $3 < t < 9$ (because the gradient is negative)

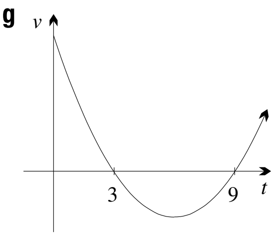
c $x = 0$ again when $t = 9$. Then $v = 0$ (because the gradient is zero) and it is accelerating to the right (because the concavity is upwards).

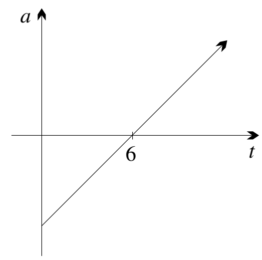
d at $t = 6$ (at the point of inflection the second derivative is zero), $x = 4$, moving to the left

e $0 \le t < 6$

f **i** $t \doteqdot 4, 12$ **ii** $t \doteqdot 10$

g





11 a $x = 4\cos\frac{\pi}{4}t$, $v = -\pi\sin\frac{\pi}{4}t$, $a = -\frac{1}{4}\pi^2\cos\frac{\pi}{4}t$

b maximum displacement: $x = 4$ when $t = 0$ or $t = 8$, maximum velocity: π m/s when $t = 6$, maximum acceleration: $\frac{1}{4}\pi^2$ m/s^2 when $t = 4$

c 40 metres, 2 m/s

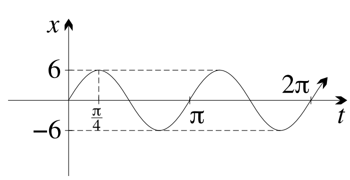
d $1\frac{1}{3} < t < 6\frac{2}{3}$

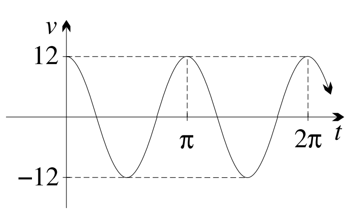
e **i** $t = 0$, $t = 4$ and $t = 8$ **ii** $4 < t < 8$

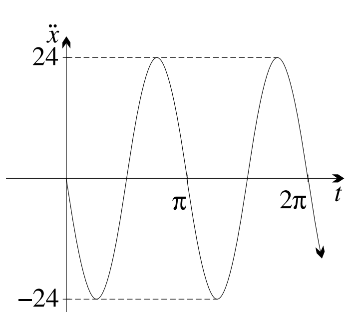
12 a $x = 6\sin 2t$

$v = 12\cos 2t$

$\ddot{x} = -24\sin 2t$







b $\ddot{x} = -4x$

c **i** $x = 0$ when $t = 0, \frac{\pi}{2}$ or π.

ii $v = 0$ when $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

iii same as **i**

d **i** $x < 0$ when $\frac{\pi}{2} < t < \pi$.

ii $v < 0$ when $\frac{\pi}{4} < t < \frac{3\pi}{4}$.

iii $\ddot{x} < 0$ when $0 < t < \frac{\pi}{2}$.

e **i** $t = \frac{\pi}{12}$

ii $t = \frac{\pi}{6}$

13 a **i** $0 \le t < 8$

ii $0 \le t < 4$ and $t > 12$

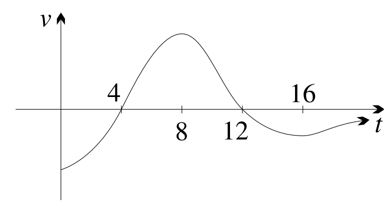
iii roughly $8 < t < 16$

b roughly $t = 8$

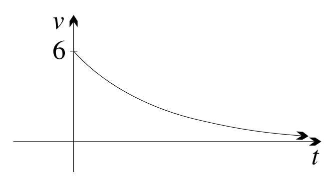
c **i** $t \doteqdot 5, 11, 13$ **ii** $t \doteqdot 13, 20$

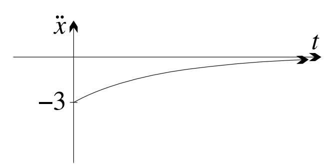
d twice **e** 17 units

f



14 a $\dot{x} = 6e^{-0.5t}$, $\ddot{x} = -3e^{-0.5t}$





b **i** downwards (downwards is positive here.)

ii upwards

c The velocity and acceleration tend to zero and the position tends to 12 metres below ground level.

d $x = 6$ when $e^{-0.5t} = \frac{1}{2}$, that is, $t = 2\log_e 2$ minutes. The speed then is 3 m/min (half the initial speed of 6 m/min) and the acceleration is $-1\frac{1}{2}$ m/min^2 (half the initial acceleration of -3 m/min^2).

e 19 minutes. When $t = 18$, $x \doteqdot 11.9985$ metres. When $t = 19$, $x \doteqdot 11.9991$ metres.

15 a $0 \le x \le 2r$

b **i** $\dfrac{dx}{d\theta} = \dfrac{2r\sin\theta}{\sqrt{5 - 4\cos\theta}}$. M is travelling upwards when $0 < \theta < \pi$.

ii M is travelling downwards when $\pi < \theta < 2\pi$.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

c The speed is maximum when $\theta = \frac{\pi}{3}$ (when $\dfrac{dx}{d\theta} = r$) and when $\theta = \frac{5\pi}{3}$ (when $\dfrac{dx}{d\theta} = -r$).

d When $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$, $\angle APC$ is a right angle, so AP is a tangent to the circle. At these places, P is moving directly towards A or directly away from A, and so the distance AP is changing at the maximum rate. Again because AP is a tangent, $\dfrac{dx}{d\theta}$ at these points must equal the rate of change of arc length with respect to θ, which is r or $-r$ when $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ respectively.

16 $\sin\alpha = \dfrac{2}{g} \doteqdot 0.204\,08$, $\alpha \doteqdot 11°47'$

Exercise 9C

1 a $x = t^3 - 3t^2 + 4$

b When $t = 2$, $x = 0$ metres and $v = 0$ m/s.

c $a = 6t - 6$

d When $t = 1$, $a = 0$ m/s^2 and $x = 2$ metres.

2 a $v = 10t$, $x = 5t^2$

b 4 seconds, 40 m/s

c After 2 seconds, it has fallen 20 metres and its speed is 20 m/s.

d It is halfway down after $2\sqrt{2}$ seconds, and its speed then is $20\sqrt{2}$ m/s.

3 a $a = -10, v = -10t - 25, x = -5t^2 - 25t + 120$

b 3 seconds, 55 m/s **c** 40 m/s

4 a i $\dot{x} = 3t^2, x = t^3$

ii $\dot{x} = -\frac{1}{3}e^{-3t} + \frac{1}{3}, x = \frac{1}{9}e^{-3t} + \frac{1}{3}t - \frac{1}{9}$

iii $\dot{x} = \frac{1}{\pi}\sin \pi t, x = -\frac{1}{\pi^2}\cos \pi t + \frac{1}{\pi^2}$

iv $\dot{x} = -12(t+1)^{-1} + 12,$
$x = -12\log_e(t+1) + 12t$

b i $a = 0, x = -4t - 2$

ii $a = \frac{1}{2}e^{\frac{1}{2}t}, x = 2e^{\frac{1}{2}t} - 4$

iii $a = 16\cos 2t, x = -4\cos 2t + 2$

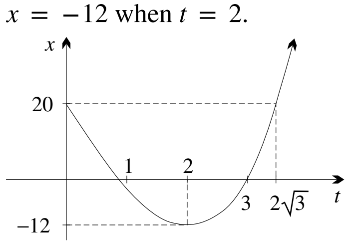
iv $a = \frac{1}{2}t^{-\frac{1}{2}}, x = \frac{2}{3}t^{\frac{3}{2}} - 2$

5 a $\dot{x} = 6t^2 - 24, x = 2t^3 - 24t + 20$

b $t = 2\sqrt{3}$, speed: 48 m/s

c $x = -12$ when $t = 2$.

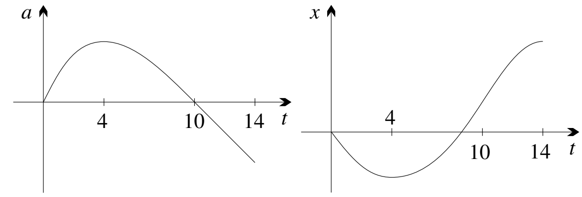
d



6 a $4 < t < 14$ **b** $0 < t < 10$ **c** $t = 14$

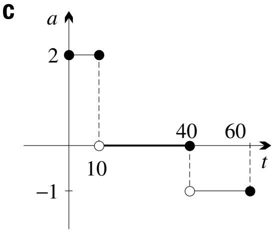
d $t = 4$ **e** $t \doteqdot 8$

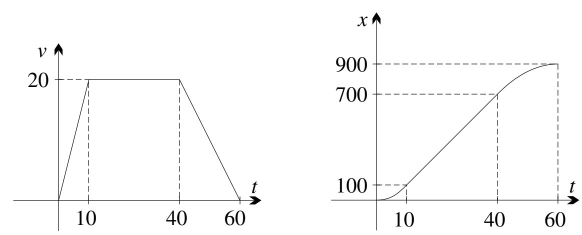
f



7 a 20 m/s

c





8 a $a = -4, x = 16t - 2t^2 + C$

b $x = C$ after 8 seconds, when the speed is 16 cm/s.

c $\dot{x} = 0$ when $t = 4$. Maximum distance right is 32 cm when $t = 4$, maximum distance left is 40 cm when $t = 10$. The acceleration is -4 cm/s^2 at all times.

d 104 cm, 10.4 cm/s

9 a $x = t^2(t-6)^2$, after 6 seconds, 0 cm/s

b 162 cm, 27 cm/s

c $\ddot{x} = 12(t^2 - 6t + 6)$, $24\sqrt{3}$ cm/s after $3 - \sqrt{3}$ and $3 + \sqrt{3}$ seconds.

d The graphs of x, v and $\ddot{x}$ are all unchanged by reflection in $t = 3$, but the mouse would be running backwards!

10 a $\ddot{x} = 6t, v = 3t^2 - 9$

b $x = t^3 - 9t + C_1$, 3 seconds

11 $e - 1$ seconds, $v = 1/e$, $\ddot{x} = -1/e^2$.
The velocity and acceleration approach zero, but the particle moves to infinity.

12 a $\dot{x} = -5 + 20e^{-2t}, x = -5t + 10 - 10e^{-2t}$,
$t = \log_e 2$ seconds

b It rises $7\frac{1}{2} - 5\log_e 2$ metres, when the acceleration is 10 m/s^2 downwards.

c The velocity approaches a limit of 5 m/s downwards, called the *terminal velocity.*

ISBN 978-1-108-76630-2

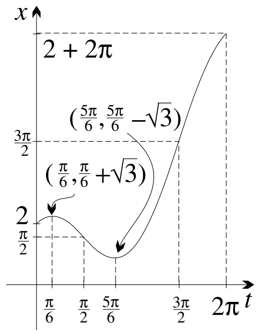
Cambridge University Press
Updated July 2022

13 a $v = 1 - 2\sin t$, $x = t + 2\cos t$

b $\frac{\pi}{2} < t < \frac{3\pi}{2}$

c $t = \frac{\pi}{6}$ when $x = \frac{\pi}{6} + \sqrt{3}$, and $\frac{5\pi}{6}$ when $x = \frac{5\pi}{6} - \sqrt{3}$, $\frac{\pi}{6} < t < \frac{5\pi}{6}$.

d 3 m/s when $t = \frac{3\pi}{2}$ and $x = \frac{3\pi}{2}$, -1 m/s when $t = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$.



e 2π metres, 1 m/s

f $4\sqrt{3} + \frac{2\pi}{3}$ metres, $\frac{1}{3} + \frac{2}{\pi}\sqrt{3}$ m/s

14 a Thomas, by 15 m/s

b $x_T = 20\log\ (t + 1)$, $x_H = 5t$

c during the 10th second, $3\frac{2}{11}$ m/s

d after 3 seconds, by 13 metres

15 a For $V \geq 30$ m/s, they collide after $180/V$ seconds, $\frac{180}{V^2}(V^2 - 900)$ metres above the valley floor.

b $V = 30\sqrt{2}$ m/s $3\sqrt{2}$ seconds

16 a $v = 5(e^{-2t} - 1)$, $x = \frac{5}{2}(1 - e^{-2t}) - 5t$

b The speed gradually increases with limit 5 m/s (the terminal velocity).

18 a $x_1 = 2 + 6t + t^2$, $x_2 = 1 + 4t - t^2$, $D = 1 + 2t + 2t^2$

b D is never zero, the minimum distance is 1 metre at $t = 0$ (t cannot be negative).

c $v_M = 5$ m/s, $12\frac{1}{2}$ metres

Exercise 9D

1 a 80 tonnes **b** When $t = 0$, $V = 0$.

c 360 tonnes **d** 20 tonnes/minute

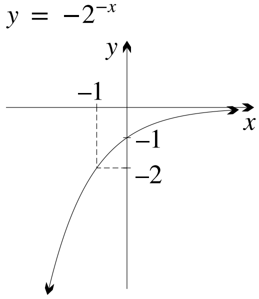
2 a 80 000 litres **b** 5 000 litres

c 0 min $\leq t \leq$ 20 min **d** 6000 litres/min

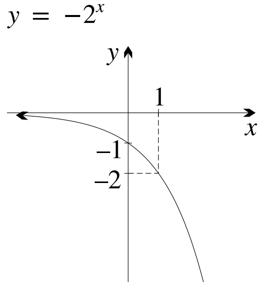
e The tank is emptying, so F is decreasing.

3 a 1500 **b** 300 **c** 15 minutes

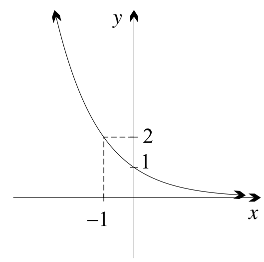
4 a $y = -2^{-x}$



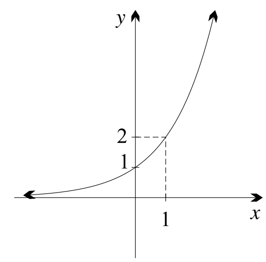
b $y = -2^{x}$



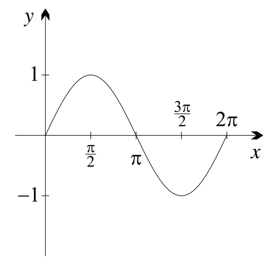
c $y = 2^{-x}$



d $y = 2^{x}$



5



a i $0 \leq x \leq \frac{\pi}{2}$

ii $\frac{\pi}{2} \leq x \leq \pi$

iii $\pi \leq x \leq \frac{3\pi}{2}$

iv $\frac{3\pi}{2} \leq x \leq 2\pi$

b i $\pi \leq x \leq 2\pi$

ii $0 \leq x \leq \pi$

6 a $h = 60e^{-\frac{t}{3}} - 30$

b 30 m/s upwards

c $h = 27.62$ m at $3\ln 2 \doteqdot 2.08$ seconds

d $h \doteqdot 10.23$ m and speed is 15 m/s downwards

e 30 m/s downwards

7 a i 12 kg/min

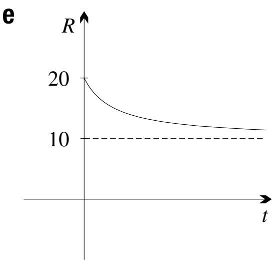
ii $10\frac{2}{3}$ kg/min

b 10 kg/min

c $\dot{R} = \frac{-20}{(1 + 2t)^2}$, $\ddot{R} = \frac{80}{(1 + 2t)^3}$

d R is decreasing at a decreasing rate

e

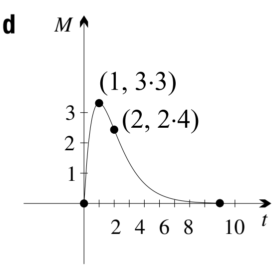


8 a $(0, 0)$ and $(9, 81e^{-9}) \doteqdot (9, 0.0)$

b $\dot{M} = 9(1 - t)e^{-t}$, $(1, 9e^{-1}) \doteqdot (1, 3.3)$

c $\ddot{M} = 9(t - 2)e^{-t}$, $(2, 18e^{-2}) \doteqdot (2, 2.4)$

d



e $t = 1$

f $t = 0$

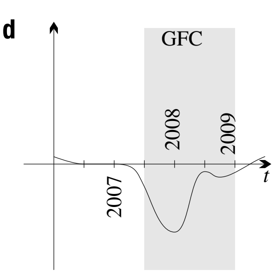
g $t = 2$

9 a The graph is steepest in January 2008.

b It levels out in 2009?

c The LIBOR reduced at a decreasing rate. It may have been mistaken as indicating the crisis was ending.

d



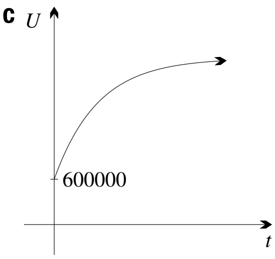
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 a Unemployment was increasing.

b The rate of increase was decreasing.

c



11 a $A = 9 \times 10^5$ **b** $N(1) = 380087$

c When t is large, N is close to 4.5×10^5.

d $\dot{N} = \frac{9 \times 10^5 e^{-t}}{(2 + e^{-t})^2}$

12 a $I = \dfrac{300t\left(2 - \frac{1}{5}t\right)}{200 + 3t^2 - \frac{1}{5}t^3}\%$ **b** $I(4) \doteqdot 6.12\%$

c $t = 0$ or 10. The latter is rejected because the model is only valid for 8 years.

13 b exponentials are always positive.

c $\phi(0) = \frac{1}{\sqrt{2\pi}}, \lim\limits_{x \to \infty} \phi(x) = 0$

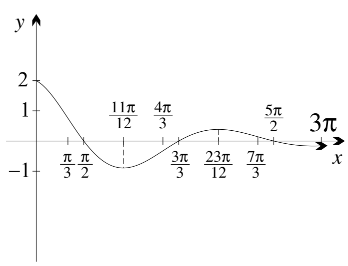
d $\phi'(x) < 0$ for $x > 0$ (decreasing)

e at $x = 1$ and $x = -1$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$.

h The curve approaches the horizontal asymptote more slowly for larger x.

14 a $y = 2$ and $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

f



Exercise 9E

1 b $1\ \text{m}^2/\text{s}$ **c** 7 metres **d** $9\ \text{m}^2$

2 a $A = \frac{1}{2}\ell^2$

c i $5\ \text{m}^2/\text{s}$ **ii** $3\ \text{m}^2/\text{s}$

d 34 metres

3 a $15.1\ \text{m}^3/\text{s}$ **b** $30.2\ \text{m}^2/\text{s}$

4 b $\frac{2}{9\pi}$ cm/s **c** $\frac{10}{\sqrt{\pi}}$ cm, $\frac{4000}{3\sqrt{\pi}}\ \text{cm}^3$

5 a $V = \frac{2}{3}\pi r^3$ **b** $\frac{1}{10\pi}$ cm/s

6 b 5 degrees per second

7 2 degrees per minute

8 a $V = 100h^2 x\ \text{cm}^3$ **b** $\dfrac{dh}{dt} = 0.1$ cm/day

10 a $-2\sqrt{1 - x^2}$

b -2 m/s — as the point crosses the y-axis it is travelling horizontally at a speed of 2 m/s.

11 a $\dfrac{2CV^2}{L^2}\ \text{m/s}^2$

c As L decreases, the speed passing the truck increases, so the driver should wait as long as possible before beginning to accelerate. A similar result is obtained if the distance between car and truck is increased. Optimally, the driver should allow both L to decrease and C to increase.

d 950 metres

12 b This is just two applications of the chain rule.

d 6

13 c $x = h = 50(\sqrt{3} + 1)$ metres

d 200 km/h

Exercise 9F

1 b $t = 4$ **c** 57 **d** $t = 2$

2 a 25 minutes **c** 3145 litres

3 a $P = 6.8 - 2\log_e(t + 1)$

b approximately 29 days

4 a $-2\ \text{m}^3/\text{s}$

b 20 s

c $V = 520 - 2t + \frac{1}{20}t^2$

d $20\ \text{m}^3$

e 2 minutes and 20 seconds

5 a no **c** $t \doteqdot 1.28$ **d** $x = \frac{5}{2}$

6 a 0 **b** 250 m/s

c $x = 1450 - 250(5e^{-0.2t} + t)$

7 a $I = 18\,000 - 5t + \frac{48}{\pi}\sin\frac{\pi}{12}t$

b $\dfrac{dI}{dt}$ has a maximum of -1, so it is always negative.

c There will be 3600 tonnes left.

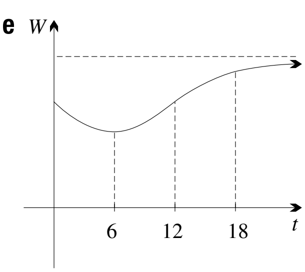
8 a It was decreasing for the first 6 months and increasing thereafter.

b after 6 months

c after 12 months

d It appears to have stabilised, increasing towards a limiting value.

e



9 a $\theta = \tan^{-1} t + \frac{\pi}{4}$

b $t = \tan\ (\theta - \frac{\pi}{4})$

c As $t \to \infty$, $\tan^{-1} t \to \frac{\pi}{2}$, and so $\theta \to \frac{3\pi}{4}$.

10 a $1200\ \text{m}^3$ per month at the beginning of July

b $W = 0.7t - \frac{3}{\pi}\sin\frac{\pi}{6}t$

11 b $k = \frac{5}{24}$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

12 a $V = \frac{1}{3}\pi r^3$ **b** $\dfrac{dr}{dt} = \dfrac{1}{2\pi r^2}$

c $t = \frac{2\pi}{3}(r^3 - 1000)$ **d** 25 minutes 25 seconds

13 a $V = \frac{\pi}{3}(128 - 48h + h^3)$

b i $A = \pi(16 - h^2)$ **ii** 1 hour 20 minutes

Chapter 9 review exercise

1 a $x = 24, x = 36$, 6 cm/s

b $x = 16, x = 36$, 10 cm/s

c $x = -8, x = -8$, 0 cm/s

d $x = 9, x = 81$, 36 cm/s

2 a $v = 40 - 2t, a = -2$, 175 m, 30 m/s, -2 m/s^2

b $v = 3t^2 - 25, a = 6t$, 0 m, 50 m/s, 30 m/s^2

c $v = 8(t - 3), a = 8$, 16 m, 16 m/s, 8 m/s^2

d $v = -4t^3, a = -12t^2$, -575 m, -500 m/s, -300 m/s^2

e $v = 4\pi \cos \pi t, a = -4\pi^2 \sin \pi t$, 0 m, -4π m/s, 0 m/s^2

f $v = 21\,e^{3t-15}, a = 63\,e^{3t-15}$, 7 m, 21 m/s, 63 m/s^2

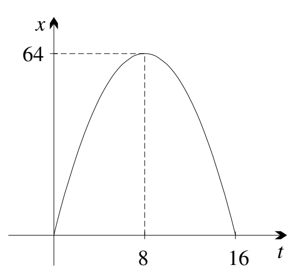
3 a $v = 16 - 2t, a = -2$

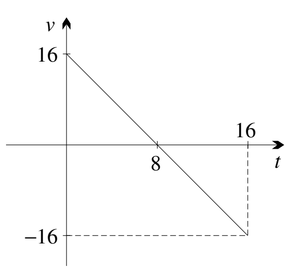
b 60 m, -4 m/s, 4 m/s, -2 m/s^2

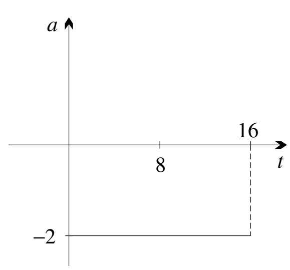
c $t = 16$s, $v = -16$ m/s

d $t = 8$ s, $x = 64$ m

e







4 a $a = 0, x = 7t + 4$

b $a = -18t, x = 4t - 3t^3 + 4$

c $a = 2(t - 1), x = \frac{1}{3}(t - 1)^3 + 4\frac{1}{3}$

d $a = 0, x = 4$

e $a = -24 \sin 2t, x = 4 + 6 \sin 2t$

f $a = -36e^{-3t}, x = 8 - 4e^{-3t}$

5 a $v = 3t^2 + 2t, x = t^3 + t^2 + 2$

b $v = -8t, x = -4t^2 + 2$

c $v = 12t^3 - 4t, x = 3t^4 - 2t^2 + 2$

d $v = 0, x = 2$

e $v = 5 \sin t, x = 7 - 5 \cos t$

f $v = 7e^t - 7, x = 7e^t - 7t - 5$

6 a $\dot{x} = 3t^2 - 12, x = t^3 - 12t$

b When $t = 2, \dot{x} = 0$.

c 16 cm

d $2\sqrt{3}$ seconds, 24 cm/s, $12\sqrt{3}$ cm/s^2

e As $t \to \infty, x \to \infty$ and $v \to \infty$.

7 a The acceleration is 10 m/s^2 downwards.

b $v = -10t + 40, x = -5t^2 + 40t + 45$

c 4 seconds, 125 metres

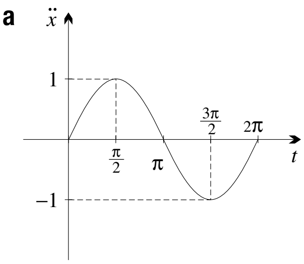
d When $t = 9, x = 0$.

e 50 m/s

f 80 metres, 105 metres

g 25 m/s

8 a



b $t = \pi$ and $t = 2\pi$

c $\dot{x} = -\cos t$

d $t = \frac{\pi}{2}$

e i $x = 5 - \sin t$

ii $x = 4$

9 a $v = 20$ m/s

b $20\,e^{-t}$ is always positive.

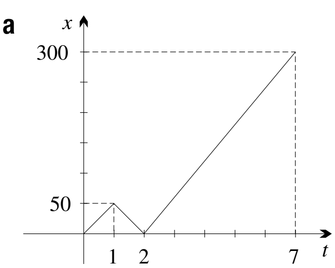
c $a = -20\,e^{-t}$

d -20 m/s^2

e $x = 20 - 20\,e^{-t}$

f As $t \to \infty, a \to 0, v \to 0$ and $x \to 20$.

10 a



b 400 km

c $57\frac{1}{7}$ km/hr

11 a $x = 20$ m, $v = 0$

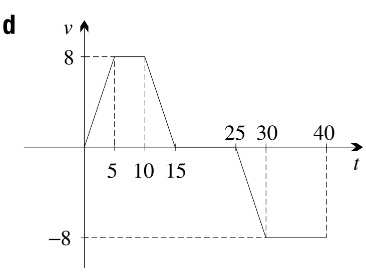
b i 8 m/s **ii** 0 **iii** -8 m/s

c i north (The graph is concave up.)

ii south (The graph is concave down.)

iii south (The graph is concave down.)

d



12 a at $t = 5$

b at $t = 12$, $0 < t < 12$, $t > 12$

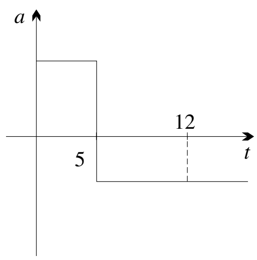
c $0 < t < 5, t > 5$

d at $t = 12$, when the velocity was zero

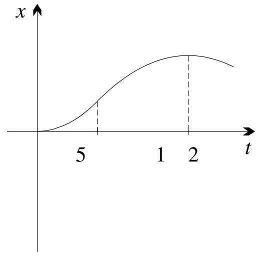
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

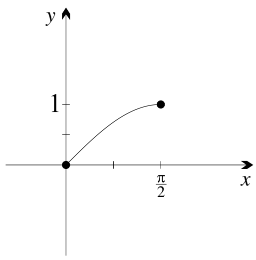
e



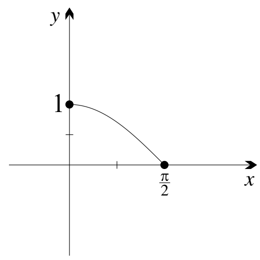
f



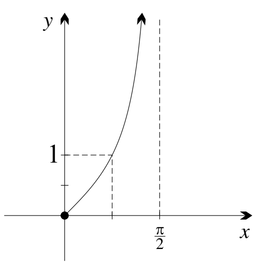
13 a i



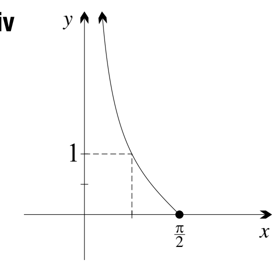
ii



iii



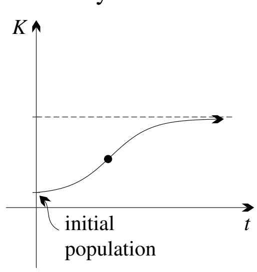
iv



b i $y = \sin x$ **ii** $y = \cos x$
iii $y = \cot x$ **iv** $y = \tan x$

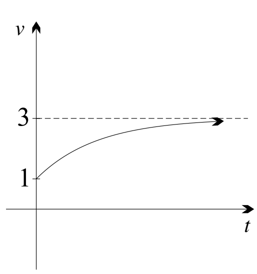
14 a Initially K increases at an increasing rate so the grpah is concave up. Then K increases at a decreasing rate so is concave down. The change in concavity coincides with the inflection point.

b



15 a 7500L
b $\dot{V} = -12(50 - 2t)$
c $\dot{V}$ is negative in the given domain.
d $\dot{V}$ is negative and $\ddot{V} = 24$ is positive, so the outflow decreases.

16 a



b $\dot{x}$ increases so it accelerates.
$\ddot{x} = \frac{2}{5}e^{-\frac{1}{5}t}$ which is always positive.
c $\lim_{t \to \infty} \dot{x} = 3\,\text{m/s}$
d $x = 3t + 10\left(e^{-\frac{1}{5}t} - 1\right)$

17 a $V = \frac{1}{5}t^2 - 20t + 500$
c $t = 50 - 25\sqrt{2} \doteqdot 15$ seconds. Discard the other answer $t = 50 + 25\sqrt{2}$ because after 50 seconds the bottle is empty.

18 a $-\frac{35}{24}$ cm/s
b $-\frac{1}{96}$ radians per second

Chapter 10

Exercise 10A

1 b $x = 30t,\ y = -5t^2 + 30t$
c $t = 6$ seconds. **d** 180 m
e $t = 3$ seconds **f** 45 m
2 b $x = 20t\sqrt{3},\ y = -5t^2 + 20t$
c i $t = 4$ seconds. **ii** $80\sqrt{3}$ m
iii 20 m
3 b $x = 10t,\ y = -5t^2 + 10t\sqrt{3}$
c i 15.9 m **ii** 12.4 m/s
4 b $x = 36t,\ y = -5t^2 + 48t$
c 146.5 m
5 a $\underset{\sim}{v} = 8\underset{\sim}{i} + (6 - 10t)\underset{\sim}{j}$
b $\underset{\sim}{r} = (8t)\underset{\sim}{i} + (6t - 5t^2)\underset{\sim}{j}$
c i 10 m/s **ii** $16\underset{\sim}{i} - 8\underset{\sim}{j}$
iii $4.8\underset{\sim}{i} + 1.8\underset{\sim}{j}$
6 a $\dot{x} = 40,\ \dot{y} = 25$
7 a $\underset{\sim}{v} = 4\underset{\sim}{i} + 4\underset{\sim}{j}$ **b** $2\underset{\sim}{i} + 0.75\underset{\sim}{j}$
9 b $x = 6t\sqrt{3},\ y = -5t^2 - 6t$
c 1 second **d** 10.4 m
10 a 1.19 seconds **b** 5.03 m
11 26 m/s
13 They collide 1 second after P_2 is projected.
14 c $\theta = 45°$ or $\theta \doteqdot 81°52'$
15 b i $\sqrt{5}$ s, $\sqrt{85}$ s
ii $20\sqrt{5} \doteqdot 44.7$ m/s in both cases, at angles of 0° and 76.0° to the horizontal.
18 c $\theta = 60°$

Exercise 10B

1 a 13.8 m **b** 18 m or 68.4 m
c $\frac{dy}{dx} = -\frac{5}{162}x + \frac{4}{3}$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

2 b **i** 192 m **ii** 20 m **iii** 22.6°
iv 6° below the horizontal
3 a $\sqrt{10}$
d **i** 9.5° above the horizontal
ii 9.5° below the horizontal
4 a $y = -\frac{1}{5}x^2$ **b** 10 m
c 76° below the horizontal
5 a $\dot{x} = 12\sqrt{3}, \dot{y} = -10t + 12,$
$x = 12t\sqrt{3}, y = -5t^2 + 12t$
d $D \doteqdot 58.7$
6 b **i** 13.5 m/s
ii 71° below the horizontal
7 b $\theta \doteqdot 28.2°$ or $61.8°$
8 b 60°
9 b $62°22'$ or $37°5'$
11 a $x = \frac{1}{2}Vt\sqrt{2}, y = -\frac{1}{2}gt^2 + \frac{1}{2}Vt\sqrt{2}$
12 c **ii** $R = 18$ metres

Chapter 10 review exercise

1 b $x = 60t\cos 40°, y = -5t^2 + 60t\sin 40°$
c **i** $t \doteqdot 7.7$ seconds.
ii 354.5 m **iii** 74.4 m
2 a 33.75 m **b** 96.58 m **c** 41.23 m/s
3 b $x = 15t, y = -5t^2 + 20t$
c $15\sqrt{2}$ m
4 a $\dot{x} = 5\sqrt{2}, x = 5t\sqrt{2}, \dot{y} = -10t + 5\sqrt{2},$
$y = -5t^2 + 5t\sqrt{2}$
b range: 10 metres, maximum height: 2.5 metres when $x = 5$
c **i** 1.6 metres
ii $\tan^{-1}\frac{3}{5}$ below the horizontal
d **i** $x = 3$
ii $\tan^{-1}\frac{2}{5}$ above the horizontal
5 a $\underset{\sim}{v} = 24\underset{\sim}{i} + (18 - 10t)\underset{\sim}{j}$
b $\underset{\sim}{r} = (24t)\underset{\sim}{i} + (18t - 5t^2)\underset{\sim}{j}$
c **i** 30 m/s
ii $96\underset{\sim}{i} - 8\underset{\sim}{j}$ **iii** $43.2\underset{\sim}{i} + 16.2\underset{\sim}{j}$
6 a $\underset{\sim}{v} = 12\sqrt{5}\underset{\sim}{i} + 24\underset{\sim}{j}$ **b** $36\sqrt{5}\underset{\sim}{i} + 27\underset{\sim}{j}$
c It is falling, because the height was 28 m after 2 seconds (or show that $\dot{y} = -6 < 0$).
7 b 11.25 m
8 a Initially, $\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5}$.
b $\dot{x} = \sqrt{5}, x = t\sqrt{5}, \dot{y} = -10t + 2\sqrt{5},$
$y = -5t^2 + 2t\sqrt{5}$
d 1 metre **e** 2 metres
f $\dot{x} = \sqrt{5}, \dot{y} = -2\sqrt{5}, v = 5$ m/s, $\theta = -\tan^{-1} 2$
g $y = 2x - x^2$
9 46 m/s
11 b $y = -\frac{4}{45}x^2 + \sqrt{3}x$
c **i** 15 m **ii** $\sqrt{3}$ seconds

Chapter 11

Exercise 11A

1 b $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$
2 b $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ or 2π
3 b $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$
4 a $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ **b** $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$
c $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}$ or $\frac{17\pi}{9}$
d $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ or 2π
5 a $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ or 2π **b** $x = \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}$ or $\frac{5\pi}{3}$
c $x = \frac{\pi}{6}, \frac{\pi}{2}$ or $\frac{5\pi}{6}$ **d** $x = \frac{2\pi}{3}, \pi$ or $\frac{4\pi}{3}$
e $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ or 2π
f $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$ or 2π
6 a $\theta = 90°, 194°29', 270°$ or $345°31'$
b $\theta = 60°, 120°, 240°$ or $300°$
c $\theta = 60°$ or $300°$
d $\theta = 22°30', 67°30', 112°30', 157°30', 202°30', 247°30', 292°30'$ or $337°30'$
e $\theta = 41°49', 138°11', 210°$ or $330°$
f $\theta = 54°44', 125°16', 234°44'$ or $305°16'$
g $\theta = 106°16'$
h $\theta = 0°, 60°, 300°$ or $360°$
i $\theta = 30°, 90°, 150°, 210°, 270°$ or $330°$
j $\theta = 45°, 63°26', 225°$ or $243°26'$
7 b $\theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$
8 b $x = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}$ or $\frac{9\pi}{5}$
9 b $\theta = 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}$ or π
10 b $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ or 2π
11 b $\theta = \frac{7\pi}{24}$ or $\frac{19\pi}{24}$
12 a $\alpha = 60°$ or $300°$
b $\alpha = 63°26', 135°, 243°26'$ or $315°$
c $\alpha = 15°, 75°, 105°, 165°, 195°, 255°, 285°$ or $345°$
d $\alpha = 180°$ or $240°$
e $\alpha = 30°, 60°, 210°$ or $240°$
f $\alpha = 60°$ or $300°$
13 b $\frac{\pi}{5}, \frac{\pi}{3}, \frac{3\pi}{5}$ or π
14 b $\theta = 160°55'$ or $289°5'$
15 c $x \doteqdot -2.571, -1.368$ or 3.939
16 e $x = \tan\frac{\pi}{10}, -\tan\frac{\pi}{10}, \tan\frac{3\pi}{10}$ or $-\tan\frac{3\pi}{10}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Exercise 11B

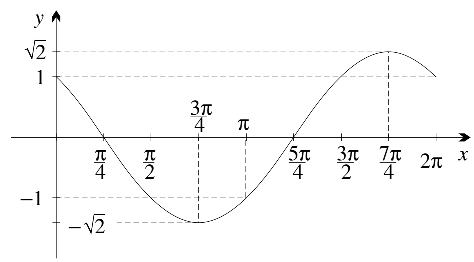
1 a $R = 2, \alpha = \frac{\pi}{3}$ **b** $R = 3\sqrt{2}, \alpha = \frac{\pi}{4}$

2 a $R = 13, \alpha \doteqdot 22°37'$ **b** $R = 2\sqrt{5}, \alpha \doteqdot 63°26'$

3 b $A = \sqrt{2}$ **c** $\alpha = \frac{\pi}{4}$

d Maximum is $\sqrt{2}$, when $x = \frac{7\pi}{4}$. Minimum is $-\sqrt{2}$, when $x = \frac{3\pi}{4}$.

e $x = \frac{\pi}{2}$ or π **f** amplitude: $\sqrt{2}$, period: 2π

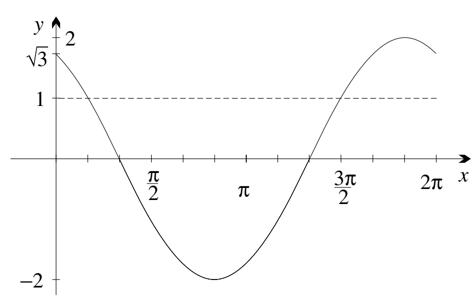


5 b $B = 2$ **c** $\theta = \frac{\pi}{6}$

d Maximum is 2, when $x = -\frac{\pi}{6}$. Minimum is -2, when $x = \frac{5\pi}{6}$.

e $x = \frac{\pi}{6}, \frac{3\pi}{2}$

f



6 c $x \doteqdot 126°52'$

7 b $x = 90°$ or $x \doteqdot 323°8'$

8 b $x = 270°$ or $x \doteqdot 306°52'$

9 a $3\sin\left(x + \tan^{-1}\dfrac{2}{\sqrt{5}}\right)$ **b** $x = 180°$ or $x \doteqdot 276°23'$

10 a $x \doteqdot 77°39'$ or $344°17'$ **b** $x \doteqdot 103°29'$ or $156°8'$

c $x \doteqdot 30°41'$ or $297°26'$ **d** $x \doteqdot 112°37'$ or $323°8'$

11 a $A = 2, \alpha = \frac{5\pi}{6}$ **b** $A = 5\sqrt{2}, \alpha = \frac{5\pi}{4}$

12 a $A = \sqrt{41}, \alpha \doteqdot 321°20'$

b $A = 5\sqrt{5}, \alpha \doteqdot 259°42'$

13 a i $2\cos(x + \frac{11\pi}{6})$ **ii** $x = \frac{\pi}{2}$ or $\frac{11\pi}{6}$

b i $\sqrt{2}\sin(x + \frac{3\pi}{4})$ **ii** $x = 0$ or $\frac{3\pi}{2}$

c i $2\sin(x + \frac{5\pi}{3})$ **ii** $x = \frac{\pi}{6}$ or $\frac{3\pi}{2}$

d i $\sqrt{2}\cos(x - \frac{5\pi}{4})$ **ii** $x = \pi$ or $\frac{3\pi}{2}$

14 a i $\sqrt{5}\sin(x + 116°34')$

ii $x = 270°$ or $x \doteqdot 36°52'$

b i $5\cos(x - 3.7851)$ **ii** $x \doteqdot 2.63$ or 4.94

15 a $x \doteqdot 313°36'$ **b** $x \doteqdot 79°6'$ or $218°59'$

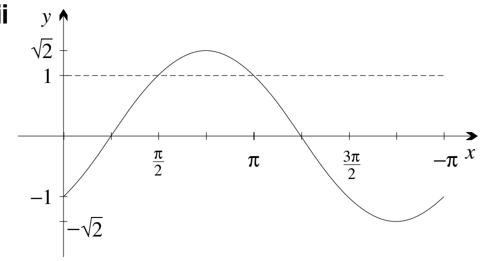
16 b $\theta = 0, \frac{3\pi}{4}, \frac{3\pi}{2}$ or $\frac{7\pi}{4}$

17 a $x = \frac{7\pi}{12}, \frac{11\pi}{12}$ **b** $x = \frac{\pi}{3}, \frac{4\pi}{3}$

c $x = 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \pi, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}, 2\pi$

18 b $x = -\frac{5\pi}{6}, -\frac{\pi}{12}, \frac{\pi}{6}$ or $\frac{11\pi}{12}$

19 a ii



iii $\frac{\pi}{2} < x < \pi$

b i $\frac{\pi}{2} \le x \le \frac{11\pi}{6}$

ii $0 \le x < \frac{\pi}{6}$ or $\frac{3\pi}{2} < x \le 2\pi$

iii $\frac{2\pi}{3} < x < \pi$ or $\frac{5\pi}{3} < x < 2\pi$

iv $0 \le x \le \frac{\pi}{12}$ or $\frac{17\pi}{12} \le x \le 2\pi$

20 b $\sin x + \sqrt{3}\cos x = 2\sin(x - \frac{5\pi}{3})$ or $2\cos(x - \frac{\pi}{6})$ or $2\cos(x + \frac{11\pi}{6})$

c $\cos x - \sin x = \sqrt{2}\cos(x - \frac{7\pi}{4})$ or $\sqrt{2}\sin(x + \frac{3\pi}{4})$ or $\sqrt{2}\sin(x - \frac{5\pi}{4})$

Exercise 11C

1 c $x = 0, \frac{3\pi}{2}, 2\pi$

2 b $x = 0, \frac{2\pi}{3}, 2\pi$

3 b $x = 90°$ or $x \doteqdot 298°4'$

4 b $x = 180°$ or $x \doteqdot 67°23'$

6 a $x = 90°$ or $x \doteqdot 12°41'$

b $x \doteqdot 36°52'$ or $241°56'$

c $x \doteqdot 49°48'$ or $197°35'$

d $x = 180°$ or $x \doteqdot 280°23'$

8 $x = 45°, 225°$ or $x \doteqdot 18.4°, 198.4°$

9 b $x \doteqdot 36°52'$

Chapter 11 review exercise

1 a $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ or 2π **b** $x = \frac{\pi}{3}, \pi$ or $\frac{5\pi}{3}$

c $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ **d** $x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

2 a $\sqrt{2}\sin(x - \frac{\pi}{4})$ **b** $x = \frac{3\pi}{4}$

3 a $2\cos(x - \frac{\pi}{6})$ **b** $x = \frac{5\pi}{6}$ or $\frac{3\pi}{2}$

4 a $3\sin(x + \tan^{-1}\frac{\sqrt{5}}{2})$ **b** $x \doteqdot 41.8°$

5 a $\sqrt{13}\cos(x + \tan^{-1}\frac{2}{3})$ **b** $x \doteqdot 40°12'$ or $252°25'$

6 $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ or 2π

8 $x \doteqdot 1.20$ or 2.87

9 b $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ or $\frac{11\pi}{6}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

Chapter 12

Exercise 12A

2 a $\frac{-1}{\sqrt{1 - x^2}}$ **b** $\frac{1}{1 + x^2}$
c $\frac{2}{\sqrt{1 - 4x^2}}$ **d** $\frac{3}{1 + 9x^2}$
e $\frac{-5}{\sqrt{1 - 25x^2}}$ **f** $\frac{-1}{\sqrt{1 - x^2}}$
g $\frac{2x}{\sqrt{1 - x^4}}$ **h** $\frac{3x^2}{1 + x^6}$
i $\frac{1}{x^2 + 4x + 5}$ **j** $\frac{1}{\sqrt{2x - x^2}}$
k $\sin^{-1}x + \frac{x}{\sqrt{1 - x^2}}$ **l** $2x\tan^{-1}x + 1$
m $\frac{1}{\sqrt{25 - x^2}}$ **n** $\frac{4}{16 + x^2}$
o $\frac{-1}{2\sqrt{x - x^2}}$ **p** $\frac{1}{2(1 + x)\sqrt{x}}$
q $\frac{-1}{1 + x^2}$

3 a 2 **b** 2 **c** 1 **d** -1

4 a Tangent is $y = -6x + \pi$, normal is $y = \frac{1}{6}x + \pi$.
b Tangent is $y = \frac{1}{\sqrt{2}}x + \frac{\pi}{4} - 1$, normal is $y = -\sqrt{2}x + \frac{\pi}{4} + 2$.

5 b $\frac{\pi}{2}$

6 a π **b** $\frac{\pi}{2}$

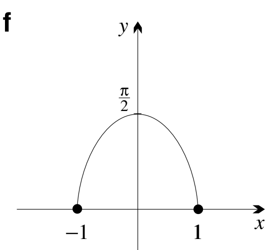
7 b concave up

9 a $\cos^{-1}x$ **b** $\frac{3e^{3x}}{\sqrt{1 - e^{6x}}}$
c $\frac{2}{\sqrt{7 + 12x - 4x^2}}$ **d** $\frac{1}{x^2 - 2x + 2}$
e $\frac{e^x}{\sqrt{1 - e^{2x}}}$ **f** $\frac{1}{2\sqrt{1 - x^2}\sin^{-1}x}$
g $\frac{1}{2x\sqrt{\log_e x(1 - \log_e x)}}$ **h** $\frac{\sin^{-1}\sqrt{1 - x}}{2\sqrt{x}} - \frac{1}{2\sqrt{1 - x}}$
i $\frac{1}{1 + x^2}$

11 a $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
b $x = \frac{1}{2}\sin y$
c $\frac{dx}{dy} = \frac{1}{2}\cos y \ge 0$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
d $\frac{dy}{dx} = \frac{2}{\sqrt{1 - 4x^2}}$

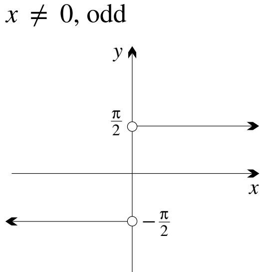
12 a $\frac{1}{\sqrt{4 - x^2}}$ **b** $\frac{-1}{\sqrt{2x - x^2}}$ **c** $\frac{1}{2(1 + x)\sqrt{x}}$

13 a $-1 \le x \le 1$, even
b The y-axis, because the function is even.
c $\frac{-2x}{\sqrt{1 - x^4}}$
e The tangents at $x = 1$ and $x = -1$ are vertical.
f

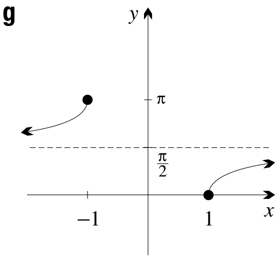


15 c $\frac{1}{45}$ rad/s

16 a $x \ne 0$, odd
c



17 a $x \ge 1$ or $x \le -1$
c They are undefined.
d When $x > 1$, $f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$, and when $x < -1$, $f'(x) = \frac{-1}{x\sqrt{x^2 - 1}}$.
e $f'(x) > 0$ for $x > 1$ and for $x < -1$.
f i $\frac{\pi}{2}$ **ii** $\frac{\pi}{2}$
g



18 a $\frac{3e^{3x}}{1 + e^{6x}}$ **b** $\frac{3x^2}{\sqrt{1 - x^6}}$
c $-\frac{1}{x\sqrt{1 - (\log_e x)^2}}$

19 a $-1 \le x \le 1$
c $g(x) = \frac{\pi}{2}$ for $0 \le x \le 1$.

20 $\tan^{-1}\frac{x + 2}{1 - 2x}$ is $\tan^{-1}x + \tan^{-1}2$ for $x < \frac{1}{2}$, and is $\tan^{-1}x + \tan^{-1}2 - \pi$ for $x > \frac{1}{2}$.

21 a domain: all real x, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, odd
c No, because $\frac{0}{0}$ is undefined.
d $f'(x) = 1$ when $\cos x > 0$, and $f'(x) = -1$ when $\cos x < 0$.

Exercise 12B

2 a $\cos^{-1}x + C$ **b** $\sin^{-1}\frac{x}{2} + C$
c $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ **d** $\sin^{-1}\frac{3x}{2} + C$
e $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$ **f** $\cos^{-1}\frac{x}{\sqrt{5}} + C$

3 a $\frac{\pi}{2}$ **b** $\frac{\pi}{8}$ **c** $\frac{\pi}{4}$ **d** $\frac{\pi}{12}$ **e** $\frac{\pi}{6}$ **f** $\frac{5\pi}{12}$

4 a $y = \sin^{-1}x + \pi$ **b** $y = \tan^{-1}\frac{x}{4} + \frac{\pi}{4}$

5 a $\frac{\pi}{3}$ **b** $\frac{\pi}{4}$

6 a $\frac{1}{2}\sin^{-1}2x + C$
b $\frac{1}{4}\tan^{-1}4x + C$
c $\frac{1}{\sqrt{2}}\cos^{-1}\sqrt{2}x + C$
d $\frac{1}{3}\sin^{-1}\frac{3x}{2} + C$
e $\frac{1}{15}\tan^{-1}\frac{3x}{5} + C$
f $\frac{1}{2}\cos^{-1}\frac{2x}{\sqrt{3}} + C$

7 a $\frac{\pi}{18}$ **b** $\frac{\pi}{12}$ **c** $\frac{2\pi}{9}\sqrt{3}$
d $\frac{5\pi}{24}$ **e** $\frac{\pi}{12}\sqrt{3}$ **f** $\frac{\pi}{120}\sqrt{10}$

8 c $\left(\frac{\pi}{12} + \frac{1}{2}\sqrt{3} - 1\right)$ unit^2

9 b $\left(1 - \frac{1}{2}\sqrt{3}\right)$ unit^2

10 b $\frac{\pi}{2}$

11 a $\frac{6x^2}{4 + x^6}$ **b** $\frac{1}{6}\tan^{-1}\frac{x^3}{2} + C$

12 a $\tan^{-1}x + \frac{x}{1 + x^2}$ **b** $\frac{\pi}{4} - \frac{1}{2}\ln 2$

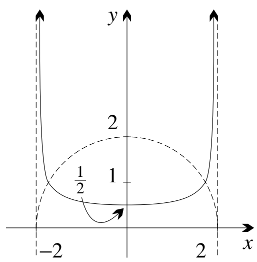
Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

13 a 0 **b** 0 **c** $\frac{3\pi}{4}$
d 0 **e** 0 **f** 18π

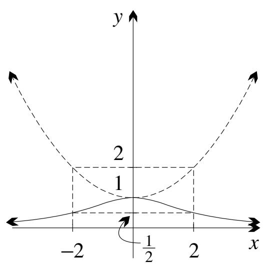
14 a i 0
b i $f(0) = 0$ and $f'(x) < 0$ for $x > 0$.
ii $\frac{\pi - 2}{8}$

15



c domain: $-2 \le x \le 2$, range: $y \ge \frac{1}{2}$, even
d $\frac{\pi}{3}$ unit^2 **e** π unit^2

16



a The y-axis, because it is an even function.
b domain: all real x, range: $0 < y \le 1$
d 0
e π unit^2
f $4\tan^{-1}\frac{a}{2}$ unit^2
g 2π unit^2

18 a $\frac{5323}{6800}$

20 a $2\tan^{-1}\sqrt{x} + C$ **b** $\tan^{-1}e - \frac{\pi}{4}$

21 b 0.153 unit^2
c The integrand is well-defined in the interval $[0, 7]$, and lies between $\frac{1}{4}$ and $\frac{1}{9}$, so the area lies between $\frac{7}{4}$ and $\frac{7}{9}$, which is much larger than the answer of 0.153 that was calculated in part **b**. The primitive, however, is undefined at two values within the interval, at $x = \frac{\pi}{2}$ and at $x = \frac{3\pi}{2}$, which renders the argument completely invalid.

22 g $\pi \doteqdot 3.092$, error $\doteqdot 0.050$

23 a $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 + \cdots + \tan^{-1}n$
b $x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2)$

Exercise 12C

2 a $\frac{2 + \sqrt{3}}{4}$ **b** $\frac{2 + \sqrt{3}}{4}$ **c** $-\frac{1}{4}$ **d** $\frac{2 - \sqrt{2}}{4}$

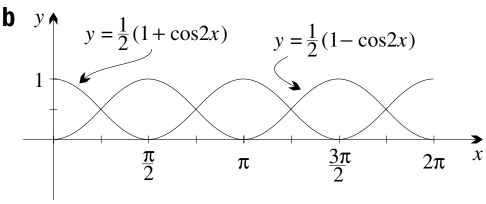
3 a $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ **b** $\frac{1}{2}x - \frac{1}{8}\sin 4x + C$
c $\frac{1}{2}x - \sin\frac{1}{2}x + C$ **d** $\frac{1}{2}x - \frac{1}{12}\sin 6x + C$

4 a $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$ **b** $\frac{1}{2}x + \frac{1}{24}\sin 12x + C$
c $\frac{1}{2}x + \frac{1}{2}\sin x + C$ **d** $\frac{1}{2}x + \frac{1}{40}\sin 20x + C$

5 a $\frac{\pi}{2}$ **b** $\frac{1}{8}(\pi + 2)$
c $\frac{1}{12}(\pi - 3)$ **d** $\frac{1}{32}(\pi + 2\sqrt{2})$
e $\frac{1}{24}(4\pi + 9)$ **f** $\frac{1}{24}(2\pi - 3\sqrt{3})$

6 a $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$
b $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C$
c $\frac{2\sqrt{2}}{3}$ **d** $\frac{\sqrt{3}}{28}$

7 b



8 a $\frac{1}{4}\sin^4 x + C$ **b** $\frac{1}{7}\sin^7 x + C$
c $-\frac{1}{6}\cos^6 x + C$ **d** $-\frac{1}{9}\cos^9 x + C$
e $-\cos e^x$ **f** $\frac{1}{5}\sin 5e^x + C$
g $-\log_e|\cos x| + C$ **h** $\frac{1}{7}\log_e|\sin 7x| + C$

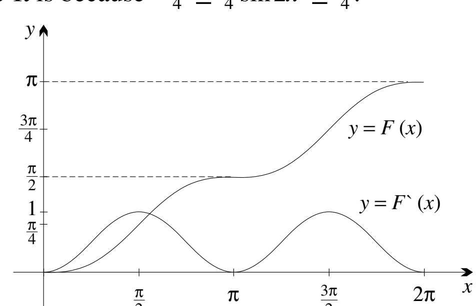
9 a $\cos x\sin x = \frac{1}{2}\sin 2x$, so $-\frac{1}{2} \le y \le \frac{1}{2}$.
b i $-\frac{1}{4}\cos 2x + C$
ii $-\frac{1}{2}\frac{1}{2}\sin^2 x + D$ using $u = \sin x$, and
$-\frac{1}{2}\cos^2 x + E$ using $u = \cos x$
c The three constants C, D and E are different. The answers in part **ii** can be reconciled with each other using the Pythagorean identity, and they can be reconciled with part **i** using the $\cos 2x$ formulae.

10 b $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$
c i $\frac{3\pi}{8}$ **ii** $\frac{1}{32}(3\pi + 8)$
d $\frac{5}{24}$

11 a $\frac{1}{2}\tan 2x - x + C$ **b** $-2\cot\frac{1}{2}x - x + C$
c $\sqrt{3} - 1 - \frac{\pi}{12}$ **d** $\frac{1}{4}\sqrt{3} - \frac{\pi}{12}$

12 a $\frac{1}{2}\tan^2 x + C$ **b** $x - \sin x + C$
c $\tan x + \sin x + C$

13 b i $x = 0, \pi$ or 2π
ii $0 < x < \pi$ or $\pi < x < 2\pi$
iii no values of x
c It is because $-\frac{1}{4} \le \frac{1}{4}\sin 2x \le \frac{1}{4}$.



Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d $(\frac{\pi}{2}, \frac{\pi}{4})$ and $(\frac{3\pi}{2}, \frac{3\pi}{4})$ are points of inflection, while $(0, 0)$, $(\pi, \frac{\pi}{2})$ and $(2\pi, \pi)$ are stationary (or horizontal) points of inflection.

f **i** $k = 3\pi$

ii $k = n\pi$, where n is an integer.

14 $\frac{1}{2}$

Exercise 12D

1 c $\frac{1}{4}(1 + x^2)^4 + C$

2 a $\frac{1}{4}(2x + 3)^4 + C$ **b** $\frac{1}{5}(1 + x^3)^5 + C$

c $\frac{-1}{1 + x^2} + C$ **d** $2\sqrt{3x - 5} + C$

e $\frac{1}{4}\sin^4 x + C$ **f** $\log_e(1 + x^4) + C$

3 c $-\sqrt{1 - x^2} + C$

4 a $\frac{1}{24}(x^4 + 1)^6 + C$ **b** $\frac{2}{9}(x^3 - 1)^{\frac{3}{2}} + C$

c $\frac{1}{3}e^{x^3} + C$ **d** $\frac{-1}{(1 + \sqrt{x})^2} + C$

e $\frac{1}{6}\tan^3 2x + C$ **f** $-e^{\frac{1}{x}} + C$

5 a $\frac{65}{12}$ **b** $\sqrt{2} - 1$ **c** $\frac{1}{3}$

d $\frac{1}{24}$ **e** 2 **f** $\frac{1}{2}(e^2 - 1)$

g $\frac{1}{10}$ **h** $\frac{\pi^4}{64}$ **i** 3

j $\frac{1}{2}\ln 3$

6 $\frac{\pi}{12}$ units^2

7 a $\log_e \frac{3}{2}$ **b** $\frac{\pi}{4}$ **c** $\frac{2}{3}$ **d** $\frac{4}{3}$

8 a $\sqrt{1 + e^{2x}} + C$ **b** $\ln(\ln x) + C$

c $-\ln(\ln \cos x) + C$ **d** $\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C$

9 a $y = \frac{1}{2}\tan^{-1} e^{2x}$ **b** $y = \sin^{-1}\frac{x}{2} + \frac{x}{2} + \frac{1}{2}$

10 b **i** $\frac{2}{\ln 2}$ **ii** $\frac{1}{5}(4\sqrt{2} - 1)$

11 a $\ln 2$ **b** $\dfrac{e}{e + 1}$

12 $\tan^{-1}\sqrt{x - 1} + C$

13 a $2\sin^{-1}\sqrt{x} + C_1$ **b** $\sin^{-1}(2x - 1) + C_2$

Exercise 12E

1 c $\frac{1}{7}(x - 1)^7 + \frac{1}{6}(x - 1)^6 + C$

2 a $\frac{2}{3}(x - 1)^{\frac{3}{2}} + 2(x - 1)^{\frac{1}{2}} + C$

b $\ln|x - 1| - \frac{1}{x - 1} + C$

3 c $\frac{2}{5}(x + 1)^{\frac{5}{2}} - \frac{2}{3}(x + 1)^{\frac{3}{2}} + C$

4 a $\frac{2}{7}(x + 1)^{\frac{7}{2}} - \frac{4}{5}(x + 1)^{\frac{5}{2}} + \frac{2}{3}(x + 1)^{\frac{3}{2}} + C$

b $\frac{4}{3}(x + 1)^{\frac{3}{2}} + 2(x + 1)^{\frac{1}{2}} + C$

5 a $(x + 2) - 4\ln(x + 2) + C$

b $\frac{1}{3}(2x - 1)^{\frac{3}{2}} + 2(2x - 1)^{\frac{1}{2}} + C$

c $\frac{3}{40}(4x - 5)^{\frac{5}{2}} + \frac{5}{8}(4x - 5)^{\frac{3}{2}} + C$

d $2(1 + \sqrt{x}) - 2\ln(1 + \sqrt{x}) + C$

6 a $\frac{49}{20}$ **b** $2\ln 2 - \frac{1}{2}$

c $\frac{8}{9}$ **d** $\frac{1}{9}$

e $\frac{128}{15}$ **f** $\frac{4}{3}$

g $4 - 6\ln\frac{5}{3}$ **h** $\frac{2517}{40}$

7 a $\sin^{-1}\frac{x + 2}{3} + C$

b **i** $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x + 1}{\sqrt{3}} + C$ **ii** $\sin^{-1}\frac{x + 1}{\sqrt{5}} + C$

iii $\frac{\pi}{6}$ **iv** $\frac{\pi}{16}$

8 b **i** $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ **ii** $\cos^{-1}\frac{x}{\sqrt{3}} + C$

iii $\frac{1}{2}\sin^{-1} 2x + C$ **iv** $\frac{1}{4}\tan^{-1} 4x + C$

v $\frac{\pi}{6}$ **vi** $\frac{\pi}{24}$

9 b **i** $\frac{x}{4\sqrt{4 + x^2}} + C$ **ii** $\frac{\pi}{12} - \frac{\sqrt{3}}{8}$

iii π **iv** $-\frac{\sqrt{25 - x^2}}{25x} + C$

v $-\frac{\sqrt{9 + x^2}}{9x} + C$ **vi** $\frac{\sqrt{3}}{8}$

10 $y = \sqrt{x^2 - 9} - 3\tan^{-1}\frac{\sqrt{x^2 - 9}}{3}$

11 $\frac{1}{3}\left(6\sqrt{3} - 7\sqrt{2}\right)$ units^2

13 b $\frac{8}{3}$

Exercise 12F

1 b 81π u^3

2 b 36π u^3

3 a 16π u^3 **b** 9π u^3 **c** $\frac{32\pi}{5}$ u^3 **d** 6π u^3

e $\frac{16\pi}{3}$ u^3 **f** $\frac{\pi}{7}$ u^3 **g** 9π u^3 **h** 16π u^3

4 a 3π u^3 **b** $\frac{256\pi}{3}$ u^3 **c** $\frac{3093\pi}{5}$ u^3 **d** $\frac{\pi}{2}$ u^3

e $\frac{256\pi}{3}$ u^3 **f** $\frac{243\pi}{5}$ u^3 **g** $\frac{16\pi}{15}$ u^3 **h** $\frac{16\pi}{3}$ u^3

5 $\frac{\pi}{2}(e^2 - 1)$ cubic units

6 $\pi\ln 2$ u^3

7 $\pi\ln 6$ u^3

8 a $\tan^2 x = \sec^2 x - 1$ **b** $\pi\left(\sqrt{3} - \frac{\pi}{3}\right)$ u^3

9 a $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ **b** $\frac{\pi^2}{4}$ u^3

10 a $\frac{296\pi}{3}$ u^3 **b** $\frac{119\pi}{6}$ u^3 **c** $\frac{625\pi}{6}$ u^3 **d** $\frac{16\pi}{105}$ u^3

11 a $\frac{\pi}{3}$ u^3 **b** $\frac{28\pi}{15}$ u^3 **c** $\frac{81\pi}{10}$ u^3 **d** $\frac{\pi}{2}$ u^3

14 71.62 mL

15 a 256π u^3 **b** 128π u^3 **c** 128π u^3

16 a $\frac{32\pi}{5}$ u^3, 8π u^3 **b** $\frac{50\pi}{3}$ u^3, $\frac{5\pi}{3}$ u^3

c 8π u^3, $\frac{128\pi}{5}$ u^3 **d** $\frac{24\pi}{5}$ u^3, $\frac{\pi}{2}$ u^3

17 b **i** $\frac{2\pi}{35}$ u^3 **ii** $\frac{\pi}{10}$ u^3

18 b $\frac{64\pi}{3}$ u^3

19 $\pi(8\ln 2 - 5)$ u^3

20 a $y = 3x$

c **i** $\frac{15\pi}{7}$ u^3 **ii** $\frac{2\pi}{5}$ u^3

21 $\frac{\pi}{2}(8\ln 2 - 3)$ u^3

23 a $\ln(\sec\theta + \tan\theta) + C$

24 a $(0, 0)$ and $(1, 1)$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c It is the cone formed by rotating the line $y = x$ from $x = 0$ to $x = 1$ about the $x-$ axis.

d $\frac{\pi}{3}$ u^3

25 c 8π u^3

Chapter 12 review exercise

1 a $\frac{3}{\sqrt{1-9x^2}}$ **b** $\frac{3}{9+x^2}$
c $\frac{1}{\sqrt{2x-x^2}}$ **d** $2x\tan^{-1}x + \frac{x^2}{1+x^2}$
e $\frac{2}{x^2+4x+8}$ **f** $\frac{-1}{\sqrt{x^4-x^2}}$

3 a They each have derivative $\frac{-1}{\sqrt{1-x^2}}$.
b The functions differ by at most a constant. This constant is zero, so $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$ for $0 \le x \le 1$.

4 a $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$ **b** $\sin^{-1}\frac{x}{\sqrt{3}} + C$
c $\frac{1}{6}\tan^{-1}\frac{2x}{3} + C$ **d** $\frac{1}{3}\cos^{-1}\frac{3x}{4} + C$

5 a $\frac{\pi}{36}$ **b** $\frac{\pi}{3}$

6 a $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$ **b** $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$
c $\frac{1}{2}x + \frac{1}{8}\sin 4x + C$ **d** $\frac{1}{2}x - \frac{1}{16}\sin 8x + C$

7 a $\dfrac{\pi}{6}$ **b** $\frac{1}{12}(\pi + 3)$

9 a $\frac{1}{6}(5x-1)^6 + C$ **b** $\frac{1}{3}(x^2+2)^3 + C$
c $\frac{-1}{x^4+1} + C$ **d** $\frac{1}{2}\sqrt{4x+3} + C$
e $\frac{1}{3}\sin^3 x + C$ **f** $\frac{1}{4}\tan^4 x + C$

10 a $\frac{1}{15}$ **b** $\frac{1}{4}$ **c** $\frac{1}{3}$
d $\frac{1}{3}$ **e** $e^2 - e$ **f** $\frac{1}{2}\ln 2$

11 a $x - 1 + \ln(x-1) + C$
b $\frac{2}{3}(x+2)^{\frac{3}{2}} - 6(x+2)^{\frac{1}{2}} + C$
c $\frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C$
d $2(4+\sqrt{x}) - 8\ln(4+\sqrt{x}) + C$

12 a $\frac{11}{30}$ **b** $4 - 3\ln 2$ **c** 36 **d** $\frac{13}{15}$

13 a $x \le 9, y \ge 0$ **b** 18 u^2
d i $\frac{81\pi}{2}$ u^3 **ii** $\frac{648\pi}{5}$ u^3

14 $4\pi\ln 2$ u^3

15 $\frac{2688\pi}{5}$ u^3

16 a $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

17 8.49 u^3

18 b $\pi(1 - \ln 2)$ u^3

19 a Minimum turning point at $(1, 2)$, maximum turning point at $(-1, -2)$.
c $\frac{9\pi}{4}$ u^3

20 b $72 - \frac{9\pi}{2}$ u^3 **c** $\frac{5004\pi}{5}$ u^3

21 a $\pi\displaystyle\int_1^3 2^{2x+2}\,dx$ **b** 180π u^3
c $\dfrac{120\pi}{\log_e 2} \doteqdot 173\pi$ u^3 The curve is concave up.

Chapter 13

Exercise 13A

1 a first **b** first
c second **d** first
e second **f** first
g first **h** second **i** second

2 a linear **b** non-linear **d** linear
f non-linear **g** linear

3 a 1 **b** 1 **c** 2 **d** 1 **e** 2
f 1 **g** 1 **h** 2 **i** 2

5 a $y = x^2 - 3x + C$
b $y = -6e^{-2x} + 4x + C$
c $y = \tan x + C$
d $y = 3\sin 2x - 3\cos 3x + C$
e $y = \frac{2}{15}(1-5x)^{\frac{3}{2}} + C$
f $y = 2\sin x^2 + C$

8 a $y = x^2 + Ax + B$
b $y = -\frac{1}{4}\cos 2x + Ax + B$
c $y = 4e^{\frac{1}{2}x} + Ax + B$
d $y = -\log|\cos x| + Ax + B$

11 a $y = x - 1$
b $y = x^2 - 3x + 2$
c $y = x^3 + 3x^2 - 9x + 7$
d $y = 2 - \cos x$
e $y = 3(e^{2x} - 1)$
f $y = 2x\sqrt{x} - 2x - 1$

12 a ii $y = \log\left|\dfrac{x}{1-x}\right| + C$
iii $y = \log\left|\dfrac{x}{1-x}\right|$
b ii $y = \log\left|\dfrac{2+x}{2-x}\right| + C$
iii $y = \log\left|\dfrac{2+x}{2-x}\right| + 1$

13 a i $2x + 2yy' = 0$
iii y' is undefined at $(3, 0)$ and $(-3, 0)$ where the tangent to the circle is vertical.
b There may be other answers that are equivalent to these.
i $y' = \frac{1}{2y}$ **ii** $\dfrac{dy}{dx} = -\dfrac{y}{x}$
iii $\dfrac{dy}{dx} = -\dfrac{9x}{16y}$ **iv** $\dfrac{dy}{dx} = \dfrac{x}{4y}$
v $\dfrac{dy}{dx} = \dfrac{y}{2y-x}$ **vi** $\dfrac{dy}{dx} = \dfrac{y-x^2}{y^2-x}$

14 b When $y = 12$, $y'' = -y = -12$.
c $f''(x)$ is the opposite of $f(x)$, so it is a reflection in the x-axis.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

15 $y = 1 + x - \log(\cos x)$

17 a $\lambda = 1$ or 3 **b** $\lambda = -1$

c No real solution.

18 a $y' - y\tan x = 0, y(0) = 1$

b $(y')^2 = y^2(y^2 - 1), y(0) = 1$

19 a $y(0) = 1$

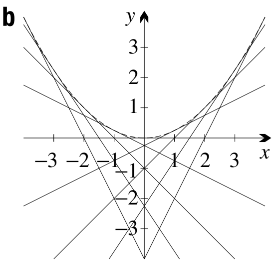
b $y'(0) = -2 \times 0 \times 1 = 0$

c i $y'' = -2y - 2xy'$ by the product rule, then substitute the expression for y'.

ii $y''(0) = -2$, so it is concave down.

d $y''' = (12x - 8x^3)y$ so $y'''(0) = 0$

20 b



They seem to form the outline of a curve (shown dotted.)

c $(2p + h, p^2 + ph)$

e $y = \frac{1}{4}x^2$; this is called a *singular* solution.

Each line in **b** is tangent to the parabola. Thus every point in the parabola is also a point in a general solution. Hence the parabola itself is also a solution.

21 $y = \dfrac{x^n}{n!} + \dfrac{A_1x^{n-1}}{(n-1)!} + \cdots + \dfrac{A_{n-1}x^1}{1!} + A_{n.}$

There are n arbitrary constants.

Exercise 13B

1 a -1 **b** 1 **c** 3 **d** $\frac{1}{2}$ **e** $\frac{1}{2}$ **f** -3

2 a

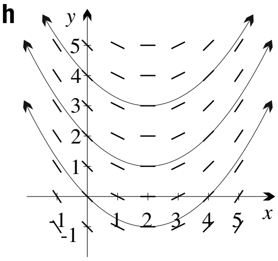
y \ x	-1	0	1	2	3	4	5
5	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
4	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
3	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
0	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
-1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

d Every entry in that column is the same.

e Every vertical line $x = k$ is an isocline.

f concave up **g** a parabola

h



3 a $y' = x - y$

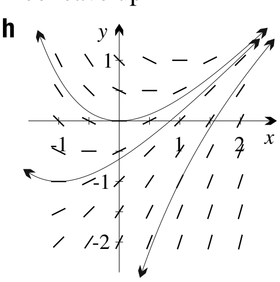
b

y \ x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
0	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
-1	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$
-2	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4

e The lines $y = x + k$ are isoclines.

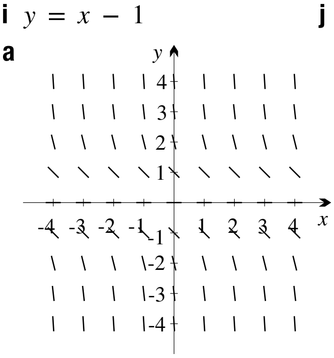
f concave up

h



i $y = x - 1$ **j** It is a solution.

4 a



i x-axis

ii any horizontal line

iii decrease to zero then increase

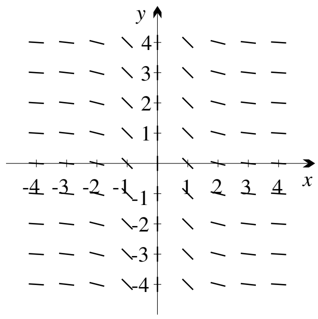
iv it is an isocline

Mathematics Extension 1 Year 12 ISBN 978-1-108-76630-2

Cambridge University Press
Cambridge Maths Stage 6

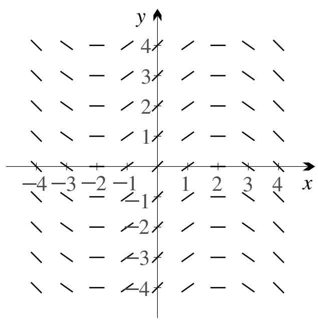
Updated July 2022

b



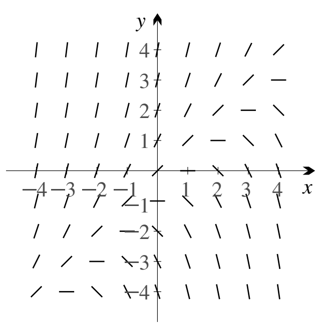
i none
ii any vertical line
iii it is an isocline
iv decrease to vertical then increase

c



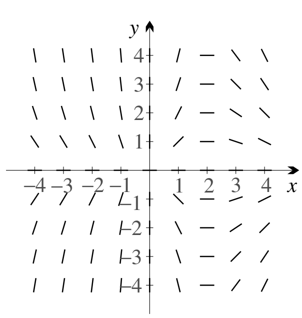
i $x = -2, 2$
ii any vertical line
iii it is an isocline
iv increase from -1 to 1 then back

d



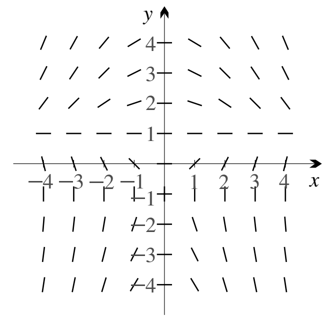
i $y = x - 1$
ii $y = x + C$
iii increase
iv decrease

e



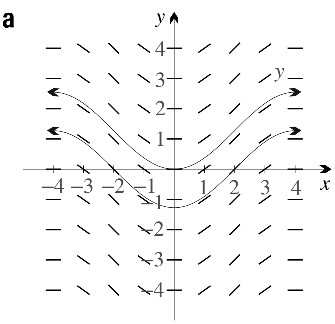
i $x = 2$
ii $y = 0$
iii increase
iv decreasing, but undefined at y-axis

f

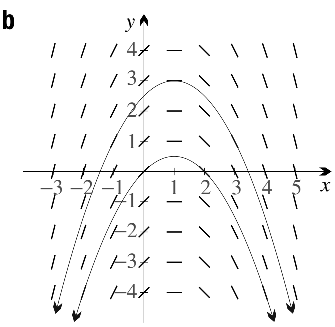


i $y = 1$ or $x = 0$
ii undefined at $y = -1$
iii decreasing, but undefined at $y = -1$
iv decrease

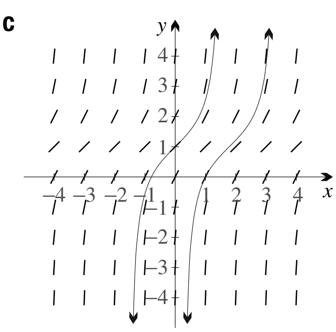
5 a



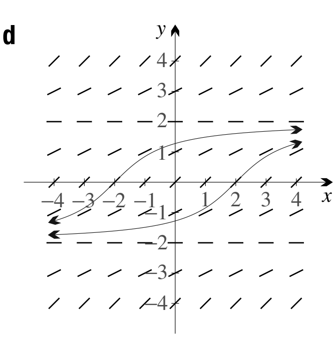
b



c



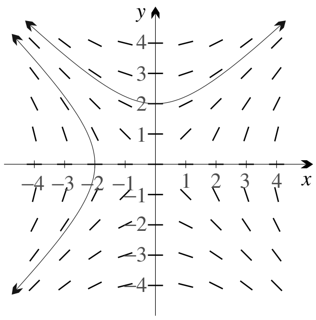
d



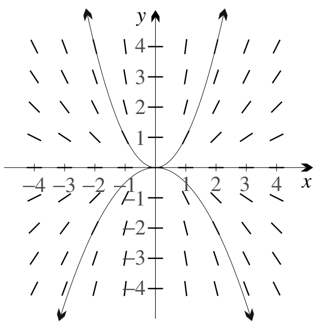
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

e



f



6 a vertical isoclines so $y' = f(x)$

b vertical isoclines so $y' = f(x)$

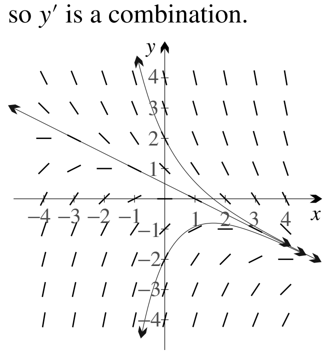
c horizontal isoclines so $y' = g(y)$

d horizontal isoclines so $y' = g(y)$

e Not all horizontals and not all verticals are isoclines, so y' is a combination.

f Not all horizontals and not all verticals are isoclines, so y' is a combination.

7



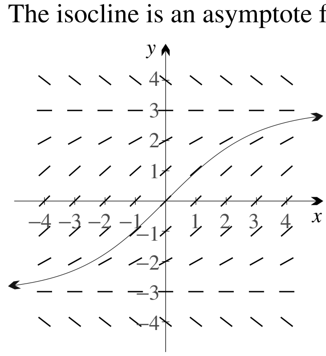
b **i** decrease

ii closer

c $y' = -\frac{1}{2}$ everywhere on that line.

e The isocline is an asymptote for each one.

8



b $y = -3, 3$ **c** yes

d **i** converge **ii** diverge **iii** yes

iv They are asymptotes for the solution curves.

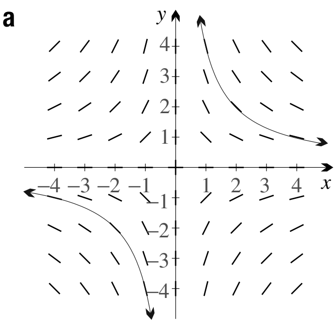
9 D. $y' = -1 - y$

10 C. $y' = \frac{1}{3}(3 - x^2)$

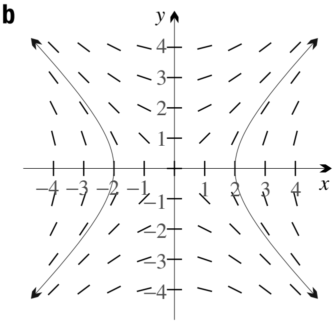
11 B. $y' = 1 - \dfrac{x}{y}$

12 a B. $y' = x - \frac{1}{2}y$ **b** D. $y' = \dfrac{-2xy}{1 + x^2}$

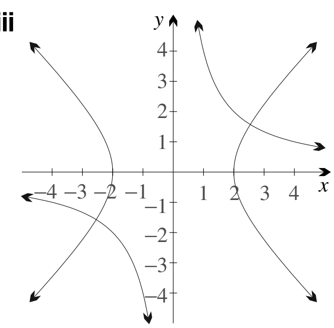
13 a



b

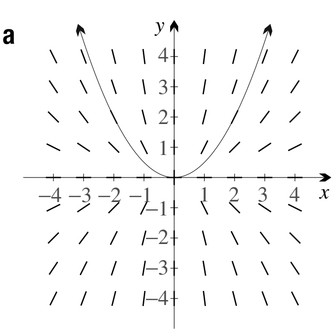


c iii

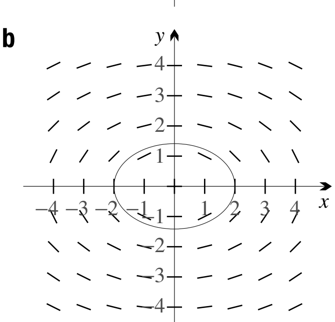


d The product of the derivatives is -1.

14 a



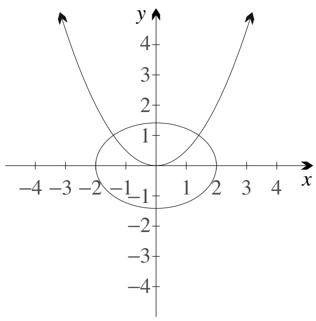
b



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

c iii



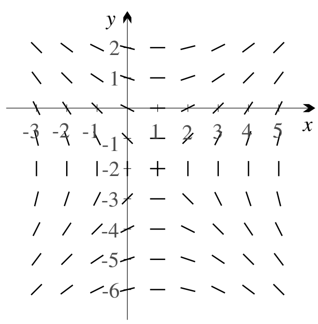
d The product of the derivatives is -1.

15 a $x^2 + y^2 = 4$

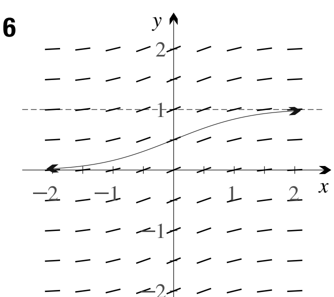
b $(x - 3)^2 + (y - 1)^2 = 4$

d i $\dfrac{dy}{dx} = \dfrac{(x - 1)}{(y + 2)}$

ii

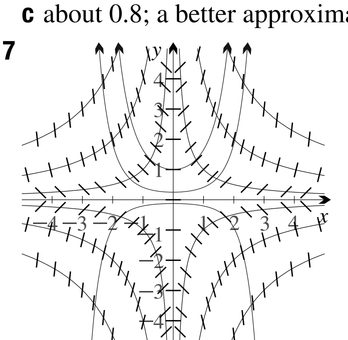


16



c about 0.8; a better approximation is 0.8413

17



a $x = 0$ and $y = 0$

b the rectangular hyperbola $xy = C$

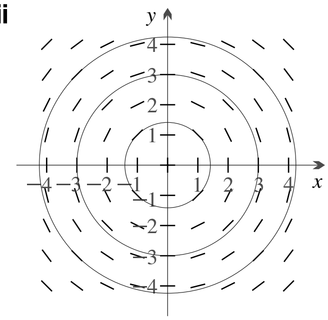
j i The exact gradient of the integral curve is known as an isocline is crossed.

ii It takes a lot of time to sketch the isoclines.

18 a i They are horizontal.

ii They are vertical.

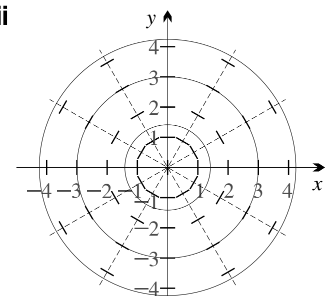
iii



iv circle

b i $y = -\frac{1}{C}x$

ii



The straight lines at 30° and 60° to the x-axis.

iii The product of the gradients is $C \times \frac{-1}{C} = -1$.

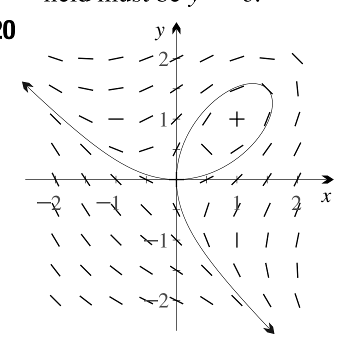
v Yes: notice that the innermost line elements almost join up to give the outline of a circle.

19 a Any two points on the same vertical line have the same x-value. Because $y' = f(x)$ is a function only of x, any two points on the same vertical line thus give the same slope, making this line an isocline.

b Any two points on the same horizontal line have the same y-value. Because $y' = g(y)$ is a function only of y, any two points on the same horizontal line thus give the same slope, making this line an isocline.

c At every point on the line, the gradient of the slope field must be $y = c$.

20



c $y = x^2$, excluding the points (0, 0) and (1, 1) where the gradient is undefined.

d $x = y^2$

f The curve crosses itself, horizontally and vertically.

g $y = -x - 1$

h $y = x$

i The equation is symmetric in x and y.

ii Swapping x and y yields the reciprocal, $\dfrac{dx}{dy}$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 13C

1 a $\displaystyle\int (y+1)\,dy = \int (x-1)\,dx$

b $(y+1)^2 = (x-1)^2 + D$

2 a $y = \log\left|\frac{1}{2}x^2 + C\right|$

b $y = \tan\,(x^4 + C)$

3 a Substitute $y = 0$, where $x \neq 0$. Then LHS and RHS are both zero.

b $y = \dfrac{1}{C + \log|x|}$

4 a $\displaystyle\int y\,dy = \int -x\,dx$ **b** $y^2 = -x^2 + D$

c $y^2 + x^2 = 4$

5 a $y^2 = x^2 + 1$

b $y = \tan\left(\frac{1}{2}(x+1)^2\right)$

c $y = \dfrac{1}{x^2+1}$

d $y = \log\,(1 + \tan x)$ (More strictly, the solution is only valid in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.)

6 a $y = -2$

b $y = Cx^2 - 2$, where $C \neq 0$

c Allow $C = 0$.

7 a $y = 0$

b $y = Ce^{-\frac{1}{2}x^2}$, where $C \neq 0$

c Allow $C = 0$.

8 a $y = \frac{C}{x} + 2$ **b** $y^2 = C(1 + x^2)$

c $y = \frac{C}{x^2}$ **d** $y = Ce^{-\cos x}$

e $y = Ce^{-3/x}$ **f** $y = Cxe^{-x}$

9 a $\cos y = 0$, that is, $y = -\dfrac{3\pi}{2}, y = -\dfrac{\pi}{2}, y = \dfrac{\pi}{2}$ and $y = \dfrac{3\pi}{2}$

b $y = \tan^{-1}(x^3 + C)$

10 a $y = 0$ **b** $y = C(x-1)^2$

c $y = (x-1)^2$

11 a $y = 1$ **b** $y = 1 + C\sec x$

c $y = 1 + \sqrt{2}\sec x$

12 a $y = \frac{1}{2}x$ **b** $y = 2\sqrt{x}$

c $y = \dfrac{4}{1+x^2}$ **d** $y = \frac{2}{x}$

e $y = e^{\sin x - 1}$ **f** $y = e^{1-\frac{2}{x}}$

13 b $y = C\log x$

14 b $y = \dfrac{4e^x}{(2+x)^2}$

15 a $1 + \cos 2x$

b $y^2 = \sin 2x + 2x + 2$

16 a $y' = u + xu'$

b i $u' = \dfrac{2+u}{x}$ **ii** $u = Cx - 2$

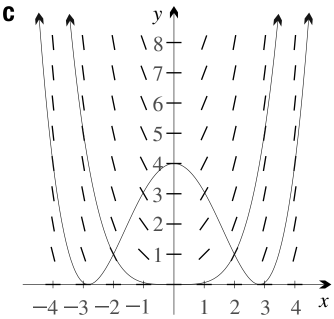
iii $y = Cx^2 - 2x$

17 a $\tan^{-1} y + \tan^{-1} x = C$

b Take the tangent of both sides and use the compound angle formula.

c $y = \dfrac{1-x}{1+x}$

18 c



d Substitute y_2 to get LHS $= \frac{1}{4}x(x^2 - 8)$ and RHS $= \frac{1}{4}x|x^2 - 8|$. These two expressions differ whenever $0 < |x| < 2\sqrt{2}$, as they have opposite sign. The correct solution is obtained by separation of variables.

19 a This is just a re-arrangement of the fundamental theorem of calculus.

b i $y = e^{-\frac{1}{2}x^2}$

ii $y_1(x) = 1 - \frac{1}{2}x^2$

$y_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4$

$y_3(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6$

$y_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8$

iii $e \doteqdot 2.7183$

Exercise 13D

1 a $y = 0$ **b** $\log|y| = -x + C$

c $y = Ae^{-x}$ **d** put $A = 0$

e $y = 2e^{-x}$

2 a $y = 0$ **b** $\dfrac{dx}{dy} = \dfrac{1}{3y}$

c $x = \frac{1}{3}\log|y| + C$ **d** $y = Ae^{3x}$ where A is a real number.

e put $A = 0$ **f** $y = -e^{3x}$

3 a $y = -3e^x$ **b** $y = e^{-2x}$

c $y = 2e^{-3x}$ **d** $y = -e^{2x}$

4 a $y = 2$

b $\log|y - 2| = -x + C$

c $y = 2 + Ae^{-x}$

d put $A = 0$

e $y = 2 + e^{-x}$

5 a $y = 1 + 2e^{-x}$ **b** $y = 1 - e^x$

c $y = -1 + 2e^{\frac{1}{2}x}$ **d** $y = 3 + e^{-2x}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

6 a $y = \dfrac{3}{1 - 6x}$ **b** $y = \dfrac{1}{1 + x}$

c $y = \tan x$ **d** $y = -\log(1 + x)$

e $y = \log(x - 2)$ **f** $y = \left(\dfrac{3 + x}{3}\right)^3$

7 a $y = Ae^{kx}$ **b** $y = 20e^{kx}$ **c** $k = -\log 2$

d $y = 20 \times 2^{-x}$, $y(3) = 2\frac{1}{2}$

8 a $y = Ae^{kx}$ **b** $y = 8e^{kx}$ **c** $k = \log \frac{3}{2}$

d $y = 8 \times \left(\frac{3}{2}\right)^x$, $y(4) = 40\frac{1}{2}$

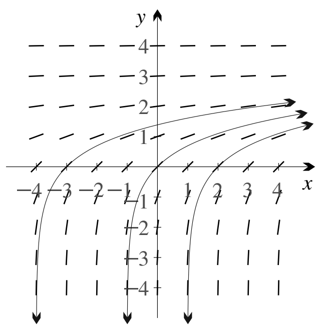
9 b $C = 0$

c ii $y = D \sin \frac{n\pi}{10} x$

10 a $y = \log(x + C)$

b Log curves with different x-intercepts.

c



iii Shift left or right.

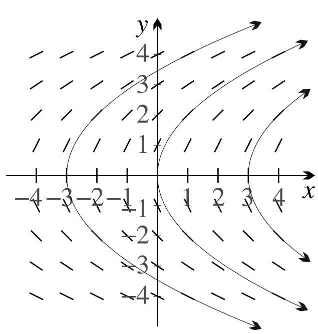
iv The isoclines are horizontal lines.

d $y = \log(x + e)$

11 a $y^2 = 4x + C$

b Concave right parabolas with vertex on the x-axis.

c



iii Shift left or right.

iv The isoclines are horizontal lines.

d $y^2 = 4x + 1$

12 a $\left(0, \frac{1}{2}\right)$ **iii** $\left(0, \frac{1}{2}\right)$

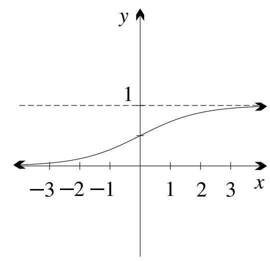
b $e^{-x} > 0$ for all x.

c 1 and 0

d $L' = \dfrac{e^{-x}}{(1 + e^{-x})^2}$ so $L' \neq 0$

e ii $L''(x) = -\dfrac{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})}{(e^{\frac{x}{2}} + e^{-\frac{x}{2}})^3}$

f



13 b i $y = 0$ and $y = 1$. **ii** $y = \dfrac{1}{1 + Be^{-x}}$

c $\log B$ to the right

d i yes: $y = 0$ **ii** yes: $y = 1$

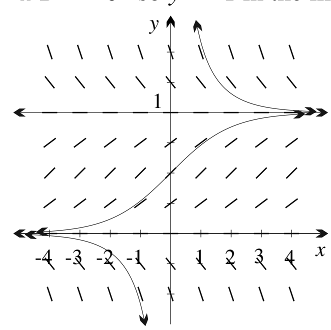
14 a $y = 0$ and $y = 1$

b $y = \dfrac{1}{1 + Be^{-rx}}$ **c** $B = \frac{1}{y_0} - 1$

e i $B \to \infty$ so $y = 0$ in the limit.

ii $B \to 0^+$ so $y = 1$ in the limit.

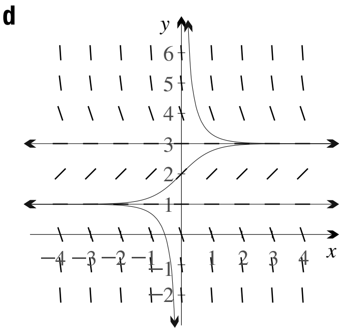
15



16 b i $y = 1$ and $y = 3$ **ii** $y = \dfrac{3 + Be^{-2x}}{1 + Be^{-2x}}$

c i $y = 3$ **ii** $y = 1$

d



e i $y'' = -2(1 - y)(2 - y)(3 - y)$

ii $(0, 2)$

17 b $v = 1 + Be^{-rx}$ **c** $y = \dfrac{1}{1 + Be^{-rx}}$

18 a $v' = 2(1 - v)$ **b** $v(0) = 0$

c $v = 1 - e^{-2x}$ **d** $y' = 1 - e^{-2x}$

e $y = \frac{1}{2} + x + \frac{1}{2}e^{-2x}$

19 a From the DE $f'(x) = g\big(f(x)\big)$ so by shifting $f'(x - C) = g\big(f(x - C)\big)$.

b Shift right by C. **c** horizontal lines

d If a graph is shifted right, its gradient at a given height is unchanged by the shift.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

20 a **ii** $v\dfrac{dv}{dy} = -y$ **b** $v^2 + y^2 = C$

c It represents a circle for $C = r^2 > 0$, a single point, the origin, when $C = 0$ and has no solutions when $C < 0$.

d $v = r\cos x,\ y = r\sin x$

e $y = r\sin(x - D)$

21 a $\cos^{-1} y = x + C$

b $C = 0$ so $\cos^{-1} y = x$

c LHS $= -\sin x$, RHS $= -\sqrt{1 - \cos^2 x} = -\sin x$

These are unequal when $\sin x < 0$.

d $y = \cos x$ with domain $0 \le x \le \pi$.

Exercise 13E

1 a **i** $a = -2,\ b = 1$ so that $y = x - 2x^2$

ii $a = -1,\ b = -1$ so that $y = -e^{-x}(\cos x + \sin x)$

iii $a = 1,\ b = -1$ so that $y = x - 1 + 3e^{-x}$

b **i** $a = \frac{1}{2},\ b = -2,\ c = -1$ so that $y = \frac{1}{2}x^2 - 2x - 1 + 4e^{-2x}$

ii $a = \frac{1}{4},\ b = \frac{5}{4},\ c = -\frac{1}{8}$ so that $y = \frac{1}{8}(2x^2 + 10x - 1) - \sin 2x$

c $\lambda = -3$ or -2 so that $y = 5e^{-3x}$ or $y = 5e^{-2x}$

2 a $R = Ae^{kt}$ **b** $A = 100$

c $k = -\frac{1}{4}\log 5$ **d** $\frac{4}{5}$ gram

3 a $H = 25 - Ae^{kt}$ **b** $H = 25 - 20e^{kt}$

c $k = -\frac{1}{10}\log 2$ **d** 43 min

4 a $\dfrac{dV}{dt} = k\pi r^2$, for some constant k.

b $\frac{dr}{dt} = \frac{1}{3}k$ **c** $r = \frac{1}{3}kt + 4$

d $k = -\frac{1}{4}$, so $r = 4 - \frac{1}{12}t$

e $V = \pi(4 - \frac{1}{12}t)^3$, for $0 \le t \le 48$

5 a $h(0) = 400$; the height decreases so $\dfrac{dh}{dt} < 0$.

b $h(t) = \frac{1}{4}(kt + 40)^2$

c $k = -1$ **d** 40 mins

e No: $h(t)$ increases for $t > 40$, which is impossible.

6 a $\dfrac{y}{x}$ **b** $\dfrac{dy}{dx} = \dfrac{y}{x}$ **c** $y = Cx$

d $x = 0$ because then y' undefined.

7 a $\dfrac{y}{x}$ **b** $\dfrac{dy}{dx} = -\dfrac{x}{y}$

c $x^2 + y^2 = r^2$

8 a y **b** $\dfrac{dy}{dx} = y$

c $y = Ae^x$ **d** $y = e^x$

9 a $P = P_0e^{kh}$ **b** $k = -\dfrac{\log 2}{4000}$

c $160\sqrt{2} \doteqdot 226$ kPa

10 a $A = (2x, 0),\ B = (0, 2y)$

b $\dfrac{dy}{dx} = -\dfrac{y}{x}$

c the hyperbola $xy = C$

11 a $C = S + Ae^{-kt}$

b $C = S + (C_0 - S)e^{-kt}$

12 b $N = \dfrac{1000}{1 + Be^{-1000kt}}$ **c** $N = \dfrac{1\,000}{1 + 24e^{-1000kt}}$

d 7.357×10^{-4} **e** 623

13 b $N = \dfrac{P}{1 + Be^{-kPt}}$

c $N = \dfrac{23.2 \times 187.5}{23.2 + 164.3e^{-187.5kt}}$

d $k = 1.702 \times 10^{-4}$ correct to four significant figures.

e $N(80) = 120.9$ million

f The mathematical model needs to be revised. Clearly the carrying capacity is much larger that the figure estimated in 1850.

14 a $N = \dfrac{P}{1 + Be^{-kPt}}$

c $k = \dfrac{1}{t_1P}\log\left(\dfrac{N_1(P - N_0)}{N_0(P - N_1)}\right)$

d $(P - N_1)^2N_2N_0 = (P - N_2)(P - N_0)N_1^2$

15 a It represents the harvest.

b $y(0) = 3$

d $y = \dfrac{12(1 - 3e^{-\frac{1}{3}t})}{1 - 9e^{-\frac{1}{3}t}}$

e $t = 3\log 3 \doteqdot 3.3$ years

f **i** $y(0) = 5$

ii $\lim\limits_{t\to\infty} y = 12$, that is, the fish population approaches a stable figure of 1200. This suggests the harvest should be stopped for one year to save the species.

16 a **i** $I = \displaystyle\int \frac{du}{u}$

ii $I = \log(u) + C = \log(\log x) + C$

b $N = e^{Ae^{kt}}$

17 $\dfrac{dM}{dt} = -$outflow $-$decay, that is

$\dfrac{dM}{dt} = -\dfrac{5M}{100} - \dfrac{10M}{100} = -\dfrac{3M}{20}$ with $M(0) = 200$,

hence $M = 200e^{-\frac{3}{20}t}$

18 a $h(t) = 20(1 + 4e^{kt})$ where $k = \frac{1}{10}\log\frac{3}{4}$

b **i** $\dfrac{dH}{dt} = k(H - 20 + t),\ H(0) = 80$

ii $\dfrac{dy}{dt} = ky + 1,\ y(0) = 60$

iii $y = -\frac{1}{k} + (60 + \frac{1}{k})e^{kt}$

$H(t) = 20 - \frac{1}{k} - t + (60 + \frac{1}{k})e^{kt}$

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

19 a $\dfrac{dy}{dt} = ry(1 - y),\ y(0) = y_0$ where $y_0 = \dfrac{N_0}{P}$

b $\dfrac{dy}{dx} = y(1 - y),\ y(0) = y_0$

c $v' = 1 - v$

d $v = 1 + Ae^{-x}$ so $y = \dfrac{1}{1 + Ae^{-x}}$

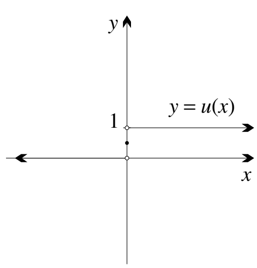
e $y = \dfrac{N_0}{N_0 + (P - N_0)e^{-x}}$

f $N = \dfrac{N_0 P}{N_0 + (P - N_0)e^{-kPt}}$

20 a The curves all have an asymptote $y = 0$ on the left and an asymptote $y = 1$ on the right, and the curves become steeper at $\left(0, \frac{1}{2}\right)$ as the value of r increases.

b **i** 1 **ii** $\frac{1}{2}$ **iii** 0

c



Chapter 13 review exercise

1 a 1st-order, linear, one arbitrary constant

b 2nd-order, two arbitrary constants

c 3rd-order, three arbitrary constants

2 a $y' = \frac{y}{x}(1 - x^2)$

b

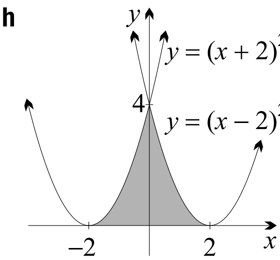
y \ x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	3	$\frac{4}{3}$	0	-3	*	3	0	$-\frac{4}{3}$	-3
$\frac{3}{2}$	$\frac{9}{4}$	$\frac{5}{4}$	0	$-\frac{5}{4}$	*	$\frac{5}{4}$	0	$-\frac{5}{4}$	$-\frac{9}{4}$
1	$\frac{3}{2}$	$\frac{5}{6}$	0	$-\frac{3}{2}$	*	$\frac{3}{2}$	0	$-\frac{5}{6}$	$-\frac{3}{2}$
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{12}$	0	$-\frac{5}{12}$	*	$\frac{5}{12}$	0	$-\frac{5}{12}$	$-\frac{3}{4}$
0	0	0	0	0	*	0	0	0	0
$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{5}{12}$	0	$\frac{5}{12}$	*	$-\frac{5}{12}$	0	$\frac{5}{12}$	$\frac{3}{4}$
-1	$-\frac{3}{2}$	$-\frac{5}{6}$	0	$\frac{3}{2}$	*	$-\frac{3}{2}$	0	$\frac{5}{6}$	$\frac{3}{2}$
$-\frac{3}{2}$	$-\frac{9}{4}$	$-\frac{5}{4}$	0	$\frac{5}{4}$	*	$-\frac{5}{4}$	0	$\frac{5}{4}$	$\frac{9}{4}$
-2	-3	$-\frac{4}{3}$	0	3	*	-3	0	$\frac{4}{3}$	3

e The lines $y = 0$, $x = 1$ and $x = -1$ are isoclines.

f $y = 0$

g Odd: y' is unchanged when x is replaced with $-x$ and y is replaced with $-y$.

h



4 a $y = 0$

b $\dfrac{dx}{dy} = \dfrac{-2}{y}$

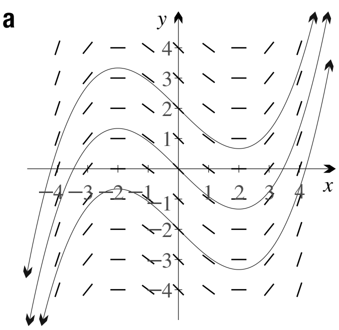
c $x = -2\log|y| + C$

d $y = Ae^{-\frac{1}{2}x}$ where A is a real number.

e put $A = 0$

f $y = 3e^{-\frac{1}{2}x}$

5 a

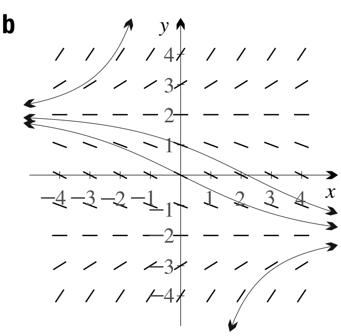


i $x = 2$ or $x = -2$

ii it is an isocline

iii decrease to -1 then increase

b



i $y = 2$ or $y = -2$

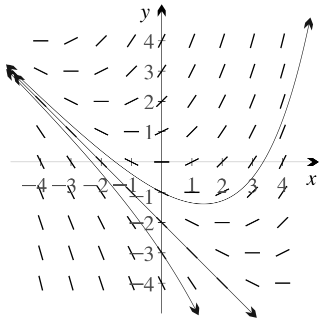
ii decrease to $-\frac{1}{2}$ then increase

iii it is an isocline

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c



i $y = -x$ **ii** decrease **iii** increase

7 a $y = \dfrac{C}{1 + x^2}$

b $(x - 1)^2 + (y + 2)^2 = C$

c $y = \dfrac{Ce^x}{x}$

8 a $y = 1 + e^{-\frac{1}{2}x}$ **b** $y = 5 - 3e^{-\frac{1}{5}x}$

9 b $y = C\sqrt{\left|\dfrac{1 + x}{1 - x}\right|}$

10 a $x \neq 0$; there are no intercepts.

b $\lim\limits_{x \to \infty} L(x) = 1$ and $\lim\limits_{x \to -\infty} L(x) = 0$

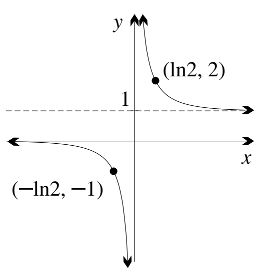
c $L(x) \to \infty$ as $x \to 0^+$ and $L(x) \to -\infty$ as $x \to 0^-$

d $(\log 2, 2)$, $(-\log 2, -1)$

e ii $L'' = \dfrac{(e^{\frac{x}{2}} + e^{\frac{x}{2}})}{(e^{\frac{x}{2}} - e^{\frac{x}{2}})^3}$

f concave up for $x > 0$, concave down for $x < 0$

g



11 C. $y' = \frac{1}{4}(x^2 + y^2)$

12 B. $y' = 1 + \dfrac{y}{x}$

13 a $y = \dfrac{2}{x}$

b $y = \log(x^2 + x - 1)$

c $y = \dfrac{-1}{1 + 2\sqrt{x}}$

14 b i $y = 0$ and $y = 1$.

ii $y = \dfrac{1}{1 + Be^{-x}}$ **iii** $y = \dfrac{1}{1 + 3e^{-x}}$

15 b i $y = 2$ and $y = 3$.

ii $y = \dfrac{3 - 4e^{-\frac{1}{5}x}}{1 - 2e^{-\frac{1}{5}x}}$ **iii** $x = 5\log\dfrac{4}{3} \doteqdot 1.44$

16 b $N = \dfrac{5}{1 + Be^{-5kt}}$ **c** $N = \dfrac{5}{1 + 4e^{-5kt}}$

d 4.418×10^{-2} **e** 233 000

17 a $y'' = 1 - 2y - 2xy''$

b $y' = 0$ and $y'' = -1$

c $y = \dfrac{1}{2}(1 + e^{-x^2})$

18 a The solutions are $y = F(x) + C$ where $F'(x) = f(x)$. Each is a vertical shift of the other.

b The solutions are $y = G^{-1}(x + C)$ where $G'(y) = \dfrac{1}{g(y)}$.

Each is a horizontal shift of the other.

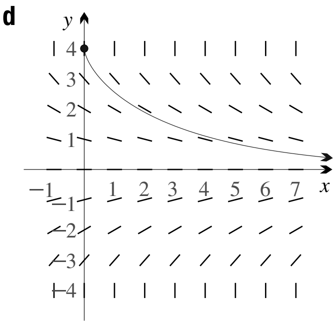
19 a $y = 1$ and $y = -1$

d $B = A + \dfrac{\pi}{2}$ (More precisely, $B = A + \dfrac{\pi}{2} + 2\pi\,\pi$, for some integer n.)

20 b $-4 \leq y \leq 4$ with $y(0) = 4$

c $y = 0$ does not satisfy $y(0) = 4$.

d



Chapter 14

Exercise 14A

1 a 300500 **b** 125

c i $d = -3$

ii $T_{35} = -2$

iii $S_n = \frac{1}{2}n(203 - 3n)$

2 a i $\frac{3}{2}$ **ii** 26375

iii $|r| = \frac{3}{2} > 1$

b i $\frac{1}{3}$

ii $|r| = \frac{1}{3} < 1$, $S_\infty = 27$

3 a \$96000, \$780000 **b** the 7th year

4 a $r = 1.05$

b Salary = \$124106, or \$1006232 if the ten salaries are first rounded and then added.

5 a i All the terms are the same.

ii The terms are decreasing.

b If $r = 0$, then $T_2 \div T_1 = 0$, so $T_2 = 0$. Hence $T_3 \div T_2 = T_3 \div 0$ is undefined.

c i The terms alternate in sign.

ii All the terms are the same.

iii The terms are $a, -a, a, -a, \ldots$.

iv The terms are decreasing in absolute value.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

6 a \$50 000, \$55 000, \$60 000, $d = \$5000$
b \$40 000, \$46 000, \$52 900, $r = 1.15$
c For Lawrence $T_5 = \$70000$ and $T_6 = \$75000$. For Julian $T_5 \doteqdot \$69960.25$ and $T_6 \doteqdot \$80454.29$. The difference in T_6 is about \$5454.

7 a i $T_n = 47000 + 3000n$
ii the 18th year
b \$71 166

8 a 12 metres, 22 metres, 32 metres
b $10n + 2$
c i 6 **ii** 222 metres

9 a 18 times **b** 1089 **c** Monday

10 a 85 000 **b** 40 000

11 a $D = 6400$ **b** $D = 7600$ **c** the 15th year
d $S_{13} = \$1092000$, $S_{14} = 1204000$

12 $r = \left(\frac{1}{2}\right)^{\frac{1}{4}}$, $S_\infty = \dfrac{F}{1 - \left(\frac{1}{2}\right)^{\frac{1}{4}}} \doteqdot 6.29F$

13 a i $-\frac{\pi}{4} < x < \frac{\pi}{4}$
ii $S_\infty = \cos^2 x$
iii When $x = 0$, the series is not a GP because the ratio $-\tan^2 x$ cannot be zero. But the series is then $1 + 0 + 0 + \cdots$, which trivially converges to 1. Because $\cos^2 x = 1$, the given formula for S_∞ is still correct.
b i $r = \cos^2 x$ **ii** $x = 0, \pi, 2\pi$
iv When $\cos x = 0$, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \cdots$, which trivially converges to 1. When $\cos x = 0$, then $\sin x = 1$ or -1, so $\operatorname{cosec}^2 x = 1$, which means that the given formula for S_∞ is still correct.
c i $r = \sin^2 x$ **ii** $x = \frac{\pi}{2}, \frac{3\pi}{2}$
iv When $\sin x = 0$, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \cdots$, which trivially converges to 1. When $\sin x = 0$, then $\cos x = 1$ or -1, so $\sec^2 x = 1$, which means that the given formula for S_∞ is still correct.

14 b at $x = 16$
c i at $x = 18$, halfway between the original positions
ii 36 metres, the original distance between the bulldozers

15 a 125 metres **b** 118.75 metres
d $a = 118.75$, $d = -6.25$, $\ell = 6.25$ and $n = 19$
e $2 \times S_{19} + 125 = 20 \times 125$, which is $2\frac{1}{2}$km.

Exercise 14B

1 a 5 **b** 14 **c** 3 **d** 15
e 4 **f** 8 **g** 14 **h** 11

2 a 13 **b** 10 **c** 8 **d** 8

3 a $\frac{T_3}{T_2} = \frac{T_2}{T_1} = 1.1$
b $a = 10$, $r = 1.1$
c $T_{15} = 10 \times 1.1^{14} \doteqdot 37.97$
d 19

4 a $r = 1.05$
b \$62 053, \$503 116
c the 13th year

5 the 19th year

6 a SC50: 50%, SC75: 25%, SC90: 10%
c 4 **d** at least 7

7 a $T_n = 3 \times \left(\frac{2}{3}\right)^{n-1}$
b 4.5 metres
c ii 16

8 a 2000 **b** 900 **c** 10 years

9 a the 10th year **b** the 7th year

10 a Increasing by 100% means doubling, increasing by 200% means trebling, increasing by 300% means multiplying by 4, and so on.
b Solve $(1.25)^n > 4$. The smallest integer solution is $n = 7$.

11 a $S_n = \dfrac{3\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}} = 9\left(1 - \left(\frac{2}{3}\right)^n\right)$
b The common ratio is less than 1. $S = 9$
c $n = 17$

12 a $\frac{1}{2}\cos\theta\sin\theta$ **b** $\sin^2\theta$

13 a $\frac{1}{\sqrt{n}}$
b No — the spiral keeps turning without bound.

Exercise 14C

1 a i \$900 **ii** \$5900
b i \$5166 **ii** \$17 166

2 a i \$5955.08 **ii** \$955.08
b i \$18 223.06 **ii** \$6223.06

3 a i \$4152.92 **ii** \$847.08
b i \$7695.22 **ii** \$4304.78

4 a \$507.89 **b** \$10 754.61

5 a $A_n = 10000(1 + 0.065 \times n)$
b $A_{15} = \$19750$, $A_{16} = \$20400$

6 a \$101 608.52 **b** \$127 391.48

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

7 a Howard — his is \$21 350 and hers is \$21 320.
b Juno — hers is now \$21 360.67 so is better by \$10.67.
8 a \$6050 **b** \$25 600 **c** 11 **d** 5.5%
9 a \$8000 **b** \$12 000 **c** \$20 000
10 \$19 990
11 a \$7678.41 **b** \$1678.41
c 9.32% per annum
12 a \$12 209.97 **b** 4.4% per annum
c Solve $10000 \times \left(\frac{1.04}{12}\right)^n > 15000$. The smallest integer solution is $n = 122$ months.
13 \$1 110 000
14 7.0%
15 a 21 years **b** 8 years and 6 months
c 14 times
16 3 years
17 Sid
18 a \$5250 **b** \$20 250
c 6.19% per annum
19 a \$40 988 **b** \$42 000
20 a i \$1120 **ii** \$1125.51
iii \$1126.83 **iv** \$1127.47
b amount $= 1000 \times e^{0.12} =$ \$1127.50
c annual compounding: \$3105.85,
continuous compounding: \$3320.12
21 a $A_n = P + PRn$
c P is the principal, PRn is the simple interest and $\sum_{k=2}^{n} {}^nC_k R^k$ is the result of compound interest over and above simple interest.

Exercise 14D

1 a i \$732.05 **ii** \$665.50 **iii** \$605
iv \$550 **v** \$2552.55
b i \$550, \$605, \$665.50, \$732.05
ii $a = 550, r = 1.1, n = 4$
iii \$2552.55
2 a i \$1531.54 **ii** \$1458.61
iii \$1389.15, \$1323, \$1260 **iv** \$6962.30
b i \$1260, \$1323, \$1389.15, \$1458.61, \$1531.54
ii $a = 1260, r = 1.05, n = 5$
iii \$6962.30
3 a i $\$1500 \times 1.07^{15}$
ii $\$1500 \times 1.07^{14}$
iii $\$1500 \times 1.07$
iv $A_{15} = (1500 \times 1.07) + (1500 \times 1.07^2) + \cdots + (1500 \times 1.07^{15})$
b \$40 332
4 a i $\$250 \times 1.005^{24}$
ii $\$250 \times 1.005^{23}$
iii $\$250 \times 1.005$
iv $A_{24} = (250 \times 1.005) + (250 \times 1.005^2) + \cdots + (250 \times 1.005^{24})$
b \$6390
5 a i $\$3000 \times 1.065^{25}$
ii $\$3000 \times 1.065^{24}$
iii $\$3000 \times 1.065$
iv $A_{25} = (3000 \times 1.065) + (3000 \times 1.065^2) + \cdots + (3000 \times 1.065^{25})$
c \$188 146 and \$75 000
6 b \$669 174.36 **c** \$429 174.36 **e** \$17 932.55
7 c iii 18
8 a \$200 000 **b** \$67 275 **c** \$630 025
d i $A_n = 100000 \times 1.1 \times ((1.1)^n - 1)$
iii 25
e $\frac{1000000}{630025} \times 10000 \doteqdot \$15872$
9 a \$360 **b** \$970.27
10 a \$31 680 **b** \$394 772 **c** \$1 398 905
11 a \$134 338 **b** \$309 281
12 \$3086
13 a \$286 593
b i \$107 355 **ii** \$152 165
14 a \$27 943.29 **b** the 19th year
15 a 18
16 The function FV calculates the value just after the last premium has been paid, not at the end of that year.
17 c $A_2 = 1.01M + 1.01^2M$,
$A_3 = 1.01M + 1.01^2M + 1.01^3M$,
$A_n = 1.01M + 1.01^2M + \ldots + 1.01^nM$
e \$4350.76 **f** \$363.70
18 b $A_2 = 1.002 \times 100 + 1.002^2 \times 100$,
$A_3 = 1.002 \times 100 + 1.002^2 \times 100 + 1.002^3 \times 100$,
$A_n = 1.002 \times 100 + 1.002^2 \times 100 + \cdots + 1.002^n \times 100$
d About 549 weeks If the balance were only accessible at the end of each week, however, the strict answer would then be 550 weeks

Exercise 14E

1 b i \$210.36 **ii** \$191.24 **iii** \$173.86
iv \$158.05 **v** \$733.51
c i \$158.05, \$173.86, \$191.24, \$210.36
ii $a = 158.05, r = 1.1, n = 4$
iii \$733.51

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

2 b **i** \$1572.21 **ii** \$1497.34

iii \$1426.04, \$1358.13, \$1293.46

iv \$7147.18

c **i** \$1293.46, \$1358.13, \$1426.04, \$1497.34, \$1572.21

ii $a = 1293.46, r = 1.05, n = 5$

iii \$7147.18

3 a **ii** $\$1646.92 \times 1.07^{14}$ **iii** $\$1646.92 \times 1.07^{13}$

iv $\$1646.92 \times 1.07$ **v** \$1646.92

vi $A_{15} = 15000 \times (1.07)^{15} - \big(1646.92 + 1646.92 \times 1.07 + \cdots + 1646.92 \times (1.07)^{13} + 1646.92 \times (1.07)^{14}\big)$

c \$0

4 a **i** $100000 \times 1.005^{240}$

ii $M \times 1.005^{239}$

iii $M \times 1.005^{238}$ and M

iv $A_{240} = 100000 \times 1.005^{240} - (M + 1.005M + 1.005^2M + \ldots + 1.005^{239}M)$

c The loan is repaid. **d** \$716.43

e \$171 943.20

5 a **i** 10000×1.015^{60}

ii $M \times 1.015^{59}$

iii $M \times 1.015^{58}$ and M

iv $A_{60} = 10000 \times 1.015^n - (M + 1.015M + 1.015^2M + \cdots + 1.015^{59}\mathrm{M})$

c \$254

6 a $A_{180} = 165000 \times 1.0075^{180} - (1700 + 1700 \times 1.0075 + 1700 \times 1.0075^2 + \cdots + 1700 \times 1.0075^{179})$

c $-\$10012.67$

7 a $A_n = 250000 \times 1.006^n - (2000 + 2000 \times 1.006 + 2000 \times 1.006^2 + \cdots + 2000 \times 1.006^{n-1})$

c \$162 498, which is more than half.

d $-\$16881$

f 8 months

8 c It will take 57 months, but the final payment will only be \$5490.41.

9 a The loan is repaid in 25 years.

c \$1226.64 **d** \$367 993

e \$187 993 and 4.2% pa

10 b \$345

11 a \$4202 **b** $A_{10} = \$6.56$

c Each instalment is approximately 48 cents short because of rounding.

12 b \$216 511

13 a \$2915.90 **b** \$84.10

14 a \$160 131.55

b $\$1633.21 < \$1650$, so the couple can afford the loan.

15 b zero balance after 20 years

c \$2054.25

16 \$44 131.77

17 b 57

18 c $A_2 = 1.005^2P - M - 1.005M,$
$A_3 = 1.005^3P - M - 1.005M - 1.005^2M,$
$A_n = 1.005^nP - M - 1.005M - \cdots - 1.005^{n-1}M$

e \$1074.65 **f** \$34 489.78

19 b $A_2 = 1.008^2P - M - 1.008M,$
$A_3 = 1.008^3P - M - 1.008M - 1.008^2M,$
$A_n = 1.008^nP - M - 1.008M - \cdots - 1.008^{n-1}M$

d \$136 262

e $n = \log_{1.008}\dfrac{125M}{125M - P}$, 202 months

Chapter 14 review exercise

1 a $a = 31, d = 13$ **b** 16 **c** 2056

2 a $\frac{1}{2}$ **b** $|r| = \frac{1}{2} < 1$ **c** $S_\infty = 48$

3 a $n = 11$ **b** $n = 99$ **c** $n = 228$ **d** $n = 14$

4 27 000 litres

5 ‘Increasing by 2000%’ means that the profit is 21 times larger. The smallest integer solution of $(1.14)^n > 21$ is $n = 24$.

6 a $r = 1.04$ **b** \$49 816, \$420 214

7 a $T_n = 43000 + 4000n$ **b** 2017

8 2029

9 a \$15 593.19 **b** \$3593.19 **c** 5.99%

10 a $\$25000 \times (0.88)^4 \doteqdot \$14992$

b \$2502 per year

c $\$25000 \div (0,88)^4 \doteqdot \$41688$

d \$4172 per year

11 b \$224 617.94 **c** \$104 617.94

d The value is \$277 419.10, with contributions of \$136 000.00.

13 a $A_{180} = 159000 \times 1.005625^{180} - (1415 + 1415 \times 1.005625 + 1415 \times 1.005625^2 + \cdots + 1415 \times 1.005625^{179})$

c $-\$2479.44$ **d** \$1407.01

14 a $A_n = 1700000 \times 1.00375^n - (18000 + 18000 \times 1.00375 + 18000 \times 1.00375^2 + \cdots + 18000 \times 1.00375^{n-1})$

c \$919 433, which is more than half.

d $-\$57677.61$ **f** 3 months

15 a $\$31.1133 \times 3000 \doteqdot \$93340$

b $100000 \div 31.1133 \doteqdot \$3214$

16 a $600000 \div 15.6221 \doteqdot \$38407$

b $15.6221 \times 60000 \doteqdot \$937326$

17 a $30.1075 \times 2000 \doteqdot \$60215$

b $11.251 \times 2000 \doteqdot \$22510$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Chapter 15

Exercise 15A

1 a categorical

b numeric and continuous. But 'height correct to the nearest mm' is numeric and discrete.

c numeric and continuous. But 'age in years' is numeric and discrete.

d categorical by party or political code. This would need to be defined carefully — if a person can be affiliated to two parties, it would not be a function.

e categorical

f categorical

g numeric and discrete

h Shoe sizes are often arranged into categories.

i These are frequently integers from 1–100, that is, numeric and discrete. If results are reported by a grade, for example, A, B, C, . . . , this might be considered categorical.

2 a median 14, mode 14, range 8

b median 10, every score is trivially a mode, range 12

c median 8, mode 3, range 12

d median 6.5, mode 4 & 6, range 6

e median 4, mode 4, range 7

f median 5.5, mode 2 & 3 & 9, range 8

3 a

score x	1	2	3	4	5	6	7	8
frequency f	4	3	4	2	1	1	1	6
cumulative	4	7	11	13	14	15	16	22

b 3.5

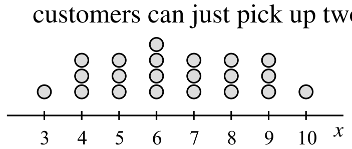
c 8

d i This is a median, but it might be more useful to use the mode in this case. It may be easier to develop a square box for four cupcakes rather than three.

ii See the previous comments. It is also common for sales to package a larger box to encourage customers to overbuy.

iii This is the mode, but if a box of four is marketed, customers can just pick up two boxes of four.

4 a



b

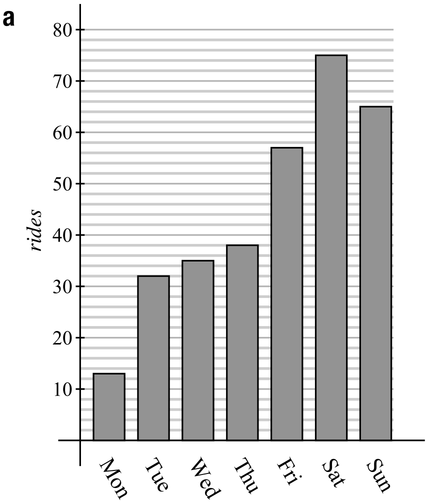
score x	3	4	5	6	7	8	9	10
frequency f	1	3	3	4	3	3	3	1
cumulative	1	4	7	11	14	18	20	21

c 6 hoops

d 6.5 hoops

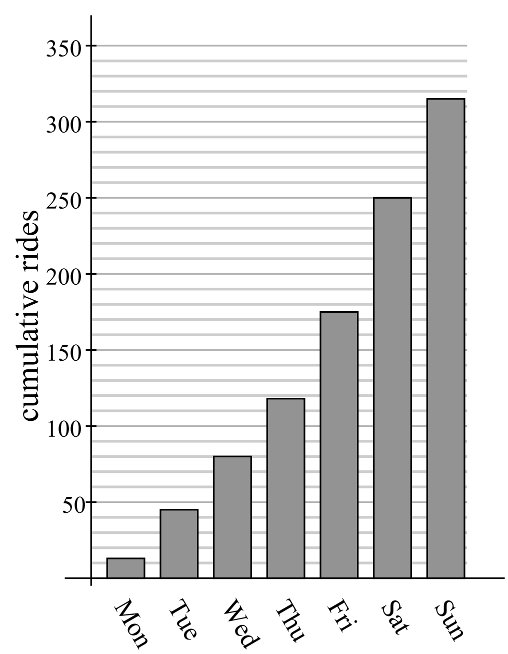
e Not really. If the scores are ordered by time, his scores improve over the sessions. This information is lost in the table and plot.

5 a



b

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
frequency	13	32	35	38	57	75	65
cumulative	13	45	80	118	175	250	315



6 a Blond hair and blue eyes. Different results might be expected in a different part of the world.

b Red hair and green eyes

c 45%

d 17%

e $25 \div 54 \doteqdot 46\%$

f $90 \div 247 \doteqdot 36\%$

g $671 \div 753 \doteqdot 89\%$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

h These two results would suggest so. Geneticists link this to various pigment genes that affect both characteristics.

i The proportion of the various eye and hair colours will vary in different genetic populations and ethnic groups. Studies such as this may be done with a relatively non-diverse population to prevent the clouding effects of differing genetics.

7 a 80

b

salad	pie	soup	panini	burger
32.5%	12.5%	8.75%	20%	26.25%

c

salad	pie	soup	panini	burger
\$130	\$60	\$70	\$96	\$168

d \$524

e It returns more money than the more popular pie option. It is probably also important for the café to include a vegetarian option on the menu to cater for such customers or for groups with such customers.

8 a In 2002 the price was \$400 thousand, and in 2017 it was \$1 million.

b Prices increased by 150% .

c \$40 thousand per year

d They will increase another $13 \times \$40\,000 = \$520\,000$ to around \$1.5 million.

e From 2014 to 2015, median house prices increased \$120 thousand.

f From 2010 to 2011, median house prices decreased \$40 thousand.

9 a 35%, 140 dogs **b** 11%

c 75% **d** 15%

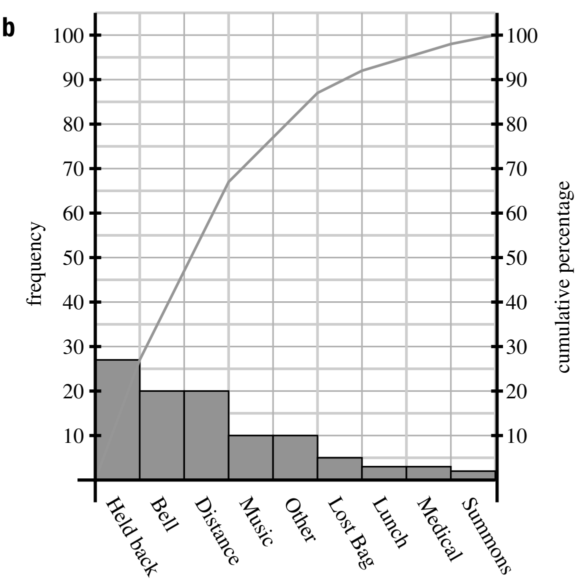
e This is quite a large category, and it may be that more investigation should be done to see if there were any other popular types of pets lumped into this category.

f Some pets may require more care and attention. For example, dogs may require frequent exercise and attention. This may give an opportunity for 'value adding' if owners are willing to pay for it. They should also consider what other pet boarding facilities are in the area, because it may be better to pick up a niche market, not covered by other pet boarding houses. Some pets may also be able to use the same types of accommodation, for example, rabbits and guinea pigs.

10 a

Reason	frequency	cumulative
Held back	27	27
Bell	20	47
Distance	20	67
Music	10	77
Other	10	87
Lost bag	5	92
Lunch	3	95
Medical	3	98
Summons	2	100

b



c The categories are arranged in descending order, so the function will be increasing (if every frequency is greater than zero), but by less at nearly every stage, causing it to curve downwards.

d 67%

e Remind teachers to release students promptly, increase the volume of the bell or the number of locations where the bell sounds, timetable students in rooms closer together where possible.

11 a 6% **b** 64% **c** 5%

d Care is needed when the graph is read in a hurry. Compare this with the Pareto chart later in this exercise where both axes are the same scale.

12 a The vertical origin is not at a 0% unemployment rate. This exaggerates the scale of the graph, which only shows a variation of 0.25%. This is still potentially significant, but it is only shown over a four-month period, so it is impossible to examine long-term trends. There are natural cycles — for

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

example, there may be a rise when school pupils enter the employment market, and a drop when Christmas provides short-term retail employment. January may be a low point in economic indicators, before businesses return from holidays and begin to hire staff.

b There has been a significant increase over this five-year period, but more questions need to be asked by someone viewing the graph. What does the vertical scale represent — is it spending per citizen or spending per household? If it is per household, have the household structures changed over the period, such as more larger households? Is this a small community, in which case the data won't be very robust to changes in population? Is the data collected from sales at local shops, and does it include tourists and people passing through — has there been an increase in tourism, and was the data collected at the same time of year (more takeaways may be sold at the height of the tourist season)? What is included in the category of 'takeaway food' — if this is a health study, takeaway salads may be considered healthier than takeaway burgers (which the graphic is trying to suggest). Finally, note that the eye interprets the increase by the size of the graphic, but in fact it is the height that holds information, suggesting a greater increase than was actually the case.

c **i** People who do not have access to the internet, or do not feel as comfortable accessing and filling in an online survey, will not be represented. This may be more prevalent amongst older demographics.

ii The group should look at other hospitals, unless they particularly want to investigate the change in costs at their local hospital. Hospital costs could be influenced by government policy increasing the staffing numbers at the hospital, by purchase of new expensive diagnostic equipment, by opening and closing particular hospital wards (possibly relocating them to other hospitals), by quality control improvements, by industrial action of staff, and so on. The group likely will want to investigate the cause of any changes to overall expenses and may want to produce graphs of particular expenses, such as doctors' fees. They need to be clear what questions they actually want to ask — for example, are they concerned that medical treatment is getting more expensive for certain sections of the community who cannot afford it?

13 a 58%

b Around 3 billion

c About 0.92 billion

d 5.2%

e It may be of some use if choosing a major world language is a consideration, but there are often other considerations in deciding what language to learn. For example, you may have relatives who speak Malay, or a girl-friend who is French, or you may want to learn Japanese because of Japan's importance to Australia's economy. Others learn languages for academic reasons, such as Latin because of its historical and linguistic importance, or Russian to study Russian literature. When deciding a languageon the number of speakers, it is probably more useful to consider the total number of speakers, not merely those who speak it as a first language — close to a billion people speak English, but only a third of them do so as a first language.

14 a **i** 15°C **ii** 30°C

b **i** 17°C **ii** 23°

c Around 6−7°C in December–January

d September and May

e November–February

f June–August

h Colour-blind readers may find the colours difficult to distinguish. Using dashes and colour also provides two visuals cues for the bulk of readers, making the graph easier to read.

15 a 30

b 73% and 27%

c Bill on Essay writing, Claire on Interpretation, Ellie on all sections.

d 40%

e 50%

f Aaron and Dion. Notice that Claire has not reached 50% in the Interpretation section.

16 a Provided that similar levels of postgraduates survive to the 55-64 age bracket.

b 61.1% **c** 21.6%

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 15B

1 a $\bar{x} = 7$, Var $= 3.6$, $s \doteqdot 1.90$

x	f	xf	$(x - \bar{x})^2$	$(x - \bar{x})^2 f$
3	1	3	16	16
5	1	5	4	4
6	1	6	1	1
7	3	21	0	0
8	2	16	1	2
9	1	9	4	4
10	1	10	9	9
Total	10	70		36

b $\bar{x} = 7$, Var $= 3.6$, $s \doteqdot 1.90$

x	f	xf	$x^2 f$
3	1	3	9
5	1	5	25
6	1	6	36
7	3	21	147
8	2	16	128
9	1	9	81
10	1	10	100
Total	10	70	526

2 a $\bar{x} = 18$, $s \doteqdot 3.67$ **b** $\bar{x} = 7$, $s \doteqdot 3.06$
c $\bar{x} = 55$, $s \doteqdot 7.58$ **d** $\bar{x} = 11$, $s \doteqdot 1.88$

3 a $\bar{x} \doteqdot 7.17$, $s \doteqdot 3.18$ **b** $\bar{x} = 5.7$, $s \doteqdot 1.73$
c $\bar{x} = 3.03$, $s \doteqdot 0.94$ **d** $\bar{x} \doteqdot 42.88$, $s \doteqdot 10.53$

4 a 34 **b** $\mu \doteqdot 3.26$, $\sigma \doteqdot 1.75$

c

class	0–2	3–5	6–8
centre	1	4	7
freq	12	18	4

d $\mu \doteqdot 3.29$, $\sigma \doteqdot 1.93$

e Information is lost when data are grouped, causing the summary statistics to change.

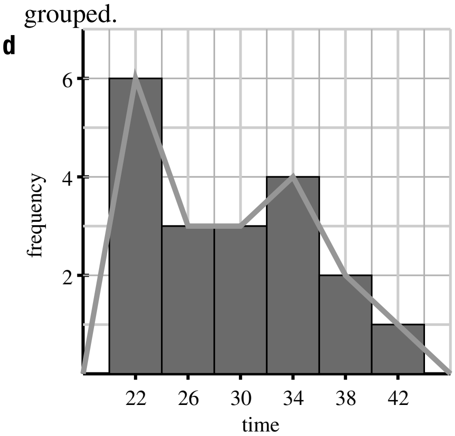
5 a 29.5

b

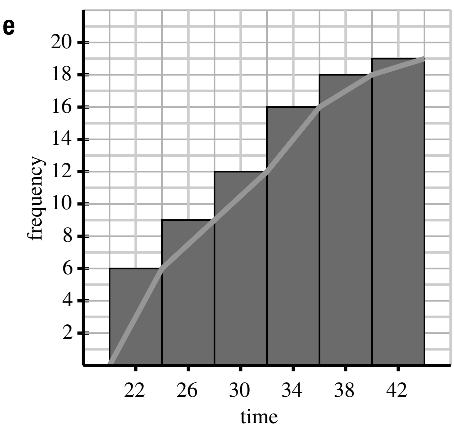
class	20–24	24–28	28–32	32–36	36–40	40–44
centre	22	26	30	34	38	40
freq	6	3	3	4	2	1
c.f.	6	9	12	16	18	19

c 30. No, because information is lost when the data are grouped.

d



e



6 a

x	152	154	155	157	158	159	162	163
f	1	2	1	1	2	3	2	3

x	164	165	166	168	170
f	2	2	3	1	1

b 162.5

c Trends are less clear when the data are not grouped, because it is less visually clear that the data are falling in certain zones on the domain.

d

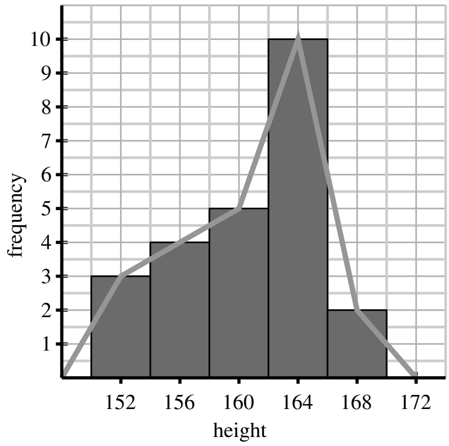
group	150–154	154–158	158–162	162–166	166–170
centre	152	156	160	164	168
freq	3	4	5	10	2

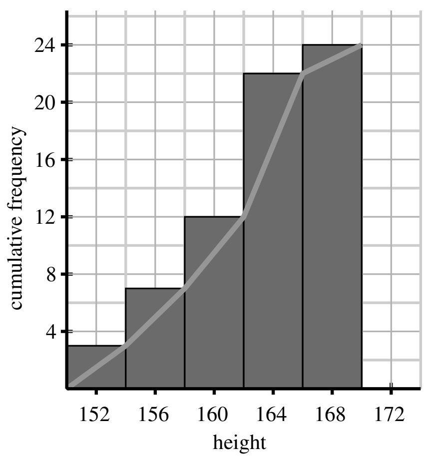
e 162

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

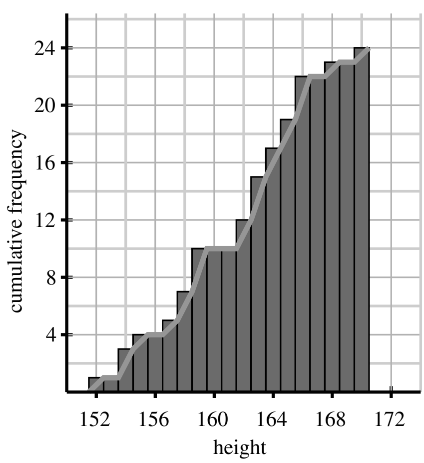
Cambridge University Press
Updated July 2022

f





h



i The cumulative frequency polygon and ogive are much less sensitive to the grouping process than the frequency histogram and ogive. The graphs in parts (g) and (h) look very similar in shape.

7 a **i** 14.3 (1 decimal place)
ii 13.7 (1 decimal place)
iii 13.6 (1 decimal place)
b 0.005%

Exercise 15C

1 a mean 6.9, median 8, mode 8, range 10
b mean 21.4, median 22.5, mode 12, range 18
c mean 5.2, median 5.5, mode 7, range 5
d mean 62.3, median 61, trimodal: 54, 61, 73, range 19

2 a $Q_1 = 7, Q_2 = 13, Q_3 = 17$, IQR $= 10$
b $Q_1 = 12.5, Q_2 = 18.5, Q_3 = 25.5$, IQR $= 13$
c $Q_1 = 7.5, Q_2 = 11, Q_3 = 18$, IQR $= 10.5$
d $Q_1 = 5, Q_2 = 8.5, Q_3 = 13$, IQR $= 8$
e $Q_1 = 4, Q_2 = 7, Q_3 = 13$, IQR $= 9$
f $Q_1 = 10, Q_2 = 15, Q_3 = 21$, IQR $= 11$
g $Q_1 = 5, Q_2 = 9, Q_3 = 13.5$, IQR $= 8.5$
h $Q_1 = 12, Q_2 = 14, Q_3 = 18$, IQR $= 6$

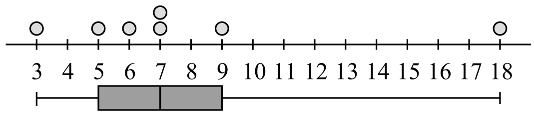
3 a $Q_1 = 4, Q_2 = 12, Q_3 = 16$, IQR $= 12$
b $Q_1 = 1, Q_2 = 6.5, Q_3 = 11$, IQR $= 10$
c $Q_1 = 7, Q_2 = 9, Q_3 = 12$, IQR $= 5$
d $Q_1 = 2.5, Q_2 = 5, Q_3 = 7$, IQR $= 4.5$
e $Q_1 = 7, Q_2 = 7, Q_3 = 10$, IQR $= 3$
f $Q_1 = 4, Q_2 = 5, Q_3 = 9$, IQR $= 5$
g $Q_1 = 2.5, Q_2 = 4, Q_3 = 9.5$, IQR $= 7$
h $Q_1 = 4.5, Q_2 = 9, Q_3 = 12$, IQR $= 7.5$

4 a Answers may differ here, but 40 and 92 are likely.
b 40, 54, 59, 69, 92
c IQR $= 15$, $Q_1 - 1.5 \times$ IQR $= 31.5$ and $Q_3 + 1.5 \times$ IQR $= 91.5$. Thus 92 is the only outlier by the IQR criterion.
d Some may identify 40 as an outlier by eye — this shows the advantage of plotting values, where it becomes evident that this score is well separated from other scores. A student receiving 40 in this cohort should be noted as someone needing extra attention and assistance.
e **i** 54, 60, 70.5, IQR $= 16.5$
ii 53.5, 58, 68.5, IQR $= 14.5$
iii 54, 59, 68.5, IQR $= 14.5$
f In this case, with a reasonably sized dataset, the middle of the data is fairly stable and removing an extreme value has only a small effect on the quartiles and IQR. With a large dataset and tightly clustered values in the middle two quarters of the data, the difference would be even smaller.
g **i** 60.8, 11.1 **ii** 61.6, 10.5
iii 59.5, 9.4 **iv** 60.3. 8.7
h 2.4 is 22% of 11.1. Any deviation from the mean is exaggerated by the standard deviation because the deviation from the mean is squared when calculating the variance.

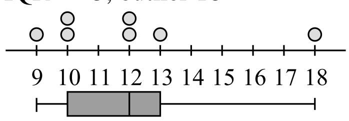
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

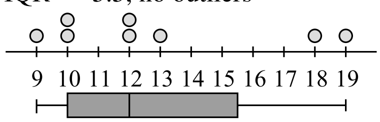
5 a i IQR $= 4$, outlier 18



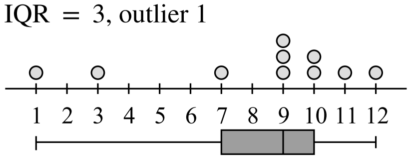
ii IQR $= 3$, outlier 18



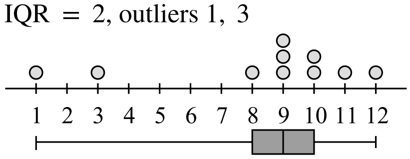
iii IQR $= 5.5$, no outliers



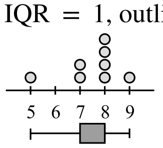
iv IQR $= 3$, outlier 1



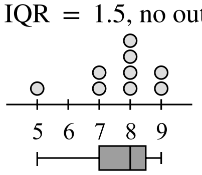
v IQR $= 2$, outliers $1,\ 3$



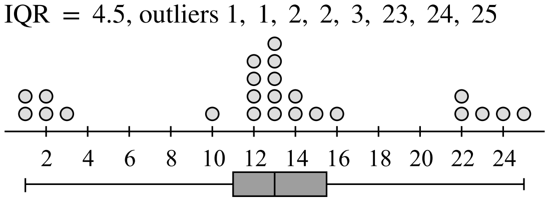
vi IQR $= 1$, outlier 5



vii IQR $= 1.5$, no outliers



viii IQR $= 4.5$, outliers $1,\ 1,\ 2,\ 2,\ 3,\ 23,\ 24,\ 25$



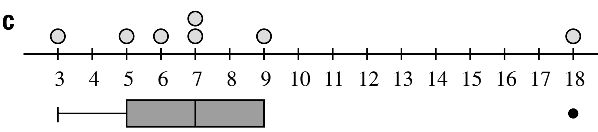
b It must be noted that some of the pathologies in these examples come about because of the small datasets. Statistics is always more accurate and reliable with a large dataset.

Generally the definition picks up the values that appear extreme on the dot plots. Notably (in these small datasets), it picks up single extreme values — if more values are a long way from the mean, they may not be marked as outliers.

Datasets with a small IQR may need a closer inspection — in part (vi) and (vii), the value at 5 is not so extreme and the datasets are not so different, yet in one case it is marked as an outlier, but in the other it is not. The final dataset has a very tight subset of data between the Q_1 and Q_3, giving a small interquartile range. This definition of outliers gives 8 values in 24 (one third of the data) as outliers. Furthermore, 23–25 are outliers, but 22 is not. The issue here is the unusual shape of the distribution.

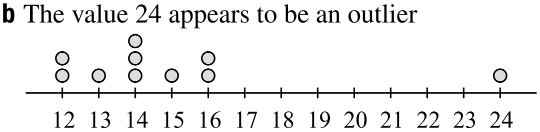
Rules such as this IQR criterion for outliers should be an invitation to inspect the values that have been flagged more closely, rather than following a rule blindly.

c



6 a $\bar{x} = 15,\ s = 3.29$

b The value 24 appears to be an outlier



c IQR $= 3$ and $Q_3 = 16$.
Because $24 > 16 + 1.5 \times 3$, this definition also labels 24 an outlier.

d $\bar{x} = 14,\ s = 1.41$

e This does not have much effect on the mean, but it has a big percentage effect on the standard deviation — removing the outlier more than halves the standard deviation. The operation of squaring $(x - \bar{x})$ means that values well separated from the mean have an exaggerated effect on the size of the variance.

f No effect at all!

g If there are significant outliers, or at least values spread far from the mean, this can have a big influence on the IQR. The IQR is a good measure if you are more interested in the spread of the central 50% of the data.

7 a Emily got less than 62

b Around 50% (and no more than 50%)

c The mathematics results were more spread out, and the centre of the data (by median) was 5 marks higher. The interquartile range of both distributions, however, was the same. Clearly the mathematics cohort has some students who perform much more

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

strongly, and others who perform much weaker, than the majority of their peers.

d Xavier was placed in the upper half of the English cohort, but in the lower half of the mathematics cohort. The English result was thus more impressive.

e **i** 45

ii The bottom 25% of English scores show a spread of 6 marks (51–57). The bottom 25% of mathematics scores show a spread of 8 marks (53−61). The spread of the lower half is now much more comparable.

8 a The results are not paired. Just becausey Genjo received the lowest score in the writing task does not mean that he received the lowest score in the speaking task. Thus we cannot answer the question, although we might make conjectures, given that Genjo is obviously struggling significantly with English.

b **i** mean 66.1, median 68, range 56

ii IQR $= 73 - 60 = 13$, 91 and 35 are outliers.

c **i** mean 64.4, median 65.5 , range 56

ii $IQR = 71 - 57.5 = 13.5$, 37 and 93 are outliers.

d It is difficult to say. Students have found the second task more challenging, evidenced by the lower mean and median. This could be due to the construction of the task, or simply because it is a type of task that some students find more difficult.

Exercise 15D

1 a **i** height **ii** weight

b **i** radius **ii** area. It is natural to think that the area of the circle is determined by the radius chosen when it is drawn, but mathematically we could write $r = \sqrt{\dfrac{A}{\pi}}$, reversing the natural relationship.

c **i** weight **ii** price. Note that the price may change when meat is bought in bulk, so there is a deeper relationship between these two quantities than simply price $=$ weight $\times$ cost per kg.

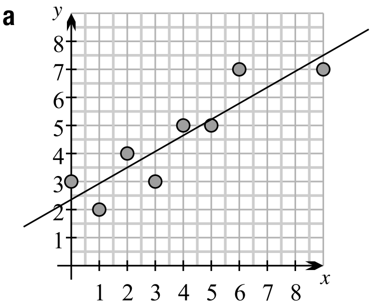
d **i** world rank **ii** placing

e **i** temperature **ii** power consumption. Power consumption increases with the use of air conditioners (higher temperatures) or heaters (colder weather).

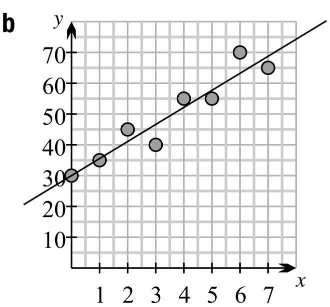
f It is natural to take x as the independent variable and y as the dependent variable. Note in this case the relationship cannot naturally be reversed, because there are multiple x-values resulting from the same y-value.

2 a strong positive
b virtually none.
c strong negative
d strong negative
e moderate positive
f weak positive
g strong negative
h strong positive
i moderate negative

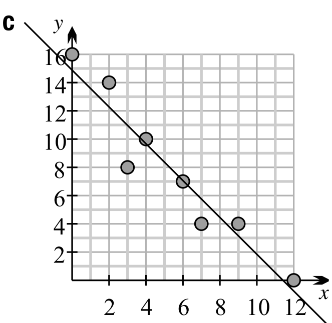
3 a



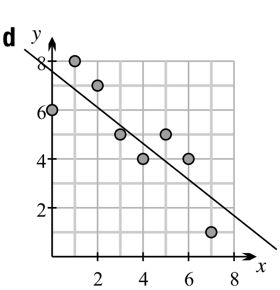
b



c



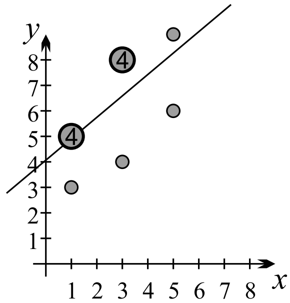
d



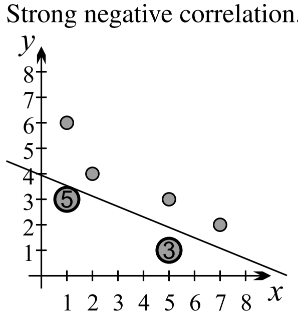
Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 a Strong positive correlation.



b Strong negative correlation.



5 a A quadratic relationship (a parabola).

b A square root. **c** A hyperbola.

d A circle. **e** An exponential.

f No obvious relationship.

6 a **ii** 6 L and 10 L

b $V = 2t$

c The V-intercept is zero. In no minutes, zero water will flow through the pipe.

d This is the flow rate of the water, 2 L/s.

e Negative time makes little sense here, because he cannot measure the volume of water that flowed for say -3 minutes.

f Experimental error could certainly be a factor, but it may simply be that the flow rate of water is not constant. It may vary due to factors in, for example, the pumping system.

g 60 L. The extrapolation seems reasonable provided that the half-hour chosen is at about the same time of day that he performed his experiment.

h 22.5 minutes

i Yasuf's experiments were all carried out in a period of several hours during the day. It may be that the flow rate changes at certain times of the day, for example, at peak demands water pressure may be lower and the flow rate may decrease. The flow rate may also be different at night — for example, the water pump may only operate during the day. More information and experimentation is required.

7 a 1000

b **i** $P = 0.9t + 5$

ii It looks fairly good.

iii Predicted $P = 13.1$, actual $P = 15.4$, so the error was 230 people.

c **i** The new model predicts $P = 16.4$, so it is certainly much better.

ii Population is growing very strongly in Hammonsville. Investigators should be looking into the cause of the growth, which may change over the next few years. For example, it may be due to a short-term mining boom. Eventually there may be other constraining factors, such as available land for housing.

d Extrapolation can be dangerous. Provided, however, that the independent variable is constrained to a small enough interval, linear predictions may well have validity. This is the idea behind calculus, where curves are approximated locally by a tangent.

8 a 99 in assessment 1, 98 in assessment 2. They were obtained by the same student, but another student also got 99 in assessment 1.

b 27 in assessment 1, 33 in assessment 2. They were the same student.

c Students getting below about 77 marks in assessment 1 do better in assessment 2, students above 77 marks in assessment 1 get a lower mark in assessment 2, according to the line of best fit. Perhaps the second assessment started easier, but was harder at the end.

d **i** 50 **ii** 65 **iv** 26

v A negative score! Clearly the model breaks down for small scores.

e $y = 0.74x + 20$

f A more accurate method would incorporate data from more than one assessment task in estimating their missing score. This is a question better tackled using standard deviation and the techniques of the next chapter.

9 a The maximum vertical difference between a plotted point and the line of best fit is about 0.8 s^2.

b It could be experimental error. For example, the string could have been twisted or released poorly, the experiment could have been incorrectly timed, or there could have been a recording error.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

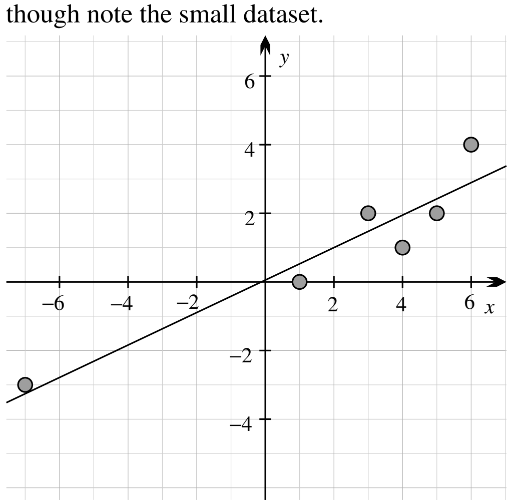
c They may have measured 10 periods and then divided by 10 before recording the length of one period. Errors could then arise if the motion was *damped*, that is, if the pendulum slowed down significantly over a short time period.

d By this model, $T^2 = \dfrac{4\pi^2}{g}L \doteqdot 4.03L$. These results are in pretty good agreement with the theory.

Exercise 15E

1 a There appears to be a fairly strong correlation, though note the small dataset.

b



c

							Sum
x	-7	1	3	4	5	6	12
y	-3	0	2	1	2	4	6
$x - \bar{x}$	-9	-1	1	2	3	4	0
$y - \bar{y}$	-4	-1	1	0	1	3	0
$(x - \bar{x})^2$	81	1	1	4	9	16	112
$(y - \bar{y})^2$	16	1	1	0	1	9	28
$(x - \bar{x})(y - \bar{y})$	36	1	1	0	3	12	53

d $(\bar{x}, \bar{y}) = (2, 1)$

e See above

f $53 \div \sqrt{112 \times 28} = 0.95$

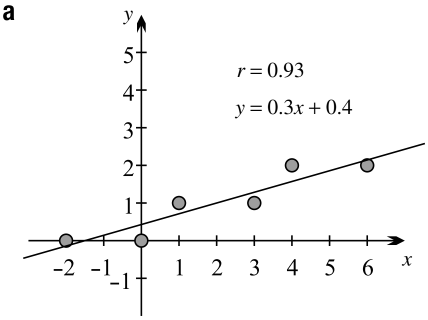
g It is a good fit.

h $53 \div 112 = 0.47$

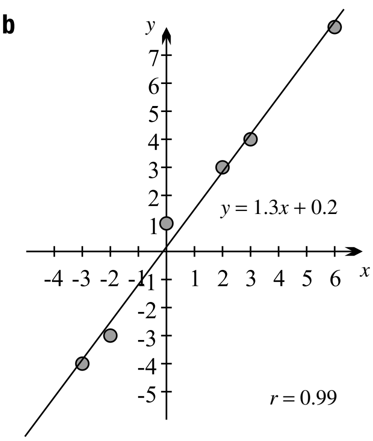
i $b = 1 - 0.47 \times 2 = 0.06$

j $y = \frac{1}{2}x + 0.$

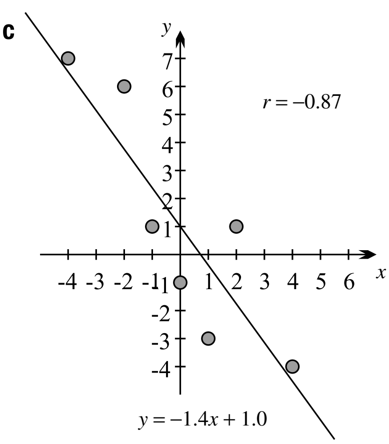
2 a



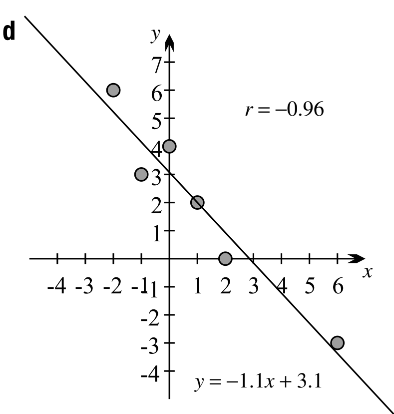
b



c



d



Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Exercise 15F

1 a $r = 0.96,\ y = 0.96x + 0.47$

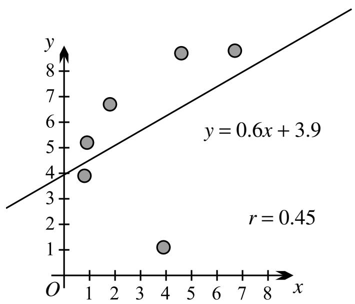
b $r = 0.79,\ y = 0.45x + 2.6$

c $r = -0.86,\ y = -1.05x + 8.75$

d $r = -0.53,\ y = -0.41x + 4.70$

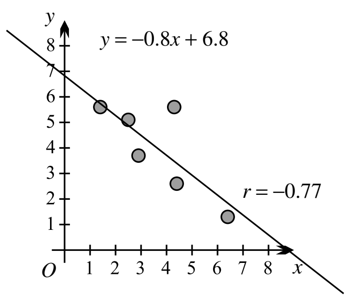
e $r = 0.96,\ y = 1.38x + 0.75$

2 a $r = 0.45,\ y = 0.58x + 3.94$



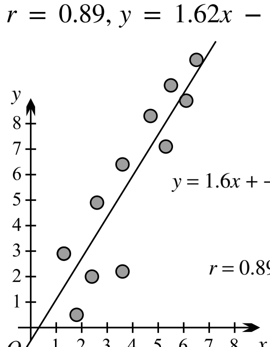
If the outlier at $(3.9,\ 1.1)$ is removed, then $r = 0.91$, $y = 0.75x + 4.43$.

b $r = -0.77,\ y = -0.78x + 6.83$



If the outlier at $(4.3, 5.6)$ is removed, then $r = -0.97,\ y = -0.89x + 6.79$.

c $r = 0.89,\ y = 1.62x - 0.51$



If the outlier at $(3.6,\ 2.2)$ is removed, then $r = 0.93$, $y = 1.61x - 0.19$.

3 Because the dataset was larger, the effect of the single outlier was mitigated by the other data points.

4 a Dataset 1:

i $y = 1x + 1.4,\ r = 0.86$

ii $y = 0.8x + 1.9,\ r = 0.79$

Dataset 2:

i $y = 0.7x + 3.0,\ r = 0.76$

ii $y = 0.7x + 2.5,\ r = 0.82$

b In all cases the correlation is strong. In part **a**, the repeated point has strengthened the correlation, but in the second example it has weakened it. Note that a strong correlation doesn't indicate that the data are correct. In part **a**, for example, leaving out 4 of the 9 points still gave a strong correlation, but a very different equation of line of best fit.

c The effect is less in the larger dataset, as expected. The gradient is unchanged (correct to one decimal place) and the y-intercept only differs by 20%, rather than by 26%. In a larger (more realistically sized) dataset, the effect would likely be less again. The effect of the repeated point will also depend on its place on the graph (central versus on the extremes of the data) and how close it is to the line of best fit.

Chapter 15 review exercise

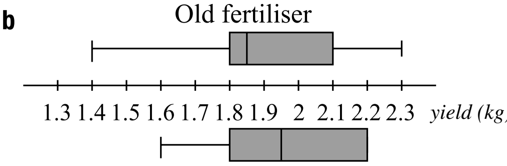
1 a mean 5, median 4.5, mode 4, range 8

b mean 15, median 15, mode 15 and 16 (it is bimodal), range 7

2 a Old fertiliser: 1.8, 1.85, 2.1,

New fertiliser: 1.8, 1.95, 2.2

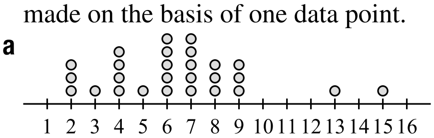
b Old fertiliser



New fertiliser

c The fertiliser does appear to increase his yield — the median yield has increased by 100 g. Probably more more data are required because the lower quarter (0–25%) shows an increase, but the maximum has reduced. These claims, however, are each being made on the basis of one data point.

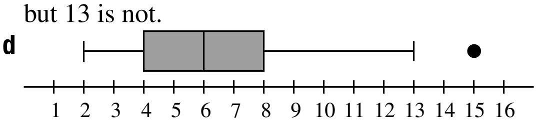
3 a



b By eye, 13 and 15 look like outliers.

c IQR $= 4$. By the IQR criterion, 15 is an outlier, but 13 is not.

d



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

4 a mean $\doteqdot 11.82\ s$, standard deviation $\doteqdot 0.537$ s

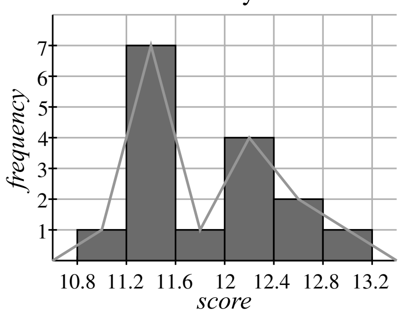
b

group	10.8–11.2	11.2–11.6	11.6–12.0
centre	11.0	11.4	11.8
freq	1	7	1

group	12.0–12.4	12.4–12.8	12.8–13.2
centre	12.2	12.6	13.0
freq	4	2	1

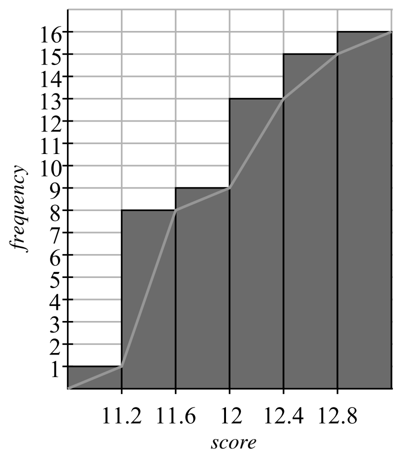
c mean $\doteqdot$ 11.85 s, standard deviation $\doteqdot$ 0.563 s. Agreement is reasonable, but as expected, the answers are not exactly the same.

d



e 0.5 seconds is a big difference in the time of a 100 metre sprint — the scale would be too coarse.

f



g The line at 50% of the data (frequency 8) meets the polygon where the sprint time is 11.6 seconds. You can confirm that this agrees with the result for splitting the grouped ordered data into two equal sets.

5 a

	first	second	Total
order entrée	45	42	87
no entreé	38	28	66
Total	83	70	153

b 153

c $87 \div 153 \doteqdot 57\%$

d 54%

e $P(\text{order entrée} \mid \text{attend first}) = 45 \div 83 \doteqdot 54\%$.
$P(\text{order entrée} \mid \text{attend second}) = 42 \div 70 = 60\%$.
No, it is not correct.

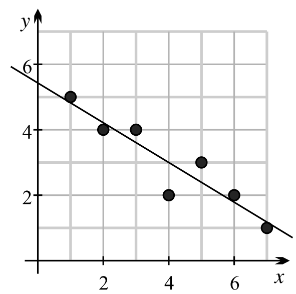
f $P(\text{attended first} \mid \text{ordered an entrée})$
$= 45 \div 87 \doteqdot 52\%$.

g $90 \times 60\% = 54$

h Those attending the first session may prefer a quick meal before heading out to the theatre or some other event. There may also be more family groups operating on a tighter budget.

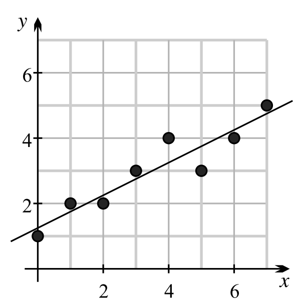
i If they can estimate the demand on certain dishes, then they may be able to prepare parts of the dish in advance, for example, preparing the garnishes or chopping the ingredients.

6 a i



ii $r = -0.93$, $y = -0.61x + 5.43$

b i



ii $r = 0.94$, $y = 0.5x + 1.25$

7 a 120 000

b 94 000, 62 000, 80 000, 80 000

c 316 000 and 79 000

d The arrivals may vary over the year because of seasonal or other effects. Government policy may consider an annual immigration quota, allowing a higher rate in one quarter to be balanced by a low rate in a subsequent quarter. As in 2000, examining the average for each quarter balances out such effects.

e 84 000

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

f It would be important to know the emigration rate of those leaving the country. The Net Overseas Migration (NOM) may be the better measure for many purposes. Other information of interest might include country of origin, destination within Australia, and whether they're intending to stay permanently or for a limited period.

g i 71 600

ii Rounding error has affected these calculations — a discrepancy in the second decimal place of the gradient is multiplied by 2000, resulting in an answer that is out by as much as $0.05 \times 2000 = 100$ thousand.

iii 84.300, which is in agreement with part **d**.

iv 660 000

v $660 \div 316 \times 100\% \doteqdot 209\%$, which is a 109% increase..

Chapter 16

Exercise 16A

1 **a** and **c**

2 a

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b **i** $\frac{3}{8}$ **ii** $\frac{1}{16}$ **iii** 0 **iv** 1

c **i** $\frac{3}{16}$ **ii** $\frac{1}{2}$ **iii** $\frac{15}{16}$ **iv** $\frac{3}{8}$

3 a

score x	1	2	3	4	5	Total
f_r	0.1	0.2	0.45	0.15	0.1	1
xf_r	0.1	0.4	1.35	0.6	0.5	2.95
$x^2 f_r$	0.1	0.8	4.05	2.4	2.5	9.85

b The sum of probabilities is 1.

c $\bar{x} = 2.95$

d The sample mean $\bar{x}$ is a measure of the centre of the dataset.

e $s^2 \doteqdot 1.15$ **f** $\sigma \doteqdot 1.07$

g The sample standard deviation s is a measure of the spread of the dataset.

h They are estimates of the mean $\mu = \mathrm{E}(X)$ and the standard deviation σ of the probability distribution.

i The sample mean $\bar{x}$ is 2.95, so after 100 throws, 295 is a reasonable estimate of the sum.

4 a $\bar{x} = 5.26$, $x \doteqdot 1.07$

b The centre of the data is about 2.3 units greater, but the spread is about the same, according to the standard deviation.

5 a 3.5 and 4

b

score x	1	2	3	4	5	6	Total
frequency f	2	4	4	8	2	0	10
$P(X = x)$	0.1	0.2	0.2	0.4	0.1	0	1
$x \times P(x)$	0.1	0.4	0.6	1.6	0.5	0	3.2
$x^2 \times P(x)$	0.1	0.8	1.8	6.4	2.5	0	11.6

c $E(X) = 3.2$

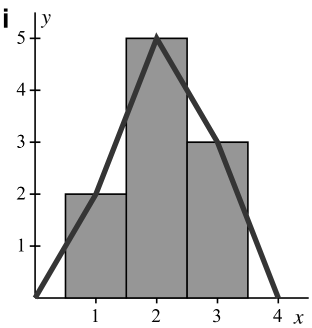
d $\mathrm{Var}(X) = 11.6 - (3.2)^2 = 1.36$

e 1.17

f It is usual to expect that for a quiz (covering recent work and including short easy questions) the marks will be high. These marks don't look impressive.

g $\mathrm{E}(X) = 16$, $\mathrm{Var}(X) = 34$, standard deviation 5.83

6 a **i**



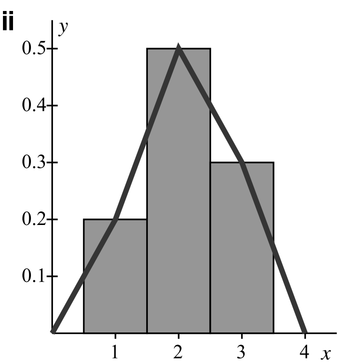
ii 10 **iii** 10

iv Both areas are the same and equal to the total frequency, that is the number of scores.

b **i**

score x	1	2	3
frequency f	2	5	3
relative frequency f_r	0.2	0.5	0.3

ii



iii 1 **iv** 1

v Both areas are the same and equal to the total 1, that is the sum of the relative frequencies. (This will only happen when the rectangles have width 1.)

ISBN 978-1-108-76630-2

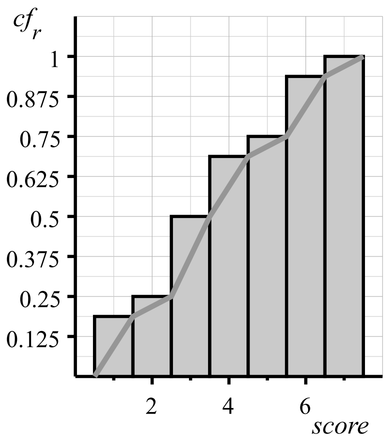
Cambridge University Press Updated July 2022

vi The relative frequencies are estimates of the probabilities. Note that both add to 1, both are non-negative, and both measure the chance that a random value will lie within the given rectangle of the histogram. A relative frequency is the *experimental* probability of an outcome, and is an *estimate* of the theoretical probability.

7 a

x	1	2	3	4	5	6	7	Total
f	3	1	4	3	1	3	1	16
f_r	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	1
cf	3	4	8	11	12	15	16	—
cf_r	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{11}{16}$	$\frac{12}{16}$	$\frac{15}{16}$	1	—

b

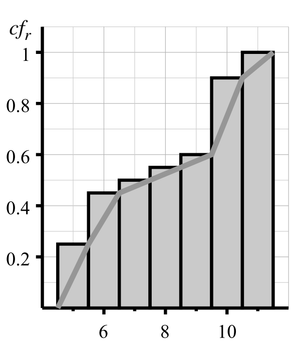


c $Q_1 = 2.5, Q_2 = 3.5, Q_3 = 5.5$

8 a

x	5	6	7	8	9	10	11	Total
f	5	4	1	1	1	6	2	20
f_r	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{6}{20}$	$\frac{2}{20}$	1
cf	5	9	10	11	12	18	20	—
cf_r	$\frac{5}{20}$	$\frac{9}{20}$	$\frac{10}{20}$	$\frac{11}{20}$	$\frac{12}{20}$	$\frac{18}{20}$	1	—

b



c $Q_1 = 5.5, Q_2 = 7.5, Q_3 = 10$

9 a $\frac{1}{4}$ **b** $\frac{3}{4}$ **c** 0.1

d 12.5% of the households have 3 or more cars, so the town planners won't recommend additional on-street parking.

e

x	0	1	2	3	4
$P(X = x)$	0.25	0.5	0.125	0.10	0.025

f The area of the histogram is exactly the sum of the probabilities, because the width of each bar is 1 in this graph.

g The triangles cut off above the polygon fit into the spaces below the polygon.

h This is an average, and is best understood by saying that for a large sample of n houses, we would expect them to have about $1.15n$ cars between them — see the next part.

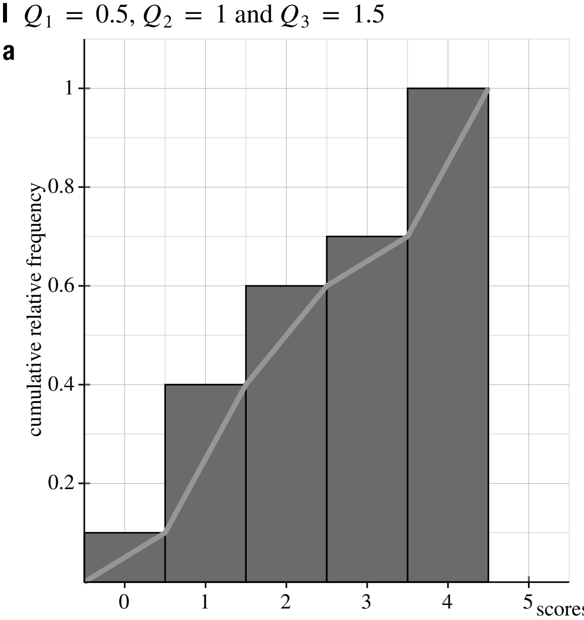
i 115 cars. We are assuming that streets in the suburb are uniform with respect to car ownership. Actually, streets closer to train stations may manage with fewer cars because people catch the train to work, more affluent streets may own more cars, people may adjust car ownership to allow for availability of off-street or on-street parking.

j

x	0	1	2	3	4
$P(X \le x)$	0.25	0.75	0.875	0.975	1

l $Q_1 = 0.5, Q_2 = 1$ and $Q_3 = 1.5$

10 a



b 3.5 **c** 1, 2

d $Q_3 \doteqdot 3.7$ **e** 3.67. They agree.

11 a median 3.5, mode 3.5

b

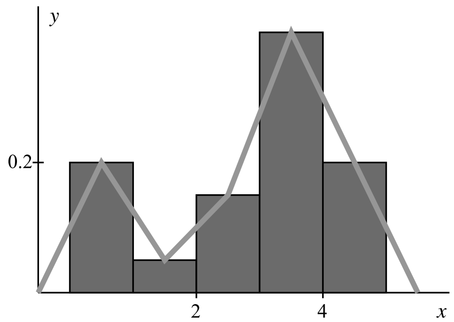
spent	0–1	1–2	2–3	3–4	4–5	Total
cc x	0.50	1.50	2.50	3.50	4.50	—
f	20	5	15	40	20	100
f_r	0.20	0.05	0.15	0.40	0.20	1

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c mean 2.85, variance 1.9275, standard deviation 1.39

d



e **i** 0.2 **ii** 0.05 **iii** 0.15
iv 0.4 **v** 0.2

f The area of the relative frequency polygon, or the area under the frequency polygon bounded by the *x*-axis (they are the same). This only happens because the rectangles have width 1.

g **i** Equally likely
ii They are twice as likely to have spent between \$3–\$4.

h $E(Y) = 4.85$, same variance

12 a The histogram covers 40 grid rectangles.

b **i** $0.3 \times 0.5 = 0.15$
ii 0.1 **iii** 0.3

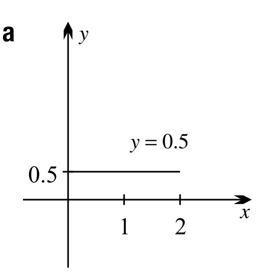
c **i** It is twice as likely to be 20°C.
ii In the class 19.25–19.75

d **i** 0.1 **ii** 0.3

e First, the histogram only records the maximum daily temperature. Secondly, it recorded 20 consectuve days, but there will be natural variation over the year, and even within a season.

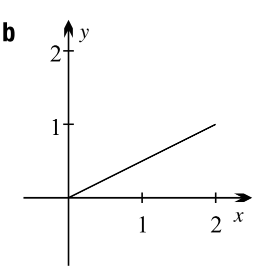
Exercise 16B

1 a



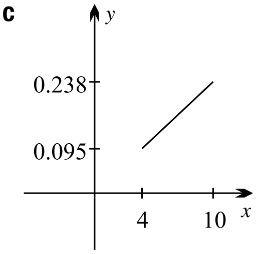
ii $\int_0^2 f(x)\,dx$
$=$ area rectangle $= 1$

b



ii $\int_0^2 f(x)\,dx$
$=$ area triangle $= 1$

c



ii $\int_4^{10} f(x)\,dx =$ area trapezium
$= \frac{1}{2} \times 6\left(\frac{4}{42} + \frac{10}{42}\right) = 1$

2 a Yes, mode is $x = 1$

b No, the integral is 3.

c No. The integral is 1, but $f(x) < 0$ if $x > 2$.

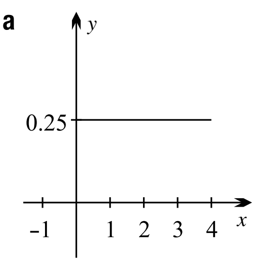
d Yes, provided that $n \geq 0$. Then mode is $x = 1$.

e Yes, mode is $x = \frac{\pi}{2}$

f Yes, mode is $x = 2$

3 b $f(x) = \frac{3}{4}(x - 3)(x - 1) < 0$, for $1 < x < 3$

4 a



b Clear from the graph

c **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$
iv 0 **v** $\frac{3}{4}$ **vi** $\frac{3}{4}$

d LHS $= \frac{1}{4}$, RHS $= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

5 a $F(x) = \frac{1}{64}x^2$

b $F(x) = \frac{1}{16}(x^3 + 8)$

c $F(x) = \frac{x}{2}(3 - x^2)$

d $F(x) = \frac{1}{e}(e^x + x - 1)$

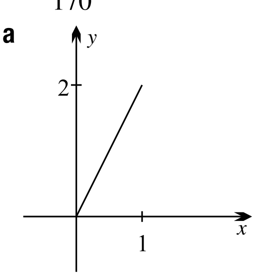
6 a $Q_2 = 4\sqrt{2}, Q_1 = 4, Q_3 = 4\sqrt{3}$

b $Q_2 = 0, Q_1 = -\sqrt[3]{4}, Q_3 = \sqrt[3]{4}$

7 a $\frac{20}{170} \doteqdot 12\%$ **b** $\frac{4\pi}{170} \doteqdot 7\%$

c $\frac{20 + 4\pi}{170} \doteqdot 19\%$ **d** $1 - \frac{20 + 4\pi}{170} \doteqdot 81\%$

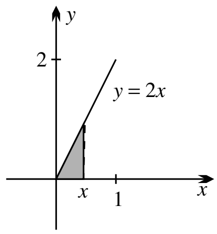
8 a



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

c **i** Area $= \frac{1}{2}x \times 2x$



d $Q_1 = \frac{1}{2}, Q_2 = \frac{1}{\sqrt{2}}, Q_3 = \frac{\sqrt{3}}{2}$

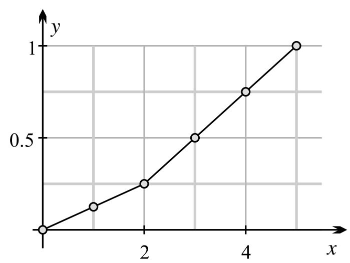
9 **a** $\frac{5}{243}$ **b** $\frac{1}{6}$ **c** $\frac{1}{10}$ **d** $\frac{1}{2}$

10 **a** Clearly $f(x) \geq 0$ for all x, and the area under the graph is $2 \times 0.125 + 3 \times 0.25 = 1$.

b

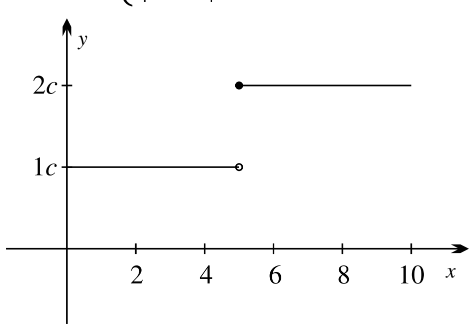
x	0	1	2	3	4	5
$P(X \leq x)$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

c



d $F(x) = \begin{cases} \frac{1}{8}x, & \text{for } 0 \leq x < 2, \\ \frac{1}{4}x - \frac{1}{4}, & \text{for } 2 \leq x \leq 5, \end{cases}$

11 **a**



b Area $= 15c$, so $c = \frac{1}{15}$

c $F(x) = \begin{cases} cx, & \text{for } 0 \leq x < 5, \\ 2cx - 5c, & \text{for } 2 \leq x \leq 5, \end{cases}$

d $P(1 < X < 7) = F(7) - F(1) = 8c = \frac{8}{15}$

12 **a** The mode is $x = 2$ (where the vertex is).

c Symmetric about $x = 2$, $P(X = 2) = 0$, and total area is 1.

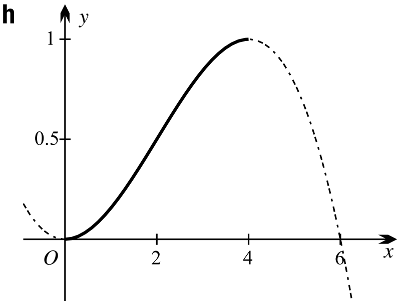
d $\frac{5}{32}$ and $\frac{27}{32}$ are complementary probabilities.

e $\frac{11}{256}$. The symmetry of the graph means that the areas are the same.

f $\frac{1}{32}x^2(6 - x)$

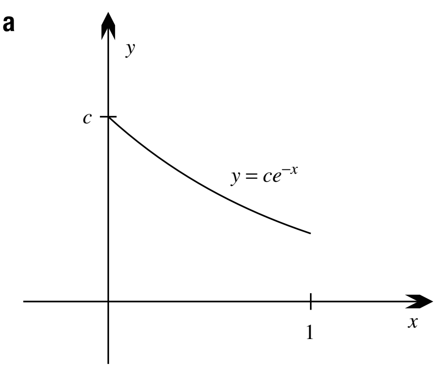
g **i** $\frac{81}{256}$ **ii** $\frac{41}{256}$ **iii** $\frac{29}{256}$ **iv** $\frac{47}{256}$

h



j $Q_1 = 1.3, Q_3 = 2.7$.

13 **a**



b $c = \frac{e}{e - 1}$

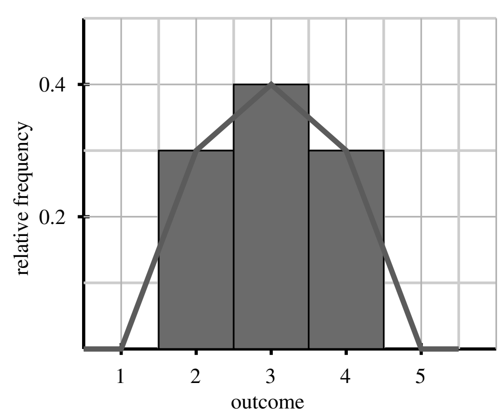
c $F(x) = \dfrac{e}{e - 1}(1 - e^{-x})$

d $Q_1 = \ln \dfrac{4e}{3e + 1}, Q_2 = \ln \dfrac{2e}{e + 1},$

$Q_3 = \ln \dfrac{4e}{e + 3}$

14 **a** See part **c**.

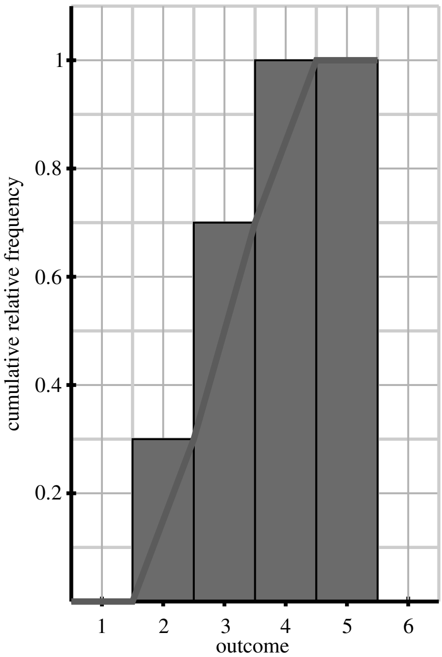
b Both areas are 1.



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c

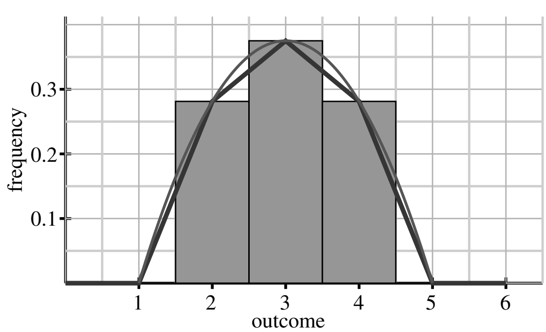


d 2.3, 3, 3.7

e i $\int_1^5 f(x)\,dx = 1$, and $f(x) \geq 0$ for $1 \leq x \leq 5$

ii

x	1	2	3	4	5
$f(x)$	0	0.28	0.375	0.28	0



iii $\frac{1}{32}(-x^3 + 9x^2 - 15x + 7)$

iv $P(X \leq 2.3) \doteqdot 0.25$, $P(X \leq 3) = 0.5$, $P(X \leq 3.7) \doteqdot 0.75$

v 2.3, 3 and 3.7 still seem good approximations.

15 b $F(x) = 1 - \dfrac{1}{x}$

c Total probability is 1.

d $\frac{4}{3}$, 2, 4

16 b $F(x) = 1 - e^{-x}$

c $Q_1 = \ln\frac{4}{3}$, $Q_2 = \ln 2$, $Q_3 = \ln 4$

17 b i This returns the square of the distance from a random point in the square to the centre of the square and circle.

ii 1 **iii** 0

c The code measures the relative frequency of points lying in the circle, that is, the probability that the point will lie in the circle. The value in cell C1 should approach π.

Exercise 16C

1 a $f(x) \geq 0$ and by area formula or integration,
$\int_0^{10} f(x)\,dx.$

b $E(X) = 5$

c Yes — in the centre of this distribution interval $[0, 10]$

d $\text{Var} = \frac{25}{3}$, $\sigma = \frac{5}{3}\sqrt{3}$

e $E(X^2) = \frac{100}{3}$ and $\text{Var} = \frac{25}{3}$

3 a The function is never negative, and the integral over $[-1, 1]$ is 1. Modes: 1 and –1.

b $E(X) = 0$ **c** $\text{Var}(X) = \frac{3}{5}$, $\sigma = \frac{\sqrt{15}}{5}$

d $\dfrac{3\sqrt{15}}{25} \doteqdot 0.46$

4 a Mode: 1, $E(X) = \frac{2}{3}$, $\text{Var}(X) = \frac{1}{18}$, $\sigma = \frac{\sqrt{2}}{6}$,
$P(\mu - \sigma \leq X \leq \mu + \sigma) = \frac{4\sqrt{2}}{9}$

b Modes: 1 and –1, $E(X) = 0$, $\text{Var}(X) = \frac{1}{2}$, $\sigma = \frac{1}{\sqrt{2}}$,
$P(\mu - \sigma \leq X \leq \mu + \sigma) = \frac{1}{2}$

c Mode: 4, $E(X) = 3$, $\text{Var}(X) = \frac{3}{5}$, $\sigma = \frac{\sqrt{15}}{5}$,
$P(\mu - \sigma \leq X \leq \mu + \sigma) \doteqdot 0.668$

5 a Yes.

b $E(X) = \frac{c}{2}$, as expected for a measure of the centre of this uniform distribution.

c $\frac{c^2}{12}$

d The answer agrees for this special case with $c = 10$.

e $E(X) = \frac{c}{2} + h = \frac{c + 2h}{2}$, variance unchanged.

f Put $h + c = k$ in the previous result: $E(X) = \frac{k + h}{2}$,
$\text{Var}(X) = \frac{(k - h)^2}{12}$

6 b $E(X) = \frac{23}{8}$, $E(X^2) = \frac{121}{12}$,
$\text{Var}(X) = \frac{349}{192} \doteqdot 1.82$

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 LHS $= \int_a^b x^2 f(x)\,dx - \int_a^b 2\mu x f(x)\,dx + \int_a^b \mu^2 f(x)\,dx.$

By the definition of a PDF,

Term 3 $= \mu^2 \int_a^b f(x)\,dx = \mu^2.$

By the formula for the mean,

Term 2 $= -2\mu \int_a^b x f(x)\,dx = -2\mu^2.$

8 b $\mathrm{E}(X) = 2.1$ **c** Agrees.

d Not only do both satisfy the condition that the area under the curve is 1, but they give the same result for the expected value.

9 b $\mathrm{E}(X) = \frac{3}{2}$, $\mathrm{E}(X^2) = 3$, $\mathrm{Var}(X) = \frac{3}{4}$

c i $1 - \frac{1}{4^3}$ **ii** $\frac{1}{8}$ **iii** $\frac{117}{1000}$

d $F(x) = 1 - \frac{1}{x^3}$

10 a $\frac{d}{dx} xe^{-x} = e^{-x} - xe^{-x}$,

so $\int xe^{-x}\,dx = -e^{-x} - xe^{-x}.$

b $\mathrm{E}(X) = 1$

c The derivative is

$(2xe^{-x} - x^2e^{-x}) + (2e^{-x} - 2xe^{-x}) - 2e^{-x}$
$= -x^2e^{-x}$,

so $\int x^2 e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

d $\mathrm{E}(X^2) = 2$ and $\mathrm{Var}(X) = 1$.

12 b Clearly $f(x) \ge 0$, and part (a) proves that the integral is 1.

c Using the formula in Box 6, $E(X) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1 + x^2}dx.$

The integrand here is odd, which could lead us to conclude that $E(X) = 0$. But this symmetry argument was developed only for integrals over finite intervals, whereas here we are integrating over the whole real line. The primitive here is $\frac{1}{2\pi}\log(1 + x^2)$, which is undefined as $x \to -\infty$ and also as $x \to \infty$, so $E(X)$ is also undefined.

d $\mathrm{E}(X^2)$ is unbounded (infinite).

Exercise 16D

1 a 0.5 **b** 0.8413 **c** 0.9972 **d** 0.9332
e 0.6554 **f** 0.9893 **g** 0.8849 **h** 1.0000

2 The total area under the curve is 1, so the areas of regions to the right and left of $z = a$ add to 1. This identity is true for any probability distribution.

a 0.5 **b** 0.1587 **c** 0.0228 **d** 0.0082
e 0.0968 **f** 0.2420 **g** 0.0548 **h** 0.0000

3 a From the even symmetry of the graph, $P(Z < -a) = P(Z > a) = 1 - P(Z \le a)$. (The result also holds for $a \le 0$, but this is not useful to us.) This result is certainly not true for all probability distributions.

b i 0.1151 **ii** 0.0107 **iii** 0.4207
iv 0.0007 **v** 0.0000 **vi** 0.2420
vii 0.0548 **viii** 0.0808 **ix** 0.5000

4 b i 0.4032 **ii** 0.4918 **iii** 0.2580
iv 0.4918 **v** 0.3643 **vi** 0.2580
vii 0.4452 **viii** 0.4032 **ix** 0.5000

c i 0.8064 **ii** 0.9836 **iii** 0.5762
iv 0.9962 **v** 0.3108 **vi** 0.8664

5 **a** & **e**, **b** & **g**, **c** & **h**, **d** & **f**

6 **a** & **c**, **b** & **g**, **d** & **f**, **e** & **h**

7 a This is evident from a graph by subtraction of areas.

b i 0.0483 **ii** 0.4100 **iii** 0.2297
iv 0.0923 **v** 0.4207 **vi** 0.1552

c i 0.9193 **ii** 0.7008 **iii** 0.9013

8 a 0.5 **b** 0 **c** 0.0359
d 0.8849 **e** 0.1151 **f** 0.3849
g 0.0359 **h** 0.8849 **i** 0.0792
j 0.8490

9 a 0.9032 **b** 0 **c** 0.3446
d 0.9554 **e** 0.9032 **f** 0.4332
g 0.2119 **h** 0.4207 **i** 0.0689
j 0.8893

10 a 0.9208 **b** 0.0792 **c** 0.6341 **d** 0.0364

12 a 50% **b** 84%

c 97.5% (Note the inaccuracy here. From the tables it should be 97.72.)

d 16% **e** 49.85% **f** 34%
g 47.5% **h** 2.35% **i** 68%
j 83.85% **k** 81.5% **l** 97.5%

13 a $b = 1$ **b** $b = 2$ **c** $b = -1$
d $b = 1$ **e** $b = 1$ **f** $b = 4$

14 a 0.6 **b** 2.3 **c** 1.2
d -0.8 **e** 1.1 **f** 2.6

15 a i $P(-1 < Z < 1) \doteqdot 68\%$

ii $P(Z < 2) \doteqdot 97.5\%$

iii $P(Z < -3 \text{ or } Z > 3) = 0.3\%$

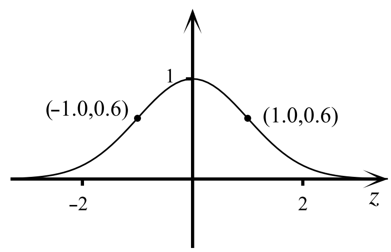
b Around 0.7 centimetres.

16 Mathematically, $P(Z = a) = \int_a^a f(x)\,dx$, which is an area of zero width. Practically, this represents the probability of getting a value exactly $Z = a$ for a continuous distribution, for example a height of exactly 1.7142435345345 … metres. In a continuous distribution, all such probabilities are zero.

 ISBN 978-1-108-76630-2 Cambridge University Press
Updated July 2022

17 a **i** all real values **ii** Even
iii $x = 0$ **iv** 1
v $z = -1$ and $z = 1$ **vi** $\left(0, \dfrac{1}{\sqrt{2\pi}}\right)$
vii There are no z-intercepts.
b **i** 0 **ii** 0 **iii** 0 **iv** 1
c $\phi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

18 c Stationary point $(0, 1)$. It is a maximum.
d Inflections at $(1, e^{-0.5})$ and $(-1, e^{-0.5})$
e $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$
f



g See the graph at the top of this exercise.

19 a **i** $P(0 \le Z \le 1) = 0.3401$
ii $P(-1 \le Z \le Z) = 0.6802$
iii The graph is concave up on $[0, 1]$ — the concavity changes at the point of inflection at $z = 1$. Thus the polygonal path of the trapezoidal rule will lie below the exact curve.
iv This is good agreement with the empirical rule (68%) and the table (0.6826).
b **i** $P(-2 < Z < 2) = 2 \times 0.4750 = 0.95$
ii $P(-3 < Z < 3) = 2 \times 0.4981 = 0.9962$

20 a $E(Z) = \int_{-\infty}^{\infty} z \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$, which is the integral of an odd function on a symmetric domain, so $E(Z) = 0$.

b $$\frac{d}{dz}\left(ze^{-\frac{1}{2}z^2}\right) = 1 \times e^{-\frac{1}{2}z^2} + z \times -ze^{-\frac{1}{2}z^2}$$
$$ze^{-\frac{1}{2}z^2} = \int e^{-\frac{1}{2}z^2}\, dz - \int z \times ze^{-\frac{1}{2}z^2}\, dz$$
$$\left[ze^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz - \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz$$
The LHS is 0, so
$$\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$
and $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz = 1$$
Thus we have shown that $E(Z^2) = 1$.

c Using the previous part,
$$\begin{aligned}\text{Var}(Z) &= E(Z^2) - E(Z)^2 \\ &= 1 - 0 \\ &= 1\end{aligned}$$

Exercise 16E

1 a $z = 1$, 1 SD above **b** $z = -2$, 2 SD below
c $z = 1$, 1 SD above **d** $z = -2$, 2 SD below
e $z = 5$, 5 SD above **f** $z = -3$, 3 SD below

2 a **i** $+2.5$ **ii** -3 **iii** $+5.5$ **iv** $+\frac{1}{4}$
b **i** iii **ii** i, iii, iv
iii ii **iv** iv
v i, ii, iii

3 a $P(Z \le 0.5)$ **b** $P(Z > 0.25)$
c $P(Z \le -1)$ **d** $P(Z \ge -1.5)$
e $P(-2 \le Z \le -0.5)$
f $P(-1.75 \le Z \le 0.25)$

4 a $P(Z \ge 0) = 0.5$
b $P(-1 \le Z \le 1) = 0.68$
c $P(Z \le 2) = 0.975$
d $P(Z \ge -2) = 0.975$
e $P(-3 \le Z \le 1) = 0.8385$
f $P(-2 \le Z \le -1) = 0.135$

5 a $P(-1 \le Z \le 3) = 0.8385$
b $P(Z \ge 1) = 0.16$
c $P(Z \ge 2) = 0.025$

6 a $P(-2.5 \le Z \le 2.5) = 0.9876$
b $P(Z \ge 1.6) = 0.0548$
c $P(Z \le -0.8) = 0.2119$
d $P(Z \ge -1.3) = 0.9032$
e $P(Z < 1.6) = 0.9452$
f $P(-2.5 < Z \le -1.5) = 0.0606$

7 a The score is above the mean.
b The score is below the mean.
c The score is equal to the mean.

8 a 69, 80
b 69, 80, 95, 50, 90, 52, 45
c 43, 45, 50, 52
d 95, 98
e It doesn't look very normal ('bell shaped'). Here is the stem-and-leaf plot:

4	3 5
5	0 2
6	9
7	
8	0
9	0 5 8

9 a **i** z-score for English (2.5) and maths (2). English is more impressive.
ii z-score for English (-0.8) and maths (-0.6). Maths is more impressive.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iii z-score for English (1.5) and maths (1). English is more impressive.

b 95% is 2.2 standard deviations above the mean. $P(Z > 2.2) \doteqdot 1 - 0.9861 \doteqdot 1.4\%$

c The mathematics mean of 62% is 0.3 English standard deviations below the English mean 65%. $P(Z > -0.3) = P(Z < 0.3) = 0.6179 \doteqdot 0.62$

10 a About 408 scores will lie within one SD from the mean, that is, in $[40, 60]$. About 570 scores will lie within two SDs from the mean, that is, in $[30, 70]$. About 598 scores will lie within three SDs from the mean, that is, in $[20, 80]$.

b **i** 415 **ii** 260 **iii** 462 **iv** 4

11 a **i** $-1,\ -1.5,\ -2$

ii 1.5 standard deviations below the mean

iii 45

iv Some assessments may be harder than others — simply averaging his other results takes no account of this.

v Jack may perform better in certain types of assessments, for example, in Biology lab experiments, or he may perform better at certain times of the year. For example, his results may improve towards the end of the year.

b z-scores 0.4, 0.625, 1, average 0.675. Jill's estimate is 71.1

Exercise 16F

1 a 97.5% **b** 84%

2 a 3 **b** 50 **c** 1630

3 a 2.5%

b $2400 \times 0.15 \div 100 = 3.6$ screws (perhaps round to 4)

4 5%

5 a 0.26% **b** 65 000

6 about 31%

7 a 1.28

b 186 cm

c **i** Interpolate between 1.6 and 1.7

ii 197 cm

8 The boxes need to be marked with a mean weight of 496.7 grams — this would probably be rounded down to 496, which is actually two standard deviations below the mean, so 97.5% of boxes weigh above this value.

9 a 69%

b **i** 33% **ii** 33%

10 a 12% **b** about 0.07%

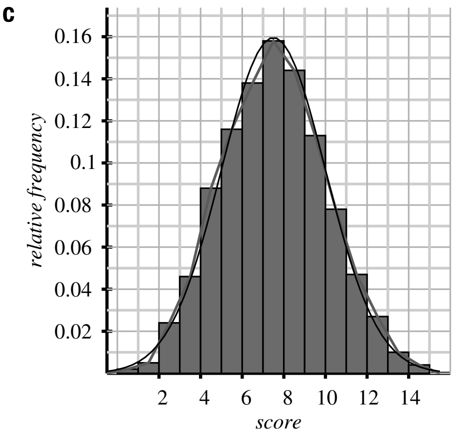
c **i** 4.5% **ii** 0.6%

11 2.5 standard deviations is 15 g, so 1 standard deviation is 6 g. The mean weight is 112 g.

Exercise 16G

1 a $\mu = 7.5, \sigma = 2.5$

c



d Either perform the experiment more than 1000 times, or average more than three random numbers at each stage.

4 c The mean should be about 5 and the standard deviation about 1.6.

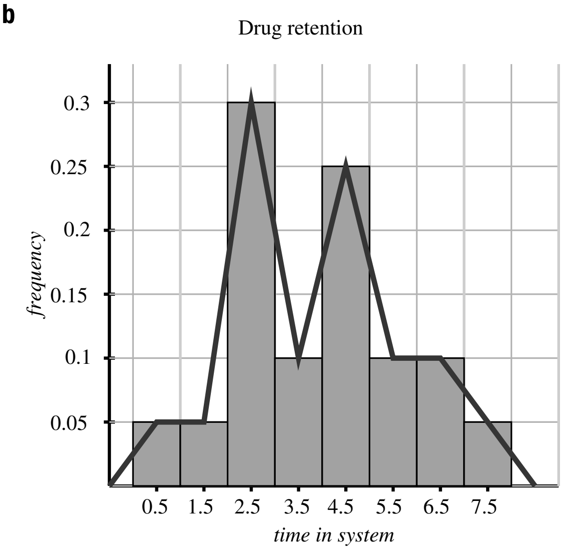
8 d $\left(0, \frac{1}{\sqrt{2\pi}}\right)$

Chapter 16 review exercise

1 a

x	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8
cc	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
f	1	1	6	2	5	2	2	1
cf	1	2	8	10	15	17	19	20
f_r	0.05	0.05	0.3	0.1	0.25	0.1	0.1	0.05
cf_r	0.05	0.1	0.4	0.5	0.75	0.85	0.95	1

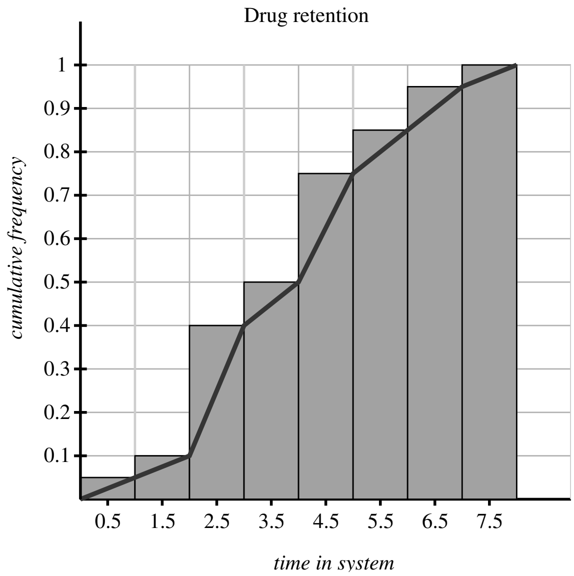
b



Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

c



d **i** 4.0 **ii** $Q_1 = 2.5, Q_3 = 5.0$
iii 6.5 **iv** 6.0

e This was only a preliminary experiment, and a larger dataset may resolve the unusual outcomes. It may be worth investigating any common links between patients falling in the two intervals associated with the two modes — perhaps different sexes react differently to the drug, perhaps it was administered differently, or perhaps the two groups are behaving differently after medication, for example, changing their levels of follow-up exercise or their food intake.

2 a False, it joins to the right end.
b False. The area under the relative frequency polygon is 1 if the rectangles each have width 1.
c True. **d** True. **e** True
f The empirical rule says 99.7%, and only applies to a normal distribution, so false in general.

3 b A uniform probability distribution with a uniform probability density function.
c $E(X) = 0$
d $E(X^2) = \text{Var}(X) = \frac{100}{3}, \sigma = \frac{10\sqrt{3}}{3}$

4 b $F(x) = \frac{1}{16}x(12 - x^2), 0 \le x \le 2$
c **i** 0.34, 0.7, 1.1 **ii** 0.85
iii 0.8 **iv** 0.78
v $P(X \le 0.4) - P(x \le 0.2) = 0.3 - 0.15 = 0.15$

5 a 0.5 **b** 0.9032 **c** 0.9282
d 0.3085 **e** 0.4207 **f** 0.4247

6 a 97.5% **b** 84% **c** 81.5% **d** 2.35%

7 a 0.6915 **b** 0.1151 **c** 0.5125 **d** 0.1760

8 a Let X be the lifetime in years of a machine.
b $P(X > 8) = P(Z > 1.3) = 9.7\%$
$P(X < 5) = P(Z < -1.07) = 14.2\%$. This is probably an unacceptable risk for the manufacturer, and they should increase the mean life, or decrease the standard deviation, or adjust the length of their advertised warranty.

Chapter 17

Exercise 17A

1 a $\frac{1}{32}$ **b** $\frac{5}{16}$ **c** $\frac{5}{32}$ **d** $\frac{31}{32}$

2 ${}^6C_2 \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right)^2$

3 a Because of the large size of the roll and the large number of males on the roll, the probability is fairly constant across the selection.
b $(0.65)^{12}$

4 a ${}^{12}C_3 \left(\frac{5}{6}\right)^9 \left(\frac{1}{6}\right)^3$ **b** ${}^{12}C_8 \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right)^8$
c ${}^{12}C_{10} \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^{10} + {}^{12}C_{11} \left(\frac{5}{6}\right)^1 \left(\frac{1}{6}\right)^{11} + \left(\frac{1}{6}\right)^{12}$

5 a 0.2009 **b** 0.7368 **c** 0.2632

6 a $\left(\frac{9}{10}\right)^{20} + {}^{20}C_1 \left(\frac{9}{10}\right)^{19} \left(\frac{1}{10}\right) + {}^{20}C_2 \left(\frac{9}{10}\right)^{18} \left(\frac{1}{10}\right)^2$
b $1 - \left(\frac{9}{10}\right)^{20}$

7 a $0.91^{10} + {}^{10}C_1 \times 0.91^9 \times 0.09$
$+ {}^{10}C_2 \times 0.91^8 \times 0.09^2$
b $1 - 0.91^{10} - {}^{10}C_1 \times .91^9 \times 0.09$

8 b HHTT, HTHT, THHT, HTTH, THTH, TTHH.
The probability that she wins is $\dfrac{6}{2^4} = \frac{3}{8}$.
c See the answer to part **b**. There are ${}^4C_2 = 6$ arrangements.
d The number of ways of choosing the two coins that are to be heads up is ${}^4C_2$.

9 0.593

10 ${}^{31}C_3 \times 0.95^{28} \times 0.05^3$

11 a **i** 0.10764 **ii** 0.11372
b 0.78549

12 a 17 **b** 7

13 a **i** 0.487 **ii** 0.031
b First, we have assumed that boys and girls are equally likely. Secondly, we have assumed that in any one family, the events ‘having a boy’ and ‘having a girl’ are independent. Thirdly, we have assumed that every child is a boy or a girl.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2

Cambridge University Press Updated July 2022

14 a The argument is invalid. Normally, mathematics books are grouped together, so that once the shelf is chosen, one would expect all or none of the books to be mathematics books, thus the five stages are not independent events. The result would be true if the books were each chosen at random from the library.

b The argument is invalid. People in a particular neighbourhood tend to vote more similarly than the population at large, so the four events are not independent. This method also oversamples small streets, which may introduce an additional bias.

15 a 0.03456 **b** 0.68256

16 a 0.409600 **b** 0.001126 **c** 0.000 885

17 a 0.0060 **b** 0.0303

18 a $\frac{3}{250}$

b i $\left(\frac{3}{250}\right)^{10}$

ii ${}^{10}C_5\left(\frac{3}{250}\right)^5\left(\frac{247}{250}\right)^5$

iii $\left(\frac{247}{250}\right)^{10} + 10\left(\frac{247}{250}\right)^9\left(\frac{3}{250}\right)$

19 a $\frac{7}{16}$

b i ${}^{8}C_3\left(\frac{9}{16}\right)^5\left(\frac{7}{16}\right)^3$

ii $1-\left(\frac{9}{16}\right)^8 - 8\left(\frac{9}{16}\right)^7\left(\frac{7}{16}\right)^1 - {}^8C_2\left(\frac{9}{16}\right)^6\left(\frac{7}{16}\right)^2$

20 a 34 **b** 22

21 a 0.2048 **b** 0.26272

c i $\dfrac{n(n-1)(n-2)(n-3)}{20\times 19\times 18\times 17}$

22 a $a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 3b^2c + 3bc^2 + 6abc$

b i 0.102 96 **ii** 0.13133

iii 0.89704

Exercise 17B

1 a The events are not independent — if it rains one day, it is more likely to rain the next day because rainy days tend to come in groups.

b Yes, and X is the number of throws that were less than 5.

c The possible number of trials is not finite, and the stages are not independent because if she wins, then the game stops.

d Let X be the number of heads turning up in 20 throws. Then X is a random variable with a binomial distribution.

e Strictly, the pens are not replaced, so the probability changes as each pen is removed and tested. If the population of pens is large, then p is almost constant with each selection, and it could be modelled with a binomial distribution.

f No. There are not two outcomes at each stage. It could be modified to ‘arrives on time’ or ‘takes less than 20 minutes’, but the events may still not be independent.

g Yes. Note that while the experiment is different at each stage, the probabilities at each stage are independent and have the same probability 0.01 of success.

2 a

heads x	0	1	2	3	4
ways	1	6	15	20	15
p	0.016	0.094	0.234	0.313	0.23
xp	0	0.094	0.469	0.938	0.938
x^2p	0	0.094	0.938	2.813	3.75

heads x	5	6	Total
ways	6	1	64
p	0.094	0.016	1
xp	0.469	0.0940	3
x^2p	2.344	0.563	10.5

b 3 heads

c $\mu = \sum xp = 3$,

variance $= \sum x^2p - \mu^2 = 10.5 - 9 = 1.5$

d Results agree.

e The distribution is symmetric, thus the centre of the distribution is exactly at the midpoint.

3 a

x	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

mode $=$ 2 or 3, mean $= \frac{5}{2}$, SD $= \frac{\sqrt{5}}{2}$

b

x	0	1	2	3	4	5
$P(X = x)$	$\frac{32}{243}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{40}{243}$	$\frac{10}{243}$	$\frac{1}{243}$

mode $=$ 1 or 2, mean $= \frac{5}{3}$, SD $= \frac{\sqrt{10}}{3}$

c

x	0	1	2	3	4	5
$P(X = x)$	0.269	0.404	0.242	0.073	0.011	0.001

mode $=$ 1, mean $\doteqdot$ 1.15, SD $\doteqdot$ 0.942

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6 ISBN 978-1-108-76630-2 Cambridge University Press Updated July 2022

4 a The expected value is 9.6 wins and the standard deviation is 2.4.

b Larry's result is 1.5 standard deviations below the mean

5 a 26%; $E(X) = 1, \sigma = 0.91$

b i 62%; $E(X) = 2, \sigma = 1.29$

ii 93%; $E(X) = 4, \sigma = 1.83$

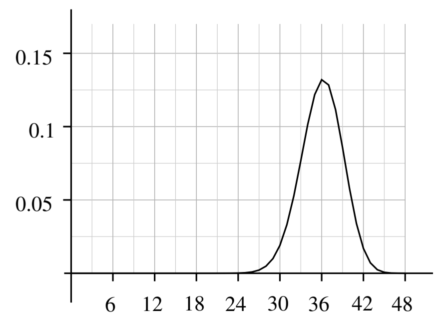
6 a right skewed

b mean 12, standard deviation 3.

c The mode is 12, which has a probability of about 0.13.

d Shade the region bounded by [6, 18] on the horizontal axis.

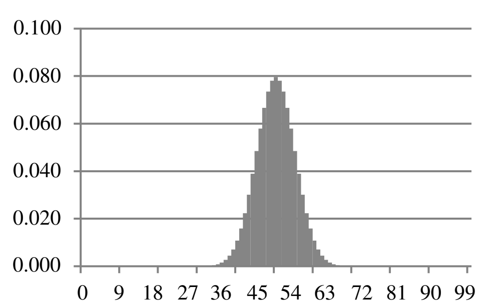
e (This is the graph for $p = 0.25$ reflected horizontally in $x = 24$.)



7 e 0.028 **f** 0.864 **g** 50 heads

h $i = 3$, and the interval is [47, 53]

i



It looks bell shaped, like a normal distribution.

8 a $E(X) = 12$. This is the number a student might expect to get right by guessing alone.

b 3 **c** 4 **d** 20, 4

e This time she is five standard deviations above the mean, which is even more unusual than her previous result.

f 75% is 36/48, which is 8 standard deviations above the mean. 60% is 60/100, which is 10 standard deviations above the mean. Thus it should be harder to get 60% in the second test than 75% in the first, just by chance. Note, however, that both results are almost impossible to achieve just by guessing.

9 Team A: $\mu = 35, \sigma = 3.2$, and 45 is 3.1σ above the mean.
Team B: $\mu = 63, \sigma = 4.3$, and 74 is 2.5σ above the mean.
Thus Team A's changes to the drug's delivery has shown stronger evidence for an improvement. Further trials should be carried out to check the validity of this result (see Section 17D on sampling).

10 a mean $= 15$, standard deviation $\doteqdot 3.57$

b That is, the probability of 14, 15 or 16 people voting WTP. This is approximately 32.5%.

11 a $\sigma^2 = np(1 - p) = -np^2 + np$

b It is a parabola, symmetric in its axis of symmetry $p = \frac{1}{2}$.

d This is the vertex, occurring halfway between the roots $p = 0$ and $p = 1$, that is, at $p = \frac{1}{2}$.

e It is clear from the quadratic graph in (a) and $\sigma = \sqrt{\sigma^2}$ has the same behaviour.

12 a 4 **b** 6 **c** 8

13 $2:\sqrt{3}$

14 d If an experiment testing a certain result is repeated enough times, it is to be expected that the hypothesis will be upheld eventually. If 99 times it fails and once it succeeds, then only publishing the success gives a skewed picture of the truth.

15 b The probability of being chosen is $p = \frac{1}{20}$, thus the mean waiting time is $\mu = \dfrac{1}{p} = 20$ time periods, or $5 \times 20 = 100$ minutes.

Exercise 17C

1 a $B(20,\ 0.3)$ **b** 10.82%

d 10.75% **e** 0.65%.

The result is very accurate!

2 a Answers for this question may vary slightly depending on the accuracy used to determine normal probability values (table or other technology).
$B(50,\ 0.5), P(18 \le X \le 20) = 8.49\%$
$np = 25, nq = 25$, normal approximation using $N(25,\ 12.5)$ is 8.46%, with percentage error 0.1%

b $B(20,\ 0.4), P(8 \le X \le 9) = 33.94\%$,
$np = 8, nq = 12$, normal approximation using $N(8,\ 4.8)$ is 34.3%, with percentage error 1%

c $B(30,\ 0.3), P(5 \le X \le 7) = 25.12\%$,
$np = 9, nq = 21$, normal approximation using $N(9,\ 6.3)$ is 23.8%, with percentage error 5%

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

d $B(40,\ 0.2)$, $P(9 \le X \le 12) = 36.36\%$, $np = 8$, $nq = 32$, normal approximation using $N(8,\ 6.4)$ is 38.3% , with percentage error 5%

e $B(22,\ 0.6)$, $P(13 \le X \le 15) = 46.59\%$, $np = 13.2$, $nq = 8.8$, normal approximation using $N(13.2,\ 5.28)$ is 46.0%, with percentage error 1%

f $B(80, 0.1)$, $P(10 \le X \le 13) = 24.98\%$, $np = 8$, $nq = 72$, normal approximation using $N(8,\ 7.2)$ is 26.8% , with percentage error 7%

g $B(500,\ 0.25)$, $P(100 \le X \le 103) = 0.84\%$, $np = 125$, $nq = 375$, normal approximation using $N(125,\ 93.75)$ is 0.9%, with percentage error 6%

h $B(200,\ 0.9)$, $P(170 \le X \le 172) = 3.39\%$, $np = 180$, $nq = 20$, normal approximation using $N(180,\ 18)$ is 3.2%, with percentage error 7%

3 a There are two possible outcomes, pink or blue.

b There are n stages, each independent, and with the same probability of success.

c If the counter is not returned, the stages of the experiment will not be independent. With the large number of counters, however, the probability will not change much, and we could approximate the experiment as binomial.

d $p = 0.6$, $\mu = np = 12$, $\sigma^2 = np(1 - p) = 4.8$, so $\sigma \doteqdot 2.19$.

e $\binom{20}{14}(0.6)^{14}(0.4)^6 \doteqdot 0.124$

f **ii** $P(13.5 < X < 14.5)$ $\doteqdot P(0.68 < Z < 1.14) \doteqdot 0.12$. Correct to two significant figures, both are 12%.

4 a $p = \frac{1320}{3000} = 0.44$

b $\mu = 6.6$, $\sigma \doteqdot 1.92$

c 14.04%

d $np = 6.6$, $nq = 8.4$. Yes.

e 14%

f Less than 1%

5 a $n > 10$ **b** $n > 20$ **c** $n > 40$
d $n > 500$ **e** $n > 20$ **f** $n > 40$
g $n > 50$ **h** $n > 11$

6 a Very large numbers are involved. A normal calculator cannot handle numbers such as $^{854}C_{76}$. Other calculating devices may not be accurate dealing with the large numbers involved. There are also a large number of cases to consider — from 60 to 76 successes.

b It is hard to get a representative sample of the whole world, because different ethnic groups will have different tendencies to colour blindness.

c $\mu \doteqdot 68.32$, $\sigma \doteqdot 7.93$,
$P(60 < X < 76)$
$= P(-1.05 < Z < 0.97) \doteqdot 68.7\%$

d $P(0.97 < Z < 1.03) = 1.5\%$

7 a $\left(\frac{3}{4}\right)^{15} \doteqdot 1\%$ **b** 99% **c** 77%

8 Interpreting this as $P(11 \le X \le 20)$, a normal approximation gives 25%.

9 $P(X > 20) = P(21 \le X \le 30) \doteqdot 17.6\%$. The underlying Bernoulli distribution is not applied with replacement, because the same person will not be in the park twice at the same gathering. If the population of Nashville is large, it should be reasonable to neglect this fact. It is also assumed that the visitors to the park are a random cross-section of Nashville. Groups with similar musical tastes may arrive together.

10 b There are still 100 trials, but the basic Bernoulli trial has changed. It could be that an extremely biased coin is tossed, or a card labelled 1 is selected (with replacement) from a pack of cards labelled 1 – 10.

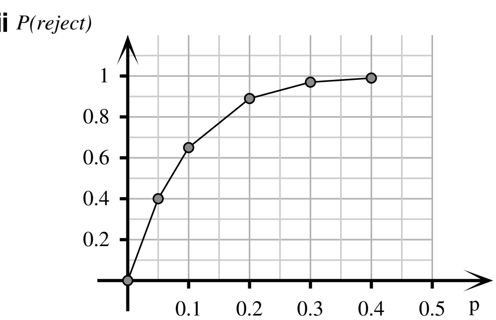
c The graphs are all bell-shaped curves. Smaller probabilities give a curve centred to the left (skewed to the right), and larger probabilities give a curve centred to the right (skewed to the left). Probabilities further from 0.5 give a narrower curve (distribution).

12 a **i** 0.4

ii

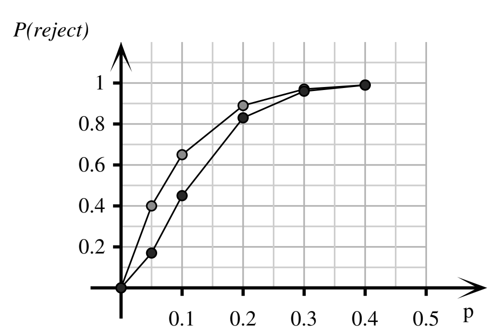
p	0	0.05	0.1	0.2	0.3	0.4
P(reject)	0	0.4	0.65	0.89	0.97	0.99

iii



b

p	0	0.05	0.1	0.2	0.3	0.4
P(reject)	0	0.17	0.45	0.83	0.96	0.99



ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c The second method is more forgiving if there are a few punnets that need to be rejected. Both methods are strongly likely to reject the batch if p is high, indeed the curves approach one another closely by the time p reaches 20%.

Exercise 17D

1 a $\frac{2}{5}$ **b** $\frac{4}{10}$ **c** $\frac{3}{4}$

2 a

X	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

b

$\hat{h}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$P(\hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

c 0.5

d It is the probability p in each Bernoulli trial, that is, it is the probability of a coin landing heads.

3 a

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
f_r	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{6}{20}$	$\frac{7}{20}$

b 0.72

c It is an estimate of the probability that a shopper chosen at random lives in the suburb.

4 a 8, 9 or 10 heads **b** 5.47%

5 a 6.4%

b 7.4% This result is surprisingly accurate because 9% of 50 is 4.5, so we are effectively applying a continuity correction.

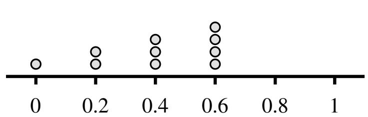
6 a $12 \div 32 = 0.375$

b 0.2

c

$\hat{p}$	0	0.2	0.4	0.6	0.8	1.0
Frequency	1	2	3	4	0	0

d



7 12%

8 a Let X record the number of hearts selected. Then $P(15.5 \le X \le 24.5) \doteqdot 75\%$

b Let $\mu = 0.25$, and $\sigma = \sqrt{0.25 \times 0.75 \div 80} \doteqdot 0.0484$. Thus $P(0.20 \le \hat{p} \le 0.30) = P(-1.03 \le Z \le 1.03)$ $= 2 \times P(Z \le 1.03) - 1 = 0.8485 \times 2$ $- 1 \doteqdot 70\%$. (The exact answer is 76% so part **a** is more accurate).

9 a 97% **b** 94%

10 a i 0.6% **ii** 3.7%

b i The mean and standard deviation for $\hat{p}$ are $\mu = 0.05$, and $\sigma = \sqrt{0.05 \times 0.95 \div 1000} \doteqdot 0.0069$. Thus $P(X < 0.03) = P(Z < -2.90)$ $= 1 - P(Z < 2.90) = 1 - 0.998 \doteqdot 0.2\%$

ii This result is significantly different from the previous claim that 5% of patients will have a reaction. They should check whether the sample was random — perhaps it consisted of patients more resistant to the side effects of the medication. Theyshould also check whether there have been any changes to the medication to reduce patient reactions. It is also possibly just chance that this result occurred, but the likelihood of this is small.

11 a $\mu = 0.6$, $\sigma^2 = 0.012$, so $\sigma = 0.1095$.

c $P(\hat{p} \le 0.4) \doteqdot P(Z \le -1.83)$ $\doteqdot 3.4\%$

d No, there is an error of 40%. The sample is too small and we are not using any continuity correction.

e $P(\hat{p} \le 0.75) \doteqdot P(Z < 1.37)$ $\doteqdot 91.5\%$

f 3.7%. The curve is flatter at the top end and varies less with $\hat{p}$. Percentage difference is also exaggerated by small values, such as at the left end of the curve.

12 b

n	1000	500	100	50	25
exact	0.9026	0.8262	0.6914	0.6641	0.6550
appr	0.896	0.8143	0.655	0.610	0.579
error	0.7	1.6	5.21	8.1	11.6

(answers may vary if you have used technology in place of the provided normal tables.)

c The accuracy improves as n increases, and is quite good for large samples.

13 For the sample proportion $\hat{p}$, $\mu = 0.5$, and $\sigma = \sqrt{0.5 \times 0.5 \div 50} \doteqdot 0.071$. Thus $P(\hat{p} > 0.6) = P(Z > 1.41) = 1 - P(Z < 1.41)$ $= 1 - 0.9207 \doteqdot 8\%$. It appears that people strongly prefer the branded version, even though it is identical. There may be an expectation that the branded version is superior, or they may prefer the packaging.

14 In the population of those with the disease, let $p = 0.3$ be the probability of a positive response to the drug. Let X be the binomial random variable

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

measuring how many respond positively to the drug in the trial of $n = 100$ and let $\hat{p}$ be the sample proportion $\hat{p} = \dfrac{X}{n}$. The mean and standard deviation of $\hat{p}$ are $\mu = p = 0.3$ and $\sigma = \sqrt{p(1-p)/n} = 0.046$. Then $P(\hat{p} \geq 40\%) = P(Z \geq 2.17) = 1.5\%$. It is unlikely to be by chance.

15 a **i** 132 **ii** 234 **iii** 526 **iv** 2101 **v** $> \dfrac{2100}{k^2}$

b **i** 101 **ii** 178 **iii** 401 **iv** 1601 **v** $> \dfrac{1600}{k^2}$

c **i** 157 **ii** 278 **iii** 626 **iv** 2501 **v** $> \dfrac{2500}{k^2}$

16 a **i** 0.205 **ii** $40 \times 0.205 \doteqdot 8$

17 a $p = \dfrac{5}{18}$

b $P(X = 0 \leq X \leq 4) = 0.3096$

c 0.2149 **d** 0.2968

e 0.2149. The sample proportion distribution is just the binomial stretched vertically by a factor n and compressed horizontally by a factor $\dfrac{1}{n}$, thus the corresponding areas will be the same.

f 0.2968. The factor $\frac{1}{40}$ corresponds to half an interval on the histogram and thus applies the same continuity correction as part **d**.

18 b **i** $\sigma \doteqdot 0.047$

ii Margin of error $= 0.09 = 9\%$, and the 95% confidence interval for the probability p is $[58\%, 76\%]$.

iii $n > 8493$

Chapter 17 review exercise

1 a ${}^4C_3 \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)$ **b** ${}^4C_2 \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^2$

2 $\left(\frac{5}{6}\right)^{15} + {}^{15}C_{14} \left(\frac{5}{6}\right)^{14} \left(\frac{1}{6}\right) + {}^{15}C_{13} \left(\frac{5}{6}\right)^{13} \left(\frac{1}{6}\right)^2$

3 0.000786

4 We need $1 - \left(\frac{3}{4}\right)^n > 0/95$.
Thus $n > \log(0.05)/\log(0.75) > 10$.
The experiment needs to be performed 11 times.

5 a $\frac{1}{8}$

b

x	0	1	2	3
$P(X = x)$	0.4488	0.3847	0.1374	0.0262

x	4	5	6
$P(X = x)$	0.0028	0.0002	0

c 26 times **d** 29 times

6 a Yes. $p = 0.5, q = 0.5$

b Yes, if success is interpreted as the player winning. Probability of success $p = \frac{1}{12}$ and of failure $q = \frac{11}{12}$.

c Yes, with experimental probability $p = 0.004$, $q = 0.996$

d No, because a Bernoulli trial must have only two possible outcomes.

7 a $4,\ 3.2,\ 1.79$ **b** $35,\ 17.5,\ 4.18$

c $4.8,\ 0.96,\ 0.98$ **d** $48,\ 28.8,\ 5.37$

e $30,\ 27,\ 5.2$ **f** $1.25,\ 0.9375,\ 0.97$

8 a $3,\ 2.85,\ 1.7$

b $P(X = 2, 3 \text{ or } 4) = 62.8\%$

c $P(0 \leq X \leq 4) \doteqdot 82\%$. The probability of rejection is 18%

d mean 1.2, standard deviation 1.08, $P(X = 0,\ 1 \text{ or } 2) = 88\%$. The probability of rejection is 12%

9 a 25.62% **b** $40, \sqrt{20}$

c $np = nq = 40 > 5$, so our criterion suggests that a normal approximation is appropriate.

d This is called a continuity correction and occurs because we are approximating a discrete distribution by a continuous curve.

e 25.6% **f** less than 0.1%

g 1.7%

10 a $\frac{1}{6}$

b If $X \sim B\left(80, \frac{1}{6}\right)$, then $P(X \geq 15) = 36\%$

11 a 0.35

b Not neccessarily.

c A binomial distribution with $n = 100$ and p equal to the (unknown) proportion of the population intending to vote for the WTP.

12 a

$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$P(\hat{p})$	$\frac{32}{3125}$	$\frac{48}{625}$	$\frac{144}{625}$	$\frac{216}{625}$	$\frac{162}{625}$	$\frac{243}{3125}$

b **i** $\frac{272}{3125}$ **ii** $\frac{992}{3125}$

c mean $\mu_{\hat{p}} = \frac{3}{5}$, variance $\frac{6}{125}$, $\sigma_{\hat{p}} = \frac{1}{25}\sqrt{30}$.

13 a $\mu = p = \frac{1}{6}$; $\sigma = \frac{1}{60}$

b The z-score is $\left(\frac{70}{500} - \frac{1}{6}\right) \div \frac{1}{6} = 1.6$ standard deviations below the mean.

14 $P(X > 0.53) \doteqdot 31\%$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Appendix B: Series and finance

14F Annuities and tables of interest factors

The calculations in Sections 14D and 14E were rather elaborate, because they required GPs to deal with the regular payments when money is invested for the future, or a loan is paid off.
Such calculations are too slow and inconvenient for the routine investment and loan decisions that people make in everyday life. Most websites now perform these calculations online, but an alternative approach is to provide tables of interest factors. With such tables, customers can see at a glance the effect of differing interest rates, and differing payments or repayments, on their finances.
These last two sections introduce such tables, and also introduce some new concepts — annuities, and their future and present value. Some important graphs that are easily calculated from these tables are presented in Questions 16 & 19 of this section, and in Questions 15 & 17 of Section 14G.

Preparation — Using tables for compound interest and depreciation

Calculations involving compound interest and depreciation are more straightforward, but using tables for them is good preparation for the rest of this section.

$$\text{Compound interest: } A_n = P(1 + R)^n \quad \text{Depreciation: } A_n = P(1 - R)^n.$$

In the tables, the principal is fixed at \$1, so the entries in the two tables are

$$(1 + R)^n \quad \text{and} \quad (1 - R)^n,$$

then in each situation we multiply the principal P by the appropriate table entry.

Example 19 **14F**

a The table of *compound interest factors* to the right shows the future value of \$1 after n units of time, or *periods*, at various rates of interest per period compounded each period.

i An amount of \$12 600 is invested at 3% p.a. interest, compounded annually. What will the value of the investment be after 7 years?

ii What amount, invested at 4% p.a. interest compounded every 3 months, will grow to \$40 000 after 2 years?

n	1.00%	2.00%	3.00%
1	1.0100	1.0200	1.0300
2	1.0201	1.0404	1.0609
3	1.0303	1.0612	1.0927
4	1.0406	1.0824	1.1255
5	1.0510	1.1041	1.1593
6	1.0615	1.1262	1.1941
7	1.0721	1.1487	1.2299
8	1.0829	1.1717	1.2668
9	1.0937	1.1951	1.3048

b The table of *depreciation factors* to the right shows the depreciated value of an item, initially worth \$1, after n years at various annual rates of depreciation.

i A car that initially cost \$56 400 depreciates at 24% p.a. How much is it worth after 7 years?

ii An 8-year-old machine has just been valued at \$12 000. If the depreciation has been 12% p.a., estimate what it cost originally.

n	12%	18%	24%
1	0.8800	0.8200	0.7600
2	0.7744	0.6724	0.5776
3	0.6815	0.5514	0.4390
4	0.5997	0.4521	0.3336
5	0.5277	0.3707	0.2536
6	0.4644	0.3040	0.1927
7	0.4087	0.2493	0.1465
8	0.3596	0.2044	0.1113

The factors in these tables are approximations, typically given to four decimal places. On the calculator, keep intermediate results in memory. Giving final answers correct to the nearest dollar is usually convenient, but never give an answer correct to more significant figures than the factor in the table.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

SOLUTION

a **i** Here $n = 7$ and rate $= 3\%$, so from the table, the factor is 1.2299.

Hence final value $= 12600 \times 1.2299$
$\doteqdot \$15497$.

ii The interest rate is 1% per 3-month period, and there are 8 periods.

Let the principal be $\$P$. Then from the table,

final value $= 1.0829P \quad (n = 8$ and rate $= 1\%)$,

so $1.0829P = 40000$
$P \doteqdot \$36938$.

c **i** Here $n = 7$ and rate $= 24\%$, so from the table, the factor is 0.1465.

Hence depreciated value $= 56400 \times 0.1465 \quad (n = 7$ and rate $= 24\%)$
$\doteqdot \$8263$.

ii Let $\$C$ be the original cost. Then from the table,

final value $= 0.3596C \quad (n = 8$ and rate $= 12\%)$,

so $0.3596C = 12000$
$C \doteqdot \$33370$.

Annuities and the future value of an annuity

The word 'annuity' is used in different ways, but it always refers to a situation where regular fixed payments are made.

The most straightforward example of an annuity is when I regularly pay a fixed instalment into an investment earning compound interest. We used GPs to deal with these investments in Section 14D, and we shall now use tables instead.

The *future value* or FV of an annuity is the value at the end of the last period. It is the sum of all the instalments, plus all the interest on all the instalments.

- The fixed *unit of time* is called the *period*.
- The interest rate must be the interest rate per period.
- Interest must be compounded at the end of each period.
- The regular fixed *payments* are also called *instalments* or *contributions*.

A convention about when payments are made

It is conventional, whenever constructing tables for annuities, that *each payment is made at the end of a period*. In particular, the first payment is not made until the end of the first period.

This is the opposite convention from what was generally used in Section 14D, but we shall adopt it in this section because the emphasis is now on the tables.

Question 17 in Exercise 14F deals with the simple correction that must be made when using tables if the payments are paid at the start of each period.

A table of interest factors for the future values of an annuity

Here is a table of *interest factors* for the future value of an annuity where a regular instalment of \$1 is paid at the end of each period, and the interest is compounded at the end of each period. We will show at the end of the section, and in Question 18 of Exercise 14F, how to generate the table.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

There is really only one such table, but it is infinite horizontally and vertically, so we must choose which interest rates to display, and how many units of time.
This table will be needed throughout Sections 14F–14G.

Future value of an annuity with instalments of \$1										
Periods	*Interest rate per period*									
n	0.25%	0.50%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	8.00%	12.00%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0025	2.0050	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600	2.0800	2.1200
3	3.0075	3.0150	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836	3.2464	3.3744
4	4.0150	4.0301	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746	4.5061	4.7793
5	5.0251	5.0503	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371	5.8666	6.3528
6	6.0376	6.0755	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753	7.3359	8.1152
7	7.0527	7.1059	7.2135	7.4343	7.6625	7.8983	8.1420	8.3938	8.9228	10.0890
8	8.0704	8.1414	8.2857	8.5830	8.8923	9.2142	9.5491	9.8975	10.6366	12.2997
9	9.0905	9.1821	9.3685	9.7546	10.1591	10.5828	11.0266	11.4913	12.4876	14.7757
10	10.1133	10.2280	10.4622	10.9497	11.4639	12.0061	12.5779	13.1808	14.4866	17.5487
11	11.1385	11.2792	11.5668	12.1687	12.8078	13.4864	14.2068	14.9716	16.6455	20.6546
12	12.1664	12.3356	12.6825	13.4121	14.1920	15.0258	15.9171	16.8699	18.9771	24.1331
13	13.1968	13.3972	13.8093	14.6803	15.6178	16.6268	17.7130	18.8821	21.4953	28.0291
14	14.2298	14.4642	14.9474	15.9739	17.0863	18.2919	19.5986	21.0151	24.2149	32.3926
15	15.2654	15.5365	16.0969	17.2934	18.5989	20.0236	21.5786	23.2760	27.1521	37.2797
16	16.3035	16.6142	17.2579	18.6393	20.1569	21.8245	23.6575	25.6725	30.3243	42.7533
17	17.3443	17.6973	18.4304	20.0121	21.7616	23.6975	25.8404	28.2129	33.7502	48.8837
18	18.3876	18.7858	19.6147	21.4123	23.4144	25.6454	28.1324	30.9057	37.4502	55.7497
19	19.4336	19.8797	20.8109	22.8406	25.1169	27.6712	30.5390	33.7600	41.4463	63.4397
20	20.4822	20.9791	22.0190	24.2974	26.8704	29.7781	33.0660	36.7856	45.7620	72.0524
21	21.5334	22.0840	23.2392	25.7833	28.6765	31.9692	35.7193	39.9927	50.4229	81.6987
22	22.5872	23.1944	24.4716	27.2990	30.5368	34.2480	38.5052	43.3923	55.4568	92.5026
23	23.6437	24.3104	25.7163	28.8450	32.4529	36.6179	41.4305	46.9958	60.8933	104.6029
24	24.7028	25.4320	26.9735	30.4219	34.4265	39.0826	44.5020	50.8156	66.7648	118.1552
25	25.7646	26.5591	28.2432	32.0303	36.4593	41.6459	47.7271	54.8645	73.1059	133.3339
26	26.8290	27.6919	29.5256	33.6709	38.5530	44.3117	51.1135	59.1564	79.9544	150.3339
27	27.8961	28.8304	30.8209	35.3443	40.7096	47.0842	54.6691	63.7058	87.3508	169.3740
28	28.9658	29.9745	32.1291	37.0512	42.9309	49.9676	58.4026	68.5281	95.3388	190.6989
29	30.0382	31.1244	33.4504	38.7922	45.2189	52.9663	62.3227	73.6398	103.9659	214.5828
30	31.1133	32.2800	34.7849	40.5681	47.5754	56.0849	66.4388	79.0582	113.2832	241.3327
31	32.1911	33.4414	36.1327	42.3794	50.0027	59.3283	70.7608	84.8017	123.3459	271.2926
32	33.2716	34.6086	37.4941	44.2270	52.5028	62.7015	75.2988	90.8898	134.2135	304.8477
33	34.3547	35.7817	38.8690	46.1116	55.0778	66.2095	80.0638	97.3432	145.9506	342.4294
34	35.4406	36.9606	40.2577	48.0338	57.7302	69.8579	85.0670	104.1838	158.6267	384.5210
35	36.5292	38.1454	41.6603	49.9945	60.4621	73.6522	90.3203	111.4348	172.3168	431.6635
36	37.6206	39.3361	43.0769	51.9944	63.2759	77.5983	95.8363	119.1209	187.1021	484.4631
37	38.7146	40.5328	44.5076	54.0343	66.1742	81.7022	101.6281	127.2681	203.0703	543.5987
38	39.8114	41.7354	45.9527	56.1149	69.1594	85.9703	107.7095	135.9042	220.3159	609.8305
39	40.9109	42.9441	47.4123	58.2372	72.2342	90.4091	114.0950	145.0585	238.9412	684.0102
40	42.0132	44.1588	48.8864	60.4020	75.4013	95.0255	120.7998	154.7620	259.0565	767.0914
41	43.1182	45.3796	50.3752	62.6100	78.6633	99.8265	127.8398	165.0477	280.7810	860.1424
42	44.2260	46.6065	51.8790	64.8622	82.0232	104.8196	135.2318	175.9505	304.2435	964.3595
43	45.3366	47.8396	53.3978	67.1595	85.4839	110.0124	142.9933	187.5076	329.5830	1081.0826

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Future value of an annuity with instalments of \$1

Periods	*Interest rate per period*									
44	46.4499	49.0788	54.9318	69.5027	89.0484	115.4129	151.1430	199.7580	356.9496	1211.8125
45	47.5661	50.3242	56.4811	71.8927	92.7199	121.0294	159.7002	212.7435	386.5056	1358.2300
46	48.6850	51.5758	58.0459	74.3306	96.5015	126.8706	168.6852	226.5081	418.4261	1522.2176
47	49.8067	52.8337	59.6263	76.8172	100.3965	132.9454	178.1194	241.0986	452.9002	1705.8838
48	50.9312	54.0978	61.2226	79.3535	104.4084	139.2632	188.0254	256.5645	490.1322	1911.5898
49	52.0585	55.3683	62.8348	81.9406	108.5406	145.8337	198.4267	272.9584	530.3427	2141.9806
50	53.1887	56.6452	64.4632	84.5794	112.7969	152.6671	209.3480	290.3359	573.7702	2400.0182
51	54.3217	57.9284	66.1078	87.2710	117.1808	159.7738	220.8154	308.7561	620.6718	2689.0204
52	55.4575	59.2180	67.7689	90.0164	121.6962	167.1647	232.8562	328.2814	671.3255	3012.7029
53	56.5961	60.5141	69.4466	92.8167	126.3471	174.8513	245.4990	348.9783	726.0316	3375.2272
54	57.7376	61.8167	71.1410	95.6731	131.1375	182.8454	258.7739	370.9170	785.1141	3781.2545
55	58.8819	63.1258	72.8525	98.5865	136.0716	191.1592	272.7126	394.1720	848.9232	4236.0050

Example 20 — 14F

On 1st January, Paul and Lucila decide that they will save for a housing deposit by putting aside \$84 000 from their two incomes on 31st December each year for 4 years. The interest rate on their annuity is 3% per annum compounded annually.

a Use the future value table to find their savings at the end of 4 years.

b If their target had been \$400 000 at the end of the 4 years, how much should each instalment have been?

c **i** If the interest rate were 12% p.a., what would the future value be?

ii If the interest rate were 12% p.a., but interest compounded monthly, what monthly instalment would have produced the future value in part **i**?

SOLUTION

a Here $n = 4$ and rate $= 3\%$, so the interest factor is 4.1836.

Hence
$$\begin{aligned}\text{future value} &= 4.1836 \times 84\,000\\ &\doteqdot \$351\,420 \quad \text{(use no more than 5 significant figures)}.\end{aligned}$$

b Here $n = 4$ and rate $= 3\%$, so the interest factor is still 4.1836.

Let $\$M$ be the annual instalment. Then the future value is $4.1836M$,

so
$$\begin{aligned}4.1836M &= 400\,000\\ M &\doteqdot \$95\,611.\end{aligned}$$

c **i** Repeating part **a** with an interest rate of 12% per annum,
$$\begin{aligned}\text{future value} &= 4.7793 \times 84000 \quad (n = 4 \text{ and rate } = 12\%)\\ &\doteqdot \$401460 \quad \text{(keep the answers in the memory)}.\end{aligned}$$

ii Let the monthly instalment be $\$M$. The unit of time is now 1 month, so the rate is 1% per month, and there are 48 months, giving an interest factor of 61.2226, and a future value of $61.2226M$.

Hence
$$\begin{aligned}61.2226M &= 4.7793 \times 84\,000\\ M &\doteqdot \$6557.\end{aligned}$$

(This should be compared with $\$84\,000 \div 12 = \$7000$ in part **i**.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 21

When Douglas started his new job, he took out a superannuation investment where he paid an instalment of \$53 000 at the end of each year of employment until he retired 10 years later. The agreed interest rate was 5% per annum.

a What did he expect the final value of this investment to be — meaning the lump sum that he withdraw and reinvested at retirement?

b After 5 years there was a sudden severe recession, and the contract was rewritten, with the interest rate dropped permanently to 1% p.a. What was the final value of Douglas's investment, and what did he lose at retirement?

SOLUTION

a Find value $= 12.5779 \times 53000$ $\quad (n = 10 \text{ and rate} = 5\%)$

$\doteqdot \$666629$ (keep in extra memory if possible).

b First calculate the value of the funds paid during the first 5 years,

value after 5 years $= 5.5256 \times 53000 \quad (n = 5 \text{ and rate} = 5\%)$.

This money then earns compound interest at 1% p.a. for the next 5 years,

so value of these funds after 10 years $= (5.5256 \times 53000) \times (1.01)^5$

$\doteqdot \$307795$ (keep in the memory).

Now calculate the value of the funds paid during the second 5 years,

value after 5 years $= 5.1010 \times 53000 \quad (n = 5 \text{ and rate} = 1\%)$

$= \$270533$.

Hence total funds after 10 years $\doteqdot 307795 + 270353$

$= \$578148$.

Doughlas lost $\$666629 - \$578148 \doteqdot \$88480$ (ignore the rounding).

A formula for the future value

Suppose that an annuity has n periods and an interest rate R per period compounded every period, with a contribution of $\$M$ made at the end of each period.

$$\begin{aligned}
\text{Value at the end of 1 period} &= M, \\
\text{value at the end of 2 periods} &= M + M(1+R), \\
\text{value at the end of 3 periods} &= M + M(1+R) + M(1+R)^2, \\
\text{value at the end of } n \text{ periods} &= M + M(1+R) + \cdots + M(1+R)^{n-1}.
\end{aligned}$$

This is a GP with first term $a = M$ and ratio $r = 1 + R$, so using the GP sum formula,

$$\text{FV} = \frac{a(r^n - 1)}{r - 1}$$

$$\text{FV} = M\,\frac{(1+R)^n - 1)}{R}.$$

This formula is not to be memorised, or even used. It is excellent when only one future value is to be calculated, and in an online calculator. But when generating a spreadsheet table of interest factors, recursive methods are clearer and more straight forward.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

9 THE FUTURE VALUE OF AN ANNUITY

- The word 'annuity' refers to a financial situation involving regular fixed *payments*, or *instalments*, or *contributions*, paid every *unit of time* or *period*.
  - The interest rate must be the interest rate per period.
  - Interest must be compounded at the end of each period.
- It is conventional, whenever constructing formulae and tables for annuities, that *each payment is made at the end of each period*.
  - This is the opposite convention from that generally used in Section 14D, which dealt with annuities involving regular savings.
- The *future value* of an annuity is its value at the end of the last period.
  It is the sum of all the instalments, plus all the interest on all the instalments.
- GPs can be used to obtain future values, as discussed in Section 14D, and a formula can be obtained,

$$\text{FV} = M\left(\frac{(1 + R)^n - 1}{R}\right) \qquad \text{(not to be memorised or used)},$$

but *a table of interest factors* is an intuitive and practical way to complete the calculations. Such a table is usually obtained recursively on a spreadsheet.

See the Enrichment Question 18 of Exercise 14F for the details of constructing a spreadsheet of interest factors.

Exercise 14F

FOUNDATION

Rounding Carry out calculations on the calculator, keeping intermediate results in memory. Giving final answers correct to the nearest dollar is usually convenient, but never give an answer correct to more significant figures than the interest factor.

1 Refer to the compound interest and depreciation tables given in worked Example 19.
(This question is only intended as an introduction to tables — otherwise use the compound interest and depreciation formulae.)
  a What does \$16 000 grow to in 5 years with interest of 2% p.a. compounded every year?
  b What grows to \$40 000 in 3 years with interest of 3% p.a. compounded every 4 months?
  c What value does a car bought for \$40 000 have 7 years later with 12% p.a. depreciation?
  d What was the original cost of machinery that is now valued at \$320 000 if it was bought 8 years ago and has been depreciating at 18% per annum?

From this point on, use the full-page table of future value interest factors given in this section.

2 An annuity has 10 annual payments of \$1000 and an interest rate 6% p.a. compounded annually.
  a Use the future value table to write down the interest factor.
  b Find the future value of the annuity.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 An annuity has n regular payments of $\$M$, and an interest rate R per period compounded every period. First write down the interest factor, and then find the future value, where:

a $n = 25, R = 8\%, M = \$2500$

b $n = 52, R = 0.5\%, M = \$600$

4 Andrea saves \$1000 at the end of each month, for 48 months, with interest rate 0.25% per month compounded monthly.

a What is the interest factor?

b How much does she have at the end?

c How much of that amount is interest?

5 Theo and Costa each save \$12 000 per year for 4 years, both investing at 12% p.a. interest.

- Theo deposits \$12 000 at the end of each year, with annual compounding.
- Costa deposits \$1000 at the end of each month, with monthly compounding.

a What are their final amounts?

b Who did better, and by how much?

6 **a** What does \$52 000 grow to in one year at 13% p.a. simple interest?

b What do weekly contributions of \$1000 grow to at 13% p.a. compounded weekly?

c Why is the answer to part **b** less than the answer to part **a**?

7 A company allows its employees to make an agreed regular superannuation payment at the end of each year of their employment, and pays them 8% p.a. interest compounded annually. Find the regular payment that an employee should agree to make so that:

a his account has \$750 000 after 12 years,

b his account has \$350 000 after 10 years.

8 A couple decide to save \$200 000 over 4 years for a housing deposit. Interest is 6% p.a.

a **i** What annual payment do they make if interest is compounded annually?

ii What monthly payment do they make if interest is compounded monthly?

b Repeat the questions in part **a** if they save the money in 3 years.

DEVELOPMENT

9 When he started his new job, Rishon took up a superannuation scheme where he contributed \$3500 at the end of each month of employment. Interest is 3% p.a. compounded monthly. Examine the future value table, and report how many months will it take for the balance of his account to exceed:

a \$70 000 **b** \$120 000 **c** \$150 000 **d** \$200 000

10 An annuity has n monthly instalments of \$1000. Examine the future value table, and report what can you say about the monthly interest rate R, compounded monthly, if:

a $n = 12$ and FV $= \$15\,500$,

b $n = 48$ and FV $= \$53\,000$,

c $n = 24$ and FV $= \$45\,000$,

d $n = 6$ and FV $= \$9000$.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

11 Eloise saved \$800 towards a new car at the end of each month for 3 years at 3% p.a. compounded monthly.

a What did she expect to have in her account at the end of the 3 years?

b After 2 years the bank lifted its interest rate to 6% p.a.

i What was the balance of her account at the end of 2 years, and what did this amount grow to under compound interest at the end of the third year?

ii What was the future value of the last 12 instalments, and what were her total funds at the end of the 3 years?

12 Ross opens a saving account with a deposit of \$24 200 at the beginning of a month. He then pays \$560 into the account at the end of each month for 15 years. Interest is 6% p.a. compounded monthly.

a Use compound interest to calculate the future value of the deposit.

b Find the future value of the regular contributions after 15 years, and hence find the final balance of the account at that time. You will need to find the future value interest factor from the following extract from a longer table.

n	0.25%	0.50%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%
177	222.2938	283.5294	481.9466	1614.1804	6204.9848	25847.6590	112579.3623	502310.7722
178	223.8495	285.9471	487.7661	1647.4640	6392.1343	26882.5654	118209.3304	532450.4185
179	225.4092	288.3768	493.6438	1681.4133	6584.8984	27958.8680	124120.7970	564398.4436
180	226.9727	290.8187	499.5802	1716.0416	6783.4453	29078.2227	130327.8368	598263.3503
181	228.5401	293.2728	505.5760	1751.3624	6987.9487	30242.3516	136845.2287	634160.1513
182	230.1115	295.7392	511.6318	1787.3896	7198.5871	31453.0457	143688.4901	672210.7604

13 Yanjun is paying \$500 at the end of every month into a bank account, and earning interest at 3% p.a. compounded monthly.

a After 1 year she withdraws half the balance to pay for a trip. What is her balance?

b After another year she withdraws \$4000 to pay a medical bill. What is her balance?

c What is her balance at the end of the third year?

14 On 1st July one year, Jack and Emily decide to save for superannuation by paying a fixed amount on 30th June each year for 15 years. The interest rate is fixed at 6% p.a. compounded annually for 15 years. They seek to have \$600 000 at the end of 15 years.

a Find the annual contribution.

b After regular payments for 8 years, their business hits trouble, and they cannot save any money for the next two years. Find what their subsequent annual payments should be:

i if they still want to reach their target of \$600 000 after the same 15 years,

ii if they extend the time to reach their target of \$600 000 by two years.

15 On 1st January, Rhys decides to save by paying \$2000 into an account at the end of each 3-month period. Interest is 4% p.a. compounded every 3 months, and he intends to continue this for 6 years.

a What does Rhys expect the future value of the annuity to be?

b After 4 years, there is a recession, and interest rates drop permanently to 2% p.a.

i If he continues saving \$2000 every 3 months, what is the final value?

ii What must he save every 3 months so that the final value in part **a** is unchanged?

iii If he continues paying \$2000 every 3 months, how many months must he save so that he has at least the final value in part **a**?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

16 [Graphs]

a A savings scheme involves regular instalments of \$1000 at 5% per period compounded every period. Let A be the future value of the annuity after n periods.

i Draw up a table of values of A as a function of n, for $n = 5, 10, 15, 20, 25, 30,$ then plot these six points on a graph.

ii Imagine the points joined by a smooth curve. How would you describe the curve?

b A savings scheme with a target of \$100 000 has n regular instalments of \$$M$ at 8% per period compounded every period. Let $T = nM$ be the total instalments paid.

i Draw up a table of values of T as a function of n, for $n = 1, 5, 10, 15, 20, 25, 30$, then plot these seven points on a graph.

ii Imagine the points joined by a smooth curve. How would you describe the curve?

c A savings scheme with target \$100 000 runs for 1 year at the very high interest rate of 24% p.a. There are n regular instalments of \$$M$ paid during the year, and interest is compounded every period. Let $T = nM$ be the total instalments paid.

i Draw up a table of values of T as a function of n, for $n = 1, 2, 3, 4, 6, 8, 12$, then plot these seven points on a graph.

ii Imagine the points joined by a smooth curve. How would you describe the curve?

ENRICHMENT

17 a On 1st January, a couple decide to save, making an annual contribution of \$$M$ every year. Interest is R% p.a. compounded annually. On 31st December n years later, they withdraw their funds. Let A be their total funds if each payment is made at the start of each year, and let F be their total funds if each payment is made at the end of each year. Explain why $A = F(1 + R)$.

b Hence use the future value table to check the answers of these questions from Section 14D and Exercise 14D — in each case, payment is made at the start of each period.

i Worked Example **13a**,
ii Question **2a iv**,
iii Question **4b**,
iv Question **14a** & **b**.

18 This question constructs a future value spreadsheet for an annuity with \$1 payments for $n = 1$ to $n = 36$ units of time at interest rates R of 6%, 7%, 8%, 9%, 10%, 11%, 12% per unit time. Keep your completed spreadsheet for later use in Exercise 14G — name it carefully.

a i Look at the table of future values in this section, but ignore the words at the top.
In the top row of your spreadsheet, enter n, 6%, 7%, 8%, 9%, 10%, 11%, 12%.
In the first column under n, enter $1, 2, 3, \ldots, 36$ (perhaps using Fill Down).

ii Explain why all the entries in the $n = 1$ row are 1.0000.

iii Explain why each entry in the $n = 2$ row is $1 + (1 + R)$.

iv Explain why each entry in the $n = 3$ row is $1 + (1 + R) \times$ (the entry above), and that this formula holds in all rows apart from the $n = 1$ row.

b i Enter 1 into the cell $n = 1$ and rate $= 6\%$, then Right Click and Format it as a number with four decimal places. Then Fill Right to the end of the row.

ii In the cell $n = 2$ and rate $= 6\%$, enter the formula 1+(1+B$1)*B2. The reason for typing the dollar sign in front of 1 is to stop that row number changing later when you are filling down.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iii Fill Right from this cell, then Fill Down from this row.

iv Check your entries where they overlap with the table in this section, and make sure that you know why the formulae in the cells are working.

c Construct the same table using the formula in Box 9 with $M = 1$, and check agreement with the other versions of the table in part b and in this section. Don't forget to use the $ sign within your cell formula before you Fill Right and Fill Down.

19 In Exercise 6G, Question 17c, you proved that $\lim_{n \to \infty} \left(1 + \frac{u}{n}\right)^n = e^u$. Use this limit, and the formula in Box 9, to prove that the horizontal asymptote in Question 16c above is

$$T = \frac{24\,000}{e^{0.24} - 1} \doteqdot 88\,480.$$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

14G The present value of an annuity

The term 'annuity' covers more than the savings schemes and superannuation dealt with in the previous section. This section extends the term to cover also:

- a pension paid regularly out of a lump sum deposited with a bank or other financial institution,
- paying off a loan by regular equal instalments.

As in Section 14F, we will use tables to facilitate calculation and avoid GPs.

Such situations require a second type of table — *present value tables*. These tables are easy to use, but need some thought to understand, because the same table can be used in several different situations with different interpretations. The same conventions apply here as with future value tables for annuitites.

- The interest rate must be the interest rate per period.
- Interest must be compounded at the end of each period.
- *Each payment is made at the end of period.*

Present value of annuities designed for saving and superannuation

We begin with the savings annuities discussed in the previous section.

Suppose that I pay a regular instalment for the purpose of saving or superannuation. The *present value* or PV of the annuity is the lump sum that I would need to invest at compound interest at the start of the first period in order to generate the future value of the annuity. That is,

$$\text{PV} \times (1 + R)^n = \text{FV} \quad \text{or equivalently} \quad \text{PV} = \text{FV} \times (1 + R)^{-n}.$$

In worked Example 21a, Douglas paid \$53000 into his superannuation fund at the end of each of 10 years, with an interest rate of 5% p.a. We calculated that the future value of his annuity at the end of these 10 years was \$666629.

Now suppose that Douglas had instead deposited a lump sum of \$$P$ in his superannuation account at the *start of the first year*, and that with compound interest this amount had grown to the same amount of \$666629. Then using the compound interest formula,

$$\begin{aligned} P \times (1.05)^{10} &= 666629 \\ P &= 666629 \div (1.05)^{10} \\ &\doteqdot \$409252. \end{aligned}$$

This is the *present value* or PV of the annuity.

You will be pleased to know that this and similar results can be obtained far more quickly if we have available a *table of present value interest factors*, such as the table printed at the top of the next page — the formula for the PV in terms of the FV is given above. The entry indexed by $n = 10$ and rate $= 5.00\%$ is 7.7217, so because his annual instalment was \$53 000,

$$\begin{aligned} \text{PV} &= 7.7217 \times 53000 \\ &\doteqdot \$409250 \,(\text{ignore the rounding error}). \end{aligned}$$

To repeat, *the present value of the annuity is the lump sum that would have yielded the future value if invested at the start of the first period under compound interest*. (Perhaps think of 'present' as having more the sense of 'initial'.)

A table of interest factors of the present value of an annuity

Here is a table of the present value of an annuity in which a payment of \$1 is made at the end of each of n periods of time. Again, there is really only one such table — it is infinite, and we must choose which interest rates, and how many units of time, to display.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Present value of an annuity with instalments of $1										
Periods	*Interest rate per period*									
n	0.25%	0.50%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	8.00%	12.00%
1	0.9975	0.9950	0.9901	0.9804	0.9709	0.9615	0.9524	0.9434	0.9259	0.8929
2	1.9925	1.9851	1.9704	1.9416	1.9135	1.8861	1.8594	1.8334	1.7833	1.6901
3	2.9851	2.9702	2.9410	2.8839	2.8286	2.7751	2.7232	2.6730	2.5771	2.4018
4	3.9751	3.9505	3.9020	3.8077	3.7171	3.6299	3.5460	3.4651	3.3121	3.0373
5	4.9627	4.9259	4.8534	4.7135	4.5797	4.4518	4.3295	4.2124	3.9927	3.6048
6	5.9478	5.8964	5.7955	5.6014	5.4172	5.2421	5.0757	4.9173	4.6229	4.1114
7	6.9305	6.8621	6.7282	6.4720	6.2303	6.0021	5.7864	5.5824	5.2064	4.5638
8	7.9107	7.8230	7.6517	7.3255	7.0197	6.7327	6.4632	6.2098	5.7466	4.9676
9	8.8885	8.7791	8.5660	8.1622	7.7861	7.4353	7.1078	6.8017	6.2469	5.3282
10	9.8639	9.7304	9.4713	8.9826	8.5302	8.1109	7.7217	7.3601	6.7101	5.6502
11	10.8368	10.6770	10.3676	9.7868	9.2526	8.7605	8.3064	7.8869	7.1390	5.9377
12	11.8073	11.6189	11.2551	10.5753	9.9540	9.3851	8.8633	8.3838	7.5361	6.1944
13	12.7753	12.5562	12.1337	11.3484	10.6350	9.9856	9.3936	8.8527	7.9038	6.4235
14	13.7410	13.4887	13.0037	12.1062	11.2961	10.5631	9.8986	9.2950	8.2442	6.6282
15	14.7042	14.4166	13.8651	12.8493	11.9379	11.1184	10.3797	9.7122	8.5595	6.8109
16	15.6650	15.3399	14.7179	13.5777	12.5611	11.6523	10.8378	10.1059	8.8514	6.9740
17	16.6235	16.2586	15.5623	14.2919	13.1661	12.1657	11.2741	10.4773	9.1216	7.1196
18	17.5795	17.1728	16.3983	14.9920	13.7535	12.6593	11.6896	10.8276	9.3719	7.2497
19	18.5332	18.0824	17.2260	15.6785	14.3238	13.1339	12.0853	11.1581	9.6036	7.3658
20	19.4845	18.9874	18.0456	16.3514	14.8775	13.5903	12.4622	11.4699	9.8181	7.4694
21	20.4334	19.8880	18.8570	17.0112	15.4150	14.0292	12.8212	11.7641	10.0168	7.5620
22	21.3800	20.7841	19.6604	17.6580	15.9369	14.4511	13.1630	12.0416	10.2007	7.6446
23	22.3241	21.6757	20.4558	18.2922	16.4436	14.8568	13.4886	12.3034	10.3711	7.7184
24	23.2660	22.5629	21.2434	18.9139	16.9355	15.2470	13.7986	12.5504	10.5288	7.7843
25	24.2055	23.4456	22.0232	19.5235	17.4131	15.6221	14.0939	12.7834	10.6748	7.8431
26	25.1426	24.3240	22.7952	20.1210	17.8768	15.9828	14.3752	13.0032	10.8100	7.8957
27	26.0774	25.1980	23.5596	20.7069	18.3270	16.3296	14.6430	13.2105	10.9352	7.9426
28	27.0099	26.0677	24.3164	21.2813	18.7641	16.6631	14.8981	13.4062	11.0511	7.9844
29	27.9400	26.9330	25.0658	21.8444	19.1885	16.9837	15.1411	13.5907	11.1584	8.0218
30	28.8679	27.7941	25.8077	22.3965	19.6004	17.2920	15.3725	13.7648	11.2578	8.0552
31	29.7934	28.6508	26.5423	22.9377	20.0004	17.5885	15.5928	13.9291	11.3498	8.0850
32	30.7166	29.5033	27.2696	23.4683	20.3888	17.8736	15.8027	14.0840	11.4350	8.1116
33	31.6375	30.3515	27.9897	23.9886	20.7658	18.1476	16.0025	14.2302	11.5139	8.1354
34	32.5561	31.1955	28.7027	24.4986	21.1318	18.4112	16.1929	14.3681	11.5869	8.1566
35	33.4724	32.0354	29.4086	24.9986	21.4872	18.6646	16.3742	14.4982	11.6546	8.1755
36	34.3865	32.8710	30.1075	25.4888	21.8323	18.9083	16.5469	14.6210	11.7172	8.1924
37	35.2982	33.7025	30.7995	25.9695	22.1672	19.1426	16.7113	14.7368	11.7752	8.2075
38	36.2077	34.5299	31.4847	26.4406	22.4925	19.3679	16.8679	14.8460	11.8289	8.2210
39	37.1149	35.3531	32.1630	26.9026	22.8082	19.5845	17.0170	14.9491	11.8786	8.2330
40	38.0199	36.1722	32.8347	27.3555	23.1148	19.7928	17.1591	15.0463	11.9246	8.2438
41	38.9226	36.9873	33.4997	27.7995	23.4124	19.9931	17.2944	15.1380	11.9672	8.2534
42	39.8230	37.7983	34.1581	28.2348	23.7014	20.1856	17.4232	15.2245	12.0067	8.2619
43	40.7212	38.6053	34.8100	28.6616	23.9819	20.3708	17.5459	15.3062	12.0432	8.2696
44	41.6172	39.4082	35.4555	29.0800	24.2543	20.5488	17.6628	15.3832	12.0771	8.2764
45	42.5109	40.2072	36.0945	29.4902	24.5187	20.7200	17.7741	15.4558	12.1084	8.2825
46	43.4024	41.0022	36.7272	29.8923	24.7754	20.8847	17.8801	15.5244	12.1374	8.2880
47	44.2916	41.7932	37.3537	30.2866	25.0247	21.0429	17.9810	15.5890	12.1643	8.2928

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Present value of an annuity with instalments of $1										
Periods	*Interest rate per period*									
48	45.1787	42.5803	37.9740	30.6731	25.2667	21.1951	18.0772	15.6500	12.1891	8.2972
49	46.0635	43.3635	38.5881	31.0521	25.5017	21.3415	18.1687	15.7076	12.2122	8.3010
50	46.9462	44.1428	39.1961	31.4236	25.7298	21.4822	18.2559	15.7619	12.2335	8.3045
51	47.8266	44.9182	39.7981	31.7878	25.9512	21.6175	18.3390	15.8131	12.2532	8.3076
52	48.7048	45.6897	40.3942	32.1449	26.1662	21.7476	18.4181	15.8614	12.2715	8.3103
53	49.5809	46.4575	40.9844	32.4950	26.3750	21.8727	18.4934	15.9070	12.2884	8.3128
54	50.4548	47.2214	41.5687	32.8383	26.5777	21.9930	18.5651	15.9500	12.3041	8.3150
55	51.3264	47.9814	42.1472	33.1748	26.7744	22.1086	18.6335	15.9905	12.3186	8.3170

Example 22

Yianni decided on 1st January to save for a car. On 31st December each year, for 3 years, he paid an annual contribution of \$12 000 at an interest rate of 3% p.a. compounded annually.

a Using the present value table, find what lump sum Yianni could have deposited, on 1st January of the first year, so that after 3 years of compound interest he would have the same amount.

b If instead he had contributed \$1000 at the end of each month (with interest compounding every month), what would the present value be?

c Confirm the answers to parts **a** and **b** the long way by using the table of future values, and then applying the compound interest formula to the present value obtained in part **a** or **b**.

d Why is the answer to part **b** greater than the answer to part **a**?

SOLUTION

a Here $n = 3$ and rate $= 3\%$, so from the present value table, the factor is 2.8286.

$$\begin{aligned} \text{Hence PV} &= 2.8286 \times 12\,000 \\ &\doteqdot \$33\,943 \quad (\text{keep in memory}). \end{aligned}$$

b The period is now 1 month, so $n = 36$ and rate $= 0.25\%$, so from the present value table, the factor is 34.3865.

$$\begin{aligned} \text{Hence PV} &= 34.3865 \times 1000 \\ &\doteqdot \$34\,387. \end{aligned}$$

c In part **a**,
$$\begin{aligned} \text{FV} &= 3.0909 \times 12\,000 \quad (n = 3 \text{ and rate} = 3\%) \\ &\doteqdot 37\,091, \end{aligned}$$

and applying compound interest to the present value,
$$\begin{aligned} \text{PV} \times (1.03)^3 &= 33\,943 \times (1.03)^3 \\ &\doteqdot \$37\,091. \end{aligned}$$

In part **b**,
$$\begin{aligned} \text{FV} &= 37.6206 \times 1000 \quad (n = 36 \text{ and rate} = 0.25\%) \\ &\doteqdot \$37\,621, \end{aligned}$$

and applying compound interest to the present value,
$$\begin{aligned} \text{PV} \times (1.0025)^{36} &= 34\,387 \times (1.0025)^{36} \\ &\doteqdot \$37\,621. \end{aligned}$$

d His contributions in part **b** have almost all earned interest for longer that in part **a**, so the corresponding lump sum at the start would need to be greater. (More frequent compounding also helps, but only in a minor way here.)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Using the present value table to calculate repayments of a loan

When a loan is repaid by regular instalments, this is also an annuity. The loan is paid out when two amounts are equal:

- the original loan plus compound interest, and
- the sum of all instalments plus compound interest on them — this is the future value.

Thus the original loan is the present value of the annuity. Present value tables are therefore the appropriate tool in this situation.

Example 23

Bill the Bodgie buys a flash car for \$80 000, with a deposit of \$32 000 and a loan of \$48 000. The loan is paid off over 4 years at 12% p.a. compound interest.

a Using the present value table, find the instalment M if he pays annually with annual compounding.
b Find the instalment M if he pays monthly with monthly compounding.
c How much more will the first method of repayment cost him?
d Check the answer to part **a** the long way by using the future value table.

SOLUTION

a Here $n = 4$ and rate $= 12\%$, so the factor is 3.0373.

$$\begin{aligned} \text{Hence } 3.0373 \times M &= 48\,000 \\ M &\doteqdot \$15\,804 \quad \text{(keep in an extra memory if available)}. \end{aligned}$$

b Here $n = 48$ and rate $= 1\%$, so the factor is 37.9740.

$$\begin{aligned} \text{Hence } 37.9740 \times M &= 48\,000 \\ M &\doteqdot \$1264 \quad \text{(keep in memory)}. \end{aligned}$$

c $$\begin{aligned} \text{Extra cost} &= 4 \times \text{part } \mathbf{a} - 48 \times \text{part } \mathbf{b} \\ &\doteqdot \$2541 \quad \text{(ignore the rounding error)}. \end{aligned}$$

d Using the future value table with $n = 4$ and rate $= 12\%$ and $M = \$15\,804$

$$\begin{aligned} \text{FV} &= 4.7793 \times 15\,804 \\ &\doteqdot \$75\,530. \end{aligned}$$

Using compound interest on the original debt of \$48 000,

$$\begin{aligned} \text{debt after 4 years} &= 48\,000 \times (1.12)^4 \\ &\doteqdot \$75\,529 \quad \text{(ignore the rounding error)}. \end{aligned}$$

Using the present value table to calculate a pension

When people retire, they often deposit a lump sum with a bank or other financial institution, with an agreement that the bank will pay them a regular pension for a certain number of years. This time the customer is receiving the instalments rather than paying them, but the contract is still an annuity.

At the end of the contract, two things are equal:

- the initial lump sum plus compound interest, and
- the sum of all instalments plus compound interest on them — this is the future value.

Thus the initial lump sum is the present value of this annuity, and we can use present value tables for calculations.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Example 24

A retiree pays a principal $\$P$ to a bank on 1st January, and on 31st December each year, for the next 20 years, the bank pays him a pension of \$35 000.

a Use the present value table to calculate P, given that the interest rate is 4% p.a. compounded annually.

b Confirm this calculation the long way by using the future value table.

SOLUTION

a Using the present value table with $n = 20$ and rate $= 4\%$,

$$\begin{aligned} P &= 13.5903 \times 35\,000 \\ &\doteqdot \$475\,661. \end{aligned}$$

b Using the future value table with $n = 20$ and rate $= 4\%$,

$$\text{future value} = 29.7781 \times 35\,000$$

so applying the compound interest formula,

$$\begin{aligned} P \times (1.04)^{20} &= 29.7781 \times 35\,000 \\ P &\doteqdot \$475\,662 \quad \text{(ignore the rounding error)}. \end{aligned}$$

Note: With pension annuities, the present value is better described as *the lump sum that I can invest now at compound interest to produce the regular pension payments for a set number of periods*. This is exactly the same thing.

Example 25

Ron retires and pays a principal of $\$P$ to a bank on 1st January. On 31st December each year, for the next 20 years, the bank pays him a pension of $\$M$. The interest rate is 8% p.a. compounded annually.

a Find the principal $\$P$ for a pension of \$45 000.

b Find the pension $\$M$ for a principal of \$500 000.

c If the principal is \$500 000, the pension is paid every 6 months, and the rate of 8% p.a. is compounded every 6 months, what is the pension, and why is it less than the pension in part **b**?

SOLUTION

a $P = 9.8181 \times 45\,000 \quad (n = 20 \text{ and rate } = 8\%)$

$P \doteqdot \$441\,810.$

b $9.8181 \times M = 500\,000 \quad (n = 20 \text{ and rate } = 8\%)$

$M \doteqdot \$50\,926.$

c $19.7928 \times M = 500\,000 \quad (n = 40 \text{ and rate } = 4\%)$

$M \doteqdot \$25\,262.$

This is \$50 524 per year, \$402 less than part **b** where he is paid annually.

It is less because half the funds are drawn out 6 months earlier than in part **b**.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

A formula for the present value

Suppose that an annuity has n periods, interest rate R compounded each period, and payment \$$M$ made at the end of each period. The present value of this annuity is the lump sum that would have yielded the future value if invested at the start of the first period under compound interest. That is,

$$\text{PV} \times (1 + R)^n = \text{FV}.$$

In the previous section, we derived the formula for the future value,

$$\text{FV} = M\left(\frac{(1 + R)^n - 1}{R}\right),$$

and combining these, $\text{PV} = M\left(\dfrac{(1 + R)^n - 1}{R(1 + R)^n}\right)$.

Again, this formula is not to be memorised or used. It is appropriate for an online calculator. But when constructing a spreadsheet present value table of interest factors, recursive methods are clearer and more straightforward.

10 THE PRESENT VALUE OF AN ANNUITY

- The word 'annuity' refers to a financial situation involving regular fixed *payments*, or *instalments*, or *contributions*, paid every *period*.
  - The interest rate must be the interest rate per period.
  - Interest must be compounded at the end of each period.
  - Each payment is made at the end of each period.
- There are various forms of annuities, including:
  - saving by regular contributions,
  - paying of a loan by regular instalments,
  - pension payments paid from an initial lump sum.
- The *present value* of the annuity is the lump sum that would have grown to the future value under compound interest.
  - The present value of a loan repaid by instalments is the original loan.
  - With an annuity paying a pension, the present value is better described as the lump sum that I can invest now at compound interest to produce the regular pension payments for a set number of periods.
- The present value and the future value are related by

  $$\text{PV} \times (1 + R)^n = \text{FV},$$

  which yields the formula

  $$\text{PV} = M\left(\frac{(1 + R)^n - 1}{R(1 + R)^n}\right) \quad \text{(not to be memorised or used)}.$$

  But *a table of interest factors* is an intuitive and practical way to complete the calculations — such a table is usually obtained recursively on a spreadsheet.

Enrichment Question 16 of Exercise 14G constructs a present value spreadsheet.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

The value of an annuity part-way through

Suppose that an annuity is paying a man a pension of \$50 000 per annum for 20 years at interest 4% p.a. compounded annually, and that he dies just after the 12th payment. What balance from the cancelled annuity goes into his estate?

The most straightforward way to calculate this is to think about an annuity starting at his death, and calculate the present value of that notional annuity.

The notional annuity runs 8 years with payment \$50000 and interest 4% p.a.

The present value table gives the interest factor as 6.7327,

$$\begin{aligned}\text{so present value} &= 6.7327 \times 50000\\ &\doteqdot \$336640, \text{ which is what is paid into his estate.}\end{aligned}$$

Example 26

Terry borrowed \$35000 for a trip, to be repaid monthly over 3 years at 24% p.a. interest compounded monthly. What is her liability just after the 8th payment?

SOLUTION

Using present values, each instalment $= 35000 \div 25.4888$

Hence for the notional 28-month annuity,

$$\begin{aligned}\text{present value} &= 21.2813 \times (35000 \div 25.4888)\\ &\doteqdot \$29222, \text{ which is what she still owes.}\end{aligned}$$

Exercise 14G

FOUNDATION

Tables These questions require the full-page table of present value interest factors given in this section, and some require also the future value table from the previous section.

Rounding As always, carry out calculations on the calculator, keeping intermediate results in memory. Giving final answers correct to the nearest dollar is usually convenient, but never give an answer correct to more significant figures than the interest factor.

1 **a** An annuity has 10 annual payments of \$1000 and interest 6% p.a. compounded annually.

  i Use the present value table to write down the interest factor.

  ii Find the present value of the annuity.

b An annuity has n regular payments of \$$M$, and interest rate R per period compounded every period. First write down the interest factor, and then find the present value, where:

  i $n = 25$, $R = 8\%$, $M = \$2500$ **ii** $n = 52$, $R = 0.5\%$, $M = \$600$

c In Questions 2–3 of Exercise 14F, you calculated the future value of each of these three annuities. For each annuity, verify that the present value would grow under compound interest to the future value, that is, that $\text{PV} \times (1 + R)^n = \text{FV}$.

2 Frank-Walter has retired happy and wealthy, and wants to buy a pension of \$120000 per year for 25 years. Interest is 2% p.a. compounded annually.

a How much must he pay into his pension fund as a lump sum?

b Add up all his payments, and find how much of what he collected was interest.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

3 Horst buys a BMW for \$110 000 with a deposit of \$30 000 and a loan of \$80 000. He repays the loan in 4 years, paying 12% p.a. interest.

a **i** If he pays annually, with annual compounding, find his annual instalment.

ii If he pays monthly, with monthly compounding, find his monthly instalment.

b In each case, find how much of what he paid was interest.

4 Annegret suddenly retires, and deposits her life savings of \$640 000 into a superannuation fund that will pay her a regular annual pension for the next 20 years. Interest is 4% p.a. compounded annually.

a Use the present value table to find her annual pension.

b Add up all her payments. How much of what she receives in the 20 years is interest?

5 Heiko wants to buy a car for long-distance travel. He has a deposit of \$22 000, and is prepared to pay it off over 4 years at a maximum of \$600 per month. Some time ago he was slow paying off a loan, so he is charged interest at 24% p.a. compounded monthly.

a Use the present value table to find the maximum that he can borrow.

b Hence find the maximum price he can pay for a car.

c If he pays the maximum, how much of his total payments will be interest?

6 Angela paid annual instalments of \$25 000 into a superannuation fund at the end of each of her 15 years in her role. Interest is 5% p.a. compounded annually.

a Use the future value table to find what she had accumulated when she left the firm.

b Use the present value table to find what lump sum she could have contributed at the start so that it would have grown to the same amount.

c Confirm your answer to part **b** by using your answer to part **a** and the compound interest formula.

DEVELOPMENT

7 Maria has retired and brought \$1 100 000 to the Pension Holdings Company so that they can pay her a regular annual pension. The interest is 3% p.a. compounded annually.

a How many full payments will she receive if she sets her annual pension at:

i \$100 000 **ii** \$80 000 **iii** \$60 000 **iv** \$42 000

b Calculate the interest on the lump sum in the first year, and hence explain what happens if she sets her annual pension at \$33 000.

8 Gordon borrows \$800 000 to set up a shop. The shop generates good profits, and he pays monthly instalments with interest rate 12% p.a. compounded monthly.

a How many full or part instalments will he need to pay if each monthly instalment is:

i \$100 000 **ii** \$50 000 **iii** \$30 000 **iv** \$20 000

b Explain what happens if his instalment is set at \$8000.

9 An annuity has n monthly instalments of \$1000. Examine the present value table, and report what can you say about the monthly interest rate R, compounded monthly, if:

a $n = 12$ and PV $=$ \$11 000,
b $n = 48$ and PV $=$ \$10 000,
c $n = 6$ and PV $=$ \$6000.
d $n = 24$ and PV $=$ \$22 000.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

10 Wilson's friend Prasad was left \$120 000 in a bequest, and his friend has invested it at 4% p.a. interest compounded annually. Wilson decides to save regularly at the end of each year so that after 6 years he will have the same amount.

a **i** What will Prasad have at the end of 6 years?

ii Hence use the future value table to find Wilson's annual instalment.

iii Check the present value of Wilson's annuity, and comment.

b Repeat part **a** if payment period and compounding period are both 3 months.

11 To answer these questions, use the method explained in worked Example 26 and above it.

a Byron borrowed money to set up a business, repaying \$12 000 per month for 4 years at 12% p.a. interest compounded monthly.

i Just over two years later he sold the business and repaid the remainder of his debt. How much did he still have to repay?

ii What was the original debt?

b Veronica used \$900 000 to set up an annual pension scheme that would run for 25 years, with interest 5% p.a. compounded annually.

i What was her annual pension?

ii Just after the 15th payment she closed the scheme and withdrew the funds. How much did she withdraw?

12 Mahmoud's friend Paul bought a car. Paul only told Mahmoud that the deposit was \$22 000 and that the instalments were \$500 per month at 12% p.a. compounded monthly.

a If Paul told Mahmoud that he would need to make more than 40 payments, what could Mahmoud conclude about the minimum cost of the car?

b If Paul told Mahmoud that 20 payments he would owe just over \$11 000, what could Mahmoud conclude that the car cost?

c If Paul told Mahmoud that the car cost between \$75 000 and \$100 000, what could Mahmoud conclude?

13 Jake has set up a pension scheme by paying a lump sum of \$700 000 with interest 6% p.a. compounded annually, and setting the pension so that the annuity runs for 20 years.

a Calculate the annual pension that Jake expects to receive.

b Just after the 5th payment, Jake's daughter falls ill in England, and he withdraws a lump sum of \$60 000 to pay for his trip there, his daughter's evacuation, and subsequent medical expenses.

i What is the new value of the annuity just after the lump sum is withdrawn?

ii If Jake reduces the subsequent pensions so that the scheme runs for the remaining 15 years, how much will his reduced pension be?

iii If instead Jake continues with the same pension, how many further full payments will he receive?

14 Susan sets up a business and borrows \$2 500 000 for equipment and fit-out. She intends to repay it by annual instalments over 15 years, with interest 8% p.a. compounded annually.

a What annual instalment is she expecting to pay?

b Just after her 5th payment, there is a disastrous drought, followed by huge bushfires, followed by a flood, followed by a deadly epidemic, and she has no profit for 2 years. The bank judges her business as sound and gives her a two-year holiday from repayments (but not from interest).

i What is her debt at the end of the 5th year?

ii What is her debt at the end of the 7th year?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

iii If she decides to pay off the loan within the original 15 years, what will her annual repayments be when she resumes payment?

iv If the repayments continue at the same amount, how many further full repayments will she make?

15 [Graphs]

a A loan is being repaid over 50 periods by regular instalments of \$1000 per period at 8% per period, compounded every period. Let L be the amount owing after n periods.

i Find the original loan, and the total interest payments.

ii Draw up a table of values of L as a function of n for $n = 0, 10, 20, 30, 35, 40, 45, 50$, then plot these eight points on a graph.

iii Imagine the points joined by a smooth curve. How would you describe the curve?

iv What is an important lesson about loans from part **i**, the table, and the graph?

b A \$100 000 loan will be paid back in n equal instalments \$$M$ at 6% per period.

i Draw up a table of values of M as a function of n, for $n = 1, 3, 5, 10, 20, 30, 40, 55$, and plot these eight points on a graph.

ii Imagine the points joined by a smooth curve. How would you describe the curve?

iii What is an important lesson about loans from the table and graph?

ENRICHMENT

16 This question constructs a present value spreadsheet for an annuity with \$1 payments for $n = 1$ to $n = 36$ units of time at interest rates R of 6%, 7%, 8%, 9%, 10%, 11%, 12% per unit time.

a **i** Look at the table of present values in this section, but ignore the words at the top.
In the top row of your spreadsheet, enter n, 6%, 7%, 8%, 9%, 10%, 11%, 12%.
In the first column under n, enter $1, 2, 3, \ldots, 36$ (perhaps using Fill Down).

ii Explain why each entry in row $n = 1$ is $1 \div (1 + R)$.

iii Explain why each entry in row $n = 2$ is $(1 + \text{entry above}) \div (1 + R)$.

iv Explain why each entry in the third row is $(1 + \text{entry above}) \div (1 + R)$, and that this formula holds in all rows apart from the row $n = 1$.

b **i** Enter 1/(1+B1) into the cell $n = 1$ and rate $= 6\%$, then Right Click and Format it as a number with four decimal places. Then Fill Right to the end of the row.

ii In the cell $n = 2$ and rate $= 6\%$, enter the formula (1+B2)/(1+B\$1). Again, the dollar sign is to stop the row number 1 changing later when you are filling down.

iii Fill Right from this cell, then Fill Down from this row.

iv Check your entries where they overlap with the table in this section, and make sure that you know why the formulae are working.

c Construct the same table using the formula in Box 10 with $M = 1$. Check agreement.

d Take a copy of the future value table that you created in the last exercise. Then apply the formula $\text{PV} \times (1 + R)^n = \text{FV}$ to create a table of present values in a third way.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 [Graphs] If you look down the columns of the present value table in this section, you will see that each column appears to be converging to a limit. For example, the 12% column appears to be converging to a number a little larger than 8.3.

- **a** **i** With $R = 12\%$, use the present value formula in the last dotpoint of Box 10 to show that this limit exists and to find its value.
  - **ii** Find a formula for the limit in terms of the general interest rate R.
- **b** Explain what this limit means in the context of pension payments paid from an initial lump sum.
- **c** Apply the result in part **a** to the graph in Question **15a**, and describe the asymptotic behaviour of the curve on the left. What does this mean for the loan?
- **d** Do the same for Question **15b**.

18 If I pay off a \$40 000 loan with monthly instalments of \$1000, where interest is 1% per month compounded monthly, what is the final instalment?

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

Answers

Exercise 14F

1 **a** \$17 666
b \$36 573
c \$16 350
d \$1 566 000

2 **a** 13.1808
b \$13 181

3 **a** 73.1059, \$182 765
b 59.2180, \$35 531

4 **a** 50.9312
b \$50 931
c \$2931

5 **a** Theo: \$57 352, Costa: \$61 223
b Costa by \$3871

6 **a** $\$52000 \times 1.13 = \$58760$
b \$55 458
c Very roughly, in part **b** half the capital has been there for less than 6 months.

7 **a** \$39 521
b \$24 160

8 **a** **i** \$45 718
ii \$3697
b **i** \$62 822
ii \$5084

9 **a** Look down the 0.25% column. The interest factor must exceed $70000 \div 3500 = 20$, which happens first after 20 months.
b $120000 \div 3500 \doteqdot 34.29$, 33 months.
c $150000 \div 3500 \doteqdot 42.86$, 41 months.
d $200000 \div 3500 \doteqdot 57.14$, 54 months.

10 **a** Look at the entries along the row $n = 12$. The factor 15.5000 would fit somewhere between 15.0258 for 4% and 15.9171 for 5%, so we can conclude that $4\% < R < 5\%$.
b $0.25\% < R < 0.50\%$
c $5\% < R < 6\%$
d $R > 12\%$

11 **a** \$30 096
b **i** \$19 762, \$20 981
ii \$9868, \$30 850

12 **a** \$59 389
b \$162 858, \$222 248

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

13 12 monthly payments have future value \$6083.20.

a \$3042

b $3134.11 + 6083.20 - 4000 \doteqdot \$5217$

c $5376.00 + 6083.20 \doteqdot \$11\,459$

14 a \$25 778

b The future value after only 8 years is \$255 134.04. Adding compound interest for the next 2 years with no payments gives \$286 668.61 at the end of 10 years.

i Five more years compound interest raise the old funds to \$383 627.27, leaving \$216 372.73 as the final when the annuity resumes, which requires contributions of \$38 384.

ii 7 years compound interest gives \$431 043.60, leaving \$168 956.40 as the final when the annuity resumes, which requires contributions of \$20 129.

15 a \$53 947

b The final value after 4 years is \$34 515.80

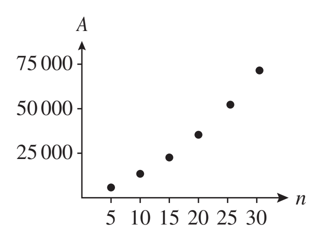
i The amount after 4 years grows to \$35 920.84, and the annuity for 2 years at 2% p.a. has final value \$16 282.80, giving \$52 204.

ii He must acquire another \$18 026.16 over the two years, requiring contributions of \$2214.

iii After 6 years, from part **i**, he has \$52 204, so the next payment of \$2000 will take him past the target of \$53 947, even without interest. Hence he must save for 6 years and 3 months.

16 a i

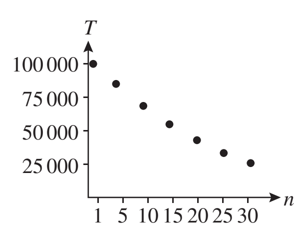
n	A
5	5527
10	12 578
15	21 579
20	33 066
25	47 727
30	66 439



ii Increasing and concave up.

b i

n	T
1	100 000
5	85 228
10	69 029
15	55 244
20	43 704
25	34 197
30	26 482



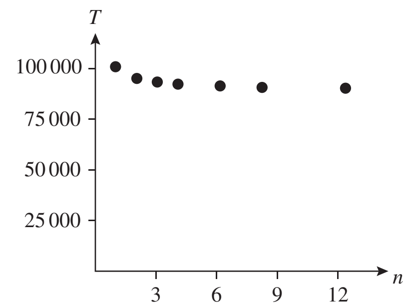
ii Decreasing and concave up. Perhaps the x-axis is an asymptote? (It is. Why?)

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

c i

n	T
1	100 000
2	94 340
3	92 410
4	91 437
6	90 457
8	89 965
12	89 471



ii Decreasing and concave up, with a horizontal asymptote.

17 a Each payment receives compound interest for one year extra, so to obtain A from F, simply multiply by $1 + R$. Alternatively, using GPs,

$$\begin{aligned} A &= M(1+R) + M(1+R)^2 + \cdots + M(1+R)^n \\ &= (1+R)(M + M(1+R)^2 + \cdots + M(1+R)^{n-1}) \\ &= (1+R)F. \end{aligned}$$

18 a ii Each contribution is added at the end of the period, so attracts no interest during that period.

iv Interest on the amount above is added, then another instalment is paid.

Exercise 14G

1 a i 7.3601

ii \$7360

b i 10.6748, \$26 687

ii 45.6897, \$27 414

2 a \$2 342 820 (interest factor 19.5235)

b sum $=$ \$3 000 000, interest $\doteqdot$ \$657 180

3 a i \$26 339 (interest factor 3.0373)

ii \$2107 (interest factor 37.9740)

b i \$25 357

ii \$21 122

4 a \$47 092 (interest factor 13.5903)

b sum $\doteqdot$ \$941 848, interest $\doteqdot$ \$301 848

5 a \$18 404 (interest factor 30.6731)

b \$40 404

c sum of payments $=$ \$28 800, interest $\doteqdot$ \$10 396

6 a \$539 465 (interest factor 21.5786)

b $10.3797 \times 25000 \doteqdot \$259493$ (interest factor 10.3797)

c $10.3797 \times 25000 \times (1.05)^{15} \doteqdot \$539466$

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

7 **a** **i** $1100000 \div 100000 = 11$, 13 payments

ii $1100000 \div 80000 = 13.75$, 17 payments

iii $1100000 \div 60000 \doteqdot 18.33$, 27 payments

iv $1100000 \div 42000 \doteqdot 26.19$, 52 payments

b interest $= 1100000 \times 0.03 = \$33000$.

She would receive the interest each year, and the capital would remain constant.

8 **a** **i** $800000 \div 100000 = 8$, 9 payments

ii 16, 18 payments

iii 26.67, 32 payments

iv 40, 52 payments

b Each month, the monthly interest is \$8000, and the payment is \$8000, so the debt remains at \$800 000.

9 **a** $1\% < R < 2\%$

b $8\% < R < 12\%$

c $R = 0\%$

d $0.5\% < R < 1\%$

10 **a** **i** \$151 838

ii \$22 891

iii \$119 999 (ignore the rounding error).

This confirms the formula $\text{PV} \times (1 + R)^n = \text{FV}$.

b We now have 1% and 24 payments.

i \$152368

ii \$5649

iii \$120000

11 **a** **i** $21.2434 \times 12000 \doteqdot \$254921$

ii $37.9740 \times 12000 \doteqdot \$455688$

b **i** $900000 \div 14.0939 \doteqdot \$63857$

ii $7.7217 \times \text{pension} \doteqdot \$493088$

12 **a** $(32.8347 \times 500) + 22000 \doteqdot \$38417$

b $11000 \div 500 = 22.0000$, which by the present value table means just over another 25 months.

So the term is 45 months, and the cost is $(36.0945 \times 500) + 22000 \doteqdot \$40047$.

c With cost \$75 000, the debt would be \$53 000, which earns interest of \$530 per month, so the debt would increase, not decrease. Hence Paul is big-noting himself.

13 **a** $700000 \div 11.4699 \doteqdot \$61029$

b **i** $(9.7122 \times \text{pension}) - 60000 \doteqdot \$532729$

ii $\text{new value} \div 9.7122 \doteqdot \$54852$

iii $\text{new value} \div \text{old pension} \doteqdot 8.73$, making 12 more full payments (17 altogether).

14 **a** $2500000 \div 8.5595 \doteqdot \$292073$

b **i** $6.7101 \times \text{instalment} \doteqdot 1959840$

ii Multiply by $(1.08)^2$ to give \$2285957 (the bank still charges compound interest)

iii $\text{new debt} \div 5.7466 \doteqdot \$397793$

iv $\text{new debt} \div \text{instalment} \doteqdot 7.83$, giving 12 more full repayments.

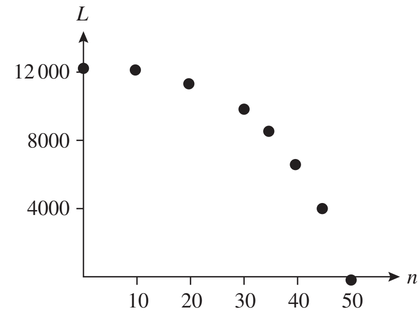
ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

15 a i original loan $= \$12\,234$, total interest $= \$37\,766$

ii

n	L
0	12 234
10	11 925
20	11 258
30	9818
35	8560
40	6710
45	3993
50	0

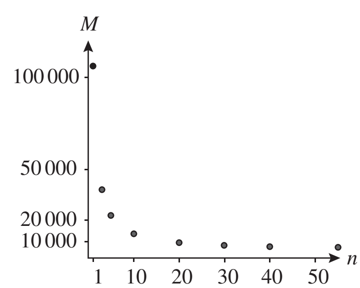


iii Decreasing and concave down. More detail in Question **17** .

iv If you ever take out a long-term loan, from then on pay everything you can possibly afford into extra payments that will reduce the loan in its early stages.

b i

n	M
1	106 000
3	37 411
5	23 739
10	13 587
20	8718
30	7265
40	6646
55	6254



ii Decreasing and concave up, with a horizontal asymptote that seems to be a little above the x-axis. The asymptote is actually at $M = 6000$, which would mean that only the interest is being paid and the loan never diminishes. See Question **17** below for more detail.

iii Do everything possible to reduce the length of a loan.

16 Let P_n be the present value of an annuity that runs for n periods.
Then by the definition of the present value,

$$\begin{aligned} P_1(1+R) &= 1, \text{ proving part ii.} \\ \text{Also } P_2(1+R)^2 &= 1 + (1+R) \\ &= P_1 + (1+R), \\ \text{so } P_2(1+R) &= 1 + P_1, \text{ proving part iii.} \\ \text{Also } P_3(1+R)^3 &= 1 + (1+R) + (1+R)^2 \\ &= P_2 + (1+R)^2, \\ \text{so } P_3(1+R) &= 1 + P_2, \text{ proving part iv.} \end{aligned}$$

Continue for all n to complete the proof of part **iv**.
Interest on the amount above is added, then another instalment is paid.

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022

17 a In the interest factor table, $M = 1$.
Dividing through by $(1 + R)^n$,
$\text{PV factor} = \dfrac{1 - (1 + R)^{-n}}{R}$.
As $n \to \infty$, $(1 + R)^{-n} \to 0$,
so the PV factor has limit $\dfrac{1}{R}$ as $n \to \infty$.
For $R = 12\%$, this gives $\dfrac{1}{0.12} = 8.3333 \ldots$.

b Let the lump sum to produce the \$1 per period pension be \$$P$ — this is the present value interest factor, and the interest earned in the first period is PR.
Now suppose that each pension payment is exactly equal to PR, so that $PR = 1$ and $P = 1/R$. Then the bank will receive in interest exactly what it pays out as pension, so the lump sum will remain at $P - PR + PR = P$ at the end of each period, and for all time into the future. Thus as $n \to \infty$, the column indexed by R has limit $1/R$.

c The PV factor has limit $\dfrac{1}{R} = \dfrac{1}{0.08} = 12.5$, so the line $L = 12\,500$ is an asymptote on the left. This means that if the loan were \$12 500, then the repayments would be all interest, and no capital.

d The PV factor has limit $\dfrac{1}{R} = \dfrac{100}{6}$, corresponding to $M = 100\,000 \div \dfrac{100}{6} = \$6000$. Instalments of \$6000 would be all interest, and no capital.

18 \$339

ISBN 978-1-108-76630-2

Cambridge University Press
Updated July 2022